I, David A Byrne, hereby submit this original work as part of the requirements for the degree of Doctor of Philosophy in Theory.

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The Harmonic Theories of Sigfrid Karg-Elert: Acoustics, Function, Transformation, Perception

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The Harmonic Theories of Sigfrid Karg-Elert: 
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ABSTRACT

This dissertation is the first comprehensive study of the harmonic theories of German composer and music theorist Sigfrid Karg-Elert (1887–1933), whose two major treatises date from the early 1930s. The dissertation’s subtitle highlights the four principal components of Karg-Elert’s theoretical project: its three-dimensional just intonation pitch space, and its acoustic derivation; its expansion of Hugo Riemann’s function theory, encompassing a variety of fifth-, third- and seventh-based chord relationships; its complete and consistent system of common-tone transformations, which operates independently from harmonic function; and finally, its ultimate presentation of the entire system as a model of harmonic perception. The appendix to the dissertation is a complete annotated German-English edition of Karg-Elert’s 1930 treatise *Akustische Ton- Klang- und Funktionsbestimmung* (“Acoustic Determination of Pitch, Chord and Function”), translated here for the first time.

Karg-Elert’s treatises synthesize three strains of thought in late nineteenth-century German theory that were previously somewhat self-contained: a model of pitch and harmonic space derived from the pure intervals of just intonation; major-minor dualism (which Karg-Elert termed *polarity*), shaped especially by the work of Arthur von Oettingen; and the concept of harmonic function, first presented in Riemann’s *Harmony Simplified* of 1893. Building on that scholarly foundation, Karg-Elert introduces several innovative ideas, including the addition of a third dimension to the pitch space, based on the pure or *concordant* seventh (4:7); a network of direct major and minor third transformations; and transformations involving the concordant seventh, which enable direct connections among dominant and half-diminished seventh chords. In total, Karg-Elert proposes 23 transformations among triads and seventh chords, all of which retain at least one common tone (conceived as a unique location in the three-dimensional just
intonation space). In many specific points of language and notation, his common-tone transformation system points forward to ‘neo-Riemannian’ theories of the 1980s and the 1990s, including seventh-chord transformations presented by Edward Gollin and Adrian Childs.

Though Karg-Elert generates and explains his pitch space using acoustical and mathematical principles, he ultimately reveals that the space is a model of perception, and that we continue to understand harmonic relations in the pure intervals of just intonation even when pitches are sounded in a different tuning system, such as 12-tone equal temperament. In addition, Karg-Elert states that the musical imagination’s ability to categorize pitches and harmonies according to their unique locations in the just-intonation space is essentially unlimited. Perhaps Karg-Elert’s most fundamental contribution lies in his belief that every pitch, chord and key can exist in multiple conceptual states. By fully embracing the infinite expanse of his three-dimensional pitch universe, Karg-Elert proposes that our understanding of every chord and key is shaped above all by the harmonic paths that connect them, even when those paths travel far from a centralized tonic.
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Above all, I thank my mother and my late father, for their unwavering love and encouragement throughout my life, not least during my many years as a graduate student. I dedicate this project to them, as it would have been impossible without their support.
On quotations and translations from German texts

This dissertation includes many quotations from German-language texts, by Karg-Elert himself, and by a number of other authors. The English translations are from three sources: by the author of this dissertation (David Byrne), by Harold Fabrikant (Australia), and by others that are cited in the bibliography.

1. All quotations from the following works are translated entirely by David Byrne:

   Hartmann 1997–98
   Hasse 1933
   Imig 1970
   Karg-Elert 1919
   Karg-Elert 1920–21 (*Die Grundlagen der Musiktheorie*)
   Karg-Elert 1921b (*Neue Bahnen der Harmonik und ihrer Lehre*)
   Karg-Elert 1930 (*Akustische Ton- Klang- und Funktionsbestimmung*)
   Oettingen 1913
   Reuter 1928
   Riemann 1872
   Riemann 1880

2. The following works by Karg-Elert have been translated into English by Harold Fabrikant, with assistance by Staffan Thuringer in the case of *Harmonologik*. This dissertation includes quotations from these translations by Fabrikant and Thuringer:

   Karg-Elert, Sceats, and Fabrikant 2000 (Karg-Elert correspondence with Godfrey Sceats)


   Karg-Elert and Fabrikant 2010 (Karg-Elert correspondence with Australian friends)

   In all of these works, Harold Fabrikant states that the English translation is “not subject to copyright. On the condition that use of, or reference to, the translation is suitably acknowledged by reference to the title of this work, and to the translators, we welcome the use of this material.” I hereby acknowledge my use of Fabrikant’s translations.

   In the case of *Harmonologik* (Karg-Elert 1931), some of the translations from Karg-Elert, Fabrikant and Thuringer 2007 have been modified by David Byrne, based on the original.

3. Other English translations from German texts are cited in the bibliography, and in many cases in the footnotes as well, with full detail on the translators.
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Chapter 1

Introduction

In the early twentieth century, the discipline of music theory (especially as practiced in Germany) experienced a change of perspective, one that reflected a fundamental shift of focus in scholarship and philosophy at large: from the external to the internal, and from the physical to the psychological. The late nineteenth-century emphasis on the sonic phenomena of music, exemplified in the acoustical writings of scholars such as Hermann von Helmholtz and Arthur von Oettingen, began to be supplanted by an understanding of music as a construct of the mind, first notably explored in the Tonpsychologie of Carl Stumpf. While much of the work of Hugo Riemann is still rooted in the physical proportions of the harmonic series, his late writings (especially his Lehre von den Tonvorstellungen of 1914–15) exhibit a new orientation toward the perception of music, rather than its sonic material. The interdependence of the objective and experiential in the understanding of music is central to the writings of Sigfrid Karg-Elert (1877–1933), a Leipzig-born composer and music theorist whose extensive writings on harmony and tonality are now little known, especially outside of Germany. This dissertation is a comprehensive study of Karg-Elert’s harmonic theories, considered both in their own right, and in relation to those of other scholars, both historical and recent. While Karg-Elert’s treatises present many ideas that are innovative and interesting in themselves, this study proposes that his work is especially relevant to modern scholars as a link between the dualist and functional theories of Oettingen and Riemann on the one hand, and recent neo-Riemannian and transformational approaches on the other.

1 Citations for these sources (and all others mentioned in this introduction) will be provided in later chapters, in which the sources are discussed frequently and in detail.
Though Karg-Elert stated that his engagement with music theory began as early as 1902, his two main treatises were only published near the end of his life: *Akustische Ton-Klang- und Funktionsbestimmung* of 1930 (“Acoustic Determination of Pitch, Chord and Function,” hereafter called *Akustische*), and *Polaristische Klang- und Tonalitätslehre* of 1931 (“Polaristic Theory of Harmony and Tonality” – subtitled *Harmonologik*). While there is some overlap of content between these two texts, their mode of presentation is quite different: *Akustische* is primarily theoretical and abstract, while *Harmonologik* is primarily analytical, with hundreds of examples from a wide range of repertoire. Both treatises synthesize and extend several important strains in late nineteenth-century German theory: detailed acoustical calculations of pitch and interval in just intonation in the manner of Helmholtz and Oettingen, the notion of major-minor dualism or polarity (influenced by both Oettingen and Riemann), and the harmonic function theory of Hugo Riemann. Rather than covering Karg-Elert’s treatises individually in a chronological manner, the dissertation is organized into several broad topics, including those listed in its subtitle: acoustics, function, transformation and perception. Each topic is the focus of an individual chapter, though there is significant interaction and overlap between the chapters; in particular, the material in each of Chapters 3, 4, 5 and 6 (respectively on acoustics, function, transformation and modulation) prepares for that in the following chapter.

Chapter 2 sets the stage for the detailed examination of Karg-Elert’s theories in subsequent chapters. After a brief biography, the chapter describes the composer’s increasing occupation with music theory after his appointment to the faculty of the Leipzig Conservatory in 1919. An overview of his theoretical writings follows, with some attention devoted to his plans for a three-part comprehensive work on harmony and many other subjects, entitled *Die Grundlagen der Musiktheorie* (“The Foundations of Music Theory”). As the chapter describes,
those plans were never fully realized, and some of the material that was to be included in Parts Two and Three of *Grundlagen* did not appear even in Karg-Elert’s final two treatises. Chapter 2 continues with a descriptive survey of the existing literature on Karg-Elert’s theories, almost all of which is in German. Chapter 2 finally considers the intellectual orientation of his theoretical project, especially as it is revealed in *Neue Bahnen der Harmonik und ihrer Lehre* (“New Paths to Harmony and its Theory”), a pamphlet published in 1921 as a prospectus and preface for the *Grundlagen* series. From several different angles, *Neue Bahnen* contrasts the notions of *Wesen* (essence, meaning) and *Erscheinung* (appearance, manifestation), a dichotomy that is central to Karg-Elert’s understanding of pitch, chord, function and perception.

Chapter 3 deals with the acoustical and mathematical derivation of Karg-Elert’s three-dimensional just intonation pitch space, from the proportions of the first seven partials of the harmonic series. After describing Karg-Elert’s ideal pitch frequency for the theoretical study of music, the chapter discusses his model of major-minor dualism (called *polarity*), which is based entirely on the mathematical principles of harmonic and arithmetic division, and thus does not depend on the physical existence of either overtones or undertones. The central sections of Chapter 3 examine in close detail the three dimensions of Karg-Elert’s pitch space: the Pythagorean (derived from pure 2:3 fifths), Didymean (from pure 4:5 major thirds), and concordant (from pure 4:7 sevenths). The inclusion of the seventh-based axis is unique to Karg-Elert, and represents one of the most innovative features of his work, one that generates new and analytically useful transformations involving dominant- and half-diminished seventh chords. His pitch space contains a potentially infinite number of unique pitches, stretching endlessly outward in three directions. To specify the location of each pitch in the space, Karg-Elert establishes middle C as a fixed center, with an ideal frequency or value of zero. He then employs acoustic
symbols to denote distances away from the central C (calculated in fifths, thirds and sevenths), as well as numerical intervals (measured in millioctaves) to distinguish between sonically similar pitches of the same name (called metharmonics) or of different names (called enharmonics). Though Karg-Elert never used the Tonnetz to illustrate locations or paths in pitch space, it is frequently employed in that capacity here, as it will be very familiar for many readers. Chapter 3 concludes with a description of the Ursprungslagen or source positions, a concept which perhaps surprisingly reaffirms that fifth-related root motions are the fundamental basis of tonality, and thus relates directly to the theories of Rameau, and to the Stufenlehre tradition.

Chapter 4 examines Karg-Elert’s adoption and revision of Riemann’s theory of harmonic function, first proposed in Riemann’s Vereinfachte Harmonielehre (“Harmony Simplified”) of 1893. It begins by describing the perceptual basis of major-minor polarity, which is rooted in Karg-Elert’s observation that any harmonic interval can be understood as an incomplete major triad or minor triad, to be mentally completed by the listener either upward (major) or downward (minor). As in Riemann, his three basic functional categories are rooted on the tonic and its two surrounding fifths, called dominant and contradominant by Karg-Elert. However, unlike that of Riemann, his model of harmonic function is fully and consistently dualistic, placing the dominant at the fifth above the tonic in the major mode, but at the fifth below tonic in minor. The three functions are exemplified by the triads on the tonic and its upper and lower fifths (collectively called the Prinzipale or principal chords); section 4.4 describes how the positive and negative energies inherent in the three triads combine to form the fundamental cadence. Chapter 4 also discusses Karg-Elert’s presentation of the 4:5:6:7 concordant seventh chord as an integrated harmony derived from a single prime, in contrast to Riemann, who explained the dominant and half-diminished sevenths as combinations of pitches derived from two different
primes (and thus representing two different functions). The remaining sections of Chapter 4 discuss the simplest alterations of the principal triads, including the ultraforms (secondary dominants and contrants), the diatonic substitutes (Riemann’s *Parallel* and *Leittonwechsel*), and the mode-shifting variants.

Chapter 5 is the longest in the dissertation, on the subject of harmonic transformation in Karg-Elert. It begins with a brief description of the “transformational attitude,” as defined in David Lewin’s *Generalized Musical Intervals and Transformations*. Riemann’s 1880 system of contextual triadic transpositions and inversions (called the *Schritte* and *Wechsel*) is studied, as an example of a transformational system (or what Lewin termed a *Generalized Interval System* or GIS). The *Schritte* and *Wechsel* are defined by the interval between the triadic roots, essentially under 12-tone equal temperament and enharmonic equivalence; as a result, they cannot be considered unique operations in a just intonation space, as they do not specify anything about the acoustic derivation of each triad’s root. In contrast, Karg-Elert’s system of harmonic transformation is based entirely in the retention of one or two acoustic common tones – in other words, each transformation retains at least one of the first chord’s pitches, in its original location. Therefore, unlike the *Schritte* and *Wechsel*, each of Karg-Elert’s transformations represents a unique and specific path in pitch space. There are 23 in total: 13 transpositions and 10 inversions, some of which involve the septimal (seventh-based) plane of pitch space. Sections 5.2 to 5.4 examine each of the 23 transformations in detail, often referring to examples from the two treatises, in order to illustrate how a single transformation can appear in multiple functional contexts. Karg-Elert’s fifth-based (section 5.2) and third-based (section 5.3) transformations are compared with analogous operations in the writings of Lewin, Brian Hyer, Richard Cohn and David Kopp; the discussion highlights how Karg-Elert’s use of language and notation often
directly prefigures neo-Riemannian concepts and terminology. Section 5.4 presents Karg-Elert’s seventh-based transformations, which exchange fifth- or third-derived pitches with those on the septimal plane, or vice versa. Some are of more theoretical interest than practical use, as they metharmonically duplicate simpler fifth- and third-based transformations. However, a particular seventh-based inversion called the \textit{Septgegenklang} or “counter-seventh chord” is revealed in Karg-Elert’s treatises to be of great potential for the analysis of passages involving dominant- and half-diminished seventh chords. The section compares his seventh-based transformations with those proposed in recent writings by Adrian Childs, Richard Bass and Edward Gollin; though all three of those authors assume equal temperament and enharmonic equivalence, their transformations involving dominant and half-diminished sevenths are quite similar to those of Karg-Elert (strikingly so in the case of Gollin). Chapter 5 concludes with a discussion of several non-common-tone harmonic relationships, including an important category called the \textit{Kollettivwechselklänge} or “collective-change chords,” which are equivalent to the hexatonic and octatonic poles of neo-Riemannian theory.

Each of Chapters 3 through 5 include numerous analytical examples with functional analysis, drawn from both the abstract progressions in \textit{Akustische} and the repertoire passages in \textit{Harmonologik}. All of those examples are short, each chosen to illustrate a specific harmonic transformation or functional interpretation. Chapter 6 examines Karg-Elert’s analyses of four longer passages of music from the nineteenth century, including two complete pieces. The four passages illustrate in different ways the concept of comma-free and comma-differing modulation, which is the primary subject matter of Chapter 6. Comma-free modulation involves keys whose initial and final tonics are both located in the \textit{Ursprungslagen} (source position) chain of fifths at the center of the pitch space; for Karg-Elert, only such comma-free tonal motions are
to be counted as true modulations. In contrast, comma-differing modulations involve motions that depart from the central fifth chain, and are thus not real modulations, but simply tonal ‘shifts’. The four analyses in Chapter 6 demonstrate how Karg-Elert’s functional notation exactly indicates comma-free or comma-differing harmonic paths in pitch space, which can be plotted on a Tonnetz. The analyses reflect how Karg-Elert was sometimes willing to accept metharmonic or enharmonic juxtapositions in order to ensure tonal closure, as in the development section of Beethoven’s *Waldstein* Sonata. In contrast, he often chooses to disregard a composer’s enharmonic notation (itself intended to provide tonal unity), when it contradicts a harmonic path defined by local common-tone connections. This is the case in both Liszt’s song *Wieder möcht’ ich dir begegnen* and Schumann’s Novelette in F major (op. 21 no. 1), in which repeated harmonic transformations drive the music into remote regions of pitch space, properly spelled (in just intonation) with double and triple accidentals. While the Liszt song is understood in the end to be tonally closed (and thus comma-free), the Schumann piece is tonally open (comma-differing), never returning to its original location in pitch space, despite its notated ending in F major. The final analysis in Chapter 6 examines the *Schlafakkorde* from Wagner’s *Die Walküre*, from two perspectives: as a middleground progression between key areas, and as a foreground progression between individual chords. The key areas outline a sequence of direct major-third transformations, much as in Brian Hyer’s 1995 interpretation of the same passage. On the other hand, the local chord progression (indicated by the function labels) suggests that the key areas are in fact related by quadruple fifths rather than major thirds, and thus comprise a comma-free modulation. Chapter 6 concludes with a brief discussion of writings by Riemann, Donald Tovey (and Eric Wen), and Daniel Harrison, which consider tonal motion from a just-intonation perspective, and thus arrive at analytical conclusions similar to those of Karg-Elert.
Even though his pitch space is explicitly derived from pure intervallic ratios, Karg-Elert was not interested in promoting just intonation in musical performance; indeed, his treatises make clear that 12-tone equal temperament is not only a practical reality, but also a necessary compromise that enables the performance of chromatic music. Instead, he regards just intonation as the basis of musical perception, stating that we understand pitch and harmonic relationships according to his model of pitch space, even when actual sounding pitches are organized in equal temperament, or some approximation of it. This is the primary topic in Chapter 7, which compares Karg-Elert’s statements on musical perception with those expressed in Riemann’s 1914–15 study *Die Lehre von denTonvorstellungen* (“On the Imagination of Tone”). Karg-Elert and Riemann share one fundamental belief: that the conceptual and perceptual basis of our tonal system is rooted in the pure intervals of just intonation. Where they sharply differ is on the extent to which the musical ear or mind will seek to distinguish between pitches of similar frequency, but of different derivation. Karg-Elert suggests that we can differentiate and categorize at least 275 distinct pitch values, which will suffice for understanding most harmonic relationships likely to occur in a piece of tonal music. In contrast, Riemann proposes that we actively seek the most ‘economic’ way of conceiving harmonic relations, and that our perception is mostly limited to a cluster of 22 tonics grouped around a primary key or center.

The conclusion (Chapter 8) considers the validity and efficacy of Karg-Elert’s system for the modern analyst and musician. It begins by revisiting four ‘critical considerations’ about his theories, raised by his student Paul Schenk in 1966. It then highlights what is likely the most fundamental and important contribution of Karg-Elert’s work: its acceptance and comprehensive exploration of the idea that every pitch, chord and key can exist in more than one conceptual state or location, defined by the harmonic paths that connect them. The dissertation finally
describes some aspects of Karg-Elert’s work that remain to be explored in future research, including the possibility of testing some of his ideas in an experimental setting.

The appendix to this dissertation is a complete German-English edition of Karg-Elert’s *Akustische Ton- Klang- und Funktionsbestimmung* of 1930, presented here in translation for the first time. The edition divides each of Karg-Elert’s original chapters into smaller sections, and provides headings which specify the subject matter in each section. In this edition, left and right pages are paired with each other. The left-side pages contain Karg-Elert’s original text, diagrams and musical examples, interspersed with English translations of the text and terminology. The right-side pages contain commentary on the text, explanation of the examples and diagrams, discussion of the sources of Karg-Elert’s ideas, connections with recent theories, and other information intended to help the reader understand the treatise. Tonnetze for many of Karg-Elert’s musical examples are provided at the end of many chapters, in order to illustrate the treatise’s acoustic data and functional harmonic analyses in a way that is visually immediate.

Many features of Karg-Elert’s work diverge greatly from modern common practice, and will likely continue to preclude its wider acceptance: his just intonation model of harmonic relations and tonality, his strict polar dualism, and his complex functional notation. In spite of those challenges, this dissertation argues that his work should be better known, in particular because it links the acoustics and dualism of Oettingen and the function theory of Riemann on one hand, with the transformational and neo-Riemannian theories of the 1980s and 1990s on the other. It is hoped that this dissertation and the accompanying English edition of *Akustische* can serve to introduce Karg-Elert’s harmonic theories to new audiences.

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2 Karg-Elert’s other major treatise, the *Harmonologik* of 1931, was translated into English in 2007 (Karg-Elert, Fabrikant and Thuringer 2007).
CHAPTER 2

Sigfrid Karg-Elert (1877–1933): his career and his theoretical writings

This chapter lays the foundation for the detailed examination of Karg-Elert’s theories in the rest of the dissertation. It begins with a brief biography, describing the composer’s increasing occupation with music theory after his appointment to the faculty of the Leipzig Conservatory in 1919. The largest part of the chapter (section 2.2) is an overview of his theoretical writings, with some attention devoted to his plans for a three-part comprehensive work on harmony and many other subjects, entitled *Die Grundlagen der Musiktheorie* (“The Foundations of Music Theory”). As the section describes, those plans were never fully realized, and some of the material that was to be included in Parts Two and Three of *Grundlagen* did not appear even in Karg-Elert’s final two treatises. Chapter 2 continues with a descriptive survey of the existing literature on Karg-Elert’s theories, almost all of which is in German. The final section considers the intellectual orientation of his theoretical project, especially as it revealed in *Neue Bahnen der Harmonik und ihrer Lehre* (“New Paths to Harmony and its Theory”), a brief but important pamphlet published in 1921 as a prospectus and preface for the *Grundlagen* series. From several different angles, *Neue Bahnen* contrasts the notions of *Wesen* (essence, meaning) and *Erscheinung* (appearance, manifestation), a dichotomy that is central to Karg-Elert’s understanding of pitch, chord, function and perception, and thus to his theories as a whole.
2.1. A brief biography

Karg-Elert’s work in music theory has been the least familiar aspect of his career, both during his life and since his death. Today, Karg-Elert is best known as a composer, primarily of music for the organ (especially the celebrated set of 66 Chorale Improvisations, op. 65), and also of notable repertoire for the flute (including the 30 Caprices, op. 107). In the concert venues of Leipzig during the first two decades of the century, Karg-Elert was frequently heard as pianist, or as a performer on the Kunstharmonium (concert harmonium), for which he wrote and arranged a large amount of music. From 1919, he was also a respected instructor of music theory and composition at the Leipzig Conservatory of Music. In spite of those varied activities, in later life Karg-Elert stated that his work on harmony and polarity would be the culmination of his career:

This work [is] my life’s purpose – my true mission … since 1902 I have worked incessantly, day and night, on this gigantic work which determines completely new spheres on each page and which is without doubt the Precepts of Harmony of the future.1

While some details of Karg-Elert’s biography have been a matter of conjecture and disagreement, the broad outlines of his life and career can be described with confidence.2 He was born as Siegfried Theodor Karg on November 21, 1877 in Oberndorf am Neckar (Baden-Württemberg, in south-western Germany). His father Johann Baptist Karg (born 1823) was an itinerant Catholic newspaper editor and publisher of Bavarian stock, while his mother Marie Friederike Ehlert (born 1839) was from a devout Lutheran north German family. He later attributed his own turbulent character to his parents’ diverse faiths and personalities: “every day I

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1 Letter from Karg-Elert to Grete Bellmont, July 12 1926 (Karg-Elert 2010, 73).
2 The most thorough biographical material on Karg-Elert is contained in Schenk 1927; Gerlach and Kaupenjohann 1984; Hartmann 2002; and Conley 2014. Much of biographical value is also contained in the collections of Karg-Elert’s correspondence, translated into English by Harold Fabrikant (Karg-Elert 2000, 2002 and 2010). This brief biographical essay draws on all of those sources.
struggle with my own double nature, the inheritance of my polar-opposite parents [polar gerichteten Eltern] – whether in religious, political or artistic matters.”

Johann Karg moved his family in 1883 to Leipzig, which remained Karg-Elert’s home for most of his life. As a boy, Karg-Elert joined the choir at the Johannis Kirche, where he wrote several choral pieces under the tutelage of the cantor, at the age of 12 or 13; he was also given a piano, though the family was not able to afford lessons. An abortive period of study at a teacher’s training school (where he learned several woodwind instruments) was followed by work as a freelance musician. In 1896, Karg-Elert was admitted as a scholarship student to the Leipzig Conservatory of Music, where his teachers included Carl Reinecke (1824–1910, composition), Salomon Jadassohn (1831–1902, music theory) and Alfred Reisenauer (1863–1907, piano). In 1900, he performed his own Piano Concerto (op. 6, now lost) with the conservatory orchestra. He extended his course for one year to study piano with Robert Teichmüller (1863–1939). Upon his graduation in 1901, Karg-Elert’s teachers were effusive in describing his abilities: notably, his final student record (Lehrer-Zeugniss) stated that “he is incredibly talented musically, and he has completed the entire theory program with the greatest success.”

In 1902, Karg-Elert accepted a post as piano teacher at the Magdeburg Conservatory; it seems that he first adopted the name “Sigfrid Karg-Ehlert” (still including the “h” which he soon dropped) while in Magdeburg, possibly to distinguish himself from a prominent Jewish businessman named Aron Karger. He moved back to Leipzig in 1903; an encounter with Norwegian composer Edvard Grieg encouraged him to abandon the life of a concert pianist, and

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3 Quoted in Schenk 1927, 6.
5 Kähne 1994, 45.
to concentrate on composition. It was during this time that Karg-Elert first discovered the Kunstharmonium, and he began to publish music for that instrument, under a contract with publisher and harmonium promoter Carl Simon. Karg-Elert soon gained fame as a harmonium performer and teacher, and he published several major instructional texts on the instrument. In 1907, Max Reger (who was at that time teaching at the Leipzig Conservatory) heard some of Karg-Elert’s harmonium works, and encouraged him to write for the organ; further inspiration in that regard came with the arrival in Leipzig of organ virtuoso Karl Straube, celebrated for his interpretations of Bach and Reger. Though Karg-Elert never became a truly proficient organist, as composer he began to dedicate himself increasingly to that instrument. He wrote a prodigious amount of music in the decade before World War I, much of it for the organ, harmonium and piano, but also a large number of songs, and several major chamber works. His compositional style moved in an increasingly modern and “exotic” direction, influenced by Debussy, Scriabin and even the early atonal works of Schoenberg.

In 1915, Karg-Elert enlisted as a soldier in the 107th Infantry Regiment; he was posted in the regiment’s excellent military orchestra, in which he mostly played the oboe, and where his colleagues included players from the famed Gewandhaus Orchestra. Notably, he befriended the flute soloist Carl Bartuzat, whose playing inspired Karg-Elert to write several significant works featuring the flute, in a strongly French-influenced style. However, his experiences playing Beethoven, Schubert, Mendelssohn and Brahms in the orchestra precipitated something of a compositional crisis: he recalled that in a fit of insecurity over his recent modernist style,

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6 Wollinger 1991.
mountains of mad piano pieces, confused orchestral sphinxes, abysmally cramped pseudo-songs and dangerously confused organ experiments wandered one day into the fire (almost 20 works!), and then I began again in C major and prayed to the muse of melody.  

Until about 1927, Karg-Elert’s compositional output decreased significantly, though this can be largely attributed to changes in his professional life, and to his occupation with music theory.

In 1919, Karg-Elert was hired by the Leipzig Conservatory as an instructor of music theory and composition, taking a position formerly held by Max Reger. Almost immediately, he began to teach his polaristic theories of harmony to his students, and his first theoretical treatise was published in 1920 and 1921 (Die Grundlagen der Musiktheorie, in two volumes). Karg-Elert’s tenure at the Conservatory was marked by conflict. His music and aesthetic outlook continued to reflect a wide range of influences from various countries, and he clashed with his colleagues Karl Straube (organ) and Hermann Grabner (theory), who espoused increasingly nationalist and conservative views. To Karg-Elert’s great distress, German organists largely ignored his organ music, as its often extravagant and colorful character was very much at odds with the ascendant Bach-inspired and German-centric Orgelbewegung (“organ movement”), led by Karl Straube and his students. In 1926, Karg-Elert complained about the situation:

Because quite a few of my works bear French or English titles, I must be ‘un-German’ and thus boycotted. Oh, how often have my friendship and sympathy towards England, France and Italy injured me, so that I am immediately denounced as Jew, traitor or Bolshevik … It is wicked.

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7 Quoted in Schenk 1927, 15.
8 Schulze 1933 is a contemporary account of the Orgelbewegung and its nationalist aims.
9 Letter from Karg-Elert to Godfrey Sceats, July 12–13 1926 (Karg-Elert 2000, 2).
However, at the time that Karg-Elert’s organ music was largely ignored in Germany, it became increasingly popular in England, the United States and Australia. As a result, in the later 1920s he wrote a number of new organ works, including his most substantial; much of that music was published in England or the United States.

For much of the 1920s, Karg-Elert’s principal occupation was the writing of his polaristic theories of harmony, which had a difficult publication history (to be discussed below), including a protracted and expensive court proceeding with his publisher. His two major treatises finally reached the market in 1930 and 1931. Karg-Elert’s last years were very eventful. In 1930, he was invited to London for a ten-concert festival of his organ music, played by the city’s leading organists. This festival brought his organ music to even wider attention in the English-speaking world, and he was invited by American concert promoters to undertake a tour of the United States and Canada as organist and composer. With his theoretical works finally published, Karg-Elert decided to accept the tour invitation, even though his abilities as organist were not those of a concert soloist. Accompanied by his daughter Katharina, he played twenty-two concerts across North America between January and March of 1932. The concerts were a popular success, though critical notices were very mixed, praising Karg-Elert’s music but often not his playing. The stress of the tour was highly detrimental to his health (already weakened by diabetes and other conditions), and though the Leipzig Conservatory promoted him to the rank of Professor upon his return home, he was unable to fully resume his duties. Karg-Elert died in his home on April 9, 1933, in his fifty-sixth year.

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10 Karg-Elert’s letters from the North American tour have been collected and translated in Karg-Elert 2002.
2.2. Karg-Elert’s theoretical writings: an overview and historical account

Though Karg-Elert stated that his active engagement with music theory began as early as 1902, his published work in the field dates almost entirely from the period beginning in 1919, when he was appointed as instructor of music theory and composition at the Leipzig Conservatory. There are three major treatises:


In this dissertation, the above three treatises will be referred to as *Grundlagen, Akustische* and *Harmonologik* respectively; their creation and dissemination histories are examined in this section. In addition to the main treatises, the following three smaller publications (listed in chronological order of publication) also discuss composition, acoustics and related matters:

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11 *Harmonologik*, ii and 63. As mentioned above, Karg-Elert acquired his first professional position (at the Magdeburg Conservatory) in 1902; according to his account, his first exploration of music theory dates from that period.

12 His appointment specified “Tonsatzlehre und musikalische Analyse” (essentially theory and analysis) and “Ausbildung des Klangbewußtseins (Musikdiktat und Gehörübungen)” (aural musicianship). He also taught a class in composition (Schinköth 1997/98, 21). On the historical and political background to the term *Tonsatz*, see Holtmeier 2004, 246–247.


*Die Ästhetik des Registrierens* begins with a brief and conventional description of basic forms, beginning with one-part to four-part song forms, before proceeding to dance forms (essentially compound song forms), rondo, sonata-allegro and fugue. This is followed by a discussion (with examples) of how the harmonium player can use registration (different stops and combinations on the instrument, creating different timbres) to help reinforce and define musical form. This short essay remained Karg-Elert’s only published work on the topic of form, aside from occasional passing comments on repertoire examples in *Harmonologik.*

*Die Grundlagen der Akustik / Eine Plauderei* is quite unusual: it is a detailed mathematical presentation of acoustic phenomena, published in a journal for harmonium enthusiasts, in the format of a dialogue between a benevolent teacher (Magister Dux) and his bright but skeptical student (Comes). Without concern for the generally amateur readership of *Der Harmoniumfreund,* Karg-Elert’s dialogue provides a very thorough discussion of frequency and wavelength, cent calculation, intervallic ratios, timbral difference due to strength.

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14 The subtitle of the journal was *Zeitschrift für Hausmusik und Kunst* (“Journal of Music and Art in the Home”). It “catered to the owners of home organs” (Applegate 2017, 268).
of overtones, and even the distinction between fifth-, third- and seventh-derived pitches and intervals (including the Pythagorean, syntonic and septimal commas). In many ways, Die Grundlagen der Akustik resembles a sketch for the material to be presented much more formally in Akustische.

Die logische Entwicklung der modernen Figuration\textsuperscript{15} is a compendium of melodic elaboration, and can be viewed as an early twentieth-century counterpart to instrumental methods of the eighteenth century, such as Johann Joachim Quantz’s Versuch einer Anweisung die Flöte traversiere zu spielen (published by Woss of Berlin in 1752).\textsuperscript{16} Like Quantz’s method, Karg-Elert’s appendix to his 30 Caprices demonstrates how a flutist (or any other musician or composer) can vary and embellish melodic-harmonic patterns. The first examples are simple and largely diatonic, but they soon become highly chromatic. By the end of the appendix, the flutist is shown how to develop patterns typical in early twentieth-century music, involving whole-tone or quartal sonorities. The last section presents elaborations of stacked fourths, widespread in the music of Karg-Elert’s time. He first carefully notes the difference between true quartal chords and those resulting from suspensions (Figure 2.2.1), then presents several rhythmic and sequential variations using quartal harmonies. The final example (Figure 2.2.2) combines

\textsuperscript{15} Appendix to Karg-Elert 1919.

\textsuperscript{16} Quantz’s 1752 Versuch is perhaps the most familiar historical text of its kind for the flute, and thus may have influenced the writing of Karg-Elert’s 30 Caprices (Karg-Elert 1919) and its appendix. However, the Quantz belongs to a venerable line of work on melodic figuration and ornamentation dating back to the sixteenth century, represented by treatises such as the Trattado de glosas (1553) for viola da gamba by Spanish-Neapolitan musician Diego Ortiz, and the Ricercate, passaggi et cadentie per potersi essercitar nel diminuir terminatamente con ogni sorte d’istrumento; et anco diversi passaggi per la semplice voce (Ricercars, passages and cadences to facilitate practicing accomplished diminutions on all kinds of instruments; and also various passages for just the voice) by Giovanni Bassano (published in Venice in 1585). Thanks are due to David Carson Berry for drawing attention to this historical background (March 2018).
altered quartal chords (i.e. set-class (016), very common in Schoenberg, Bartók and many others) with whole-tone and regular quartal sonorities:

Figure 2.2.1. Chords of the fourth, in 3 to 5 parts (Karg-Elert 1919, 35)

Figure 2.2.2. Elaboration combining regular and altered fourth-chords with whole-tone chords (Karg-Elert 1919, 35).

**Die logische Entwicklung der modernen Figuration** is Karg-Elert’s only work dedicated to the study of melody. It is likely that similar examples of melodic elaboration would have been included in the projected Part III of *Grundlagen*, which is discussed in the following pages.
a. Karg-Elert’s first treatise: *Die Grundlagen der Musiktheorie* (1920-21)

Published shortly after Karg-Elert began teaching theory and composition at the Leipzig Conservatory in 1919, it seems likely that *Die Grundlagen der Musiktheorie* was written with two aims: to help establish his reputation as a scholar and teacher, and for use as classroom teaching material that would reflect his own theoretical views and priorities. The front matter of Karg-Elert’s *Grundlagen* boldly announces an ambitious work in three volumes (Figure 2.2.3):

**Heft I:**
Elementareinführung: Noten-, Schlüssel-, Intervallen- und Skalenlehre

**Heft II:**
Akkord-, Salz- und Generalbaßlehre [Monismus]
Die Grundzüge der Akustik und Tonpsychologie
Das Phänomen der harmonischen Polarität und die daraus resultierende „Moderne Harmonielehre“

**Heft III:**
Rhythmik und Metrik
Melodik und Figuration
Einführung in die Polyphonie. Grundriß der Formen

Part I:
Guide to rudiments: study of notes, keys, intervals and scales

Part II:
Study of chords, voice leading and figured bass [monism]
Basic features of acoustics and music psychology
The phenomenon of harmonic polarity, and the resultant “modern theory of harmony”

Part III:
Rhythm and meter
Melody and figuration

**Figure 2.2.3.** *Die Grundlagen der Musiktheorie*, plan for three volumes (*Grundlagen I*, p. II)
Karg-Elert’s planned three-volume curriculum begins with the familiar and the traditional: a survey of rudiments in Part I, followed by a study of basic harmony and voice leading in the conventional manner (here called “monism,” borrowing Georg Capellen’s umbrella term for the opposite of harmonic dualism)\(^{17}\) in the beginning of Part II. He then proposes to introduce the student to elements of acoustics, as well as *Tonpsychologie* (surely a nod to Carl Stumpf’s two-volume work on music cognition, of the same title).\(^{18}\) All of that is to prepare for Karg-Elert’s “modern theory of harmony,” and the concept of harmonic polarity. In the event, the above plan was never fully realized within the *Grundlagen* series, or even in Karg-Elert’s final two treatises. Though some of what is listed under Part II was included in the second (and last) published volume of *Grundlagen*, the majority of it only appeared in *Akustische* and *Harmonologik*. The material listed under Part III was never published by Karg-Elert in any form, and may never have been written at all (though the appendix to the *30 Caprices* did provide a thorough demonstration of modern melodic figuration, as discussed above).

Part I of *Grundlagen* opens with a preface that describes the pedagogical plan and rationale for the entire project. Karg-Elert defines his overarching goal, which is nothing less than a complete explanation of harmony, including that of his time:

> From my perspective as a modern composer, I want to build bridges of understanding concerning the complete and consistent development of the wondrous world of harmony – from the most basic to the ultra-complicated – using examples from the repertoire. The striking revelations of natural laws serve only as proof for the correctness of modern harmonic development. Speculation and empiricism should be mutually supportive and complementary.\(^{19}\)

This statement reveals four significant themes that apply to Karg-Elert’s work as a whole:

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\(^{17}\) Capellen 1905.
\(^{18}\) Stumpf 1883/1890.
\(^{19}\) *Grundlagen* I, p. IV.
a) That the development of harmony has been “consistent,” and that therefore the entire gamut of harmony can be understood using a consistent analytical system

b) That the development of harmony can be illustrated using repertoire examples

c) That “natural laws” (i.e. acoustic principles) can explain harmonic relationships

d) That subjective and objective valuations of music are not exclusive; instead, they reinforce each other, and point to similar conclusions.

In many ways, these four points can be regarded as general conclusions to be drawn from Karg-Elert’s theoretical project. The final point is perhaps the most important, as it touches on a most basic question in music theory: is music best understood through research into external physical and acoustic phenomena (what we might call “Nature”), or into psychological processes of sensation and perception? This dichotomy has been a central issue in the study of music since the ancient Greek philosophical battles between the Pythagoreans and the Aristoxenians, and it was exemplified in the late nineteenth and early twentieth centuries by the work of physicist Hermann von Helmholtz (1821–1894) on the one hand, and of psychologist Carl Stumpf (1848–1936) on the other. Karg-Elert expands on the matter of “speculation and empiricism” in the preface to Part II of Grundlagen (discussed in section 2.4).

Part I is explicitly described as “the material for my elementary theory classes at the Leipzig Conservatory.” As a guide to rudiments such as notes, intervals and scales, it is “designed to be practical for the widest possible audience [music students in schools, seminaries and conservatories as well as autodidacts].” Here is the table of contents for Part I (Figure

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21 A useful summary, comparison and assessment of their work is Green and Butler 2002.
22 Grundlagen I, p. III
23 Ibid.
2.2.4), which is largely traditional in material and approach, though a few sections (especially Chapters 7 and 16) point forward to Karg-Elert’s polaristic theories:

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**Figure 2.2.4. Grundlagen I – table of contents**

In 1921, Karg-Elert published *Neue Bahnen der Harmonik und ihrer Lehre* [New Paths to Harmony and its Theory]. It was first released as a separate pamphlet, though its introductory essay was also included as the preface to Part II of *Grundlagen*. *Neue Bahnen* is an important document, as its introductory essay provides the most significant insight into the philosophical position of Karg-Elert’s entire project; this essay will be discussed in section 2.4. The second half of *Neue Bahnen* is a complete outline and table of contents for Part II of *Grundlagen*, which was to be the “modern theory of harmony” listed in the three-volume plan. The outline in *Neue Bahnen* provides the best glimpse into Karg-Elert’s vision for a comprehensive theory of harmony, as his published works never fully realized that vision. The complete outline is

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24 Karg-Elert 1921b.
reproduced in the Appendix to Chapter 2; translations of Karg-Elert’s chapter headings are
provided below. Part II is divided into three large sections (Abteilungen):

Section 1 (Erste Abteilung):
Chord and harmony, in the monistic fashion. [Figured bass and scale-step labelling.]

Section 2 (Zweite Abteilung):
The foundations of harmonic polarity, in objective and subjective evaluation

Section 3 (Dritte Abteilung):
Practical textbook of polaristic harmony.
Dissonances as natural harmonic combinations.
Atonal pitch systems.

Here are the chapter headings for Section 1 of Part II (Figure 2.2.5):

Section 1: Chord and harmony, in the monistic fashion. [Figured bass and scale-step labelling.]
Chapter 23: Building simple chords in mechanical fashion
Chapter 24: Figured bass notation
Chapter 25: Scale-step labelling. Chord steps in natural major and minor, and in the altered scales.
Chapter 26: Voice leading. Changes of register and position.
Chapter 27: The three basic principles of voice leading.
Chapter 28: Three-part writing in the simplest harmonic settings.
Chapter 29: Four-part writing in the simplest harmonic settings.
Chapter 30: Cadential chords of altered quality \{i.e. major V in minor keys, minor IV in major keys\}
Chapter 31: Dominant seventh chords.
Chapter 32: Seventh chords on the other scale steps.
Chapter 33: Ninth chords.
Chapter 34: Alterations \{i.e. raised or lowered chord tones\}
Chapter 35: Modulation
Chapter 36: Non-harmonic tones [Chords of motion]
Appendix: Contradictions between figured bass and scale-step labels

Figure 2.2.5. Grundlagen II, section 1 – chapter headings

As can be seen from the chapter headings, Section 1 of Grundlagen Part II features
essentially standard material, much like that in a mainstream harmony textbook such as
Jadassohn’s Harmonielehre: the construction and qualities of triads and other chords (presented
in the usual “monistic” fashion), figured bass notation, voice leading and resolution in four-part
texture, and basic diatonic harmony (explained and analyzed using Stufen or scale-step Roman
numerals). Karg-Elert explained why he (as a “convinced polarist”) chose to teach traditional theories of harmony before his own system:

It seems inconsistent that I – a convinced polarist – would devote such attention to the old-fashioned figured bass and the sterile formalism of scale-step labelling (Stufnenbezeichnung)…But I feel the need for students to internalize this primitive-naïve “theory” because 1. it is of historical importance, and is also of practical use for knowledge and mastery of the literature from the figured-bass era, and 2. so that the student, on the basis of his own perception and experience, can critically evaluate and decide for himself between the different systems (= the old scale-step labelling, Riemann’s “dual” and my “polar” function values).  

Though Karg-Elert was certainly convinced of the rightness of his own theories (and their superiority to “sterile formalisms” such as the Stufenlehre tradition), it is notable that he regards the musician’s perception and judgment as the ultimate arbiters of his work’s validity. The published edition of Grundlagen Part II includes all of Section 1, organized exactly as shown in the Neue Bahnen outline. It also includes an appendix of musical examples (Notenbeispiele) which are referenced in the text; Section 1 encompasses the first 125 musical examples.

Section 2 of Part II presents acoustical, mathematical and conceptual information intended to prepare the reader for Karg-Elert’s polaristic system of harmony: the Pythagorean (fifth-based) and Didymean (third-based) pitch systems, the dualistic concept of major and minor, overtones and undertones, relationships between chords, and the idea of harmonic function. Karg-Elert mentions his predecessors, all contributors to the development of polaristic harmony: the ancient Greeks, Zarlino, Rameau, Oettingen and Riemann. Figure 2.2.6 lists the chapter headings for Section 2 of Grundlagen Part II, as outlined in Neue Bahnen:

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25 Neue Bahnen, iv
Comparing this outline with the three-volume plan (Figure 2.2.3), the main omission is any mention of the term Tonpsychologie (music psychology). Chapter 38 does discuss how the musical imagination (or “ear”) understands each interval as an incomplete harmony, conceived either up or down from the prime (i.e. the generating tone); it also asserts that we perceive pitch relationships in terms of just intonation, even when performed in equal temperament. (These topics will be discussed at length in later chapters of this dissertation.) Otherwise, section 2 is mostly devoted to acoustics (especially the mathematical presentation of the Pythagorean and Didymean systems), a description of polarity, and an introduction to harmonic function (comparing Riemann’s system with his own). The published edition of Grundlagen Part II includes all of section 2, organized as in the outline from Neue Bahnen. All of the material from section 2 later appeared in much-expanded form in Akustische and Harmonologik.

Section 3 finally initiates the study of harmony using Karg-Elert’s polaristic system. The outline for section 3 in Neue Bahnen (Figure 2.2.7, on the next page) presents an ambitious array of topics. The outline clearly indicates the broad trajectory of the project: after a study of diatonic tonality (including some basic chromatic alterations), the plan turns first to harmony in modal systems, and then to features of expanded chromatic tonality in the nineteenth century (mediant and more distant harmonic relations, modulations), eventually leading to the “abandon of tonality” and the adoption of “atonal pitch systems.” The outline reflects Karg-Elert’s view of
harmonic style as an essentially linear and consistent continuum from modality to atonality; it also demonstrates his goal to examine the range of historical harmonic idioms, using a unified analytical system.

**Section 3: Practical textbook of polaristic harmony**

Chapter 42: Principal harmonies in the natural systems \(\text{i.e. major and minor}\)

Chapter 43: Energetic curves and the principal of cadential weight

Chapter 44: The tempered (altered) contradominants. Altered contradominant seventh chords.

Chapter 45: The substitutes or principal representatives in the key. The substitutes in musical contexts.

Chapter 46: The problematic \(\text{i.e. diminished}\) triad

Chapter 47: Diatonic twins and triplets. Twins in the diatonic sequence.


Chapter 49: Other diatonic chords. Chords of motion.

Chapter 50: The Neapolitan sixth chord.

**Expansion of tonality**

Chapter 51: General explanation of concepts of tonality.

Chapter 52: Prefix and suffix chords

Chapter 53: The principal variants. Seventh-additions to the tonic in Dorian and Mixolydian.

Chapter 54: Typical features of the Phrygian and Lydian systems. Church modes and exotic scales.

Chapter 55: General overview of harmony in the church modes.

Chapter 56: Third-related substitutes.

Chapter 57: Collective-change chords.

Chapter 58: Distant harmonic relationships surrounding the boundaries of tonality

Chapter 59: Harmonic shifts \(\text{abrupt modulation}\)

Chapter 60: Modulation

Chapter 61: Abandon of tonality

Afterword: the exhaustion of the entire harmonic continuum

**Dissonances as natural harmonic combinations**


Chapter 63: Narrow and wide alterations \{of harmonic tones\}

Chapter 64: The Italian ninth chord \(\text{i.e. the dominant seventh with minor ninth}\)

Chapter 65: The diminished seventh chord

**Atonal pitch systems**

Chapter 66: The pentatonic (pure five-tone) system

Chapter 67: The hexatonic (six-tone equal-tempered) whole tone system

Chapter 68: Quartal chords (expansion of chapter 66)

Chapter 69: Atonal chords with many pitches

Appendix: prospectus on third-, quarter- and fifth-tones \{microtonality\}

**Figure 2.2.7**. Outline and chapter headings for *Grundlagen* II, section 3 (from *Neue Bahnen*)

This ambitious outline for the “modern theory of harmony” was not fully realized in the published Part II of *Grundlagen*, which in fact ends after Chapter 43. Chapter 42 is quite extensive: though it presents only the three principal chords in major and minor (i.e. the tonic and its two surrounding dominants), it also includes the only real discussion in all of Karg-Elert’s
work on the interaction of harmony, phrase and rhythm.\textsuperscript{27} The following example from Chapter 42 (Figure 2.2.8) is representative of Karg-Elert’s approach to the topic:

![Figure 2.2.8](image)

**Figure 2.2.8.** Analysis of metric weight and harmonic rhythm in the main theme from Emile Waldteufel’s waltz “Sirenenzauber,” op. 154 (*Grundlagen* II, p. 197 and example 127).

Above the melody itself (Example 127), Karg-Elert indicates the apparent four-bar phrasing, defined by the repetition and variation of the opening motive; the pitches marked “NB” are non-harmonic tones. The harmonic rhythm does not follow the four-bar melodic division, except possibly in the last four-bar group. Karg-Elert states that “harmonic change is essentially marking of meter”\textsuperscript{28} – in other words, harmonic rhythm can define relative metric strength. Elsewhere in the same chapter, Karg-Elert proposes that “\textit{D} and \textit{C} [i.e. the dominant and sub-dominant] have an unmistakable tendency toward the metrically strong \textit{T} [tonic].”\textsuperscript{29} The

\textsuperscript{27} Schröder 2014 discusses some aspects of rhythm and meter in Karg-Elert, as presented in *Grundlagen* Chapter 42.

\textsuperscript{28} *Grundlagen* II, 196.

\textsuperscript{29} Ibid., 203.
Waldteufel analysis illustrates this principle: the 16-bar melody is renotated as eight hypermeasures in 6/4 meter. The original bars are marked 1 through 4, but “1” (signifying a strongly-weighted measure) is always displaced by one hypermeasure from the melodic phrasing. This reinterpretation places the first move to the dominant on a strong hypermeasure, and likewise the returns to tonic near the end. The acceleration of harmonic rhythm in bars 6 and 7 creates a “backlog of energy” (Energetische Stauung), released and relaxed in the final measure (Ruhe, Entspannung). Both the “end-accented” multi-levelled rhythmic analysis (illustrating the hypermetric structure) and the use of stratified crescendo and decrescendo wedges (to show both local direction and the large-scale momentum toward the final measure) strongly recall Riemann’s theories of rhythm and meter. Chapter 42 of Grundlagen Part II is significant in Karg-Elert’s work as a whole, as the topics of harmonic rhythm and phrase rhythm are elsewhere almost entirely ignored.

As mentioned above, Grundlagen Part II ends abruptly after Chapter 43 (“Energetic curves and the principal of cadential weight”), and therefore does not proceed beyond the three basic functions in major and minor. The Neue Bahnen table of contents specifies numbered “models and repertoire examples” (Muster- und Literaturbeispiele) for Chapters 42 to 55, encompassing 571 examples in total; however, exactly how much of the material after Chapter 43 was completed or drafted by 1921 is impossible to say. The abrupt cessation of the Grundlagen series was likely due to economic difficulties. The period from 1921 to 1924 saw

30 The terms “hypermeter” and “hypermeasure” were introduced in Cone 1968. However, the grouping of measures into hierarchical accent patterns was recognized in the eighteenth and nineteenth centuries, in treatises including Mattheson 1739 (which associated different types of caesura with punctuation marks in vocal music), Kirnberger 1779 (which conceived some measures as if they were strong or weak beats), and Weber 1821 (which presents a model of “rhythms of a higher order”). Zbikowski 2002 (pp. 319–320) summarizes Weber’s ideas.

31 Caplin 2002 provides a summary of Riemann’s views on rhythm and meter.
massive hyperinflation in Germany, and costs related to publishing became completely impractical. Karg-Elert complained about the situation in a letter of January 1923:

Manuscript paper per sheet 100 marks (but it actually rises daily, so when this letter reaches you will cost 160—200 marks) [In 1914, one sheet (Bosworth) cost 3½ pfennigs] which is a difference of 3,000 to 5,000 times!, compared with the pre-war price. At present, an engraver at C. G. Röder receives 1,250 marks per hour. Formerly he got 1.20 marks, but what’s the use to him of his 1,000-fold increase when the average price rise amounts to 2,000.\textsuperscript{32}

Speka Musikalienverlag (the publisher of Grundlagen) went bankrupt soon afterward,\textsuperscript{33} and Karg-Elert’s first treatise has never been reprinted. The rest of the “modern theory of harmony” outlined in Neue Bahnen would be delayed by almost a decade, appearing only in Akustische and Harmonologik; the material on “atonal pitch systems” (planned for Chapters 66 to 69) was not included even in the two final treatises.

\section*{b. 1922-1930: between Grundlagen and the final treatises}

During the 1920s, Karg-Elert’s occupation with music theory continued without a break. He developed and refined his theories, both on paper and in his classes. He also attempted to promote his theoretical work in various ways, and sought avenues for publication. The period is very well documented in Karg-Elert’s letters to friends in Australia and England. A letter from December 1923 describes an opportunity for promotion that never came to fruition, again due to economic austerity, or possibly to interpersonal intrigues between Karg-Elert and his Conservatory colleague and frequent opponent, Karl Straube:

\textsuperscript{32} Letter from Karg-Elert to Grete Bellmont and A.E.H. Nickson, January 13 1923 (Karg-Elert 2010, 30).

\textsuperscript{33} Attested in a letter from Karg-Elert to Godfrey Sceats, February 24 1930 (Karg-Elert 2000, 19). A search in multiple catalogues suggests that the two volumes of Grundlagen (plus Neue Bahnen and the notebook of musical examples) were in fact the only publications ever issued by Speka Musikalienverlag.
At the International Congress of Music, 20th–25th October this year, in which America was to have been strongly represented, I was to have delivered my completely newly developed pathways, “Precepts on the Polarity of Harmony” and my “Theory of Musical Cells” (biogenetic: the evolution of the single tone to atonality). The Congress had to be cancelled as financial support was inadequate. But what was the end of the story? Straube and ever again Straube diverted the whole affair “from the back” into his own hand. No-one at all had been asked.\textsuperscript{34}

In the same letter (which is over thirty pages in length) Karg-Elert enthusiastically considers the possibility of having his polaristic harmony translated and published in England:

> But your lovely Christmas-letter contains a proposal which is so very tempting…I have always thought that my Precepts of the Polarity of Harmony, which in simple fashion solves all conceivable problems of tonality and sound…should be very accommodating, attractive reading material, especially spiritually, for English-speaking nations! I also believe it will have an excellent result from publication . . . for this tremendous discovery of mysterious polarity transforms the entire teaching of harmony up until now and at last establishes agreement with practice…I experience it with my students, when they arrive stuffed full of Jadassohn’s dead documents — unload the whole mess within 10 minutes and — fresh and free of prejudice get into the tonal philosophy of cells and cosmic polarity…An English edition would be an act of culture! Surely it would also be well worthwhile financially! …I would try to interest Novello in this work; Dr. Hull would be an enthusiastic, influential advocate! …in any case I could dare to risk the entire work in the only-authorised translation by you, Madame Bellmont, to be pledged against the travelling- and running-costs!\textsuperscript{35}

This excerpt reflects not only Karg-Elert’s boundless confidence in his theories, but also his work’s notable reputation in England. Novello published a sizable number of Karg-Elert’s organ compositions, starting with his \textit{Three Impressions}, op. 72 (1909). “Dr. Hull” was the English writer, organist and composer Arthur Eaglefield Hull (Doctor of Music from Oxford), who in

\textsuperscript{34} Letter from Karg-Elert to “my Australian friends” (i.e. Bellmont and Nickson), December 20 1923 (Karg-Elert 2010, 39).

\textsuperscript{35} Ibid. (Karg-Elert 2010, 52). “Madame Bellmont” is Margarete Bellmont (née Wienskowitz, 1870–1927), a German-born pianist and teacher who settled in Melbourne, Australia. She was a friend of organist A.E.H. Nickson, who promoted Karg-Elert’s music in Australia, and corresponded with the composer since 1913. From the early 1920s, Bellmont translated all of the correspondence between Nickson and Karg-Elert; in 1926 she visited Karg-Elert in Leipzig, and had some lessons with him.
1913 published the first full English-language article on Karg-Elert’s music. With connections such as these, publication of Karg-Elert’s theories in England was perhaps not too far-fetched, though no further steps in that direction were evidently taken.

The next mention of the polarity project comes from July 1926, by which time the publication process was apparently well advanced. He had signed a contract with respected Leipzig publisher F. E. C. Leuckart, but the terms of the contract had fallen into dispute:

But this work — my life’s purpose — my true mission — has had disastrous misfortune: 380 engraving plates, on which the international firm C. G. Röder has worked for exactly a whole year, and which are ready for printing (they look ravishingly beautiful!), are still lacking about 80 pages on atonality, and on quarter- and sixth-tones — and now that the work has become considerably larger than originally agreed upon, my publisher C. F. Leuckart (Martin Sander) has annulled the contract, thus declaring it invalid. Since 1902 I have worked incessantly, day and night, on this gigantic work which determines completely new spheres on each page and is without doubt the Precepts of Harmony of the future, — and now … I stand before nothing! No-one knows what will happen: I cannot offer the work to another publisher as 380 plates are already complete and belong to the publishing house. But the publisher is unable to start doing something with the torso (or central subject matter) and furthermore he will not. The engraver and lithographer however is demanding his approximately 5,000 marks manufacturing costs. He cannot be recompensed by me but only by the publisher, who in turn makes me liable for having exceeded the agreed limit. Well, a lawsuit in our dear Fatherland can run for 2–3 years.

It is evident that just as in his compositions, Karg-Elert was unable to restrain his tendency toward excess in his theoretical work; it is not surprising that Martin Sander (director of F. E. C. Leuckart) balked at printing a treatise that might be almost five hundred pages in length. The ensuing lawsuit did run for several years, further delaying publication.

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36 Hull 1913. Hull 1915 includes six brief examples from three of Karg-Elert’s published works. 37 Karg-Elert 2010, 82-83.
In the meantime, Karg-Elert was of course teaching his theories at the Conservatory, as attested by surviving classroom notes\(^{38}\) taken by one of his composition students, Sigfrid Walther Müller (1905–1946). In addition, his theories began to attract enthusiastic attention from two young teachers of music theory hired by the Conservatory in the 1920s: Fritz Reuter (1896–1963; student of Hugo Riemann, hired in 1921) and Paul Schenk (1899–1977; student of Karg-Elert in the early 1920s, hired as faculty in 1925). In December 1927, Karg-Elert wrote to his English friend Godfrey Sceats of his continuing legal issues with his publisher, and though discouraged about his own treatise, he considered a new avenue for publication:

My dear friend Paul Schenk (formerly a pupil and now my assistant) asks me for my approval of his request to you. It concerns my textbook on harmonic polarity. My own work will probably never be published; it is 9/10 finished but the publisher does not want to release it as it has become too expensive. This remains a judicial matter…\(^{39}\)

Paul Schenk’s “request” was to see if a textbook on Karg-Elert’s theories that Schenk had recently written (apparently with his former teacher’s approval) could possibly be published in England. Sceats’ efforts were unsuccessful:

It is most regrettable that you have been unable to find a home for the sensational, unique, theoretical work by my pupil Paul Schenk, for it would be something radically new for the English world of music. It is only through Schenk that these ideas could be disseminated in England, for my great work (on which I have worked without break for 26 years), due for release in Autumn, may not appear in foreign countries under my name a second time. After the publication of my “Harmonic Polarity,” no work of my pupils may be published if it deals with the same subject matter.\(^{40}\)

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\(^{38}\) “Musiktheoretische Notate von Sigfrid Walther Müller” (Schinköth 1997–98, 201–222). Müller studied at the Leipzig Conservatory between 1923 and 1926; his classroom notes are in content and organization very similar to passages from *Harmonologik*.

\(^{39}\) Letter from Karg-Elert to Godfrey Sceats, December 22 1927 (Karg-Elert 2000, 16). Sceats (1888–1966) was an English organist and linguist, and a tireless advocate for Karg-Elert’s music; he published several articles on the composer, and his collection of Karg-Elert-related documents (including correspondence) is now held at the British Library in London.

\(^{40}\) Letter from Karg-Elert to Godfrey Sceats, April 5 1928 (Karg-Elert 2000, 17).
Schenk’s work was titled *Lehrbuch des polaren Harmonik, mit einem Anhang von Analyse-Beispielen von Sigfrid Karg-Elert* [Textbook of polaristic harmony, with an appendix of analytical examples by Sigfrid Karg-Elert]. Schenk himself unsuccessfully applied to have it published by Breitkopf und Härtel;\(^\text{41}\) it never appeared in print, and the current status of the typescript is unknown.\(^\text{42}\) After 1945, Schenk wrote several textbooks on harmony and modulation that engage in varying degrees with Karg-Elert’s methods.

In 1928, Fritz Reuter published a brief introduction to Karg-Elert’s functional approach: *Harmonieaufgaben nach dem System Sigfrid Karg-Elerts* [Harmony exercises after the system of Sigfrid Karg-Elert].\(^\text{43}\) Published by C. F. Kahnt of Leipzig, it is dedicated to “Sigfrid Karg-Elert with gratitude.” Reuter explained his modest ambitions:

> The following collection of exercises is not a textbook. As a result, I have omitted almost all explanation. I do not want to anticipate the brilliant innovator and author of this polar system of harmony, when his own work will soon appear in print.\(^\text{44}\)

In a booklet of only 35 small pages (each page about eight inches square), Reuter demonstrates virtually all of Karg-Elert’s function labels and chordal transformations, plus many types of dissonant sonorities. Comparing Reuter’s table of contents (*Figure 2.2.9*, next page) with that for *Grundlagen* Part II section 3 (*Figure 2.2.7* and Appendix to Chapter 2), one can see that Reuter’s brief handbook manages to provide a comprehensive overview of Karg-Elert’s theories, albeit in a very compressed and somewhat superficial manner.

\(^{41}\) Schröder 2011, 218.
\(^{42}\) According to Schenk’s student Franziska Seils, the typescript still existed as of 1996 (Seils 1996/97, 173). Gesine Schröder states that it was owned by Carlferdinand Zech (1928–1999), a musicologist and choir director at the University of Halle, though she notes that the typescript’s current location is unknown (Schröder 2011, 218).
\(^{43}\) Reuter 1928.
\(^{44}\) Ibid., 2.
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<td>55</td>
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</tbody>
</table>

**Figure 2.2.9.** Table of contents to Fritz Reuter’s *Harmonieaufgaben nach dem System Sigfrid Karg-Elerts* (Reuter 1928)
The biggest issue with Reuter’s 1928 handbook is that in order to avoid pre-empting Karg-Elert’s own forthcoming treatise, Reuter omitted all mention of the three-dimensional pitch space that is the foundation of Karg-Elert’s work. This is problematic, as it can lead the reader to believe that Karg-Elert’s approach is nothing more than a quirky revision and expansion of Riemannian functional notation. Here is Reuter’s example of a comma-differing modulation (Figure 2.2.10) – that is, a modulation in which the initial and final keys differ acoustically by one or more syntonic commas (in addition to the change of tonic):

![Figure 2.2.10. Comma-differing modulation (first 4 chords), from Reuter 1928 (p. 19)](image)

The first functional analysis involves a change of key, from C major to E major; the lower two are entirely in E major and C major respectively, and use mediant transformations of the tonic instead of changing the key. Each version of the functional analysis equally indicates the crucial point: that the E major chord (bar 2) will be a syntonic comma lower (Kommafehler) than the opening C major. This is because the E major chord is rooted not on the E four fifths (i.e. 2:3 times 4) above C, but on the E one major third (4:5) above C. Those two E’s are functionally and conceptually distinct in Karg-Elert’s theories, a topic explained at great length in his two final treatises. Reuter does mark the moments of comma difference, but his booklet never explains what comma differences are, or what they signify. For a reader new to Karg-Elert’s theories, Reuter’s Harmonieaufgaben likely raised as many questions as it answered; however,
for someone well-versed in those theories, it is a useful reference guide to Karg-Elert’s chordal transformations and functional notation. In the early 1950s, Reuter published a much more thorough and detailed textbook\textsuperscript{45} on Karg-Elert’s polaristic harmony.

c. *Akustische* (1930) and *Harmonologik* (1931): similar material, different approaches

As late as February 1930, Karg-Elert’s letters still suggest that he was preparing a single treatise, the “modern theory of harmony” on which he had laboured for two decades:

You know that I have worked for about 21 years without interruption on my Relativity-teachings (Harmonic Polarity). The first publisher became bankrupt; with the second I was advised to enter into a legal dispute and have since had more and more years of frightful proceedings, which have passed through many appeals and deadlines. How will this catastrophe come out eventually?\textsuperscript{46}

However, about six weeks later Karg-Elert announced that a book he called “Acoustic Polarity” was set to be released:

…this “tapeworm” of a legal case has cost me over 2,000 and I haven't a single Mark in reserve. It is no longer a case of saving for extreme circumstances. (A comment in passing: my Acoustic Polarity will still be coming on the market before Easter.)\textsuperscript{47}

The lawsuit with F. E. C. Leuckart was finally resolved by June 1930, and Karg-Elert could confirm not only the publication of a work on acoustics, but also of his harmony treatise:

I think that you know how my legal action finished (German Supreme Court). It was settled such that I must pay a great sum to the plaintiff, another to the lawyer and yet another to the Court … My great life’s work, “The Logic of Harmony and Polarity” will be published in Autumn!! My “Acoustics” has just come out! You will have the first printed copy!\textsuperscript{48}

\textsuperscript{45} Reuter 1952.
\textsuperscript{46} Letter from Karg-Elert to Godfrey Sceats, February 24 1930 (Karg-Elert 2000, 19).
\textsuperscript{47} Letter from Karg-Elert to Godfrey Sceats, April 3 1930 (Karg-Elert 2000, 21).
\textsuperscript{48} Letter from Karg-Elert to Godfrey Sceats, June 30 1930 (Karg-Elert 2000, 27).
Akustische was indeed published in 1930 (by Leipzig firm Carl M. F. Rothe), but Leuckart did not ultimately release Harmonologik until early 1931. A final quote from Karg-Elert’s letters reveals not only that he viewed his theoretical work as his magnum opus, but also his awareness that his treatise would likely gain only a small audience:

At the 15th of Jan. I sold my organ-sinfomy {Symphony in F sharp minor, op. 143} to C.F. Peters, and finally my child of sorrow “Polaritätslehre” (Harmonologie) was published {by F. E. C. Leuckart}. I worked at this work since 1902 without a pause. Unfortunately the book is very expensive and therefore it will not become a very spread book! But my work will not go with me into the tomb /i.e. “to the grave”/. I know, that I have gained by this book a place in the history of music!!

The reasons for the division of Karg-Elert’s harmonic theories into two treatises remain unclear. It may simply have been because Leuckart demanded that their treatise be reduced in length; as published, Harmonologik contains 330 pages, which is somewhat less than the “380 engraving plates” Karg-Elert reported in July 1926. While Harmonologik does discuss the acoustic basis of the harmonic theories in some detail, it lacks the depth and systematic coverage found in Akustische; one might assume that some of that acoustics material was among the pages excised from Harmonologik. It may also be that in early 1930, Karg-Elert was still unsure if Leuckart would issue anything of his work at all, and he therefore jumped at the opportunity to publish part of his work in a different form. In any case, Karg-Elert’s view of the relative importance of his two treatises is clear: in the table of contents to Harmonologik, Akustische is described as “a supplement” (Ergänzung) to the larger work. Indeed, Akustische’s 104 pages of (meticulously) handwritten text and diagrams will likely strike the reader as less significant than the 330 pages of beautifully printed material found in Harmonologik.

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49 Letter from Karg-Elert to Godfrey Sceats, March 31 1931 (Karg-Elert 2000, 31). Karg-Elert wrote this letter in his imperfect English; normally, he and Sceats corresponded in German.
Close study of both treatises reveals that while their subject matter is broadly similar, their intellectual focus and modes of presentation are quite different. Akustische is very much a work of speculative music theory: after a discussion of practical and ideal pitch and tuning (Chapters 1 and 2), it defines and examines each dimension of the three-dimensional pitch space in turn (in Chapters 3, 5 and 9), and then presents the harmonic and functional relationships that are characteristic of each dimension (especially Chapters 4, 6 through 8, and 10). Akustische’s verbal commentary is usually quite succinct, and most concepts are demonstrated using a combination of acoustic data (specifying the exact location of each pitch in the pitch space), one or more diagrams, and an abstract musical example (with functional analysis) written by Karg-Elert. It contains no passages from repertoire, and little discussion of how Karg-Elert’s theories compare with more traditional methods, or those of his near contemporaries. In contrast, Harmonologik is oriented much more toward musical practice. It features lengthy and lively verbal explanations of each topic, often relating his ideas to earlier methods. The acoustic and conceptual pitch space is presented in a less formal (and less organized) manner, scattered among various chapters of Part One (entitled “Basic introduction to the entire work”). In Parts Two and Three (on the polaristic system of harmony), emphasis is generally placed on the functional meaning of harmonic relationships; the acoustic basis of those relationships is often left implicit, to be understood from the function labels. Perhaps the greatest strength of Harmonologik is its wealth of functional analysis of repertoire, ranging from “Sumer is icumen in” (13th century) to Schoenberg and Scriabin, plus excerpts from Karg-Elert’s own compositions. It clearly aims to present the complete history of harmonic development as a consistent linear process: one which manifests the “natural” phenomenon of harmonic polarity, and which can be demonstrated through cumulative examination of the literature.
The detailed examination of acoustics, function and transformation in Chapters 3 to 5 of this dissertation will often refer to *Akustische* as primary material, because concepts are often presented in more concise and abstract form in the first treatise. Those technical discussions will then be bolstered by repertoire examples and analyses drawn from *Harmonologik*. The latter introduces some graphic representations of harmonic relationships that are not found in *Akustische*; on the other hand, *Akustische* includes some detail on chord types that was omitted in the larger treatise.

Except for an ambivalent review of *Harmonologik* by Karl Hasse\(^{50}\) (discussed below in section 2.3) published just after Karg-Elert’s death in early 1933, his treatises attracted almost no significant critical attention. All of the manuscripts of his theoretical writings (plus those of many compositions) were lost during the war years, most likely due to Allied bombings of Leipzig in 1943 and 1944 that destroyed much of the city, including most of the publishing houses.\(^{51}\) The disappearance of the manuscripts was confirmed in a 1947 letter from Karg-Elert’s daughter to his student Paul Schenk.\(^{52}\) Except for rare copies held by a few libraries around the world, Karg-Elert’s treatises remained mostly inaccessible for over seventy years. In 2004, Peter Ewers Verlag (Paderborn) published a volume entitled *Sigfrid Karg-Elert: Die theoretischen Werke*, containing facsimile reprints of *Akustische* and *Harmonologik*, and of the 1925 pamphlet *Orgel und Harmonium*.

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\(^{50}\) Hasse 1933.

\(^{51}\) The Leipzig buildings of F. E. C. Leuckart (publisher of *Harmonologik*) were destroyed by bombing in 1943; the firm reopened in Munich after the war (Musikverlagswiki 2017). Carl M. F. Rothe (publisher of *Akustische*) was absent from the Leipzig business register of March 1941 (Musikverlagswiki 2017).

Since 2007, part of Karg-Elert’s theoretical work has finally been available in English. Australian organist and keyboard player (and professional radiologist) Dr. Harold Fabrikant has been a champion of Karg-Elert’s music for many years. Fabrikant has translated and published three collections of Karg-Elert’s letters, some of which have been quoted above. With the help of translator Staffan Thuringer and graphic designer Terry Truman, in 2007 Fabrikant produced and self-published a German-English edition of *Harmonologik*, under the title “Precepts on the Polarity of Sound and Tonality.” It is still currently available for purchase in book and CD-ROM formats. The translation is for the most part very literal; it purposely does not attempt to interpret or explicate, leaving that for the reader. It is an elegant edition, placing the original German and the English translation on opposite pages, enabling easy comparison. Though *Harmonologik* remains a difficult and somewhat diffuse treatise, Fabrikant’s edition certainly makes its complexities more accessible to the English speaker. This dissertation hopes to fulfill a similar role for *Akustische*, which appears in an annotated German-English edition in the Appendix.

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2.3. *A survey of literature about Karg-Elert’s theories*

Not surprisingly, almost all of the published commentary on Karg-Elert’s theories has been in German. The only significant critical discussion in the years immediately following publication was an April 1933 review of *Harmonologik* by composer, organist and Reger pupil Karl Hasse\(^5\) (1883–1960), published just after Karg-Elert’s death in *Zeitschrift für Musik* (since the mid-1920s increasingly a venue for nationalist and anti-modernist ideologies).\(^5\) Hasse praises some of the ideas in Karg-Elert’s treatise, though at length he criticizes Karg-Elert’s statements on Max Reger’s “mediantic” style, and also his sympathies with impressionist music. Hasse concludes that *Harmonologik* is a fine addition to the literature of music theory, but has little relevance to musical practice:

> If Karg-Elert confined himself to his main objective, which is to tabulate his functional harmonies and substitutions, without intervening in practical music so thoroughly and intrusively, his book would be welcomed with unrestrained gratitude as an exceptionally important contribution to systematic music theory. But as he seeks to combine the incompatible – namely pure harmony \(i.e. \textit{just intonation}\) with the practical study of chords and part writing – strong objections must be made.\(^5\)

While Hasse’s question about the relevance of Karg-Elert’s theories for practical music making is a valid one to ask, his review suggests that his understanding of the theories remained somewhat superficial, focused primarily on functional notation.

\(^{54}\) Hasse 1933.
\(^{55}\) Sachs 1970, 75–76. From November 1923, the subtitle of *Zeitschrift für Musik* was “Kampflätter für deutsche Musik und Musikkultur” (roughly “fighting paper for German music and musical culture”).
\(^{56}\) Hasse 1933, 345.
During the period of National Socialism in Germany (1933–1945), Karg-Elert’s theories and music essentially vanished from the scene; the composer was widely (and erroneously) believed to be Jewish,\(^{57}\) and so his work would have been officially condemned. His polaristic theories were removed from the curriculum at the Leipzig Conservatory, replaced by Hermann Grabner’s simplified monistic version of function theory,\(^{58}\) and a new focus on the German Volkslied. Even Karg-Elert’s friends and supporters Paul Schenk and Fritz Reuter moved into the nationalist camp, writing music and textbooks for the Deutsche Jugendbewegung (“German youth movement”).\(^{59}\) It goes without saying that no further discussion of Karg-Elert’s theories appeared in print during this period. As previously mentioned, both Schenk and Reuter re-engaged with Karg-Elert’s theories in the early 1950s, in treatises published in East Germany.

Perhaps the most important assessment of Karg-Elert’s theories was written by Paul Schenk himself, and published in a 1966 collection of essays entitled Beiträge zur Musiktheorie des 19. Jahrhunderts.\(^{60}\) Most of Schenk’s article is devoted to summarizing and explaining the basic components of the theories: the distinction of fifth-, third- and seventh-derived pitches; functional relationships between fifth-, third- and seventh-derived harmonies; comma-free and comma-differing modulation; and the sources of dissonance. Schenk concludes with four “critical considerations” (Kritische Erwägungen),\(^{61}\) which can be summarized as follows:

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\(^{57}\) Karg-Elert was listed in Das musikalische Juden-ABC (Rock and Brückner 1935), though the later Lexikon der Juden in der Musik (Stengel and Gerigk 1940) explicitly removes him from the list of Jewish musicians.

\(^{58}\) Schröder 2014, 3.

\(^{59}\) Holtmeier 2004, 257.

\(^{60}\) Schenk 1966.

\(^{61}\) Ibid., 159-161.
1. Karg-Elert’s presentation of minor as the strict polar counterpart of major creates a pure Aeolian mode, which has been mostly irrelevant to musical practice since the seventeenth century. It is therefore difficult to reconcile Karg-Elert’s theory of the minor mode with musical practice.

2. Stemming from the tradition of Pythagoras, Karg-Elert’s theories are rooted in number as a symbol of universal order. But mathematics is not the essence of music; theories must contend not only with natural phenomena and acoustic calculation, but also with psychological principles.

3. Hugo Riemann’s conception of harmony is essentially static and vertical, and the same applies equally to Karg-Elert. The true meaning of harmonic function is not limited to vertical phenomena, but also encompasses processes of dynamic movement and kinetic energy.

4. In spite of their innovations, Karg-Elert’s theories are fundamentally grounded in the major-minor system of tonality, and are thus inadequate for the understanding of the music of his time, such as Schoenberg, Stravinsky and Krenek.

Schenk raises important questions about the practicality and validity of Karg-Elert’s work; these questions will be revisited in the concluding chapter of this dissertation. Schenk’s 1966 essay reflects how he had distanced himself from significant aspects of his teacher’s theories. While some of Schenk’s earlier texts\textsuperscript{62} retain much of Karg-Elert’s just intonation-based polar approach, his final textbook\textsuperscript{63} displays very little of his mentor’s influence.

Renate Imig’s 1973 study of post-Riemann functional notation\textsuperscript{64} compares Riemann’s analytical system with those of his contemporaries and successors, including Hermann Erpf,\textsuperscript{65} Karg-Elert, Hermann Grabner,\textsuperscript{66} and Wilhelm Maler.\textsuperscript{67} Imig begins with a thorough description of Riemann’s theories, usefully detailing how Riemann subtly changed his explanations of

\textsuperscript{62} Schenk 1953 and 1954.
\textsuperscript{63} Schenk 1976.
\textsuperscript{64} Imig 1973.
\textsuperscript{65} Erpf 1927.
\textsuperscript{66} See especially Grabner 1944, but also Grabner 1923 and 1935.
\textsuperscript{67} Maler 1931, or its later revision (Maler 1950).
various harmonic relationships in successive publications. Imig then discusses the other functional systems in slightly less detail but with great clarity, frequently using the Tonnetz\textsuperscript{68} to map harmonic relations and transformations. However, she makes almost no mention of the most distinctive aspect of Karg-Elert’s functional system: its basis in a three-dimensional just-intonation pitch space. In her book’s conclusion, Imig discusses how the most widely-used texts by Grabner and Maler employ simplified function labels, adopt a monistic conception of major and minor, and eschew all concern for issues of acoustics or intonation.

By far the most prolific writer on Karg-Elert has been Günter Hartmann, who wrote his 1985 dissertation\textsuperscript{69} on Karg-Elert’s organ works. This dissertation was published in 2002 (in a revised and much expanded form) as *Sigfrid Karg-Elert und seine Musik für Orgel*.\textsuperscript{70} It is much more than a survey of the organ music; in particular, it provides the most detailed and documented biography of the composer, citing and reproducing letters, and obscure early articles from German journals. Hartmann’s scope is very wide; he devotes considerable attention to Karg-Elert’s rejection by the Bach-inspired Orgelbewegung (“organ movement”),\textsuperscript{71} which closely reflected nationalist and conservative sentiments in German society. Hartmann’s other

\textsuperscript{68} The archetypal just-intonation version of the fifth-third Tonnetz is found in Oettingen 1866 (see Figure 3.4.4 below). However, Oettingen himself drew on similar graphic representations by earlier theorists, notably that in Weber 1821. Mooney 1996 describes in depth the historical development of the Tonnetz.

\textsuperscript{69} Hartmann 1985.

\textsuperscript{70} Hartmann 2002.

\textsuperscript{71} A contemporary account of the Orgelbewegung’s ideology is Schulze 1933. This overtly nationalist statement (published in Zeitschrift für Musik) culminates with a manifesto on “Church Music in the Third Reich,” calling for “struggle against the subversive powers of liberalism and individualism.” This manifesto was endorsed by more than thirty significant organists, church music directors, professors and composers; at the head of the list is Dr. Karl Straube – famed organist, cantor at the Thomaskirche in Leipzig, and Karg-Elert’s colleague and frequent rival at the Leipzig Conservatory.
major book is *Karg-Elerts Harmonologik: Vorstufen und Stellungnahmen*,\(^{72}\) published in 1999; this remains to date the only monograph on Karg-Elert’s theoretical work. The subtitle of this book (“precursors and reactions”) aptly denotes its strengths, and also its weaknesses. Hartmann draws connections between Karg-Elert and many historical writers, going back to the ancient Greeks. He then divides more recent scholars into supporters and opponents of Karg-Elert’s concepts. Hartmann’s sharply polemical tone is unfortunate, as it detracts from his frequently insightful commentary. However, the principal flaw is that while Hartmann has much to say on the perceived opposition to Karg-Elert from other theorists, he does not really attempt to explain Karg-Elert’s work itself, or to consider its significance on its own terms. Nonetheless, Hartmann’s 1999 book remains very useful, especially as it reproduces two of Karg-Elert’s theoretical writings,\(^{73}\) and a number of early biographical articles.

The other dissertation related to Karg-Elert’s theories is Hermann Bergmann’s “*Harmonie und Funktion in den Klavierwerken von Sigfrid Karg-Elert (1877–1933)*,”\(^{74}\) written in 1991. The first part of this dissertation includes a useful summary of Karg-Elert’s functional notation; however, like Imig, Bergmann barely discusses the issue of just intonation and its ramifications for harmonic function and tonality. The second (and larger) part of Bergmann’s study divides Karg-Elert’s works for piano into several stylistic categories (“neo-Baroque,” “Impressionistic,” “modern” and so on), based on their type of harmonic vocabulary; surprisingly, this second part makes very little use of Karg-Elert’s analytical methods, and therefore seems somewhat disconnected from the first part of the dissertation.

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\(^{72}\) Hartmann 1999.

\(^{73}\) Karg-Elert 1928 and 1930.

\(^{74}\) Bergmann 1991. A summary of the dissertation was published as Bergmann 1997/98.
The *Karg-Elert-Gesellschaft* was established in Heidelberg in 1984, and is still active today. This organization promotes the performance, publication and dissemination of Karg-Elert’s work. In addition to annual meetings (which have featured concerts and lectures, including several presentations on Karg-Elert’s theories), the *Gesellschaft* has to date published twelve volumes of *Mitteilungen* (“notes”), most recently in 2015. While the majority of articles in the *Mitteilungen* focus on Karg-Elert’s biography, on specific works, or on performance of the composer’s music, a few items related to Karg-Elert’s theories have been published, including articles by Günter Hartmann and Hermann Bergmann (mentioned above). The most significant publication by the *Karg-Elert-Gesellschaft* is their *Mitteilungen* of 1997-98, which is a collection of papers presented at the November 1996 meeting in Leipzig. The theme of that meeting was “Sigfrid Karg-Elert and his Leipzig students,” and the publication\(^75\) includes papers relating to Karg-Elert’s theories by Bergmann and Hartmann, plus articles on Paul Schenk (by Franziska Seils) and on Fritz Reuter (by Günther Eisenstadt). Several of the papers in the 1997–98 *Mitteilungen* are referenced in this dissertation, and cited under their authors’ names.

Mention must be made of scholarly work relating to Karg-Elert by Gesine Schröder, who has taught music theory and musicology at the *Hochschule für Musik und Theater “Felix Mendelssohn Bartholdy” Leipzig* (the successor to Karg-Elert’s institution) from 1992 to 2012, and since that time at the *Universität für Musik und darstellende Kunst Wien* (Vienna). Schröder has published on a wide variety of subjects, including notably on music theory in Leipzig, and its dissemination in other countries. Her articles include “*Farb-Ton-Figuren,*”\(^76\) a brief exploration of visual formatting in the music theories and harmonium works of Karg-Elert; and a study of

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\(^75\) Schinköth 1997–98.

\(^76\) Schröder 1995.
Karg-Elert’s orchestration of a piano suite by Bizet.\textsuperscript{77} Of greater consequence is Schröder’s article on the teaching of music theory at the Leipzig Conservatory in the 1920s and 1930s by Schenk, Reuter and Grabner, and the spread of their work in central and eastern Europe.\textsuperscript{78} Schröder’s 2014 article on aspects of rhythm and meter presented in Karg-Elert’s Grundlagen\textsuperscript{79} suggests that his harmonic theories may have been influential on Russian music theory, in the work of prominent Russian theorist Yuri Kholopov.\textsuperscript{80} Schröder’s most recent project on the spread of central European music theory in China\textsuperscript{81} points to the influence of Paul Schenk, who went to China in 1959 to work with local conservatory teachers.

To date, very little has been written in English on Karg-Elert’s theories. William Mickelsen’s study of Riemann\textsuperscript{82} concludes with a brief survey of post-Riemann function theories, including that of Karg-Elert; Mickelsen’s survey is based explicitly on the work of Renate Imig, and amounts to little more than a comparison of functional notation. Daniel Harrison’s comments are a bit more extensive, and also more penetrating. In his survey of harmonic dualism in the 1994 book Harmonic Function in Chromatic Music, Harrison names Karg-Elert as a “radical” among Riemann’s successors, and even cites Harmonologik as an example of “crackpot music theory.” Nonetheless, Harrison is enthusiastic about Karg-Elert’s work, both for its “imaginative analytical insights” into chromatic music, and for the

\textsuperscript{77} Schröder 1996/97.
\textsuperscript{78} Schröder 2011.
\textsuperscript{79} Schröder 2014.
\textsuperscript{80} The function labels for mediant transformations in Kholopov’s Garmoniya (Kholopov 2003) do strikingly resemble Karg-Elert’s. However, the question of the possible influence of Karg-Elert on Russian music theory requires further research. One interesting detail: after his studies at the Moscow Conservatory, Kholopov was stationed as a soldier in East Germany from 1955 to 1958 (http://kholopov.ru). This was exactly the period when Schenk and Reuter were the leading music theorists in East Germany, and promoting their Karg-Elert-based texts.
\textsuperscript{81} Schröder 2017.
\textsuperscript{82} Mickelsen 1977, 90 and 94.
“astonishingly sophisticated and effective examples that he contrived.” Finally, Richard Cohn’s 2012 book *Audacious Euphony* cites Karg-Elert as the originator of two concepts familiar in neo-Riemannian theory: the *Terzgleicher* (“common third,” equivalent to David Lewin’s SLIDE transformation), and the *Kollektivwechselklang* (“collective change chord,” of which the two types are equivalent to the hexatonic and octatonic poles of Cohn and Adrian Childs). The recent *Oxford Handbook of Riemannian and Neo-Riemannian Music Theories* contains only one mention of Karg-Elert, named by Ludwig Holtmeier as a purveyor of “dualistic parochialism.”

As this summary of previous literature has suggested, there remains a need for a comprehensive examination of Karg-Elert’s harmonic theory, considered primarily on its own merits. In addition, the many notable affinities between Karg-Elert’s work and recent neo-Riemannian theories have not yet been addressed at all. This dissertation hopes to fill these gaps, and in the process to raise awareness (especially in the English-speaking world) of Karg-Elert as a significant figure in early twentieth-century music theory.

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84 Cohn 2012.
85 Ibid., 66, 155 and 166. Connections between Karg-Elert and recent transformational theories are discussed in detail in Chapter 5 of this dissertation.
86 Holtmeier 2011, 8.
2.4. The intellectual orientation of Karg-Elert’s theories

An attempt to summarize Karg-Elert’s harmonic theories in a single sentence might say that they combine the just-intonation acoustic apparatus and strict major-minor dualism of Arthur von Oettingen (1836–1920) with the harmonic function theory of Hugo Riemann (1849–1919), enriched by his own ideas which include an infinite three-dimensional pitch space (based on the natural fifth, third and seventh) and an expanded array of chordal transformations that operate in those three dimensions. That statement is of course overly reductive, but it pinpoints the two primary influences on his work (as Karg-Elert repeatedly acknowledged in his treatises), and also summarizes what is most innovative about it. Karg-Elert’s specific engagement with the legacy of Oettingen and Riemann – mathematical music theory, acoustics, just intonation, dualism (or polarity), harmonic function, transformation – will be discussed in later chapters and sections that deal with those topics. This section will instead examine Karg-Elert’s broader intellectual orientation, as it is revealed in the preface to *Neue Bahnen*.

*Neue Bahnen* is an important document, as its lengthy preface provides the most significant insight into the philosophical positions behind Karg-Elert’s entire project. Perhaps not surprisingly for a man who stressed the “double nature” of his own personality and outlook, most of the preface centers around a set of oppositions, which Karg-Elert discusses in turn: tradition and modernity, theory and practice, “theory and theory” (meaning between different theories), empiricism and speculation, monism and polarism, objectivity and subjectivity, and most fundamentally *Urwesen* (principle, ideal) and *Erscheinung* (manifestation, appearance). This last pairing is a central and recurring theme in Karg-Elert’s work, especially in relation to the concepts of polarity, harmony and function. The question of principle versus manifestation also informs Karg-Elert’s discussion of the other oppositions listed above.
Karg-Elert begins the *Neue Bahnen* preface by informing the reader that his work is “not for those who swear by rigid “Tradition”…nor for young firebrands…who can only find their salvation in the modish triumphs of the avant-garde.”\(^87\) He states that during his time, a wide “discrepancy between practice and theory” had arisen, in large part due to “the rules and formulas of inflexible music theories…which reinforced the dogmas of one school or another.” Many theorists had blamed the “un-naturalness of the moderns [*Unnatur der Moderne]*” for the ever-growing chasm between theory and practice. In contrast, Karg-Elert calls for the discovery of new theories that can deal with modern works, which “with closer acquaintance reveal ever-increasing naturalness [*Natürlichkeit*] and “self-evidence” [*Selbstverständlichkeit*].” Based on such statements, it is clear that Karg-Elert views modern musical styles not as drastic departures from what had come before, but as logical and natural developments of earlier idioms. In turn, theories of modern music and harmony should grow out of the work of earlier scholars.

Karg-Elert recounts cases of the “anti-art terror of theory [*Kunstfeindlicher Terror der Theorie]*,” when great composers such as Mozart, Beethoven, Wagner and Liszt had been condemned by contemporary scholars as “lacking in theoretical knowledge,” or the spectacle of Anton Bruckner telling his theory classes that “those are the rules, but I don’t write that way.” The result is “a gloomy picture: creative artists and theorists as antipodes instead of complements, opponents instead of colleagues.” Equally troubling to Karg-Elert was the tendency for theorists to pronounce that “there is only one correct theory.” Instead, “all good theories must be valued as necessary building blocks in the temple of knowledge. Just as we treasure the individuality and originality of creative artists, so too must we credit theorists for their subjective differences and diverging opinions on musical phenomena.” What matters to

\(^87\) All quotations in section 2.4 are translations from *Neue Bahnen* (Karg-Elert 1921b).
Karg-Elert is that theory (regardless of method or outlook) and practice reinforce each other, as they are different manifestations of fundamental musical principles.

He then considers the question of empiricism and speculation in the study of music, and in particular of harmony. This dichotomy has been a central issue since the time of the ancient Greeks, with the Pythagoreans regarding music as a reflection of cosmic mathematical truths, and the Aristoxenians traditionally viewed as more reliant on the judgment of the ear, and on musical practice.\footnote{Litchfield 1988, 51.} Karg-Elert describes a split between those who focus solely on “the analysis of secondary manifestations” (the empirical) and those concerned mostly with studying “the principles of natural basic phenomena” (the speculative). In the first category, he places most standard harmony textbooks, which are derived from practice, and serve the practical musician; such books cannot be classified as works of theory, as they generally ignore the origins of musical events, which are simply exemplars of broader principles. On the other hand, works of great scientific precision and wisdom too often fail to recognize the subjective insights of the practical musician. As a result, many of those musicians (“who believe that everything in creative art is intuition and inspiration”) assert the right to reject scientific research as irrelevant. Karg-Elert proposes that “the study of harmony comes alive when it links empiricism and speculation, joins analysis and synthesis, and becomes prophecy [Prophetie].” The prophetic nature of a good theory (one that reconciles speculation and practice) may lie in its ability to identify future musical developments that can logically arise out of basic principles.

According to Karg-Elert, basic musical principles have their origin in Nature. This was certainly not a unique viewpoint; indeed, the association of music and nature has been central to
much musical discourse since the seventeenth century. For Karg-Elert, nature is ultimately reflected in music in the form of acoustic and mathematical relationships: “questions about the sources of harmonic phenomena are answered by acoustics – the revelation of nature through harmonic proportions.” To demonstrate this point, he notes that “the natural basis of the major and minor triads was discovered by theorists in the proportions of string lengths, long before the inner ear of future artists conceived those actual harmonies!” The necessity of acoustics for the study of music is evident when we remember that a musical pitch is itself a manifestation of a mathematical frequency. Karg-Elert states that “our senses are inseparably bound to Nature, in which mathematical forces are greatly abundant. If we subjectively perceive a sonic event to be natural, it must without exception reflect objective natural principles… Most musicians feel that the linkage of tone and number results in a vile devaluation of the former. But no! Both reveal their origin in a single root – both “live” through the will of Nature!”

All of these statements make clear that for Karg-Elert, the mathematical and acoustic explanation of music does not negate or contradict its perceptive and sensual understanding; instead, they reinforce each other, as they reflect the same principles. To explore this idea, he observes that harmonic polarity (the symmetrical opposition between major and minor) is not only defined by mathematical proportions, but also perceived by the musical imagination. As he explains, “each harmonic {i.e. consonant} interval equally implies two additional tones, completing either a major or a minor triad. These imagined additional tones or harmonic complements exhibit a strict symmetrical polarity. The will of an interval to be a harmonic

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89 Two items in Music Theory and Natural Order (Clark and Rehding 2001) provide a useful overview of how “nature” has been conceived in relation to music: the introduction written by Clark and Rehding (1–13), and Daniel Chua’s “Vincenzo Galilei, modernity and the division of nature” (17–29).
consonance is a natural phenomenon, revealed in the mathematical proportions of the primary triad.” Thus according to Karg-Elert, major-minor polarity is not merely a mathematical conceit imposed onto practical music (as Karl Hasse’s 1933 review of *Harmonologik* accused); rather, it is rooted in nature, and therefore can be both understood objectively and perceived subjectively.

*Neue Bahnen*’s preface concludes with a contemplation of cosmic infinite polarity, featuring the most enigmatic diagram in all of Karg-Elert’s work (*Figure 2.4.1*):

![Diagram](image)

*Figure 2.4.1.* *Urklangkomplex als Ewigkeitssonanz:* the original harmonic complex as the sound of infinity (*Neue Bahnen*, page XVIII)
Exactly what Karg-Elert intends to demonstrate in this diagram is a matter of speculation, but two fundamental points are quite evident. One is that the totality of pitches, harmonies and harmonic relationships (Ton, Klang, Klangkomplex) that are available to the practical musician are “but a cross-section of the infinite range of the sound world”; in other words, the music that we can perceive (or the “subjectively bounded sonic field [subjektiv begrenzte Klangsphäre]”) is just a manifestation of boundless general principles. The other point is that the complete range of pitch and harmonic relationships has a center (Zentrum), and that all other values extend endlessly in reciprocal (polar) fashion from that center. Based on the above diagram, one might easily connect Karg-Elert’s theories with the ancient musica universalis, or the Pythagorean doctrine of the “harmony of the spheres.” However, on a slightly more mundane level, the diagram points to some basic features of Karg-Elert’s model of pitch space, which is examined in detail in Chapter 3 of this dissertation.
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CHAPTER 3

Acoustics

This chapter examines the acoustical and mathematical derivation of Karg-Elert’s three-dimensional just intonation pitch space, from the proportions of the first seven partials of the harmonic series. After describing Karg-Elert’s ideal pitch frequency for the theoretical study of music, the chapter discusses his model of major-minor dualism (called polarity), which is based entirely on the mathematical principles of harmonic and arithmetic division, and thus does not depend on the physical existence of either overtones or undertones. The central sections examine in close detail the three dimensions of Karg-Elert’s pitch space: the Pythagorean (derived from pure 2:3 fifths), Didymean (from pure 4:5 major thirds), and concordant (from pure 4:7 sevenths). The inclusion of the seventh-based axis is unique to Karg-Elert, and represents one of the most innovative features of his work, as it implies new transformations involving dominant- and half-diminished seventh chords. His pitch space contains a potentially infinite number of unique pitches, stretching endlessly outward in three directions. To specify the location of each pitch in the space, Karg-Elert establishes middle C as a fixed center, with an ideal frequency or value of zero. He then employs acoustic symbols to denote distances away from the central C, as well as numerical intervals (measured in millioctaves) to distinguish between sonically similar pitches of the same name (called metharmonics) or of different names (called enharmonics). Though Karg-Elert never used the Tonnetz to illustrate locations or paths in pitch space, it is frequently employed in that capacity here, as it will be familiar for many readers. The chapter concludes with a description of the Ursprungslagen or source positions, which comprise the chain of fifths extending outward in both directions from the central C, and which in Karg-Elert’s estimation form the basis of harmonic function and tonality.
3.1. Defining practical and ideal pitch levels

_Akustische_ begins in a similar way to countless other works on acoustics, with a definition of pitch in terms of frequency and wavelength (Akustische Chapter 1). Karg-Elert then embarks on a detailed description of practical tuning pitch levels (Ak 2.1 to 2.4), and of how they have varied widely since the sixteenth century, generally ascending to $a^1 = 440$ Hertz, which was widespread but not yet standardized in 1930. Karg-Elert notes that “the assignment of a normative tuning pitch level is an arbitrary act” (Ak 2.1), and indeed a discussion of tuning pitch levels may seem irrelevant to the study of harmony, function and polarity. The immediate purpose of that discussion is to introduce _Ak_ 2.5, where the author establishes an ideal, constant fundamental pitch level for the physical and theoretical study of music. This hypothetical pitch is $C_6$, assigned a frequency of 1 Hertz (one vibration per second). The concept is borrowed ultimately from the work of Joseph Sauveur (1653-1716), who in 1713 proposed a _son fixe_ (“fixed tuning”) of middle C ($c^1$) = 256 Hz. The appeal of this pitch level to a mathematician such as Sauveur is evident, as it ensures that C (in all octaves) is a power of 2 (Figure 3.1.1):

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1 References to chapters and sections in _Akustische_ are headed “Ak,” followed by the section numbers used in the annotated translation. References to _Harmonologik_ are headed “Harm,” followed by the page numbers. Finally, references to _Grundlagen_ are headed “Grundlagen,” followed by the volume (I or II) and the page numbers.
2 As noted in _Ak_ 1.6, Karg-Elert uses Helmholtz pitch notation to specify octaves. Middle C is c¹, and the A above middle C is therefore a¹ (= 440 Hz). This dissertation likewise uses Helmholtz notation to name pitches in specific octaves. Hertz (Hz) is the unit of frequency, measuring the number of complete vibrations (cycles) per second.
3 In 1939, an Anglo-German conference organized by the Acoustic Committee of Radio Berlin and the British Standard Association recommended that the standard pitch should be set at $a^1 = 440$ (Lloyd 1949, 75). This pitch level was reaffirmed by the International Organization for Standardization in 1955, and again in 1975 (Cavanagh n.d., 3). It remains in wide use today in North America and the United Kingdom, though many orchestras in continental Europe tune to $a^1 = 442$ to 445 – see Franz Nistl’s list of international orchestral tuning frequencies (Nistl 2007). 
4 Haynes 2002, 42.
\[ C_6 = 1 \text{ Hz} \quad c^6 = 128 \text{ Hz} \]
\[ C_5 = 2 \text{ Hz} \quad c^5 = 256 \text{ Hz} \]
\[ C_4 = 4 \text{ Hz} \quad c^4 = 512 \text{ Hz} \]
\[ C_3 = 8 \text{ Hz} \quad c^3 = 1024 \text{ Hz} \]
\[ C_2 = 16 \text{ Hz} \quad c^2 = 2048 \text{ Hz} \]
\[ C_1 = 32 \text{ Hz} \quad c^1 = 4096 \text{ Hz} \]
\[ C = 64 \text{ Hz} \quad c^0 = 8192 \text{ Hz} \text{ etc.} \]

**Figure 3.1.1.** Frequencies of C as exponents of 2 (see also the table in *Ak 2.6*)

This tuning level (now often called *scientific pitch* or “Sauveur pitch”) places \( a^1 \) at 430.54 Hz, which was somewhat higher than average levels in early eighteenth century France, and thus was not widely adopted by practicing musicians of that time. In the late nineteenth century, a group of Italian musicians would advocate a very similar level (\( a^1 = 432 \text{ Hz} \)), in order to reduce vocal strain caused by rising pitch levels in operatic performance.\(^5\) Scientific pitch has been frequently cited by music theorists and physicists as an ideal pitch level,\(^6\) and its adoption by Karg-Elert is in a similar idealistic vein, though he also seems to favor it over the higher pitch levels of his time on aesthetic grounds. Discussing gramophone recordings in which pitch levels approached \( a^1 = 468 \text{ Hz} \), he complains that “as a result, the silvery glittering key of B major becomes a colorless C major” (*Ak 2.4*). This comment reveals two notable points about Karg-Elert’s conception of absolute pitch frequencies, and their connection with keys and tonality. The first is that he seems to associate keys with specific affective or sensual attributes, as was widespread in the eighteenth and early nineteenth centuries.\(^7\) The other (and more problematic) point is that he equates a perceived change of key with a specific change in pitch frequency. A

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\(^5\) Haynes 2002, 353. This pitch level has been called “Verdi pitch,” though according to Haynes, Giuseppe Verdi did not think that tuning down to \( a^1 = 432 \text{ Hz} \) would be practical, and so he did not actively promote it.

\(^6\) Ibid., 42.

\(^7\) Steblin 2002 is a survey of key characteristics discussed during the Baroque, Classical and early Romantic periods.
listener who has perfect absolute pitch (and who is almost exclusively experienced in one standard pitch level) might directly correlate pitch frequency and key in that manner; however, for all others, the association of pitch frequency and key is approximate at best. Is Karg-Elert actually suggesting that the musical imagination links real physical frequencies with pitch names, harmonies and keys, in an essentially fixed manner?

Perhaps the real sense of Karg-Elert’s comment – and indeed of his entire discussion of practical and ideal pitch levels – is revealed by his description of C major as “colorless”: something neutral, to which all other keys are compared and contrasted. Since at least the time of Guido d’Arezzo (and the celebrated acrostic hymn *Ut queant laxis*), C has acted as a central or starting pitch class. The centrality of C is of course built into the system of major-minor tonality, as C major is the key (or diatonic collection) without sharps or flats. In addition, C is frequently a boundary pitch class, marking the lower or upper range limits for many instruments, as Karg-Elert notes in the chart in *Ak* 2.6. It is likely for all of these reasons that Sauveur chose the pitch class C (rather than B or C#, or anything else) for his *son fixe*, to be linked with the powers of 2. As will be discussed in section 3.7, Karg-Elert normally places C (specifically middle C) at the center of his infinite pitch space, and in turn at the center of harmonic relations as a whole. If C is to be privileged as a central focus of the pitch and harmonic universe, then the choice of $C_6 = 1$ Hz as a hypothetical fundamental can be readily understood.

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8 When references are not preceded by *Ak*, they refer to other sections in the dissertation.  
9 In a few instances, Karg-Elert places D at the center, in order to illustrate symmetry between flats and sharps. See Figures 3.3.1 and 3.3.2 below. Those examples were surely inspired by Oettingen’s use of D as axis of inversive symmetry, sometimes indicated as a D-clef (see for example Oettingen 1913, 46).
3.2. **Polarity: its logical and mathematical justification**

Karg-Elert’s model of pitch space is entirely based in just intonation. It contains a potentially infinite number of distinct pitches, all derived from different combinations of pure intervals, which comprise frequencies that relate to each other in various whole-number ratios. The derivation of pitches from pure interval ratios will be discussed in sections 3.3 to 3.5. First, we must consider how the pure intervals are themselves defined in Karg-Elert’s treatises. There are two possible methods to calculate interval ratios: division of a relatively longer unit (representing a relatively low generating pitch), and multiplication of a relatively shorter unit (representing a relatively high generating pitch). As will be seen, the two methods produce identical ratios and intervallic series, but generated in opposite directions.

The division method is of course very familiar, as it has been central to the study of intervals and acoustics since the time of the ancient Greeks, in the division of the monochord.\(^\text{10}\) The complete single string of that instrument (which was called *kanōn* in Greek) produced a relatively low pitch, depending on its length. Using the monochord, Greek musicians such as Pythagoras found that by dividing the string in half (1 : 1/2), the resulting interval (called the octave) sounded like a near duplication of the whole. The next interval to be discovered was the twelfth, created by dividing the string in thirds (1 : 1/3). Through successive division into proportions following the number series 1/2, 1/3, 1/4... (a process called *harmonic division*), more intervals were discovered, such as the pure major third at 1 : 1/5. The twenty-one-fold harmonic division of a string sounding the pitch C (labelled [1] to denote that it is the complete unit) creates the following succession of pitches (**Figure 3.2.1**):

\(^{10}\) Creese 2010 is a detailed study of the monochord’s use in ancient Greek science and music theory, from the earliest sources to the time of Ptolemy (1st century AD).
Figure 3.2.1. Pitches created by the twenty-one-fold division of $C = \frac{1}{4} (Harmonologik, 9)$

The acoustic symbols placed above many of the ratios in Figure 3.2.1 will be explained in sections 3.3 to 3.5. Successively multiplying 1 by $1/2$ ($1/2, 1/4, 1/8, 1/16, \text{etc.}$) create octaves above the fundamental $C$. Likewise, multiplying other ratios by $1/2$ creates other octave series; for example, $1/3, 1/6, 1/12, 1/24 \ldots$ results in a series of octave-related Gs. Figure 3.2.1 will be immediately recognized as the first twenty-one pitches in the overtone series, manifested in practical music making in natural harmonics on string instruments, or in the process of overblowing on most woodwind and brass instruments.\textsuperscript{11} The consonant major triad based on the (octave-duplicated) fundamental C is found in the simple ratios $1/4 : 1/5 : 1/6$; for that reason, the major consonance has been widely regarded as a “natural” sonority. The case of the minor triad is more problematic, and the question of its origin has been a central one in music theory since at least the time of Zarlino. Simply put, the minor triad cannot be easily derived from the harmonic division of a string length. While it is outlined in the overtone series at $1/10 : 1/12 : 1/15$, those non-adjacent ratios have been considered too complex, and too remote from the

\textsuperscript{11} The clarinet is an exception; it only produces the “odd” pitches of the overtone series ($1/1, 1/3, 1/5 \ldots$) due to its closed cylindrical bore.
generating pitch to be a convincing source of the minor triad. Helmholtz concluded that minor triads are artificial and dissonant alterations of major triads, and are “generally inferior in harmoniousness to major triads...in the minor chords, the Third does not belong to the compound tone of its fundamental note.”¹² In other words, the third of the minor triad clashes with the fifth overtone of the root, creating a feeling of “roughness” or dissonance. Many nineteenth-century theorists sought alternative ways to explain the minor triad, which they still perceived to be a consonant and “natural” entity; as Karg-Elert notes, “the problem of the minor consonance is not soluble by simple string division...” *(Harmonologik, 8)*

Theorists such as Hauptmann, Oettingen and Riemann (plus many of their successors, including Karg-Elert) found the solution to the problem of the minor triad in various models of major-minor dualism, all of which are essentially rooted in the fact that the intervals of the major triad appear in inverted order in the minor triad.¹³ Riemann names Zarlino as the first to specify this fact, directly quoting his 1558 treatise *Le instituzioni harmoniche*:

The two types of harmony are distinguished by the position of the third which divides the fifth either harmonically or arithmetically. All of the diversities of harmony depend on the distinction of these two formations.¹⁴

Harmonic division (following the number series) of a string sounding C produces a major triad:

```
<table>
<thead>
<tr>
<th>1</th>
<th>1/2</th>
<th>1/3</th>
<th>1/4</th>
<th>1/5</th>
<th>1/6</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>c</td>
<td>g</td>
<td>e²</td>
<td>e¹</td>
<td>g¹</td>
</tr>
</tbody>
</table>
```

¹² Helmholtz/Ellis 1885, 216.
¹³ Numerous sources examine the history of harmonic dualism, and compare different models of dualism. These sources include Harrison 1994, Klumpenhouwer 2002, and Rehding 2003. Riemann described the development of dualism from his perspective in Riemann 1879 (translated by Benjamin Steege in Gollin and Rehding 2011), and again at greater length in Book III of Riemann 1898 (translated in Mickelsen 1977).
Arithmetic division (into equal-sized parts) of the same string creates a minor triad, downward:

\[
\begin{array}{ccccccc}
1/6 & 2/6 & 3/6 & 4/6 & 5/6 & 6/6 (= 1) \\
g^1 & g & c & G & E_b & C \\
\end{array}
\]

Zarlino did not extend the process of division beyond the number six, as he stated that all consonances are contained within the proportions 1: 2: 3: 4: 5: 6 (called the Senario), which when sounded together create the most perfect harmony.\(^{15}\) However, if the process was extended to proportions beyond six, the harmonic division of a string would continue the familiar overtone series, while arithmetic division would generate its exact symmetrical inversion.

The arithmetical division of a long string length is equivalent to the multiplication of a short string length. The latter is the basis of the Messel theory, which Riemann adopted as early as 1878 as an explanation of an inverted overtone series, and thus of the minor triad.\(^{16}\) Riemann borrows the Messel concept from medieval Arabic and Persian music theory, and he attributes it (after Kiesewetter’s 1842 treatise Die Musik der Araber) to the Persian mathematician and scholar Mahmud al-Schirazi (1236-1311, also known as Qutb al-Din al-Shirazi). However, Riemann also proposes that the Messel concept may be much older than that, possibly dating from before the time of Arabic scholar al-Farabi (c. 872-950). A Messel\(^{17}\) is a short string length which is multiplied to create a longer string; Riemann states that “the longer string usually comprises twelve Messel,” as shown in the following (Figure 3.2.2):

\(^{15}\) Riemann 1898, 370.
\(^{16}\) Riemann 1878, 77–85.
\(^{17}\) Messel is not a standard German word, but it conveniently recalls the verb messen, which means “to measure” or “to quantify.” See below on the Arabic origin of the word.
Figure 3.2.2. A string length ($\alpha - \beta$) comprised of twelve equal-sized Messel (I – XII), from Riemann 1878, 77

If the single Messel length $\beta$ - I sounds the pitch $g^3$, the multiplication of the Messel produces the following pitch series (Figure 3.2.3):

Figure 3.2.3. The pitch series produced by the multiplication of a Messel sounding $g^3$, from Riemann 1878, 80
The line above the Eb (Messel V and its octave duplication at Messel X) indicates that the pitch differs from its fifth-derived equivalent by a syntonic comma, as will be discussed in detail in section 3.4. The asterisks on the A and Db (Messel VII and XI) indicate that the natural tuning of those pitches differs from 12-tone equal temperament. Figure 3.2.3 shows that Messel multiplication generates an inversion of the overtone series, including the minor triad within Messel IV to VI – the symmetrical counterpart of the major triad created by the overtone ratios 1/4 : 1/5 : 1/6.

For many years, Riemann searched for evidence of a physical undertone series,\textsuperscript{20} a sonic manifestation of the Messel concept. He was of course unsuccessful in this endeavor, and he finally rejected undertones in his 1905 article entitled “The Problem of Harmonic Dualism”: I openly confess that the pseudo-logic of this undertone series constructed from several overtone series had even me fooled for a long time, and can still be detected in my earliest writings on harmonic theory.\textsuperscript{21}

After admitting that undertones are not a true acoustic phenomenon, Riemann also rejected overtones as a source of consonant harmony, as they can only generate the minor triad in the complex proportions 10 : 12 : 15. He explains the problem as follows:

\textsuperscript{20} Riemann first proposed the existence of a physical undertone series in Riemann 1875. Klumpenhouwer notes that Helmholtz had previously defined undertones not as “a series of harmonic partials emitted or extended ‘downwards’ from a fundamental,” but as “the patterns of fundamentals associated with a particular partial” – in other words, the same concept as Oettingen’s phonicity (see Klumpenhouwer 2002, 464).

\textsuperscript{21} Riemann 1905 (translated by Ian Bent in Gollin and Rehding 2011, 176).
Restriction to a single mode of mathematically defined explanation for the two relationships necessary for either chord individually incurs complicated numbers:

<table>
<thead>
<tr>
<th></th>
<th>Major 4 : 5 : 6</th>
<th>Minor 6 : 5 : 4</th>
<th>Frequencies</th>
<th>String lengths</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequencies</td>
<td>10 : 12 : 15</td>
<td>15 : 12 : 10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

i.e., the definition in terms of frequencies represents the minor chord as a less simple formulation, while that in terms of string lengths represents the major chord likewise. This simple comparison teaches us that the minor consonance is more correctly derived from the relative size of the sound waves (string lengths, pipe sizes), the major consonance on the other hand is derived from the relative speeds of vibration.\(^\text{22}\)

Thus, in 1905 Riemann essentially returned to the Messel theory (the arithmetical multiplication or division of a sounding body) as a logical and mathematical explanation of both major and minor (and consequently of dualism), with no recourse to physical overtones or undertones at all:

The principle in major is increase in speed of vibration (ascent toward the next-related tones), the principle in minor is growth in wavelength (descent toward the next-related tones). Consequently, the two are numerically best expressed through the same simple number series:

\[
\begin{array}{cccccc}
\text{Minor}\ &\ 5 &\ 4 &\ 3 &\ 2 &\ 1 \\
\text{Major}\ &\ 2 &\ 3 &\ 4 &\ 5 \\
\text{(relative string lengths)} &\text{(relative frequency)}^{23}
\end{array}
\]

More recent scholarship has shown that Riemann’s “Messel theory” assumes more than is contained in the original Arabic concept. In the second edition of the Harvard Dictionary of Music, Willi Apel notes that Messel is a Germanization of mathal, “a term used in Arab theory to indicate fractions of the type \((n + 1)/n\) (e.g., 4/3, 5/4, etc). All intervals represented by such fractions were considered consonant by the Arabs.”\(^\text{24}\) Thus, in Arabic music theory, the

---

\(^{22}\) Riemann 1905 (translated by Ian Bent in Gollin and Rehding 2011, 181).

\(^{23}\) Ibid., 181.

\(^{24}\) Apel 1969: 523.
consonant 4:5 third was simply one of many *mathal* intervals, and was not privileged as a component of the consonant major or minor triad, as suggested by Riemann.

Karg-Elert’s mathematical model of major-minor dualism, which he called polarity (*Polarität*), is essentially the same as that promoted by Riemann in 1905, and likewise does not involve any mention of physical overtones or undertones. First, he discussed the reciprocal relationship between frequency and wavelength (i.e. string length) in *Akustische* 1.3, and demonstrated it mathematically in *Ak* 2.6. He then explains the Messel concept in *Ak* 3.2 and 3.3, and again in *Harmonologik* pp. 8–9; he almost certainly took the concept directly from Riemann’s work. **Figure 3.2.4** is the example from *Harmonologik*, which uses a string length sounding d³ as the Messel unit (*Messeleinheit*) or source (*Quelle*):

![Figure 3.2.4. Karg-Elert’s generation of the minor triad from Messel multiplication (Harm, 8)](image)

Some details included in Figure 3.2.4 will be discussed in section 3.4: the distinction between *Prime* (generating pitch) and *Grundton* (chord root), and the acoustic symbols placed next to the pitches B♭ and G. After demonstrating the Messel concept, Karg-Elert further describes the symmetry between major and minor, using increasingly metaphysical language:

In terms of wave division, the minor chord is an equally unnatural form, just as the major chord is in terms of wave multiplication…Nature’s revelations are both forms, her elements are ordered by polarity, symmetrical-reciprocal. The major chord is revealed as micro-cosmic, the minor chord as macro-cosmic. (*Harmonologik*, 8)
The latter statement reflects the reciprocity between frequency and wavelength: major is “micro-cosmic” as it is generated through increase in frequency (and thus a decrease in wavelength), and the opposite is true for “macro-cosmic” minor.

Like Riemann, Karg-Elert rejects overtones as the source of the consonant triad:

Overtones are only the natural upper partials of a tone rich in colour…their specific combinations determine the tone colour of fundamental pitches (full, flat, pointed, hollow, nasal, shrill)…It is a basic error of certain textbooks on harmony, which regard overtones as proof for the natural form of ‘the consonance’, and also for the non-existence of the natural minor consonance. It would be equally one-sided to regard ‘the consonance’ as a natural entity, based only on the primary values of wavelengths. This would prove that the major consonance does not exist in a natural sense! (Harm, 11)

As discussed above in section 2.4, and reflected in the passages from Harmonologik just quoted, Karg-Elert considered the polarity of major and minor to be a basic principle rooted in Nature. As such, polarity is not only a logical construct or acoustic phenomenon, to be explained using mathematical ideas such as the Messel concept; it is also to be perceived and understood by the musical ear in a subjective or sensual manner, through practical experience of harmony and tonality. The perceptual basis of polarity in harmony will be examined in section 4.1 of this dissertation.
3.3. Pythagorean (canonic) pitch derivation

Though in *Akustiche* Chapter 12 Karg-Elert briefly considers the possibility of pitches generated from complex pure intervals such as the natural eleventh (4:11) and thirteenth (4:13), for practical purposes he derives pitches from three simple ratios: the pure fifth (2:3), third (4:5) and seventh (4:7). Pitches and intervals involving the latter three ratios are presented in separate chapters in *Akustische* (Chapters 3, 5 and 9 respectively), and also less systematically at the beginning of *Harmonologik* (pp. 3–17). Models of pitch relations based on pure fifths and thirds have of course been widespread since the ancient Greeks, and were still a basic element in nineteenth-century music theory, featuring prominently in the work of Helmholtz, Oettingen and Riemann. As will be described shortly, Karg-Elert essentially borrows the fifth-third pitch model from his predecessors, particularly from Oettingen. What is unique to Karg-Elert among his contemporaries is his inclusion of the pure seventh as an integral consonant interval, one which can generate new harmonies, and define new harmonic relationships. In effect, Karg-Elert’s model of pitch space is three-dimensional; this is likely the most consequential innovation in his work, one with significant ramifications both for immediate chord-to-chord connections and for larger-scale tonal trajectories. Pitch derivation from pure fifths is the primary subject matter in this section; derivation from thirds and sevenths is described in sections 3.4 and 3.5 respectively.

Citing the mathematician Pythagoras (*Ak* 3.1 and *Harm* p. 3), Karg-Elert notes that the ancient Greeks used the monochord to discover the “natural phenomena” of the pure octave and twelfth, in the string length ratios of 1 : 1/2 and 1 : 1/3. The difference of the twelfth and octave is the perfect fifth, which has the ratio 1/2 : 1/3, or more simply 2:3. From the intervals of the octave and fifth, the Greeks derived “all of the generally recognized intervals” – meaning a
complete chromatic semitone collection, and all of the intervals contained therein. Pure fifths can be linked to create an endless pitch series, in which no two members will have an octave relationship. Pitch collections generated entirely from pure fifths, plus the tuning systems that result from fifth derivation, are called Pythagorean. Alternatively, Karg-Elert often uses the term canonic, stemming from the Greek name of the monochord (κανόν). The following diagram from Harmonologik (Figure 3.3.1) illustrates the process, generating pitch collections of different sizes by incrementally adding pure fifths above and below a central D.\footnote{The linear format of Figure 3.3.1, and its organization around the central pitch D, resembles the “line of fifths” in Julian Hook’s Musical Spaces and Transformations (forthcoming). Hook’s Figure 1.18 diagrams an unconfirmed fifths space in much the same way as Karg-Elert; each fifth is labelled with a positive or negative number, exactly matching Karg-Elert’s dot above or below the note name.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.3.1.png}
\caption{Pythagorean (fifth-derived) collections, generated above and below D (Harm, 3)}
\end{figure}

The numbers inside the brackets in Figure 3.3.1 indicate the cardinality of the canonic pitch collections, which are illustrated in more detail in the next example (Figure 3.3.2):
Figure 3.3.2. Pythagorean (canonic) pitch collections (*Harmonologik*, 4)

The layout of Figure 3.3.2 highlights the melodic tendencies of each added pair of tones (marked by the arrows), disposed symmetrically (or *polaristically*) around the central D (the axis of symmetry between flats and sharps). The three-note collection comprises the principals (*Prinzipale*), namely the center or tonic and its surrounding fourths. This is followed by five-note (pentatonic), seven-note (diatonic) and larger collections. The 13-note collection is a complete chromatic one; the pitches six fifths below and above the D centre (A♭ and G♯, which have three flats and sharps respectively) are classified as enharmonic in 12-tone equal temperament (12-TET for short), but are acoustically and conceptually distinct in the Pythagorean system, separated by the Pythagorean comma (to be described presently).
Throughout *Akustische* (and to a lesser extent in *Harmonologik*), Karg-Elert uses acoustic symbols (*akustische Sigeln*) to label the derivation of individual pitches – that is, their location in pitch space. Fifth-derived pitches are denoted using dots, placed above the note name for fifths above the center, or below the note name for fifths below the center, as shown in Figure 3.3.1:

For example, D♭ (“des” in German pitch nomenclature) is seven fifths below D, while D♯ (“dis”) is seven fifths above D. The central pitch itself is often circled, to indicate from where the other pitches are calculated. The dots do not indicate specific octaves above or below the center, as octave equivalence is assumed (*Ak 3.4*). Thus, the notes in the above example are really pitch classes, and the dots denote their fifth-based derivation in pitch-class space. Dots can also be placed above or below pitches on a staff, as in this example from *Akustische* 3.4 (Figure 3.3.3):

**Figure 3.3.3.** Pitches generated above and below a central pitch C, with proportions reflecting relative frequency ($S_z$) and wavelength ($W_{lt}$)

In most cases, Karg-Elert places C at the center of his pitch space; examples in which a different pitch is the center (as in Figures 3.3.1 and 3.3.2) are carefully identified in Karg-Elert’s treatises. In both this dissertation and the German-English annotated edition of *Akustische*, the notation
\( \mathbf{P}_x \) is used in place of Karg-Elert’s dots, to indicate that pitch \( P \) is \( x \) fifths above or below the central pitch (\( C \) unless specified), as follows:

- \( G_{(1)} = G \) that is 1 fifth above \( C \)
- \( F_{(-1)} = F \) that is 1 fifth below \( C \)
- \( D_{(2)} = D \) that is two fifths above \( C \), and so on.

