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ABSTRACT

The dissertation is focused on analyzing nonlinear time-varying dynamic model of hypoid geared rotor system with emphasis on more detailed gear mesh modeling and nonlinear interactions among different driveline components. The main contents consist of two sessions including enhanced gear mesh modeling and coupling effect from misaligned propeller shaft as well as bearing clearance. Detailed contents are listed as follows:

Firstly, an enhanced time-varying load dependent hypoid gear mesh generation model that comprises of effective mesh point, stiffness, line-of-action (LOA) and transmission error (TE) is proposed. The model accounts for instantaneous torque variation within each shaft rotation cycle by employing a three-dimensional (3D) interpolation scheme to construct the mesh parameter surfaces. The variation of mesh parameters within one pinion shaft rotation cycle due to time-varying external loading conditions is analyzed based on the model formulated. Comparison of the analysis results from the proposed model with those calculated from constant load dependent theory under medium to heavy load conditions shows only minor differences due to the small effect of torque fluctuation on dynamic responses when mean torque values are high. This implies that the time-varying mesh parameters calculated with mean torque assumption is sufficiently practical for medium to heavy loads. On the other hand, comparative results also show that the proposed model yields much more reasonable dynamic response predictions than the traditional stationary mesh model in light load cases. This is because the effect of torque fluctuations and existence of complex torque forms are more prominent in light loading conditions. Comparative study of external excitation orders reveals that high order excitation tend to have more significant influence on dynamic responses especially under light torque load conditions.
Secondly, an enhanced hypoid gear mesh model that incorporates Hertzian impact damping function is presented. Two types of impact damping models, namely viscous and non-viscous type based on first principle of mechanics are compared to previous empirical damping models with constant damping coefficient. Parametric studies are performed for both steady state and transient analysis to investigate the impact damping effect under different load conditions or physical parameters of meshing gear pair. Comparative study between viscous and non-viscous damping models are also performed to find their damping effect on dynamic response under different load levels. It is demonstrated that impact damping model can significantly reduce the amplitude of dynamic mesh force. Non-viscous damping model has more significant effect on dynamic response under heavy torque load due to greater elastic deformation. Furthermore, it is found that impact damping can turn double sided impact into single sided impact and suppress response peaks in certain mesh frequency range during speed ramp up.

Thirdly, detailed modeling of a universal joint that connects driveline propeller shaft and pinion shaft is proposed to demonstrate the effect of shaft misalignment on hypoid gear dynamic response with pinion mass unbalance considered. A nonlinear 14-DOF lumped parameter model based on coupled multi-body dynamics is used for case study. Driving speed and torque fluctuation as well as external bending moment generated when driven through universal joint are then analyzed with different misalignment angle. Bending moment components acting on the driven pinion shaft can exert additional load on pinion shaft bearings. Besides, shaft misalignment and mass unbalance excitation will interact with gear internal excitation caused by transmission error (TE). Spectrum analysis are conducted to investigate the components in dynamic response. Finally, dynamic load on bearings are evaluated which is found to be more sensitive to shaft misalignment and mass unbalance excitations compared to gear dynamic response along effective line-of-action.
Finally, nonlinear interaction between time-varying hypoid gear mesh and bearing support is investigated in this study. Mesh parameters are time-varying due to complex tooth profile of hypoid gear. Bearing stiffness is formulated based on real geometry and instantaneous orbital position of rolling elements. Linear model is firstly analyzed to study the modal frequency and mode shape variations under different stiffness ratio between gear mesh and bearing support. Then, nonlinear analysis is conducted to compare the differences between linear and nonlinear dynamic response based on specific nonlinear conditions of geared rotor system. It is found that the coupling between hypoid gear mesh and bearing support can be either strong or weak depending on the ratio between mesh stiffness along line-of-action (LOA) and bearing stiffness in radial direction. Parametric studies indicate that dynamic mesh force is sensitive to bearing clearance for certain stiffness ratio. Spectrum analysis further reveals complex nonlinear behavior due to loss of contact in meshing gear teeth. Dynamic force changes on actual bearing locations due to bearing clearance are evaluated. It is found that bearing radial clearance has influence on structure-borne noise transmission for a complete hypoid gear transmission system due to its effect on gear dynamic response and actual bearing loads.
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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_c )</td>
<td>gear backlash</td>
</tr>
<tr>
<td>( c_m )</td>
<td>empirical mesh damping coefficient</td>
</tr>
<tr>
<td>( c_i )</td>
<td>impact damping</td>
</tr>
<tr>
<td>([C])</td>
<td>system damping matrix</td>
</tr>
<tr>
<td>( e_l, e_u )</td>
<td>translational loaded/unloaded static transmission error</td>
</tr>
<tr>
<td>( e_r )</td>
<td>eccentricity in redial direction</td>
</tr>
<tr>
<td>( \vec{f}_i )</td>
<td>normal contact force vector</td>
</tr>
<tr>
<td>( f(\delta - e) )</td>
<td>non-linear displacement function</td>
</tr>
<tr>
<td>( F )</td>
<td>net gear contact force</td>
</tr>
<tr>
<td>( {F_{bxm}, F_{bym}, F_{bxm}, M_{xm}, M_{ym}} )</td>
<td>bearing force vector</td>
</tr>
<tr>
<td>([G])</td>
<td>velocity gyroscopic matrix</td>
</tr>
<tr>
<td>([G_a])</td>
<td>acceleration gyroscopic matrix</td>
</tr>
<tr>
<td>( h_p, h_g, )</td>
<td>directional rotation radius vector of pinion and gear</td>
</tr>
<tr>
<td>([H(\omega)])</td>
<td>FRF matrix</td>
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<tr>
<td>( I_p, I_g )</td>
<td>mass moment of inertia of pinion and gear</td>
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<tr>
<td>( k_m )</td>
<td>equivalent mesh stiffness</td>
</tr>
<tr>
<td>([k_p], [k_g])</td>
<td>pinion/gear supporting stiffness matrices</td>
</tr>
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<td>( k_E, k_L, )</td>
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<tr>
<td>([K])</td>
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<tr>
<td>( L )</td>
<td>pinion end to mass center distance</td>
</tr>
<tr>
<td>( m_e, m_p, m_g )</td>
<td>equivalent, pinion and gear mass</td>
</tr>
<tr>
<td>([M])</td>
<td>mass matrix</td>
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<tr>
<td>( N )</td>
<td>number of contact cells</td>
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<tr>
<td>( \vec{n}_i )</td>
<td>line of action unit vector</td>
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<tr>
<td>( {q} )</td>
<td>general coordinate vector</td>
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<td>( R )</td>
<td>gear ratio</td>
</tr>
<tr>
<td>( \vec{r}_{lm} )</td>
<td>mesh point vector</td>
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<tr>
<td>( r )</td>
<td>mean value ratio</td>
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</tbody>
</table>
\( r_b, r_d \) roller and bearing inner raceway curvature radius
\( S_i \) pinion \((l = p)\)/gear \((l = g)\) coordinate system
\( t \) time
\( T_p, T_g \) load on pinion and gear
\( p \) \( \delta_d - e_U \)
\( x, y, z \) translational coordinates
\( \alpha \) shaft misalignment angle
\( \beta \) impact damping coefficient
\( \delta_d \) dynamic transmission error
\( \{\delta_{xm}, \delta_{ym}, \delta_{zm}, \beta_{xm}, \beta_{ym}\} \) bearing displacement vector
\( \lambda_i \) directional rotation radius
\( \omega \) frequency
\( \zeta \) empirical damping ratio
\( \theta \) rotational displacement
\( \Phi \) modal shape
\( \Omega_z \) mean shaft speed
\( \psi_j \) orbit position of rolling element
\( \varphi \) transition coefficient
\( \gamma_L \) bearing radial clearance

**Subscripts**

- \( B \) bearing
- \( E, D \) engine or driver
- \( R \) reference node
- \( S \) shaft
- \( cont \) gear contact parameter
- \( e \) equivalent value
- \( ext \) external parameter
- \( g \) gear
\( i \)  
index of contact cell

\( \text{int} \)  
internal parameter

\( k \)  
stiffness

\( l \)  
pinion \((l = p)\) and gear \((l = g)\)

\( m \)  
mean value

\( n \)  
natural mode

\( p \)  
pinion

\( s \)  
geared-rotor system

\( x, y, z \)  
translational coordinates

1  
non-viscous type

2  
viscous type

\textit{Superscripts}

\( \cdot \)  
derivative w.r.t. time

\( \rightarrow \)  
vector quantities
Chapter 1. Introduction

Hypoid and spiral bevel gear transmissions are widely used in industries today for transmitting rotational motion and torque between non-parallel or non-intersecting shafts. Specifically, its applications can be found in automotive systems, off-highway vehicles, wind turbines or other industrial machineries with speed reducing mechanisms. However, the mechanical vibration and noise problems caused by such mechanism have frequently been an annoyance and detrimental to life span and efficiency of transmission components such as gear pairs and roller element bearings supporting them. The noise and vibration problems are mainly excited by the time-varying gear mesh parameters and are then transmitted and amplified by structural resonances. To design a more durable, efficient and quiet powertrain requires an in-depth study of geared rotor system dynamics and more accurate understanding of the dynamic response considering specific operating condition.

From the available published works in gear dynamics, it is found that most of the studies are limited to parallel axis gears like spur or helical gears. In addition, previous analytical methods are also limited to low degree-of-freedom system and based on a lot assumptions that are unrealistic when analyzing more complex cases. Furthermore, the study of hypoid or bevel geared rotor system dynamics is far less mature as compared to vibratory behavior of spur or helical gears. This is because of their complicated geometry that introduces strong time and spatial-varying characteristics and dynamic coupling between parameters in different coordinates or components of the system.

Therefore, the goal of this dissertation is to establish a more accurate modeling of hypoid geared rotor system by improving the mesh model functions and considering specific nonlinear factors such as impact damping, propeller shaft misalignment and unbalance as well as the
nonlinear coupling effect between gear mesh and external excitation. Finally, the work is expected to enhance the understanding of hypoid and bevel gear dynamics and lay a foundation for future research work that can lead to more quiet and durable transmission designs.

1.1. Literature Review

With increasing requirement for more efficient and durable gear transmission, engineers have made consistent effort in improving gear designs that cover a wide range of topics including dynamics, tooth wear, contact analysis, fatigue life prediction, power loss and efficiency. The following literature review will be mainly focused on dynamics of hypoid geared rotor system and tooth contact analysis.

According to published literature (Brecher et al., 2011), the gear dynamic response is generated by combined influence of internal and external excitations. The internal excitation is caused by gear transmission error either considering or not considering gear tooth elasticity obtained by tooth contact analysis (TCA). On the other hand, the external excitations are due to speed or torque fluctuation at driving or driven end of the geared rotor system.

Transmission error is evaluated by deviation of tooth surface from ideal theoretical profile, and has been shown to be one of the root causes of most gear vibration and noise problems (Litvin, F. L., and Zhang, Y., 1991). If tooth load and its elastic deformation is not considered, kinematic transmission error can be evaluated by analyzing imperfections in the gear tooth surface geometry assuming no misalignment is present. Because of its complexity, the mathematical expression of hypoid and bevel gear tooth surface is generated from specified cutter settings of machine tools. Two machining methods: face-milled (Litvin and Gutman, 1981; Litvin et al., 1989; 1995; Fan and Wilcox, 2005; Fan 2006, 2007) and face-hobbed gears (Litvin and Chaing et al., 1990; Fan and Wilcox, 2005; Fan 2006, 2007; Vimercati, 2007) are commonly used and have been
extensively studied for deriving detailed expression for tooth geometrical characteristics. By properly adjusting machine tool settings or applying typical profile modification like tip relief, kinematic transmission error can be reduced in the attempt to improve dynamic performance at design stage. This method is especially suitable for reducing gear noise under light torque load, but complex load conditions over a wide range is ignored.

If on the other hand, gear tooth elasticity is considered, loaded transmission error can be determined by employing a loaded TCA based on contact mechanics such as Hertzian theory, finite element analysis, semi-analytical or combined method (Vijayakar, 1991; Piazza and Vimercati, 2007). Generally, the loaded TCA is expected to demonstrate a more accurate prediction of gear dynamic performance. In some studies, the analytical results were also compared to experimental data (Handschtuh and Bibel, 1999; Gosselin, Guertin, Remond and Jean, 2000; Litvin, Fuentes and Hayasaka, 2006) and good correlation were observed in their case studies which indicated effectiveness of analytical modeling method in predicting gear dynamic response.

System dynamic response and noise radiation are indication of the sensitivity of geared rotor system towards internal and external excitations and have been among major concerns during gear application in different fields. A series of research work to study the dynamics of parallel axis gear pairs (Ozguven and Houser, 1988; Kahraman and Singh, 1990, 1991, 1994, 1996; Lim and Singh, 1990, 1991; Blankenship and Singh, 1995; Velex and Maatar, 1996; Raclot and Velex, 1999;) is available, which focused on mesh interface and nonlinear coupling of the geared rotor system. The relationship between gear mesh excitation and dynamic response of the system found in their work indicates that performance of hypoid gear dynamic modeling can be potentially evaluated by studying a rear axle system excited by gear mesh.
Apart from those early stage simplified analytical modeling work in parallel axis gear, some basic experimental studies in rear axle are also found where mathematical methods were further compared to practical test data. Remmers (1971) analyzed a single degree-of-freedom dynamic model for a rear axle gear system with infinite mesh stiffness to predict the pinion resonance and performed experiments to confirm the existence of vibration peaks. Kiyono et al. (1981) derived a 2 degrees-of-freedom (DOF) vibration model for bevel gear pair and the time-varying line-of-action (LOA) was simplified as a pure sine wave. Nakayashiki et al. (1983) examined the dynamic coupling effect between gear torsional vibration and supporting bearing stiffness. Abe and Hagiwara (1990) performed experimental study on a driveline system with hypoid gear pair by speed sweeping up. Their results showed that by adding inertia disk on either of the side flanges of the final drive, axle gear noise can be reduced in some cases by changing the vibration mode. Hirasaka et al. (1991) proposed an experimental method to study the gear body and driveline sensitivity to unit transmission error of a specific gear design. Forces at the contact point were measured after applying acceleration and deceleration on pinion and ring gear separately using special device. Significant effect of torsional vibration on hypoid gear dynamic response was found. Brecher et al. (2011) found the correlation between gear dynamic transmission error and driveline noise level by a new method of measuring the transmission error at each assembly step to analyze the combined effect of driveline components on noise radiation. However, most of these analytical or experimental work are either limited to parallel axis gears or assuming torsional 2-DOF gear mesh model.

More recently, a more realistic mathematical framework for studying hypoid/bevel gear dynamics was proposed by Cheng and Lim (Lim and Cheng, 1999; Cheng, 2000). The tooth contact analysis is based on exact tooth geometry which generates a set of time-varying mesh
parameters such as mesh stiffness, mesh damping, mesh point and line of action (LOA). Clearance type backlash is also considered and is applied together with transmission error to form nonlinear time-varying dynamic coupling parameters at the gear mesh.

Jiang (2002) derived a 2-DOF nonlinear time-varying (NLTV) dynamic model for hypoid gear pair by applying the Harmonic Balance Method (HBM) and numerical integration. The time-varying mesh parameters are expressed by perfect sinusoid. Wang and Lim (2002) then extended the work into a multi-point gear mesh model seeking to apply individual coupling at each contacting tooth pair. Wang and Lim (2007) further included asymmetry effects on time-varying mesh parameters of each tooth pair based on torsional dynamic model. They also extensively analyzed and demonstrated the significance of out-of-phase vibratory mode in gear dynamic response and attempted to improve dynamic performance of gear by tuning the out-of-phase mode. Peng (2010) developed a 14-DOF coupled multi-body dynamic model for hypoid geared rotor system that considers large rotation displacement of pinion or gear shaft along with small perturbation movement of gear body. Time-varying parameters change with exact angular position instead of strictly following time value. In addition, good correlation was attained between dynamic mesh force and noise radiation of a real off-highway vehicle gearbox. Later, Yang (2012) enhanced the previous analytical modeling methodology by using Multi-term Harmonic Balance method (MHBM) and studied response stability of a 2-DOF meshing gear pair. The effect of driveline components on gear dynamic characteristics are also introduced by analyzing the coupling effect from time-varying bearing stiffness, elastic housing and propeller shaft.

However, from the reviewed literature listed above, studies on hypoid and bevel gear dynamics are still very limited as compared with significantly more published works in parallel axis gear research. Most of the previous hypoid or bevel gear studies are limited to insufficient
consideration of nonlinear coupling of gear mesh, and the lack of the study on nonlinear dynamic interaction between meshing gear body and surrounding components such as driving/load units, propeller shaft and bearings. These factors can either have direct impact on hypoid gear dynamic response through dynamic coupling between gear pair and other driveline components, or can change mesh parameters within each rotation cycle by generating torque or speed fluctuation. Therefore, such gaps in hypoid gear dynamics need further study for a more comprehensive and deeper understanding of the dynamic performance of hypoid geared rotor system. The attempt to fulfill this gap in the literature is the basis for this dissertation research.

1.2. Motivation, Scope and Objectives

According to those published literatures reviewed above, it can be seen that studies on hypoid or bevel gear dynamics are far less in-depth than the more common parallel axis gears such as spur gears or helical gears. Based on the fact that hypoid or bevel gear mesh is far more complex due to its tooth surface geometry, many assumptions and approximations made in previous work cannot be directly applied to hypoid gear case. Thus, more detailed and accurate modeling is needed for achieving valid and realistic prediction of the dynamic performance.

Therefore, this dissertation will introduce an enhanced nonlinear time-varying (NLTV) geared rotor system model for better understanding of nonlinear coupling effect between meshing gear pairs and amongst the major driveline components. The main purpose is to evaluate effect of different external excitations on gear dynamic response and its interaction with time-varying mesh parameters, to further study the nonlinear impact phenomenon between gear teeth during light torque load, and also to examine the coupling between hypoid gear mesh and other components
with nonlinearity included. To achieve these objectives, tasks to be accomplished are listed and explained below:

a) Develop an enhanced load dependent time-varying mesh model based on linear interpolation whose parameter can change with specific pinion rotation angle and can reflect the variation of any external load conditions within a certain amplitude range. Both linear and nonlinear cases are covered to analyze the significance of nonlinear factor and coupling effect. Extensive parametric studies are performed to compare the differences between enhanced model and previous model that only considers constant load condition.

b) Establish impact damping model for hypoid gear mesh. Use detailed function instead of empirical coefficient to express nonlinear damping force of the mesh coupling between pinion and gear teeth. The change in damping value over the entire meshing process is evaluated. The differences between non-viscous and viscous damping functions are highlighted by performing a series parametric studies considering torque level, backlash nonlinearity and load condition.

c) Introduce shaft misalignment into gear shaft-bearing system by detailed modeling of universal joint and analyze the combined effect of misalignment and mass unbalance on dynamic response. Different cases are compared to investigate their influence on specific speed range. Sensitivity of gear dynamic response and bearing dynamic load towards external excitation from either misalignment or mass unbalance is also analyzed.

d) Expand the hypoid geared rotor system by incorporating clearance type nonlinearity from bearing support. Potential key factors that have most contribution to the coupling between gear mesh and bearing nonlinearities are investigated. Cause of softening and hardening in dynamic response is explained through comparative case study.
1.3. Organization of Proposal

In Chapter 1, a general introduction of this dissertation is presented by discussing a list of relevant literatures reviewed and demonstrating the motivations, scope and objectives. Also, the expected results are discussed.

In Chapter 2, previous time-varying mesh model is further enhanced by incorporating instantaneous external torque effect on gear mesh. Comparison study is made between enhanced model and previous analysis under different levels of fluctuating torque excitation.

In Chapter 3, two types of Hertzian impact damping models including non-viscous and viscous type are introduced for incorporation into the nonlinear time-varying mesh model. The damping coefficient is decided by both velocity and displacement of two meshing teeth as well as the inertia of pinion/gear body. Parameter studies are carried out under steady state and transient conditions.

In Chapter 4, propeller shaft misalignment and pinion mass unbalance effect is considered in hypoid geared rotor system. External excitations from universal joint including torque/speed fluctuation and bending moment are imposed at the driving end of pinion shaft. Comparison studies are performed to evaluate model sensitivity towards specific excitations due to misalignment and mass unbalance.

In Chapter 5, nonlinear coupling between hypoid gear pair and bearings is studied by introducing clearance type nonlinearity in the bearing stiffness functions. Both weak and strong coupling cases are compared and evaluated based on different nonlinear conditions. The study provides an in-depth explanation of nonlinear coupling effect in a hypoid geared-rotor system with multiple clearances.
In Chapter 6, the addressed topics and achievements expected are concluded and summarized. Future tasks are also briefly explained.
Chapter 2. Time-varying Torque Load Dependent Hypoid Gear Mesh and Dynamic Analysis

2.1. Introduction

Hypoid gears are widely applied in automotive industry for transmitting rotational motion and power between two perpendicular non-intersecting shafts. However, this type of gears are often sources of axle vibration and noise concerns such as gear whine and rattle which may be affected by the combined effect of external torque fluctuation and internal gear mesh transmission error. Thus, gaining a more complete understanding of the hypoid gear meshing characteristics and dynamics is essential in mitigating this type of noise and vibration problems.

