I, Heather May, hereby submit this original work as part of the requirements for the degree of Master of Science in Mathematical Sciences.

It is entitled:
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Student's name: Heather May

This work and its defense approved by:

Committee chair: Herbert Halpern, Ph.D.
Committee member: Donald French, Ph.D.
Committee member: Benjamin Vaughan, Ph.D.
Wavelet-Based Image Processing

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Heather May

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Committee Chair: Herbert P. Halpern, Ph.D.
Abstract: This thesis will guide you through the uses of wavelets in image processing. A review of Fourier transforms, $z$ Transforms, multiresolution analysis, and subband coding. A main advantage of discrete wavelet transform (DWT) is the ability to decompose images in the space/time-frequency domain. The diverse applications for wavelets can be seen in biometrics, medical imaging, statistical analysis, and even current technology for images such as the JPEG 2000.
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1. Introduction

We live in a world of signals that have a physical meaning or convey attributes of a system. Every living organism relies on signals to survive. Digital signal processing (DSP) is a method of transforming signals into data that a human can understand. Digital data is discrete and may be represented as sequences of numbers or symbols. Signal processing operates on a signal to extract the approximations and details that can be analyzed to produce useful information. A signal may vary in spatial form or with time. So, in other words, signals are either discrete or continuous functions of independent variables.

Digital signals or discrete signals are widely used and continue to appear in homes through electronic devices, the Internet, and mobile communication. A few things that have affected us as a society in the past decade was the conversion of local television stations and mobile phones from analog signals to digital signals. The analog signals also known as continuous time and continuous amplitude signals are converted to digital signals to be processed by computers among other digital devices. Digital systems are advantageous due to the capability of processing multiple types of operations whereas analog signals need a system for each type of operation. Digital signals rely on numerical calculations that always produce the same result.

Signals can be categorized into dimensions. A one dimensional signal is typically modeled as a function of time, i.e. $f(t)$ or $x(t)$, can represent the electrical signal in telecommunications, the daily maximum temperature or annual rainfall at
a specific location. Photographic images are represented as two dimensional signals with horizontal and vertical components.

Wavelets are useful for analyzing the local irregular behavior of functions that are not smooth, such as the spike in temperature for a given location. A wavelet is an oscillatory function that either vanishes outside the boundary or decays rapidly to zero. Wavelets are waves that last for a finite period of time or are not always prevalent as time persists. Recently SIAM announced that wavelets were used to develop an application to detect potholes on city streets. One main advantage of discrete wavelet transform (DWT) is the ability to decompose images in the space-frequency domain. This is a major advantage to industrial standards for still images (e.g., .JPEG) and motion pictures (e.g., .MPEG) which are based on the discrete cosine transform (DCT) that only provide frequency decomposition.

There are multiple sub-fields in DSP such as audio processing, biomedical signal processing, communication signal processing, sonar signal processing, statistical signal processing and video processing. The diverse applications are left to your imagination as this thesis will delve into image processing in the finite spatial domain through a combination of wavelets and other transforms.
2. Digital Signal Processing

An analog signal can either be a continuous time or a discrete time function. An analog signal is discrete when the input is discretized but the output is still an analog signal. Digital signals occur when both the independent and dependent variables are discretized. Consider integer values for the input and allocate the values to an analog/digital converter which quantizes the input. Thereafter convert this output into binary for transmission. This binary output is considered the digital signal. However, realize that binary is not the only option for the output.

Processing occurs to change the signal into a more desirable form. For example, a telephone conversation is transmitted by digital signals but is then processed back into analog form, speech, so the two people on the line can understand one another. To understand DSP a knowledge of discrete time signals, the sampling process, discrete time systems, and linear time invariant systems are needed.

Most images are recorded and processed in the time domain or spatial domain. The spatial domain refers to the collection of pixels composing an image. Thus spatial domain processing involves mathematical operations applied directly on these pixels. However, it is at times convenient and even efficient to process images in the frequency domain. High-frequency components generally depict edge pixels and low-frequency components depict the interior pixels of an object. Sometimes there is a need to utilize both the frequency and the time domain. In digital image processing the intensity values are finite and discrete which leads to the next section on discrete time signals.
2.1. Discrete Time Signals. A sequence of numbers represents the discrete
time signal. For example,

\[(2.1) \quad \{x[n]\}_{n \in \mathbb{Z}}\]
corresponds to the amplitude of the signal at specific instances, \(n\), over the open
unit intervals, \((n, n+1)\). Some of the significant discrete time signals are shown
below.

![Figure 1. Discrete Time Signals](image-url)
Start with the fundamental unit impulse function:

\[ \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \quad (2.2) \]

Translation of the unit impulse function gives the delayed impulse function:

\[ \delta[n - k] = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases} \quad (2.3) \]

Another imperative impulse function is the unit step function:

\[ u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad (2.4) \]

Transform the unit step function into a sum of the discrete unit impulse or the delayed unit impulse functions since

\[ \delta[n] = u[n] - u[n - 1] \quad (2.5) \]

then it follows that

\[ \sum_{n=-\infty}^{n} \delta[k] = \sum_{k=0}^{\infty} \delta[n - k] = 1. \quad (2.6) \]

This sum of translates of the unit impulse function indicates the notation for the unit ramp function:

\[ r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad (2.7) \]

as the sum of the delayed impulse function multiplied by the amplitude of the signal. Hence,

\[ r[n] = \sum_{k=0}^{\infty} k \delta[n - k] \quad (2.8) \]
the unit ramp function implies another way to express discrete time signals. This leads to the sifting property that defines any discrete time signal as

\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]. \]

Therefore, the sifting property produces the value of the function at the impulse location.

2.2. Discrete Time Systems. Many discrete time systems stem from continuous time systems. Discrete time signals are typically samples of continuous time signals that are transformed through an analog/digital converter.

2.2.1. Sampling Process. Let \( x(t) \) be a continuous time signal. If one samples the continuous function every \( T \) seconds to measure the value of the output. Then \( T \) is the sampling interval. The discrete time signal is defined as

\[ x[n] = x(nT) \]

for integer values \( n \). Therefore the sampling frequency or sampling rate is \( f_s = \frac{1}{T} \).

Some common types of sampling for discrete time signals are up-sampling and down-sampling denoted \( \uparrow m \) and \( \downarrow m \), respectively. For example, if one down-samples by 2, \( \downarrow 2 \), this takes the odd samples and converts them to zeros and re-indexes the nonzero entries. This down-sampler outputs every other input at half the rate of the input. Likewise, if one up-samples by 3, \( \uparrow 3 \), this re-indexes the input by placing \( 3 - 1 = 2 \) zeros between them. The up-sampler is an invertible operation but during down-sampling one looses the information.

2.2.2. Filters. A filter is a mathematical operation that maps an input sequence to an output sequence. Filters tends to boost or attenuate specific frequency ranges. A few types of filters are lowpass, highpass, and bandpass filters. Lowpass filters
reduce high frequencies while boosting low frequencies. Highpass filters attenuate low frequencies permitting high frequencies through. Whereas, bandpass filters allow a band of frequencies through that lie between two specific frequencies. The process of filtering in the spatial domain is obtained through convolution. Some important properties of discrete time systems are linearity, causality, and time invariance.