As noted above, the outer pitches in the 13-note Pythagorean chromatic collection are 12 fifths apart from each other. Such pitch pairs are considered enharmonic in 12-TET, but are acoustically distinct in Pythagorean tuning. The difference between two pitches such as \( G_{b(-6)} \) and \( F_{#(6)} \) is called the Pythagorean comma, which has the intervallic ratio of 524288: 531441.

But exactly how large an interval is the Pythagorean comma? Because ratios involving very large numbers are unwieldy, another method of measuring small intervals is needed. Instead of the familiar cent (1/1200 of an octave) developed and promoted by Alexander Ellis,\(^{26}\) Karg-Elert uses the millioctave (1/1000 of an octave) to precisely quantify and compare intervals (\( Ak 3.6 \)). According to John Biddell Airy, the division of the octave into 1000 parts was first proposed by English astronomer and mathematician John Herschel.\(^{27}\) It was also used in the 1891 treatise on acoustics and just intonation Das mathematisch-reine Tonsystem by Carl Eitz (1848-1924).\(^{28}\) However, Karg-Elert most likely borrowed it from Arthur von Oettingen’s Das duale Harmoniesystem.\(^{29}\) Oettingen and Karg-Elert use the Greek letter \( \mu \) as a symbol for millioctave. Intervals expressed in millioctaves are binary (base-2) logarithms of intervallic ratios, multiplied by 1000. For example:

\(^{26}\) Helmholtz/Ellis 1885, 41.
\(^{27}\) Airy 1871, 222.
\(^{28}\) Eitz 1891, 5.
\(^{29}\) Oettingen 1913, 249–253.
The interval \( n \) (measured in \( \mu \)) between two pitches in the ratio \( a:b \) is calculated as follows:

\[
n = 1000 \times \log_2(a:b)
\]

Using the pure fifth (2:3) as an example:

\[
1000 \times \log_2(2:3) = 1000 \times 0.5849625 = 585 \mu
\]

The binary logarithm is almost always fractional; both Oettingen and Karg-Elert round it off to the nearest whole number.

The same interval in cents would be calculated \( n = 1200 \times \log_2(a:b) \). For the perfect fifth:

\[
1200 \times \log_2(2:3) = 1200 \times 0.5849625 = 702 \text{ cents}
\]

To convert millioctaves to cents, multiply the \( \mu \) value by 1.2:

\[
\text{Fifth} = 585 \mu \times 1.2 = 702 \text{ cents}
\]

To convert cents to millioctaves, divide by 1.2:

\[
\text{Fifth} = 702 \text{ cents} : 1.2 = 585 \mu
\]

It is likely that Karg-Elert’s fundamental reason for adopting the millioctave rather than the cent is to divorce the acoustic calculation of intervals from 12-tone equal temperament (for which the cent is eminently suited, as the tempered semitone is 100 cents). However, the millioctave also provides a more practical advantage: as Karg-Elert describes in Ak 3.7, adding and subtracting basic Pythagorean intervals to calculate other Pythagorean intervals is considerably easier in millioctaves than in cents, because the basic Pythagorean intervals in millioctaves are multiples of 5 or 10.

For example, if we know

Fifth (2:3) \( 585 \mu \) (instead of 702 cents) and its inversion:

Fourth (3:4) \( 415 \mu \) (instead of 498 cents)

then calculating other Pythagorean intervals becomes very simple:
Whole tone (8:9)  Fifth – fourth = 585 – 415 = 170 μ (instead of 204 cents)
Major third (64:81)  Two whole tones = 170 + 170 = 340 μ (instead of 408 cents)

To invert an interval calculated in μ, simply subtract the μ value from 1000 (the octave):

Minor sixth (81:128) = Inversion of major third  
\[1000 – 340 = 660 \mu\]

Inversions can be used to calculate μ values for polar-counterpart intervals in the Pythagorean system. For example: what are the μ values for F♯(6) and G♭(6), in relation to C? Begin with:

\[F^{#}(6) = \text{six fifths above } C = 585 \times 6 = 3510 = 510 \mu\]

In the above product of six fifths (3510), the thousands-place (3) indicates the interval of three octaves (3 * 1000). Since octave equivalence is assumed, the thousands-place can always be eliminated in μ calculation. Therefore, the interval of six fifths is notated as 510 μ, not as 3510.

If F♯(6) is known to be 510 μ, then G♭(6) will be its inversion:

\[G^{b}(6) = \text{six fifths below } C = 1000 – 510 = 490 \mu.\]

Negative values are not used in μ calculation: instead of -510 μ, use 490 μ.

In *Akustische* 3.10 and 3.11, Karg-Elert calculates ratios and μ values for most intervals contained in a 25-tone Pythagorean system with C at its center. This system spans 12 fifths above and below C: from D♭(12) to B♯(12), which are respectively a Pythagorean comma below and above C. The Pythagorean comma is the interval of 20 μ, which exists between any two pitches that are 12 fifths apart (considered enharmonically equivalent in 12-TET).

The smallest intervals in the 25-tone Pythagorean system are the following:

| Apotome (chromatic semitone, augmented unison) | 2048:2187 | 95 μ |
| Limma (diatonic semitone, minor second) | 243:256 | 75 μ |
| Pythagorean comma (diminished second) | 524288:531441 | 20 μ |
The presentation of the Pythagorean system in Akustische Chapter 3 concludes with a diagram comparing interval sizes in that system and in the 12-tone equal-tempered 12-tone system (Ak 3.15 and 3.16). He finds that the two systems can substitute for each other quite well, as their intervals never differ by more than 20 µ (the Pythagorean comma).

In Akustische Chapter 4, Karg-Elert explains that because the Pythagorean system is completely derived from a single interval (the 2:3 fifth), it cannot be a source of harmony, which is based in the consonant 4:5 third. Instead, all Pythagorean intervals are fundamentally melodic and horizontal in tendency and function: “without exception, all diatonic melodic structures reflect canonic values” (Ak 4.1). Melodies contain consistent equal whole tones (always 8:9), and a subtle difference between diatonic and chromatic semitones (Limma and Apotome). Karg-Elert describes on several occasions how melody and harmony are derived from different systems: melody from the Pythagorean (fifth-based), and harmony from the Didymean (fifth- and third-based) and concordant (seventh-based) systems. This distinction has ramifications for pitch relations in different musical contexts: for example, the acoustic status of non-harmonic tones that embellish harmonies, or the status of triadic simultaneities in polyphonic music. Karg-Elert explores the latter point in Ak 4.9, in which the following melodic passage (Figure 3.3.4) contains only Pythagorean intervals, as indicated by the µ values:

![Figure 3.3.4](image.png)

**Figure 3.3.4.** Melodic line containing Pythagorean fourths (510 µ), chromatic semitones (95 µ) and diatonic semitones (75 µ)
When the above melodic fragment is placed in a polyphonic setting (as indicated by the brackets in the next example), Karg-Elert states that “the \{Pythagorean\} intervallic sizes remain valid…the following is nothing other than a combination of self-sufficient, independent voices” (Figure 3.3.5):

![Figure 3.3.5. Pythagorean “pseudo-triadic” simultaneities in a polyphonic texture (Ak 4.9)](image)

If the above passage is understood entirely as a confluence of three independent melodies as Karg-Elert suggests, then all melodic intervals in all three voices will be consistently canonic (fifth-derived). As a result, all vertical simultaneities will also comprise only canonic intervals (Ak 4.9 and 4.10). The apparent E major and F minor triads that occur at the two dotted horizontal lines are not true harmonic complexes, but pseudo-triads that contain the Pythagorean major third (64:81) rather than the harmonic third (4:5). Karg-Elert suggests that such pseudo-triads should actually be regarded as dissonant: “The Greek, mathematically oriented theory of music counts the Pythagorean third as a dissonance. They were correct, in that the concept of consonance and dissonance is of course a harmonic evaluation” (Harmonologik, 6). True harmonic complexes include the pure 4:5 major third (called the syntonic third), which defines the second axis of Karg-Elert’s pitch space, to be discussed in the following section.
3.4. Didymean (syntonic) pitch derivation

As just described, Karg-Elert stated that the fifth-based or Pythagorean system is the basis of melody, and of the horizontal dimension of music. Conversely, harmony (the vertical dimension) is fundamentally derived from the interval of the pure major third (4:5), described in *Ak* 4.3 as “the primary determinant of harmony, the syntonic blending third.” Karg-Elert attributes the discovery of the 4:5 pure third to “Didymus of Alexandria (c. 60 BCE),” a name usually given to the Greek grammarian Didymus Chalcenterus (c. 63 BCE – 10 CE). In the 2nd and 3rd centuries CE, Ptolemy and Porphyry named *Didymos ho mousikos* (“Didymus the musician”) as their source for the 4:5 major third. Scholars now generally believe that the Didymus named by Ptolemy and Porphyry was in fact a different man, a mid-1st century CE Roman grammarian and musician from the time of Nero. Karg-Elert may have read about Didymus in August Wilhelm Ambros’ *Geschichte der Musik*. In any case, Karg-Elert often uses the term Didymean (*didymische*) to name the system of pitch relations based on the pure third; other synonymous terms for the same system are harmonic (*harmonische*) and especially syntonic (*syntonische*), named after the syntonic comma (see below). Karg-Elert also names “the Persians and Arabs” (*Ak* 5.1) as another source of the Messel ratio 4:5.

The following diagram from *Akustische* 5.1 illustrates the generation of the consonant triads (Figure 3.4.1):

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31 Richter 2017.
32 Ambros 1862, 308.
Figure 3.4.1. Consonant triads generated within the first six partials above C and below e³

The *Prime* is the generating pitch of a consonant harmony: the lowest note of a major triad, and the highest note of a minor triad (conceived in root position). The concept of prime thus reflects the polaristic nature of harmony. In contrast, a triad’s *Grundton* or root (see Figure 3.2.4 above) is always its lowest pitch, in both major and minor. Descriptions and names for the minor triad vary among different models of harmonic dualism. Oettingen named minor triads after their generating pitch (that is, their highest pitch), and classified them as “phonic” based on his concept of *phonicity*, wherein the pitches of the minor triad share the generating pitch-class (not the fundamental) as a common overtone. ³³ Thus, the A minor triad was called “E phonic” by Oettingen, and labelled as eᵪ, with the small circle indicating that the triad is generated below the E. ³⁴ Riemann also named minor triads by their upper generating pitch, and used almost the same labelling system as Oettingen: A minor is notated as eᵪ instead of eᵩ. ³⁵ In contrast to Oettingen and Riemann, Karg-Elert names minor triads and keys in the traditional manner, after their roots rather than their primes; the triads outlined in Figure 3.4.1 are called C major and A minor. This may seem inconsistent, especially for someone who stressed the consistency of his polaristic theory of harmony, and repeatedly denounced Riemann’s minor-key compromises between dualism and monism (see section 4.3 below). Karg-Elert explains that the individual tones in a

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³³ Oettingen 1866, 27–35 and 44–47.  
³⁴ Ibid., 46.  
³⁵ Riemann 1880, 3.
consonant harmony are “basically oscillating substances, subject to the law of gravity: heavier matter predominates over lighter matter, longer waves predominate over shorter waves. Thus also do the deeper tones overwhelm the higher ones…regardless of what harmonic value these have” (*Harmonologik*, 51). The notion of weight is reciprocal to that of energy: “the major triad is a symbol of harmony, increasing in energy and decreasing in weight, while the minor triad is a symbol of harmony, increasing in weight and decreasing in energy.” In other words: major triads strive upwards from the prime, and minor triads strive downward from the prime, but both have the greatest weight in their lowest notes. Karg-Elert concludes this discussion of *Prime* and *Grundton* by invoking the concept of ideal (*Wesen*) and manifestation (*Erscheinung*), discussed above in section 2.4: the prime is the “harmonic prime” or sound root of consonant harmony considered as a category or ideal, while the root (*Grundton*) is the “chordal prime” of a specific concrete manifestation of consonant harmony.

In Figure 3.4.1, the harmonic 4:5 third is indicated in two ways: using a closed notehead, and with a “comma” or short diagonal. In many examples in *Akustische* (but much less frequently in *Harmonologik*), Karg-Elert uses closed noteheads to indicate pitches that are Didyme (third-derived) rather than Pythagorean (fifth-derived) in origin. In the C major triad, the E is a pure third above C (4:5, or 64:80), instead of four fifths above C (64:81). The difference between these two versions of E is the interval 80:81, called the *syntonic comma*. To be exact, the E in the C major triad is a syntonic comma lower than the E four fifths above C. The syntonic comma is the interval of 18 µ: very slightly smaller than the Pythagorean comma. *Akustische* 5.4 lists syntonic comma differences between various Pythagorean and Didyme intervals, beginning with the major third:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Value</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pythagorean (64:81)</td>
<td>340 µ</td>
<td></td>
</tr>
<tr>
<td>Didyme (4:5)</td>
<td>322 µ</td>
<td>Difference: 18 µ (syntonic comma)</td>
</tr>
</tbody>
</table>
In the major triad, the third is above the prime, and lowered by one syntonic comma (in relation to the fifth-derived pitch of the same name). Karg-Elert’s acoustic symbol for this is a “lowering comma” (*Senkungskomma*) or grave accent (‘), written above the note name or notehead. In contrast, in the minor triad the third is below the prime, and is raised by one syntonic comma. The acoustic symbol is a “raising comma” (*Hebungskomma*) or acute accent (´), written below the note name or notehead, as shown in Figure 3.3.5. These symbols were first introduced in an English acoustics treatise from 1876 by R.H.M. Bosanquet. However, Karg-Elert may have borrowed the symbols from Oettingen’s *Das duale Harmoniesystem*, which denotes syntonic commas using lines above and below note names (see Figure 3.4.4 below), but employs Bosanquet’s comma symbols next to noteheads, and also within key signatures, as in the following from a Palestrina motet (Figure 3.4.2):

![Figure 3.4.2. Oettingen’s key signatures with syntonic comma symbols (Oettingen 1913, 31)](image)

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36 Bosanquet 1876, 11.
37 Oettingen 1913, 7.
In the Palestrina passage, the key is B-flat major, indicated as usual by the two flats. The pitch classes D, G and A are marked with lowering commas, because they are the pure thirds above B♭, E♭ and F respectively, and thus will be a syntonic comma (18 µ) lower than their canonic counterparts. Karg-Elert calls syntonic-comma altered pitches “variants” (Varianten), as they acoustically diverge from their fifth-derived counterparts. Unlike Oettingen, Karg-Elert does not place comma symbols in key signatures, likely because the acoustic identity of pitch classes can vary widely within a piece of music (see Ak 5.6 and many other sections): for example, in the key of B-flat major, D can possibly enter as the pitch four fifths above B♭ instead of the third above B♭, depending on the harmonic context. In any case, Karg-Elert would likely regard a polyphonic passage such as Figure 3.4.2 as a confluence of melodies, and therefore comprised entirely of Pythagorean intervals, without syntonic commas.

The next step in the construction of Karg-Elert’s pitch space is the derivation of pitches from combinations of pure fifths and thirds (described in Ak 5.5 and 5.6). Such combinations were present in the Oettingen example in Figure 3.4.2, which contained the following:

- D = one third above the tonic B♭
- G = one fifth below + one third above the tonic B♭
- A = one fifth above + one third above the tonic B♭

In this dissertation (and also in the annotated edition of Akustische), the acoustic derivation of pitches will be notated as \( P_{(x,y)} \), meaning that pitch P is located x fifths and y thirds from the central pitch. When that central pitch is C, it is indicated as C(0). The value for x is equivalent to Karg-Elert’s dots, and the value for y is equivalent to his commas. The following example from Ak 5.6 demonstrates the acoustic derivation for three different versions of D (Figure 3.4.3):
Figure 3.4.3. Three different versions of D, with acoustic derivation and $\mu$ values (Ak 5.6)

In Figure 3.4.3, C(0) is the center. The first D is Pythagorean (canonic), located two pure fifths above C(0). The other two D’s are derived from combinations of fifths and thirds, as indicated by the acoustic symbols:

- $D_{(2,0)} = 2$ fifths above C(0) $\mu = 170$
- $D_{(-2,1)} = 2$ fifths below + 1 third above C(0) $\mu = 152$
- $D_{(6,-1)} = 6$ fifths above + 1 third below C(0) $\mu = 188$

The $\mu$ values specify the syntonic comma differences: $D_{(-2,1)}$ is one syntonic comma (18 $\mu$) lower than the canonic pitch $D_{(2,0)}$, while $D_{(6,-1)}$ is one syntonic comma higher than $D_{(2,0)}$. Remember: a third above the generating pitch is a syntonic comma lower, and a third below the generating pitch is a syntonic comma higher. In Figure 3.4.3, all of the D’s are notated with closed noteheads, though only the latter two are syntonic-comma variant pitches. The concept of the Ursprungsline or “source registers” (and its use of specific octaves) is discussed in section 3.7.
Pitches derived from combinations of fifths and thirds define a two-dimensional model of pitch space that is a central element in German harmonic theory, one which has been graphically represented in various diagrams, now collectively called the *Tonnetz*. Many recent American scholars have explored in detail the origins and historical development of the fifth-third Tonnetz, from the initial formulation of a fifth-third pitch space by mathematician Leonhard Euler, to tables of harmonic relations created by Gottfried Weber, and finally to the acoustically-based grid designs presented in treatises by Ernst Naumann, Oettingen, Ottokar Hostinsky and Riemann, among others. At its root, the Tonnetz is a schematic representation of acoustic pitch derivation and intervals. However, when those pitches and intervals are combined into harmonies (normally consonant triads), the Tonnetz becomes a map of harmonic perception and understanding, graphically and metaphorically reflecting tonal distances traversed in music.

Though recent American formulations of the Tonnetz by Richard Cohn and others are adapted for 12-tone equal temperament, original versions of the Tonnetz were based in just intonation, with endless series of fifths and thirds, as in the classic diagram from Oettingen’s 1866 treatise *Harmoniesystem in dualer Entwicklung* (Figure 3.4.4):

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38 See also sections 2.1.5 and 2.1.6 of Lewin 1987, which examine the derivation and representation of pitches in a just intonation space created from pure fifths and thirds.
40 Described in Gollin 2006.
41 The most comprehensive description of the Tonnetz and its historical development is Mooney 1996, especially with regard to the work of Riemann. Also see Cohn 2011, and Gollin 2011.
42 Cohn 1997.
The notation “$5^m \ 3^n$” refers to pitch derivation: the $n$ axis denotes the number of fifths (labelled “3” after the third overtone), and the $m$ axis denotes the number of pure thirds (“5” after the fifth overtone). The numbers in the $n$ axis (shown here from -8 to 8) indicate the number of fifths away from the central pitch, which in this diagram is C. The numbers in the $m$ axis (2 to -2 shown here) indicate the number of thirds from the central horizontal row. The pitches are of course named in German: double sharps (such as fisis, cisis…) and double and triple flats (bb, eses, asas…bbb) are notated as such, without enharmonic adjustment. All pitches in the grid are acoustically and conceptually distinct, even those with the exactly same names, such as the different versions of C in each horizontal row. Finally, the lines above and below the pitch names in all but the central row ($5^0$) indicate syntonic comma differences. Oettingen’s notation here is slightly counter-intuitive: each line written above a note represents one third above the center, which accordingly lowers the pitch by a syntonic comma. Conversely, each line below represents a third below the center, and is thus raised by a syntonic comma. Oettingen’s later
use of lowering and raising comma symbols (adopted by Karg-Elert) is preferable, as they point in the direction of the syntonic comma difference (down or up).

Oettingen’s *Das duale Harmoniesystem* includes a slightly different version of his fifth-third Tonnetz (Figure 3.4.5):

![Figure 3.4.5](image)

**Figure 3.4.5.** “Note spellings with logarithms of frequencies”: Tonnetz with intervals calculated in millioctaves (Oettingen 1913, 250)

In this diagram, the central pitch is D; this was usually favored by Oettingen, as D is the axis of symmetry between flats and sharps. Apart from the shift of center, the middle five rows are the same as in Figure 3.4.4, ranging horizontally from -8 to +8 fifths, and vertically from 2 to -2 thirds. The added partial rows at the top and bottom are +3 and -3 thirds from the center, and range horizontally from -4 to +4 fifths. The numerical values are intervals from the center (000), calculated in millioctaves (µ). While the fifths and thirds could extend infinitely in both directions, Oettingen limits his map to these 103 distinct pitches, which include D in five
101 different syntonic-comma variants, plus its enharmonics Ebb and C# in values as close as 2 µ (a micro-interval called the schisma). The central group of nine columns (containing 63 pitches) formed the basis of Oettingen’s just-intonation harmonium or Orthotonophonium,43 which was built in 1914, and contained 53 pitches per octave, omitting ten pitches from the central group of 63 as shown in this diagram (Figure 3.4.6):

![Figure 3.4.6](image)

Figure 3.4.6. “Array of pitches in the 53-note dual just intonation instrument, with frequency logarithms” (Oettingen 1913, 267)

Unlike Oettingen, Karg-Elert did not advocate for the invention and use of just intonation keyboards or other instruments; in fact, in Akustische 11.5 Karg-Elert criticized Oettingen’s

43 Oettingen 1917 is an exhaustive description of this instrument, and its theoretical and acoustic foundations. Its history and production is also described in Goldbach 2007.
Orthotonophonium as being too limited as a just intonation instrument, as it (for example) equates the pure seventh (807 µ) with the augmented sixth (814 µ). Karg-Elert also did not include Tonnetz diagrams in his treatises, even though he would have been very familiar with them from the work of Oettingen and Riemann. This is likely because the Tonnetz cannot easily represent pitches in the three dimensions of Karg-Elert’s pitch space: fifths, thirds and pure sevenths. In place of Tonnetze, Karg-Elert employed acoustic symbols to specify the acoustic derivation of pitches, thus pinpointing their location in pitch space, and reflecting their distance from the center. Later chapters of this dissertation (as well as the annotated edition of Akustische) will frequently employ Tonnetz diagrams, especially to highlight how Karg-Elert’s harmonic function labels reflect pitch-space trajectories in a very exact manner. The Tonnetze to be used here are essentially the same as Figure 3.4.4, with C as central pitch, pure fifths in horizontal rows, and pure thirds in vertical columns. Additional “spokes” will be added when necessary to indicate seventh-derived pitches (see section 3.5), and the harmonies that they generate.
3.5. *Concordant (septimal) pitch derivation*

The final element in Karg-Elert’s pitch space is the pure natural seventh (4:7), which is identified as an integral harmonic interval:

I defend the chord of the dominant seventh as a natural and integral structure, which I call ‘concordant’. It is found in close position within the third octave of partials [i.e. 4:5:6:7], and it includes the natural proportions involving the prime numbers 1:3, 1:5 and 1:7.

(*Harmonologik, 18*)

In *Ak* 9.2, he states that “only this value [i.e. the pure 4:7 seventh] melds and ‘con cords’ with the dominant triad”; for that reason, Karg-Elert calls the 4:7 interval the *concordant* seventh. It is considerably smaller than both the Pythagorean and Didymean minor sevenths (see *Ak* 9.2):

<table>
<thead>
<tr>
<th>Interval</th>
<th>µ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concordant (4:7)</td>
<td>807</td>
</tr>
<tr>
<td>Pythagorean (9:16 = inversion of 8:9 major second)</td>
<td>830</td>
</tr>
<tr>
<td>Didymean (5:9 = inversion of 9:10 major second)</td>
<td>848</td>
</tr>
</tbody>
</table>

The pure seventh is also smaller than the minor seventh in 12-TET (833.33 µ). Due to its wide divergence from most practical tuning systems, many theorists have rejected the 4:7 seventh as a viable harmonic interval, even though it can be justified acoustically due to its prominent presence in the harmonic series, and mathematically as a ratio of low whole numbers. Zarlino famously stated that the *Senario* (the proportions 1:2:3:4:5:6) encompasses all the consonant intervals, and outlines the perfect harmony. While a few notable musicians such as Tartini\(^\text{44}\) and Kirnberger\(^\text{45}\) considered the natural seventh as a possible source of the dominant seventh chord, most of Zarlino’s successors eschewed the 4:7 interval, and sought alternative explanations of the dominant seventh chord.\(^\text{46}\)

\(^{46}\) Vogel 1991 is an exhaustive historical survey of the natural seventh, both as an interval in practical music, and as a matter of speculation for theorists and mathematicians. The book is an expansion of Vogel’s 1954 doctoral dissertation “*Die Zahl Sieben in der spekulativen Musiktheorie*” (University of Bonn).
Several of Karg-Elert’s predecessors expressed considerable enthusiasm for the natural seventh. Helmholtz noted that it is consonant to a significant extent: “the subminor seventh 4:7 is very often more harmonious than the minor sixth 5:8.” However, he also stated that the four-note chord 4:5:6:7 is unsuitable as a consonant harmony, due to the dissonant effect of its other constituent intervals: “the subminor seventh when combined with other consonances in chords produces intervals which are all worse than itself, as 6:7, 5:7, 7:8 etc., and it is consequently not used as a consonance in modern music.”47 In his Tonpsychologie, Carl Stumpf stated that the natural seventh fuses with other harmonic intervals at least as much as do thirds and sixths: “At most, the so-called natural seventh (4:7) might even blend a little more than the others.”48 Stumpf further suggests that “it is conceivable and even likely, that the tonal proportions at the lowest level of fusion [i.e. intervals involving the natural seventh] will increasingly be perceived more subtly, and may be exploited more in practice; in other words, that 4:7 (7:8), 6:7 (7:12) and the like will be regarded as consonances. This would of course result in a complete transformation of our musical system…”49 Riemann went even further, stating that the natural 4:5:6:7 dominant seventh chord is unequivocally an acoustic consonance, and “exceeds even the major chord in equal temperament in physical euphony.” However, in spite of its consonant quality, Riemann rejects the natural seventh because it cannot be reconciled with musical practice, which is based in equal temperament: “4:5:6:7 is a musical consonance, [but] it can never be accepted from science by art…it cannot make sense to introduce an untempered seventh next to a tempered third and tempered fifth.”50 Riemann instead explained the dominant seventh

47 Helmholtz/Ellis 1885, 195.
48 Stumpf 1890, 135.
49 Ibid., 177.
50 Riemann 1879 (translated by Benjamin Steege in Gollin and Rehding 2011, 75).
chord as a major triad built on the fifth above the tonic, plus the “characteristic addition” of the fifth below the tonic;\textsuperscript{51} the minor seventh interval in this case is the Pythagorean 9:16 (the inversion of the 4:9 major ninth). For his part, Oettingen did not mention the natural seventh (or the dominant seventh chord) at all in his 1866 treatise Harmoniesystem in dualer Entwicklung. His 1913 Das duale Harmoniesystem explains the dominant seventh in a manner similar to Riemann, as a dissonant combination of pitches from two different consonant triads, and containing a Pythagorean minor seventh interval.\textsuperscript{52} For example (Figure 3.5.1), in the key of C major, the dominant seventh combines a major triad on G (notated by Oettingen as $g^+$) with the F from F minor (notated in dualistic fashion as $c^0$):

\[
\begin{align*}
g & - h & - d - f = g^+ + c^0 \\
g & - c & - e - g = c^+
\end{align*}
\]

**Figure 3.5.1.** Oettingen’s explanation of the dominant seventh as a mixture from two different consonant triads (Oettingen 1913, 67)

Uniquely among his contemporaries, Karg-Elert regarded the natural seventh as a consonant (or ‘concordant’) harmonic interval, and fully incorporated it into his model of pitch space and harmonic relations. He rejected the notion of the dominant seventh as a dissonant combination of pitches from two different triads. He explained the dominant seventh as an integrated consonance by referring not only to the mathematical simplicity of the 4:5:6:7 proportions, but also to perception and practical music-making. In Harmonologik, he recounts the experience of a “budding student of harmony” (likely Karg-Elert himself):

\textsuperscript{51} Riemann 1893, 55. See section 4.6 below.
\textsuperscript{52} Oettingen 1913, 67.
Prof. Krehl\textsuperscript{53} indicated, to a budding student of harmony, the natural seventh as ‘harmonically just as unnatural as a “simply hideous” sounding element’. Quite soon [literally two or three days] after that the very conscientious candidate could be convinced by two organ stops…and by a horn quartet, what the state of affairs is with the ‘unnatural’ naturalness of the harmonic seventh…tuned pure, it blends with the fundamental! (Harmonologik, 19)

He also explains how musicians choose to play the natural seventh in certain contexts:

In the Eroica (Trio from the Scherzo), [conductor Artur] Nikisch always left the seventh ‘plain’ in the second horn, i.e. played as a natural tone; this tone, notated B-flat (on the E-flat horn sounding as D-flat) is slightly but noticeably flat in relation to the octave or fifth. This chord merges quite obviously into a tonal unity (‘concordance’) …Many brass players with sensitive hearing know this quite well, for they pitch the seventh in unaccompanied ‘horn passages’ differently from when they play in combination with tempered instruments, or when the tone changes its harmonic function. (Harm, 15)

Here is the passage just described, from Beethoven’s Symphony No. 3 (Figure 3.5.2):

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure352.png}
\caption{Natural dominant seventh in Beethoven, Symphony No. 3 (II, mm. 229-239)}
\end{figure}

The following example from Ak 9.1 illustrates the natural concordant seventh as an integral harmonic tone, added above a G major triad, and below a D minor triad (Figure 3.5.3):

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure353.png}
\caption{The concordant seventh chord (Akustische 9.1)}
\end{figure}

\textsuperscript{53} Stephan Krehl (1864-1924) was a music theorist and composer, author of Riemann-influenced textbooks, and professor at the Leipzig Conservatory of Music for many years.
Karg-Elert denotes seventh-derived pitches in two ways: using wedge acoustic symbols (\(\vee\) for sevenths above the central pitch, \(\wedge\) for sevenths below), and diamond-shaped noteheads. The concordant seventh (807 \(\mu\)) differs from the Pythagorean minor seventh (830 \(\mu\)) by a comma of 63:64 or 23 \(\mu\), which is slightly larger than the Pythagorean comma. This interval is most often simply called the *septimal comma*, though it is also known more specifically as Archytas’ comma, as there are in fact several small comma intervals involving the natural seventh.\(^{54}\) Karg-Elert named it the “Leipzig comma” (*Ak* 9.3), after his lifelong city of residence.

Though concordant sevenths are typically included when triads have certain harmonic functions (especially dominant, but also altered subdominant, as will be discussed in section 4.6 of this dissertation), any consonant triad can include its concordant seventh, generated above the prime in major, or below the prime in minor (as shown in Figure 3.5.3). As a result, any pitch can be derived from a combination of pure fifths, thirds and sevenths. Karg-Elert indicates such derivation using his three acoustic symbols (dots, commas and wedges), as shown in the following example from *Ak* 9.1 (Figure 3.5.4):

\[\text{Figure 3.5.4. Concordant seventh chords, with acoustic symbols for each pitch (Ak 9.1)}\]

\(^{54}\) Fokker 1955.
The numerical data in Figure 3.20 illustrate how the interval sizes (Abstände) in the concordant seventh chord continually decrease: 322 → 263 → 222 µ. To denote pitch derivation involving sevenths as well as fifths and thirds, this dissertation (and the annotated edition of Akustische) will use the notation \( P_{(x,y,z)} \), which means that pitch \( P \) is located \( x \) fifths, \( y \) thirds and \( z \) pure (4:7) sevenths away from the central pitch, most often \( C(0) \). For example:

- \( B♭_{(0,0,1)} \) is the concordant seventh above \( C(0) \)
- \( D_{(0,1,1)} \) is the concordant seventh above \( E_{(0,1)} \)
- \( F♯_{(0,1,-1)} \) is the concordant seventh below \( E_{(0,1)} \)

If no seventh is shown in the coordinates, then the pitch is entirely fifth- and/or third-derived. The Tonnetz can also display seventh-derived pitches, using “spokes” that extend diagonally from the generating pitch, and represent a three-dimensional pitch space (Figure 3.5.5):

**Figure 3.5.5.** Tonnetze for concordant seventh chords, including the natural seventh

The features of the three-dimensional pitch space will be discussed further in section 3.7.
German scholar Martin Vogel (1923-2007) promoted an identical model in several books\textsuperscript{55}, and his \textit{On the Relations of Tone} includes a just-intonation three-dimensional \textit{Tonnetz}.\textsuperscript{56} Dutch organist and physicist Adriaan Fokker (1887-1972) also explored the harmonic possibilities of the fifth-third-seventh pitch space, in articles dating back to the 1950s.\textsuperscript{57} More recently, Edward Gollin has written about properties of three-dimensional \textit{Tonnetze}, with particular focus on the representation of transformations between (0258) tetrachords (i.e. the dominant and half-diminished seventh chords).\textsuperscript{58} Though Gollin’s model is based in equal temperament, his seventh-chord transformations (and his notation for inversional chord-tone exchanges) are very similar to those of Karg-Elert, as will be discussed in section 5.4.

\textsuperscript{55} Especially in Vogel 1991, but also Vogel 1975. Vogel acknowledges Karg-Elert’s inclusion of the pure seventh as a generating harmonic interval: “Sigfrid Karg-Elert, with his polaristic theory of harmony, came closest of all other harmony teachers to the tone space of the fifth-third-seventh system” (Vogel 1975, 294). Karg-Elert’s pitch space is also briefly discussed in Vogel 1991 (pp. 237–238).

\textsuperscript{56} Vogel 1975, 125.

\textsuperscript{57} Fokker 1955.

\textsuperscript{58} Gollin 1998. Gollin’s three-dimensional space (and its Tonnetz representation) is discussed further in Chapter 4 of Julian Hook’s \textit{Musical Spaces and Transformations} (forthcoming); see Hook’s Figure 4.17.
3.6. Metharmonics and enharmonics

In standard musical practice (based in 12-tone equal temperament), enharmonicism is above all a matter of notation. In 12-TET, pitch pairs such as G♯ and A♭ are sonically identical; while their different spellings imply different melodic or harmonic functions in tonal music, they are nonetheless understood to be equivalent in certain respects. Enharmonic reinterpretation in 12-TET enables common tricks of chromatic harmony, such as remote key shifts involving diminished seventh chords, or modulations exploiting the sonic identity of the German sixth and dominant seventh chords. In some cases, enharmonic respelling has no harmonic meaning at all, but is simply employed to aid legibility, by avoiding complex accidentals and key signatures. Karg-Elert calls this practice “apparent enharmonicism,” since it is “chosen simply for reasons of more convenient legibility; conceptually, a fundamental revaluation of the true values does not occur” (Ak 8.7). A familiar case of such apparent enharmonicism is Schumann’s Lied “Widmung” (from Myrthen, op. 25), which is originally in the key of A-flat major; the middle section is notated in E major, but that notation really represents F-flat major, the flat submediant (♭VI), connected directly to the main key by the common tone A♭ (respelled as G♯).

In just intonation, the type of enharmonicism familiar in 12-TET does not exist, as every pitch is distinct in derivation and intonation. Instead of pitch frequencies having multiple names as in 12-TET, in just intonation pitch names can have multiple frequencies, due to their differing acoustic origins. While complex accidentals can be used in some cases to distinguish between similar pitches of different derivation (such as C♯, D and E♭♭), it is impossible to use standard flats and sharps to accurately indicate all such differences. There will remain cases where a single pitch name (such as D) will have multiple derivations and tunings. Many instances can be
found on Oettingen’s *Tonnetze* (Figures 3.4.4 and 3.4.5), and Figure 3.4.3 presented the following three different versions of D:

\[
D_{(2,0)} = 2 \text{ fifths above } C(0) \quad 170 \mu \\
D_{(-2,1)} = 2 \text{ fifths below } + 1 \text{ third above } C(0) \quad 152 \mu \\
D_{(6,-1)} = 6 \text{ fifths above } + 1 \text{ third below } C(0) \quad 188 \mu 
\]

Karg-Elert calls these like-named but distinct pitches metharmonics (*Metharmose*), a term referring to any pitches of exactly the same name that have different acoustic derivation. He borrowed the term from Oettingen’s *Das duale Harmoniesystem*, where it has exactly the same meaning.\(^{59}\) While the term was original to Oettingen, theorists and musicians have recognized the concept for centuries. In a just-intonation diatonic collection derived from fifths and thirds, \(D_{(2,0)}\) is acoustically distinct from \(D_{(-2,1)}\) – they differ by a syntonic comma (18 \(\mu\)). In other words, in a just-intonation C major, the fifth of V is not identical to the root of II (Figure 3.6.1):

\[\begin{align*}
D^1 & = \{170\} \\
D^{(-2,1)} & = \{152\} \\
\end{align*}\]

**Figure 3.6.1.** Metharmonic versions of D in the fifth/third diatonic collection (*Ak 6.3*)

Like the two versions of D just described, metharmonic pitches will always differ from each other by one or more syntonic commas, by one or more septimal commas, or by some combination of both commas; there should never be two different fifth-related pitches of exactly the same name, except possibly in cases of apparent enharmonicism, where note spelling does

\(^{59}\) Oettingen 1913, 51.
not accurately reflect pitch derivation. The following are three more metharmonic versions of D, with their μ values:

\[
\begin{align*}
D_{(6,2)} &= 2 \text{ thirds above } Gb_{(6,0)} & 134 \mu \\
D_{(0,1,1)} &= 1 \text{ seventh above } E_{(0,1)} & 129 \mu \\
D_{(0,0,-1)} &= 1 \text{ seventh below } C(0) & 193 \mu
\end{align*}
\]

The six different versions of D that have been listed are no more than two syntonic commas or one septimal comma distant from the center C(0); numerous additional metharmonics of D are possible, even without going too much further from the center. The widely differing μ values for D (a range of more than 60 μ) reflect how a single pitch class can have very different harmonic functions, depending on musical context. **Figure 3.6.2** tabulates four different metharmonic versions of F♯ (with their acoustic symbols and μ values), and presents some harmonic contexts (indicated by the function labels) in which each version of F♯ is most likely to occur:

![Figure 3.6.2](image)

**Figure 3.6.2.** Four different metharmonic versions of F♯, in typical harmonic contexts (*Ak 8.6*)

\( T \) is tonic in major keys, and \( L \) is tonic in minor keys. The other harmonic function labels in Figure 3.6.2 will be described in Chapters 4 and 5 of this dissertation.

Karg-Elert describes how musicians will sometimes “metharmonically” adjust the tuning of a single pitch “when a tone of unchanging name changes its harmonic function” (*Harm*, 15), as in the following example written for four horns (**Figure 3.6.3**):
In the first chord, the B♭ is the concordant seventh of C7, and therefore should be played as the natural seventh B♭(0,0,-1), the seventh overtone of C (played without valves). However, in the second chord the B♭ becomes a tonic, and should be played as B♭(2,0), the eighth overtone of B♭ (played with the first valve). The two versions of B♭ differ by a septimal comma (23 µ).

According to Karg-Elert, the primary significance of metharmonicism is “the conceptual equalization of comma-differing tones or chords that have the same name” (Ak 8.1). This statement can be understood from two different directions. On the one hand, it can be read as the mental cancelling of comma differences, and as the identification of pitches performed with slightly differing frequencies under the umbrella of a single pitch name; this reading accords very well with equal temperament, which evolved precisely in order to eliminate comma differences. Alternatively, the musical ear may continue to perceive comma differences between metharmonic pitches, but will actively work to reconcile such pitches when they are juxtaposed, while still recognizing the subtle sense of acoustic “shift” that they engender. This second view reflects a just-intonation perspective, and not surprisingly it is how Karg-Elert understands metharmonic juxtapositions in harmonic contexts (Figure 3.6.4):
In Figure 3.6.4, the sequence in rising diatonic steps is indicated by the brackets above the upper system. The sequence is in C major, and the function labels relate to that key. The lower system analyzes the progression in terms of the *Ursprungslagen* or “source positions” in just-intonation pitch space; the *Ursprungslagen* concept will be examined in section 3.7. What matters to the present discussion are the pitch pairs marked by dotted lines, which occur at the junctures marked (Met.) for metharmonic, and also noted with a break (/). They are metharmonic pairs, their pitches always differing by one syntonic comma:

\[
\begin{align*}
D_{(2,1)} & \rightarrow D_{(2,0)} & F_{(-1,0)} & \rightarrow F_{(3,-1)} & B_{(3,2)} & \rightarrow B_{(1,1)} \\
F_{(-1,0)} & \rightarrow F_{(3,-1)} & A_{(-2,1)} & \rightarrow A_{(3,0)} & D_{(-2,1)} & \rightarrow D_{(2,0)} \\
\end{align*}
\]

The two [B, D, F] diminished triads (bars 1 and 4) are dissonant alterations of consonant triads:

- Bar 1, beat 4: \( F_{(3,-1)} \) in place of \( F^\#_{(2,1)} \)
- Bar 4, beat 1: \( B_{(-3,2)} \) in place of \( B^b_{(-2,0)} \)
Such metharmonic shifts in a diatonic passage like Figure 3.6.4 are absolutely necessary to ensure tonal closure; without making metharmonic adjustments, beginning and ending in the exact same key (without syntonic comma slippage) would become very difficult, as pitches and harmonies would travel through pitch space ever further from the starting point. This is the sense behind one of Karg-Elert’s more elusive statements: “metharmonics always reveal themselves as if ‘metaphorically speaking’, when open or infinite chord spirals become closed, backward-tracing chord rings” (Ak 8.4). As will be seen in chapter 6 of this dissertation, Karg-Elert reveals how some passages which appear to be tonally unified and closed “rings” (based on their notation) are in fact tonally open “spirals,” as their opening and closing keys are not identical, but rather metharmonic (comma-differing) keys of the same name.

For Karg-Elert (again following Oettingen’s Das duale Harmoniesystem), true enharmonics are similar pitches that have different acoustic derivations, as well as different names.\(^{60}\) Enharmonics will usually differ by one or more syntonic commas, though that is not always the case – for example, the fifth-related enharmonics G♯\(_{8,0}\) and A♭\(_{-4,0}\). Compared to metharmonics which abound in all types of harmonic progression, true enharmonics are quite rare, as they are necessarily quite distant from each other in pitch space. Some examples are:

- 12 fifths apart (1 Pythagorean comma): G♯\(_{8,0}\) and A♭\(_{-4,0}\)
- 3 pure thirds apart (3 syntonic commas): G♯\(_{0,2}\) and A♭\(_{0,1}\)
- 4 fifths + 2 thirds apart (2 syntonic commas): G♯\(_{8,0}\) and A♭\(_{4,-2}\)

The following enharmonics involve 2 syntonic commas and one septimal comma:

- 2 fifths, 2 thirds and 1 seventh apart: G♯\(_{2,1,-1}\) and A♭\(_{0,-1}\)

---

\(^{60}\) Oettingen 1913, 104.
The following example (Figure 3.6.5) demonstrates a true enharmonic, in a highly chromatic progression that modulates through several keys:

Tonnetz: 1 2 (1) 3 4 5 6 7 8 9 10

\[
\begin{align*}
C\# & \quad D^b \\
F(6,0) & \quad A(7,1) & \quad C(8,2) & \quad G(9,2) & \quad B_b(10,3) & \quad D(11,4) & \text{major}
\end{align*}
\]

**Figure 3.6.5.** Modulatory passage beginning in F\# major, ending on D\# major (*Ak 8.8*)

Details of the function labels and the notation for modulation will be explained in later chapters.

Pertinent to this discussion is the acoustic identity of Chords 2 and 10, marked above the staff:

the C\# major chord in bar 1 (which should contain E\#), and the D\# major chord in bar 4. They are true enharmonics of each other, not simply because of their different names, but also because their acoustic derivation is very different (4 fifths + 4 thirds apart from each other):

- **C\# major chord (bar 1, beat 2):**
  \[
  C(7,0) \quad E(7,1) \quad G(8,0)
  \]
  95 \(\mu\) 435 \(\mu\) 680 \(\mu\)

- **D\# major chord (bar 4, beat 3):**
  \[
  D(11,4) \quad F(11,3) \quad A_b(12,4)
  \]
  147 \(\mu\) 487 \(\mu\) 732 \(\mu\)
The distance between the two enharmonic chords (which includes four syntonic commas) is highlighted on the Tonnetz for the passage (Figure 3.6.6):

![Tonnetz Diagram](image)

**Figure 3.6.6.** Tonnetz for Figure 3.6.5, highlighting the distance between enharmonic chords 2 (C♯ major) and 10 (Db major)

As described previously, many examples of enharmonic respelling are simply “apparent,” and do not reflect real differences of acoustic pitch derivation. Both metharmonics and enharmonics (real and apparent) play significant roles in chromatic modulations, which Karg-Elert classifies as comma-free or comma-differing (described in Chapter 6 of this dissertation). The foregoing discussion should make clear that to precisely account for metharmonic and enharmonic differences in harmonic passages (without simply tabulating the acoustic derivation or υ value for every pitch), an analytical system that can accurately reflect trajectories in just-intonation pitch space is needed. With this purpose in mind, Karg-Elert adopted Riemann’s method of functional analysis, altered and expanded in various ways to accommodate harmonic relations in three dimensions. Before functional relationships can be examined, we need to specify the nature of the center of Karg-Elert’s pitch space, to which all other pitches and harmonies are related.
3.7. A three-dimensional, infinite and fixed pitch space

The previous sections in this chapter have described the construction of a three-dimensional pitch space, in which each axis (fifths, thirds and sevenths) is potentially infinite, generating a theoretically endless number of acoustically distinct pitches. In *Harmonologik*, Karg-Elert suggests that for analytical purposes, a pitch space of 275 pitches would encompass most (though not all) harmonic relationships generally found in musical passages: a horizontal fifth-axis spanning 25 pitches with different note names from E♭ to C♯, plus ten different metharmonics (syntonic and septimal comma variants) for each of those 25 Pythagorean pitches, making 275 pitches in total (or really 275 pitch classes, due to octave equivalence). In Appendix A of *Akustische*, Karg-Elert lists 303 distinct pitches within the octave from c¹ to c², including most of the 275 outlined in *Harmonologik*, 22 equal-tempered pitches, plus some additional double-sharp and double-flat pitches. On the question of how many pitch combinations are possible, Karg-Elert chided “certain preachers on Atonality,” who enthusiastically pointed to the “over 479 million different possibilities of grouping” in the 12-tone collection. He states that “if instead of 275 original values I take about 30 only – values which any normal ear is easily able to differentiate…then I obtain by permutation such gigantic numbers, against which the 4¾ hundred millions appear a bagatelle!” Indeed, the number of permutations in a 30-tone pitch collection is 30!, or approximately 15.5 septillion (15.5 * 10²⁴); the harmonic combinations possible in a 275-note collection will approach infinity. In a practical pitch space containing at least 275 distinct pitches, the analyst will surely ask: from

---

61 *Harmonologik*, 30.
62 Ibid., 43. Karg-Elert’s target here is likely Josef Matthias Hauer, who repeatedly cited the figure of 479,001,600 permutations of the 12-tone collection (see for example Hauer 1925, 12).
63 Ibid., 44.
which point in the space should pitch and harmonic relationships be calculated? How is the center of the space defined?

Karg-Elert places the major third $c^1 + e^1$ (middle C and the third above) at the center of his “infinitely-wide harmonic space” (Ak 6.1). He calls this third “the harmonic prime cell in its source register” (Figure 3.7.1):

![Figure 3.7.1](image)

**Figure 3.7.1.** The “harmonic prime cell in its source register” (Ak 6.1)

In *Harmonologik*, Karg-Elert notes that this specific third lies in the middle of standard instrumental pitch ranges: “I start *a priori* from a symbol of harmony (the natural third!), and put it in the middle of our tonal range that serves practical music” (Figure 3.7.2):

![Figure 3.7.2](image)

**Figure 3.7.2.** The “central third” in the center of the practical pitch range (*Harmonologik*, 27)

Thus, the placement of $c^1$-$e^1$ at the center is at least partially motivated by its location in the middle of the practical pitch range. But why does Karg-Elert place $c^1$ and $e^1$ at the center of his pitch space, instead of the apparent midpoint $d^1$? Since the work of Arthur von Oettingen was a
significant influence on Karg-Elert’s theories (as the latter repeatedly acknowledged)\(^6^4\), one might expect Karg-Elert to follow Oettingen’s lead, and place D at the center (see Figure 3.4.5). Karg-Elert recognizes Oettingen’s main point: that D is the axis of symmetry between flats and sharps, and thereby reflects major-minor dualism in terms of musical notation (see Figures 3.3.1 and 3.3.2, which do use D as center for that purpose). However, if d\(^1\) is to be adopted as center between c\(^1\) and e\(^1\), a problem arises: what is the acoustic derivation of that D? In other words, what pitch actually is at the midpoint of the c\(^1\)-e\(^1\) syntonic third? The prime cell is a 4:5 harmonic third (322 \(\mu\)), and is the source of the C major and A minor triads (to be discussed in section 4.4 of this dissertation). The Aequator (161 \(\mu\)) at the midpoint of the prime cell is neither canonic D\(_{(2,0)}\) at 170 \(\mu\), nor syntonic D\(_{(-2,1)}\) at 152 \(\mu\). In fact, the Aequator is not an actual pitch in the fifth-third-seventh system – instead, it is the conceptual axis of symmetry for the prime cell, and thus for the entire pitch space. Instead of the hypothetical Aequator D (which does not exist in reality, and thus cannot generate harmonies), Karg-Elert’s center comprises two real pitches that together generate the major and minor triads. He presents several other criticisms of Oettingen’s D axis, which he regards “incomprehensible”:

1. “The allegedly ‘absolute’ unisons are degraded to relative ninths”
2. “D major and G minor are regarded as central keys”
3. “C major climbs upwards (!) to A minor (via G major), or A minor climbs downwards to C major (via D minor)”

He diagrams these criticisms in the following example (Figure 3.7.3):

\(^6^4\) The earliest such acknowledgment of Oettingen’s influence is in the preface to Grundlagen Part I (p. III), where Karg-Elert describes Oettingen as “much more systematic” than Riemann.
On each of the above criticisms:

1. By “absolute unisons,” Karg-Elert is referring to the C major and A minor triads, which as polar counterparts of the same diatonic collection should share exactly the same major third C and E (with the same acoustic derivation). However, if d\textsuperscript{1} is the center, the pitch classes C and E are generated as two fifths (or “relative ninths”) on either side. C (the lower ninth) would generate a major triad, while E (the upper ninth) would generate a minor triad. The pitch classes C and E between the two triads would be metharmonics, not acoustically equal. On the other hand, the c\textsuperscript{1}-e\textsuperscript{1} harmonic pitch cell generates C major and A minor triads that share identical pitches.

2. If the single pitch d\textsuperscript{1} acts as prime of both a major and minor triad, a D major / G minor system would result. Unlike C major and A minor, the keys of D major and G minor do not share a pitch collection.

3. If D is center, the acoustic distance between C major and A minor (already expressed in criticism no. 1) means that they share no common tones. Major must travel up to minor (and vice versa) through a chain of diatonic thirds, instead of via a direct common tone connection (Figure 3.7.4):

**Figure 3.7.3.** Karg-Elert’s criticisms of Oettingen’s D center (*Harmonologik, 27*)

**Figure 3.7.4.** The “simultaneous third” center generating the “central harmonies” C major and A minor (*Harmonologik, 28*)
Another likely reason for Karg-Elert’s placement of c1-e1 is his view of C major/A minor as a neutral or “colorless” key (Ak 2.4), to which all other keys (such as the “silvery glittering key of B major”) are to be compared and related. In effect, Karg-Elert’s harmonic space is fixed, always based around the c1-e1 cell and the keys of C major/A minor, regardless of the key of the actual music under consideration. Evidence of his fixed view of pitch space can be found in Figure 3.7.5, which is the same as Figure 3.6.5:

**Figure 3.7.5.** Modulatory passage beginning in F# major, ending on Db major (Ak 8.8)

The opening key of this example is F# major. Most modes of harmonic and tonal analysis would adopt that key as a central point, to which all other pitches and keys would be compared. But Karg-Elert specifies that the opening tonic pitch is F#(6,0) – six fifths above C(0). While neither C major nor A minor are emphasized in this passage, the music is nonetheless imagined to begin from a position that is already distant from an *a priori* C-E fixed center, and then to proceed ever further from that center (see the *Tonnetz* in Figure 3.6.6). This example (like many other examples in both *Akustische* and *Harmonologik*) demonstrates that for Karg-Elert, harmonic
relationships operate in a fixed pitch space, centered on an unchanging C-E cell. Instead of conceiving the harmonic space around whichever pitch is the primary tonic of the music under consideration, Karg-Elert first locates the music’s primary tonic in relation to the fixed central cell, and then follows the harmonic trajectory of the music from that location.

In Figure 3.7.1, Karg-Elert described the cell c¹-e¹ as the “harmonic prime cell in its source position” (Ak 6.1). The concept of “source positions” (Ursprungslagen in German) is derived from the Pythagorean series of fifths that extends in both directions from middle C. Each pitch in that fifth series is imagined as the prime of a major triad, or as the third of a minor triad. Figure 3.7.6 (from Ak 6.12) has the C major and A minor triads at the center, with major triads in rising fifths to the right, and minor triads in falling fifths to the left:

![Figure 3.7.6](image1)

The idea of the Ursprungslagen is that each major and minor triad “naturally” exists in one specific “source position” along the fifth-chain, centered around C major and A minor. For example, the Ursprungslage of G major is one fifth above C, while that of C minor is three fifths below A minor. Karg-Elert uses registral placement on the staff to denote fifth distances from the center; this is why distant triads such as F♯ major and E♭ minor are notated in extreme high and low ranges. In fact, the Ursprungslagen is a model of distance in pitch space, very much like the horizontal fifth-axis of the Tonnetz (Figure 3.7.7):

![Figure 3.7.7](image2)
Figure 3.7.7. *Tonnetz* of the *Ursprungslagen* triads from Figure 3.7.6

Where the *Tonnetz* uses horizontal space to indicate distance from the center C(0), the *Ursprungslagen* employs register. To be clear: the *Ursprungslagen* do not represent the actual octaves of pitches as they occur in a musical passage. Rather, they represent the acoustic derivation of pitches in pitch space, regardless of the octaves in which they are stated. When we say that a G major triad is in its *Ursprungslage* or source position, we mean that it contains the following specific pitches: $G_{(1,0)}$, $B_{(1,1)}$, $D_{(2,0)}$. The key point to remember is that each *Ursprungslage* triad is generated by the fifth-chain generated from C(0): either as the prime of the major triads, or as the third of the minor triads. Other triads that do not include a pitch from the central series of fifths are comma variants, and therefore not in their *Ursprungslagen*. Figures 3.7.6 and 3.7.7 do not illustrate all of the source positions: both the major and minor triads extend continuously in both directions horizontally from the center. In all cases, a passage’s opening tonic is assumed to be a source position triad (see Figure 3.6.5). Therein lies the major analytical point of the *Ursprungslagen* concept: the source positions represent the pitch-space locations of “primary tonal centers,” as noted in Figure 3.7.6. Examples in later chapters will further demonstrate the concept of the source positions, which is best understood in conjunction with the analysis of harmonic function, the focus of chapter 4.
CHAPTER 4

Function

This chapter examines Karg-Elert’s adoption and revision of Riemann’s theory of harmonic function, which was first proposed in Riemann’s *Vereinfachte Harmonielehre* (“Harmony Simplified”) of 1893. It begins by describing the perceptual basis of major-minor polarity, which is rooted in Karg-Elert’s observation that any harmonic interval can be understood as an incomplete major triad or minor triad, to be mentally completed by the listener either upward (major) or downward (minor). As in Riemann, his three basic functional categories are rooted on the tonic and its two surrounding fifths, called dominant and contradominant by Karg-Elert. However, unlike that of Riemann, his model of harmonic function is fully and consistently dualistic, placing the dominant at the fifth above the tonic in the major mode, but at the fifth below tonic in minor. The three functions are exemplified by the triads on the tonic and its upper and lower fifths (collectively called the *Prinzipale* or principal chords); section 4.4 describes how the positive and negative energies inherent in the three triads combine to form the fundamental cadence. Chapter 4 also discusses Karg-Elert’s presentation of the 4:5:6:7 concordant seventh chord as an integrated harmony derived from a single prime, in contrast to Riemann, who explained the dominant and half-diminished sevenths as combinations of pitches derived from two different primes (and thus representing two different functions). The remaining sections of this chapter discuss the simplest alterations of the principal triads: the ultraforms (secondary dominants and contrants), the diatonic substitutes (Riemann’s *Parallel* and *Leittonwechsel*), and the mode-shifting variants.
4.1. Polarity: its perceptual basis

For Karg-Elert, the “natural” phenomenon of major-minor polarity was inextricably linked with harmonic relationships. Section 3.7 described how the harmonic prime cell (the harmonic third c₁ – e₃) is placed at the center of the three-dimensional pitch space (Figures 3.7.1 and 3.7.2), and how that prime cell in turn generates the C major and A minor consonances in polaristic fashion (Figure 3.7.4). As we have seen, Karg-Elert relies to a large extent on mathematical data and acoustic concepts to explain and justify polarity. However, he stated that his initial glimpses of polarity were prompted by his own perceptions of harmonic relationships in practical music, without any contribution from acoustics or mathematics, and before he knew any of his predecessors’ dualistic theories. In Harmonologik, he recalls that he arrived independently at the “embryo of polarity” right after he completed his Conservatory studies:

Even in 1902/03 I had intuitively recognized polarity, without at that time even knowing any syllable of Zarlino, Rameau or the masters of the most recent past and present: without having heard or read Hauptmann, Oettingen, Riemann (although I had studied for 5 years with a celebrated ‘theoretical’ leading light, and on top of that at Hauptmann’s former domain!)

In his Conservatory theory classes with Jadassohn, Karg-Elert had been thoroughly schooled in the Stufenlehre (scale-step) tradition² of harmonic labelling, which he admits is used in many “outstanding treatises on harmony, e.g. by Halm, Louis and Thuille, [Eugen] Schmitz and

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¹ Harmonologik, 63. The “leading light” to whom Karg-Elert refers rather disparagingly is Salomon Jadassohn, whose Harmonielehre (Jadassohn 1883) was a standard text in German schools of music for many years.

² The term Stufenlehre (“scale step methods”) generally refers to harmonic analysis based on the chords generated by each note or “step” (Stufe) of the diatonic scale, symbolized using Roman numerals. The work of Simon Sechter (1853) is perhaps most representative of Stufenlehre, though it is rooted in the work of earlier theorists such as Rameau (1722) and Gottfried Weber (1821). It is also a fundamental element in Schenkerian theory, and is still the basis of harmonic pedagogy in North America, the United Kingdom and many other countries.
Schenker”³ (Harmonologik, 62). In spite of his respect for those authors, he states that the scale-step labelling of harmonies is conceptually flawed, because it “proceeds from a melodic element [i.e. the scale] and thus renounces the natural development of harmony.” One must remember that for Karg-Elert, melody and harmony are derived from different sources: from Pythagorean and Didymean pitch intervals respectively. Scale-step methods are monistic, labelling chords in the major and minor modes in the same manner. Karg-Elert notes that from a scale-step perspective, major and minor “appear as quite dissimilar” (Figure 4.1.1):

![Scale-step labels in C major and A natural minor](image)

**Figure 4.1.1.** Scale-step labels in C major and A natural minor (Harmonologik, 62)

In Figure 4.1.1, the three basic chords (I, IV and V) are notated as whole notes, and their labels are printed larger than those for subsidiary chords (II, III, VI and VII), which are notated as half or quarter notes. Both C major and A natural minor contain three major triads and three minor triads, exchanging qualities on the following steps: I, III, IV, V, VI and VIII (= I). However, this major-minor correspondence is broken by the diminished triad [B D F], which is linked with D minor (II) and G major (VII) triads. The divergence between major and minor is heightened when scale degree ^7 is raised in minor, resulting in “only two basic minor triads (I and IV), but a basic major triad V, an augmented III (which however in practice must be perpetually restored to a regular major III), and two diminished triads (II and VII).” The

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³ Halm 1900; Louis and Thuille 1907; Schmitz 1911; Schenker 1906.
confusion is obvious!” (Harmonologik, 62) Karg-Elert then lists several features of tonal music which are noted in many Stufenlehre textbooks, and which reveal “the natural will” for polarity:

- The authentic V – I cadence is typical of major, and the IV – I plagal cadence is typical of minor. However, the different scale-step labels do not reflect this correspondence.
- Just as V in minor keys is often a major triad, so IV in major keys is often a minor triad. Again, the chord labels leave this harmonic symmetry “completely veiled.”
- The concept of relative major and minor keys is polaristic, not monistic: the relative of C major is down a third (A minor), but the relative key of A minor is up a third (C major).

In Karg-Elert’s opinion, such clear manifestations of harmonic polarity were hidden “by the scale-steps themselves.” He described his inchoate perceptions of polarity in greater detail:

The correspondence between the E major chord in A minor and the F minor chord in C major {i.e. major V in minor keys and minor IV in major keys], the E minor chord in A major and the D major chord in A minor {minor V in major keys and major IV in minor keys}…all the contrasting correspondence between Dorian and Mixolydian, as well as between Lydian and Phrygian…forced itself upon me categorically, without my initial possession of the means to render this “felt” analogy tangible. I felt only instinctively that a ‘something’ veils the secret relations between the major V and the minor IV…”

He reported that his first attempt to express these perceptions in notation involved a symmetrical arrangement of major and minor, and “polaristic” Stufen labels (Figure 4.1.2):

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4 Harmonologik, 63.
In Figure 4.1.2, the *Stufen* labels for major triads have beams, while those for minor and diminished triads do not. Most obviously, the labels in major lean to the right, while those in minor lean to the left; as will be seen, Karg-Elert retained this graphic opposition in his later harmonic function labels. In this system, the three basic chords (I, IV and V) exchange qualities between major and minor; the labels for those basic chords are printed larger than those of the subsidiary chords (II, III and VI, which are relatives of IV, V and I respectively, denoted by the arrows). The diminished triad is labelled as VII in both keys. The possible chromatic alterations to VII (F♯ and B♭) reflect polarity between modal cadences: “the Dorian cadence corresponds to the Mixolydian cadence, just as the Phrygian corresponds to the Lydian” (*Harmonologik*, 64).

Finally, the correspondence between the major dominant in minor keys and the minor subdominant in major keys is now reflected in notation, as both are labelled in Karg-Elert’s *Stufen* system as IV. The system is fully and consistently dualistic, demonstrated by the V → I progression in each key: [GM → CM] in C major, and [Dm → Am] in A minor. The major-key dominant is a fifth above the tonic, while the minor-key dominant is a fifth below.

With modifications to accommodate more chromatic harmonies, Karg-Elert’s polaristic *Stufen* system could serve well in harmonic analysis, if one accepts the dualistic opposition of major and minor. However, an unescapable feature of the system is that the Roman numerals I to VII still represent scale degrees, and thus are fundamentally melodic in origin. Based on his comments above, for Karg-Elert dualism (or polarity) was motivated entirely by harmonic rather than melodic perceptions: the symmetrical intervallic structure of the “double consonance” (the major and minor triads), and symmetrical correspondences between chords in major and minor such as those that he described. One might argue generally that *Stufen* labels are designed for monistic models of major and minor, in which the direct linkage of harmonies and scale degrees
(melodic tones) can work in the same way in both modes; indeed, a big advantage of traditional scale-step methods is that they enable an analogous understanding of major and minor. In contrast, a dualistic system of harmonic analysis should ideally be derived from purely harmonic precepts, not necessarily dependent on specific scale degrees in either major or minor. Karg-Elert found such a system in Riemann’s harmonic function theory, which he said he first discovered “in about 1906” (Harmonologik, 64). As will be described in section 4.3, Karg-Elert found Riemann’s functional approach to be “full of contradictions and gaps,” as it attempts to partially reconcile with a monistic concept of minor. In spite of these perceived flaws, he always acknowledged the importance of Riemann’s work, and the central role of harmonic function in his own theories. Karg-Elert’s presentation of the three basic functions (largely borrowed from Riemann), as well as simple alterations and substitutes of those functions, are the primary focus in this chapter.

In Harmonologik (pp. 11–12), Karg-Elert discusses “the perception of polarity through harmonic imagination” (Harmonievorstellung). By itself, “every interval is without quality, neutral, gravitationally balanced” – that is, every interval can equally be part of an “upward” major or “downward” minor chord (Figure 4.1.3):

![Figure 4.1.3. “Gravitationally balanced” intervals, part of major or minor chords (Harm, 11)](image)

When we hear a harmonic interval (one that belongs to a consonant triad or concordant seventh chord), we will mentally supply the note(s) needed to complete the harmony. Each harmonic interval can be understood as representing a major or a minor triad, or their corresponding concordant seventh chords (the dominant and half-diminished sevenths). Figure 4.1.4 illustrates
the process of “harmonic imagination”: the sounded intervals are on the left, and the mentally completed harmonies are on the right.

Figure 4.1.4. Intervals filled in polaristically through “harmonic imagination” (Harm, 12)

The first three lines of Figure 4.1.4 demonstrate how the ear can understand harmonic intervals polaristically, as belonging to either a major or a minor triad; the last two lines illustrate the same process with regard to dominant and half-diminished seventh chords. The arrows in the right-hand examples reflect the generation of the mentally-completed chords, either upward or downward from the harmonic prime. The closed noteheads indicate the imagined pitches that complete the chords, and that give the intervals the harmonic identity that was previously undefined. As Karg-Elert notes, “how superficial, therefore, to call the isolated large third ‘major’, and the isolated small third ‘minor’! Each interval belongs equally well to the major and minor consonance” (Harmonologik, 12). While the purpose of the Figure 4.1.4 example is to demonstrate that “all imagined supplemental pitches are symmetrically polar in relation to the
sounded intervals,” another feature of the example is that its notation reveals a transformational perspective, displaying significant affinities with recent neo-Riemannian concepts and concerns. That aspect of the Figure 4.1.4 example will be examined in Chapter 5, which focuses on connections between Karg-Elert’s work and transformational and neo-Riemannian theories. For now, it will suffice to say that Figure 4.1.4 suggests that harmonic transformation is not simply a relationship between chords actually present in a musical passage; it is also a matter of perception, acting in the realm of “harmonic imagination.”

4.2. Function: harmony as ideal (Wesen), chord as manifestation (Erscheinung)

Riemann himself never provided a clear definition of exactly what harmonic function is; indeed, he used the term Funktion very rarely, even in his 1893 landmark text Harmony Simplified (Vereinfachte Harmonielehre), whose subtitle is “The Theory of the Tonal Functions of Chords” (Die Lehre von den tonalen Funktionen der Akkorde). In the Introduction to Harmony Simplified, Riemann proposes two general principles that “explain and develop the title of the book”:

I. “There are only two kinds of clangs [Klänge]: overclangers and underclangers {i.e. major and minor triads}. All dissonant chords are to be conceived, explained and indicated as modifications of overclangers and underclangers.”

II. “There are only three kinds of tonal functions [tonale Funktionen der Harmonie] (significance within the key [Bedeutungen innerhalb der Tonart]), namely tonic, dominant and subdominant. In the change of these functions lies the essence of modulation.” (Riemann 1893, 9)

The second of Riemann’s principles does mention the term Funktion, but it does not really define the concept. Instead, he states that function is exemplified in the already familiar categories of tonic, dominant and subdominant, originally defined by Rameau as the pillars of the perfect
(authentic) and imperfect (plagal) cadences. Brian Hyer begins his recent essay “What Is a Function?” with a detailed reading of Riemann’s two principles, particularly focusing on the second principle’s equation of Funktion with Bedeutung, which Hyer translates as “meaning” rather than “significance.” Hyer traces Riemann’s use of Bedeutung or “meaning” back to Musikalische Logik of 1872 (his first major theoretical writing), which already describes three basic chords (Tonika, Überdominante and Unterdominante), and examines “the different meanings [Bedeutungen] of these chords in relation to one another, their logical meaning [Bedeutung] in musical structure.” This early statement already contains the basic seeds of the harmonic function concept: that there are three fundamental harmonies, and that they relate to each other in logical ways. Musikalische Logik locates the paradigm of “harmonic logic” in the große Kadenz (“great cadence”), which combines the plagal and authentic cadences: as C-F-C-G-C [I-IV-I-V-I], or with the central return to tonic elided as C-F-G-C [I-IV-V-I]. Following Hauptmann, Riemann describes the Kadenz in dialectic terms: “thetic is the tonic [I], antithetic the underdominant [IV], synthetic the dominant [V].” From this perspective, tonic is revealed as relatively stable and constant, while the two dominants are relatively unstable, defined on the one hand by their need to return to the tonic, and on the other hand by their different purposes (functions) within the cadence. Riemann’s second principle in Harmony Simplified refers to essentially the same concept: harmonies acquire function (“significance/meaning within the

5 Rameau 1722, Book II, 57 and 65. See Figures 4.6.2 and 4.6.3 below.
6 Hyer 2011.
7 Ibid., 95.
8 Riemann 1872, 2. The progressions discussed in Musikalische Logik are almost exclusively in major keys, and there is no evidence of dualism in the article. This suggests that from an early stage in his development, Riemann considered the cadence (I-IV-V-I) to apply to both major and minor in the same way: V as the upper dominant, and IV as the lower dominant (subdominant).
9 Hauptmann 1853.
10 Riemann 1872, 3.
key”) according to whether they express tonic stability or dominant/subdominant instability. The question then arises: what is the relationship between function and chord? Can the functional cadence $T – S – D – T$ (see Figure 4.2.1) be essentially equated with $I – IV – V – I$, allowing for the modifications of each “overclang and underclang” described in Riemann’s first principle?

The nature of the correspondence between function and chord is difficult to describe precisely. The first demonstration of harmonic function in the Introduction to Harmony Simplified suggests that the correspondence between the three functions and the three basic chords ($I$, $IV$ and $V$) is very close indeed (Figure 4.2.1):

![Figure 4.2.1. The three tonal functions in C major and A minor (Riemann 1893, 8)](image)

In both major and minor, the “principal clang” or tonic ($T$) is surrounded by its “nearest related clangs” a fifth above and below, respectively the dominant ($D$) and subdominant ($S$). Minor triads are preceded by a circle superscript ($\circ$), and alterations from minor to major are indicated by the plus ($+$) sign. There is no symmetrical opposition or difference in the main functions between major and minor, in spite of Riemann’s dualistic model of the harmonic triads; this apparent inconsistency was criticized by several writers, not least by Karg-Elert (see section 4.3 below). The crucial point here is that Riemann’s first example of the three harmonic functions seems to equate them with the $I$, $IV$ and $V$ chords, in both major and minor. However, the example also illustrates how the basic triads can be modified, in the alternation of $S$ and $\circ S$ in
major, and $D$ and $D^+$ in minor. While those modifications are limited to changes of triad quality, they nonetheless suggest that further modifications of each basic chord are possible within each function, as implied by Riemann’s first principle in Harmony Simplified. Therefore, while the three tonal functions are fundamentally represented by the I, IV and V triads, they can also be represented by a range of other chords that relate in varying ways to those triads.

Karg-Elert explores the distinction between function and chord by returning to the basic opposition from the Preface to Neue Bahnen, namely between ideal or essence ($Wesen$) and manifestation or appearance ($Erscheinung$). In an appendix to Section 1 of Grundlagen Part II (pp. 157–160), Karg-Elert examines “inconsistencies in the figured bass/scale-step labelling system.” He opens the discussion with the following general statement:

Figured bass notation is a primitive shorthand for chord structure. It has nothing to do with the ‘essence’ ($Wesen$) of harmonies; rather, it represents only an external counting of chord tones above a notated bass. It is thus a purely mechanical process. Scale-step labelling identifies basic chords, considered exclusively as structures of stacked thirds, ordered according to their relative height following the scale. Common practice usually presents both types of label at the same time, resulting in confusion, as the chord symbols often suggest a contradictory double meaning. (Grundlagen II, 157)

Karg-Elert proceeds to describe several examples of such “double meanings,” including the following common problem (Figure 4.2.2):

![Figure 4.2.2. IV – I motion in C major, with soprano passing tone D (Grundlagen II, 157)](image)

See section 2.4 of this dissertation.

Karg-Elert describes the next two examples in prose; Figures 4.2.2 and 4.2.3 realize the examples in staff notation.
Most analysts (especially those influenced by the more linear approach of Heinrich Schenker) would likely not provide a new Roman numeral label where the soprano moves to D. Instead, one might indicate the soprano motion as 5 – 6 over the sustained F in the bass, melodically extending the IV chord. Karg-Elert recognizes the linear purpose of the D, and in turn the function of the chord it creates: “the soprano D is merely a passing tone linking IV and I, and so the chord F-A-D – which strives or “tends” toward C-G-E – is understood in the sense of F-A-C (or IV)” (Grundlagen II, 157). The analytical mismatch arises between the figured bass number “6” (labelling the sixth D above the bass F), and the Roman numeral IV (which should not contain D at all): “the labelling of the chord F-A-D as a sixth-chord is correct, but in the sense of a dissonance [applied to IV] requiring resolution, not as an inversion of II [D minor].” Even if the passing D minor chord is labelled as II⁶, its status as an embellishment or extension of IV is clear. In other words, both F-A-C and F-A-D represent the same harmonic function – in this case subdominant – though strictly speaking the change of soprano requires a change of Stufen label. The traditional notation marks the specific chordal manifestations, but confuses their harmonic meaning or essence (Wesen).

Another familiar case of notational “double meaning” is the conundrum of the “tonic six-four”: the second inversion of the C major triad, in the key of C major. Karg-Elert describes two different contexts for that chord (Figure 4.2.3):

![Figure 4.2.3](image)

Figure 4.2.3. Two different contexts for the “tonic six-four” in C major (Grundlagen II, 159)
In both cases, the figured bass for the G-C-E chord is “6/4,” indicating the intervals above the bass tone G. In the first case, the C and E are clearly upper neighbor tones, embellishing the underlying G major (V) harmony. For that reason, changing the Roman numeral to I at the “tonic six-four” would be misleading – but strictly speaking, the label “V 6/4” denotes D-G-B, not G-C-E. In the second case, the same combination of pitches (G-C-E) extends the tonic (I) through bass arpeggiation, and is correctly labelled as I 6/4. One might feel that Karg-Elert is being overly pedantic, as the meaning of the 6/4 chord is very clear in both contexts. His point is that the function or meaning is indeed obvious, but that the figured bass and scale-step labels introduce unnecessary confusion and ambiguity, contradicting each other in the first case.

What is needed is analytical notation that begins with the meaning or ‘essence’ of each harmony, rather than its specific ordering of intervals (figured bass), or its position on the tonic scale (Stufen labels). For Karg-Elert, therein lies the significance of harmonic function:

Completely different is ‘functional analysis’, which understands all chordal structures in terms of their fundamental harmonic essence; it recognizes the most complex dissonances as logical developments of basic chords, differentiates between real and apparent consonances and dissonances, and relates the incredible richly-developed life of harmony in its entirety back to the fundamental form of the cadence. (Grundlagen II, 160)

He describes harmony and chord as different aspects of a single principle:

Harmony is content, and chord is perceptible form; both correspond with each other like ‘thought’ and ‘word’; one can speak of the metaphysics of harmony, and of the physics of chord. (Grundlagen II, 181)

In Harmonologik, Karg-Elert extends his thoughts in a similar vein:

Harmony is to be understood as completely abstract: devoid of space and weightless. Its form of representation in the concrete world of manifestation is the chord. If one strips the chord of all physical forms (‘manifestation’), then only the concept of harmony (‘essence’) remains. (Harmonologik, 50)
The labels for the three basic functions reflect the essence or ideal of harmony; supplemental notation can indicate the features of specific chordal manifestations. Karg-Elert demonstrates this once again using the “tonic six-four,” now including function labels (Figure 4.2.4):

![Figure 4.2.4.](image)

**Figure 4.2.4.** Two different contexts for the C major six-four chord, in C major (Harmon, 50)

Karg-Elert’s annotation for Figure 4.2.4 reads:

The ‘chord’ at A and B is the same (“six-four chord”), but the harmonic evaluation is fundamentally different:

A) It is a C major harmony [and thus labelled as tonic]
B) It stands for G major harmony [and thus labelled as dominant]

The dots in example A indicate inversions (see Figure 4.4.11 below), and the notation in example B denotes linear modification of the dominant (D), in the voices E → D and C → B.

In his 1994 dualism-inspired study of *Harmonic Function in Chromatic Music*, Daniel Harrison describes harmonic function in a way that accords well with Karg-Elert’s discussion of harmony (or function) as ideal or essence, and chord as manifestation. His definition seeks
to reconcil[e] two different strains of thinking emanating from Riemann. One strain holds that function is an abstract impression of tonal attitude. It can profitably be thought of as a category, or set, to which tones and chords belong…although elements may be distinct, they are members of this same set, and it is this common membership that is sensed and valued…The other strain is more concrete. A function is essentially a primary triad and those chords derived from it under certain, specified operations.13

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13 Harrison 1994, 38.
Harrison’s first strain expresses the notion of harmonic meaning or essence, while his second strain focuses on the range of chordal manifestations possible within each function. One might say that at its best, functional harmonic analysis balances these two strains of thinking: it labels individual chords with great specificity, but never loses sight of their fundamental meanings within tonal progressions.

4.3. **Karg-Elert on Riemann’s theory of harmonic function**

Though Karg-Elert’s treatises refer to various of Riemann’s concepts (including the system of *Schritte* and *Wechsel* relations or transformations first proposed in Riemann 1880),¹⁴ Karg-Elert’s debts to Riemann are largely in the realm of harmonic function, and are thus primarily derived from Riemann 1893. As mentioned previously, Karg-Elert openly and effusively acknowledged his debts to Riemann’s theory of harmonic function:

> Function theory is a ‘logic of harmony’ and the most outstanding form of harmonic analysis. Riemann is to be highly respected…to have set up the fundamental principle of function remains a great feat in the history of harmonic theory. (*Harmonologik*, 63)

Though Karg-Elert stated that he perceived the phenomenon of polarity well before he knew any of Riemann’s theories (see section 4.1), he recognized their galvanizing effect on his own work:

> I doubt very much whether I would have found this path of development {i.e. harmonic polarity} if I had not learnt Riemann’s function theories in about 1906. (*Harm*, 64)

Yet from the start, Karg-Elert found significant “contradictions and gaps” in Riemann’s work:

> It is painful for me, nevertheless, to admit that I must oppose Riemann’s theory in very many points. I felt that from the first reading, which so clearly revealed the nature of function to me, even if it nevertheless had to be opposed in so many instances. (ibid.)

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¹⁴ See section 5.1 of this dissertation for more on Riemann’s *Schritte* and *Wechsel*.
Karg-Elert’s most basic difference with Riemann concerns his $S$ and $D$ functions in minor, which are disposed respectively below and above the tonic, exactly as in major (see Figure 4.2.1), in spite of the dualistic model of the triads, and the presence of various dualistic features within Riemann’s functional system. Karg-Elert was not the first to criticize Riemann for this apparent inconsistency, which is the main topic in this section.

To prepare for his critique, Karg-Elert first discusses the tendencies and energies that exist between harmonies. In Grundlagen Part II (pp. 182-184), he explains that “just as melodic motions establish the whole and half steps as normal forms, so do harmonic events establish chord motions by fifth. The entire life of harmony has its fundamental form and source in the cadence, as recognized by the great master J. Ph. Rameau.” In this statement, Karg-Elert confirms the basis of tonal harmony in fifth relationships, and in the specific progression of the tonal cadence, in which “two opposing elements” operate:

1) Goal harmonies ($Zielklang$) as stable
2) Tendency harmonies ($Tendenzklang$) as “labile” or unstable

The $Zielklang$ is clearly the tonic, the stable harmony to which all others relate and return. Harmonic tendencies can be understood and perceived in two different senses:

a) In a spatial manner: rising and falling motions, higher and lower, treble and bass. In spatial terms, major and minor are analogous – they are “monistic.”

b) In an energetic manner: positive and negative, strong and weak. In terms of harmonic energy, major and minor are opposed – they are “polaristic.”

Karg-Elert tabulates the foregoing ideas in Figure 4.3.1, which also incorporates the notion of tonic ($Tonika$), plus the harmonies on the fifths above and below the tonic:
Figure 4.3.1. Spatial and energetic directions in major and minor (Grundlagen II, 183)

Figure 4.3.1 suggests that the spatial realm is concerned with objective pitch relationships such as fifths above and below the tonic, and therefore operates in the same way in both major and minor. In contrast, energetic tendencies among harmonies (considered wesentlich, in terms of their functional essence) mimic the energies contained within the major and minor triads themselves: upward or “positive” in major, and downward or “negative” in minor (see section 3.4). Therefore, in major keys the positively-charged harmony is on the over-fifth, but in minor keys is on the under-fifth. All of this makes logical sense within a dualistic model of harmony and harmonic relationships. Karg-Elert states that “so far, all harmony texts have named the dominant entirely after the spatial principle” – that is, the dominant has always been identified simply as the fifth above, in both major and minor. He notes that Oettingen’s names for the
principal harmonies in major and minor are different in each mode, but still indicate over and under fifths in the standard spatial manner as ober and unter (Figure 4.3.2):

![Diagram of principal harmonies in major and minor]

**Figure 4.3.2.** Oettingen’s principal harmonies in major and minor (Grundlagen II, 183: based on Oettingen 1913)

The arrows in Figure 4.3.2 simply reflect the upward or downward generation of major and minor triads, not energetic tendencies. However, Oettingen does classify the energetic tendencies of the principal harmonies in the same way as Karg-Elert: Oberdominante (in major) and Unterregnante (in minor) as “strong” (stark), and Unterdominante and Oberregnante as “weak” (schwach). As will be seen in section 4.4, Karg-Elert adopts Riemann’s names and labels for the harmonic functions (with two significant alterations), but disposes them in strict polar fashion according to their energetic tendencies, rather than their spatial (pitch) locations: for Karg-Elert, the dominant (D) is always the “positive” or “strong” harmony, at the fifth above the tonic in major, but at the fifth below the tonic in minor.

Karg-Elert’s most basic criticism of Riemann’s functional system concerns its mixing of dualistic and monistic relationships between major and minor (Figure 4.3.3): “Riemann’s diatonic third relationships [Terzverwandten] demonstrate polarity between major and minor, but he adopts parallelism in the fifth relationships [Quintverwandten]!” (Neue Bahnen, XI)

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15 Oettingen 1913, 49–50.
Once again: major and minor triads are indicated in Riemann’s system as $^+$ and $^o$. The diatonic thirds (or parallel chords, labelled $Tp$ and $Sp$ in Figure 4.3.3) display polarity: the tonic parallel in C major is A minor (a diatonic third below), while the tonic parallel in A minor is C major (a diatonic third above). In contrast, Riemann’s three basic functions are disposed in the same way in major and minor. The resulting system “swings between polarity and parallelism, in a most illogical way…its function symbols translate scale-step labels into letters”\(^{16}\) (Figure 4.3.4):

The arrows indicate the connections between the principal harmonies ($T, S, D$) and their parallels ($Tp, Sp, Dp$); the dotted lines between the parallel chords reflect the polarity between major and minor. In contrast, the principal harmonies themselves are equated with I, IV and V, in the same

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\(^{16}\) *Neue Bahnen*, XII.
way in both modes. Finally, the diminished triad ([B D F] in C major and A minor) is labelled as an incomplete dominant seventh ($D^7$) in major, but as an incomplete subdominant with added under-seventh ($S_{IV}$) in minor. Riemann’s compromises between dualism and monism likely made his function theory more approachable to the average musician and student, and thus helped its reception and dissemination. However, for Karg-Elert such compromises smacked of logical inconsistency, and represented a partial retreat from analysis of harmonic meaning to simple labelling of chordal manifestations.

Karg-Elert’s critique of Riemann’s apparently non-dualistic functions echoes a similar one made in 1904 by Dutch musicologist Ary Belinfante, who like Karg-Elert pointed out that if the dominant is based on the fifth of an upward-generated major triad, then logically the dominant should be based on the fifth of a downward-generated minor triad (Figure 4.3.5):

$$
\begin{align*}
S & \\
T & \\
D & \\
\text{Major: } & f^+ & c^+ & g^+ \\
\text{Minor: } & o_g & o_c & o_f
\end{align*}
$$

Figure 4.3.5. A strictly dualistic system of the basic functions, after Belinfante

Riemann responded to Belinfante in his 1905 “The Problem of Harmonic Dualism,” stating that

These names (the function labels Tonic, Subdominant and Dominant) are not at all mine, but rather have been widely used since Rameau. I have properly retained these terms for the same reasons I have retained the designations major, minor, parallel, Grundton, and a whole host of others.19

This appeal to common practice is not especially convincing, as the traditional terms listed had “all received a radically new meaning from Riemann.”20 However, Riemann continues by

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17 Belinfante 1904; summarized in Klumpenhouwer 2011, 201.
18 Klumpenhouwer 2011, 201.
19 Riemann 1905. Translated by Ian Bent in Gollin and Rehding 2011, 184.
20 Rehding 2003, 79.
reminding his critics of the fully dualistic system of *Schritte* and *Wechsel* (from Riemann 1880), which still guided his approach to harmonic relationships even after the introduction of the functions, and ensured that the dominant in major and the dominant in minor are in no way to be conceived alike:

The particular form of my notational system in which *the prime of the dual minor tonic is the same as the prime of the dominant* makes such a conclusion impossible.\(^{21}\)

In major, the prime of D (dominant) is the fifth of T (tonic): $C \rightarrow G \rightarrow B\ D$

In minor, the prime of D (dominant) is the prime of T (tonic): $A \rightarrow E \rightarrow G\#\ B$

From this perspective, Riemann’s functional system can in fact be called dualistic. The opposition between major and minor in Riemann’s system is reflected not in the roots of the fifth-related chords, but in the transformations that link T and D: *Quintschritt* (“fifth step”) in major, but *Gegenquintschritt* (“counter fifth step”) in minor.\(^{22}\) In his close reading of Riemann’s 1905 response to Belinfante, Henry Klumpenhouwer notes that “by bringing *Schritte* and *Wechsel* into the picture, [Riemann provides] the basis for a reasonable structural refutation of Belinfante’s criticism.”\(^{23}\) Klumpenhouwer shifts the focus from the triads themselves to the *Schritt* transformations that connect them, as follows (Figure 4.3.6):

\(^{21}\) Riemann 1905. Translated by Ian Bent in Gollin and Rehding, 185.

\(^{22}\) Riemann 1880, 7–13.

\(^{23}\) Klumpenhouwer 2011, 203.
In Figure 4.3.6, Q is *Quintschritt*, which ascends by fifth in major, and descends by fifth in minor. Its inverse is *Gegenquintschritt*, labelled –Q. The major key is displayed in the first line, and minor in the second. The third line reverses the perspective of the second: the transformations are now the same as in major, but they converge onto the tonic instead of diverging out from it, reflecting Hauptmann’s views of dualist chord structure: “whereas in major, the formative relationships extend from the Grundton (or here the tonic) to other parts of structure, in dual minor the same relationships extend to the Grundton (or tonic) from other parts of structure.”

In the functional system proposed by Belinfante (and Karg-Elert), the prime of the dominant is always the fifth of the tonic, in both major and minor. The Belinfante/Karg-Elert system is thus monistic in terms of Riemann’s Q and –Q transformations (Figure 4.3.7):

**Figure 4.3.6.** Klumpenhouwer’s diagram of *Schritte* in Riemann’s functional system (Klumpenhouwer 2011, 203)

![Diagram of Schritte in Riemann’s functional system](image)

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**Figure 4.3.7.** Monistic transformations in Belinfante and Karg-Elert’s three-function system

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24 Klumpenhouwer 2011, 204.
In sum, Belinfante and Karg-Elert characterize Riemann’s system of the three functions as monistic because of its analogous arrangement of fifths in major and minor. However, Riemann’s system is dualistic in terms of the transformations that connect the fifths to the tonic. The exact opposite is true of the Belinfante/Karg-Elert system. Klumpenhouwer accuses Belinfante (and by extension Karg-Elert) of “projecting monist expectations of how hard dualisms must operate.”

Klumpenhouwer’s defence of dualism in Riemann’s functional system essentially involves a transformational change of focus, from the objects themselves (the basic triads that represent the three functions) to the operations that link the objects. Karg-Elert would surely defend Belinfante (and himself) by referring to the model illustrated in Figure 4.3.1: the functions exhibit polarity between the modes in terms of their energetic tendencies, matching the energies projected within the major and minor triads themselves.

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25 Klumpenhouwer 2011, 205.
4.4. The principal harmonies, and the fundamental cadence in major and minor

As described in section 3.7, Karg-Elert places the “prime harmonic cell” $c^1 – e^1$ at the center of his pitch space, illustrated in Figure 3.7.1 (reproduced here as Figure 4.4.1):

![Figure 4.4.1](image)

**Figure 4.4.1.** The “harmonic prime cell in its source register” (*Ak* 6.1)

“In a symmetrically polar manner, the prime cell generates the double consonance” (*Ak* 6.2), namely the C major and A minor triads (Figure 4.4.2):

![Figure 4.4.2](image)

**Figure 4.4.2.** The generation of the major and minor triads from the prime cell (*Ak* 6.2)

*Akustische* 6.3 completes the diatonic space by adding thirds above and below the two triads, up to the point of pitch-class duplication and syntonic comma difference (Figure 4.4.3):

![Figure 4.4.3](image)

**Figure 4.4.3.** The completed diatonic space, generated above and below the prime cell (*Ak* 6.3)

The eight-note diatonic space outlined in Figure 4.4.3 encompasses the basic triads that represent the three harmonic functions in C major and A minor (Figure 4.4.4):
Figure 4.4.4. The three harmonic functions in C major and A minor (Ak 6.4)

Figure 4.4.4 presents Karg-Elert’s function labels, which use normal letters in major (T, D and C), and inverted letters in minor (L, D and C). This is one of many examples of how Karg-Elert’s notation visually reflects the polar opposition between major and minor. The figure highlights the “dominating harmonic direction” (dominierende Klangrichtung) in both modes: upward in major, downward in minor. The dominant (D or C) “connects with the tonic in its dominating direction,” and therefore its prime lies a fifth above T in major, but a fifth below L in minor (as shown by the arrows). In the opposite direction from the tonic lies the contradominant (labelled C or C, and often abbreviated as contrant), at the fifth below tonic in major, and above tonic in minor. To a greater degree than “subdominant,” the term “contradominant” stresses polar or symmetrical opposition from the dominant. Karg-Elert’s functional system is strictly dualistic, in how C and D relate as fifths above or below T in the two modes. Collectively, Karg-Elert called the three functions (and their basic triadic representations) the Prinzipale – short for Prinzipalklänge, or principal harmonies. The following illustration from Harmonologik (Figure 4.4.5) demonstrates the energetic tendencies in the three-function system, previously discussed in section 4.3.

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26 In the text of this dissertation, the inverted function labels L, D and C do not slant to the left as they do in Karg-Elert’s treatises, because of font limitations.
The dominants with added sevenths above and below (labelled $\bar{D}$ and $\bar{A}$, and called concordant sevenths by Karg-Elert) will be discussed in section 4.6. In Figure 4.4.5, the slurs indicate common tones between the tonic and the other functions: $T$ prime becomes $C$ fifth, and $T$ fifth becomes $D$ prime (in both modes). The dominants are “strong” (stark) and “open” (offen), continuing forward in the direction of the tonic triad; the contrants are “weak” (schwach), and point backward from the tonic. The triangles at the bottom indicate the generating direction of the triads: $\Delta$ for major, $\nabla$ for minor. (There is a graphic error in the C major example: the contrant F major triad should be symbolized as $\bar{D}$, not as $\bar{A}$.)

As discussed in section 3.4, major triads increase in energy as they ascend from the prime: in Figure 4.4.5, the normal energetic charge present in the major tonic is shown as two positive signs ($++$), which increases in the dominant ($+++$), but decreases in the contrant ($+$). Minor triads decrease in energy as they descend from the prime, and so energetic charges in minor triads are indicated using negative signs: ($--$) for the tonic, increasing in the dominant ($---$) and decreasing in the contrant ($-$). The following discussion will examine how the energies projected within the basic functions determines the trajectory of the fundamental cadence, which Karg-Elert describes as the source
of all harmonic progression: “the entirety of harmonic life (harmonische Leben) is a great expansion of the cadence, and functional analysis knows only three basic labels, which are symbols of the cadence” (Grundlagen II, 185).

Both Akustische and Harmonologik present the principal harmonies succinctly, with little discussion of how each harmony functions within in the cadence. Akustische 6.4 specifies that the “normal cadence” is authentic in major, and plagal in minor (Figure 4.4.6):

![Normal cadences in major and minor (Ak 6.4)](image)

**Figure 4.4.6.** Normal cadences in major and minor (Ak 6.4)

Karg-Elert uses the terms “authentic” and “plagal” in their traditional sense: authentic cadences feature a descending fifth/ascending fourth root motion (G \(\rightarrow\) C), while plagal cadences have an ascending fifth/descending fourth root motion (D \(\rightarrow\) A). However, functionally both cadences are \(D \rightarrow T\), due to the polar opposition of the functions in each mode. The \(D \rightarrow T\) cadence is considered normal, as the dominant has the strongest pull back to the tonic, due to the leading-tone motions to the prime (indicated by the lines in Figure 4.4.6: B \(\rightarrow\) C in C major, F \(\rightarrow\) E in A minor). The following example from Harmonologik (Figure 4.4.7) indicates energetic curves (Kurven) or charges in the cadence; the contrant (C) provides relaxation, while the dominant (D) provides an increase of intensity that is resolved in the return to tonic. The \(D \rightarrow T\) cadence is again classified as authentic in major but plagal in minor, based on the root motions:
Though the examples in Figure 4.4.7 do not begin with tonic, they otherwise replicate Riemann’s complete \( \textit{Große Kadenz} \) (see section 4.2): \( I - IV - I - V - I \), or \( T - C - T - D - T \). Karg-Elert’s A minor version is of course strictly polar, with \( D \) a fifth below \( T \). He discusses the cadence in some detail in \textit{Grundlagen} Part II, Chapter 42 (and its accompanying musical examples). He notes that the contrant’s normal tendency is to return to tonic, just like the dominant: “\( C \) and \( D \) connect with equal strength, but with opposing tendencies, to their central harmony” (\textit{Grundlagen} II, 197-198). Figure 4.4.7 illustrates how \( C \) and \( T \) connect through their common tone (fifth of \( C \) = prime of \( T \)), and also how the contrant’s weakened energetic charge proceeds back to the tonic’s normal level. The central tonic can be elided in the cadence, juxtaposing contrant and dominant: \( T - C - D - T \). However, Karg-Elert cautions that “it must be noted that \( C \) has no tendency toward \( D \), only singular direction toward \( T \)” (Ibid., 197). In the “two-sided cadence” (\textit{zweiseitige Kadenz}), the two dominants converge onto the tonic from opposite directions (Figure 4.4.8):

![Diagram of two-sided cadence]

Figure 4.4.8. The two-sided cadence (\textit{Grundlagen} II, 197)
The following annotated example from a Schubert waltz (Figure 4.4.9) features the two-sided cadence; the rising and falling horizontal lines indicate rising energy in $T \rightarrow D$, and falling energy in both $T \rightarrow C$ and $D \rightarrow T$. Karg-Elert explains the role of the contrant as follows:

The first phrase ends with an open cadence [half cadence on $D$]...The melody of the second phrase demands a beginning and ending on the tonic. One-bar alternation of $T$ and $D$ (with an ending on $T$) is not possible in four bars. If $D$ is to be reserved for the final resolution, a harmonic gap is opened between $T$ and $D$. This is filled by $C$ as a counter-tension, which brings about the two-sided cadence. (Grundlagen II, 197)

Figure 4.4.9. Melody from a Schubert waltz in C major, with functional analysis and energetic directions (Grundlagen II, example 129)

As will be seen in later sections and chapters, Karg-Elert’s functional analyses often focus on local harmonic features, carefully tabulating each pitch in each vertical sonority. However, he sometimes reveals his clear awareness of harmonic progression at multiple levels of structure, especially when more complex progressions elaborate the harmonic cadence. In Grundlagen Part II (pp. 211-212), Karg-Elert discusses the first eight bars from the second movement of Beethoven’s Piano Sonata in A major, op. 2 no. 2 (Figure 4.4.10):

Figure 4.4.10. Beethoven: Piano Sonata in A major, op. 2 no. 2 (opening of movement II)
Karg-Elert notes that the “harmonic basic function” of measures 1-8 is “conventional” – in other words, they expand the standard cadence. His functional analysis (Figure 4.4.11) specifies the identity of each local chord ("im Kleinen"), as well as the cadential progression that underlies the passage ("im Großen"):

![Figure 4.4.11. Two-level analysis of an expanded and elaborated harmonic cadence (Grundlagen II, example 141a)](image)

The notation preceding $D$ and $T$ in the last two bars indicates linear elaboration of those chords, as will be discussed in chapter 7 of this dissertation. A few of the function labels in the local-level analysis are preceded by dots, which indicate triadic inversions as follows (Figure 4.4.12):

![Figure 4.4.12. Dot notation for triad inversions (Grundlagen II, 189)](image)

To be clear, this inversionsal notation is to be understood in the traditional way, in both major and minor: root position of a minor triad has the lowest note in the bass, not the dual prime. Karg-Elert uses the dot notation for inversions quite frequently in Grundlagen, but much less so in Akustische and Harmonologik – see Figure 4.4.7, which does not indicate inversions at all, as is usual in the later treatises. Six-four chords of linear derivation (such as the cadential 6/4) are not notated using dots, but instead with alteration symbols similar to those in the last two bars of Figure 4.4.11. These symbols will be examined in Chapter 7.
The standard harmonic cadence is $T - C - D - T$, in both major and minor. The following is an example in A natural minor (Figure 4.4.13):

![Figure 4.4.13](image)

*Figure 4.4.13.* The “natural minor cadence” (*Grundlagen II*, example 144c)

As in Figure 4.4.9, this is a two-sided cadence, with no direct connection between $\mathfrak{C}$ and $\mathfrak{D}$. The final $\mathfrak{D} \rightarrow \mathfrak{L}$ motion is a plagal cadence, which as Karg-Elert said, is typical of minor. However, authentic cadences (by descending fifth/ascending fourth) are also possible in minor, as in the following example in C natural minor (Figure 4.4.14):

![Figure 4.4.14](image)

*Figure 4.4.14.* Plagal ($\mathfrak{D} \rightarrow \mathfrak{L}$) and authentic ($\mathfrak{C} \rightarrow \mathfrak{L}$) cadences in minor (*Grund II*, ex. 149)

The passage in Figure 4.4.14 presents a complete reversal of Riemann’s *Große Kadenz*: the initial expansion of tonic is effected by the dominant in the plagal cadence motion $\mathfrak{D} \rightarrow \mathfrak{L}$, while the more conclusive expansion and return is defined by the authentic cadence $\mathfrak{C} \rightarrow \mathfrak{L}$. Karg-Elert remarks that “the progression $TDCT$ is less common. It no longer has the transparently
natural character of the true cadence formula [i.e. $T C D T$]; it is perceived as artificial, deliberate.”

In Figure 4.4.14, Karg-Elert’s polaristic minor-key function labels may strain the patience of one who sees and hears a C natural minor version of the “normal” cadence $I – IV – I – V – I$, punctuated by a $^5 \rightarrow ^1$ bass motion. Indeed, one might question whether the “normal” minor-key cadence (Figure 4.4.13) sounds more natural than the “artificial” one (Figure 4.4.14), especially because of the bass motions in each case. In any case, neither of the natural minor cadences resemble those usually found in minor-key music, which involve the raised scale degree $^7$. This divergence from standard tonal practice is resolved to a considerable extent by the concept of the contrant variant, to be discussed in section 4.5.

The following table (Figure 4.4.15) summarizes Karg-Elert’s descriptions of the cadence, involving only the unaltered basic triads for the three functions:

<table>
<thead>
<tr>
<th>Natural, normal</th>
<th>MAJOR</th>
<th>MINOR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T – C – D – T$ (authentic)</td>
<td>$\perp – C – D – \perp$ (plagal)</td>
</tr>
<tr>
<td>Artificial</td>
<td>$T – D – C – T$ (plagal)</td>
<td>$\perp – D – C – \perp$ (authentic)</td>
</tr>
</tbody>
</table>

**Figure 4.4.15.** The authentic and plagal cadences, in major and minor

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$^27$ *Grundlagen* II, 226.
4.5. Variants of the principal harmonies

Karg-Elert describes how the contrant (C) differs from the tonic and dominant, in that its normal triadic quality goes against the position of its root in relation to the tonic. That is, in major keys C is located a fifth below T, but is nonetheless a major (↑) triad; in minor keys, C is located a fifth above T, but is a minor (↓) triad. Figure 4.5.1 illustrates how in each mode, all three functional triads are of the same type, even though C and D are located symmetrically opposite from the tonic:

Figure 4.5.1. Generation of the three basic triads in major and minor (Harmonologik, 34)

Karg-Elert then explains that “the inconsistency of the contradominant is ‘evened out’ if the under-fifth is accompanied by the under-third” – in other words, if the major-key contrant is a minor triad, and the minor-key contrant is a major triad (Figure 4.5.2):

Figure 4.5.2. Minor contrant in major, and major contrant in minor (Harmonologik, 35)
Karg-Elert calls these altered versions of C “tempered contrants” (temperierte Contrante) as they “temper” the directional inconsistency in the normal contrants illustrated in Figure 4.27. However, to avoid possible confusion between the term “tempered” and equal temperament, this dissertation (and the annotated edition of Akustische) instead translates temperierte Contrante as contrant variant, as the term “variant” denotes a change of quality from major to minor, or vice versa. The label for the contrant variant is lowercase $c$ in major, and $\varnothing$ in minor (Figure 4.5.3):

![Contrant variants in C major and A minor (Ak 6.7)](image)

**Figure 4.5.3.** Contrant variants in C major and A minor (Ak 6.7)

The concept of the contrant variant addresses one of Karg-Elert’s earliest perceptions of symmetry or polarity in harmonic relationships, namely between major V in minor keys, and minor IV in major keys (see Figure 4.1.2). The chromatically altered pitches featured in those two cases of modal mixture (G♯ in A minor and A♭ in C major) are polar counterparts, disposed symmetrically around the harmonic prime cell C-E. The temperierte Contrant is classified as a variant (Variant), which switches the contrant’s normal triad quality from major to minor, or vice versa. The change from normal to variant triad quality is reflected in the change from uppercase to lowercase: $C \rightarrow c$ in major, or $\varnothing \rightarrow \varnothing$ in minor. Another important feature of variants is that their acoustic derivation differs from their Pythagorean or Ursprungslage source position.
counterparts by a syntonic comma. For example, in relation to C(0), here is the acoustic data for the regular and variant contrants in C major and A minor:

\[
\begin{align*}
\text{C in C major: } & \ F(-1,0) \ A(-1,1) \ C(0) \\
\text{c in C major: } & \ F(-1,0) \ A\flat_{0,1} \ C(0) \\
\text{C in A minor: } & \ E(0,1) \ G(1,0) \ B(1,1) \\
\text{c in A minor: } & \ E(0,1) \ G\sharp(0,2) \ B(1,1)
\end{align*}
\]

This is the data for the *Ursprungslage* (source position) counterparts to the contrant variants:

\[
\begin{align*}
\text{F(-5,1) \ A\flat_{-4,0} \ C(-4,1) } & \\
\text{E(4,0) \ G\sharp(4,1) \ B(5,0) }
\end{align*}
\]

The following Tonnetze (Figure 4.5.4) illustrate the acoustic data just provided, indicating the syntonic commas acquired by the contrant variants, and their distance from the source positions:

**Figure 4.5.4.** The acoustic locations for the contrant variants in C major and A minor

Karg-Elert often uses closed noteheads (see Figure 4.5.3) to indicate the syntonic comma-differing thirds that are present in all variant triads.
As will be discussed in Chapter 6, contrant variants act frequently as pivot chords in comma-differing modulations. This is because they closely resemble the dominants of their parallel keys (Figure 4.5.5):

Figure 4.5.5. Comparison of dominants and contrant variants in harmonic (mixed) C major and C minor, and a diagram of functional energies in those keys (Harmonologik, 74)

Figure 4.5.5 demonstrates that the dominant (D) of C major contains the same pitch classes as the contrant variant (ơ) of C minor: both are [G B D]. Likewise, the contrant variant (ơ) of C major contains the same pitch classes as the dominant (D) of C minor: both are [F A♭ C]. However, the syntonic comma differences are still present, as the dominants are source position triads (related by fifth to their tonics), while the contrant variants are not. The first Tonnetz in Figure 4.5.4 illustrates the pitch-space difference between the two F minor triads. The diagram at the right of Figure 4.5.5 illustrates how the contrant variants create mixed versions of the modes, often called harmonic major and harmonic minor: the major mode now contains major T and D but minor ơ, while the minor mode contains minor L and ơ but major ơ. In terms of functional energies, the contrant variants substitute “a strong negative harmony in place of a weak positive harmony” (Harmonologik, 74), and thus provide a feeling of opposition or tension that is largely absent from the natural “unmixed” modes. Of course, most minor-key music uses a mixed mode, with raised scale degree ^7 in most dominant-function harmonies. The contrant variant can explain both major V in minor keys, and minor IV in major keys (Figure 4.5.6):
Figure 4.5.6. Parallel passages outlining cadences in C major and A minor, featuring the contrant variant (Harmonologik, 74)

Karg-Elert describes the unaltered dominant as “natural,” and the contrant variant as “artificial” (künstlich); his annotation to Figure 4.5.6 notes the “parallel imitation” between C major and A minor, but the “non-conformance of function”: the C major passage unfolds the natural cadence $T-c-D-T$, while the A minor version has the artificial cadence $L-D-c-L$. As in the case of Figure 4.4.14 above, one might regard this functional non-conformance purely as a stubborn consequence of Karg-Elert’s strictly polar system, especially in light of the analogous melodies and bass progressions in Figure 4.5.6. Nonetheless, the concept of the contrant variant is a valuable and elegant one, as it highlights the correspondence between the most frequent modal alterations in major and minor. The contrant variant most often appears within a cadential progression, leading to $D$ or $T$; however, the regular and variant contrants can be juxtaposed directly to create a cross relation (Querstand), as in the following familiar passage by Beethoven (Figure 4.33):

Figure 4.5.7. Beethoven: Piano Sonata in C minor, op. 2 no. 1 (movement I exc. – Harm, 75)
While the contrant variant is the most common and important variant of the three basic triads, the tonic and dominant can also appear as variants, in both major and minor keys. Karg-Elert calls the dominant variants “church dominants” \((\text{Kirchendominanten})\), as they are the normal dominants in two of the Church modes, namely Mixolydian and Dorian (Figure 4.5.8):

![Figure 4.5.8. Dominant variants in the Mixolydian and Dorian modes (\textit{Ak} 7.8)](image)

As indicated in Figure 4.5.8, C Mixolydian is equivalent to C major with lowered scale degree \(^\flat7\), while A Dorian is A natural minor with raised \(^\natural6\). While the contrant in Mixolydian and Dorian is the same as in major and minor respectively, the dominant changes quality to minor in Mixolydian, and to major in Dorian. This change of quality is again indicated by a change from uppercase to lowercase: \(D \rightarrow d\) in major, and \(\overline{\text{G}} \rightarrow d\) in minor. The scale degrees that define the Mixolydian and Dorian dominants (\(\downarrow 7\) in Mixolydian, and \(\uparrow 6\) in A Dorian) are polar equivalents, disposed symmetrically around the prime harmonic cell C-E. In relation to major and natural minor, Mixolydian and Dorian can be considered mixed modes, analogous to the harmonic major and minor described above, because their three basic triads are of different qualities. The following graphic from \textit{Harmonologik} (Figure 4.5.9) demonstrates the functional systems in Mixolydian and Dorian, and the opposition of functional energies in those modes:
Figure 4.5.9. Basic triads and functional energies in C Mixolydian and A Dorian (*Harm*, 178)

Figure 4.5.9 illustrates how the regular dominants of C major and A minor (that is, the “strong” and “open” chords) are replaced in Mixolydian and Dorian by their variants, which are “weak” (*schwach*) and “closed” (*geschlossen*), as they oppose the energetic direction of the tonics and contrants. Therefore, Mixolydian and Dorian are described as “doubly-sided weak” (*zweiseitig schwach*) – an observation that accords well with the significant decrease of directional tendency in those modes, owing to the lack of semitone resolution to the primes: the tonality-defining semitones B → C and F → E are replaced by the whole tones B♭ - C and F♯ - E. The last statement in Figure 4.5.9 (“the strong sides {i.e. the regular dominants of major and minor} are replaced by the weak contradominants of the variant keys”) suggests that the Mixolydian and Dorian dominants can be regarded as instances of simple modal mixture:

- **d** of C Mixolydian ~ **C** of C minor  
  [G B♭ D]
- **d** of A Dorian ~ **C** of A major  
  [D F♯ A]
Those equivalences are only approximate, as they metharmonically differ from each other by a syntonic comma. Regular contrants (C and G) are fifth-derived from their tonics, and are thus in the Ursprungslagen (source positions). The dominant variants of the parallel (variant) keys differ from those source position triads, as shown on the following Tonnetze (Figure 4.5.10):

![Tonnetze diagram]

**Figure 4.5.10.** Syntonic comma differences between dominant variants and the contrants of parallel keys

One might feel that assigning functions to triads in modes such as Mixolydian or Dorian is misguided, as the medieval modal system was fundamentally melodic rather than harmonic in origin. However, Karg-Elert is especially interested in modal variants of the basic functions that occur in the music of his time, such as the following from the music of Grieg (Figure 4.5.11):
Figure 4.5.11. Dominant variants in two passages by Edvard Grieg (Harmonologik, 184)

The keys of the two passages in Figure 4.5.11 are A minor and A major respectively, and Grieg’s harmony is largely traditional, and thus suitable for functional analysis. The dominant-variants are modal inflections, borrowed from A Dorian and A Mixolydian. Such modal borrowings are very common in much nineteenth- and early twentieth-century music, as Karg-Elert remarked:

In the works of Chopin (particularly the Mazurkas), Liszt, Brahms, Dvorák, Grieg, MacDowell, Debussy, Sibelius, Sinigaglia, Respighi, the Russian innovators – in so far as they reveal an emphatic national character, and in all exotic or pseudo-exotic pieces – unmistakable influences of the church modes markedly stand out. (Harmonologik, 179)

As will be examined in section 5.3 of this dissertation, Karg-Elert discusses the variants of the principal triads in the context of mediant (major- and minor-third) transformations (see Akustische 7.8). This is because “all principal variants are valid as bissonances of the counter-mediants” (Harmonologik, 201). The terms ‘bissonance’ and ‘counter-median’ will be explained in due course. In simpler language: the variants of the principal triads are also diatonic substitutes of the major-third lower mediants, as shown in Figure 4.5.12 (next page).
For example: C minor is both the tonic variant of C major ($T \rightarrow t$), and also the Leittonwechsel of A♭ major ($T_M$, the lower mediant of C major), or the Parallel (i.e. relative) of E♭ major ($D_M$, the lower mediant of G major). Transformational relationships involving mediants and counter-mediants will be discussed in detail in section 5.3. For now, the relevant point is that the principal variants are often found in passages of more thorough modal mixture, together with the counter-mediants. In familiar Stufenlehre terms: in major keys, minor I, IV and V often occur together with bVI, bIII or bII.
4.6. The concordant seventh and ninth chords

Section 3.5 examined how for Karg-Elert, the dominant seventh chord is an integral harmonic entity, derived entirely above or below a single prime pitch, and featuring the intervallic proportions 4:5:6:7. He calls the natural seventh (4:7) interval the concordant seventh, as it “melds and ‘concords’ with the dominant triad” (Ak 9.2). The concordant seventh can not only be generated above a major triad, but also in polar fashion below a minor triad. Figure 4.4.5 (reproduced here as Figure 4.6.1) demonstrated the addition of the concordant seventh to the dominant, in C major and A minor:

The addition of the concordant seventh is indicated as a line above or below the dominant function label: $\overline{D}$ or $\overline{A}$. On the staff, Karg-Elert often notates concordant sevenths using diamond-shaped noteheads, in order to reflect the septimal derivation of those pitches (and thus their separation from the basic fifth-third pitch space). The dotted lines added to the “Dominante” triangles graphically suggest that the addition of the concordant seventh to the dominant intensifies the latter’s already “strong” (stark) energetic charge – more positive in major, more negative in minor. Rameau described how the dominant seventh contains a stronger
tendency toward tonic resolution than the dominant without seventh, because of the tritone between the leading tone (called the “major dissonance”) and the chordal seventh (called the “minor dissonance”). In his *Traité de l’harmonie* of 1722, Rameau includes the dominant seventh in his perfect cadence (*cadence parfaite*), in both major and minor (Figure 4.6.2):

![Figure 4.6.2. Rameau’s *cadence parfaite* in C major and C minor (Rameau 1722, Book II, 57)](image)

In his explanation of the imperfect cadence (*cadence irregulière*, or what is now called a plagal or $S \rightarrow T$ cadence), Rameau adds a sixth above the subdominant, in both major and minor. Like

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28 Rameau 1722, Book II, 56. Lester 2002 provides a thorough overview of Rameau’s harmonic theories, including the concepts of perfect and imperfect cadences.
the added minor seventh above the dominant in the perfect cadence, the added sixth creates a dissonance that requires resolution, and that increases the pull toward the tonic (Figure 4.6.3):

![Image](image_url)

**Figure 4.6.3.** Rameau’s *cadence irregulière* in C major and C minor (Rameau 1722, II, 65)

Taking his cue from Rameau, Riemann discusses four “characteristic dissonances which are usually associated with the dominants” (Riemann 1893, 55). In all four cases, the added tones are taken from a different function than that of the base triad:

a) For the major dominant (*D*), the added root (lowest note) of the subdominant.  
   In C major: [G B D + F]. In A minor: [E G# B + D].  
   Function label = *D*7.

b) For the major subdominant (*S*), the added fifth of the dominant.  

c) For the minor subdominant (*s*S): in minor keys, the added (dual) prime of the minor dominant; in major keys, the added fifth of the dominant.  
   In A minor: [B + D F A]. In C major: [D + F Ab C].  
   Function label = *s*S7.

d) For the minor dominant (*s*D): the added root (lowest note) of the minor subdominant.  
   In A minor: [D + E G B]. Function label = *s*D7.
**Figure 4.6.4** demonstrates how Riemann’s characteristic dissonances combine pitches from different functions:

![Diagram](Image)

**Figure 4.6.4.** Riemann’s characteristic dissonances, derived from multiple functions (Riemann 1893, 56)

In cases A and C, the characteristic dissonance is a minor seventh, labelled as “7” when added above a major triad, and as “VII” when added below a minor triad. In cases B and D, the dissonance is a major sixth, labelled as “6” when added above a major triad, and as “VI” when added below a minor triad. Riemann’s added-sixth chords, and Karg-Elert’s formulation of them, will be discussed in section 4.8. Riemann’s added minor-seventh chords are dominant sevenths (case A) and half-diminished sevenths (case C), which are symmetrical counterparts of each other, and are the two chord types described by Karg-Elert as concordant sevenths. In contrast to Riemann’s differing function labels ($D^7$ and $S^{VII}$), Karg-Elert’s polaristic system indicates both cases A and C as dominants with added natural sevenths: $\overline{D}$ in major keys, and $G$ in minor keys (see Figure 4.6.1).
Karg-Elert’s derivation of the dominant seventh ($\overline{D}$ and $\underline{D}$) differs sharply from Riemann’s. Instead of being a dissonant sonority combining pitches from two different functions (and thus derived from two different primes), Karg-Elert’s concordant seventh is a unified and unidirectional harmony, generated entirely above or below a single prime. The term ‘concordant’ signifies a quality somewhere between consonance or dissonance: while the concordant seventh is a pleasing and ‘blending’ sonority, as attested by musicians from Tartini to Riemann himself (see section 3.5), it nonetheless contains a higher degree of tension than the simple major or minor triad (Figure 4.4.5), and thus has a strong forward pull toward resolution. The following Tonnetze (Figure 4.6.5) illustrate the unified derivation of Karg-Elert’s dominant sevenths, compared with the mixing of different functions in Riemann’s $D^7$ and $S^{VII}$.

**Figure 4.6.5.** The differing derivation of dominant and half-diminished sevenths in Karg-Elert and Riemann
Because the contrant variants (c and Ə) are metharmonic counterparts of the dominants from the parallel mode (see Figure 4.5.5), they too often generate their concordant sevenths, in the same way as the dominants (Figure 4.6.6):

As shown in Figure 4.6.6, the contrant variant in major keys is a minor triad, and so its concordant seventh is added below the prime; its function label is c. In minor keys, the concordant seventh is added above the contrant variant, and so its function label is Ə with a line above. The annotations under the staves in Figure 4.6.6 compare Karg-Elert’s function labels with those of Riemann, as well as typical Stufen notation using lowercase for minor and diminished triad. They illustrate how Riemann’s notation closely matches the Stufen labels, though it does reflect the very similar pitch content and function of the iv and iiø7 chords. In contrast, the strict polarity of Karg-Elert’s system highlights the symmetry between the concordant sevenths in major and minor, and between the dominant and half-diminished seventh chords. The following passage from Akustische demonstrates that symmetry (Figure 4.6.7):
Figure 4.6.7. A passage featuring concordant sevenths, in C major and A minor (Ak 6.8)

The dominant sevenths in C major and A minor share a tritone that resolves in the same way in both keys: B → C and F → E. The altered-contrant sevenths in the two keys contain tritones that are enharmonics of each other: D → E and Ab → G in C major, D → C and G♯ → A in A minor. (See also Figure 4.5.6, which is very similar to Figure 4.6.7). Neither traditional Stufen nor Riemann’s functional system indicate these symmetries, which can be readily heard and felt by the listener. In this instance, Karg-Elert’s polaristic function labels more closely reflect harmonic perceptions than other methods. The intensity and forward direction provided by the concordant sevenths is exemplified by the opening bars of Schumann’s “Ich grolle nicht” (Figure 4.6.8), a cadential progression in which the weak positive energy of C is replaced by the strong negative charge of G, followed immediately by the strong positive one of D♭, and the necessary release of tension at the return of T:

Figure 4.6.8. Schumann: “Ich grolle nicht,” opening measures (Harmonologik, 75)
While the addition of the concordant seventh is associated especially with the dominant and the contrant variant, any consonant triad can potentially generate its concordant seventh. A hallmark of some late nineteenth- and early twentieth-century composers and styles is the increasing preference for concordant sevenths (or extensions thereof) over simple triads. The following abstract example from chapter 11 of *Akustische* (Figure 4.6.9) is representative:

Figure 4.6.9. A chromatic passage consisting entirely of concordant sevenths (*Ak* 11.6)

The tonic of both passages is understood to be C major, largely due to its position at the center of the pitch space. The relevant point here is that all chords (even the tonics) include their concordant sevenths; instead of writing horizontal lines above or below each function label to indicate the sevenths, Karg-Elert simply notes “*mit Septimen*” (with sevenths). In a passage such as Figure 4.6.9, the sevenths no longer have any clear harmonic or voice-leading function. The passage demonstrates that while the concordant seventh normally suggests dominant or contrant variant function in tonal music, over time it gradually lost its traditional functional attributes, and in some styles virtually replaced the simple triad as the basic harmonic sonority.
The concordant seventh chord can generate an added major ninth: above the prime of a major chord, or below the prime of a minor chord. As with the concordant seventh, concordant ninth chords usually have dominant or contrant variant function (Figure 4.6.10):

![Concordant Ninth Chords](image)

**Figure 4.6.10.** Concordant ninth chords (*Bikordanzen*) in C major and A minor (*Harm*, 77)

The two lines above or below the function labels indicate that both the concordant seventh (4:7) and major ninth (4:9) from the prime are present. The added major ninth is Pythagorean, lying two fifths above/below the prime, or one fifth above/below the chordal fifth ("Quinte der Quinte" – see the arrows in Figure 4.6.10). While dominant ninths use only pitches from the diatonic key, contrant-variant ninths overstep the diatonic boundaries of the key: the altered-contrant ninth in C major \([C A\flat F D B\flat]\) adds \(B\flat(2,0)\), while the contrant-variant ninth \([E G\# B D F\#]\) of A minor introduces \(F\#(2,1)\). Both can be considered as modal inflections (respectively Mixolydian and Dorian) that enrich their keys without leaving them altogether.