Modeling of gear dynamics has been performed for many years (Ozguven and Houser, 1988) in which gear mesh coupling is simplified as a spring-damping element connecting two meshing gears. This type of modeling depends on the ability to conduct an accurate loaded tooth contact analysis (LTCA). Tooth backlash designed for lubrication or interference avoidance is another form of clearance nonlinearity that can significantly affect gear dynamic response. This type of nonlinear factor was extensively analyzed for spur gear pairs (Kahraman and Singh, 1990) and strong nonlinear coupling effect was seen under different gear mesh conditions. In addition, time-varying mesh stiffness was incorporated in the gear dynamic modeling and comparative study was conducted to analyze the influence of time-varying parameters on dynamic response (Kahraman and Singh, 1991). The method was extended to helical gear train analysis (Kahraman, 1994). Studies on the combined effect of clearance type nonlinearity and time-varying mesh parameters (Kahraman and Blankenship, 1996; Lim and Singh, 1991; Blankenship and Kahraman, 1995) demonstrated complex behavior in lightly loaded case such as single or double sided impact
(Litvin et al., 1996), quasi-periodic, subharmonic or chaotic responses (Kahraman and Singh, 1990; Kahraman and Singh, 1991) and jump discontinuity due to softening or hardening during speed sweep (Kahraman and Blankenship, 1997). The effectiveness of dynamic models were further validated by comparison between direct numerical integration and analytical modeling method such as multi-tem harmonic balance method (HBM) (Al-shyyab and Kahraman, 2005; Ma and Kahraman, 2005, 2006) or finite element method (Ambarisha and Parker, 2007). Based on tooth profile generation methods (Litvin, 2004), dynamic performances can be improved by optimizing transmission error through adjusting machine settings (Lin, Tsay and Fong, 1998; Fong and Tsay, 1991) or profile modifications (Litvin, Lian and Kapelevich, 2000). However, most of these studies are focused on parallel axis gears such as spur and helical gears. For hypoid gear mesh modeling, there is only limited research work published because of the strong nonlinear coupling characteristics and time-varying mesh parameters due to the complex surface geometry of tooth profile (Cheng and Lim, 2000). In addition, the time-varying torque condition also may have significant effect on gear dynamic response which calls for a more accurate hypoid gear mesh modeling method for complicated load cases.

Early studies on hypoid gear mesh behavior based on extending parallel gear mesh analysis are highly approximated (Fujii and Suzuki, 1981; Rautert and Kollmann, 1989; Sugita and Asai, 1991). These models featuring time and spatial-invariant parameters are estimated empirically by employing simple geometry. With combined surface integral and finite element method, quasi-static loaded tooth contact analysis (LTCA) was conducted to solve a three-dimensional contact problem (Vijayakar, 1991, 2003). Later, hypoid gear mesh model was developed, which applies three-dimensional LTCA to extract mesh parameters from exact tooth geometry (Cheng and Lim, 1998). Recently, hypoid gear mesh and dynamic analysis was performed using either relative
torsional displacement (Jiang, 2002; Wang, 2002) or pure-vibration model (Wang, 2007) and served as the basis for further in-depth hypoid gear dynamic analysis. Then, a coupled multi-body dynamic model (Peng, 2010) was developed that considers large rotation displacement of pinion shaft with emphasis on constant load dependent and time-varying gear mesh characteristics. However, how mesh parameters change with pinion rotational angle and time-varying torque within one mesh cycle was not explained.

In this paper, a gear mesh model that is capable of reflecting the effect of external torque variation is developed to achieve a more reasonable prediction of hypoid gear dynamics under time-varying load conditions. The main focus of this study is to propose a time-varying load dependent gear mesh model for use in performing hypoid gear dynamic analysis. The proposed formulation investigates the variation of mesh parameters with pinion rotation angle and instantaneous torque value. Both linear and nonlinear dynamic responses are simulated and compared under time-varying torque load conditions with different mean values and fluctuation speeds. Change in the gear mesh parameters due to torque variation within one mesh cycle as well as shaft rotation cycle are analyzed. Comparison between proposed time-varying load dependent gear mesh model and previous constant load dependent model is performed to find the balance between efficiency and accuracy in generating the relevant gear mesh parameters.

### 2.2. Hypoid Gear Mesh

The mesh coupling between pinion and gear body illustrated in Figure 2.1 (a) is expressed as an effective spring-damping element generated from LTCA assuming constant load. Mesh parameters such as mesh point, line-of-action (LOA), mesh stiffness and transmission error (TE)
are load dependent and time varying as the gear pair rolls due to the interaction between pinion and gear teeth which differs hypoid gear mesh from spur or helical gears. Variation of contact area and instantaneous load distributions are the root causes of those varying parameters.

To formulate time-varying load dependent gear mesh model, a combined surface integral and finite element method (Vijayakar, 1991) was applied. For the points near the contact area, surface integral is used to synthesize the relative displacement between points in a highly accurate way, while finite element method is used to predict element deformation beyond a certain distance from the contact area including gear body, shaft bearing or other surrounding structures. Load distribution and angular TE are simulated using simplex type algorithm (Vijayakar, 2003) under specified nominal torque loads. For each mesh cycle, load distribution and TE are indexed by rolling angle or time step. As is shown in Figure 2.1 (b), the contact area illustrated in hypoid gear pair can be divided into N contact cells depending on instantaneous load distribution. Then for each cell in the area, the position vector, normal vector as well as normal contact force vector on the cell can be specified accordingly where y-axis is defined as pinion rotational axis and thus the combined contact force vectors in three directions can be calculated:

\[
F_x = \sum_{i=1}^{N} (\bar{n}_{ix} \cdot \bar{f}_i), 
F_y = \sum_{i=1}^{N} (\bar{n}_{iy} \cdot \bar{f}_i), 
F_z = \sum_{i=1}^{N} (\bar{n}_{iz} \cdot \bar{f}_i) 
\]

(2.1)

The equivalent total force is:

\[
F_{\text{cont}} = \sqrt{F_x^2 + F_y^2 + F_z^2},
\]

(2.2)

Then the effective mesh stiffness can be written as:

\[
k_m = \frac{F_{\text{cont}}}{(e_L - e_U)}
\]

(2.3)

where \(e_L\) and \(e_U\) are loaded and unloaded translational TE derived from angular TE (\(e_{La}\) and \(e_{Ua}\)).
\[ e_L = e_{La} \cdot \dot{\lambda}_y \]  
\[ e_U = e_{Ua} \cdot \dot{\lambda}_y \]  

(2.4)  
\[ (2.5) \]

Here \( \dot{\lambda}_y \) is the directional rotation radius around y axis. The rotation radii around three coordinate axes can be written as:

\[ \dot{\lambda}_x = y_m n_z - z_m n_y \]  
\[ \dot{\lambda}_y = z_m n_x - x_m n_z \]  
\[ \dot{\lambda}_z = x_m n_y - y_m n_x \]  

(2.6)  
\[ (2.7) \]  
\[ (2.8) \]

where \((n_x, n_y, n_z)\) is the coordinate of effective LOA calculated from surface normal vector of each cell in contact area:

\[ n_x = \frac{F_x}{F_{cont}}, \quad n_y = \frac{F_y}{F_{cont}}, \quad n_z = \frac{F_z}{F_{cont}} \]  

(2.9)

the equivalent mesh point \((x_m, y_m, z_m)\) is derived from contact force and moment:

\[ y_m = \frac{\sum_{i=1}^{N} r_{iy} f_i}{N} \]  
\[ x_m = \frac{(M_x + F_x y)}{F_y}, \quad \sum_{i=1}^{N} f_i, \quad z_m = \frac{(M_y + F_y z)}{F_x} \]  

(2.10)  
\[ (2.11) \]  
\[ (2.12) \]  
\[ (2.13) \]

\[ M_x = \sum_{i=1}^{N} f_i \left[ n_{ix} r_{iy} - n_{iy} r_{ix} \right] \]  
\[ M_y = \sum_{i=1}^{N} f_i \left[ n_{iy} r_{iz} - n_{iz} r_{iy} \right] \]  
\[ M_z = \sum_{i=1}^{N} f_i \left[ n_{iz} r_{ix} - n_{ix} r_{iz} \right] \]
Figure 2.1 (c) illustrates the mesh coupling based on above equations. Notice that the synthesis of all the mesh parameters are generated at each pinion roll angle under specific torque load. Mesh stiffness accounts for both tooth (contact and bending) and gear body flexibility. Mean mesh stiffness generated from constant load dependent mesh model is shown in Figure 2.2 (a) where the mesh stiffness at certain torque load is the mean value. On the other hand, the 3D parametric surface of time-varying load dependent mesh stiffness demonstrated in Figure 2.2 (b) is capable of describing specific stiffness variation within one mesh cycle.

Figure 2.1, Hypoid gear illustration (Peng, 2010): (a) hypoid gear body, (b) contact cells, mesh position vector and normal vector, (c) effective mesh coupling
Both loaded and unloaded TE are applied in deriving the mesh stiffness as expressed by Equation. (2.3). Unloaded ($\epsilon_0$) and loaded ($\epsilon_L$) TE can be used in nonlinear or linear mesh model as imposed displacement excitation of gear pair. The unloaded TE indicates manufacturing or assembly error. The loaded TE generated from LTCA also includes elastic deformation in gear tooth and gear body during gear mesh which means it is synthesized from manufacturing error and nonlinear mesh stiffness. The comparison between unloaded and loaded TE is shown in Figure 2.3 (a)-(b). It is observed that unlike unloaded TE that varies over a relatively small amplitude range, the loaded TE demonstrates clear time-varying load dependent characteristics. The changing of absolute value of loaded TE along with the rising of torque is unidirectional.
Mesh point and LOA are applied to describe the position and direction of effective mesh stiffness. Because of its complex surface geometry, the mesh point, LOA and effective rotation radii of hypoid gear pair are substantially time varying when the gear pair is operating which would generate additional excitation to the entire geared rotor system. In order to account for the effect of time-varying external torque on mesh parameters synthesized from quasi-static tooth contact analysis, 3D surface interpolation method is also applied to generate instantaneous parametric values during operation. The pre-synthesized parameter surfaces are shown in Figure 2.4 with \( (R_{px}, R_{py}, R_{pz}) \) as mesh point coordinates (Figure 2.4 (a)-(c)), \( (N_{px}, N_{py}, N_{pz}) \) as LOA coordinates (Figure 2.4 (d)-(f)) and \( (\lambda_{px}, \lambda_{py}, \lambda_{pz}) \) (Figure 2.4 (g)-(i)) as directional rotation radii. Pinion rolling angle and instantaneous torque value at that angular position are used as indexes for defining exact values. The shapes of parameter surfaces indicate that mesh parameters are sensitive to rotation angle and torque variation in light torque range. On the other hand, the slopes in medium...
and heavy torque range are more flat which implies that variations of mesh parameters are small when torque level is high enough. In addition, it can also be observed that mesh parameter fluctuation within one mesh cycle reduces with higher external mean torque level. It can be seen that the time-varying load dependent nature of hypoid gear mesh parameters is an important internal excitation that is expected to have significant influence on dynamic response especially in light load cases.
2.3. Geared Rotor System

As is illustrated in Figure 2.5, a 14-DOF lumped parameter dynamic model based on coupled multibody dynamics (Peng, 2010) is applied to stand for a hypoid geared rotor system. The engine/driver and load are represented by rigid bodies with one rotation DOF (θ_E and θ_L) respectively because the other DOFs are assumed to be decoupled. The gear pair including pinion and gear bodies are also modeled as rigid bodies and each has 6 DOFs (translational x_l, y_l, z_l, and rotational θ_{lx}, θ_{ly}, θ_{lz}, where l = p / g). The coordinate set is relative to either pinion or gear body’s local inertial reference frame (Figure 2.1 (a)). The lumped support under pinion and gear are simplified as a set of spring and damper elements. The rotational DOFs of the lumped parameter dynamic model such as θ_{ps}, θ_{gs}, θ_E, θ_L are large rotational displacements and are subject to torsional dynamics of driveline. The driving and load end of the geared rotor system are connected to the
gear pair by torsional spring and damper element. Driving torque \( T_E \) and load torque \( T_L \) can be either constant or time-varying depending on practical cases. Therefore, the equations of motion generated from Lagrange formulation can be expressed as:

\[
I_E \ddot{\theta}_E + c_{pyr}(\dot{\theta}_E - \dot{\theta}_p) + k_{pyr}(\theta_E - \theta_p) = T_E
\]

(2.14)

\[
M_{px} \ddot{x}_p + c_{pxr} \dot{x}_p + k_{pxr} \dot{x}_p = -n_{px} F_m
\]

(2.15)

\[
M_{py} \ddot{y}_p + c_{pyr} \dot{y}_p + k_{pyr} \dot{y}_p = -n_{py} F_m
\]

(2.16)

\[
M_{pz} \ddot{z}_p + c_{pzr} \dot{z}_p + k_{pzr} \dot{z}_p = -n_{pz} F_m
\]

(2.17)

\[
I_{px} \ddot{\theta}_x + c_{pxr} \dot{\theta}_x + k_{pxr} \dot{\theta}_x - I_{px} \ddot{\theta}_y - I_{py} \dot{\theta}_y - \lambda_{px} F_m
\]

(2.18)

\[
I_{px} \ddot{\theta}_y + c_{pyr} (\dot{\theta}_y - \dot{\theta}_E) + k_{pyr} (\theta_y - \theta_E) = -\lambda_{px} F_m
\]

(2.19)

\[
I_{px} \ddot{\theta}_z + c_{pzr} \dot{\theta}_z + k_{pzr} \dot{\theta}_z + I_{px} \ddot{\theta}_y + I_{py} \dot{\theta}_y + \lambda_{px} F_m
\]

(2.20)

\[
M_{gx} \ddot{x}_g + c_{gxr} \dot{x}_g + k_{gxr} x_g = n_{gx} F_m
\]

(2.21)

\[
M_{gy} \ddot{y}_g + c_{gyr} \dot{y}_g + k_{gyr} y_g = n_{gy} F_m
\]

(2.22)

\[
M_{gz} \ddot{z}_g + c_{gze} \dot{z}_g + k_{gze} z_g = n_{gz} F_m
\]

(2.23)

\[
I_{gx} \ddot{\theta}_x + c_{gxr} \dot{\theta}_x + k_{gxr} \dot{\theta}_x - I_{gy} \ddot{\theta}_y - I_{gz} \ddot{\theta}_z = \lambda_{gx} F_m
\]

(2.24)

\[
I_{gy} \ddot{\theta}_y + c_{gyr} (\dot{\theta}_y - \dot{\theta}_L) + k_{gyr} (\theta_y - \theta_L) = \lambda_{gy} F_m
\]

(2.25)

\[
I_{gz} \ddot{\theta}_z + c_{gze} \dot{\theta}_z + k_{gze} \dot{\theta}_z + I_{gy} \ddot{\theta}_y + I_{gz} \ddot{\theta}_z = \lambda_{gz} F_m
\]

(2.26)

\[
I_L \ddot{\theta}_L + c_{gvr} (\dot{\theta}_L - \dot{\theta}_r) + k_{gvr} (\theta_L - \theta_r) = T_L
\]

(2.27)

The combined expression of equation of motions can be condensed into matrix form as:

\[
[M]\{\dot{x}\} + [C]\{\ddot{x}\} + [K]\{x\} + [G]\{\hat{x}\} + [G]\{x\} = \{F\}
\]

(2.28)

The fourteen generalized coordinates including two rotational DOFs of engine/driver and load, six DOFs of pinion or gear body coordinates can be expressed as:
\{x\} = \{\theta_E, x_p, y_p, z_p, \theta_{px}, \theta_{py}, \theta_{pz}, x_g, y_g, z_g, \theta_{gx}, \theta_{gy}, \theta_{gz}\}^T \tag{2.29}

The lumped mass matrix includes both mass and moment of inertia of the system:

\[ [M] = \text{diag}[I_E, M_p, M_p, I_{px}, I_{py}, I_{pz}, M_g, M_g, I_{gx}, I_{gy}, I_{gz}, I_L] \tag{2.30} \]

Gyroscopic matrix \([G]\) are extracted from the complete expression of equation of motion above. The definition can be found in reference literatures (Genta, 2005).

The system stiffness matrix can be written as:

\[
[K] = \begin{bmatrix}
k_E & [K_p] \\
[K_g] & k_L
\end{bmatrix} \tag{2.31}
\]

The stiffness elements in the matrix are lumped at the generalized coordinates accordingly. The damping matrix can be either modal damping or component proportional damping based on empirical damping ratio. The system excitation vectors are expressed as:

\[
\{F\} = \begin{bmatrix} T_E, h_p F_m, -h_g F_m, -T_L \end{bmatrix}^T \tag{2.32}
\]

where \(T_E\) and \(T_L\) are external excitation acting on engine/driving end and load end respectively. The external excitation can be either constant as was commonly assumed in previous studies or time-varying that can be written as:

\[
T_E = T_m \star \left[ 1 + a \cdot \cos \left( n \dot{\theta}_{py} \cdot t \right) \right] \tag{2.33}
\]

where \(T_m\) is mean torque value, \(a\) is the percentage of torque fluctuation and \(n\) is the excitation order number.
The internal vibratory excitations originate from kinematic TE which is a displacement type excitation. The coordinate transformation vectors \( h_p \) and \( h_g \) in internal excitation component can be defined as:

\[
h_l = \{n_{ix}, n_{iy}, n_{iz}, \lambda_{ix}, \lambda_{iy}, \lambda_{iz}\}, \quad l = p, g
\]

(2.34)

The vector consists of effective LOA directional cosine vector \( Z_{lm}(n_{ix}, n_{iy}, n_{iz}) \) and directional rotation radii written as:

\[
\lambda_{ix} = y_{im} n_{iz} - z_{im} n_{iy}, \quad \lambda_{iy} = z_{im} n_{ix} - x_{im} n_{iz}, \quad \lambda_{iz} = x_{im} n_{iy} - y_{im} n_{ix}, \quad l = p, g
\]

(2.35)

Therefore, hypoid gear dynamic transmission error (DTE) along LOA can be denoted as:

\[
\delta_d = h_p (x_p, y_p, z_p, \theta_{p}, \theta_{g}, R, R_{p}, R_{g})^T - h_g (x_g, y_g, z_g, \theta_{g}, 0, 0)^T
\]

(2.36)

The gear ratio \( R \) is applied to find relative displacement between engaging teeth that is expressed as:

\[
R = N_g / N_p
\]

(2.37)

where \( N_p \) and \( N_g \) are tooth numbers of pinion and gear respectively.

The dynamic mesh force acting on effective mesh point along LOA can be expressed as:

\[
F_m = \begin{cases} 
    k_m \cdot (\delta_d - e_U - b_c) + c_m \cdot (\dot{\delta}_d - \dot{e}_U) & \delta_d - e_U > b_c \\
    0 & -b_c < \delta_d - e_U < b_c \\
    k_m \cdot (\delta_d - e_U + b_c) + c_m \cdot (\dot{\delta}_d - \dot{e}_U) & \delta_d - e_U < -b_c 
\end{cases}
\]

(2.38)

where \( b_c \) denotes backlash nonlinearity and \( k_m \) is time-varying load dependent mesh stiffness and effective mesh damping respectively. The mesh damping \( c_m \) here is assumed to be viscous type based on empirical constant damping ratio which can be written as:

\[
c_m = \frac{2 \zeta_m}{\omega_n} \cdot k_m
\]

(2.39)
\[ \omega_n = \sqrt{\frac{k_m}{m_e}} \]  

(2.40)

where \( m_e \) is the equivalent mass of the gear pair. For linear case \( (b_e = 0) \), the dynamic mesh force can be simplified as:

\[ F_m = k_m \cdot (\delta_d - e_U) + c_m \cdot (\dot{\delta}_d - \dot{e}_U) \]  

(2.41)

In order to solve both non-linear and linear time-varying load dependent model, time domain numerical integration method that applies Runge Kutta 4-5\(^{th}\) algorithm is used at a series of nominal rotation speeds (mesh frequency). A torque imposed step speed sweep simulation is performed which attains steady state dynamic response under constant mesh frequency. The initial state for next step is based on the final stage of current state variable values. Root mean square (RMS) is then calculated from time history results to generate gear dynamic response.
2.4. Example Case

A set of face-milled hypoid gear pair is applied in the case study. The gear geometry and dynamic system parameters are listed in Table 2.1. It is assumed that no tooth surface modification or assembly error is considered when generating tooth geometry. Gear mesh friction effect is also not included. Different load levels ranging from light to heavy are applied to compare dynamic responses based on constant and time-varying load dependent mesh model.