2.2.3. Linearity. In general, a filter is linear when scaling the input corresponds to scaling the output. Let

\[ y[n] = H(x[n]) \] (2.11)

where \( H(\cdot) \) is the filter. Then a discrete time system in linear if and only if,

\[ H(cx[n]) = cH(x[n]) \] (2.12)

and

\[ H(x_1[n] + x_2[n]) = H(x_1[n]) + H(x_2[n]) \] (2.13)

where the constant \( c \in \mathbb{R} \) and \( x[n], x_1[n], x_2[n] \) are sequences.

2.2.4. Time Invariance. Similarly, recognize a filter as time invariant if translation of the input, \( x[n] \), in the time domain corresponds to translation of the output, \( y[n] \). Hence, a discrete time system is time invariant if and only if,

\[ H(x[n-k]) = y[n-k] \quad \forall k \in \mathbb{Z}. \] (2.14)

Note. The discrete time system could be a function of length and the properties still hold.
2.2.5. \textit{Causality}. Causality occurs when the output of the system at some instant, \( n = l \), does not depend on \( x[l+k] \) for \( k = 1, 2, \ldots \), which is input that occurs after \( x[l] \). Thus, a discrete time system is causal if and only if,

\begin{equation}
H(x_1[l]) = H(x_2[l]), \quad \text{for } x_1[l] = x_2[l], l < l + k.
\end{equation}

In other words, a filter is causal if the current input only depends on itself or previous input. Furthermore, a filter is causal if it does not respond to the signal until the signal is received. Therefore,

\begin{equation}
h[k] = 0 \quad \text{for } k < 0, k \in \mathbb{Z}
\end{equation}

since the response is zero.

2.3. \textit{Linear Time Invariant Systems}. If a filter is both linear and time invariant then we can express it as a convolution operation:

\begin{equation}
y[n] = \sum_{k \in \mathbb{Z}} h[k] x[n-k] = x[n] * h[n].
\end{equation}

One can relate this to the previous section on discrete time signals. As the coefficients, \( h[\cdot] \), are the impulse responses of the system. From equation 2.9 one has

\begin{equation}
x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k],
\end{equation}

and one can rewrite the output as

\begin{equation}
y[n] = H \left( \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \right) = \sum_{k=-\infty}^{\infty} H(\delta[n-k]).
\end{equation}

Recall that \( x[k] \) is a constant so that the linearity of the filter, \( H(\cdot) \), implies

\begin{equation}
y[n] = \sum_{k=-\infty}^{\infty} x[k] H(\delta[n-k]).
\end{equation}
Denote $H(\delta[n - k]) = h_k[n]$ as the impulse response to the system for $n = k$. Since one has not only a linear but also a time invariant system the following

$$H(\delta[n]) = h_0[n] = h[n]$$

holds and equation 2.20 transforms into

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k].$$

Since this is a convolution operation one can rewrite the above equation as

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k] h[k] = h[n] * x[n].$$

Therefore, the linear time invariant system relies on the filtered version of the unit impulse response.

Although many discrete time systems stem from continuous time systems, in image processing, the information is given as a discrete system. It is unnecessary to convert the data from a continuous time system to a discrete time system. However, the concepts introduced in this section, linearity, causality, and time invariance, leads to an understanding of how a signal can be decomposed and reconstructed. The next section will introduce continuous transforms in order to relay the importance of translations, dilations, and convolutions in one and two dimensions.
3. Understanding Continuous Transforms

This chapter introduces some of the similarities between the Fourier transform and the continuous wavelet transform that are often used in image processing for converting an image from spatial domain to frequency domain. This document uses the convention that time domain signals are denoted by lowercase letters and the frequency domain signals by uppercase letters.

3.1. Laplace Transform. Given a function $x(t)$ of the continuous time variable $t$, the Laplace transform, denoted by $X(s)$, is a function of the complex variable $s = \sigma + i\omega$ defined by

$$X(s) = \int_{0}^{\infty} x(t) e^{-st} \, dt \quad \sigma \in \mathbb{R}, \ s \in \mathbb{C}$$

(3.1)

such that

$$\exists \sigma \in \mathbb{R}, \int_{0}^{\infty} x(t) e^{-\sigma st} \, dt < \infty$$

(3.2)

Conversely, one uses the inverse Laplace transform to compute $x(t)$,

$$x(t) = \frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{\sigma - i\omega}^{\sigma + i\omega} X(s) e^{st} \, ds$$

(3.3)

Properties. The following properties hold for Laplace transforms

1. Linearity

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s)$$

for constants $a$ and $b$. 
3. UNDERSTANDING CONTINUOUS TRANSFORMS

(2) Time translation (shifting)

\[ x(t - a)y(t - a) \leftrightarrow X(s)e^{-ias} \forall a > 0 \]

(3) Frequency translation

\[ x(t)e^{at} \leftrightarrow X(s - a) \forall a \in \mathbb{R} \forall a \in \mathbb{C} \]

(4) Time dilation (scaling)

\[ x(at) \leftrightarrow \frac{1}{a}X\left(\frac{s}{a}\right) \forall a \in \mathbb{R} \]

(5) Time differentiation

\[ X^{(n)}(t) \leftrightarrow (s)^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{k-1}(0) \]

(6) Frequency differentiation

\[ (t)^{n}x(t) \leftrightarrow (-1)^{n} \frac{d^{n}X(s)}{ds^{n}} \]

(7) Time Integration

\[ X^{(-1)}(t) = \int_{0}^{t} x(\tau) \ d\tau \leftrightarrow \frac{X(s)}{s} \]

(8) Frequency Integration

\[ \frac{x(t)}{t} \leftrightarrow \int_{0}^{\infty} X(s) \ ds \]

(9) Convolution 1D

\[ x(t) * y(t) \leftrightarrow X(s)Y(s) \]

\[ x(t)y(t) \leftrightarrow X(s) * Y(s) \]
(10) **Convolution 2D**

\[
z(s,t) = x \ast y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(a,b) y(s-a,t-b) \ da \ db
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(s-a,t-b) y(a,b) \ da \ db
\]

\[
= y \ast x
\]

(11) **Correlation**

\[
z(s,t) = x \circ y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^*(a,b) y(s+a,t+b) \ da \ db
\]

where \(x^*\) denotes the complex conjugate of \(x\).

The Laplace transform is introduced to help clarify the difference between convolutions, correlations, dilations, and translations, as well as their similarities and differences with respect to time and frequency. This arises again with the derivation of a wavelet.

**3.2. Fourier Transform.** Since an image is considered to vary spatially one can use a Fourier transform. The Fourier transform converts the spatial intensity image into its frequency domain by decomposing the image into orthogonal functions. This gives a continuous form that one may use to digitize images in discrete time.