The dominant ninths of relative keys contain the same pitch classes (see Figure 4.6.10):

- C major \((\rightarrow)\) G B D + F + A
- A minor \(G + B + D F A (\leftrightarrow)\)

It is for that reason that Karg-Elert names the concordant ninth chord as a “bicordance” (*Bikordanz*): it can be derived by extending either a major triad or a minor triad. However, Karg-Elert clearly states that the concordant ninth is a “natural uniformly-directed” entity: the prime
generates the fifth, which in turn generates the ninth, all in the same direction. The dominant ninths of relative keys are actually metharmonics, with completely different acoustic origins:

\[
\begin{align*}
\text{C major} & : G_{(1,0)} \quad B_{(1,1)} \quad D_{(2,0)} \quad F_{(1,0,1)} \quad A_{(3,0)} \\
\text{A minor} & : G_{(-3,1)} \quad B_{(-1,1,-1)} \quad D_{(-2,1)} \quad F_{(-1,0)} \quad A_{(-1,1)}
\end{align*}
\]

This Tonnetz (Figure 4.6.11) compares the dominant ninth chords in C major and A minor:

![Tonic and dominant ninth chords in C major and A minor](image)

**Figure 4.6.11.** Tonic and dominant ninth chords in C major and A minor

Due to their metharmonic similarity, dominant ninths from relative keys can be reinterpreted as each other, and thereby used as pivot chords in comma-differing modulations (see Chapter 6).

Ninth chords appear frequently with omitted primes (Figure 4.6.12):

![Ninth chords with omitted primes](image)

**Figure 4.6.12.** Dominant and altered-contrant ninths with omitted primes (*Harmonologik*, 77)

The omission of the prime is indicated by the small circle next to the function label: to the bottom left for major triads (prime at the bottom), and to the upper right for minor triads (prime at the top). In Figure 4.6.12, the second chord (dominant ninth of A minor) should have an “x”
at the pitch a\(^1\), which is the omitted prime. Karg-Elert notes that the ninths with omitted primes resemble (Ähnlichkeit mit) dominant sevenths in other keys; those resemblances are once again metharmonic rather than acoustically exact. For example, the prime-omitted dominant ninth in C major is metharmonically similar to the (complete) dominant seventh in A minor:

\[
\begin{align*}
\text{C major} & \quad B_{(1,1)} \quad D_{(2,0)} \quad F_{(1,0,1)} \quad A_{(3,0)} \\
\text{A minor} & \quad B_{(-1,1,-1)} \quad D_{(-2,1)} \quad F_{(-1,0)} \quad A_{(-1,1)}
\end{align*}
\]

Because of these metharmonic similarities, incomplete concordant ninths can be reinterpreted as dominant sevenths of other keys in comma-differing modulations.

Like concordant sevenths, concordant ninths or Bikordanzen are not considered dissonant, because they are entirely derived from a single prime, manifesting the natural intervallic proportions 4:5:6:7:(8):9. However, a specific dissonant alteration of the concordant ninth is very common: the so-called “Italian ninth chord” (Italienischer None), which has a minor ninth above or below the prime, and which can appear with or without the prime. Figure 4.6.13 illustrates the complete and prime-omitted versions of the minor ninth chord:

![Figure 4.6.13](image)

**Figure 4.6.13.** Major and minor concordant ninth chords in C major and A minor (*Harm*, 78)

The minor ninth chords are dissonant alterations of the concordant ninths: the natural 4:9 ninth interval has been artificially lowered or raised by a Pythagorean chromatic semitone. The raised or lowered ninths are indicated by the “hook” symbols above or below the function labels. The
dominant-seventh with minor ninth above is a very common sonority; in Karg-Elert’s system, it appears as the dominant minor ninth in major keys, or as the altered-contrant minor ninth in minor keys. Its symmetrical inversion (a minor ninth below a half-diminished seventh) is much less familiar; it appears in Karg-Elert’s system as the altered-contrant minor ninth in major keys, or as the dominant minor ninth in minor keys (see Figure 4.6.13). *Harmonologik* does not provide a repertoire example of the half-diminished seventh with minor ninth below; however, it is featured prominently at the climax of the first movement in Mahler’s Symphony No. 2, three bars before the recapitulation and the return to the C minor tonic (Figure 4.6.14):

![Figure 4.6.14. Mahler: Symphony No. 2 in C minor, first movement (three bars before the recapitulation at rehearsal 21)](image)

The passage in Figure 4.6.14 concludes a lengthy retransition, standing on the dominant of C minor. In traditional usage, the first bar will be analyzed as an applied dominant tonicizing V of C minor, while the last two bars are the dominant of C minor. However, because the dominant root G is held as a pedal in both the top and bottom voices, one might understand the harmony in a polaristic manner, both from below and from above. The first chord can be heard upwards as [(D) F♯ A C Eb]: a dominant minor ninth of G major, but with its prime D omitted, and with the G pedal sustained above and below. Alternatively, it can be heard downwards as [G Eb C A F♯], which for Karg-Elert is a complete altered-contrant minor ninth chord in G major.
The second bar in the Mahler passage is a stack of thirds, spanning the entire C harmonic minor collection: [G B D F A♭ C E♭ G]. If the G major triad is assumed as the chord’s functional basis (owing to its registral placement at the bottom of the chord), then it can be understood in Karg-Elert’s system as a contrant-variant minor ninth [G B D F A♭] of C minor, with added eleventh and thirteenth [C + E♭]. However, viewed polaristically, the chord contains the dominant minor ninth of C minor: [C A♭ F D B], with added eleventh and thirteenth below [G + E♭]. In the final bar, the C is extinguished, leaving a contrant-variant minor ninth of C minor [G B D F A♭] with added eleventh E♭.

If the prime is omitted from the minor ninth (indicated in Figure 4.6.13 by the small circle), the remaining pitches create a diminished seventh chord. The figure demonstrates that in C major, the diminished seventh chord [B D F A♭] can be created in two ways— as a dominant minor ninth with omitted prime [(G) B D F A♭], or as an altered-contrant minor ninth with omitted prime [B D F A♭ (C)]. Those two diminished-seventh chords are metharmonics of each other, displaying “comma-differing similarity” (*kommadifferierende Ähnlichkeit*). Figure 4.6.13 indicates how for Karg-Elert, the functional meaning of the diminished seventh chord depends on its acoustic derivation, which in turn depends on musical context. Karg-Elert’s complex formulation of the diminished seventh chord is representative of his theories of dissonance, which are presented in chapter 11 of *Akustische*. Detailed examination of those theories lies beyond the scope of this dissertation, and should be the focus of a separate study in the future.

Karg-Elert notes that “the ninth-chords, which are distinguished by a softer, more pleasant, more mellifluous and sensual character…form with the ‘twins’…the basic elements of a hybridized Impressionistic style” (*Harmonologik*, 79). The ‘twins’ (*Zwillinge*) are diatonic
seventh chords, which will be discussed in section 4.8. *Harmonologik* provides several examples of ninth chords in the Impressionist and pre-Impressionist literature. The following authentic cadence from a song by Frederick Delius (Figure 4.6.15) demonstrates the efficacy of the contrant variant ninth as a functional concept:

![Figure 4.6.15. Delius: "Das Veilchen" (“The Violet,” 1900), final cadence (Harm p. 81)](image)

While the identity of the tonic and dominant in Figure 4.6.15 are very clear, traditional methods of analysis would struggle with the pre-dominant F9 chord. Karg-Elert reveals it to be a simple contrant variant (C minor), with concordant seventh and ninth (A and F). Heard from the bass up, the function of the F9 chord is ambiguous; heard polaristically, its function becomes clear.

The following example (Figure 4.6.16) from the second (“Serenade”) of Mussorgsky’s *Songs and Dances of Death* (1875) demonstrate concordant ninth chords in minor:

![Figure 4.6.16. Mussorgsky: “Serenade” from Songs and Dances of Death, excerpts (Harm p. 81)](image)
Excerpt A is a simple alternation of tonic and dominant in E-flat minor; the first dominant chord (Ab minor) is unadorned, but the second adds the concordant seventh and ninth below, creating a Db9 chord. As in the Delius example above, traditional Stufen methods would struggle to identify that chord. Excerpt B is very similar, but the tonic is now E-flat major. Karg-Elert provides two functional analyses. The first relates back to the original key of E-flat minor: the dominants are still in that key, but the tonics are now major, indicated as tonic variants (♮). The second analysis is in E-flat major, and thus alternates between tonic and contrant variant. Of this and similar examples in Harmonologik, Karg-Elert remarks that “the hybridization (major? minor? or major + minor) is particularly characteristic” (Harmonologik, 81). In other words: because the concordant ninth is itself a hybrid major/minor “bicordant” sonority, music that features the concordant ninth will often display a hybrid or mixed sense of mode.
4.7. **Enriching the key: the ultraforms, and prefix and suffix chords**

Though the idea of a secondary or applied dominant was recognized by various scholars at least as far back as Jérôme-Joseph Momigny, it was the work of Hugo Riemann that established the secondary dominant as a mainstay of analytical practice. In *Harmony Simplified*, Riemann first discusses the “dominant of the dominant” and the “subdominant of the subdominant” as self-sufficient, stand-alone harmonies. Their function labels are doubled and overlapped $D$ and $S$ symbols, as shown in **Figure 4.7.1**:

![Figure 4.7.1. Stand-alone double dominants and subdominants (Riemann 1893, 101)](image)

In example A, the double dominant proceeds to the regular dominant, creating a tonicization of G major; likewise, in example C the double subdominant leads to the regular $\text{oS}$. However, this is not necessary; double dominants and subdominants can proceed to or from other harmonies in the key, as shown in examples B and D. To specify that chords function as local dominants or subdominants of other harmonies, Riemann introduced the concept of the *intermediate cadence*, in which “more than one chord is to be referred to a following chord as tonic (for the time being).” Riemann uses parentheses to indicate such applied or secondary chords, which can include subdominants as well as dominants. The rule is: “chord signs in brackets are not to be

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30 Momigny 1821. Damschroder 2008 (pp. 151–152) examines Momigny’s tendency to incorporate a variety of chromatic chords within in a central key, instead of frequent modulation.
31 Riemann 1893, 128.
understood as relating to the principal key, but as circumscribing the chord immediately following the bracket, as tonic.” He provides the following example (Figure 4.7.2):

**Figure 4.7.2.** Intermediate cadences (i.e. secondary dominants and subdominants) in C major and A minor (Riemann 1893, 129)

In some cases, the applied chords follow rather than precede the local tonic, as a kind of harmonic suffix. Riemann indicates such cases using a backward arrow (Figure 4.7.3):

**Figure 4.7.3.** Applied chords following their local tonic (Riemann 1893, 130)

Finally, applied chords are sometimes neither preceded or followed by their local tonic, as in the following example (Figure 4.7.4):
In this case, the applied chords in bar 2 relate to A minor (the tonic parallel or $T_p$ in C major). However, the expected A minor arrival in bar 3 is replaced deceptively by a subdominant major seventh ($S^7<$), leading back to C major. The $T_p$ label is placed in square brackets to show that A minor is a non-present, imagined tonic: “the harmony indicated within the angular brackets is the merely imagined tonic of the preceding intermediate cadence indicated in the rounded brackets, and is itself not introduced at all, but skipped over.”

Karg-Elert’s presentation of applied dominants and subdominants is largely similar to that of Riemann, distinguishing between chords applied to local tonics, and stand-alone double dominants and subdominants. The following example from Akustische 6.9 (Figure 4.7.5, on the next page) illustrates the distinction, as well as Karg-Elert’s analytical notation for the two types. The double dominant ($DD$) and double contrant ($CC$) are called ultradominant and ultracontrant, as they “extend beyond the boundaries (Grenze) of the key, ‘over’stepping its space or region” ($Ak$ 6.9); that is, they introduce non-diatonic pitches, such as the $B_b$ and $F#$ in the keys of C major or A minor. Karg-Elert explains that when the ultraforms act as applied dominants or contrants of a local tonic, “they have direct linear connections with the regular dominant or contrant, [and] they are understood as prefix (Einführung) or suffix (Ausführung) chords.”

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32 Riemann 1893, 130.
33 Akustische 6.9.
in Riemann, prefix and suffix applied chords are placed in parentheses; Karg-Elert adds a tie below to indicate prefixes, and a tie above to indicate suffixes.

Figure 4.7.5. Ultradominant and ultracontrant, and prefix and suffix chords (Ak 6.9)

The upper half of Figure 4.7.5 shows that the ultraforms extend the chain of fifth-related triads surrounding the tonic; they are therefore Ursprungslagen or source position chords, related by fifth to the principal harmonies. In the lower half of the figure, the prefix and suffix chords are linked by acoustic common tone to their local tonics; for example, in the C major progression C and C(C) share F(−1,0). In contrast, there is no acoustic common tone between the two ultraforms, which contain metharmonic versions of the same pitch class:

C major: \[ EC \quad B^\flat(-2,0) \quad D(-2,1) \quad F(-1,0) \quad BD \quad D(2,0) \quad F^\sharp(2,1) \quad A(3,0) \]

The metharmonic gap occurs where a break ("\) is indicated in the Figure 4.7.5 progressions.

So-called “real” ultradominants and ultracontrants have no direct linear connection to the tonic, unlike the regular dominant and contrant which they can replace in harmonic progressions.
Karg-Elert’s function labels for the stand-alone ultraforms reflect how the original dominant or contrant are omitted or “struck through”; he states that the original chords are “not actually present, but latently implied through transformation” (Figure 4.7.6):

![Figura 4.7.6. “Real” ultraforms: “isolated, non-consecutive” (Ak 6.9)](image)

In Figure 4.7.6, the root motions are entirely by whole step, as they skip over the regular D and C. The ultraforms each contain one pitch from the regular harmonies: specifically, CC contains the prime of C, and DD contains the fifth of D. The ultraforms thus retain an audible connection to the “latently implied” principal harmonies, and so the progressions in Figure 4.7.6 can be understood as modifications of the complete authentic cadence T–C–T–D–T.

Prefix and suffix chords are not limited to the simple D and C triads, but can also include dominant and contrant-variant sevenths, as shown in the following (Figure 4.7.7):

![Figure 4.7.7. Concordant sevenths and contrant variants as prefix and suffix chords (Harm, 83)](image)
In Figure 4.7.7, the upper staff is in C major, and the lower staff is its polar counterpart in A minor. Example A includes a prefix dominant seventh, while example D features a prefix contrant variant. Examples B and C involve added-sixth chords, which will be examined in section 4.8. Finally, Example E demonstrates various prefix and suffix chords, including dominant and altered-contrant sevenths. In all examples, acoustic common tones are indicated using ties. Once again, the two ultraforms share a metharmonic pitch class: the comma-differing versions of D, indicated in the penultimate measure.

Figure 4.7.7 demonstrates how the ultraforms and other prefix/suffix chords enrich the key by introducing various chromatic pitches, but without endangering the key-defining status of the main tonic and the other principal harmonies. Where Karg-Elert goes notably beyond Riemann in terms of chromatic expansion is in the range of possible modifications to the ultraforms. The following example (Figure 4.7.8, next page) demonstrates several possibilities, some of which introduce pitches far from the diatonic key, and substantially blur the sense of tonality. Examples A through D are in C major, while examples E and F are in A minor:

**Figure 4.7.8.** Modified ultraforms (*Harmonologik*, 84)
The following explains the modifications of the ultraforms in Figure 4.7.8:

a) In C major. The ultradominant includes its concordant seventh: [D F♯ A C]. There is no tie linking the two C’s in the soprano, as they are acoustically distinct: C(2,0,1) vs. C(0).

b) The ultradominant is a concordant ninth [(D) F♯ A C E], with omitted prime [D]. Once again, the two C’s are acoustically distinct, exactly as in example A.

c) The ultracontrant [B♭ D F] appears as its minor variant [B♭ D♭ F], with concordant seventh below [G]. The variant shift is denoted by the lowercase label (E♭). The G’s are acoustically distinct: G(-1,0,-1) vs. G(1,0).

d) The ultracontrant is a minor variant as in example C, but now with the addition of the major ninth below [E♭]. Again, the G’s are acoustically distinct, exactly as in example C.

e) Now in A minor. The penultimate chord is what Karg-Elert calls a Zwitter (“hybrid”): a major/minor combination of the regular contrant [E G B] and the contrant variant [E G♯ B]. The concordant seventh [D] is included, and the prime [E] is omitted, as shown by the circle. The function label for the Zwitter combines the regular and contrant variants (Œ and œ), the latter placed inside the former.

f) This complex example includes all of the modifications from examples A through E (excepting the Zwitter). The resulting stream of concordant sevenths and ninths includes almost the entire chromatic scale (excepting G♯/A♭), and the sense of key is greatly suspended, at least until the last three chords.

By means of the regular and modified ultraforms, as well as prefix and suffix chords, extensive chromatic expansion of the key becomes possible and explainable. However, while the harmonic relationships presented in Figure 4.7.8 are typical of late nineteenth or early twentieth century music, Karg-Elert notes that the ultraforms are also common in much pre-Baroque music, in which “the ultradominants very often replace the dominants” (Harm, 84).

The following phrase from an anonymous a cappella “dance-song from the 14th century” (possibly in a sixteenth-century setting) illustrates the point (Figure 4.7.9):
Figure 4.7.9. Example of ultraforms and prefix/suffix chords in early music (Harm, 85)

The isolated DD and CC (bars 2 and 3) replace the more “tonal” D (D major) and C (C major), which could harmonize the same upper-voice melody. While the mode of this phrase would likely be Mixolydian, the harmonies include various pitches from outside the G Mixolydian collection. In bars 4-5, the ultradominant (A major) is itself tonicized by its dominant (E major) – a “triple dominant” of G major. The chord roots outline a purely Pythagorean series of fifths, and so the chords are all in their source positions (Ursprungslagen), as shown in Figure 4.7.10:

Figure 4.7.10. Tonnetz indicating the chain of fifth-related triads in Figure 4.7.9

The preceding example suggests that the idea of ultradominant/ultracontrant may not be limited to the fifths on either side of the regular dominant and contrant, and that fifth-transformations of basic triads can be reiterated. A transformational perspective on fifth-related chords (such as the ultraforms) will be presented in section 5.2 of this dissertation.
4.8. The diatonic substitutes, twins and triplets

In *Harmony Simplified*, Riemann describes how a new class of functional chords can be derived from “the omission of the fifth in the chords of the sixth…namely, those arising from the addition of the characteristic dissonance to the contra-fifth clangs (*S* and *DVI*)…by omission of the fifth a clang apparently of the opposite mode results.”34 The chords in question are cases B and D of the characteristic dissonances, demonstrated above in Figure 4.6.4. *S* is a major triad with added major sixth above, while *DVI* is a minor triad with added major sixth below:

C major \[ S^6 = [F A C + D] \]
A minor \[ D^{VI} = [D + E G B] \]

If the fifths are omitted from those chords, they create triads of the opposite quality to the mode:

C major \[ S^6 = [F A (C) + D] \rightarrow [F A D] = D^\text{minor} \]
A minor \[ D^{VI} = [D + (E) G B] \rightarrow [D G B] = G^\text{major} \]

Riemann calls such triads “feigning consonances” (*Scheinkonsonanzen*), because while they sound as consonant major and minor triads, they are derived from the characteristic dissonances, which mix pitches from two different functions (see Figure 4.6.4). Because the *Scheinkonsonanz* stands “in the relation in which tonics of parallel [i.e. relative] keys stand to each other (F major and D minor, E minor and G major)…we will call it the parallel clang.”35 Each of the basic functional triads (*T*, *S* and *D*) have parallels, indicated by *p* placed after the function label: *T*<sub>p</sub>, *Sp*, *Dp*. Riemann explains how the parallels introduce a pitch from a different functional triad:

“In *T* – *Tp* the newly added note is third of *S*,
in *S* – *Sp* the newly added note is fifth of *D*…
in *D* – *Dp* the newly added note is third of *T*.

In *o*T – *oTp* the newly added note is third of *oD*,
in *oD* – *oDp* the newly added note is fifth (V) of *oS*…
in *oS* – *oSp* the newly added note is third of *oT*.” (Riemann 1893, 73-74)

34 Riemann 1893, 71.
35 Ibid.
Based on the newly-added notes just described, Riemann states that “the parallel clang appearing after the principal clang, therefore, always indicates the anticipation of an element of the harmony which follows logically…In the cadence of the pure major key,

- \( Tp \) enters between \( T \) and \( S \)
- \( Sp \) enters between \( S \) and \( D \)
- \( Dp \) enters between \( D \) and \( T \). In the cadence of the pure minor key,

- \( oTp \) between \( oT \) and \( oD \)
- \( oDp \) between \( oD \) and \( oS \)
- \( oSp \) between \( oS \) and \( oT \).”  

(Riemann 1893, 74)

The parallels thus appear in the following cadences in C major and A minor, “which may be designated as normal” (Figure 4.8.1):

Figure 4.8.1. Parallels connecting principal harmonies in the cadence (Riemann 1893, 74)

One important detail to note is that in just intonation, \( Sp \) does not in fact contain the fifth of \( D \) (and nor does \( oDp \) contain the dual fifth of \( oS \)). All of those chords include the pitch class D, but while \( Sp \) and \( oS \) contain \( D_{-2,1} \), \( D \) and \( oDp \) include \( D_{2,0} \). This syntonic comma difference is endemic to the just-intonation diatonic collection (see Figure 3.6.1), as Riemann was surely aware; evidently Riemann chose to ignore such comma differences in his example of the parallels, in favor of the comma levelling provided by equal temperament. As one might expect, Karg-Elert meticulously takes comma differences into account, as will be discussed below.
Riemann defines the *Leittonwechsel* ("leading-tone change") as "the step from one clang to the change-clang of its plain leading-note" — that is, the step from a triad to the triad of opposite quality whose (dual) prime is the (dual) leading-tone of the first triad. As with the parallels, the Leittonwechsel is a symmetrical or polaristic relationship: C major → E minor, and vice versa. Riemann first explains that the Leittonwechsel is manifested between a triad and a parallel of a different function, such as $T - Dp$ or $o^T - o^Sp$, as in the following (Figure 4.8.2):

![Figure 4.8.2. Leittonwechsel in C major ($T - Dp$) and A minor ($o^T - o^Sp$) (Riemann 1893, 76)](image)

Riemann then explains "another possible way of deriving minor chords [and vice versa]...the (figurative) replacing of a prime by the minor contra-second." When applied to major triads, this replacement is indicated as $II<;$ applied to minor triads, it is indicated as $2>$ (Figure 4.8.3):

![Figure 4.8.3. Replacing a triadic prime by its "minor contra-second" (Riemann 1893, 79)](image)

In this manner, Riemann creates the Leittonwechsel through direct transformation of a single functional triad, rather than between a triad and a parallel of different function. Instead of the

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36 Riemann 1893, 76.
37 Ibid., 79.
cumbersome notation of Figure 4.8.3, Riemann indicates Leittonwechsel substitutions using wedges: < is written through major-triad function labels, while > is written through minor-triad labels. He demonstrates that because parallels and Leittonwechsel are of the opposite quality to the mode, they enable modulation between parallel (i.e. relative) keys. The following passages begin in C major, but modulate to A minor through parallels or Leittonwechsel (Figure 4.8.4):

![Figure 4.8.4. Parallel-key modulation through parallels and Leittonwechsel (Riemann 1893, 81)](image)

Karg-Elert’s description of the parallels and Leittonwechsel (known collectively as the diatonic substitutes) owes much to that of Riemann. He explains in *Akustische* 6.10 that the some of the substitutes “result from the fusion of two principal harmonies, from which the peripheral chord-tones have been detached” (Figure 4.8.5):

![Figure 4.8.5. Creations of substitutes through fusion of $C + T$, or $T + D$ (Ak 6.10)](image)

(v.) = forward (vorwärts)   (r.) = backward (rückwärts)
For that reason, Karg-Elert classifies the diatonic substitutes as “bissonances” (*Bisonanzen*): two incomplete triads of like quality, fused together into a new triad of opposite quality. Unlike the principal triads which manifest the harmonic proportions 4:5:6, the bissonances exhibit the more remote proportions 10:12:15. The major-key substitutes are minor triads, but derived from the upward series of partials; conversely, the minor-key substitutes are major triads, but derived from the downward partials (Figure 4.8.6):

\[C \text{ major: } T^1 \text{ or } D_p\]
\[A \text{ minor: } L_L \text{ or } C^p\]

**Figure 4.8.6.** Acoustic derivation of the diatonic substitutes (*Harmonologik*, 86)

One notable difference from Riemann is in the labelling of both parallel and Leittonwechsel. Because each parallel “lies in the backward direction from its principal,” the symbol “\(p\)” is placed as a subscript in major keys, but as a superscript in minor keys. Conversely, because each Leittonwechsel “lies in the forward direction from its principal, the letter “\(L\)” is placed as a superscript in major keys, and as a subscript in minor keys. Figure 4.8.5 illustrates how \(C^l\) and \(T_p\), as well as \(T^l\) and \(D_p\), contain exactly the same pitches in both major and minor, with no comma differences. Even though the parallels combine pitches from different functions, Karg-Elert does not follow Riemann in describing the diatonic substitutes as *Scheinkonsonanzen* or “feigning consonances,” or as added-sixth chords with omitted fifths (those are labelled differently by Karg-Elert, as will be seen below).
While $C^\flat/T_p$ and $T^\flat/D_p$ hover between $C$ and $T$ or $T$ and $D$ respectively (see Figure 4.8.5), the “outer substitutes” $C_p$ and $D^\flat$ reach or even breach the boundaries of the diatonic key, as “their formation assumes the latent existence of the ultraforms” (Figure 4.8.7):

![Figure 4.8.7. $C_p$ and $D^\flat$ at the boundaries of the diatonic key (Ak 6.10)](image)

$C_p$ combines pitches from $C$ and $CC$, while $D^\flat$ combines pitches from $D$ and $DD$. Therein lies another difference with Riemann, already described above: the (dual) fifth of $C_p$ is not identical with the (dual) fifth of $D$. Instead, the two pitches are metharmonics, differing by a syntonic comma. Whenever $C_p$ leads directly to $D$ (as is very common in diatonic progressions), metharmonic adjustment is necessary, as in the following passage by Wagner (Figure 4.8.8):

![Figure 4.8.8. Juxtaposition of $C_p$ and $D$, with metharmonic B♭’s (Harmonologik, 98)](image)

In the Parsifal passage, $C_p$ contains B♭(-5,1), while $D$ contains the source-position B♭(-2,0) – a syntonic comma difference of $18 \mu$. Notably, the B♭ is sustained in the bass: the metharmonic adjustment must therefore be made within a single pitch and voice. In musical practice, the actual pitch frequency of the bass B♭ is unlikely to change; the metharmonic adjustment will thus
be realized as a mental change of orientation, reflecting the pitch’s change of function. In addition to the metharmonic B♭’s, the D♭’s also differ from each other, now by a septimal comma: $\text{Cp}$ contains the Pythagorean $\text{Db}_{(-5,0)}$, while $\text{D}$ contains the septimal (concordant) pitch $\text{Db}_{(-3,0,1)}$. The point of metharmonic shift is indicated in Figure 4.8.8 by a break (||).

Because $\text{CL}$ and $\text{Tp}$ are acoustically identical (and the same is true for $\text{TL}$ and $\text{Dp}$), the functional identity of those harmonies is dependent on musical context. Two basic uses of the diatonic substitutes are as neighbouring chords (Wechsler) expanding a single function, and as connecting chords between different principals as described by Riemann (see Figure 4.8.1). Those two usages are demonstrated in Figures 4.8.9 and 4.8.10 respectively:

![Figure 4.8.9](image1.png)

**Figure 4.8.9.** Diatonic substitutes as neighbor chords within a function (*Harmonologik, 89*)

![Figure 4.8.10](image2.png)

**Figure 4.8.10.** Diatonic substitutes making “connection between principals” (*Harmonologik, 89*)

Both Figures 4.8.9 and 4.8.10 are in C major. The dotted lines in Figure 4.8.9 indicate the identical pitch content between $\text{CL}/\text{Tp}$ and $\text{TL}/\text{Dp}$; their function is determined by the surrounding
principal harmonies. In Figure 4.8.10, the Leittonwechsel are described as “strong” substitutes, because they proceed in an upward or forward direction from the major-key principal chords; conversely, the parallels are “weak” substitutes because they proceed downwards from the principals, in the opposite direction from the major mode. The choice of $T^l$ and $T_p$ (instead of $D_p$ and $C^l$) may be explained in two ways. Firstly, whenever possible, diatonic substitutes are understood by Karg-Elert to extend the function of a preceding chord. Secondly, changes of function are more likely to be perceived on metrically strong beats rather than weak ones. The same factors influence the functional analysis of the progressions in Akustische 6.11, which illustrate how diatonic substitutes can elaborate a simple cadential pattern in terms of harmonic and rhythmic activity.

The juxtaposition of diatonic substitutes and the contrant variant creates chromatic relationships that are common in later nineteenth century music (Figure 4.8.11):

![Figure 4.8.11. Juxtapositions of substitutes and the contrant variant, in C major (Harm, 90)](image)

The annotations above Figure 4.8.11 indicate the relationships between the chord roots: minor third ($Nebenmediante$), major third ($Mediante$), diatonic semitone ($Leitklang$) and tritone ($Tritonante$). The first two of those terms also refer to third-based transformations; the successions $[Dm \rightarrow Fm]$ and $[Am \rightarrow Fm]$ can be derived through direct transformation of a single triad, and therefore can be analyzed within a single function (see section 5.3 below). The $Leitklang$ and $Tritonante$ relationships cannot be created via direct transformation of a single
triad (at least not without a large number of steps), and so they are labelled using different functions.

Karg-Elert notes that the diatonic substitutes “cannot make a concordance, as their sonic direction and origins reject it” (Harmonologik, 92) – that is, they do not generate concordant sevenths, because they are not true harmonic triads, but rather fusions of pitches from two triads. This does not mean that sevenths are never added to a diatonic substitute, but rather that the addition of the concordant seventh changes its function, from diatonic substitute to a dominant or contrant variant of a different key (Figure 4.8.12):

Figure 4.8.12. Conversion of diatonic substitutes to applied dominants, through addition of the concordant seventh (Harmonologik, 92)

The annotation states that such functional conversion “is possible for any substitute whatsoever.”

As the preceding examples have illustrated, the diatonic substitutes create new chromatic relationships within the key. This expansion is increased when prefix and suffix chords are applied to the substitutes, as in the following example from Akustische 6.12 (Figure 4.8.13):

Figure 4.8.13. Prefix chords applied to the diatonic substitutes, in C major (Ak 6.12)
Figure 4.8.13 includes much of the chromatic scale, including the enharmonic and acoustically distinct pitch pairs C♯ / D♭ and A♯ / B♭. Notably, the most key-expanding chords are altered-contrant sevenths (chords #2, 3 and 4), which introduce syntonic comma differences. Karg-Elert provides an Ursprungslagen (source positions) analysis of the passage, placing each chord in a specific register that reflects its distance in pitch space from the tonic C major (Figure 4.8.14):

![Figure 4.8.14. Ursprungslagen (source positions) analysis of Figure 4.8.13 (Ak 6.12)](image)

All of the concordant sevenths are omitted from the analysis, as none of the chords are connected via seventh-based transformations (see section 5.4). The chords that are notated entirely in open noteheads are in their source positions, related by fifth to the tonic C major. The thirds of the contrant variants (notated as closed noteheads) differ by one or more syntonic commas from their source position counterparts. Much like the Ursprungslagen analysis, the Tonnetz (Figure 4.8.15, next page) for Figure 4.8.13 is also a graphic representation of harmonic distances in pitch space. While all of the principal chords and their diatonic substitutes are on the same vertical level (reflecting their fifth-based relationships and lack of syntonic commas), the contrant variants are on different levels. The largest harmonic leap is between bars 2 and 3, from F major (C) to F♯ major (chord #4, contrant variant of D♮). Once again, all concordant sevenths are omitted from the Tonnetz, as they do not generate any of the harmonic relationships, or alter the harmonic path in pitch space.
Both the source position analysis and the Tonnetz demonstrate that a passage like Figure 4.8.13 can introduce a wide array of chromatic pitches and chords without greatly weakening the overall sense of tonal coherence or centricity. Karg-Elert summarized this process in an enigmatic statement: “Tonality widens, the key is destroyed” (Die Tonalität weitet sich, die Tonart wird zerstört). It suggests that for Karg-Elert, a key (Tonart) is mainly limited to a specific diatonic collection, while tonality (Tonalität) is a tonally-centralized organization of a diverse and wide-ranging harmonic space. This topic will be examined further in Chapter 6, which deals especially with comma-free and comma-differing modulation.

Though Karg-Elert’s analytical system does provide a way to indicate chordal inversions (see Figure 4.4.12), for the most part the analyses in both Akustische and Harmonologik do not indicate inversions; the focus is primarily on the acoustic derivation and functional meaning of the chords, rather than on their specific voicings or positions. However, the voicing of certain chords can have functional implications. For example, in the key of C major: should the chord [F A D] be understood as an inversion of II (or Sp in Riemann’s system), or as an added-sixth

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38 Harmonologik, 93.
alteration of IV (S)? This question was raised by Karg-Elert in *Grundlagen* (see Figure 4.2.2). While both will be understood as having subdominant (contrant) function, should the subtle distinction between the two chords be reflected in the analytical notation? Riemann explicitly defined the subdominant parallel (Sp) as a subdominant added-sixth chord (S⁶) with omitted fifth; therefore, in C major the pitch combination {D, F, A} will normally be labelled in Riemann’s system as Sp, regardless of its inversion. In contrast, Karg-Elert’s system carefully distinguishes between root-position parallels, parallel-seventh chords, and complete or incomplete added-sixth chords (Figure 4.8.16):

![Figure 4.8.16. Contrant parallels and added sixth chords in C major and A minor (Harm, 72)](image)

The figure suggests that Cₚ specifically denotes a root-position parallel of C, generated downward from the same prime/third cell (namely [F A]) as C itself. The contrant with added sixth [F A C + D] is labelled as C^₃; the triangle indicates the combination of the fifth (⁻) and the added sixth (^). Karg-Elert’s C^₃ is thus equivalent to Riemann’s S⁶. When the fifth is omitted from the added-sixth chord, the symbol becomes C^; the missing lower edge indicates the omission of the fifth. Clearly, Cₚ and C^ contain the same pitch classes. The difference is largely one of derivation: Cₚ is understood as a symmetrical flip of C around the prime/third [F A], while C^ is a linear alteration of C itself [C → D]. In *Akustische* 11.3, Karg-Elert classifies the added sixth (like all other linear alterations of regular chord tones) as a dissonance; in contrast, he
describes the parallels as “bissonances,” which are fusions of two different triads, but which essentially acquire the status of new consonant triads. Karg-Elert’s analyses suggest that the choice of parallel or incomplete added-sixth (such as $C_p$ or $C^\$) is largely a matter of chordal inversion. For example, he describes the “Viennese Classical typical cadence-form” (Figure 4.8.17), which features the “sixth-chord of the contradominant in major (thus an apparent contradominant-parallel) before the 6-4 chord of the dominant” (Harmonologik, 96):

Figure 4.8.17. $C^\$ in the “Viennese Classical typical cadence-form” (Harmonologik, 96)

One important question raised by Karg-Elert’s comment on Figure 4.8.17: exactly how is $C^\$ an “apparent contradominant-parallel”? More specifically: do $C^\$ and $C_p$ contain acoustically-identical pitches? The diagram in Harmonologik (Figure 4.8.16) suggests that $C^\$ and $C_p$ are indeed acoustically identical, differing only in the registral arrangement of the pitches. However, in Akustische Karg-Elert lists the added sixth as a dissonant linear alteration, which would be purely melodic and Pythagorean in origin (see Akustische 11.1 and 11.2).

Figure 4.8.16 lists one additional version of the contrant: the minor seventh [D F A C] or $C\#$, which Karg-Elert calls the ‘contrant-parallel twin’ (Contrant-Parallelzwilling). It fuses the complete contrant ($C$) and contrant-parallel ($C_p$) into a single four-note harmony; the line above the “p” indicates that the fifth of $C$ is retained along with the complete $C_p$. Parallel twins are minor-minor seventh chords, in which the outside interval is a 9:16 Pythagorean minor seventh. Both parallels and Leittonwechsel of all three functions can appear as twins (Figure 4.8.18):
The next example (Figure 4.8.19) outlines all of the diatonic seventh chords in C major and A minor, most of which are parallel or Leittonwechsel twins (Zwillinge). The latter are major-major seventh chords, usually built on tonic ($T_L$) or contrant ($C_L$). The figure also demonstrates how the twins can be chromatically changed into concordants (applied dominant and altered-contrant sevenths of other harmonies).
The most common twin chord is the contrant-parallel twin ($C\text{P}$), which Karg-Elert remarks “had already attained independence as an isolated harmonic form”\textsuperscript{39} in the classical period. The following example (Figure 4.8.20) demonstrates different forms of the contrant (and their different function labels) in the authentic cadence:

![Figure 4.8.20](image)

**Figure 4.8.20.** Different forms of the contrant in the authentic cadence (*Harmonologik*, 157)

While other diatonic sevenths are most commonly found in “closed key cycles” such as sequences of descending fifths, in the nineteenth century they were increasingly used as chords in their own right, not least in “Viennese waltz-music,” where added sixths and sevenths abound. The analysis in Figure 4.8.21 labels the chords as complete added sixths rather than as parallel twins, due to the registral arrangement of the chords, and the choice of bass pitches:

![Figure 4.8.21](image)

**Figure 4.8.21.** Added-sixth chords in “Viennese waltz-music” (*Harmonologik*, 161)

\textsuperscript{39} *Harmonologik*, 158.
Perhaps the most pervasive use of twins can be found in the late Romantic and Impressionist repertoires, in which seventh chords often replace simple triads as basic harmonic sonorities (Figures 4.8.22 and 4.8.23). The example by French composer Gabriel Grovlez (1879-1944) is a useful demonstration of how parallel-twins and added sixths of the same function (such as $T\bar{P}$ and $T^\chi$, or $C\bar{P}$ and $C^\chi$) share pitch content, but are labelled differently due to their inversions.

**Figure 4.8.22.** Excerpt from Debussy’s *La demoiselle élue* (1888), featuring twins and concordant ninths (*Harmonologik*, 162)

**Figure 4.8.23.** A demonstration of twins (root position) and added sixths (inversions) (*Harmonologik*, 164)
The final diatonic sonority to be discussed here is the triplet (*Drilling*), a five-note chord which Karg-Elert describes in two ways. He first defines it as “the combination of three fifth-related principals {i.e. $C + T + D$}, with their peripheral chord tones omitted” (Figure 4.8.24):

![Figure 4.8.24. The triplet (*Drilling*) derived from three different functional triads (*Ak* 11.3)]

It is also explained as “an amalgamation of a principal chord and both of its substitutes,”\(^{40}\) such as $C + Cp + Cl$, which reflects how it is labelled in root position (Figure 4.8.25):

![Figure 4.8.25. Diatonic triplet (first chord) and other added-note sonorities (*Ak* 11.3)]

The label for the contrant triplet (first chord in Figure 4.8.25) places the “$L$” above the “$p$,” as the lowest note is the root of the parallel. The first chord in bar 3 ($C^{AL}$) contains the same pitches as the initial triplet, but it is instead analyzed as a fusion of the contrant added sixth ($C^{A}$) and the contrant Leittonwechsel ($C^{AL}$), because of the change of inversion. The final chord ($T^{AL}$) is very similar, with the tonic pitch C sustained in the bass (as indicated by the horizontal line). The first

\(^{40}\) *Akustische* 11.3.
triplet in common currency was the contrant-triplet, found when $\chi^3$ is supported by subdominant harmony, as in the following cadence from a choral work by Peter Cornelius (Figure 4.8.26):

![Figure 4.8.26. Authentic cadence featuring the contrant triplet (Harmonologik, 170)](image)

Like the twins, triplets appear as basic sonorities in the music of the Impressionists and pre-Impressionists, as in the following familiar passage by Grieg (Figure 4.8.27):

![Figure 4.8.27. Expansion of C to its triplet, in Grieg’s “Wedding Day at Troldhaugen,” from Lyric Pieces, op. 65 (Harmonologik, 170)](image)

Except for the principal variants such as the contrant variant (section 4.5), which introduce chromatic pitches and syntonic comma difference, the elaborations presented in the foregoing chapter are mostly diatonic, and thus usually do not alter the basic functional identities of individual chords. Chapter 5 will revisit many of those diatonic elaborations, to study how Karg-Elert’s presentation of them exhibits a transformational perspective. The chapter will then examine harmonic relationships and transformations that are derived from pure thirds and sevenths; these introduce further chromatic pitches into the key, and thereby expand the boundaries of tonality to a much greater extent than the diatonic elaborations presented thus far.
Chapter 5

Transformation

This chapter is the longest in the dissertation, on the subject of harmonic transformation in Karg-Elert’s theories. The opening section begins with a description of the “transformational attitude,” as defined in David Lewin’s *Generalized Musical Intervals and Transformations*. Riemann’s 1880 system of contextual triadic transpositions and inversions called the *Schritte* and *Wechsel* is then studied, as an example of a transformational system, and also of what Lewin termed a *Generalized Interval System* or GIS. The *Schritte* and *Wechsel* are defined by the interval between the triadic roots, essentially under 12-tone equal temperament and enharmonic equivalence; as a result, they cannot be considered unique operations in a just intonation space, as they do not specify anything about the acoustic derivation of each triad’s root. In contrast, Karg-Elert’s system of harmonic transformations is based entirely in the retention of one or two acoustic common tones – each transformation retains at least one of the first chord’s pitches, in its original acoustic location. Therefore, each of Karg-Elert’s transformations represents a unique and specific path in pitch space. There are 23 in total: 13 transpositions and 10 inversions, some of which involve the septimal (seventh-based) plane of pitch space. Sections 5.2 to 5.4 examine each of the 23 transformations in detail, often referring to examples from the two treatises, in order to illustrate how a single transformation can appear in multiple functional contexts. Karg-Elert’s fifth-based (section 5.2) and third-based (section 5.3) transformations are compared with analogous operations in the writings of Lewin, Brian Hyer, Richard Cohn and David Kopp; the discussion highlights how Karg-Elert’s use of language and notation often directly prefigures neo-Riemannian concepts and terminology. Section 5.4 presents Karg-Elert’s seventh-based transformations, which exchange fifth- or third-derived pitches with those on the
septimal plane, or vice versa. While some are of more theoretical interest than practical use, a particular seventh-based inversion called the \textit{Septgegenklang} or “counter-seventh chord” is revealed to be of great potential for the analysis of passages involving dominant- and half-diminished seventh chords. Section 5.5 compares his seventh-based transformations with those proposed in recent writings by Adrian Childs, Richard Bass and Edward Gollin; though all three of those authors assume equal temperament and enharmonic equivalence, their transformations involving dominant and half-diminished sevenths are quite similar to those of Karg-Elert (strikingly so in the case of Gollin). The chapter concludes with a discussion of several non-common-tone harmonic relationships, including the \textit{Kollektivwechselklänge} or “collective-change chords,” equivalent to the hexatonic and octatonic poles of neo-Riemannian theory.

The primary goal of this chapter is to specify how Karg-Elert’s work relates to some basic features of transformational and neo-Riemannian theories, and thereby to highlight its interest and relevance for modern readers.

5.1. **Harmonic transformation systems in Riemann and Karg-Elert**

Though the concept of harmonic function is now familiar and widely accepted, the practice of functional harmonic analysis derived from the methods of Riemann and his successors has been mainly limited to central and eastern European countries, including Russia.\footnote{Schröder 2011 traces the dissemination of the music theory prevalent at the Leipzig Conservatory in the 1920s and early 1930s (especially functional harmonic analysis) into central, northern and eastern Europe.}

For recent North American theorists and scholars, Riemann’s primary influence does not stem from his theories of harmonic function, but rather from his earlier system of triadic relationships
known as the Schritte and Wechsel. In two seminal writings, David Lewin adapted certain of Riemann’s Schritte and Wechsel relations, recasting them within a new perspective influenced by mathematical group theory. Lewin’s general approach (which encompassed far more than connections among triads) is broadly known as transformational theory, and its central text is Lewin’s *Generalized Musical Intervals and Transformations* (henceforth GMIT). Inspired by Lewin’s transformational and group-theoretic perspective, and more specifically by analyses in GMIT of triadic progressions in passages from Wagner’s music, other American theorists including Brian Hyer and Richard Cohn proposed transformations (and networks of transformations) among triads, with the aim of understanding triadic progressions in chromatic music that diverge from the common practice. At the same time, several scholars (notably Henry Klumpenhouwer) revisited Riemann’s own work, and examined how the complete collection of Schritte and Wechsel can be formulated as a transformational system. The work of Hyer, Cohn, Klumpenhouwer and others is known collectively as neo-Riemannian, because their work has developed out of some basic operations invented by Riemann (albeit as revived and formulated by Lewin), such as the Parallel and Leittonwechsel.

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4 Cohn 1996, 9–13 provides a useful overview of the rationale for his triadic transformations.
6 Richard Cohn has described his discomfort with the term ‘neo-Riemannian’, which in his view “gives too much credit to Hugo Riemann” at the expense of Lewin’s contribution (Cohn 2012, xiii). Cohn also believes that the label ‘neo-Riemannian’ is both prescriptive and imprecise: while Dmitri Tymoczko proposed that acceptance of dualism is integral to the neo-Riemannian project (Tymoczko 2009), the label itself has been applied to general analytical approaches that retain little connection to Riemann, to a specific group of transformations, and even to musical passages involving triadic progressions that can be explained using those transformations.
Lewin’s introduction to *GMIT* provides the best (and most frequently cited) expression of a transformational perspective or “attitude.” He imagines points or objects *s* and *t* in a musical space, and a motion or distance *i* from *s* to *t* (Figure 5.1.1, which is Lewin’s Figure 0.1):

![Figure 5.1.1](image)

**Figure 5.1.1.** Lewin’s Figure 0.1, illustrating a transformational perspective (Lewin 1987, xxix)

While *s* and *t* can be various musical objects, the concept is most easily understood if they are imagined as pitches. One might say that *i* is the interval from pitch *s* to pitch *t*; this is a fairly static understanding, one that focuses primarily on the different pitch identities of *s* and *t*. Lewin then reverses the perspective, focusing more on the motion or distance *i* than on the objects *s* and *t*: “I am at *s*; what characteristic transformation do I perform in order to arrive at *t*?” This statement conveys the idea of transformation as path or journey, and reflects a kinetic view of musical relationships. Chapter 2 of *GMIT* considers twelve different manifestations of Figure 5.1: six involving pitches, and six involving durations. To encompass many types of musical objects and transformations, Lewin introduces the concept of the *Generalized Interval System*:

A Generalized Interval System (GIS) is an ordered triple \((S, \text{IVLS}, \text{int})\), where *S*, the space of the GIS, is a family of elements, IVLS, the group of intervals for the GIS, is a mathematical group, and int is a function mapping \(S \times S\) into IVLS, all subject to the two conditions (A) and (B) following:

(A): For all *r*, *s* and *t* in *S*, \(\text{int}(r,s)\text{int}(s,t) = \text{int}(r,t)\).

(B): For every *s* in *S* and every *i* in IVLS, there is a unique *t* in *S* which lies the intervals *i* from *s*, that is a unique *t* which satisfies the equation \(\text{int}(s,t) = i\).8

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7 Lewin 1987, xxxi.
8 Lewin 1987, 26.
The collection of pitch classes in 12-tone equal temperament\(^9\) clearly illustrates the GIS concept. The family of elements S is the collection itself, comprising the twelve pitch classes from C to B (or 0 to 11 in standard integer notation). The group of intervals (IVLS) are the integers from 0 to 11 (or all integers mod 12), and the function (or transformation) \(\text{int}\) is “the number of hours clockwise from \(s\) to \(t\) on a 12-hour clock.”\(^{10}\) The system constitutes a GIS, as it satisfies conditions A and B; each member of the family S lies a unique mod 12 pitch-class interval from every other member in S.

### a. Riemann’s *Schritte* and *Wechsel*

The group of intervals in a GIS, and the functions or transformations that are typical in that system, will of course depend on the nature of the family of elements and the musical objects contained within. In Riemann’s system of *Schritte* and *Wechsel* (presented in Riemann 1880),\(^{11}\) the family of objects is the collection of major and minor triads; for the most part, 12-tone equal temperament and enharmonic equivalence can be implicitly assumed, as will be discussed. *Schritte* (“steps”) transpose consonant triads by specified intervals, in the generative direction of the triads themselves: upward for major triads, and downward for minor triads. As a result, all *Schritte* maintain triad quality. The following table (Figure 5.1.2, next page) lists the twelve *Schritte*, ascending from the smallest interval of transposition to the largest. The first column provides the Uniform Triadic Transformation (UTT)\(^{12}\) that corresponds to each *Schritt*.

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9. This case is discussed in Lewin’s section 2.1.3 (Lewin 1987, 17).
11. The conceptual basis of Riemann’s *Schritt/Wechsel* system lies in Oettingen 1866, which proposed four generic transformations among triads; see Engebretsen 2008, 117–121. Riemann’s *Schritte* and *Wechsel* are discussed in detail in numerous sources, including Imig 1973, 54–64; Klumpenhouwer 2002; Kopp 2002, 66–74; Hook 2002, 78–82; and Engebretsen 2011.
UTTs can represent dualistic operations in a monistic fashion. The plus sign (+) indicates that the UTT retains mode, while a minus sign (-) indicates mode change. The numerals specify the interval by which the triad’s (monistic) root is transposed upward; the first numeral applies to major triads, and the second numerals applies to minor triads. The third column lists references to each Schritt in Riemann 1880. Finally, the last two columns apply each Schritt to C major and C minor.

<table>
<thead>
<tr>
<th>UTT (Hook 2002)</th>
<th>Schritt name</th>
<th>References (Riemann 1880)</th>
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<th>From Cm</th>
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<td>Gegenleittonschritt</td>
<td>78</td>
<td>CM → D♭M</td>
<td>Cm → Bm</td>
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<td>Cm → B♭m</td>
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<td>CM → E♭M</td>
<td>Cm → A♭m</td>
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<td>Cm → A♭m</td>
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<td>CM → F♭M</td>
<td>Cm → Gm</td>
</tr>
<tr>
<td>&lt;+6,6&gt;</td>
<td>Tritonussschritt</td>
<td>80</td>
<td>CM → F♮M</td>
<td>Cm → Gm</td>
</tr>
<tr>
<td>ditto</td>
<td>Gegentritonussschritt</td>
<td>81</td>
<td>CM → G♭M</td>
<td>Cm → F♭m</td>
</tr>
<tr>
<td>&lt;+7,5&gt;</td>
<td>Quintschritt</td>
<td>7, 70</td>
<td>CM → G♭M</td>
<td>Cm → Fm</td>
</tr>
<tr>
<td>&lt;+8,4&gt;</td>
<td>Gegenterzschritt</td>
<td>72</td>
<td>CM → A♭M</td>
<td>Cm → Bm</td>
</tr>
<tr>
<td>&lt;+9,3&gt;</td>
<td>Kleinterschritt(^\text{13}) (Sextschnitt)</td>
<td>27, 74</td>
<td>CM → A♭M</td>
<td>Cm → E♭m</td>
</tr>
<tr>
<td>&lt;+10,2&gt;</td>
<td>Gegenganztonschritt</td>
<td>76</td>
<td>CM → B♭M</td>
<td>Cm → Dm</td>
</tr>
<tr>
<td>&lt;+11,1&gt;</td>
<td>Leitonschritt</td>
<td>27, 79</td>
<td>CM → B♭M</td>
<td>Cm → D♭m</td>
</tr>
</tbody>
</table>

Figure 5.1.2. The complete system of Schritte or contextual transpositions

\(^{13}\) The layout and content of this table resemble those in Klumpenhouwer 2000, 168 and Hook 2002, 80–81. Figures 5.1.2 and 5.1.3 replace Klumpenhouwer’s term Sextschnitt with Riemann’s original term Kleinterschritt, which is “the relationship between a tonic and the chord of the same quality at its major sixth” (Riemann 1880, 74).
The *Schritte* are fully dualistic, as the interval of transposition is upward in major and downward in minor, matching the generation of the triads. Because Riemann provides no distinct *Schritte* for augmented-interval steps such as \([\text{CM} \to \text{C\#M}]\), \([\text{CM} \to \text{D\#M}]\), \([\text{CM} \to \text{G\#M}]\) or \([\text{CM} \to \text{A\#M}]\), or diminished-interval steps such as \([\text{CM} \to \text{F\flat M}]\) or \([\text{CM} \to \text{C\flat M}]\), one may conclude that Riemann’s system implies almost complete enharmonic equivalence. The only exception is between *Tritonusschritt* \([\text{CM} \to \text{F\#M}]\) and *Gegentritonusschritt* \([\text{CM} \to \text{G\flat M}]\), marking the threshold of enharmonic identity. Though Riemann’s conception of pitch space ostensibly remained rooted in just intonation throughout his career (see section 7.2 below), his presentation of the *Schritte* makes no mention of just intonation or syntonic comma differences; the motion \([\text{CM} \to \text{DM}]\) is always described as *Ganztonschritt*, regardless of whether the D major triad is based on \(D_{(2,0)}\) or \(D_{(2,1)}\) – that is, two fifths above C, or one syntonic third above \(B\flat_{(2,0)}\). Therefore, for practical purposes the *Schritte* operate in 12-tone equal temperament, and every triad is connected to every other like-quality triad by a unique interval.

Riemann’s *Wechsel* (“changes”) are tabulated in Figure 5.1.3 (next page). They are contextual inversions rather than transpositions: they switch the triad quality from major to minor, or vice versa. The basic *Wechsel* is the *Seitenwechsel* (“lateral change”), which inverts a triad around its (dual) prime: C major \([\text{C E G}]\) becomes ‘C dual minor’ or F minor \([\text{C A\flat F}]\), and ‘G dual minor’ or C minor \([\text{C E\flat G}]\) becomes G major \([\text{G B D}]\). All of the other *Wechsel* can be understood as the corresponding transposition or *Schritt*, followed by *Seitenwechsel*. For example, Ganztonwechsel is Ganztonschritt plus Seitenwechsel; from C major, that is \([\text{CM} \to \text{DM} \to \text{Gm}]\). Unlike the *Schritte* which are bijections (operating in one direction only), the *Wechsel* are involutions: that is, each *Wechsel* is its own inverse.
<table>
<thead>
<tr>
<th>UTT (Hook 2002)</th>
<th>Wechsel name</th>
<th>References (Riemann 1880)</th>
<th>From CM</th>
<th>From Cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;,-0,0&gt;</td>
<td>Quintwechsel</td>
<td>15, 71</td>
<td>CM → Cm</td>
<td>Cm → CM</td>
</tr>
<tr>
<td>&lt;,-1,11&gt;</td>
<td>Doppelterwechsel</td>
<td>81</td>
<td>CM → Cm</td>
<td>Cm → CbM</td>
</tr>
<tr>
<td>ditto</td>
<td>Gegenterwechsel</td>
<td>73</td>
<td>CM → Dm</td>
<td>Cm → BM</td>
</tr>
<tr>
<td>&lt;,-2,10&gt;</td>
<td>Kleinterwechsel (Sextwechsel)</td>
<td>21, 74</td>
<td>CM → Dm</td>
<td>Cm → BbM</td>
</tr>
<tr>
<td>&lt;,-3,9&gt;</td>
<td>Gegenganztonwechsel</td>
<td>76</td>
<td>CM → Ebm</td>
<td>Cm → AM</td>
</tr>
<tr>
<td>&lt;,-4,8&gt;</td>
<td>Leittonwechsel</td>
<td>22, 79</td>
<td>CM → Em</td>
<td>Cm → AbM</td>
</tr>
<tr>
<td>&lt;,-5,7&gt;</td>
<td>Seitenwechsel</td>
<td>10, 70</td>
<td>CM → Fm</td>
<td>Cm → GM</td>
</tr>
<tr>
<td>&lt;,-6,6&gt;</td>
<td>Gegenleittonwechsel</td>
<td>79</td>
<td>CM → F#m</td>
<td>Cm → GbM</td>
</tr>
<tr>
<td>&lt;,-7,5&gt;</td>
<td>Ganztonwechsel</td>
<td>23, 76</td>
<td>CM → Gm</td>
<td>Cm → FM</td>
</tr>
<tr>
<td>&lt;,-8,4&gt;</td>
<td>Gegenkleinterwechsel (Sextwechsel)</td>
<td>75</td>
<td>CM → Abm</td>
<td>Cm → EM</td>
</tr>
<tr>
<td>&lt;,-9,3&gt;</td>
<td>Terzwechsel</td>
<td>19, 72</td>
<td>CM → Am</td>
<td>Cm → EbM</td>
</tr>
<tr>
<td>&lt;,-10,2&gt;</td>
<td>Gegeng quintwechsel</td>
<td>16, 71</td>
<td>CM → BbM</td>
<td>Cm → DM</td>
</tr>
<tr>
<td>&lt;,-11,1&gt;</td>
<td>Tritonuswechsel</td>
<td>24, 81</td>
<td>CM → Bm</td>
<td>Cm → DbM</td>
</tr>
</tbody>
</table>

**Figure 5.1.3.** The complete system of *Wechsel* or contextual inversions

Like the *Schritte*, the *Wechsel* imply enharmonic equivalence; Riemann does not even specify a *Wechsel* counterpart to the *Gegentritonusschritt*. However, there is again one exception: the *Doppelterwechsel* (“double-third change”), which is enharmonically equivalent with the *Gegenterwechsel*. Riemann remarks that the former transformation “contains a common tone (the third), which makes it easier to conceive”\(^{14}\) than the latter, which has an enharmonic ligature such as [E / Fb] or [Eb / D#]. There is again no reference to intonation or comma differences, and thus 12-TET can be assumed in the *Wechsel* system. In totality, Riemann’s system of *Schritte* and *Wechsel* almost qualify as a GIS, in which the family of objects is the collection of 24 major and minor triads under 12-TET and enharmonic equivalence, the group of intervals comprises

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\(^{14}\) Riemann 1880, 81.
the integers from 0 to 11, and the transformations are *Schritt* and *Wechsel*. All that is needed to complete the mathematical group is an identity *Schritt* (the mode-retaining counterpart to the *Seitenwechsel*), which is included in Henry Klumpenhouwer’s reworking of the system.\(^{15}\)

Perhaps the crucial characteristic of the *Schritte* and *Wechsel* is that they are contextual: they are defined by the interval between the roots of the two triads, and do not depend at all on the presence of a governing key or tonal center. Therefore, they essentially have no functional implications. To be sure, Riemann’s text often makes clear that he continued to understand triadic relationships within a tonal context; for example, he defines *Kleinterzschritt* as “the relationship between a tonic (*Tonika*) and the chord of the same quality at its major sixth.”\(^{16}\) In addition, he described *Quintschritt* as a *Dominantschritt* (“dominant-step”),\(^{17}\) an expression of harmonic function that is meaningless without the presence of a corresponding tonic. But in practice, any *Schritt* or *Wechsel* can be applied to any major or minor triad, regardless of tonality. A key motivation of Lewin’s work involving triads (plus that of his neo-Riemannian successors) is the contention that the music of Wagner and other late-nineteenth century composers feature triadic successions that present new patterns, diverging from the tonal and functional norms of common practice. The “free-floating” *Schritte* and *Wechsel* can reflect local triadic connections in a manner similar to how set-theoretical operations relate pitch groups in non-tonal music.

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\(^{15}\) Klumpenhouwer 1994 and 2002.

\(^{16}\) Riemann 1880, 74.

\(^{17}\) Ibid., 7.
b. Karg-Elert’s system of common-tone transformations

Many of Riemann’s *Schritte* and *Wechsel* do not represent a traceable path or journey through triadic space. This is especially true when the source and goal triads do not share any common tones; for example, while *Ganztonschritt* [CM → DM] or [Cm → B♭m] is clearly a whole tone transposition, the term does not reveal anything about the pitch-space locations of the triads, or the path traversed during the transformation. While the question of path tracing may not be very significant in an equal-tempered pitch space, it is of great consequence in just intonation, where the specific pitch-space locations of each triad define each transformation. If just-intonation pitch differences are to be carefully observed, a simple transposition such as *Ganztonschritt* is not sufficiently specific: the pitch-space journey C(0) major → D(2,0) major differs greatly from C(0) major → D(-2,1) major.

When the source and goal harmonies share one or more common tones, the pitch-space path is clear, as the common tones will normally be heard as acoustically identical. In Figure 5.1.4 (next page, previously cited as Figure 4.1.4), Karg-Elert suggests that when we hear a harmonic interval (i.e. one belonging to a consonant triad or concordant seventh chord), our harmonic imagination fills in the pitches needed to complete the chords. As indicated by the arrows in the figure, the mental completion process reflects our innate sense of harmonic polarity, as in each case, the sounded intervals can represent either an upward-striving major chord, or a downward-striving minor chord. When we hear an interval such as the perfect fifth C - G, we mentally perform the inversion indicated by the chord-tone numbers. If C is the prime (1), then G will be the fifth (5), and the chord will be completed by the third (3) E. However, if G is the prime (1), then C will be the fifth (5), and the chord will be completed by the third (3) E♭. Both triadic completions are equally implied by the perfect fifth C - G.
Though the intention of Figure 5.1.4 is to illustrate the perception of major-minor polarity, it also demonstrates transformations between triads and concordant sevenths that share two common tones. The first three lines involve major and minor triads that respectively share a perfect fifth, major third or minor third. The modern reader will recognize them immediately as the familiar P (parallel), R (relative) and L (Leittonwechsel) transformations, now established as basic operations in neo-Riemannian theory. Each is an inversion or Wechsel, and the numbers next to the staves in Figure 5.1.4 indicate the inversive exchange of common tones, a topic to be examined in detail presently. The right column of staves indicates the common tones using horizontal lines; the figure thus graphically emphasizes common-tone retention, in a manner that prefigures many recent descriptions of P, R and L. One difference between Karg-Elert’s

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18 Cohn 1997 presents a formal and generalized model of the PRL family of operations.
19 A representative example is the introductory paragraph of Cohn 1997.
example and the PRL family of operations is that the changing pitches (in the right side of the figure) exhibit no notable preference for parsimonious voice leading\(^{20}\); the added pitches that complete the triads (right side of the figure) are placed in octaves that emphasize major-minor polarity, rather than smooth linear connection.

Karg-Elert states that “all chords that share at least one common tone are related. Prime, third, fifth and seventh relationships arise from those common tones. These relationships can retain or change chord quality.”\(^{21}\) To be clear, Karg-Elert is referring to the retention of common tones that have identical derivation in just-intonation pitch space; pitches that are metharmonics or enharmonics of each other (see section 3.6) cannot serve as true common tones. Transpositions retain chord quality, while inversions change quality. In Figure 5.1.4 and throughout his treatises, Karg-Elert uses Arabic numerals to indicate *common-tone transformations*, which are the reinterpretations of tones shared between two triads or two concordant sevenths. The members of a major triad or dominant seventh are labelled 1 (prime), 3, 5 and 7, while the members of a minor triad or half-diminished seventh are labelled as †, €, ⊃ and ⊲, conceived dualistically from the highest pitch downward. In Figure 5.1.4, the \( \mathbf{P} \) transformation \([\text{CM} \leftrightarrow \text{Cm}]\) is labelled as 1 - ⊃ and 5 - †, as the prime and fifth exchange chord-tone roles. \( \mathbf{R} \) \([\text{CM} \rightarrow \text{Am}]\) exchanges the prime and third (1 - € and 3 - †), while \( \mathbf{L} \) \([\text{CM} \rightarrow \text{Em}]\) exchanges the third and fifth (3 - ⊃ and 5 - €). Figure 5.1.4 concludes with two

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\(^{20}\) Cohn 1997 (especially 1–2 and 12–17) describes how smooth or parsimonious voice leading has been central to much of the theory and practice of Western music, and how it specifically defines the PRL family of operations.

\(^{21}\) Harmonologik, 52. David Kopp’s transformational system (fully presented in Chapter 7 of Kopp 2002) is also based in common-tone retention between triads. His system of third relations and transformations displays numerous points of similarity with that of Karg-Elert (see section 5.3 below). However, Kopp’s system is to a large extent non-dualistic, and based in 12-TET and enharmonic equivalence, and thus diverges from Karg-Elert’s in some fundamental ways.
transformations between concordant seventh chords that exchange two chord tones, one of which is the seventh:

\[
F\#7 \leftrightarrow F^\#_7 \quad 1 - \nabla \text{ and } 7 - \uparrow \\
Bb7 \leftrightarrow D^\#7 \quad 3 - \nabla \text{ and } 7 - \varepsilon \quad (5 - \varepsilon \text{ is metharmonic – see below})
\]

As in the PRL group, these two seventh-based transformations are Wechsel or inversions, in which two chord tones exchange roles, and the qualities of the chords change.

Karg-Elert’s notation for common-tone transformations is often somewhat redundant, with the numbers simply flipped in minor; for example, the transposition that maps the (dual) prime onto the third is notated as 1 - 3 in major (as in [CM \(\rightarrow\) AøM]), and as 1 - 3 in minor [Cm \(\rightarrow\) Em]. To generalize and simplify Karg-Elert’s notation for common-tone transformations (both transpositions and inversions), this dissertation will use lowercase Roman numerals to label the members of triads and concordant sevenths, conceived dualistically: i (prime), iii (5:4 major third above/below the prime), v (3:2 perfect fifth above/below the prime), vii (7:4 concordant seventh above/below the prime). Transpositions that retain at least one common tone can be expressed as \(T_{x \rightarrow y}\), which is the transposition that maps dual chord-tone x onto dual chord-tone y; for example, \(T_{i \rightarrow \text{iii}}\) indicates both Karg-Elert’s 1 – 3 (major) and \(\uparrow - \varepsilon\) (minor).

The sixteen possible transpositions (in each mode) can be reduced to thirteen, as the identity operation (here called \(T_I\)) encompasses four of them. Figure 5.1.5 (on the next page) tabulates the transpositions that retain at least one common tone, and applies each transposition to C major and C7 (both of which are generated from the prime C(0)), and also to C minor and Aø7 (both generated from the prime G_{(1,0)}). When the transposition does not involve the concordant

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22 This notation for dualistic chord tones is based on that proposed in Gollin 1998, modified here to match Karg-Elert’s chord-tone numerals; see section 5.4 below.
seventh (vii), the product is shown as a major or minor triad; however, the concordant seventh can be added to any triad, without altering the basic transposition. Each transposition is unique, including the pairs $T_{iii \rightarrow v}/T_{v \rightarrow vii}$ and $T_{v \rightarrow iii}/T_{vii \rightarrow v}$, whose products differ metharmonically, as shown by the acoustic data for the new primes in columns 4 and 7 of the figure. Each transposition retains only one pitch from the original chord, except of course for the identity $T_1$, which retains all pitches.

<table>
<thead>
<tr>
<th>Transposition $(T_{x \rightarrow y})$</th>
<th>Major chord-tone transformations</th>
<th>$T_{x \rightarrow y}$ applied to CM or C7 [prime = C(0)]</th>
<th>New prime after $T_{x \rightarrow y}$</th>
<th>Minor chord-tone transformations</th>
<th>$T_{x \rightarrow y}$ applied to Cm or Aø7 [prime = G(1,0)]</th>
<th>New prime after $T_{x \rightarrow y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$ (identity)</td>
<td>$1 \rightarrow 1, 3 \rightarrow 3, 5 \rightarrow 5, 7 \rightarrow 7$</td>
<td>CM (or C7)</td>
<td>C(0)</td>
<td>$\uparrow \rightarrow \uparrow, \varepsilon \rightarrow \varepsilon, \bar{\varepsilon} \rightarrow \bar{\varepsilon}, \ \uparrow \rightarrow \uparrow$</td>
<td>Cm (or Aø7)</td>
<td>G(1,0)</td>
</tr>
<tr>
<td>$T_{iii \rightarrow v}$</td>
<td>$1 \rightarrow 3$</td>
<td>AbM</td>
<td>$\uparrow \rightarrow \varepsilon$</td>
<td>Em</td>
<td>B(1,1)</td>
<td></td>
</tr>
<tr>
<td>$T_{v \rightarrow vii}$</td>
<td>$1 \rightarrow 7$</td>
<td>FM</td>
<td>$\uparrow \rightarrow \varepsilon$</td>
<td>Gm</td>
<td>D(2,0)</td>
<td></td>
</tr>
<tr>
<td>$T_{iii \rightarrow vi}$</td>
<td>$3 \rightarrow 1$</td>
<td>EM</td>
<td>$\varepsilon \rightarrow \uparrow$</td>
<td>Abm</td>
<td>B(1,-1)</td>
<td></td>
</tr>
<tr>
<td>$T_{iii \rightarrow vii}$</td>
<td>$3 \rightarrow 7$</td>
<td>F#7</td>
<td>$\varepsilon \rightarrow \varepsilon$</td>
<td>Ebø7</td>
<td>Ebø7</td>
<td></td>
</tr>
<tr>
<td>$T_{v \rightarrow i}$</td>
<td>$5 \rightarrow 1$</td>
<td>GM</td>
<td>$\bar{\varepsilon} \rightarrow \uparrow$</td>
<td>Fm</td>
<td>C(0)</td>
<td></td>
</tr>
<tr>
<td>$T_{v \rightarrow iii}$</td>
<td>$5 \rightarrow 3$</td>
<td>EbM</td>
<td>$\bar{\varepsilon} \rightarrow \varepsilon$</td>
<td>Am</td>
<td>E(0,1)</td>
<td></td>
</tr>
<tr>
<td>$T_{v \rightarrow vii}$</td>
<td>$5 \rightarrow 7$</td>
<td>A7</td>
<td>$\bar{\varepsilon} \rightarrow \uparrow$</td>
<td>Cø7</td>
<td>B(0,0,1)</td>
<td></td>
</tr>
<tr>
<td>$T_{vii \rightarrow vi}$</td>
<td>$7 \rightarrow 1$</td>
<td>Bø7</td>
<td>$\downarrow \rightarrow \uparrow$</td>
<td>Bø7</td>
<td>A(1,0,-1)</td>
<td></td>
</tr>
<tr>
<td>$T_{vii \rightarrow i}$</td>
<td>$7 \rightarrow 3$</td>
<td>Gø7</td>
<td>$\downarrow \rightarrow \varepsilon$</td>
<td>Dø7</td>
<td>Cø7</td>
<td></td>
</tr>
<tr>
<td>$T_{vii \rightarrow v}$</td>
<td>$7 \rightarrow 5$</td>
<td>Ebø7</td>
<td>$\downarrow \rightarrow \bar{\varepsilon}$</td>
<td>F#ø7</td>
<td>E(2,0,-1)</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 5.1.5.** Transpositions that retain at least one common tone

Figure 5.1.4 illustrated how inversions that retain at least one common tone are expressed by Karg-Elert as chord-tone exchanges between major and minor. Once again, his notation is very specific but somewhat redundant; for example, neo-Riemannian $P$ (parallel) is indicated as both $1 - \bar{\varepsilon}$ and $5 - \uparrow$. This notation is here simplified as $I_x$, which indicates the inversion that
maps dual chord-tone x onto dual chord-tone y, and vice versa.\textsuperscript{23} The chord tones x and y are again notated as i (prime), iii (third), v (fifth) or vii (concordant seventh); the two chord tones will of course belong to chords of opposite quality. For example, P [CM ↔ Cm] can be expressed as \( I_v^i \), which maps 1 onto \( \Xi \) and 5 onto \( \Gamma \). When two chords are inverted around a single common tone, x and y will be the same; for example, Riemann’s Seitenwechsel [CM ↔ Fm] can be expressed as \( 1 – \Gamma \), or as \( I_v^i \). The sixteen possible inversions are reduced to ten unique ones, due to the exchanges of chord tones. Figure 5.1.6 (on the next page) tabulates the ten inversions, and applies them to CM/C7 and Cm/Aø7, as in the table of transpositions. Once again, concordant sevenths can be freely added to major or minor triads, without altering the inversions. Note that \( I_{\text{vii}}^{\text{ii}} (3 = \Xi, 7 = \Xi) \) and \( I_v^{\text{iv}} (5 = \Xi) \) produce primes that are metharmonics of each other: the primes have the same name, but they differ by two fifths, one third and one concordant seventh. While one might assume that \( I_{\text{vii}}^{\text{ii}} (3 = \Xi, 7 = \Xi) \) also implies the common-tone exchange \( 5 = \Xi \) (see the last line of Figure 5.1.4 above), the chordal fifths under \( I_{\text{vii}}^{\text{ii}} \) are in fact metharmonics of each other, with different acoustic derivations in just intonation. As a result, \( I_{\text{vii}}^{\text{ii}} \) does not include or imply \( I_v^{\text{iv}} \), and so Karg-Elert’s notation of \( 5 – \Xi \) in the last line of Figure 5.1.4 (which illustrates \( I_{\text{vii}}^{\text{ii}} \)) is not wholly accurate. In totality, Figure 5.1.4 demonstrates the following five inversions: \( I_v^i \) (neo-Riemannian P), \( I_{\text{vii}}^i \) (R), \( I_v^{\text{iv}} \) (L), \( I_{\text{vii}}^i \) and \( I_{\text{v}}^{\text{iv}} \).

\textsuperscript{23} This notation for inversional chord-tone exchange is inspired by that first proposed in Lewin 1977, used frequently in Lewin 1987 (see for example p. 51), and especially by the version of it featured in Gollin 1998.
Inversion (I^γ_x)

<table>
<thead>
<tr>
<th>Inversion (I^γ_x)</th>
<th>Chord-tone transformations (major and minor)</th>
<th>I^γ_x applied to CM or C7 [prime = C(0)]</th>
<th>New prime after I^γ_x</th>
<th>I^γ_x applied to Cm or Aø7 [prime = Gø(0,0)]</th>
<th>New prime after I^γ_x</th>
</tr>
</thead>
<tbody>
<tr>
<td>I_1^γ</td>
<td>1 = ↑</td>
<td>Fm</td>
<td>C(0)</td>
<td>Gm</td>
<td>G(1,0)</td>
</tr>
<tr>
<td>I_2^γ</td>
<td>1 = E, 3 = ↑</td>
<td>Am</td>
<td>E(0,1)</td>
<td>EbM</td>
<td>Eb(1,1)</td>
</tr>
<tr>
<td>I_3^γ</td>
<td>1 = ♭, 5 = ↑</td>
<td>Cm</td>
<td>G(1,0)</td>
<td>CM</td>
<td>C(0)</td>
</tr>
<tr>
<td>I_4^γ</td>
<td>1 = 7, 7 = ↑</td>
<td>Cø7</td>
<td>B♭p(0,0,1)</td>
<td>A7</td>
<td>A(1,0,1)</td>
</tr>
<tr>
<td>I_5^γ</td>
<td>3 = E</td>
<td>C#m</td>
<td>G♯(0,2)</td>
<td>C♯M</td>
<td>C♯(1,2)</td>
</tr>
<tr>
<td>I_6^γ</td>
<td>3 = ♭, 5 = E</td>
<td>Em</td>
<td>B♭(1,1)</td>
<td>AbM</td>
<td>Ab(0,1)</td>
</tr>
<tr>
<td>I_7^γ</td>
<td>3 = 7, 7 = E</td>
<td>Eø7</td>
<td>D(0,1,1)</td>
<td>F7</td>
<td>F(1,1,1)</td>
</tr>
<tr>
<td>I_5^γ</td>
<td>5 = ♭</td>
<td>Gm</td>
<td>D(2,0)</td>
<td>FM</td>
<td>F(1,0)</td>
</tr>
<tr>
<td>I_6^γ</td>
<td>5 = 7, 7 = ♭</td>
<td>Gø7</td>
<td>F(1,0,1)</td>
<td>D7</td>
<td>D(0,0,1)</td>
</tr>
<tr>
<td>I_7^γ</td>
<td>7 = ♭</td>
<td>B♭ø7</td>
<td>A♭(0,0,2)</td>
<td>B7</td>
<td>B(1,0,2)</td>
</tr>
</tbody>
</table>

Figure 5.1.6. Inversions that exchange one or two common tones

In sum, Karg-Elert’s treatises propose 23 unique common-tone transformations involving triads and concordant sevenths; he also specifies several other relations that do not involve common tones, some of which will be examined in section 5.5. In addition to the numerical notation for common-tone transformations, Karg-Elert also provides descriptive names for many of them. Like Riemann’s Schritte and Wechsel, many of these names do not explicitly reflect tonality or functional meaning; for example, T_{vii→i} (7 → 1 or 7 → ↑) is called Konkordant (transposed by a concordant seventh), and I_{iiii}^γ (3 = E) is called Terzgleich (“same third”). All of Karg-Elert’s names for transformations will be outlined in full later in this chapter.

Unlike the common-tone transpositions and inversions which are completely contextual, the functions T, C and D are entirely defined by the presence of a tonic or center. As discussed in Chapter 4, the three functions can be understood as categories, exemplified by their basic triads; all other chords are classified as modifications of one of the basic triads, and then

---

24 The first example in Kopp 2002 (Table 1.1, page 2) illustrates the 13 common-tone triadic relationships listed in Figures 5.1.5 and 5.1.6. Kopp’s system does not deal with chordal sevenths, and so his table does not include Karg-Elert’s ten transformations that involve the seventh.
accordingly assigned to its functional category. Harmonic function theories suggest an alternative view of transformation: that of transformation as mutation, rather than as path or journey. With regard to Lewin’s Figure 5.1.1, rather than thinking about “what transformation takes me from s to t,” we might ask “how much of s remains in t.” Consider the category of tonic (T), which in C major is exemplified by the basic triad [C E G]. Following Karg-Elert’s definition of chordal relationship through common-tone retention, one might propose that any chord that retains at least one pitch from [C E G] is a recognizable mutation of that triad, and will therefore continue to express at least some degree of tonic function in the key of C major. One might then develop labels that explain all of the common-tone mutations as modifications of T. Some such labels are very familiar, such as Tp for [A C E] in C major, or Karg-Elert’s r for [C Eb G]; both of those triads share two common tones with the tonic triad [C E G], and so assigning tonic function seems logical and intuitive. However, it is impossible to assert a direct or one-to-one correlation between specific transformations and function labels, because many chords contain pitches that belong to two of the basic functional triads. While the three basic triads are not likely to change their functions in any case, the problem of functional assignment becomes more acute with other chords, such as the diatonic substitutes. As discussed in section 4.9, Tp contains the same pitches as Cl, and the same is true of Tcl and Dp. In all such cases, the choice of function depends on contextual factors, such as the identity of surrounding harmonies, and rhythmic placement within the measure. The following three sections examine a selection of Karg-Elert’s analyses, chosen to highlight how a single common-tone transformation can suggest multiple functional interpretations.
5.2. Prime- and fifth-based transformations

Part One of Harmonologik includes a chart that describes and demonstrates all 23 of the chordal transformations that retain at least one common tone (listed above in Figures 5.1.5 and 5.1.6). The chart is entitled Allgemeine Übersicht der Klangverwandtschaften auf Grund der Tongemeinschaft, or “general overview of harmonic relationships based on common tones.”

The chart has four sections, dealing in turn with prime, fifth, third and seventh-based relations or transformations. Each section is subdivided into two parts, outlining relations between chords of the “same quality” (gleichgeschlechtlich) and of “different quality” (gegengeschlechtlich) – in other words, transpositions followed by inversions. The transformations are indicated using Karg-Elert’s numerical notation for chord tones, previously shown in Figures 5.1.4 to 5.1.6. The chart also provides Karg-Elert’s names for most of the transformations. Finally, it demonstrates each transformation in staff notation in the keys of C major and A minor, with accompanying function labels in those keys. The chart thereby links the 23 common-tone transformations with their most typical functional interpretations. It is among the most important and useful sections in all of Karg-Elert’s theoretical work, as it nicely summarizes much of the content discussed and demonstrated at length in the rest of Harmonologik, and also in Chapters 4, 6 and 10 of Akustische. The chart does not explicitly specify the pitch-space locations or trajectories of the chords in each transformation; however, because all of them retain at least one pitch from the tonic triad (C major or A minor), pitch-space locations can be easily determined, and then indicated using acoustic data, or plotted on a Tonnetz. This section (as well as sections 5.3 and 5.4) will reproduce each section of Karg-Elert’s chart, and then discuss each transformation in detail, in the order they are listed. To a large extent, the prime- and fifth-based transformations to

\[25\] Harmonologik, 52-54.
be examined presently are linked directly with the three basic functions; therefore, the analytical findings in this section are on the whole similar to those discussed in Chapter 4. However, when fifth-based transpositions are reiterated to form chains of dominants and contrants, they begin to lose their connection with the nominal tonal center, and sometimes have no sense of key at all. They thereby become emancipated from the functions, exhibiting a more purely transformational agency, as will be illustrated in the final repertoire examples in this section.

a. Prime-based transformations

Karg-Elert’s chart first outlines *Primverwandtschaft*: relationships or transformations involving the prime (I in major, and ♭ in minor). There are only two possibilities (Figure 5.2.1):

![Diagram of Prime-based transformations]

A. Prime-based transformations

a) **Same quality** (transpositions)

[same chords] T₁ (identity) [same chords]

b) **Different quality** (inversions)

[CM → Fm] Counter-chord, or altered contrant [Am → EM]  

![Figure 5.2.1. Prime-based transformations (Harmonologik, 52)]
The first is not named or demonstrated by Karg-Elert, as it does not involve a change of chord at all. It is simply an identity operation (*gleiche Klänge* or “same chords”), in which the prime stays prime, without changing chord quality. In Figure 5.1.4, identity was labelled as T₁, which encompasses the chord-tone transformations \(1 = 1, \ 3 = 3\) and \(5 = 5\) (or their minor-key versions). It is the transpositional counterpart to the *Gegenklang* or \(1 = \uparrow\) inversion that follows. The inclusion of the identity relationship is purely motivated by a desire for symmetry and completeness; it suggests that Karg-Elert conceived his array of common-tone transformations as a complete and balanced system. David Lewin labelled the T₁ operation as IDENT,²⁶ while Brian Hyer and David Kopp use the symbol I.²⁷

The other prime-based transformation is the inversion \(I_1^i\) or \(1 = \uparrow\). It is first called *Gegenklang* or “counter-chord,” a name borrowed from Riemann 1893²⁸ that highlights both the change of quality from major to minor (or vice versa), and the inversion or “flip” around the prime. Karg-Elert might also have called it *Primgleicher* or “same prime” – a term that he did not actually use, but which corresponds to his names for the other inversions around a single chord-tone (*Quintgleicher, Terzgleicher* and *Septgleicher*). It is equivalent to Riemann’s *Seitenwechsel* (“lateral change”), and also to Oettingen’s *Wechsel*.²⁹ Richard Cohn uses N or *Nebenverwandt*³⁰ (meaning “neighbor-related”), a term originally introduced by Carl Weitzmann to denote the \(I_1^i\) relationship between triads of opposite quality.³¹ Like the PRL family, N is an involution; it is also dualistic, operating in opposite directions in major and minor: [CM \(\leftrightarrow\) ___]

²⁶ Lewin 1987, 176.
²⁸ Riemann 1893, 44.
²⁹ Engebretsen 2008, 121.
³⁰ Cohn 1998, 290.
³¹ Weitzmann 1853, 16.
Fm] and [Am ↔ EM]. Unlike the PRL operations, N retains only one common tone, and thus it does not exhibit parsimonious voice leading in the same manner as those operations. N expresses in a single label what is a ternary operation in the PRL family: either PLR or RLP.

David Kopp also uses a unary label: F, which indicates that the root transposes down by a fifth, and the triad changes quality. A crucial difference is that Kopp’s F is non-dualistic – it always indicates to transpose the (monistic) root down a fifth, in both major and minor. Therefore, it is not an involution: [CM → Fm] is F, but [Fm → CM] is its inverse F⁻¹.32 Kopp’s F/F⁻¹ is thus not conceived as an inversion around the dualistic prime like Iₕ, but rather as a fifth transposition followed by a change of mode; in that respect, it closely resembles earlier fifth-based transformations in the work of Lewin and Hyer, which will be examined below.

In contrast to Gegenklang, Seitenwechsel and N which do not reflect tonality or function at all, Karg-Elert’s most common name for Iₕ reflects a change of function: temperierte Contradominant, or contrant-variant. This is because the most common instance of the transformation is the progression T → c, discussed in detail in section 4.5. Iₕ can also appear as D → t, an authentic cadence motion that replaces the normal tonic with its variant. In most contexts, Iₕ will result in a change of function. However, because Iₕ-related triads share a common prime, one can conceive of situations in which Iₕ does not express a change of function. An example is shown in Figure 5.2.2 (next page), in which the F minor triad (ostensibly the contrant-variant or c) clearly acts as a neighbor chord to the C major tonic; the sustained common prime in the bass is the main element that suggests continued tonic function.

32 Kopp 2002, 170.
Figure 5.2.2. $I^i$ as a linear embellishment, in a function-retaining context

b. Fifth-based transpositions

The second section of Karg-Elert’s chart deals with Quintverwandtschaft (fifth relations), or transformations involving the dual chord tones $v$ and $i$ (Figure 5.2.3):

Figure 5.2.3. (English version on the next page)
B. Fifth-based transformations

a) Same quality (transpositions)

\[
\begin{array}{ccc}
F \leftrightarrow C & G & \text{Principals, or dominants} \\
\text{CM} & \text{FM} & \text{GM} \\
\text{C} & \text{T} & \text{D} \\
\begin{array}{c}
5 \leftrightarrow 1 \\
5 \rightarrow 1 \\
\end{array}
\end{array}
\]

b) Different quality (inversions)

\[
\begin{array}{ccc}
\text{Primary:} & \text{Secondary:} \\
\text{CM} \leftrightarrow \text{Cm} & \text{CM} \leftrightarrow \text{Gm} & \text{CM} \leftrightarrow \text{Gm} \\
\text{Am} & \text{AM} & \text{DM} \\
\text{T} & \text{t} & \text{T} & \text{d} & \text{I}_b^T & \text{I}_b^d \\
\begin{array}{c}
1 \leftrightarrow 2 \\
5 \leftrightarrow 4 \\
\end{array}
\end{array}
\]

**Figure 5.2.3.** Fifth-based transformations (*Harmonologik*, 52)

The fifth-based transpositions $T_{i\rightarrow v}$ and $T_{v\rightarrow i}$ produce the two dominants that surround the tonic, respectively $C$ (contradominant) and $D$ (dominant). Together with the tonic ($T$), they comprise the principals ($Prinzipale$), or the three basic functional triads. Karg-Elert’s original chart illustrates how $T_{v\rightarrow i}$ manifests both as $T \rightarrow D$ and as $C \rightarrow T$, in both modes. He does not provide “function-free” names for the two fifth-transpositions, as they are so closely linked with the functional categories themselves. The English version in Figure 5.2.3 slightly alters the perspective of the original chart, illustrating the two motions of a fifth away from the tonic: $T \rightarrow D$ is $T_{v\rightarrow i}$, and $T \rightarrow C$ is $T_{i\rightarrow v}$. It is difficult to conceive a situation in which the products of these transpositions (acting as true harmonies rather than as linear simultaneities) would retain the
original chord’s function; therefore, $T_i \rightarrow v$ and $T_v \rightarrow i$ are classified here as function-changing transformations.

Different labels for the two fifth-transpositions reflect the idea of dominant function to varying degrees. Riemann’s names focus entirely on the fifth interval: $T_v \rightarrow i$ is *Quintschritt* (“fifth step”), while $T_i \rightarrow v$ is its inverse *Gegenquintschritt* (“counter fifth step”). They can be applied freely to any triad, and have no overt tonal or functional implications beyond those suggested by the root motions themselves. David Lewin defined his **DOM** (“dominant”) transformation not in the standard sense as ‘chord Q is the dominant of chord P’, but in reverse as ‘P becomes the dominant of chord Q’; this reversal allows “the dominants to point to their tonics” in a dynamic way, matching the functional and voice-leading energies inherent in the dominant.\(^{33}\) In C major, Lewin’s **DOM** is thus [CM $\rightarrow$ FM] rather than [CM $\rightarrow$ GM]; the latter transformation is labelled either as the inverse **DOM’** (in the sense of the motion I $\rightarrow$ V away from the tonic), or as **SUBD** (“subdominant,” in the sense of a IV $\rightarrow$ I plagal return to the tonic). Lewin’s transformations are not equivalent to $T_i \rightarrow v$ or $T_v \rightarrow i$, as they are non-dualistic, operating in the same directions in major and natural minor: **DOM** is both [CM $\rightarrow$ FM] and [Em $\rightarrow$ Am]. The typical minor-key authentic cadence [Em $\rightarrow$ Am] is (**DOM**)(**PAR**), or “becomes dominant + change of mode.”\(^{34}\) Brian Hyer’s **D** and **D**\(^{-1}\) are also non-dualistic, and are essentially the same as **DOM** and **DOM’** respectively. Kopp’s system borrows **D** and **D**\(^{-1}\) from Hyer, with exactly the same meaning. Kopp’s **F** and **F**\(^{-1}\) (described above) are the same as **D** and **D**\(^{-1}\), but also change triad quality; therefore, **F** is equivalent to Lewin’s (**DOM**)(**PAR**), or Hyer’s **DP** (both of which mean transpose down a fifth + change of mode). Finally, Richard Cohn’s system eschews fifth

\(^{33}\) Lewin 1987, 176-177.  
\(^{34}\) Ibid., 229.
transpositions altogether, because they can be expressed as the compound operations $\text{RL}$ (for $T_{i\rightarrow i}$) and $\text{LR}$ (for $T_{v\rightarrow i}$). The omission of fifth-transpositions lends Cohn’s system an elegant consistency and simplicity, and also helps to remove it further from diatonic tonality and function; indeed, a central theme in Cohn’s work is that chromatic music often develops its own harmonic logic, and features triadic relations that depart from those of the common practice. In contrast, Karg-Elert would argue strongly that direct fifth-based transformations must be retained, because in his model of harmonic space, fifths outline the principal source positions ($Ursprungslagen$) of all primary keys (see section 3.7), and are thus privileged as the principal basis of tonality.

c. Fifth-based inversions

The first fifth-based inversion on the chart is $I_p^i$, or $5 = \uparrow / 1 = \check{2}$, which changes the mode of a triad to its parallel major or minor $[\text{CM} \leftrightarrow \text{Cm}]$. The transformation is therefore called “parallel” in English, and is abbreviated as $\text{PAR}$ by Lewin, and as $\text{P}$ by Hyer and other neo-Riemannian scholars. Somewhat confusingly, the term $\text{Parallel}$ has a different meaning in German: it refers to the relationship or transformation called “relative” major/minor in English, and is thus neo-Riemannian $\text{R}$, or $[\text{CM} \leftrightarrow \text{Am}]$. The German meaning is reflected in Riemann’s functional notation for the diatonic substitutes (see section 4.9): in the key of C major, the A minor triad is often understood as $T_p$, or tonic parallel. Karg-Elert follows Riemann’s practice, reserving the term $\text{Parallel}$ (and the function-label subscript $p$) for the switch to relative major or minor. Both Riemann and Karg-Elert use the term $\text{Variant}$ to

35 Chapter 1 of Cohn 2012 discusses various rationales for “alternative views of triadic chromaticism” (p. 11), engaging both with the diatonic tradition and with more recent ideas about pitch space and harmonic syntax.
describe the change to the opposite mode.\(^{36}\) In Karg-Elert’s functional system, the variant mode switch is shown as a change from uppercase (which indicates the normal triad quality for the mode) to lowercase (for the variant triad quality). As always, normal-facing letters are used in major, while backward-facing letters are used in minor. Therefore, the minor tonic in major keys is \(t\), and the major tonic in minor keys is \(\tilde{t}\).

The \(I^{\prime}_p\) or variant relationship is classified in Figure 5.2.3 as a primary (primäre) fifth-based inversion, as it exchanges the chordal fifth and prime. It is significant that the Karg-Elert indicates the variant switch by a simple change of letter case, rather than by appending a postscript such as “\(v\)” (as in Riemann 1918, to be described below). His notational choice suggests that the variant transformation preserves a higher degree of the original triad’s functional identity than the diatonic substitutes, which are shown as postscripts (\(p\) and \(L\)). Indeed, the variant transformation retains almost the entire original triad: both prime and fifth, plus a chromatic-semitone alteration of the third. As a result, \(I^{\prime}_p\) is here classified as a function-retaining transformation. It must be emphasized that for Karg-Elert, the term Variant reflects not only the major/minor alteration of the third, but also the acquisition of syntonic-comma difference. The thirds of all variant triads differ from their Pythagorean or Ursprungslande (source position) counterparts by a syntonic comma (see section 4.5 above). Therefore, while the variant transformation suggests a high degree of functional retention, it also implies that a degree of artificiality is introduced, altering the normal fifth-derived “natural” harmonic order. To reflect this artificial quality, Karg-Elert often notates comma-variant pitches using closed note heads, to contrast with the open note heads for source-position pitches (see Figures 4.5.3 and 4.5.8, and the Ursprungslagen analysis in Figure 4.8.14).

\(^{36}\) Riemann 1893, 195.
In his 1880 treatise, Riemann described the $I_v$ or variant relationship as *Quintwechsel* ("fifth change"), which is *Quintschritt* plus *Seitenwechsel*: [CM $\rightarrow$ GM $\rightarrow$ Cm], or [Am $\rightarrow$ Em $\rightarrow$ AM]. He later incorporated the variant switch into his functional system, in several different ways. In *Harmony Simplified*, $^o$ and $^+$ are appended to function labels to indicate variants in major and minor keys; the minor subdominant in a major key is $^oS$, while the major dominant in a minor key is $D^+$ (see Figure 4.2.1). Those superscripts are normally added only to unaltered functions. Another method used mostly with function labels that are already altered (such as the diatonic substitutes) is to indicate the lowering or raising of the chordal third; raising a minor triad’s third is notated as $III^<$, while lowering a major triad’s third is notated as $^3>$. In the key of C major, the A major triad can be labelled as $Tp^{III<}$ (tonic parallel with raised third), though much more likely in Riemann’s practice is $(D)[Sp]$, or dominant of the subdominant parallel (i.e. V of II). Alexander Rehding has noted that for some time after *Harmony Simplified*, Riemann generally hesitated to alter a function label more than once, especially with the parallel and Leittonwechsel substitutes.

In his last publications, Riemann relaxed his earlier views on multiple alterations, and also introduced some new concepts and labels. One notable late addition is $v$ for the variant mode switch, first proposed in the preface to the sixth edition of the *Handbuch der Harmonielehre*. Discussing the celebrated minor Neapolitan chord ($\bar{\text{ii}}$) in measure 80 of Schubert’s Impromptu D. 899 no. 3, Riemann labels it as $^oS<v$, which is the variant of the Leittonwechsel of the minor subdominant: in the key of G major, that is CM ($S$) $\rightarrow$ Cm ($^oS$) $\rightarrow$ A$bM$ ($^oS<$) $\rightarrow$ A$bM$ ($^oS<v$). **Figure 5.2.4** reproduces the passage, with Riemannian functional

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37 Rehding 2003, 58.
38 Riemann 1918, xvii.
analysis, in the key of G major. Both of the applied dominants (mm. 77 and 79) lead to minor variants of the expected major-triad resolutions.

![Musical notation]

**Figure 5.2.4.** Schubert: Impromptu, D. 899 no. 3, mm. 75-80 (in a G major edition), with function labels after Riemann 1918

In her detailed study of Riemann’s functional system, Renate Imig states that “a closer look at the idea of Variant reveals that its theoretical conception is incompatible with Riemann’s dualistic foundations…The main tone [Hauptton] of a minor triad is the fifth, the highest note of the triad…on the other hand, a minor variant is derived from the lowest note of its corresponding major triad.” Imig thus suggests that minor variants are not true harmonic minor triads, but simply mode-switched versions of major triads. Riemann himself described how variants “do not

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39 Schubert’s Impromptu is originally in G-flat major, but Riemann’s discussion is based on a G major edition of the piece, likely that by Franz Liszt and Sigmund Lebert (published by J. G. Cotta of Stuttgart in 1870). Figure 5.10 quotes from that edition. Rings 2011 is centered on Riemann’s analysis of this passage, contrasted with his own Lewin- and Cohn-inspired readings.

40 Imig 1973, 51.
create true modulations, but rather a sudden lightening or darkening of an existing tonic." The acoustic distinction in just intonation is revealed in the following data, which compares the tonic variant in C major with its fifth-derived or source position equivalent:

\[
\begin{align*}
T \text{ in C major} &= \quad C(0) \quad E_{(0,1)} \quad G_{(1,0)} \\
\tilde{t} \text{ in C major} &= \quad C(0) \quad E_b_{(1,-1)} \quad G_{(1,0)} \quad \text{(variant)} \\
\mathcal{L} \text{ in C minor} &= \quad C_{(-4,1)} \quad E_b_{(-3,0)} \quad G_{(-3,1)} \quad \text{(source position)}
\end{align*}
\]

Though variants share the same lowest and highest notes with their source triads, Karg-Elert’s chord-tone transformations demonstrate that a monistic understanding of variants is not required: the two common tones simply exchange roles (1 → 2 and 5 → 1), and the resulting triadic inversion creates the chromatic-semitone alteration of the third (3 → 3). Neo-Riemannian labels for the \( I^l_p \) or variant transformation are described in essentially the same manner. Both Lewin’s \textbf{PAR} and neo-Riemannian \textbf{P} (featured in the systems of Hyer, Cohn and Kopp) are inherently dualistic, as they are based on the common-tone inversionsal exchange of the root and fifth. \textbf{P} maximizes common-tone retention and voice-leading parsimony, and is thus included in Cohn’s reductive PRL family of operations.

The last of the fifth-based inversions illustrated in Figure 5.2.3 is \( I^b_p \) or \( 5 \leftrightarrow 2 \), which inverts a triad around its dual fifth: [CM → Gm] or [Am → DM]. Karg-Elert classifies it as a secondary (\textit{secundär}) fifth-based inversion, signifying that the chord tone in question transforms into the dual fifth – not the prime, as in the primary inversion \( I^l_p \) (see above). His name for this transformation is \textit{Quintgleicher} (“same fifth”); the \textit{-gleich} inversions always flip around a single chord tone (see \( I^l_p \) or \textit{Primgleicher}, described above). Because \( I^b_p \) appears most frequently as the motion \( T \rightarrow d \), it can also be called \textit{Dominant-variant} – a name that reveals how \( I^b_p \) is equivalent

\[\text{Riemann 1916, 1166. Quoted in Imig 1973, 51.}\]
to $T_{v \rightarrow i}$ (Dominant) + $l^v_p$ (Variant). Karg-Elert also describes the products of $l^v_p$ as Kirchendominanten (“church dominants”), because the dominant-variant ($d$) in major is equivalent with the dominant in the Mixolydian mode, while $d$ in minor is equivalent with the dominant in Dorian (see Figures 4.5.8 to 4.5.11). Though $l^v_p$ is a fifth-based inversion, Karg-Elert introduces the dominant-variant in Akustische 7.8, within the context of third-based transformations; this is because all variants can be derived as diatonic substitutes of the counter-mediants (major-third lower mediants), and thus often appear in conjunction with mediator transformations of the basic triads (see section 4.5 above). For example, the dominant-variant ($d$) is equivalent to the dominant countermediant-Leittonwechsel ($D_M^d$); in the key of C major, that is $GM \rightarrow EbM (D_M) \rightarrow Gm (D_M^d)$. Because $l^v_p$ will almost always appear either as $T \rightarrow d$ or $C \rightarrow t$, and not as a mutation of a single function, it is classified as a function-changing transformation.

The corresponding operation in Riemann 1880 is Ganztonwechsel, which is Ganztonschnitt (transposition up or down a whole tone), followed by Seitenwechsel (inversion around the prime). Riemann’s name is of course accurate, in terms of dual root motion: the progression of $[GM \rightarrow Dm]$ is $g^+ \rightarrow ^a a$ in Riemann’s Klangschlüssel notation. He states that Ganztonwechsel is often found in the cadential pre-dominant to dominant motion, “in place of the parallel-fifths-prone whole tone step (Ganztonschnitt)”; that is, II – V in place of IV – V. Riemann’s example of Ganztonwechsel in the authentic cadence is provided in Figure 5.2.5 with dualistic chord symbols, plus functional analysis in the manner of Riemann 1893, and of Karg-Elert. Riemann uses a tie to denote the common tone D, which is the dual fifth around which the Ganztonwechsel-related G major and D minor triads are inverted.
The functional analysis in Figure 5.2.5 is not included in Riemann 1880, but the meaning of each chord in the progression is clear. But there is a problem with Riemann’s example: unless 12-tone equal temperament is assumed, the D minor and G major chords do not actually share a common tone. The D’s tied together in Figure 5.2.5 are in fact metharmonics, differing by a syntonic comma, as shown in Figure 5.2.6. The inversion $I_v$ requires that the dual fifth (v) be an identical pitch; $I_v$ can only be equated with Ganztonwechsel if equal temperament is specified.

In just intonation, $Cp$ and $D$ do not share an acoustically exact dual fifth; thus, the progression $Cp \rightarrow D$ is not a manifestation of the $I_v$ or Quintgleicher transformation, while $T \rightarrow d$ and $C \rightarrow t$ are, as they invert around the dual fifth.
Lewin and Hyer express $I^\nu$ as a two-step transformation, similar to the idea of Dominant-Variant discussed above. Both Lewin’s **DOM’ PAR** and **SUBD PAR** involve a transposition up a fifth, followed by a change of mode; the former suggests motion from the tonic to the dominant variant ([CM $\rightarrow$ Gm] in C major), while the latter implies motion from the subdominant to the tonic variant ([FM $\rightarrow$ Cm] in C major). Hyer represents both as $D^{-1} P$, labelling that conveys less functional meaning than in Lewin. Kopp combines $D^{-1}$ and $P$ into the unary transformation $F^{-1}$, which involves both a transposition up a fifth and a change of mode. It must be stressed that all of the transformations just described combine a non-dualistic fifth motion upward (in both major and minor) with a dualistic mode-change operation: Hyer’s $D^{-1} P$ is both [CM $\rightarrow$ Gm] and [Cm $\rightarrow$ GM], but they invert around different dual chord tones – the former is $I_\%$ and the latter is $I^\nu_\%$. Karg-Elert’s unary *Quintgleicher* ($I^\nu_\%$) requires three steps in Cohn’s PRL system: either **LRP** or **PRL**, which arrive at exactly the same pitch-space destination, but travel via different paths. Cohn’s operations are of course dualistic, as they are based entirely on inversion operations:

\[
\textbf{LRP:} \quad \text{CM} \rightarrow \text{Em} \rightarrow \text{GM} \rightarrow \text{Gm} \quad \text{Am} \rightarrow \text{FM} \rightarrow \text{Dm} \rightarrow \text{DM}
\]
\[
\textbf{PRL:} \quad \text{CM} \rightarrow \text{Cm} \rightarrow \text{E}_b\text{M} \rightarrow \text{Gm} \quad \text{Am} \rightarrow \text{AM} \rightarrow \text{F}_#\text{m} \rightarrow \text{DM}
\]

The three-step process is dictated by Cohn’s requirement for voice-leading parsimony. Though the PRL-family can accurately represent all triadic relationships, the case of $I^\nu_\%$ is one where one might fault Cohn’s system, as it requires more steps to explain a diatonic fifth-based harmonic relationship like [CM $\rightarrow$ Gm] than a chromatic third-based one like [CM $\rightarrow$ EM] or **LP**.
d. **Emancipating the fifth-transformations from the three functions**

Sections 4.4 to 4.6 provided numerous examples of Karg-Elert’s prime- and fifth-based transformations, involving the three basic functional triads \((T, C, D)\) and their variants \((t, c, d)\). Those examples demonstrate that transformations involving dual chord tones \(i\) and \(v\) (and in particular the transpositions \(T_{v\rightarrow i}\) and \(T_{i\rightarrow v}\)) are very closely intertwined with the three functional categories, to the point where one might question whether the transformations have any real identity that is separate from the functions. One way in which fifth-based transformations can be emancipated from the three functions, and from a centralized tonality, is through reiteration. The repetition of inversions like \(I^I_t\) (PLR or RLP), \(I^v_t\) (P) or \(I^v_v\) (LRP or PRL) is superfluous, since each inversion is its own inverse, and will thus replicate the original triad when repeated. On the other hand, repeated fifth-transpositions continually produce new triads, leading further away from the source in a unidirectional manner along one of the fifth-axes of the pitch space. The first instances of reiterated fifth-transpositions in Karg-Elert’s are the ultradominant \((\text{DD})\) and ultracontrant \((\text{CC})\), which are essentially borrowed from Riemann 1893 (see section 4.7). Figure 4.7.6 (reproduced here as Figure 5.2.7) demonstrated the “isolated, bounding” or stand-alone versions of the ultraforms:

![Figure 5.2.7](image)

**Figure 5.2.7.** “Real” (i.e. stand-alone) ultradominant and ultracontrant \((Ak 6.9)\)
Akustische 6.9 describes how the ultraforms “latently imply through transformation” the regular dominant and contrant, with which they share one common tone. Their functional notation indicates that the regular triads are replaced by their doubled versions: $C \rightarrow EC$ and $D \rightarrow DD$. Therefore, they can be readily understood as doubled fifth-transpositions: $DD$ is $D^2$ or $T_{(v \rightarrow i)}^2$, while $EC$ is $C^2$ or $T_{(i \rightarrow v)}^2$. The duplication of $C$ and $D$ prefigures repeated operations such as Lewin’s (DOM) (DOM), or repeated compound operations in the PRL-family such as $<LR>^2$ for $DD$, or $<RL>^2$ for $CC$. However, one might argue that the ultraforms are still tied to the functional categories, as they share a common tone with the regular dominant or contrant. A further step toward the “unshackling” of the fifth-transformations is illustrated by the mixed ultraforms, previously shown in Figure 4.61 (in part reproduced here as Figure 5.2.8):

![Figure 5.2.8](image)

**Figure 5.2.8.** Ultracontrant-variants ($Ec$), in examples C and D (Harmonologik, 84)

The relevant chords in Figure 5.2.8 are the ultracontrant-variants ($Ec$) in examples C and D, which are both in the key of C major. If $CC$ is $BbM$, then $Ec$ is its variant $Bbm$; both include the concordant seventh below (G), while example D also adds the ninth below ($Eb$). The function label $Ec$ reflects a three-step transformation from the tonic: $T_{i \rightarrow v} + T_{i \rightarrow v} + I_{v}$, or FM ($C$) $\rightarrow$ $BbM$ ($EC$) $\rightarrow$ $Bbm$ ($Ec$) in C major. The equivalent three-part process would be (DOM) (DOM)
(PAR) in Lewin, $D^2P$ in Hyer, $F^2P$ in Kopp, or $(RL)^2P$ in Cohn. As discussed previously, the fifth-transpositions in Lewin, Hyer and Kopp are not dualistic, and so $DOM/D/F$ would be replaced by their respective inverse operations in minor keys.

Ultradominant-variants ($Dd$) are also possible (see the first example in Akustische 7.8), and once again represent a three-step transformation: $T_{v\rightarrow i} + T_{v\rightarrow i} + 1^l_v$, or $GM (D) \rightarrow DM (DD) \rightarrow Dm (Dd)$ in C major. $Dd$ differs metharmonically from $Cp$, as it is the variant of $DD$, not the parallel of $C$ – see the Tonnetz in Figure 5.2.9, which plots both triads in the key of C major.

The equivalent operation to $T \rightarrow Dd$ in the PRL system is $(LR)^2P$, which contrasts with the simpler $RLR$ for $T \rightarrow Cp$. Both the ultracontrant-variant ($Cc$) and ultradominant-variant ($Dd$) still share one common tone with the regular contrant and dominant, and so they will likely retain those respective functions.

![Figure 5.2.9](image)

**Figure 5.2.9.** Metharmonic difference between $Cp$ and $Dd$, in C major

In several passages, Karg-Elert presents reiterated fifth-transpositions that largely eliminate the centralizing power of the tonic, and limit functional meaning to immediate chord-to-chord connections. **Figure 5.2.10** demonstrates two paths from C major to A major – through
upward transposition by three fifths, or through common-tone connections with both C major and its contrant (F major):

Figure 5.2.10. Two paths from C major to A major (*Harmonologik*, 144)

The latter path (in the second half of the figure) retains both the tonic’s third (E) and the contrant’s third (A), as shown by the ties which connect the common tones; the A major triad is labelled as the upper mediant of the contrant (CM), though it could also be understood as a transformation of the tonic, as will be discussed in section 5.3. Because of the common-tone connections, the path is described as *Nahverwandtschaft*, or a “near relationship.” On the other hand, the pattern of rising fifths in the first path is called *Fernverwandtschaft* or a “distant relationship,” because the A major triad shares no acoustic common tones with either the tonic C major, or with the regular dominant (G major). In both parts of Figure 5.2.10, all triads are written in octaves that reflect their distance in pitch space from the C major tonic; the fifth-related triads are all in their source positions or *Ursprungsagen* (see section 3.7), while the mediant of F major (CM) is a variant of A minor (Cp), containing the comma-different C4\((1,2)\). The analysis of the first path outlines a triple dominant transformation, or T\((v\rightarrow l)^3\). The reiteration of D closely resembles chains of compound operators in the PRL-system. In Cohn’s system, the first path is equivalent to (LR)3, while the second path can be simplified as RP, bypassing the motion to the contrant F major.
While the A major triad in the first path of Figure 5.2.10 is clearly the dominant of the preceding D major triad (and is labelled as such), it no longer projects any particular function in the original key of C major. As a result, it may be said that the thrice-repeated label D no longer represents a true functional category such as dominant, but simply indicates a fifth-transposition from the preceding chord. The following analysis from Harmonologik (Figure 5.2.11) illustrates the point even more clearly. The passage is the first eight measures of the Rhapsody op. 79 no. 2 by Brahms, which Karg-Elert describes as “the so-called ‘G minor’ Rhapsody, which only commits itself to that key in later development.”42 He begins not from the nominal key of G minor, but from the opening E-flat major triad, which is understood as a tonic (T):

![Musical notation of Brahms' Rhapsody op. 79 no. 2](image)

**Figure 5.2.11.** Analysis of Brahms, Rhapsody op. 79 no. 2 (Harmonologik, 145)

The annotations inside the large bracket at the bottom mark the overall progression in the first phrase (mm. 1-4), from E♭ major to G major (Es dur → G dur), and its sequential repetition a major third higher in the second phrase (mm. 5-8) from G major to B major (G dur → H dur). A

42 Harmonologik, 145.
traditional *Stufen* interpretation might label those harmonic stations as $1 \rightarrow \text{III}_b$ in $E_b$ major, followed by the same in $G$ major. Alternatively, the final $B$ major might be understood enharmonically as $b\text{VI}$ of $E_b$ major, assuming 12-TET. Karg-Elert forcefully rejects such an interpretation, stating that there are no mediants (common-tone major-third relationships) at all, and no comma differences (*keine Medianten! keine Kommadifferenzen!*). Instead, all of the chords in the passage are related by fifth to the opening $E_b$ major, creating a “pure principal progression” (*reine Prinzipalentwicklung*) – a chain of fifths, matching the triadic source positions (*Ursprungslagen*). The first phrase jumps from the tonic $E_b$ major to its ultradominant $F$ major, followed by two more ascending fifths to $G$ major: $T \rightarrow \text{DD} \rightarrow (D) \rightarrow (D)$. The second phrase does the same, starting from $G$ major. From bar 3 onward, the labels $(D)$ and $(\text{DD})$ do not express any particular tonic; they simply indicate the fifth-transpositions that lead from chord to chord. Overall, the passage modulates from $E_b$ major to the $B$ major that is eight fifths higher, as indicated by the numbers under the function labels. The following Tonnetz (*Figure 5.2.12*) illustrates the chain of fifths in the passage, using the Arabic numerals from the figure to denote the distance of eight fifths from the opening chord:

![Tonnetz](image)

*Figure 5.2.12. Tonnetz for the Brahms, Rhapsody in G minor example (Figure 5.2.11)*
The Brahms passage in Figure 5.2.11 is an example of a *comma-free modulation*, in which the source and goal triads or keys are related by fifths, without syntonic-comma displacement. Comma-free modulations will be examined in detail in Chapter 6. The point to be made here is that repeated fifth-transpositions can enable rapid comma-free changes of key. Just as dominants or contrants can be reiterated in extended chains, the same is true of the ultraforms. *Figure 5.2.13* illustrates chains of ultracontrants and ultradominants, modulating down twelve fifths from F♯ major to G♭ major (part A), or their relative minors (part B). The triangles (Δ for major, ∨ for minor) indicate the quality of the opening triad, and the number “12” under the horizontal line indicates the number of fifths downward. The whole-tone scales in the bass are notated without enharmonic adjustments, as their opening and closing pitches differ by a Pythagorean comma. However, because the source and goal triads are located twelve fifths apart on the chain of fifths (with no syntonic-comma differences), the modulations are classified as comma-free. The labels ČČ and ĎĎ are wholly contextual in meaning, and do not define the opening triads as functional tonics.

*Figure 5.2.13.* Chains of ultracontrants and ultradominants (*Harmonologik*, 142)
The following transitional passage from Act III of Puccini’s *Madama Butterfly* (Figure 5.2.14) features a similar chain of major triads descending by whole tone, beginning with the E major triad in the second measure. E major is labelled as tonic (T) simply because it is the starting point of the chain, not because it (or for that matter any other triad) acts as a functional center.

Descending whole steps:  

\[
T \quad CC \quad CC \quad CC \quad CC
\]

**Figure 5.2.14.** Extended ultracontrant chain in Puccini, *Madama Butterfly* (Act III)

In Figure 5.2.14, Puccini introduces an enharmonic notational shift in bar 8 (E major instead of F-flat major), as is typical in passages involving the whole-tone collection. While extended ultraform chains like those in Figure 5.2.13 and 5.2.14 are not very common, the next example (Figure 5.2.15) demonstrates how a shorter chain can rapidly move to a distant key, which is then confirmed through a diatonic progression and cadence. The passage modulates down six fifths from C major to G-flat major through a chain of ultracontrants; the latter triad becomes a new tonic in bar 2, and is then extended diatonically to the final cadence.
Figure 5.2.15. Comma-free modulation from C major to Gb major (Harmonologik, 143)

The progression in Figure 5.2.15 illustrates once again how labels such as $C, D, CC$ and $DD$ can sometimes denote essentially non-tonal and non-functional fifth transpositions (mm. 1-2), even while they continue to reflect their traditional functional categories in other contexts (mm. 2-4).

5.3. Third-based transformations

Relationships between triads whose roots lie major or minor thirds apart are of great importance in tonal music, second only to fifth relations. This is due in part to the presence of the major and minor thirds within the consonant triad, which in turn promote common-tone connections between third-related triads. Though certain transpositions and inversions involving thirds are built into the network of diatonic triads, others introduce chromatic pitches that expand the pitch content of the major-minor system. This is particularly true in nineteenth-century music from Beethoven and Schubert onward, which increasingly featured chromatic third relationships between triads and key areas. Analytical systems that are fundamentally rooted in the diatonic collection (such as Stufenlehre methods) had to be enriched in various ways, in order to encompass and represent chromatic third relationships. Likewise, Riemann’s functional
system is based in the contention that even highly chromatic music is still governed by tonality, and that all chromatic chords continue to express one of the fifth-related functions. On the other hand, recent scholars such as Richard Cohn have argued that nineteenth-century chromatic music developed its own types of harmonic logic and syntax that are distinct from those of the common practice, and that are defined in no small part by the prevalence of chromatic third relations. Accordingly, neo-Riemannian methods often represent fifth and third relationships with equal ease and directness, with little concern for centralized tonality or functional meaning. Karg-Elert’s theories inhabit an intermediate position between those two approaches. As this section will describe, Karg-Elert developed a comprehensive system for labelling and understanding third relationships between triads, prefiguring in many ways those of recent transformational and neo-Riemannian theorists. On the other hand, he continued to integrate third relationships within the fifth-based functional system. Some of his statements described chromatic thirds as an attractive but dangerous force, with the power to corrupt the fifth-based “natural” framework of tonality. As will be discussed below, his views on third relationships are influenced to a considerable extent by his model of just-intonation pitch space.

a. David Kopp’s classification of third relations

Relationships between triads whose roots are major or minor thirds apart can be categorized in several ways. The two most basic criteria are the direction of the third motion (above or below the source triad), and the quality or size of the motion (major or minor third). David Kopp’s 2002 treatise on Chromatic Transformations in Nineteenth-Century Music (which

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43 See Example 1 in Cohn 1996, and the discussion in pp. 9–13. Cohn was by no means the first to describe a “second practice” of chromatic tonality; two notable earlier explorations of the subject are Proctor 1978, and Kinderman and Krebs 1996.
is still the most thorough and detailed study of third relations) outlines several useful ways to classify and compare different third relations. He first explains how there are sixteen possible mediant (major or minor third) relationships between triads: “eight from a major tonic, eight from a minor tonic.” His Figure 1.2 provides the following list of the eight third-related triads away from either C major or C minor, organized by direction and quality:

- Major 3rd above = EM, Em
- Minor 3rd above = EbM, Ebm
- Major 3rd below = AbM, Abm
- Minor 3rd below = AM, Am

Kopp calls the first two pairs upper mediants, and the second two pairs lower mediants; he uses the term *mediant* to describe both major and minor third relationships, in both major and minor keys. He then divides the mediants into three subcategories: relative, chromatic and disjunct. The relative mediants belong to the diatonic key, and are specifically “fundamental chords of the relative mode”; in C major, the relative mediants are Am and Em (i.e. I and natural V of the relative key A minor), while in C minor the relative mediants are EbM and AbM (I and IV of the relative key). Kopp’s discussion of the relative mediants lists two additional criteria for classifying third relationships: the degree of common-tone retention with the tonic (in this case two common tones), and whether the quality of the source triad changes or not (in this case, the quality changes from major to minor, or vice versa). The relative mediants connect to the source triad by “one diatonic change of either a semitone or a whole tone”, in conjunction with the retention of two common tones, the stepwise diatonic motion defines voice-leading parsimony,

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45 Ibid.
46 Ibid.
47 Ibid.
as described by Cohn. The two relative mediants are the diatonic substitutes (parallel and Leittonwechsel), which result from neo-Riemannian R and L. However, Kopp’s model is more closely tied to the Stufenlehre tradition. He introduces the terms upper relative mediant (URM) and lower relative mediant (LRM) to denote the diatonic mediants. The labels URM and LRM reflect their positions above or below the tonic according to the diatonic scale, and are therefore analogous to monistic Stufen labels rather than dualistic R and L transformations. URM is diatonic III and LRM is diatonic VI, in both major and minor:

<table>
<thead>
<tr>
<th>Key / source triad</th>
<th>URM</th>
<th>LRM</th>
</tr>
</thead>
<tbody>
<tr>
<td>C major</td>
<td>Em</td>
<td>Am</td>
</tr>
<tr>
<td>A minor</td>
<td>CM</td>
<td>FM</td>
</tr>
</tbody>
</table>

Kopp’s second category is that of the chromatic mediants, which introduce one or two pitches from outside of the source triad’s diatonic collection, while retaining one common tone from the source triad. In each case, the chromatic mediants preserve the mode of the source triad, to which they are related by transposition rather than inversion. Kopp’s labels reflect their placement above and below the source triad (upper and lower), and whether the chromatic pitches are raised or lowered in relation to the source triad’s diatonic collection: upper flat mediant (UFM), upper sharp mediant (USM), lower flat mediant (LFM) and lower sharp mediant (LSM). In major keys, the two flat mediants contain lowered roots, while in minor keys the two sharp mediants contain raised roots. The chromatic mediants in C major and A minor are the following:

<table>
<thead>
<tr>
<th>Key / source triad</th>
<th>UFM</th>
<th>USM</th>
<th>LFM</th>
<th>LSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>C major</td>
<td>EbM</td>
<td>EM</td>
<td>AbM</td>
<td>AM</td>
</tr>
<tr>
<td>A minor</td>
<td>Cm</td>
<td>C#m</td>
<td>Fm</td>
<td>F#m</td>
</tr>
</tbody>
</table>
Kopp’s final category is that of the **disjunct mediants**, which share no common tones with the original triad. The upper and lower disjunct mediants (UDM and LDM) are of the opposite quality to the source triad, and are specifically the modal variants of the chromatic mediants with altered roots:

<table>
<thead>
<tr>
<th><strong>Key / source triad</strong></th>
<th><strong>UDM</strong></th>
<th><strong>LDM</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>C major</td>
<td>E♭m</td>
<td>A♭m</td>
</tr>
<tr>
<td>A minor</td>
<td>C♯M</td>
<td>F♯M</td>
</tr>
</tbody>
</table>

b. **The mediants: major-third transpositions**

Karg-Elert’s chart of third-based common-tone transformations includes Kopp’s diatonic and chromatic mediants; the disjunct mediants are not on the chart, as they are not considered by Karg-Elert to be directly related to the source triad, as no common tones are retained. The chart first divides the third-based transformations into those that retain or change triad quality: *gleichgeschlechtlich* (same quality) or *gegengeschlechtlich* (opposite quality), or transpositions and inversions respectively. Within those two categories, the transformations are ordered according to the reinterpretation of the common tone – whether the dual third (iii) transforms with the prime (i), the fifth (v), or inverts around itself (iii).
Here is Karg-Elert’s chart of the third-based transpositions (Figure 5.3.1):

C. Third-based transformations

a) Same quality (transpositions)

Primary:

\[
\begin{array}{c|c|c}
T_M & T & T^M \\
3 & 1 & 3 \\
\end{array}
\]

Mediants (major-third relations)

\[ \begin{align*}
T_{ii \to iii} & \text{ (counter-median)} \\
T_{iii \to i} & \text{ (median)}
\end{align*} \]

Secondary:

\[
\begin{array}{c|c|c}
T_P & T & T^P \\
5 & 3 & 5 \\
\end{array}
\]

Neighbor-mediants (minor-third relations)

\[ \begin{align*}
T_{iii \to v} & \text{ (counter-neighbour-median)} \\
T_{v \to iii} & \text{ (neighbour-median)}
\end{align*} \]

\[ T_P / T^P = \text{parallel-variants} \]

\[ T^P / T_P = \text{variant-parallels} \]

Figure 5.3.1. Major- and minor-third transpositions (Harmonologik, 54)
The relations outlined in Figure 5.3.1 are the same as Kopp’s chromatic mediants. The category of Medianten (“mediants”) comprises the major-third transpositions $T_{\text{iii} \to \text{i}}$ or Mediant, which transposes according to the generative direction of the source triad (up a major third for major triads as in $[CM \to EM]$, and down a major third for minor triads as in $[Am \to Fm]$), and $T_{i \to \text{iii}}$ or Gegenmediant (“counter-median”), which transposes by a major third opposite to the generative direction of the triad (as in $[CM \to A\bar{m}]$ or $[AM \to C\#m]$). They are described as the primary (primär) third-transpositions, as the dual third (iii) maps onto the new prime (i), or vice versa. Like its English equivalent, the term Mediant traditionally denotes the third Stufe (III), which is normally of the opposite quality to the tonic in both the major and minor diatonic modes. The diatonic mediant is at the major third above the tonic in major, but a minor third above the tonic in minor. Conversely, in a dualistic system the chordal third (iii) is always a major third (or harmonic 4:5 third in just intonation) away from the prime (i); accordingly, the mediant can be conceived in a dualistic system as consistently a major third from the tonic.