Table 2.1 Hypoid gear geometry settings and system parameters

<table>
<thead>
<tr>
<th>Gear Data</th>
<th>Pinion</th>
<th>Gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth</td>
<td>10</td>
<td>43</td>
</tr>
<tr>
<td>Spiral angle (rad)</td>
<td>0.803</td>
<td>0.591</td>
</tr>
<tr>
<td>-------------------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>Pitch angle (rad)</td>
<td>0.295</td>
<td>1.269</td>
</tr>
<tr>
<td>Face width (m)</td>
<td>0.048</td>
<td>0.168</td>
</tr>
<tr>
<td>Type</td>
<td>Left hand</td>
<td>Right hand</td>
</tr>
<tr>
<td>Loaded side</td>
<td>Concave</td>
<td>Convex</td>
</tr>
<tr>
<td>Offset (m)</td>
<td>0.0318</td>
<td></td>
</tr>
</tbody>
</table>

**System Parameters**

<table>
<thead>
<tr>
<th></th>
<th>Engine/Driver</th>
<th>Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsional moment of inertia (kg-m²)</td>
<td>0.0055</td>
<td>0.1</td>
</tr>
<tr>
<td>Mass (kg)</td>
<td>11.48</td>
<td>49.52</td>
</tr>
<tr>
<td>Torsional moment of inertia (kg-m²)</td>
<td>0.0083</td>
<td>0.5233</td>
</tr>
<tr>
<td>Bending moment of inertia (kg-m²)</td>
<td>0.0332</td>
<td>0.5</td>
</tr>
<tr>
<td>Axial support stiffness (N/m)</td>
<td>1.0E8</td>
<td>1.0E8</td>
</tr>
<tr>
<td>Lateral support stiffness (N/m)</td>
<td>3.8E8</td>
<td>3.8E8</td>
</tr>
<tr>
<td>Torsional stiffness of shaft (Nm/rad)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bending shaft-bearing stiffness (Nm/rad)</td>
<td>1.0E7</td>
<td>1.6E7</td>
</tr>
<tr>
<td>Mesh damping ratio</td>
<td>0.06</td>
<td></td>
</tr>
<tr>
<td>Support component damping ratio</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>Gear backlash (mm)</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>

### 2.4.1 Time-varying Load Dependent Mesh Analysis

Mathematically, mesh parameters are nonlinear functions of both rolling angle and external torque expressed as:

\[
k_m = k_m(\theta_P, T_E), \quad L_m = L_m(\theta_P, T_E), \quad R_m = R_m(\theta_P, T_E), \quad e_L = e_L(\theta_P, T_E), \quad e_U = e_U(\theta_P)
\]  

(2.42)

Four load cases are used for case study including: 30 Nm with/without 80% fluctuation and 80 Nm with/without 50% fluctuation. The fluctuation speed is assumed to be in 1st shaft order. As can be seen from Figure 2.6 – 2.10, mesh parameter variation in first mesh cycle is illustrated as 3D curves on the parameter surfaces and variation of parametric values in one shaft rotation cycle are also compared. From comparison results in Figure 2.6 and Figure 2.7, one can see that the variation of
effective mesh stiffness (Figure 2.6) and loaded TE (Figure 2.7) under the effect of 1st shaft order torque fluctuation demonstrate clear periodic variation in one shaft rotation cycle. Accordingly, parameter values in one mesh cycle show significant discrepancy under different fluctuation cases. Compared to the results of other mesh parameters including directional rotation radii, LOA and effective mesh point (Figure 2.8 – 2.10), it is found that mesh stiffness and loaded TE are more sensitive to torque fluctuation especially in light torque load level. The observation of mesh parameter variation under different circumstances indicates that external load conditions such as mean torque value, torque fluctuation percentage and fluctuation speed can have profound effect on dynamic response.

Figure 2.6 Mesh stiffness variation: (a) in 1st mesh cycle with: 30 Nm mean torque load, 80 Nm mean torque load, 30 Nm with 80 % fluctuation, 80 Nm with 50 % fluctuation, (b) in one pinion shaft rotation cycle with: 30 Nm mean torque load, 30 Nm with 80 % torque fluctuation
Figure 2.7 Loaded transmission error variation: (a) in 1st mesh cycle with: 30 Nm mean torque load, 80 Nm mean torque load, 30 Nm with 80 % fluctuation, 80 Nm with 50 % fluctuation, (b) in one pinion shaft rotation cycle with: 30 Nm mean torque load, 30 Nm with 80 % torque fluctuation
Figure 2.8 Directional rotation radii variation: (a) in 1st mesh cycle with: 30 Nm mean torque load, 80 Nm mean torque load, 30 Nm with 80% fluctuation, 80 Nm with 50% fluctuation, (b) in one pinion shaft rotation cycle with: 30 Nm mean torque load, 30 Nm with 80% torque fluctuation
Figure 2.9, Normal vector variation: (a) in 1st mesh cycle with: — 30 Nm mean torque load, —— 80 Nm mean torque load, —— 30 Nm with 80 % fluctuation —— 80 Nm with 50 % fluctuation.
fluctuation, (b) in one pinion shaft rotation cycle with: \(--\) 30 Nm mean torque load, \(\textcolor{red}{-\cdots-}\) 30 Nm with 80 \% torque fluctuation
2.4.2 Dynamic Analysis

To begin with, the effectiveness of time-varying load dependent mesh model proposed above is proved by comparing its dynamic responses with those generated from constant load dependent mesh model under same time-invariant load conditions. Two constant load cases including 60 Nm and 90 Nm are studied. For constant load dependent model, quasi-static LTCA program (Vijayakar, 2003) is used to generate load distribution by specifying the target torque loads, nominal operation speed, gear design and machine settings that extracted from SPA file. The results of stress distribution are then derived into a set of mesh parameters assuming constant or mean value. For time-varying load dependent model, mesh parameters are defined by direct interpolation on a pre-synthesized 3D parameter surface with pinion rolling angle and instantaneous torque value as indexes. Therefore, mesh parameters generated are capable of
reflecting external torque variation within on mesh cycle. From the comparison results shown in Figure 2.11, it can be seen that the gear dynamic responses of the two mesh models are similar. Slight difference can be eliminated by employing more advanced interpolation method or adding more data points onto the mesh parameter surfaces. Comparison results indicate that the proposed time-varying load dependent mesh model can generate accurate mesh parameters compared to those directly derived from quasi-static LTCA in constant torque load cases.

Empirically, gear dynamic response under light or heavy load condition perform differently. For lightly loaded cases, tooth loss of contact and impact frequently occurs frequently. Although in this case, the mean torque can be small, external torque fluctuation is usually high which tends to have significant influence on mesh parameters during operation. In addition, coupling effect between external excitation from torque fluctuation and internal mesh excitation caused by kinematic TE are also expected due to system nonlinearity. As a result, the spectrum of gear
dynamic responses can include different components which is another focus in the following analysis. On the other hand, heavy loaded condition is commonly seen in high speed or acceleration in on-highway or heavy loaded operations of off-highway vehicles. Dynamic responses under this condition are mostly continuous without jump nonlinearity because there’s hardly any tooth contact loss during speed sweep and the dynamic factor is less than 1 indicating linear characteristics.

The combined effect of gear mesh and torque fluctuation is a major source of tonal gear whine noise. Earlier studies are mostly focused on mesh order excitation whereas parametric study on external excitation is limited. In order to analyze the system dynamic performance under different load conditions, two groups of cases are selected empirically: a) 30 Nm nominal driving torque with 80% fluctuation and 80 Nm torque with 50% fluctuation are selected for light loaded case, b) 200 Nm nominal driving torque with 20% fluctuation magnitude is chose for medium loaded case and 1000 Nm with 10% fluctuation is studied for heavy loaded case.

A. Light Loaded Case

In light loaded gear mesh operation, under the combined effect of internal and external excitation, constant loss of tooth contact can be observed. The geared rotor system demonstrates strong nonlinear characteristics originated from two factors: backlash and mesh parameter nonlinearity. Backlash can be described as a clearance type non-linear function between relative displacement and elastic force (Comparin, and Singh, 1989). The presence of backlash can induce intensive dynamic load on tooth surface and cause severe noise and vibration problems. The other nonlinear factor originates from mesh parameter variation explained above.
Two groups of results are shown in Figure 2.12 and Figure 2.13 respectively with slight difference in mean torque and fluctuation percentage. For the first simulation (Figure 2.12), a driving torque with 30 Nm mean value and 24 Nm variant value (80% fluctuation) at 1st pinion shaft order is imposed at the driving end of geared rotor system. Clear difference is found between responses of NLTV and LTV models in nonlinear region where response curve is above the mean mesh force line. For NLTV model, clear jump phenomenon can be seen in a broad frequency range while the response curve of LTV model is continuous accompanied by larger amplitude and frequency of resonance peaks. This indicates that in light loaded case, the system demonstrates strong nonlinear characteristics which is similar to earlier studies on gear backlash nonlinearity (Wang, 2007; Peng, 2010).

Comparison between dynamic responses from time-varying and constant load dependent mesh model for both NLTV and LTV case demonstrates significant discrepancy due to mesh parameter variation in one mesh cycle. In LTV model where mesh parameter variation is the only nonlinear factor, responses from time-varying and constant load dependent model are similar in most mesh frequency range except the response peak around 1400 Hz. On the other hand, in NLTV case where backlash nonlinearity is incorporated, discrepancy between the responses from the two mesh models can be seen in a wide range of frequency. In lower frequency range (<2500 Hz), the gear dynamic response amplitude predicted by time-varying load dependent mesh model is slightly higher than that by constant load dependent model. When mesh frequency sweeps up, the dynamic response generated from constant load dependent model becomes higher than time-varying load dependent model especially around the resonance peak at 2700 Hz. The results indicate that compared with time-varying load dependent mesh model that defined by instantaneous rolling
angle and torque value, previous mesh model that based on mean torque load alone tend to overestimate the response amplitude in higher mesh frequency range.

Another group of simulations under light loaded condition is shown in Figure 2.13 which applies 80 Nm mean torque with 40 Nm (50% fluctuation) torque variation. Similar observation can be obtained compared with previous case where responses in LTV cases are similar whereas those in NLTV demonstrate significant difference. Discrepancies between responses from two mesh models in NLTV case mainly occur in two frequency ranges (1300~2200 Hz and 2900~3600 Hz) where the response curves go beyond the mean mesh force line. This implies that nonlinear response is more sensitive to the variation of mesh parameters within one mesh cycle. Moreover, diminished difference in responses of NLTV case is seen compared to previous case illustrated in Figure 2.12. The reason is that with increased torque level, slopes of mesh parameter surfaces become more flat and parameter values are less sensitive to torque variation.
Figure 2.12 Dynamic response at 30 Nm nominal torque load with 80% fluctuation of magnitude from different models: — constant load dependent NLTV, — time-varying load dependent NLTV — constant load dependent LTV, — time-varying load dependent LTV

Figure 2.13 Dynamic response at 80 Nm nominal torque load with 50% fluctuation of magnitude from different models: — constant load dependent NLTV, — time-varying load dependent NLTV — constant load dependent LTV, — time-varying load dependent LTV

In order to further study the effect of torque fluctuation on gear mesh response based on time-varying load dependent gear mesh model and investigate the existing different types of response components, two sets of Fourier spectra plots and Poincare maps are shown in Figure 2.14 and Figure 2.15. In first simulation presented in Figure 2.14 where 80 Nm nominal torque with no fluctuation was imposed at the driving end, clear quasi-periodic route to chaos can be observed that is commonly seen in light loaded cases (Kahraman and Singh, 1990): First, period one solution is found in low mesh frequency range such as 1250 and 1350 Hz where Poincare map
shows single point; then the solution bifurcates into a quasi-periodic type when mesh frequency is increased to 1500 Hz and the Poincare map turns into a closed orbit. With a further increase in mesh frequency, strong nonlinear response featuring tooth contact loss is presented as a broadband spectrum of chaotic solution. For the second set of simulation (Figure 2.15), a time-varying torque segment with the magnitude of 50% fluctuation of mean torque value was applied. Strong nonlinear responses can be observed due to interaction between internal TE and external shaft order excitation. By comparing to previous case in Figure 2.14, one can find clear side bands in local region of mesh order response that results from modulation of the two excitations. In addition, shaft order excitation illustrated by a spike at shaft rotation frequency with side band in local frequency range can be seen throughout the selected nonlinear response range (Figure 2.15 (a)-(c)). Accordingly, those period-one solutions in first set of simulation (Figure 2.14 (e)-(f)) turn into quasi-static or chaotic responses (Figure 2.15 (e)-(f)) and such broad band response can worsen tonal noise and sound quality problems.
Figure 2.14 Fourier transform spectra and Poincare map plot for dynamic response of 80 Nm with no external torque fluctuation
Figure 2.15 Fourier transform spectra and Poincare map plot for dynamic response of 80 Nm with 50% magnitude of fluctuation
B. Medium and Heavy Loaded Case

Gear dynamic responses under medium (200 Nm) and heavy load conditions (1000 Nm) are shown in Figure 2.16 and Figure 2.17 respectively. Smaller nonlinear response range where response curve goes beyond mean mesh force line can be observed compared to light loaded case. For the curves below the mean mesh force line, there’s no jump discontinuity in the response indicating no loss of contact during that mesh frequency range. On the other hand, responses in nonlinear response range demonstrate clear softening effect due to loss of tooth contact. From the results in Figure 2.16 and Figure 2.17, it is also seen that in medium and heavy loaded cases, the difference in the responses generated from time-varying and constant load dependent mesh model mostly appears in nonlinear response range. The discrepancy caused by applying time-varying load dependent mesh model is much smaller compared to light loaded case because with increased mean torque level, mesh parameters are less sensitive to external torque variation as indicated by Figure 2.4. The comparative study on response from time-varying and constant load dependent mesh model implies that under medium and heavy load level, dynamic response generated from constant load dependent mesh model can be an accurate prediction of system performance. The reason is that for higher nominal torque level, mesh parameters such as effective mesh stiffness, TE, LOA and mesh point are less sensitive to external torque variation.
Figure 2.16 Dynamic response at 200 Nm nominal torque load with 20% fluctuation of magnitude from different models: -- constant load dependent NLTV, \(\circ\) time-varying load dependent NLTV, \(--\) constant load dependent LTV, \(\times\) time-varying load dependent LTV

Figure 2.17 Dynamic response at 1000 Nm nominal torque load with 10% fluctuation of magnitude from different models: -- constant load dependent NLTV, \(\circ\) time-varying load dependent NLTV, \(--\) constant load dependent LTV, \(\times\) time-varying load dependent LTV
C. **Shaft Order Parametric Study**

The effect of torque fluctuation speed on dynamic response of linear and non-linear time-varying load dependent model is analyzed in this section. Two load cases including 30 Nm with 80% fluctuation and 200 Nm with 20% fluctuation in magnitude are selected. From comparison results in Figure 2.18 and Figure 2.19, it is seen that fluctuation with higher shaft order tend to cause discrepancy between LTV and NLTV model in a wider mesh frequency range. For 30 Nm torque with 80% fluctuation shown in Figure 2.18, the difference between responses of LTV and NLTV models under 2nd shaft order fluctuation starts at 1000 Hz whereas for 4th shaft order excitation, the difference can be observed throughout the entire frequency range of interest. This indicates that fluctuation with higher shaft order will have more extensive coupling effect with gear internal excitation due to increased external excitation speed. In addition, for both LTV and NLTV, response peak with largest amplitude tend to shift towards lower mesh frequency when fluctuation with higher shaft order is employed. Similar conclusion can be drawn from Figure 2.19 where 200 Nm nominal torque with 20% fluctuation is imposed at the driving end. Difference between responses from LTV and NLTV model under same load conditions mostly occurs in nonlinear response range due to loss of tooth contact. By comparing the responses from 2nd and 4th shaft order excitation, it is found that for both LTV and NLTV model, responses at 1700 Hz are subject to 2nd shaft order excitation whereas the response peaks at 700 Hz are generated by 4th shaft order excitation. The analysis reveals that torque fluctuation with high shaft order (∋ 4th) has significant influence on dynamic responses in low mesh frequency range (<1500 Hz) where responses with high peak amplitude can be found. On the other hand, lower shaft order excitation tend to affect dynamic responses in middle or high mesh frequency range.
Figure 2.18 Dynamic response at 30 Nm nominal torque load with 80% fluctuation of magnitude from different shaft order excitation: 2nd shaft order NLTV, 2nd shaft order LTV, 4th shaft order NLTV, 4th shaft order LTV
2.5. Conclusions

A time-varying load dependent gear mesh model based on 3D parametric surface interpolation is proposed through which variation of mesh parameters due to external torque fluctuation can be accounted for within one mesh cycle. Mesh parameters generated from both time-varying and constant load dependent mesh model are employed in dynamic analysis. Comparative studies are performed applying NLTV and LTV models under different mean torque levels and fluctuation percentages. For light load condition, dynamic response of geared rotor system is more sensitive to mesh parameter variation in one mesh cycle compared to medium or heavy load cases. Backlash nonlinearity that causes tooth loss of contact can introduce strong coupling effect between external torque excitation and internal mesh excitation. Therefore, external torque fluctuation has more profound influence on nonlinear response of the system than linear response. Based on time-varying load dependent mesh model, frequency shift and magnitude changes can be observed between responses from NLTV and LTV mesh. In addition, side band modulation caused by external excitation can be found in FFT spectrum in nonlinear response frequency range. These studies imply that applying the proposed mesh model can obtain reasonable dynamic response predictions. On the other hand, under medium or heavy load level, dynamic response from constant load dependent mesh model can be a practical approximation because mesh parameter variation due to torque fluctuation in one mesh cycle is negligible when external mean torque level is high enough. Parametric study on torque fluctuation speed
demonstrates that external excitation with higher speed can have stronger coupling with internal excitation and will generate response peaks in low frequency range.
Chapter 3. Modeling of Impact Damping Effect on Hypoid Gear Dynamic Response

3.1. Introduction

Gear pair transmission is commonly applied in modern industrial technology because of its high efficiency and compact structure. Nonlinear frequency response models of parallel axis gears such as spur gears and helical gears were developed and extensively studied in the past several decades (Ozguven and Houser, 1988; Kahraman and Singh, 1990; Kahraman and Singh, 1991). Hypoid gear set is also widely used in power transmissions for certain type of applications such as rear axles of vehicles for transmitting motion in perpendicular directions. However, publications focusing on hypoid gear pairs are less prevalent as compared to those on study of spur or helical gears. The reason is partly because of the complex geometry and meshing mechanism in hypoid gears due to curvature feature of tooth profile as well as time and spatial varying mesh parameters such as mesh stiffness, mesh damping, line–of–action (LOA) and mesh point.

In recent years, a series of studies were carried out: Cheng and Lim (1998, 2000) established an accurate NLTV model for hypoid gear dynamic analysis based on accurate loaded tooth contact analysis (LTCA) results of exact tooth profile; Jiang (2002) conducted basic analysis on a torsional vibration model for hypoid gear pairs; Wang (2007) focused on dynamics of multi-point gear mesh and tuning of out-of-phase mode for better dynamic performance; Peng (2010) proposed a multi-body dynamic model for complete geared rotor system that accounts for large rotation displacement; Yang (2012) extensively analyzed the coupling between meshing gear pair and other surrounding components such as time-varying bearing and elastic housing structure. These work provided an effective analytical modeling framework for more in-depth study on
hypoid gear modeling and dynamic analysis. A significant factor that affects gear dynamics is the existence of backlash due to tooth profile deformation, vibration, misalignment or the inherent design for lubrication and assembly purposes, which can cause loss of tooth contact, reduced stability and system nonlinearity. To better understand this phenomenon, an extensive study is needed for analyzing operating tooth impact.

In the study of vibro-impact in hypoid gear pairs, damping is considered as one of the key mesh parameters that describes the mesh coupling. Conventionally, damping function used in gear mesh is based on constant damping coefficient defined empirically, but an accurate approximation of its value is hard to attain especially in hypoid gear. Moreover, classical damping function that uses simple damping coefficient does not satisfy boundary conditions of two impact bodies. Therefore, a more detailed study of damping modeling is needed to achieve a more reasonable and practical prediction of hypoid gear dynamic response considering backlash nonlinearity.

Nonlinear impact damping model was first extensively studied by Hunt and Crossley (1975) based on basic work of Dubowsky and Freudenstein (1971) to study the impact phenomenon between two mass blocks in contact. Then, numerical solutions generated from this model are compared with experimental results (Veluswami and Crossley, 1975; Azar and Crossley, 1977; Padmanabhan and Singh, 1995) to demonstrate the effectiveness of the impact damping function developed. However, the impact damping function in their work is still a highly approximated model that assumes a pure linear relationship between damping coefficient and damping force. In addition, the damping coefficient that considers only surface deflection was shown to be unrealistic and inaccurate because the approaching speed as the tooth pair comes into engagement is another important factor that may contribute to the coefficient value (Khulief and Shabana, 1987). Herbert and McWhannell (1977) studied and compared four widely used impact damping models:
Crossley, nonlinear Crossley, Direct and Dubowsky models. In their work, these four models are grouped into two types: nonlinear and linear models. Either type can be used depending on specific speed and load conditions. Later, impact damping models are applied in spur gear dynamic analysis by Lee and Wang (1983). Their analysis shows that both Direct model (non-viscous type) and Dubowsky model (viscous type) are suitable for gear mesh modeling in generating reasonable dynamic response. More recently, Kim, Rook and Singh (2005) examined nonlinear impact damping model for a spur gear pair using three analytical methods and numerical integration. Good correlation among results from different methods were obtained which indicated validity of the model proposed by Crossley. However, most previous studies are limited to simple impacting mass blocks or parallel axis gear mesh analysis. There has not been a related study so far on evaluating the effect of impact damping on hypoid gear dynamics. This paper attempts to address this gap in the literature.