The frequency is \(\nu = 1/T\) for a periodic continuous time signal

\[
x(t) = x(t-T) \forall T
\]

and the angular frequency is \(\omega = 2\pi \nu\).
3. UNDERSTANDING CONTINUOUS TRANSFORMS

**Lemma 1. (Fourier’s coefficients)** A complex periodic function \( x(t) \) can be represented as the linear sum of complex exponentials as follows:

\[
x(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega t}
\]

such that

\[
\alpha_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-in\omega t} dt
\]

are constants dependent on the period \( T \). Therefore using Euler’s identities

\[
e^{\pm in\omega t} = \cos n\omega t \pm i \sin n\omega t
\]

one gets the series expansion

\[
x(t) = \alpha_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)
\]

where

\[
a_n = 2Re[\alpha_n] \quad \text{and} \quad b_n = -2Im[\alpha_n]
\]

**Theorem 3.1. (Fourier’s Transform)** Assume that \( x(t) \) the continuous time signal and the following integrals exist then its Fourier transform is

\[
X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt
\]

and the inverse Fourier transform is

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega
\]
where

\[ X(\omega) = \text{Re}[\omega] + i\text{Im}[\omega] \]

\[ = \sqrt{\text{Re}[\omega]^2 + \text{Im}[\omega]^2} \]

\[ = |X(\omega)| e^{i\theta} \]

with the phase angle

\[ \theta = \arctan \frac{\text{Im}[\omega]}{\text{Re}[\omega]} \]

and note the domain of the Fourier transform is the frequency domain.

Furthermore, the magnitude of the Fourier transform, \(|X(\omega)|\), is real and called the Fourier spectrum of the continuous time signal. Whereas, the square of the Fourier transform, \(X(\omega)^2\), is known as the energy spectrum. [18] These one can relate to the energy of the discrete Fourier transform and wavelet decomposition in later sections.

Utilizing the sifting property from 2.9 one can find a constant in the frequency domain. This occurs via the Fourier transform of the unit impulse function at \(t = 0\) in the spatial domain.

\[ X(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-i\omega t} \, dt \]

\[ = e^{-i\omega 0} \]

\[ = 1 \]

Similarly, the Fourier transform of the unit impulse function at \(t = t_0\) is

\[ X(\omega) = \int_{-\infty}^{\infty} \delta(t-t_0)e^{-i\omega t} \, dt \]

\[ = e^{-i\omega t_0} \]

\[ = \cos(\omega t_0) - i \sin(\omega t_0) \]
by the sifting property and Euler’s identity.

First recall the concept of convolution in one dimension 2.23 for discrete time signals. So for two continuous functions \( g(t) \) and \( h(t) \) the convolution is defined as

\[
(3.14) \quad g(t) * h(t) = \int_{-\infty}^{\infty} g(t)h(t - \tau) \, d\tau
\]

One will now show that the product of two Fourier transforms in the frequency domain is equivalent to the Fourier transform of the convolution of two functions in the spatial domain.

\[
H(\omega)G(\omega) = H(\omega) \int_{-\infty}^{\infty} g(\tau)e^{-i\omega \tau} \, d\tau
= \int_{-\infty}^{\infty} g(\tau)H(\omega)e^{-i\omega \tau} \, d\tau
= \int_{-\infty}^{\infty} g(\tau) \left( \int_{-\infty}^{\infty} h(t - \tau)e^{-i\omega t} \, dt \right) \, d\tau
= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} g(t)h(t - \tau) \, d\tau \right)e^{-i\omega t} \, dt
= X(g(t) * h(t))
\]

Similarly, convolution in the frequency domain is equivalent to multiplication in the spatial domain. Convolution will prove to be vital in filtering the frequency domain, discrete wavelet transforms, and the fast wavelet transform.

The Fourier transform \( X(\sigma, \tau) \) of a continuous function \( x(s, t) \) in two dimensions is given by

\[
X(\sigma, \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(s, t)e^{-i\omega(\sigma s + \tau t)} \, ds \, dt
\]

and the inverse Fourier transform is

\[
x(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\sigma, \tau)e^{-i\omega(\sigma s + \tau t)} \, d\sigma \, d\tau
\]
3.3. Continuous Wavelet Transform. A wavelet is a function whose integral is zero. Wavelets are multi-scale transformations capable of analyzing sequences, functions, and images at a variety of levels. As shown below, the mother wavelets and father wavelets in two dimensions are defined by means of the outer product. These bases functions are obtainable via dilation and translation of a mother wavelet $\Psi(x)$ by amounts $\tau$ and $s$, respectively:

$$\Psi_{\tau,s} = \left\{ \psi\left(\frac{x - \tau}{s}\right) | (\tau, s) \in \mathbb{R} \times \mathbb{R}^+ \right\}$$

The dilation $\tau$ and translation $s$ allows the localization of the wavelet transform in frequency and time. The continuous wavelet transform (CWT) of a continuous function $x(t)$ can be defined as

$$w(\tau, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} x(t) \psi^*\left(\frac{t - \tau}{s}\right) dt$$

where $x(t)$ is an input signal in the time domain so in reference to the previous section this is the continuous time signal and

$$\psi^*\left(\frac{t - \tau}{s}\right)$$

is the complex conjugate of

$$\psi\left(\frac{x - \tau}{s}\right).$$

Similarly, one can also define the continuous wavelet transform from a uniformly sampled sequence
\( \{x_1, x_2, \ldots \} = \{x(t_0), x(\Delta + t_0), \ldots \} \)

as

\[
(3.20) \quad w(\tau, s) = \frac{1}{\sqrt{s}} \sum_{k=1}^{n} x_k \psi^* \left( \frac{\Delta(k - \tau)}{s} \right).
\]

Furthermore, to obtain the inverse CWT use the convolution of \( w(\tau, s) \) and \( \Psi_{\tau,s}(t) \) to acquire

\[
(3.21) \quad x(t) = \frac{1}{C_\psi} \int_s \int_\tau w(\tau, s) \frac{1}{s^2} \Psi_{d,s}(t) \ d\tau \ ds.
\]

where \( C_\psi \) is a constant dependent on the wavelet.

This occurs since the convolution by Laplace transform in the time domain corresponds to a product in the frequency domain. Conversely, the convolution in the frequency domain corresponds to a product in the time domain. Moreover,

\[
(3.22) \quad x(t) y(t) \longleftrightarrow X(s) \ast Y(s)
\]

\[
(3.22) \quad x(t) \ast y(t) \longleftrightarrow X(s) Y(s)
\]

The concepts introduced in this section will contribute to understanding the relationships necessary for image processing in the discrete scale-translation domain.
4. Understanding Discrete Transforms

Moving away from the continuous transforms, recall the sifting property \ref{eq:2.9} for the impulse response, and extend this to two dimensions. The \(2-D\) discrete impulse is defined as

\begin{equation}
\delta[m, n] = \begin{cases} 
1, & m = n = 0 \\
0, & \text{elsewhere}
\end{cases}
\end{equation}

4.0.1. The Sifting Property. For \(x[m, n]\) is

\[
\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[m, n] \delta[m - k, n - l] = x[k, l]
\]

which as before produces the value of the function at the impulse location.

Next recall the properties of the linear time invariant systems and let the input \(x[n] = z^n\). It follows that the output of a linear time invariant discrete system with impulse response \(h[n]\) is

\begin{equation}
y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k] = \sum_{k=-\infty}^{\infty} z^k h[n - k]
\end{equation}

and by the property of convolution this becomes

\begin{equation}
y[n] = \sum_{k=-\infty}^{\infty} z^{n-k} h[k] = z^n \sum_{k=-\infty}^{\infty} z^{-k} h[k].
\end{equation}

The summation in the end result of the above equation is the well known \(z\) transform.