Each of the three primary triads ($T$, $C$, $D$) can be transformed into its mediant and counter-median, as shown inside the vertical brackets in Figure 5.3.2. Line A is in C major, while line B is in A minor:

![Figure 5.3.2. Mediant and counter-median transformations (Harmonologik, 200)](image-url)
The figure indicates syntonic-comma variant pitches using closed noteheads; it highlights the fact that mediants always include one comma-variant pitch, while counter-medians always include two comma-variant pitches. The medians replace diatonic source position harmonies: for example, both the tonic mediant ($T^M$) and the dominant counter-median ($D_M$) are chromatic substitutes for the tonic Leittonwechsel ($T^L$)/dominant parallel ($D_p$). The presence of comma-variant pitches infuses the medians with what Karg-Elert described as an artificial and “unnatural” quality, one which notably influenced his views on the “mediant” style, as will be examined later in this section. In spatial (registral) terms, the placement of the $M$ postscript next to the function label seems non-dualistic at first glance: the upper major third is always indicated as a superscript, and the lower major third as a subscript, in both major and minor modes. However, polarity is clearly reflected in the location of the $M$ in relation to the inverted function labels in minor. Mediant ($T_{iii\rightarrow i}$) is notated as $T^M$ in major, and as its visual inverse in minor: $L_M$. Gegenmediant ($T_{i\rightarrow iii}$) is $T_M$ in major, and $L^M$ in minor.

In addition to the basic functional triads, the ultraforms can also be transformed into their medians and counter-mediants, as shown in the peripheral measures in Figure 5.3.2 (outside the central bracket). The ultracontrant mediant ($E_C^M$) is metharmonically similar with the ultradominant ($D_D$); in C major, both are D major triads, but in different pitch-space locations. Likewise, the ultradominant counter-median ($D_{DM}$) is metharmonically similar with the ultraconrtant ($E_C$): in C major, both are B♭ major triads, though the latter is a source-position harmony, while the former contains two syntonic-comma variant pitches. The ultracontrant counter-median ($E_C^M$) and ultradominant mediant ($D_D^M$) are called Tritonanten, as they are both a tritone away from the tonic ($T$); in C major, that is G♭ major and F♯ major respectively.
Mediant transformations can be combined with other alterations, in particular with the parallels ($p$): see the discussion of the Terzgleich, later in this section. Mediant transformations can also be reiterated, as illustrated in Figure 5.3.3 (next page). The example begins and ends in Db major, and is thus tonally closed. Chord 2 is the tonic mediant ($T^M$) of DbM; it is then reinterpreted as a new tonic for chords 2 through 5. Chord 3 is the tonic mediant in the secondary key of F major. The acoustic data below the example highlights the common-tone transformation featured in the first three chords, namely $3 \rightarrow 1$ or $T_{iii} \rightarrow i$. Those chords present a double mediant, or ultramediant transformation: DbM ($T$) $\rightarrow$ FM ($T^M$) $\rightarrow$ AM ($T^{MM}$). The acoustic data also specifies that the ultramediant is lowered by two syntonic commas, in relation to its source position counterpart. The A major triad in Figure 5.3.3 is notated correctly, as it is the major-third transposition above F major; understanding it as an enharmonically-respelled B♭ major triad ($bVI$ in Db major) may seem plausible from a mono-tonal perspective, but it would disregard the pitch-space identities of the common tones. The spelling of the pitches (especially the C$\sharp$ instead of Db) visually represents the effect of the ultramediant progression on the listener: of a two-fold “elevation” or leap upward through harmonic space.

Figure 5.3.3. Ultramediant transformations, connecting adjacent chords (Akustische 8.7)
The modal variants of the principal harmonies \((t, c, d)\) do not transform into their mediants; this is because in some cases they duplicate a simpler operation, and in other cases because they eliminate all common-tone connections. The “tonic variant mediant” \((t^M)\) is superfluous, as it is simply the tonic \textit{Leittonwechsel}: in C major, CM \((T) \rightarrow \text{Cm} (t) \rightarrow \text{Em} (t^M = T^l)\). On the other hand, the “tonic variant counter-median” \((t^M)\) is a new triadic relationship, but one which no longer retains any common tones with the source triad: CM \((T) \rightarrow \text{Cm} (t) \rightarrow \text{A} \overline{\text{m}} (t^M)\). For that reason, Karg-Elert does not include it among the mediants, but rather in a different category of transformations called the \textit{Kollektivwechselklänge} (“collective-change chords”), to be discussed in section 5.5.

Karg-Elert’s \textit{Mediant} and \textit{Gegenmediant} transformations are equivalent with Riemann’s \textit{Terzschritt} and \textit{Gegenterzschritt} respectively. Though Riemann’s explanation of those \textit{Schritte} does not explicitly highlight common-tone retention (or mention anything about acoustic pitch-space relationships), he states that the two major-third transpositions are “directly intelligible” \((direct \text{ verständlich})\) in relation to a tonic, and have the “power of closure” \((Schlusskraft)\) when they return to the tonic.\(^{49}\) The latter statement implies that major-third transpositions can inherently project some degree of non-tonic (i.e. dominant or subdominant) function. However, beginning with \textit{Harmony Simplified}, Riemann mostly understood mediants and counter-mediants as secondary dominants of parallel chords, rather than as direct transformations of the functional triads; in C major, the E major triad is normally \((D)[Tp]\), or the dominant of A minor. In most of his work, Riemann did not introduce symbols for direct major-third relations, even though he later reaffirmed that such progressions were “clearly comprehensible…not only possible, but

\(^{49}\) Riemann 1880, 72.
very fruitful.\textsuperscript{50} His hesitation to endow the major-third transpositions with their own symbols may be due to their functional ambiguity, which will be examined further later in this section. In the preface to the sixth edition (1918) of the \textit{Handbuch der Harmonielehre}, Riemann finally introduced new symbols for mediants: $3^+$ and $^93$ for the major and minor triads at the major third above the tonic, $III^+$ and $^9III$ for the triads at the major third below the tonic.\textsuperscript{51} These are stand-alone symbols, not attached to function labels; Riemann evidently felt unwilling to attach the major-third relations any particular harmonic function. The new symbols are presented at greater length in Riemann 1919 (his last published writing),\textsuperscript{52} in conjunction with numerical symbols for the tonic and the two dominants as well (\textbf{Figure 5.3.4}):

\begin{align*}
3^+ &- 1^+ - 5^+ = e^+ - c^+ - g^+ \\
^9III &- ^9I - ^9V = ^9e - ^9e - ^9a
\end{align*}

\textbf{Figure 5.3.4.} Numerical symbols for triads on the prime, third and fifth (Riemann 1919, 185)

In both major and minor keys, 1, 3 and 5 indicate the prime and its upper major third and fifth, while I, III and V indicate the prime and its lower major third and fifth; the superscripts $^+$ and $^9$ denote major or minor triad quality. In Figure 5.3.4, the minor triads are named according to their dual primes, so $^9I$ in the lower row is the A minor triad (notated as $^9e$). In this final work, Riemann questioned whether “the major-third harmonies [Terzklänge] can express a dominant meaning in a similar way as the fifth-relations”; he admitted that the answer to the question eluded him, and was “not closely at hand in the present work.”\textsuperscript{53} Regarding the new symbols in

\begin{itemize}
\item \textsuperscript{50} Riemann 1900, 127. Quoted in Imig 1973, 447.
\item \textsuperscript{51} Riemann 1918 (see Imig 1973, 45).
\item \textsuperscript{52} Riemann 1919, 185.
\item \textsuperscript{53} Riemann 1918, 191–192.
\end{itemize}
Riemann 1919, Renate Imig notes that they were “surely not without consequences for
Riemann’s wider theoretical project.” Indeed, they imply a thorough rethinking of the three-
function system: while 1/I and 5/V continue to assert the prevalence of fifth-relations, 3/III is
presented as a separate type of harmonic relationship, different from but equal in validity and
importance to the fifths. Riemann’s new nomenclature also suggests some rapprochement with
scale step theory, though the symbols are still dualistic; 5 and V can essentially replace D and S,
though they are now entirely defined as the fifth steps above and below the tonic, and not as
functional categories derived from the cadential progression. While Karg-Elert borrowed much
from Riemann’s theories, he did not adopt the new symbols for third relations presented in
Riemann 1918 and 1919; instead, Karg-Elert developed his own notation that incorporated
mediant relations within the earlier three-function system.

In the neo-Riemannian systems of Hyer and Cohn, Mediant (or Terzschritt) is LP, and
Gegenmediant (Gegenterzschritt) is PL. They are two-step operations, representing an equal
pitch-space distance to the fifth-transpositions LR (dominant) and RL (contrant). They are fully
dualistic, and are thus equivalent to Karg-Elert’s unary transformations, though they are of
course completely independent of function or tonal meaning. In Akustische 7.4, Karg-Elert
defines Mediant in patently neo-Riemannian terms: “same as variant of Leittonwechsel,” or
exactly LP. Strangely, he does not proceed to define the Gegenmediant as Leittonwechsel of

54 Imig 1973, 48.
55 Erpf 1927 (written concurrently with Karg-Elert’s treatises, but published slightly earlier)
replaces Riemann’s S and D function labels in a manner that recalls Riemann 1919’s 5 and V
symbols. Erpf calls the upper dominant (Oberdominant) D⁺ or D⁰ (for major and minor triad
qualities), and the lower dominant (Unterdominant or subdominant) D⁻ or D⁰. One major
difference with Riemann is that Erpf’s system is for the most part non-dualistic, working in the
same way in major and minor keys; see Imig, 240-241. Harrison 1994 (pp. 310–311) discusses
the vestigial influence of Riemann’s dualism in Erpf’s treatise.
variant (i.e. in C major, CM → Cm → AbM), but rather as “Terzgleich of parallel,” or neo-Riemannian R + SLIDE; in C major, that is CM (T) → Am (Tp, or R) → AbM (Terzgleich or SLIDE of A minor). In David Kopp’s transformational system, the major-third transpositions are non-dualistic, operating in the same way in major and minor keys: M (major third below) and M⁻¹ (major third above). Like Karg-Elert’s M transformations, they are unary operations that preserve mode. Thus, they are always chromatic medians: M is the lower flat mediant (LFM: [CM → A♭M] or [Am → Fm]), and M⁻¹ is the upper sharp mediant (USM: [CM → EM] or [Am → C♯m]). Because they are monistic, they closely resemble chromatic mediant and submediant Stufen: M is basically a motion to bVI (in major) or VII♭ (in minor), while M⁻¹ is a motion to III♯ (major) or ♯III (minor).

c. The auxiliary mediants: minor-third transpositions

The secondary third-based transpositions on Karg-Elert’s chart (see Figure 5.3.1) are collectively called the Nebenmedianten, meaning “neighbor” or auxiliary mediants. They are minor-third transpositions involving the chordal third (iii) and fifth (v). Karg-Elert defines the two transpositions more specifically, in terms that are once again strikingly neo-Riemannian. T_{iii→v} is called Parallelvariant, which is exactly RP; T_{v→iii} is Variantparallel, or PR. The “explanation of terminology” (Erklärung der Wortbezeichnung) is illustrated in Figure 5.3.5. As in Figure 5.3.2 above, syntonic-comma variant pitches are shown as closed noteheads; the parallel-variant includes a comma-variant third, while the variant-parallel includes a comma-variant third.

---

56 SLIDE is a transformation between two triads of opposite mode that are inverted around a common third; it was proposed and named in Lewin 1987, 178. It is identical to Karg-Elert’s Terzgleich (“common third”), described later in this section.
variant prime and fifth. At the right of the figure, Karg-Elert notes that the variant-parallel is a chromatic transformation (specifically the *Terzgleich* or *SLIDE*) of the Leittonwechsel.

The *M* symbol for the mediants does not reflect their incremental pitch-space journey: it does not reveal that *Mediant* is equivalent to Leittonwechsel plus variant, for example. In contrast, the functional notation for the auxiliary mediants does reflect their derivation, as demonstrated in Figure 5.3.5. The parallel-variant is notated as a shift from the regular diatonic parallel to its modal variant, by changing the “*p*” postscript to uppercase: $T \rightarrow Tp \rightarrow Tp$.

Likewise, the variant-parallel is indicated by appending the uppercase *P* to the lowercase variant function label: $T \rightarrow t \rightarrow t^P$. The uppercase *P* is necessary with the variant-parallel, as the lowercase *p* always indicates an inversion rather than a transposition (see section 4.9, and also later in this section). In all cases, the placement of the *P* indicates the spatial direction of the transposition, in both major and minor: *P* as superscript denotes the minor third above the tonic, while *P* as subscript denotes the minor third below the tonic.

**Figure 5.3.5.** *Nebenmedian ten* (auxiliary mediants) in C major and A minor (*Harm*, 200)
In Riemann 1880, the parallel-variant \((T_{\text{iii}} \rightarrow \text{a})\) is called \textit{Kleinterzschritt} (see Figure 5.1.2 and its accompanying footnote), and variant-parallel \((T_{\text{a}} \rightarrow \text{iii})\) is called \textit{Gegenkleinterzschritt}. As with the major-third transpositions, parallel-variants are usually understood in \textit{Harmony Simplified} as secondary dominants of parallels from other functions; for example in C major, the A major triad is most often \((D) [Cp]\). However, variant-parallels are usually understood as such: in C major, the Eb major triad is most often labelled as \(^aTp\), meaning the parallel of the tonic variant, as in \([CM (T) \rightarrow Cm \ (\text{a}T) \rightarrow EbM \ (\text{a}Tp)]\). Thus, while Karg-Elert’s variant-parallel differs from Riemann’s in terms of functional notation, they are conceptually identical. In the neo-Riemannian systems of Hyer and Cohn, parallel-variant is \textit{RP}, and variant-parallel is \textit{PR}. As described earlier in this section, Karg-Elert employs very similar language to explain the auxiliary mediants. In Kopp’s system, the minor-third transpositions are non-dualistic, operating in the same way in major and minor keys: \(m\) (minor third below) and \(m^1\) (major third above). Like Karg-Elert’s auxiliary mediant transformations, they preserve mode, and are thus always chromatic mediants: \(m\) is the lower sharp mediant ([CM \(\rightarrow\) AM] or [Am \(\rightarrow\) F\#m]), and \(m^1\) is the upper flat mediant ([CM \(\rightarrow\) EbM] or [Am \(\rightarrow\) Cm]). Kopp’s monistic \(m\) and \(m^1\) closely resemble chromatic mediant and submediant \textit{Stufen}: \(m\) is essentially a motion to \(VI\#\) (in major) or \(#VI\) (in minor), while \(m^1\) is a motion to \(b\)III (major) or III\# (minor).
d. Third-based transpositions and functional meaning

Like the diatonic substitutes (parallel and Leittonwechsel) of the principal harmonies, the mediants and auxiliary mediants of different functions duplicate and overlap with each other. For example, in the key of C major:

\[
\begin{align*}
E \text{ major} &= TM \text{ or } DP \\
A_{b} \text{ major} &= TM \text{ or } c^{\#} \\
A \text{ major} &= TP \text{ or } C^{M} \\
E_{b} \text{ major} &= t^{p} \text{ or } D_{M}
\end{align*}
\]

The preceding label pairs do not indicate metharmonic (comma-differing) versions of a triad, but rather different transformational derivations of exactly the same harmonies, which always retain at least one acoustic common tone with C major. The E major triad contains both the third of C major (\(T\)) and the fifth of G major (\(D\)); its functional multiplicity is therefore not surprising. As with the diatonic substitutes (see Figures 4.8.10 and 4.8.11), the choice of functional assignment with mediants and auxiliary mediants will ultimately depend on context, and in particular on factors such as melodic voicing and rhythmic placement. The following C major example drawn from Riemann 1900 (Figure 5.3.6) illustrates the point. The third-derived harmonies are marked NB:

\[
\begin{align*}
\text{CM: } & T \quad TM \quad D \quad T \quad T \quad DP \quad T \quad T \quad c^{p} \quad T \\
& [DP?] \quad [TM?] \quad [T?] \quad [T?] \quad [T?] \quad [T?] \quad [T?] \quad [T?] \\
\end{align*}
\]

**Figure 5.3.6.** Third-related harmonies in context (Riemann 1900, 127), with analysis
In the first passage, the E major triad precedes a dominant (D), and so it is not needed to itself serve as the dominant; therefore, it is labelled in the functional analysis as a mediant extension of the tonic, in part due to the repetition of the tonic’s third (E-natural) in the upper voice. On the other hand, in the second passage, E major evidently serves as the dominant, as it contains the leading tone (B-natural) in the upper voice, and because it is placed on the upbeat, leading back to tonic in the following measure; it is therefore labelled as a parallel-variant transformation of the dominant (D_P). The third passage is similar to the second: the Gegenmediant A♭ major triad is understood as a transformed contrant (c^p) instead of tonic (T_M), due to its placement on the weak beat surrounded by strong-beat tonics, and because of the prominent ^6 scale degree in the upper voice, which suggests contrant-variant (c) function.

The following two examples by Berlioz and Schubert (with analysis by Karg-Elert) shed further light on the question of functional assignment with mediants and counter-mediants; they also reveal Karg-Elert to be a perceptive and subtle analyst of harmony, concerned both with a listener’s local harmonic perceptions from chord to chord, and with how individual chords function on a larger scale within complete phrases and structures.

![Figure 5.3.7. “Mediant oscillations” (Mediantenpendelungen) in Berlioz (Harmonologik, 252)](image-url)
In the passage from Berlioz’ *Requiem* (Figure 5.3.7), the key is D♭ major. The first two measures are analyzed as an extended tonic, including the motion to the tonic mediant ($T^M$). The F major chord might have been labelled instead as a dominant parallel-variant ($D_P$) – indeed, that interpretation may seem more convincing, due to its weak-beat rhythmic placement, and because of the lower-neighbor motion to the leading tone (C-natural) in the bottom voice. The rationale for Karg-Elert’s analysis of measure 1 is not revealed until measure 4, in which the notated D major triad is understood to represent E♭♭ major (Eses dur), the contrant counter-median ($C_M$) of D♭ major. The counter-median “sinking” in measure 4 (indicated by the downward arrow) balances the mediant “rising” in measure 1 (indicated by the upward arrow). As described in chapter 4, Karg-Elert was very much interested in the energetic tendencies of the consonant triads, and of different harmonies within the key. Figure 5.3.7 illustrates how his analyses were often informed by ideas of energy and tension, and concerned with “expressive and coloristic effects” such as the “mystical, enraptured” quality of the Berlioz passage.

In the passage (Figure 5.3.8, next page) from Schubert’s celebrated Lied “*Du bist die Ruh*” (D. 776, op. 59 no. 3), the heading describes the “counter-median as chromatic substitute for the diatonic parallel.” Indeed, at the moment where the first two verses move to the tonic parallel C minor ($T_P$), the third verse substitutes the tonic counter-median C♭ major ($T_M$), on the magical word *Augenzelt* – literally “eye-tent,” an idiom coined by the poet Friedrich Rückert to convey the sheltering of the singer’s rapture within the beloved’s gaze.57

---

Karg-Elert provides two analyses of the progression: one that changes to the key of the tonic counter-mediant for four measures in the third verse, and one that remains entirely in the key of E♭ major. In the first interpretation shown immediately under the passage, the C♭ major triad at –zelt becomes a new local tonic, ending with a dominant mediant (D^M) of that key in the third-last measure, which then itself serves as a dominant pivot (D) back to E♭ major. Next, Karg-Elert suggests a “latent cadence” progression that is “possibly still more logical” than the first version: both the C minor and C♭ major triads are now analyzed as transformations of the contrant (Ab major), specifically as C^ against c^P. This allows the harmonic shift in the third verse to be understood within a monotonal framework, highlighting how mediants and auxiliary mediants can provide new colour and perspective within an expanded tonality, without changing to a new key.
e. Third-based inversions

The next section of the chart illustrates the third-based inversions (Figure 5.3.9):

b) Different quality (inversions)

\[
\begin{align*}
\text{Primary:} & & \text{Secondary:} & & \text{Tertiary:} \\
[\text{CM} \leftrightarrow \text{Am}] & & [\text{CM} \leftrightarrow \text{Em}] & & [\text{CM} \leftrightarrow \text{C#m}] \\
\text{Parallel} & & \text{Leittonwechsel} & & \text{\textit{Terzgleich} ("common third")} & & [\text{Am} \leftrightarrow \text{AbM}] \\
T & & T^l & & T & & T^{\text{Mediant-parallel}} \\
3 \leftrightarrow \text{I} & & \text{I}^{\text{viii}} & & \text{I}^{\text{iii}} & & \text{I}^{\text{iii}} \\
1 \leftrightarrow \varepsilon & & 5 \leftrightarrow \varepsilon & & 3 \leftrightarrow \varepsilon & & \varepsilon \leftrightarrow 3 \\
\end{align*}
\]

\textbf{Figure 5.3.9.} Third-based inversions (Harmonologik, 53)
The primary and secondary third-based inversions are the familiar diatonic substitutes, the parallel ($I_{iii}^1$) and Leittonwechsel ($I_{v}^{iii}$). They are two of the basic neo-Riemannian operations, named REL and LT by Lewin, or R and L by Hyer and Cohn. Their centrality in the latter’s system is based in part on their production of parsimonious voice leading, namely the retention (or inversionsal exchange) of two common tones, and the stepwise motion of the remaining voice. In a similar way, Karg-Elert’s chart uses horizontal lines on the staff to indicate the retention of the two common tones, and the numbers specify the chord-tone exchanges in each inversion. David Kopp employs R in place of L, and lowercase r in place of R; they operate in exactly the same way as the corresponding neo-Riemannian transformations, and thus constitute one of the few remaining vestiges of dualism in Kopp’s system, along with P for parallel (mode change) and S for Terzgleich or SLIDE (see below). Kopp’s alternate labels were chosen as “an explicit sign of the correspondence between major- and minor-third relative mediants, analogous to that of the chromatic mediants”59 – that is, the transformational pair R and r for the diatonic mediants corresponds with M and m for the chromatic mediants. In both pairs, the uppercase letters denote major-third relations, and the lowercase letters indicate minor-third relations. The correspondence is not perfect, because while M and m are non-dualistic transpositions (and thus have inverse operations $M^{-1}$ and $m^{-1}$), R and r are dualistic inversions, and are thus their own inverses. The parallel and Leittonwechsel are often function-retaining transformations, as they each retain two pitches from the original triad. However, as described in section 4.8 and elsewhere, the diatonic substitutes of different functions duplicate each other, and thus their functional assignment depends on musical context. As the diatonic substitutes and their functional meanings were discussed at length in chapter 4, they are not examined further here.

The tertiary third-based inversion \( I_{iii}^{lll} \) is a mode-changing operation that inverts around a common third, as in \([CM \rightarrow C\sharp m] \) or \([\text{Am} \rightarrow A\flat M] \). As a result, it is called *Terzgleicher* or “same third,” analogous to the *Quintgleicher* \( I_v^p \) discussed in section 5.2, and the *Septgleicher* \( I_{vll}^{lll} \) to be discussed in section 5.4. David Kopp has noted that the first volume of A. B. Marx’s *Die Lehre von der musikalischen Komposition* likely contains the earliest mention of the \( I_{iii}^{lll} \) relationship; Marx lists it among the triadic pairs that share one or more common tones, and that can therefore serve as the source and goal keys of abrupt modulations, or in direct chordal juxtapositions separated only by a pause.\(^{60}\) In Riemann 1880, the \( I_{iii}^{lll} \) operation is named *Doppelterzwechsel* or “double-third change”: using Riemannian dualistic triad labels, that is \(c^+ \rightarrow ^o g^\sharp \text{ or } ^o e \rightarrow ab^+ \). In spite of the strong common-tone connection created by the shared third, \( I_{iii}^{lll} \) is normally understood as a quite distant chromatic relationship in the analysis of tonal music; Cohn has noted that \( I_{iii}^{lll} \)-related triads can never belong to a single diatonic scale, and thus will always be in a complex relationship within a diatonic system.\(^{61}\) In Riemann 1893, the *Doppelterzwechsel* is described tortuously as “the combination of the parallel clang of the contra-clang with the parallel clang of the tonic,” as in \(^o Sp \rightarrow + Tp \text{ or } + Dp \rightarrow ^o Tp \).\(^{62}\) Karg-Elert was likely the first to characterize \( I_{iii}^{lll} \) as an inversion around the common third, as acknowledged by Cohn.\(^{63}\) The inversion is best known as David Lewin’s *SLIDE*, a name that reflects the parallel downward or upward semitone sliding of the prime and fifth.\(^{64}\) It requires three steps in Cohn’s transformational system, either *RPL* or *LPR*; while the intervening stations differ, each of those two paths leads to the exact same destination in triadic space.

\(^{61}\) Cohn 2012, 64.
\(^{62}\) Riemann 1893, 100.
\(^{63}\) Cohn 2012, 64.
\(^{64}\) Lewin 1987, 178.
Karg-Elert notes that the *Terzgleich* most often arises as a *Mediantparallele* or mediant-parallel transformation of a basic functional triad; this is notated as “$\text{MP}$” attached to a function label. Figure 5.3.10 demonstrates the mediant-parallels of the basic functional triads in C major and A minor, plus the ultradominant mediant-parallel ($\text{DDMP}$) in each key; *Terzgleich* transformations of the ultraforms are indeed possible, though they occur quite infrequently.

![Terzgleicher der Principale](image)

**Figure 5.3.10.** *Terzgleicher* of the principal harmonies in C major and A minor (*Akustische*, 42)

The figure demonstrates how the *Terzgleich* always contains a syntonic-comma variant prime and fifth, indicated using closed noteheads. The *Terzgleich* is often understood as a function-retaining transformation, due to the retention of the original triad’s third. Karg-Elert also notes that the *Terzgleich* also appears as function-retaining between “the parallel and the counter-median of the same principal,” as in $\text{TP} - \text{TM}$ or $\text{DP} - \text{DM}$ (*Harmonologik*, 201). However, the *Terzgleich* can also arise between triadic pairs from different functions such as $\text{TP} - \text{cP}$ or $\text{tP} - \text{DP}$, and therefore can also be a function-changing transformation. The functional multiplicity of the *Terzgleich* is illustrated in the following three examples, drawn from the Romantic repertoire.
The first passage (Figure 5.3.11) is from Schumann’s “Mit Myrten und Rosen” (op. 24 no. 9), from the song cycle Liederkreis. It juxtaposes the tonic (T) triad of D major with its Terzgleich transformation D# minor (labelled T<sub>Mp</sub>), with the common third F# in the melody:

The analysis of the D# minor triad as a transformed tonic is convincing, due largely to the common tone F# in the melody, but also to the alternation with the regular tonic on the measure’s stronger beats. Both T and T<sub>Mp</sub> are tonicized on their preceding weak beats; the A# major triad (Ais dur) is the local contrant-variant of D# minor (dis moll). In addition to the Terzgleicher between T and T<sub>Mp</sub>, the analysis highlights the Chromonant (mode-retaining chromatic semitone shift) relationship between A major and A# major. The following Tonnetz (Figure 5.3.12) outlines the progression in Figure 5.3.11, and highlights how the vocal-part semitone oscillation around the common third (E# → F# ↔ G) is in fact symmetrical in terms of pitch-space location, as indicated by the dotted arrows.

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65 See section 5.5 for more on the Chromonant, and on other harmonic relations that do not retain common tones.
Figure 5.3.12. Tonnetz for Schumann, “Mit Myrten und Rosen” (Figure 5.3.11)

Much of Schubert’s Lied “Der Doppelgänger” (from Schwanengesang, D. 957) features a repeated four-measure bass and harmonic pattern, with occasional chromatic alterations of the second and fourth bars (Figure 5.3.13). Beginning with the approach to the final climax (from m. 43), the pattern breaks, as shown in Karg-Elert’s analysis (Figure 5.3.14, next page):

Figure 5.3.13. Schubert, “Der Doppelgänger” (D. 957 no. 13) – Thema ostinato, in B minor
Figure 5.3.14. Schubert, “Der Doppelgänger” – analysis from m. 43 to the end (Harm, 245-246)

With the chromatic ascent in the bass (mm. 43-46), the tonic B minor (T) becomes its parallel D major (D dur), which is then brusquely juxtaposed with its Terzgleicher D♯ minor (dis moll) in measure 47. As in the Schumann example above, the common third F♯ is retained in the vocal part, creating an audible linear connection. The difference in this passage is that the Terzgleich arises not as a direct transformation of a regular tonic triad, but between two different transformed versions of the tonic, namely Tp and TM. D♯ minor is followed by its own contrant-variant A♯ major (mm. 47-48, repeated in mm. 49-50). Karg-Elert’s annotation under the
example notes that A♯ major (Ais dur) relates to the tonic B minor (h moll) as an enharmonic Terzgleicher: A♯ major contains C♯, which is an enharmonic of the D-natural in the B minor triad. The same enharmonic juxtaposition also occurs in mm. 50-51, marked in Figure 5.3.14 as a wavy line between the two pitches. Of course, the two pitches are quite distant from each other in just intonation pitch space. The alternative Septgleicher (“same seventh”) transformation between A♯ major and B minor will be explained later, in section 5.4.

The final example in this section is from Act Three of Wagner’s Parsifal. It highlights the tonal and functional multiplicity of the Terzgleich, and of third-based harmony in general. The passage begins in E minor; it then moves through a four-bar chromatic passage featuring Terzgleich pairs to settle on an apparent contrant-variant of D minor, though the next section begins in B♭ major. The score of the passage is provided in Figure 5.3.15 (continues on the next page). Karg-Elert’s analysis of the Terzgleich sequence (Figure 5.3.16, on the next page) considers both D minor and B♭ major as possible tonics.

(Fig. 5.3.16 begins)

![Music notation image]

Figure 5.3.15. Wagner, Parsifal – passage from Act III (continued on next page)
Dm:  

\[ \begin{array} {c}
\text{\textbf{Em:} } \\
\end{array} \]

\[ \begin{array} {c}
\text{\textbf{B}b\textbf{M:} } D \\
T \\
T_M \\
T \\
\end{array} \]

**Figure 5.3.15.** Wagner, *Parsifal* – passage from Act III (vocal score pp. 273-74)

**Figure 5.3.16.** Wagner, *Parsifal* – *Terzgleicher* in passage from Act III (*Harmonologik*, 285)
In Figure 5.3.16, the initial E minor tonic is reinterpreted as a contrant mediant-parallel \((C^{\text{Mp}})\) in B\(b\) major, effecting the modulation to that key. The two-measure *Terzgleich* sequence of \(C^{\text{Mp}} - C\) and \(Dp - D\text{M}\) is marked by the brackets under the function labels. The figure illustrates how the common tone inversion \(\mathcal{E} \rightarrow 3\) alternates between the upper voice and the bass. Though the sequence is exactly transposed down a whole tone in the third and fourth measures, the analysis of the repetition differs from the first statement: while the first *Terzgleich* pair is understood as a mediant-parallel transformation of a principal triad, the repetition is analyzed instead as a diatonic parallel and a counter-median. The different labels in each pair are determined by their tonal meaning in the key of B\(b\) major: while the triads in both pairs are \(I_{\text{III}}^{\text{III}}\) or *Terzgleich*-related with each other, their harmonic functions in B\(b\) major (B dur) operate independently from the sequential repetition. Karg-Elert also considers D minor (\(d\) moll) as an alternate key, inside the square brackets on the upper line of the functional analysis. D minor is quite plausible in measures 2 to 5: the D\(b\) major triad in measure 4 is a tonic mediant-parallel \((T^{\text{Mp}})\), or direct *Terzgleich* of D minor. At the bottom of Figure 5.3.16 is a possible analysis in the original key of E minor; while the first *Terzgleich* motion to Eb major is easily explained as a direct mediant-parallel of the tonic \((D^{\text{Mp}})\), the rest of the passage must refer to the ultradominant D minor \((\mathcal{A} \mathcal{D})\), and so the analyses in B\(b\) major and D minor are preferable.

**f. Der Mediantenstil – the “mediantic style”**

For Karg-Elert, the third-based transformations described in the foregoing section represent “the romantic type in harmony, as opposed to the dominant, which corresponds to the classical style” (Harmonologik, 203). While chromatic harmonies and distant triadic relationships can certainly arise through purely fifth-based progressions (as demonstrated in the
passage from Brahms’ Rhapsody in G minor in Figure 5.2.8), mediants and auxiliary mediants
greatly expand the potential for the chromatic enrichment of tonality, and for journeys to distant keys. Much of Chapter 13 of Harmonologik (which is over one hundred pages in length)
consists of a chronological catalogue of examples (with annotation and functional analysis) of
third-based harmony, or what Karg-Elert dubbed “the mediantic style” (Der Mediantenstil). This
catalogue is likely the most extensive collection of functional harmonic analysis of repertoire
ever published, and as such is one of the most valuable features of Harmonologik. It is preceded
by an essay\(^{66}\) that surveys the historical development of the style, which begins with “amazingly
bold” precursors in works by Gesualdo and Monteverdi, and isolated examples from Bach to
Mozart and Haydn. In the middle-period works of Beethoven, third-relationships are featured
frequently, especially between thematic groups, and between movements; however, Karg-Elert
notes that Beethoven’s late style “is by far more diatonic than chromatic; color recedes behind
monumental architecture.” The latter statement’s opposition of diatonicism/architecture and
chromaticism/color reveals much about Karg-Elert’s evaluation of harmonic space, as will be
discussed presently. The “full fruition” of the mediantic style is found in Schubert, in which
Karg-Elert states that “mediant tonality…becomes the specific feature of his harmony.” His
successors Schumann, Berlioz and Chopin introduce further elaborations, especially in the use of
mediant and auxiliary-median sequences. Such third-chains are used to excess by Liszt, in
whose works “nearly every harmonic cliché appears in ensuing repetition at the third”; nonetheless, Karg-Elert finds that Liszt at his best “manages true mediant miracles which were to
be powerful stimulus for Wagner, Bruckner, Grieg, Debussy and Scriabin.” The “point of
culmination is unquestionably reached” in the music of Wagner, in which mediant relationships

\(^{66}\) Harmonologik, 204–206. All of the quotations in this paragraph are drawn from that essay.
create endless “strung-out curves” of decentralized harmony. In contrast to Wagner’s continuous transformations, Bruckner’s “steplike, graded mediants” appear at the level of the phrase in immediate juxtaposition, and often prefer minor-third steps to major thirds. Finally, in the music of Max Reger, “we find everything that is possible in tonal resolution … only just in excessive accumulation.” Though Karg-Elert praises Reger for “the introduction of previously unused minor mediants” (meaning minor triads in major-third relationships), he states that the “mighty” third-based harmonic idioms of Wagner and Bruckner “stand sharply against the unruly, rampantly vagrant, timeless but lame mediants of Reger.” The catalogue of third-based harmony in Harmonologik Chapter 13 includes passages by all of the above composers (plus others such as Brahms, Wolf and Richard Strauss), and ends with a brief selection of excerpts from Karg-Elert’s own compositions, including a string quartet that is now lost and otherwise unknown.67

Based on the amount of analytical attention devoted to third-based harmony in Harmonologik (almost one third of the entire treatise) and Akustische, and also on the evidence of his own compositions, it is clear that Karg-Elert was highly attracted to “the mediantic style.” It is probably accurate to state that no other theorist before the “neo-Riemannian era” developed a method for analyzing third-based harmony that is as comprehensive and systematic as that of Karg-Elert. Nonetheless, some of his statements suggest that he considered third-relations to be

67 Examples 689 to 691 in Harmonologik (pp. 306–308) are listed as passages from a work entitled Streichquartett Nr. 2 “Klärung,” op. 152, for string quartet and voice. No such work was ever performed or published, and no manuscripts survive (Gerlach 1984, 124); whether it was ever completed is now unknown. Example 691 contains the unattributed text fragment “beug’ dich zurück…und trinke fein geschwinde” (“lean back…and drink well and swiftly”), set in Sprechstimme in the manner of Schoenberg’s Pierrot Lunaire, op. 21. The text is from the poem Der bittre Kelch (“The bitter chalice”) by German poet Gustav Schüler (1868–1938). Karg-Elert set a number of poems by Schüler during World War I (the composer’s most avant-garde period); the complex musical style of examples 689 to 691 suggests that they may originally date from that period as well, in spite of the late opus number.
inferior to fifth-relations, and a direct threat to musical coherence. A recurring motif is the
collection of mediant and other chromatic harmonies with the idea of color, mentioned above
in opposition to diatonic “architecture” in the late works of Beethoven. Karg-Elert states that
during the nineteenth century, color eventually took precedence over all other elements:

Color dominates the design. At first only a mild glaze and occasional retouching, the
color-moment gradually becomes the primary means of expression (in the Impressionist
style) until finally, at the beginning of this century, veritable orgiastic blossoms bloom
(partly by Reger!) and through colour-inbreeding [Farben-Inzucht] this brings about an
absolute sterility and supersaturation. 68

Another passage in Harmonologik describes Karg-Elert’s conflicting views about third-based
harmony in more detail, and points to their acoustic source:

[Mediants] broaden the key to a boundless tonality, but at the same time they also carry
within themselves the seeds for destruction of tonality, as through intersection and
enharmonic levelling of distant relatives, the tonic loses its unambiguous centralisation. 69

As described in section 3.7, the central horizontal “beam” of Karg-Elert’s pitch space is an
endless chain of pure fifths. The major and minor triads built on this central fifth-chain are
privileged in the space, regarded to be in the Ursprungslagen or source positions for each triad
(Figures 3.7.6 and 3.7.7), representing the “primary tonal centers” within a key. All triads that
are not located on the fifth-chain are syntonic-comma variants, “artificial” substitutes which
occupy positions in the pitch space that rightfully belong to the source position triads (see Figure
5.3.2, or Figures 4.8.13 and 4.8.14). Therefore, while third-based harmonies inject variety and
colour into the pitch and harmonic palettes of the key, their comma-variant pitches ensure
motion away from the stability of the fifth-chain, and can thus cause the tonic to lose its
“unambiguous centralization.” Indeed, in a boundless multi-dimensional just intonation pitch

68 Harmonologik, 204.
69 Ibid., 216.
space, the cumulative effect of third-based harmony can prevent the music from returning to its original tonic triad (in its original source position): a progression, movement or entire piece may seem to begin and end in the same notated key, but its harmonic trajectory may in fact end in a metharmonic (same-name) variant of the starting key, in a completely different location in pitch space. This issue is the central concern of Karg-Elert’s theory of comma-free and comma-differing modulation, which is the principal focus of chapter 6.

5.4. **Seventh-based transformations**

Section 3.5 examined the septimal axis of Karg-Elert’s pitch space, derived from the interval of the 4:7 natural seventh (called the *concordant* seventh). As described above, the addition of the seventh to the pitch space is an innovation that is unique to Karg-Elert among his contemporaries, one that presents significant ramifications for connections between harmonies, and for harmonic functions within the key. Pitches that are located a natural seventh below or above a fifth/third-derived pitch exist on a different acoustic and conceptual plane; if just intonation is to be observed, seventh-derived pitches cannot be equated with fifth- or third-derived pitches of the same name. As a result, chords that contain seventh-derived (or concordant) pitches will have distinct functions within the key, requiring new methods of classification and analysis. Perhaps because the seventh axis was such a radical expansion of the Oettingen/Riemann fifth-third pitch space, Karg-Elert’s treatises provide only a partial exploration of its possibilities. One particular problem was that of harmonic function: how can seventh-derived chords (with pitches located in the seventh axis) be incorporated into the three-function tonal system, which is conceptually rooted in the fifth-third space?
The difficulty of grappling with seventh-derived harmony is reflected in the relative brevity of the material on the topic in the two treatises. Two chapters in Akustische (Chapters 9 and 10, ten pages in total) present all of the possible transformations among concordant sevenths, though two of them are much emphasized over the others. One very short chapter in Harmonologik (three pages) describes a single seventh-based inversion; the other transformations are excluded entirely. The most practical consequence of Karg-Elert’s septimal axis is that it implies common-tone transformations between concordant seventh chords – the dominant and half-diminished sevenths that display the intervallic proportions 4:5:6:7 (see section 3.5). In recent years, several neo-Riemannian scholars have proposed transformations between (0258) seventh chords\textsuperscript{70} that closely resemble those in Karg-Elert. Their models will be examined later in this section, and compared with those in Akustische and Harmonologik. First, this section will outline all of Karg-Elert’s seventh-based common-tone transformations, beginning as before with transpositions, followed by inversions. The discussion reveals that some of the seventh-transformations are of more theoretical interest than practical use, as they metharmonically mimic simpler fifth- and third-based operations. On the other hand, at least two of the seventh-based inversions are fundamentally new, and one (the Septgegenklang or “counter-seventh chord”) is of particular analytical significance.

\textsuperscript{70} That is, the set of dominant and half-diminished seventh chords under 12-TET and enharmonic equivalence. As will be discussed later in this section, most recent work on transformations involving those chords assumes 12-TET and enharmonic equivalence, and thus operates within a different conceptual framework than Karg-Elert’s.
a. Seventh-based transpositions

The following (Figure 5.4.1) lists the seventh-based common-tone transpositions; the English version continues on the next page.

D. Seventh-based transformations

a) Transpositions (same quality)

Primary:
Concordants

[D7 ↓ C7 → B♭7]  
7 ↓ 1  
T_vii→i

[Em7 ↓ F♯7 → G♯7]  
\[ \downarrow \rightarrow \]  
T_vii→v

Used with complete seventh chords.  
If the sevenths are omitted, likely be replaced by ultradominants.

Secondary:  
[no name]

[A7 ↓ C7 → Eb7]  
7 ↓ 5  
T_v→vii

[A♭7 ↓ F♭7 → D♭7]  
\[ \downarrow \rightarrow \]  
T_vii→v

Likewise, simpler to replace with auxiliary medians.
Tertiary:
Tritonants

\[
\begin{align*}
[F#7 & \leftrightarrow C7 \rightarrow Gb7] \\
7 & \leftrightarrow 3 \\
7 & \rightarrow 3 \\
[Cø7 & \leftrightarrow Fø7 \rightarrow Bø7] \\
\text{T}_{\text{iii} \rightarrow \text{vii}} \\
\text{T}_{\text{vii} \rightarrow \text{iii}} \\
\end{align*}
\]

Understood much more simply as ultradominant mediants (2Q, 1T), or as ultracontrant counter-mediants (-2Q, -1T)

Figure 5.4.1. Seventh-based transpositions (Harmonologik, 53)

Karg-Elert names both major and minor triads according to the Grundton or lowest pitch: A minor is called A minor (a moll), not dual “E minor” as in Riemann and Oettingen. When the concordant seventh is added either above a major triad or below a minor triad (see section 3.5), the chord is still named after the triad’s Grundton. Therefore, when A minor is made concordant by the addition of the seventh F# below, Karg-Elert still calls the chord “A minor” rather than F# half-diminished seventh. In the original German version of Figure 5.4.1, the chords listed under the secundär (secondary) transposition are named as triads, but they should be understood as concordant sevenths: dominant sevenths in the key of C major (left side), and half-diminished sevenths in A minor (right side). In the English version of the figure, all seventh chords are named in the standard modern manner: C major with concordant seventh above is labelled C7, and A minor with concordant seventh below is labelled as F#ø7. This deviation from Karg-Elert’s practice aims to make his seventh-chord transformations more familiar and immediate for the modern reader, and in turn to enable comparison with recent writings on seventh chords. Nonetheless, one must remember that all of the concordant seventh chords to be discussed here contain the 4:7 pure seventh interval, and often other pitches located on the septimal plane of pitch space.
In the two *Konkordant* transpositions, either the concordant seventh becomes the new prime (*T*<sub>vii→i</sub>), or vice versa (*T*<sub>i→vii</sub>). Under *T*<sub>vii→i</sub>, the new root, third and fifth are septimal (seventh-axis) pitches, and the new seventh is a double-septimal pitch (i.e. the pure seventh of a pure seventh). The common tone is printed in **boldface**:

$$T_{vii \rightarrow i} \quad [C(0) \ E_{(0,1)} \ G_{(1,0)} \ B_{(0,0,1)}] \rightarrow [B_{(0,0,1)} \ D_{(0,1,1)} \ F_{(1,0,1)} \ A_{(0,0,2)}]$$

Under *T*<sub>i→vii</sub>, the new root, third and fifth are septimal pitches, while the new seventh is a fifth/third-based pitch, as it was the original chord’s prime:

$$T_{i \rightarrow vii} \quad [C(0) \ E_{(0,1)} \ G_{(1,0)} \ B_{(0,0,1)}] \rightarrow [D_{(0,0,-1)} \ F^\#_{(0,1,-1)} \ A_{(1,0,-1)} \ C(0)]$$

The following Tonnetz (**Figure 5.4.2**) illustrates the two *Konkordanten*, from C major:

![Tonnetz Diagram](image)

**Figure 5.4.2.** The *Konkordant* transpositions, from C major

In *Akustische* Chapters 9 and 10, Karg-Elert sometimes uses diamond-shaped noteheads to indicate septimal pitches (see Figure 3.5.3). Unfortunately, he does not employ this useful notation consistently in *Akustische*, and it does not appear at all in *Harmonologik*. In terms of root motion, the *Konkordant* transpositions resemble the ultracontrant and ultradominant transpositions. Both the *Konkordant* (*T*<sub>vii→i</sub>) and the ultracontrant (*T*<sub>i→v</sub>)² transpose a dominant seventh upward by a notated minor seventh: *[C7 → B♭7]* in both cases. However, the roots of the B♭7 chords in each transposition differ by a septimal comma: B♭<sub>(-2,0)</sub> for the ultracontrant, and
B\textsubscript{b}(0,0,1) for the Konkordant. In the annotation to the Konkordanten in Figure 5.4.1, Karg-Elert states that when the concordant sevenths are not actually present, [CM \rightarrow B\textsubscript{b}M] will most likely be understood as an ultracontrant. However, when the sevenths are present, the Konkordant transformation becomes a possibility. As in many such cases, voice leading and the registral placement of the common tone can play a crucial role in determining the transformation.

Figure 5.4.3 presents two manifestations of the progression \([C7 \rightarrow B\textsubscript{b}7 \rightarrow C7]\):

\begin{itemize}
  \item[a)] Ultracontrant (T_{i \rightarrow v})^2
  \item[b)] Konkordant (T_{vii \rightarrow i})
\end{itemize}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{ultracontrant_konkordant}
\caption{Comparison of ultracontrant and Konkordant}
\end{figure}

In Figure 5.4.3a, all voices move in parallel motion, as is typical in the music of Debussy and other Impressionist composers. The neighbouring bass motion from C to B\textsubscript{b} and back dictates the motion of the other voices, and therefore can be regarded as the crucial element in the progression. For that reason, the B\textsubscript{b}7 chord is analyzed as an ultracontrant. While all of the chords contain a B\textsubscript{b}, the progression’s voice leading does nothing to suggest that B\textsubscript{b} must be understood as an acoustically-exact common tone. On the other hand, Figure 5.4.3b sustains the B\textsubscript{b} as an upper-voice pedal, strongly implying that it is indeed a true common tone. In that case, the B\textsubscript{b}7 chord should be analyzed as the Konkordant or T_{vii \rightarrow i} of C7. The A\textsubscript{b} in the B\textsubscript{b}7 chord is now a double-septimal pitch: the seventh of the seventh of C major, or A\textsubscript{b}(0,0,2). Karg-Elert provides no examples of the Konkordant transpositions in either of his treatises; they will surely
be limited to cases where the common concordant seventh is emphasized in some way, as in Figure 5.4.3b. Notably, Karg-Elert also does not propose any functional notation for the \textit{Konkordant} transpositions. This omission suggests that Karg-Elert faced challenges incorporating the new septimal-axis chords into the theory of the three functions, which is based in the fifth/third space.

The secondary seventh-transpositions (\(T_{v \rightarrow vii}\) and \(T_{vii \rightarrow v}\)) are even more scarce; Karg-Elert does not even provide a name, let alone any examples in his treatises. Under \(T_{v \rightarrow vii}\), the new root, third and fifth are septimal (seventh-axis) pitches, while the new seventh is a fifth/third-based pitch, as it was the original chord’s fifth. \(T_{v \rightarrow vii}\) also produces a metharmonic pair (underlined), between the original chord’s third and the new chord’s fifth:

\[
T_{v \rightarrow vii} \quad [C(0) \ E_{(0,1)} \ G_{(1,0)} \ Bb_{(0,0,1)}] \quad \rightarrow \quad [A_{(1,0,-1)} \ C_{(1,1,-1)} \ E_{(2,0,-1)} \ G_{(1,0)}]
\]

Under \(T_{vii \rightarrow v}\), the new root, third and fifth are septimal pitches, while the new seventh is a double-septimal pitch. It also produces a metharmonic pair, between the original chord’s fifth and the new chord’s third:

\[
T_{vii \rightarrow v} \quad [C(0) \ E_{(0,1)} \ G_{(1,0)} \ Bb_{(0,0,1)}] \quad \rightarrow \quad [E_{b(-1,0,1)} \ G_{(0,1,1)} \ Bb_{(0,0,1)} \ D_{b(-1,0,2)}]
\]

The following Tonnetze (Figure 5.4.4) illustrate \(T_{v \rightarrow vii}\) and \(T_{vii \rightarrow v}\) from C major:

![Figure 5.4.4. \(T_{v \rightarrow vii}\) and \(T_{vii \rightarrow v}\) transpositions from C major](image-url)
The rarity of $T_v\to vii$ and $T_{vii}\to v$ in musical practice is once again because they metharmonically duplicate simpler fifth- and third-based transformations, in this case the auxiliary mediant (minor third) transpositions. Both $T_v\to vii$ and the counter-auxiliary-mediant ($T_{iii}\to v$) are notated as $[C7 \to A7]$, but the roots of the A7 chords differ by a septimal comma: $A_{(1,0)}$ for $T_{iii}\to v$ versus $A_{(1,0,-1)}$ for $T_v\to vii$. The most plausible manifestation of $T_v\to vii$ would place the common tone in one of the outer voices, making the common-tone connection clearly audible. As with the Konkordants, Karg-Elert does not provide functional notation for $T_v\to vii$ or $T_{vii}\to v$.

The final pair of seventh-based transpositions are called the tritonants ($Tritonanten$), because they transpose concordant seventh chords by a diminished fifth.71 Under $T_{iii}\to vii$, the new root, third and fifth are septimal (seventh-axis) pitches, while the new seventh is a fifth/third-based pitch, as it was the original chord’s third. It also produces an enharmonic pair (italicized), between the original chord’s seventh and the new chord’s third:

$$T_{iii}\to vii \ [C(0) \ E_{(0,1)} \ G_{(1,0)} \ B_{b_{(0,0,1)}}] \ \to \ \ [F^\#_{(0,1,-1)} \ A^\#_{(0,2,-1)} \ C^\#_{(1,1,-1)} \ E_{(0,1)}]$$

Under $T_{vii}\to iii$, the new root, third and fifth are septimal pitches, while the new seventh is a double septimal pitch. It also produces an enharmonic pair, between the original chord’s third and the new chord’s seventh:

$$T_{vii}\to iii \ [C(0) \ E_{(0,1)} \ G_{(1,0)} \ B_{b_{(0,0,1)}}] \ \to \ \ [G_{b_{(-1,0,1)}} \ B_{b_{(0,0,1)}} \ D_{b_{(0,0,1)}} \ F_{b_{(-1,0,2)}}]$$

---

71 As will be discussed in section 5.5, Karg-Elert also uses the term Tritonant in a less specific manner, to describe any augmented-fourth transposition between two triads or concordant sevenths of the same quality, regardless of pitch-space locations or functional meaning.
The following Tonnetz (Figure 5.4.5) illustrates the Tritonant transpositions from C major:

![Tritonant Transpositions Diagram](image)

**Figure 5.4.5.** The Tritonant transpositions, from C major

In the chart (Figure 5.4.1), Karg-Elert notes that the tritons are “understood much more simply as ultradominant mediants, or as ultraconrant counter-mediants.” While the ‘simplicity’ of the latter transformations is debatable, they are fifth- and third-based, and thus operate within the normal functional pitch space. On the other hand, a passage might emphasize the common tone in a manner that overpowers the tritone root motion. Though Karg-Elert’s treatises do not include any examples of the Tritonanten, a celebrated case is the oft-repeated tritone oscillation between A♭7 and D7 in the Coronation Scene of Mussorgsky’s *Boris Godunov* (Figure 5.4.6):

![Mussorgsky Example](image)

**Figure 5.4.6.** Mussorgsky, *Boris Godunov* (ed. Rimsky-Korsakov): Prologue, scene II (m. 1-7)
The tonal centre of the passage is surely C, owing to its repetition in the bass, its registral placement in the upper and lower voices of the Ab7 and D7 chords (beginning in measure 3), and its rhythmic placement at the beginning of each four-note woodwind figure (measure 7). C major is also the principal key of Scene II as a whole. If C major is the tonic, the Ab7 and D7 chords can be labelled functionally as $T_M$ and $DD$ respectively (with added concordant sevenths). The transformation $T_M \rightarrow DD$ is the ultradominant-mediant: two fifths above plus one third above, as the acoustic data in Figure 5.4.1 indicates. But that interpretation is problematic and unconvincing for the Mussorgsky passage, as $T_M$ and $DD$ do not share any acoustic common tones: their C’s are in fact metharmonics, differing by a septimal comma:

$$
T_M = \begin{array}{c} \text{Ab}(0,-1) \ C(0) \ Eb(-1,-1) \ Gb(0,-1,1) \end{array}
\quad DD = \begin{array}{c} D(2,0) \ F#(2,1) \ A(3,0) \ C(2,0,1) \end{array}
$$

If the tonic pitch C(0) is to be understood as a true common tone in the Mussorgsky, then Ab7 $\rightarrow$ D7 must be a $T_{iii\rightarrowvii}$ transposition, with the following acoustic locations:

$$
\text{Ab7} = \begin{array}{c} \text{Ab}(0,-1) \ C(0) \ Eb(-1,-1) \ Gb(0,-1,1) \end{array}
\quad D7 = \begin{array}{c} D(0,0,-1) \ F#(0,1,-1) \ A(1,0,-1) \ C(0) \end{array}
$$

The Tonnetze in Figure 5.4.7 contrasts the ultracontrant-mediant and $T_{iii\rightarrowvii}$ interpretations of Ab7 $\rightarrow$ D7, and illustrates how only the latter transformation retains an acoustic common tone.
Figure 5.4.7. Two interpretations of the progression [Ab7 → D7], from Mussorgsky (Fig. 5.4.6)

In the chart of seventh-based transpositions (Figure 5.4.1), the Tritonanten are given function labels (Figure 5.4.8):

The “K” stands for Konkordant, which here refers to the concordant seventh interval (vii). It faces in the same direction as the function label, because the transformation retains chord quality (i.e. it is a transposition, not an inversion). The placement of the horizontal line near the function label is significant, as it indicates which note of the original tonic (T) is retained in the transposition. For the F#7 chord, the line at the middle of the T shows that the chord contains the original tonic’s third (iii); on the Gb7 chord, the line above the T shows that the chord contains the original tonic’s seventh (vii). This type of notation is very specific, and might have been applied to the other seventh-based transpositions. However, it is unwieldy and difficult to use, and it never appears anywhere else in Karg-Elert’s treatises. Indeed, he devoted very little attention to the seventh-based transpositions as a whole, beyond listing them in the chart of transformations.
b. The **Septgegenklang**

The following (Figure 5.4.9) demonstrates the first of the seventh-based inversions: the *Septgegenklang*, described by Karg-Elert as “the most important seventh relationship.”

![Diagram ofSeptgegenklang](image)

*Figure 5.4.9. The Septgegenklang (Harmonologik, 53)*

The primary seventh-inversion $I_{vii}^l$ is aptly called the *Septgegenklang* or “seventh-counter-chord,” as it inverts a concordant seventh chord around its outer chord tones, the prime and concordant seventh (i and vii). In Chapter 14 of *Harmonologik*, Karg-Elert also calls it *Gegenkonkordant* or counter-concordant, analogous to the *Gegenklang* (see Figure 5.2.1). Due to the pair of acoustic common tones, $I_{vii}^l$ is usually a function-retaining operation. As shown in Figure 5.4.10 (next page), Karg-Elert’s notation for $I_{vii}^l$ attaches the letter $K$ (for *Konkordant*) to the function label, facing in the opposite direction from the label to indicate the change of quality (except in the case of the contrant-variant). The $K$ is attached to the line above or below the function label, to denote the retention of the original chord’s concordant seventh; in turn, another...
horizontal line is added below or above the $K$, to indicate that the original chord’s prime is now
the new seventh. In the figure, the numerals indicate chord-tone transformations:

![Diagram](image)

**Figure 5.4.10.** *Septgegenklänge* of the basic functions in C major and A minor (*Harm*, 309)

The *Septgegenklang* transformation shifts all but one of the original chord’s pitches into the
septimal plane, as illustrated by the acoustic data for $\overline{T} \rightarrow \overline{T} \mathbf{K}$ in C major. The two common
tones are printed in boldface:

$$\overline{T} = \begin{array}{c}
C(0) \ E_{(0,1)} \ G_{(1,0)} \ B_{(0,0,1)} \\
\end{array}$$

$$\overline{T} \mathbf{K} = \begin{array}{c}
C(0) \ E_{(-1,0,1)} \ G_{(0,-1,1)} \ B_{(0,0,1)} \\
\end{array}$$

The following Tonnetz (**Figure 5.4.11**) demonstrates the *Septgegenklang* transformations of the
tonic in C major and A minor:

![Diagram](image)

**Figure 5.4.11.** *Septgegenklänge* ($I^n_{\text{vil}}$) in C major and A minor

Karg-Elert found the *Septgegenklang* to be by far the most important and analytically
fruitful of the seventh transformations; indeed, the discussion of seventh relations in Chapter 14
of *Harmonologik* focuses entirely on the *Septgegenklang*. Its importance lies in its ability to
“greatly simplify the harmonic analysis of complex cases” involving concordant sevenths.\textsuperscript{72} He describes in further detail: “the counter-concordants often enter as vastly simpler substitutes for the ‘ultracontrant-variants’, ‘counter-median-variants’ and ‘variant-parallel-variants’ (!).\textsuperscript{73} One might question how a transformation involving the septimal axis can be in any way simpler than even multi-step ones that remain in the fifth/third pitch space, such as those listed in the latter quotation. The key to understanding what Karg-Elert means lies again in the connecting power of acoustic common tones. Figure 5.4.12 highlights how the Septgegenklang retains the prime of a function’s basic triad; as a result, pairings of dominant and half-diminished seventh chords can be heard as “latent cadences,” in simple $T : C$ or $T : D$ relationships. Each cadential progression is first shown in its unaltered form (in closed noteheads), and then with one of the chords in its Septgegenklang transformation, as indicated in the functional analysis.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.4.12.png}
\caption{Septgegenklang transformations in cadential progressions (Harm, 309)}
\end{figure}

At the bottom of the figure, Karg-Elert specifies how the Septgegenklang progressions might be understood alternatively (“\textit{statt}”), using combinations of mediant and variant transformations. While the latter remain in the fifth/third space, they do not retain any common-tone connections.

\textsuperscript{72} Harmonologik, 309.
\textsuperscript{73} Ibid.
with the original functional triads, and thus negate the latent sense of cadence that is strongly suggested by the bass motions. The following diagram (Figure 5.4.13) illustrates the differences in pitch-space connection and distance between the three versions of [CM $\rightarrow$ Gø7] cited in Figure 5.4.12, graphically demonstrating how the Septgegenklang enables a “simpler,” more direct understanding of the progression.

![Diagram](image)

**Figure 5.4.13.** Three functional interpretations of [C7 $\rightarrow$ Gø7], from Figure 5.4.12

Not surprisingly, Karg-Elert finds several examples of the Septgegenklänge in the music of Wagner, which abounds in dominant and half-diminished seventh chords. The following discussion of passages in *Tristan und Isolde* (Figure 5.4.14, next page) is worth citing in full, due to its clear explanation and contrast of fifth, third and seventh transformations, and its characterization of the Septgegenklang as a dramatic symbol of life and death (*Leben und Tod*):
Figure 5.4.14. Discussion of *Septgegenklänge* in Wagner’s *Tristan und Isolde* (Harm, 54)

The following pages provide an English translation of Figure 5.4.14, with Tonnetze in place of the musical examples:

[CM → Ebm] can be traversed as follows through first-order steps [i.e. fifth-relations] – what a journey!
Steps of the second order [i.e. third-relations] shorten the path:

![Diagram](image)

But Wagner saw another way, that he wrote – no, had to write – at the following moment in *Tristan* (last act):

**Tristan:** “The light expires!...

**Isolde!** [he dies]

What perception! The flame is upended, the symbol of death. This Eb minor chord [i.e. Cø7] can be nothing other than a *Septgegenklang*. It is not a “chord progression” or a “harmonic connection” in the normal sense; it is a symbol!

Some measures later [12 bars later, to be exact], the same idea recurs [in a passage in G minor]:
Isolde: “Isolde has come, with faithful Tristan to die”  (Life and Death)

Gm: \( \dd{c} \) \( \dd{cK} \)

Figure 5.4.14. Tristan analysis (English version), with Tonnetze

A notable feature of the Septgegenklang transformation is that the concordant seventh can be omitted from either the source chord or the transformed one, or even from both chords. In the first passage from Tristan in Figure 5.4.14, the source chord is simply C major, not C7. The underlying seventh-based transformation is still in effect, even though the seventh is not actually sounded in the music. The following passage from Wagner's Parsifal (Figure 5.4.15) features a descending fifths sequence from G\( ^\# \) (Gis) to the tonic D major; the sequence is made explicitly audible in the bass, supported by a 7-10 soprano/bass linear intervallic pattern.

Figure 5.4.15. Wagner, Parsifal – Septgegenklänge in a descending fifths sequence (Harm, 310)
Karg-Elert’s annotation under the functional analysis notes that each chord could simply be a major triad (*Dur Akk.*) or dominant seventh, creating a cycle of secondary dominants. Instead, Wagner replaces the first major triad/dominant seventh in each pair with its *Septgegenklang*, as indicated by the function labels, and also marked “x” under the analysis. The second chord in each pair is a simple triad, without its concordant seventh. The Tonnetz for the passage (Figure 5.4.16) illustrates how the passage descends “horizontally” along the chain of fifths (i.e. the source position triads), but replaces some of the fifths with their seventh-counter-chords.\(^\text{74}\)

![Figure 5.4.16. Tonnetz for Wagner, Parsifal passage (Figure 5.4.15)](image)

Reflecting how the music of Debussy was strongly influenced by that of Wagner, Karg-Elert directly follows the *Tristan* and *Parsifal* passages with two examples of *Septgegenklang* transformations in Debussy’s *Prélude à l’après-midi d’un faune*. The first example (Figure 5.4.17, next page) contains three short excerpts, from bars 3-5 at the beginning of the work (excerpt A), from two bars after rehearsal 2 (excerpt B), and from seven bars after rehearsal 1 (excerpt C). The work as a whole is in E major, and Karg-Elert’s functional analysis of excerpts B and C are in that key.

\(^{74}\) Childs 1998 studies an almost identical passage from *Parsifal*, ending on D-flat major; see his Example 1 and Figure 1.
Karg-Elert does not provide a functional interpretation of the opening (excerpt A); this may be because the first chord to be heard is not the tonic triad in E major, but rather A♭ø7 (i.e. C♭ minor with concordant seventh below). Indeed, one might argue that the opening (considered in isolation) suggests C♭ minor quite strongly, as the flute solo begins on C♭5, and reiterates that pitch frequently. Therefore, the C♭m/A♭ø7 chord is here interpreted as tonic (I) in C♭ minor, with concordant seventh below. The following chord is notated as B♭7, but Karg-Elert notes that it is in fact A♭7 (Ais dur statt B dur), which is a direct Septgegenklang transformation of A♭ø7. From that perspective, the chordal pair in excerpt A maintains tonic function (I → I[K]), through the retention of the common tones G♯ and A♯ (renotated enharmonically in the second chord).

The progression in excerpt B [Eø7 → B9] is a simple motion from tonic to the dominant ninth (T → D), but the tonic is replaced by its Septgegenklang (T[K]). Excerpt C is more complex,
featuring three concordant seventh chords: $[G^#7 \rightarrow Bø7 \rightarrow A^#7]$. In context, the passage does not clearly express the tonic key of E major; C# major is more likely, though the passage’s sense of key is fluid and ambiguous. Nonetheless, Karg-Elert orients his functional analysis in the global key of E major, which can accommodate all three chords. The initial chord $[G^#7]$ is labelled as tonic mediant ($T^M$) in E major, even though in context it sounds more like a dominant of C# major. The choice of E major as tonic reference is likely motivated by the second chord [Bø7], which is understood as the Septgegenklang of B7, the dominant of E major. The final chord [A^#7] is the ultradominant-mediant ($BD^M$) in E major – quite far removed from the tonic, but still within its functional orbit. **Figure 5.4.18** provides Tonnetze for all three excerpts.

![Figure 5.4.18](image-url)
In Figure 5.4.17, all three excerpts involved complete seventh chords. In contrast, the following passage (Figure 5.4.19) near the end of *L’après-midi d’un faune* features only triads:

![Musical notation](image)

### Tonnetz (Fig. 5.4.21):

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
</table>

**Figure 5.4.19.** *Septgegenklang* transformations among triads (*Harmonologik*, 310)

The basic progression is a pattern of whole tones to and from the tonic E major, sounded on the main beats, and indicated by the boxes on Figure 5.4.20: [EM → DM → CM → DM → EM]. In functional terms, it is a descent from tonic to the double ultracontrant and back: $T \rightarrow CC \rightarrow (CC) \rightarrow CC \rightarrow T$. Three of the intervening weak-beat chords are minor triads, but they are understood as incomplete half-diminished sevenths. The two C minor chords are analyzed as incomplete AØ7 chords, and are thus labelled as *Septgegenklang* transformations of A7 (the dominant of DM). Likewise, the B♭ minor triad is an incomplete GØ7 chord, the *Septgegenklang* of G7 (the dominant of CM). The passage is rewritten in **Figure 5.4.20**, which replaces the seventh-counter-chords with their untransformed versions; Karg-Elert’s annotation acts as a quasi-Ramellian *basse fondamentale* (shown in the bass staff), revealing that the first five chords form a chromaticized descending fifths sequence, much like the *Parsifal* passage in Figure 5.4.15:
Finally, Figure 5.4.20 outlines the original passage on a Tonnetz, demonstrating its underlying “horizontal” motion along the chain of fifths (again, much like the Parsifal example).

Figure 5.4.20. Rewritten version of Figure 5.4.19, reversing the Septgegenklänge

All of the preceding examples illustrate the directness and analytical utility of the Septgegenklang transformation, not only in passages involving dominant and half-diminished seventh chords, but also in those involving triads that do not clearly relate to each other diatonically, or even as mediants and countermediants. Based on its possibilities, it is not surprising that the discussion of seventh-based transformations in both of Karg-Elert’s treatises focuses almost entirely on the Septgegenklang.
c. Other seventh-based inversions

In contrast to the Septgegenklang, Karg-Elert says very little about the remaining seventh-inversions, beyond listing them in the chart as follows (Figure 5.4.22):

**Secondary:**
Dominant seventh-counter-chords

\[
\begin{align*}
[C7 \leftrightarrow Gø7] & \\
7 & \leftrightarrow 5 \\
5 & \leftrightarrow 7 \\
\end{align*}
\]

\[I_v^{\text{vii}}\]

Simpler as ultracontrant-variant
Simpler as dominant-variant-parallel-variant
Very useful linking chord between \(T - \overline{D} / T - D\)

**Tertiary [no name]:**

\[
\begin{align*}
[C7 \leftrightarrow Eø7] & \\
7 & \leftrightarrow 3 \\
3 & \leftrightarrow 7 \\
\end{align*}
\]

\[I_v^{\text{vii}}\]

Simpler as Quintgleicher \((I_v^{v})\)

**Quaternary:**
Septgleicher (“same seventh”)

\[
\begin{align*}
[C7 \leftrightarrow Bbø7] & \\
7 & \leftrightarrow 3 \\
\end{align*}
\]

\[I_v^{\text{vii}}\]

The last and purest consequence of seventh-relations

**Figure 5.4.22.** Seventh-based inversions \((Harmonologik, 53–54)\)
The secondary inversion $I_{vii}^v$ (called the “dominant seventh-counter-chord” or Dominantenseptgegenklang) denotes a quite common relationship between two triads or concordant sevenths: $[C7 \rightarrow Gø7]$. However, the two chords are much less likely to be understood as direct transformations of each other, but rather as representatives of two different functions, as in the progression $\overline{T} \rightarrow \overline{DX}$ (see the first example in Figure 5.4.14). Whenever possible, Karg-Elert prefers to label seventh-transformations as Septgegenklänge, and change function as needed, rather than use one of the other seventh-based inversions. The chart notes that $I_{vii}^v$ closely resembles the ultracontrant-variant ($Ec$), and also a putative “dominant-variant-parallel-variant” ($d^p$, or neo-Riemannian LRPRP). All are notated as $[C7 \rightarrow Gø7]$ or $[F^\#7 \rightarrow B7]$, but their products differ metharmonically from each other, as shown by the following acoustic data (in the key of C major):

$$
\begin{align*}
T &= C(0) \quad \text{E}_{(0,1)} \quad \text{G}_{(1,0)} \quad \text{Bb}_{(0,0,1)} \\
I_{vii}^v &= \text{G}_{(1,0)} \quad \text{Bb}_{(0,0,1)} \quad \text{Db}_{(1,-1,1)} \quad \text{F}_{(1,0,1)} \\
Ec &= \text{G}_{(-1,0,-1)} \quad \text{Bb}_{(-2,0)} \quad \text{Db}_{(-1,-1)} \quad \text{F}_{(-1,0)} \\
d^p &= \text{G}_{(3,-1,-1)} \quad \text{Bb}_{(2,-1)} \quad \text{Db}_{(3,-2)} \quad \text{F}_{(3,-1)}
\end{align*}
$$

(this chord is also $D^r$ in C major)

Figure 5.4.23 illustrates the different pitch-space trajectories of the above three transformations:

![Diagram](image)

Figure 5.4.23. Comparison of $I_{vii}^v$ and two metharmonic counterparts

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Karg-Elert does not provide any repertoire or analytical examples of \( I_v^{\text{ll}} \), though the chart states that it can be a “very useful linking chord between \( T - D / L - A \).” The following example (Figure 5.4.24) tests that idea, in C major:

![Figure 5.4.24. \( I_v^{\text{ll}} \) of tonic as a linking chord to the dominant]

As described above, the Gø7 chord in the figure can be analyzed as the Septgegenklang of the dominant (G7), and can be labelled as such (\( D\chi \)). Figure 5.4.24 demonstrates a case in which the Gø7 chord may be better understood as a \( I_v^{\text{ll}} \) transformation of the tonic, due especially to its purpose as an approach to the dominant, and also to its weak-beat rhythmic placement. One might argue that the spelling of the chromatic pitches detracts from the purported “leading” function of the chord: in standard tonal practice, B♭ and D♭ do not normally lead upward to B-natural and D-natural.\(^{75}\) However, the chord’s ability to link tonic and dominant is illustrated vividly on the following Tonnetz in C major (Figure 5.4.25, next page), which highlights how all three chords in Figure 5.4.24 share \( G(1,0) \) as an acoustic common tone:

---

\(^{75}\) On the other hand, one can also argue that because the flats and sharps of standard 12-TET-based notation can only approximate the seventh-derived pitches in the second chord of Figure 5.4.24, they should not be expected to resolve in the traditional manner.
The tertiary seventh-based inversion $I^{\text{III}}_{\text{vII}}$ does not have a name, and is not demonstrated in any examples in the treatises. In C major and A minor, it is $[C7 \leftrightarrow E\varnothing7]$ and $[F\#\varnothing7 \leftrightarrow D7]$ respectively. As Karg-Elert’s chart notes, $I^{\text{III}}_{\text{vII}}$ is meharmonically equivalent with the Quintgleicher ($I^v_v$), discussed above in section 5.2. As a result, most instances of the above chord pairings will be understood as the simpler fifth/third-based Quintgleicher transformation, except possibly when the common tones $iii$ or $vii$ are emphasized in some way in a passage. The different pitch-space trajectories of $I^{\text{III}}_{\text{vII}}$ and $I^v_v$ are specified in the following acoustic data in C major, and plotted on the Tonnetz (Figure 5.4.26):

\[
T \text{ in C major} = \begin{array}{cccc}
C(0) & E(0,1) & G(1,0) & B_b(0,0,1) \\
E(0,1) & G(-1,0,1) & B_b(0,0,1) & D(0,1,1) \\
E(2,0,-1) & G(1,0) & B_b(2,-1) & D(2,0) \\
\end{array}
\]

\[
I^{\text{III}}_{\text{vII}} = \begin{array}{cccc}
E(0,1) & G(-1,0,1) & B_b(0,0,1) & D(0,1,1) \\
E(2,0,-1) & G(1,0) & B_b(2,-1) & D(2,0) \\
\end{array}
\]

\[
I^v_v \text{ (Quintgleicher)} = \begin{array}{cccc}
E(0,1) & G(-1,0,1) & B_b(0,0,1) & D(0,1,1) \\
E(2,0,-1) & G(1,0) & B_b(2,-1) & D(2,0) \\
\end{array}
\]

Figure 5.4.25. Tonnetz for Figure 5.4.24

Figure 5.4.26. Comparison of $I^{\text{III}}_{\text{vII}}$ and $I^v_v$, in C major
The final seventh-based inversion is $I_{vii}^{vii}$, which is called the *Septgleicher* (“same seventh”) as it inverts around a common concordant seventh. Karg-Elert states that it is “the last and purest consequence of the seventh relations,” because it does not imitate any fifth- or third-based transformation, except through enharmonic equivalence. In C major and A minor, $I_{vii}^{vii}$ is $[C7 \leftrightarrow Bbø7]$ and $[F#ø7 \leftrightarrow G#7]$ respectively. The products of $I_{vii}^{vii}$ are quite remote from the original chord, as they contain three pitches on the second septimal plane (i.e. they are separated from the fifth-third plane by two septimal commas). The following acoustic data and Tonnetz (Figure 5.4.27) specifies the pitch-space distance traversed by $I_{vii}^{vii}$, in C major:

\[
\begin{align*}
T \text{ in C major} &= \quad C(0) \quad E_{(0,1)} \quad G_{(1,0)} \quad B_{b(0,0,1)} \\
I_{vii}^{vii} (Septgleicher) &= \quad B_{b(0,0,1)} \quad D_{b(-1,0,2)} \quad F_{b(0,-1,2)} \quad A_{b(0,0,2)}
\end{align*}
\]

**Figure 5.4.27.** $I_{vii}^{vii} (Septgleicher)$ transformation of C major

Though Karg-Elert provides no repertoire examples of the *Septgleicher*, there are two abstract progressions in *Akustische* 10.7. The first example (Figure 5.4.28) begins with the juxtaposition of Aø7 and G7, respectively the tonic variant (t) and dominant (D) in C major. This pairing is an $I_1^t$ or *Gegenklang* transformation, inverting around the common dual prime $G_{(1,0)}$. Then, both chords are converted into their *Septgegenklänge*, flipping the chord qualities: [Aø7 $\rightarrow$ A7] and
[G7 → Gø7]. The resulting pairing [Gø7 : A7] is a Septgleicher or $I_{VII}^\text{aug}$ relation, as the common tone $G_{(1,0)}$ is now the concordant seventh of both chords.

![Gø7 as common tone](image)

**Figure 5.4.28.** A Septgleicher ($I_{VII}^\text{aug}$) relation arising through Septgegenklänge

The following Tonnetze (Figure 5.4.29) demonstrate the chord pairs in the preceding example:

<table>
<thead>
<tr>
<th>Gegenklang ($I_1$): [Aø7 → G7]</th>
<th>Septgleicher ($I_{VII}^\text{aug}$): [Gø7 → A7]</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Gegenklang diagram]</td>
<td>![Septgleicher diagram]</td>
</tr>
</tbody>
</table>

**Figure 5.4.29.** Tonnetze for Figure 5.4.28

The other example of the Septgleicher in Akustische (Figure 5.4.30) reflects how Karg-Elert sometimes relaxed his fastidious observation of comma differences, in order to explain certain harmonic juxtapositions. The pairing of [F♯7 → Eø7] is analyzed in A major (see Akustische 10.7 for the complete passage, which is quite complex). While both chords have E as
their concordant seventh, the acoustic symbols specify that the E’s differ from each other by 41 μ; see also the Tonnetz in Figure 5.4.31, which represents the acoustic data. Karg-Elert thus calls the harmonic relationship a *metharmonische Septgleicher* – not a true $I^{vi}$ transformation due to the lack of acoustic common tones, but mimicking it through metharmonic identification of the concordant sevenths.

![Figure 5.4.30](image1)

**Figure 5.4.30.** A metharmonic *Septgleicher* – similar to $I^{vi}$, but no acoustic common tones

![Figure 5.4.31](image2)

**Figure 5.4.31.** Tonnetz for Figure 5.4.30

When metharmonic differences are disregarded as in Figure 5.4.29, the resulting harmonic relationships can no longer be regarded as true transformations, as defined by Karg-Elert in terms of acoustic common-tone retention. Section 5.5 will discuss other harmonic relations that do not retain common tones, some of which have the same names as the common-tone transformations.
d. Seventh transformations in Karg-Elert and in the neo-Riemannian literature

Since the mid 1990s, a number of American theorists have proposed transformations among dominant and half-diminished seventh chords (both within each chord type, and between the chord types). With one exception\textsuperscript{76} to be discussed below, most of their transformations are defined in terms of parsimonious voice leading: the number of common tones, and the total amount of semitone motion. Much of this work is built on a concept presented in David Lewin’s 1996 article “Cohn Functions,” itself based on work by Jack Douthett.\textsuperscript{77} Lewin proposes two relations called DOUTH1 and DOUTH2. The former connects two sets X and Y “if Y can be obtained by discarding some member pitch class x of X, and then picking up some new pitch class y one semitone away from the discarded x.”\textsuperscript{78} Neo-Riemannian P and L are examples of DOUTH1, as they retain two pitches from a triad, and the remaining voice moves by a semitone; R is not DOUTH1, as the non-stationary voice moves by a whole tone. DOUTH1 is also found between half-diminished/dominant seventh chords and minor-minor seventh chords, as in \([\text{Aø7} \rightarrow \text{Am7}]\) or \([\text{Am7} \rightarrow \text{A7}]\), but not directly among \((0258)\) chords.\textsuperscript{79} More applicable to the latter seventh chords is DOUTH2, in which set “Y can be obtained by discarding two member pitch classes \(x_1\) and \(x_2\) of X, and then picking up two new pitch classes \(y_1\) and \(y_2\), where \(y_1\) lies one semitone away from the discarded \(x_1\) and \(y_2\) lies one semitone away from the discarded \(x_2\).”\textsuperscript{80}

The cycles of dominant and half-diminished seventh chords contained within each octatonic

\textsuperscript{76} Gollin 1998.
\textsuperscript{77} Lewin 1996.
\textsuperscript{78} Lewin 1996, 206.
\textsuperscript{79} Many of the networks in Douthett and Steinbach 1998 connect (0258) tetrachords with minor-minor seventh chords.
\textsuperscript{80} Lewin 1996, 207.
collection are DOUTH2-related, as in for example \([C\#7 \rightarrow E7 \rightarrow G7 \rightarrow Bb7]\), and \([A\flat7 \rightarrow C\flat7 \rightarrow D\#\flat7 \rightarrow F\#\flat7]\). In addition, each dominant or half-diminished seventh chord is DOUTH2-related to eight chords of the opposite type. For example, \(F\flat7\) (an enharmonic respelling of the famed Tristan chord) is DOUTH2-related with \(E7, F7, G7, Ab7, Bb7, B7, C\#7\) and \(D7\).\(^{81}\) Of course, the DOUTH2 relation assumes 12-TET and enharmonic equivalence: many of the chord pairings just listed involve enharmonic common tones (of different names), and no consideration is given to possible metharmonic (comma) differences between common tones of the same name.

Transformations among (0258) seventh chords proposed by Adrian Childs\(^{82}\) and Richard Bass\(^{83}\) closely resemble the DOUTH2 relation, as they likewise involve the retention of two common tones (again assuming 12-TET and enharmonic equivalence), and the semitone motion of the other two pitch classes, creating a total displacement of two semitones, or what Jack Douthett termed a \(P_2\) relation.\(^{84}\) The more comprehensive system is that of Childs, which includes both transformations among dominant and half-diminished seventh chords separately, and also between the two chord types. He proposes two types of transformation. \(S_m(n)\) changes chord quality, and is thus an inversion, and an involution (it is its own inverse). The two pitch classes that create interval class \(m\) are held in common, while the two pitch classes that form interval class \(n\) move by semitone in similar (actually parallel) motion. Childs’ other transformation \(C_{m(n)}\) retains chord quality, and is thus a transposition; it is not involutional, except for the tritone transposition \(C_6(5)\). The two pitch classes that create interval class \(m\) are

\(^{81}\) Lewin 1996, 207. See also Gauldin 2001, which develops out of Lewin’s discussion of DOUTH2 and the Tristan chord.

\(^{82}\) Childs 1998.

\(^{83}\) Bass 2001.

\(^{84}\) Lewin 1996, 206. In Douthett and Steinbach 1998, the \(P_2\) relation is revised, becoming the \(P_{m,n}\) relation, in which \(m\) and \(n\) refer to the number of half-step and whole-step motions.
again held in common, while the two pitch classes that form interval class n move by semitone, but now in contrary motion. Figure 5.4.32 reproduces Childs’ Figure 5, which illustrates all nine of the $S_{m(n)}$ and $C_{m(n)}$ transformations among (0258) seventh chords:

![Figure 5. A system of transformations for dominant and half-diminished seventh chords (set class 4-27). + and – refer to dominant and half-diminished qualities, respectively. F+ and F– are taken as the initial chords in each example. Notes which are held constant have open noteheads, while those that move are represented by filled-in noteheads.](image)

**Figure 5.4.32.** Childs 1998, 186 – transformations among (0258) seventh chords

Childs notes that “these nine transformations represent all of the possible $P_2$-relations among individual members of set class 4-27 [(0258)].” As shown in the figure, a few of the retained common tones (notated as open noteheads) are actually enharmonic equivalents.

Richard Bass’ transformations nominally operate only among half-diminished seventh chords, and so they are transpositions that retain chord quality. Once again, two pitch classes are held in common; his labels specify the interval class of the two moving pitch classes, plus the interval class to which they move. His first transformation is called ic 4-2, in which the two pitch classes that form interval class 4 converge inward by semitones to interval class 2, as in $[F_{ø7} \rightarrow D_{ø7}]: F$ and $A_b$ are retained, and the moving voices are $[C_b \rightarrow C]$ and $[E_b \rightarrow D]$. Bass’ other transformation is ic 5-5, in which the two pitch classes that form interval class 5 move by semitones in contrary motion to a different interval class 5 (in other words, from a perfect fifth to

---

85 Childs 1998, 185.

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a perfect fourth, or vice versa). An example is \([Fø7 \rightarrow Bø7]\), in which F and C\(_b\)/B are retained, and the moving voices are \([A_b \rightarrow A]\) and \([E_b \rightarrow D]\). The inverse operation to ic 4-2 is simply ic 2-4, while ic 5-5 is an involution. Bass’ ic 4-2 is equivalent to Childs’ C\(_{3(4)}\), while ic 5-5 is equivalent to C\(_{6(5)}\). Though Bass explicitly limits his focus to half-diminished seventh chords, his two transformations can also connect dominant sevenths to each other: \([C7 \rightarrow E_b7]\) is ic 4-2, \([C7 \rightarrow A7]\) is ic 2-4, and \([C7 \rightarrow F\#7]\) is ic 5-5.

Childs’ six \(S_m(n)\) transformations can be equated directly with six of Karg-Elert’s inversions that retain two common tones; three of them are equivalent to neo-Riemannian \(L\), \(R\) and \(P\), with the addition of sevenths above or below the triad. Childs’ \(C_m(n)\) transformations correspond to three of Karg-Elert’s seventh-based transpositions, but both metharmonic and enharmonic equivalence must be assumed. The following chart (Figure 5.4.33, next page) compares the seventh-chord transformations in Childs 1998 and Bass 2001 with their counterparts in Karg-Elert.
<table>
<thead>
<tr>
<th>Childs 1998</th>
<th>Bass 2001</th>
<th>Karg-Elert</th>
<th>Cohn</th>
<th>From C7</th>
<th>From F#7</th>
</tr>
</thead>
<tbody>
<tr>
<td>S(2)</td>
<td>n/a</td>
<td>I^vii_{vii} (Septgegenklang)</td>
<td></td>
<td>[C7 → Cø7]</td>
<td>[F#ø7 → F#7]</td>
</tr>
<tr>
<td>S(3)</td>
<td>n/a</td>
<td>I^vii_{vii} (Leittonwechsel)</td>
<td>L</td>
<td>[C7 → Cø7]</td>
<td>[F#ø7 → F7]</td>
</tr>
<tr>
<td>S(4)</td>
<td>n/a</td>
<td>I^vii_{vii}</td>
<td></td>
<td>[C7 → Gø7]</td>
<td>[F#ø7 → B7]</td>
</tr>
<tr>
<td>S(5)</td>
<td>n/a</td>
<td>I^vii_{vii} (Parallel)</td>
<td>R</td>
<td>[C7 → F#ø7]</td>
<td>[F#ø7 → C7]</td>
</tr>
<tr>
<td>S(6)</td>
<td>n/a</td>
<td>I^vii_{vii} (Variant)</td>
<td>P</td>
<td>[C7 → Aø7]</td>
<td>[F#ø7 → A7]</td>
</tr>
<tr>
<td>S(7)</td>
<td>n/a</td>
<td>I^vii_{vii} (Septgleicher)</td>
<td></td>
<td>[C7 → Bøø7]</td>
<td>[F#ø7 → G#7]</td>
</tr>
<tr>
<td>C(2)</td>
<td>IC 2-4</td>
<td>T_{vii} → vii</td>
<td></td>
<td>[C7 → A7]</td>
<td>[F#ø7 → Aø7]</td>
</tr>
<tr>
<td>C(4)</td>
<td>IC 4-2</td>
<td>T_{vii} → v</td>
<td></td>
<td>[C7 → Eø7]</td>
<td>[F#ø7 → Døø7]</td>
</tr>
<tr>
<td>C(5)</td>
<td>IC 5-5</td>
<td>T_{vii} → vii, enharmonically (Tritonanten)</td>
<td>[C7 → F#ø7]</td>
<td>[F#ø7 → Cø7]</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>[C7 → Gøø7]</td>
<td>[F#ø7 → Bøø7]</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 5.4.33.** Comparison of transformations in Childs 1998, Bass 2001 and Karg-Elert

Among modern writings, Edward Gollin’s 1998 article “Some Aspects of Three-Dimensional Tonnetze”\(^\text{86}\) displays the most significant affinities with Karg-Elert’s model of chordal transformation. In particular, the article explores properties of a three-dimensional pitch space in which the planes or axes are derived from the intervals of the (0258) tetrachord: the perfect fifth, major thirds and minor seventh. That pitch space is identical to Karg-Elert’s in all respects, except that Gollin assumes 12-TET, and so his axes comprise chains of equal-tempered intervals. Just as the two-dimensional equal-tempered Tonnetz forms the space of a torus, a three-dimensional equal-tempered Tonnetz “occupies the closed, unbounded volume of a hypertorus in 4-dimensional space.”\(^\text{87}\) **Figure 5.4.34** reproduces Gollin’s Figure 2, which represents a region of his (0258) Tonnetz; the three axes are constructed from chains of major thirds (a-axis),

\(^\text{86}\) Gollin 1998.
\(^\text{87}\) Gollin 1998, 197.
perfect fifths (b-axis) and minor sevenths (c-axis). Though equal temperament is assumed, each pitch class will appear in multiple locations in the space, in all three planes. Karg-Elert’s justly-intoned pitch space can be represented in the same way, but with the addition of symbols to each pitch node to indicate syntonic and septimal comma differences.

Figure 2. A region within an (0258) Tonnetz

**Figure 5.4.34.** Gollin 1998, 198 – a region of a three-dimensional (0258) Tonnetz

After describing the three-dimensional equal-tempered (0258) space, Gollin explains how it “emphasizes certain operations within a larger group of transformations based on their common-tone retention properties.” The operations that are the main focus in the article are the inversions that share at least one common tone; Gollin does not discuss transpositions between (0258) tetrachords of the same quality, though he recognizes that they are possible in his space, and that they belong to the “larger group” of (0258) Schritte and Wechsel transformations that “is identical in structure (or isomorphic) to the S/W group acting on harmonic triads.” To label

---

88 Figures 2, 3 and 4 from Gollin 1998 are reproduced here with permission of the author (February 2018).
90 Gollin 1998, 203. As discussed above in section 5.2, Riemann’s system of Schritte and Wechsel (and especially its modern reformulation by Klumpenhouver) is based in 12-TET and enharmonic equivalence. In contrast, Karg-Elert’s common-tone transpositions move chords by pure fifths, thirds and sevenths, the intervals in his just-intonation space.
contextual inversions between (0258) tetrachords, Gollin proposes notation for the chord tones that is inspired by Moritz Hauptmann’s “designation of triadic chord tones” (Figure 5.4.35):

![Figure 5.4.35](image)

**Figure 5.4.35.** Gollin 1998, 200 – “neo-Hauptmannian” tokens for the chord tones in (0258)

In the figure, line A displays Hauptmann’s dualistic labels for the tones in a consonant triad: the prime or *Einheit* (“unity”) is I, the fifth is II, and the third is III. The arrows indicate the different energetic conception of the major and minor triads in Hauptmann, and how II and III relate to I: in major, the *Einheit* “has a perfect fifth and major third,” while in minor the Einheit “is a perfect fifth and a major third to the tones below.” Gollin revises Hauptmann’s labels for the (0258) tetrachord in line B of the figure, naming the four chord tones i (prime), ii (major third), iii (fifth) and iv (minor seventh). As in Hauptmann, Gollin’s labels are dualistic. Line C generalizes the system for any three-dimensional Tonnetz: the notation (u,v,w) refers to distances in the three axes of the space.

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91 This concept (from Hauptmann 1853) is discussed in Klumpenhouwer 2002, 459–462.
92 Gollin 1998, 199.
Gollin uses his “neo-Hauptmannian tokens” to label contextual inversions between dominant and half-diminished seventh chords in the format $I^x_y$, where $x$ and $y$ are the dual chord tones (i, ii, iii or iv) that are exchanged. In his Figure 4a (Figure 5.4.36), Gollin specifies six ‘edge-flips’ around one edge of the tetrachord [C, E, G, Bb]. In these six inversions, $x$ and $y$ are different dualistic chord tones, and so each inversion retains two common tones.

![Figure 4a. Six ‘edge-flips’ about a nexus tetrachord, (C,E,G,Bb), within an [0258] Tonnetz](image)

**Figure 5.4.36.** Gollin 1998, 201 – (0258) inversions around a pair of common tones

As discussed earlier, this dissertation has adopted Gollin’s lowercase Roman numeral labels for dualistic chord tones, and his $I^x_y$ format for contextual inversions. The only difference to his notation is the replacement of his [i, ii, iii and iv] with [i, iii, v and vii], which match Karg-Elert’s Arabic numerals for chord tones [$1/\flat, 3/\natural, 5/\natural, 7/\natural$]. The following chart (Figure 5.4.37) demonstrates that Gollin’s six ‘edge-flips’ are exactly equivalent to six of Karg-Elert’s common-tone inversions; three of them are the PRL operations as applied to the base triads of the (0258) tetrachords, while the other three exchange the chordal seventh ($iv/\natural$).
### Gollin 1998 vs Karg-Elert

<table>
<thead>
<tr>
<th>Gollin 1998</th>
<th>Karg-Elert</th>
<th>Cohn</th>
<th>From C7</th>
<th>From F#ø7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{ii}^I$</td>
<td>$I_{ii}^I$</td>
<td>R</td>
<td>$[C7 \rightarrow F#ø7]$</td>
<td>$[F#ø7 \rightarrow C7]$</td>
</tr>
<tr>
<td></td>
<td>(Parallel)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I_{iii}^I$</td>
<td>$I_v^I$</td>
<td>P</td>
<td>$[C7 \rightarrow Aø7]$</td>
<td>$[F#ø7 \rightarrow A7]$</td>
</tr>
<tr>
<td></td>
<td>(Variant)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I_{iv}^I$</td>
<td>$I_{vii}^I$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(Septgegenklang)</td>
<td></td>
<td>$[C7 \rightarrow Cø7]$</td>
<td>$[F#ø7 \rightarrow F#7]$</td>
</tr>
<tr>
<td>$I_{iii}^{II}$</td>
<td>$I_v^{III}$</td>
<td>L</td>
<td>$[C7 \rightarrow C#ø7]$</td>
<td>$[F#ø7 \rightarrow F7]$</td>
</tr>
<tr>
<td></td>
<td>(Leittonwechsel)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I_{iv}^{II}$</td>
<td>$I_{vii}^{II}$</td>
<td></td>
<td>$[C7 \rightarrow Eø7]$</td>
<td>$[F#ø7 \rightarrow D7]$</td>
</tr>
<tr>
<td>$I_{iv}^{III}$</td>
<td>$I_v^{IV}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$[C7 \rightarrow Gø7]$</td>
<td>$[F#ø7 \rightarrow B7]$</td>
</tr>
</tbody>
</table>

**Figure 5.4.37.** Comparison of Gollin’s ‘edge-flips’ and inversions in Karg-Elert

Gollin’s Figure 4b ([Figure 5.4.38](#)) displays four ‘vertex-flips’, which invert an (0258) tetrachord around a single pitch. For that reason, x and y in $I_v^X$ are the same dualistic chord tone:

![Figure 4b. Four ‘vertex-flips’ about a nexus tetrachord (C,E,G,B♭), within an [0258] Tonnetz](image)

**Figure 5.4.38.** Gollin 1998, 201 – (0258) inversions around a single pitch
Once again, Gollin’s ‘vertex-flips’ correspond exactly with four of Karg-Elert’s common-tone inversions, specifically the –gleich or “same” transformations (Figure 5.4.39):

<table>
<thead>
<tr>
<th>Gollin 1998</th>
<th>Karg-Elert</th>
<th>From C7</th>
<th>From F#ø7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I^{i}_{i}$</td>
<td>$I^{i}_{i}$</td>
<td>[C7 → Dø7]</td>
<td>[F#ø7 → E7]</td>
</tr>
<tr>
<td></td>
<td>(Gegenklang / Primgleich)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I^{ii}_{ii}$</td>
<td>$I^{iii}_{i}$</td>
<td>[C7 → A#ø7]</td>
<td>[F#ø7 → Ab7]</td>
</tr>
<tr>
<td></td>
<td>(Terzgleich)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I^{iii}_{iii}$</td>
<td>$I^{v}_{v}$</td>
<td>[C7 → Eø7]</td>
<td>[F#ø7 → D7]</td>
</tr>
<tr>
<td></td>
<td>(Quintgleich)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I^{iv}_{iv}$</td>
<td>$I^{vii}_{vii}$</td>
<td>[C7 → Bbø7]</td>
<td>[F#ø7 → G#7]</td>
</tr>
<tr>
<td></td>
<td>(Septgleich)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 5.4.39.** Comparison of Gollin’s ‘edge-flips’ and inversions in Karg-Elert

Gollin notes that whereas his $I^{ii}_{ii}$ and $I^{iv}_{iv}$ “are the same tetrachord in our curved modular space”93 under 12-TET, “this would not be true in a just-intoned version”94 such as that of Karg-Elert, in which $I^{iii}_{iii}$ and $I^{vii}_{vii}$ differ enharmonically. The case of Gollin’s $I^{v}_{v}$ and $I^{iii}_{iii}$ (Karg-Elert’s $I^{iii}_{iii}$ and $I^{v}_{v}$) is similar: their products are in fact spelled identically, but they differ metharmonically in just intonation. Barring the significant differences that derive from their respective tuning systems, Gollin’s three-dimensional pitch space and seventh-chord inversions are essentially the same as Karg-Elert’s; all of the latter’s common-tone inversions are represented in Gollin’s Figure 4.

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94 Ibid., 205.
5.5. Non-common-tone chord relations

Sections 5.2 to 5.4 have described how common-tone retention is the basic determinant in Karg-Elert’s chordal transformations. When an operation retains at least one pitch from the original chord in the new chord, it is possible to uniquely specify its path in pitch space. The same is not consistently true when no common tones are retained. As discussed in section 5.1, an operation like Riemann’s *Ganztonschritt* [CM → DM] does not represent a unique path in pitch space: in Cohn’s PRL system, it could be *(LR)^2* or **RLRP**, among other possibilities. While such distinctions are largely moot under 12-TET and enharmonic equivalence, they are of crucial importance in just intonation, in which each pitch is acoustically and conceptually distinct. Chords that do not share common tones are usually not defined by Karg-Elert as direct transformations of each other, but rather as indirect relations between representatives of different functions. His treatises introduce a variety of names for such non-common-tone relations, often reflecting the interval between the original and new dual primes, as in Riemann’s *Schritte* and *Wechsel*. This section will first examine a number of Karg-Elert’s non-common-tone relations, as presented in two excerpts from *Harmonologik*. It then turns to the *Kollektivwechselklänge* or “collective-change chords,” an important concept that is based entirely in the semitone (or parsimonious) motion of all chord tones, and which encompasses the hexatonic and octatonic poles of neo-Riemannian theory.

a. Leitklang, Chromonant and Tritonant

The following table (Figure 5.5.1, next page) compares Riemann’s *Schritte* or contextual transpositions with their counterparts in Karg-Elert. As discussed in section 5.1, the *Schritte* imply 12-TET, and almost entirely imply enharmonic equivalence.
<table>
<thead>
<tr>
<th>UTT (Hook 2002)</th>
<th>Riemann 1880</th>
<th>Karg-Elert</th>
<th>From CM</th>
<th>From Cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;+0,0&gt;</td>
<td>Identity</td>
<td>Identity*</td>
<td>CM → CM</td>
<td>Cm → Cm</td>
</tr>
<tr>
<td>&lt;+1,1&gt;</td>
<td>Gegenleitonschritt</td>
<td>Gegenleitklang*</td>
<td>CM → D♭M</td>
<td>Cm → Bm</td>
</tr>
<tr>
<td></td>
<td>ditto</td>
<td>Chromonant</td>
<td>CM → C♯M</td>
<td>Cm → C♭m</td>
</tr>
<tr>
<td>&lt;+2,10&gt;</td>
<td>Ganztonschritt</td>
<td>Ultradominant: (T_{v→i})^2</td>
<td>CM → DM</td>
<td>Cm → B♭m</td>
</tr>
<tr>
<td></td>
<td>Gegenkonkordant: T_{i→vii}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&lt;+3,9&gt;</td>
<td>Gegenkleinterschritt</td>
<td>Variantenparallel: T_{v→ii}</td>
<td>CM → E♭M</td>
<td>Cm → Am</td>
</tr>
<tr>
<td></td>
<td>(Gegensextschritt)</td>
<td>[unnamed]: T_{ii→v}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&lt;+4,8&gt;</td>
<td>Terzschritt</td>
<td>Mediant: T_{iii→i}</td>
<td>CM → EM</td>
<td>Cm → A♭m</td>
</tr>
<tr>
<td>&lt;+5,7&gt;</td>
<td>Gegenquintschritt</td>
<td>Contra(domin)ant: T_{i→v}</td>
<td>CM → F♭M</td>
<td>Cm → Gm</td>
</tr>
<tr>
<td>&lt;+6,6&gt;</td>
<td>Tritonusschritt</td>
<td>Tritonante: T_{ii→vii}</td>
<td>CM → F♯M</td>
<td>Cm → G♯m</td>
</tr>
<tr>
<td></td>
<td>Gegentritonusschritt</td>
<td>Gegentritonante: T_{vii→iii}</td>
<td>CM → G♭M</td>
<td>Cm → F♭m</td>
</tr>
<tr>
<td>&lt;+7,5&gt;</td>
<td>Quintschritt</td>
<td>Dominant: T_{v→i}</td>
<td>CM → G♭M</td>
<td>Cm → F♭m</td>
</tr>
<tr>
<td>&lt;+8,4&gt;</td>
<td>Gegenterzschritt</td>
<td>Gegenmediant: T_{iii→i}</td>
<td>CM → A♭M</td>
<td>Cm → B♭m</td>
</tr>
<tr>
<td>&lt;+9,3&gt;</td>
<td>Kleinterzschritt (Sextschritt)</td>
<td>Parallelvariant: T_{v→i}</td>
<td>CM → A♭M</td>
<td>Cm → E♭m</td>
</tr>
<tr>
<td></td>
<td>[unnamed]: T_{v→vii}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&lt;+10,2&gt;</td>
<td>Gegenganztonschritt</td>
<td>Ultracontrant: (T_{i→v})^2</td>
<td>CM → B♭M</td>
<td>Cm → D♭m</td>
</tr>
<tr>
<td></td>
<td>Konkordant: T_{ii→i}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&lt;+11,1&gt;</td>
<td>Leitonschritt</td>
<td>Leitklang</td>
<td>CM → B♭M</td>
<td>Cm → D♭m</td>
</tr>
<tr>
<td></td>
<td>ditto</td>
<td>Gegenchromonant*</td>
<td>CM → C♯M</td>
<td>Cm → C♭m</td>
</tr>
</tbody>
</table>

**Figure 5.5.1.** The Schritte and their counterparts in Karg-Elert

Once again, the first column provides the Uniform Triadic Transformation (UTT)\(^\text{95}\) that corresponds to each Schritt. The third column lists the relations in Karg-Elert’s work that correspond to each Schritt. The majority are common-tone transformations, listed here by name and also in the format T\(x_{→}\)y. The crucial difference between the Schritte and the common-tone transpositions is that the latter refer to unique paths in pitch-space, while the former do not; for example, Karg-Elert’s Mediant is specifically the transposition that maps the dual third (iii) onto the prime (i), while Riemann’s Terzschritt describes any motion from C major to E major, regardless of pitch-space locations. In four cases, a single Schritt corresponds to two common-

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\(^95\) Hook 2002.
tone transformations: one that operates in fifth/third space, and another that involves the septimal plane. There are no common-tone transpositions that replicate Riemann’s Leitonschritt and Gegenleitonschritt, which transpose a triad by a major seventh above and below respectively. Karg-Elert proposes a very similar term: the Leitklang or “leading chord,” which is identical with the Leitonschritt. Its opposite would logically be called Gegenleitklang (“counter leading chord”), though that term never appears in Karg-Elert’s treatises; it is therefore marked with an asterisk in Figure 5.5.1. Karg-Elert distinguishes between diatonic and chromatic semitone transpositions: the latter is called the Chromonant, which is “a chromatic shift up or down of a complete chord, as in A♭ major → A major.”96 Its opposite would be called Gegenchromonant, though it too does not appear in Karg-Elert’s writings. The Leitklang and Chromonant are always transpositions, and thus retain mode quality. They do not represent unique paths in pitch space; like Riemann’s Schritte, the terms simply indicate relations based on the interval between their primes.

In section 5.4, the Tritonante was listed as a specific seventh-based transposition, namely $T_{iii \rightarrow vii}$, which transposes a triad or concordant seventh by an augmented fourth, then displaced a septimal comma. Its inverse is the Gegentritonante or $T_{vii \rightarrow iii}$, which transposes by a diminished fifth and a septimal comma. In addition to their specific seventh-based meanings, Karg-Elert uses Tritonante and Gegentritonante to refer to any tritone transpositions, regardless of their pitch-space locations. In that more general sense, they are synonymous with Riemann’s Tritonuschritt and Gegentritonuschritt respectively.

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96 Harmonologik, 202.
The following table (Figure 5.5.2) compares Riemann’s *Wechsel* or contextual inversions with their counterparts in Karg-Elert:

<table>
<thead>
<tr>
<th>UTT (Hook 2002)</th>
<th>Riemann 1880</th>
<th>Karg-Elert</th>
<th>From CM</th>
<th>From Cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;-,0,0&gt;</td>
<td>Quintwechsel</td>
<td>Variant: $I_v^{I}$</td>
<td>CM $\rightarrow$ Cm</td>
<td>Cm $\rightarrow$ CM</td>
</tr>
<tr>
<td>&lt;-,1,11&gt;</td>
<td>Doppelterwechsel</td>
<td>Terzgleicher: $I_v^{III}$</td>
<td>CM $\rightarrow$ Cm</td>
<td>Cm $\rightarrow$ CbM</td>
</tr>
<tr>
<td></td>
<td>Gegenterwechsel</td>
<td>Septgleicher: $I_v^{VI}$</td>
<td>CM $\rightarrow$ Dm</td>
<td>Cm $\rightarrow$ BM</td>
</tr>
<tr>
<td>&lt;-,2,10&gt;</td>
<td>Kleinterwechsel (Sextwechsel)</td>
<td>Contrant-Parallel; Ultradominant-Variant</td>
<td>CM $\rightarrow$ Dm</td>
<td>Cm $\rightarrow$ BbM</td>
</tr>
<tr>
<td>&lt;-,3,9&gt;</td>
<td>Gegenganztonwechsel</td>
<td>Septgegenklang: $I_v^{VI}$</td>
<td>CM $\rightarrow$ EbM</td>
<td>Cm $\rightarrow$ AM</td>
</tr>
<tr>
<td>&lt;-,4,8&gt;</td>
<td>Leittonwechsel</td>
<td>Leittonwechsel: $I_v^{VI}$</td>
<td>CM $\rightarrow$ Em</td>
<td>Cm $\rightarrow$ A#M</td>
</tr>
<tr>
<td>&lt;-,5,7&gt;</td>
<td>Seitenwechsel</td>
<td>Gegenklang / Primgleicher / Contrant-Variant: $I_i$</td>
<td>CM $\rightarrow$ Fm</td>
<td>Cm $\rightarrow$ GM</td>
</tr>
<tr>
<td>&lt;-,6,6&gt;</td>
<td>Gegenleittonwechsel</td>
<td>Tritonant-Variant (Kollektivwechselklänge II)</td>
<td>CM $\rightarrow$ F#m</td>
<td>Cm $\rightarrow$ GbM</td>
</tr>
<tr>
<td>&lt;-,7,5&gt;</td>
<td>Ganztonwechsel</td>
<td>Quintgleicher / Dominant-Variant: $I_v^{v}$  [unnamed]: $I_v^{VI}$</td>
<td>CM $\rightarrow$ Gm</td>
<td>Cm $\rightarrow$ FM</td>
</tr>
<tr>
<td>&lt;-,8,4&gt;</td>
<td>Gegenkleinterwechsel (Gegensextwechsel)</td>
<td>Gegenmediant-Variant (Kollektivwechselklänge I)</td>
<td>CM $\rightarrow$ A#m</td>
<td>Cm $\rightarrow$ EM</td>
</tr>
<tr>
<td>&lt;-,9,3&gt;</td>
<td>Terzwechsel</td>
<td>Parallel: $I_v^{III}$</td>
<td>CM $\rightarrow$ Am</td>
<td>Cm $\rightarrow$ EbM</td>
</tr>
<tr>
<td>&lt;-,10,2&gt;</td>
<td>Gegenquintwechsel</td>
<td>Dominantenseptgegenklang: $I_v^{VI}$</td>
<td>CM $\rightarrow$ BbM</td>
<td>Cm $\rightarrow$ DM</td>
</tr>
<tr>
<td>&lt;-,11,1&gt;</td>
<td>Tritonuswechsel</td>
<td>Leitklang-Variant*</td>
<td>CM $\rightarrow$ Bm</td>
<td>Cm $\rightarrow$ DbM</td>
</tr>
</tbody>
</table>

**Figure 5.5.2.** The *Wechsel* and their counterparts in Karg-Elert

Ten of the *Wechsel* correspond to Karg-Elert’s common-tone inversions, keeping in mind that the latter indicate specific pitch-space trajectories. Several of the non-common-tone inversions are described simply as transpositions followed by *Variant* (change of mode). The *Tritonant-Variant* and *Gegenmediant-Variant* are new harmonic relations, not possible through direct common-tone transformation; they are classified as *Kollektivwechselklänge* or “collective-change chords,” a topic that will be discussed shortly. The table proposes the *Leitklang-Variant*
and Gegenleitklang-Variant, as they logically complete the listing; however, they never appear in Karg-Elert’s writings.

The following abstract progression from Harmonologik (Figure 5.5.3) demonstrates Leitklang, Chromonant and Tritonant relations, in addition to common-tone transformations. The upper voice descends through a chromatic scale, outlining a “whole tone downward progression on the strong-beat chords” (ganztönige Abwärtsschreitung der schwerzeitigen Klänge):

Because the progression begins on a C major triad, the functional analysis adopts that triad as tonic, and stays in C major throughout. However, Karg-Elert fully recognizes that “the key is completely erased”97 in the passage; the function labels no longer express a true sense of tonal

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97 Harmonologik, 216.
meaning or syntax in C major, but rather serve to denote locations in pitch space in relation to
the opening chord. Acoustic common tones are indicated as ties in the figure. In addition to the
function labels, Karg-Elert identifies relations between each pair of adjacent chords, including
the following:

- **Gegentritionante**
  - GM → D♭M, [BM → FM], [AM → E♭M]
- **Tritonante** (mm. 2-3)
  - [FM → BM]
- **Chromonant** (m. 3)
  - [BM → B♭M]
- **Leitklang** (mm. 3-4)
  - [B♭M → AM]

The three instances of the **Gegentritionante** (here meaning a generic diminished-fifth
transposition) arise from different harmonic relationships. All of them change function, but in
different ways: \( D \rightarrow C_M, D^M \rightarrow C \), and \( C^M \rightarrow D_M \). Each relation denotes a specific journey in
pitch space; as illustrated on the Tonnetz for the passage (**Figure 5.5.4**), the first two are
analogous (chords 3 → 4 and 6 → 5), but the last is different (chords 8 → 2).

**Figure 5.5.4.** Tonnetz for Figure 5.5.3
b. The Kollektivwechselklänge

For the most part, Karg-Elert’s discussion of chord relations and transformations is not explicitly concerned with smooth voice leading, or indeed with voice leading at all, beyond the retention and reinterpretation of common tones. An exception is the Kollektivwechselklang or “collective-change chord,” which is defined entirely in terms of the semitone motion of all chord tones. The primary type (Type I) changes a major triad into a minor one, or vice versa. Karg-Elert describes the process: “the outside pitches [i.e. the prime and fifth] of a triad move by semitones in contrary motion; the third slides by semitone in the direction that turns a major triad into a minor triad, or a minor triad into a major triad.”98 The process is demonstrated in the following example (Figure 5.5.5):

![Figure 5.5.5. Kollektivwechselklänge Type I (Harmonologik, 202)](image)

In other words, the two voices that form interval class 4 (major third) move in parallel motion, while the remaining voice moves in the opposite direction. In all cases, one of the semitone motions is by chromatic semitone (same letter name), while the other voices move by diatonic semitone. The semitone motions in Figure 5.5.5 reflect transformational rather than literal voice

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98 Harmonologik, 202.
leading; the specified motions do not need to be realized at the musical surface through register, instrumentation or any other means. The roots of the triads are related by a major third or augmented fourth; for example, the relation is between C major and G♯ minor / A♭ minor, and between A minor and C♯ major / D♭ major. Both spellings of the goal triad are equally possible, as the Kollektivwechselklang concept (henceforth KWK) does not specify pitch-space locations or trajectories. The choice of spelling will depend on the functional meaning of each chord within the active key. There is one misprint in Figure 5.5.5: in the fourth example, the F-natural should move to F♯ instead of E♯, creating the progression D minor (d moll) to F♯ major (Fis dur) rather than A♯ minor (ais moll). The Type I KWK relation is equivalent to Richard Cohn’s hexatonic pole, a term referring to the pair of triads that is “maximally disjunct” within an equal-tempered hexatonic system.99

Karg-Elert names three functional contexts for the KWK Type I, each of which reflects a specific trajectory in pitch space:

a) Between a triad and its counter-mediant-variant, as in \( T \rightarrow T_m \)

b) Between a triad and its dominant’s mediant-parallel, as in \( T \rightarrow D^{Mp} \)

c) Between the counter-mediant and Leittonwechsel of the same function, as in \( T_M \rightarrow T^L \)

The first case is the counter-mediant-variant (\( T_m \)), which is the counter-mediant (\( T_M \)) followed by a change of mode. It is thus directly related to the third-transformations; however, Karg-Elert’s treatises discuss it separately from the counter-mediant and other common-tone third relations, because it no longer retains any pitches from the original chord. The change of mode is indicated by the switch from uppercase \( M \) to lowercase \( m \). The progression \( T \rightarrow T_m \) should be

99 The concept of the hexatonic pole is introduced in Cohn 1996, 19. Cohn 2004 explores the “tonal signification” of the hexatonic pole in music, theory and psychology of the late nineteenth and early twentieth centuries.
notated as a descending root motion by major third, as in \([CM \rightarrow Ab\, m]\). The second context reflects the spelling \([CM \rightarrow G\#m]\), and involves a change of function, as in \(T \rightarrow D^{Mp}\); in C major, that is \(CM \rightarrow GM \rightarrow BM \rightarrow G\#m \rightarrow D^{Mp}\). The latter chord is the Terzgleich or SLIDE of the dominant \((GM \rightarrow G\#m \text{ in C major})\). Finally, the third case also properly indicates a root motion by augmented fourth: in C major, \(T_M \rightarrow T^L\) is \([Ab\, M \rightarrow Em]\). The following Tonnetze (Figure 5.5.6) displays the three contexts for \(KWK\) Type I just described, in the key of C major.

![Figure 5.5.6](image)

**Figure 5.5.6.** Three functional contexts for Kollektivwechselklang Type I, in C major

The next two analytical examples from Harmonologik will demonstrate the functional multiplicity of the \(KWK\) Type I. The first is from Act III of Wagner’s Parsifal (Figure 5.5.7):
The initial *KWK* succession [EM \(\rightarrow\) Cm] is first analyzed in E major as \(T \rightarrow T_m\); the C minor triad is then reinterpreted as dominant parallel (\(Dp\)) in A\(b\) major, the new tonic that ends the succession. However, the bass progression [E – C – A\(b\)] suggests an alternate analysis entirely in the key of C major (“auf C”), in which the tonic variant (\(t\)) of that key is surrounded by its upper and lower mediants (\(T^M\) and \(T_M\)). Both interpretations are equally logical, and they define the exact same trajectory in pitch space (though the location of the initial E major triad will be displaced by a syntonic comma if C major is adopted as the tonic). At a later point, the passage is repeated a whole tone lower, starting from D major: [DM \(\rightarrow\) B\(b\)m \(\rightarrow\) G\(b\)M]. Karg-Elert’s first analysis begins in the same way as the original passage, with *KWK* Type I as \(T \rightarrow T_m\). The G\(b\) major triad is the *Ultragegenmediante* (\(T_{MM}\)) of D major: transposed downwards by two major thirds. Perhaps finding the latter relation to be too distant from the initial chord, Karg-Elert then considers that the G\(b\) major triad may in fact be F\# major, the regular mediant (\(T^M\)) of D major;
in that case, the B♭ minor triad would really be A♯ minor, which can be understood either as
tonic mediant-Leittonwechsel ($T^{ML}$)\textsuperscript{100} of D major, or as the dominant parallel ($Dp$) of F♯ major.
The following Tonnetz (Figure 5.5.8) plots the two analyses of the D major passage. It reveals
that the second version follows a more direct path, linking the initial D major to the final F♯
major through their common tone.

Figure 5.5.8. Tonnetz for the passage in Figure 5.5.5, beginning on D major

But why does Karg-Elert propose two possible analyses for the progression when it begins on D
major, but not when it begins on E major? The answer is likely due to the limitations of standard
pitch notation: while changing B♭ minor to A♯ minor in the second progression would not be too
rare or remote, rewriting C minor as B♯ minor in the first progression (in order to end on G♯
major) would be considered positively bizarre: [B♭, D♭, F♯]. Therefore, while the analysis of
$T \rightarrow (Dp)T^M \rightarrow T^M$ might be the most direct and convincing, the analyst may be unlikely to
propose it in keys that would require difficult accidentals, or unusual enharmonic respelling.