This study extends the impact damping study to hypoid gear dynamic analysis based on applying a time-varying load dependent mesh model. A 14 degrees-of-freedom (DOF) coupled multibody dynamic model is formulated, which incorporates large rotational displacement effect. The needed impact damping model is derived applying time-varying mesh parameters generated from a quasi-static LTCA that based on combined methods of finite element and surface integral assuming no misalignment or assembly error. Comparative studies are conducted to examine differences and similarities of impact damping models proposed here and previous simplified model applying constant coefficient. Effects of non-viscous and viscous type damping on dynamic response are compared to demonstrate their damping characteristics related to speed and surface deflection under different load levels in both steady state and transient analysis.
3.2. Mathematical Model:

3.2.1. Gear Mesh:

Mesh parameters of hypoid or bevel gear pairs (Figure 3.1(a)) vary substantially over one mesh cycle due to complex tooth geometry, variations in contact area and load distribution. To quantify this time-varying effect, the pinion roll angle is used as time index in simulating changes in the mesh parameters.

Mesh coupling applied in gear dynamic analysis is commonly expressed as a spring-damper component connecting the pinion and gear bodies. To synthesize the hypoid gear mesh model, a three-dimensional tooth contact analysis program (Vijayakar, 1991, 2003) is employed to perform quasi-static LTCA. The analysis combines the finite element (FE) formulation with the surface integral method to generate surface contact results including load distribution and transmission error (TE). The condensation of large quantity of LTCA results to generate the mesh model and equivalent mesh coupling as shown in Figure 3.1(b) are described next.
From the LTCA results, the contact area consists of N cells. The net contact force is calculated by:

\[
F_{\text{cont}} = \left( F_x^2 + F_y^2 + F_z^2 \right)^{1/2}
\]  

(3.1)

The detailed expression for force vectors in three directions can be listed as:

\[
F_x = \sum_{i=1}^{N} (n_{ix} g_i), \quad F_y = \sum_{i=1}^{N} (n_{iy} g_i), \quad F_z = \sum_{i=1}^{N} (n_{iz} g_i)
\]  

(3.2)

where \( n_i = (n_{ix}, n_{iy}, n_{iz}) \) is the surface normal vector of each contact cell in the global coordinate system and \( g_i \) is normal force.

The effective LOA \( (L_m(n_x, n_y, n_z)) \) is given by:

\[
n_x = \frac{F_x}{F_{\text{cont}}}, \quad n_y = \frac{F_y}{F_{\text{cont}}}, \quad n_z = \frac{F_z}{F_{\text{cont}}}
\]  

(3.3)

The net moment vectors due to tooth surface contact are expressed as:
\[ M_x = \sum_{i=1}^{N} \left[ g_i \left( n_{ix} r_{iy} - n_{iy} r_{ix} \right) \right] \]  

(3.4)

\[ M_y = \sum_{i=1}^{N} \left[ g_i \left( n_{ix} r_{iz} - n_{iz} r_{ix} \right) \right] \]  

(3.5)

\[ M_z = \sum_{i=1}^{N} \left[ g_i \left( n_{iy} r_{iz} - n_{iz} r_{iy} \right) \right] \]  

(3.6)

The effective mesh point is defined as \( R_m(x_m, y_m, z_m) \) where:

\[ y_m = \frac{\sum_{i=1}^{N} (r_{iy} g_i)}{\sum_{i=1}^{N} g_i}, \quad x_m = \frac{(M_z + F_x y)}{F_y}, \quad z_m = \frac{(M_y + F_y y)}{F_x} \]  

(3.7)

Mesh stiffness value can be calculated from both loaded and unloaded transmission errors (TE) projected onto the effective LOA:

\[ k_m = \frac{F_m}{(e_L - e_U)} \]  

(3.8)

All mesh parameters derived above such as mesh point \( o_m \), LOA \( L_m \), mesh stiffness \( k_m \) and transmission error (TE) are synthesized at each pinion rolling angle indexed by time.

### 3.2.2. Geared Rotor System

The hypoid geared rotor system is represented by a 14-DOF lumped parameter model comprised of an engine/driver, a load element and a hypoid gear pair (Peng, 2010). The pinion and gear are rigid body elements with 6 DOFs including translational and rotational coordinates. The engine and load elements each has one rotational DOF along their respective rotating shaft axis. The stiffness and damping elements of pinion and gear body represent the shaft and bearing supports. The coordinates of pinion \((x_p, y_p, z_p, \theta_{p}, \theta_{p}, \theta_{p2})\) and gear \((x_g, y_g, z_g, \theta_g, \theta_g, \theta_{g2})\) are defined relative to their local inertial reference frames illustrated in Figure 3.2 with the origin of
the local coordinate systems at the centroid of pinion and gear bodies respectively. Notice that the rotational coordinates relative to the shaft axis $\theta_p, \theta_g, \theta_E$ and $\theta_L$ are large rotational displacements.

Therefore, the equations of motion of the 14-DOF system can be written as:

\[ I_E \ddot{\theta}_E + c_{rpm} \left( \dot{\theta}_E - \dot{\theta}_p \right) + k_{rpm} (\theta_E - \theta_p) = T_E \]  \hspace{1cm} (3.9)

\[ M_q \ddot{x}_p + c_{qp} \dot{x}_p + k_{qp} x_p = -n_px_m \]  \hspace{1cm} (3.10)

\[ M_q \ddot{y}_p + c_{qy} \dot{y}_p + k_{qy} y_p = -n_py_m \]  \hspace{1cm} (3.11)

\[ M_q \ddot{z}_p + c_{qz} \dot{z}_p + k_{qz} z_p = -n_zx_m \]  \hspace{1cm} (3.12)

\[ I_{rpm} \ddot{\theta}_p + c_{rps} \dot{\theta}_p + k_{rps} \theta_p - I_{rpm} \ddot{\theta}_E + I_{rpm} \dot{\theta}_p - I_{rpm} \theta_E \dot{\theta}_p = -\lambda_{px} F_m \]  \hspace{1cm} (3.13)

\[ I_{rpm} \ddot{\theta}_p + c_{rps} \dot{\theta}_p + k_{rps} \theta_p - I_{rpm} \ddot{\theta}_E + I_{rpm} \theta_E \dot{\theta}_p = -\lambda_{py} F_m \]  \hspace{1cm} (3.14)

\[ I_{rpm} \ddot{\theta}_p + c_{rps} \dot{\theta}_p + k_{rps} \theta_p - I_{rpm} \ddot{\theta}_E + I_{rpm} \theta_E \dot{\theta}_p = -\lambda_{pz} F_m \]  \hspace{1cm} (3.15)

\[ M_{rg} \ddot{x}_g + c_{rgx} \dot{x}_g + k_{rgx} x_g = n_{gx} F_m \]  \hspace{1cm} (3.16)

\[ M_{rg} \ddot{y}_g + c_{rgy} \dot{y}_g + k_{rgy} y_g = n_{gy} F_m \]  \hspace{1cm} (3.17)

\[ M_{rg} \ddot{z}_g + c_{rgz} \dot{z}_g + k_{rgz} z_g = n_{gz} F_m \]  \hspace{1cm} (3.18)

\[ I_{rsg} \ddot{\theta}_s + c_{rsg} \dot{\theta}_s + k_{rsg} \theta_s - I_{rsg} \ddot{\theta}_g + I_{rsg} \ddot{\theta}_s - I_{rsg} \theta_g \dot{\theta}_s = \lambda_{gs} F_m \]  \hspace{1cm} (3.19)

\[ I_{rsg} \ddot{\theta}_g + c_{rsg} \dot{\theta}_g + k_{rsg} \theta_g = \lambda_{gs} F_m \]  \hspace{1cm} (3.20)

\[ I_{rsg} \ddot{\theta}_g + c_{rsg} \dot{\theta}_g + k_{rsg} \theta_g + I_{rsg} \ddot{\theta}_s + I_{rsg} \theta_g \dot{\theta}_s = \lambda_{gs} F_m \]  \hspace{1cm} (3.21)

\[ I_L \ddot{\theta}_L + c_{rgl} \dot{\theta}_L + k_{rgl} (\theta_L - \dot{\theta}_l) = T_L \]  \hspace{1cm} (3.22)

The above equations can be transformed into matrix form as:

\[ [M] \{x\} + [C] \{\dot{x}\} + [K] \{x\} + [G] \{\ddot{x}\} + [G_z] \{x\} = \{F\} \]  \hspace{1cm} (3.23)
The lumped mass matrix is written as:

\[
[M] = \text{diag}[I_E, M_{ip}, M_{ig}, I_{ip}, I_{ip}, M_{ig}, M_{ig}, I_{ip}, I_{ip}, I_L]
\]  (3.25)

The system stiffness matrix is represented as:

\[
[K] = \begin{bmatrix}
[K_E] \\
[K_p] \\
[K_g] \\
[K_L]
\end{bmatrix}
\]  (3.26)

The matrix \([K_p]\) and \([K_g]\) are the effective shaft-bearing stiffness supporting pinion and gear respectively. In addition, \([G]\) in Equation 3.23 is the first gyroscopic matrix associated with pinion/gear rotational velocities and \([G_a]\) is the second gyroscopic matrix associated with rotational accelerations. The excitation force which consists of external and internal excitation vector is represented as:

\[
\{F\} = \begin{bmatrix}
T_E, Q_p \cdot F_m^x, -Q_g \cdot F_m^x, -T_L
\end{bmatrix}^T
\]  (3.27)

\[Q_l = \{n_{lx}, n_{ly}, n_{lz}, \lambda_{lx}, \lambda_{ly}, \lambda_{lz}\}, \quad l = p, g\]  (3.28)

where \(T_E\) and \(T_L\) are the external driving and load torques. The coordinate transformation vectors \(Q_p\) and \(Q_g\) are applied to project the mesh force along onto the LOA vector consisting of directional cosine vectors \(L_l = \{n_{lx}, n_{ly}, n_{lz}\}\) and directional rotation radii can be expressed as:

\[
\begin{align*}
\hat{\lambda}_{lx} &= y_{im}n_{lz} - z_{im}n_{ly}, \\
\hat{\lambda}_{ly} &= z_{im}n_{lx} - x_{im}n_{lz}, \\
\hat{\lambda}_{lz} &= x_{im}n_{ly} - y_{im}n_{lx}
\end{align*}
\]  (3.29)

where \(l\) indicates if the parameters are measured in pinion \((l = p)\) or gear \((l = g)\) local reference coordinates. The dynamic transmission error along the LOA vector is given by:
where: \( R = \frac{\lambda_{gy}}{\lambda_{py}} \) is the gear ratio.

Figure 3.2 Lumped parameter hypoid geared rotor system

Backlash is designed into gear pairs for better lubrication and avoiding gear teeth interference. However, its existence can induce tooth separation and impact (Figure 3.3). The dynamic mesh force (DMF) function in a gear pair with backlash is a non-linear function of dynamic transmission error (DTE) as is shown in Figure 3.4. Under different working conditions, double sided or single sided impact can be observed as plotted in Figure 3.5. Here, the speed and displacement are measured along the effective LOA vector. The nonlinear dynamic mesh force comprised of both elastic and damping forces acts at the effective mesh point along LOA which can be derived into three expressions depending on the relationship between transmission error and backlash:
Accordingly, the linear model without gear backlash can be simplified as:

\[
F_m = \begin{cases} 
  k_m \cdot (\delta - e_U - b_c) + c_m \cdot (\dot{\delta} - \dot{e}_U) & \text{if } \delta - e_U > b_c \\
  0 & \text{if } -b_c < \delta - e_U < b_c \\
  k_m \cdot (\delta - e_U + b_c) + c_m \cdot (\dot{\delta} - \dot{e}_U) & \text{if } \delta - e_U < -b_c 
\end{cases}
\] (3.31)

where \( k_m \) is effective mesh stiffness and \( e_U \) is the translational kinematic TE derived from angular kinematic TE. Damping element \( c_m \) in Equation 3.32 stands for fluid-induced drag damping effect from driveline and can be expressed as:

\[
c_m = 2\xi_m \cdot k_m / \omega_n
\] (3.33)

\[
\omega_n = \sqrt{k_{amm}} / m_c
\] (3.34)

where \( \xi_m \) is the empirical damping ratio whose value is small so that the viscous damping effect from driveline is negligible. Numerical integration method applying explicit 4/5-th order Runge-Kutta algorithm is used to solve the equation of motion presented above.

Figure 3.3 Parameters of meshing gear pair (Cheng, 2000)
Figure 3.4 Illustration of nonlinear function
3.2.3. Impact Damping

1. Constant Coefficient

Nonlinear impact damping model was first proposed by Hunt and Crossley (1975) to overcome non-physical artifact of classical linear damping model assumption. The damping force from solid body impact is determined by contact stiffness, Hertizian compliance exponent, dynamic displacement and velocity. By applying this impact damping model in hypoid gear mesh, the gear impact damping force can be written as:

\[ f_i(p) = k_m \cdot f^q(p) \cdot \beta \cdot \dot{p} \]  

(3.35)

where \( \beta \) is the impact damping coefficient as a function of the characteristics of two impact bodies which is assumed to be constant. In gear mesh case, the value of Hertizian compliance exponent value \( q \) is 1 (Lee and Wang, 1983). The nonlinear stiffness function of gear mesh model is synthesized as \( k_m \cdot f(p) \) and \( f(p) \) as illustrated in Figure 3.3 can be written as:

\[ f(p) = \begin{cases} 
    p - b_c & p > b_c \\
    0 & -b_c < p < b_c \\
    p + b_c & p < -b_c 
\end{cases} \]  

(3.36)

\[ p = \delta_d - e_u \]  

(3.37)

By adding the nonlinear elastic force and impact damping force, the combined internal excitation of hypoid gear mesh can be derived as:
\[ f_n(p) = k_m \cdot f(p) + f_i(p) = \begin{cases} 
 k_m \cdot (p - b_c) \cdot (1 + \beta \dot{p}) & p > b_c \\
 0 & -b_c < p < b_c \\
 k_m \cdot (p + b_c) \cdot (1 + \beta \dot{p}) & p < -b_c 
\end{cases} \quad (3.38) \]

Therefore, the updated mean mesh force can be described as:

\[ F_m = \begin{cases} 
 k_m \cdot (p - b_c) + (c_m + c_i) \cdot p & p > b_c \\
 0 & -b_c < p < b_c \\
 k_m \cdot (p + b_c) + (c_m + c_i) \cdot \dot{p} & p < -b_c 
\end{cases} \quad (3.39) \]

where \( c_i \) is the impact damping function defined as:

\[ c_i = \beta \cdot k_m \cdot f(p) \quad (3.40) \]

When gear teeth are in contact, the dynamic mesh force generated is mostly governed by mesh stiffness \( k_m \) and impact damping coefficient \( \beta \). Viscous damping coefficient \( c_m \) stands for other damping factors in the driveline structure. If backlash is zero \( (b_c = 0) \), the dynamic mesh force expression can be simplified as:

\[ F_m = k_m \cdot p + (c_m + c_i) \cdot \dot{p} \quad (3.41) \]

2. **Enhanced Impact Damping Model**

Although it is convenient to model damping function using constant damping coefficient as widely used in gear mesh modeling as well as in the Hunt & Crossley’s constant coefficient impact damping model mentioned above, it is difficult to accurately estimate the coefficient value that is in fact speed and load dependent. In order to model the impact damping in a more precise way instead of using single constant coefficient, an enhanced damping model will be introduced here by considering the time varying parameters of hypoid gear mesh.

A  **Non-viscous Type**
Lee and Wang (1983) put forward a mesh damping function considering the speed and load 
dependent nature of impact damping to satisfy the hysteresis boundary conditions. In their 
formulation, if there is no contact (\(-b_c < p < b_c\)) or relative speed is zero (\(\dot{p} = 0\)), the damping 
coefficient value should also be zero. The equivalent impact damping force based on Hertizan 
contact theory is then formulated as:

\[
f_i(p) = \mu_i \cdot f''(p) \dot{p}
\]

(3.42)

where \(n = 1\), and \(\mu_i\) is the non-viscous damping factor. Referring to the work by Herbert and 
McWhannell (1977), the expression of damping factor can be written as:

\[
\mu_i = \frac{6(1-\alpha) \cdot k_m}{[(2\alpha-1)^2 + 3]V_i}
\]

(3.43)

\[
\alpha = -\frac{V_o}{V_i} = 1 - 0.026V_i^{1/3}
\]

(3.44)

where \(\alpha\) is the coefficient of restitution for approaching and departing speed ratio. It is clear that 
this damping function satisfies the boundary condition of damping force, that is at the beginning 
of impact (no surface deflection) or when the two impact bodies reach maximal deflection (relative 
speed is zero), damping force will be zero. Hence, the non-viscous impact damping function can 
be expressed as:

\[
c_i = \mu_i \cdot f(p)
\]

(3.45)

This type of damping function depends on both relative speed and surface deflection of two 
meshing teeth. It is easy to notice that in this enhanced impact damping function, 
\[
\frac{6(1-\alpha)}{[(2\alpha-1)^2 + 3]} \cdot \frac{1}{V_i}
\]
is the detailed expression for constant coefficient \(\beta\) in Hunt and Crossley’s model. This non-
viscous impact damping model was applied into spur gear impact dynamic analysis which showed reasonable results (Yang and Lin, 1987).

B  Viscous Type

The viscous type damping is commonly used in gear mesh representation that assumes a constant damping ratio as is briefly introduced in earlier studies. However, such simplified model will generate maximum damping force at the moment of contact which does not comply with the boundary condition of practical case. Therefore, it is necessary to introduce an enhanced viscous damping model that also preserves the basic characteristics of viscous type damping model. An enhanced viscous damping function (Lee and Wang, 1983) is applied to meet the requirement of classical Kelvin-Voigt model. This impact damping function consists of a viscous damping factor $\mu_2$ and a transition function $f_i(p)$ written respectively as:

$$
\mu_2 = \frac{2m_c \cdot k_m |Ln\alpha|}{\sqrt{(Ln\alpha)^2 + \pi^2}}
$$

(3.46)

$$
f_i(p) = \begin{cases} 
\exp\left\{\left[p - b_c - \varphi\right] - \left|p - b_c - \varphi\right| \cdot \frac{\eta}{\varphi}\right\} & p > b_c \\
0 & -b_c < p < b_c \\
\exp\left\{\left[-p-b_c-\varphi\right] - \left|-p-b_c-\varphi\right| \cdot \frac{\eta}{\varphi}\right\} & p < -b_c
\end{cases}
$$

(3.47)

where $\varphi$ is transition coefficient that defines the width of transition range between initial and final stage of impact process and $\eta$ is the constant value. It follows that the viscous type impact damping function can be written as:
Note that the viscous type impact damping function does not include tooth deflection which is different from non-viscous type damping function as described in Equation 3.45. This enhanced viscous type impact damping model may be able to yield a more accurate prediction of the damping force by reflecting changes of damping coefficient during different stages of tooth impact. Furthermore, the enhanced viscous damping function fits the boundary condition defined by damping force that is zero at the beginning and ending of impact occurrences. Also, because of no contact, it reaches maximum and minimum during working and return stroke respectively.

### 3.3. Computational Results

In this section, three impact damping models introduced above will be analyzed to evaluate their effect on hypoid gear dynamics. Damping models are divided into two groups including non-viscous and viscous types. Comparative studies are firstly conducted within each group to show the merits of detailed damping model over those using constant damping coefficient of the same type. Then, non-viscous and viscous type damping models are compared under different torque levels. Parameters of gear pair geometry and system dynamic parameters are listed in Table 3.1. Hypoid gear mesh parameters are generated from quasi-static LTCA performed at different constant torque levels.

<table>
<thead>
<tr>
<th>Gear data</th>
<th>Pinion</th>
<th>Gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth</td>
<td>10</td>
<td>43</td>
</tr>
<tr>
<td>Spiral angle (rad)</td>
<td>0.803</td>
<td>0.591</td>
</tr>
<tr>
<td>Pitch angle (rad)</td>
<td>0.295</td>
<td>1.269</td>
</tr>
<tr>
<td>Face width (m)</td>
<td>0.048</td>
<td>0.168</td>
</tr>
<tr>
<td>------------------------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td>Type</td>
<td>Left hand</td>
<td>Right hand</td>
</tr>
<tr>
<td>Loaded side</td>
<td>Concave</td>
<td>Convex</td>
</tr>
<tr>
<td>Offset (m)</td>
<td>0.0318</td>
<td></td>
</tr>
</tbody>
</table>

### System parameters

<table>
<thead>
<tr>
<th></th>
<th>Engine/Driver</th>
<th>Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsional moment of inertia (kg(\cdot)m(^2))</td>
<td>0.0055</td>
<td>0.1</td>
</tr>
<tr>
<td>Mass (kg)</td>
<td>11.48</td>
<td>49.52</td>
</tr>
<tr>
<td>Torsional moment of inertia (kg(\cdot)m(^2))</td>
<td>0.0083</td>
<td>0.5233</td>
</tr>
<tr>
<td>Bending moment of inertia (kg(\cdot)m(^3))</td>
<td>0.0332</td>
<td>0.5</td>
</tr>
<tr>
<td>Axial support stiffness (N/m)</td>
<td>1.0E8</td>
<td>1.0E8</td>
</tr>
<tr>
<td>Lateral support stiffness (N/m)</td>
<td>3.8E8</td>
<td>3.8E8</td>
</tr>
<tr>
<td>Torsional stiffness of shaft (Nm/rad)</td>
<td>1.0E7</td>
<td>1.6E7</td>
</tr>
<tr>
<td>Bending shaft-bearing stiffness (Nm/rad)</td>
<td>1.0E7</td>
<td>1.6E7</td>
</tr>
<tr>
<td>Mesh damping ratio</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>Support component damping ratio</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>Gear backlash (mm)</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>

### 3.3.1. Constant Coefficient Non-viscous Damping

The impact damping model based on constant damping coefficient is applied in hypoid gear mesh without the influence of backlash nonlinearity to evaluate the system sensitivity to impact damping coefficient \(\beta\) alone. Two load cases including 80 Nm and 150 Nm are analyzed. The parametric results for \(\beta\) are shown in Figure 3.6. It can be seen that the peak amplitude decreases with increasing \(\beta\) as expected. In 150 Nm torque load case, because of larger deflection, the effect of non-viscous type impact damping shows more influence on gear dynamic response in both frequency (a-b) and time domain (c-d) results. Next, the combined effect of impact damping and backlash nonlinearity is analyzed. It can be observed in Figure 3.7 that by including stiffness nonlinearity \(f(\theta)\) of gear mesh, the effect of impact damping function on the dynamic response...
at the resonance peak around 1500 Hz is diminished due to loss of tooth contact while dynamic response in low mesh frequency range is still significantly affected.