4.1. \(z\) Transform. The \(z\) transform of the discrete time sequence \(h[n]\) is denoted, \(X(z)\). One defines the \(z\) transform as

\begin{equation}
X(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}.
\end{equation}
Let \( x[n] \) be the input sequence and let \( x_u[n] \) be the output after up-sampling by \( m, \uparrow m \). Then the \( z \) transform of \( x_u[n] \) is

\[
X(x_u[n]) = \sum_{n=-\infty}^{\infty} x_u[n] z^{-n} = \sum_{k=-\infty}^{\infty} x[k] z^{-mk}
\]

where \( x[k] \) occurs at \( mk \). Denote the \( z \) transform of \( x_u[n] \) as \( X_u(z) \) which is equivalent to \( X(z^m) \). Furthermore, up-sampling is invertible so that \( X(z) = X_u(z^{1/m}) \).

One can utilize the \( z \) transform in two dimensions by computing the transform on the rows and the columns separately.

### 4.2. Discrete Fourier Transform.

The discrete Fourier Transform (DFT) is a derivation of a continuous time function \( x(t) \). Take \( N \) samples over a sampling interval, say \( T \), that is \( N_T : \{ t = 0, T, 2T, \ldots, (N-1)T_0, t = 0, T, 2T, \ldots, (N-1)T_n \} \) to form a sequence so that \( x(t) = x(t_0 + N_T) \).

So if one chooses \( t_0 = 0, \omega = 2\pi \nu \) for \( \nu = 0, 1, \ldots, N-1 \) then one can find the discrete Fourier Transform as follows

\[
X(\nu) = \sum_{t=0}^{N-1} x(t) e^{-i2\pi \nu t/N}
\]

and the inverse DFT is

\[
x(t) = \frac{1}{N} \sum_{\nu=0}^{N-1} X(\nu) e^{i2\pi \nu t/N} \quad (t = 0, 1, \ldots, N-1)
\]

Similarly, one can define the two dimensional version that is used for images. Let \( x(s, t) \) and \( X(\sigma, \tau) \) denote the spatial and frequency domains, respectively. Then the DFT is

\[
X(\sigma, \tau) = \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} x(s, t) e^{-i2\pi (\sigma s/M + \tau t/N)} \quad \text{for} \quad (\sigma = 0, 1, \ldots, M-1 \text{ and } \tau = 0, 1, \ldots, N-1)
\]
and the inverse DFT is

\[(4.9)\]
\[x(s, t) = \frac{1}{MN} \sum_{\sigma=0}^{M-1} \sum_{\tau=0}^{N-1} X(\sigma, \tau)e^{2\pi i (\sigma s/M + \tau t/N)} \text{ for } (s = 0, 1, \ldots, M-1 \text{ and } t = 0, 1, \ldots, N-1).\]

Therefore, to obtain a result in the spatial domain, one must re-transform the information using the inverse Fourier Transform.

Furthermore, the following properties hold for Fourier transforms:

**Properties.**

1. **Linearity**
   \[ax_1(t) + bx_2(t) \leftrightarrow aX_1(\omega) + bX_2(\omega)\]
   for constants \(a\) and \(b\).

2. **Symmetry**
   \[x(t) \leftrightarrow X(\omega)\]
   \[X(t) \leftrightarrow 2\pi x(-\omega)\]

3. **Time translation (shifting)**
   \[x(t - a) \leftrightarrow X(\omega)e^{-i\omega a}\]

4. **Frequency translation**
   \[x(t)e^{iat} \leftrightarrow X(\omega - a)\]

5. **Time dilation (scaling)**
   \[x(at) \leftrightarrow \frac{1}{|a|}X(\frac{\omega}{a}) \quad \forall a \in \mathbb{R}\]
(6) **Time differentiation**

\[
\frac{d^n x(t)}{dt^n} \leftrightarrow (i\omega)^N X(\omega)
\]

(7) **Frequency differentiation**

\[
(-it)^n x(t) \leftrightarrow \frac{d^n X(\omega)}{d\omega^n}
\]

(8) **Complex conjugate**, denoted by \(*\)

\[
x^*(t) = a(t) - ib(t) \implies x^*(t) \leftrightarrow X^*(-\omega)
\]

(9) **Parseval's relation**

\[
\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \left| X(\omega) \right|^2
\]

One may consider Parseval’s relation as an energy relation. Notice the similarity between the right hand side of the relation and the physicist’s equation for kinetic energy \(\Delta K = \frac{1}{2}mv^2\). Thus both are proportional to the sum of the squares. Hence the energy density of the continuous time signal is equal to the sum of squares of the energy densities of its Fourier components.

Now one will extend some of the properties to two dimensions. Let’s sample a continuous function \(x(y, z)\) to form a digital image, \(x(s, t)\), such that one takes \(M\) and \(N\) uniform samples of \(y\) and \(z\), with separations between samples of \(Y\) and \(Z\), respectively. This implies that the separation in the spatial domain is \(\Delta s = \frac{1}{MY}\) and \(\Delta t = \frac{1}{NZ}\), respectively.

The symmetry property can be conveyed by the sum of its even and odd parts, \(2k\) and \(2k + 1\) for \(k \in \mathbb{Z}\), respectively

\[
x(s, t) = x_{2k}(s, t) + x_{2k+1}(s, t)
\]
for

\[(4.11) \quad x_{2k}(s, t) = \frac{x(s, t) + x(-s, -t)}{2} = x_{2k}(-s, -t)\]

and

\[(4.12) \quad x_{2k+1}(s, t) = \frac{x(s, t) - x(-s, -t)}{2} = -x_{2k+1}(-s, -t).\]

However, for an \(M \times N\) image, negative indices are irrelevant so one shifts the above definitions to

\[(4.13) \quad x_{2k}(s, t) = x_{2k}(M - s, N - t) \quad \text{and} \quad x_{2k+1}(s, t) = -x_{2k+1}(M - s, N - t).\]

In addition, discrete functions are odd when all its samples sum to zero. Therefore, if one has the sum of the product of an even and an odd function such as

\[(4.14) \quad \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} x_{2k}(s, t)x_{2k+1}(s, t) = 0\]

since the product of these functions is odd.

For real functions, \(x(s, t)\), the conjugate of the DFT in two dimensions is

\[(4.15) \quad X^* (\sigma, \tau) = \left( \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} x(s, t)e^{-i2\pi(\sigma s/M + \tau t/N)} \right)^* = \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} x^*(s, t)e^{i2\pi(\sigma s/M + \tau t/N)}
= \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} x(s, t)e^{i2\pi(\sigma s/M + \tau t/N)}
= \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} x(s, t)e^{-i2\pi(-\sigma s/M - \tau t/N)}
= X(-\sigma, -\tau)\]

since \(x^*(s, t) = x(s, t)\) for real functions.
The time and frequency translation properties becomes

\[(4.16) \quad x(s - s_0, t - t_0) \iff X(\sigma, \tau)e^{-i2\pi(s_0 \sigma / M + t_0 \tau / N)}\]

and

\[(4.17) \quad X(\sigma - \sigma_0, \tau - \tau_0) \iff x(s, t)e^{i2\pi(s_0 \sigma / M + t_0 \tau / N)}\]

which shifts the origin of the DFT in the spatial and frequency domains.