\textsuperscript{100} Figure 5.5.5 erroneously labels the A♯ minor triad as $T^{Mp}$, when $T^{ML}$ is correct.
The next example from Richard Strauss’ opera *Elektra* (Figure 5.5.9) further demonstrates how the *KWK* Type I frequently blurs the sense of key, and invites enharmonic reinterpretation:

![KWK Type I](image)

**Figure 5.5.9.** *Kollektivwechselklang* Type I in Richard Strauss, *Elektra* (*Harmonologik*, 300)

As in the *Parsifal* passage, Karg-Elert provides multiple functional analyses, in various keys. Strauss’ original notation is provided on the upper staff, beginning on F♯ major, and ending on its dominant C♯ major. The first two chords are the *KWK* Type I: [F♯M → Dm], understood in F♯ major as $T \rightarrow T_m$, as expected for a descending major third root motion. The upper analysis is “centralized” (*zentralisiert*), entirely in the key of F♯ major; it is for the most part convincing, as the phrase ends on the dominant of that key. The only chord that is particularly remote from F♯ major is the G minor triad in measure 2, labelled as the contrant-median-variant ($C_m$), which is a *KWK* type I relation from the regular contrant B major. The second analysis is “decentralised”
(dezentralisiert): the initial triad and key are reinterpreted as G♭ major, but the final chord is still C♯ major (the dominant of F♯ major). The change of key results in different functional meanings for the KWK-related chords (D minor and G minor). Karg-Elert provides a third possible analysis, with the first measure in B♭ major (B dur Bereich), and the second in D major (D dur Bereich) – though those two triads do not actually appear in the passage. The rationale for the last analysis is likely because it eliminates the most complex chordal transformations, using only diatonic substitutes and the regular mediants. One common thread in all three functional analyses is that Karg-Elert evidently wants to hear an expansion of tonic (T) in the first bar, followed by a C → D cadential motion in measure 2; this likely explains why the D minor triad in the G♭ major version is analyzed as tonic mediant-Leittonwechsel (TML), instead of the slightly more common (and equally correct) dominant mediant-parallel or Terzgleich (DMP).

None of the analyses of Figure 5.5.9 are wholly convincing, especially from the perspective of a listener that knows nothing of specific note spellings. The problem with the first analysis concerns the upper voice melody: why would a listener hear the first upper-voice pitch in measure 2 as B♭, immediately after two statements of A♯ in the same register and voice? In the second “decentralised” version, the issue is the relationship between the initial and final chords: surely one would tend to hear a tonic-to-dominant motion in the same key, in spite of the chromatic excursion in measure 2. The third analysis is also problematic for much the same reason: why would one choose to understand the passage in B♭ major and D major, when the first and last chords sound like T and D in F♯ major or G♭ major?

The problems in Karg-Elert’s analyses of Figure 5.5.9 raise the issue of perception, and of aural/experiential versus visual/intellectual understanding of harmonic relations. The basic
question is one of what should take precedence in the analysis of tonality and function – the composer’s pitch notation, or the chord-to-chord relations and common-tone transformations that follow the “harmonic logic” of pitch space? Ideally, the two should agree with each other; however, Karg-Elert’s treatises highlight numerous cases in which notation suggests tonal unity and closure, but the surface-level harmonic trajectory reveals otherwise. This is the primary topic of chapter 6 of this dissertation, which deals with Karg-Elert’s theory of comma-free and comma-differing modulation.

The secondary type of Kollektivwechselklang (abbreviated here as KWK Type II) is a mode-changing tritone relationship between triads: [CM \(\rightarrow\) F\#m / G\#m], or [Cm \(\rightarrow\) F\#M / G\#M]. Accordingly, Karg-Elert calls it Tritonant(e)-Variant(e). As with KWK Type I, the relation is derived from the semitone motion of all chord tones, with one voice moving in contrary motion from the others. To create a tritone relationship, one needs to begin with concordant sevenths rather than simple triads. Figure 5.5.10 illustrates the process:

![Figure 5.5.10. Derivation of the Kollektivwechselklänge from concordant sevenths (Harm, 202)](image)

The first two measures of Figure 5.5.10 still demonstrate KWK Type I, but now the source triad moves to a four-note chord. The outer voices (prime and fifth) move by diatonic semitones in contrary motion as in Figure 5.5.5, but now the third splits into its surrounding diatonic
semitones: in C major, that is [E → F] and [E → D♯].\footnote{The splitting (and subsequent fusing) of the chordal third into its surrounding semitones recalls similar processes presented in Callender 1998; see his Figure 5 (p. 224).} The resulting chord is enharmonically equivalent to a concordant seventh, as Karg-Elert indicates: in C major, “A♭ or G♯ minor triad with seventh below (♯).” The added chordal seventh (F-natural) may be omitted, leaving the \(KWK\) Type I relation between simple triads: \([CM \rightarrow A♭m / G♯m]\). The derivation of the \(KWK\) Type II (in the second half of Figure 5.5.10) works in a similar way, starting from a concordant seventh such as C7 or F♯ø7. The three pitches that form a diminished triad (i.e. the chordal third, fifth and seventh) move together by diatonic semitone, in parallel motion: in C major, that is \([E \rightarrow D♯], [G \rightarrow F♯]\) and \([B♭ \rightarrow A]\). The remaining voice (i.e. the prime) moves by diatonic semitone in the opposite direction: \([C \rightarrow D♭]\). The resulting four-note chord is enharmonically equivalent to a concordant seventh, and so the \(KWK\) Type II relation between concordant seventh chords is \([C7 \rightarrow D♯ø7], \text{ or } [F♯ø7 \rightarrow E♭7]\). Karg-Elert’s annotation below the figure states that if the sevenths are omitted, the triads that remain are related by tritone: \([CM \rightarrow F♭m], \text{ or } [Am \rightarrow E♭M]\). This root-interval relationship explains the other designation for \(KWK\) Type II as \textit{Tritonant-Variant}, which is a tritone transposition followed by \textit{Variant} (change of mode).

The primes of \(KWK\) Type II-related chords are a chromatic semitone apart, as shown in Figure 5.5.11. Based on the interval between the primes, Riemann calls the relation \textit{Gegenleittonwechsel} (see Figure 5.5.2).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_5_11.png}
\caption{\textit{KWK} Type II-related (\textit{Tritonant-Variant}) triadic pairs \textit{(Harmonologik, 202)}}
\end{figure}
The following excerpt (Figure 5.5.12) from Edward Elgar’s oratorio *The Apostles* demonstrates how “the secondary *Kollektivwechselklänge* enters into the cadential progression in the place of the contradominant, of which it is the *Terzgleicher.*”\(^{102}\) The chords do not include their sevenths, and so the chord roots in each pair outline the tritone. The analysis indicates that the A minor and B minor triads are the *Terzgleicher* (i.e. mediant-parallel) transformations of the regular contrant (Ab major) and dominant (Bb major) respectively. Thus, the basic progression at its core is simple and entirely fifth-based: \(T \rightarrow C \rightarrow D\). 

\[
\begin{align*}
KWK \text{ Type II:} & \\
\text{Substituting for:} & \\
\end{align*}
\]

\[
\begin{align*}
\text{[E}b\text{M} \rightarrow \text{Am]} & & \text{[FM} \rightarrow \text{Bm]} \\
\text{As dur} = & & \text{B dur Akk.}
\end{align*}
\]

**Figure 5.5.12.** *KWK Type II (Tritonant-Variant) harmonies in Elgar, The Apostles (Harm, 302)*

The *KWK* Type II relation is equivalent to the *octatonic pole*, a term coined by Adrian Childs as a counterpart to Cohn’s hexatonic pole.\(^{103}\) It refers to the tritone-related pair of triads (of opposite mode) that are maximally distant in the IC3-cycle outlined in the octatonic collection. Richard Cohn has commented on how Karg-Elert “identified the affinity” of the hexatonic and

\(^{102}\) *Harmonologik*, 302.

\(^{103}\) Childs 1998, 187.
octatonic poles, in their similar voice-leading derivation: “both progressions combine
inversionally related species...[and] involve upshifting in all-but-one voice, offset by
downshifting in the remaining voice.”

c. “Free tonality”

In the final chapter of Harmonologik, Karg-Elert discusses what he calls die aufgehobene
Tonalität (“suspended tonality”), or more succinctly as Freitonalité (“free tonality”). This is
not a specific style or harmonic idiom, but simply any music in which “harmonic events no
longer indicate any connection to a tonal center.” He discusses some specific elements familiar
in the music of his contemporaries: parallel chord shifting in the manner of Debussy and his
school; the adoption of complete enharmonic equivalence (and thus the levelling of comma
differences); and “atonal complexes,” referring to chromatic alterations of triads and seventh
chords, plus chords with added tones. However, the chapter mostly focuses on the use of
triads and concordant seventh chords in non-functional contexts. Most of the examples in the
chapter are by Karg-Elert himself; the majority are abstract progressions, but several passages
from his works (both published and unpublished) are featured. None of the examples include
functional analysis, which is of course meaningless in the absence of tonality. Instead, Karg-
Elert uses labels such as Chromonant and Tritonant to indicate root-interval motions between

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104 Cohn 2012, 155.
105 Chapter 15 of Harmonologik (312–327).
106 Harmonologik, 312.
107 Neither Harmonologik nor Akustische discuss non-tertian chord construction, which was
listed under “atonal pitch systems” in the plans for the Grundlagen series (see Figure 2.2.7).
Chapter 2 of this dissertation quoted a letter by Karg-Elert of July 1926, lamenting that he would
not be able to publish “about 80 pages on atonality, and on quarter- and sixth-tones.”
adjacent chords. **Figure 5.5.13** is a representative example of *Freitonalität* involving triads and seventh chords. If the concordant sevenths are omitted, the resulting succession is:

\[
\text{[Am} \rightarrow \text{EbM} \rightarrow \text{Bb} \rightarrow \text{EM} \rightarrow \text{Cm} \rightarrow \text{DM} \rightarrow \text{Bb} \rightarrow \text{CM]}\]

**Figure 5.5.13.** An abstract example of “free tonality” (*Harmonologik*, 312)

Each step involves a change of mode (indicated as *Variant* in the analysis). Karg-Elert states that the succession in Figure 5.5.13 is “theoretically functional, as it can be quite simply related to several key centers such as C major, Eb major, etc. But those keys will never be perceived as such! The chords proceed in a linear detachment-pattern [i.e. avoiding common-tone connections] and prefer distant relationships, thus precluding centralized grouping [in a key].”

The brackets above the staff indicate *Kollektivwechselklänge* Type I (*pr.*) and Type II (*sec.*), and the brackets and labels below the staff specify relationships between adjacent chords.

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108 *Harmonologik*, 312.
The last example to be cited in this chapter (Figure 5.5.14) is an intriguing fragment: the only surviving passage of a vocal chamber work by Karg-Elert, now otherwise lost.\textsuperscript{109} It begins with a sequence of alternating Kollektivwechselklänge Types I and II, in a free-tonal context:

![Figure 5.5.14. Excerpt from Karg-Elert, Musik für 5 Instrumente und Gesang (Harm, 303)](image)

The final examples in both Akustische and Harmonologik stretch beyond the orbit of centralized tonality, to the point where it is no longer possible to assign functional meaning to each chord, or even to accurately locate every pitch in the harmonic space. Based on Karg-Elert’s common-tone transformations and their functional meanings, one might state that as long as harmonic paths in the space can be exactly specified, some degree of tonality will likely remain active. The concept of key (and of motion between keys) is complicated in Karg-Elert’s work by the fact that there are hundreds of potential tonics, each in a unique location in harmonic space. The topics of tonality, modulation and harmonic trajectory are explored in Chapter 6 of this dissertation.

\textsuperscript{109} This missing work is listed as W 62 in Sonja Gerlach’s Karg-Elert catalogue (Gerlach 1984, 125). Its text (“you will not die a flower’s easy death”) is from the poem “Ich möchte hingehn” (“I want to pass away”) by German poet Georg Herwegh (1817–1875); see Buchheim 1881, 303.
Chapter 6

Tonality and modulation

The bulk of this chapter comprises four brief essays on nineteenth-century works, based on analyses in Harmonologik. The selected passages explore the theory of comma-free and comma-differing modulation, which is explained in the chapter’s opening section. Comma-free modulation involves keys whose initial and final tonics are both located in the Ursprungslagen (source position) chain of fifths at the center of the pitch space; for Karg-Elert, only such fifth-based tonal motions are to be counted as true modulations. In contrast, comma-differing modulations involve motions that end outside of the central fifth chain, and are thus not real modulations, but simply tonal ‘shifts’. The essays illustrate that the distinction between comma-free and comma-differing modulation is not always absolute; they also raise questions about the nature of tonality and closure in a just intonation universe. As will be seen, Karg-Elert is sometimes willing to accept metharmonic or enharmonic juxtapositions in order to ensure tonal closure, as in the development section of Beethoven’s Waldstein Sonata. However, he often chooses to disregard a composer’s enharmonic notation (itself intended to provide tonal unity), when it contradicts a harmonic path defined by local common-tone connections. This is the case in both Liszt’s song Wieder möcht’ ich dir begegnen and Schumann’s Novelette in F major, in which repeated harmonic transformations drive the music into remote regions of pitch space. The final passage from Wagner’s Die Walküre is examined from two perspectives: as a middleground progression between key areas, and as a foreground progression between individual chords. The chapter closes with a brief discussion of writings by other authors that explore “non-conformist notions”\(^1\) of enharmonicism and tonality in nineteenth-century music.

\(^1\) A reference to the title of Harrison 2002.
6.1. **Comma-free and comma-differing modulation**

Under 12-tone equal temperament and enharmonic equivalence, there are for practical purposes twelve major triads or tonics, and twelve minor triads or tonics. As a result, there are 576 possible relationships or motions among those 24 tonics (24 * 24, including each tonic with itself). While that number is sizable, it is an insignificant portion of the potentially infinite number of key relationships and modulations that exist in a just-intonation harmonic universe. Even if one limits the just-intonation space to the 275 distinct pitches that Karg-Elert found sufficient for understanding most passages, the number of possible relations is exceedingly high: the space contains 480 triads in distinct acoustic locations (48 major and minor triads in each of 10 comma-variant rows), and so the number of relationships between those triads or tonics is (480 * 480), or 230,400. When comma differences are stringently observed as in Karg-Elert’s theories, the concepts of modulation (understood in a general sense as a motion between two tonics or keys) and tonal unity are greatly problematized, in comparison to their relative simplicity under 12-TET and enharmonic equivalence. In just intonation, one can no longer simply speak of a modulation from C major to D minor, as each of those tonics exists in multiple (metharmonically differing) locations in the space. It even becomes impossible to say that all passages that begin and end in C major are tonally unified in the strictest sense, as a progression or modulation may conclude at an acoustically-different C major than the opening triad.

When faced with the multiplicity of possible keys in just intonation, almost all modes of analysis have chosen to assume metharmonic equivalence between keys of the same name. As Riemann stated, the “identification of acoustical values that differ by a syntonic comma is simply

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2 See section 3.7, and *Harmonologik*, 30.
3 This number includes identity relations, between each triad in the space and itself.
indispensable to our musical hearing**⁴ – and more to the point, such identification is indispensable to standard analytical models of tonality. With regard to modulation, Riemann further proposed that “the possibility (even the necessity) of the exchange of coinciding enharmonic tone-values intrudes upon us from all sides as soon as modulation strays into regions that allow a simpler understanding, more in the central area of the tonal system.”⁵ As will be discussed in chapter 7, Riemann argued that the musical ear/mind always prefers the most ‘economic’ mode of understanding relations among tones and harmonies. Karg-Elert often proceeds from the opposite viewpoint, stating that “Nature does not seek closure.”⁶ This aphorism reflects how his harmonic analyses follow an organic and non-predictive process, rather like the growth of a plant or tree: beginning from the initial chord (which may or may not be the piece’s primary tonic), he follows the local common-tone-based transformations and juxtapositions wherever they happen to lead, tracing the pitch-space path to its conclusion. In some cases, the path manages to lead back to the initial chord, in its original acoustic location; Karg-Elert termed such paths tonal “rings” or “circles.” More frequently, the journey does not end exactly where it began, even if the notated key is the same as at the opening; Karg-Elert called these paths tonal “spirals,” more akin to irregular processes of proliferation and growth in Nature. To distinguish between different key relations, Karg-Elert developed his theory of comma-free and comma-differing modulation, which is the principal subject matter in this chapter.

In Chapter 8 of Harmonologik, Karg-Elert states that “modulation (Modulation) means tonal transformation [Wandlung]. This excludes abrupt juxtapositions of distantly related keys,

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⁴ Riemann/Wason/Marvin 1992, 100.
⁵ Ibid., 109–110.
⁶ Harmonologik, 101.
which are better described as ‘shifts’ [Rückung]…It is obvious that no modulation without harmonic movement is possible, yet every transition or harmonic movement is not necessarily a modulation.”

He describes modulation in a more detailed yet rather cryptic fashion: “True modulation is comma-free tonal transformation. Directed motion via the tonal cadence of the source key. Linear progression on the basis of the principal triads.”

Karg-Elert is saying that true modulation involves keys that are ultimately related by the harmonic step featured in the tonal cadence (tonalen Kadenz), or that separating the principal triads – namely, the perfect fifth. In contrast, key changes effected by chromatic harmonies that involve syntonic-comma displacement are not true modulations. He criticized Max Reger’s Beiträge zur Modulationslehre, which demonstrates modulations between all (12-TET) major and minor keys, using the smallest possible number of chords: “Modulations they are not, but rather shifts, buckling of tonality [Tonalitätsknickungen], or diagonal harmonic movements …all its cited examples spin around in circles, or spiral up and down to chords derived from comma difference (mediants, ultramediants…”

In Akustische 7.1, Karg-Elert writes that “comma-free modulation is Pythagorean in basis; that is, the harmonic stations that connect the principal initial key and the principal goal key [not simply from chord to chord] belong to the chain of fifth-relations. To be sure, variants can be included at will, but they must be only fleeting in character [as for instance in a cadential prefix], and cannot themselves initiate further modulatory processes.” For practical purposes, this means that if a passage’s initial and final tonics are related by fifths (i.e. they are both in their source positions), they are not displaced by any syntonic (or septimal) commas, and so the

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7 Harmonologik, 112.
8 Ibid., 115.
9 Reger 1903.
progression or modulation is classified as comma-free. Almost anything can happen in between those two tonics, including momentary syntonic-comma displacement caused by variants, mediant, or other harmonies that include comma-different pitches; what matters is that both the opening and closing keys are in the chain of fifth-relations. If a comma-variant harmony does shift the final tonic outside of the source positions, the modulation will be *comma-differing*.

The first two examples in *Akustische* Chapter 7 clearly demonstrate the concept of comma-free and comma-different modulation. Both passages modulate from D minor to B minor. The passages begin from the same D minor triad, in the same pitch-space location. As described in section 3.7, a passage’s initial tonic is always assumed to be one of the source position (*Ursprungslage*) triads, on the central chain of fifth-related triads. Therefore, the initial D minor tonic is the relative minor of F major, located one fifth below the central C major triad. The first example (Figure 6.1.1) is a comma-free modulation to B minor:

![Musical score](image)

**Figure 6.1.1.** Comma-free modulation from D minor to B minor (*Akustische* 7.1)
The passage in Figure 6.1 first moves from the D minor tonic to its contrant parallel C major (Cₚ), itself tonicized by the preceding two chords. C major then acts as a pivot to B minor, becoming the dominant Leittonwechsel (Gₜ) in the new key, which is then confirmed by the Gₜ → ⁰ → ⁷ final cadence. Each of the three main chords in the passage (D minor, C major and B minor) is preceded by its contrant variant (Cₚ or Gₜ), which always includes a syntonic-comma variant third (shown as closed noteheads in the Ursprungslagen analysis above the figure). The main harmonies (chords 2, 5 and 8) are all located on the principal fifth-chain; they progress in an “eastward” horizontal direction in the pitch space, as demonstrated on the Tonnetz for the passage (Figure 6.1.2). The tonicizing contrant variants (chords 1, 3 and 7) momentarily depart from the fifth-chain, but do not impede the directed motion of the progression. An important factor in comma-free modulation is that the pivot chord between the old and new keys does not create what Karg-Elert calls a ‘variant trick’ (to be described presently). In Figure 6.1.1, the C major pivot (chord 5) is a diatonic substitute in both D minor and B minor, and thus does not include a comma-different pitch. The modulation traverses a distance of three fifths higher on the source-position chain: [Dm → Am → Em → Bm].

Figure 6.1.2. Tonnetz for Figure 6.1.1
The second example (Figure 6.1.3) is a comma-differing modulation from D minor to B minor. Unlike the comma-free version which passed through an intermediary key (C major) along the fifth-chain, this modulation is very brief and abrupt, rather like those in Reger 1903:

![Tonnetz chords diagram](image)

**Figure 6.1.3.** Comma-differing modulation from D minor to B minor (*Akustische* 7.1)

The annotation to the right of Figure 6.3 states that “this alleged modulation is achieved by means of a variant trick [inside the dotted rectangle]. The result: comma difference.”\(^{10}\) A variant trick (*Variantentrick*) occurs when a chord in one key that contains one or more syntonic comma-variant pitches (such as a variant, mediant or auxiliary mediant) acts in the other key as a non-comma-variant harmony (such as a principal triad, parallel or Leittonwechsel). In Figure 6.1.3, the pivot chord is A major (chord 2), which is the contrant-variant (♮) in D minor – a comma-variant chord that departs from the central fifth-chain. It is then interpreted in B minor as a contrant-parallel (♮♮), which leads to a ♯ C cadence in the new key. The problem is that what follows the pivot chord does not eliminate the comma difference; instead, chords 3 and 4 proceed along the same comma-different chain, and so the final B minor chord is not in its source position (i.e. three fifths higher than D minor, as in Figure 6.1.1). Instead, the goal B

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\(^{10}\) *Akustische* 7.1.
minor is the variant-parallel (ทร) of the opening D minor, sharing D(2,1) as an acoustic common
tone, as shown on the Tonnetz for the passage (Figure 6.1.4):

![Tonnetz for Figure 6.1.3]

Figure 6.1.4. Tonnetz for Figure 6.1.3

While one might question the extent to which the musical imagination can perceive syntonic
comma differences in harmonic passages, it is safe to say that a listener will likely experience the
modulations in Figures 6.1.1 and 6.1.2 quite differently. Where the first can be described as
gradual and methodical in its harmonic path (effected by fifth-based local cadential
progressions), the second is abrupt, creating a feeling of deception or ‘trickery’ through the
juxtaposition of A and A♯ between chords 2 and 3, and the switch of F to F♯ between the initial
and final tonics. Whereas Figure 6.1.1 outlines a true harmonic motion or ‘tonal transformation’
by three fifths, Figure 6.1.2 essentially ‘goes nowhere’, simply exchanging the tonic D minor for
its direct auxiliary-median transform B minor, with its syntonic-comma displacement.

The next two examples are longer abstract progressions from Akustische that further
demonstrate comma-free and comma-differing modulations, and the role of specific pivot chords
in each type of modulation. Each example travels from D♭ major to C♯ minor; in just intonation,
the tonics of those keys are of course distinct, not to be enharmonically equated. **Figure 6.1.5** is a comma-free (*kommarein*) modulation from D♭ major to C♯ minor. Karg-Elert’s annotation of 9½ *Schritte* is in error, as the modulation actually travels 8½ steps higher on the chain of fifths: eight fifths above the initial D♭ major is A major, and the extra half-step is to its *Leittonwechsel* C♯ minor. The *Urspungslage* analysis under the passage indicates that the music traverses two intermediate regions: B♭ major (*B dur Bereich*) and G major (*G dur Bereich*). The *Ursprungsangen* also show that the passage contains only four chords that involve syntonic commas, namely the contrant-variants (chords 5, 7, 10 and 13). Crucially, the pivot chords between the keys (chords 4, 8 and 11) are in all cases diatonic substitutes (*D*<sub>L</sub>, *Cp*) or principal triads (D) in both the old and new keys, thereby avoiding syntonic-comma ‘variant tricks’.

**Figure 6.1.5.** Comma-free modulation from D♭ major to C♯ minor (*Akustische* 7.3)
The Tonnetz for the preceding passage (Figure 6.1.6) is an analogue of the *Ursprungslage* analysis, illustrating the journey of $8\frac{1}{2}$ fifths upward from the opening $D\flat$ major. Where Karg-Elert employs rising pitch register to indicate the journey in pitch space, the Tonnetz employs rightward motion along the horizontal axis from the source (chord 2) to the goal (chord 12):

![Tonnetz Diagram](image)

**Figure 6.1.6.** Tonnetz for Figure 6.1.5

The next passage also modulates from $D\flat$ major to $C\#$ minor, but now with the path “shortened by means of variants” (Figure 6.1.7), and is therefore comma-differing:

![Musical Example](image)

**Figure 6.1.7.** Comma-differing modulation from $D\flat$ major to $C\#$ minor (*Akustische* 7.3)
Karg-Elert analyzes the passage in Figure 6.1.7 in two ways. The first begins from $D\#_b$ major as the initial tonic. There are two variant tricks, each involving the reinterpretation of a diatonic parallel ($Tp$) as a contrant-variant ($c$) in the key a major third higher:

Chord 2 ($B\#$ minor): $Tp$ of $D\#_b$ major $\rightarrow c$ of $F$ major
Chord 4 ($D$ minor): $Tp$ of $F$ major $\rightarrow c$ of $A$ major

Each of these variant tricks knocks the harmonic path into a different fifth “lane,” running parallel to the central source-position chain, but each displaced by a syntonic comma. Two variant tricks = two syntonic commas. Karg-Elert’s Ursprungslage analysis (in the second half of Figure 6.1.7) is entirely in $F$ major; the choice of that key is likely because it lies midway between its counter-median $D\#_b$ major (chord 1) and mediant $A$ major (chord 5). It indicates that the final $C\#$ minor triad is the dominant mediant-parallel ($D^{Mp}$) of $F$ major, and that it contains two syntonic-comma variant pitches. The following Tonnetz (Figure 6.1.8) follows the second analysis, by placing the $F$ major tonic (not the opening $D\#_b$ major triad) in the fifth chain.

![Figure 6.1.8. Tonnetz for Figure 6.1.7](image-url)
The following passage by Brahms (Figure 6.1.9) was previously cited as Figure 5.2.8, as an example of reiterated fifth transpositions. As Karg-Elert’s annotation indicates, the three principal chords (Eb major, G major and B major) are not mediants of each other. Rather, the progression ascends through a series of eight fifths from Eb major, without comma differences, outlining a “pure principal-succession” (i.e. a chain of dominants). It is another example of comma-free modulation, as shown on the accompanying Tonnetz (Figure 6.1.10). The numbers in both Karg-Elert’s analysis and the Tonnetz specify the distances in fifths from the source triad, illustrating the unidirectional trajectory in pitch space from the initial chord.

Figure 6.1.9. Comma-free modulation in Brahms, Rhapsody op. 79 no. 2 (Harmonologik, 145)

Figure 6.1.10. Tonnetz for Figure 6.1.9
6.2. Beethoven, Sonata in C major, op. 53 ("Waldstein") – first movement

In his essay on “the mediantic style,” Karg-Elert remarked that Beethoven’s middle-period music “displays a remarkable predilection for sudden “shifts” and enharmonic reinterpretations…which often result from third relationships.”\(^{11}\) A case in point is the first movement of the *Waldstein* Sonata in C major (op. 53), which “poses a host of mediant- and apparent mediant-problems. In the first movement, C major is continually put aside, and only at the beginning and end, plus some episodes in the development, is it established as a true tonic.”\(^{12}\) The first part of the principal theme (mm. 1-13) is firmly in the main key, ending on the dominant. Always attuned to acoustic differences, Karg-Elert notes that the connection “between G major and Bb major [mm. 4-5] is not a third relationship, but a triple fifth-relationship” (Figure 6.2.1); the D’s in the two chords differ by a syntonic comma, and the acoustic separation is marked by the brackets \[\] between the \(D\) and \((C)\) function labels. The measure numbers for each harmony are provided, allowing easy comparison with the score.

\[\text{Figure 6.2.1. Beethoven: Sonata op. 53, I (mm. 1-13), analysis from Harmonologik (p. 240)}\]

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\(^{11}\) *Harmonologik*, 205.

\(^{12}\) Ibid., 240.
“The [first] group is harmonically altered and formally extended, as follows” (Figure 6.2.2).

The passage (mm. 15-34) ends on a prolonged B major harmony, which acts as a transition (mm. 23-28), medial caesura\(^{13}\) (m. 29) and fill (mm. 29-34), preparing the secondary theme in measure 35. B major enters as the contrant-variant (\(\mathfrak{o}\)) of E minor, which is displaced by a syntonic comma. However, the repetitive figures, continuous energy gain and harmonic acceleration in mm. 23-28 rhetorically mark B major as a dominant preparation; it is thus “reinterpreted as D, in the mediant-type E major.” As in mm. 4-5, there is a metharmonic separation between G major (\(D\)) and D minor (\(C_p\)), which contain syntonic comma-different D’s.

Karg-Elert notes that “the following 40 bars (secondary theme and following material) are in E major = \(T^H\) of the opening C major tonic (\(T\)). The return (“bridge”) then nullifies the mediant!”\(^{14}\) This statement is accurate, as the secondary key of E major is indeed the mediant of C major, located one major third above the opening triad; the modulation involves a variant trick, reinterpreting the contrant-variant of E minor (mm. 23-30) as the dominant of E major (mm. 31-34). Thus, the initial modulation is comma-differing. The nullification of the mediant takes place during the “bridge” (mm. 74-86), which follows the essential expositional cadence\(^{15}\) in bar

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\(^{13}\) Hepokoski and Darcy 2006, 23–25.

\(^{14}\) Harmonologik, 240.

\(^{15}\) Hepokoski and Darcy 2006, 120–124.
and leads back to the opening –ey of C major. The E major tonic \( T \) is transformed into its variant E minor \( t \), reinterpreted as \( T^l \) of C major in its original position (Figure 6.2.3):

![Musical notation](image)

Figure 6.2.3. Beethoven: Sonata op. 53, I (mm. 74-87) – bridge back to C major

The real significance of Karg-Elert’s analysis lies in his discussion of the development, which “as always initiates a strong modulatory life (starkes modulatorisches Leben).” He remarks that the development features a “strong counter-mediant emphasis, to balance the mediant shift in the exposition.” To be exact, the development does not include lengthy passages in the actual counter-mediant key, namely A\( \text{♭} \) major. However, most of the development does descend into a ‘flatward’ region of pitch space, and mm. 104-111 are largely oriented around F minor (the relative of A\( \text{♭} \) major). The following Ursprungslagen analysis (Figure 6.2.4, from
Harmonologik p. 240) reduces the entire development, which begins with the original tonic C major. Measure numbers have been added, along with Tonnetz stations (see Figure 6.2.5 below). Most harmonies in the passage are included, except for immediate repetitions; however, some detail is omitted, especially in mm. 107-111, which expand the key of F minor.

The following Tonnetz (Figure 6.2.5) diagrams Karg-Elert’s reduction of the development section. It graphically demonstrates how most of the section proceeds in a flatward direction from the opening C major (chord 1), as far as its lower Chromonant C♭ major (chord 10):
The key moment occurs in measures 124 to 127, where an “enharmonic derailment” (enharmonische Entgleisung) connects Eb minor (chord 12) directly to F# major (chord 13), equating Bb with A# and Gb with F#, and thereby erasing not only the enharmonic differences, but also the difference of two syntonic commas. The purpose of the enharmonic shift is to ensure closure in the opening C major, in its source position. Karg-Elert notes that “such enharmonic derailments become more and more frequent in the music of Schubert, Chopin, Liszt, Wagner, Bruckner and Reger, finally overturning tonality itself.” The analysis in Figure 6.2.4 suggests that if the F# major chord in mm. 126-127 was properly understood as Gb major (as dictated by the preceding Eb minor triad), the succession that follows would continue to descend into further flatward regions. This would result in a resolution to Dbb major at the recapitulation, and thus a lack of tonal closure. This alternative path is illustrated in Figure 6.2.6, which removes the enharmonic derailment in m. 126. The final tonic is Dbb major (chord 17), which is two syntonic commas higher than the opening C major.

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16 Harmonologik, 240.
17 Ibid.
Figure 6.2.6. Beethoven: Sonata op. 53, I (development) – without enharmonic derailment

The lesson to be learned from the foregoing analysis of Beethoven’s op. 53 is that even for Karg-Elert, who fully recognizes and accepts syntonic comma differences between pitches and triads, the desire for tonal unity and closure can override local chord-to-chord common-tone connections. He chooses to respect Beethoven’s notation faithfully, by accepting the jarring enharmonic derailment of $[E\flat_m \rightarrow F\sharp M]$ in mm. 124-127, rather than to negate the return of the original tonic at the recapitulation. Karg-Elert’s analysis highlights the radical and dramatic agency of the enharmonic shift, a moment that traditional modes of analysis would tend to classify as a convention of chromatic usage, or simply as a consequence of equal temperament.
6.3. Liszt: “Wieder möcht’ ich dir begegnen” (Lied, 1860)

This song for voice and piano sets a love poem by Liszt’s friend, the composer and writer Peter Cornelius (1824-1874). Here is the poem, plus an English version by John Bernhoff:\textsuperscript{18}

\begin{multicols}{2}
\begin{flushleft}
Wieder möcht' ich Dir begegnen, \\
Wieder schauen Deinen Blick; \\
Aber was auch mein Geschick, \\
Deine liebe Seele will ich segnen.
\end{flushleft}
\begin{flushright}
Could I once again caress thee
Whom my heart has ne'er forgot!
But whatever be my lot,
For thine angel soul I'll pray, and bless thee.
\end{flushright}

\begin{flushleft}
Leben möcht' ich Dir zu Füßen, \\
Blumen streuen vor Dich hin, \\
Aber, ob ich ferne bin, \\
Deine liebe Seele will ich grüßen.
\end{flushleft}
\begin{flushright}
At thy feet would I be kneeling,
Strew thy path with flow'rs of May,
Yet though I be far away,
To thine angel soul my thoughts are stealing.
\end{flushright}

\begin{flushleft}
Blieb' ich ewig auch vertrieben, \\
Meinem reinsten Glücke fern, \\
Deine Seele ist mein Stern, \\
Deine liebe Seele will ich lieben.
\end{flushleft}
\begin{flushright}
Be a home on earth denied me,
Tho' from thee I wander far,
Let thy spirit be my star,
Let thine angel soul, love, guard and guide me!
\end{flushright}
\end{multicols}

The form of Liszt’s setting can be described as modified strophic: the first two verses are set to identical music, while the third verse begins differently, but ends in a similar way to the first two. Each verse is preceded by an identical piano prelude of four measures, expanding the tonic key of E major. The music of each verse contains two distinct parts. The first two lines are set in quadruple meter, in a calm recitative style with a sparse chordal background. In contrast, the last two lines of each verse are in triple meter, supported by pulsing eighth notes in crescendo and accelerando. Notably, the first half of each verse is mostly diatonic, while the second half features sequences of rising thirds. Karg-Elert’s discussion of the song focuses mostly on those sequences, and their effect on tonal unity and closure. \textbf{Figure 6.3.1} contains his analysis of the second half of verse 1 (mm. 9-19); the same music concludes verse 2 (mm. 28-38).

\textsuperscript{18} http://www.lieder.net/lieder/get_text.html?TextId=76360. John Bernhoff’s translation is in the public domain, and was included with the Oliver Ditson publication of the song (Boston, 1911). The translation matches the rhyme scheme of the original poem, but often not its imagery.
Figure 6.3.1. Liszt, “Wieder möcht’ ich dir begegnen,” mm. 9-19 (Harmonologik, 268)

The passage in Figure 6.3.1 outlines a cycle of ascending minor thirds, which in equal temperament can be described as a symmetrical division of the octave, notated as [EM \rightarrow GM \rightarrow BbM \rightarrow C#M \rightarrow EM]. Though the final E major is preceded by a cadential series of fifths, the overall minor-third cycle is clear. Karg-Elert analyzes the first two steps in the cycle as a motion from tonic (T) to dominant counter-median (DM), with each chord introducing its concordant seventh. He is faced with a choice in mm. 13-14, which is notated not as a minor-third ascent like the first two steps in the sequence, but enharmonically as an augmented-second root motion [BbM \rightarrow C#M]. If Liszt’s notation is to be respected strictly, the result is an “enharmonic derailment” or pitch-space ‘quantum leap’ like that featured in the development of Beethoven’s
op. 53: the C♯ major triad would be the triple dominant of E major, very distant from the B♭ major triad in bar 13. On the other hand, if C♯ major is understood as D♭ major, the third step in the sequence can be heard in the same way as the first two steps. Karg-Elert chooses the latter option: mm. 13-14 are labelled as $T \rightarrow D_M$, which is [B♭M → D♭M]. The latter triad then initiates a chain of dominants, reaching a cadence that is notated in E major, but which is really in F♭ major in pitch-space, as illustrated on the Tonnetz (Figure 6.3.2). The entire verse thus outlines a comma-differing modulation, from the opening source-position E major to its enharmonic counterpart F♭ major, three major thirds lower. The first verse is understood not as a tonally closed “ring,” but rather as a tonally open “spiral.”

![Tonnetz for Liszt, “Wieder möcht’ ich dir begegnen,” verse 1](image)

**Figure 6.3.2.** Tonnetz for Liszt, “Wieder möcht’ ich dir begegnen,” verse 1
Verse 2 (mm. 20-38) contains exactly the same music as Verse 1. If the second verse’s opening E major triad is identified with the F♭ major final cadence of Verse 1, then Verse 2 would conclude in the key of G♭♭ major (i.e. six major thirds below the original E major). However, Karg-Elert recoils from that possibility, instead beginning the second verse anew from the first verse’s original E major tonic. Verse 3 (mm. 39-58) begins differently from the others, with a new recitative section that tonicizes the relative key of C♯ minor. Its second half starts from a notated E major, and features a sequence of rising major thirds. Karg-Elert’s analysis of this passage (mm. 47-58) is provided in Figure 6.3.3.

Analyzed as:  
EM  G♯M  B♯M  
Tonnetz: 1  2  3

Figure 6.3.3. Liszt, “Wieder möcht’ ich dir begegnen,” mm. 47–58 (Harmonologik, 268–9)
Once again, in equal temperament the sequence in Figure 6.3.3 can be understood simply as a symmetrical division of the octave, with one of the major-third steps notated enharmonically as a diminished fourth, as in Liszt’s score: [EM → A♭M → CM → EM]. The enharmonic shift is made explicit in the composer’s notation, which switches from G♯ major in bar 48 to A♭ major in the following measure. However, Karg-Elert’s analysis chooses a different path, respecting the common-tone connections from chord to chord, rather than Liszt’s notated enharmonic shift. The first major-third step (mm. 47-48) is labelled as a motion from contrant-variant-parallel (cP) to tonic (T), in the key of G♯ major. The next two steps in the sequence are analyzed in the same way, and thus reflect the progression [EM → G♯M → B♭M → D♭M]. As in the first verse, the final step in the sequence is preceded by a chain of applied dominants, leading to the final cadence. Figure 6.3.4 illustrates the progression of verse 2 in pitch space, considered in isolation; its path moves ‘polaristically’, in the opposite direction from the first verse: “where the first verse ends in F♭ major (Fes dur), the second ends in D♭ major (Disis dur).”¹⁹ Verse 2 thus outlines another comma-differing modulation, from E major to another enharmonic counterpart.

Figure 6.3.4. Tonnetz for Liszt, “Wieder möcht’ ich dir begegnen,” verse 2

¹⁹ Harmonologik, 269.
As his analysis of the *Waldstein* development section demonstrated, Karg-Elert often finds a way to ensure tonal return to the opening tonic, even when local harmonic progressions suggest otherwise. He concludes the Liszt analysis with the following reduction of the entire song (Figure 6.3.5), which illustrates that “if the second verse begins with the key that ends the first verse, so the entire progression moves from E major to Fb major, then back to E major.”

![Tonnetz diagram](image)

**Figure 6.3.5.** Liszt, “*Wieder möcht’ ich dir begegnen,***” tonal plan (*Harmonologik*, 269)

This interpretation requires hearing verse 2 as an exact repetition of verse 1, beginning again from the opening E major. While the minor-third (*Nebenmedianten*) sequence in verse 1 plunges the music into the faraway region of Fb major, the major-third (*Medianten*) sequence in verse 3 rises from the depths to recapture the original tonic pitch E in the C major triad in bar 51 (shown as an open notehead in Figure 6.3.5), ultimately ensuring tonal closure in the source-position E major of the song’s opening measures. Thus, while each verse individually enacts a comma-differing modulation, the entire song outlines a unified, comma-free tonal plan. The following Tonnetz (Figure 6.3.6) illustrates the song’s circular, tonally closed “ring” scheme:

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20 *Harmonologik*, 269.
In sum, Karg-Elert’s Liszt analysis reconciles his penchant to follow local harmonic motions wherever their common-tone connections lead in pitch space, with his lingering desire for large-scale unity and closure.
6.4. Schumann: Novellette in F major, op. 21 no. 1

Karg-Elert begins his analysis with the following comment: “This over-popular piece is tonally rather confused (just as Schumann’s harmony and rhythms often reveal themselves after analysis).”\(^{21}\) The Novelette’s form resembles a seven-part rondo, with the following tonal plan:

<table>
<thead>
<tr>
<th>SECTION</th>
<th>MEASURES</th>
<th>KEY</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1-20</td>
<td>FM $\rightarrow$ DbM $\rightarrow$ AM</td>
</tr>
<tr>
<td>B1 (Trio I)</td>
<td>21-48</td>
<td>FM</td>
</tr>
<tr>
<td>A</td>
<td>49-60</td>
<td>FM</td>
</tr>
<tr>
<td>C (Trio II)</td>
<td>61-81</td>
<td>DbM</td>
</tr>
<tr>
<td>A</td>
<td>82-85</td>
<td>DbM</td>
</tr>
<tr>
<td>B2 (Trio I)</td>
<td>86-113</td>
<td>AM</td>
</tr>
<tr>
<td>A</td>
<td>114-137</td>
<td>FM</td>
</tr>
</tbody>
</table>

The opening of each section is provided in the first example in *Harmonologik* (Figure 6.4.1):

![Figure 6.4.1](image)

**Figure 6.4.1.** Schumann: Novelette op. 21 no. 1 – form and themes (*Harmonologik*, 253)

\(^{21}\) *Harmonologik*, 253.
Karg-Elert first examines the key relationships between the sections: the main key is clearly F major, and one might assume the subsidiary keys of Db major and A major to be its counter-median (\(T_M\)) and mediant (\(T^M\)) respectively, creating a balanced and coherent tonal scheme. However, that interpretation once again assumes equal temperament and enharmonic equivalence, as a closed cycle of ascending major thirds must include one step spelled as a diminished fourth: [FM \(\rightarrow\) DbM \(\rightarrow\) AM \(\rightarrow\) FM]. Karg-Elert instead analyzes each change of key as a motion to the counter-median [i.e. a transposition to the major third below], resulting in the following open progression of major thirds, without enharmonic adjustment: [FM \(\rightarrow\) DbM \(\rightarrow\) Bbm \(\rightarrow\) Gbm].

The opening A section (mm. 1-20) outlines the same tonal plan as the entire piece: [FM \(\rightarrow\) DbM \(\rightarrow\) AM], returning abruptly to F major for the B section (m. 21). As Karg-Elert states, that sequence “might ex abrupto be identified as \(T \rightarrow T_M \rightarrow T^M\), but the linear progression of the harmony absolutely rejects this interpretation! Instead, the stations are as follows” (Figure 6.4.2); the same passage is provided in score in Figure 6.4.3 (next page).

![Harmonic analysis of mm. 1–21](image)

**Figure 6.4.2.** Schumann, Novelette op. 21 no. 1 – harmonic analysis of mm. 1–21

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22 *Harmonologik*, 253.
Figure 6.4.3. Schumann: Novelette, op. 21 no. 1 (section A and beginning of B1, mm. 1–23)

The functional analysis in Figure 6.4.2 illustrates how measures 5 to 8 follow a descending sequence of fifths; Schumann never actually states the minor triads listed in the analysis, but his progressions do imply and tonicize those keys. After a phrase in a stable $D_b$
major (mm. 9-12), the descending-fifth sequence from mm. 5-8 recurs in bars 13 to 16, now leading into ever darker and more obscure keys. The composer’s notation continues in a flatward direction as far as possible, even including G♭ minor and G♭ minor (mm. 15-16). Schumann finally yields to the norms of notational practice, switching to sharps in measure 17; however, this enharmonic switch can be regarded as purely for the sake of legibility. Because measures 13-20 are an exact transposition of mm. 5-12, the true harmonic meaning of mm. 17-20 is revealed when the flats are restored. The notated F♯ minor in m. 17 is actually G♭ minor, and the section ends not in the notated A major, but in B♭ major (Bes dur). Karg-Elert specifies that “between F, D♭ and B♭ major…are no mediant relationships. Rather, the progression is absolutely fifth-based…. The chord-to-chord motions define the entire modulation to B♭ major as comma-free, moving flatward by eight fifths along the chain of source-position triads, as illustrated by the numbers in Figure 6.4.2, and in the following Tonnetz (Figure 6.4.4). Only the final motion into the B1 section [B♭M → G♭M, notated as AM → FM] is a true counter-median transposition, one that finally leaves the central fifth-chain and introduces syntonic-comma difference.

![Tonnetz for Schumann, Novelette op. 21 no. 1, mm. 1–21](Figure 6.4.4)

Unlike the opening A section, the other sections of the Novelette are tonally closed, staying entirely within a single key, in spite of momentary chromatic events. This is even true of
the second and third statements of the A material. Measures 53-60 recall the sequential passage of mm. 5-8, but altered to remain in the notated key of F major; mm. 82-85 only recall mm. 1-4, entirely in Db major. In contrast to the descending-fifth trajectory of the first A section, the rest of the piece outlines a chain of counter-mediant, as each change of key is an abrupt downward transposition of a major third, as in mm. 20-21. If one follows the implications of Karg-Elert’s analysis of the first section to their ultimate conclusion, the following tonal plan results:

<table>
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<th>KEY</th>
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<td>GbM ( = FM)</td>
</tr>
<tr>
<td>A</td>
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<td>GbM</td>
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<td>C (Trio II)</td>
<td>61-81</td>
<td>EbM ( = DbM)</td>
</tr>
<tr>
<td>A</td>
<td>82-85</td>
<td>EbM</td>
</tr>
<tr>
<td>B2 (Trio I)</td>
<td>86-113</td>
<td>CbbM ( = AM)</td>
</tr>
<tr>
<td>A</td>
<td>114-137</td>
<td>AbbbB ( = FM)</td>
</tr>
</tbody>
</table>

The tonal plan of the entire piece thus unfolds a chain of counter-mediant, descending ever further into extreme ‘flat’ regions of pitch space, never returning to the opening key:

\[
\begin{align*}
FM & \rightarrow DbM \rightarrow BbM \rightarrow GbM \rightarrow EbM \rightarrow CbbM \rightarrow AbbbB \\
T & \rightarrow TM \rightarrow (TM) \rightarrow (TM) \rightarrow (TM) \rightarrow (TM) \rightarrow (TM)
\end{align*}
\]

This is a radically “non-conformist” interpretation of the Novelette’s tonal plan, one that chooses to reject the composer’s notated key areas, to follow instead the local harmonic progressions in the opening section, and the abrupt major-third shifts between the other sections. One might wonder why Karg-Elert was willing to accept Beethoven’s “enharmonic derailment” in order to ensure tonal closure and unity in the Waldstein analysis, but completely rejects all the notated enharmonic shifts in the Schumann. His description of the latter’s effect on the listener may reveal the answer: “this piece unfolds a virtually endless wandering into previously unknown locations. Very easy to understand from station to station, its beginning and end are made
invisible to each other by shadowy distances.” In contrast to the tonally-closed Classical “architecture” of the Beethoven, the open-ended tonal scheme of the Schumann reflects the Romantic motif of the journey, one without a discernible destination: “The reader [of literature] understands, and will immediately recognize here a typical case of the Romantic essence: all boundaries of possibility stretch into infinity…” Karg-Elert concludes his discussion of Schumann’s Novelette by considering both the benefits and the deficits of 12-TET:

The technical description of such boundless expanses faces insurmountable problems. The equal-tempered 12-semitone system of course eliminates all metharmonic and enharmonic conflicts (comma differences), but it also re-routes wide harmonic paths, turning straight lines into backward-turning curves. The perception of absurd distances is thus relegated to the realm of fantasy [NB: also a typically Romantic notion!].

In this last statement, Karg-Elert acknowledges that 12-TET and enharmonic equivalence greatly simplify the understanding of music such as the Schumann Novelette, and are indeed necessary to ensure the tonal unity that is expressed in the composer’s notation. But while most modern analysts will reject the notion of “modulating” from F major to A♭♭♭♭ major (keys which are sonically identical in equal temperament), it is much harder to deny the validity of Karg-Elert’s interpretation of the opening section’s tonal structure, which is for the most part confirmed by Schumann’s notation. His analysis of the A section nicely captures its feeling of a continuous unidirectional path through boundless pitch space; in contrast, a standard equal-tempered view of the A section’s key areas (and in turn those of the whole piece) as a symmetrical cycle of equal-sized major thirds turns Karg-Elert’s wayward “straight lines” into streamlined “backward-turning curves,” and a distant journey into a much shorter circular route.

23 Harmonologik, 253.
24 Ibid.
6.5.  Wagner: ‘Schlafakkorde’ from *Die Walküre* (Act III, scene 3)

Karg-Elert’s analysis of the famous ‘Schlafakkorde’ (“sleep chords”) from the third act of *Die Walküre* is brief, consisting of the progression in its most frequently cited appearance (mm. 1617-1625 of Act III), plus complete functional analysis (in two levels), and a listing of the transformations that link each pair of adjacent chords. The passage features four transformations in total: the common-tone transpositions *Nebenmediante* (as $T \to T^p$ or $T^m \to D$) and *Dominante* (consistently as $D \to T$), plus the non-common-tone semitone transpositions *Leitklang* and *Chromonante* (see section 5.5). Karg-Elert’s analysis is provided in Figure 6.5.1 (next page).

Between the passage itself and the functional analysis, triad labels (based on Karg-Elert’s function labels, to be discussed presently) and Tonnetz stations have been inserted. The functional analysis interprets every chord in the passage as a major triad; this includes the diminished sevenths (the last chord in each two-bar segment), plus the dissonant chord 7, which is labelled as a the dominant of G major with semitone-raised root ($D \to E_b$) and both regular and raised fifth ($A$ and $B_b$). With one exception, Wagner’s spelling for each chord is taken at face value; only chord 2 is reinterpreted enharmonically as $C_b$ major (*Ces dur*) instead of $B$ major. This single change allows Karg-Elert to analyze measures 1-2 and 5-6 as exact transpositions of each other, and thus to label the first chord pair [$A_bM \to C_bM$] as a *Nebenmediant* (minor-third) transposition, analogous to [$CM \to E_bM$] in bar 5, and [$G#M \to BM$] in bar 7.
Figure 6.5.1. Wagner: ‘Schlafakkorde’ from Die Walküre, Act III scene 3 (Harmonologik, 280)
In addition to the chord-to-chord analysis, Figure 6.5.1 examines the transformations that link the main chords in the passage: the first chord in each two-bar segment, each of which is understood as a local tonic. E major occurs twice (measures 3 and 9), and thus can be regarded as the primary tonic for the passage, as indicated by Karg-Elert’s network of arrows, in which all other keys project outward from E major. That part of the example closely resembles a Lewinian transformational network, as shown in Figure 6.5.2. The only difference is that Karg-Elert has E major pointing to C major twice, for reasons that will be discussed below (see Figure 6.5.7).

![Network of arrows for the Schlafakkorde keys, and a Lewinian version](image)

**Figure 6.5.2.** The network of arrows for the Schlafakkorde keys, and a Lewinian version

From the perspective of the network in Figure 6.5.2, all of the key areas in the Schlafakkorde are related by mediant and counter-mediant transformations, and are thus linked by acoustic common tones, as shown in the following Tonnetz (Figure 6.5.3):

---

26 This interpretation is suggested in Karg-Elert’s heading to Figure 6.5.1: “mediant-grouped chromatic scale,” which refers to the upper-voice descent and the major-third succession.
Figure 6.5.3. Tonnetz for the Figure 6.28 transformational network

Except for the fact that Karg-Elert distinguishes between the triads/keys of A♭ major and G♯ major, the network outlined in Figures 6.5.2 and 6.5.3 in some ways resembles that presented in Brian Hyer’s transformational analysis of the same passage, in an early article of neo-Riemannian theory.\(^{27}\) Hyer’s Example 5 is reproduced in Figure 6.5.4 (next page). In order to demonstrate the cyclic properties of his transformational group, Hyer appends four measures from earlier in Act III which begin the *Schlafakkorde* progression on E major; this is why his example jumps from measure 1538 to measure 1617 (where Karg-Elert’s example begins).

Figure 6.5.4. Hyer 1995, 112 – transformational analysis of the Schlafakorde
Hyer’s transformations first link the downbeat chords every two measures, in the order they appear in the music. The transformation is consistently PL, which transposes each major triad/key down a major third (sometimes spelled as a diminished fourth, as Hyer’s system assumes equal temperament and enharmonic equivalence). When PL indicates a true descending major-third transposition (i.e. one that is notated as such), it is equivalent to Karg-Elert’s Gegenmediant or $T_{i\rightarrow\text{iii}}$ transposition, usually labelled as $T_M$. Hyer’s Example 5 then identifies the ascending major-third LP transformation that links the passage’s main triads, on every fourth downbeat: $[\text{EM} \rightarrow \text{AbM} \rightarrow \text{CM} \rightarrow \text{EM}]$. When LP indicates a true ascending major-third transposition, it is equal to Karg-Elert’s Mediant or $T_{\text{ii} \rightarrow \text{i}}$ transposition, usually labelled as $T^M$. A Tonnetz for Hyer’s networks would look the same as in Figure 6.5.3, except that the Ab major and G♯ major triads should be conformed enharmonically, creating a loop instead of a plane.

An interesting aspect of Karg-Elert’s Schlafakkorde analysis that is not explicitly discussed in Harmonologik is that the local chord-to-chord progression outlines a very different tonal journey than that implied by the network of major-third-related keys in Figure 6.5.2. Based on the local function labels in Figure 6.5.1, Figure 6.5.5 plots each chord of the passage on a Tonnetz. The numbers for the Tonnetz stations are provided in Figure 6.5.1.

![Figure 6.5.5. Tonnetz for the Schlafakkorde, based on the functional analysis in Figure 6.5.1](image-url)
And here is the same Tonnetz, with all but the tonics or key areas removed (Figure 6.5.6):

Figure 6.5.6. Tonnetz from Figure 6.5.5, showing only the tonics/key areas

Figures 6.5.5 and 6.5.6 reveal that if one follows the path defined by the local chord progressions, the keys of A♭ major, C major and E major are in fact not related by mediant transformations or connected by acoustic common tones, as shown in Figure 6.5.3. Instead, those keys are actually fifth-related to each other: C major (bar 5) is four fifths above the opening A♭ major, and the ultimate goal triad of E major (bar 9) is another four fifths higher. All three triads are in their source positions, and so the entire passage presents a comma-free modulation. The intervening E major (m. 3) and G♯ major (m. 7) tonics can be regarded as counter-mediant (T_M) prefixes to the main fifth-related keys. Notably, the E major triad in bar 3 is not to be equated with the final goal key, either acoustically or in terms of its functional status within the passage. The following network (Figure 6.5.7) illustrates the transformations that operate in the passage, as revealed in Figure 6.5.6; the principal keys are at the bottom, and their prefixes above.

Figure 6.5.7. Transformational network for the Schlafakkorde keys, based on Figure 6.5.6
Perhaps the main lesson to be learned from Karg-Elert’s discussion of the *Schlafakkorde* is that as in all methods of harmonic and tonal analysis, the relationships to be discovered will depend on the objects that are chosen for comparison. When the focus is directed to the larger-scale connections between key areas, a cycle of mediant-related major thirds (and thus a comma-differing modulation) seems apparent and logical. In contrast, detailed observation of the local harmonic succession reveals that the principal tonics are actually related by quadruple fifth rather than major third, and thus that the overall modulation is in fact comma-free. The former interpretation (Figure 6.5.2) is more straightforward and more normalized, as it essentially classifies each tonic as equal members in a chain of major thirds. The latter (Figure 6.5.7) is more complex and arguably more interesting, as it suggests a degree of hierarchy among the key areas: the principal (comma-free) keys are related by fifth-based transformations, while the subsidiary (comma-different) keys are related to the principal ones by third-based transformations.
6.6. Other analytical discussions of key relations in a just-intonation space

It is no exaggeration to state that most methods of harmonic analysis would simply not work without the metharmonic and enharmonic identification of triads and keys provided by the equal-tempered 12-tone system. Even analysts such as Hugo Riemann who strongly believed in the natural derivation of intervals (and thus in just intonation) generally assumed 12-TET and enharmonic equivalence, as shown by his system of Schritte and Wechsel (see section 5.1). Therefore, few published analyses of chromatic music faithfully observe metharmonic and enharmonic differences in the manner of Karg-Elert. However, there are some notable exceptions, in the writings of Karg-Elert’s contemporaries, and in more recent scholarship. This section will briefly discuss three analytical writings that consider how tonal relationships can be understood in a non-equal-tempered space.

A celebrated test case for the study of enharmonicism is the third movement (Funeral March) of Beethoven’s Piano Sonata op. 26. Alexander Rehding has described how this movement’s unorthodox tonal plan and abrupt enharmonic shifts attracted the attention of several nineteenth-century scholars, including Weitzmann, Oettingen and Riemann. Karg-Elert also briefly analyzes the movement’s opening section in Harmonologik. Riemann discussed the movement’s opening section in two writings: briefly in the Skizze of 1880, and at greater length in the second volume of his analytical survey of the Beethoven piano sonatas, published in 1918 and 1919. The latter text’s main analytical example is provided in Figure 6.6.1, which outlines the movement’s first twenty-one bars; the quoted melody is actually a moving inner voice.

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28 Rehding 2011, 116 refers to Weitzmann 1861, 28; Oettingen 1866, 143; Riemann 1880, 82. The passage is also discussed in Imig 1973, 35–6; and Rehding 2011, 116–8.
29 Harmonologik, 238–9.
30 Riemann 1880, 82.
Section II (mm. 9-16) is a near-exact repetition of Section I (mm. 1-8), transposed up a minor third. Section III (labelled by Riemann as Zwischen-Halbsatz or “transitional half-section”) is largely a dominant preparation for the return to A♭ minor in bar 21. The tonal scheme of measures 1-21 can be summarized as follows:

<table>
<thead>
<tr>
<th>Section I</th>
<th>Section II</th>
<th>Section III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(mm. 1-8)</td>
<td>(mm. 9-16)</td>
<td>(mm. 17-21)</td>
</tr>
<tr>
<td>Notation:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Abm → C♭M</td>
<td>Bm → DM</td>
<td>Do7 → Eb7 → Abm</td>
</tr>
<tr>
<td>Enharmonic:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Abm → C♭M</td>
<td>C♭m → E♭♭M</td>
<td>E♭♭♭♭♭7 → F♭♭♭♭♭7 → B♭♭♭♭♭m</td>
</tr>
</tbody>
</table>

In 1880, Riemann described the opening section as follows: “In the Funeral March of the Sonata op. 26, Beethoven replaces C♭ minor [in bar 9] with B minor for simplification, and then

---

32 Figure 6.6.1 was reproduced from Rehding 2011, 117; see his Example 4(a).
modulates through D major, B♭ major and E♭ major back to A♭ minor; but without the enharmonic switch from C♭ minor to B minor, the already stated A♭ minor would actually be B♮♭♭ minor.” Riemann’s 1918-19 analysis does not address the issue of the enharmonic shift in mm. 8-9, choosing instead to modulate several times to accommodate the notated keys (see the boxes in Figure 6.6.1, which were added to Riemann’s examples in Rehding 2011). The lack of attention paid to the enharmonic shift likely reflects how by 1918, Riemann had essentially accepted the reality and efficacy of equal temperament and enharmonic equivalence, as will be discussed further in chapter 7 of this dissertation.

In his 1931 Companion to the Beethoven Pianoforte Sonatas, British analyst and composer Donald Tovey remarked on a passage near the end of the development section in the first movement of the Piano Sonata in F minor, op. 57 ‘Appassionata’. Tovey stated:

“The bass [in bar 120] is on C-flat = B-natural. The next two bars bring it to C. The key of C-flat minor is written as B minor; once more we are in an enharmonic circle – in the same direction as the previous one – so we will not enquire into the relation of E-quadruple flat to the home dominant, but will attend to serious matters.”

Eric Wen has closely examined and discussed Tovey’s “whimsy,” in an article that compares its radical just-intonation perspective (reminiscent of Karg-Elert’s discussion of the Schumann Novelette) to Beethoven’s enharmonically-shifting notation, and Schenker’s analysis of the movement. Essentially, Tovey notes that the sonata contains two “enharmonic derailments,” one at the beginning of the development (m. 65), and the other near the end (m. 120):

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33 Riemann 1880, 82. Rehding 2011, 118 provides a Tonnetz for Riemann’s 1880 analysis.
34 Rehding 2011, 117.
35 Tovey 1931, 180. Quoted in Wen 2011, 82.
36 Wen 2011.
End of exposition (m. 64) \(\rightarrow\) Beginning of development (m. 65)

\[A_b\text{ minor} \rightarrow G\#\text{ minor (} = A_b\text{ minor)}\]

Bar 119 \(\rightarrow\) Bar 120

\[G_b\text{ major} \rightarrow B\text{ minor (} = C_b\text{ minor)}\]

If the first enharmonic shift in bar 65 is removed, the tonal trajectory of the entire development would be drastically altered. Wen traces the chord-to-chord progression of the development starting from \(A_b\text{ minor}\), in order to test Tovey’s remark; he finds that the section would indeed culminate in bar 122 on a chord of \(E_{\#\#}\text{ major}\), enharmonically equivalent to the dominant of the work’s main key of F minor.\(^{37}\) Of course, Tovey forgets his passing reference to E-quadruple flat as quickly as he makes it, and proceeds to discuss the sonata’s tonal structure in a more conventional manner. Nonetheless, his remark reveals his awareness of how enharmonic shifts can mask local harmonic perceptions, in order to enable large-scale tonal coherence. In addition, it displays an affinity for tonal relations that extend beyond the equal-tempered universe, one which accords well with Karg-Elert’s analytical perspective.

Probably the most thorough exploration of key relations in an “unconformed” (i.e. non-12-TET and non-enharmonically equivalent) harmonic space is Daniel Harrison’s 2002 article “Nonconformist Notions of Nineteenth-Century Enharmonicism.”\(^{38}\) The article views enharmonicism as “that feature of late nineteenth-century music” that skirts “the margins of ‘sameness’ and ‘difference’”\(^{39}\) – which is in fact the central concern of Karg-Elert’s theory of comma-free and comma-differing modulation, and its distinction between metharmonically-different keys of the same name and notation. Harrison’s analyses of passages from Beethoven to Mahler and Richard Strauss primarily focus on key relationships between sections, and not on

\(^{37}\) Wen 2011, 83 (see the reduction in his Example 9).
\(^{38}\) Harrison 2002.
\(^{39}\) Ibid., 116.
the local chord-to-chord connections that are typical of Karg-Elert’s practice. Nonetheless, Harrison treats keys in much the same way as Karg-Elert treats individual chords: linking them (when possible) via common tones, and then following the paths they forge through a harmonic space that “encourages nomadic movement, and not the permanent settlement of the conceptual idea of ‘A’, ‘Eb’, etc.”

Harrison employs Tonnetze to display key relations and paths. In contrast to the equal-tempered diagrams favoured by his neo-Riemannian colleagues, each of the chords or keys in Harrison’s Tonnetze is considered unique in derivation, and distinct from each other. However, he is not especially concerned with just intonation or comma differences: “we too can imagine tonal relations independent of the materialist, acoustical aspect of the Tonnetz and not bother with the assumption of just intonation.” Differences between keys that differ enharmonically (or between two keys of the same name) are characterized in terms of conceptual distance: “we can still maintain, however, the uniqueness of all objects in the Tonnetz as expressed by their unique co-ordinates with respect to some origin.”

Harrison’s views align closely with Karg-Elert’s statements on just intonation as a model of perception and understanding, a topic that will be examined further in Chapter 7.

A representative example of Harrison’s approach is his analysis of the first movement of Liszt’s Piano Concerto No. 1 in E flat. His Tonnetz for the movement (his Figure 6) is reproduced in Figure 6.6.2. The diamond-shaped “lozenges” represent key areas (all major in this case), articulated by a cadence or a degree of emphasis at the numbered measures. Perfect fifth relations are located on the horizontal rows, with major and minor thirds on the diagonals.

All of the displayed keys in Harrison’s Tonnetz conform to Liszt’s notation, with one exception: Harrison’s F♯ major passage at measure 79 is notated in G major. The movement’s culmination at measure 86 is an orthographic oddity: while the orchestral parts are written in Eb major, the solo piano is in fact notated in D♯ major, with a key signature of five sharps (D♯ minor), adding F♯ and other accidentals as needed. Harrison’s interpretation of the G major passage at bar 79 as F♯ major is surely dictated by the goal key at bar 86: F♯M is the upper mediant (Tʿ in Karg-Elert’s system) of D♯ major. Figure 6.35 recalls Karg-Elert’s analysis of the Liszt song “Wieder möcht’ ich dir begegnen,” in which verse 1 is deemed to modulate from
E major to F♭ major (see Figure 6.3.2), and verse 3 (in isolation) from E major to D♯ major (Figure 6.3.4). Harrison concludes his discussion of the Concerto No. 1 movement by endowing the simultaneous keys of E♭ major and D♯ major with narrative meaning: “the orchestra [E♭M] settling calmly with steady quavers and crotchets into an apparently ‘old’ key while the piano [D♯M] strains ahead with semiquaver sextuplets towards the ‘new’.”41 The piano’s excursion into a “nonconformist” key thus invokes the idea of Romantic ‘wandering’ in search of new tonal vistas, as Karg-Elert noted in his analysis of the Schumann Novelette.

Harrison’s article ends by considering the problem of how objects in music (pitches, chords, keys) are linked with names (letters, accidentals). In 12-TET, the problem is largely non-existent: the name ‘E♭’ refers to a fixed pitch-class in the twelve-semitone system, regardless of its intonation, derivation or context. However, in a just-intonation space (or any other “unconformed” space in which all pitches and keys are considered distinct), a single name such as ‘E♭’ refers to many different objects, out of a potentially infinite number of such objects. The fundamental question is: “how do these new objects relate to those notational objects [names, categories] through which we habitually study music?”42 The issue is ultimately not a matter of acoustics, but rather one of understanding and perception. Karg-Elert’s views on these topics are the primary focus in Chapter 7 of this dissertation.

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41 Harrison 2002, 139-140.
42 Ibid., 152.
Chapter 7

Perception

The reader of Karg-Elert’s treatises (and in particular of Akustische) is immediately presented with a plethora of acoustical calculations, finely distinguishing between intervals that can differ by as little as one millioctave (1.2 cents).\(^1\) In Harmonologik, the same reader will learn that the potentially infinite number of distinct pitches in his three-dimensional space can practically be reduced to 275, encompassing the pitches generally found in most musical passages. Even if millioctave values are omitted (as is the case in most of Harmonologik), the constant use of acoustic symbols to denote pitch derivation reflects how Karg-Elert’s understanding of harmony and tonality depends on differentiation between pitches of similar frequency, and often given the same name. Not long after delving into Karg-Elert’s treatises, the reader will surely ask: what do all of the acoustic symbols and millioctave calculations actually signify? Is the author proposing that musicians should adopt a just-intonation system with 275 or more notes per octave in performance, based on the ‘natural’ origins of that system? Or, does Karg-Elert’s acoustic information intend to represent how we mentally classify and connect pitches, harmonies and keys? This chapter considers these questions and others, in an examination of how Karg-Elert’s model of pitch and harmonic space relates to the mental processing and understanding of music, or what is often described as the perception of music.

The field of music perception has its roots in behavioral and cognitive psychological research from the early and mid-twentieth century, and has developed to a great extent in recent decades, buoyed by a wide array of experiments involving human subjects, as well as clinical

\(^1\) See Akustische 12.5, and also Akustische Appendix A.
research into brain activity measured by advanced medical technologies. Rather than to engage with this vast and ever-growing body of modern scholarship, this chapter has a much more modest aim: to examine Karg-Elert’s statements on the perception of pitch and harmonic relationships, and to compare them with views expressed in an early, seminal writing in the fields of music perception and cognition – the first part of Hugo Riemann’s *Lehre von den Tonvorstellungen* ("Study on the Imagination of Tone") published in 1914-15. The chapter is organized into three sections, each dealing with a specific issue raised by Karg-Elert’s theories. As this chapter will describe, Karg-Elert and Riemann share one fundamental belief: that the conceptual and perceptual basis of our tonal system is rooted in the pure intervals of just intonation. Where they differ is on the extent to which our ear/mind will seek to distinguish between pitches of similar frequency, but of different derivation.

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2 Gjerdingen 2002 provides a useful history and overview of the fields of music psychology and cognition, with some emphasis on their earlier stages of development. Much modern scholarship is disseminated in journals such as *Music Perception* and *Psychology of Music*, and also in a growing number of conferences and symposia around the world.


4 In this chapter, the hybrid term “ear/mind” refers to the agent of perceptual activity.
7.1. On the meaning of small differences in frequency

As described in Chapter 3, Karg-Elert’s treatises (in particular *Akustische*) frequently label locations in just-intonation space not only with acoustic symbols to indicate pure fifths, thirds and sevenths, but also as intervals in millioctaves, calculated from the central pitch C(0). Appendix A of *Akustische* tabulates 300 distinct pitches within the octave, many of which differ from each other by as little as a schisma or 2 µ (2.4 cents), as in the following pair:

\[
\begin{align*}
F_b(-8,0) & \quad \text{eight fifths below } C(0) = 320 \mu \text{ (384 cents)} \\
E_{(0,1)} & \quad \text{one third above } C(0) = 322 \mu \text{ (386.4 cents)}
\end{align*}
\]

Is Karg-Elert suggesting that the ear/mind can actively distinguish between pitch frequencies that differ by such a minute amount? In *Harmonologik*, he seems to answer this question in the negative: “in a tonal succession, half a syntonic comma (= 1/20 tone) should be the ultimate that can be sensed as a tonal shift.”\(^5\) In other words, he states that the smallest pitch difference that can be clearly perceived is 9 µ (10.8 cents), which is half of a syntonic comma. He does not mention any source for that information; a more recent study suggests that “changes in pitch larger than 3.6 Hz can be perceived in a clinical setting.”\(^6\) Regardless of the accuracy of Karg-Elert’s views on pitch differentiation, one will surely wonder about the significance of the 2 µ interval between the two pitches listed above, which is smaller than the human ear can objectively distinguish. Based on that limitation: do we mentally erase the schisma difference between \(F_b(-8,0)\) and \(E_{(0,1)}\), and simply understand them as equivalent pitches, in spite of their considerable pitch-space distance from each other in relation to C(0)? Karg-Elert’s acoustic symbols and millioctave calculations make patently clear that all pitches in his space are unique in derivation and conception, and thus must not be equated with each other, even with pitch pairs.

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\(^{5}\) *Harmonologik*, 31.

\(^{6}\) Olson 1967, 258–251.
whose ideal frequencies differ by an inaudibly small amount. Therefore, the significance of the
distinction between $F_\flat(-8,0)$ and $E_{(0,1)}$, or any other pair of similar-sounding pitches, must lie
beyond the physical. Indeed, Karg-Elert’s preface to *Harmonologik* states that all of the acoustic
data in his treatises are simply intended as “analogies between musical and mathematical
proportions” that reveal an “interior nature…that had become obvious to me long before, through
sensitive, naïve empathy for living, practical music.” This statement implies that his millioctave
calculations and acoustic symbols reflect internal, conceptual distinctions rather than external
phenomena.

Karg-Elert’s apparent lack of concern for physical pitch frequencies is reflected in a
major difference between his treatises and those by his predecessors of similar acoustic
exactitude, such as Helmholtz, Bosanquet and Oettingen. The latter scholars all described in
great detail the design and construction of just-intonation harmoniums, and promoted their use in
performance (see Figure 3.4.6 for a schematic of Oettingen’s 53-note-per-octave harmonium,
called the *Orthotonophonium*). In contrast, Karg-Elert’s treatises say almost nothing about the
prospect of just intonation in performance, beyond a few enthusiastic comments on the use of the
natural seventh partial in brass playing. As an expert pianist and keyboard player, he would
have fully recognized that 12-tone equal temperament was a pervasive and likely irreversible
reality in modern musical performance. The point of the lengthy description of historical pitch

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7 *Harmonologik*, iii.
8 Helmholtz/Ellis 1885; Bosanquet 1876; Oettingen 1913 and 1917. Vogel 1975 is in the
tradition of these 19th-century scholars; not only does it analyze passages of tonal music in the
context of a three-dimensional just-intonation pitch space that is identical to Karg-Elert’s, it also
advocates and describes the use of various wind and fretted string instruments that can accurately
produce pitches in that space.
9 See section 3.5, and Figure 3.18.
levels in Chapter 2 of *Akustische* is to stress that the connection between frequency and pitch name is essentially arbitrary. This fact is clearly illustrated in Karg-Elert’s discussion of the standard tuning pitch a¹ (A above middle C), which historically varied in frequency from 374 to 567 Hertz, or almost a perfect fifth.¹⁰ While the practical frequency range of each pitch in a stable tuning system (whether equal-tempered, just intonation or of any other type) is much narrower, Karg-Elert’s point is still valid: any pitch can be performed within a reasonably small range of frequencies, and still be recognized as that pitch. As noted above, F♭(−8,0) should ideally be sounded 2 µ lower than E(0,1), based on their derivation and distances from C(0). However, because the association of frequency and pitch category (or name) is at best approximate, it does not matter if the two pitches are actually sounded as identical frequencies (as on a piano), or if they differed by a bit more than 2 µ, or even if F♭(−8,0) is sounded a little higher than E(0,1). The significance of the 2 µ difference is not a matter of actual, physical sound; rather, it symbolizes a difference in how the pitches are perceived, based on their harmonic context, as will be described in the following section.

Much like Karg-Elert, Riemann asserts in his *Lehre von den Tonvorstellungen* that the association between frequency and pitch name (category) is only approximate, and will vary “according to the standard of tuning”:

> A tone of 430 vibrations per second is, with respect to its tonal effect, as definite as is possible in notation; whether it is to be written or named as A or A♭ depends on habituation to a particular standard of tuning.¹¹

---

¹⁰ *Akustische* 2.3.
¹¹ Riemann/Wason/Marvin 1992, 95.
Accordingly, Riemann proposes that any distinctions made between pitches of similar frequency
do not strictly represent the frequencies themselves, but rather the perceptual status or “mental
interpretation” of those pitches within the operating musical system:

For the differences in naming tones as flats or sharps have nothing to do with absolute
pitch level but merely result from the internal construction of our tonal system and
notation...To qualify an F♯ chord or a Gb chord – which are identical in our tempered
system – as the one or the other is purely a matter of mental interpretation and not
dependent on varying intonation.12

The latter statement reveals how Riemann and Karg-Elert share some fundamental views on the
relationship between sounded frequencies and perceived tonal meaning, and on the nature of the
tonal system as a mental construct. Riemann 1914/15 does not explicitly discuss the size of the
smallest intervals that can be perceived; rather, the latter part of the Lehre defines the boundaries
of a conceptual pitch space that Riemann calls “the residence of chords and keys,” to be
described below. However, quite early in the Lehre Riemann reveals his essential view on the
perception of small intervalllic differences: “cases that will occupy us later differ...in that the
error is so small that the imagination essentially ignores it”;13 in other words, the smallest
distinctions between similar pitches are in Riemann’s view levelled out, in favor of conceptual
equivalence. Riemann continues the point:

Our musical imagination does not allow itself to be thrown around aimlessly through
superficial tonal attractions but, on the contrary, has its own will, which it enforces
continually an everywhere in the sense of a centralization, a simplification of tonal
relations.14

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12 Riemann/Wason/Marvin 1992, 95.
13 Ibid., 90.
14 Ibid.
Riemann’s entire perspective is informed by a general “Principle of the Greatest Possible Economy for the Musical Imagination,” proposed near the beginning of the Lehre:

Let attention be drawn here to the definite inclination of the interpreting mind to find its way easily through the confusion of endless possibilities of tonal combinations (in melody and harmony) by means of preferring simple relationships over more complicated ones. This Principle of the Greatest Possible Economy for the Musical Imagination moves directly toward the rejection of more complicated structures, where other possible meanings suggest themselves that weigh less heavily on the powers of interpretation (even though this conflicts strongly with actual intonations).¹⁵

The question of exactly what for Riemann constitutes a “simple relationship” or “the greatest possible economy” is only answered later in the Lehre, in its presentation of the “residence of chords and keys.” As will be seen below, his model differs sharply from that of Karg-Elert, for whom the simplest and most “economic” harmonic relationships involve the retention of acoustic common tones whenever possible, as discussed at length in chapters 5 and 6. What is immediately clear in the “Principle” is that Riemann will reject “complicated structures,” replacing them with what he views as simpler modes of understanding and representation.

7.2. On just intonation as a model of ‘harmonic logic’

The opening paragraph of Akustische 13.3 contains perhaps the definitive statement in all of Karg-Elert’s work on the question of pitch frequency, just intonation and perception:

Our inner and specifically musical ear is endowed with the wondrous ability to hear in a selective and relative manner – that is, out of the myriad of approximate tonal and chordal events that we perceive as possible tonal structures, we instinctively use our imagination (Vorstellung) to reconceive each unstable tone and chord in terms of its intended value, even in the objective absence of exact pure tuning [as long as its bounds are not greatly exceeded].¹⁶

¹⁵ Riemann/Wason/Marvin 1992, 88.
¹⁶ Akustische 13.3. The underlined words are in the original text.
There is a lot to unpack in this statement, and it will be fruitful to examine it closely. One of its first points is that we process musical stimuli in a “relative manner” – that is, rather than hearing pitches and chords as autonomous objects, we understand them in relation to each other. (The notion of ‘absolute’ and ‘relative’ hearing will be discussed further below.) The statement also suggests that we understand pitch events not only in relation to each other, but also in relation to a set of “intended values,” essentially fixed categories or locations into which our musical imagination will place all sounded tones and chords. Karg-Elert does not explicitly define the nature of the “possible tonal structures” that can be perceived, though the preceding chapters of Akustische make clear that those structures are derived from the intervals that define the three axes of the pitch space – the pure fifth, third and seventh. Finally, the statement affirms that we continue to process and organize pitch events according to the values of “exact pure tuning,” even when its frequencies are not precisely realized in performance.

In Chapter 9 of Harmonologik, Karg-Elert expands on the question of ideal and sounded pitch values by making an analogy with the drawing of geometrical shapes in art:

The essence (Wesen) of the naturally-given form is in no way altered by artificial stylization, nor by a necessary compromise on practical grounds…If a painter excludes assistance by mechanical aids (ruler, protractor, compass) and because of this his straight lines, angles, curves and circles do not precisely match the exact geometric (and thus naturally-given) forms, the observer experiences nevertheless the shapes portrayed in their natural, pure form. He satisfies himself with approximate values, ‘in the sense of the ideational archetype’.17

The “ideational archetypes” are the unique locations in the just-intonation pitch space, which can be represented by “analogies” such as acoustic symbols, positions on a Tonnetz, or numerical intervals calculated from a fixed center like C(0). They may or may not be exactly realized in performance, but they are always perceived in their “natural, pure forms.” Karg-Elert concludes:

17 Harmonologik, 19.
The countless deviations from the exactly pure forms – inevitable in the tempered 12-semitone system – are by no means questionable in the ‘artistic’ sense, as we still experience the levelled values in a naturally differentiated way, by virtue of our conscious ability to perceive, and our musical logic…comma-different tones and chords are of course levelled in equal temperament, but they are still experienced in different values.\(^\text{18}\)

The latter two statements confirm Karg-Elert’s conviction that even when sounded pitches only approximate the ideal frequencies that define the “naturally-given” (\textit{naturgegeben}) just-intonation pitch space, we will perceive and understand those pitches in their ideal values. Though we may not be able to physically hear minute differences of frequency (i.e. less than 9 \(\mu\)), our ‘musical logic’ can classify sonically-similar pitches according to their different locations in pitch space.

But what are the criteria under which our musical imagination can distinguish between a wide variety of tones and chords, such as the 275 pitches per octave that Karg-Elert defined as sufficient for understanding most musical passages? Some insight into this question is conveyed in his discussion of ‘absolute’ and ‘relative’ pitch, in a passage entitled \textit{Die harmonologische Tonalitätsempfindung}, or “The Harmonological Perception of Tonality.”\(^\text{19}\) He first states that “absolute (= perfect) pitch is purely physically (acoustically) set.” This seems to contradict his statements on the arbitrary association of pitch name and frequency. He explains that absolute pitch “evaluates the single sound as a thing in itself,” explicitly citing Immanuel Kant’s concept of the \textit{Ding an sich}, which refers to the \textit{a priori} factual existence of an object, free of its representation or appearance in the mind. Kant describes the concept as follows:

\(^{18}\) Harmonologik, 20.  
\(^{19}\) Harmonologik, 211–212. The quotations in this paragraph are from that passage.
And we indeed, rightly considering objects of sense as mere appearances, confess thereby that they are based upon a thing in itself, though we know not this thing as it is in itself, but only know its appearances, viz., the way in which our senses are affected by this unknown something.\textsuperscript{21}

An ‘absolute’ listener with perfect pitch will tend to understand pitches and chords as isolated ‘things-in-themselves’, categorizing them in the most familiar or convenient way possible. The following example (\textbf{Figure 7.2.1}) demonstrates the point:

\textbf{Figure 7.2.1.} An “absolute pitch” hearing of two analogous chord pairs (\textit{Harmonologik}, 212)

The annotation to the figure describes how the “absolute ear” will understand the first pair as a \textit{Leitklang} or major seventh progression [FM $\rightarrow$ EM], because both triads are familiar objects in the standard tonal system. But when the pair is transposed down a semitone, the ‘absolute’ listener will likely identify it not as [EM $\rightarrow$ D$\#$M], but as the \textit{Chromonant} or diminished-octave pair [EM $\rightarrow$ E$\flat$M], simply because the triad of D$\#$M occurs rarely in common practice. This mode of hearing is well served by our standard tonal system based in equal temperament, which tends to equate unusual chords such as D$\#$M with common ones such as E$\flat$M, even though they will likely occur in very different keys or harmonic contexts.

\textsuperscript{21} Kant 1783, section 32. The English translation is by Paul Carus (1902), 75.
Unlike the isolated objects perceived by the absolute listener, a relative mode of hearing assesses each object in relation to its context, as Karg-Elert describes:

Relative pitch, on the other hand, is physio-psychologically capable of perceiving subjective divergences where objective analogies exist; it does not – as does perfect pitch – evaluate the sound as a thing in itself (Ding an sich) but as a component of a process, a harmonic event in which ‘conditioning and being conditioned’ (Bedingen und Bedingtsein) are causally inseparable. (Harmonologik, 211)

This rich statement contrasts Kant’s ‘thing in itself’ (representing the “absolute ear”) with Friedrich Schelling’s concept of Bedingen or “conditioning,” which the German philosopher defined in Vom Ich als Princip der Philosophie (“On the I as Philosophical Principle”) as:

Bedingen (to condition) means that activity through which something becomes a thing, bedingt (conditioned) that which is made into a thing, from which it follows that it could not be posited through itself as a thing, which is to say, that an unbedingtes Ding (unconditioned thing) is a contradiction in terms.\(^\text{22}\)

Schelling is essentially saying that (pace Kant) nothing can exist in isolation, free from its mental representation or ‘conditioning’. Likewise, Karg-Elert states that no tone or chord can have any meaning in isolation, separate from its context: each musical object simultaneously conditions, and is conditioned by, the other objects that surround it, in a “causally inseparable” relationship. For example, in the second chord pair presented in Figure 7.2.1, a relative or contextual hearing may accept the notationally-unusual progression [EM $\rightarrow$ D$\#$M] more readily, perceiving the second chord’s root as a leading tone to the first chord.

But Karg-Elert rightly notes that for the latter pair in Figure 7.2.1, “even relative pitch cannot differentiate between E – D$\#$, E – Eb and F$\flat$ - Eb major…only from the tonal validity of a whole phrase could one draw an a posteriori conclusion, by means of harmonologic ability. In dubious cases, the inner ear will register simultaneously enharmonic double values and

\(^{22}\) Schelling 1795, 166. Translated in Frank 2012, 81.
afterwards – when the tonal problem has yielded a tonally recognizable determination – cancel one of the two interpretations, which would not lead to the intended goal. In other words, the harmonic meaning of an isolated pair of chords or pitches is often impossible to determine, and will only be confirmed by its continuation. The following example (Figure 7.2.2) illustrates the point: the harmonic meaning of chords 3 through 5 remains ambiguous (and thus enharmonically undetermined) until the entrance of chord 6, which connects with the continuation in C major.

Note that chords 2 and 3 are the second pair from Figure 7.2.1, now in a larger harmonic context.

Retroactively determined as:

Figure 7.2.2. Retroactive determination of harmonic meaning (Harmonologik, 212)
In Figure 7.2.2, chord 6 serves as the agent of harmonic ‘conditioning’. In progression A, chord 6 is retroactively revealed to be B♭ major rather than A♯ major, as only the former can easily connect with the succession in C major that follows; in turn, the B♭ major triad provides harmonic meaning and identity for the preceding chords 3 to 5, allowing the listener to reject one of the possible enharmonic interpretations (see the crossed-out chords) in favor of the other. In progression B, chord 6 will be clearly understood as E minor rather than F♭ minor, as the former connects to the following C major via two common tones; it in turn determines the identity of the preceding chords, including chord 3 as the unusual D♯ major rather the familiar E♭ major. Figure 7.2.2 vividly makes the case not only for the necessity of “relative hearing” in understanding harmonic progressions, but also for the validity and autonomy of chords such as D♯ major, A♯ major and F♭ minor, which are often enharmonically levelled out in standard harmonic systems. Therein lies the true significance of Karg-Elert’s just-intonation pitch space: in order to recognize the functional difference between sonically similar (or identical) chords such as E♭ major and D♯ major in harmonic progressions such as those in Figure 7.2.2, one must carefully differentiate between their pitches. When one takes into account the many possible chordal transformations and relations described in Chapter 5, Karg-Elert’s array of 275 or more pitches per octave becomes easier to accept as a model of harmonic perception and understanding.

Riemann’s *Lehre von den Tonvorstellungen* agrees with Karg-Elert on one fundamental point: “that we imagine tonal relationships thoroughly in the sense of pure tuning is beyond question.”24 His detailed discussion of “the residence of chords and keys” (i.e. his model of pitch space) begins with his version of “the well-known table of tonal relationships” – the

familiar two-dimensional just intonation space derived from pure fifths and thirds, described in Oettingen 1866 and other writings. Figure 7.2.3 provides Riemann’s original diagram, plus Wason and Marvin’s English version.\textsuperscript{25}

\textbf{Figure 7.2.3.} The Tonnetz in Riemann 1914–15, and the version in Riemann/Wason/Marvin 1992

\textsuperscript{25} The original is from Riemann 1914–15, 20, and is reproduced in Riemann/Wason/Marvin 1992, 102 (along with the English version shown here), with permission of the authors.
As in Oettingen’s Tonnetz (see Figure 3.4.4), each pitch in Riemann’s diagram (represented by a diamond) is acoustically and conceptually distinct. Riemann uses dashes above and below pitches in the opposite manner to Oettingen: while the latter’s dashes denote thirds above and below the central “pure fifth row,” Riemann’s dashes indicate the number of syntonic commas by which pitches are either raised (written above the note) or lowered (below the note). 26 Figure 7.2.3 is oriented around the central triad and diatonic collection of C major, which is highlighted in the middle; it intends to illustrate “the determination of every interval according to fifth or third successions, and discloses for every multiply-determined interval the simplest and closest derivation [from C].” 27 Based on the figure, one can surmise that according to Riemann, a total of 53 distinct pitches can encompass every interval in C major, understood in its “simplest and closest derivation”: a central row of eight fifth-related pitches surrounding C, plus six additional rows located no further than three thirds above or below the center. Note that the top two rows contain fewer than eight pitches, omitting ‘exotic’ entities like B♭(4,2), G×(3,3) and D×(4,3), which Riemann considered superfluous, as will be seen below.

Riemann’s pitch space is certainly not an equal-tempered one: not only does it distinguish between enharmonic pitches such as B♭, A♯ and C♭♭, it also includes many metharmonic pairs (pitches of the same name in comma-different locations). The space includes 41 major triads and 39 minor triads, and thus should be able to accommodate the majority of harmonies likely to occur in C major. Nonetheless, it is more restricted than Karg-Elert’s space, especially in the horizontal breadth of its fifth-rows. For that reason, certain of Karg-Elert’s analytical findings

26 As described in section 3.4, each third above the central fifth-row is lowered by a syntonic comma, while each third below the central row is raised by a syntonic comma. Thus, while Oettingen’s dashes represent thirds above and below the center (which acquire syntonic comma differences), Riemann’s dashes directly indicate the syntonic-comma raising or lowering.

are impossible in Riemann’s space, such as the “sharpward” comma-free modulation of eight fifths from E♭ major to B major in the Brahms Rhapsody in G minor (see Figures 5.2.11 and 5.2.12), or the similar “flatward” sequence of fifths from G minor to G♭ minor in Schumann’s Novelette in F major (see Figures 6.4.3 and 6.4.4). As will be recalled, those analytical findings were based on the local common-tone transformations between adjacent chords, and thus reflect an essentially local and contextual mode of hearing. A more restricted pitch space such as that in Riemann’s Lehre will require the metharmonic and/or enharmonic equivalence of at least some chords in passages like the Brahms and Schumann examples, and will thus result in the “centralization” of tonal relations that Riemann found desirable.

In fact, Riemann’s “Law of the Greatest Possible Economy for the Musical Imagination” prompts a further reduction of the pitch space shown in Figure 7.2.3. He proposes that “only the keys of the first and second fifths above and below the central keys C major and A minor count as purely fifth-related.”28 Therefore, the Law dictates that in C major, the root of an A major triad will not be identified as three fifths above C, but always as one third above F. The Lehre proceeds to specify a single location in the space for each major and minor triad, ending with C♭ major and C♯ major. The resulting space contains only 22 pitches (Figure 7.2.4), of which Riemann states: “I will not maintain that with this the totality of all imaginable tones is described, but the tonics (primes) certainly are.” Thus, the 22 pitches imply the existence of 22 major triads and 22 minor triads; these include several enharmonic pairs, but only one metharmonic pair (based on the two comma-different D’s).

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Riemann suggests that six additional pitches (not tonics) may be encountered on rare occasions: C♯, G♯, B in the dominants of D minor and A♯ minor, and F♯, E♭, and A♭ in the minor subdominants of G♭ major and C♭ major. Riemann concludes that retaining any more than a total of 28 distinct pitches “makes little sense”:

D♭ minor and G♭ minor can certainly occur as keys on occasion for short stretches of time (in pieces in D♭ major and G♭ major); however, these keys scarcely bring any new tones with them. Higher major keys with sharps (G♯ major and D♯ major) are hardly imaginable without enharmonic transformation, and may remain out of the discussion, as easy as they are ultimately to describe schematically.²⁹

As is often the case in Riemann’s theories, his “residence of chords and keys” as a mode of harmonic perception exhibits a compromise between an idealistic model based on pure intervals, and common musical practice. Though he states in the Lehre’s conclusion that “we imagine tonal relationships thoroughly in the sense of pure tuning,” his Figure 7.2.4 suggests that our powers of pitch imagination must be quite limited, and will thus prefer metharmonic or

²⁹ Riemann/Wason/Marvin 1992, 104.
enharmonic equivalence over the more distant relationships proposed in Karg-Elert’s work. For Riemann, the “simplest” and most “economic” mode of perceiving harmony is one that is centralized and “global,” understanding each triad in the closest possible relationship to the center, as in most standard methods of understanding harmony. In contrast, Karg-Elert’s method favors direct local common-tone connections between chords over proximity to the central tonic; indeed, he would likely regard the unidirectional chains of fifths outlined in the Brahms G minor Rhapsody or the Schumann F major Novelette as the “simplest” possible harmonic progressions that can be perceived, even when they traverse a much wider region of the pitch universe.

6.3. **On 12-tone equal temperament**

Both Riemann and Karg-Elert recognize that 12-tone equal temperament is a necessary practical compromise, one whose frequencies are similar enough to various distinct pitches in just intonation that they can replace the latter in performance, and still be understood in their pure “intended values.” For example, Riemann’s *Lehre von den Tonvorstellungen* describes how “it has long been understood and established that we definitely notice the difference in tuning [in C major] between the fifth[-derived] E and third[-derived] E. But it is likewise established that we can be satisfied with an average of the two, as equal-tempered, 12-tone tuning offers.”

Similarly, Karg-Elert notes in Chapter 4 of *Akustische* that because fifth- and third-derived pitches cannot substitute for each other due to their sizable tuning differences, “the most convenient compromise is provided by tempered tuning.”

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31 *Akustische* 4.11.
In the final section of *Akustische*, Karg-Elert explains that “the musical and practical significance of equal-tempered tuning resides in the removal of all commas, and the artificial closure of endless tonal circles by means of metharmonics and enharmonics.” Karg-Elert demonstrates his final point in his analysis of the development section of Beethoven’s *Waldstein* Sonata (see section 6.2): in order to ensure closure in the initial C major tonic (in its source position), Karg-Elert allows the “enharmonic derailment” or juxtaposition of E♭ minor and F♯ major, requiring the conceptual reconciliation of G♭(2,-1) and F♯(2,1), as shown in Figures 6.2.4 and 6.2.5. However, while he fully realizes that such enharmonic pitches will sound identical in performance (thus sonically “smoothing out” the juxtaposition), his theories suggest that enharmonic pairs are not fully merged into a single conceptual pitch category. Rather, the ear/mind will still perceive the difference between the enharmonic pitches at the moment of juxtaposition, but will mentally reconcile that difference in order to ensure tonal closure. Though Riemann and Karg-Elert agree that equal temperament can represent the pitch values of just intonation in performance, they differ on the number of distinct values which the ear/mind will seek to recognize, as discussed in section 7.2. In Riemann’s pitch space as shown in Figure 7.2.4, the reduction to 22 primes implies a limited perspective for the tonal imagination, approaching the centralization and normalization of 12-tone equal temperament. In that respect, Riemann’s pitch space of 1914/15 links back with his 1880 system of *Schritte* and *Wechsel*, which is more explicitly oriented toward equal temperament and enharmonic equivalence, as discussed in section 5.1.
Perhaps a fruitful way to consider the status of equal temperament and just intonation in the perception of tonal and harmonic relations is to imagine equal temperament as a neutral sonic canvas, onto which the ear/mind can project a myriad of ‘harmonological’ interpretations, including those from a limited just-intonation system like that of Riemann, or a much more complex and expansive one like that of Karg-Elert. Because 12-tone equal temperament contains only twelve evenly-spaced pitch frequencies, it is likely well suited to accommodate a variety of tonal and harmonic perceptions, probably better than any tuning system which contains smaller or less uniform intervals. Karg-Elert concludes *Akustische* with a statement that contrasts the sonic reality and compromise of equal temperament with the wide-ranging selective powers of the imagination: “And thus do we open our ears to interpret pitches sounded in equal temperament as melodic [canonic] or syntonic entities, through the strength of our abilities of selection and perception.”

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32 *Akustische* 13.3.
Chapter 8

Conclusion

The preceding chapters have examined in detail the four principal components of Karg-Elert’s harmonic theories, as listed in the dissertation’s title: his three-dimensional just intonation space and its acoustic framework, his adaptation and expansion of Riemann’s function theory to encompass a variety of fifth-, third- and seventh-based chord relationships, his comprehensive system of common-tone transformations, and finally his presentation of the entire system as a model of harmonic perception. What remains to be considered is the validity and efficacy of Karg-Elert’s system for the modern analyst and musician. This concluding chapter begins by revisiting four ‘critical considerations’ about his theories, raised by his student Paul Schenk in 1966. It then highlights the most fundamental and important contribution of Karg-Elert’s work, one which resonates well with recent transformational theories: its acceptance and comprehensive exploration of the idea that every pitch, chord and key can exist in more than one conceptual state or location, defined by the harmonic paths that connect them. The dissertation concludes by describing some aspects of Karg-Elert’s work that remain to be explored in future research, including the possibility of testing some of his ideas in an experimental setting.

As mentioned in section 2.3, the most important and thorough discussion of Karg-Elert’s theories in the existing literature is an essay by his student and faculty colleague Paul Schenk, published in 1966.¹ Schenk’s knowledge and experience with Karg-Elert’s methods was both extensive and multifaceted: not only did he learn the methods directly from their creator during the time when they were still in development, he later taught them in German music schools for over forty years, beginning in 1925. Therefore, Schenk’s 1966 essay was colored not only by his

¹ Schenk 1966.
intellectual appraisal of the theories themselves, but also by his experience of teaching them (later in simplified form) to a generation of music students, most of whom were not prospective theorists or composers. After describing the most important features of Karg-Elert’s theories, Schenk’s essay concludes with four “critical considerations” (*Kritische Erwägungen*)\(^2\), essentially problems in Karg-Elert’s work that in Schenk’s estimation reduced its validity and usefulness for modern musicians and analysts. The following are Schenk’s four criticisms, summarized as in section 2.3:

1. Karg-Elert’s presentation of minor as the strict polar counterpart of major creates a pure Aeolian mode, which has been mostly irrelevant to musical practice since the seventeenth century. It is therefore difficult to reconcile Karg-Elert’s theory of the minor mode with musical practice.

2. Stemming from the tradition of Pythagoras, Karg-Elert’s theories are rooted in number as a symbol of universal order. But mathematics is not the essence of music; theories must contend not only with natural phenomena and acoustic calculation, but also with psychological principles.

3. Hugo Riemann’s conception of harmony is essentially static and vertical, and the same applies equally to Karg-Elert. The true meaning of harmonic function is not limited to vertical phenomena, but also encompasses processes of dynamic movement and kinetic energy.

4. In spite of their innovations, Karg-Elert’s theories are fundamentally grounded in the major-minor system of tonality, and are thus inadequate for the understanding of the music of his time, such as Schoenberg, Stravinsky and Krenek.

The following paragraphs evaluate each of Schenk’s criticisms in turn, considering them from the perspective of Karg-Elert’s work itself, as well as that of standard musical practice.

Schenk’s first point appears to focus on a very specific matter: that the polar inversion of the major mode is a pure minor or Aeolian mode, lacking the raised seventh scale degree that is typical in most musical practice since the seventeenth century. One might wonder why Schenk

is so concerned with a seemingly insignificant detail; after all, it is no more difficult to indicate a change from minor to major in Karg-Elert’s functional notation than in any other analytical system. Perhaps the true sense of Schenk’s criticism is not really about the lack of a leading tone in the pure minor scale, but rather that Karg-Elert’s strict dualistic opposition of harmonic relationships in major and minor is difficult to integrate into standard musical practice, and in particular into the study and teaching of harmony. Section 4.3 contrasted the strict and consistent polar dualism of Karg-Elert’s functional system with Riemann’s partial compromise with traditional monistic theories such as the familiar Stufenlehre. It cannot be denied that Karg-Elert’s functional notation is more difficult to use than Riemann’s in minor keys, in which the analyst must not only remember to reverse the placement of dominant and contrant in relation to the tonic, but also to write the function labels themselves backward or upside-down, and to invert the vertical placement of alterations such as mediants ($M$) that are attached to the function labels. Likely more than even the forbidding abundance of acoustic data and symbols in Karg-Elert’s work (especially in Akustische), its polaristic analytical notation will be a major stumbling block for most readers. In order to address this challenge, it may be possible to create a revised or simplified version of Karg-Elert’s function labels. One change might involve accepting Riemann’s compromise with monism described in section 4.3, namely to define the dominant as the fifth above the tonic, and the contrant (subdominant) as the fifth below the tonic, in both major and minor keys. That change would likely have no effect (beyond the notation itself) on any of Karg-Elert’s insights into tonal structure, or paths in harmonic space. However, even that single change would require a significant revision of his common-tone transformation system, which is completely and consistently dualistic.
Schenk’s second criticism is understandable but somewhat misguided, as it suggests that Karg-Elert’s theories are concerned only with acoustic phenomena and their numerical representation, at the expense of the psychological and perceptual experience of music. While one cannot deny that his pitch space reflects acoustic principles such as the pure intervals of the harmonic series (expressed numerically as harmonic ratios and ideal frequencies), Karg-Elert carefully explains that this information does not strictly represent actual frequencies and sounded intervals; rather, it expresses ideal proportions that are perceived in the ear/mind, and thus manifests a psychological model of music, not a physical one. It is of course true that Karg-Elert’s statements on pitch and harmonic perception are not supported by experimental evidence; instead, they simply assert that we understand tones and chords within a framework that ultimately is derived from physical principles. Karg-Elert would likely respond to Schenk’s criticism by stating that both acoustic and psychological phenomena are different manifestations of universal ‘truths’, which can be represented mathematically.

Schenk’s third ‘critical consideration’ notes that Karg-Elert’s theories are almost entirely concerned with harmony and the ‘vertical’, and very little with ‘horizontal’ musical domains such as melody, counterpoint, voice leading and rhythm. This is a fair criticism; in spite of some interesting insights into phrase rhythm and hypermeter in Chapter 42 of Grundlagen (see Figure 2.8), and some discussion of melodic elaboration in early 20th-century harmonic idioms in Die logische Entwicklung der modernen Figuration (see Figures 2.2.1 and 2.2.2), one must admit that Karg-Elert’s published treatises deal mostly with harmonic function, tonality and chordal transformation, as their titles explicitly indicate. The prospectus for the three-volume Grundlagen series (see Figure 2.2.3) suggests that Karg-Elert was in fact interested in all musical domains; however, his extant work does have a singular focus, and is in that respect more limited
than that of his contemporaries such as Riemann (who wrote extensively on rhythm, form and many other topics), Heinrich Schenker and Arnold Schoenberg. Notwithstanding Karg-Elert’s concentration on harmony, it is perhaps unfair to state that he was not concerned with notions of dynamic motion and kinetic energy, which can be conveyed through harmonic means. In his analyses of progressions and modulations that travel large distances in pitch space (such as the Schumann Novelette in F major, discussed in section 6.4), a sense of dynamic projection and direction is clearly palpable. In addition, his descriptions of polarity in the major and minor triads, and also among the principal functional triads in the major and minor modes, constantly refer to upward- and downward-striving energies, as discussed in sections 4.3 and 4.4.

Schenk’s final point states that because Karg-Elert’s theories are explicitly rooted in the major and minor consonances and the notion of harmonic function, they are not suitable for the analysis of music that is not tonal, such as that of the 20th-century composers that Schenk mentioned. This point is largely accurate; though Karg-Elert envisioned his entire project as a complete “modern theory of harmony,” his published theories assume the existence not only of tonal centers, but also of the traditional fifth-based functional categories of tonic, dominant and contrant. It is notable that the last chapter of Harmonologik (on the subject of Freitonalität or “free tonality,” featuring free-floating triads, seventh chords and whole-tone sonorities) almost entirely abandons functional notation, or any mention of keys or pitch centers – in other words, the last chapter exhibits little of Karg-Elert’s analytical methods. His plans for the complete Grundlagen series included chapters on free tonality, atonality and even microtonality (see Figure 2.2.7); unfortunately, those topics are largely absent from his published treatises. Therefore, Schenk is essentially correct to state that Karg-Elert’s methods are not very applicable
for the analysis of 20th-century music, beyond that of transitional composers such as Reger, Richard Strauss, Debussy, Scriabin, and Karg-Elert himself.

One can point to many significant innovations in Karg-Elert’s harmonic theories, such as his complete array of direct third-based transformations (which foreshadows similar arrays in more recent work such as that of David Kopp), his concept of the “collective-change chords” or Kollektivwechselklänge (which prefigures the hexatonic and octatonic poles of neo-Riemannian theory), and in particular his inclusion of the seventh-axis in the harmonic space, plus common-tone transformations that involve the concordant seventh. However, Karg-Elert’s most important and fundamental contribution to the understanding of harmony and tonal structure lies in his recognition that pitches, chords and keys are not fixed entities. Instead, his theories proceed from the view that the ‘ontological’ status or meaning of any tone or chord depends on its relationship to other tones and chords, and in particular on the paths that connect every tone and chord. This idea is of course inherent to a just-intonation harmonic space, in which every pitch class exists in multiple locations, and can be reached via multiple routes. As described in chapter 7, both Karg-Elert and Riemann believed that the ear/mind perceives pitch and harmony according to the proportions of just intonation – but unlike Riemann who promoted a “Law of the Greatest Possible Economy,” Karg-Elert fully embraced the multiplicity of locations in his space, and eagerly accepted the possibility of a modulation from E major to F♯ major (see the Liszt analysis in section 6.3), or even between two tonics that have the same name, but are of different derivation. Like Karg-Elert’s acoustic symbols and milloctave calculations, a central purpose of the original just-intonation Tonnetz was to represent a potentially infinite number of distinct pitch locations. However, many recent theorists have chosen to incorporate 12-tone equal temperament and enharmonic equivalence into their Tonnetze, conforming an endless plane into
a torus, and placing each pitch class in a single location. The adoption of 12-TET has enabled the development of cyclical transformational networks such as those of Douthett and Steinbach, which have been proven useful for the understanding of chromatic music. On the other hand, one may feel that the modern equal-tempered Tonnetz tends to hide or even to negate the original grid’s ability to illustrate multiple paths between two pitches or chords. As Daniel Harrison has pointed out, it is not strictly necessary to retain just intonation and comma differences in order to conceive each pitch class and chord in multiple locations. Nonetheless, in order to precisely distinguish between locations in pitch space (and thus between harmonies in the space), one needs a method of naming them. More than any theorist before him, Karg-Elert provided both a model for understanding and harmony and tonality in terms of paths in pitch space, and also a method for precisely specifying those paths using analytical notation.

The primary aims of this dissertation have been to explain the most important features of Karg-Elert’s harmonic theories, and then to relate them to recent transformational and neo-Riemannian methods. Some facets of Karg-Elert’s work have been largely excluded from this study, and thus await further research and explication. Perhaps the most significant topic that is not discussed here is his theory of dissonance, presented in full in Chapter 11 of Akustische, but only in piecemeal fashion in Harmonologik. One of its central ideas is that all linear alterations of consonant chords (such as raised or lowered chord tones, or added non-harmonic tones) are Pythagorean in origin – that is, they are entirely fifth-derived, and thus may clash with syntonic (third-derived) harmonic tones. This idea raises problems for certain familiar dissonant chords

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3 Cohn 1997, 17–21.
4 Douthett and Steinbach 1998. All of their networks of triads and seventh chords, such as “HexaCycles” (from Cohn 1996), “OctaTowers” and “Cube Dance,” require 12-TET and enharmonic equivalence.
such as the diminished seventh, which Karg-Elert explains in a very complicated manner in *Akustische* 11.7 to 11.9; though he exhaustively describes three ways in which diminished sevenths can be derived from consonant chords, he does not quite clarify the issue of exactly where the pitches of diminished seventh chords are located in the pitch space. Though Karg-Elert’s theory of dissonance is presented in a less polished (and less complete) manner than other topics in his work, it nonetheless provides interesting insight into various common sonorities, and is thus worthy of future study.

Another topic of considerable interest that is only briefly summarized in Chapter 2 is that of the reception of Karg-Elert’s theories. While it is certainly fair to say that Karg-Elert’s work has not had a major impact on the field of music theory writ large, it did achieve some measure of recognition and success in East Germany, beginning in the 1950s: directly in the form of Karg-Elert-based textbooks and other writings by his pupils Paul Schenk and Fritz Reuter, and indirectly through the influence of their students, some of whom were still teaching into the 1990s and beyond. Some research on the reception of Karg-Elert has been undertaken in Germany during the past two decades, and is still in progress. As mentioned in section 2.3, Gesine Schröder has published on music theory in Leipzig during the early twentieth century, and her articles have discussed the influence of Schenk and Reuter, both in German conservatories and beyond. In addition, German scholar Jonathan Gammert has been writing a dissertation on the reception of Karg-Elert’s theories; he has recently published an article on the work of Reuter and Karg-Elert, viewed as a response to ideological and political criticism published in East German music journals.6

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One additional direction for further research into Karg-Elert’s theories would be to put his just-intonation space into practice, by creating computer-generated recordings of examples from his treatises in which every pitch would be precisely sounded at its ideal frequency. Such recordings might then be used in a variety of experiments, to test some of Karg-Elert’s assertions about pitch perception, such as the ear/mind’s ability to distinguish between metharmonic and enharmonic pitches, or to experience different harmonic paths between two chords or keys in different ways. Though Karg-Elert’s treatises clearly state that accuracy of pitch frequency is not strictly required for understanding harmony, it would nonetheless be interesting to observe the extent to which hearing music performed in the ideal frequencies of Karg-Elert’s pitch space might affect the perception of harmonic paths, and in turn of tonal unity and closure.

This dissertation hopes to shine new light on the harmonic theories of Karg-Elert, in particular for English-speaking readers. Many of his views on harmonic relations and the nature of tonality are quite radically different from those that are widely accepted today. In addition, his strict polar dualism and complex functional notation will likely continue to preclude the wider knowledge and acceptance of Karg-Elert’s theories. In spite of those challenges, his work should be better known in the music theory community, not only for what Daniel Harrison called its “imaginative analytical insights” and “astonishingly sophisticated and effective examples,” but also because it occupies an intriguing position in the history of twentieth-century music theory, building on the foundational work of predecessors such as Oettingen and Riemann, but also pointing forward in specific ways to the transformational and neo-Riemannian theories of the 1980s and 1990s. If this study and the accompanying English translation of Akustische can successfully introduce Karg-Elert’s harmonic theories to a new audience, they will serve their essential purpose.
BIBLIOGRAPHY


Kant, Immanuel. 1783. Prolegomena zu einer jeden künftigen Metaphysik, die als Wissenschaft wird auftreten können. Riga: Johann Friedrich Hartknoch. Translated by Dr. Paul Carus (1902) as Prolegomena to any Future Metaphysics. Chicago: Open Court Publishing.


Kirnberger, Johann Philipp. 1766. *Clavirübungen (4. Sammlung).*


Das Musikalische Juden-ABC. Munich: H. Brückner.

Signals and Systems for Speech and Hearing. 2nd edition. Leiden: BRILL.

Some Aspects of Musical Politics in Pre-Nazi Germany.” Perspectives of New Music 9.1: 74–95.


Tovey, Donald Francis. 1931. *A Companion to Beethoven’s Pianoforte Sonatas*. London: The Associated Board of the Royal Schools of Music.


Zarlino, Gioseffo. 1558. Le institutioni harmoniche. Venice.

APPENDIX

Sigfrid Karg-Elert:

_Akustische Ton- Klang- und Funktionsbestimmung_ (1930)

Annotated German-English edition, translated by David A. Byrne
ON THE FORMAT OF THIS EDITION

Sigfrid Karg-Elert’s *Akustiche Ton- Klang- und Funktionsbestimmung* was published in 1930 by Carl M. F. Rothe (Leipzig). In its original form, the treatise is quite brief: only 104 pages of text and examples, plus the preface and table of contents. However, the pages are very densely packed, especially in chapters 7 through 11, in which the organization does not always make clear where one section or topic is finished and the next begins. Karg-Elert provides few headings for different topics within chapters; this lack of headings makes reading and comprehension more difficult than is necessary, and does not readily allow for cross-referencing between different parts of the text. One goal in this annotated German-English edition has been to help the reader by dividing each chapter into smaller sections, and by providing headings which specify the subject matter in each section. The title pages for each chapter include a list of headings. In addition, the annotations often refer back to previous sections, reminding the reader of concepts or examples presented earlier in the treatise.

In this edition, left and right pages are paired with each other; for that reason, this edition will be best read using a program such as Adobe Acrobat or Reader, with left and right pages viewed side by side. The left-side pages contain Karg-Elert’s text, diagrams and musical examples, interspersed with English translations of the text and terminology. All of Karg-Elert’s material is reproduced in its original order, though no attempt has been made to retain the original pagination. Instead, the material is now organized into sections as described above. The right-side pages contain commentary on the text, explanation of the examples and diagrams, discussion of the sources of Karg-Elert’s ideas, demonstration of connections with more recent theories, and other information that will help the reader to understand Karg-Elert’s material.
At the end of chapters 3 to 11, this edition provides Tonnetz diagrams for many of Karg-Elert’s musical examples. It is hoped that these Tonnetze will help illustrate the treatise’s often complex acoustic data and functional harmonic analyses in a way that is visually immediate, and that will be familiar to many readers. In the text, the abbreviation TN is printed to the right of each example or diagram that has an accompanying Tonnetz. The reader will also see that a number of errors in Karg-Elert’s text and examples are marked ERRATUM, and enclosed in boxes; the annotations provide corrections for those errors.

The English translation has two goals, which may at times conflict with each other. Most importantly, it strives to explain Karg-Elert’s ideas in a detailed yet clear manner, using English equivalents of his own terminology when appropriate. On the other hand, it also hopes to convey something of Karg-Elert’s colorful and energetic use of language, even when discussing comparatively dry subject matter such as historical tuning levels. While I believe that the translation largely achieves these two goals, there will surely remain some passages where an alternate translation is possible or desirable.

David A. Byrne
Akustische Ton-Klang- und Funktionsbestimmung

Die Polarität der naturgegebenen Ton- und Klangproportionen

Ein Beitrag zu jeder Lehre von der Harmonik und musikalischen Akustik

Sigfrid Karg-Elert
(Landeskonservatorium zu Leipzig)

Carl M.F. Rothe, Leipzig
Acoustic Determination of Pitch, Chord and Function

The polarity of natural pitch and chord relations

A contribution to the study of harmony and acoustics

Sigfrid Karg-Elert

(Leipzig Conservatory of Music)

Annotated German-English edition, translated by David A. Byrne

Carl M.F. Rothe, Leipzig

(1930)
In place of a preface

Number is the essence of all things.  

[Pythagoras]

Musical relationships strike me to be virtually identical with the basic proportions of Nature. The concept of mathematics provides the most fully valid evidence of the natural ideal; internal coherence, in sympathy with the universe, is its basis.

Pure mathematics is the vision of the universal intellect. It appears in the form of music as a revelation, as the ideal made tangible.

[Novalis]
Canonics  (Pythagorean pure fifth relations, arranged symmetrically around D)
Syntonics  (pure third, fifth and seventh relations, arranged symmetrically around D)
Equal temperament  (12-tone chromatic, arranged symmetrically around D)
1. KAPITEL
Allgemeines über Schwingungszahl und Wellenlänge.
Umwandlung der $S_z$ in $W$ - Werte.

2. Die zeitlich schwankende und konstante \textit{absolute} Tonhöhe.

3. \textbf{Kanonsiche Partial- und Messelwerte.}
Tabellen der 2er und 3er Potenzen.
Die Millis Oktave und die \textit{Einheiten} der pythagoräischen Werte.
Tabelle: Trickrechnung.


5. \textbf{Die syntonische Terz. Terzpotenzen und \textit{Produkte}.}
Obersicht der zunächst gebräuchlichen Kommata.

6. \textbf{Die \textit{praktisch-musikalische} Bedeutung der syntonischen Werte.}

7. \textbf{Die komm但reine und kommodifferierende Modulation.}
Medianten und Nebenmedianten. Schematische Übersicht.

8. M"ixtnormose und Ennormose. \textit{[Scheinbare und wirkliche Ennormose]}


10. \textbf{Die \textit{praktisch-musikalische} Bedeutung der natürlichen Septimenwerte.}
Gegenkondanten.

11. \textbf{Das Prinzip der Dissonanz.}

12. \textbf{Übersicht der natürlichen Partialtöne in rel. Sz. 1 bis 32 \( C - C^3 \)}
Messelwerte in rel. \( W \) 1 bis 32 \( C^3 - C \).

13. \textbf{Die gleichschwebende Temperatur.}

\textbf{ANHANG}
\textit{Generalübersicht von 303 unterschiedlichen Werten innerhalb einer Oktave}
Vergleichende Funktionswerte der C-Dur- und c-Moll-Tonalität.
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Conversion between frequency and wavelength values.

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The millioctave and the μ values of the Pythagorean intervals.  
Tables. Calculation shortcuts.

Chapter 4.  The practical and musical significance of the canonic values.

Chapter 5.  The syntonic third. Multiple thirds and third cycles.  
Summary of common commas (introduced to this point).

Chapter 6.  The practical and musical significance of the syntonic values.

Chapter 7.  Comma-free and comma-differing modulation.  
Mediants and neighbor mediants. Schematic overview.

Chapter 8.  Metharmonics and enharmonics. [Apparent and true enharmonics.]

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Chapter 10. The practical and musical significance of the seventh-based values. 
Counter-concordants.

Chapter 11. The principle of dissonance.

Chapter 12. Summary of the first 32 overtones as relative frequencies (C – c3)  
Summary of the first 32 Messel values as relative wavelengths (c3 – C)

Chapter 13. Equal temperament.

APPENDIX Comprehensive summary of 303 distinct pitch values within the octave. 
Comparative functional values in the keys of C major and A minor.

The text, musical examples, diagrams and headings were handwritten by 
Erhard Henniger (Leipzig Ost 28, Wurzner Str. 174)  
Telephone number: 67376
1. KAPITEL

ALLGEMEINES ÜBER SCHWINGUNGSZAHL UND WELLENLÄNGE

Chapter 1

On frequency and wavelength
Chapter 1
On frequency and wavelength

1.1. Temporal and spatial definition of absolute pitch
1.2. Frequency [F]: half or whole vibrations
1.3. Wavelength [W], and the relationship between wavelength and frequency
1.4. Wavelength and organ pipe length
1.5. The propagation velocity [P] of sound
1.6. Conversion between frequency and wavelength values
1.1. **Temporal and spatial definition of absolute pitch**

Die absolute Höhe eines beliebigen Tones wird bestimmt

zeitlich: durch die Dauer seiner [halben oder ganzen] Schwingungsperiode,

Als Zeitausseinheit wird die Sekunde,
as Raummaßeinheit wird das Meter [resp. Centimeter] angenommen.

The absolute pitch of any sounded tone can be defined

temporally: by the duration of its [half or whole] individual vibration periods
spatially: by the length of its [half or whole] waves

The second will be adopted as the unit of time, and the meter (or centimeter) as the unit of length.

1.2. **Frequency [F]: half or whole vibrations**


Frequency [F] measures the number of half or whole vibrations in one second. In Germany, Austria and Scandinavia, one vibration is understood to comprise a complete back and forth oscillation of the vibrating body [a whole or “double vibration”]. In contrast, in France, England and America, one vibration comprises only one back or forth motion [a single or “half vibration”]. F 440 in the German counting method = 880 in the French-English method.
1.1. **Temporal and spatial definition of absolute pitch**

(When this space is blank, no additional notes are provided.)

1.2. **Frequency [F]: half or whole vibrations**

Karg-Elert’s German abbreviations are here translated to English:

- Sz $\rightarrow$ F (frequency)
- W $\rightarrow$ W (wavelength)
- F $\rightarrow$ P (propagation velocity of sound).