Figure 3.6 Dynamic mesh force of linear model with 80 Nm (a, c) and 150 Nm (b, d) nominal driving torque with: \( \beta = 0 \), \( \beta = 0.2 \), \( \beta = 0.6 \); (c) 1550 Hz mesh frequency response, (d) 1650 Hz mesh frequency response;
3.3.2. Enhanced Non-viscous Impact Damping

Constant driving torque and step speed sweep up are applied to study the steady state response of the hypoid geared rotor system applying the proposed enhanced non-viscous impact damping model that process more detailed expression for the damping coefficient instead of a constant value. The same parameters of gear pair tooth geometry and dynamic system are used (Table 3.1). The comparison of dynamic mesh force (DMF) along LOA with and without impact damping functions under different mean torque loads are shown in Figures 3.8 and 3.9. Under 150 Nm torque load, due to the combined effect of mesh stiffness variation and tooth surface deflection, non-viscous type demonstrates more significant influence on the dynamic response as compared to the 50 Nm load case. Figures 3.10 and 3.11 show the diminishing effect of non-viscous impact damping on gear mesh response in low speed range as well as nonlinear response range. The
displacement and velocity are measured at effective mesh point along LOA. It can be observed in Figures 3.10(a) and 3.11(a) that the displacements of working and return strokes decrease under the influence of impact damping as indicated by tooth deflection value changes during operation. In addition, tooth deflection or pinion rotation speed is also reduced. This effect would cause response peak shift as is clearly seen in Figure 3.9. For nonlinear response speed range where tooth contact loss occurs frequently, impact damping can affect the response by turning double-sided tooth impact into single-sided tooth impact in certain range as is shown in Figures 3.10(b) and 3.11(b).

The root-mean-square (RMS) value of mesh damping force generated from constant coefficient and enhanced non-viscous impact damping model during step speed sweep up are compared in Figure 3.12(a). Accordingly, overall dynamic responses under the influence of two non-viscous type damping models are illustrated in Figure 3.12(b). The value of $\beta$ in Equation 3.40 is specified by ensuring that the dynamic responses from the two impact damping models are similar in most mesh frequency range. As can be seen in Figure 3.12(a), the damping force generated from enhanced non-viscous impact damping model is much lower than that from constant coefficient model. This is because the impact damping coefficient in enhanced model is based on coefficient of restitution instead of a constant empirical value which can also explain the notable difference in Figure 3.12(b). Figure 3.13 demonstrates more extensive comparison between the introduced two non-viscous type damping models. It can be seen that the response peaks at 550 Hz and 800 Hz of enhanced damping model is close to that from constant damping model with coefficient $\beta$ equal to 2.0 while the response of enhanced damping model in the mesh frequency higher than 1500 Hz is similar to the constant damping model with 0.2 coefficient value. The comparison further implies that the damping effect from enhanced damping model
depends on mesh frequency and response nonlinearity and constant damping model can obtain reasonable prediction of dynamic response in a limited speed range.

Figure 3.8 Dynamic response with 50 Nm driving torque with: response with no impact damping, response with enhanced non-viscous type impact damping

Figure 3.9 Dynamic response with 150 Nm driving torque with: response with no impact damping, response with enhanced non-viscous impact damping
Figure 3.10 Time domain response with 50 Nm driving torque with: \(-\cdots\cdots\) response with no impact damping, \(-\cdots\cdots\) response with enhanced non-viscous impact damping; (a): response at 500 Hz mesh frequency, (b): response at 1350 Hz mesh frequency
Figure 3.11 Time domain response with 150 Nm driving torque with: -- response with no impact damping, response with enhanced non-viscous impact damping; (a): response at 550 Hz mesh frequency, (b): response at 1650 Hz mesh frequency
Figure 3.12 Comparing two non-viscous type damping model: (a) damping force comparison, (b) comparison of gear dynamic response applying two non-viscous damping model separately with:

constant coefficient model , enhanced non-viscous type model

Figure 3.13 Comparing between two non-viscous type damping model with: b=0.2 , b=0.6, b=1.0, b=1.5, b=2.0, enhanced non-viscous type model
3.3.3. Enhanced Viscous Impact Damping

Dynamic response of hypoid gear model applying enhanced viscous impact damping model are plotted and compared with dynamic response computed from the model with non-viscous damping model as shown in Figure 3.14. Two torque load cases including 40 Nm (Figure 3.14(a)) and 150 Nm (Figure 3.14(b)) are used for this case study. It can be seen from the results that in the light load case (40 Nm), both types of impact damping model demonstrate similar effect on dynamic response. In resonance peak frequency range (1400 Hz - 1800 Hz), calculations using viscous impact damping has slightly more diminishing effect on the amplitudes than calculations from non-viscous type. This is because the energy dissipation predicted using viscous damping model is higher than non-viscous damping model. However, it is noticed in Figure 3.14(b), when the applied torque load increases (150 Nm), because of larger surface deflection, non-viscous damping effect is more evident than viscous type especially from 500 Hz to 1500 Hz frequency range. In the local area of response peak, the response amplitude are similar due to frequent tooth contact loss where the influence from impact damping is diminished. The results can be further shown by comparison study of damping forces in Figure 3.15(a) where clear differences between the two types of damping forces can be observed. From the results in Figure 3.15(b), it is seen that by using previous viscous damping model that applies constant damping coefficient, the dynamic response amplitude tend to be under estimated. On the other hand, the enhanced viscous damping model proposed in this study will yield more reasonable predictions by considering the time-varying factors in damping model as well as in gear mesh parameters.
Figure 3.14 Comparison between two types of impact damping models in frequency domain with different driving torque with: 
- no damping, 
- non-viscous type damping model, 
- viscous type damping model; (a): 40 Nm case, (b): 150 Nm case
3.3.4. Transient Analysis

Ramp speed sweeps are commonly employed in noise and vibration testing of geared transmission systems. Therefore, another major concern of analyzing impact damping function is to study its performance under continuous ramp speed sweep. Based on the earlier analysis, the proposed enhanced impact damping model will be applied for both types of damping in the following case study.

First, a constant driving torque (80 Nm or 150 Nm) is applied at driving end of the hypoid geared rotor system, then a speed is imposed which continuously changes from initial pinion rotation speed up to a maximum value corresponding to 3500 Hz mesh frequency. The predicted transient frequency domain results obtained by RMS calculation are presented in Figure 3.16 and Figure 3.17. Time domain responses processed by short-time FFT method are shown in Figure 3.18 and Figure 3.19.

As can be seen in Figures 3.16(a) and 3.16(b), for the case of non-viscous damping, significant discrepancy can be found below 3500 rpm and in the speed range of 9000 - 10000 rpm. As for viscous damping case, effect of impact damping on peak amplitude is similar while its response in lower speed range is close to that without impact damping effect. The influence of two types of impact damping models are further compared in Figures 3.16(c) and 3.16(d). Results show that in light torque load case (80 Nm), non-viscous damping has more effect on dynamic response of hypoid gear in the low speed range. In resonance peak speed range, both impact damping models
demonstrate similar effect due to loss of tooth contact. Similar results can be observed in Figure 3.18. Stabilization effect that turns double-sided tooth impact into single-sided tooth impact can be found in Figures 3.18(e) and 3.18(f).

From case study results in Figure 3.17 and Figure 3.19, it is found that when torque load is increased to 150 Nm, damping effect of non-viscous type impact damping model is more salient than 80 Nm case while viscous damping model has limited influence on both frequency and time domain response. In addition to response amplitude decreases, peak shift is also observed in non-viscous damping case especially for peaks with largest amplitude. To understand this observation, the influence of impact damping model on pinion rotation speed is presented in Figure 3.20. It shows that during ramp speed sweep up, the pinion rotation speed is diminished under impact damping effect which follows the same trend as that of steady state analysis, especially in those local areas of different peaks in dynamic response of hypoid geared rotor system.
Figure 3.16 Dynamic response from continuous ramp speed sweep up with 80 Nm driving torque in frequency domain with 

- no impact damping, 
- non-viscous impact damping, 
- viscous impact damping
Figure 3.17 Dynamic response from continuous ramp speed sweep up with 150 Nm driving torque in frequency domain with
no impact damping, non-viscous impact damping, viscous impact damping

Figure 3.18 Dynamic response from continuous ramp speed sweep up with 80 Nm driving torque in time domain with
no impact damping, non-viscous impact damping, viscous impact damping
Figure 3.19 Dynamic response from continuous ramp speed sweep up with 150 Nm driving torque in time domain with no impact damping, non-viscous impact damping, and viscous impact damping.
Figure 3.20 Speed fluctuation results during ramp up with no impact damping, non-viscous impact damping.

3.4. Conclusions

In this study, a new approach to model hypoid gear mesh damping is proposed that introduces two different types of impact damping functions including non-viscous and viscous types. Both models can predict damping force that satisfies boundary conditions of meshing gear teeth. Subsequently, a more accurate geared rotor dynamic system model is formulated that can reveal the impact mechanism by taking into account the deflection and velocity of meshing gear teeth when generating damping coefficient.

Results show that the impact damping models can reduce resonance peak amplitudes of hypoid gear dynamic response. In addition, peak shift can be observed because of the nonlinearity of impact damping functions and their effect on pinion rotation speed. The main difference between non-viscous and viscous impact damping function is that the non-viscous damping model is affected by tooth surface deflection during operation while viscous damping model is mostly
dependent on impact velocity. In other words, the non-viscous type damping model is more sensitive to the torque imposed at driving end of hypoid geared rotor system. Finally, results of transient response study by continuous speed ramp up demonstrate that non-viscous type impact damping has significant effect on the responses in low speed range by suppressing response peaks and reducing amplitude values. On the other hand, both viscous and non-viscous damping has similar damping effect on higher speed range and can turn double-sided impact into single-sided impact at certain speed range.
Chapter 4. Effect of Shaft Misalignment on Hypoid Gear Pair Driven through a Universal Joint

4.1 Introduction

Hypoid and bevel gear dynamics is one of the main causes of axle vibration and noise problem which is widely seen in automotive and off-highway vehicle industry. Two factors including internal and external excitations contribute to gear dynamics. The internal excitation is caused by gear transmission error (TE). Time varying gear mesh characteristics such as mesh point, line-of-action (LOA), mesh stiffness and damping comprise the coupling between meshing gear pair and TE is defined as the difference between ideal and real position of gear pair during rotation. Gear backlash is considered as clearance type nonlinearity in the system. Mathematic framework can be found in previous works concerning hypoid gear dynamics (Cheng and Lim, 2000; Wang, 2002; Wang, 2007) which is mainly based on lumped parameter model and focused on effect of internal excitation as well as mesh parameter variation on gear dynamic response. However, the external excitation accompanied with large shaft rotation displacement and driveline coupling dynamics is not extensively studied. Because of the existence of nonlinear factors caused by gear backlash, there exist interaction between high frequency gear internal excitation and shaft rotational excitation. By applying coupled multi-body dynamics in nonlinear gear mesh model (Peng, 2010), a 14-DOF lumped parameter model of hypoid/bevel geared rotor system can be numerically solved under different load conditions. External excitation study of hypoid/bevel gear was limited to driveline torsional dynamics such as external torque fluctuation. Later, propeller shaft bending effect (Yang, 2012) on driveline dynamics was studied using component mode synthesis, but the universal joint was simplified as simple supported boundary condition assuming
perfect alignment with pinion shaft. However, in rotating machinery like propeller and pinion shafts which combines large-torsional displacement, misalignment and mass unbalance are among the most common faults (Sudhakar and Sekhar, 2009) which have been ignored in hypoid gear dynamic analysis in the past.

Shaft misalignment refers to the condition where driving and driven shaft are not in perfect alignment with each other. Even perfect alignment is achieved at first place, rotating preload in radial direction caused by improper assembly, heat or lubrication is inevitable which can cause misalignment in shaft coupling during operation. Therefore, a connector such as universal joint that can transmit torque from driving shaft to driven shaft with the capability of accommodating such misalignment should be applied. The connection misalignment can cause additional external excitation on hypoid/bevel gear pair when driving torque is passing through universal joint. The basic mathematic modeling of shaft misalignment (Ota, Kato and Sugita, 1985) can be found which established torque and speed relationship between two rotating shafts connected by universal joint with or without friction. Later, stability analysis of disk rotor system with coupling misalignment was analyzed (Mazzei, Argento and Scott, 1999; Desmidt, Wang and Smith, 2002). Those works indicated that such coupling would result in torque and speed fluctuation and bending moment acting on the driven shaft and impose additional loads on bearings. Experimental studies (Zhao et al., 2012; Xu and Marangoni, 1994) also showed good correlation between analytical study and test results.

Rotor unbalance occurs when the center of mass such as pinion or gear body is not coincide with rotation center. In fact, the unbalance in rotating system is also hard to avoid because of manufacturing error, unequal material density distribution, material run out or eccentricity (Blankenship and Singh, 1995). Unbalance would cause additional load on bearings that support
pinion or gear shaft due to centrifugal force. Such excitation caused by rotor unbalance is usually at shaft order which is also called synchronous excitation. If there also exists misalignment effect, the unbalance excitation could be at certain order of shaft rotational speed and can be detected by spectral analysis. With the combined effect of misalignment between driving and driven shaft and pinion/gear body unbalance, the universal joint that connects propeller shaft and pinion shaft can become a major vibration source (Xu and Marangoni, 1994).

However, most previous study either deals with simple perfectly aligned rotor disk system, or are more focused on system stability performance considering shaft rotor dynamics. Study of the effect of misalignment and mass unbalance by means of universal joint modeling in geared-rotor system is rarely seen. In-depth analysis is needed to evaluate the sensitivity of hypoid geared-rotor system towards the combined effect of propeller shaft misalignment and pinion mass unbalance which is commonly seen in industry and practical applications.

Therefore, in order to build a dynamic system capable of evaluating vibration that results from shaft misalignment and mass unbalance, universal joint connection will be analyzed in detail for a 14-DOF hypoid geared rotor system for more accurate modeling of driveline. Coupled multi-body dynamic model will be applied to account for shaft rotational displacement. Pinion shaft is driven by propeller shaft built by finite element method. Mass unbalance due to eccentricity is applied on pinion body. The effect of shaft bending moment due to shaft misalignment and mass unbalance will be analyzed by evaluating changes in gear dynamic response as well as the dynamic bearing load. The study seeks to provide a more accurate prediction of dynamic response of hypoid geared rotor system and investigate system sensitivity towards shaft misalignment. Furthermore, the formulation of additional load generated on pinion shaft will help conducting more reliable bearing design.
4.2 Mathemtic Modeling

4.2.1 Mesh model

Mesh coupling between pinion and gear body is represented as a spring damper element. In this study, the gear mesh model is obtained by loaded tooth contact analysis (LTCA) (Vijayakar, 2003) based on a combined method of semi-analytical theory and 3D finite element analysis (Vijayakar, 1991). The hypoid gear pair geometry and contact cells on tooth surface are shown in Fig. 4.1 where $f_i$ is normal contact force acting on contact cell $i$, $r_i^{(1)}$ and $r_i^{(2)}$ are cell’s position vectors in pinion and gear coordinates. Unit vector $n_i$ represents its normal direction. Therefore, the combined mesh force of all contact cells concerned is formulated as:

$$F_j = \sum_{i=1}^{N} n_{ij} f_i, \quad (4.1)$$

$$F = \sqrt{\sum_{j} F_j^2}, \quad (j = x, y, z), \quad (4.2)$$

The equivalent line-of-action (LOA) is calculated by:

$$n_j = F_j / F, \quad (j = x, y, z), \quad (4.3)$$

The coordinate of effective mesh point $R_m(x_m, y_m, z_m)$ is obtained by:

$$x_m = (M_z + F_x y_m) / F_y, \quad (4.4)$$

$$y_m = \frac{\sum_{i=1}^{N} f_i^{(1)}}{\sum_{i=1}^{N} f_i}, \quad (4.5)$$
\[
\begin{align*}
z_m &= (M_y + F_z x_m) / F_s, \quad (4.6) \\
\end{align*}
\]

Where \( \{M_x, M_y, M_z\} \) is the effective total moment vector due to contact force. The mesh stiffness is given as:

\[
k_m = F / (e_L - e_U). \quad (4.7)
\]

where \( e_L \) and \( e_U \) are loaded and unloaded translational transmission error (TE) projected along LOA.

\[
\begin{align*}
[M_s] \{\ddot{x}_s\} + [C_s] \{\dot{x}_s\} + [K_s] \{x_s\} + [G] \{\ddot{x}_s\} + [G_a] \{x_s\} &= \{F_s\} \\
\end{align*}
\]

\textbf{4.2.2 Hypoid Geared Rotor System Formulation}

The 14-DOF coupled multi-body dynamic model for hypoid geared rotor system is illustrated in Fig. 4.2 which comprises pinion, gear, engine and load component. The complete differential equations of motion can be written as:
where \( \{x_s\} \) is the system’s displacement vector in both translational and rotational coordinates which can be written as:

\[
\{x_s\} = \{\theta_e, x_p, y_p, z_p, \theta_{pe}, \theta_{pp}, \theta_{pe}, x_g, y_g, z_g, \theta_{ge}, \theta_{gp}, \theta_{ge}, \theta_{gl}\}^T
\]

(4.9)

Notice that torsional displacement of pinion, gear, engine and load (\( \theta_{pe}, \theta_{gp}, \theta_{ge}, \theta_{gl} \)) are large angular displacements which are subject to shaft torsional dynamics. Accordingly, lumped mass matrix of the hypoid geared rotor system is:

\[
[M_s] = \text{diag}[M_p, M_p, M_p, I_{pe}, I_{pp}, \ldots, M_g, M_g, M_g, I_{ge}, I_{gp}, \ldots, I_{gl}]
\]

(4.10)

The coordinate system of lumped parameter model is different from that of gear mesh model so a transformation matrix is used when mesh parameters are applied in the dynamic model. The stiffness matrix shown in Eq. 4.11 consists of torsional stiffness (\( k_e \) and \( k_i \)), condensed shaft-bearing support stiffness matrices for pinion (\( \begin{bmatrix} K_p \end{bmatrix} \)) and gear body (\( \begin{bmatrix} K_g \end{bmatrix} \)).

\[
[K_s] = \text{diag}[k_e, \begin{bmatrix} K_p \end{bmatrix}, \begin{bmatrix} K_g \end{bmatrix}, k_i]
\]

(4.11)

The form of damping is assumed to be proportional type and can be written as:

\[
[C_s] = \zeta_s \cdot [K_s]
\]

(4.12)

where \( \zeta_s \) is shaft-bearing system damping ratio.

Two gyroscopic matrix associated with absolute rolling speed and acceleration respectively are also included in system equation of motion. They can be expressed as:
The excitation force vector consists of both internal and external components. The internal excitation is generated by TE while external excitations are caused by other driveline components. The combined expression of excitation can therefore be expressed as:

\[
\{F\} = \{F_{int}\} + \{F_{ext}\} \tag{4.15}
\]

The internal excitation vector is written as:

\[
\{F_{int}\} = \left\{0, h_p \cdot F_m, h_g \cdot F_m, 0\right\} \tag{4.16}
\]

where \(\{h_p\}\) and \(\{h_g\}\) are the directional transforming vectors which consist of the directional cosines and directional rotational radii. They are applied to project time varying mesh force along LOA to generalized coordinate directions of pinion or gear body. The detailed expression of projection vector is:

\[
\{h_l\} = \{n_{lx}, n_{ly}, n_{lz}, \lambda_{lx}, \lambda_{ly}, \lambda_{lz}\}, \quad l = p, g \tag{4.17}
\]

The directional rotation radius \(\lambda_l = \left(\lambda_{lx}, \lambda_{ly}, \lambda_{lz}\right)\) for pinion \((l = p)\) or gear \((l = g)\) are defined as:

\[
\lambda_{lx} = y_p n_{lz} - z_p n_{ly}, \tag{4.18}
\]
\[ \lambda_y = z_p n_{ix} - x_p n_{iz}, \quad (4.19) \]
\[ \lambda_z = x_p n_{iy} - y_p n_{iz}, \quad (4.20) \]

where vector \( \{n_{ix}, n_{iy}, n_{iz}\} \) stands for LOA vector and \( \{x_i, y_i, z_i\} \) is mesh point coordinate.