Recall the concept of the Fourier spectrum and energy from continuous Fourier transforms. In addition, recall that a Fourier transform can be written as a sum of its real and imaginary parts. Thus, in two dimensions one has

\[(4.18) \quad X(\sigma, \tau) = |X(\sigma, \tau)|e^{i2\pi(s_0 \sigma / M + t_0 \tau / N)}
= \sqrt{Re[\sigma, \tau]^2 + Im[\sigma, \tau]^2}e^{i2\pi(s_0 \sigma / M + t_0 \tau / N)}\]

for the Fourier spectrum and the energy (power) spectrum is

\[(4.19) \quad |X(\sigma, \tau)|^2 = Re[\sigma, \tau]^2 + Im[\sigma, \tau]^2.\]

Notice that the largest contribution comes from the \(\sigma = \tau = 0\) frequency term as

\[(4.20) \quad X(0, 0) = MN \frac{1}{MN} \sum_{s=0}^{M-1} \sum_{t=0}^{N-1} x(s, t)e^0
= MNx_{avg}(s, t)\]

is typically large depending on the dimensions of the image.

In Mathematica one uses the `ImagePeriodogram` to show the power spectrum of the DFT. The following figure depicts how the portion around the origin of the DFT encompasses the largest values of the power spectrum. These values appear brighter in the image of the periodogram.
These will be compared to the spectrum of the wavelet transform in later sections.

4.3. Discrete Cosine Transform. The Discrete Cosine Transform (DCT) of the sequence $x[n]$ is denoted $C[k]$ for $k = 0, \ldots, N-1$. One defines the DCT as

$$ C[k] = a[k] \sum_{n=0}^{N-1} x[n] \cos \left( \frac{2n+1}{2N} k \pi \right) \quad \text{for} \quad k = 0, \ldots, N-1 $$
where

\[
 a[k] = \begin{cases} 
 N^{-1/2} & \text{for } k = 0 \\
 \left(\frac{2}{N}\right)^{1/2} & \text{elsewhere} 
\end{cases}
\]

and the inverse DCT is

\[
 x[n] = \sum_{k=0}^{N-1} a[k] C[k] \cos\left(\frac{2n+1}{2N} k \pi\right) \quad \text{for } n = 0, \ldots, N - 1
\]

where

\[
 a[k] = \begin{cases} 
 N^{-1/2} & \text{for } k = 0 \\
 \left(\frac{2}{N}\right)^{1/2} & \text{elsewhere} 
\end{cases}
\]

Notice that for \( k = 0 \) one has \( C[0] = N^{-1/2} \sum_{n=0}^{N-1} x[n] \) the average value of the sequence as we can see in the top left corner of Figure 3.

![Figure 3. DCT Basis Vectors for \( N = 8 \)](image)

For images one utilizes the two dimensional transform. This is accomplished by performing the one dimensional transform to the rows and the columns separately.

The transform of the matrix \( x[m, n] \) is denoted \( C[k, l] \) for \( k = 0, \ldots, M - 1 \) and \( l = \ldots, L - 1 \).
0, \ldots, N - 1. One defines the two dimensional DCT as

\begin{equation}
C[k, l] = a[k] a[l] \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x[m, n] \cos\left(\frac{2m + 1}{2M} k\pi\right) \cos\left(\frac{2n + 1}{2N} l\pi\right) \tag{4.25}
\end{equation}

where

\begin{equation}
a[l] = \begin{cases} 
N^{-1/2} & \text{for } l = 0 \\
\left(\frac{2}{N}\right)^{1/2} & \text{elsewhere}
\end{cases} \tag{4.26}
\end{equation}

is the same computation as in the one dimensional version of the DCT. The inverse DCT is

\begin{equation}
x[m, n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} a[k] a[l] C[k, l] \cos\left(\frac{2n + 1}{2N} k\pi\right) \cos\left(\frac{2m + 1}{2M} l\pi\right) \tag{4.27}
\end{equation}

which is separable. Thus the 2 - D DCT can be computed successively as 1 - D DCTs on the rows and columns of an image.

To produce the figure 4 one converts the image to bytes by utilizing the floor function

\begin{equation}
\lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \leq x\} \tag{4.28}
\end{equation}

of the product of the image data with 256. Then one computes the DCT of the data and keeps \( \left\lfloor N\sqrt{\pi}\right\rfloor \times \left\lfloor M\sqrt{\pi}\right\rfloor \) non-zero values and truncates the remaining values. In Mathematica one utilizes the \texttt{ArrayPlot} command to produce an image based on the data from the DCT. Note that the original image had close to eight hundred thousand, nonzero pixel values and after truncation the following images have significantly less. However, more nonzero values due to truncation leads to more distortion of the picture.
The DCT compresses an image into $8 \times 8$ blocks [10] and then files them consecutively. Block-wise transform coding quantizes the transform coefficients through entropy coding and then either stores them or transmits them. The quantized coefficients then use the inverse transform to reconstruct the block.

A disadvantage of the DCT is the block effect that can be seen on a television during a storm which comes from corrupted files. This block effect occurs as the bit-rate decreases. A solution that prevents the block effect, smooths the boundaries, and does not increase the number of transform coefficients, is the subband
transform. The subsequent section on discrete wavelet transforms examines the subband transform. The JPEG 2000 standard encodes the source data through a forward discrete wavelet transform, quantizes the transform, encodes the quantization by its entropy and the compressed image data is the result. The decoder of the JPEG 2000 is an exact reverse process. In the next section, the coefficients of the mother(translation) wavelets and the father(scaling) function of the discrete wavelet transform are derived in order to delve into the various levels of a process and analyze the signal or image.
5. Discrete Wavelet Transforms

Discrete Wavelet Transforms (DWT) are useful in data analysis, data compression, image processing and signal coding. Instead of applying a decomposition over the entire image, the DWT is applied multiple times over portions of the image from the spatial domain to the scale-translation domain and finally back to the spatial domain. The DWT reveals the frequency of the patterns in an image in addition to the variation of the patterns throughout the image.

Utilization of linear algebra distinguishes the relationship between the original signal and the discrete wavelet coefficients.

5.0.1. Father Scaling Function Discretization. The father scaling function \( \varphi \) has basis vectors,

\[
\varphi_{j,k}(t) = 2^{j/2} \varphi(2^j t - k) \quad \text{for} \quad j \in \mathbb{N} \quad \text{and} \quad k = 1, 2, \ldots, 2^j
\]

that are in the vector space

\[
V_j = \overline{\text{span}} \{ \varphi_{j,k}(t) \}_{k=1,2,\ldots,2^j} \quad \text{for} \quad t \geq 0
\]

where \( \text{span} \) denotes the span of the basis vectors for some specific \( j = j_0 \). Therefore, \( \varphi(t) \in L^2(\mathbb{R}) \), is square integrable. Notice that the \( k \) determines position along the \( x \)-axis since it is the translation parameter and \( j \) determines the width along the \( x \)-axis as it is the scale. In addition, the \( 2^j \) controls the amplitude of the scaling function.