The “German counting method” is now standard worldwide: F 440 = 440 Hertz, which is a common orchestral tuning pitch in many countries.
1.3. **Wavelength [W], and the relationship between wavelength and frequency**

A complete wavelength [W] comprises the spatial distance between two rising or falling nodes, or between two wave peaks or troughs [peaks and troughs can be also called “bellies”]. Wavelength is defined by the temporal duration of the vibration periods; it stands in the same relationship to the vibration speed [that is: the smaller the time-span between identical vibrations, the shorter the periods, and so the smaller or shorter are the waves]. In contrast, the numerical values for frequency and wavelength stand in an inverse (reciprocal) relationship:

- **Relatively:**
  - High F = Low W = Higher pitch
  - Low F = High W = Lower pitch

Intervals illustrating the relationships between F and W values:

Frequency = \[
\frac{4}{5} = \frac{9}{5} \\
\frac{8}{9} = \frac{9}{8}
\]

Wavelength = \[
\frac{5}{4} = \frac{9}{6} \\
\frac{9}{8} = \frac{9}{6}\]
1.3. *Wavelength* \( \lambda \), and the relationship between wavelength and frequency
1.4. **Wavelength and organ pipe length**

Just as the practice in other countries is to count frequency in half vibrations, so it is common among organ builders to name tones in terms of half-wavelengths [that is, in “feet” lengths]. This stems from the fact that an open flue pipe approximately corresponds in length to half of a sound wave:

<table>
<thead>
<tr>
<th>Organ pipe length:</th>
<th>32'</th>
<th>16'</th>
<th>10 2/3'</th>
<th>8'</th>
<th>6 2/5'</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sound wave length:</td>
<td>64'</td>
<td>32'</td>
<td>21 1/3'</td>
<td>16'</td>
<td>12 4/5'</td>
</tr>
</tbody>
</table>

1.5. **The propagation velocity [P] of sound**

If a foot-length is understood as 33 5/32 cm, a 32-foot open organ pipe corresponds with a sound wave of 33 5/32 * 64 cm = 21 1/4 m in length. That is one sixteenth of the distance that a sound wave travels in one second, at an air temperature of 16° [Celsius]:

340.00 m

which is called the “propagation velocity” of sound.
1.4.  \textit{Wavelength and organ pipe length}

1.5.  \textit{The propagation velocity [P] of sound}

Karg-Elert does not explain or cite his source for 1 foot $= 33 \ 5/32$ cm. It approximates an obsolete French/south-west German definition of the foot as one third of a meter, or 33.333 cm (Fevrier 2014). Karg-Elert’s foot length conveniently provides a multiple of 10 for the propagation velocity, making conversions between F (frequency) and W (wavelength) much easier.

In 1959, the international foot was standardized as 30.48 cm (not 33 $5/32$ cm). Accordingly, the speed of sound (or “propagation velocity”) is now most frequently expressed as $343.2 \text{ m per second}$, rather than Karg-Elert’s 340 m. In addition, the propagation velocity is now usually defined as the speed at which a sound wave travels through dry air at 20~\text{degrees} Celsius, not 16°C (Sengpiel Audio 2017).
1.6. Conversion between frequency and wavelength values

**UMRECHNUNG DER \( S \) IN \( W \)'S WERTE**

Dividiert man die bekannte Schwingungszahl \( S \) eines Tones durch die Fortpflanzungsgeschwindigkeit \( F \), so erhält man die zuvor unbekannte Wellenlänge \( W \) jenes Tones.

Als bekannt sei angenommen: die Wellenlänge \( 4 \frac{1}{4} \) m für das große \( E_1 \),
die Schw.-Zahl 96 für das große \( G_1 \);
im ersten Falle soll die Schwingungszahl - im letzten dagegen die Wellenlänge ermittelt werden:

\[
\frac{F}{W} = \frac{340}{4 \frac{1}{4}} = 80 \\
\frac{F}{S} = \frac{340}{96} = \frac{3}{1/3}
\]

If one divides the Propagation Velocity \([P = 340]\) by the known frequency \([F]\) of a tone, one can calculate the previously unknown wavelength \([W]\) of that tone.

If one divides the Propagation Velocity \([P = 340]\) by the known wavelength \([W]\) of a tone, one can calculate the previously unknown frequency \([F]\) of that tone.

Beginning with the following as known: \( W 4 \frac{1}{4} \) m for the pitch \( E_1 \), \( F 96 \) for the pitch \( G_1 \).

For \( E_1 \), we need to calculate the frequency: \( P 340 \) divided by \( W 4 \frac{1}{4} = F 80 \)
For \( G_1 \), we need to calculate the wavelength: \( P 340 \) divided by \( F 96 = W 3 \frac{1}{3} \)
1.6. Conversion between frequency and wavelength values

To denote octaves, Karg-Elert uses **Helmholtz pitch notation** (presented in Helmholtz 1875). This method uses both uppercase and lowercase letters for note names, with accent marks as subscripts and superscripts:

![Helmholtz pitch notation image](Public domain image, from the Wikipedia page “Helmholtz pitch notation”)

The following chart (Helmholtz/Ellis 1885, 30) lists octave ranges in Helmholtz pitch notation, plus frequencies (in Hertz) for the notes of the C major scale in each octave:

<table>
<thead>
<tr>
<th>Notes</th>
<th>Contra Octave C\text{\textsubscript{1}} to B\text{\textsubscript{1}}</th>
<th>Great Octave C\text{\textsubscript{2}} to B\text{\textsubscript{2}}</th>
<th>Unaccented Octave C\text{\textsubscript{3}} to C\text{\textsubscript{5}}</th>
<th>Once accented Octave C\text{\textsubscript{4}} to C\text{\textsubscript{5}}</th>
<th>Twice accented Octave C\text{\textsubscript{6}} to C\text{\textsubscript{7}}</th>
<th>Thrice accented Octave C\text{\textsubscript{8}} to C\text{\textsubscript{9}}</th>
<th>Four times accented Octave C\text{\textsubscript{10}} to C\text{\textsubscript{11}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>33</td>
<td>66</td>
<td>182</td>
<td>264</td>
<td>628</td>
<td>1056</td>
<td>2112</td>
</tr>
<tr>
<td>D</td>
<td>37.125</td>
<td>74.25</td>
<td>148.5</td>
<td>297</td>
<td>604</td>
<td>1188</td>
<td>2376</td>
</tr>
<tr>
<td>E</td>
<td>41.25</td>
<td>82.5</td>
<td>165</td>
<td>330</td>
<td>660</td>
<td>1320</td>
<td>2640</td>
</tr>
<tr>
<td>F</td>
<td>44</td>
<td>88</td>
<td>176</td>
<td>352</td>
<td>704</td>
<td>1408</td>
<td>2816</td>
</tr>
<tr>
<td>G</td>
<td>49.5</td>
<td>99</td>
<td>198</td>
<td>396</td>
<td>792</td>
<td>1584</td>
<td>3168</td>
</tr>
<tr>
<td>A</td>
<td>55</td>
<td>110</td>
<td>220</td>
<td>440</td>
<td>880</td>
<td>1760</td>
<td>3520</td>
</tr>
<tr>
<td>B</td>
<td>61.875</td>
<td>123.75</td>
<td>247.5</td>
<td>495</td>
<td>990</td>
<td>1980</td>
<td>3960</td>
</tr>
</tbody>
</table>

Instead of accent marks, Karg-Elert uses **numerical** subscripts and superscripts:

$$C_2, C_1, C, c, c^1, c^2, etc.$$  

The following compares Helmholtz pitch notation with scientific pitch notation (SPI), proposed in 1939 by the Acoustical Society of America (Young 1939), and now used widely to specify octave ranges:

- **Helmholtz:** C\text{\textsubscript{2}} C\text{\textsubscript{1}} C c c\text{\textsuperscript{1}} c\text{\textsuperscript{2}} c\text{\textsuperscript{3}} c\text{\textsuperscript{4}} c\text{\textsuperscript{5}}
- **SPI:** C0 C1 C2 C3 C4 C5 C6 C7 C8
PITCH

Frequency E₁
Wavelength in m 80
Wavelength in feet 4 1/4'
Organ pipe length 6 2/5'

That is: 5 times
That is: 1/5

of subcontra C (C₂):

Frequency 16
Wavelength in m 21 1/4 m
Wavelength in feet 64'
Organ pipe length 32'

2 octaves lower
1.6. (continued)
Chapter 2

Historically variable and constant "absolute" pitch levels
Chapter 2
Historically variable and constant “absolute” pitch levels

2.1. Assignment of normative tuning pitch frequencies
2.2. Choir, chamber and cornett pitch levels (16\textsuperscript{th} to 18\textsuperscript{th} centuries)
2.3. Fluctuating frequencies of the tuning pitch a\textsuperscript{1} (1582 to 1770)
2.4. Post-1885 rising of tuning pitch frequencies
2.5. A constant fundamental pitch for the theoretical study of music
2.6. A summary of frequencies and wavelengths for the total pitch range
2.1. Assignment of normative tuning pitch frequencies

As with the definition of units of space and length, the assignment of a normative tuning pitch level is an arbitrary act.

In 1859, the Paris Academy established the one-line A [i.e. a¹] at 870 single [half] or 435 double [whole] cycles per second as the obligatory standard "normal tuning pitch" for France and its colonies. At that time, Germany and Austria employed the "high" tuning for a¹ at 440 cycles. In 1885 at the "World Tuning Congress" in Vienna, an international commission of instrument makers, instrumentalists, conductors and pedagogues adopted the Parisian tuning as the standard for all countries, (allegedly!) valid in perpetuity.

France and England continue to measure frequency in half vibrations, while Germany and Austria measure frequency in complete vibrations:

\[ a¹ = 870 \text{ or } 435 \]

\[ | = \text{half vibrations} \quad l = \text{complete vibrations} \]

Prior to these international rulings on the standardization of absolute pitch, a truly chaotic situation existed between and even within individual countries!
2.1. Assignment of normative tuning pitch frequencies

Much of Karg-Elert’s information in sections 2.1 to 2.3 can be found in Alexander J. Ellis’ monograph “On the History of Musical Pitch” (Ellis 1880). Ellis translated and annotated Helmholtz’s 1863 treatise Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik (Helmholtz/Ellis 1885).

Ellis cites the 1859 Parisian standard of 435 Hz as the “French Diapason Normal,” and 440 Hz as “Scheibler’s Stuttgart Standard” of 1834 (Ellis 1880, 305). Scheibler’s frequency for a¹ was also listed in Helmholtz 1875.
2.2. Choir, chamber and cornett pitch levels (16th to 18th centuries)

We can find telling evidence of the erstwhile unfathomable differences in pitch levels through comparison of existing historical instruments of unchanging pitch [woodwind and brass instruments, organ flue pipes, etc.], and through comparison of the widely varying lengths for wind instruments and organ pipes specified in published and handwritten historical texts on instrument making.

Between the 16th and 18th centuries, there existed in the musically-cultured nations [Italy, France, Germany, the Low Countries, England] at first two, and later even three fundamentally different “common” pitch levels, that independently from each other experienced various phases of rise and fall.

The lowest tuning was called choir pitch, which made singing in higher registers easier for the church choir and congregation. It was almost a whole tone lower than the brilliant chamber tuning used by soloists and chamber musicians. Still higher [a semitone higher than chamber pitch, or a minor third higher than choir pitch] was the “sharp” cornett pitch, used by cornett players, tower musicians, herald trumpeters and city musicians.
2.2. Choir, chamber and cornett pitch levels (16th to 18th centuries)

A detailed description of these pitch levels (and much else on the history of pitch) can be found in Haynes and Cooke, "Pitch" (Grove Music Online), and also in Haynes 2002.
2.3. Fluctuating frequencies of the tuning pitch $a^1$ (1582 to 1770)

The tuning pitch was $a^1$; however, reliable research from original sources has provided evidence that in the aforementioned time [according to sources dating between 1582 and 1770], this tuning pitch $a^1$ fluctuated between 374[!] and 567[!] cycles. That is a difference of about a fifth! [374 : 567 approximates the ratio $2 : 3 = \text{interval of a fifth}$].

374 is a frequency between our present-day $f#^1$ and $g^1$
567 is a frequency between our present-day $c^2$ and $d^2$

<table>
<thead>
<tr>
<th>Parisian tuning [tempered]</th>
<th>$f#^1$</th>
<th>365.789825</th>
<th>374</th>
<th>387.5415</th>
</tr>
</thead>
<tbody>
<tr>
<td>Physical [pure]</td>
<td></td>
<td>364.5</td>
<td></td>
<td>384.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parisian tuning [tempered]</th>
<th>$c^2$</th>
<th>548.0652</th>
<th>567</th>
<th>580.6554</th>
</tr>
</thead>
<tbody>
<tr>
<td>Physical [pure]</td>
<td></td>
<td>546.75</td>
<td>576.0</td>
<td></td>
</tr>
</tbody>
</table>
2.3. *Fluctuating frequencies of the tuning pitch a¹ (1582 to 1770)*

These tuning pitch frequencies are cited in Ellis 1880 (p. 305). 374 Hz is listed as the “low church pitch” from the Hospice Comtesse (1700), and 567 Hz is listed as the “high chamber pitch” in Praetorius (“De Organographia” of 1619, part II of *Syntagma musicum*). Ellis names his sources for these and many other tuning frequencies.

The Parisian tunings are based on a¹ = 435 Hz, as described in 2.1. Karg-Elert’s values for the physical or “pure” tunings are calculated from the following frequency values: C₆ = 1 and C₁ = 32. See 2.5, and the chart in 2.6.
2.4. Post-1885 rising of tuning pitch frequencies

In defiance of the insufficiently-accepted standardization of the various tuning pitch levels [1885 Vienna World Tuning Congress], disagreement on this matter began to spread once again, as early as 1910.

For the last fifteen years in America, the normal tuning pitch has been established [without a congressional ruling!] at \( a^1 = 880 \) single cycles [which is identical to the pre-Paris German “high” pitch = \( a^1 = 440 \) complete cycles]. A large proportion of German instrument makers (including piano factories) have followed the American example, tuning their instruments “brightly” [\( = 440 \) up to about 468 (!)]. Already, this distuning has frequently reached a semitone (\( b^\# = a^1 \)), especially in string quartets and pianos.

The writer is himself familiar with many such outrageously over-pitched “solo” instruments. In addition, gramophones set at 78 revolutions per minute almost invariably produce a pitch level at which \( a^1 = 468 \) cycles, and is thus equivalent to \( b^\# \). The source of this monstrosity is the recording industry’s desire to endow instruments with the highest possible degree of brightness and luminosity [the ancient principle of the cornett pitch!] – as a result, the silvery glittering key of B major becomes a colorless C major ….. Difficile est satiram non scribere!
2.4. Post-1885 rising of tuning pitch frequencies

In 1939, an Anglo-German conference organized by the Acoustic Committee of Radio Berlin and the British Standard Association recommended that the standard pitch should be set at $a^1 = 440$ (Lloyd 1949, 75). This pitch level was reaffirmed by the International Organization for Standardization in 1955, and again in 1975 (Cavanagh, 3). It remains in wide use today in North America and the United Kingdom, though many orchestras in continental Europe tune to $a^1 = 442$ to 445 – see Franz Nistl’s list of international orchestral tuning frequencies (Nistl 2007).

“It is difficult not to write satire” – a motto from the first Satire by 2nd century Roman writer Juvenal. This seems to have been one of Karg-Elert’s favorite phrases, as he quotes it again in section 12.3, in what may be a critique of Schoenberg’s *Harmonielehre*. 
2.5. A constant fundamental pitch for the theoretical study of music

Die sogenannte physische, theoräischen Studien zugrunde liegende Stimmung ist konstant. Sie basiert auf dem hypothetischen Urton C₆, dessen Schwingungszahl 1 ist und dessen Wellenlänge 340.00 m beträgt. Er selbst bleibt dem menschlichen Gehörsorgan verborgen, da wir erst dann einen Ton empfinden, wenn die periodisch-konstanten Lufterschütterungen (die sind gleichmäßig alternierende Luftverdichtungen und -verdünnungen) so rasch aufeinander folgen, dass sie nicht mehr getrennt empfunden werden können, sondern zu einem unendlichen Ganzen verschmolzen. Dieser Übergang von zahlbaren Einzelstücken zum geschlossenen Summen der Brummtöne findet etwa bei 12 bis 18 Schwingungen ein [Ende der Ultrasubcontra-bis untere Region der Subcontra-Oktaoe].

Alle nur denkbaren Töne, deren Anzahl endlos ist, gehen direkt oder indirekt auf diesen Urton zurück;

- direkt: als Teilwerte der Urwelle, - oder als einfache Vielfachen der Schwingungseinheit;
- indirekt: als einfache Vielfachen der Teilwelle, - oder als Teilwerte der einfachen Schwingungsvielfachen. [Siehe später]

For the so-called physical, theoretical study of music, the basic pitch level is constant. It is based on a hypothetical fundamental tone C₆, whose frequency is 1, and whose wavelength is 340.00 m. This pitch is itself hidden to the human ear, which can first sense pitch when the periodically-constant fluctuations in the air [equal, alternating compressions and relaxations] follow each other so quickly that they can no longer be perceived individually, and are instead fused into an unbroken whole. This transition from numerable individual vibrations to a cohesive hum begins at around 12 to 18 cycles [the upper range of the ultrasubcontra-octave to the lower range of the subcontra-octave].

All conceivable pitches, whose number is infinite, proceed directly or indirectly from this fundamental tone:

Directly: as fractions of the fundamental, or as simple multiples of its frequency
Indirectly: as simple multiples of fractional wavelengths, - or as fractions of the simple multiples of the frequency. [See later chapters.]
2.5. *A constant fundamental pitch for the theoretical study of music*

The hypothetical fundamental pitch $C_6 = 1$ is derived from the work of Joseph Sauveur, who in 1713 proposed a standard tuning or *son fixe* of middle C ($C^1$) = 256. It ensures that C (in all octaves) is a power of 2 (see section 3.5 below), and places $a^1$ at 432 Hz. Sauveur’s tuning (now often called “scientific pitch” or “Sauveur pitch”) was not adopted by musicians of his time, though it has been frequently cited by acousticians and physicists as an ideal pitch level (Haynes 2002, 42).

“For auditory signals and human listeners, the accepted range is 20 to 20000 Hz, the limits of human hearing” (Rosen 2013, 163).

The acoustic derivation of “all conceivable pitches” is the primary topic of the entire treatise, and especially of chapters 3, 5, 9 and 12.
### 2.6. A summary of frequencies and wavelengths for the total pitch range

Nächstehende Übersicht der Oktavebereiche zeigt für die verschiedenen = C x = :

1. Die absolute Schwingungszahl.
2. Die absolute Wellenlänge in irdischen Fußmaßen.
3. im Oktavequivalentmaß.
4. Wellenlänge in Meter.
5. Die letzte Rubrik verweist auf die Höhen- und Tiefengrenzen einiger Instrumente:

<table>
<thead>
<tr>
<th>Key</th>
<th>Frequency</th>
<th>Oktavefuß</th>
<th>Wavelength [m]</th>
<th>Instrumentalgrößen</th>
</tr>
</thead>
<tbody>
<tr>
<td>C3</td>
<td>327.68</td>
<td>3³</td>
<td>0.0104</td>
<td>Orgel [C' im 1']</td>
</tr>
<tr>
<td>C4</td>
<td>163.84</td>
<td>3²</td>
<td>0.0208</td>
<td>Wurl. [unbestimmt]</td>
</tr>
<tr>
<td>C5</td>
<td>819.20</td>
<td>3¹</td>
<td>0.0415</td>
<td>Flöte</td>
</tr>
<tr>
<td>C6</td>
<td>409.60</td>
<td>3⁰</td>
<td>0.0830</td>
<td>Flöte</td>
</tr>
<tr>
<td>C7</td>
<td>204.80</td>
<td>3⁻¹</td>
<td>0.1660</td>
<td>Flöte</td>
</tr>
<tr>
<td>C8</td>
<td>102.40</td>
<td>3⁻²</td>
<td>0.3320</td>
<td>Flöte</td>
</tr>
<tr>
<td>C9</td>
<td>512</td>
<td>2</td>
<td>0.6640</td>
<td>Flöte</td>
</tr>
<tr>
<td>C10</td>
<td>256</td>
<td>4</td>
<td>1.3281</td>
<td>Flöte</td>
</tr>
<tr>
<td>C11</td>
<td>128</td>
<td>8</td>
<td>2.5562</td>
<td>Flöte</td>
</tr>
<tr>
<td>C12</td>
<td>64</td>
<td>16</td>
<td>5.1125</td>
<td>Violoncello</td>
</tr>
<tr>
<td>C14</td>
<td>16</td>
<td>64</td>
<td>21.25</td>
<td>Orgel (→ Klavier A₁)</td>
</tr>
<tr>
<td>C15</td>
<td>8</td>
<td>128</td>
<td>42.5</td>
<td>Orgel (mehrfach disp)</td>
</tr>
</tbody>
</table>

(English translation on next page)
The following summary of the octave ranges for the different “C”s includes:

1. The absolute frequency (F) in cycles per second (Hz)
2. The absolute wavelength (W) in actual feet
3. The organ pipe length
4. Wavelength in meters
5. The last column specifies the upper (↑) and lower (↓) range limits of certain instruments

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>W in feet</th>
<th>Organ pipe</th>
<th>W in meters</th>
<th>Range limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>c8</td>
<td>32768</td>
<td>1/32</td>
<td>1/64’</td>
<td>0.0104…</td>
<td>↑ Galton whistle</td>
</tr>
<tr>
<td>c7</td>
<td>16384</td>
<td>1/16</td>
<td>1/32’</td>
<td>0.0208…</td>
<td>↑ Organ [c4 in 1’]</td>
</tr>
<tr>
<td>c6</td>
<td>8192</td>
<td>1/8</td>
<td>1/16’</td>
<td>0.0415…</td>
<td>↑ Violin harmonics</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(indistinct)</td>
</tr>
<tr>
<td>c5</td>
<td>4096</td>
<td>1/4</td>
<td>1/8’</td>
<td>0.0830…</td>
<td>↑ Piccolo {and piano}</td>
</tr>
<tr>
<td>c4</td>
<td>2048</td>
<td>1/2</td>
<td>1/4’</td>
<td>0.1660…</td>
<td>↑ Flute</td>
</tr>
<tr>
<td>c3</td>
<td>1024</td>
<td>1</td>
<td>1/2’</td>
<td>0.33203…</td>
<td>↑ Soprano voice</td>
</tr>
<tr>
<td>c2</td>
<td>512</td>
<td>2</td>
<td>1’</td>
<td>0.66406…</td>
<td>↑ Bassoon, horn {not</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>really!}</td>
</tr>
<tr>
<td>c1</td>
<td>256</td>
<td>4</td>
<td>2’</td>
<td>1.32812(5)</td>
<td>↓ Flute</td>
</tr>
<tr>
<td>c</td>
<td>128</td>
<td>8</td>
<td>4’</td>
<td>2.5625</td>
<td>↓ Viola</td>
</tr>
<tr>
<td>C</td>
<td>64</td>
<td>16</td>
<td>8’</td>
<td>5.3125</td>
<td>↓ Cello</td>
</tr>
<tr>
<td>C1</td>
<td>32</td>
<td>32</td>
<td>16’</td>
<td>10.625</td>
<td>↓ Double bass,</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>contrabassoon</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>[← Piano A2]</td>
</tr>
<tr>
<td>C2</td>
<td>16</td>
<td>64</td>
<td>32’</td>
<td>21.25</td>
<td>↓ Organ</td>
</tr>
<tr>
<td>C3</td>
<td>8</td>
<td>128</td>
<td>64’</td>
<td>42.5</td>
<td></td>
</tr>
<tr>
<td>C4</td>
<td>4</td>
<td>256</td>
<td>128’</td>
<td>85</td>
<td></td>
</tr>
<tr>
<td>C5</td>
<td>2</td>
<td>512</td>
<td>256’</td>
<td>170</td>
<td></td>
</tr>
<tr>
<td>C6</td>
<td>1</td>
<td>1024</td>
<td>512’</td>
<td>340</td>
<td></td>
</tr>
</tbody>
</table>

NOTE: as mentioned, pitches outside the frequency range of 20 to 20000 Hz are normally inaudible to the human ear. The Galton whistle (invented in 1876 by Sir Francis Galton, 1822-1911) could produce frequencies in the range of 5000 to over 42000 Hz; it was used in the training of animals, especially dogs (Galton 1883, 26-27).
Chapter 3

Canonic [Pythagorean] partial and Messel values
Chapter 3
Canonic [Pythagorean] partial and Messel values

3.1. Pythagorean derivation of intervals from the natural fifth and fourth
3.2. Arabic/Persian derivation of intervals through Messel multiplication
3.3. Diagram comparing the Pythagorean and Messel derivation methods
3.4. Dot notation for fifth-derived canonic [Pythagorean] pitches
3.5. Table of powers of 3 (fifths) and 2 (octaves)
3.6. The millioctave (µ)
3.7. The millioctave and µ units for simple intervals
3.8. The Limma, Apotome, and Pythagorean comma
3.9. The practical advantage of using µ calculation
3.10. Complete table of [over]fifth-derived interval values, within an octave
3.11. Complete table of [under]fifth-derived interval values, within an octave
3.12. Calculation shortcuts
3.14. Example: µ values of chromatic and diatonic semitones and enharmonics
3.15. Comparative overview of Pythagorean and equal-tempered 12-tone scales
3.16. Explanation of the foregoing table
3.1. **Pythagorean derivation of intervals from the natural fifth and fourth**

Using the monochord, the great mathematician Pythagoras discovered the natural phenomena of the octave and twelfth in the string-length proportions $1:1/2$ and $1:1/3$. The difference of these intervals automatically produces the fifth and its (octave) inversion the fourth. From the natural, basic values of the octave and twelfth, Pythagoras derived all of the generally recognized intervals.

3.2. **Arabic/Persian derivation of intervals through Messel multiplication**

The Arabs and Persians also employed the monochord in order to measure the natural intervals. However, in polar opposite to the Greek pitch derivation, which arrived at the higher partial values through progressive division of a relatively longer string length – the Arabs and Persians derived lower Messel values through progressive multiplication (that is, the addition of equal complete units) of a relatively smaller string length.
3.1.  Pythagorean derivation of intervals from the natural fifth and fourth

3.2.  Arabic/Persian derivation of intervals through Messel multiplication

Riemann and Karg-Elert understand a Messel to be a short length of string used in canonic calculations. The Messel is multiplied to create a longer string: most often, a longer string comprises twelve Messel. The “Messel theory” is first described in Riemann 1878 (pp. 77-85), where Riemann (quoting Kiesewetter’s 1842 treatise Die Musik der Araber) attributes it to the 14th-century theorist Mahmud al-Schirazi; however, Riemann proposes that the Messel concept may be much older, possibly dating from before the time of al-Farabi (c. 925). The multiplication of the Messel provides values equal to an inversion of the overtone series, and thus generates a minor triad. Riemann continued to cite the “Messel theory” as a logical and mathematical explanation of the minor triad and of harmonic dualism, even after he admitted that undertones were not a physical reality. Karg-Elert almost certainly borrowed the Messel concept directly from Riemann, and used it in exactly the same way.

In the second edition of the Harvard Dictionary of Music, Willi Apel notes that Messel is a Germanization of mathal, “a term used in Arab theory to indicate fractions of the type (n + 1)/n (e.g., 4/3, 5/4, etc). All intervals represented by such fractions were considered consonant by the Arabs” (Apel 1969, 523). Thus, in Arabic music theory, the consonant 5:4 third was simply one of the mathal intervals, not necessarily to be privileged as a component of the consonant major or minor triad.
3.3. **Diagram comparing the Pythagorean and Messel derivation methods**

Griechisches Prinzip = Greek concept
Arabisches Prinzip = Arabic concept
Abszisse = abscissa (location on horizontal axis)
Ordinate = ordinate (location on vertical axis)
Von der Einheit zu den Teilen (Partialen) = from the unit to the fragments (partials)
Von der Messeinheit zu den Messelvielheiten = from single to multiple Messel units
3.3. *Diagram comparing the Pythagorean and Messel derivation methods*
3.4. Dot notation for fifth-derived canonic [Pythagorean] pitches

My acoustic symbols denote fifth-derived tones by using dots:

' = 1 over-fifth from C(0) = G

.. = 2 under-fifths from C(0) = B♭, and so on

The primary octaves are not marked. Octave ranges can appear as a result of exponents of the fifth. The fifth-power $3^2 = 9''$ falls in the octave range $8 - 16$; therefore, $8:9''$ will be understood as a whole tone, and $9'' : 16$ as a Pythagorean minor seventh.
Instead of Karg-Elert’s dots, the translation uses the notation \( X(n) \) to indicate that pitch \( X \) is \( n \) fifths above or below the central pitch \( C \), as follows:

\[
\begin{align*}
G(1) &= G \text{ that is 1 fifth above } C. \\
F(-1) &= F \text{ that is 1 fifth below } C. \\
D(2) &= D \text{ that is two fifths above } C, \text{ and so on.}
\end{align*}
\]

In later chapters, one or two more coordinates will be added, to denote positions in multidimensional pitch space involving pure thirds and/or sevenths, as well as fifths.

This paragraph’s linkage of partials and fifths needs some clarification. The twelfth above the fundamental is the third partial, and is thus symbolized by 3. From C, the third partial or 3 is \( G(1) \), the first over-fifth (‘). Two fifths (‘‘) above \( C \) is \( D(2) \), which is equal to \( 3 \times 3 = 9 \) (the ninth partial). See the next section for a full demonstration.

While over- and under-fifths occupy a given register in relation to the generating pitch (later modeled in Karg-Elert’s *Ursprungslage* or “source positions”), they can be transposed into different octaves or ranges without losing their fifth-derived identity. In 3.10 and 3.11, Karg-Elert reduces the large compound intervals produced by adding over- and under-fifths, in order to derive a 25-note Pythagorean scale spanning one octave.
3.5. **Table of powers of 3 (fifths) and 2 (octaves)**

### Table of Powers of 3 (Fifths)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$Punkt$</th>
<th>$Duodezime$</th>
<th>Akust. Kennzahl</th>
<th>$3^{\text{rd}} = 3$ bwz. $\frac{3}{5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\cdot$</td>
<td>$2$</td>
<td>$\cdot$</td>
<td>$3^2 = 9$</td>
</tr>
<tr>
<td>3</td>
<td>$\cdot$</td>
<td>$3$</td>
<td>$\cdot$</td>
<td>$3^3 = 27$</td>
</tr>
<tr>
<td>4</td>
<td>$\cdot$</td>
<td>$4$</td>
<td>$\cdot$</td>
<td>$3^4 = 81$</td>
</tr>
<tr>
<td>5</td>
<td>$\cdot$</td>
<td>$5$</td>
<td>$\cdot$</td>
<td>$3^5 = 243$</td>
</tr>
<tr>
<td>6</td>
<td>$\cdot$</td>
<td>$6$</td>
<td>$\cdot$</td>
<td>$3^6 = 729$</td>
</tr>
<tr>
<td>7</td>
<td>$\cdot$</td>
<td>$7$</td>
<td>$\cdot$</td>
<td>$3^7 = 2,187$</td>
</tr>
<tr>
<td>8</td>
<td>$\cdot$</td>
<td>$8$</td>
<td>$\cdot$</td>
<td>$3^8 = 6,561$</td>
</tr>
<tr>
<td>9</td>
<td>$\cdot$</td>
<td>$9$</td>
<td>$\cdot$</td>
<td>$3^9 = 19,683$</td>
</tr>
<tr>
<td>10</td>
<td>$\cdot$</td>
<td>$10$</td>
<td>$\cdot$</td>
<td>$3^{10} = 59,049$</td>
</tr>
<tr>
<td>11</td>
<td>$\cdot$</td>
<td>$11$</td>
<td>$\cdot$</td>
<td>$3^{11} = 177,147$</td>
</tr>
<tr>
<td>12</td>
<td>$\cdot$</td>
<td>$12$</td>
<td>$\cdot$</td>
<td>$3^{12} = 531,441$</td>
</tr>
</tbody>
</table>

$n$. Punkt gleich $n$. Duodezime = $n$ dots equals $n$ twelfths  
Akustische Kennzahl = acoustic number

### Table of Powers of 2 (Octave Ranges)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$Oktavbereich$</th>
<th>Akust. Kennzahl</th>
<th>$1 : 2$ bwz. $1 : \frac{3}{5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$2 : 4 , \frac{3}{5} : \frac{3}{4}$</td>
</tr>
<tr>
<td>3</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$4 : 8 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>4</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$8 : 16 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>5</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$16 : 32 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>6</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$32 : 64 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>7</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$64 : 128 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>8</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$128 : 256 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>9</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$256 : 512 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>10</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$512 : 1,024 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>11</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$1,024 : 2,048 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>12</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$2,048 : 4,096 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>13</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$4,096 : 8,192 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>14</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$8,192 : 16,384 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>15</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$16,384 : 32,768 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>16</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$32,768 : 65,536 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>17</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$65,536 : 131,072 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>18</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$131,072 : 262,144 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>19</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$262,144 : 524,288 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
<tr>
<td>20</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$524,288 : 1,048,576 , \frac{3}{4} : \frac{3}{4}$</td>
</tr>
</tbody>
</table>

$n$. Oktavbereich = $n$th octave range  
Akustische Kennzahl = acoustic number
3.5. **Table of powers of 3 (fifths) and 2 (octaves)**

The powers of 3 are listed at the right: 3, 9, 27, 81…
Those powers represent a pitch series in ascending twelfths (or fifths if octave-reduced):

- \(3 = G\)
- \(9 = D\)
- \(27 = A\)
- \(81 = E\), etc.

Those powers of 3 are also the partials (overtones) that produce the specified pitches, in pure Pythagorean tuning.

The powers of 2 are listed at the right: 2, 4, 8, 16, 32…
Those powers represent a series of ascending octaves; the numbers are also the partials that produce those octaves.

As discussed in 2.5, using powers of 2 to represent octaves and octave ranges stems from Joseph Sauveur’s adoption in 1713 of \(c^1 = 256\) Hz as a *son fixe* (“fixed sound”). Karg-Elert adopts \(C_6 = 1\) as the fundamental source of his pitch space.
3.6. The millioctave (µ)

Before the complete table of the 25-tone canonic scale can be derived, we must now introduce the principle of the 1000-part octave [Milli-octave, µ (Greek letter for “m,” pronounced “mi”) which is the abbreviation for “mio” = unit of the millioctave]. In addition to F (frequency) and W (wavelength) values, the table provides µ values for intervals.
3.6. *The millioctave (µ)*

The millioctave (1/1000 of an octave) is an alternative unit for division of the octave, used here instead of the much more familiar cent (1/1200 of an octave). Karg-Elert likely borrowed it from Arthur von Oettingen’s *Das duale Harmoniesystem* (Oettingen 1913, 249-253), though it was also used in the 1891 acoustics treatise *Das mathematisch-reine Tonsystem* by Carl Eitz (1848–1924). According to John Biddell Airy, the division of the octave into 1000 parts was first proposed by English astronomer and mathematician John Herschel (Airy 1871, 222).
3.7. **The millioctave and µ units for simple intervals**

Die Millioctave und die µ Einheiten

Die Logarithmen der relativen Schwingungszahlen auf Basis 2 veranschaulichen die Raumwerte innerhalb einer in 1000 Teile zergliederten Oktave.

Dann ist:

\[ S \frac{2}{3} \; \text{od.} \; \text{W} \frac{2}{3} = \text{Oktave} = \frac{1000}{\mu} \] (min)

\[ S \frac{1}{2} = \text{Quinte} = \frac{585}{\mu} \]

\[ S \frac{2}{3} = \text{Quarte} = \frac{415}{\mu} \]

Die Oktave ist gleich der Summe der Quinte und Quarte:

\[ \left( \frac{3}{2} \cdot \frac{12}{6} \right) = \text{Oktave} = \frac{1000}{\mu} \]

Der Ganzton resultiert aus der Differenz zwischen Quinte und Quarte:

\[ \left( \frac{3}{2} - \frac{9}{8} \right) = \text{Ganzton} = \frac{170}{\mu} \]

The base-2 logarithms of the frequency ratios provide their positions within the octave divided into 1000 parts (i.e. their value in µ):

We begin with:

- F 2/1 or W 1/2 = Octave = \(\frac{1000}{\mu}\)
- F 3/2 or W 2/3 = Fifth = \(\frac{585}{\mu}\)
- F 4/3 or W 3/4 = Fourth = \(\frac{415}{\mu}\)

The **octave** is exactly the sum of the fifth and fourth:

\[ \frac{3}{2} \cdot \frac{4}{3} = 12/6 = 2/1 \; \text{F} \]

In µ-values:

\[ 585 \; \text{(fifth)} + 415 \; \text{(fourth)} = 1000 \; \mu \; \text{(octave)} \]

The **whole tone** results from the difference between the fifth and fourth:

\[ \frac{3}{2} : \frac{4}{3} = 9/8 \; \text{F} \]

In µ-values:

\[ 585 \; \text{(fifth)} - 415 \; \text{(fourth)} = 170 \; \mu \; \text{(whole tone)} \]
3.7. The millioctave and μ units for simple intervals

The interval n (measured in μ) between two pitches in the ratio a : b is calculated as:
\[ n = 1000 \times \log_2(a:b) \]

The same interval in cents would be calculated \[ n = 1200 \times \log_2(a:b) \]
Karg-Elert rounds off the logarithmic values to the nearest whole number.

To convert millioctaves to cents, simply multiply by 1.2:
Fifth = 585 μ x 1.2 = 702 cents
To convert cents to millioctaves, divide by 1.2.

It is likely that Karg-Elert’s fundamental reason for adopting the millioctave rather than the cent is to divorce the acoustic calculation of pitch as much as possible from 12-tone equal temperament (for which the cent is eminently suited, as the tempered semitone is 100 cents). However, the millioctave also provides a more practical advantage: adding and subtracting basic Pythagorean intervals to calculate other Pythagorean (as opposed to tempered) intervals is considerably easier in millioctaves than in cents, as the basic intervals in millioctaves are multiples of 5 or 10. For example:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Millioctaves (μ)</th>
<th>Cent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fifth (3:2)</td>
<td>585</td>
<td>702</td>
</tr>
<tr>
<td>Fourth (4:3)</td>
<td>415</td>
<td>498</td>
</tr>
<tr>
<td>Major third (81:64)</td>
<td>340</td>
<td>408</td>
</tr>
<tr>
<td>Whole tone (9:8)</td>
<td>170</td>
<td>204</td>
</tr>
</tbody>
</table>
The **minor third** is the difference between the fourth and whole tone:

\[
\frac{4}{3} : \frac{9}{8} = \frac{32}{27} = 245 \mu \text{ (minor third)}
\]

The **major third** is exactly the sum of two whole tones:

\[
\frac{9}{8} \times \frac{9}{8} = \frac{81}{64} = 340 \text{ (major third)}
\]

The major and minor thirds added together form a fifth:

\[
\frac{81}{64} \times \frac{32}{27} = \frac{2592}{1728} = \frac{3}{2} = 585 \text{ (fifth)}
\]
3.7. (continued)
### 3.8. The Limma, Apotome, and Pythagorean comma

The **diatonic** (small) semitone ["**Limma**"] is the difference between the fourth and the major third:

\[
\frac{4}{3} \quad : \quad \frac{81}{64} = \frac{256}{243} \quad = \quad 75 \text{ (Limma)}
\]

*) Diatonic = "passing," back through neighbouring scale steps.

The **chromatic** (large) semitone ["**Apotome**"] is the difference between the whole tone and the Limma:

\[
\frac{9}{8} \quad : \quad \frac{256}{243} = \frac{2187}{2048} \quad = \quad 95 \text{ (Apotome)}
\]

The **Pythagorean comma** is the difference between the Apotome and Limma:

\[
\frac{2187}{2048} \quad : \quad \frac{256}{243} = \frac{531441}{524288} \quad = \quad 20 \text{ (Pythagorean comma)}
\]
3.8. *The Limma, Apotome, and Pythagorean comma*

Diatonic semitones (such as the Limma) have different letter names: e.g. C♯ and D.

Chromatic semitones (such as the Apotome) have the same letter name: e.g. C and C♯.

The Pythagorean comma (20 µ) is the difference between two pitches that are 12 fifths apart, such as C and B♯, or C and D♭♭.
3.9. The practical advantage of using $\mu$ calculation

We do not need to explain at length the immense advantages of using the $\mu$-calculation, in comparison to the other methods outlined above. The previous examples speak for themselves.

Division and multiplication of mathematical ratios correspond {respectively} to subtraction and addition in $\mu$-calculations.

\[
\frac{a}{b} \text{ (math.)} = \frac{a}{b} \text{ (}$\mu$\text{)} \parallel a \cdot b \text{ (math.)} = a + b \text{ (}$\mu$\text{)}
\]

Alles weitere ergibt sich aus nachstehender Tabelle:

We do not need to explain at length the immense advantages of using the $\mu$-calculation, in comparison to the other methods outlined above. The previous examples speak for themselves.

Division and multiplication of mathematical ratios correspond {respectively} to subtraction and addition in $\mu$-calculations.

\[
\frac{a}{b} \text{ (math.)} = \frac{a}{b} \text{ (}$\mu$\text{)} \parallel a \cdot b \text{ (math.)} = a + b \text{ (}$\mu$\text{)}
\]

Everything else {the other intervals} is provided on the following table {3.10}:
3.9. **The practical advantage of using μ calculation**
### Übersichtstabelle

**3.10. Complete table of [over]fifth-derived interval values, within an octave**

**Übersichtstabelle**
der Quintpotenzen innerhalb je eines Oktavbereichs
[pythagoräische resp. kanonische Intervalleinteilungen]

[Pythagorean or canonic interval-values]

<table>
<thead>
<tr>
<th>Over-fifths and their inversions:</th>
<th>Calculated from C₀</th>
<th>Interval in mio (µ)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1)</strong> Pure fifth</td>
<td>2:3</td>
<td>G Bolivia</td>
</tr>
<tr>
<td>Pure fourth</td>
<td>3:4</td>
<td></td>
</tr>
<tr>
<td><strong>2)</strong> Major second</td>
<td>8:9</td>
<td>D Bolivia</td>
</tr>
<tr>
<td>Minor seventh</td>
<td>9:16</td>
<td></td>
</tr>
<tr>
<td><strong>3)</strong> Major sixth</td>
<td>16:27</td>
<td>A Venezuela</td>
</tr>
<tr>
<td>Minor third</td>
<td>27:32</td>
<td></td>
</tr>
<tr>
<td><strong>4)</strong> Major third</td>
<td>64:81</td>
<td>E Venezuela</td>
</tr>
<tr>
<td>Minor sixth</td>
<td>81:128</td>
<td></td>
</tr>
<tr>
<td><strong>5)</strong> Major seventh</td>
<td>128:243</td>
<td>B Venezuela</td>
</tr>
<tr>
<td>Minor second (Limma)</td>
<td>243:256</td>
<td></td>
</tr>
<tr>
<td><strong>6)</strong> Augmented fourth (tritone)</td>
<td>512:729</td>
<td>F Bolivia</td>
</tr>
<tr>
<td>Diminished fifth</td>
<td>729:1024</td>
<td></td>
</tr>
</tbody>
</table>

---

1) Pure fifth  
2) Major fourth  
3) Major second  
4) Minor seventh  
5) Minor sixth  
6) Major sixth  
7) Major seventh  
8) Minor second (Limma)  
9) Augmented fourth (tritone)  
10) Diminished fifth
7) Augmented unison (Apotome) 2048:2187 \[ C_f \]
   Diminished octave 2187:4096 \[ 95 \mu \] 905 \mu
8) Augmented fifth 4096:6561 \[ G_f \]
   Diminished fourth 6561:8192 \[ 680 \mu \] 320 \mu
9) Augmented second (Hiatus) 16384:19683 \[ D_f \]
   Diminished seventh 19683:32768 \[ 265 \mu \] 735 \mu
10) Augmented sixth 32768:59049 \[ A_f \]
    Diminished third 59049:65536 \[ 850 \mu \] 150 \mu
11) Augmented third 131072:177147 \[ B_f \]
    Diminished sixth 177147:262144 \[ 435 \mu \] 565 \mu
12) Diminished second (Pythagorean comma) 524288:531441 \[ 20 \mu \]
    (smaller than an octave)
    Diminished ninth 531441:1048576 \[ higher than C_f \] 980 \mu
3.11. Complete table of [under]fifth-derived interval values, within an octave

The wavelength (W) values of the under-fifths and their inversions are the inverses of the frequency (F) values of the over-fifths and their inversions. For this reason, the W-values are placed under the dividing line. Here, the under-fifths are changed by octave transposition into over-fourths, to make possible a clearer comparison with the over-fifth intervals.

(chart begins on next page)
3.11. Complete table of fifth-derived interval values, within an octave

The table in this section provides exactly the same intervals and $\mu$ values as that in 3.10., but now calculated using fifths below C(0).
### Under-fifths and their inversions:

<table>
<thead>
<tr>
<th>Under-fifth</th>
<th>Interval in mio (µ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4 : 3)</td>
<td>(415) µ</td>
</tr>
<tr>
<td>(3 : 2)</td>
<td>(585) µ</td>
</tr>
<tr>
<td>(3 : 2)</td>
<td>(830) µ</td>
</tr>
<tr>
<td>(3 : 2)</td>
<td>(75) µ</td>
</tr>
<tr>
<td>(3 : 2)</td>
<td>(490) µ</td>
</tr>
</tbody>
</table>

### Calculated from \(C_0\)

<table>
<thead>
<tr>
<th>Interval in mio (µ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(415) µ</td>
</tr>
<tr>
<td>(585) µ</td>
</tr>
<tr>
<td>(830) µ</td>
</tr>
<tr>
<td>(75) µ</td>
</tr>
<tr>
<td>(490) µ</td>
</tr>
</tbody>
</table>

### Interval in mio (µ)

1) Pure fourth  \(F_{4(1)}\)  \(415\) µ  
2) Pure fifth  \(Bb_{7(2)}\)  \(585\) µ  
3) Minor seventh  \(E_{7(3)}\)  \(830\) µ  
4) Major second  \(E_{7(3)}\)  \(170\) µ  
5) Minor third  \(Ab_{7(4)}\)  \(245\) µ  
6) Major sixth  \(Ab_{7(4)}\)  \(7555\) µ  
7) Minor sixth  \(Db_{7(5)}\)  \(340\) µ  
8) Major third  \(Db_{7(5)}\)  \(925\) µ  
9) Minor second (Limma)  \(G_{7(6)}\)  \(490\) µ  
10) Major seventh  \(G_{7(6)}\)  \(510\) µ  

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7) Diminished octave
   Augmented unison (Apotome)
   $C\sharp(7)$ 905 µ
   $95$ µ

8) Diminished fourth
   Augmented fifth
   $F\flat(8)$ 320 µ
   680 µ

9) Diminished seventh
   Augmented second
   $B\flat\flat(9)$ 735 µ
   265 µ

10) Diminished third
    Augmented sixth
    $E\flat\flat(10)$ 150 µ
    850 µ

11) Diminished sixth
    Augmented third
    $A\flat\flat(11)$ 565 µ
    435 µ

12) Diminished ninth
    Diminished second (Pythagorean comma)
    $D\flat\flat(12)$ 980 µ
    20 µ
If the tone D is used in the preceding tables as the generating prime \textit{instead of C}, the reciprocity between $\#$ and $\flat$ becomes evident:

$$
\begin{array}{c|c|c}
1& \#& 1 \flat \\
5& \#& 5 \flat \\
\end{array}
$$
If D is used as an axis of symmetry, sharps and flats (as applied to pitches) are indeed reciprocal:

<table>
<thead>
<tr>
<th>E♭</th>
<th>B♭</th>
<th>F</th>
<th>C</th>
<th>G</th>
<th>D</th>
<th>A</th>
<th>E</th>
<th>B</th>
<th>F♯</th>
<th>C♯</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd♭</td>
<td>1st♭</td>
<td>1st #</td>
<td>2nd #</td>
<td>etc.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Due to this systematic exchange of flats and sharps when D is used as an axis of symmetry, Arthur von Oettingen used it consistently in his work in order to illustrate polarity between major and minor (or what Oettingen termed tonic and phonic). In his 1913 treatise *Das duale Harmoniesystem*, Oettingen uses a D-clef (the letter “D” written on the middle staff line) in many examples (such as p. 46), always to emphasize major/minor dualism.
3.12. Calculation shortcuts

TRICKRECHNUNG

Es ist ratsam, bei größeren Quintpotenzen nicht umständliche Multiplikation (= Addition) mit der 585 zu vermeiden, etwa 4 * 585 = (340) sondern man fasse gleich 2 Punkte als Wert zusammen und multipliziere mit 2 (2Q = 2 * 170 = 340).

Diese Quadrupelquinten merke man sich ebenfalls als Einheitswert, der oft genug umständliche Rechnungen vereinfacht:

\[ 585 \times 2 = 2 \cdot 340 = 680 \quad \text{oder} \quad 585 \times 3 = 3 \cdot 340 = 1020. \]

Günstigere Verhältnisse weisen sich stets durch Plus- und Minusdifferenzen von 2 Punkten aus, also wird in der Wertung stets zur bereits bekannten Zahl 170 hinzugefügt resp. von der bereits bekannten Zahl 170 abgezogen:

\[ \text{Entspricht} \quad \text{zu} \quad \text{von} \quad \text{ab} \quad \text{zu} \quad \text{ab} \quad \text{zu} \quad \text{ab} \]

\[ \text{Entspricht} \quad \text{zu} \quad \text{von} \quad \text{ab} \quad \text{zu} \quad \text{ab} \quad \text{zu} \quad \text{ab} \]

Instead of performing laborious multiplication [= addition] of fifth-powers using the figure 585 [for example: 4Q = 4 * 585 = (2340)], one can use 2Q = 170 [i.e. the whole tone] as a basic unit for multiplication/addition. For example:

\[ 4Q = 2 \times (2Q) = 2 \times 170 = 340. \]

The quadruple fifth [i.e. 4Q = 340] can likewise be used as a unit to simplify calculations:

\[ 8Q = 2 \times 4Q = 2 \times 340 = 680 \quad \text{and} \quad 12Q = 3 \times 340 = 1020 \quad (= \text{Pythagorean comma}) \]

Whole-tone distances can be indicated as plus- or minus-differences of 2 dots (2Q), and so are calculated by adding or subtracting the familiar \( \mu \) value of 170:

\[ \begin{align*}
\text{Db}_5 \quad \text{Eb}_3 \quad \text{F}_1 \quad \text{F}_{-1} & = 415 \\
\text{Eb}_3 \quad 415 - 170 & = 245 \\
\text{Db}_5 \quad 245 - 170 & = 95 \quad \text{or} \quad 415 - 340 = 95
\end{align*} \]

\[ \begin{align*}
\text{G}_1 \quad \text{A}_3 \quad \text{B}_5 \quad \text{G}_1 & = 585 \\
\text{A}_3 \quad 585 + 170 & = 755 \\
\text{B}_5 \quad 755 + 170 & = 925 \quad \text{or} \quad 585 + 340 = 925
\end{align*} \]
3.12. *Calculation shortcuts*

“Q” refers to fifths – the same as Karg-Elert’s dots.

The whole tone (170 µ) is equal to two fifths (2 * 585 µ = 1170). To reduce a µ-value by one or more octaves, simply subtract 1000 or a multiple of 1000, as in 1170 → 170, or 2340 → 340.
3.13. Example: calculation of \( \mu \) value for [canonic] G-flat and B double-flat

We can quickly compute the \( \mu \)-value for G\( \flat \) [in three different ways]:

1. G\( \flat \) = Major 3\( \text{rd} \) below B\( \flat \)
   
   \[ \text{G}\( \flat \) = 1000 - 170 = 830 \quad \{\text{C minus whole tone}\} \]
   
   \[ \text{G}\( \flat \) = 830 - 340 = 490 \quad \{\text{B}\( \flat \) minus major third}\]  

2. G\( \flat \) = Limma above F
   
   \[ \text{F} = 415 + 75 \quad \{\text{limma}\} \quad = 490 \]

3. G\( \flat \) = Apotome below G
   
   \[ \text{G} = 585 - 95 = 490 \quad \{\text{apotome}\} \]

We can quickly compute the \( \mu \)-value for B\( \flat \):  

1. B\( \flat \) = Apotome below B\( \flat \)
   
   \[ \text{B}\( \flat \) = 1000 - 170 = 830 \quad \{\text{C minus whole tone}\} \]
   
   \[ \text{B}\( \flat \) = 830 - 95 = 735 \quad \{\text{B}\( \flat \) minus whole tone}\]  

2. B\( \flat \) = Limma above A\( \flat \)
   
   \[ \text{A}\( \flat \) = 1000 - 340 = 660 \]
   
   \[ \text{B}\( \flat \) = 660 + 75 = 735 \]

3. B\( \flat \) = Comma below A
   
   \[ \text{A} = 755 - 20 = 735 \]

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3.13. **Example: calculation of $\mu$ value for [canonic] G-flat and B double-flat**
3.14. Example: $\mu$ values of chromatic and diatonic semitones and enharmonics

EXAMPLE: $\mu$-values of chromatic and diatonic semitones and enharmonics

If you know the $\mu$-value of a given base-tone such as $D_{(2)} = 170$, shortcuts greatly simplify the calculation of its diatonic and chromatic semitones, and of its enharmonics:

In relation to $D_{(2)} = 170$

Limmata: $C#_{(7)} = 170 - 75 = 95$  $Eb_{(3)} = 170 + 75 = 245$

Apotome: $D#_{(9)} = 170 + 95 = 265$  $Db_{(5)} = 170 - 95 = 75$

Commas: $C\flat_{(14)} = 170 + 20 = 190$  $Es\flat_{(10)} = 170 - 20 = 150$

Using similar tricks, other seemingly difficult calculations can be performed almost instantly, allowing for sufficient practice and musical facility.
3.14. Example: $\mu$ values of chromatic and diatonic semitones and enharmonics

**ERRATUM:** C# (“cis”) should have 7 dots instead of 5, as it is C#(7), or seven fifths above C(0). The millioctave value of 95 is correct.
3.15. Comparative overview of Pythagorean and equal-tempered 12-tone scales

COMPARATIVE OVERVIEW between the three-interval symmetrical-polar 25- and 12-tone unequally-divided Pythagorean scale, and the 24-tone equally-divided quarter-tone scale or the 12-tone equally-divided half-tone scale of the tempered system.

Abstände stets gleichmäßig = (quarter-tone and semitone) intervals always equal (lines C and D)
3.15. *Comparative overview of Pythagorean and equal-tempered 12-tone scales*

The *mechanische Hälfte* or “mechanical axis” of the 25-tone Pythagorean scale is exactly halfway between G♭ and F♯, and therefore is not an actual pitch in the Pythagorean 25-tone system. Conversely, the midpoint of the equal tempered scale is an actual pitch (F♯/G♭). In the 25-tone Pythagorean scale, B♯ is a Pythagorean comma higher than C, and D♭♭ is a comma lower than C. Similarly, C♯ is a comma higher than D♭, and so on. For any two pitches that are considered enharmonically equivalent in 12-tone equal temperament, the pitch with the “lower” letter name is a comma higher in the Pythagorean system.
3.16. **Explanation of the foregoing table**

**ERKLÄRUNG DER VORSTEHENDEN TABELLE:**

In den Oktavbereich $$\text{C} - \text{G}$$ sind 12 Ober > 12 Unterkintöne projiziert; sodaß sich eine 25-stufige Skala ergibt.

1. 2. 3. bedeutet 1 2 3 Ober = \[\frac{1}{2}, \frac{2}{3}\] t. 2. 3. Unterkintone von \[\text{G}\]

- **Ganztone** (d. s. oktavreduzierte Doppelquinten)
- **Apotomen** = chromatische Halbtöne (d. s. oktavreduzierte Quinttöne)
- **Limma** = diatonische Halbtöne (d. s. 5, ...)

Die Gruppierung zeigt sinnfällige symmetrische Polarität.

Die Bogen (Notensystem) verketten je 6 Töne in ganztönigen Abständen: aufwärts vom unteren \text{C} nach oberen (über \text{C} liegend), abwärts vom oberen \text{G} nach unterem (unter \text{C} liegend). Die zweiseitige Überschreitung der Oktavgrenzen beträgt je ein pyth Komma, d. i. ca. ein enger Neuntelton = 20 μ ζ Abstand.

A Die aufsteigende Tonkette zeigt ungleichgroße Intervallfolgen. Es wechseln KommA < 20 μ und Πyken < 55 μ : doppelverminderte Terzen mit = Limma < 75 μ ab.

Die Gruppierung zeigt sinnfällige symmetrische Polarität (vgl. die Reihenfolge von unten und oben nach der Mitte zu).

B Die Reduzierung dieser 25 ungleichtufigen Skala auf eine 12-stufige ist offensichtlich unmöglich. Wechseln 6 Limmata mit 6 Apotomen gleichmäßig ab, so ergibt sich am Schluß eine Komma = Plusdifferenz gegenüber der Oktave; wechseln dagegen 5 Apotomen mit 7 Limmata, so ist \text{t}, die Symmetrie der Folge gestört und 2 ν ergibt sich am Schluß eine Komma = Minusdifferenz gegenüber der Oktave.

C Die temperierte Vierteltonskala porziert die Oktave in 24 gleichgroße Teile von je 91 5/12 μ Größen.


Stellvertreter ein temperierter Halbtön (91 5/12 μ) ein Limma (75 μ), so beträgt die Plusdifferenz nur 6 μ, substituieren dagegen eine Apotome (95 μ), so ergibt sich eine Minusdifferenz von 13 3/12 μ, also noch immer stark hinter dem Komma zurückbleibend.

Die temperierten Viertelton (91 5/12 μ) kommen den - in unmittelbarer Folge nicht recht glaubhaften - Πyken (55 μ) einigermaßen nahe. Die Differenz ist also die gleiche, wie zwischen der Apotome und dem Temp. Halbtön: nämlich 13 3/12 μ.

Alles weitere findet später seine Erläuterung.

(English translation on next page)
3.16. *Explanation of the foregoing table*
Explanation of the preceding table:

12 over-fifths and 12 under-fifths are reduced within the octave C → C, creating a 25-tone scale.

1. 2. 3. refer to over-fifths from C.  
1. 2. 3. refer to under-fifths from C.

Whole tones (i.e. octave-reduced double-fifths)  
Apotome = chromatic semitones (i.e. octave-reduced 7th fifth)  
Limmata = diatonic semitones (i.e. octave-reduced 5th fifth)

The arrangement illustrates the evident symmetrical polarity.

The curved arrows (on the staff) link 6 whole tones, from the lower C to the upper B# (which lies higher than C), and from the upper C to the lower D (which lies lower than C). This over-stepping of the octave boundary comprises a Pythagorean comma [about a narrow ninth of a tone = interval of \(20\) µ.]

A. The rising pattern indicates the unequal interval sequence, which alternates commas (\(20\) µ) and Pyklen (\(55\) µ = doubly diminished 3rds) with Limmata (\(75\) µ). The arrangement illustrates the symmetrical polarity [observe how the sequence reads the same from the bottom or the top, and meets in the middle].

B. Reducing this unequally-divided 25-tone scale to a 12-tone scale is clearly impossible. If we substitute (in equal measure) six Apotome for six Limmata, the final result is one comma greater than the octave. On the other hand, if we substitute seven Limmata for five Apotome: 1) the symmetry of the scalar sequence is disturbed, and 2) the final result is one comma less than the octave.

C. The tempered quarter-tone scale divides the octave into 24 equal segments, each \(41\frac{2}{3}\) µ in size.

D. The tempered half-tone scale divides the octave into 12 equal portions of \(83\frac{1}{2}\) µ. This scale approximates the canonic values remarkably closely. The differences are indicated by the dotted “V” symbols [between lines B and C], placed throughout under the commas of the Pythagorean scale.

If we replace an equal-tempered semitone (\(83\frac{1}{2}\) µ) with a Limma (\(75\) µ), the plus-difference is only \(8\frac{1}{3}\) µ. In contrast, if we substitute an Apotome (\(95\) µ), the minus-difference is \(13\frac{2}{3}\) µ, still substantially less than a comma.

The equal-tempered quarter tones (\(41\frac{2}{3}\) µ) roughly approximate Pyklen (\(55\) µ) [in immediate sequence not truly plausible]. The difference is the same as that between the Apotome and the equal-tempered semitone: namely \(13\frac{2}{3}\) µ.

Everything else will be further explained below.
Line A contains the 25-tone Pythagorean scale. *Pyklen* (55 µ) are intervals that are notated as doubly diminished thirds (such as B♯ and D♭). *Limmata* (75 µ) are diatonic semitones (such as D and Eb, or E and F).

Line B attempts to create a 12-tone scale from the 25-tone Pythagorean scale, which is impossible without compressing or stretching the octave by a comma. *Apotome* are chromatic semitones, such as C and C♯.

Line C is an equal tempered 24-tone (quarter-tone) scale.

Line D is the familiar 12-tone equal tempered scale. Karg-Elert highlights the fact that the equal-tempered scale is a good substitute for the Pythagorean, as the values between the scales diverge by only 20 µ (the Pythagorean comma, or about a ninth of a tone). However, as Chapter 4 will discuss, Karg-Elert regards the Pythagorean scale as purely melodic in nature, completely distinct from harmonic values.

“in immediate sequence not truly plausible” – in other words, a continuous cycle of *Pyklen* (55 µ) will never create a true octave (1000 µ).
Chapter 3: Tonnetze

\(C\) is \(C(0)\): the center of the series of pure fifths, extending infinitely in both directions. 
\(C(0)\) is always at the center of the pitch space, unless specified otherwise. The fifth series extending from \(C(0)\) is shaded, in order to distinguish it from other fifth-rows in the fifth-third Tonnetz.

3.10. Complete table of [over]fifth-derived interval values, within an octave

Top row of numbers (in parentheses): fifths above \(C(0)\)

Bottom row of numbers: acoustic values in \(\mu\) (millioctaves). All \(\mu\) values are rounded to the nearest whole number, and are reduced within an octave by omitting the thousands place – for example, the value for \(B\#\) (12) is given as 20 instead of 1020 \(\mu\).

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
<th>(11)</th>
<th>(12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C)</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0)</td>
<td>585</td>
<td>170</td>
<td>755</td>
<td>340</td>
<td>925</td>
<td>510</td>
<td>95</td>
<td>680</td>
<td>265</td>
<td>850</td>
<td>435</td>
<td>20</td>
</tr>
</tbody>
</table>

3.11. Complete table of [under]fifth-derived interval values, within an octave

Top row of numbers: fifths below \(C(0)\), indicated as negative numbers

Bottom row: values in \(\mu\)

<table>
<thead>
<tr>
<th></th>
<th>(-12)</th>
<th>(-11)</th>
<th>(-10)</th>
<th>(-9)</th>
<th>(-8)</th>
<th>(-7)</th>
<th>(-6)</th>
<th>(-5)</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D#)</td>
<td>980</td>
<td>565</td>
<td>150</td>
<td>735</td>
<td>320</td>
<td>905</td>
<td>490</td>
<td>75</td>
<td>660</td>
<td>245</td>
<td>830</td>
<td>415</td>
</tr>
<tr>
<td>(A#)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
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</tr>
<tr>
<td>(F#)</td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<td></td>
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<td></td>
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</tr>
</tbody>
</table>

Together, the Tonnetze in 3.10 and 3.11 comprise the 25-note Pythagorean scale described in 3.15.
Chapter 4

The practical and musical significance of the canonic values
Chapter 4
The practical and musical significance of the canonic values

4.1. The three-tone collection, and the whole tone as basic melodic cell
4.2. The five-tone collection (pentatonic)
4.3. Major- and minor-like non-harmonic groupings
4.4. Typical pentatonic melodic forms (rotations)
4.5. The heptatonic collection: the Greek and Latin modes
4.6. Modal degrees as chromatic variants of major or minor scale degrees
4.7. A fundamental error in most discussions of pure-scale interval ratios
4.8. The distinction between melodic and harmonic interval ratios
4.9. Example: major- and minor-like canonic simultaneities in polyphony
4.10. Comparison of canonic and harmonic values (from 4.9)
4.11. The compromise provided by tempered tuning (from 4.9)
4.1. **The three-tone collection, and the whole tone as basic melodic cell**

Without exception, all diatonic melodic structures reflect canonic values.

The three-tone collection results from the linkage of two fifths:

![Diagram of three-tone collection](image)

Projected within the octave D – D, the following pitch structures are created:

a) The frames of the Greek diatonic tetrachords (further described below)
b) The three principal degrees of the major and minor keys, the basses of the elements of the cadence, the pillars of tonality.

The descending or ascending whole tone *(boxed in the above two examples)* will be recognized as the basic cell of melodic motion. This pitch interval is by no means an arbitrarily [randomly] adopted measure. Rather, it outlines a span provided by nature, one that will be perceived as “natural” without any specific calculation – and thus simply and unconsciously.
4.1. *The three-tone collection, and the whole tone as basic melodic cell*

Here, Karg-Elert is stating that melody is fundamentally canonic or Pythagorean (fifth-based), while harmony is syntonic or Didymean (fifth and third-based), as we will see in Chapters 5 and 6. This opposition will be further examined in Chapter 11, which discusses the topic of non-harmonic dissonance.

The breath mark and hiatus between IV and V in example B highlights how the step from G to A is actually an octave reduction of the two fifths surrounding D: the over-dominant and under-dominant. This clearly recalls Rameau’s *Traité de l’harmonie*, which explained step progressions such as IV to V as elided third- and fifth-progressions: “whenever it is permissible to have the fundamental bass ascend a tone or a semitone, the progression of a third and a fourth is always implied” (Rameau 1722, 234).
4.2. The five-tone collection (pentatonic)

The five-tone collection [“pentatonic”] adds a further link to each of the peripheral fifths in the three-tone scale:

[Arranged in a linear manner] five anhemitonic, pentatonic scales:

This tone sequence resembles that of the black keys of the piano [of course without regard for equal temperament]. The pentatonic is the basis of the music of the Chinese, Javanese, Celts, Scots and various colored North American peoples, and is above all clearly palpable in Gregorian chant.

Due to the absence of the purposefully-directed semitone [anhemitonic], the scale lacks a precise central focus.
4.2. *The five-tone collection (pentatonic)*

*Einiger farbigen nord-amerikanischer Stämme* is Karg-Elert’s way to describe North Americans of non-European heritage, primarily indigenous peoples and African-Americans. The term *farbigen* (“colored”) is insensitive, but very typical of its time.
4.3. **Major- and minor-like non-harmonic groupings**

While major- and minor-like groupings are possible *in the Pythagorean scale*,

they are under no circumstances to be understood as actual harmonic structures, as they do not contain the primary determinant of harmony: the syntonic blending third (4:5).

4.4. **Typical pentatonic melodic forms (rotations)**

These are not transpositions of each other, but rather “shifted” by {pentatonic} steps.
4.3. **Major- and minor-like non-harmonic groupings**

Note that the placement of the dots indicates the direction from the center, which is D in this example: C is two fifths below D, while E is two fifths above D. G is one fifth below D. The interval between C and E in this example is a Pythagorean major third (81:64, or 340 µ), which Karg-Elert states cannot create true harmonic structures (generated only from the syntonic 4:5 major third, as described in Chapter 5). The accent ’ is Karg-Elert’s symbol for the syntonic third, which is explained in section 4.7.

4.4. **Typical pentatonic melodic forms (rotations)**

In fact, they are transpositions of each other within a mod-5 pentatonic set.
4.5. **The heptatonic collection: the Greek and Latin modes**

By adding the next pair of fifths $F_{-3} \leftrightarrow C_{-2}$ and $E_2 \rightarrow B_3$, we create the heptatonic collection, material of the Greek scales and their Western counterparts: the Catholic church modes.

![Musical diagrams of Greek modes]

(Greek) Dorian  (Greek) Phrygian  (Greek) Lydian

(Greek) Hypodorian  (Greek) Hypophrygian  (Greek) Hypolydian

[Hypo = “under,” meaning that Tetrachord I lies under Tetrachord II.]

The medieval Latin Church modes [read from bottom to top] correspond to the Greek scales in their overall arrangement:

Latin: Ionian  Dorian  Phrygian  Lydian  Mixolydian  Aeolian
Greek: Lydian  Phrygian  Dorian  Hypolydian  Hypophrygian  Hypodorian

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4.5. *The heptatonic collection: the Greek and Latin modes*

The heptatonic collection described here is of course the familiar diatonic collection, containing two semitones and one tritone.

Karg-Elert here shows only the diatonic genus of the Greek tetrachords. The first three scales indicate the tone of disjunction between the tetrachords, while the “hypo” scales involve conjunct tetrachords. See section 12.5 for more on the Greek tetrachords.
4.6. Modal degrees as chromatic variants of major or minor scale degrees

\[
\begin{align*}
\text{Dur} &= \text{major} & \text{Moll} &= \text{minor} \\
\text{verstärkte Typen} &= \text{intensified forms} & \text{zur Norm erhoben} &= \text{established as normal} \\
\text{gemischte Typen} &= \text{mixed forms}
\end{align*}
\]

Canonic proportions [“regular” units]:

\[
\begin{align*}
\land &= \text{always } 9/8 = \frac{170}{18} \mu \\
\lor &= \text{always } 256/243 = \frac{75}{73} \mu \quad \text{(Limma)}
\end{align*}
\]

The “characteristic” notes in each mode that are marked \( \star \) can be understood as chromatic variants of pitches from transposed normal scales:

- B in F Lydian, instead of B♭ in F major
- F in E Phrygian, instead of F♯ in E minor
- F in G Mixolydian, instead of F♯ in G major
- B in D Dorian, instead of B♭ in D minor
4.6. *Modal degrees as chromatic variants of major or minor scale degrees*

The Ionian and Aeolian scales were “established as normal” over the course of the 17th century. Lydian and Phrygian are “intensified forms” because they each contain an interval that is altered in the same direction as the basic mode is generated: Lydian is major (generated upward) with raised \(^4\), while Phrygian is minor (generated downward) with lowered \(^2\). Mixolydian and Dorian are “mixed forms” because they each contain an interval that is altered in the opposite direction from the basic mode’s generation: Mixolydian is major with lowered \(^7\), while Dorian is minor with raised \(^6\).
4.7. **A fundamental error in most discussions of pure-scale interval ratios**

Most textbooks of the physical sciences, and the acoustical chapters of countless books on musical rudiments, intervals and harmony perpetuate a fundamental error with respect to the proportions of the “pure scales.”

They present the following alleged proportions:

```latex
\begin{align*}
\text{\[ C \quad \text{D} \quad \text{E} \quad \text{F} \quad \text{G} \quad \text{A} \quad \text{B} \]} & \quad \begin{bmatrix}
\frac{9}{8} & \frac{10}{9} & \frac{16}{9} & \frac{10}{9} & \frac{9}{8} & \frac{16}{15} & \frac{9}{8} & \frac{16}{15}
\end{bmatrix} \\
\text{\[ \text{[gr]} \quad \text{[kl]} \quad \text{[kl]} \quad \text{[gr]} \quad \text{[kl]} \]} & \quad \begin{bmatrix}
\frac{10}{9} & \frac{9}{8} & \frac{16}{9} & \frac{15}{8} & \frac{9}{8} & \frac{16}{15} & \frac{10}{9} & \frac{9}{8}
\end{bmatrix}
\end{align*}
```

\[ \text{\symbol{`}} \text{= symbol for the syntonic pure (4:5) major third} \]

\[ \text{gr.} \text{= gross (large whole-tone)} \]

\[ \text{kl.} \text{= klein (small whole-tone)} \]

[There are two errors, if the ratios in A minor are to correspond with the ratios in C major!]

```latex
\begin{align*}
\text{\[ \text{Dur} \]} & \quad \begin{bmatrix}
\frac{9}{8} & \frac{16}{15} \\
\frac{9}{8} & \frac{10}{9} & \frac{16}{15} & \frac{9}{8} & \frac{10}{9} & \frac{16}{15}
\end{bmatrix} \\
\text{\[ \text{Moll} \]} & \quad \begin{bmatrix}
\frac{9}{8} & \frac{16}{15} \\
\frac{9}{8} & \frac{10}{9} & \frac{16}{15} & \frac{9}{8} & \frac{10}{9} & \frac{16}{15}
\end{bmatrix}
\end{align*}
```

\[ \text{Dur} = \text{major} \]

\[ \text{Moll} = \text{minor} \]
4.7. A fundamental error in most discussions of pure-scale interval ratios

Here is the first explanation of the accent ′, which indicates the syntonic third (4:5) above or below the central pitch.

Beginning in this chapter, the acoustic derivation of pitches will usually be notated as $X_{(m,n)}$, meaning that pitch $X$ is located $m$ fifths and $n$ thirds away from the central pitch (which is C(0) unless specified otherwise).

For example:
$E_{(0,1)}$ is no fifths + one pure 5:4 third above C(0).
$A_{b(0,-1)}$ is no fifths + one 5:4 third below C(0).
$B_{(1,1)}$ is one fifth above + one third above C(0).

The syntonic third (5:4) is the primary topic in Chapter 5.

The “fundamental error” that Karg-Elert describes in this section concerns the division of the major third C – E in the key of C major, and then the same interval in A minor. If A minor is conceived simply as the dualist mirror image of C major, its prime or generating pitch must be E. The following intervals result, as shown in the first diagram:

C major: $C(0) \rightarrow D_{(2,0)} \rightarrow E_{(0,1)}$  A minor: $E(0) \rightarrow D_{(2,0)} \rightarrow C_{(0,-1)}$

According to this view, the intervals C $\rightarrow$ D and D $\rightarrow$ E are of different sizes in C major and A minor. However, the other major thirds F $\rightarrow$ A and G $\rightarrow$ B are divided identically in both scales; this creates a logical and acoustic inconsistency.

Karg-Elert states that in order for C major and A minor to truly correspond in a reciprocal manner, they must contain all of the same intervals. This will lead him to propose a prime major-third cell as the center of both major and minor (see 6.1).
4.8. **The distinction between melodic and harmonic interval ratios**

“The distinction between melodic and harmonic interval ratios

“Pure scales” can never include changing sizes of whole tone (9/8 = \( \frac{170}{1} \) and 10/9 = \( \frac{152}{1} \)) and wide semitones (16/15 = \( \frac{93}{1} \))! [Wide semitones lead automatically to raised ♯ and lowered ♫ values].

The above ratios apply only to harmonic values: that is, if the individual members of the three major and minor principal harmonies are seemingly presented in linear form. [See the acoustical symbols in the above example {4.7.}: \( \backslash \) = lowered upper {major} third, \( \triangleright \) = raised upper third]. Such forms result from a (practical) principle of intonation: adjustment to overriding harmonic entities. However, they are not “pure melodic values.”

The polyphonic style primarily requires melodic independence, and therefore purity and equality of the individual voices: equally-sized, energetic whole tone steps and strongly-directed leading tones: in other words, narrow semitones.

Harmonic complexes have a secondary meaning if they do not exist in and of themselves, but instead result from the simultaneity of independently moving lines. In such cases, the unity and purity of the harmony is subordinated to the equal status of the individual voices.
4.8. The distinction between melodic and harmonic interval ratios

As in section 4.1, Karg-Elert clearly distinguishes between “melodic” (canonic or fifth-derived) and “harmonic” (syntonic or third-derived) pitch values. This distinction is further discussed in Chapter 11.
4.9.  Example: major- and minor-like canonic simultaneities in polyphony

(four-tone collection)

This melodic passage can only be understood as a consistent pattern of pure fourths and normal whole-tone steps. A passing semitone inserted between G and A or between A and G must be G♯ in the first instance, and A♭ in the second:
4.9. **Example: major- and minor-like canonic simultaneities in polyphony**

The two melodic fragments described here (C – G – G♯ - A and E – A – Ab – G) are bracketed in the polyphonic passage that immediately follows (next page).
These intervallic sizes remain valid in polyphonic writing. Thus, the following is nothing other than a combination of self-sufficient, independent voices:

The E major-type chord created by the polyphonic web contains the following canonic values:

\[ E_4 \quad G_b^\#_8 \quad B_5 \]

The F minor-type chord created by the polyphonic web contains the following canonic values:

\[ F_1 \quad A_b^\#_4 \quad C_0 \]
4.9. (continued)

The two simultaneities described here (indicated by the dotted vertical lines in the musical example) are not true harmonic complexes, but rather pseudo-triads containing only Pythagorean pitch values, resulting from the coincidence of the three polyphonic lines.
4.10. **Comparison of canonic and harmonic values (from 4.9)**

The pure harmonies, which resolve to A minor and C major, are syntonic:

\[ F_{(-1,0)} \quad A_{b(0,-1)} \quad C_{(0)} \quad E_{(0,1)} \quad G_{(0,2)} \quad B_{(1,1)} \]

In *mio*-values, for comparison:

<table>
<thead>
<tr>
<th></th>
<th>415</th>
<th>660</th>
<th>1000</th>
<th>1340</th>
<th>1680</th>
<th>1925</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td>f</td>
<td>as</td>
<td>C</td>
<td>e</td>
<td>gis</td>
<td>h</td>
</tr>
</tbody>
</table>

natural melodic values *{purely fifth-based}*

\[ F_{(-1,0)} \quad A_{b(-4,0)} \quad C_{(0)} \quad E_{(4,0)} \quad G_{(8,0)} \quad B_{(5,0)} \]

\[ \text{Difference: } 18 + 18 - 36 - 18 - (\text{in } \mu) \]
4.10. **Comparison of canonic and harmonic values (from 4.9)**

The slurs indicate generated pitches: the fifths below C(0) and above E_{(4,0)} in ex. A (canonic), and the fifths and thirds above and below C(0) and E_{(0,1)} in ex. B (syntonic).
4.11. The compromise provided by tempered tuning (from 4.9)

Due to the notable differences, neither of the above tunings can adequately substitute for the other. The most convenient compromise is provided by "tempered" tuning:

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>A♭</th>
<th>C</th>
<th>E</th>
<th>G♯</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+6.5</td>
<td>+6.3/3</td>
<td>+13/3</td>
<td>+13/3</td>
<td>+13/3</td>
<td>+13/3</td>
</tr>
<tr>
<td>Differenz zu A</td>
<td>+6.5</td>
<td>0</td>
<td>-6.3/3</td>
<td>-12/3</td>
<td>-8.5/3</td>
<td></td>
</tr>
<tr>
<td>Differenz zu B</td>
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<td>+12.5/3</td>
<td>+22.5/3</td>
<td>+13.5/3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4.11. The compromise provided by tempered tuning (from 4.9)

“Difference from A,” “difference from B” = differences from lines A and B in 4.10, which indicate pure canonic and syntonic values respectively.
Chapter 4: Tonnetze

4.10. **Comparison of canonic and harmonic values (from 4.9)**

<table>
<thead>
<tr>
<th>A</th>
<th>F(_{(-1,0)})</th>
<th>A(_b)(-4,0)</th>
<th>C((0))</th>
<th>E(_{(4,0)})</th>
<th>G(_{(8,0)})</th>
<th>B(_{(5,0)})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\frac{415}{415})</td>
<td>(\frac{660}{660})</td>
<td>(1000)</td>
<td>(\frac{1340}{1340})</td>
<td>(\frac{1680}{1680})</td>
<td>(\frac{925}{925})</td>
</tr>
</tbody>
</table>

natural melodic values \{purely fifth-based\}

<table>
<thead>
<tr>
<th>B</th>
<th>F(_{(-1,0)})</th>
<th>A(_b)(0,-1)</th>
<th>C((0))</th>
<th>E(_{(0,1)})</th>
<th>G(_{(0,2)})</th>
<th>B(_{(1,1)})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\frac{415}{415})</td>
<td>(\frac{678}{678})</td>
<td>(1000)</td>
<td>(\frac{1322}{1322})</td>
<td>(\frac{1644}{1644})</td>
<td>(\frac{907}{907})</td>
</tr>
</tbody>
</table>

natural harmonic values \{fifth- and third-based\}

Difference: \(18 + 18 - 36 - 18\) (in \(\mu\))

Line A (canonic):

<table>
<thead>
<tr>
<th>Ab</th>
<th>Eb</th>
<th>Bb</th>
<th>F</th>
<th>C</th>
<th>G</th>
<th>D</th>
<th>A</th>
<th>E</th>
<th>B</th>
<th>F#</th>
<th>C#</th>
<th>G#</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1,0)</td>
<td>(-1,0)</td>
<td>(-1,0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(5,0)</td>
<td>(4,0)</td>
<td>(5,0)</td>
<td>(5,0)</td>
<td>(0,0)</td>
<td>(0,0)</td>
</tr>
</tbody>
</table>

Line B (syntonic):

NOTE: this is a fifth-third Tonnetz, with pure fifths \((2:3)\) on the horizontal rows, and pure syntonic major thirds \((4:5)\) on the vertical columns.

The numerical subscripts and superscripts added to pitches indicate their vertical position on the thirds axis:

\(E^1 = E_{(0,1)}\), one major third above C\((0)\)
\(D^b_{-1} = D^b_{(-1,-1)}\), one major third below F\(_{(-1,0)}\)

As will be discussed in Chapter 5, thirds above or below canonic (Pythagorean) pitches are raised or lowered by one or more syntonic commas. All pitches on the just intonation Tonnetz are acoustically unique.
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5. KAPITEL

DIE SYNTONISCHE TERZ

Chapter 5

The syntonic third
Chapter 5
The syntonic third

5.1. The syntonic third (4:5)
5.2. Acoustical symbols for over- and under-thirds
5.3. Comparative overview of Pythagorean and Didymeant thirds
5.4. Syntonic comma differences between Pythagorean and Didymeant intervals
5.5. Calculating µ values using pure canonic fifths and pure syntonic thirds
5.6. Example - different µ values for three pitches, all called D
5.7. Multiple-third values
5.8. Comparison of pitch values in the Pythagorean and Didymeant systems
5.9. Semitones in the Didymeant system
5.10. Summary of common commas (not including those previously listed)
5.1. The syntonic third (4:5)

Using the monochord, the Greek musician and mathematician Didymus of Alexandria (c. 60 BCE) discovered the natural primary third in the string-length ratio 1/4 : 1/5. The primary third was calculated as the Messel ratio 4 : 5 by the Persians and Arabs – possibly without knowledge of Didymus’ writings.
5.1. The syntonic third (4:5)

The name “Didymus of Alexandria” is usually given to the Greek grammarian Didymus Chalcenterus (late 1st century BCE). In the 2nd and 3rd centuries CE, Ptolemy and Porphyry named Didymos ho mousikos (“Didymus the musician”) as their source for the 4:5 major third. Scholars now generally believe that Ptolemy and Porphyry’s source was actually a mid-1st century CE Roman grammarian and musician named Didymus, from the time of Nero (Richter 2017). Karg-Elert may have read about Didymus in August Wilhelm Ambros’ Geschichte der Musik (Ambros 1862, 308). See Chapter 3 for a description of the Messel ratios.
5.2. Acoustical symbols for over- and under-thirds

Prime = the central generating pitch (C in C major, E in A minor)
rel. Sz. = relative frequency (in relation to the prime)
rel. Wl. = relative wavelength

My acoustical symbols indicate the relative (that is, in relation to canonic thirds):

- Lowered over-third by \( \downarrow \) (lowering comma)
- Raised under-third by \( \uparrow \) (raising comma)
5.2. *Acoustical symbols for over- and under-thirds*

The **prime** is the central or generating pitch of a mode, conceived dualistically: C in C major, and E in A minor. Prime is not the same concept as **root**, which is always the lowest pitch of a chord, or the starting note of a scale. Karg-Elert names keys and scales not according to their primes, but to their **roots**: the polar equivalent to C major is A minor, rather than (for example) Oettingen’s E phonic (Oettingen 1866, 46).

“Lowered” and “raised” refer to syntonic comma differences, calculated in relation to Pythagorean pitches – see 5.4.
5.3. **Comparative overview of Pythagorean and Didymean thirds**

Comparative overview of the Pythagorean [canonic] and Didymean [syntonic] thirds:

**In frequency ratios:**

Top row: 4 [canonic] fifths (actually twelfths) above the fundamental pitch C.
Second row: 1 pure [syntonic] third above C, replicated in several octaves. Enger = narrower
Bottom row: 6 octaves from the fundamental pitch C1, producing the reference pitch 64 (C6).

**In wavelength ratios:**

Top row: 6 primary octaves descending from the fundamental pitch E6 (wavelength or Messel 1).
Second row: 1 pure [syntonic] third below E, replicated in several octaves.
Third row: 4 [canonic] fifths (actually twelfths) below the fundamental pitch E.
5.3. **Comparative overview of Pythagorean and Didymean thirds**

This example illustrates the difference between the Pythagorean major third (64 : 81) and the pure syntonic or Didymean third (64 : 80, which reduces to 4 : 5).

The ratio of 80 : 81 is the syntonic comma (see 5.4).

This example also illustrates the difference between the Pythagorean and Didymean thirds, but now calculated in polar fashion below the prime e\(^4\).
5.4. **Syntonic comma differences between Pythagorean and Didymean intervals**

| Pythagorean major third | | Didymean major third |
|-------------------------|-------------------------|
| \( \frac{5}{4} \) | \( \mu = 340 \) | \( \frac{5}{4} \) | \( \mu = 322 \) |

<table>
<thead>
<tr>
<th>Pythagorean whole tone</th>
<th>Didymean whole tone</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{5}{4} )</td>
<td>( \mu = 340 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pythagorean minor third</th>
<th>Didymean minor third</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{5}{4} )</td>
<td>( \mu = 340 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pythagorean minor seventh</th>
<th>Didymean minor seventh</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{16}{9} )</td>
<td>( \mu = 263 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pure perfect fifth</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{3}{2} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Narrow (so-called false) fifth</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{20}{27} )</td>
</tr>
</tbody>
</table>

Each pair differs by \( \mu = 18 \) = the syntonic comma.
As described in 3.7, the \( \mu \)-values are calculated using the base-2 logarithms of the frequency ratios (as always, the 1:2 octave = 1000). The values listed here (especially the syntonic third at 322 \( \mu \)) are used to facilitate \( \mu \) calculations for other intervals.

The “narrow (so-called false) fifth” (567 \( \mu \)) is the interval between \( A_{3,0} \) and \( E_{0,1} \).

The **syntonic comma** (18 \( \mu \)) is slightly smaller than the Pythagorean comma (20 \( \mu \)). It is the ratio 80 : 81, which is the difference between two pitches that are four pure fifths and a pure major third apart. For example, using the two values for E from 5.3:

<table>
<thead>
<tr>
<th></th>
<th>( E_{4,0} )</th>
<th>( E_{0,1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Canonic:</strong></td>
<td></td>
<td>340 ( \mu )</td>
</tr>
<tr>
<td><strong>Syntonic:</strong></td>
<td>( E_{0,1} )</td>
<td>322 ( \mu )</td>
</tr>
<tr>
<td><strong>Difference =</strong></td>
<td></td>
<td>18 ( \mu ) = one syntonic comma</td>
</tr>
</tbody>
</table>
5.5. Calculating µ values using pure canonic fifths and pure syntonic thirds

Every {over- or under-} fifth-tone can generate its own over- or under-third:

\[
\text{Jeder Quintton kann seine Ober- oder Unterterz heraushilden:}
\]

\[
\begin{array}{c}
\text{ak. } \frac{3}{5} & = \mu \left(585 + 322\right) = 907 \\
\text{ak. } \frac{3}{5} \text{ resp. } \frac{5}{3} & = \mu \left(585 - 322\right) = 263 \\
\text{ak. } \frac{16}{15} & = \mu \left(1000 - 907\right) = 93 \\
\text{ak. } \frac{9}{5} & = \mu \left(170 + 322\right) = 492 \\
\text{ak. } \frac{9}{5} & = \mu \left(3 \cdot 170\right) = 510
\end{array}
\]

In comparison with the {Pythagorean} fifth-tones of the same name,

Over-thirds \( \nearrow \) are lowered by 18 µ (subtracted)

Under-thirds \( \searrow \) are raised by 18 µ (added)

First row: \( B_{(1,1)} = \) one fifth \( (585 \mu) \) above + one third \( (322 \mu) \) above \( C(0) \)

Second row: \( E_{b(1,-1)} = \) one fifth above + one third below \( C(0) \)

Third row: \( D_{b(1,-1)} = \) one fifth below + one third below \( C(0) \) \( 1000 - 585 - 322 = 93 \mu \)

Fourth row: \( A_{(1,-1)} = \) one fifth below + one third above \( C(0) \) \( 1000 - 585 + 322 = 737 \mu \)

This is the normal value for \( A \) in the diatonic scales of \( C \) major and \( A \) natural minor.

Fifth row: \( F_{(2,1)} = (585 \times 2) + 322 = 492 \) (reduced from 1492).

\( F_{(6,0)} = 585 \times 6 = 510 \) (reduced from 3510). Difference = syntonic comma (18 µ).

Sixth row: \( G_{b(2,-1)} = 1000 - (585 \times 2) - 322 = -492 = 508. \)

\( G_{b(6,0)} = 1000 - (585 \times 6) = -2510 = 490. \) Difference = syntonic comma (18 µ).

They are the polar counterparts of the fifth-row values for \( F \), around the central pitch \( C(0) \).
5.5. Calculating µ values using pure canonic fifths and pure syntonic thirds

N.B.: pitches with an accent (’) written above are thirds above a canonic pitch, but are a syntonic comma (18 µ) lower than their canonic counterparts – and vice versa (accent below = third below, raised by a syntonic comma). In co-ordinate notation: B_{(1,1)} is a syntonic comma lower than B_{(5,0)}, while E_{b(1,-1)} is a syntonic comma higher than E_{b(-3,0)}. 
5.6. **Example – different $\mu$ values for three pitches, all called D**

$\mu = 170, \quad \mu = -18 = 152, \quad \mu = 170 + 18 = 188$

*Ursprunglage* = source positions
*Von kl. C aus* = in relation to C(0)
5.6. **Example – different µ values for three pitches, all called D**

\[
\begin{align*}
\mu 170 &= D_{(2,0)}, \text{ which is 2 fifths above } C(0) \\
\mu 152 &= D_{(-2,1)}, \text{ which is 1 third above canonic } B^\flat_{(2,0)} \\
\mu 188 &= D_{(6,-1)}, \text{ which is 1 third below canonic } F^\sharp_{(6,0)}
\end{align*}
\]

The *Ursprungslagen* or “source positions” are the Pythagorean (canonic) pitches fifth-derived from the prime C(0), in both directions. The *Ursprungslage* for C(0) is middle C, which is in the center of practical pitch space. Every pitch in the Pythagorean fifth-chain is conceived in its specific fifth-derived octave or “source position,” even if the actual music states that pitch in some other octave or register. Therefore, the *Ursprungslage* for D\(_{(2,0)}\) is d\(^2\), or two fifths above middle C. The other two Ds in this diagram differ from D\(_{(2,0)}\) by a syntonic comma; therefore, they are not *Ursprungslage* pitches, but rather *Varianten* (syntonic-comma variants), which are indicated by the black noteheads. See 6.12 for more on the *Ursprungslagen* and *Varianten*.
5.7. **Multiple-third values**

Unlike multiple fifths, multiple thirds cannot generate a scale. While the octave is not approached enharmonically until the 12th fifth, the 3rd third already approximates the octave. But while the 12th fifth exceeds the 7th octave by 20 µ, the 3rd third is 34 µ smaller than the octave.

\[
\begin{array}{cccc}
C(0) & E_{(0,1)} & G^\#_{(0,2)} & B^\#_{(0,3)} \\
\text{ak.} & 1 & 5 & 25 & 125-128 \\
\mu & 0 & 322 & 644 & 966 - 1000 \\
\text{Diff.} & 34 = \text{sog. kleine Diesis}
\end{array}
\]

34 µ = minor diesis

As shown by the acoustical symbols, the difference between \( \text{and } \) equals three syntonic commas \( = 3 \times 18 = 54 \) µ.

\[
\begin{array}{cccc}
\text{Quintenpotenzen} & \text{Terzpotenzen} \\
\text{\includegraphics[width=0.5\textwidth]{quintenpotenzen.pdf}} & \text{\includegraphics[width=0.5\textwidth]{terzpotenzen.pdf}}
\end{array}
\]

Quintenpotenzen = multiple fifths
Terzpotenzen = multiple thirds
5.7. *Multiple-third values*

Multiple fifths can generate the pentatonic (see 4.2), diatonic (4.5) and chromatic (3.10 and 3.11) collections.

20 $\mu$ = Pythagorean comma = difference between two pitches 12 fifths apart
34 $\mu$ = minor diesis = 125 : 128 = difference between two pitches 3 major thirds apart

This diagram compares $\mu$ values for multiple thirds above C(0):

<table>
<thead>
<tr>
<th></th>
<th>C(0)</th>
<th>E(0,1)</th>
<th>G*(0,2)</th>
<th>B*(0,3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Canon:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>36</td>
<td>54</td>
<td></td>
</tr>
</tbody>
</table>

Differences

54 $\mu$ = one syntonic comma per major third
5.8. **Comparison of pitch values in the Pythagorean and Didymean systems**

Instead of the Pythagorean comma, the Didymean system introduces four different enharmonic values:

**PYTHAGOREAN**

<table>
<thead>
<tr>
<th></th>
<th>Ratio</th>
<th>( \mu ) comparison</th>
<th>Difference in ( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>C(0) and B(\sharp)(_{(1,2,0)})</td>
<td>531441 / 524288</td>
<td>1020 / 1000</td>
<td>+20 (Pyth. comma)</td>
</tr>
</tbody>
</table>

**DIDYMEAN**

<table>
<thead>
<tr>
<th></th>
<th>Ratio</th>
<th>( \mu ) comparison</th>
<th>Difference in ( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>C(0) and B#(8,1)</td>
<td>32984 / 32805</td>
<td>1002 / 1000</td>
<td>+2 (Schisma)</td>
</tr>
<tr>
<td>B#(4,2) and C(0)</td>
<td>2048 / 2025</td>
<td>1000 / 984</td>
<td>-16 (Diaschisma)</td>
</tr>
<tr>
<td>B#(6,3) and C(0)</td>
<td>128 / 125</td>
<td>1000 / 966</td>
<td>-34 (minor diesis)</td>
</tr>
<tr>
<td>B#(4,4) and C(0)</td>
<td>648 / 625</td>
<td>1000 / 948</td>
<td>-52 (major diesis)</td>
</tr>
</tbody>
</table>

\( D\# \) lies the same distance from C as B\#, but in the opposite direction. Therefore, if D\# replaces B\#, the above relations are respectively inverted: minus differences become plus differences, and vice versa. Comparing D\# to B\#, the four intervals specified above are naturally doubled: 4 \[ \{32 \ 68 \ 104\]  

\[ \{104\] \( \mu \) = i.e. more than 2/3 of a whole tone

\( B\#(4,4) : D\#(4,4) \)
5.8. **Comparison of pitch values in the Pythagorean and Didymean systems**

Schisma (2 μ) = difference between the Pythagorean and syntonic commas
Diaschisma (16 μ) = difference between the syntonic comma and the schisma
Minor diesis (34 μ) = two syntonic commas minus a schisma
Major diesis (52 μ) = three syntonic commas minus a schisma

104 μ = two major diesis
The note B♯ in different tonal contexts (using the 4 Didymean values from the previous page):

First row: B♯ (1002 µ) as (pure) third of a G♯ major triad, whose root is G♯(8,0) in the key of C♯ major. Since the root of this G♯ major chord is 8 fifths above C, its Ursprung or source position is four octaves higher (4 Okt. höher) than written.

Second row: B♯ (984 µ) as third of a G♯ major triad, whose root is G♯(4,1) in the key of E major. Since the root of the E major chord is 4 fifths above C, its Ursprung is two octaves higher (2 Okt. höher) than written.

Third row: B♯ (966 µ) as third of a G♯ major triad, whose root is G♯(0,2) in the key of C major. The Ursprung of this G♯ major triad is in the notated octave, since it is third-derived directly from C(0).

Fourth row: B♯ (948 µ) as third of a G♯ major triad, whose root is G♯(4,3) in the key of A♭ major. Since the root of the A♭ major tonic is 4 fifths below C(0), its Ursprung is two octaves lower (2 Okt. tiefer) than written.
5.8. (continued)

First row: $B^\#_{(8,1)}$  1002 µ
Second row: $B^\#_{(4,2)}$  984 µ
Third row: $B^\#_{(0,3)}$  966 µ
Fourth row: $B^\#_{(-4,4)}$  948 µ

These are the four $B^\#$s just described above, with their intervals calculated from C(0). The adjacent $B^\#$s all differ from each other by a syntonic comma = 18 µ. The dark noteheads indicate variants = pitches that differ from their canonic or Ursprungslage counterparts by one or more syntonic commas.

Karg-Elert notes to “see later for the functional values” (Funktionswerte siehe später) – see especially the discussion of mediants (third-based transformations) in Chapter 7.
5.9. **Semitones in the Didyme system**

In the Pythagorean system, the diatonic semitone $\frac{243}{256} = \frac{\text{B}(5,0)}{\text{C}(0)} = \mu 75 = \text{Limma}$

In the Didyme system, the diatonic semitone $\frac{15}{16} = \frac{\text{B}(1,1)}{\text{C}(0)} = \mu 93 = \text{Leading tone}$

They differ from each other by a syntonic comma (18 µ).

The Pythagorean Apotome $\frac{\text{C}}{\text{C}^\#(7,0)} = \frac{2048}{2187} = \mu 95 \mu$ has two different corresponding intervals in the Didyme system:

- a) **Major chroma** $\frac{\text{C}}{\text{C}^\#:3,1} = \frac{128}{135} = \mu 77$
- b) **Minor chroma** $\frac{\text{C}}{\text{C}^\#:1,2} = \frac{24}{25} = \mu 59$  Difference $= 18 \mu$
Diatonic semitones (including the Limma and leading tone) have different letter names.

*Leitton* (“leading tone”) is the interval of 93 µ, which exists between $B_{1(1)}$ and C. It is 15:16, the interval between the 15\textsuperscript{th} and 16\textsuperscript{th} partials.

$B_{1(1)}$ is the third of $G_{1(0)}$, and thus the *Leitton* interval is also the usual leading-tone to tonic motion ($^7 \rightarrow ^8$) in diatonic harmony.

Chromatic semitones (which include the Apotome and both chromas) have the same letter name.
5.10. Summary of commas (not including those previously listed)

<table>
<thead>
<tr>
<th>Schisma 2</th>
<th>gr. Chroma 77</th>
<th>minus</th>
<th>Limma 75 ( = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apotome 95</td>
<td>minus</td>
<td>Leitton 93 ( = 2 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Diaschisma 16</th>
<th>Limma 75</th>
<th>minus</th>
<th>kl. Chroma 59 ( = 16 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leitton 93</td>
<td>minus</td>
<td>gr. Chroma 77 ( = 16 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Komma 18</th>
<th>gr. Chroma 77</th>
<th>minus</th>
<th>kl. Chroma 59 ( = 18 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Apotome 95</td>
<td>minus</td>
<td>gr. Chroma 77 ( = 18 )</td>
<td></td>
</tr>
</tbody>
</table>

| kl. Diesis 34 | Leitton 93 | minus | kl. Chroma 59 ( = 34 ) |

Kl. = minor
Gr. = major
Komma = syntonic comma
Leitton = leading tone
5.10. Summary of commas (not including those previously listed)

Examples of each comma, using semitones above and below C(0):

- $C^\#_{(3,1)} = 77 \mu$ minus $D_b(5,0) = 75 \mu = \text{Schisma (2 }\mu\text{)}$
- $C^\#_{(7,0)} = 95 \mu$ minus $D_b(1,-1) = 93 \mu = \text{Schisma (2 }\mu\text{)}$
- $D_b(5,0) = 75 \mu$ minus $C^\#_{(-1,2)} = 59 \mu = \text{Diaschisma (16 }\mu\text{)}$
- $D_b(1,-1) = 93 \mu$ minus $C^\#_{(3,1)} = 77 \mu = \text{Diaschisma (16 }\mu\text{)}$
- $C^\#_{(3,1)} = 77 \mu$ minus $C^\#_{(-1,2)} = 59 \mu = \text{Synt. comma (18 }\mu\text{)}$
- $C^\#_{(7,0)} = 95 \mu$ minus $C^\#_{(3,1)} = 77 \mu = \text{Synt. comma (18 }\mu\text{)}$
- $D_b(1,-1) = 93 \mu$ minus $C^\#_{(-1,2)} = 59 \mu = \text{Minor diesis (34 }\mu\text{)}$
5.5. **Calculating μ values using pure canonic fifths and pure syntonic thirds**

These are the eight pitches for which μ values are calculated:

\[
\begin{array}{cccccccc}
B^1 & F & C & G & D & A^1 & E^1 & B^1 \\
597 & 492 & 507 & 737 & 432 & 597 & 492 & 507 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}
\]

The eight pitches comprise four pairs, disposed symmetrically around C(0):

- A\(_{-1,1}\) and E\(_{1,1}\)
- B\(_{1,1}\) and D\(_{-1,-1}\)
- F\(_{2,1}\) and G\(_{-2,-1}\)
- F\(_{6,0}\) and G\(_{-6,0}\)

5.6. **Example – different μ values for three pitches, all called D**

\[
\begin{array}{cccccccc}
D^1 & A & E & B & F^1 & C^1 & G^1 & D^1 \\
152 & 170 & 170 & -18 & 152 & 170 & 188 & 152 \\
170 & 170 & 170 & 170 & 170 & 170 & 170 & 170 \\
G & F & C & G & D & A & E & B \\
D & 152 & 170 & 170 & 170 & 170 & 170 & 170 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}
\]

D\(_{-2,1}\) and D\(_{6,-1}\) are disposed symmetrically around D\(_{2,0}\).
5.7. Multiple-third values

Quintenpotenzen = multiple fifths above C(0), to the right
Terzpotenzen = multiple thirds above C(0), upward

5.8. Comparison of pitch values in the Pythagorean and Didymeian systems

C(0) and five values for B♯:

B♯(12,0) is a Pythagorean comma (20 µ) above C(0).
Moving diagonally left and up from B♯(12,0): each value for B♯ is a syntonic comma (18 µ) lower than the previous one.
5.9. Semitones in the Didymean system

Two values for B and three for C♯, used in calculating μ values for semitones:

5.10. Summary of commas (not including those previously listed)

Chromatic and diatonic semitones above C(0) used in this section:
Chapter 6

The practical and musical significance of the syntonic values
Chapter 6
The practical and musical significance of the syntonic values

6.1. The pure third as prime cell of harmonic space
6.2. The generation of the double consonance
6.3. The completion of diatonic space
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6.10. The substitutes: Leittonwechsel (L) and parallel (p)
6.11. Example of substitutes and contrant variants
6.12. Expansion of tonality through cadential elaboration of the substitutes
6.1. The pure third as prime cell of harmonic space

By now, it will be sufficiently apparent that harmonic complexes can only emerge out of the contribution of the (primary) pure major third...

Therefore, I adopt the pure “natural third” C E as the prime cell of the infinitely-wide harmonic space. Its true midpoint is an ideal \( \text{Equator} \) = Equator (a whole tone (170 \( \mu \)) minus \( \frac{1}{2} \) of a syntonic comma (9 \( \mu \)) from either side of the primary third cell = 161 \( \mu \)). The prime cell indicates the center or so-called source position (\textit{Ursprungslage}) of our usable tonal system:

Harmonic prime cell in its source position (\textit{Ursprungslage}):

\[ \text{Aequator} = \text{equator (the midpoint in the prime-third cell)} \]
The pure third as prime cell of harmonic space

Karg-Elert generates the consonant triads not from a single prime pitch, but from a major-third prime cell. In almost all cases, Karg-Elert places the C-E prime cell at the center of the pitch space; acoustic values for all other pitches are calculated in relation to the C-E prime cell. In effect, Karg-Elert views pitch space as a fixed field centered on the C-E prime cell, irrespective of the tonal center in any specific piece of music.

The Ursprungslage (source position) of the prime cell is middle C (c⁴) and its major third (e⁴).

The Aequator (161 µ) is the midpoint of the prime cell (the pure major third, or 322 µ). Note that the midpoint is neither canonic D₂,0 at 170 µ, or syntonic D(-2,1) at 152 µ. Instead, the Aequator is the conceptual axis between those two Ds, and is not an actual acoustic pitch.
6.2. The generation of the double consonance

From the theories of frequency and wavelength, we know that the proportions involving the third assume the real or latent presence of the fifth:

This is confirmed by our instinctive sense for harmony: \( \text{C E} \) is understood to represent the harmonies of C major and A minor. In C major, our harmonic imagination supplies the upper G [ = fifth above C]; in A minor, the lower A [fifth below E] is supplied.

In a symmetrically polar manner, the prime cell generates the double consonance:
6.2. *The generation of the double consonance*

In the harmonic series, the third (partial 5) implies the presence of the fifth (partial 3).

Likewise, the prime-cell third implies the complete major and minor triads.

The term “double consonance” refers to the major and minor triads, generated in polar fashion upward and downward from the prime cell C-E.

The graphic symbol combines two overlapping triangles: Δ for the C major triad, V for A minor.

322 $\mu$ = syntonic major third
263 $\mu$ = syntonic minor third
322 + 263 = 585 $\mu$ = perfect fifth
6.3. The completion of diatonic space

The minor third assumes the presence of the natural pure (major) third:

\[
\begin{align*}
\{ &c \} \text{ will be perceived as harmonic representative of A minor or F major.} \\
\{ &e \} \text{ will be perceived as harmonic representative of C major or E minor.}
\end{align*}
\]
6.3. The completion of diatonic space

Just as the major third can represent both the major and minor triads (6.2), the minor third can also represent (or generate) both major and minor triads.

The graphic symbols consist of overlapping triangles: Δ for major triads, and ∇ for minor. The second symbol combines all four triads: F major, A minor, C major and E minor.

322 μ = syntonic major third
263 μ = syntonic minor third
322 + 263 = 585 μ = perfect fifth
The major thirds $\# a \# g b$ in turn imply their harmony-expanding fifths $\times d^\natural$:

But two octaves are only $2000$. The two Ds differ by $18 \mu$ (= syntonic comma).

The upper D is understood as $\# c^\#$ (2 upper-fifths) from C, while the lower D is $\# c$, which is 2 fifths below E$_{(1,0)}$.

$\# c^\# = 9 \quad \# c = 5/9 \quad 9 = \frac{80}{9} = \frac{8 \times 8}{9} = \frac{81 : 80}{9} = \text{syntonic comma}$

(within the octave $8 \rightarrow 16$)

It is much simpler to calculate the difference in $\mu$:

$$\# c^\# = \frac{170}{170} \quad \downarrow \quad \# c = \frac{322}{170} = \frac{152}{152} = \frac{170}{152} = 18$$

$= \text{syntonic comma}$
Here, Karg-Elert completes diatonic pitch space, but with eight tones instead of seven, as it contains two distinct values for D. The syntonic comma presents an age-old problem in purely-tuned diatonic harmony: $D_{(2,0)}$ is acoustically distinct from $D_{(-2,1)}$. In other words: in a just intonation C major, the fifth of V is not identical to the root of II.

Karg-Elert follows Hauptmann and Oettingen by fully embracing the syntonic comma difference between the two Ds, and highlighting their functional and conceptual separation (Hauptmann 1853, 24-25; Oettingen 1866, 69). See section 13.3 for more on the musical ear’s ability to perceive and distinguish between comma-different pitches.
6.4. **The major and minor systems and their principal harmonies**

This natural eight-tone scale with seven different note-names, whose lowest and highest pitches differ by a syntonic comma, establish one major and one minor principal system [principal = main harmony].

The tonic is the central harmony, to which both of the other principal harmonies are most closely related.

The “dominating” [controlling] harmonic direction is exactly that implied by the tendency of the tonic itself:

In major ↑, “forward” or “dominating” implies ascending ➤ direction.
In minor ↓, “forward” or “dominating” implies descending ◀ direction.

Consequently, I call “dominant” the harmony that connects with the tonic in its dominating direction [i.e. fifth of the tonic = prime of the dominant].

Thus, in the upward-harmonic (major) system D lies above T, while in the downward-harmonic (minor) system A lies below J.
6.4. The major and minor systems and their principal harmonies

Note that Karg-Elert’s function labels lean in the direction of the mode: “forward” (to the right) in major, and “backward” (to the left) in minor. In this translation, major-key labels will lean right (T and D), but minor-key labels will not lean at all (Ⅰ̲ and Ⅶ̲), due to font limitations.

“lowest and highest pitches” = D(−2,1) and D(2,0), discussed in 6.3
“principal system” = a tonic and its two dominants (D and C)
In the “contrary” or opposite direction from the tonic lies the contrant [abbreviation for contra-dominant], which lies

In the **upward**-harmonic (major) system, C is **below** the tonic: \[ C ^ { \text{maj} } \]

In the **downward**-harmonic (minor) system, \( \exists \) is **above** the tonic:

**Normal cadences specific to each system:**

- major = authentic
- minor = plagal
Contradominant (or contrant) is Karg-Elert’s term for the subdominant. Its function symbol is C in major, and C in minor. To a greater degree than “subdominant,” the term “contradominant” stresses polar or symmetrical opposition from the dominant. It is cognate with Rameau’s sous-dominante – the fifth below the tonic (Rameau 1726, 38).

In this figure, the arrows indicate the “dominating harmonic direction” (ascending in major, descending in minor). [Ⅵ] and [Ⅲ] indicate the primes or generating pitches, in C major and A minor respectively. (7) and (7) indicate the natural (pure) sevenths above and below their respective roots, to be discussed in detail in section 6.5.

The terms authentic and plagal are to be understood in the traditional way, in terms of root motions by fifth: in an authentic cadence the chord roots descend by a fifth (G → C), while in a plagal cadence the roots ascend by a fifth (D → A). Functionally, both cadences are D → T.
6.5. **The dominant and its concordant seventh**

Only the dominants can be concordant, i.e. they generate their natural (pure) sevenths $\frac{7}{5}$ and $\frac{3}{8}$ as integral chordal components.

For everything else related to sevenths, see the chapter on that topic.
6.5. **The dominant and its concordant seventh**

**Concordant** is Karg-Elert’s descriptive term for major or minor triads with their natural (4 : 7) sevenths, and also for interval involving the natural seventh, which is symbolized here in two ways: by the numeral “7,” and by the wedges (\(\vee = 7^{\text{th}}\) above major triad; \(\wedge = 7^{\text{th}}\) below minor triad).

The line above or below the dominant (\(D\)) function label indicates the presence of the concordant seventh. The concordant seventh is notated above the label in major, and below the label in minor.

“relative (rel.) Primen” (relative primes) = A and G, respectively the primes (generating pitches) of the dominants in A minor and C major, symmetrically disposed around the prime cell C-E.

Note that the sizes of the thirds progressively decrease: 322 \(\rightarrow\) 263 \(\rightarrow\) 222 \(\mu\).

See Chapters 9 and 10 for a detailed discussion of the natural sevenths (concordants).
6.6. The contrant variant

The characteristic trait of the contrants is that they:
- Lie under $T$ in major keys, but nonetheless as upward harmonies (major triads)
- Lie over $L$ in minor keys, but nonetheless as downward harmonies (minor triads)

A = over-third of lower-fifth

G = under-third of upper-fifth
6.6. The contrant variant

Due to Karg-Elert’s strict polarity, the contrant in minor keys (i.e. what is called the dominant in monistic systems) is a minor triad – E minor, in the key of A minor. This contrasts with Riemann, for whom the dominant in minor keys is normally a major triad, reflecting traditional practice (Riemann 1893, 8).
The contrant tends to adapt its harmonic direction to its harmonic position, i.e.

- when it lies below $T$, it generates its lower third
- when it lies above $T$, it generates its upper third,

...to create an integral unidirectional entity:

**In functional terms, these harmonies are called:**

*Contrant-variants, or altered ("tempered")* **contrants**

They are indicated in function notation by a lowercase $c$ or $\text{c}$ [capital letters = major triads in major keys, and minor triads in minor keys; lowercase letters = minor triads in major keys, and major triads in minor keys.]

**)"tempered" in its original basic meaning: "mediated, softened, mixed, equalized" [namely between major and minor]. The use of the term here does not refer to tempered tuning."
In other words: the contrant in major keys is often a minor triad, reflecting its harmonic position (Klanglage) below the tonic. This is a very familiar example of primary mixture: the use of iv in place of IV in major keys. Conversely, the contrant in minor keys is often a major triad, reflecting its harmonic position above the tonic: using major V as dominant in minor keys.

To avoid possible confusion between the term “tempered” and equal temperament, this translation uses the term contrant variant for Karg-Elert’s temperierte Contrant. The term Contrant-Variant reflects not only that the contrant’s normal chord quality is switched from major to minor (or vice versa), but also that its third is a variant pitch, differing from its canonic counterpart by a syntonic comma. In the musical example in 6.7 (and in many subsequent examples), variant pitches are shown as closed noteheads.
6.7. The concordant seventh of the contrant variant

The contrant variants are “similar” [but not identical!] to the dominants of their parallel keys, and as such can likewise be concordant, i.e. they can generate their natural sevenths.

*) similar to C in C minor

*) similar to D in A major
6.7. The concordant seventh of the contrant variant

For example: the contrant variant in C major is similar (but not identical) to the dominant in C minor. Both are F minor triads, but they are acoustically distinct:

\[
\begin{align*}
\text{e in C major: } & F_{(-1,0)} \quad A\flat_{(0,-1)} \quad C(0) \\
\text{D in C minor: } & F_{(-5,1)} \quad A\flat_{(-4,0)} \quad C_{(-4,1)}
\end{align*}
\]

Due to their similarity, the contrant variant can generate a natural seventh, just like a dominant. The second example illustrates the polar opposite: the contrant variant in A minor approximates the dominant in A major, as both are E major triads.

Though it is a consequence of Karg-Elert’s strict polar dualism, this example makes an elegant correlation between the two most common chromatic alterations in diatonic tonality: raised \(^7\) in minor keys, and lowered \(^6\) in major keys. In equal temperament, these two scale degrees are enharmonically equivalent, though conceptually very distinct.

The line in the function symbol \(\bar{D}\) (first introduced in 6.5) indicates the addition of the concordant natural seventh (4:7). When contrant variants (e) generate their natural sevenths (as just discussed), they likewise have a line above or below the function label.
6.8. **Polarity and the contrant variant**

Ab\(_{(0,1)}\) in C major is intervallically equivalent to G\#\) in A minor:

\[
\begin{align*}
\text{µ intervals: } & \quad \mu = \text{minor diesis} \\
& \quad \text{Together } = 966 \mu = \text{a minor diesis}
\end{align*}
\]

Pure melody requires:

```
All generated from C (1)
```

But in pure harmony:

```
Sevenths (see later)
```

The only possible compromise is provided by equal-tempered tuning.
6.8. **Polarity and the contrant variant**

Note that in many of his musical examples, Karg-Elert often uses ties to indicate common tones – meaning exact common tones in just-intonation pitch space, with the same acoustic and functional value (not simply any two pitches with the same name or register).

The variant thirds A♭ and G♯ are symmetrically disposed around the prime cell C-E.

966 μ is the octave minus a minor diesis (34 μ), as described in section 5.7.

This pitch succession is the upper voice of the preceding example, interpreted as a purely melodic line. Karg-Elert states that the semitones (and all other intervals) must reflect Pythagorean fifth relations.

However, considered entirely as members of harmonies, the pitches display a mixture of fifth-, third- and seventh-based acoustic values. “See later” = see Chapters 9 and 10.

Equal temperament provides a practical compromise between canonic and syntonic values. However, we will see in Chapter 13 that Karg-Elert understands all pitches in terms of their pure acoustic values, even when they are performed in equal temperament.
If the dominants generate their own dominants – and the contrants their own contrants - the resulting forms are double- or ultradominants and ultracontrants, which extend beyond the boundaries of the key, “over”stepping its space or region. If they have direct linear connections with the regular dominant or contrant, they are understood as prefix or suffix chords [for which the function symbol is placed in parentheses, and connected to the simple principal Klang by ties below for prefixes, or above for suffixes].

![Grenze](image1)

**ERRATUM:**

Without a direct linear connection, such chords are true “ultraforms,” symbolized by doubled function labels with the simple D, C, D or C struck through. [Striking through indicates: not actually present, but latently implied through transformation].

Isolated, non-consecutive forms: [real ultradominants and ultracontrants]
6.9. *The ultradominants and ultracontrants*

The “ultraforms” exceed the “boundaries of the key” by introducing the non-diatonic pitches B♭ and F♯, which are polaristically disposed around the C-E prime cell. In C major, B♭ is the prime of the ultracontrant and F♯ is the third of the ultradominant. In A minor, F♯ is the prime of the ultracontrant, and B♭ is the third of the ultradominant.

Prefix and suffix ultraforms are notated with parentheses, as follows:

- **C(C)** = ultracontrant as suffix to **C**
- **(D)D** = ultradominant as prefix to **D**

As noted in 6.8, the ties indicate acoustically-exact common tones. The divider || between contrant (C) and dominant (D) suggests a conceptual gulf between those functions, perhaps due to the lack of common tones; significantly, **C(C)** and **(D)D** contain acoustically and functionally distinct Ds (not tied together). Finally, the slurs above the upper voice connect **T** to **C**, and **D** to **T**, while again separating **C** from **D**. The marking Parenthesen suggests that the ultraforms act as insertions or expansions within the basic **T – C – D – T** progression.

**ERRATUM:** the first **T** chord should have **C** in the tenor voice, rather than the erroneous **B**.

This statement suggests that the ultra-forms can actually substitute for their corresponding principal harmonies in chordal progressions. These “real” ultraforms (notated in the translation as **CC** and **DD**) substitute for the regular **C** and **D**. They are “non-consecutive” because they are not linked as suffix or prefix to the regular **C** or **D**.
The pitches $\text{B}_\flat : F$ (i.e. the two new pitches introduced by the ultraforms) exhibit a fundamentally different relationship than $\text{A}_\flat : E$ or $\text{C} : G\sharp$ from Example No. {?}

\[
\text{A}_\flat (0, -1) \rightarrow (\text{C}(0)) \rightarrow \text{E}(0, 1) \\
\text{Third + third} = 2 \times 322 = 644
\]

\[
\text{C}(0) \rightarrow \text{E}(0, 1) \rightarrow \text{G}(0, 2) \\
\text{Third + third} = 2 \times 322 = 644
\]

\[
\text{B}_\flat (-2, 0) \rightarrow \text{F}(2, 1) \\
\text{Fifth + fifth + fifth + fifth + third} \\
4 \times 585 = (2)340 \\
+ 322 = 662
\]

\[
\text{B}_\flat (-2, 0) \rightarrow \text{F}(2, 1) \\
\text{Third + fifth + fifth + fifth + fifth} \\
4 \times 585 = (2)340 \\
+ 322 = 662
\]
Karg-Elert is surely referring to the musical example in 6.8., where the pitch pairs Ab – E and C – G♯ (both notated as augmented fifths) are found in the final cadences in C major and A minor.

The pitch pair B♭ and F♯ (found between the two ultraforms) is also notated as an augmented fifth, though its acoustic value (662 µ) and derivation is different, as specified here.
6.10.  **The substitutes: Leittonwechsel (l) and parallel (p)**

Just as every tonic is surrounded by two fifth-related principal harmonies, every principal harmony is surrounded by two diatonic third-related chords, which act as its representatives ["substitutes"]. They result from the fusion of two principal harmonies, from which the peripheral chord-tones have been detached.

The substitute that lies in the **forward** direction from its principal is called \( X = \text{Leittonwechsel} \ (l) \)

The substitute that lies in the **backward** direction from its principal is called \( X = \text{Parallel} \ (p) \)

The substitutes are of the opposite qualities to their relative principal chords.

\[(v.) = \text{forward (vorwärts)} \quad (r.) = \text{backward (rückwärts)}\]
This passage recalls Riemann’s concept of *Klangvertretung* or “chord representation,” by which any chord (not only those described here) is characterized “as a more or less equal representative of some tonic, dominant or subdominant” (Riemann 1893, 141). The concept of the diatonic substitutes is taken directly from Riemann’s *Harmony Simplified*, though the postscript $L$ for the *Leittonwechsel* is original to Karg-Elert.

Karg-Elert does not invoke Riemann’s explanation of the diatonic substitutes as *Scheinkonsonanzen* or “feigning consonances” (Riemann 1893, 56). For Riemann, the substitutes sounded consonant but were actually dissonant, as they result from the “characteristic dissonance” of the sixth. Karg-Elert later uses the term *Scheinkonsonanz* with a quite different sense - see 11.2.

This example clearly indicates how the diatonic substitutes have a potential double function. $C^L$ and $T_p$ are acoustically equal, as are $T^L$ and $D_p$. The choice of function depends on context, to be further demonstrated by the example in 6.11.
Very important [and relevant to comma-free modulation] are the diatonic “outer” substitutes:

\[ \begin{array}{c}
C_p & \rightarrow & D & \parallel & D^\# & \rightarrow & D^P \\
\text{Contrant-parallel} & | & \text{Dominant-Leittonwechsel} & | & \text{Contrant-parallel}
\end{array} \]

Their formation assumes the latent existence of the ultraforms:

\[ \begin{array}{c}
C_p & \rightarrow & D & \rightarrow & D^\# & \rightarrow & D^P \\
\text{(v.) = forward (vorwärts)} & | & \text{(r.) = backward (rückwärts)}
\end{array} \]
6.10. (continued)

The diatonic “outer” substitutes (namely \( Cp \) and \( D^l \)) point toward the ultraforms (\( CC \) and \( DD \) respectively), which are the next links in the chain of pure canonic fifths. “Comma-free modulation” involves keys related by pure fifths, as will be discussed in Chapter 8.