The dynamic mesh coupling between meshing gear pair is modeled by mesh stiffness \( (k_m) \), mesh damping \( (c_m) \), backlash \( (b) \) and kinematic transmission error \( (e_U) \). The non-linear dynamic mesh force which acts at the effective mesh point along LOA can be expressed as:

\[ F_m = k_m \cdot f(\delta - e_U) - c_m \cdot (\dot{\delta} - \dot{e}_U) \quad (4.21) \]

where \( f(\delta - e_U) \) is the nonlinear function with backlash \( b \) as the nonlinear factor which can be formulated as:

\[
f(\delta - e_U) = \begin{cases} 
\delta - e_U - b, & \delta - e_U \geq b, \\
0, & -b < \delta - e_U < b, \\
\delta - e_U + b, & \delta - e_U \leq -b,
\end{cases} \quad (4.22)
\]

The translational dynamic transmission error \( (\delta) \) in Eq. 4.21 indicates the displacement difference between pinion and gear along LOA during operation which can be written as:

\[ \delta = \{h_p\}^T \cdot \{x_p, y_p, z_p, \theta_{px}, \theta_{py}, \theta_{pz}\} - \{h_g\}^T \cdot \{x_g, y_g, z_g, \theta_{gx}, \theta_{gy}, \theta_{gz}\} \quad (4.23) \]

In order to incorporate large rotation displacement of pinion and gear while take into account time-varying characteristics of mesh parameters, a more generic equation is then formulated as:

\[ \delta = \{h_p\}^T \cdot \{x_p, y_p, z_p, \theta_{px}, \theta_{py} - \theta_{gy} \cdot R, \theta_{pz}\} - \{h_g\}^T \cdot \{x_g, y_g, z_g, \theta_{gx}, 0, \theta_{gz}\} \quad (4.24) \]

Where \( R = \lambda_{gy} / \lambda_{py} \) is the nominal gear ratio and \( (\theta_{py} - \theta_{gy} \cdot R) \) is the instantaneous pinion rotational TE. Because of the complex geometry of hypoid gear tooth surface, the coupling
characteristics including mesh stiffness, mesh damping, effective mesh point and LOA vary significantly when gear is rotating. Therefore, exact mesh parameters are dependent on instantaneous pinion or gear’s rolling angular position described by large rotational displacement of pinion or gear shaft. The external excitation part of force vector assuming no driveline coupling effect from other components or system assembly error incorporated can be defined as:

\[ \{F_{\text{ex}}\} = \{T_e, 0, 0, -T_l\} \]  \hspace{1cm} (4.25)

where \( T_e \) and \( T_l \) are engine/driving torque and load torque of the lumped parameter model respectively.

Figure 4.2 Illustration of 14-DOF lumped parameter model
4.2.3 Propeller Shaft Misalignment

In previous session, a 14-DOF lumped parameter model based on nonlinear time-varying (NLTV) coupled multi-body dynamics is built. Next, the effect of universal joint will be included with focus on its kinematics and torque relationship between propeller shaft and pinion shaft as well as its effect on lateral vibration of pinion shaft. It is assumed that in current study, only angular misalignment is considered.

A complete model the comprises of beam element (Fig. 4.3) and universal joint (Fig. 4.4) is illustrated in Fig 4.5 which consists of engine, propeller shaft, pinion shaft and a universal joint that connects them. Timoshenko beam element is used for building shaft model. Stiffness matrix and mass matrix including both translational inertia mass \((M_t)\) and rotational inertia mass \((M_r)\) are shown below. Gyroscopic effect of shaft element is not included since its value is too small compared to that of pinion or gear body in system model. More details about finite element modeling for shaft and coupling methods can be found in Genta’s work (2005).

\[
M_t = \frac{\rho Al}{420(1+\phi)^2} \begin{bmatrix} m_1 & l m_2 & m_3 & -l m_4 \\ l^2 m_5 & l m_4 & -l^2 m_6 \\ m_1 & -l m_2 \\ l^2 m_5 \end{bmatrix}
\] (4.26)

\[
M_r = \frac{\rho I_y}{30l(1+\phi)^2} \begin{bmatrix} m_7 & l m_8 & -m_7 & l m_8 \\ l^2 m_9 & -l m_8 & -l^2 m_{10} \\ m_7 & -l m_8 \\ l^2 m_9 \end{bmatrix}
\] (4.27)

\[
K = \frac{EI_y}{l^2(1+\phi)} \begin{bmatrix} 12 & 6l & -12 & 6l \\ (4+\phi)l^2 & -6l & (2-\phi)l^2 & 12 \\ -6l & (2-\phi)l^2 & 12 & -6l \\ (4+\phi)l^2 & -6l & (2-\phi)l^2 & 12 \end{bmatrix}
\] (4.28)
The universal joint applied is presented in Fig. 4.4. It is assumed that no friction or any type of power loss is incorporated. The misalignment angle is $\alpha$. Unit vector $\vec{k}$ indicates the center line direction of propeller (driving) shaft $z$ and $\vec{k}_c$ is the centerline direction of pinion (driven) shaft $z_a$. $\theta_e$ and $\theta_d$ describe the rotation angle of propeller and pinion shaft respectively. Since the two pins of a universal joint are always orthogonal to each other, the relationship can be described as:

\[ p(\theta_e) \cdot q(\theta_d) = 0 \quad \text{where:} \]

\[ p(\theta_e) = \cos \alpha \cos \theta_e \cdot \vec{i} + \sin \theta_e \cdot \vec{j} + \sin \alpha \cos \theta_e \cdot \vec{k} \]  

\[ q(\theta_d) = -\sin \theta_d \cdot \vec{i} + \cos \theta_d \cdot \vec{j} \]  

According to the research work by Hiroshi Ota and Masayoshi Kato (1985), the above basic relationship can be further derived into the following forms:

\[ \tan \theta_d = \frac{\tan \theta_e}{\cos \alpha}, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \]  

\[ \sin \theta_d = \frac{\sin \theta_e}{\sqrt{H(\theta_e)}} \]  

\[ \cos \theta_d = \frac{\cos \alpha \cos \theta_e}{\sqrt{H(\theta_e)}} \]  

\[ H(\theta_e) = 1 - \sin^2 \alpha \cdot \cos^2 \theta_e \]  

It is assumed that the two pins are rigid. Therefore, during shaft rotation, only the torque ($T_a$) that is perpendicular to both pins are transmitted because no friction or deformation is considered in this case. Based on the kinematic relationship of two shafts connected by universal joint, the driving torque vector acting on the pinion shaft can be expressed as:
\[ T_{\alpha} = T_{\alpha} \cdot [p(\theta_c) \times q(\theta_c)] \] (4.36)

According to the coordinate system of universal joint in Fig. 4.4, the torque value on propeller shaft and pinion shaft can be calculated by projecting \( T_{\alpha} \) on their rotation axis respectively:

\[ T_{\alpha e} = T_{\alpha} \cdot k = T_{\alpha} \frac{\cos \alpha}{\sqrt{H(\theta_c)}} \] (4.37)
\[ T_{\alpha d} = T_{\alpha} \cdot \bar{k}_a = T_{\alpha} \sqrt{H(\theta_c)} \] (4.38)

Accordingly, the torque relationship can be further derived as:

\[ T_{\alpha d} = \frac{T_{\alpha e} \cdot H}{\cos \alpha} \] (4.39)

The bending moment on both shafts connected by universal joint can be expressed through the same method as:

\[ M_{\alpha e} = -T_{\alpha} \frac{\sin \alpha \sin \theta_c}{\sqrt{H(\theta_c)}} \] (4.40)
\[ M_{\alpha d} = T_{\alpha} \sin \alpha \cos \theta_c \] (4.41)

It can be expected that these in-plane moments will have significant influence on lateral vibration of two shafts and thus exert additional dynamic load on bearings that support them. If no power loss is considered, the speed and torque relationship can be expressed as:

\[ \frac{\dot{\theta}_d}{\dot{\theta}_c} = \frac{\cos \alpha}{H(\theta_c)} = \frac{T_{\alpha e}}{T_{\alpha d}} \] (4.42)

In order to show harmonic components and also for easy mathematical operation in the following analysis, the relationship in Equation 4.42 can be expanded into (Xu and Marangoni, 1994):
\[ \frac{\dot{\theta}_d}{\dot{\theta}_e} = 1 + 2 \sum_{n=1,2,...} e^n \cos 2n\omega t \quad \text{where} \]

\[ e = \cos \alpha \sum_{k=1}^{n} C_{2k-1} \left( \frac{\sin \alpha}{2} \right)^{2k} \]

(4.43)

(4.44)

When torque is transmitted from engine (driving shaft) to misaligned pinion shaft through universal joint, the relationship between the two angular velocity can be described as follows in harmonics form:

\[ \frac{\dot{\theta}_d}{\dot{\theta}_e} = 1 + A_2 \cos 2\theta_e + A_4 \cos 4\theta_e + \ldots + A_{2n} \cos 2n\theta_e \]

(4.45)

\[ A_{2n} = 2e^n, \quad n=1, 2, 3… \]

(4.46)

Acceleration at the driving end of pinion shaft can be written as:

\[ \ddot{\theta}_d = -2A_2 \omega^2 \sin 2\omega t - 4A_4 \omega^2 \sin 4\omega t - \cdots - 2nA_{2n} \omega^2 \sin 2n\omega t \]

(4.47)

Rotational displacement of pinion shaft is expressed as:

\[ \theta_d = \omega t + \frac{A_2}{2} \sin 2\omega t + \frac{A_4}{4} \sin 4\omega t + \ldots + \frac{A_{2n}}{2n} \sin 2n\omega t \]

(4.48)

Therefore, based on Yang’s work (2012), the translational force on pinion body due to bending moment at the driving end of pinion shaft can be calculated and the overall excitation from shaft misalignment acting on pinion can be formulated in vector form as:

\[ F_{\text{mis}} = \begin{bmatrix} 0 & \cdots & 0 & \frac{M_{\dot{\theta}L}}{L} \end{bmatrix}^T \]

(4.49)

In the above equations, \( \alpha \) is the misalignment angle. It is obvious that if engine and pinion are in perfect alignment (\( \alpha = 0 \)), there will be no difference between engine speed and pinion speed. On the other hand, if misalignment angle \( \alpha \) doesn’t equal zero, both angular velocity and acceleration
depend on pinion angular position. Furthermore, driving torque from the engine will split into two components when passing through universal joint with misalignment. The external excitation induced by misalignment bending moment will not only affect gear mesh response, but also exert external loads on the rolling element bearings that support pinion or gear body which should be taken into consideration at design stage.

Figure 4.3 Timoshenko beam element

Figure 4.4 Universal joint and coordinate system
4.2.4 Unbalance from Eccentricity

Hypoid/bevel gear eccentricity effect was firstly studied by Peng (2010) with emphasis on its influence on gear transmission error (Fig. 4.6). In his study, translational kinematic TE was modified to approximate the eccentricity effect in a more efficient way compared to considering eccentricity directly in LTCA. However, mass unbalance of pinion or gear body, as one of important consequences is not included which is a usual type of external excitation. Therefore, to account for mass unbalance effect of geometry eccentricity and study the combined effect of both shaft misalignment and unbalance, the external excitation need to be further modified.

As is illustrated in Fig. 4.7, the unbalance force acting on point \( e_{rp} \) can be divided into normal and tangential force vectors in complex form:

\[
F_u = F_{un} + i \cdot F_{ut} \tag{4.50}
\]

\[
F_{un} = -m_p e_{rp} \cdot \dot{\theta}_p^2 \tag{4.51}
\]
The tangential part of unbalance force can be calculated based on Euler equation in x and z direction. In the case of combined effect of shaft misalignment and unbalance, both tangential and normal component affects the external excitation. The overall external excitation under shaft misalignment and pinion mass unbalance in x and z direction (Fig. 4.8) can be expressed as:

\[
F_x = \left( \frac{M_{\theta d}}{L} + m_p e_{rp} \cdot \dot{\theta}_p^2 \right) \sin \theta_p - m_p e_{rp} \cdot \ddot{\theta}_p \cdot \cos \theta_p \tag{4.53}
\]

\[
F_z = \left( \frac{M_{\theta d}}{L} + m_p e_{rp} \cdot \dot{\theta}_p^2 \right) \cos \theta_p + m_p e_{rp} \cdot \ddot{\theta}_p \cdot \sin \theta_p \tag{4.54}
\]
4.2.5 Combined Shaft-bearing Stiffness Modeling

The combined shaft-bearing structure can be included in analysis by deriving equivalent stiffness function based on static finite element method. In this case, 2-bearing overhung mounted configuration shown in Fig. 4.9 is applied to both pinion and gear shafts. The bearing stiffness function can be either simple support using empirical stiffness value or full stiffness matrix considering practical geometry of each roller bearing supporting the shaft (Lim and Singh, 1990, 1991). The load-displacement relationship can be simplified as:
\[ \{ F_{sys} \} = [K_{sys}] \{ \Delta_{sys} \} \]  

(4.55)

The combined shaft bearing matrix can be separated into three parts including reference point, shaft element and bearing stiffness matrix. The above equation can be re-written in a more detailed form as:

\[
\begin{bmatrix}
F_R \\
F_S \\
F_B
\end{bmatrix} =
\begin{bmatrix}
K_{RR} & K_{RS} & K_{BB} \\
K_{SR} & K_{SS} & K_{SB} \\
K_{BR} & K_{BS} & K_{BB}
\end{bmatrix}
\begin{bmatrix}
\Delta_R \\
\Delta_S \\
\Delta_B
\end{bmatrix}
\]

(4.56)

Where \( F_j \) and \( \Delta_j \ (j = R, S, B) \) refer to the load and displacement of reference node, other free DOFs on the shaft and bearing outer raceway at actual locations. Since it is assumed that load is only acting on the reference node and the housing is rigid, both \( F_S \) and \( \Delta_B \) are zero vectors. By applying unit load in 5 DOFs at the reference node except the torsional DOF around the shaft rotation axis, equivalent shaft-bearing stiffness can be derived. Displacement on the reference node and other nodes on shaft can be expressed as:

\[
\begin{bmatrix}
\Delta_R \\
\Delta_S
\end{bmatrix} =
\begin{bmatrix}
K_{RR} & K_{RS} \\
K_{SR} & K_{SS}
\end{bmatrix}^{-1}
\begin{bmatrix}
F_R \\
0
\end{bmatrix}
\]

(4.57)

The equivalent dynamic bearing load at actual locations is written as:

\[
\{ F_B \} = [K_{BR} \quad K_{BS}] \{ \Delta_R \quad \Delta_S \}^T
\]

(4.58)
4.3 Comparative Analysis and Case Study

4.3.1 Hypoid Gear Dynamic Response Analysis

The geometry data and system parameters for the geared rotor system illustrated in Fig. 4.2 are listed in Table 4.1. The driving torque at engine is set to be 80 Nm. Both linear and nonlinear time-varying model are used. Shaft misalignment between propeller shaft and pinion shaft is considered by introducing torque and speed fluctuation as well as additional in-plan moment as explained in previous sessions. Mass unbalance due to eccentricity is assumed to be on pinion body.

Table 4.1 Hypoid gear geometry and system settings

<table>
<thead>
<tr>
<th>Gear data</th>
<th>Pinion</th>
<th>Gear</th>
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<tbody>
<tr>
<td>Number of teeth</td>
<td>10</td>
<td>43</td>
</tr>
<tr>
<td>Spiral angle (rad)</td>
<td>0.803</td>
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<td>Pitch angle (rad)</td>
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<tr>
<td>Face width (m)</td>
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<td>0.168</td>
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<tr>
<td>Type</td>
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<td>Right hand</td>
</tr>
<tr>
<td>----------------------</td>
<td>-----------</td>
<td>------------</td>
</tr>
<tr>
<td>Loaded side</td>
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<td>Convex</td>
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<tr>
<td>Offset (m)</td>
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**System parameters**

<table>
<thead>
<tr>
<th></th>
<th>Engine</th>
<th>Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsional moment of inertia (kg-m²)</td>
<td>0.0055</td>
<td>0.1</td>
</tr>
<tr>
<td>Pinion Mass (kg)</td>
<td>11.48</td>
<td>49.52</td>
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<tr>
<td>Torsional moment of inertia (kg-m²)</td>
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<tr>
<td>Bending moment of inertia (kg-m²)</td>
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<td>Axial support stiffness (N/m)</td>
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<td>Lateral support stiffness (N/m)</td>
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<td>Torsional stiffness of shaft (Nm/rad)</td>
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<td>Bending shaft-bearing stiffness (Nm/rad)</td>
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</tr>
<tr>
<td>Support component damping ratio</td>
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</tr>
<tr>
<td>Gear backlash (mm)</td>
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<td></td>
</tr>
</tbody>
</table>

The torque and speed fluctuation on pinion shaft when constant torque and speed are transmitted through a universal joint with misalignment are shown in Fig. 4.10. It is observed that the amplitude of fluctuation increases with larger misalignment angle. According to gear mesh analysis above, torque fluctuation will cause variation in mesh parameters under specific mesh frequency. Fig. 4.11 shows a 3D surface of load dependent and time-varying mesh stiffness. The value is dependent on instantaneous torque and angular position. It can be noticed that mesh stiffness is more sensitive to external torque variation in lightly loaded case. Mesh stiffness fluctuation due to 20 degree shaft misalignment over one shaft rotation cycle is shown in Fig. 4.12 where clear discrepancy can be observed. In addition, speed variation will further affect the mesh parameters by alternating instantaneous rotation angle at specific moment when there exists shaft misalignment.
Figure 4.10 Torque (a) and speed (b) fluctuation on pinion shaft with shaft misalignment considered:

- 0 degree,
- 5 degree,
- 10 degree,
- 15 degree,
- 20 degree,
- 25 degree,
- 30 degree
In order to demonstrate the external excitation effect of shaft misalignment and pinion mass unbalance on hypoid gear dynamic response, dynamic analysis under 80 Nm nominal driving torque assuming constant mesh parameters is conducted and the results are shown in Fig. 4.13. Mesh parameters are approximated using their mean value so that dynamic response is caused by external excitation alone. Responses from three excitation cases are compared and it is observed
that pinion mass unbalance excitation has more significant effect in higher mesh frequency range.
This is due to large tangential and radial forces caused by eccentricity in high speed range. On the
other hand, the hypoid gear pair is more subject to shaft misalignment excitation in low frequency
range. This is because the bending moment excited mostly depends on mean driving torque,
misalignment angle and propeller shaft rotation angle. At mesh frequency of 1300 Hz, it is noticed
that a mode is excited by the coupled effect of misalignment and pinion mass unbalance.

![Graph showing dynamic mesh force vs mesh frequency](image)

Figure 4.13 Dynamic response under mean mesh parameters with different misalignment and unbalance conditions: - - - - - 0 degree misalignment with pinion mass unbalance, --- 20 degree
misalignment, - - - - 20 degree misalignment with pinion mass unbalance

The above simulation shows dynamic response due to misalignment and unbalance alone.
To analyze their interaction with gear internal excitation, time-varying mesh parameters are
considered in the following analysis. Fig. 4.14 presents the parametric analysis of misalignment
angle and its interaction with gear internal excitation based on linear time-varying model during step speed weep up. As can be seen, the combined excitation generates four major response peaks at 1120 Hz, 1660 Hz, 2140 Hz and 3240 Hz. The amplitude of the two peaks at 520 Hz and 2240 Hz increases with larger misalignment angle applied which indicates that these modes are subject to shaft assembly angle. Next, pinion mass unbalance is also introduced. From case study results shown in Fig. 4.15, it can be seen that mass unbalance mostly affects the response in high mesh frequency range after 2000 Hz which is expected according to previous simulation results. In addition, dynamic response peak at 2240 Hz is mostly dominated by mass unbalance excitation which is at $2n$ ($n = 1, 2, 3, ...$) times of shaft order according to the expression in Equation 4.45.

Figure 4.14 Dynamic response of linear time-varying model with different misalignment angle: - - - - - - -

0 degree, 10 degree, 15 degree, 20 degree
Figure 4.15 Dynamic response of linear time-varying model with different misalignment and unbalance conditions: \( \cdots \cdots \) 0 degree misalignment, \( \cdots \cdots \cdots \) 20 degree misalignment, \( \cdots \cdots \cdots \) 20 degree misalignment with pinion mass unbalance

The results in Fig. 4.14 ~Fig. 4.15 show that interaction among gear internal excitation, shaft misalignment and pinion mass unbalance can be significant without considering gear loss of contact. If backlash is incorporated, strong nonlinear interactions is expected to exist. The dynamic response of nonlinear time-varying mesh model is plotted in Fig. 4.16 where the influence of shaft misalignment covers a wider frequency range compared to linear case. It is also noted that before 2000 Hz, shaft misalignment has dominating effect on the overall response compared to mass unbalance indicating that it is a high order excitation and can have strong coupling effect with gear internal excitation. However, with higher mesh frequency, mass unbalance excitation amplitude grows much higher than misalignment and starts to significantly affect the overall response. Since the simulation is based on step sweep, the effect of mass unbalance mostly comes from steady state centripetal force in radial direction.
In Fig. 4.16, the dynamic response above the mean mesh force line is an indication of occurrence of contact loss between meshing teeth. Both misalignment and mass unbalance effect are seen to have expanded the speed range of tooth contact loss. Spectrum analysis is therefore needed to have a deeper understanding of their influence on frequency component in response at certain rotation speed. The comparative analysis results are demonstrated in Fig. 4.17 using waterfall plot. From baseline simulation in Fig. 4.17 (a), it can be concluded that complicated frequency spectrum commonly exists in nonlinear response range that is due to gear backlash. Consequently, tonal noise and sound quality will be worsen under similar noise level. In Fig. 4.17 (b), complex spectrum covers a wider area where strong modulation and broad band response can
be observed. The nonlinear responses mainly locate in approximately three speed ranges: 400 Hz ~ 600 Hz, 1100 Hz ~ 1800 Hz, 2300 Hz ~ 3500 Hz.