Furthermore the inner product,

\[
\langle \varphi_{j,k}(t), \varphi_{j,k}(t) \rangle = \int_0^1 \varphi_{j,k}(t)^2 \, dt = 1,
\]
due to the normalization constant $2^{j/2}$. The normalization constant is used later in this section to compute the energy of a wavelet transform.

5.0.2. *Mother Wavelet Discretization.* Similarly, the mother wavelet $\psi$ has basis vectors,

$$
\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k) \quad \text{for} \quad j \in \mathbb{N} \text{ and } k = 1, 2, \ldots, 2^j
$$

that are in the vector space

$$
W_j = \text{span} \{ \psi_{j,k}(t) \}_{k=1,2,\ldots,2^j} \quad \text{for} \quad t \geq 0
$$

with

$$
\langle \psi_{j,k}(t), \psi_{j,k}(t) \rangle = 1.
$$

Moreover,

$$
V_j \subseteq V_{j+1} \quad \text{and} \quad W_j \subseteq V_{j+1}
$$

such that

$$
V_{j+1} = V_j \oplus W_j.
$$

Wavelets capture information at multiple resolutions using dilates and translates. Dilation occurs between resolutions while translation occurs across a resolution.

5.1. Multiresolution Analysis. Localization in time, high frequencies, and low frequencies are important in identifying specific information about data. Often transient data requires time resolution, whereas frequency resolution is desirable for lower frequency ranges. However, high frequencies need more frequency resolution and smaller sampling intervals. Therefore, multirate digital signal processing is necessary.
Multiresolution Analysis (MRA) incorporates analyzing data at different resolutions.

**Theorem 5.1.** The sequence of nested subspaces \{V_j\}_{j \in \mathbb{Z}} \subset L^2(\mathbb{R}) is a MRA if the following properties hold

1. **Density**
   \[ \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}) \]

2. **Segmentation**
   \[ \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \]

3. **Orthonormality** There exists \phi(t) such that,
   \[ V_0 = \text{span}\{\phi(t-n)\}_{n \in \mathbb{Z}} \]

4. **Scale Invariance** If \( f(t) \in V_j \) then \( f(2^{-j}t) \in V_0, \forall j \in \mathbb{Z} \)

5. **Shift Invariance** If \( f(t) \in V_0 \) then \( f(t-n) \in V_0, \forall n \in \mathbb{Z} \)

and there exists a wavelet function, \psi(t) \in L^2(\mathbb{R}), such that \{\psi(2^j t - n)\}_{j \in \mathbb{Z}, n \in \mathbb{Z}} spans L^2(\mathbb{R}).

When analyzing images at times, it is beneficial to look at multiple resolutions to decipher between high-contrast, low-contrast, large, or small objects in a single photograph. In fact, both analysis and synthesis filters are needed in DSP, which is called the filter bank. Imaging techniques that relate to multiresolution analysis are signal processing through filter banks, up-sampling, down-sampling, subband coding, and the Haar transform.
5.2. Finding Coefficients of Mother and Father Wavelets. Since wavelets are multi-scale transformations capable of analyzing sequences, functions, and images at a variety of levels, it is sufficient to define the components of the most basic wavelet known as the Haar wavelet. Therefore, the definition of the mother wavelet and father scaling functions arises.

5.2.1. The Haar Mother Wavelet. The Haar mother wavelet is

\[
\psi(t) = \begin{cases} 
1 & t \in \left[0, \frac{1}{2}\right) \\
-1 & t \in \left(\frac{1}{2}, 1\right] \\
0 & \text{elsewhere}
\end{cases}
\]

The mother wavelet is necessary to calculate the wavelet coefficients, \(d_j\). These can be obtained by the following

\[
d_j = 2^{j/2} \int x(t) \psi_{j,k}(t) \, dt
\]
where

\( \psi_{j,k}(t) = 2^j \psi(2^j t - k) \) for \( j, k \in \mathbb{Z}, k = 1, 2, \ldots, 2^j \)

where \( W_j = \text{span} \{ \psi_{j,k}(t) \} \).

5.2.2. The Haar Father Wavelet. The Haar father scaling function is

\[
\varphi(t) = \begin{cases} 
1 & t \in [0, 1] \\
0 & \text{elsewhere} 
\end{cases}
\]

The father scaling function is used to calculate the scaling coefficients denoted as \( a_j \). These can be obtained by the following

\[
a_j = \int x(t) \varphi_{j,k}(t) \, dt
\]
where

\[(5.14) \quad \varphi_{j,k}(t) = 2^{j/2} \varphi(2^j t - k).\]

This notation is useful to relate filter banks, convolution, scaling, and wavelet coefficients through multi-resolution analysis.

5.2.3. **Scaling Function.** Recall equation 5.1 where \(j, k \in \mathbb{Z}\) and \(\varphi(t) \in L^2(R)\) for some specific \(j = j_0\). Thus for \(x(t) \in V_{j_0}\) the result is

\[(5.15) \quad x(t) = \sum_n a_n \varphi_{j_0+1,n}(t)\]

Since \(V_j \subseteq V_{j+1}\) if \(\varphi_{j,k} \in V_j\) then it follows that

\[(5.16) \quad \varphi_{j,k}(t) = \sum_n a_n \varphi_{j+1,n}(t)\]

which is a sum of the product of the scaling coefficients, \(a_n = h_{\varphi}(n)\) and the scaling functions \(\varphi_{j+1,n}(t) \in V_{j+1}\). Reference to equations 5.16 and 5.1 with \(j = k = 0\) derives

\[(5.17) \quad \varphi_{0,0}(t) = \sum_{n=0}^1 a_n \varphi_{1,0}(t) = \sum_{n=0}^1 h_{\varphi}(n) \varphi_{1,0}(t) = \sum_n h_{\varphi}(n) \sqrt{2} \varphi(2t - n) = \varphi(t)\]

which is known as the dilation equation in multi-resolution analysis.

Therefore, in general

\[(5.18) \quad \varphi_{j,k} = \sum_n h_{\varphi}(n) 2^{(j+1)/2} \varphi(2^{j+1} t - n)\]

for the scaling function.
5.2.4. Wavelet Function. Similar to the scaling functions the set \( \{ \psi_{j,k}(t) \} \) of wavelets is defined as

\[
\psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k)
\]

as given in equation 5.4. If \( x(t) \in W_j \) then

\[
x(t) = \sum_n d_n \psi_{j,n}(t)
\]

where \( d_n = h_\psi(n) \) are the wavelet coefficients.