For example, in C major:

\( Cp \) contains the minor third \([D F]\), which latently implies \( CC \) \([B\flat D F]\).

\( D^l \) contains the major third \([D F^\#]\), which latently implies \( DD \) \([D F^\# A]\).
6.11. Example of substitutes and contrant variants

= sonically identical, functionally different
= unique, occurring only once.
6.11. Example of substitutes and contrant variants

*Kl.* and *gr.* (“small” and “large”) refer to the minor and major thirds in the bass lines.

The open noteheads indicate the changing pitches that create the parallel and *Leittonwechsel* chords: whole tone motions ((mesh)) create parallel chords, while semitone motions (stalk) create *Leittonwechsel* chords.

The regular ties mark acoustically-identical common tones within a single function; the dotted ties indicate acoustically-identical common tones that bridge different functions.

Finally, the boxed functions on the downbeats of mm. 1-3 govern their entire measures; the notation succinctly indicates chordal hierarchy, reinforced by rhythmic placement.

This example suggests that the functional meaning of parallel and *Leittonwechsel* chords is normally determined by the function of the preceding chord. In the C major passage, the A-minor triad in m. 1 is *C\l\l*, as it follows *C*; in contrast, the A-minor triad only two beats later (m. 2) is *Tp*, as it follows *T*. The two A-minor triads are acoustically identical in just intonation; nonetheless, they express different functions due to their context.
In **pure melodic** terms = narrow semitone steps (**canonic**).

\[
\begin{align*}
F\# & \rightarrow G \leftarrow Ab : Bb \rightarrow A \leftarrow G\#
\end{align*}
\]

\[
\begin{align*}
F\#(8,0) & \quad G(1,0) \quad Ab(-4,0) \quad G\#(8,0) \quad A(3,0) \quad Bb(-2,0) \\
75 + 75 + 20 + 75 + 75 \mu &= 320 \mu \\
(4 \text{ Limmata} + 1 \text{ Pyth. comma})
\end{align*}
\]

In **harmonic** terms = more precise, wider semitone steps (**syntonic**).

\[
\begin{align*}
F\#(2,1) & \quad G(1,0) \quad Ab(0,1) \quad G\#(0,2) \quad A(1,1) \quad Bb(-2,0) \\
93 + 93 - 34 + 93 + 93 \mu &= 338 \mu \\
(\text{minor diesis}) & \quad (4 \text{ leading tones} - 1 \text{ minor diesis})
\end{align*}
\]

Difference = \(+ 18 \mu\)

\(\text{(syntonic comma)}\)
ERRATUM: instead of “4 Leittöne – 1 synt. Komma,” it should read “4 Leittöne – 1 kl. Diēsis.”

These are the bracketed upper-voice motions in the two preceding musical examples.
6.12. Expansion of tonality through cadential elaboration of the substitutes

Tonality is expanded by means of cadential prefix chords applied to the substitutes:

The surprising key-expanding B♭-minor and F♯-major chords in C major

and B-major and E♭-minor chords in A minor

are without exception contrant variants: two octaves away from their principal source positions (Ursprungslagen), and differing from them by a syntonic comma.
6.12. *Expansion of tonality through cadential elaboration of the substitutes*

This example is a $T – C – D – T$ cadential progression enriched by diatonic substitutes, which are themselves embellished by prefix and suffix chords (applied $C$ and $D$), introducing further chromaticism.

This is the polar equivalent progression in A minor, though the voice leading is different.

See the next page for a further demonstration of the *Ursprungslagen* concept.
Ursprungsagen (source positions) of the principal key-centers:

The source positions of the above chord progressions in C major and A minor illustrate that the Eb-minor, B♭-minor, B-major and F♯-major triads are variants, located in the registers of the E♭-major, B♭-major, B-minor and F♯-minor triads respectively.

C major example above:  A minor example above:
6.12. (continued)

Here is the first clear presentation of the *Ursprungslagen* or source positions for triads and keys, extending by canonic pure fifths in either direction from the central C major/A minor. Each triad/key is notated on the staff in a specific octave, reflecting the number of fifths away from the center. However, all pitches in the *Ursprungslagen* should be understood as *pitch classes*, which retain their fifth-derived identity regardless of their registral placement in musical passages.

This visually complex example refers back to the passages at the beginning of 6.12, showing each chord in its *Ursprungslage*. Each chord is rewritten in a specific octave to indicate its source position. The pitches written entirely in open noteheads are in their source positions; in other words, they are stations along the chain of fifths, not displaced by any syntonic commas. Conversely, the pitches written using closed noteheads are *variants*, displaced by a syntonic comma. For example: the A major triad exists in the position of A minor (*A moll*), with C♯(-1,1) substituting for C(0). All slurs and ties indicate acoustically equivalent pitches. The beamed chord groups express the functions shown by the function labels.

The following pitch pairs in the examples differ by 16 μ (diaschisma):

- In C major: D♭(-1,-1) / C♯(3,1) and B♭(-2,0) / A♯(2,2)
- In A minor: F♯(2,1) / G♭(-2,-1) and D♯(1,2) / E♭(-3,0)

The difference between B♭(-2,0) at [830] μ and A♯(2,2) at [814] μ = 16 μ = Diaschisma
Chapter 6: Tonnetze

6.1. *The pure third as prime cell of harmonic space*

Harmonic prime cell in its source position (*Ursprungslage*):

The prime cell is $C = C(0)$, plus its major third above $= E_{(0,1)}$.

The *Ursprungslage* or source position for the prime cell is middle C ($c^1$) and E ($e^1$) - right in the middle of the practical pitch range.

6.2. *The generation of the double consonance*

The double consonance = the major and minor triads, generated above and below the prime harmonic cell:

The prime cell is $C = C(0)$, plus its major third above $= E_{(0,1)}$.

The *Ursprungslage* or source position for the prime cell is middle C ($c^1$) and E ($e^1$) - right in the middle of the practical pitch range.
6.3. The completion of diatonic space

The completed 8-note diatonic collection:

The 8-note diatonic scale contains $D_{(2,0)}$ and $D_{(-2,1)}$, differing by a syntonic comma:
6.4. The major and minor systems and their principal harmonies

The three principal harmonies in C major:

The three principal harmonies in A minor:

In the version of the Tonnetz used here, right triangles outline consonant triads:

Major triad

Minor triad
6.5. The dominant and its concordant seventh

The concordant seventh is the natural minor seventh (4:7), generated above or below the prime, usually of a dominant \( V \), but also of a contrant variant \( v \) (see 6.6).

On the Tonnetz, the concordant seventh above the prime is shown a diagonal, extending up and to the right of the prime. The seventh below the prime is a diagonal extending down and to the left. No acoustic subscripts/superscripts will be attached to concordant sevenths; their acoustic derivation should be apparent from the connecting lines:

- In C major:

- In A minor:

The sevenths add a third axis to the grid, one which can potentially extend continuously in both directions, creating an infinite three-dimensional pitch space. Unless the concordant sevenths themselves generate harmonic relationships or chordal transformations (see chapters 9 and 10), they usually do not influence our understanding of harmonic progressions. Therefore, concordant sevenths will usually be omitted from the Tonnetz.

The acoustic and musical significance of seventh-derived pitches is explored fully in Chapters 9 and 10.
6.6. **The contrant variant**

![Diagram of minor triad in major and major triad in minor]

C major

\[
\begin{array}{c}
D^1 & A^1 & E^1 & B^1 \\
B_b & F & C & G
\end{array}
\]

A minor

\[
\begin{array}{c}
F^2 & C^2 & G^2 & D^2 \\
B_b & F & C & G
\end{array}
\]

6.7. **The concordant sevenths of the contrant variant**

In C major:

![Musical notation and diagram]

In A minor:

![Musical notation and diagram]

As in 6.5, the extra “spokes” are the concordant natural sevenths added to the dominants (D) and contrant variants (c).
6.8. Polarity and the contrant variant

These Tonnetze are identical to those for 6.7. The ties in the musical examples indicate acoustic common tones: C(0) in C major, and E(0,1) in A minor.

6.9. The ultradominants and ultracontrants

These Tonnetze apply for all musical examples in 6.9.
6.10. *The substitutes: Leittonwechsel (L) and parallel (p)*

Diatonic substitutes of principal harmonies in C major:

Parallel (p) substitutes in C major:

Leittonwechsel (L) substitutes in C major:

Diatonic substitutes of principal harmonies in A minor:

Parallel (p) substitutes in A minor:

Leittonwechsel (L) substitutes in A minor:
6.11. Example of substitutes and contrant variants

\[\text{\textbullet} = \text{sonically identical, functionally different} \]
\[\times = \text{unique, occurring only once}\]

C major:
Tonnetz chords:

\[1 \quad 1 \quad 2 \quad 2\]

A minor:
Tonnetz chords:

\[1 \quad 1 \quad 2 \quad 2\]

NOTE: the concordant sevenths (in the last bar) are not shown on this Tonnetz.
6.12. Expansion of tonality through cadential elaboration of the substitutes

C major:

Tonnetz chords:

NOTE: if a triad (triangle) does not have a function label or chord number, it does not occur in the passage.

A minor:

Tonnetz chords:
6.12. Expansion of tonality through cadential elaboration of the substitutes (continued)

This example demonstrates the concept of the Ursprungslagen or source positions:

Each of these major or minor triads contains one pitch that belongs to the fifth-chain generated from C(0): the primes of the major triads, or the thirds of the minor triads.

This Tonnetz displays the same 14 triads as the above example:

Both the example and the Tonnetz show only a selection of the Ursprungslagen triads.

Both the Tonnetz and the Ursprungslagen are models of distance in pitch space. Where the Tonnetz uses horizontal space to indicate distance from the center C(0), the Ursprungslagen employs register. The central C major and A minor triads are generated from the prime cell C-E, in the middle of the pitch space. They are in the middle of the Tonnetz, and their source positions are based around middle C. Triads that are many fifths away from C(0) – such as Eb minor or F# major – are represented in extreme registers, just as they are horizontally distant from the center on the Tonnetz.

To be clear: the Ursprungslagen do not represent the actual octaves of pitches as they occur in a musical passage. Rather, they represent the acoustic derivation of pitches in pitch space, regardless of the octaves in which they are stated. When we say that a G major triad is in its Ursprunglage or source position, we mean that it contains the following specific pitches: G(1,0)  B(1,1)  D(2,0).
6.12. Expansion of tonality through cadential elaboration of the substitutes (continued)

This is an Ursprungslegen analysis of the passages from the start of 6.12. Each pitch and chord is renotated in an octave that represents distance (in fifths) from the center C(0):

C major: A minor:

The tonic chords (T and \( T \)) are centered around middle C. Registral distance from the center reflects harmonic distance from the tonic, in terms of fifths above or below.

Triads that contain only open noteheads are in the Ursprungslegen. Pitches notated as closed noteheads are **variants**: a syntonic comma away from their Ursprungslegen counterparts. Triads that contain variant pitches are **not** in their source positions.

The Tonnetze for the two passages were provided above. Here is the C major Tonnetz once more:

The Ursprungslegen chords are on the same horizontal level. The variant chords (numbered 2, 3 and 4) are on different levels, as they contain pitches that are a syntonic comma higher or lower than the canonic (Ursprungslegen) counterparts. Thus, the Ursprungslegen and the Tonnetz model both acoustic information and distance between chords in similar ways.
Chapter 7

Comma-free and comma-differing modulation
Chapter 7
Comma-free and comma-differing modulation

7.1. Comma-free (Pythagorean) modulation
7.2. Comma-differing (syntonic) modulation
7.3. Further examples of comma-free and comma-differing modulation
7.4. Mediants and neighbor-mediants
7.5. Schematic overview of mediant substitutions, applied to the cadence
7.6. Mediant substitutions of the ultraforms
7.7. Comparison of the 20 triads in the mediant system with the same principals
7.8. Functional values of mediant and counter-median major triads in minor, and minor triads in major

Addendum to 7.8: counter-mediator-variants
7.1. Comma-free (Pythagorean) modulation

Comma-free modulation is Pythagorean in basis; that is, the harmonic stations that connect the principal initial key and the principal goal key [not simply from chord to chord] belong to the chain of fifth-relations. To be sure, variants can be included at will, but they must be only fleeting in character [as for instance in a cadential prefix], and cannot themselves initiate further modulatory processes.
7.1. Comma-free (Pythagorean) modulation

“variants” = triads with chromatically-altered thirds that acquire a syntonic comma. See 5.6 and 6.12.
For example: modulation from D minor to B minor [tonics in their principal source positions]

H moll Tonartbereich = B minor collection or scale
d moll Tonartbereich = D minor collection or scale

varierte Terz = varied (altered) third, indicated using a closed notehead
statt A moll, A dur = instead of A minor, A major
alte Tonart erlischt = The old (initial) key expires
neue Tonart wird gültig = The new key becomes valid (active)

Die beiden {circled}Ds at the beginning and end are in no way identical; if the initial D is understood as {tonal center}, we jump to → A → E → B → F#, whose lower third is the final D! That is four fifths minus a syntonic third:

4 fifths; simplified = 2 whole tones = 2 \times 170 = 340

1 third subtracted

\[ - 322 = 18 \text{ Difference: syntonic comma} \]

Do not misunderstand: the above modulation is comma-free. The difference of 18 \mu between the [initial] fifth-derived D and the [final] third-derived D must be recognized. It is nothing less than the hallmark of comma-free tonal transformation, as it indicates that there is absolutely no tonal connection between the tonics of D minor and B minor!
7.1. (continued)

On the upper grand staff, the triads in the progression are shown in their *Ursprungslagen* (see section 6.12). The opening D minor tonic triad contains D\(_{(2,1)}\), while the closing tonic of B minor contains D\(_{(2,0)}\); the two Ds differ by a syntonic comma, and are therefore written in different registers in the *Ursprungslagen* on the upper system.

This example is the first to contain a modulation from one key to another (in this case from D minor to B minor). The C major chord in the penultimate bar (beat 1) is the pivot chord, where Karg-Elert notes that “the old key expires” and “the new key becomes active.” In the functional analysis, the pivot chord is assigned two function labels, separated by a curve; the upper label applies to the old key, and the lower label applies to the new key.

While the two Ds differ by a syntonic comma, both the initial and final tonics are in their *Ursprungslagen*, located on the chain of canonie fifth relations. Therefore, the modulation is classified as *comma-free*. 
7.2. Comma-differing (syntonic) modulation

If the D minor and B minor triads [in 7.1] were to contain an identical D, then the B minor triad must be a direct transformation [namely a variant-parallel] of the D minor triad, or the D minor triad must be a direct transformation [a parallel-variant] of the B minor triad:

\[ \times = \text{variant or mediant alteration of triads in their source positions} \]

Comma-differing modulation: from D minor to B minor

This alleged modulation is achieved by means of a variant trick:

Result: comma difference.

In truth, entirely in B minor:

The variant-deception can engender further fruits: B major can substitute for the B minor triad! And so on.
7.2. **Comma-differing (syntonic) modulation**

As in previous examples, the closed noteheads indicate chromatic variants. The horizontal line links the acoustic common tone D\(_{(-2,1)}\). The lowercase “p” labels the diatonic parallel substitutes, while the uppercase “P” indicates a variant alteration of the parallel; see the material on mediant in 7.4.

In **comma-differing modulation**, the opening and destination keys differ from each other by one or more syntonic commas.

In this example, the opening tonic is D\(_{(-2,1)}\), but the closing tonic is B\(_{(-3,2)}\), which acquires a syntonic comma.

A “variant trick” is a functional reinterpretation involving a variant chord such as the contrant variant. In the previous passage, the A major triad enters as a variant chord – the contrant variant (\(\sigma\)) in D minor – but is reinterpreted as a non-variant chord: the contrant-parallel (\(\sigma^p\)) of B minor. While such harmonic tricks readily enable any number of key changes, they result in syntonic comma difference, and therefore engender a feeling of “deception.”
Or: from B-flat minor to C# major

3 variant tricks = 3 comma differences

The above is a twisted and agonized chromaticization of the following harmless basic form:

Moll-Var. = minor variant of B♭ major
Dur-Var. = major variant of A minor
Terzgleicher = “common third”: a mode-shifting triadic transformation that keeps the same third (see below)
Chromonanten = “chromonants”
In passages that modulate frequently, the tonic chords (labelled \( T \) or \( \downarrow \)) indicate the key areas.

This chord progression presents an interesting test of Karg-Elert’s ideas, and also of our aural capability to process enharmonic equivalence (in equal temperament). In the upper voice: does the ear enharmonically equate the opening soprano D-flat and the closing C#, even at the end of the chromatic progression? For Karg-Elert, they differ by three syntonic commas (54 µ), and are thus functionally distant from each other. However, since they are repeatedly sounded in the same voice and octave, the ear may nonetheless equate the two pitches enharmonically.

The *Terzgleicher* is identical to David Lewin’s SLIDE triadic transformation (Lewin 1987, 178). While the same transformation also results from Riemann’s *Doppelterzwechsel* (Riemann 1880, 81), Karg-Elert presents the relationship in the same manner as Lewin: the third remains constant, while the outside perfect fifth shifts by a semitone. This description is likely original to Karg-Elert.

*Chromonants* are triadic shifts in which all three pitches of the original triad are moved up or down by a *chromatic* semitone. Chromonants always have the same letter name, and are of the same quality as the original triad (major or minor). For example: the chromonants of C major are C\(^\#\) major and C\(_b\) major; the chromonants of A minor are A\(^\#\) minor and A\(_b\) minor.
7.3. Further examples of comma-free and comma-differing modulation

Diatonic [7 ½ fifth-steps ↑]:
D minor to F# major (via B minor), comma-free.

Chromatic:
D minor to F# major

Diatonic [9 ½ fifth-steps ↑]:
D-flat major to C# minor (via B-flat major and G major), comma-free.

Erroratum:

The closing tonics of the two progressions differ by two commas: F#(6,0) vs. F#(2,2).

Bereich = scale, collection, region
Des dur = D-flat major  B dur = B-flat major  G dur = G major  Cis moll = C# minor

[D♭(5,0)  F(5,1)  A♭(4,0)] -------- comma-free --------→ [C♯(3,1)  E(4,0)  G♯(4,1)]
7.3. Further examples of comma-free and comma-differing modulation

Two modulations from D minor to F# major. The first (“diatonic”) is comma-free, as the opening and closing tonics are both in their Ursprungsagen.

7 fifth-steps up (i.e. toward the “sharp” direction) would be from D minor to D# minor; the extra half fifth-step up is from D# minor to its parallel F# major. The modulation traverses the keys of D minor, B minor and finally F# minor.

The second (“chromatic”) involves two variant tricks, and the addition of two syntonic commas. It is a chromaticised variant of the basic diatonic progression [Dm → Am → C7 → FM].

**ERRATUM:** a diatonic comma-free modulation from Db major to C# minor would be $8 \frac{1}{2}$ fifth-steps up, not $9 \frac{1}{2}$. It is $8$ fifth-steps up from Db major to A major, plus one more half fifth-step to C# minor.

This staff shows the preceding comma-free modulation from Db major to C# minor, with each chord notated in relation to the Ursprungsagen. The harmonic distance of eight and a half fifth-steps is illustrated by the extreme shift in the source positions. As Karg-Elert’s annotations indicate, the passage travels through the regions (*Bereiche*) of Db major, Bb major and G major, before settling in C# minor.
The same as above (D-flat major to C# minor), shortened by means of variants:

Basic: \( d \quad Bb \quad (F \quad d) \quad a \quad C \quad G \quad C \)

Keys: \( DbM \quad FM \quad AM \quad Cm \quad All \text{ in } F \text{ major} \)

Comma-free from A minor to A-flat minor (by means of ultradominants) = 7 fifth-steps ↓

Middle voice \( \text{circled pitches} \) = descending Pythagorean whole-tone series

\[ C(0) \rightarrow Bb(-2,0) \rightarrow Ab(-4,0) \rightarrow Gb(-6,0) \rightarrow Fb(-8,0) \]
7.3. (continued)

Karg-Elert provides two very different functional analyses for this comma-differing modulation from D♭ major to C♯ minor. The first (at the left) treats the initial D♭ major triad as the opening tonic, and traverses three different keys: F major, A major and finally C♯ minor (see the tonic function labels). This first analysis involves three variant tricks, reinterpreting tonic-function chords as contrant variants (or vice versa).

The second analysis attempts to understand the entire progression in F major (the boxed tonic). All chords are notated in their Ursprungslagen. The final C♯ minor triad is labeled as D^{Mp}, or the upper mediant-parallel of the dominant: CM (D) → EM (D^{M}) → C^{#} m (D^{Mp}). This complex designation is perceptible in at least one important sense: the C♯ minor triad contains E_{(0,1)}, which is the third of C major (i.e. the dominant of the F major tonic). The C♯ minor triad is the Terzgleich or SLIDE transformation of C major, in which only the third E is unaltered (and thus shown as an open notehead). A full discussion of mediants follows, in 7.4.

This progression uses a chain of ultradominants in place of dominants; the result is rapid comma-free motion to the “flat” side of pitch space.
7.4.  Mediants and neighbor-mediants

Every principal harmony can be substituted by two chord-pairs of the same type: one pair related by major thirds, and one pair related by minor thirds.
7.4. *Mediants and neighbor-mediant*

In other words: each principal harmony has two mediant and two counter-mediant transformations (to be described in this section).
Major-third relations are called **mediants** when they proceed **forward** from the principal harmony. That is:

- In major, \(\uparrow\) **above** the principal harmony
- In minor, \(\downarrow\) **below** the principal harmony

Major-third relations are called **counter-mediants** when they proceed **backward** from the principal harmony. That is:

- In major, \(\downarrow\) **below** the principal harmony
- In minor, \(\uparrow\) **above** the principal harmony

Formula:

- **Mediant** = Principal third becomes mediant prime [same as variant of Leittonwechsel]
- **Counter-median** = Principal prime becomes mediant third [same as Terzgleich of parallel]

**NOTE:** the filled note-heads indicate pitches that are chromatic alterations of their source-position diatonic counterparts.
The following chart compares Karg-Elert’s description of mediant (major third) relations with that of Riemann’s *Schritte* and *Wechsel* (Riemann 1880), and also with neo-Riemannian triadic transformations (based on Cohn 1997 and Lewin 1987):

<table>
<thead>
<tr>
<th>KARG-ELERT</th>
<th>RIEMANN 1880</th>
<th>NEO-RIEMANNIAN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mediant ((M))</td>
<td><em>Terzschritt</em></td>
<td>LP, or SLIDE + R</td>
</tr>
<tr>
<td>Counter-median ((\bar{M}))</td>
<td><em>Gegenterzschritt</em></td>
<td>PL, or R + SLIDE</td>
</tr>
</tbody>
</table>

In Riemann 1880 (p. 72), these relationships are defined in terms of distance between the chordal roots (*Terz*), and whether or not the chord switches from major to minor (*Schritt* or “step” indicates no change). In contrast, Karg-Elert explains the mediants first in terms of chord tone transformation (“principal third becomes mediant prime”), and then as compound functional alterations (“variant of Leittonwechsel,” which is exactly neo-Riemannian LP). Karg-Elert's transformations are explained in a manner that often directly matches the neo-Riemannian operations, though it is curious that he defines the counter-median as “*Terzgleich* of parallel” (i.e. R + SLIDE) rather than “Leittonwechsel of variant” (PL), which would also keep the same acoustic common tone (principal prime \(\rightarrow\) mediant third).

As before, Karg-Elert uses closed noteheads to indicate variant pitches that acquire a syntonic comma. This example shows that mediants include one variant pitch, while counter-mediants include two.
Minor-third relations are called **neighbor-mediant**s, or “aspiring” mediants of the neighboring principal harmonies.

![Diagram of musical relationships]

- **dur** = major
- **moll** = minor

---

**Parallel** = relative major or minor (Neo-Riemannian $R$)  
**Variant** = parallel major or minor (Neo-Riemannian $P$)  

**NOTE:** the filled note-heads indicate chromatic variant alterations, displaced by a syntonic comma.

---

“**Backward**”:
Parallel-variant in major $\uparrow$ = a minor third below the principal triad  
Parallel-variant in minor $\downarrow$ = a minor third above the principal triad

“**Forward**”:
Variant-parallel in major $\uparrow$ = a minor third above the principal triad  
Variant-parallel in minor $\downarrow$ = a minor third below the principal triad
E♭ major ("Es dur") and A major ("A dur") are the neighbor-mediants (Nebenmedianten) of C major, indicated by the curved arrows. The straight arrows show that the neighbor-mediants are also the “converging” (zustrebende) mediants of the neighboring harmonies: A major (C\textsuperscript{M} in C major) and E♭ major (D\textsubscript{M}) lead respectively from the contrant (F major) and dominant (G major), converging onto the tonic.

Once again, similarities between Karg-Elert’s explanations of the neighbor-mediants and neo-Riemannian compound transformations are readily apparent. He defines the lower neighbor-median of C major as \( T_p \), or the “tonic parallel variant”: CM \( \rightarrow \) Am \( \rightarrow \) AM. This compound transformation is identical to neo-Riemannian RP. The other neighbor-variant (E♭ major) is \( t^p \) or “tonic variant parallel,” which is equivalent to neo-Riemannian PR. The corresponding relationships in Riemann 1880 (p. 74) are the Kleinterzschritt (CM \( \rightarrow \) AM) and Gegenkleinterzschritt (CM \( \rightarrow \) E♭M), which are defined in terms of chordal root distance by minor thirds, rather than as transformations of basic functions.

This example also explicitly matches the central concern of neo-Riemannian theory for voice-leading parsimony (Cohn 1997). The horizontal lines mark the two common tones that link each triad first with its parallel, and then with the succeeding variant (or vice versa). Each pair of horizontal lines equates to an edge on the Tonnetz. Similarly, neo-Riemannian dual transformations such as RP and PR involve a total of two moving voices.
Parallelvariant = parallel-variant
vorwärts = forward
stets = always

Variantenparallel = variant-parallel
rückwärts = backward
7.4. – section B (continued)

Parallelvarianten = neo-Riemannian RP  e.g. CM \rightarrow Am \rightarrow AM

Variantenparallelen = neo-Riemannian PR  e.g. CM \rightarrow Cm \rightarrow \overline{E}kM
7.5. Schematic overview of mediant substitutions, applied to the cadence
ERRATUM: in the $C \rightarrow T$ progression, the final bass note should be C rather than D.

By “the cadence,” Karg-Elert is referring equally to $D \rightarrow T$ and $C \rightarrow T$. All of the mediant substitutions are regarded as manifestations of one of those two basic cadential progressions. Any chord can be substituted by one of its mediants, though in these charts Karg-Elert only substitutes one chord in each cadence.

Lines mark semitonal tendency tone resolutions, and ties mark acoustically exact common tones (prime $\rightarrow$ fifth in $D \rightarrow T$, and fifth $\rightarrow$ prime in $C \rightarrow T$). Mediant substitutions can erase some or even all of these linear connections, as shown in $C^M \rightarrow T$ (AM $\rightarrow$ CM). This suggests that while tendency tone resolutions and common tones are typical of cadential progressions, they are not strictly essential.

Note that some of the dominant ($D$) chords and their mediant substitutions are concordant: that is, they become dominant seventh chords, by generating their natural sevenths. This is indicated by the macron (horizontal line) above or below the function label.

These charts also illustrate how many triads can be interpreted as mediants of different functions, depending on their context. For example: in the key of C major, the E major triad can appear as $T^M$ (tonic mediant) or $D_P$ (dominant-parallel-variant).
What extraordinary richness of close chord connections arises through the introduction of mediant! The key is expanded into a tonality, which remains centralized in spite of the powerful enrichment of chords. In a latent functional sense, the essentially canon boundaries \{ C \leftrightarrow D \} and \{ A \leftrightarrow C \} have not fundamentally been overstepped.
“The key is expanded into a tonality” (Die Tonart weitet sich zur Tonalität) – a pregnant phrase, stated without explanation. It suggests that tonality requires the enrichment of the three basic functions by means of chromatic substitutions and alterations such as mediants. A key is defined simply by the diatonic scale contained within the canonic boundaries of $C \rightarrow T \rightarrow D$; a tonality is a chromatic universe.
7.6. Mediant substitutions of the ultraforms

The ultraforms overstep the tonal boundaries – and so the B-flat major and D major triads (as $\text{CC}$ and $\text{DD}$) make possible a greater linear expansion from the tonal center than can the E-flat, A-flat, D-flat, A, E and B major triads (as mediants!) [See the tabular overview on the next page].

If the ultradominants and ultracontrants generate their centrifugal mediant extremities,
### 7.6. Mediant substitutions of the ultraforms

The ultraforms ($CC$ and $DD$) overstep the tonal boundaries because their **major-third prime cells** B♭–D and D–F♯ contain B♭ and F♯, which do not belong to the diatonic collections (of C major and A minor in this case). Conversely, all of the mediants and counter mediants (such as E♭ major, A♭ major, etc) contain one common tone with the **prime cells** of their corresponding basic functions. In that sense, the ultraforms expand the pitch collection away from the tonal center to a greater extent.

The mediants of the ultraforms introduce the tritones on either side of the tonic:

In C major:

- $CC_M = \text{counter-median of } B♭M = G♭M$
- $DD_M = \text{median of } DM = F♯M$

In A minor:

- $CC_M = \text{counter-median of } B♭m = D♭m$
- $DD_M = \text{median of } Gm = E♭m$
the chord collection expands to the point of enharmonic difference = tritone/counter-tritone.

**ERRATA:**

**Major chords**
- Top row = mediant $\searrow = [18] \downarrow$ lowered
- Middle row = principal chords = normal
- Bottom row = counter-mediants $\nearrow = [18] \uparrow$ raised

**Minor chords**
- Top row = counter-mediants $\nearrow = [18] \uparrow$ raised
- Middle row = principal chords = normal
- Bottom row = mediants $\searrow = [18] \downarrow$ lowered

\( \nearrow \) = mediants; \( \searrow \) = neighbor-mediants; \( \parallel \) = variants; \( \frac{2}{3} \) = Terzgleich [Mediant-parallel]
ERRATA: the two function symbols at the top right of this diagram should be exchanged. In the key of A minor: $EC^M$ is $D^\#$ minor ("dis"), and $Ec$ is B major ("H").

While this diagram somewhat resembles a Tonnetz, it is strictly a map of triadic relations, not a map of acoustic pitch space. For once, Karg-Elert provides a detailed explanation of the various lines, but not of the brackets. The 18 triads in the inner brackets are those that contain at least one pitch from one of the principal chords’ prime cells: F or A from the contrant, C or E from the tonic, G or B from the dominant. Though Bm and BbM also contain pitches from one of the prime cells, they also contain non-diatonic pitches: respectively D-F$^\#$ and Bb-D. As they are direct mediant transformations of the dominant, they are placed within the outer bracket. The triads that are located completely outside of the brackets are the aforementioned ultraforms and their mediant transformations.

Finally, it should be noted that these triadic maps include at least one of each chromatic major and minor triad, in equal tempered enharmonic space. All of the triads on the C major map (left side) are entirely contained within this Tonnetz segment, encompassing six fifths and four thirds:

\[
\begin{array}{cccccc}
F^\# & C^\# & G^\# & D^\# & A^\# \\
D & A & E & B & F^\# & C^\#
\end{array}
\]

\[
\begin{array}{cccccc}
Bb & F & C & G & D & A \\
Gb & Db & Ab & Eb & Bb & F
\end{array}
\]
It should be clear that $F\sharp/\text{Gb}$ or $D\sharp/E\flat$ do not differ by a Pythagorean comma, as the distance of 12 fifths has not been traversed.

The **minor diesis** is the difference between three *major* thirds and the octave:

For example, $\text{Gb}_{(-2,1)} = \text{Prime of } \mathcal{C} \mathcal{C}_M$

$F\sharp_{(2,2)} = \text{Third of } \mathcal{C}_P$

\[
\begin{align*}
508 & \quad (= 1000 \text{ minus } 492) \\
474 & \quad (644 \text{ minus } 170)
\end{align*}
\]

Difference: $34 \mu = \text{minor diesis}$

Alternatively, $F\sharp$ is the third of $\mathcal{D} \mathcal{D}$, which is also the prime of $\mathcal{D} \mathcal{D}^M$:

\[
\begin{align*}
508 & \quad (= 1000 \text{ minus } 492) \\
492
\end{align*}
\]

Difference $= 16 \mu = \text{Diaschisma (4 fifths + 2 thirds)}
The two pitches compared in this calculation (G♭ = 508 µ and F♯ = 474 µ) are highlighted on the Tonnetz segment shown above.

See section 5.7 on the minor diesis.
7.7. Comparison of the 20 triads in the mediant system with the same principals
7.7. *Comparison of the 20 triads in the mediant system with the same principals*

These are the 20 mediant transformations possible in C major (top two systems) and A minor (lower two systems). The triads on this chart are ordered semitonally (ascending from C major, then descending in A minor). Each chord is notated in a specific octave, in relation to the *Ursprungslagen*. As with the triadic maps in 7.6., there is at least one triad built on each note of the 12-note chromatic scale. The only triads that belong to the chain of canonic fifths (i.e. that are in the *Ursprungslagen*) are the principal triads (*T, C, D*) and the ultraforms (*BD* and *EC*). All mediants and counter-mediants contain one or two variant pitches that are either raised (↑) or lowered (↓) by a syntonic comma (18 µ).
7.8. **Functional values of mediant and counter-median major triads in minor, and minor triads in major**

Terzgleicher der Principale = Terzgleicher (“common-third chords”) of the principal harmonies

Varianten der Principale = mode-changing variants of the principal harmonies

Contrant-median-parallel (C\textsubscript{Mp} or C\textsubscript{Mp}) = “tritonal variant” of the tonic.

Ultradominant-median-parallel = “Leittonggleicher” (same leading tone):

B is the leading tone of C major, and also of the D\# minor triad.
7.8. **Functional values of mediant and counter-median major triads in minor, and minor triads in major**

Until now, Karg-Elert has discussed only mediant major triads in major keys, and minor mediant triads in minor keys. This example lists triads of the opposite mode that are generated as mediants and counter-mediants.

These mediants are described in two ways: first as compound transformations of a principal harmony, and then as unitary chordal transformations. For example: in C major, C♯m is labeled as $T^{Mp}$ (“tonic mediant parallel,” equivalent to neo-Riemannian LPR), but is also explained as the *Terzgleich* of the tonic (“same third,” i.e. SLIDE of CM).

It may seem surprising that Karg-Elert discusses the simple *Variant* (neo-Riemannian P) in the context of mediant transformations. Indeed, the change to lowercase in the function labels does not explicitly reflect a mediant relationship. However, mode-changing variants are acoustically equivalent to the *Leittonwechsel* of the counter-median (applied to the same function). For example, in C major:

Tonic variant (t): $CM \rightarrow Cm$

*Leittonwechsel* of the tonic counter-median ($T_{M^L}$): $CM \rightarrow A♭M \rightarrow Cm$

In that respect, variants relate to the mediant transformations.

By “tritonal variant,” Karg-Elert means the triad of opposite mode that is a tritone away from the tonic.
In C major: contrant (FM) $\rightarrow$ mediant (AM) $\rightarrow$ parallel (F♯m) = tritonal variant of C major.

Ultradominant-median-parallel is $DD^{Mp}$.
In C major: ultradominant (DM) $\rightarrow$ mediant (F♯M) $\rightarrow$ parallel (D♯m).

$B_{(1,1)}$ is indeed the leading tone in C major, but also in D♯ minor, whose (dual) prime is A♯. Thankfully, the concept of *Leittongleicher* rarely appears in Karg-Elert’s writings, as it is difficult to mentally connect two chords that share a *non-present* leading tone (neither the C major nor the D♯ minor triad actually contains its shared leading tone B).
Dominant-variants = “Church dominants”

\[
d = \text{[in major]} \text{ Mixolydian dominant (variant of regular } D) \\
\mathcal{A} = \text{[in minor]} \text{ Dorian dominant (variant of regular } G) \\
\]

Mixolydian: ↓7
Dorian: ↑6

In contrast, the “defining characteristic” pitches in Lydian and Phrygian are best understood as the thirds of the ultradominants [“diatonic” in nature, and thus not belonging to this category \{of variants\}):

Lydian: ↑4
Phrygian: ↓2
7.8. (continued)

Once again, simple *Varianten* (i.e. neo-Riemannian P) fall under the umbrella of mediant transformations, because variant = *Leittonwechsel* of counter-mediant. In C major:

\[ G \text{ minor is } d \text{ (dominant-variant), but also } D_M^\downarrow: GM \rightarrow EbM \rightarrow Gm \]

This example describes how the dominant variants can be regarded as Mixolydian and Dorian modal inflections.

The characteristic scale degrees of Lydian (raised 4) and Phrygian (lowered 2) are derived from the ultraforms \( (BB\) and \( CC \) respectively), rather than from variants of principal triads.
ADDENDUM to 7.8: counter-median-variants

In Chapter 7, Karg-Elert does not include an additional category of mediant transformations: the Gegenmedianten-Varianten, or counter-median-variants. They never actually appear in the main body of the text, but they are listed in Appendix B as follows:

C major:

\[
\begin{array}{c}
\text{C major:} \\
C_m & T_m & D_m
\end{array}
\]

A minor:

\[
\begin{array}{c}
\text{A minor:} \\
C_m & T_m & D_m
\end{array}
\]

They are variants of the Gegenmedianten (counter-mediants): their thirds are raised in major, or lowered in minor, thus changing the original triad’s quality.

Karg-Elert introduces \( T_m \) in section 11.6, where it is classified as a Kollektivwechselklang or “collective-change chord”: all voices move by semitone away from the source chord, as in

\[
\text{C major } \rightarrow \text{ G}\sharp \text{ minor } \quad \text{ or } \quad \text{C major } (T) \rightarrow \text{ A}\flat \text{ minor } (T_m)
\]

See section 11.6 for more on the collective-change chords.

The Gegenmedianten-Variant transformation is equal to Riemann’s Gegenkleinterzwechsel (Riemann 1880, 75). It creates what Richard Cohn called the hexatonic pole (Cohn 1996) – two triads of different qualities (major and minor), which combine to create a (014589) hexatonic collection. In terms of voice leading, all pitches move by a semitone (i.e. “collective change”). Hexatonic poles are created by either neo-Riemannian PLP or LPL, in 12-tone equal temperament. For example:

PLP: \( \text{CM } \rightarrow \text{ Cm } \rightarrow \text{ AbM } \rightarrow \text{ Abm} \) \quad \text{Am } \rightarrow \text{ AM } \rightarrow \text{ C}\sharp \text{m } \rightarrow \text{ C}\sharp \text{M} \\
LPL: \( \text{CM } \rightarrow \text{ Em } \rightarrow \text{ EM } \rightarrow \text{ G}\##m \) \quad \text{Am } \rightarrow \text{ FM } \rightarrow \text{ Fm } \rightarrow \text{ D}\flat \text{M}

In just intonation, these transformations are of course not equivalent. In C major, A\flat minor is the tonic counter-median-variant (\( T_m \)). Conversely, G\# minor would be the tonic mediant-Leittonwechsel (\( T_{ML} \)), or more likely the dominant mediant-parallel (\( D_{MP} \)). Therefore, Karg-Elert’s Gegenmedianten-Variant is equivalent to neo-Riemannian PLP, but not LPL.
Chapter 7: Tonnetze

7.1. Comma-free (Pythagorean) modulation

Comma-free modulation from D minor to B minor
(Top staff: Ursprungslage analysis)

This modulation is **comma-free**, as both the original D minor tonic (chord 2) and the final B minor tonic (chord 8) are in their Ursprungslagen.

The variant chords (1, 3 and 7, which contain syntonic-comma variant pitches) are not in the Ursprungslagen, and deviate from the central horizontal plane on the Tonnetz.
7.2. **Comma-differing (syntonic) modulation**

Comma-differing modulations (shifts) from D minor to B minor, and vice versa

Starting in:

<table>
<thead>
<tr>
<th>D minor</th>
<th>B minor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The chords rotate around the common tones: \( D_{(2,1)} \) in the D minor passage, \( D_{(2,0)} \) in the B minor passage.

These are both **comma-differing** modulations, as the final chord in each passage is a variant, containing one or more pitches that differ from their canonic counterparts. In other words: the final chords are not in their *Ursprungslage*.

Another comma-differing modulation from D minor to B minor:

207

1234

Tonnetz chords:

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( D_{(2,1)} \) \( A_{(2,0)} \) \( E_{(1,0)} \) \( B_{(1,0)} \)
7.2. (continued)
Comma-differing modulation from B♭ minor to C# major:

Tonnetz chords:
1  2  3  4  5  6  7

The above is a “twisted and agonized chromaticization” of the following:

Tonnetz of the modulation, matching the Ursprungsland analysis:

Ursprungsland analysis of the chromaticization:
7.3. Further examples of comma-free and comma-differing modulation

Diatonic \[7 \frac{1}{2}\text{ fifth-steps} \uparrow\]:
D minor to F# major (via B minor), comma-free.

Chromatic:
D minor to F# major

Tonnetz for the diatonic (comma-free) modulation:

Both tonics (chords 1 and 10) are in their Ursprungslagen, so the modulation is comma-free.

Tonnetz for the chromatic (comma-differing) modulation:

The final F# major tonic (chord 4) is located two horizontal levels above the original tonic (chord 1), reflecting the double syntonic comma difference.
Diatonic [9 \( \frac{1}{2} \) fifth-steps ↑]:
D-flat major to C# minor (via B-flat major and G major), comma-free

As noted in the text, this modulation actually travels 8\( \frac{1}{2} \) steps rather than 9\( \frac{1}{2} \). Those steps are “up” in the Ursprungslage (gradual increase in register), and to the right on the Tonnetz. As both tonics (chords 2 and 12) are in their source positions, the modulation is comma-free (kommarein).  
7.3. (continued)

D-flat major to C# minor, shortened by means of variants (i.e. comma-differing):

Basic: d B♭ (F d) a C G C

Tonnetz chords: 1 2 3 4 5 6 7 (6)

Comma-free from A minor to A-flat minor (by means of ultradominants) = 7 fifth-steps ↓

Tonnetz chords: 1 2 3 4 5 6 7
7.4. Mediants and neighbor-mediants

Top system: C major
Bottom system: A minor

C major:
Top layer: mediants
Middle layer: principal harmonies
Bottom layer: counter-mediants

A minor:
Top layer: counter-mediants
Middle layer: principal harmonies
Bottom layer: mediants
7.4. (continued)

Top system: C major
Bottom system: A minor

NOTE: the Tonnetze for section 7.4 also apply to the example in 7.5.
7.6. **Mediant substitutions of the ultraforms**

C major:  

A minor:  

These Tonnetze display the mediant and variant substitutions of the ultraforms (DD and CC), boxed on the outside corners of the diagram above. All of the triads listed in the above diagrams can be found on these Tonnetze, or on those for section 7.4.
7.8. **Functional values of mediant and counter-median major triads in minor, and minor triads in major**

Top system: C major  
Bottom system: A minor

As discussed in the text, the mediant-parallels ($^M_P$) are also Terzgleicher of the principal harmonies. The variants are also Leittonwechsel of the counter-median ($^M_L$).
Chapter 8

Metharmonics and enharmonics
Chapter 8
Metharmonics and enharmonics

8.1. Metharmonics and enharmonics
8.2. The problematic (i.e. diminished) triad as substitute for $D_e$ and $C_C$
8.3. Metharmonics in the diatonic sequence
8.4. Metharmonic equivalence in the harmonic cadence
8.5. Metharmonics and the reinterpretation of variants and mediants
8.6. Acoustic comparison of metharmonically equivalent pitches
8.7. Apparent and true enharmonics
8.8. Enharmonics and metharmonics in chromatic sequences
8.9. Chromatic sequences and tonal closure
8.1. Metharmonics and enharmonics

Under these headings, we refer to the conceptual equalization of comma-differing tones or chords that have the same name [metharmonic], or the equalization of enharmonically-differing tones or chords [i.e. with different names].

Ursprungslagen der Prinzipe = source positions of the principal triads
Ursprungslagen der Varianten (Medianten) = source positions of the variants (mediants)

Metharmonics:

\[ \text{about } \pm 18 \mu \text{ lower than the (corresponding) principal harmony} \]

\[ \text{about } \pm 18 \mu \text{ higher than the (corresponding) principal harmony} \]
8.1. Metharmonics and enharmonics

The term Metharmose (metharmonic) is borrowed from Arthur von Oettingen’s Das duale Harmoniesystem (Oettingen 1913, 51). Metharmonics are pitch pairs have exactly the same name, but different acoustic derivation: for example, D\(_{(2,0)}\) and D\(_{(2,-1)}\). Metharmonics will always differ by one or more syntonic commas (except possibly when note spelling does not accurately reflect acoustic derivation).

Enharmonics (Enharmose) are not simply pitch pairs that are similar but with different names. In Oettingen’s and Karg-Elert’s usage, enharmonics are similar pitch pairs that have different acoustic derivations, as well as different names. Enharmonics will usually differ by one or more syntonic commas, though that is not always the case – for example, G\(\#_{(8,0)}\) and A\(b_{(-4,0)}\), which are both canonic (Ursprungsage) pitches.

This example includes several metharmonic pitch pairs. On the upper system, all pitches and chords are principal harmonies in A minor/C major, in their Ursprungsagen. On the lower system, all of the dominant-function chords (first three in A minor, last three in C major) have been transformed as variants or mediants (see 7.4). As in previous examples, the black noteheads indicate variant pitches, which differ from their Ursprungsage equivalents by a syntonic comma (18 µ).

The dotted diagonals link metharmonic pitch pairs (variants first, then Ursprungsage pitches):

B\(_{(3,2)}\) and B\(_{(1,1)}\)  \hspace{2cm} F\(#_{(-2,2)}\) and F\(#_{(2,1)}\)

B\(_{b_{(2,-1)}}\) and B\(_{b_{(2,0)}}\)  \hspace{2cm} F\(_{(3,-1)}\) and F\(_{(-1,0)}\)

The horizontal double lines indicate Variant chord transformations (i.e. variant thirds), while the horizontal dotted lines indicate Terzgleich transformations (variant primes and fifths).
8.2. The problematic (i.e. diminished) triad as substitute for $D^\flat$ and $\acute{C}$

The problematic (diminished) triad $\{B \ D \ F\}$ is a dissonant substitute for both the B-flat major triad and the B minor triad.

The minor thirds $\{B : D\}$ and $\{D : F\}$ are to be understood as $6/5$ and $5/6$ respectively. They invoke both their lower relative primes as over-harmonies (i.e. major triads) and their upper relative primes as under-harmonies (minor triads), which they “summon” in double form.

The forms marked $\{B\}$ will be discussed in a later chapter (“concordant types”).
8.2. *The problematic (i.e. diminished) triad as substitute for $D^7$ and $CC$*

First half in C major, second half in A minor. In both keys, the [B, D, F] triad presents a problem: is the pitch D identical with that in the preceding $Cp$ chord, or in the following $D$ chord? In both keys, those two Ds are distinct: in C major, $Cp$ contains $D_{(2,1)}$ while $D$ contains $D_{(2,0)}$. Therefore, in order to remain in the same key, two comma-different (i.e. metharmonic) Ds are necessary.

Therein lies the “problematic” nature of the diminished triad: what is the acoustic status of its third? By inference, we might consider it as a metharmonic pivot, linking the comma-different Ds in the $Cp$ and $D$ chords.

The diminished triad is called “dissonant” not only because of its diminished fifth interval, but more importantly because it combines pitches from two different functions: $D^7$ and $CC$ (in both C major and A minor).

German pitch nomenclature:

| H = B-natural | Fis = F♯ | B = B♭ |

See chapter 9 on pure-seventh concordant pitch relationships.
8.3. Metharmonics in the diatonic sequence

\[\text{Metharmose in der diativen Sequenz}\]

\[\text{Schematischer Satz} = \text{schematic pattern}\]
\[\text{Ursprungslagen} = \text{source positions}\]

\(\text{„} \) indicates lowering by a semitone.
\(\text{—} \) indicates raising by a semitone.

\(\text{Statt fis = instead of } F^\#\)
\(\text{Statt b = instead of } B_b\)

\(\text{(Met.) = metharmonic shift between comma-differing tones having the same name}\)

\(\text{and } \underline{\text{—} \text{—}} \underline{\text{—} \text{—}} \underline{\text{—} \text{—}} \underline{\text{—} \text{—}} \) = metharmonic pitch pairs, differing by a syntonic comma of 18 µ.
8.3. **Metharmonics in the diatonic sequence**

In this C major example, the acoustic status of all pitches is clear, due to their harmonic context. In bar 1 (beats 3 and 4), the two Ds are metharmonically distinct, as discussed in 8.2.: \( D_{-2,1} \) and \( D_{2,0} \). The diminished triad (beat 4) is a chromatic variant of B minor, which is (C) of \( \text{Dp} \). The downward “hook” at the top of the (C) function label indicates that its fifth has been lowered by a chromatic semitone (\( F \sharp \rightarrow F \)). The upward hook at the bottom of the (C) label in bar 4 indicates that its prime has been raised by a chromatic semitone (\( B \flat \rightarrow B \)). This hook notation for chromatically raised chord tones is presented more fully in Chapter 11 (see 11.2).

As in 8.2, the dotted diagonals mark the metharmonic pitch pairs, which are:

\[
D_{-2,1} \text{ and } D_{2,0} \quad F_{-1,0} \text{ and } F_{3,-1} \quad A_{-2,1} \text{ and } A_{3,0} \quad B_{1,1} \text{ and } B_{3,2}
\]

This is the polar equivalent of the previous example, in A minor.

The metharmonic pitch pairs in this passage are:

\[
B_{1,1} \text{ and } B_{3,-2} \quad D_{2,0} \text{ and } D_{-2,1} \quad G_{1,0} \text{ and } G_{-3,1} \quad F_{3,-1} \text{ and } F_{-1,0}
\]
8.4. Metharmonic equivalence in the harmonic cadence

More crucial than the linkage of all identically-named tones [G G, B B, D D, F F, A A] is the harmonic purity of closed cadential groups. These metharmonics break the continuity of a uniformly directed chord sequence at moments of definite repose, but they do not alter the actual direction of the cadence itself.

The problematic sonorities $B \underline{D} \ F\sharp$  
$B_\flat \ D \ F$  
and the D-minor triad as substitute for the D-major triad, or G-major triad as substitute for the G-minor triad now become comprehensible.

These metharmonics always reveal themselves as if “metaphorically speaking,” when open or infinite chord spirals become closed, backward-tracing chord rings.
Essentially, Karg-Elert is stating that while metharmonic shifts between pitches of the same name (such as those illustrated in 8.2 and 8.3) might break a uniform trajectory through the *Ursprungslagen* (i.e. in a consistent “flatward” or “sharpward” direction), they do not disturb the tonal unity of a passage. In fact, to ensure tonal unity and closure (what Karg-Elert calls chord “rings” instead of “spirals”), metharmonic shifts are inevitable and indispensable.
8.5. Metharmonics and the reinterpretation of variants and mediants

Metharmonics always appear when fifth- and third-types are interchanged, and thus whenever variants or mediants are reinterpreted as principal harmonies:

\[
\begin{align*}
\text{E} & \quad \text{G} \\
\text{B} & \quad \text{C}
\end{align*}
\]

The above passage includes a comma-free modulation \textit{first seven chords} from C major to the key of the 4th upper-fifth = E major \textit{at NB1}:

\[
\begin{align*}
\text{E}_{(4,0)} & \quad \text{G}_#_{(4,1)} & \quad \text{B}_{(5,0)}
\end{align*}
\]

If this E major triad generates its own counter-median \textit{the C major triad at NB2}, \( E_{(4,0)} \) will naturally be held in common:

\[
\begin{align*}
\text{E}_{(4,0)} & \quad \text{G}_#_{(4,1)} & \quad \text{B}_{(5,0)} \\
\text{C}_{(4,1)} & \quad \text{E}_{(4,0)} & \quad \text{G}_{(5,1)}
\end{align*}
\]

This C major triad is the \textit{Terzgleich} (common-third) of C# minor, and is a syntonic comma \((18 \mu)\) higher than the initial chord.

If such a short passage would be slowly repeated four times (in chorale style) by a choir singing purely syntonically \textit{without re-adjusting to the original tuning at the beginning of each repeat!}, the result would be an ample elevation of about a half-tone at the fifth appearance of the C-major triad! \( [5 \times 18 \mu = 90 \mu] \)
8.5. Metharmonics and the reinterpretation of variants and mediants

“When fifth- and third-types are interchanged” – here Karg-Elert refers to pivot chords in modulations, specifically to the reinterpretation of a variant or mediant (“third-type”) as a principal harmony (“fifth-type”). In the example, beat 3 of bar 2 is such a pivot chord.

ERRATUM: bar 2 (beat 2), the bass note should be B (root of the dominant in E major), not A.

Bar 3, beat 1: the E is indeed an acoustic common tone (from E major), but now the C and G are a syntonic comma different from the opening chord: C(4,1) and G(5,1) instead of C(0) and G(1,0).

After four repetitions of mm. 2-3, the C major chord in m. 3 would have these pitches:
\[
\begin{align*}
& C_{(16,-4)} \quad E_{(16,-3)} \quad G_{(17,-4)} \\
& 69 \mu \quad 412 \mu \quad 675 \mu
\end{align*}
\]

…which is very close to the following D-flat major chord (C\(_M\) in the key of C major):
\[
\begin{align*}
& D_{b(1,-1)} \quad F_{(1,0)} \quad A_{b(0,-1)} \\
& 93 \mu \quad 413 \mu \quad 678 \mu
\end{align*}
\]
In reverse, a quadruple repetition of the following results in a sinking of about a large semitone:

\[
\begin{align*}
4 \times \text{zu wiederholen} &= \text{repeat four times}
\end{align*}
\]

This is a comma-free modulation from C major to F minor: \( F_{(5,1)} \rightarrow A_{(4,0)} \rightarrow C_{(4,1)} \). If the F minor triad generates its own contrant variant (i.e. the final C major triad), it can be understood as nothing other than \( C_{(4,1)} \rightarrow E_{(4,2)} \rightarrow G_{(3,1)} \). This chord is a syntonic comma (18 μ) lower than the opening.

Hence, the fifth C-major triad (after four repetitions) is lowered by \( 5 \times 18 \mu = 90 \mu \).

In musical practice, many directors of first-rate a cappella choirs are sufficiently aware of the fact of such dis- and de-tuning, sometimes not to be avoided. The root of such tuning fluctuations is [as in the above examples] choral and technical respect for compositionally intrinsic comma differences!
In a performance that strictly observes syntonic tuning, the final C major chord (after four repetitions) would contain these pitches:

\[
\begin{align*}
C_{(-16,4)} & \quad E_{(-16,5)} & \quad G_{(-15,4)} \\
910 \mu & \quad 232 \mu & \quad 495 \mu
\end{align*}
\]

… which is very close to the following B major triad (\(D^M\) in the key of C major):

\[
\begin{align*}
B_{(1,1)} & \quad D^\#_{(1,2)} & \quad F^\#_{(2,1)} \\
907 \mu & \quad 229 \mu & \quad 492 \mu
\end{align*}
\]
Another example of comma-free modulation, and of metharmonically-related pitches:

Comma-free: C maj $\rightarrow$ C min  
Double metharmonics  
Comma-free: B maj $\rightarrow$ C maj

A)  $G_{(3,1)} (D_{(3,2)} D_{(2,1)})$  mediant of Eb major = variant of G minor

B)  $B_{(3,2)} (D\#_{(3,3)} F\#_{(2,2)})$  ultramediant of Eb major = chromonant of Bb major

Chromonant = upper or lower chromatic semitonal transformation of an Ursprungslage chord

From the double-comma-charged B major, the progression takes the canonic road back to a C major triad, one that inherits both of the lowering commas as a legacy from the B major triad…

Von diesem doppelt komma belasteten H-Durakk aus führt der kanonische Weg nach einem C-Dur-Akk., der die beiden Senkungskommate vom H-Durakk, als Erbe übernimmt…

Ausgangstonart = opening key of C major
Endakkord = final chord of C major

$C(0)$  $E(0,1)$  $G(1,0)$  
$C(8,2)$  $E(8,3)$  $G(7,2)$

Difference: a double comma ($\frac{36}{\mu}$ = about a fifth of a whole tone)
8.5. (continued)

Bars 1-2 (first 7 chords): the modulation is comma-free. The pivot chord (m. 1 beat 3) does not incur a syntonic comma, as it does not introduce any variant (metharmonic) pitches.

In contrast, the middle section introduces two syntonic commas, due to the two variant reinterpretations.

Just as an ultradominant is a double dominant (DD), an ultramediant is a double mediant. In the key of E♭ major, the ultramediant (TMM) is B major.

Chromonant is Karg-Elert’s term for a triad in which all three pitches are variants: a chromatic semitone above or below the pitches of an Ursprungslage harmony. In this example:

B major: \[B_{(-3,2)} \ D_{(-3,3)}^{#} \ F_{(-2,2)}^{#}\] is a chromonant substitute for

B♭ major: \[B_{♭(-2,0)} \ D_{(-2,1)}^{♭} \ F_{(-1,0)}^{♭}\] which is in the Ursprungslage.

The last two bars of the example above (last eight chords) themselves provide a comma-free modulation from B major to C major. However, they do not erase the two syntonic commas previously acquired. This diagram is very close to a Tonnetz, graphically illustrating the chord-to-chord progress through pitch space.
8.6. Acoustic comparison of metharmonically equivalent pitches

4 metharmonically differing F-sharps, and 4 metharmonically differing G-flats:

The 8 differing values [all identical in equal temperament], compared to each other:

Synt. = Syntonic
Pyth. = Pythagorean
Kl. = small (minor)
Gr. = large (major)
Doppelkomma = double syntonic comma
cia. Halbtondifferenzen = differences of almost a semitone
8.6. *Acoustic comparison of metharmonically equivalent pitches*

Each system begins with the canonic (*Ursprungslage*) $F\sharp$ or $G_b$, and then proceeds to their metharmonics with one, two and three syntonic commas.

The key signatures reflect the keys in which the specified pitches are most likely to occur, and the function labels indicate the chords that will most likely contain those pitches in each key. All of the keys are major (as indicated by the regular function labels), except for the last two bars of the first system, which are in G minor and E-flat minor respectively (see the inverted function labels).

This chart specifies the intervals between the metharmonic versions of $F\sharp$ and $G_b$ just described. Not surprisingly, the largest intervals of 70 and 88 µ (which are about a semitone) involve the pitches with three syntonic commas: $F\sharp$ at $456\,\mu$ (in the key of $G_b$ major!), and $G_b$ at $544\,\mu$ (in the key of $F\#$ major!).
8.7. Apparent and true enharmonics

An “apparent” enharmonic is one whose notation is chosen simply for reasons of more convenient legibility; conceptually, a fundamental revaluation of the true values does not occur.

(The examples are of a schematic type; the mediantic excursions should be imagined as large groups [alternative measures, or the like]...)

Counter-mediantic episode (insertion)  Mediantic episode (insertion)

In C Dur weiter = continues in C major
8.7. *Apparent and true enharmonics*

As a reminder: for Karg-Elert, true enharmonics are pitches of different names that also have different acoustic derivation (usually differing by one or more syntonic commas). Therefore, enharmonics are only “apparent” if they are purely a notational convenience, substituting for an acoustically correct note spelling (often one involving unusual accidentals such as double flats or sharps).

These first two examples (both in C major) do not involve any enharmonics (true or apparent)! They simply prepare for the next part of 8.7, which transposes the same two passages to D-flat major and B major.
The notation [of the previous two examples] is intelligible enough. But if the first example was in D-flat major (first example on this page) instead of C major, the central episode would shift to the counter-median ant B♭ major, and later return (once again mediantly) to the principal key of D-flat major. Instead of the logical but unusable notation in B♭ major, its enharmonic equivalent (A major) is usually employed. However, actual enharmonics do not occur; the functional analysis clearly indicates the meaning of the B♭ major insertion.

The inverse is found when the second example is transposed to B major (second example on this page). C major : E major corresponds to B major : D♯ major. In a harmonologic sense, E♭ major is out of the question; this extrinsic enharmonic exchange results has a very different motivation – to visually clarify an unusual notation.

\[
\begin{align*}
\text{B♭ major: Notated in A major} & \quad \text{D♯ major: Notated in E-flat major} \\
\text{ Functions:} & \quad \text{B♭, E♭, F♭, B♭ chords} \quad \text{D♯, A♯, G♯ chords}
\end{align*}
\]
The counter-mediant ($T_M$) in D-flat major is B♭ major, with the following pitches:

$$\text{B♭}_5 \text{ D}_5 \text{ F}_5$$

The notation in A major is simply for convenience and legibility; it does not introduce an acoustic shift such as a syntonic comma, and is therefore only an apparent enharmonic.

In a similar way, the second passage modulates from B major to its mediant ($T_M$) D♯ major, which contains these pitches:  

$$\text{D♯}_5 \text{ F♯}_5 \text{ A♯}_5$$

Once again, there is no true enharmonic; the notation in E♭ major (instead of D♯ major) is purely for convenience.
The A major chord in D-flat major and the Eb major chord in B major are not always necessarily enharmonic substitutes for the chords of B♭ major and D♯ major. They can appear in a very plausible manner as $T^{MM}$ and $T_{MM}$ respectively [see the following], which certainly do not occur in the preceding examples.
8.7. (continued)

$T^{MM}$ and $T_{MM}$ are ultramediant transformations. There is no enharmonic notation here; the spelling of each chord exactly reflects its acoustic derivation and harmonic function.

In spite of its heading (provided by Karg-Elert), this entire section does not provide a single example of a true enharmonic! One is presented in section 8.8.
8.8. **Enharmonics and metharmonics in chromatic sequences**

If the above phrase is repeated at \( \Phi \), a real enharmonic is revealed; that is, the chord of D\( \overline{b} \) major would be equated with a chord of C\( \overline{b} \) major (i.e. as D of the opening chord), but with the addition of a “large diesis” \([\text{52}]\). Instead, the above piece continues as follows [next page]:

---

**ERRATA:**

\[ \text{Offensichtliche Nebenmediantensequenz in klarer Funktionsbezeichnung} = \text{a straightforward neighbor-median sequence, with clear functional analysis} \]

**Schwerzeitige Stationen = strong-beat “stations” (points of arrival)**

\[ \text{Triff bei } \Phi \text{ Wiederholung der Phrase eine solitie effektive Enharmose vor, d.h. der } \text{D}\overline{b}\text{-Durakkord wird mit einem } \text{C}\overline{b}\text{-Durklong (als } D \text{ des Anfangsakkordes) gleichgesetzt; es tritt mithin Kassierung der } \text{großen Diesis } \] \[\text{[52]}\] \text{ein. Verläuft dagegen obiges Stück in folgender Weise, :}

---

**TN**
8.8. Enharmonics and metharmonics in chromatic sequences

This passage is almost identical to bars 5-8 from Karg-Elert’s “Adagio, alla Bruckner” (No. 22 from 33 Portraits for harmonium, op. 101, published by C. F. Peters in 1924).

ERRATA: in bar 1 (beat 2), the dominant (D) chord should contain E§ instead of E.
In bar 4 (beat 2), the bass note should be F instead of E.

The two chords in question are true enharmonics of each other (different spelling and acoustic derivation, in this case differing by four syntonic commas):

- C♯ major chord (bar 1, beat 2):
  - C♯(7,0) E♯(7,1) G♯(8,0)
  - [95 µ 435 µ 680 µ]

- Db major chord (bar 4, beat 3):
  - Db(11,-4) F(11,-3) Ab(12,-4)
  - [147 µ 487 µ 732 µ]

If the passage was repeated, the opening F♯ major triad would be understood enharmonically as G♭ major, with these acoustic values:

- G♭(10,-4) B♭(10,-3) Db(11,-4)
  - [562 µ 902 µ 147 µ]
The passage does return to F# major, but it is a comma-lower ed F# major (differing by 18 µ).

The aforementioned enharmonic (large diesis = 52↑) is eliminated…

…at the return; instead, the lesser difference of 18µ↑ arises, and therefore a metharmonic:

ERRATUM:

F♯(6,0) major  Gb♯(8,4) major  F♯(10,-1) major

Enharmose = enharmonics
Metharmose = metharmonics
8.8. (continued)

This continuation is different from that in Karg-Elert’s op. 101, no. 22. The harmonic progression here eliminates the enharmonic difference between the Db and C♯ triads, though the return to F♯ major still differs from the original opening by one syntonic comma (18 µ).

This diagram summarizes the acoustic journey of the preceding two passages:

- Opening tonic = F♯(6,0)  510 µ
- Tonic if repeated = Gb(10,-4)  562 µ
- Final tonic = F♯(10,-1)  528 µ  (ERRATUM: this should have 10 dots, not 12)

The two F♯s are metharmonics, and both are enharmonic with the Gb(10,-4).
8.9. Chromatic sequences and tonal closure

All “mechanically” repeating sequences of mediantic or variant type lead to tonal infinity [in terms of inherent comma displacement]. If tonal unity [i.e. a centralized tonal circle] is desired, effective enharmonics are indispensable. In such cases, the point of ligature is marked by a “diaschismatic (16)” or “minor diesic (34)” step.

Example 8.9.1: medianten

Upper voice: G A B C♯ D♯ E♭ F G
(1,0) (-1,1) (1,1) (-1,2) (1,2) (1,-1) (-1,0) (1,0)

Mediantan = mediants
Tonalitätische Entgleisung = tonal “side-steps”
3 gr. Terzen = three major thirds
Auf G-[es] bezogen = beginning from G-flat (as prime = 0)
Diff. gegenüber Anfang = difference in comparison to the opening
Oberstimme = upper voice
Summa = sum (of the large and small whole tones)
8.9. Chromatic sequences and tonal closure

In other words: if a mediant transformation is applied repeatedly, the original chord will never recur, without “effective enharmonics” = true enharmonics that effect tonal closure.

Example 8.9.1: mediants

The first example illustrates how a repeated mediant sequence will not create tonal closure: it proceeds from G♭ major to its ultra-ultramediant (TMMM) F♯ major. The spelling of those two enharmonic keys reflects the lack of tonal closure.

The second example is like the first, but in C major. However, the passage is now understood as tonally closed: the initial and final C major tonics are acoustically identical, and therefore the two intervening mediants must be interpreted as TM and TM, instead of TM and TMM. The enharmonic shift occurs in bar 2 (beat 3): while the sequential repetition of bar 1 would suggest a major triad on D♯(1,2) = 229 µ, tonal closure requires a major triad on E♭(1,-1) = 263 µ. These are true enharmonics, differing by a minor diesis (34 µ), which effect the tonal return.
Example 8.9.2: neighbor mediants

Neben-Medianten = neighbor mediants (i.e. minor thirds)
Tonalitätische Entgleisung = tonal "side-slips"
4 kl. Terzen = four minor thirds
Auf fis bezogen = beginning from F# (as prime = 0)
Auf C-Dur bezogen = beginning in C major
Diff. gegenüber Anfang = difference in comparison to the opening
Summa = sum
4 Quinten = four perfect fifths
Tonalitätisch geschlossen = tonally closed (i.e. ending in exactly the same key as the beginning)
Example 8.9.2: neighbor mediants

As in example 8.9.1, the first passage is tonally open, modulating from F♯ major to G♭ major. This time, the modulation is by a sequence of neighbor mediants (minor thirds). The spelling of each chord reflects its harmonic analysis, and the lack of tonal closure.

The second passage is the same neighbor-median sequence, but now in a tonally-closed C major. The note spellings suggest that the enharmonic shift happens in bar 4, which is notated in A major instead of the expected B♭ major. However, the harmonic analysis reflects a different view. Instead of focusing on the repeated harmonic pattern (as in the key-changing analysis of the first passage), it interprets every chord as a functional harmony in C major. The Entgleisung or “derailment” at bar 3 (beat 1) suggests that an enharmonic shift to F♯ major might have been perceived at that point, though it would have broken the harmonic trajectory in the approach to the final C major. The different semitone sizes in the upper voice (59, 93 and 111 µ) result from the harmonic progression; the upper voice is here not a true melody (which would contain canonic values), but simply a succession of harmonic values.
Example 8.9.3: leading-tone sequence

Leittönige Sequenz: radikale Entgleisung bei gleichen Funktionen =
Leading-tone sequence: radical shifting through identical functions

Gegenterz = “counter-third,” i.e. major thirds below the indicated canonic Ursprungslagen

\begin{align*}
b &= B_b \\
Fis &= F^\# \\
Ges &= G_b \\
Gis &= G^\# \\
As &= A_b \\
Cis &= C^\#
\end{align*}

\begin{align*}
Des &= D_b \\
Es &= E_b \\
Fes &= F_b \\
Bes &= B_b \\
Eses &= E_b \\
Geses &= G_b \\
Ceses &= C_b \\
Deses &= D_b \\
Aseses &= A_b \\
Beses &= B_b \\
Eseses &= E_b
\end{align*}

\begin{align*}
\text{C}^\#_{(7,0)} \text{ major : E}^{bb}_{(3,5)} \text{ major corresponds to } \text{C}(0) \text{ major : E}^{bb}_{(4,5)} \text{ major} \\
\text{(enharmonically = C major)}
\end{align*}

which would constitute an incremental comma elevation of $50 \mu$!

In this case, the steps \textit{in the upper voice} remain uniformly equal:

\begin{align*}
G^\# : A : B_b : C_b : D_b : E_b \\
93 & 93 & 93 & 93 & 93
\end{align*}

\begin{align*}
= 465 \mu & \quad \text{Pure fourth } = 415 \\
\text{Difference: } 50 \mu \uparrow
\end{align*}
Example 8.9.3: leading-tone sequence

In this leading-tone sequence (i.e. rising by diatonic semitones), the harmonic analysis illustrates the repeated harmonic pattern, and the acquisition of a syntonic comma at each key shift. The note spelling accurately reflects the journey through pitch space: a modulation from C♯ major to E♭♭♭♭ major. Those two keys are enharmonic (differing by 50 µ), and thus the passage is tonally open.

Here, the above modulation is reconceived in C major, simply to make the calculation of 50 µ easier.

In the upper voice, the semitones can all be of the same size, as the passage is tonally open.
Example 8.9.4: tonally closed leading-tone sequence

If tonal unity is to be preserved, i.e. if the chord progression is to be closed in circular fashion, the plus-difference of 50 µ must be eroded during the course of the passage. In such cases, the [enharmonic] ligatures must be lowered twice: once by a diaschisma (16), and once by a minor diesis (34).

\[
\begin{align*}
16 \downarrow + 34 \downarrow &= 50 \downarrow \\
50 \uparrow - 50 \downarrow &= 0
\end{align*}
\]

Keys: C#M DM C major! C# major

kl. Diēsis = minor diēsis
Enharm. = enharmonic
reine Quarte = pure perfect fourth
Example 8.9.4: tonally closed leading-tone sequence

This example has the difficult task of realizing two goals:
1) tonal closure, in exactly the same C# major as in the first bar
2) a functional interpretation of every chord, changing keys when needed

To realize tonal closure, the upper two voices include two enharmonic “ligatures” (of 34 and 16 μ), as indicated. In addition, the rising semitones in the lower voice are of differing sizes (59 and 77 μ instead of 93 μ) at the points of enharmonic ligature.

In the harmonic analysis, bars 3 to 6 (beat 1) are interpreted in C major – a surprise, as the bars contain neither the C major triad nor its dominant! Other functional interpretations of these measures are certainly possible, and one might dispute whether any functional analysis of bars 3-6 is truly meaningful. It seems as if Karg-Elert wanted to enclose the two enharmonic ligatures within a single key area, like a parenthesis. In any case, the note spellings and the function labels correspond exactly.

ERRATUM: the function labels for bar 6 (beat 1) should be DD\textsuperscript{M} / C, not DD\textsuperscript{M} / T.
In C major, the minor diesis [with large chroma] and the diachisma [with small chroma] appear accordingly in other positions. The end result is of course the same as above:

kl. Diësis = minor diësis        Sopran = soprano (upper voice of the progression)
Unterstimme = lower voice (of the progression)        gr. (kl.) Chroma = large (small) chroma
Leitton = leading-tone (i.e. semitone of 93 µ)        reine Quarte = pure perfect fourth

It should be patently obvious that only the artificially-levelling twelve-semitone tempered tuning can help resolve the multifarious dilemma of the ever-present threat of comma slippage.
8.9. (continued)

This is the same leading-tone sequence as the preceding example, but now in C major. The overall result is the same: tonal closure is effected by means of two enharmonic ligatures in the upper voices, and different sizes of semitone in the lower voice.

However, the harmonic analysis is now very different: it attempts to understand every chord in C major, without brief detours to other keys (though an alternate interpretation of mm. 2-3 in C♯ major is suggested). Staying entirely in C major places the enharmonic ligatures in different locations, compared to the C♯ major version.

One of the major lessons to be learned from the preceding examples is that for Karg-Elert, note spelling and harmonic function are very closely intertwined. The different harmonic analysis of the C♯ major and C major examples in 8.9. suggest that functional interpretation depends in part on the notation of the pitches. On the other hand, Karg-Elert would surely recommend that note spellings must be carefully chosen in order to reflect the intended harmonic meaning (and not simply for convenience in performance). See section 13.3 for more on note spelling and harmonic intention.

Karg-Elert ends the chapter with what is indeed obvious: the practical necessity of equal temperament in the performance of chromatic music. However, as he will propose in Chapter 13, the musical ear perceives and understands all of the complex metharmonic and enharmonic relationships and “comma slippages” discussed in Chapter 8, even when music is performed in 12-tone equal temperament.
Chapter 8: Tonnetze

8.1. Metharmonics and enharmonics

Ursprungslagen der Prinzipale = source positions of the principal triads
Ursprungslagen der Varianten (Medianten) = source positions of the variants (mediants)

Metharmonics:

about 18 µ lower than the (corresponding) principal harmony

about 18 µ higher than the (corresponding) principal harmony

The function labels apply to the keys of A minor (left group) and C major (right group). Pitches with the same name are metharmonics, differing by a syntonic comma (18 µ):
8.2. The problematic (i.e. diminished) triad as substitute for D\textsuperscript{1} and E\textsubscript{C}

C major:  

A minor:  

The “problem” with the diminished triad: what is the acoustic identity of its third?  
In C major: is it D\textsubscript{(2,1)} from C\textsubscript{p}, or D\textsubscript{(2,0)} from D?  
In A minor: is it D\textsubscript{(2,0)} from E\textsubscript{p}, or D\textsubscript{(-2,1)} from A?  

On the Tonnetz, in C major:

In A minor:

8.3. Metharmonics in the diatonic sequence

These are the pitches in the metharmonic pairs from the two examples:
8.5. Metharmonics and the reinterpretation of variants and mediants

First time: 1 2 3 4 5 (4) (5) 6 7 8 9
Second time: 10 (9) (10) 11 12 13 14
Third time: 15 (14) (15) 16 17 etc.

If the above passage is sung purely syntonic (i.e. without adjusting back to the tuning of chord 1), each repetition will be a syntonic comma higher, because of the mediant transformation at chords 5 → 6, then again at 10 → 11, 15 → 16, etc.
8.5. (continued)

First time: 1 2 3 4 5 6 7
Second time: (7) 8 9 10 11 12 13
Third time: (13) 14 etc.

4 x zu wiederholen = repeat four times

If the above passage is sung purely syntonically, each repetition will be a syntonic comma lower, based on the final chord of the phrase (chord 7, 13, etc.):
8.5. (continued)

Comma-free: C maj $\rightarrow$ C min
Double metharmonics
Comma-free: B maj $\rightarrow$ C maj

A) $G_{(3,1)}$, $B_{(3,2)}$, $D_{(2,1)}$  mediant of Eb major = variant of G minor
B) $B_{(3,2)}$, $D^\flat_{(3,3)}$, $F^\flat_{(2,2)}$  ultramediant of Eb major = chromonant of Bb major

Chromonant = upper or lower chromatic semitonal transformation of an Ursprungslage chord

Chords 1-7: comma-free modulation from C major to C minor
Chords 7-11: modulation to B major (chord 11), lowered by two syntonic commas
Chords 11-18: comma-free modulation from B major to C major

The Tonnetz illustrates how the final C major (chord 18) is two syntonic commas lower than the opening C major (chord 1):
8.6. Acoustic comparison of metharmonically equivalent pitches

This Tonnetz indicates the metharmonic versions of F# and Gb shown above:

\[
\begin{array}{cccccccccccc}
F\# & C & G & D & A & E & B & F & C & G & D & A & E & E\\
\text{D} & \text{A} & \text{E} & \text{B} & F\# & C & G & D & A & E & B & F & C\\
\text{Bb} & F & C & G & D & A & E & B & F & C & G & D & A & A\\
\text{Gb} & \text{Db} & \text{Ab} & \text{Eb} & \text{Bb} & \text{F} & \text{C} & \text{G} & \text{D} & \text{A} & \text{E} & \text{B} & F & \text{Gb}\\
\text{Ebb} & \text{Bbb} & \text{Fb} & \text{Cb} & \text{Gb} & \text{Db} & \text{Ab} & \text{Eb} & \text{Bb} & \text{F} & \text{C} & \text{G} & \text{D} & \text{Gb}\\
\text{Cbb} & \text{Gbb} & \text{Db} & \text{Ab} & \text{Ebb} & \text{Bbb} & \text{Fb} & \text{Cb} & \text{Gb} & \text{Db} & \text{A} & \text{Eb} & \text{Bb} & \text{Gb}\\
\text{Abb} & \text{Ebb} & \text{Bbb} & \text{Fb} & \text{Cb} & \text{Gbb} & \text{Db} & \text{Ab} & \text{Eb} & \text{Bb} & \text{Fb} & \text{Cb} & \text{Gb} & \text{Gb}\\
\end{array}
\]
8.7. **Apparent and true enharmonics**

NOTE: these first two examples contain no enharmonics (true or apparent).

Example 1: the counter-medianctic insertion (chords 4-7, in Ab major) moves down one horizontal level on the Tonnetz.

Example 2: the mediantic episode (chords 4-7, in E major) moves up one horizontal level on the Tonnetz.
8.7. (continued)

*B♭ major:* Notated in A major

*D♯ major:* Notated in E-flat major

Tonnetz:

- Functions: B♭ g♭ E♭ Fb B♭ chords
- Tonnetz: T  C  D  T₉  C  D  T
- Funktionen: Bes. ges. Eis. Es. Eis. Klänge

These two progressions are exactly the same as those on the previous page, but now transposed up a semitone to D♭ major (Example 1), or down a semitone to B major (Example 2). They demonstrate the concept of *apparent enharmonics*, as follows:

**Example 1:** the counter-mediantic insertion is really in B♭ major, but is notated enharmonically in A major for convenience. The Tonnetz indicates the true harmonic progression, which is exactly the same as it was in C major.

**Example 2:** the mediantic insertion is really in D♯ major, but is notated enharmonically in E♭ major for convenience. The Tonnetz indicates the true harmonic progression, which is exactly the same as it was in C major.
8.7. (continued)

A major chord as $T^{MM}$ in $D_b$ major:  

$E\bar{b}$ major chords as $T_{MM}$ in $B$ major:

The spelling of each chord reflects its location in pitch space, in the keys of $D_b$ major and $B$ major. They are cited to contrast with the previous page, in which $A$ major and $E\bar{b}$ major are apparent enharmonics, standing for $B_b$ major and $D\#$ major respectively.
### 8.8. Enharmonics and metharmonics in chromatic sequences

**Tonnetz:**

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>(1)</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
</table>

Offensichtliche Nebenmediantensequenz in klarer Funktionsbezeichnung =

*In a straightforward neighbor-median sequence, with clear functional analysis*

Schwerzeitige Stationen = strong-beat “stations” (points of arrival)

The Tonnetz illustrates the sequence of descending neighbor mediants (minor thirds), and the lowering of 4 syntonic commas:

Chords 2 and 10 are **true enharmonics** of each other, as they have different spelling and different acoustic derivation.
8.8. (continued)

Continuation of the previous example:

Tonnetz chords: 10 9 11 12 13 14 15 16 17

End of the last example

Beginning of the example

F(11,-3)    E(12,-2)    D(10,-2)    C(11,-1)    F(10,-1)    vs.    F(6,0)

(opening)

Durakk. = major triad    dagegen Anfang = in relation to the beginning

This is a Tonnetz for the entire passage, beginning in F♯ major (chord 1):

The original tonic (chord 1) and the final tonic (chord 17) are both F♯ major, but they differ by a syntonic comma. The two keys are metharmonics of each other.
8.9. Chromatic sequences and tonal closure

Example 8.9.1: mediants

Sequence of ascending mediants (major thirds), beginning in G♭ major:

Tonnetz chords: 1 2 3 4 5 6 7

The same sequence, but now tonally closed, beginning and ending in C major:

Tonnetz chords: 1 2 3 4 5 6 7

Upper voice: G A B C♯ D4 E♭ F G
(1,0) (-1,1) (1,1) (-1,2) (1,2) (1,-1) (-1,0) (1,0)

Ex. 1 (from G♭ major): Ex. 2 (in C major):

Ex. 1: the sequence could repeat infinitely, shown on the Tonnetz as an unbroken ascent.
Ex. 2: to ensure closure in C major, the sequence substitutes a diminished fourth step instead of a major third in bar 2 (chords 3-5, E major → A♭ major). The tonic C major is surrounded by both its mediant (chord 3) and counter-median (chord 5).
Example 8.9.2: neighbor mediants

Sequence of ascending neighbor mediants (minor thirds), beginning in F♯ major:

Tonnetz chords:    1  2  3  4  5  6  7  8  9

The same sequence, but now tonally closed in C major:

Tonnetz chords:    1  2  3  4  5  6  7  8  9

Ex. 1 (from F♯ major):

Ex. 2 (in C major):

Ex. 1: the sequence could repeat infinitely, on the Tonnetz as an unbroken diagonal
Ex. 2: the Tonnetz shows that chord sequences 1 → 2, 5 → 6 and 7 → 8 are analogous, even though their function labels are not (instead, they all relate to the key of C major). Chords 3 → 4 break the sequential pattern: to ensure closure in C major, chord 4 is Cp (rooted on D(2,1)), instead of Dd (rooted on D(2,0)) which would have continued the sequence. The break is indicated in the musical example, inside the box.
Example 8.9.3: leading-tone sequence

Leittonige Sequenz: radikale Entgleisung bei gleichen Funktionen =
Leading-tone sequence: radical shifting through identical functions
Gegenterz = “counter-third,” i.e. major thirds below the indicated canonic Ursprungslagen
Enh. = “enharmonic” = the usual spellings for these chords, omitting double and triple flats

The Tonnetz clearly illustrates the ascending leading-tone sequence (repeating every other chord), and shows that Chord 13 is a true enharmonic of Chord 1, raised by five syntonic commas:
Example 8.9.4: tonally closed leading-tone sequence

This is another ascending leading-tone sequence – like Example 8.9.3, but now tonally closed in C# major:

Tonnetz: 1 2 3 4 5 6 7 8 9 10 11 12 (1)

Keys: C#M DM C major! C# major

To ensure tonal closure, two real enharmonic shifts are necessary: at chords 6 → 7 (using A♭ → G# and C♭ → B), and at chords 10 → 11 (B♭ → A♯ and D♭ → C♯). Those shifts are highlighted as distant leaps on the Tonnetz:
(Example 8.9.4: tonally closed leading-tone sequence – continued)

Here is the same tonally-closed leading-tone sequence, now transposed to C major. The harmonic analysis is now entirely in C major:

Tonnetz: \begin{align*}
1 & \ 2 & \ 3 & \ 4 & \ 5 & \ 6 & \ 7 & \ 8 & \ 9 & \ 10 & \ 11 & \ 12 & \ (1) \\
\end{align*}

To ensure tonal closure, this example also includes two real enharmonic shifts, but now at different places in the sequence: at chords 3 $\rightarrow$ 4 (using F $\rightarrow$ E$\#$ and A$\flat$ $\rightarrow$ G$\#$), and at chords 8 $\rightarrow$ 9 (C$\flat$ $\rightarrow$ B and A$\flat$ $\rightarrow$ G$\#$).

Once again, the enharmonic shifts are highlighted as distant leaps on the Tonnetz:
Chapter 9

The natural, primary, syntonic seventh
Chapter 9
The natural, primary, syntonic seventh

9.1. The natural concordant seventh (4:7)
9.2. Various theoretical views on the seventh
9.3. Interval sizes involving the natural seventh

NOTE for this chapter, and subsequent chapters:

Karg-Elert uses vertical wedges to denote natural seventh (4:7) intervals:
\( \triangledown \) (written above the notehead or note name) = seventh above the prime
\( \triangledown \) (written below the notehead or note name) = seventh below the prime
Natural sevenths (called concordants) are also notated as diamond-shaped noteheads.

To indicate sevenths, the translation uses the notation \( N(x,y,z) \), which means that pitch N is located x fifths, y thirds and z pure (4:7) sevenths away from the central C(0).

For example:
\( B^b_{(0,0,1)} \) is the concordant seventh above C(0)
\( D_{(0,1,1)} \) is the concordant seventh above E\(_{(0,1)}\)
\( F^#_{(0,1,-1)} \) is the concordant seventh below E\(_{(0,1)}\)

If no seventh is specified in the coordinates, then the pitch is entirely fifth- and/or third-derived.
9.1. **The natural concordant seventh (4:7)**

![Diagram of the natural concordant seventh in C major and A minor](image)

- **C major**
- **A minor**

Rel. Sz. = relative frequency

rel. W. = relative wavelength
9.1. The natural concordant seventh (4:7)

\( \checkmark \) is Karg-Elert’s symbol for the natural, syntonc pure 4:7 seventh above a given note. 
\( \wedge \) indicates a pure seventh below a given note.

Karg-Elert also sometimes uses diamond-shaped noteheads to indicate pure sevenths.

In the C major example: F is the pure seventh above G. 
In the A minor example: B is the pure seventh below A.

The addition of the concordant seventh to the dominant (\( \overline{D} \)) was introduced in 6.5.
Von c = 0 aus: starting from C(0)
Abstände: intervals, measured in μ
Diff. der Intervallgrößen: difference in intervallic size, measured in μ

It is worth emphasizing the natural size of the seventh \( \left[ \begin{array}{c} 4^7 : 4_i \end{array} \right] = 807 \, \mu \), in order to properly understand its character and its range of possible relations.
9.1. (continued)

This diagram presents the seventh chords built above each note of the C major triad, and below each note of the A minor triad. In both keys, C is C(0). Notably, the sizes of the thirds decrease progressively: $322 \rightarrow 263 \rightarrow 222 \mu$. Together, they sum to $807\mu$, which is the interval of the concordant 4:7 seventh.
9.2. Various theoretical views on the seventh

Many music theorists conceive of the {minor} seventh as:

a) the difference between the ninth {partial} and the double octave = 9 : 16
Conversely, others view it as:

 b) the difference between the fifth and ninth {partials} = 5 : 9
A maverick even believes that we must accept the seventh as:

c) 6 : 11 (∆)

That is, instead of 4 : 7(√) or 807 μ:

\[
\begin{align*}
\text{A)} &= 4 : 7 \frac{9}{2} \\
\text{B)} &= 4 : 7 \frac{3}{2} \\
\text{C)} &= 4 : 7 \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
in \mu \text{ Werten} & \quad 807 & \quad 830 & \quad 848 & \quad 869 \\
\text{A)} & \quad 807 & \quad \frac{222}{2} & \quad \frac{245}{2} & \quad \frac{263}{2} & \quad \frac{284}{2}
\end{align*}
\]

The concordant minor third (the difference between the pure fifth and the pure seventh) will be recognized as \(\frac{222}{2}\) μ in size: 807 – 585 = \(\frac{222}{2}\). In the other cases described above, the minor third appears as follows:

In case A, as 830 – 585 = \(\frac{245}{2}\) (Pythagorean minor third!)
In case B, as 848 – 585 = \(\frac{263}{2}\) (Didymean minor third!!)
In case C, as 869 – 585 = \(\frac{284}{2}\) (!!!)
9.2. Various theoretical views on the seventh

When Karg-Elert discusses “the seventh” (die Septime), he means the concordant 4:7 interval, unless specified otherwise.

The identity of this “maverick” (Außenseiter) is unclear; it might be Schoenberg, based on a passage from his Harmonielehre (1911) on the derivation of the C major scale, which Karg-Elert found objectionable (for more on this, see below in section 12.3).

is Karg-Elert’s acoustic symbol for the 11th partial. He also uses a similar descending hook (attached to a function label) to indicate a chord tone that has been lowered by a chromatic semitone (see 11.2). More information on the 11th and 13th partials is found in chapter 12.
The seventh G/F in different values [harmonic derivations]:

It is clear that the seventh 4:7 / 7:4 = 807 μ is uniquely and only possible as a syntonic dominant seventh. **Only this** value melds and “concords” with the dominant triad.

If the [pure] seventh replaces a fifth- or third-value, or a fifth- or third-exponent, or a product of these values, a comma difference of 23 μ [plus or minus] will invariably arise.
9.2. (continued)

This example demonstrates possible harmonic contexts for the four different values of the minor seventh just described, beginning with the concordant 4:7 interval (807 µ). The last (874 µ, which is option C above) is impractical as a harmonic minor seventh, as it is much “too wide” (zu weit!) Nonetheless, in Chapter 12 Karg-Elert considers the potential of the 11th partial as a harmonic interval.

“fifth- or third-exponent” = a multiple fifth or third, such as A_{(3,0)} or G^\sharp_{(0,2)}.

23 µ is what Karg-Elert calls the “Leipzig comma” (Leipziger Komma) - a term he introduces in 9.3. As described here, 23 µ is the difference between a seventh-derived interval and its canonic counterpart of the same name. The Leipzig comma is of course named after Karg-Elert’s lifelong city of residence.
9.3. Interval sizes involving the natural seventh

The three intervals contained in the natural tetrachord [i.e. the concordant dominant seventh]:

\[
\begin{align*}
C(0) & \quad E_{(0,1)} & \quad G_{(1,0)} & \quad B_{b(0,0,1)} & \quad D_{(0,0,-1)} & \quad F_{(1,0)} & \quad A_{b(1,-1)} & \quad C(0) \\
000 & \quad 322 & \quad 585 & \quad 807 & \quad 193 & \quad 415 & \quad 678 & \quad 1000
\end{align*}
\]

ERRATUM:

Concordant seventh (4:7) = 807 µ
Concordant diminished fifth (5:7) = 485 µ \{not 515 µ!\}
Concordant minor third (6:7) = 222 µ

These intervals have a modally double meaning: that is, they can be interpreted as part of either an upward harmony \{dominant seventh\} or a downward harmony \{half-diminished seventh\}:

\[
\begin{align*}
\text{Upward \{dominant 7th\} harmony} & \quad \text{Downward \{half-dim. 7th\} harmony} \\
C(0) + B_{b(0,0,1)} & \quad C(0) \\
000 & \quad \text{E}_{(0,1)} \quad G_{(1,0)} \quad B_{b(0,0,1)} \\
\quad \text{322} & \quad \quad \text{585} & \quad \quad \text{807} & \quad \quad \text{000} & \quad \quad \text{222} & \quad \quad \text{485} & \quad \quad \text{807} & \quad \quad \mu \\
E_{(0,1)} + B_{b(0,0,1)} & \quad C(0) \\
000 & \quad \text{E}_{(0,1)} \quad G_{(1,0)} \quad B_{b(0,0,1)} \\
\quad \text{322} & \quad \quad \text{585} & \quad \quad \text{807} & \quad \quad \text{000} & \quad \quad \text{222} & \quad \quad \text{485} & \quad \quad \text{807} & \quad \quad \mu \\
G_{(1,0)} + B_{b(0,0,1)} & \quad C(0) \\
000 & \quad \text{E}_{(0,1)} \quad G_{(1,0)} \quad B_{b(0,0,1)} \\
\quad \text{322} & \quad \quad \text{585} & \quad \quad \text{807} & \quad \quad \mu
\end{align*}
\]
9.3. *Interval sizes involving the natural seventh*

**ERRATUM:** the correct value for the concordant diminished fifth is $485 \, \mu$ (confirmed in the next example). It is the sum of the 5:6 minor third (263 $\mu$) and the 6:7 concordant minor third (222 $\mu$) – see 9.1.

In this example, the common tones (which include the concordant seventh at 807 $\mu$) are tied together. The circled notes are the primes of each chord.
Typical intervals involving natural sevenths:

**Concordant sevenths**

\[ \text{C}(0) \rightarrow \text{Bb}(0,0,1) \]
\[ \text{E}(0,1) \rightarrow \text{D}(0,1,1) \]
\[ \text{G}(1,0) \rightarrow \text{F}(1,0,1) \]
\[ 807 - 0 = 807 \]
\[ (1)129 - 322 = 807 \]
\[ (1)392 - 585 = 807 \]

**Concordant dim. fifths**

\[ \text{E}(0,1) \rightarrow \text{Bb}(0,0,1) \]
\[ \text{G}(1,0) \rightarrow \text{Db}(1,-1,1) \]
\[ 807 - 322 = 485 \]
\[ (1)070 - 585 = 485 \]

**Concordant minor thirds**

\[ \text{G}(1,0) \rightarrow \text{Bb}(0,0,1) \]
\[ \text{E}(1,0) \rightarrow \text{G}(4,1,1) \]
\[ 807 - 585 = 222 \]
\[ 544 - 322 = 222 \]

*Leipzig comma differences*:

\[ \text{Bb}(0,0,1) / \text{Bb}(2,0) \]
\[ 807 = 830 - 23 \]
\[ (1)129 = 152 - 23 \]
\[ (1)392 = 415 - 23 \]

\[ \text{Db}(1,-1,1) / \text{Db}(4,-1,1) \]
\[ (10)70 = 93 - 23 \]
\[ 485 = 508 - 23 \]
\[ 222 = 245 - 23 \]
9.3. (continued)

The Leipzig comma of 23 µ is marked in this diagram (and that on the next page) as ♩, and concordant (seventh-based) intervals are marked ♫. 

All pitches and µ values are calculated as usual from C(0).
(Smaller intervals involving the concordant seventh):

Whole tones: 
- wide (7:8) = \( \frac{193}{\text{D}_{(0,0,-1)} \text{ D}_{(2,0)}} = 170 + 23 \)
- narrow (32:35) = \( \frac{129}{\text{D}_{(0,1,1)} \text{ D}_{(2,0)}} = 170 - 23 \)

Diatonic semitone (20:21) = \( \frac{70}{\text{D}_{(1,0,1)} \text{ E}_{(0,1)}} = 93 - 23 \)
- e.g. \( (1)392 - (1)322 \)
- (10)70 – (10)0

("Leipzig gliding tone")

Chromatic semitone (14:15) = \( \frac{100}{\text{E}_{(0,1)} \text{ E}_{(1,0,1)} = 77 + 23} \)
- e.g. 322 – 222
- 585 – 485

("Leipzig chroma” = 1/10 of an octave)

Diminished minor third (25:28) = \( \frac{163}{\text{B}_{b(0,0,1)} \text{ G}_{#(0,2)}} = 263 - 100 \)
- e.g. 807 – 644
The direct translation of *verminderte Kleinterz* is indeed “diminished minor third.” It is unclear if Karg-Elert intended simply “diminished third,” or if he purposely used the redundant term to distinguish this interval (25:28) from the Pythagorean diminished third of 150 µ or 59049:65536 (see section 3.10).
Chapter 9: Tonnetze

9.1. The natural concordant seventh (4:7)

The natural concordant seventh (4:7)

The dominant with concordant seventh

in C major:

in A minor:

First introduced in section 6.5, the concordant seventh is the interval 4:7, generated above or below the prime, usually of a dominant (D) or altered contrant (c).

On the Tonnetz, no acoustic subscripts/superscripts are attached to concordant sevenths; their acoustic derivation above or below the prime should be apparent from the connecting short diagonals.
9.1. (continued)

Von $c = 0$ aus: starting from $C(0)$

Abstände: intervals, measured in $\mu$

Diff. der Intervallgrößen: difference in intervallic size, measured in $\mu$

The above diagram presents the concordant seventh chords built above each note of the C major triad, and below each note of the A minor triad. In both keys, C is $C(0)$.

The following Tonnetz outlines all of the above pitches and chords:
9.3. Interval sizes involving the natural seventh

<table>
<thead>
<tr>
<th>Interval</th>
<th>Upward {dominant 7\textsuperscript{th}} harmony</th>
<th>Downward {half-dim. 7\textsuperscript{th}} harmony</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(0) + B_{b(0,0,1)}$</td>
<td>$C(0)$ $E_{(0,1)}$ $G_{(1,0)}$ $B_{b(0,0,1)}$</td>
<td>$C(0)$ $E_{b(-1,0,1)}$ $G_{b(0,-1,1)}$ $B_{b(0,0,1)}$</td>
</tr>
<tr>
<td></td>
<td>000 322 585 807 $\mu$</td>
<td>000 222 485 807 $\mu$</td>
</tr>
<tr>
<td>$E_{(0,1)} + B_{b(0,0,1)}$</td>
<td>$E_{(0,1)}$ $G_{(1,0)}$ $B_{b(0,0,1)}$</td>
<td>$E_{(1,0)}$ $G_{(-1,1,1)}$ $B_{b(0,0,1)}$ $D_{(0,1,1)}$</td>
</tr>
<tr>
<td></td>
<td>000 322 585 807 $\mu$</td>
<td>322 544 807 (1)129 $\mu$</td>
</tr>
<tr>
<td>$G_{(1,0)} + B_{b(0,0,1)}$</td>
<td>$E_{(0,1)}$ $G_{(1,0)}$ $B_{b(0,0,1)}$</td>
<td>$G_{(1,0)}$ $B_{b(0,0,1)}$ $D_{b(-1,-1,1)}$ $F_{(1,0,1)}$</td>
</tr>
<tr>
<td></td>
<td>000 322 585 807 $\mu$</td>
<td>585 807 (10)70 (1)392 $\mu$</td>
</tr>
</tbody>
</table>

These Tonnetz illustrate the chord pairs that share the featured concordant intervals:

$C(0) + B_{b(0,0,1)}$; $E_{(0,1)} + B_{b(0,0,1)}$; $G_{(1,0)} + B_{b(0,0,1)}$;

Chord-tone exchanges between the chord pairs:

$7 = \acute{\flat}$ and $1 = \natural$  $3 = \natural$ and $7 = \acute{\natural}$  $5 = \natural$ and $7 = \natural$
Chapter 10

The practical and musical significance of the seventh values
Chapter 10
The practical and musical significance of the seventh values

10.1. Concordant sevenths generated by dominants and contrant variants
10.2. Syntonic vs. canonic values in harmony involving concordant sevenths
10.3. Counter-concordants [ = counter seventh chords]
10.4. Comparative overview of the 7 semitonal values
10.5. Comparison of fifth- and seventh-relations in different contexts
10.6. Counter-concordant substitutions in fifth-based sequences
10.7. Septimengleicher (“same-seventh chords”)
10.8. Tritone variants in the $D : T$ cadence
10.9. Summary of seventh relations
10.1. Concordant sevenths generated by dominants and contrant variants

Every dominant, mediant and contrant variant can generate its own pure seventh, in the chord’s controlling direction \(i.e.\) above a major triad, and below a minor triad.
10.1. Concordant sevenths generated by dominants and contrant variants

Top staff in C major, bottom staff in A minor. The Schrittverste (step sizes) are the μ-values for the upper- or lower-voice resolutions marked by diagonal lines (for minor seconds, at 70 μ) or angles (for major seconds, at 129 μ). In each pair, the first pitch is the concordant seventh of a dominant (D) or contrant variant (c).

ERRATA:
In the third C major example: the E-major triad (with seventh) should be labelled D_P, not D_M.
In the third A minor example: the F-minor triad (with seventh) should be labelled D_P, not D_M.
The acoustic information for both chords is correct.
10.2. **Syntonic vs. canonic values in harmony involving concordant sevenths**

The pure lines of the following melodic passages {labelled A and B} of course outline Pythagorean relations:
10.2. *Syntonic vs. canonic values in harmony involving concordant sevenths*

Upper row of numbers = interval sizes in $\mu$.
Lower row of numbers = $\mu$ values for the pitches, in relation to $C(0)$.
The dots above/below each melodic pitch indicate distance in fifths from $C(0)$.

Upper row of numbers = $\mu$ values for the pitches, in relation to $C(0)$.
The dots above/below each melodic pitch indicate distance in fifths from $C(0)$.
Lower row of numbers = interval sizes in $\mu$. 
However, if harmony takes precedence, one must respect its specific pure relations. The linear melodies decay into a series of integral chord members. The melodies are canonically impure: that is, they are altered in favour of syntonic exactitude [compare line A above with line Ab below, and line B with line Bb].

im Satz = within a harmonic progression
in Ursprungslagen = in source positions
Now, melodic line A (from above, in closed noteheads) is presented within a C major harmonic context.

The chords from line Aa are shown in relation to the *Ursprungalagen*. Here, the top row of numbers are μ-values for line A, now calculated as harmonic tones (indicated by the acoustic symbols). The lower numbers are the interval sizes in μ. The arrows on the staff indicate Leipzig comma (23 μ) differences.

Melodic line B (from above, in closed noteheads) now occurs as the bass line within an A-minor harmonic context.

Once again, the top row of numbers are μ-values for line B, now calculated as harmonic tones (as indicated by the acoustic symbols). The lower numbers are the interval sizes in μ.
10.3. Counter-concordants [ = counter-seventh chords]

Von großer Bedeutung für die tonalitatische Erweiterung sind die Klangformen mit simultanem Septimenrahmen:

Of great importance for the expansion of tonality are harmonies that share a seventh-frame {i.e. between the prime and the concordant seventh}:

um 23 µ tiefer/hoher als = about 23 µ lower/higher than

Concordant chroma differ by 100 µ (that is, a tenth of an octave):
10.3. Counter-concordants \( \Delta = \text{counter-seventh chords} \)

Counter-concordants (or counter-seventh chords) share a minor-seventh frame, and two acoustically-identical common pitches. The concordant seventh of the major triad becomes the prime of the minor triad \((7 \Rightarrow \bar{1})\), and vice versa \((1 \Rightarrow \bar{7})\). The chord quality switches from dominant seventh to half-diminished, and vice versa. See section 10.7 for an example of counter-seventh chords with a different common-tone connection.

The first two examples are in C major, and the second two are in A minor. Here are the pitches in the first C major example:

\[
\begin{align*}
&G_{(1,0)} \ B_{(1,1)} \ D_{(2,0)} \ F_{(1,0,1)} \Rightarrow \ F_{(1,0,1)} \ D_{b(1,-1,1)} \ B_{b(0,0,1)} \ G_{(1,0)} \\
&585 \ 907 \ 170 \ 392 \ 392 \ 70 \ 807 \ 585 \ \mu
\end{align*}
\]

This is indeed 23 \(\mu\) lower than the Ursprungslage equivalent:

\[
\begin{align*}
&F_{(-1,0)} \ D_{b(-1,-1)} \ B_{b(-2,0)} \ G_{(-1,0,-1)} \\
&415 \ 93 \ 830 \ 608 \ \mu
\end{align*}
\]

\(\overline{D}\) indicates the counter-concordant transformation of \(\overline{D}\) in which the prime becomes the seventh, and vice versa. The second and fourth examples show that the same transformation can be applied to contrant variants \(\overline{C}\), as they approximate the dominants of their parallel keys, and can therefore also generate a concordant seventh (see section 6.7).

Concordant chroma \((100 \mu) = \text{chromatic semitone in which one pitch is seventh-derived}\)
### 10.4. Comparative overview of the 7 semitonal values

<table>
<thead>
<tr>
<th></th>
<th>DIATONIC SEMITONES</th>
<th>CHROMATIC SEMITONES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fifth-related</td>
<td>Limma = B(_{(5,0)}) : C(0)</td>
<td>Apotome = B(<em>{(5,0)}) : B(</em>{b(2,0)})</td>
</tr>
<tr>
<td></td>
<td>256 / 243 = (\frac{73}{\mu})</td>
<td>2187 : 2048 = (\frac{95}{\mu})</td>
</tr>
<tr>
<td>Third-related</td>
<td>Leading tone = B(_{(1,1)}) : C(0)</td>
<td>Minor chroma = B(<em>{(1,1)}) : B(</em>{b(2,1)})</td>
</tr>
<tr>
<td></td>
<td>16 / 15 = (\frac{59}{\mu})</td>
<td>25 / 24 = (\frac{59}{\mu})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Major chroma = B(<em>{(1,1)}) : B(</em>{b(2,0)})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>135 / 128 = (\frac{59}{\mu})</td>
</tr>
<tr>
<td>Seventh-related</td>
<td>Gliding tone = B(<em>{(1,1)}) : C(</em>{(2,0,1)})</td>
<td>Acoustic chroma = B(<em>{(1,1)}) : B(</em>{b(0,0,1)})</td>
</tr>
<tr>
<td></td>
<td>63 / 60 = (\frac{70}{\mu})</td>
<td>15 / 14 = (\frac{100}{\mu})</td>
</tr>
</tbody>
</table>

One will recognize that the seventh-derived semitonal values closely approximate the canonic values (differing by only \(\frac{5}{\mu}\)), providing narrow diatonic and wide chromatic steps.
10.4. Comparative overview of the 7 semitonal values

Karg-Elert notes that just as the fifth-derived Limma and Apotome sum to a Pythagorean whole tone (170 µ), so do the seventh-derived gliding tone (70 µ) and the acoustic chroma (100 µ).

The “acoustic chroma” (14:15, or 100 µ) is called the *Leipziger Chroma* in 9.3., and also in Appendix A. It is called the *konkordante Chroma* in 10.3. Karg-Elert does not seem to use the term *akustischer Chroma* elsewhere.
10.5. Comparison of fifth- and seventh-relations in different contexts

C major:                                                                 A minor:

fifth-related     seventh-related     fifth-related     seventh-related
quintverwandt    septverwandt        quintverwandt    septverwandt

TN

\[ \text{23 \mu Diff.} \]

\[ \text{23 \mu Diff.} \]
10.5. **Comparison of fifth- and seventh-relations in different contexts**

In the two C major examples, the pitch-class D has different acoustic values, and therefore the D7 chord has different functions. In the fifth-related example, the D7 chord is the ultradominant, and C\(_{(2,0,1)}\) is the natural seventh of D\(_{(2,0,0)}\). In the seventh-related example, the C(0) is held constant, as an acoustic common tone. The D in the D7 chord is D\(_{(0,0,-1)}\), which is the counter-concordant (\(\mathbf{K}\)) of the contrant variant (the F minor triad, labelled \(c\)).

The \(\mu\) calculations show that the two Ds and the two Cs differ from each other by the Leipzig comma (23 \(\mu\)).
10.6. Counter-concordant substitutions in fifth-based sequences

The chromatic cadence can be greatly enriched by means of counter-concordants, substituting on alternating steps in a chain of fifth relations:

Keys: Eb F G A Bb C D E

Statt = in place of
Durakk. = major triad

Tritt ein = enters a
Mollakk. = minor triad
10.6. **Counter-concordant substitutions in fifth-based sequences**

The upward arrows (↑) indicate major triads, which are assigned tonic (T) or ultradominant (DD) function in the labelled keys. The downward arrows (↓) indicate counter-concordant substitutes (half-diminished in quality) for dominant sevenths or altered-contrant sevenths. The upper system presents all chords in their Ursprungslagen.

The brackets above the upper staff indicate the sequential repetition, including the root motions by fifth. The first example retains the ascending fifth pattern in the bass, even though every other chord has been substituted by its counter-concordant seventh.
The same \textit{as the previous example}, in strict polar inversion:

\begin{center}
\includegraphics[width=\textwidth]{example.png}
\end{center}

\textit{Keys:} Fm, Em, Dm, Cm, Bm, Am, Gm, (Fm)
10.6. (continued)

Functionally, this passage is an exact polar equivalent of the previous one, though there are some additional differences. In the first half: each downbeat tonic (\(T\)) is a minor triad, but the D minor and C minor chords are treated in a “concordant” manner, as they contain added minor sevenths below their primes. Every upbeat chord is a counter-concordant substitute of a dominant seventh. Each chord is shown in its *Ursprungslage*.

In the second half, the chords that were counter-concordants in the first half become tonics, and vice versa (as indicated by the solid and dotted arrows). The tonics are placed on downbeats. Every chord contains its minor seventh above or below the prime; the result is a continuous stream of dominant and half-diminished seventh chords, with no clearly audible tonality.

The voice leading is very different than in the previous passage; neither of the outer voices emphasize the sequential repetition, or the root motions of a fifth.
10.7. Septimengleicher ("same-seventh chords")

Counter-seventh chords can clarify sequences of remotely-related chords, with great simplicity:

The bass part is a linear chromatic ascent to its goal, the under-seventh (A♯) of the C♯ minor triad.

The G minor and F♯ major triads are metharmonic Septgleicher:

(Difference = 41 µ = ca. a quarter tone)
10.7. Septimengleicher ("same-seventh chords")

To understand this example, first ignore the lower voice (marked “bassoon”), which is completely non-harmonic, except for the final pitch A#. The primary chords are the major triads, which are assigned tonic (T) function; they outline a pattern of descending minor thirds: C → A → F# → D#. In the upper voice, the first three notes of each beat arpeggiate the tonic triads. The intervening minor triads are analyzed as incomplete counter-seventh chords (Septgegenklänge), substituting for the major triads of C, E, F# and A#. Notably, the omitted tone in each counter-seventh chord is the concordant seventh itself – except for the A# in the lower voice at the end of bar 2, which is the under-seventh (Unterseptime) of the C#m chord.

The first half of both bars here feature the same counter-seventh transformation previously seen in 10.3, 10.5 and 10.6, where the concordant seventh becomes the prime (7 → 1), and vice versa (1 → 7). The other two counter-sevenths (G minor and C# minor) are connected to their preceding chords by a different common-pitch connection, namely 5 → 7 and 7 → 5.

ERRATUM: the function of the F# major triad at the beginning of bar 2 is C^M (upper mediant of C), not C_M.

Here is the acoustic information for all eight chords. The arrows indicate the exchange of acoustically-exact common pitches:

Chord-tone exchanges:
- 1 → ① 5 → ① 1 → ① 5 → ① 7 → ① 7 → ① 7 → ①

This example provides the acoustic information for the 5th and 4th chords above (in that order). The two Es differ by 41 μ: E_{(-2,2,1)} at 281 μ, and E_{(0,1)} at 322 μ.

Karg-Elert calls these two chords metharmonische Septgleicher ("metharmonic same-sevenths"), as their sevenths have the same name, but are not acoustically identical.
Comma-free Septimengleicher arise when contrant variants become counter-concordants:

\[ G(1,0) \text{ as common tone} \quad A(1,1) \text{ as common tone} \]

Chords: \( A\text{ø7} \quad G7 \quad G\text{ø7} \quad A7 \quad A7 \quad B\text{ø7} \quad B7 \quad A\text{ø7} \)

\[ \text{TN} \]

\[ \text{ohne Tausender = “without thousands.” The digits in parentheses are only included in order to show the calculation of 807 µ.} \]

\[ \text{Die oktavverkreuzte Doppelseptime (2000 – (2 \times 807)) ergibt eine stark überstreckte Große} \]
\[ \text{terz = 386 [d. s. 2 weite Ganztöne], die nur einen 6tel Ton Abstand zur Quarte hält. (415 min} \]
\[ \text{386 = 29, 29 ist in 170 ca. 6 mal enthalten).} \]

The octave-reduced double seventh \((2000 – (2 \times 807))\) produces a greatly over-stretched major third = 386 [i.e. two wide whole tones], only a sixth of a tone away from a fourth. \((415 – 386 = 29\) which is included in 170 approx. 6 times).
10.7. (continued)

To understand this statement (and how it relates to the example), first remember that contrant variants (c or C) as well as dominants (D or D) can include their concordant sevenths.

In the first part (chords 1-4):
Aø7 is contrant variant (c) in G major, and G7 is dominant (D) in C major. They share G(1,0) as prime.
Gø7 is counter-seventh of G7, and A7 is counter-seventh of Aø7. These two chords are Septimengleicher, as they share G(1,0) as concordant seventh.

In the second part (chords 5-8):
A7 is contrant variant (c) in D minor, and Bø7 is dominant (D) in A minor. They share A(-1,1) as prime.
B7 is counter-seventh of Bø7, and Aø7 is counter-seventh of A7. These two chords are Septimengleicher, as they share A(-1,1) as concordant seventh.

In both of the Septimengleicher pairs, the chord tone transformation is 7 = 7.

The interval of 386 µ occurs between the primes of the Septimengleicher just described:

\[
\begin{align*}
A_{(1,0,-1)} & \text{ and } F_{(1,0,1)} \\
778 & - 392 = 386 \mu
\end{align*}
\]

\[
\begin{align*}
B_{(-1,1,-1)} & \text{ and } G_{(-1,1,1)} \\
930 & - 544 = 386 \mu
\end{align*}
\]

415 µ = canonic perfect fourth
386 µ = “over-stretched major third” between the primes of the Septimengleicher
170 µ = canonic whole tone. One sixth of a whole tone is 170 divided by 6 = 28.33 µ.
10.8. **Tritone variants in the D : T cadence**

Tritone variants result from a transformation of the direct cadence $D : T$ or $C : G$, in which the dominant is substituted by its counter-seventh, and the tonic by its mediant:

**Triads:**
- $G$
- $C$
- $E$
- $d$
- $a$
- $B$
- $f$

**Sequences (of tritone variants):**

- Sequence descending by semitones
  - $B \bar{m}$
  - $E$
  - $A \bar{m}$
  - $E$
  - $A \bar{m}$
  - $D$

- Sequence descending by whole tones
  - $B \bar{m}$
  - $E$
  - $A \bar{m}$
  - $D$
  - $G \bar{m}$

- Sequence ascending by semitones
  - $B$
  - $Fm$
  - $Cm$
  - $C\#$

- Sequence ascending by whole tones
  - $B$
  - $Fm$
  - $C\#$
  - $Gm$
  - $D\#$

**Erratum:**

Correcting the notation for the sequences of tritone variants.
10.8. **Tritone variants in the D : T cadence**

Tritone variants are chord pairs whose roots (not their dual primes!) lie a tritone apart; in addition, both chords are variants (i.e. they are not Ursprungslage harmonies). In this example, the tritonal variant chord pairs are marked by the X.

The primes of the tritonal variants are actually a semitone apart: F and E in the first example, B and C in the second. Therefore, Riemann calls this relationship Gegenleittonwechsel (Riemann 1880, 79).

**ERRATUM:** in the B7 chord (second part), the B should be written as a diamond-shaped note, as it is B\(_{(1,1,-1)}\) = the concordant seventh of the Bø7 chord.

Instead of function labels, Karg-Elert here provides analysis of chord tones in the outer voices. Major triads use regular numbers (1, 3, 5, 7), while minor triads use inverted numbers (†, Ɛ, ɛ, ¥). Throughout these two examples, the upper voice contains the prime (1 or †) of each chord; the primes in each tritone-variant pair are a semitone apart.

The chord labels are for the basic triads, with the tritone-variant pairs in boxes. All chords contain their concordant sevenths.

The long arrows indicate chromatic descents or ascents.
### 10.9. Summary of seventh relations

Chords sharing $B_{b(0,0,1)}$ as common tone

<table>
<thead>
<tr>
<th>Chord</th>
<th>gegenüber $B_{b(0,0,1)}$</th>
<th>gesenkt (in μ)</th>
</tr>
</thead>
</table>
| $\begin{array}{c}
v \\ b \\ g \\ b \\
\end{array}$ | $\begin{array}{c}
v \\ b \\ g \\ b \\
\end{array}$ | $\begin{array}{c}
v \\ b \\ g \\ b \\
\end{array}$ |
| $\begin{array}{c}
v \\ g \\ b \\
\end{array}$ | $\begin{array}{c}
v \\ g \\ b \\
\end{array}$ | $\begin{array}{c}
v \\ g \\ b \\
\end{array}$ |

**gegenüber** = *in comparison to*

**gesenkt** = *lowered (in μ)*
10.9. Summary of seventh relations

Each chord pair (at the left in each row) shares $B_{b(0,0,1)}$ as an acoustic common tone. These chord pairs display all possible relationships between two concordant seventh chords that share at least one acoustic common tone.

<table>
<thead>
<tr>
<th>Seventh-related pairs sharing $B_{b(0,0,1)}$ as common tone</th>
<th>Chord-tone exchanges</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>C7 $\rightarrow$ Bb7:</strong></td>
<td></td>
</tr>
<tr>
<td>$B_{b(0,0,1)}$, $D_{(0,1,1)}$, $F_{(1,0,1)}$ $[A_{b(0,0,2)}]$</td>
<td></td>
</tr>
<tr>
<td>C(0) $E_{(0,1)}$, $G_{(1,0)}$, $B_{b(0,0,1)}$</td>
<td>$7 \rightarrow 1$</td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ Gm (Eø7):</strong></td>
<td></td>
</tr>
<tr>
<td>$[E_{(0,1)}]$ $G_{(-1,1,1)}$, $B_{b(0,0,1)}$, $D_{(0,1,1)}$</td>
<td></td>
</tr>
<tr>
<td>C(0) $E_{(0,1)}$, $G_{(1,0)}$, $B_{b(0,0,1)}$</td>
<td>$3 \rightarrow \natural$ and $7 \rightarrow \natural$</td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ Eb7:</strong></td>
<td></td>
</tr>
<tr>
<td>$E_{b(-1,0,1)}$, $G_{(-1,1,1)}$, $B_{b(0,0,1)}$ $[D_{b(-1,0,2)}]$</td>
<td></td>
</tr>
<tr>
<td>C(0) $E_{(0,1)}$, $G_{(1,0)}$, $B_{b(0,0,1)}$</td>
<td>$7 \rightarrow 5$</td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ Eb$\natural$ (Cø7) = counter-seventh chords</strong></td>
<td></td>
</tr>
<tr>
<td>$[C(0)]$ $E_{b(-1,0,1)}$, $G_{(0,1,1)}$, $B_{b(0,0,1)}$</td>
<td></td>
</tr>
<tr>
<td>C(0) $E_{(0,1)}$, $G_{(1,0)}$, $B_{b(0,0,1)}$</td>
<td>$1 \rightarrow \natural$ and $7 \rightarrow \natural$</td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ Gb7:</strong></td>
<td></td>
</tr>
<tr>
<td>$G_{b(0,1,1)}$, $B_{b(0,0,1)}$, $D_{b(-1,1,1)}$ $[F_{b(0,1,2)}]$</td>
<td></td>
</tr>
<tr>
<td>C(0) $E_{(0,1)}$, $G_{(1,0)}$, $B_{b(0,0,1)}$</td>
<td>$7 \rightarrow 3$</td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ Bb$\natural$ (Gø7):</strong></td>
<td></td>
</tr>
<tr>
<td>$[G_{(1,0)}]$ $B_{b(0,0,1)}$, $D_{b(-1,1,1)}$, $F_{(1,0,1)}$</td>
<td></td>
</tr>
<tr>
<td>C(0) $E_{(0,1)}$, $G_{(1,0)}$, $B_{b(0,0,1)}$</td>
<td>$5 \rightarrow \natural$ and $7 \rightarrow \natural$</td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ Db$\natural$ (Bø7) = Septimengleicher</strong></td>
<td></td>
</tr>
<tr>
<td>$B_{b(0,0,1)}$, $D_{b(-1,0,2)}$, $F_{b(0,1,2)}$, $A_{b(0,0,2)}$</td>
<td></td>
</tr>
<tr>
<td>C(0) $E_{(0,1)}$, $G_{(1,0)}$, $B_{b(0,0,1)}$</td>
<td>$7 \rightarrow \natural$</td>
</tr>
</tbody>
</table>

Karg-Elert’s chart does not include the concordant sevenths for the second chord in the first six pairs, though the sevenths are logically implied, and are thus included here. The chord-tone exchanges listed in the second column apply to the complete seventh chords. For comparison, Karg-Elert provides acoustic information for metharmonic equivalent chords; these equivalents contain only canonic (fifth-based) and syntonic (fifth- and third-based) pitches.
The comma differences (from the previous chart) of 5, 28, 41, 46 and 64 are calculated using the following subtractions and additions:

Leipzig comma (23) minus syntonic comma (18) = 5

```
5
```

```
+ 41
```

In minor, these relationships are of course polar [with raised instead of lowered values].
10.9. (continued)
Chapter 10: Tonnetze

10.3. Counter-concordants [ = counter-seventh chords]

C major: A minor:

um 23 µ tiefer/hoher als = about 23 µ lower/higher than

The chord-tone exchange in the counter-seventh chords is 7 → 1 and 1 → 5.
The principal chords (with their concordant sevenths) are outlined in regular triangles,
while the counter-seventh chords are shown as dotted triangles (to reflect their location
in the third dimension of the fifth-third-seventh pitch space). The major or minor triads
included in each counter-seventh chord are generated from the concordant seventh.