Figure 4.17 Waterfall plot of dynamic mesh force response: (a) without misalignment, (b) with misalignment
The frequency spectrum is further analyzed using FFT as is shown in Fig. 4.18. Fig. 4.18(a) shows that the response peak at 500 Hz is mainly in 2nd shaft order due to misalignment excitation. Side bands are also seen near the mesh order responses. In Fig. 4.18(b), it can be seen that the effect of shaft misalignment and mass unbalance on response amplitude at 1300 Hz is limited. Instead, complex side bands caused by response modulation are observed which indicates that the dynamic responses from 1200 Hz to 1700 Hz are mostly subject to gear internal excitation. On the other hand, the peak at 2250 Hz is excited by both shaft misalignment and mass unbalance as shown in Fig. 4.18(c). Therefore, the broad band response is caused by three factors including gear backlash nonlinearity, speed/torque fluctuation expressed by Equation 4.45 and bending moment. For response in higher speed range, the coupling between misalignment and unbalance is limited. An example is illustrated in Fig. 4.18(d) where misalignment affects the response by creating side bands in the local area of multiple super and sub harmonics of mesh order responses generated by gear mesh excitation. The introduction of mass unbalance only generates shaft order response and responses in most frequency spectrum are not distorted. This implies that in high speed range, either misalignment or mass unbalance can be studied separately without losing much accuracy because the coupling between these two types of external excitations are weak.
4.3.2 Dynamic Bearing Load Analysis

The analysis above mainly focuses on dynamic response of a 14-DOF lumped parameter model under the influence of different excitation cases. The response amplitude is measured at effective mesh point along LOA. One of the main objectives of this research is to evaluate the dynamic load changes on bearings during operation to help guide the design when shaft
misalignment or mass unbalance is incorporated. Moreover, as an important transferring path between hypoid gear pair and housing which is also a main source of noise radiation, dynamic load on each rolling element bearing is an important indication of noise and vibration performance. The dynamic reactions of the support will be further studied in this part and the results are expected to characterize the dynamic load on each bearing of the system that are transferred to the gear housing. The configuration of pinion and gear shaft as well as bearings are illustrated in Fig. 4.19. The bearing properties are assumed as simply supported and bearing geometry or stiffness function analysis is beyond the scope of current study. Driving torque is set to be 50Nm and other system parameters are listed in Table 4.1.

According to the results shown in Fig. 4.20, the effect of shaft misalignment on gear dynamic response curve of shaft-bearing model is significant which is similar to that of lumped parameter model analyzed above especially the response peak generated at around 520 Hz. On the
contrary, the influence from pinion mass unbalance is mostly suppressed. No clear peaks created by mass unbalance are found in the mesh frequency range of interest which indicates that the gear dynamic response in this case is more subject to shaft misalignment effect. To prove this, FFT analysis (Fig. 4.21) is conducted for responses in two main nonlinear response frequency ranges. It is found that only weak side band responses are caused by mass unbalance whereas modulation or side bands excited by misalignment are more dominant in response.

Figure 4.20 Dynamic response of full shaft-bearing nonlinear time-varying model with different misalignment and unbalance conditions: 0 degree misalignment, 20 degree misalignment, 20 degree misalignment with pinion mass unbalance
Figure 4.21 FFT results of 0 degree misalignment, 20 degree misalignment, 20 degree misalignment with pinion mass unbalance at mesh frequencies of: (a) 840 Hz; (b) 1320 Hz

The bearing reaction forces in radial direction at their actual locations are plotted in Fig. 4.22 which are found to be profoundly affected by shaft misalignment and pinion mass unbalance compared to dynamic mesh force results in Fig. 4.20. It is observed that dynamic forces on pinion bearings (bearing 1 and 2) are more sensitive to misalignment and mass unbalance than those on gear gearings (bearing 3 and 4) because these two excitations are directly acting on pinion shaft. According to subfigure (a) and (b), with increased mesh frequency during step speed sweep up, both shaft misalignment and pinion mass unbalance generate higher reaction force amplitude. Strong side band responses excited by shaft misalignment are found in the local area of response peaks that caused by hypoid gear mesh. However, the effect of misalignment on gear bearing forces are limited as can be seen in Fig. 4.22(c) and (d) where its influence mainly exists in the mesh frequency range where gear contact loss occurs. This observation further proves that backlash nonlinearity is one of the key factors affecting the coupling between shaft misalignment and gear mesh excitation. In high mesh frequency range, reaction forces on both pinion and gear
bears are mostly subject to pinion mass unbalance because its force amplitude is sensitive to 
speed changes. The analysis implies that although pinion mass unbalance has negligible influence 
on gear dynamic response, its effect on bearing dynamic forces are significant which should be 
taken into consideration when selecting bearing models for a combined shaft-bearing system.

![Comparison of reaction forces on actual bearing locations:](Figure 4.22) 

0 degree misalignment, 20 degree misalignment, 20 degree misalignment with pinion mass unbalance; (a) 
bearing 1 on pinion shaft, (b) bearing 2 on pinion shaft, (c) bearing 3 on gear shaft, (d) bearing 4 on gear 
shaft
4.4 Conclusions

This study demonstrates a method of incorporating shaft misalignment and mass unbalance effect in analyzing hypoid gear dynamic response through detailed modeling of universal joint connecting propeller and pinion shaft. The cross pin of universal joint is assumed to be rigid and no power loss factors such as friction or damping are considered in shaft connection. Mechanism of torque and speed transmission through universal joint is explained. It is found that both torque and speed fluctuation exist on pinion shaft when constant driving torque on propeller shaft is transmitted to pinion shaft. In addition, a secondary moment on pinion shaft is generated due to misaligned connection which will cause lateral vibration on hypoid gear pair. The excitations from shaft misalignment are even multiple frequencies of shaft rotation speed. Therefore, strong coupling can be observed in a wide mesh frequency range through spectrum analysis. Backlash nonlinearity will enhance the coupling among shaft misalignment, mass unbalance and gear mesh excitations. The predicted additional dynamic load on actual location of pinion and gear bearings are expected to help facilitate the design and selection of tapered rolling element bearings which are found to be sensitive to shaft rotation speed and conditions of universal joint connection.
5.1. Introduction

Gear dynamic response caused by combined effect of loaded transmission error and external torque is a main source of structure borne noise of a gear transmission system. It is of significant value to develop an accurate gear shaft-bearing system to analyze the underlying mechanism of complex interaction between different components especially when there exists clearance type nonlinearity in either gear or bearing model. Therefore, an accurate prediction of the severity of system dynamic problems can be obtained at an early design stage.

dynamic model for hypoid gear shaft-bearing system. Instead of analyzing pure vibration or relative motion of meshing gear pair, the model is derived to include large rotational motion of pinion and gear body. Time-varying mesh parameters of the model is based on instantaneous rotation angle of pinion or gear shaft. The model can thus be used for either steady state or transient dynamic analysis.

Bearing radial clearance is another nonlinear effect commonly seen in geared shaft bearing system which can interact with backlash nonlinearity of gear pair. Kahraman and Singh (1991) studied nonlinear interaction and route to chaos of a spur geared rotor-bearing system with multiple clearance. Later, Gurkan and Ozguven (2007) studied the interaction between gear backlash and bearing clearance and their relationship with mesh and bearing stiffness. Harsha and Kankar (2004) investigated the combined effect of internal radial clearance and surface waviness on non-linear response of balanced rotor supported by ball bearings. Tiwari et al (2000) focused on the effect of radial bearing clearance on dynamic behavior of balanced rotor. Theodossiades and Natsiavas (2000) considered both bearing clearance and gear mesh backlash in analyzing periodic and chaotic phenomenon in dynamic response of load dependent gear mesh model. Later, Guo and Parker (2010) applied the bearing clearance type non-linearity in planetary gear dynamic analysis and investigated root cause of softening and hardening effect in response. However, bearing clearance type non-linearity has not been considered in hypoid gear analysis by far due to complex interaction between gear mesh and bearing. In addition, most nonlinear bearing stiffness functions applied are based on time-invariant function without taking into account the changing of orbital position of rolling elements which has been proved to have certain effect on hypoid gear dynamic response (Liew and Lim, 2005; Yang, 2012).
Therefore, the purpose of this study is to further extend the previous work on hypoid gear dynamics by studying the coupling effect between gear backlash and bearing radial clearance nonlinearities. Nonlinear behavior of gear response and actual load on bearing will also be investigated through parametric study to analyze system sensitivity to different nonlinear conditions.

5.2. Mathematical Modeling of Nonlinear Geared Rotor System

A generic 14-DOF coupled multi-body dynamic model of a hypoid geared rotor system applied in this study is illustrated in Figure 5.1. The system consists of an engaging gear pair, engine, load and supporting shaft-bearing elements. The pinion/gear body is assumed as rigid body with lumped stiffness and damping as the mesh interface.

Figure 5.1 14-DOF lumped parameter model of hypoid geared rotor system (Peng, 2010)
5.2.1 Gear Mesh Modeling and Tooth Contact Analysis

A loaded tooth contact analysis that based on semi-analytical and finite element analysis (Vijayakar, 1991, 2003) is applied. The gear pair geometry and mesh mechanism is illustrated in Figure 5.2 where the contact area is divided into N cells where $r_i(r_{ix}, r_{iy}, r_{iz})$ and $n_i(n_{ix}, n_{iy}, n_{iz})$ are position and normal vector of an individual cell and $f_i$ is the cell contact force in normal direction. Therefore, the combined mesh force can be written as:

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$ (5.1)

$$F_s = \sum_{i=1}^{N} n_i f_i, \ (s = x, y, z)$$ (5.2)

The coordinates of LOA vector $L(n_s, n_y, n_z)$ can be expressed as:

$$n_s = \frac{F_s}{F}, \ (s = x, y, z)$$ (5.3)

Position of effective mesh point $R(r_x, r_y, r_z)$ in gear mesh coordinate system is calculated as:

$$r_x = \frac{(M_x + F_x r_y)}{F_y}$$ (5.4)

$$r_y = \frac{(M_x + F_y r_z)}{F_z}$$ (5.5)

$$r_z = \frac{(M_y + F_z r_x)}{F_z}$$ (5.6)

where $M_{x/y/z}$ is the moment caused by combined load vectors of each contact cell in contact area about coordinate axis.

Transmission error (TE) is the internal excitation of gear mesh which is the deviation of gear body rotational motion from ideal position. Translational TE is calculated by projecting loaded and unloaded angular transmission error along LOA. Then, the gear mesh stiffness can be derived as:
\[ k_m = \frac{F}{e_L - e_U} \]  

(5.7)

The above time-varying mesh parameters generated from loaded tooth contact analysis represent the coupling between engaging gear pair and will be used in coupled multi-body dynamic model of geared rotor system.

![Illustration of loaded tooth contact analysis](image)

Figure 5.2 Illustration of loaded tooth contact analysis

### 5.2.2 Multi-body Dynamic Model of Hypoid Gear Shaft-bearing System

The complete dynamic system studies is shown in Figure 5.1 which is a 14-DOF lumped parameter model. The equation of motion can be written in matrix form as:

\[
[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} + [G]\{\ddot{x}\} + [G_a]\{x\} = \{F\} 
\]  

(5.8)

\[
[M] = \text{diag} \left[ I_D, M_p, M_p, I_{px}, I_{py}, M_g, M_g, I_{gx}, I_{gy}, I_{gz}, I_L \right] 
\]  

(5.9)

where \( I_D \) and \( I_L \) stand for mass moment of inertia of engine and load respectively. In addition, \( M_l \) and \( l_{lm} \), \( (l = p, g; m = x, y, z) \) are the body mass and moment of inertia of pinion/gear. Accordingly, the system displacement vector is:

\[
{x} = \{\theta_D, x_p, y_p, z_p, \theta_{px}, \theta_{py}, x_g, y_g, z_g, \theta_{gx}, \theta_{gy}, \theta_{gz}\}^T 
\]  

(5.10)
The torsional coordinates of pinion/gear body, engine and load are large rotational displacement (Peng, 2010). Notice that the coordinate system of the complete geared rotor system is different from that of tooth contact analysis illustrated in Figure 5.1. Therefore, coordinate transition is needed when applying mesh parameters into system dynamic analysis.

The stiffness matrix consists of input ($k_D$) and output ($k_L$) shaft torsional stiffness and lumped shaft-bearing support stiffness on pinion ($[K_p]$) or gear ($[K_g]$) shaft. The system damping matrix $[C]$ is derived from component support damping model. The system stiffness matrix can be expressed as:

$$
[K] = \begin{bmatrix}
    k_D & [K_p] \\
    [K_p] & k_L \\
\end{bmatrix}
$$

(5.11)

The force vector consists of internal and external excitations which can be defined as:

$$
\{F\} = \{T_D, h_p \cdot F_m, -h_g \cdot F_m, -T_L\}^T + \{F_{ex}\}
$$

(5.12)

where $T_D$ and $T_L$ are torque at driving and load end. Coordinate transition vector $h_l (l = p, g)$ is used for projecting mesh force onto pinion or gear local coordinate system. It consists of normal vector and rotational radii of effective mesh point:

$$
h_l = \{n_{ix}, n_{iy}, n_{iz}, \lambda_{ix}, \lambda_{iy}, \lambda_{iz}\}
$$

(5.13)

The mesh force $F_m$ acts at effective mesh point along effective LOA and is derived from dynamic transmission error ($\delta$) and static transmission error ($e_U$):

$$
F_m = k_m \cdot f(\delta - e_U) - c_m \cdot (\dot{\delta} - \dot{e}_U)
$$

(5.14)
where $k_m$ is the mesh stiffness calculated from loaded tooth contact analysis and $c_m$ is the mesh damping ratio. Dynamic transmission error $\delta$ is expressed as:

$$\delta = h_p^T \cdot \{x_p, y_p, z_p, \theta_{px}, \theta_{py}, \theta_{pz}\} - h_g^T \cdot \{x_g, y_g, z_g, \theta_{gx}, \theta_{gy}, \theta_{gz}\}.$$  

(5.15)

where $f(\delta - e_c)$ is the clearance type nonlinear function comprised of three stages depending on the relationship between $\delta - e_U$ and backlash $b_c$:

$$f(\delta - e) = \begin{cases} 
\delta - e_U - b_c, & \delta - e_U \geq b_c, \\
0, & -b_c < \delta - e_U < b_c, \\
\delta - e_U + b_c, & \delta - e_U \leq -b_c,
\end{cases} \quad (5.16)$$

### 5.2.3 Nonlinear Time-varying Bearing Stiffness Model

The clearance type non-linear rolling element bearing model is illustrated in Figure 5.3 where $\{F_{bim}, F_{bym}, F_{bzm}, M_{xm}, M_{ym}\}$ is the load acting on the bearing assuming it can rotate freely around z-axis of the local coordinate system. Accordingly, $\{\delta_{xm}, \delta_{ym}, \delta_{zm}, \beta_{xm}, \beta_{ym}\}$ is the translational and rotational displacement of bearing. Based on previous studies (Liew and Lim, 2005), bearing stiffness matrix can be generated using partial differential methods assuming rigid raceways:

$$[K_b] = \begin{bmatrix} 
\frac{\partial F_{bim}}{\partial \delta_{jm}} & \frac{\partial F_{bim}}{\partial \beta_{jm}} \\
\frac{\partial F_{bym}}{\partial \delta_{jm}} & \frac{\partial F_{bym}}{\partial \beta_{jm}} \\
\frac{\partial F_{bzm}}{\partial \delta_{jm}} & \frac{\partial F_{bzm}}{\partial \beta_{jm}} \\
\frac{\partial M_{bim}}{\partial \delta_{jm}} & \frac{\partial M_{bim}}{\partial \beta_{jm}} \\
\frac{\partial M_{bym}}{\partial \delta_{jm}} & \frac{\partial M_{bym}}{\partial \beta_{jm}} \\
\frac{\partial M_{bzm}}{\partial \delta_{jm}} & \frac{\partial M_{bzm}}{\partial \beta_{jm}} 
\end{bmatrix}.$$  

(5.17)
Bearing stiffness variation is caused by changing orbital position of rolling elements in raceway during operation. Assuming that interaction between rolling element and raceway is pure motion, the position of rolling elements in raceway can be written as:

\[
\psi_s(t) = \frac{1}{2} \left( 1 - \frac{r_b}{r_d} \cos(\alpha_0) \right) \Omega_z t + \frac{2\pi(s-1)}{Z}, s = 1,2,...Z,
\]

where \( r_b \) is element radius, \( r_d \) is the pitch radius of raceway, \( \alpha_0 \) is unloaded contact angle and \( \Omega_z \) is the shaft speed.

Radial clearance type nonlinearity is considered in bearing stiffness model (Figure 5.3) which can be described using non-linear displacement functions \( f_{bq}(x_{iq}) \) where \( l = p, g \); \( q = x, z \). Therefore, the equation of motion in radial translational direction can be re-written as:

\[
M_{iq} \ddot{x}_{iq} + c_{iq} \dot{x}_{iq} + k_{iq} f_{bq}(x_{iq}) = n_{iq} F_m, \quad l = p, g; \quad q = x, z
\]

The detailed expression of non-linear function based on quasi-static contact analysis is (Kahraman and Singh, 1991):

\[
f_{bx}(x_{iq}) = \left\{ \begin{array}{ll}
\sum_{j} \left( (x_{iq} \cos \psi_j - \gamma L) \right)^n \cdot \cos \psi_j, & x_{iq} > \gamma L \\
0, & -\gamma L < x_{iq} < \gamma L \\
-\sum_{j} \left( |x_{iq}| \cos \psi_j - \gamma L \right)^n \cdot \cos \psi_j, & x_{iq} < -\gamma L \\
\end{array} \right.
\]

where \( \gamma L \) is bearing radial clearance, \( \psi_j \) is the angular position of jth rolling element, \( Z \) is the total number of rolling element under load conditions, \( n \) is the power coefficient defined by Hertzian contact: for ball baring \( n = 1.5 \), for roller bearing \( n = 10/9 \). According to previous studies, the bearing nonlinear displacement function can be approximated by piecewise linear function similar to mesh non-linear function shown in Equation 5.16.
5.2.4 Quasi-static Shaft-bearing Model

The configuration of combined shaft-bearing model used is illustrated in Figure 5.4. The gear/bearing geometry data and system parameters are listed in Table 5.1. Shaft finite element model is built from static beam element and is combined with bearing stiffness model to synthesize effective lumped support stiffness matrix relative to lumped point. Hypoid gear pair which is modeled as rigid body is mounted at the reference point. For computational simplicity and convenience, it is assumed that the bearing stiffness is not influenced by operating loads. Therefore, the load displacement relation of the combined shaft-bearing system can be written in matrix form as:
\[
\begin{bmatrix}
\Delta_R \\
\Delta_S
\end{bmatrix}_{5 \times n} = \begin{bmatrix} K_{RR} & K_{RS} \\ K_{SR} & K_{SS}
\end{bmatrix}^{-1} \begin{bmatrix} F_R \\
0
\end{bmatrix}_{5 \times n}
\]

The 5 by 5 matrix \([\Delta_R]\) stands for the displacement at reference point in five DOFs except the torsional displacement around rotation axis. The 5 DOF matrix \([\Delta_S]\) stands for the displacement at other nodes. The external force expressed as \([F_R]\) is acting on reference point. For calculation simplicity, the force on reference point is set to be unit load. Therefore, the equivalent time-varying lumped stiffness matrix can be written as:

\[
K_{RR} = [\Delta_R]^{-1}
\]

(5.22)

The stiffness is calculated at each mesh frequency and specific roller orbit position. An example of x-direction translational stiffness of lumped support on pinion is illustrated in Figure 5.5. The stiffness value in Figure 5.5 (a) is dependent on different position status of rolling elements as shown in Figure 5.5 (b).

Figure 5.4 Configuration of pinion shaft-bearing model
5.2.5 Corresponding Linear Time-invariant Model

To investigate the relationship between vibration mode and non-linear effect on dynamic response, linear geared rotor system is used by ignoring gear backlash and bearing radial clearance. Based on modal superposition method and by solving eigenvalue problem, the dynamic compliance matrix (Cheng, 2000) can be expressed as:

\[
[H(\omega)] = \sum_{r=2}^{14} \frac{\Phi_r \Phi_r^T}{(\omega_r^2 - \omega^2 + 2i\zeta_r \omega \omega_r)} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad i = \sqrt{-1} \tag{5.23}
\]

where \(\omega_r\) is natural frequency, \(\Phi_r\) is the corresponding frequency mode and \(\zeta_r\) is the modal damping ratio expressed as:

\[
\zeta_r = \Phi_r^T [C_r] \Phi_r / 2\omega_r \tag{5.24}
\]

The linear dynamic mesh force along effective LOA can be written as:
\[ F_o(t) = k_m(\delta_d(t) + \epsilon_L(t)) + c_m(\dot{\delta}_d(t) + \dot{\epsilon}_L(t)) \] (5.25)

5.3. Comparative Analysis and Discussion

In current study, 80 Nm is applied at driving end of the model. Detailed system settings including gear pair, bearing geometry and other dynamic system parameters are listed in Table 5.1. In order to analyze the nonlinear coupling between different components and its effect on dynamic response, the following study is divided into two steps: First, time-invariant model is analyzed to compare the coupling effect between gear mesh and bearing support under different stiffness value. Then, nonlinear time-varying gear and bearing model are incorporated in shaft-bearing model for evaluating effect of bearing clearance on actual bearing load during speed sweep.