Since \( W_j \) is orthogonal to \( V_1 \) the inner product

\[
\langle \varphi_{j,k}(t), \psi_{j,l}(t) \rangle = 0
\]

for all \( j, k, l \in \mathbb{Z} \). Therefore, relate the union of scaling and wavelet subspaces as

\[
V_{j+1} = V_j \oplus W_j
\]

where \( W_j \subseteq V_{j+1} \) also. Hence,

\[
\psi(t) = \sum_n h_\psi(n) \sqrt{2} \varphi(2t - n)
\]

for the wavelet function coefficients \( h_\psi(n) \) and by orthogonality

\[
h_\psi(n) = (-1)^n h_\varphi(1 - n).
\]

Let’s delve into the connection between the subspaces \( V_j \) and \( W_j \) and the inner products of the orthonormal basis that are equivalent to the coefficients. In addition, assume that \( x(t) \) is a real, continuous, function. Previously, the detail coefficients were defined as

\[
d_j = \int x(t) \psi_{j,k}(t) \, dt
\]

\[
= \langle x(t), \psi_{j,k}(t) \rangle
\]
and the approximation coefficients as

\[ a_j = \int x(t) \varphi_{j_0,k}(t) \, dt \]

(5.26)

\[ = \langle x(t), \varphi_{j_0,k}(t) \rangle . \]

Since \( x(t) \in L^2(R) \) and \( V_{j+1} = V_j \oplus W_j \) the space of square integrable functions can be written as

\[ L^2(R) = V_0 \oplus W_0 \oplus V_1 \oplus \ldots \]

(5.27)

\[ = V_1 \oplus W_1 \oplus V_2 \oplus \ldots \]

\[ = V_2 \oplus W_2 \oplus W_3 \oplus \ldots \]

or in other words it is a sum of sums. Therefore, by equations 5.15 and 5.20 with \( a_n = a_{j_0} \) and \( d_n = d_j \) this develops

\[ x(t) = \sum_n a_{j_0}(n) \varphi_{j_0,k}(t) + \sum_{j=j_0}^{\infty} \sum_n d_j \psi_{j,k}(t) \]

(5.28)

the sum of the sum of the product of the approximation coefficients and the scaling function and the sum of the sum of the product of the detail coefficients and the wavelet function. Thus the \( 1 - D \) Haar transform for \( \varphi_{0,k}(t) \in V_0 \) can be written as

\[ \varphi_{0,k}(t) = \frac{1}{\sqrt{2}} \varphi_{1,2k}(t) + \frac{1}{\sqrt{2}} \varphi_{1,2k+1}(t) \]

(5.29)

Observe that this can be considered a mapping of a continuous function into a sequence of numbers. The next section covers subband coding and filter banks.
5.3. **Subband Coding.** This section considers multiscale analysis of a sequence through *subband coding*. Subband coding decomposes an image into bandlimited components known as subbands. Essentially, the goal of subband encoding is perfect reconstruction. However, the result depends on the analysis and synthesis filters.

5.3.1. *Filter Banks.* Input the sequence into an *analysis filter bank* composed of high pass filters and low pass filters denoted, \( h_0(n) \) and \( h_1(n) \), respectively. The high pass filter (HPF) reveals the details by removal of the low frequencies through a difference process. Whereas, the low pass filter (LPF) eliminates the high frequencies, which tends to smooth the data, as a result of an average process. Figure 5 represents a two-band filter bank that contains two analysis *finite impulse response* (FIR) filters and produces two half-band subbands, \( x_{LP}[n] \) and \( x_{HP}[n] \).

\[
x[n] \ast h_0(n) \downarrow 2 \rightarrow x_{LP}[n] \\
x[n] \ast h_1(n) \downarrow 2 \rightarrow x_{HP}[n]
\]

**Figure 5.** Analysis Filter Bank

Thus the analysis filter bank divides the signal into lowpass and highpass subbands. Whereas, the synthesis filter bank reconstructs the data using either the same or different lowpass and highpass filters denoted, \( g_0(n) \) and \( g_1(n) \), respectively.
Figure 6 represents a two-band filter bank that contains two synthesis FIR filters that reconstructs the two half-band subbands, \(x_{LP}[n]\) and \(x_{HP}[n]\) into the output sequence \(y[n]\).

Next relate the two-band filter banks and subband encoding to the wavelet subspaces. Let \(x(t) \in V_1\) such that

\[
(5.30) \quad x(t) = \sum_{n=-\infty}^{\infty} x[n] \varphi(2t - n).
\]

Recall from equation 5.8 that we can decompose the space \(V_1\) into

\[
(5.31) \quad V_1 = V_0 \oplus W_0
\]

orthogonal subspaces. The analysis filter bank in figure 5 and down-sampling denoted, \(\downarrow 2\), decomposes the sequence into the projections of \(x[n]\) onto \(V_0\) and \(W_0\). The projection of \(x[n]\) onto \(V_0\) is the output of the low-pass filter, \(h_0[n]\), followed by down-sampling which is denoted \(x_{V_0}[n]\). Thus the approximation of the sequence, \(x[n]\), is \(x_{V_0}[n]\) which is found by convolution of the input with the low-pass filter and downsampling.
Similarly, the projection of \( x[n] \) onto \( W_0 \) is the output of the high-pass filter, \( h_1[n] \), followed by down-sampling which is denoted \( x_{W_0}[n] \). Thus the details, or the high frequencies, of the sequence, \( x[n] \), are \( x_{W_0}[n] \). These projections are known as the subbands that represent the input signal. These subbands are half the length of the original sequence.

To reconstruct the original sequence the subbands must go through an up-sampler denoted, \( \uparrow 2 \), and then through the synthesis filter banks. Up-sampling doubles the length of the subbands. The up-sampled \( x_{W_0}[n] \) is then sent through the high-pass synthesis filter, \( g_1[n] \). Simultaneously, the up-sampled \( x_{V_0}[n] \) is sent through the low-pass synthesis filter, \( g_0[n] \). These outputs are then added together to give the final output denoted, \( \hat{x}[n] \).

In the case of perfect reconstruction

\[
\hat{x}[n] = x[n]
\]

such as when

\[
g_0[n] = (-1)^n h_1[n] \quad \text{and} \quad g_1[n] = (-1)^{n+1} h_0[n].
\]

Figure 7 is a depiction of analysis and synthesis filter banks that produce a perfect reconstruction of the input.
5.4. Fast Wavelet Transform. Recall the concepts of analysis filter banks and down-sampling and find the discrete wavelet coefficients using inner products. Derivation of the Fast Wavelet Transform (FWT) occurs by manipulating the multi-resolution dilation equation

\begin{equation}
\varphi(t) = \sum_n h_\varphi(n) \sqrt{2} \varphi(2t - n) \tag{5.34}
\end{equation}

first scale the equation by \(2^j\)

\begin{equation}
\varphi(2^j t) = \sum_n h_\varphi(n) \sqrt{2} \varphi(2(2^j t) - n) \tag{5.35}
\end{equation}

then translate the equation by \(k\)

\begin{equation}
\varphi(2^j t - k) = \sum_n h_\varphi(n) \sqrt{2} \varphi(2(2^j t - k) - n) \tag{5.36}
\end{equation}

finally change the indices to \(l = 2k + n\)

\begin{equation}
\varphi(2^j t - k) = \sum_l h_\varphi(l - 2k) \sqrt{2} \varphi(2(2^j t - k) - (l - 2k)) \\
= \sum_l h_\varphi(l - 2k) \sqrt{2} \varphi(2^{j+1} t - 2k - l + 2k) \\
= \sum_l h_\varphi(l - 2k) \sqrt{2} \varphi(2^{j+1} t - l). \tag{5.37}
\end{equation}

Similarly, scaling and translating the wavelet function gives

\begin{equation}
\psi(2^j t - k) = \sum_l h_\psi(l - 2k) \sqrt{2} \varphi(2^{j+1} t - l). \tag{5.38}
\end{equation}