C major: $\bar{D} \rightarrow D\overline{N}$

A minor: $\bar{A} \rightarrow A\overline{K}$
10.5. Comparison of fifth- and seventh-relations in different contexts

C major: fifth-related

\[
\begin{array}{c}
E^1 \rightarrow B^1 \rightarrow F^\# \rightarrow C^1 \\
C(0) \rightarrow C_{(2,0,1)}
\end{array}
\]

C major: seventh-related

A minor: fifth-related

\[
\begin{array}{c}
A^1 \rightarrow E^1 \rightarrow B^1
\end{array}
\]

A minor: seventh-related

\[
\begin{array}{c}
G^1 \rightarrow D^1 \rightarrow A^1 \rightarrow E^1 \\
E_{(-2,1,-1)} \rightarrow E_{(0,1)}
\end{array}
\]

C(0) is a common tone throughout

E_{(0,1)} is a common tone throughout
10.6. **Counter-concordant substitutions in fifth-based sequences**

Tonnetz: 1 2 3 4 5 6 7 1 2 3 4 5 6 7 8

Example 1:

Example 2:
The same \textit{as the previous example}, in strict polar inversion:

Tonnetz: 1 2 3 4 5 6 7 1 2 3 4 5 6 7 8

Keys: F\sharp m Em Dm Cm Bm Am Gm (Fm)

Example 1:

Example 2:
10.7. Septimengleicher (“same-seventh chords”)

Major: C A F# D#

Minor: eb g a c#

(Counter-seventh of: C E F# A#)

The strong beats outline a descending neighbor mediant (minor third) progression:
10.7. (continued)

The passage elaborates the neighbor mediant progression with counter-seventh chords:

The G minor and F# major triads are metharmonic *Septgleicher* (same-seventh chords): their concordant sevenths are both E’s, but they differ by 41 µ:

\[ E_{(2,2,1)} \quad 281 \ \mu \quad E_{(0,1)} \quad 322 \ \mu \]
Comma-free Septimengleicher arise when altered contrants become counter-concordants:

Example 1: \( G_{(1,0)} \) as common tone  
Chords: A\( \phi \)7 G7 G\( \phi \)7 A7

Example 2: \( A_{(-1,1)} \) as common tone  
A7 B\( \phi \)7 B7 A\( \phi \)7

Example 1: \( G_{(1,0)} \) as common tone  
A\( \phi \)7 and G7  
Their counter-concordants = A7 and G\( \phi \)7

A7 and G\( \phi \)7 are comma-free Septimengleicher (same-seventh chords) – the seventh of both chords is \( G_{(1,0)} \).

Example 2: \( A_{(-1,1)} \) as common tone  
A7 and B\( \phi \)7  
Their counter-concordants = A\( \phi \)7 and B7

B7 and A\( \phi \)7 are comma-free Septimengleicher; the seventh of both chords is \( A_{(-1,1)} \).
10.8. **Tritone variants in the D : T cadence**

Tritone variants result from a transformation of the direct cadence D : T or A : L in which the dominant is substituted by its counter-seventh, and the tonic by its mediant:

C major: $\bar{D} \rightarrow T$

A minor: $\bar{D}K \rightarrow T^M$

B♭ minor + E major = tritone variants

F minor + B major = tritone variants
10.8. (continued)

Sequences \{of tritone variants\}:

\[
\begin{align*}
Bom & \quad E \\
Am & \quad Eb \\
A\hspace{-1pt}m & \quad D
\end{align*}
\]

Sequence descending by semitones

Karg-Elert does not provide functional analysis for these tritone-variant sequences. Instead, the numbers above and below the staves indicate the chord tones in the outer voices: always primes in the soprano, but various chord tones in the bass.

The lack of functional analysis may stem from a problem presented between the second and third chords (E7 $\rightarrow$ Aø7): is their shared E an acoustically identical common tone?

If it is, we can begin the analysis with $D\overline{K} \rightarrow T^M$ in C major, as in the example that opens 10.8. All the other chords are then labeled in relation to C major:

\[
\begin{align*}
\text{Tonnetz:} & \quad 1 & 2 & 3 & 4 & 5 & 6 \\
\text{(CM):} & \quad D\overline{K} & T^M & C_p & D_M & C\checkmark & C^M
\end{align*}
\]
In the preceding example, the Tonnetz highlights how chord motions $1 \rightarrow 2$ and $5 \rightarrow 6$ are analogous ($5 \rightarrow 6$ is a whole tone lower than $1 \rightarrow 2$). Chords $3 \rightarrow 4$ are different, as neither chord is understood as a counter-seventh; instead, both are analyzed as fifth- or third-based transformations of principal harmonies ($C$ and $D$ respectively).

The above analysis seems plausible (and comparatively straightforward), as it directly connects chords $3$ and $4$ to chords $1$ and $2$ by means of acoustic common tones. However, it is by no means the only possible interpretation. The following connects chord $4$ ($E_b7$) directly to chord $5$ ($A_b\varnothing7$), using the common tone $E_b(-1,0,1)$:

Chord $4$ ($E_b7$) is now rooted on $E_b(-1,0,1)$, the seventh of $F(-1,0)$. As a result, the entire chord is seventh-derived:

$$E_b(-1,0,1) \quad G_{(-1,1,1)} \quad B_b(0,0,1) \quad D_b(-1,0,2)$$

The $D_b$ is two concordant sevenths above $F(-1,0)$. While this may seem far-fetched, it is in fact a logical relationship between two concordant seventh chords (see 10.9). In this interpretation, chord $4$ would be the altered contrant of chord $5$, or $(c)C\varnothing$. 
### 10.9. Summary of seventh relations

<table>
<thead>
<tr>
<th>Seventh-related pairs sharing $B_b(0,0,1)$ as common tone</th>
<th>Chord-tone exchanges</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>C7 $\rightarrow$ B♭7:</strong></td>
<td></td>
</tr>
<tr>
<td>$E_b(0,0,1)$ $D_c(0,1,1)$ $F(1,0,1)$ $[A♭(0,0,2)]$</td>
<td>7 $\rightarrow$ 1</td>
</tr>
<tr>
<td>C(0) $E(0,1)$ $G(1,0)$ $B_b(0,0,1)$</td>
<td></td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ Gm (Eø7):</strong></td>
<td></td>
</tr>
<tr>
<td>$[E(0,1)]$ $G(1,0,1)$ $B_b(0,0,1)$ $D(0,1,1)$</td>
<td>3 $\rightarrow$ $\natural$ and 7 $\rightarrow$ $E$</td>
</tr>
<tr>
<td>C(0) $E(0,1)$ $G(1,0)$ $B_b(0,0,1)$</td>
<td></td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ E♭7:</strong></td>
<td></td>
</tr>
<tr>
<td>$E_b(1,0,1)$ $G(1,1,1)$ $B_b(0,0,1)$ $[D♭(1,0,2)]$</td>
<td>7 $\rightarrow$ 5</td>
</tr>
<tr>
<td>C(0) $E(0,1)$ $G(1,0)$ $B_b(0,0,1)$</td>
<td></td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ Eb (Cø7) = counter-seventh chords</strong></td>
<td></td>
</tr>
<tr>
<td>$[C(0)]$ $E_b(1,0,1)$ $G_b(0,1,1)$ $B_b(0,0,1)$</td>
<td>1 $\rightarrow$ $\natural$ and 7 $\rightarrow$ $\natural$</td>
</tr>
<tr>
<td>C(0) $E(0,1)$ $G(1,0)$ $B_b(0,0,1)$</td>
<td></td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ G♭7:</strong></td>
<td></td>
</tr>
<tr>
<td>$G_b(0,1,1)$ $B_b(0,0,1)$ $D♭(1,1,1)$ $[F♭(0,1,2)]$</td>
<td>7 $\rightarrow$ 3</td>
</tr>
<tr>
<td>C(0) $E(0,1)$ $G(1,0)$ $B_b(0,0,1)$</td>
<td></td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ B♭m (Gø7):</strong></td>
<td></td>
</tr>
<tr>
<td>$[G(1,0)]$ $B_b(0,0,1)$ $D♭(1,1,1)$ $F(1,0,1)$</td>
<td>5 $\rightarrow$ $\natural$ and 7 $\rightarrow$ $E$</td>
</tr>
<tr>
<td>C(0) $E(0,1)$ $G(1,0)$ $B_b(0,0,1)$</td>
<td></td>
</tr>
<tr>
<td><strong>C7 $\rightarrow$ D♭m (B♭ø7) = Septimengleicher</strong></td>
<td></td>
</tr>
<tr>
<td>$B_b(0,0,1)$ $D♭(1,0,2)$ $F♭(0,1,2)$ $A♭(0,0,2)$</td>
<td>7 $\rightarrow$ $\natural$</td>
</tr>
<tr>
<td>C(0) $E(0,1)$ $G(1,0)$ $B_b(0,0,1)$</td>
<td></td>
</tr>
</tbody>
</table>

Tonnetze are provided on the following pages.
10.9. (continued)

C7 → B♭7:

C7 → Gm (E∅7):

C7 → Eb7:

C7 → Eb m (C∅7) = counter-seventh chords
Both chords have $B_{b(0,0,1)}$ as their concordant seventh. The entire $B_{b7}$ chord is derived from that pitch, and is thus seventh-derived:

$$B_{b(0,0,1)} \ D_{b(-1,0,2)} \ F_{b(0,-1,2)} \ A_{b(0,0,2)}$$

This is the only instance in *Akustische* where Karg-Elert generates double concordant sevenths – thereby adding another layer to the third dimension of pitch space. (See also the final Tonnetz and discussion in 10.8., which suggests a similar case).
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Chapter 11

The principle of dissonance
Chapter 11
The principle of dissonance

11.1. The canonic values of non-harmonic tones
11.2. Chords of motion, and coincident harmonies
11.3. Coupling principal triads: twins and triplets
11.4. Augmented triads and their resolutions
11.5. Augmented sixth chords
11.6. Collective-change chords
11.7. The diminished seventh chord: acoustic derivation
11.8. The diminished seventh chord: functional meaning
11.9. The diminished seventh chord: resolutions for all three types, in different keys
11.1. The canonic values of non-harmonic tones

“Chords of motion” are understood as chordal formations that result when a melodically-driven voice becomes detached from a complete harmonic complex. Whether approaching or departing from, passing through or suspended into an existing harmonic structure, a single voice can create various distinct non-syntonic tensions.

As long as the individual tones of a melodic voice are not “captured” as integral members of the harmonic masses – and thus transformed into syntonic values – those “non-harmonic” prefix and suffix tones retain their Pythagorean values:

In a harmonic sense, C♯ is the third of the A major triad, D♯ the third of B major, E the third of C major, F♯ the third of D major, A the third of F major, and A♭ the third of F minor:

Lowered: C♯\(_{0,1}\) or C♯\(_{3,1}\), D♯\(_{2,2}\), E\(_{0,1}\), F♯\(_{2,1}\), A\(_{1,1}\)
Raised: A♭\(_{0,1}\)

However, these syntonic values are in no way present, as all of the above tones are simply melodic departures from the harmonic tones D and G, and therefore are of primary canonic type:

high, bright: C♯\(_{7,0}\), D♯\(_{9,0}\), E\(_{4,0}\), F♯\(_{6,0}\), A\(_{3,0}\) and
low, dull: A♭\(_{4,0}\)
11.1. *The canonic values of non-harmonic tones*

Karg-Elert is stating that when a melodic tone belongs to an underlying harmony, its acoustic value will match that of the harmonic tone. On the other hand, melodic pitches that do not belong to the harmony retain their canonic (Pythagorean) values. In bar 1, the C♯ and D♯ are linear alterations of the D, and are thus dissonant pitches that are canonic in origin. **Linear alteration** of chord tones is one of two principal sources of dissonance discussed in this chapter. The other source of dissonance is **functional mixture**, to be discussed below.

Karg-Elert comments below on the melodic B in bar 2, beat 4 (marked ?).

These are the syntonic (harmonic) values for the circled non-harmonic tones in the passage above. Karg-Elert states that these harmonic values are not present in the example. Instead…

…the passage contains the canonic pitches listed here, as non-harmonic tones.
The leading tone B in measure 2 is problematic: while as third of the G major harmony [note that the tenor drops the dominant third!] B has a syntonic value, as a striving tone in the kinetic sense it must unquestionably be described as a limma, as it also is in any case in the penultimate measure, as a motion within the C major harmony.
While we may likely perceive the soprano B in bar 2 (beat 4) as a chord tone in the dominant harmony, it can also be understood in melodic/Pythagorean terms as an escape tone to the C. This suggests that the canonic or syntonic identity of melodic pitches may sometimes be ambiguous.
11.2. Chords of motion, and coincident harmonies

Chords of motion are created by means of:

1. Neighbor tones
2. Double neighbor tones
3. Passing tones on weak beats
4. Unprepared prefix tones
5. Anticipations
6. Suspensions *(prepared or unprepared)* on strong beats
11.2. *Chords of motion, and coincident harmonies*

All of the pitches marked X are non-harmonic, linear alterations of chord tones. They are dissonant, and will be therefore be canonic (Pythagorean) in origin.

In this example (and the following one as well), the numbers 1 to 6 label the types of non-harmonic tones.
Chords of motion can appear as real dissonances, or as feigning consonances:

Bewegungsklänge können real dissonant oder scheinkonsant auftreten:

Considering each quarter-note simultaneity in itself, the following harmonic complexes result:

Betrachtet man jedes Viertel für sich, so ergeben sich folgende harmonische Komplexe:

Akkorde = triads  Funktionsnoten = functions

However, in terms of “harmonic” function, most of the above “chords” have no independent value; rather, they act much more as “mobilizations” of the following harmonies:

Im „harmonischen“ Funktionssinne dagegen wird der größere Teil der „Akkorde“ gar nicht selbst ständig gewertet; sie fungieren vielmehr als „Bewegungsbildungen“ folgender Harmonien:

Spezifiziert bezeichnete Bewegungsklänge = chords of motion notated in specific detail. The symbols are not explained here!

⇒ Only those harmonies that are indicated by function symbols are truly syntonic entities!
11.2 (continued)

Scheinkonsonanz (“feigning” or “apparent” consonance) is an important concept in Riemann’s theories: a Scheinkonsonanz sounds like a consonant triad, but is in fact conceptually dissonant since it replaces a normal chord tone with a dissonant substitute. For example: in C major, the tonic parallel (Tp) is C-E-A. It sounds consonant, but it replaces the fifth (G) of the tonic triad with its “characteristic dissonance” of the added sixth (A), and is therefore a Scheinkonsonanz (Riemann 1895, 71).

Karg-Elert uses the term scheinkonsonant rarely, and with a different sense to Riemann. Here, it seems to refer to vertical simultaneities that are consonant triads, but which result from melodic elaboration rather than true harmonic motion. As these melodic elaborations involve non-harmonic tones, they create “feigning consonances.”

The passage is in C major, and every vertical sonority is analyzed in that key (but with an excursion to A minor in bars 4-7). A good example of a Scheinkonsonanz is the F major triad in bar 2 (beat 2); while it is a principal harmony in C major (and therefore not an “apparent consonance” in the Riemannian sense), it arises purely as a weak-beat melodic motion, not from a true harmonic movement (see the following reduction).

This reduction of the above passage outlines the true harmonic progression, with each chord shown in relation to the Ursprungslagen.

Under this grand staff, Karg-Elert analyzes the linear alterations, introducing three new symbols without explanation!

∧ or ∨ = raise or lower by a whole tone. In bar 1 (beat 2), the root and third stay the same, but the fifth is raised: G → A. In bar 3 (beat 2), the fifth of the C (contrant) is raised: C → D.

∩ or ∪ = raise or lower by a diatonic semitone. In bar 2 (beat 2), the root and third stay the same, but the fifth is raised: E → F.

¬ = lower by a chromatic semitone. Upward hook = raise by a chromatic semitone.

Bar 3, beat 3 = all pitches of the contrant F major triad are altered melodically. The root is lowered by a diatonic semitone (∪): F → E. The third is both raised by a diatonic semitone (∩) and lowered by a whole tone (∨): A → B♭ and A → G. The fifth is raised by a chromatic semitone: C → C♯. The resulting simultaneity is a diminished seventh chord, but Karg-Elert treats it here as a non-harmonic weak-beat elaboration of the contrant.
All so-called “coincident harmonies” result from the simultaneous sounding of linearly motivated individual voices; they therefore demonstrate canonic [asyntonic] values. In addition, all feigning consonances (suspended first- and second-inversion triads, structures with augmented sixthse/harmonic natural sevenths, and so on) must be considered dissonant.
Once again, Karg-Elert suggests that *Scheinkonsonanz* refers to consonant-sounding simultaneities that result from melodic elaboration. Notably, Karg-Elert states that augmented sixth chords can be feigning consonances, as they are similar to concordant seventh chords (which are classified as **consonant**). See 11.5 for more on augmented sixth chords.

Karg-Elert clearly reiterates what has been stated above: linear alterations of chord tones (raised or lowered by semitones or whole tones) “demonstrate canonic (Pythagorean) values.” However, some examples later in this chapter reveal inconsistencies on this point; in particular, there are cases where function labels and specified acoustic information seem to contradict each other. Such cases will be discussed below.
11.3. **Coupling principal triads: twins and triplets**

Another source of dissonance is the coupling of two (or possibly three) individual harmonies, which can be in complete or incomplete voicing.

![Musical notation image](image.png)
11.3. Coupling principal triads: twins and triplets

Zwillinge or “twins” combine two harmonies, of different functions. They provide our first example of functional mixture, which is another main source of dissonance.

The first two examples (bars 1-4) feature combinations of two incomplete chords. The small circles indicate which chord tone(s) are omitted; when the circle is placed in the middle of the function label (as in bars 1-2), the chordal third is omitted. In bars 3-4, the two circles indicate that the dominant seventh (D or G) lacks both the third and fifth. The stacking of the function labels matches the vertical placement of the pitches.

The last bar of the upper system features minor-minor and major-major seventh chords, which combine two (complete) versions of a single principal harmony (not two different functions). The Parallelzwilling combines a principal harmony and its parallel transformation (in this case T + Tp); Leittonzwilling combines a principal harmony and its Leittonwechsel (in this case T and T♭). See the example on the next page for a demonstration of the parallel and leading-tone twins.
The coupling of principal chords and their diatonic substitutes create parallel and leading tone twins:

\[
\text{ohne Vorzeichnung} = \text{no key signature (as in the example)}
\]

\[
\text{Vorzeichnung} = \text{key signature (added to the example)}
\]

\[
P.Z. = \text{Parallelzwilling (parallel twin, combining a principal harmony and its parallel)}
\]

\[
L.Z. = \text{Leittonzwilling (leading-tone twin, created by combining a principal harmony and its Leittonwechsel)}
\]

\[
\mu \text{ values of the parallel twin}
\]

\[
\mu \text{ values of the leading-tone twin}
\]

\[
\mu \text{ values of } \square
\]

\[
\mu \text{ values of } \bar{D}
\]
11.3. (continued)

This passage contains a sequence of three chords (indicated by the brackets above the staff), repeated in descending diatonic steps.

These rows identify the chord types, in different key signatures from 0 to 3 sharps, and then with 1 or 2 flats. With the key signatures of no sharps/flats, 2 sharps and 2 flats, the key is assumed to be minor, due to the presence of half-diminished seventh chords (labelled as $\text{A}$). Conversely, the keys with 1 and 3 sharps and 1 flat are assumed to be major, due to the dominant seventh chords (labelled as $\text{D}$). Minor-minor sevenths are parallel twins (P.Z.), and major-major sevenths are leading-tone twins (L.Z.); harmonic functions are not specified here.

The dynamic markings above the μ values indicate the symmetrical interval sizes in the parallel and leading-tone twins, contrasted with the unidirectional increase or decrease of interval size in the dominants with concordant seventh (refer to 9.1).
The diatonic “triplet” results from the combination of the three fifth-related principals \(i.e. T + C + D\), with their peripheral chord-tones omitted:

\[
\begin{array}{c|c|c|c}
C & T & D \\
\end{array}
\]

It is an amalgamation of a principal chord and both of its \{diatonic\} substitutes \{for example, \(T + T^p + T^l\)\}:

\[
\begin{array}{c|c|c}
C & T \\
\end{array}
\]

\(C = \text{F major}\)

\(+ C^l = \text{A minor}\)

\(+ C^p = \text{D minor}\)
The minor ninth chord (i.e. minor-minor seventh + major ninth) or diatonic Drilling (“triplet”) is first explained as a case of **functional mixture**, using a combination of pitches from three different functions: $C + T + D$.

However, it is also a combination of three versions of a single function, such as $T + Tp + Tl$.

The function labels for Drilling can vary in subtle ways, and seem to reflect chord spacing. The example is in C major:

$$ C_p^l = C + C^l + Cp \quad \text{i.e. F major + A minor + D minor} $$

$$ C^\Delta = C + C^l + \text{added sixth (}\Delta\text{)} \quad \text{i.e. F major + A minor + D} $$

While $C^\Delta$ has exactly the same pitch classes as $C_p^l$, the D is not from $Cp$, but is rather a dissonant added sixth above F(1,0): the canonic pitch D(2,0) instead of D(2,1).

$$ C_M^\Lambda = C_M + C_M^l + \text{added sixth} \quad \text{i.e. D}^\flat \text{ major + F minor + B}^\flat $$

This example presents a problem: what is the acoustic status of the added sixth B$^\flat$? If it is understood as a linear alteration, we would expect it to be the canonic pitch B$^\flat_{(2,0)}$. But that very pitch is found in $C_M^p$, suggesting that the “p” should be included in the function label. Or, we might conceive the added sixth above D$^\flat_{(2,-1)}$ as B$^\flat_{(2,-1)}$, analogous to the previous chord. This case illustrates how the acoustic derivation of linearly-altered chord tones can be ambiguous, and is not always clear from the function labels.

$$ T^\Lambda = T + T^l + \text{added sixth} \quad \text{i.e. C major + E minor + A} $$

The horizontal line at the bottom of the $T$ label confirms that the tonic prime (C) is in the bass. This chord presents us with another issue: the A is specified here as $A_{C(1,1)}$, the root of the tonic parallel ($Tp$). However, the function label does not include the “p,” and instead indicates the A as a non-harmonic added sixth. Based on previous statements, this added sixth should be conceived as the canonic pitch $A_{C(3,0)}$, not as syntonic $A_{C(1,1)}$. Once again, there is a degree of inconsistency between the dissonant (canonic) added sixth and the consonant (syntonic) root of the parallel. This detail is also relevant to the next example.
If the pitch A belongs to a principal D major triad \( \text{as in the final chord in this example} \), it becomes the fifth-type \( A_{(3,0)} \), and thereby renounces its affinity with the third types \( B_{(1,1)} \) and \( E_{(0,1)} \):

\[
\text{Nonenverwandter des Fundaments} = \text{ninth-related to the fundamental (i.e. the D major triad, generated by the ninth above the fundamental C).}
\]

\[\text{[A more complex example of chord couplings and substitutions]:}\]
11.3. (continued)

This example highlights the distinction between tertian extensions of a single function on the one hand, and polychords on the other. In the first bar, the pitch A is drawn from the tonic parallel \( Tp \), and is thus the syntonic pitch \( A_{(1,1)} \). However, the final chord (bar 2) is a polychord, combining the C major tonic (with concordant seventh B\(_b\)) and its ultradominant (the D major triad, or \textit{Nonenverwandter des Fundaments}). The A in the ultradominant is canonic \( A_{(3,0)} \), and therefore distinct from syntonic pitches like B\(_{(1,1)}\) and E\(_{(0,1)}\).

Though this progression does not really establish any particular key, the functional analysis is in C major. Chord 4 is a \textit{Zwitter} or “hybrid”: the coupling of a function and its own variant, such as \( D \) and \( d \). In this example, the function label \( \hat{D} \) indicates the hybrid contrant of the following C minor (in other words, the combination of C and e, or G major and minor). The \textit{Zwitter} chord should actually have a line below its function label, as it includes its concordant natural seventh (F-natural). However, Karg-Elert does not always carefully indicate concordant sevenths, especially when almost every chord includes its seventh, as in this progression.

\[
\begin{align*}
\hat{C} & = T + \text{added sixth (\( \nabla \)) + concordant seventh (\( \check{\ } \)) + diatonic semitone-lowered ninth (\( \cup \))} \\
\hat{D} & = D + \text{diatonic semitone-lowered fifth (\( \cup \)) + added sixth (\( \nabla \)) + concordant seventh + ninth }
\end{align*}
\]

Chords 6 and 7 are counter-concordants, as discussed in 10.3. The final chord \( T^M \) is a \textit{Tonika-Vertreter}: a mediant substitute for (or transformation of) the tonic C major chord. Its function symbol should include a line to show its concordant seventh.

The reduction at the right notates the progression in relation to the \textit{Ursprungslagen}. The most notable detail is that chord 1 (a polychordal combination of C7 + A major) includes two versions of E: the third of C major or E\(_{(0,1)}\), and the fifth of A major or E\(_{(4,0)}\). The brackets shown on chords 1, 2 and 4 reflect their polychordal derivation.
11.4. Augmented triads and their resolutions

All chords that contain an augmented fifth result from third relationships:

\[
\begin{align*}
\text{Dur} & = \text{major triads} \\
\text{übermäβ} & = \text{augmented triads} \\
\text{Moll} & = \text{minor triads}
\end{align*}
\]
11.4.  Augmented triads and their resolutions

This diagram suggests that augmented triads combine the syntonic major thirds from two Terzgleich-related triads, such as A♭ major and A minor, or C major and C♯ minor:

\[
\begin{align*}
A♭ & : C & Eb + A & : C & E

C & : E & G & + & C♯ & : E & G#
\end{align*}
\]

As the triads are Terzgleicher of each other, they result from third (mediant) relationships. They can represent different functions, or different versions of the same function:

\[
\begin{align*}
A♭ & : C & Eb & = & T_M & or & c_P \\
A & : C & E & = & T_p & or & C_L
\end{align*}
\]

Therefore, augmented triads can be understood as resulting from functional mixture. However, Karg-Elert’s function labels usually explain augmented triads as linear alterations of chord tones, as will be seen below.

The augmented triads in groups 1 to 4 (middle rows) include various enharmonic spellings. The major thirds in augmented triads are all syntonic thirds (322 µ).
Every major or minor chord can be preceded by any of four distinct types of augmented triad; likewise, every augmented triad can participate in four distinct types of cadential motion.

Type I (strongest) one common tone, two leading tones in the dominating direction
(plus a third leading tone in the case of $\bar{D}$ or $\underline{C}$)

Type II (weak) two common tones, plus one counter-leading-tone

Type III two counter-leading-tones (in parallel motion), plus one minor third leap [or a leading tone to the natural seventh]

Type IV (harsh) one leading tone to the fifth [and a metharmonic common tone with the seventh], or two whole steps when there is no natural seventh.

In C major, demonstrating Types I to IV:

In A minor, demonstrating Types I to IV:
11.4. (continued)

Explanation of the C major example:

**Type I** has the clearest and most familiar harmonic function: a dominant \((D)\) with a fifth raised by a chromatic semitone \((D \rightarrow D\#)\).

\[ \text{\underline{\text{I}}} \quad D \quad \text{with raised fifth.} \quad \text{\underline{\text{I}}} \quad D \quad \text{with raised fifth and concordant seventh} \]

The ties in the example indicate common tones, and the lines indicate leading-tone motions.

**Type II** is functionally ambiguous: either an contrant variant \((c)\) with a diatonic-semitone lowered root \((F \rightarrow E)\), or tonic parallel \((Tp)\) with chromatic-semitone lowered root \((A \rightarrow Ab)\). Karg-Elert seems to favor the first interpretation as contrant variant, due to the motion from Ab to G \((\text{^6\#} \rightarrow 5)\) in C major).

**Type III** has contrant function: either C with diatonic-semitone raised fifth \((C \rightarrow D\#)\), or \(Cp\) with chromatic-semitone lowered root \((D \rightarrow Db)\). Karg-Elert favors the first interpretation. The angles indicate voices that move by an interval larger than a semitone.

**Type IV** is especially ambiguous, and Karg-Elert suggests three possible interpretations: as minor dominant \((d)\) with diatonic-semitone lowered root \((G \rightarrow F\#)\), as ultradominant \((DD)\) with diatonic-semitone raised fifth \((A \rightarrow B\#)\), or as ultracontrant \((CC)\) with chromatic-semitone raised fifth \((F \rightarrow F\#)\). He seems to prefer the first analysis as minor dominant, due to the \(^2 \rightarrow 1\) and \(^7 \rightarrow 1\) motions to the tonic.

This is the A minor polar equivalent to the preceding example.
With the function of these basic triads:

```
C  F  Cm  E  Cm  B  C  D  Cm  Gm  C  Am  C  Dm  Cm  Bm
```

Kadenz = cadence ending on the tonic (most often D → T, but also C → T)
Halbkadenz = half cadence (normally T → D)
Harmoniesprünge = harmonic leap (i.e. not by fifth or third)
This example demonstrates different resolutions of the augmented triad spelled \([C, E, G\#]\) to various major and minor triads (some with concordant sevenths), in terms of the four types just described. Karg-Elert also specifies the major or minor triads from which the augmented triads (marked “x”) are derived, in order to help determine their function.

Type I is clearly cadential: \(D \rightarrow T\).

Type II is described as a half cadence (Halbkadenz), a term not previously introduced in this treatise, and not explained here. Karg-Elert previously described Type II as an alteration of the cadential motion \(c \rightarrow T\), or of the parallel motion \(Tp \rightarrow T\). Here, the annotation “C\#m \rightarrow E” might suggest \(T^k\) (with lowered root) \(\rightarrow D\) in A major: a half cadence in that key.

Types III and IV are classified as *Harmoniesprünge* (harmonic leaps) because the underlying root motions are by step rather than by fifth or third. In the two examples of Type IV, the dotted line on the staff indicates the metharmonic (i.e. non-equivalent) common tone to the concordant seventh of the second chord.
11.5. Augmented sixth chords

As with all augmented forms, the \textit{gapped} twins \textit{i.e. containing a Hiatus or augmented second} also result from third relationships.

The augmented sixth is “enharmonically” distinguished from the natural seventh:

\begin{align*}
\text{Rel. Sz} &= \text{frequency ratios} \\
\text{d.i. (das ist) } &= \text{that is}
\end{align*}

\begin{align*}
The \textit{different} values B_{\{0,0,1\}} &= 807 \text{ and } A^{\sharp}_{\{2,2,0\}} = 814 \text{ are both realized as } 811 \text{ on the so-called “Orthotonophonium” [a supposedly pure harmonium with 53 pitches in the octave!] Thereby the functional opposition between contrant (C) and dominant (D) is fully invalidated. In truth, the C major chord with } A^{\sharp} \text{ functions as C in } G \text{ major [as } A^{\sharp}_{\{2,2\}} \text{ rises to the G-major third } B_{\{1,1\}} \text{], while the C major chord with } B_{\flat} \text{ functions as D in } F \text{ major [as } B_{\flat}_{\{0,0,1\}} \text{ falls to the F-major third } A_{\{1,1\}} \text{].}
\end{align*}

\text{Sz = Schwingungszahl (frequency)}
11.5.  Augmented sixth chords

The focus here is entirely on the German augmented sixth: a consonant triad plus an augmented sixth. The Italian would be the same type with an omitted fifth, while the French lowers the fifth by a diatonic semitone.

The augmented sixth interval (814 µ) is equal to two fifths plus two syntonic major thirds:

\[(585 \times 2) + (322 \times 2) = (1)170 + 644 = 814 \mu\]

The acoustic data indicate that while the augmented sixth (814 µ) is indeed distinct from the concordant seventh (807 µ), the difference is small (only 7 µ)…

Nonetheless, the difference of 7 µ is too much for Karg-Elert to overlook, as he specifies it as a limitation of Arthur von Oettingen’s 53-tone Orthotonophonium reed organ (first built in 1914, and described fully in Oettingen’s final treatise on music, entitled Die Grundlagen der Musikwissenschaft und das duale Reininstrument (Oettingen 1917).

The intervals of resolution differ by a larger amount: 70 µ for the concordant seventh, versus 93 µ for the augmented sixth. Those resolutions are indicated by the arrows.

\[\overset{\wedge}{\text{C}}\]  = contrant (C) + added augmented sixth

This function label indicates the augmented sixth as a linear alteration of the fifth, raised by an augmented second (or Hiatus). However, Karg-Elert notes above that the augmented sixth “results from third relationships,” which is indeed the case:

\[\text{C E G} = \text{contrant (C) of G major} \quad \text{F}^\# \text{A}^\# \text{C}^\# = \text{dominant-mediant (D}^M\text{) of G major}\]

Thus, the augmented sixth chord is derived from functional mixture, even if such mixture is not reflected in the function labels. Karg-Elert confirms this point on the next page.
The so-called augmented “sixth” should be properly understood as an octave-reduced “augmented thirteenth.”

Type I: Cadential form
1 common tone, 2 leading tones converging onto the third

Type II: Mediantic form
1 common tone, 2 leading tones to the fifth; a chroma to the third

Type III: Leading-tone form
no common tones; 4 leading tones: 1 to the prime, 3 moving in parallel motion to the complete triad

traditional jazz cadence

TN
This example illustrates how augmented sixth chords combine pitches from diverse functions. The first three examples are in C major, and the last three are in A minor.

In C major:

- Contrant \((C)\) + added augmented sixth, F major + D\(^\#\)
- Parallel of contrant variant \((c^P)\) + aug. sixth, A\(^b\) major + F\(^\#\)
- Counter-mediant of contrant \((C_M)\) + aug. sixth, D\(^b\) major + B-natural

Explanation of the C major examples:

**Type I** is a \(C \rightarrow T\) cadential motion; the added augmented sixth heightens the pull to the tonic.

**Type II** contains the notes of the German augmented sixth as it is normally defined: scale degrees \(^\#6, 1, b3\) and \(4\) (of a major key). The function label does reflect this definition, as \(c^P\) is rooted on \(^\#6\). However, in Karg-Elert’s C major example it does not resolve in the traditional way to a dominant (with or without cadential 6/4); instead, it acts as a common-tone prefix or embellishment of the tonic chord. (“Am Meer” from Schubert’s Schwanengesang begins with this motion, with exactly the same key, voicing and doubling as Karg-Elert’s example.)

**Type III** is labelled as a contrant counter-mediant \((C_M)\), which it is rooted on \(^b2\). However, the tritone interval F + B (and its contrary-motion resolution) strongly suggests dominant function. The B is indeed the third of the dominant \((D) = B_{(1,1)}\) at 907 \(\mu\). But the F is the third of \(D_{b(-1,-1)}\), and thus is not acoustically equivalent to the concordant seventh of \(\overline{D}\):

\[
\begin{align*}
F \text{ in } C_M &= F_{(-1,0)} \\
&= 415 \mu \\
F \text{ in } D &= F_{(1,0,1)} \\
&= 392 \mu
\end{align*}
\]

While augmented sixth chords always combine pitches from different functions in this way, Karg-Elert’s function labels are based on the underlying complete triad.
11.6. Collective-change chords

The two types of collective-change chords
In concordant form, the two types result from third- and seventh relations:

{Type I}

Doppelkomma-Differenz = double comma difference
Kommadifferent = comma-differing
Kommarein = comma-free
11.6 Collective-change chords

The basic attribute of collective-change chord pairs is that all voices move by semitone; they share no common tones. They change (Wechsel) quality from major to minor, or vice versa.

Type I collective-change chords can result from third (mediant) relationships, such as:

C major \( \to \) G\# minor \((T \to D^{Mp})\)  
C major \( \to \) A\# minor \((T \to T_m)\)

A minor \( \to \) D\# major \((\, \downarrow \to \, A_{Mp})\)
A minor \( \to \) C\# major \((\, \downarrow \to \, J^m)\)

They can also result from seventh relationships, specifically the counter-concordants:

C major \( \to \) A\# minor \((T \to C\, \uparrow)\)  
C major \( \to \) A\# minor \((T \to C\, \uparrow)\)  
i.e. A\# minor is counter-concordant of F major

Type I collective-change chords are equivalent to Richard Cohn’s hexatonic pole (Cohn 1996; see the addendum to section 7.8).

Here is detail on the first system, which is in C major:

#1. Modulation from C major to A\# major, using D\# as a pivot. The first two chords are Type I collective-change chords. \( D^{Mp} \) contains E\#\( (1,2,-1) \), G\#\( (0,2) \), B\( (1,1) \), D\#\( (1,2) \). The final tonic is A\#\( (3,2) \), so the modulation acquires a double syntonic comma.

#2. Modulation from C major to E\# minor, using \( T_m \) as a pivot. The first two chords are also Type I collective-change chords, though the different spelling of the second chord reflects its different function and key of resolution. \( T_m \) contains F\#\( (1,-1) \), A\#\( (0,-1) \), C\#\( (1,-2) \), E\#\( (1,-1) \). The final tonic is E\#\( (1,-1) \), so the modulation acquires a syntonic comma.

#3. Modulation from C major to its double-ultracontrant A\# major. There are no pivot chords, and no “variant tricks.” The first two chords are Type I collective-change chords; the second chord (F\#7) then acts as the counter-concordant \( D\, \uparrow \) of F7, or the dominant of B\# major (or \( C\# \) of C major). The same pattern is repeated a whole tone lower in measures 2-3. The modulation is comma-free (Kommarein), as the final chord’s root is A\#\( (4,0) \).

The lower system is the polar counterpart in A minor, though the voice leading is different.
Type II [concordant - tritonal variants]

mit Septimen = [each chord] with natural sevenths   im Sinne von = with the function of
11.6. (continued)

As shown here, **Type II collective-change** chord pairs are concordant (*konkordierend*) – that is, both chords in the pair contain their natural sevenths. Once again, all voices move by a semitone: the three notes that form a diminished triad move together in parallel, while the remaining voice moves in the opposite direction. The chord quality switches from dominant seventh to half-diminished seventh, or vice versa. They are called *tritonantische Varianten*, as their chord roots are a tritone apart, and are also variants (as there is one comma-different tone between the chords, shown here as a black notehead). The brackets above the top staff indicate the Type II collective-change chord pairs.

The functional analysis of both passages is in C major. Once again, the Type II collective-change chords involve third (mediant) relationships. The triads listed under the function labels (mostly F, C and G major) reflect the basic harmonic functions (*C, T* and *D* respectively).

The concordant sevenths are not indicated at all in the labels, though Karg-Elert specifies *mit Septimen* (with sevenths). If all of the sevenths were omitted, the analysis would be the same.
True collective-change chords are markedly dissonant entities created by the fourfold rise and fall of prefix or suffix leading tones.

Both types I and II of the collective-change chords are derived from the combination of two distinct harmonies $[C_M + D^m]$, or $\omega^m$ and $D_M$:

The dissonant character of the collective-change chords is evident!
In place of the syntonic minor third $G^#/B$ ($A_{b}/C_{b}$), or $F/A_{b}$ ($E^#/G^#$) $= 263$ is the syntonic hiatus $A_{b(0,1)}/B_{(1,1)}$ or $F_{(-1,0)}/G_{(0,2)} = 229$.
11.6. (continued)

**ERRATA**: the last two bars in the first system should be headed “II,” as they are Type II collective-change chords. In bar 4, the second chord should not contain A in the bass – the correct chord is [C, Eb, Gb, G#].

The triangles denote major (Δ) and minor (∨) triads or concordant sevenths.

The curves under the middle chords show the voice leading: diatonic semitones up (○) or down (∪). In Type I, chord tones 1 and 3 move down, while chord tones 3 and 5 move down; chord tone 3 splits into a diminished third. In Type II, chord tones 3, 5 and 7 (which form a diminished triad) move together, while chord tone 1 (the prime) moves in the opposite direction.

What Karg-Elert calls “true” collective-change are “markedly dissonant entities” derived from **functional mixture**, combining prefix and suffix leading tones derived from different functions, as shown in the lower example.

Here is Type I in C major (bar 1 of the example):

- C → B ○
- E → D♯ and E → F ○ and ∪
- G → Ab ○

This example illustrates that the B and D♯ are from $D^M$, while the F and Ab are from $C_M$. The pitch pairs are acoustically distant from each other: $B_{(1,1)}$ $D^*_{(1,2)}$ $F_{(-1,0)}$ $Ab_{(0,-1)}$

Here is Type II in C major:

- C → Db ○
- E → D♯ ○
- G → F♯ ○
- Bb → A ○

The Db is from $C_M$, while the other three pitches are from $D^M$.

The collective-change chords on the previous pages are spelled differently than these “true” versions – they do not contain augmented or diminished intervals. Therefore, their acoustic values are somewhat different; they replace the syntonic hiatus or augmented second (229 µ) with a syntonic minor third (263 µ). Accordingly, they can be explained as transformations of a single function, as shown on the previous page.
11.7. The diminished seventh chord: acoustic derivation

The octave scale can be divided into three minor thirds. If all minor thirds are of the same size, the following values result:

First group = Pythagorean (canonic) minor thirds at $\frac{245}{12}$ $\mu$ (= three perfect fifths, octave-reduced)
Second group = consonant (syntonic) minor thirds at $\frac{263}{12}$ $\mu$
Third group = concordant (seventh-derived) minor thirds at $\frac{222}{12}$ $\mu$

In the context of tonal harmony, none of these four-note groups can create unified harmonic structures, as three intervals of the same size can never exist within the “collection” of a single, centralized key.
11.7. The diminished seventh chord: acoustic derivation

These diagrams are perfect fifth/minor third Tonnetze, in which all horizontal segments of four adjacent pitches are diminished seventh chords. They are of course just-intonation Tonnetze: all horizontal note pairs are true minor thirds (with no enharmonic respelling, as is usual with diminished seventh chords in equal temperament). The diagrams differ in the interval size/acoustic derivation of the minor thirds, as indicated.

The vertical lines indicate perfect fifths (always 585 \(\mu\)). The diagonal lines indicate other intervals: major thirds, major seconds, diatonic semitones, chromatic semitones. The sizes of those intervals vary: they are canonic in the first group, syntonic in the second group, and concordant in the third. Appendix A provides \(\mu\) values for almost all of these pitches.

A reminder of German pitch nomenclature used here:

- Fisis = F\(: =\ A\)is = A\(\uparrow\)
- Cis = C\(\uparrow\)
- B = B\(\downarrow\)
- Des = D\(\downarrow\)
- Fes = F\(\downarrow\)
- His = B\(\uparrow\)
- Dis = D\(\uparrow\)
- Fis = F\(\downarrow\)
- Es = E\(\downarrow\)
- Ges = G\(\downarrow\)
- Bes = B\(\uparrow\)
- Eis = E\(\uparrow\)
- Gis = G\(\uparrow\)
- H = B
- As = A\(\downarrow\)
- Ces = C\(\downarrow\)
- Eses = E\(\downarrow\)

**ERRATA** in these diagrams:
Second group: the interval size of the consonant (syntonic) minor third is \(=\)263 \(\mu\), not 265 \(\mu\)
Third group, second row: the last two pitch names should be “ges” and “bes”

This is a challenging statement whose meaning is not easy to extract. The main point is that a diminished seventh chord that contains three minor thirds of equal size (such as all of those on the preceding chart) cannot exist within a single centralized key, with all of its correct canonic and syntonic intervals. Instead, if a diminished seventh chord is to have true harmonic function and status, it must combine minor thirds of different sizes and acoustic derivations, as will be discussed in this section.
The *melodic* repetition of identical intervals of course produces the following:

However, these linear values certainly do not comply with syntonic demands; the minor thirds do not “blend” with each other in either consonant [263 μ] or concordant [222 μ] terms.
11.7. (continued)

Here, Karg-Elert creates Pythagorean diminished seventh chords on C♯ and F♯, containing equal canonic minor thirds (245 µ). Chords containing only canonic minor thirds do not blend syntonically, and therefore cannot serve as harmonic structures (see 4.3 and 4.8).

This type of diminished seventh chord can be called **PPP**, refers to the three Pythagorean minor thirds (245 µ). In total, section 11.7 introduces five different versions of the diminished seventh, containing the following intervallic patterns:

<table>
<thead>
<tr>
<th>PPP</th>
<th>SSS</th>
<th>SPS</th>
<th>CSC</th>
<th>SCS</th>
</tr>
</thead>
<tbody>
<tr>
<td>P =</td>
<td>S =</td>
<td>C =</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pythagorean (canonic) minor third</td>
<td>consonant (syntonic) minor third</td>
<td>concordant (seventh-derived) minor third</td>
<td>245 µ</td>
<td>263 µ</td>
</tr>
</tbody>
</table>

As **PPP** (described here) is not a harmonic structure, it will not be seen again in the treatise.
Consonant minor thirds [263 \( \mu \)]:

- \( C^\#(1,2) \ E(0,1) \) Belongs to the mediant A major, or the counter-mediant C\(^\#\) minor triad
- \( E(0,1) \ G(1,0) \) Belongs to the principal C major or E minor triad
- \( G(1,0) \ Bb(2,-1) \) Belongs to the counter-mediant Eb major, or the mediant G minor triad

In a diminished seventh chord of this type, the interval sizes are uniform:

\[
\begin{align*}
C^\#(1,2) & \quad E(0,1) & \quad G(1,0) & \quad Bb(2,-1) \\
\text{cis} & \quad \text{cis} & \quad \text{cis} & \quad \text{cis} \\
263 & \quad 263 & \quad 263 & \quad 263 \\
\end{align*}
\]

However, a chord of this type is functionally untenable, as:

- \( C^\#(1,2) + E(0,1) \) resolves as contrant variant to D minor, but \( G(1,0) + Bb(2,-1) \) does not, and
- \( G(1,0) + Bb(2,-1) \) resolves as contrant variant to D major, but \( C^\#(1,2) + E(0,1) \) does not!

[See the following diagram]
Next, Karg-Elert discusses a diminished seventh containing three syntonic minor thirds (263 \( \mu \)), which will be called SSS.

To create an SSS diminished seventh chord from the above minor thirds, a combination of two different functions (both in mediant transformations) is necessary, as shown here.

This chord is “functionally untenable,” as it contains pitch pairs from two different keys, namely D minor and D major:

\[
\begin{align*}
C_\#(1,2) + E(0,1) & \text{ are in the contrant variant of D minor, and} \\
G(1,0) + Bb(2,-1) & \text{ are in the contrant variant of D major}
\end{align*}
\]

D minor and D major are not equivalent keys in any way, as their roots are acoustically distinct: \( D_{(2,1)} \) in D minor, and \( D_{(2,0)} \) in D major.
D minor  D major

\[
\begin{align*}
G_{(3,1)} & \quad Bl_{(2,0)} & \quad C_{(1,2)} & \quad E_{(0,1)} & \quad G_{(1,0)} & \quad Bl_{(2,-1)} & \quad C_{(3,1)} & \quad E_{(4,0)} \\
\text{ohne Oktavziffer} = \text{octave-reduced (without octave designation)}
\end{align*}
\]
11.7. (continued)

This example demonstrates the previous point. Look at the middle four pitches, which form an SSS diminished seventh (all 263 $\mu$ intervals):

\[
C^\#_{(-1,2)} + E_{(0,1)} \text{ resolve to D minor, while }
\]
\[
G_{(1,0)} + Bb_{(2,-1)} \text{ resolve to D major}
\]

This example also introduces a different type of diminished seventh chord, created within a single key by combining minor thirds from the dominant (D) and the contrant variant (e). Look at the first four pitches (D minor), then the last four pitches (D major):

<table>
<thead>
<tr>
<th></th>
<th>D minor:</th>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$C^#_{(-1,2)}$</td>
<td>$E_{(0,1)}$</td>
<td>$G_{(-3,1)}$</td>
</tr>
<tr>
<td></td>
<td>59</td>
<td>322</td>
<td>567</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>D major:</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C^#_{(3,1)}$</td>
<td>$E_{(4,0)}$</td>
<td>$G_{(1,0)}$</td>
</tr>
<tr>
<td></td>
<td>77</td>
<td>340</td>
<td>585</td>
</tr>
</tbody>
</table>

The minor thirds in these chords are not of equal size, as will be shown on the next page.
The minor thirds of the contrant variant and the dominant can work together to define tonal closure; but the diminished seventh degenerates into a transferred interval, as it simply results from the circuitous octave reduction and inversion of an augmented ninth.

The interval sizes within these doubly-cadential tetrachords:

\[
\begin{align*}
\text{D minor} & : & \text{D major} \\
C#(1,2) & E(0,1) & G(3,1) & Bb(2,0) & C#(3,1) & E(4,0) & G(1,0) & Bb(2,-1)
\end{align*}
\]

The Pythagorean minor third [245 µ] breaks the structure.
This example demonstrates the type of diminished seventh just introduced, combining pitches from the contrant variant (c) and dominant (D) of a single key.

When the pitches are shown in relation to the Ursprungslagen, the gap of an augmented ninth highlights the functional distance between the thirds from the contrant variant and dominant.

This type of diminished seventh contains both syntonic and Pythagorean minor thirds (263 and 245 µ respectively), and will be called SPS. It is described as “doubly-cadential” because it integrates two cadential motions: \( c \rightarrow T \) and \( D \rightarrow T \).
If two Leittonwechsel-related chords each generate their natural sevenths, and their primes are omitted, combined they form a diminished-seventh chord with the following interval sizes:

\[
\begin{align*}
C^\#_{(1,1,-1)} & \quad E_{(0,1)} & \quad G_{(1,0)} & \quad Bb_{(0,0,1)} \\
& & & \\
\end{align*}
\]

This form cannot be used cadentially, as \([C^\#_{(1,1,-1)} \quad E_{(0,1)} \quad G_{(1,0)} + \text{implied } B_{(1,0)}] \) resolves to \(B\) minor, while \([E_{(0,1)} \quad G_{(1,0)} \quad Bb_{(0,0,1)} + \text{latent } C_{(0)}] \) resolves to \(F\) major.

In the first case (the key of \(B\) minor), \(807\) is much too low; it should be \(837 = A^\#_{(0,2,-1)}\), which is a Leipzig semitone (“gliding tone”) below \(B_{(1,1)} = 837 : 907 = 70\).

In the second case (the key of \(F\) major), \(100\) is far too high to “glide” to \(C(0)\); it should be \(70 = D^\#_{(1,-1,1)}\), which is a Leipzig semitone (“gliding tone”) above \(C(0) = 70 : 0\).

stets = consistently
Gleittöne = “gliding tones”
Leittöne = leading tones (i.e. leading to the prime)
Here, Karg-Elert describes a different type of diminished seventh chord, containing syntonic and **concordant** minor thirds (263 and 222 $\mu$ respectively). It can be labelled **CSC**.

The two Leittonwechsel-related chords here are $T$ and $T^L$ in C major: i.e. C major and E minor. Both chords generate their concordant sevenths, and the primes are omitted:

(C) $\boxed{\begin{array}{c} E \ G \ B^b + C^\sharp \ E \ G \end{array}} \ (B)$

The function of the **CSC** diminished seventh chord is highly ambiguous, as its constituent pitches resolve to different tritone-related keys.

This example demonstrates the interval sizes and resolutions found in the most common and important variety of diminished seventh chord, to be introduced on the next page.
This is the only case in which a unified [continuous] i.e. single-functioned diminished seventh chord can be described as a natural seventh chord [concordance] with the prime omitted, and with lowered upper- or under-ninth.

\[
\begin{align*}
A^\#_{(0,2,1)} & \quad C^\#_{(1,1,-1)} & \quad E_{(0,1)} & \quad G_{(1,0)} & \quad B_{(0,0,1)} & \quad D_{(1,-1,1)}
\end{align*}
\]

erratum:

enhar. Oktave = enharmonic octaves
konsonierend = consonant [syntonic]
konkordierend = concordant [seventh-based]
11.7. (continued)

Finally, Karg-Elert now presents the version of the diminished seventh chord that will be most familiar: as a concordant seventh chord with omitted root and added minor ninth. The two lines above/below the \( D \) function label indicate the concordant seventh and added ninth. The downward/upward hook indicates that the ninth is lowered/raised by a chromatic semitone.

Note that the minor ninth is a concordant (seventh-derived) pitch, as it is a minor third \((263 \mu)\) above/below the concordant seventh.

**ERRATA**: in the first chord in the B minor example, the lowest diamond-shaped note should be \( A^\# \) (not A-natural). There should also be a small circle to the upper right of the \( D \) function label, to indicate the omission of the prime (B).

B minor: \[ A^\#_{(0,2,-1)} \quad C^\#_{(1,1,-1)} \quad E_{(0,1)} \quad G_{(1,0)} \]
\[
837 \quad 100 \quad 322 \quad 585 \mu
\]

F major: \[ E_{(0,1)} \quad G_{(1,0)} \quad B_{b(0,0,1)} \quad D_{b(1,-1,1)} \]
\[
322 \quad 585 \quad 807 \quad 70 \mu
\]

Intervals: \[
263 \quad 222 \quad 263 \quad 263 \mu = \text{syntonic minor third} \\
222 \mu = \text{concordant minor third}
\]

Therefore, the diminished seventh chord described here is of type SCS.

**ERRATUM**: \( D_{b}(“\text{des”)} \) in the last diagram should have only one dot. It is \( D_{b(1,-1,1)} \): the concordant natural seventh of \( E_{b(1,-1)} \), and a syntonic minor third above \( B_{b(0,0,1)} \).

The SCS diminished seventh chord is “unified” and “continuous,” as all of its pitches are generated from a single prime, express a single function, and resolve to the same key. Its function is dominant (\( D \)) or contrant variant (\( c \)), as will be described at the beginning of section 11.8.
11.8. The diminished seventh chord: functional meaning

Each of the tetrachords shown on page 71 [i.e. the beginning of 11.7] as a chain of three minor thirds can function as prefix or suffix chords to each individual triad.

First row, in C major

As contrant variant: \( B_{(1,1,1)} \ D_{(0,0,1)} \ F_{(1,0,0)} \ A_{b_{(0,1,1)}} \) \( D_{(0,0,1)} = 193 \text{ m} \) (raised by a Leipzig comma)

As dominant: \( B_{(1,1,0)} \ D_{(2,0,0)} \ F_{(1,0,1)} \ A_{b_{(2,-1,1)}} \) \( D_{(2,0,0)} = 170 \text{ m} \) (Ursprungslage)

Second row, in A minor

As dominant: \( G^\#_{(2,2,-1)} \ B_{(1,1,1)} \ D_{(2,1,0)} \ F_{(1,0,0)} \) \( D_{(2,1,0)} = 152 \text{ m} \) (lowered by a syntonic comma)

As contrant variant: \( G^\#_{(0,0,2)} \ B_{(1,1,0)} \ D_{(0,1,1)} \ F_{(1,0,1)} \) \( D_{(0,1,1)} = 129 \text{ m} \) (lowered by a syntonic comma + a Leipzig comma)
11.8. The diminished seventh chord: functional meaning

All of these four diminished seventh chords are of the SCS variety described at the end of 11.7: a concordant seventh with omitted root and added minor ninth. They are Karg-Elert’s Type Ia and Type Ie diminished sevenths (see the next page).

Such diminished seventh chords can have two different functions:

\[ \overline{c} \] = contrant variant (c) with omitted prime, concordant seventh and added minor ninth

\[ \overline{D} \] = dominant (D) with omitted prime, concordant seventh and added minor ninth

The function depends on the acoustic values of the pitches, and on musical context, as will be demonstrated below.

Type Ia has dominant function, while Type Ie has contrant variant function (see below).

This is the acoustic data for the pitches in each chord, depending on its function.

At the right, the four different µ values for the pitch class D are provided.

Leipzig (concordant) comma = 23 µ  
Syntonic comma = 18 µ
I. **Primary type** (applying respectively to C major $T$, or to A minor $\Lambda$)

Bewegungsform = neighboring form  
od.(oder) = or (listing different functional meanings for Type Ic)
Karg-Elert defines three types of diminished seventh chord (not the intervallic varieties described in 11.7). The three types can be described as follows:

Type I (primary) = dominant (D) or contrant variant (c) function
Type II (secondary) = common-tone diminished seventh, embellishing the tonic
Type III (tertiary) = linear embellishment of the dominant/contrant variant

Each of the three types is further divided into subcategories, according to functional and contextual meaning. There is some correlation between these subcategories and the intervallic construction of the diminished seventh chords (as described in 11.7). However, the provided Ursprungslagen analysis (top system) suggests that some examples are not true diminished sevenths as defined in 11.7, but simply chords of motion (Bewegungsformen) mixing pitches from different intervallic varieties of diminished seventh.

Type I (primary) = dominant (D) or contrant variant (c) function

Explanation of first system of examples, which are in C major:

Ia. Intervallic construction: SCS. Dominant with concordant seventh and added minor ninth. The dominant function is determined by its approach from the ultradominant (DD).

Ib. Intervallic construction: mixed. The A♭ here is not a minor ninth above the prime, but simply a diatonic-semitone raising of the prime (shown as М, attached to the D symbol). It is a “neighboring form,” a linear decoration. The entire measure is an extension of the dominant.

Ic. Intervallic construction: SPS, combining pitches from the diminished seventh combines pitches from the dominant and the contrant variant. The function label combining D and c seems most appropriate, though Karg-Elert provides alternative labels that do not combine functions.

Id. Intervallic construction: mixed. Another “neighboring motion”: the prime of the contrant variant is lowered by a diatonic semitone (C → B). The entire measure is an extension of the contrant variant.

Ie. Intervallic construction: SCS. The function as contrant variant (rather than dominant) is determined by its approach from the ultra-contrant (EC).
II. Secondary type (applying respectively to C major $T$, or to A minor $\underline{\lambda}$)

The structures labeled $\beta$ employ a value that differs from the tonic prime by a comma of 23 $\mu$. In such cases, the common tone $C - C$ or $E - E$ is replaced by motion of a comma:
11.8. (continued)

Type II (secondary) = common-tone diminished seventh, embellishing the tonic

The Type II diminished seventh always includes the tonic prime (C in C major, or E in A minor); in a few cases, the Type II diminished seventh contains a metharmonic of the prime. The other pitches are often derived from the dominant-median (D^M), so that function label is frequently used, in spite of the common-tone link with the tonic. Like Type I, Type II is divided into five subcategories, based on the acoustic derivation of the pitches (as indicated in the Ursprungslagen analysis). The function labels are meant to reflect the acoustic derivation.

In the C major examples (top system), the diminished seventh chords are marked X. The chords are not labelled with their subcategories. This may not be a simple omission by the publisher; there seem to be some confusion between the function labels and the Ursprungslagen acoustic analysis, which creates some issues in classifying the diminished seventh chords.

Here is an explanation of the four C major passages, and their function labels:

#1. Type IIa. Intervallic construction: SCS. Dominant-median (D^M) with concordant seventh and minor ninth. The C in this chord is C_{(2,0,1)} at 977 µ, which is metharmonic with the tonic C(0). In addition, the diminished seventh contains A_{(1,1,1)}, which is metharmonic with the final A_{(-1,1)}.

#2. Type IIb. Intervallic construction: mixed. Dominant-median (D^M) with raised root (B → C) and concordant seventh. According to Karg-Elert’s annotation, the two Cs connected by a dotted arrow (977 and 0 µ) are metharmonics, differing by a Leipzig comma of 23 µ. However, the Ursprungslagen analysis for Type IIb shows that the C is C(0), which is the tonic prime (not a metharmonic). That makes more sense, since the C is a linear alteration of B, and should therefore be a canonic pitch. Accordingly, the dotted arrow indicating the metharmonic should perhaps be omitted.

#3. This example is somewhat perplexing, as it does not clearly match any of the five subcategories. The third function label seems to be incorrect: it should likely be Tp, which is the tonic parallel (A minor) with concordant seventh (F^#). The diminished seventh is labelled as a tonic parallel (Tp) with the prime raised by a whole tone (Δ) and also lowered by a semitone (♭): E → D♭ and E → F♭. The diminished seventh appears to be of Type IID, which is the same as the next example, though here it is a modification of the tonic A minor. Its intervallic construction is mixed.

#4. Type IID. Intervallic construction: mixed. Contrant (C) with semitone-raised root (F → F♭) and added augmented sixth (D^♯).
III. **Tertiary type** (applying respectively to C major $T$, or to A minor $\underline{\lambda}$)

The tertiary type is never cadential. The diminished seventh and its resolution have the same function. The concordant {actually consonant} minor third of both chords is identical (double common tones). The prime expands out by a whole tone and also contracts in by a semitone [that is, “within” the chordal space], thus replacing the prime; therefore, the natural seventh and lowered ninth are added to the remaining third and fifth.

<table>
<thead>
<tr>
<th>G major</th>
<th>F minor</th>
<th>D minor</th>
<th>E major</th>
</tr>
</thead>
</table>

**ERRATUM:**

Der tertiäre Typ kadenziert überhaupt nicht. Verminderter Septimenakkord und Auflösungsklang haben die gleiche Funktion; die konkordante Kleinterz beider Klänge ist identisch (Doppel-Ligatur); die Prime tritt ganzförmig aus und leitförmig ein (d.h., in den Akkordrahmen) resp. die Prime wird unterdrückt, dafür tritt die Septime und engaligierte Nona zur leiteten Terz und Quinte.
ERRATUM: the minor third that is held in common is the consonant or syntonic minor third (i.e. the third and fifth of a consonant triad), not the “concordant” minor third as stated here.

As described here, the Type III diminished seventh is a melodic/linear embellishment of a dominant ($D$) or contrant variant ($c$); they extend a function, rather than change or establish a function. In the case of a major triad, the prime is both raised by a diatonic semitone ($\uparrow$) and lowered by a whole tone ($\nabla$); the opposite is true for minor triads. These alterations are attached to the function label, as shown in these examples.

Karg-Elert does not say anything about the acoustic derivation of the Type III diminished seventh. As dissonant linear alterations of the prime, we might assume that the altered pitches (notated as quarter notes) are canonic in origin. The function labels suggest this interpretation as well.
### 11.9  The diminished seventh chord: resolutions for all three types, in different keys

<table>
<thead>
<tr>
<th></th>
<th>E♭</th>
<th>G♭</th>
<th>B</th>
<th>D</th>
<th>F</th>
<th>A♭</th>
<th>C♭</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
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</table>

Kadenziert als I zum Fis-Dur Akkord = resolves as diminished-7th chord Type I to F♯ major
Geht als III zum Cis-Dur Akkord = moves as diminished-7th chord Type III to C♯ minor
12 Dur- und 12 Mollakkorde = 12 major and 12 minor triads

In the same way, the diminished seventh chords formed by the other chains of minor thirds [beginning with B♭ and F♯] can also lead to these 24 triads.

Further dissonant formations cannot be exhaustively discussed here. Without exception, they can be readily understood as fusions or combinations of structures previously discussed. Their analysis remains within the realm of pure harmony.
11.9.  *The diminished seventh chord: resolutions for all three types, in different keys*

A reminder of the three types of diminished seventh (from 11.8):
Type I (primary) = dominant/contrant variant function
Type II (secondary) = common-tone diminished seventh (with tonic prime as common tone)
Type III (tertiary) = linear embellishment of the dominant/contrant variant

Chord A = E\# G\# B D
- Resolves as Type I to F\# major and F\# minor
- Resolves as Type II to D major and A\# minor
- Resolves as Type III to C\# major and B minor

Chord B = G\# B D F
- Resolves as Type I to A major and A minor
- Resolves as Type II to F major and C\# minor
- Resolves as Type III to E major and D minor

Chord C = B D F A♭
- Resolves as Type I to C major and C minor
- Resolves as Type II to A♭ major and E minor
- Resolves as Type III to G major and F minor

Chord D = D F A♭ C♭
- Resolves as Type I to E♭ major and E♭ minor
- Resolves as Type II to C♭ major and G minor
- Resolves as Type III to B♭ major and A♭ minor

The resolutions listed here include 12 major and 12 minor triads.
Chapter 11: Tonnetze

11.3. Coupling principal triads: twins and triplets

Parallelzwilling (parallel twin) – combining $T$ and $T_p$, or $L$ and $L_p$

The following applies to both C major ($T + T_p$) and A minor ($L + L_p$):

Leittonzwilling (leading-tone twin) – combining $T$ and $T^L$, or $L$ and $L_L$

C major ($T + T^L$) A minor ($L + L_L$)

The dotted lines separate the two constituent triads.
11.3. Coupling principal triads: twins and triplets (continued)

Drilling (“triplet”) - an amalgamation of a principal chord and both of its diatonic substitutes (for example, $T + T'$):

$$C^l = C + C^l + C_p$$

$$C^{al} = C + C^l + \text{added sixth } (\Delta)$$

The dissonant added sixth is the canonic pitch $D_{(2,0)}$, or three fifths above $F_{(1,0)}$, the root of the contrant.

$$C^M_{al} = C^M + C^l + \text{added sixth } (\Delta)$$

A question: is the added sixth the canonic pitch $B_{(2,0)}$, which is actually the root of $C^M$? Or is it $B_{(2,1)}$, which would be three fifths above the root (as in the previous chord), but is not strictly speaking a canonic pitch? Karg-Elert does not resolve this issue.

$$T_{al} = T + T' + \text{added sixth}$$

Karg-Elert’s acoustic analysis suggests:

But his function label suggests:
11.3. Coupling principal triads: twins and triplets (continued)

On the last chord of the previous example: while the A is indicated as an added sixth (Δ) in the function label, the acoustic information specifies that it is A_{-1,0}, the root of the tonic parallel (Tp). The function label and the acoustic data thus contradict each other: according to statements in 11.1 and 11.2, linear alterations such as the added sixth are canonic pitches, while harmonic tones retain their syntonic values. Thus, in spite of his earlier statements, this example suggests that the root of the parallel and the added sixth were not always acoustically distinct in Karg-Elert’s estimation.

The same issue is raised by the next example. Is the A in the first bar drawn from the tonic parallel (Tp), or is it an added sixth?

Nonenverwandter des Fundaments = ninth-related to the fundamental (i.e. the D major triad)

According to Karg-Elert’s acoustic analysis, the A in the first bar is A_{-1,1}, which belongs to the tonic parallel (Tp), or to the contrant (C). In contrast, the top note A in the final chord is A_{3,0}, which is the fifth of the ultradominant (B_D), rooted on D_{2,0}:

Karg-Elert does not provide function labels for the last example, so the apparent discrepancies between labels and acoustic data are left unresolved.
11.3. Coupling principal triads: twins and triplets (continued)

{A more complex example of chord couplings and substitutions}:

This passage contains three polychords:

Chord #1 (Tonnetz 1 + 4)       C dominant seventh + A major
Chord #2 (Tonnetz 2 + 4)       G dominant seventh + A major
Chord #4 (Tonnetz 2 + 5)       G dominant seventh + G minor

Karg-Elert calls the latter chord a Zwitter or “hybrid,” combining major and minor.
**11.4. Augmented triads and their resolutions**

All augmented triads contain two syntonic major thirds:

Karg-Elert describes four types of augmented triad, defined by their harmonic function, and by the voice-leading motions that resolve to the tonic:

Type I (strongest) one common tone, two leading tones in the dominating direction (plus a third leading tone in the case of \(*\overrightarrow{F}^7\) or \(*\overrightarrow{G}\))

Type II (weak) two common tones, plus one counter-leading-tone

Type III two counter-leading-tones (in parallel motion), plus one minor third leap [or a leading tone to the natural seventh]

Type IV (harsh) one leading tone to the fifth [and a metharmonic common tone with the seventh], or two whole steps when there is no natural seventh.

In C major, demonstrating Types I to IV:

Tonnetze for these examples are on the next page.
11.4.  Augmented triads and their resolutions (continued)

Type I:
Dominant with raised fifth
(chromatic semitone: D → D♯)

Type II:
Altered contrant with lowered root
(diatomic semitone: F → E)

Type III:
Contrant with raised fifth
(diatomic semitone: C → D♭)

Type IV:
Dominant-variant (d) with lowered root
(diatomic semitone: G → F♯)

The dotted arrows indicate stepwise voice-leading in resolution to the tonic.
The slashes indicate diatomic pitches that are raised or lowered in the augmented triads.

Karg-Elert provides two alternate function labels for the Type 4 augmented triad:

- ultradominant (BD, or D major) with diatomic-semitone raised fifth (A → B♭). This would have the same acoustic values, as shown on the above Tonnetz.
- ultracontrant (CC, or B♭ major) with chromatic-semitone raised fifth (F → F♯). This would have different acoustic values altogether: B_{♭(2,0)} D_{(2,1)} F_{♯(2,2)}.
11.4. Augmented triads and their resolutions (continued)

In A minor, demonstrating Types I to IV:

Type I:
Dominant with lowered root
(chromatic semitone: D → Db)

Type II:
Altered contrant with raised fifth
(diatonic semitone: B → C)

Type III:
Contrant with lowered root
(diatonic semitone: E → D♯)

Type IV:
Dominant-variant (d) with raised fifth
(diatonic semitone: A → B♭)
11.5. **Augmented sixth chords**

The concordant seventh and the augmented sixth are acoustically similar, but distinct:

Concordant seventh chord and tonic resolution (in F major):

![Diagram of concordant seventh chord and tonic resolution (in F major)](image)

Augmented sixth chord and resolution to tonic (in G major):

![Diagram of augmented sixth chord and resolution to tonic (in G major)](image)

The interval of the augmented sixth is 814 µ, which is equal to two fifths and two syntonic major thirds (see the circled pitches on the Tonnetz). Both pitches in the augmented sixth interval normally resolve by diatonic semitone in opposite directions, converging onto the same pitch (in this case, C → B and A♭ → B).
11.5. Augmented sixth chords (continued)

Karg-Elert describes three types of augmented sixth chord, defined by their harmonic function, and by their voice-leading motions in resolution to the tonic:

Type I: Cadential form
- 1 common tone, 2 leading tones converging onto the third

Type II: Mediantic form
- 1 common tone, 2 leading tones to the fifth; a chroma to the third

Type III: Leading-tone form
- No common tones; 4 leading tones: 1 to the prime, 3 moving in parallel motion to the complete triad

Note that in minor keys, augmented sixth chords are minor triads with an augmented sixth below the prime, and acoustically resemble the minor-key concordant seventh (i.e. the half-diminished seventh).
11.6. **Collective-change chords**

**Type I – C major examples**

<table>
<thead>
<tr>
<th>Example 1</th>
<th>Example 2</th>
<th>Example 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tonnetz:</strong></td>
<td>1 2 3 4</td>
<td>1 2 3 4</td>
</tr>
</tbody>
</table>

Example 1

Collective-change chords: 1 and 2

Modulation acquires two syntonic commas

Example 2

Collective-change chords: 1 and 2

Modulation acquires one syntonic comma

Example 3

Collective-change chords: 1 + 2 and 3 + 4

Modulation is comma-free
11.6. Collective-change chords (continued)

Type I – A minor examples

<table>
<thead>
<tr>
<th>Example 1</th>
<th>Example 2</th>
<th>Example 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tonnetz:</td>
<td>1 2 3 4</td>
<td>1 2 3 4 5</td>
</tr>
</tbody>
</table>

Example 1
Collective-change chords: 1 and 2

Example 2
Collective-change chords: 1 and 2

Example 3
Collective-change chords: 1 + 2 and 3 + 4

Modulation acquires two syntonic commas

Modulation acquires one syntonic comma

Modulation is comma-free
11.6. Collective-change chords (continued)

Type II: tritonal variants (examples in C major)

Example 1

Type II collective-change chord pairs: 1 + 2, 3 + 4, 5 + 6

Example 2

Type II collective-change chord pairs: 1 + 2, 3 + 4, 5 + 6
11.6. *Collective-change chords (continued)*

What Karg-Elert calls “true” collective-change chords are dissonant sonorities, created by raising or lowering all of a chord’s pitches by diatonic semitones.

In the following example, the upper system demonstrates the raising and lowering of chord tones, in close position. The lower system presents the same voice leading in more open position; the Type I chords resolve to the tonic, while the Type II chords resolve to the dominant.

![Musical notation image](image)

The following *Ursprungslen* analysis illustrates that the above dissonant sonorities combine pitches from two different functions \([C_M \text{ and } D_M^m, \text{ or } C_M \text{ and } D_M]\). They are thus another example of functional mixture as source of dissonance.

![Ursprungslen image](image)

See the Tonnetze on the following page.
11.6. Collective-change chords (continued)

Linear collective-change chords, derived through functional mixture:

C major, Type I

A minor, Type I

The circled notes are those included in the linear collective-change chords. The dotted arrows indicate the semitonal voice-leading motions to the tonic.

The very different configuration of these Tonnetze from those earlier in 11.6 reflect how collective-change chords can result either from harmonic relationships and transformations (which can be explained functionally like those on preceding pages), or purely from linear alterations (like those on this page).
### 11.7. The diminished seventh chord: acoustic derivation

In this section, Karg-Elert describes five different varieties of diminished seventh chord, defined by the size and sequence of their minor thirds:

<table>
<thead>
<tr>
<th>PPP</th>
<th>SSS</th>
<th>SPS</th>
<th>CSC</th>
<th>SCS</th>
</tr>
</thead>
<tbody>
<tr>
<td>P = Pythagorean (canonic) minor third 245 µ</td>
<td>S = consonant (syntonic) minor third 263 µ</td>
<td>C = concordant (seventh-derived) minor third 222 µ</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The first two types have equal-sized minor thirds. First, the **PPP** variety:

As Karg-Elert explains, no diatonic collection contains three minor thirds of equal size. In addition, the Pythagorean minor third (245 µ) does not blend in a harmonic manner, and so the PPP diminished seventh will only occur as a melodic sequence.

Also with equal-sized minor thirds is the **SSS** diminished seventh, which combines pitches from the contrant mediant (C<sup>M</sup>) and the dominant counter-mediant (D<sub>M</sub>):
The diminished seventh chord: acoustic derivation (continued)

The pitches of the SSS diminished seventh appear on the Tonnetz as an unbroken diagonal (see the circled pitches):

However, the pitches of the SSS diminished seventh resolve to two different keys:

Therefore, the SSS diminished seventh is “functionally untenable.”
11.7. *The diminished seventh chord: acoustic derivation (continued)*

The SPS diminished seventh contains two syntonic minor thirds (denoted by the short diagonals), separated by a Pythagorean minor third (i.e. the three fifths from G to E):

The SPS diminished seventh combines pitches from two different functions (within a single key): dominant (D) and altered contrant (c).

The SPS diminished seventh is “doubly cadential,” as the pitches from both the dominant and the altered contrant resolve to the tonic, all in the same key:
Next to be discussed is the CSC diminished seventh chord, which contains two concordant minor thirds and one syntonic minor third. It is created by combining two Leittonwechsel-related concordant seventh chords with their primes omitted:

\[(C) \ E G Bb + C\# E G (B)\]

On the Tonnetz, all four pitches appear within the space of a Leittonzwilling (see 11.3):

The CSC diminished seventh is even more “functionally untenable” than the SSS variety, as its constituent pitch pairs resolve to two different tritone-related keys:

Therefore, if it will occur at all, the CSC diminished seventh is most likely to arise as a confluence of linear motions, and not as a functional harmonic entity.
11.7. The diminished seventh chord: acoustic derivation (continued)

The last and most important variety of diminished seventh chord is SCS, with two syntonic minor thirds (E → G and B♭ → Db, indicated here as dotted lines) separated by one concordant minor third (between G and B♭):

The SCS diminished seventh is a concordant seventh chord with a chromatic-semitone lowered/raised ninth (indicated by the hook attached to the function label) and an omitted prime (indicated by the small circle).

The SCS diminished seventh is “unified and continuous,” because all of its pitches are generated by a single prime, express a single function, and resolve within the same key.
11.8. *The diminished seventh chord: functional meaning*

The SCS diminished seventh is based on either the dominant (D) or the altered contrant (c), as illustrated by the following:

*First row, in C major*

As altered contrant:
*As dominant:*

\[ B_{(1,1,-1)} \ D_{(0,0,-1)} \ F_{(-1,0,0)} \ Ab_{(0,-1,0)} \]

\[ B_{(1,1,0)} \ D_{(2,0,0)} \ F_{(1,0,1)} \ Ab_{(2,-1,1)} \]

*Second row, in A minor*

As dominant:

\[ G_{(2,2,-1)} \ B_{(1,1,-1)} \ D_{(-2,1,0)} \ F_{(-1,0,0)} \]

\[ G_{(0,0,2)} \ B_{(1,1,0)} \ D_{(0,1,1)} \ F_{(1,0,1)} \]
11.8. *The diminished seventh chord: functional meaning (continued)*

Type I (primary) = dominant (D) or altered contrant (c) function

_Ursprungslagen_ and functional analysis of Type I, subcategories A to E:

C major examples:

**Type Ia**

**Type Ib**

**Type Ic**

**Type Id**

**Type Ie**
11.8. *The diminished seventh chord: functional meaning (continued)*

**Type I (primary)**

A minor examples (refer back to the *Ursprungrlagen* analysis on the last page):

Type Ia

Type Ib

Type Ic

Type Id

Type Ie
11.8. The diminished seventh chord: functional meaning (continued)

Type II (secondary) = common-tone diminished seventh, embellishing the tonic

Ursprungslagen and functional analysis of Type II, subcategories A to E:

C major examples of Type II (subcategories not specified):

Example 1: Type IIa

Example 2: Type IIb

Example 3: most likely Type IId

Example 4: Type IId
11.8. *The diminished seventh chord: functional meaning (continued)*

**Type II (secondary)**

A minor examples of Type II (subcategories not specified):

Example 1: Type IIa

Example 2: Type IIa? (see note below)

Example 3: most likely Type IIe (see the note below)

Example 4: Type IId

Example 2: the Tonnetz takes Karg-Elert’s acoustic data at face value, using two different metharmonic values for the pitch class E. However, as discussed in the text, the function label (and the *Ursprungslagen* analysis above) suggest Type IIb.

Example 3: the diminished seventh is derived note by note from the tonic parallel, which suggests Type IIe.
11.8. The diminished seventh chord: functional meaning (continued)

Type III (tertiary) = linear embellishment of the dominant/altered contrant

Example 1:  
G major  F minor

Example 2:  
D minor  E major

Example 1

Example 2
[This page is intentionally blank.]
Chapter 12

Summary of the \textit{first 32} partials (with their acoustic symbols)
in frequency ratios and $\mu$-values [generated from C]
Chapter 12
Summary of the \textit{first 32} partials (with their acoustic symbols) in frequency ratios and µ-values [generated from C]

12.1. Summary of the first 32 partials, upward from the fundamental pitch C
12.2. On the preceding tables, and on the 11\textsuperscript{th} and 13\textsuperscript{th} partials
12.3. On the supposed derivation of scales and harmony from overtones
12.4. The special case of the pure seventh as integral harmonic interval
12.5. The semischisma (1 µ)
### 12.1. Summary of the first 32 partials, upward from the fundamental pitch C

<table>
<thead>
<tr>
<th>Partial</th>
<th>Frequency ratio</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. Partialton</td>
<td>rel. Sz: 2</td>
<td>0 = 1000</td>
</tr>
<tr>
<td>3. ([\text{primäre Quinte}])</td>
<td>(\frac{5}{3})</td>
<td>1,58496 gekürzt: 1585*</td>
</tr>
<tr>
<td>4. ([2^1])</td>
<td>4</td>
<td>2000</td>
</tr>
<tr>
<td>5. ([\text{primäre Terz}])</td>
<td>(\frac{5}{3})</td>
<td>2,32193 (\mu): 2322*</td>
</tr>
<tr>
<td>6. ([3 \cdot 2])</td>
<td>6</td>
<td>2585</td>
</tr>
<tr>
<td>7. ([\text{primäre Septime}])</td>
<td>(\frac{7}{3})</td>
<td>2,80735 (\mu): 2807*</td>
</tr>
<tr>
<td>8. ([2^3])</td>
<td>8</td>
<td>3000</td>
</tr>
<tr>
<td>9. ([3^2])</td>
<td>9 ((2 \times 1585))</td>
<td>3170</td>
</tr>
<tr>
<td>10. ([5 \cdot 2])</td>
<td>(\frac{10}{3}) ((\text{siehe unter 5}))</td>
<td>3322</td>
</tr>
<tr>
<td>11. ([\text{prim. verm. Quinte}])</td>
<td>(\frac{11}{3})</td>
<td>3,45943 gekürzt: 3459*</td>
</tr>
<tr>
<td>12. ([3 \cdot 2^2])</td>
<td>(\frac{12}{3}) ((\text{siehe unter 6}))</td>
<td>3585</td>
</tr>
<tr>
<td>13. ([\text{prim. überm. Quinte}])</td>
<td>(\frac{13}{3})</td>
<td>3,70044 (\mu): 3700*</td>
</tr>
<tr>
<td>14. ([7 \cdot 2])</td>
<td>(\frac{14}{5}) ((\text{siehe unter 7}))</td>
<td>3807</td>
</tr>
<tr>
<td>15. ([3 \cdot 5])</td>
<td>(\frac{15}{5})</td>
<td>1,585 + 2322 = 3907</td>
</tr>
</tbody>
</table>

**Notes:**
- Partialton = overtone, partial
- gekürzt = reduced to a whole number
- 3. Okt (dritte Oktave) = third octave (above the fundamental C)
- prim. (primäre) = primary, not an octave duplication of a lower partial. Also denotes the prime numbers.
- Terz = [syntonic] major third
- verm. (vermässige) = diminished
- überm. (übermässige) = augmented
- Quinte = fifth
- Siehe unter 5 = see the 5th overtone (which has the same acoustic value, as it is an octave duplication)
12.1. Summary of the first 32 partials, upward from the fundamental pitch C

The \( \mu \) values for each partial are given in unreduced form, and then reduced (gekürzt) to the nearest whole number (boxed). The first digit of a four-place \( \mu \) value denotes its octave in the series of partials (1, 2, 3, 4 or 5); it is normally omitted (i.e. the value is octave-reduced).

The underlined “primary” (Primäre) partials (3, 5, 7, 11, 13, etc) are the prime numbers; their frequency ratios cannot be replicated using 2 (the octave) and/or 3 (the Pythagorean fifth).

Karg-Elert’s acoustic symbols for the third, fifth and seventh partials are now very familiar:

Dots (.) = canonic fifths (2:3)
Accents (‘) = syntonic major thirds (4:5)
Wedges (\( \tilde{\text{v}} \)) = concordant natural sevenths (4:7)

The fourth octave introduces two new primary partials and their acoustic symbols:

\[ 11_{\tilde{\text{v}}} = \text{the natural eleventh partial (8:11), at 459 } \mu. \]
It is here called the primary diminished fifth, and is therefore spelled as a G\( \bar{b} \) instead of an F\#.
As discussed earlier, Karg-Elert also uses a downward hook to denote lowering of any pitch by a chromatic semitone (see section 11.2); such usage does not necessarily indicate the 11\(^{th}\) partial.

\[ 13_{\tilde{\text{v}}} = \text{the natural thirteenth partial (8:13), at 700 } \mu. \]
It is here called the primary augmented fifth, and is therefore spelled as a G\# instead of an A\( \bar{b} \).
Karg-Elert also uses an upward hook to denote raising a pitch by a chromatic semitone; such usage does not necessarily indicate the 13\(^{th}\) partial.
<table>
<thead>
<tr>
<th>Partial</th>
<th>Frequency ratio</th>
<th>µ</th>
</tr>
</thead>
<tbody>
<tr>
<td>16. Partialton $2^4$</td>
<td>$S_2: 16$</td>
<td>4.000</td>
</tr>
<tr>
<td>17. $\times$ [primäres Chroma]</td>
<td>17</td>
<td>4.08746 gekürzt: 4.087</td>
</tr>
<tr>
<td>18. $\times$ [$3^2 \cdot 2$]</td>
<td>18</td>
<td>(siehe unter 9) 4.170</td>
</tr>
<tr>
<td>19. $\times$ [primärer Hiatus]</td>
<td>19</td>
<td>4.24793 gekürzt: 4.248*</td>
</tr>
<tr>
<td>20. $\times$ [$5 \cdot 2^2$]</td>
<td>20</td>
<td>(siehe unter 5) 4.322</td>
</tr>
<tr>
<td>21. $\times$ [$3 \cdot 7^1$]</td>
<td>21</td>
<td>1.585 + 2.807</td>
</tr>
<tr>
<td>22. $\times$ [11 \cdot 2]</td>
<td>22</td>
<td>(siehe unter 11) 4.459</td>
</tr>
<tr>
<td>23. $\times$ [primärer Tritonus]</td>
<td>23</td>
<td>4.52356 gekürzt: 4.524*</td>
</tr>
<tr>
<td>24. $\times$ [$3 \cdot 2^3$]</td>
<td>24</td>
<td>(siehe unter 3) 4.585</td>
</tr>
<tr>
<td>25. $\times$ [$5^2$]</td>
<td>25</td>
<td>(2 \times 2,322)</td>
</tr>
<tr>
<td>26. $\times$ [$13 \cdot 2$]</td>
<td>26</td>
<td>(siehe unter 13) 4.720</td>
</tr>
<tr>
<td>27. $\times$ [$3^3$]</td>
<td>27</td>
<td>(3 \times 1,585)</td>
</tr>
<tr>
<td>28. $\times$ [$7 \cdot 2^2$]</td>
<td>28</td>
<td>(siehe unter 14) 4.807</td>
</tr>
<tr>
<td>30. $\times$ [$3 \cdot 5 \cdot 2$]</td>
<td>30</td>
<td>(siehe unter 15) 4.907</td>
</tr>
<tr>
<td>31. $\times$ [primärer Leitton]</td>
<td>31</td>
<td>4.95420 gekürzt: 4.954</td>
</tr>
<tr>
<td>32. $\times$ [$2^5$]</td>
<td>32</td>
<td>5.000</td>
</tr>
</tbody>
</table>

Chroma = chromatic semitone  
Tritonus = tritone (i.e. three whole tones)  
Leitton = leading tone  
Hiatus = augmented second  
überm. Sexte = augmented sixth  
For other German terms, see the previous page
12.1. (continued)

The fifth octave provides further primary partials, with new acoustic symbols introduced here for the first time in the treatise:

\[ 17 \] = primary chroma (16:17), at 87 µ

\[ 19 \] = primary Hiatus (augmented second or 16:19), at 248 µ

\[ 23 \] = primary tritone (16:23), at 524 µ

\[ 29 \] = primary augmented sixth (16:29), at 858 µ

\[ 31 \] = primary leading tone (16:31), at 954 µ

The symbol for the augmented sixth \[ \] was in fact introduced earlier (in section 11.5 on augmented sixth chords), as an attachment to function labels. However, it was not presented there as the 29th partial, but as a double fifth plus a double third (814 µ).

The following two pages are the polar equivalent to the preceding chart, descending from c4.
The same, in Messel lengths (wavelengths), downward from c4
12.2. **On the preceding tables, and on the 11th and 13th partials**

On the preceding tables:

As long as backward-looking practitioners of musical acoustics continue to deny the importance of the seventh partial as an integral component of the dominant, a detailed examination of higher partials in terms of their syntonic values is superfluous.

11′ and 13′ already appear as overtones on the *non plus ultra* organ [1283 stops from 128′ to 1/4′] at the Convention Hall in Atlantic City (New Jersey):

\[
2 \frac{10}{11} \text{′} = \frac{32}{11} \quad [\text{11th partial of a 32′ stop}],
\]

\[
1 \frac{5}{11} \text{′} = \frac{16}{11} \quad [\text{11th partial of a 16′}],
\]

\[
1 \frac{3}{13} \text{′} = \frac{16}{13} \quad [\text{13th partial of a 16′}],
\]

and \(8/11\)′ (belonging to an 8′).

Natural horns and trumpets that have a low fundamental pitch and narrow bore – and are therefore rich in usable overtones – employ the 11th and 13th partials as makeshift substitutes for the pure fourth and major sixth.

However, when these tones are blown on natural instruments without bending their intonation, even a single hearing makes all too clear “how” the so-called pure fourth and major sixth sound... the sixth is much too low [nearly G♭\(_{5}(8,0)\)], while the fourth is much too high [close to G♭\(_{1}(4,0,1)\)].
12.2. On the preceding tables, and on the 11th and 13th partials

The Atlantic City building is now called the Boardwalk Hall Auditorium. Its organ was built by the Midmer-Losh Organ Company between 1929 and 1932, and is today still counted as the largest in the world, with seven manuals and over 33,000 pipes. Due to extensive damage from a hurricane in 1944, and subsequent neglect, the instrument has been only partially playable for many decades. Restoration is now underway, with completion scheduled for 2023 (Boardwalk Hall website).

Therefore, if a quasi-major scale is derived from the fourth octave of partials, the eleventh as \(^4\) and the thirteenth as \(^6\) will be too far from the pure intervals to be usable.
The 11th partial (the supposed “F” fourth) is 459 from C – and the 13th partial (the alleged “A” major sixth) is 700 in μ units.

\[
\begin{array}{cccccccc}
& \text{F}_{(-1,0)} & \text{Gb}_{(-4,0,1)} & \text{Gb}_{(-6,0)} & \text{G}^\#_{(8,0)} & \text{G}^\#_{(2,1,-1)} & \text{G}^\#_{(6,0,-1)} & \text{A}_{(-1,1)} & \text{A}_{(0,0)} \\
\text{ μ} & 4.15 & 4.67 & 4.90 & 6.80 & 6.85 & 703 & 737 & 755 \\
\end{array}
\]

11. Partialton (etwa ges) = 11th partial, roughly Gb
angeblich = allegedly

13. Partialtone, etwa gis = 13th partial, roughly G#
Abstand = range, interval

The difference between 340 and 241 = 99 (!) Consequently, one can hardly call the interval between the 11th and 13th partials 241 a “major third.” In fact, it is 22 μ smaller than the syntonic minor third (263 μ), and 4 μ smaller than the Pythagorean minor third (245 μ).
12.2. (continued)

$340 \mu = \text{Pythagorean major third (i.e. four fifths)}$

$241 \mu = \text{the interval between the 11}^{\text{th}} \text{ and 13}^{\text{th}} \text{ partials}$
There are harmony textbooks [that appear to be full of wisdom, and which have found unquestioned success], which claim in all sincerity that scales – both diatonic and chromatic – are derived from the overtone series (!). In addition: all harmony results from scales (!!!). [A greater degree of dilettantism could never be achieved!]

“The diatonic scale appears in the fourth octave of overtones; then in the next (5th) octave the chromatic scale is found…”

“The major and minor triads are simultaneities drawn from these scales made of whole tones and semitones.”

Good grief! Now that is a simple solution to the harmonic problem!

First line = supposed 7-tone (angeblich 7-tönig) diatonic scale, in the fourth octave of overtones
Second line = supposed 12-tone (angeblich 12-tönig) chromatic scale, in the fifth octave of overtones
12.3. On the supposed derivation of scales and harmony from overtones

It seems that Karg-Elert’s main target here is Schoenberg’s *Harmonielehre*, and specifically its derivation of the C major scale from “the primary components of a fundamental tone and its nearest relatives” (aus den wichtigsten Bestandteilen eines Grundtons und seiner nächsten Verwandten) – i.e. by combining pitches from the overtone series of C and its fifths F and G:

\[
\begin{array}{c|c}
\text{Grundton} & \text{Oberöte} \\
F & f \ldots c \ldots f \cdot a \cdot c \cdot (es) \cdot f \cdot g \cdot a \cdot b \cdot c \cdot usw. \cdot f \cdot usw. \\
C & c \ldots g \ldots c \cdot e \cdot g \cdot (b) \cdot c \cdot d \cdot e \cdot f \cdot g \cdot usw. \\
G & g \ldots d \cdot g \cdot h \cdot d \cdot (f) \cdot g \cdot a \cdot h \cdot c \cdot d \\
\end{array}
\]

(Schoenberg 1922, 22)

In his copy of Schoenberg’s *Harmonielehre*, Karg-Elert wrote “Heavens! What a dilettante!” (Himmel! ist das dilettantisch!) on this very page (Hartmann 1996–97, 165). Karg-Elert was surely troubled by the lack of acoustic exactitude in the resultant C major scale: its apparently free substitution of Eb\(_{-1,0,1}\) and E\(_{0,1}\) as well as B\(_b\) (11\(^{th}\) overtone of F!), B\(_b\)\(_{0,0,1}\) and B\(_1\)\(_{1,1}\) on the third and seventh Stufen, and by the presence of both F\(_{-1,0}\) and F\(_{1,0,1}\) as well as both A\(_{-1,1}\) and A\(_{2,0}\).

However, Schoenberg’s *Harmonielehre* does not derive the C major scale entirely from the overtones of C, as suggested here. In addition, Karg-Elert’s two quotations are not from Schoenberg; their source is uncertain.
Presumably C♯ = D♭, and D♯ = E♭, and so on. That means a further convenient simplification of theory! The fact that certain overtones (14, 21, 27, 29, 31) had to be ignored did not arouse the scruples of this pioneer of “simplified harmony”...

“Unfortunately,” μ-calculation is sufficiently objective to consign the abundant folly of such ideas to the pillory. The “diatonic and chromatic scales” that are extracted from the overtone series exhibit the following amusing changes in interval size:

<table>
<thead>
<tr>
<th>Whole tones</th>
<th>Semitones</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ganztöne:</strong></td>
<td><strong>Halbton (diat.)</strong>:</td>
</tr>
<tr>
<td>c − d = 170</td>
<td>a − f = 137</td>
</tr>
<tr>
<td>d − e = 152</td>
<td>h − c = 93</td>
</tr>
<tr>
<td>f − g = 126</td>
<td>cis − d = 83</td>
</tr>
<tr>
<td>g − a = 115</td>
<td>dis − e = 75</td>
</tr>
<tr>
<td>a − h = 207</td>
<td>fis − g = 67</td>
</tr>
<tr>
<td>h − cis = 180</td>
<td>gis − a = 56</td>
</tr>
<tr>
<td>cis − dis = 161</td>
<td>a − b = 107</td>
</tr>
<tr>
<td>e − fis = 202</td>
<td>(vergl. Ganztöne:</td>
</tr>
<tr>
<td>fis − gis = 120</td>
<td>h − c = 93</td>
</tr>
<tr>
<td>b − c = 193</td>
<td>cis − d = 83</td>
</tr>
</tbody>
</table>

In contrast, here are the canonic scale values:

<table>
<thead>
<tr>
<th>Vergleiche dagegen die kanonischen Skalenwerte:</th>
</tr>
</thead>
<tbody>
<tr>
<td>170 + 75</td>
</tr>
<tr>
<td>Ganzton — ist — Apotome + Limma</td>
</tr>
<tr>
<td>(chrom) (diat)</td>
</tr>
<tr>
<td>The whole tone — equals — Apotome + Limma</td>
</tr>
<tr>
<td>(chromatic semitone) (diatonic semitone)</td>
</tr>
</tbody>
</table>

880
A reminder of German pitch nomenclature used here:

Cis = C♯   Dis = D♯   Fis = F♯   Gis = G♯   B = B♭   H = B
The fate of the chords on scale degrees II, IV and VI in major (C₄, C and C₈, or S₄ and S) is particularly amusing. Because the monist pioneers of “simplified overtone harmonic theory” reject all undertone projections as absurd, they must deem invalid the very concept of the subdominant (or contrant) and its substitutes: D F A would be conceived in terms of an “over-dominant,” derived from the 9th, 11th and 13th overtones.

The following certainly appears well-founded:

\[
\begin{array}{cccccc}
\text{I} & \text{II} & \text{III} \\
{c} & {e} & {h} & {d} & {f} & {a} & {c} \\
1 & 4 & 5 & 8 & 9 & 11 & 13 & 16
\end{array}
\]

Partialtöne = partials (with acoustic symbols)

But how abstruse are the interval sizes that result from this “uniform overtone theory”:

ERRATUM:

The “major” 3rd F/A is smaller than the “minor” 3rd A/C (!!)

Minor triads in major:

And also here: the “minor” 3rd D/F is larger than the “major” third F/A (!!)

It is hard not to write satire... (Juvenal)
Here, Karg-Elert mentions the term Unterton (“undertone”) for the first and only time in the text. Unlike Riemann, Karg-Elert never promoted the idea of a physical undertone series; instead, Karg-Elert followed Riemann’s later use of the Messel concept as a logical and mathematical explanation of the minor triad.

The Roman numerals here are Stufen, in the key of C major.

ERRATUM: In this C major listing of Stufen, the Roman numerals IV and V should be reversed.
12.4. The special case of the pure seventh as integral harmonic interval

As previously described, the 11th and 13th partials were occasionally used on natural trumpets and horns before the introduction of valves, but such primary tones could only be employed in ornaments and fleeting passages, more in melodic “gestures” than as purely harmonic entities.

Sometimes 11 was made to sound like a pure F or F#, and 13 like a pure G#/Ab or A, by lowering or raising the pitch. But a pure, full-voiced harmonic F and A required the use of an instrument in F, on which this third is found as the 8th and 10th partials.

The case of the seventh partial is intrinsically different. It only seems “impure” when it is used in place of a third- or fifth-derived pitch, or in the context of tempered tunings. How wonderfully integrated and unified sounds the natural seventh, as part of a seventh chord played on horns or trombones. It only seems “impure” or “too low” in comparison with a tempered pitch of the same name, played on a valved brass or woodwind instrument.
12.4. *The special case of the pure seventh as integral harmonic interval*

Players of unvalved brass instruments used various methods to alter the natural tuning of overtones: embouchure adjustment, using the hand in the bell (for horns), and sometimes tiny holes (on trumpets).
12.5. The semischisma (1 \( \mu \))

Theoretically, the interval 31 : 32 arises in the Greek enharmonic tetrachord:

\[
\begin{align*}
\text{Diatonic} & : \text{Chromatic} : \text{Enharmonic} \\
\text{pythagoräisch} & : \text{chromatisch} : \text{eharmonisch} \\
\frac{8}{9} : \frac{5}{6} : \frac{4}{5} & : \frac{9}{8} : \frac{256}{243} : \frac{31}{30} \\
\text{Diff.} & = 1 \mu \\
\text{Ein 1000stel Oktave} & = \text{Semischisma}
\end{align*}
\]

a) intervals calculated as ratios

- \text{pythagoräisch} = \text{Pythagorean (fifth-based)}
- \text{pyth. Ganzton} = \text{Pythagorean whole tone (170 \( \mu \))}
- \text{did. Halbton} = \text{Didymean semitone (93 \( \mu \))}

\( \mu \)) intervals calculated in \( \mu \)

- \text{didymisch} = \text{Didymean (third-based)}
- \text{did. Ganzton} = \text{Didymean whole tone (152 \( \mu \))}
- \text{ein 1000stel Oktave} = \text{one 1000th of an octave (Semischisma)}
12.5. *The semischisma (1 µ)*

Karg-Elert cites the traditional Pythagorean division of the diatonic tetrachord (which uses only canonic intervals), and Didymus’ divisions for the chromatic and enharmonic tetrachords (using syntonic intervals). Karg-Elert’s source for these ratios is unknown. Ptolemy’s *Harmonics* describes these and other divisions of the tetrachords; see Hagel 2009, 189.

It would seem that Karg-Elert discusses the Greek enharmonic tetrachord and the interval 31:32 in order to cite a historical instance of the interval of 1 µ: the *semischisma*, or the unit in the octave divided into 1000 parts. Indeed, in the Table of Contents to *Harmonologik* (Karg-Elert 1931), Karg-Elert refers to *Akustische* as “Die 1000-teilige Oktave.” While a just-intonation pitch space of course contains an infinite number of possible pitches (especially if higher partials such as the 11th and 13th are brought into play), it seems that Karg-Elert found a division of the octave into 1000 parts to be specific enough for most analytical situations! Nonetheless, Appendix A lists numerous pitch pairs that differ by less than 1 µ.
Chapter 13

Equal temperament
Chapter 13
Equal temperament

13.1. The conversion of pure frequency ratios into tempered values (absolute frequencies)
13.2. The 12 equal-tempered intervals calculated in μ
13.3. The practical significance of equal-tempered tuning, and the ear’s powers of perception
13.1. **The conversion of pure frequency ratios into tempered values (absolute frequencies)**

If we know the absolute frequency of a pitch – for example, the tuning pitch $a^1$ at 435 cycles – the frequency of $b^1$ as (-1,-1) of $A$ can be calculated as 435 multiplied by $16/15$.

If the prime is $a^1$ (435 Hz):

\[ a^1 = 435 \text{ Sz} \parallel b = \frac{435 \times 16}{15} = 466 \text{ Schwingungen} \]

**ERRATA:**

\[ gis = \frac{435 \times 15}{16} = 642.5 \text{ Hz} \]
\[ g = \frac{435 \times 8}{9} = 348.4 \text{ Hz} \]
\[ h = \frac{435 \times 5}{9} = 241.5 \text{ Hz} \]

Schwingungen (Sz) = cycles (Hz)  als ... von $A$ = ... in relation to $A$
13.1. The conversion of pure frequency ratios into tempered values (absolute frequencies)

\( a^1 = 435 \text{ Hz} \) was the standardized Parisian tuning pitch (see section 2.1).

Note that these calculations are in Hertz (not \( \mu \)), and that they are of just rather than equal-tempered intervals. 16:15 is a just minor second, and 9:8 is a just major second.

ERRATA
The first three of these calculations are slightly off the mark:
\[
\begin{align*}
435 \times 16 &= 6960 : 15 = 464 \\
435 \times 15 &= 6525 : 16 = 407 \frac{13}{16} \\
435 \times 8 &= 3480 : 9 = 386 \frac{2}{3} \text{ (the result is correct!)}
\end{align*}
\]

Karg-Elert cites these frequency calculations (using just-intonation whole-number ratios) in order to contrast them with the fractional ratios required to calculate frequencies in equal temperament (see the next page).
The mathematical ratios of the equal-tempered intervals require the use of decimals, as follows:

<table>
<thead>
<tr>
<th>Interval Description</th>
<th>Ratio (X)</th>
<th>Ratio (X+1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 semitone above X (minor second)</td>
<td>1.00000</td>
<td>1.05946</td>
</tr>
<tr>
<td>2 semitones above X (major second)</td>
<td></td>
<td>1.12246</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1.18921</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>1.25992</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>1.33484</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>1.41421</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>1.49831</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>1.58740</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>1.68179</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>1.78180</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td>1.88775</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>2.00000</td>
</tr>
</tbody>
</table>

Die mathematischen Proportionen der temperierten Werte verhalten sich in Dezimalen ausgedrückt, folgendermaßen:

<table>
<thead>
<tr>
<th>X zu seinem nächsthöheren Halbton (kl. Sekunde od. Chroma)</th>
<th>1.00000 : 1.05946</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. höheren Halbton (gr. Sekunde, resp. vermind. Sekunde)</td>
<td>1.12246</td>
</tr>
<tr>
<td>3. (kl. Terz, resp. überm. Sekunde)</td>
<td>1.18921</td>
</tr>
<tr>
<td>4. (gr. Terz, resp. vermind. Quartie)</td>
<td>1.25992</td>
</tr>
<tr>
<td>5. (Quarte, resp. überm. Terz)</td>
<td>1.33484</td>
</tr>
<tr>
<td>6. (überm. Quarle, resp. vermind. Quinte)</td>
<td>1.41421</td>
</tr>
<tr>
<td>7. (Quinte)</td>
<td>1.49831</td>
</tr>
<tr>
<td>8. (kl. Sexte, resp. überm. Quinte)</td>
<td>1.58740</td>
</tr>
<tr>
<td>9. (gr. Sexte)</td>
<td>1.68179</td>
</tr>
<tr>
<td>10. (kl. Septime, resp. überm. Sexte)</td>
<td>1.78180</td>
</tr>
<tr>
<td>11. (gr. Septime, resp. vermind. Oktave)</td>
<td>1.88775</td>
</tr>
<tr>
<td>12. (Oktave)</td>
<td>2.00000</td>
</tr>
</tbody>
</table>
The intervals paired together are of course exactly the same size in equal temperament.
For example: the actual frequency of an equal-tempered A#/B♭ can be calculated as follows, starting from the prime a set at 435 cycles:

\[
A = 435 \times 1.05946 = \frac{460.86510}{\text{equal-tempered } A#/B♭} \approx 461
\]

[c] is a tempered {minor} third from [i.e. 3 semitones above] the tuning pitch a:

\[
a = 435 \times 1.18921 = \frac{517.30635}{\text{equal-tempered } c}
\]

Compare c as (1,-1) from a: 435 x 6 = 2610 : 5 = 522 = C as syntonic minor third from A

The difference is apparent, and is also revealed by \( \mu \) calculation:
Syntonic minor third = 263 \( \mu \)  \quad \text{Equal-tempered minor third} = \frac{1}{4} \text{octave} = 250 \( \mu \)
Difference = 13 \( \mu \)
If you know the frequency for pitch X, you can calculate the equal-tempered intervals above pitch X as follows:

\[ \text{frequency of } X \text{ multiplied by interval ratio (from the preceding chart)} \]
13.2. The 12 equal-tempered intervals calculated in $\mu$

The octave is $1000$. One half of an octave = diminished fifth or augmented fourth = 500
One third of an octave = major third = $\frac{333}{2}$
Two thirds of an octave = augmented fifth or minor sixth = $\frac{666}{2}$
One quarter of an octave = minor third or hiatus (augmented second) = 250
Three quarters of an octave = major sixth = $\frac{750}{2}$

The difference between the minor and major thirds = $333 \frac{1}{3} - 250$
The difference between the major and minor sixths = $750 - 666 \frac{2}{3}$

The whole tone = $83 \frac{1}{3} \times 2 = 166 \frac{2}{3}$
Major third $333 \frac{1}{3} : 2 = 166 \frac{2}{3}$

The intervals above C:

The intervals above C:

<table>
<thead>
<tr>
<th>C</th>
<th>C#/Db</th>
<th>D</th>
<th>D#/Eb</th>
<th>E</th>
<th>F</th>
<th>F#/Gb</th>
<th>G</th>
<th>G#/Ab</th>
<th>A</th>
<th>A#/Bb</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\text{Die 12 temporierten Intervalle in } \mu \text{-Werken.}
\]

\[
\text{Die Oktave ist } 1000; \text{ die Hälfte einer Oktave = verm. Quinte resp. übermäßige Quarte = 500 ein drittel Oktave = große Terz = } \frac{333}{2}; \text{ zweidrittel Oktave = überm. Quinte = kl. Sexte = } \frac{666}{3}; \text{ ein viertel Oktave = kleine Terz resp. Hiatus = 250, dreiviertel Oktave = gr. Sexte = } \frac{750}{2}.
\]

\[
\text{Differenz zwischen kl. und gr. Terz: } \frac{333}{2} + \frac{250}{3} = \frac{833}{3}; \text{ Halbton.}
\]

\[
\text{Ganzton: } \frac{833}{3} \times 2 = \frac{1666}{3} \text{ od. große Terz } \frac{333}{2} : 2 = \frac{1666}{3};
\]
13.2. The 12 equal-tempered intervals calculated in $\mu$

The $\mu$ values for several equal-tempered intervals that divide the octave symmetrically are very easy to remember: 500, 250, 750 $\mu$. The equal-tempered perfect fourth and fifth are much less easily memorable in $\mu$.

All semitones in the equal-tempered 12-tone scale are $83 \frac{1}{3} \mu$.

The brackets indicate 5 equal divisions of the octave (from top to bottom): the tritone, major thirds, minor thirds, whole tones and finally semitones.
13.3. The practical significance of equal-tempered tuning, and the ear's powers of perception

The musical and practical significance of equal-tempered tuning resides in the removal of all commas, and the artificial closure of endless tonal circles by means of metharmonics and enharmonics. That no interval other than the octave can be described as pure is a drawback that is surely not trivial. However, our inherently and specifically musical ear is endowed with the wondrous ability to hear in a selective and relative manner – that is, out of the myriad of approximate tonal and chordal events that we perceive as possible tonal structures, we instinctively use our imagination to reconceive each unstable tone and chord in terms of its intended value, even in the objective absence of exact pure tuning [as long as its bounds are not greatly exceeded].

The boundaries of this ability are unique to each individual – and the “intended pure values” are very ambiguous in complex cases…

Considered in isolation, the dissonances in examples a) and b) would be recognized as augmented triads, but it would be impossible to specify their “intended” exact meaning, as C E G♯ would be as equally possible as C E A♭ or B ♭ D♯ E♭, and so on. The connection to the following F major triad retroactively reveals the “intended” meaning of the dissonance. If the entrance of the second dissonance at b) implies an exact sequence [a] altered C major triad → F major, b) the same progression a whole tone higher…], the dissonance would be perceived as D F♯ A♭ - but in fact that is not what is “intended.” Again, the meaning of the dissonance is only revealed retroactively as D F♯ B♭ after the entrance of the readily understood E♭ major triad.
Here is the perhaps the crucial point of *Akustische*, and a constant in all of Karg-Elert’s writings: that the musical ear (*Ohr*) or imagination (*Vorstellung*) conceives all pitch relationships (both melodic and harmonic) according to the complex and infinite patterns of natural or just intonation, regardless of the actual tuning system used in performance.

Here is another central point: that the mental processing of pitch relationships depends entirely on musical context. This is especially important to the concept of harmonic function: one cannot make a one-to-one connection between pitch groups and functions, such as to say that all augmented triads are dominants with a chromatic-semitone raised fifth – or even that [A C E] in C major is in all cases the tonic parallel (*Tp*).
If the consonant triads are omitted, the dissonances could no longer be recognized as “intentional” cadential chords. The notion of “intended” meaning becomes a total illusion:

\[
\begin{array}{c|c}
G^4/Ab & Bb/A^\# \\
D^\#/E & Gb/F^\# \\
B^\#/C & D/C^\#
\end{array}
\]

In addition: isolated and in direct succession, these chained double major thirds sound passive, faded, tired, slack, morbid, impotent and extremely soft. But as soon as a cadential progression is initiated, their effect is unmistakably reversed: they become energized, directed, active, expectant.
When the augmented triads are isolated, we cannot know the acoustic identity of the pitches—thus, any enharmonic spelling will do. Such pitch relationships have no harmonic intention or meaning, according to Karg-Elert.
This is due to our subjective interpretation or perception of syntonic combinations on the one hand, and of directed individual voices on the other:

\[
\begin{align*}
C(0) & \quad E_{(1)} \quad C_{(0,2)} \quad B_{(0)} \quad D_{(1,1)} \quad F_{(2,1)} \\
\text{[and similarly also all other forms!]} & \\
\text{[in gleicher Weise auch alle anderen Formen]} & 
\end{align*}
\]

This six-note sonority is a decidedly syntonic polychord, in which the underlying canonic fifth relationship is consequently suppressed [the ninth as “pure” double-fifth being replaced], and the typically syntonic elements – the narrow third and seventh – contributing solely to the formation of the sonority. The result is a kind of collapsed structure.
These are the acoustic values for the six-note chord, understood purely as a harmonic entity (i.e. ignoring the tendencies of its individual pitches). It is essentially a concordant seventh on C; the concordant seventh $Bb_{(0,0,1)}$ generates its own augmented triad, creating a polychord.
But in a linear sense, C E G♯ is a tendency chord that leads to the F major consonant triad, and likewise the F♯ (even as part of the C chord) tends upward to G. These tendency tones will not be confused for their corresponding harmonic values – they are true Limmata.

Melodic:  
\[ E_{(4,0)} \quad G♯_{(8,0)} \quad F♯_{(6,0)} \]
\[ 340 \mu \quad 680 \mu \quad 510 \mu \]

Harmonic:  
\[ E_{(0,1)} \quad G♯_{(0,2)} \quad F♯_{(0,2,1)} \]
\[ 322 \mu \quad 644 \mu \quad 451 \mu = \text{very close to the 11th partial!} \]

Equal-tempered:  
\[ E \quad G♯ \quad F♯ \]
\[ 333\frac{1}{3} \quad 666\frac{2}{3} \quad 500 \]

And thus do we “open our ears” to interpret pitches sounded in equal temperament as melodic or syntonic entities, through the strength of our abilities of selection and perception.
13.3. (continued)

If the individual tendencies of the pitches are considered, we may understand them as melodic rather than harmonic – that is, in their canonic rather than syntonic values. From that perspective, the distinction between melodic and harmonic pitches is not always cut and dried.

Equal temperament provides a useful compromise between the melodic and syntonic values, as demonstrated here. While equal temperament can be faulted for erasing acoustic subtleties such as syntonic comma differences, Karg-Elert would likely say that it is the artificial or “neutral” levelling effect of equal temperament which enables the ear to process and interpret the true pitch relationships in the music – pitch relationships that are potentially infinite in number and kind.
**APPENDIX A**

**POLARITÄTISCHE GENERALTABELLE ALLER BISHER BEKANNT GEWORDENEN WERTE**

**COMPREHENSIVE POLARISTIC TABLE OF ALL PREVIOUSLY KNOWN ACOUSTIC VALUES**

Glossary for this table:
- Didym. = Didymean (i.e. sytonic)
- Dominantnone = dominant ninth
- Doppelkomma = double comma
- Doppelterz = double major third
- Dreiviertelton = three-quarters of a tone (equal-tempered)
- Ganzton = whole tone
- Gleitton = gliding tone
- gr. (grosse) = large
- Halbton = semitone
- harm. = harmonic (i.e. sytonic)
- Hiatus = augmented second
- Hoch = high
- kl. (kleine) = small
- Kleinterz = minor third
- konk. = concordant (pure seventh-based)
- Leipz. (Leipziger) Komma = Leipzig comma (23 µ) – the difference between canonic and seventh-derived pitches of the same name
- Leitton = leading tone
- Messelwert = Messel value (partials calculated downward)
- Naturseptime = natural seventh {4:7}
- Normalganzton = normal whole tone [i.e. Pythagorean]
- Oktave = octave
- Partialwert = partial
- pyth. = Pythagorean (canonic or fifth-based)
- Quarte = fourth
- Quinte = fifth
- rein = pure
- Schnittpunkt, von dem aus Reziprozität der Werte eintritt = central point from where the values exchange symmetrically
- Septime = seventh
- synt. = sytonic (fifth- and third-based)
- temp. (temperiert) = equal-tempered
- Terz = major third
- tief = low
- Tripelkomma = triple comma
- Tritonus = tritone [i.e. three whole tones]
- verm. = diminished
- Viertelton = one quarter-tone (equal-tempered)
- weiter Ganzton = wider whole tone

*) The asterisks denote pitch pairs (of different acoustic derivation) that have identical µ-values. The differences in these pairs result from rounding off logarithmic calculations, and are consistently less than 1 µ (semischisma).
<table>
<thead>
<tr>
<th>No.↑</th>
<th>Pitch, with acoustic symbol</th>
<th>Name of the interval</th>
<th>↑µ</th>
<th>↓µ</th>
<th>Name of the interval</th>
<th>No.↓</th>
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<td>305</td>
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<td>Oktave</td>
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<td>302</td>
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<td>998</td>
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<td>995</td>
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<td></td>
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</table>
*) The identical mio-values for tones of different acoustic derivation result from rounding off logarithmic calculations. The differences between such pairs are consistently less than 1 µ (semischisma).
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**APPENDIX B: GENERAL OVERVIEW** of all functional values, within the tonality of C major and A minor

### I. Diatonic

**Prinzipale** = *principal harmonies* (6.4)

- **Diatonic Prinzipale**

  - Dominating direction = *dominating direction* (i.e. up in major, down in minor)

- **Ultrante** = *ultraforms* (6.9)

  - Chords that step outside of the diatonic boundaries of the key (indicated by the empty brackets)

- **Paralleklänge** = *parallel chords* (6.10)

  - Weak substitutes, moving backwards from their principal chords

- **Leittonwechselklänge** = *leading-tone change chords* (6.10)

  - Strong substitutes, moving forwards from their principal chords

**Lydische Parallele** = parallel chord of the “Lydian” ultradominant in C major

**Phrygische Parallele** = parallel chord of the “Phrygian” ultradominant in A minor
APPENDIX B

Note that the dominants are shown here with their concordant sevenths (indicated by diamond-shaped noteheads, and by the line above/below the $D$ function label). Those sevenths are characteristic but optional.

As with the dominants, both ultraforms are shown here with their concordant sevenths. Those sevenths are once again optional.

The parallel chords are weak substitutes for the principal chords, as they move away from the dominating direction of the mode (i.e. up for major and down for minor).

The leading-tone change chords are strong substitutes for the principal chords, as they move toward the dominating direction of the mode.
II. Chromatic

Prinzipal-Varianten = principal-variants (6.6, 7.8)

Mixolydisch = Mixolydian (specifically, the typically Mixolydian lowered ^7)
Dorisch = Dorian (specifically, the typically Dorian raised ^6)

Parallel-Varianten = parallel-variants (7.4)

rückwärts zu ihren Prinzipalen = moving backwards from their principal chords

Variant-Parallelen = variant-parallels (7.4)

Vorwärts zu ihren Prinzipalen = moving forwards from their principal chords
Mixolydische Parallele = parallel chord of the “Mixolydian” dominant-variant (d) in C major

Medianten = medians (7.4)

Vorwärts zu ihren Prinzipalen = moving forwards from their principal chords
Leitklang = “leading chord,” or triad formed on the leading tone, belonging to the mode’s principal triad type
Tritonante = “tritone chord,” or triad formed on the pitch a tritone away from the tonic (in the mode’s dominating direction), of the mode’s principal triad type
APPENDIX B (continued)

The closed noteheads denote variant pitches, which are one or more syntonic commas away from their Ursprungslage (canonic) counterparts. The altered contrants and ultracontrants (e and Ec) include their concordant sevenths (shown as diamond noteheads).

The closed noteheads indicate that the third of each chord is a variant pitch.

Both the root and fifth are variant pitches, while the third is an Ursprungslage canonic pitch.

Logically, dp in the minor mode would be the dorische Parallele = parallel chord of the “Dorian” dominant-variant (d), which is D major (in the key of A minor).

Once again, the third of each chord is a variant pitch.
Gegenmedianten = Counter-mediants (7.4)  
(proceeding backward from their principal chords)

Gegenleitklang = “counter leading chord,” or triad formed on the counter-leading tone, of the principal triad type
Gegentritonante = “counter tritone chord,” or triad formed on the pitch a tritone away from the tonic (in the mode’s non-dominating direction), of the mode’s principal triad type

Terzgleicher (d.s. Chromonanten der Prinzipalvarianten = same-third chords (i.e. chromonants of the principal-variants) (7.8)

Chromonanten = chromonants, i.e. triads in which all pitches are shifted up or down by a chromatic semitone

Gegenmedianten-Varianten = counter-median-t-variants (addendum to 7.8)

Gegenkonkordanten (septverwandt) = counter-concordants (seventh-related) (10.3)

ERRATUM:

Ich danke meinem Schüler und Freund Roland Müller für seine Mitarbeit.

Sigfrid Karg-Elert.

I thank my student and friend Roland Müller for his assistance.

Sigfrid Karg-Elert
APPENDIX B (continued)

The root and fifth of each chord are variant pitches.

In the key of C major, the tonic variant (t) is C minor, whose chromonant is indeed C# minor. However, note that it is derived through the mediant rather than the principal variant: its function is labelled as $T^{Mp}$, or tonic mediant-parallel.

Once again, the root and fifth of each chord are variant pitches.

As discussed in the addendum to 7.8, the Gegenmedianten-Varianten appear only in this appendix. They could be described as Chromanten der Parallelklänge, or chromonants of the diatonic parallel chords (e.g. Am $\rightarrow$ Abm).

The diamond-shaped noteheads indicate the concordant sevenths.

**ERRATUM:** in C major, $\overline{7}\not A$ must have G-flat, not G-natural.