Table 5.1 Design parameters: (a) Gear geometry, (b) Dynamic system data, (c) Bearing data

<table>
<thead>
<tr>
<th>Gear data</th>
<th>Pinion</th>
<th>Gear</th>
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<tbody>
<tr>
<td>Number of teeth</td>
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<td>43</td>
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<tr>
<td>Spiral angle (rad)</td>
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<td>Pitch angle (rad)</td>
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<td>Face width (m)</td>
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<td>Type</td>
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<td>Loaded side</td>
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(a) Gear geometry

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<th>System parameters</th>
<th>Engine/Driver</th>
<th>Load</th>
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<td>Properties</td>
<td>Engine/Driver</td>
<td>Load</td>
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<tr>
<td>Torsional moment of inertia (kg-m2)</td>
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<td>Mass (kg)</td>
<td>Pinion</td>
<td>Gear</td>
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<td></td>
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<tr>
<td>Torsional moment of inertia (kg-m²)</td>
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<td>Bending moment of inertia (kg-m²)</td>
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<td>3.8E8/6.8E8</td>
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<td>Torsional stiffness of shaft (Nm/rad)</td>
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(b) Dynamic system data

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<tr>
<th>Bearing parameters</th>
<th>Bearing#1&amp;2</th>
<th>Bearing#3&amp;4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Rollers</td>
<td>20</td>
<td>21</td>
</tr>
<tr>
<td>Pitch Radius (m)</td>
<td>0.049</td>
<td>0.061</td>
</tr>
<tr>
<td>Contact Angle (rad)</td>
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<td>0.262</td>
</tr>
<tr>
<td>Axial Displacement (m)</td>
<td>3.5E-6/5.5E-6</td>
<td>5E-6/1.5E-5</td>
</tr>
<tr>
<td>Length of Roller (m)</td>
<td>0.023</td>
<td>0.027</td>
</tr>
</tbody>
</table>

(c) Bearing data

5.3.1 Nonlinear Time-invariant Dynamic Analysis

The modal frequency and mode shapes of 14-DOF linear time-invariant model are shown in Figure 5.6 with bearing stiffness set to be 3.8E8 N/m. The 5th and 11th (Figure 5.6(b)) mode are pure transverse modes in either x or z direction. Previous studies (Wang, 2007) indicated that these modes will not show in gear dynamic response because they’re not out-of-phase mode. These transverse modes are mostly affected by bearing clearance. For the modes next to the pure transverse mode (Figure 5.6(c)), coupling effect can be seen where both transverse and out-of-phase modes exist. It can be seen that in this bearing stiffness case, natural frequencies of transverse mode and torsional mode are away from each other.
The dynamic responses under different nonlinear conditions are compared in Figure 5.7. Results show that bearing clearance nonlinearity mostly affects response peaks in frequency range lower than 1000 Hz, whereas the effect from gear backlash nonlinearity dominates those peaks in higher frequency range. Jump discontinuities that occur at 4\textsuperscript{th} and 5\textsuperscript{th} peak is due to gear backlash nonlinearity. Accordingly, bearing clearance causes the discontinuities at the 2\textsuperscript{nd} and 3\textsuperscript{rd} peak
which can also be seen in lumped bearing force results illustrated in Figure 5.8(a) and Figure 5.8(b). Among those peaks of gear dynamic response, only 4\textsuperscript{th} peak (12\textsuperscript{th} mode) is subject to both bearing clearance and gear backlash. Therefore, natural modes of this case are weakly coupled and the interactions between non-linearities of different components are limited. Non-linearity effects from either bearing or gear mesh can be analyzed separately without losing much accuracy.

Figure 5.7 Gear dynamic response of weak coupling case: \(--\), linear response; \(-\), no gear backlash, non-linear bearings; +, with gear backlash, non-linear bearings.
Corresponding force at pinion lumped support: , linear response; , no gear backlash, non-linear bearings; +, with gear backlash, non-linear bearings.

Figure 5.9 shows the average bearing contact percentage in one shaft rotation cycle. It can be seen that for current weak coupling case, when bearing clearance is added, bearing still remains in full contact in a wide speed range. Partial contact mainly exits in two frequency ranges of resonance peaks shown in Figure 5.7 where hardening effect can be observed when partial contact turns into full contact at 1400 Hz. When gear backlash is also introduced into the system, a third bearing partial contact speed range from 2480 Hz to 2720 Hz is generated that causes softening effect and jump discontinuity at 2500 Hz in Figure 5.7. Changes in both radial and axial lumped bearing force also reflect the effect of gear backlash nonlinearity as is shown in Figure 5.8. The analysis implies that lateral vibration in either pinion or gear body is large and can overcome the bearing clearance in a wide speed range. Therefore, the influence from bearing nonlinearity on gear dynamic response along LOA is limited for current system.
Figure 5.9 Bearing contact percentage within one shaft rotation cycle: , no gear backlash, non-linear bearings; , with gear backlash, non-linear bearings.

Poincare maps of two nonlinear cases are plotted in Figure 5.10 where y axis is the radial velocity at mesh point. For Figure 5.10(a), only the gear backlash nonlinearity is considered, with no bearing clearance. It is observed that under the effect of backlash, the period-one solution turns into chaotic at 1250 Hz. Then, chaotic motion changes into period-two solution at 1400 Hz and becomes period one again at 1550 Hz. Similar changes can be seen in higher speed range. Figure 5.10(b) shows the case when both gear and bearing nonlinearity exists. Comparison between the two cases indicates that introducing bearing clearance only slightly expands the speed range of chaotic motion and no significant changes are found in the range above 2500 Hz. This demonstrates that for 3.8E8 N/m bearing stiffness case, nonlinear behavior in response is mainly caused by gear backlash that introduces softening into the dynamic response. In Figure 5.11 are
shown comparative results of FFT analysis based on time domain response with first harmonic excitation. From Figure 5.11(a), it is seen that system response at 1370 Hz with bearing clearance considered has peaks at $\omega$ and $\omega/2$ where $\omega$ is the fundamental excitation frequency. When gear backlash is added, side bands can be observed which indicates quasi-periodic motion in response. In addition, the response amplitude at same mesh frequency is much lower compared to bearing nonlinearity case because frequency of peak is reduced due to softening effect. Results in Figure 5.11(b) further validate that nonlinear behavior in high mesh frequency range is mainly generated from gear backlash.

Figure 5.10 Poincare map of radial velocity of weak coupling case: (a) with gear backlash; (b) with both gear backlash and bearing clearance
Figure 5.11 Corresponding FFT spectra of dynamic mesh force; (a) 1370 Hz, (b) 2630 Hz with:

- - - - - , linear response; - - - - , no gear backlash, non-linear bearings; - - - - , with gear backlash, non-linear bearings.

Next, bearing stiffness value is set to 6.8E8 N/m and natural frequencies as well as mode shapes are listed in Figure 5.12. It can be seen that natural frequencies of transverse mode (8th mode) and out-of-phase mode (7th mode) are closer to each other. In addition, mode shapes such as 12th mode are more dependent on transverse mode in radial direction. This indicates that nonlinearity in either bearing or gear mesh will have more significant influence on the dynamic response of combined shaft-bearing system and strong coupling between bearing clearance and gear backlash is expected in nonlinear analysis. Therefore, in this case, all nonlinearities should be considered simultaneously.
In Figure 5.13 are shown gear dynamic responses under different nonlinear conditions. When bearing clearance is incorporated alone, response in a wider mesh frequency range is affected compared to previous weak coupling case. Jump discontinuity is found in three modes in frequency range from 500 Hz to 1500 Hz accompanied by peak shifting to the left. In addition, a 4\textsuperscript{th} peak is generated in that region and most discontinuity caused by bearing clearance demonstrate hardening effect. When gear backlash is included along with bearing clearance, 10\textsuperscript{th} and 12\textsuperscript{th} modes are further affected and show softening effect with peaks bend to the left. Similar effect can be seen on lumped bearing forces shown in Figure 5.14(a) and 5.14(b).
Figure 5.13 Gear dynamic response of strong coupling case: - - - - - , linear response; -- , no gear backlash, non-linear bearings; +, with gear backlash, non-linear bearings.

Figure 5.14 Corresponding force at pinion lumped support: - - - - - , linear response; -- , no gear backlash, non-linear bearings; +, with gear backlash, non-linear bearings.
Bearing contact percentage during speed sweep up is plotted in Figure 5.15. With higher bearing stiffness applied, partial contact of bearings covers a wider speed range and contact percentage within one shaft rotation cycle drops significantly at resonance frequency range. Comparison with previous contact percentage plot implies that influence of bearing clearance on system dynamic response depend on bearing stiffness value. Resonance peak shift and jump discontinuity in partial contact frequency range in Figure 5.13 reflects the bearing clearance effect on dynamic response. Figure 5.15 also indicates that the new peak generated by introduction of bearing clearance is due to a sudden rise in bearing contact percentage at 1160 Hz which causes hardening in the response. Similarly, jump discontinuity at 12th mode occurs when partial contact changes to full contact in speed range from 1250 Hz to 1370 Hz. When gear mesh nonlinearity is also included, bearing contact percentage is further reduced in partial contact region while the speed range of major partial contact is not changed. The analysis above shows that bearing loss of contact as well as natural frequency and mode shapes can significantly affect the coupling between gear and bearing nonlinearities.
Figure 5.15 Bearing contact percentage within one shaft rotation cycle: , no gear backlash, non-linear bearings; , with gear backlash, non-linear bearings.

Poincare map of two nonlinear cases are plotted and compared in Figure 5.16. For nonlinear system with linear bearing model (Figure 5.16(a)), it can be seen that period-one solution first changes into quasi-periodic at 860 Hz, then goes through period-two and period-one. The second routes to chaos starts at 1430 Hz where period-one solution becomes chaotic instantly and unstable response covers a wider frequency range. This implies that the response peak of 12\textsuperscript{th} mode is more subject to gear backlash than that of 10\textsuperscript{th} mode. The chaotic solution is significantly expanded when bearing clearance is also included along with gear backlash as is demonstrated in Figure 5.16(b). This supports the above claim of strong coupling existing between gear and bearing nonlinearities in the speed range from 830 Hz to 1670 Hz. Bifurcation at 2510 Hz in both Figure 5.16(a) and 5.16(b) shows that the peak at 2600 Hz in Figure 5.13 is more dependent on torsional mode and the jump discontinuity is mostly caused by nonlinearity in gear mesh. Fourier spectra of
two major response peaks are compared in Figure 5.17 under different nonlinear conditions. As is illustrated in Figure 5.17(a), when bearing clearance is introduced alone, corresponding spectrum has both super and sub-harmonic response peaks at $\omega/2$ and $3\omega$ respectively, where $\omega$ is the fundamental frequency. When gear backlash is also considered, a series of subharmonic peaks are excited that indicate nonlinear interaction between system components. For responses at 1460 Hz shown in Figure 5.17(b), the combined effect of both bearing and gear nonlinearity generates both super and sub-harmonic side bands which consists of more complicated response components than the response at 860 Hz under same condition. This observation further validates the claim that nonlinear coupling has more significant effect on the resonance peak of the 12th mode.

Figure 5.16 Poincare map of radial velocity of strong coupling case: (a) with gear backlash; (b) with both gear backlash and bearing clearance
In order to further evaluate response sensitivity to bearing clearance, parametric study is performed assuming no gear backlash and the results are shown in Figure 5.18. It can be observed that with increased clearance, resonance peak frequencies shift to lower value due to softening. Furthermore, the amplitude and location of the jump generated by bearing nonlinearity is most sensitive to clearance value compared with other discontinuity jumps. This jump occurs around the 11th mode shown in Figure 5.12(b) which is a pure transverse vibration mode. This implies that this response peak between 10th and 12th mode is dominated by transverse vibration of the system and bearing contact loss has significant influence on its amplitude and frequency location.
Figure 5.18 Effect of bearing clearance on gear dynamic response (no backlash): , linear response; , 2E-6 m bearing clearance; , 6E-6 m bearing clearance; , 1E-5 m bearing clearance.

5.3.2 Nonlinear Time-varying Dynamic Analysis

The study above is based on time-invariant model for both gear mesh and bearing to identify cause and effect of the nonlinear coupling and jump discontinuity of the dynamic response. In this part, gear backlash and bearing clearance are applied in a practical hypoid geared rotor system to study nonlinear interaction in coupled multi-body dynamic model. Parameters of hypoid gear and bearing geometry are listed in Table 5.1. Configuration of pinion and gear shafts are illustrated in Figure 5.19. Detailed analysis of time-varying coupled multibody dynamic model of hypoid geared-rotor system can be found in reference (Yang, 2012). The purpose here is to analyze changes in actual bearing forces under the influence of multiple nonlinearities with different bearing stiffness values.
Dynamic mesh forces of first bearing stiffness case are compared in Figure 5.20. In this case, the axial bearing preload is set to be 3.5E-6 mm for bearing 1 and 2; 5E-6 mm for bearing 3 and 4. Except the observable peak shift due to nonlinear interaction between gear and bearing, the amplitude changes of the response along LOA is limited especially on the resonance peaks. The Fourier spectrum analysis is shown in Figure 5.21. When gear backlash is introduced alone, period-two and quasi-periodic solutions are seen at 1520 Hz and 2060 Hz respectively implying that gear non-linearity has more effect on the last peak compared to the second last one within the speed range of interest. When bearing clearance is also included in the model, its interaction with gear mesh non-linearity generates more complicated response components on both peaks and such coupling effect seems to be a major cause of peak shift in Figure 5.20.
Figure 5.20 Dynamic response of full shaft-bearing model (low bearing stiffness): ——, with gear backlash and linear bearing; ——, with gear backlash and bearing clearance.

Figure 5.21 Corresponding FFT spectra of dynamic mesh force (low bearing stiffness case); (a) 1520 Hz, (b) 2060 Hz with: ——, with gear backlash, linear bearing stiffness; ——, with gear backlash, non-linear bearing stiffness.
On the other hand, the dynamic forces on bearings are more sensitive to bearing clearance especially for the bearings closer to meshing gear pair as shown in Figure 5.22. With the introduction of bearing clearance in the system, radial reaction force on bearing 1 increases accompanied by frequent jump discontinuities which demonstrates similar trend reported in reference (Kahraman and Singh, 1991). This indicates that bearings closer to meshing gear pair such as bearing 1 and 3 in Figure 5.19 are dictated by lateral vibration and therefore more subject to bearing clearance in system whereas the changes in force amplitude on other bearings are small.

Figure 5.22 Actual dynamic load on pinion bearings with: ----------- , linear response; ---- , with gear backlash, linear bearing stiffness; ------ , with gear backlash, non-linear bearing stiffness.

Then consider the same system with higher bearing stiffness by increasing bearing axial pre-load according to Table 5.1. As is seen from Figure 5.23, an increase in bearing support
stiffness can enhance the non-linear coupling in current case. The introduction of bearing clearance causes strong coupling with existing gear backlash nonlinearity which not only changes frequency of resonance peak, but also suppress peak amplitude significantly. It is also shown that gear dynamic response is mostly subject to nonlinearity in the speed range from 650 Hz to 2000 Hz which can be further proved by FFT results in Figure 5.24. Fourier spectrum analysis shows similar response at 2060 Hz for cases with and without bearing clearance. The response at 1520 Hz is dominated by harmonic response under the effect of gear backlash alone and strong nonlinearities are introduced when bearing clearance also exists. This implies that for mesh frequency beyond 2000 Hz, the dynamic mesh force is mostly dictated by gear nonlinearity while the effect from bearing clearance is insignificant.

Figure 5.23 Dynamic response of full shaft-bearing model (high bearing stiffness): , with gear backlash and linear bearing ; , with gear backlash and bearing clearance
Figure 5.24 Corresponding FFT spectra of dynamic mesh force (high bearing stiffness case); (a) 1520 Hz, (b) 2060 Hz with: , with gear backlash, linear bearing stiffness; , with gear backlash, non-linear bearing stiffness.

The reaction force on pinion bearing 1 is profoundly subject to bearing clearance in radial directions as shown in Figure 5.25. It is observed that the influence of bearing non-linearity covers a wider speed range compared to previous case which indicates a stronger coupling in the frequency range from 650 Hz to 2000 Hz. From the analysis on both gear dynamic response and bearing reaction force, it can be concluded that for current time-varying coupled multi-body dynamic model of hypoid shaft-bearing system, an increase in bearing stiffness can enhance the modal coupling and non-linear interaction between different components and will have more significant effect on dynamic response.
Figure 5.25 Actual dynamic load on pinion bearings with: ---------, linear response; ---, with gear backlash, linear bearing stiffness; - - - - , with gear backlash, non-linear bearing stiffness.

5.4. Conclusions

Non-linear coupling between hypoid gear and bearing is extensively studied by including both gear backlash and bearing clearance into coupled multi-body dynamic system. Parametric study on bearing stiffness reveals that the coupling between nonlinearities can be either weak or strong which can decide whether or not gear dynamic response is sensitive to bearing clearance. Those key factors are listed below:

1) Natural frequencies of transverse and torsional mode of geared-rotor system

2) Contribution of transverse mode in out-of-phase mode shape

3) Average bearing contact percentage within one shaft rotation cycle
The non-linear coupling can be strong if: a) transverse mode frequency is close to that of torsional mode, b) transverse mode has significant contribution to gear out-of-phase mode, c) bearing partial contact occurs frequently during speed sweep. In this case, both gear dynamic mesh force and actual bearing force can be sensitive to either gear backlash or bearing clearance and such nonlinearity will cause jump discontinuity in response. In most cases, gear nonlinearity will introduce softening due to loss of tooth contact whereas bearing nonlinearity can cause either softening or hardening depending on fluctuation of contact percentage in a certain speed range. The analysis on time-varying full shaft-bearing model demonstrates that increased bearing stiffness will enhance the nonlinear coupling and the introduction of bearing clearance tend to reduce the response amplitude and affect resonance frequency of dynamic mesh force. In addition, dynamic forces on bearings that are closer to meshing gear pair are found to be more subject to bearing nonlinearity especially in radial directions. Results in this study are based on speed sweep up analysis, while the frequency sweep down analysis also shows similar trends which is not shown here for the sake brevity. Based on the study above, part of the future work will include parametric study on input torque condition by incorporating torque fluctuation at driving end. Time-varying load dependent model developed previously will also be applied for a more accurate prediction of the dynamic performance of system.
Chapter 6. Conclusions and Expected Future Studies

6.1 Conclusions

The primary goal of this dissertation is to perform a more in-depth study of the nonlinear dynamics of hypoid geared rotor system applying a lumped parameter model that includes new gear mesh functions and nonlinear factors from other driveline structures that may have significant influence on gear dynamics response. Using this proposed model, the nonlinear coupling effect is the focus in evaluating gear dynamics. The proposed analytical methods aim to help transmission product design with good dynamic performance.

Main research achievements to date are listed below:

a) A time-varying torque load dependent hypoid gear mesh model that is based on three-dimensional parametric surface interpolation is introduced to demonstrate the effect of time-varying external torque on gear mesh. The mesh parameters emphasize instantaneous torque load value change within one pinion shaft rotation cycle which is expected to yield a more accurate hypoid gear dynamic response prediction when external torque fluctuation exists. Comparison study further demonstrate the effect of drive torque on dynamic response of gear pair.

b) A new approach to model gear mesh damping is proposed which introduces two different types of impact damping functions including non-viscous and viscous type. Both functions can predict a damping force that satisfies boundary conditions of a pair of gear teeth in contact. Instead of relying on empirical experience to decide damping ratio that is difficult to measure, the parameters such as coefficient of restitution in damping functions are
evaluated systematically. These parameters in the damping functions can be easily evaluated by measuring rotation speed and rotational radius of meshing gear pair.

c) An enhanced multi-body dynamic model of geared rotor system is developed, which incorporates propeller shaft misalignment and gear mass unbalance. The effect can be described as additional bending moment from driving end, pinion rotation speed and torque fluctuation. The modeling methodology is expected to help understand the sensitivity of nonlinear gear dynamics and actual bearing forces to commonly occurring external excitation factors.

d) Analytical modeling of a hypoid geared rotor system is formulated which incorporates both gear backlash and bearing radial clearance to fill the gap of lacking analytical study in multi-clearance hypoid geared rotor bearing system. Nonlinear and time-varying dynamic interactions of gear-shaft-bearing systems is analyzed. It is found that changing supporting bearing stiffness can significantly affect the coupling between gear mesh and bearing nonlinearities.

### 6.2 Expected Future Studies

a) Conduct experimental testing and correlate the results with simulation predictions based on proposed time-varying load dependent mesh model under fluctuating driving torque.

b) Incorporate multiple misaligned shafts in driveline connected by universal joints and analyze the effect of friction in connections on dynamic response.

c) Perform extensive parametric analysis on a hypoid geared-rotor system with multiple clearances under different external load conditions and evaluate system sensitivity towards external excitations.
BIBLIOGRAPHY


Fan, Q. and Lowell W., (2005), “New developments in tooth contact analysis (TCA) and loaded TCA for spiral bevel and hypoid gear drives,” AGMA.