Recall the definition of the wavelet function 5.19 and the detail coefficients 5.25 then by substitution the detail coefficients are

\begin{equation}
d_j(k) = \int x(t) 2^{j/2} \psi(2^j t - k) \ dt. \tag{5.39}
\end{equation}
Then due to the right hand side of equation 5.38 and the definition of the approximation coefficients in 5.26 for \( j_0 = j + 1 \) this becomes

\[
d_j(k) = \int x(t)2^{j/2} \left( \sum_l h_\psi(l - 2k)\sqrt{2}\psi(2^{j+1}t - l) \right) \, dt = \sum_l h_\psi(l - 2k) \left( \int x(t)2^{j/2}2\psi(2^{j+1}t - l) \, dt \right) = \sum_l h_\psi(l - 2k)a_{j+1}(l).
\]

(5.40)

Observe that the detail coefficients at scale \( j \) are based on the approximation coefficients at scale \( j + 1 \). Furthermore, recall equations 5.4 and 5.25 then the equation for the detail coefficients becomes

\[
d_\psi(j, k) = \sum_l h_\psi(l - 2k)a_\phi(j + 1, l).
\]

(5.41)

Similarly, substitution and equations 5.14 and 5.26 leads to

\[
a_j(k) = \sum_l h_\psi(l - 2k)a_{j+1}(l)
\]

(5.42)

which can be written as

\[
a_\phi(j, k) = \sum_l h_\phi(l - 2k)a_\phi(j + 1, l).
\]

(5.43)

The next portion provides the equations for the FWT.

The FWT uses the analysis filter bank thus employing filtering via convolution and downsampling. This is shown by manipulating equations 5.41 and 5.43 for the
coefficients. Since

\[ d_\psi(j, k) = \sum_l h_\psi(l - 2k)a_\varphi(j + 1, l) \]

\[ = \sum_l h_\psi(-2k + l)a_\varphi(j + 1, l) \]

\[ = \sum_l h_\psi(-(2k - l))a_\varphi(j + 1, l) \]

\[ = a_\varphi(j + 1, m) * h_\psi(-m) \]

is convolution of the \( j + 1 \) scale coefficients with the high pass filter for every \( 2k \) samples. Similarly,

\[ a_\varphi(j, k) = \sum_l h_\varphi(l - 2k)a_\varphi(j + 1, l) \]

\[ = \sum_l h_\varphi(-2k + l)a_\varphi(j + 1, l) \]

\[ = \sum_l h_\varphi(-(2k - l))a_\varphi(j + 1, l) \]

\[ = a_\varphi(j + 1, m) * h_\varphi(-m) \]

is the same process that occurs after the low pass filter and downsampling which gives the approximation coefficients at the \( j \) scale. This FWT analysis filter bank is shown in figure 8

In constructing wavelets one used a two band filter bank. In

\[ a_\varphi(j + 1, m) \]

\[ \downarrow 2 \]

\[ *h_\varphi(-m) \]

\[ \downarrow 2 \]

\[ a_\varphi(j, k) \]

\[ d_\psi(j, k) \]

\[ \downarrow 2 \]

\[ *h_\psi(-m) \]

\[ \downarrow 2 \]

\[ Figure 8. FWT Analysis Filter Bank \]
particular, when the analysis and synthesis wavelets and scaling functions are the same they are called orthogonal filter banks. Filter banks with different analysis and synthesis wavelets and scaling functions are called biorthogonal filter banks.

5.5. Discretizing Wavelet Coefficients. If one acquires $M$ samples of the continuous function $x(t)$ that starts at an arbitrary $t = t_0$ for $t \geq 0$ then one gets the discrete function

$$x[n] = x(t_0 + n\Delta t) \text{ for } n = 0, 1, \ldots, M - 1$$

Hence, recalling equations 5.1 and 5.4 in addition to multiplying by the normalization factor $\frac{1}{\sqrt{M}}$ one has the forward DWT coefficients

$$a_\varphi(j_0, n) = \frac{1}{\sqrt{M}} \sum_n x[n] \varphi_{j_0, k}(n)$$

and

$$d_\varphi(j, k) = \frac{1}{\sqrt{M}} \sum_n x[n] \varphi_{j, k}(n) \text{ for } j \geq j_0$$

where we have changed the notation for the coefficients $a_{j_0}(n) = a_\varphi(j_0, n)$ and $d_j(n) = d_\varphi(j, k)$ to clarify the parent function they stem from and the $k^{th}$ location. Thus by equation 5.28 the inverse DWT is

$$x[n] = \frac{1}{\sqrt{M}} \sum_k a_\varphi(j_0, n) \varphi_{j_0, k}(n) + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k d_\varphi(j, k) \psi_{j, k}(n).$$

The next section introduces the one dimensional Haar transform and its relation to low pass and high pass filters. In addition, it covers filter banks more in depth, implements the one dimensional Haar transform, and relates the discrete wavelet transform to the conservation of energy. The next section introduces the one dimensional Haar transform and its relation to low pass and high pass filters. In
addition, it implements the one dimensional Haar transform, and relates the discrete wavelet transform to the conservation of energy.
5.6. **1D Haar Transform.** An analog signal with \( N \) samples is given by the vector \( \mathbf{x} \in \mathbb{R}^N \) where
\[
\mathbf{x} = \{x_1, x_2, \ldots, x_N\} = \{x(t_1), x(t_2), \ldots, x(t_N)\}.
\]
(5.50)

Assume that the dimension of the signal \( N \) is even; if it is odd, make it even by padding the signal with a constant, typically zero. The level \( k \) (L-k) Haar transform is a mapping from the original signal \( x \) to \( 2^k \) sub-signals, \( a^k \) and \( d^k \) for \( k = 1, 2, \ldots, N/2 \), of equal length, \( N/2^k \).

5.6.1. **Low Pass Filter.** The first sub-signal, \( a^1 \) occurs via the low pass filter that multiplies the running average by \( 2^{1/2} \) and its values are given by
\[
(5.51) \quad a_k = \frac{x_{2^k} + x_{2^{k-1}}}{2} \frac{1}{\sqrt{2}} = \frac{x_{2^k} + x_{2^{k-1}}}{\sqrt{2}} \quad \text{for} \quad k = 1, 2, \ldots, N/2
\]

5.6.2. **High Pass Filter.** The second sub-signal, \( d^1 \) follows from the high pass filter that multiplies the running difference by \( 2^{1/2} \) and its values are given by
\[
(5.52) \quad d_k = \frac{x_{2^{k-1}} - x_{2^k}}{2} \frac{1}{\sqrt{2}} = \frac{x_{2^{k-1}} - x_{2^k}}{\sqrt{2}} \quad \text{for} \quad k = 1, 2, \ldots, N/2
\]

Moreover, the L-k Haar transform is found by calculating the L-1 transform on \( a^{k-1} \).

To understand filter banks, up-sampling, and down-sampling, take a sequence \( x[n] \) and perfectly reconstruct the sequence using two filter banks. Such as example 5.1 which uses a module for the previous and most basic wavelet, the Haar Wavelet to decompose the signal and example 5.3 that reconstructs the signal. The Haar LPF and HPF are \( \frac{1}{\sqrt{2}} \{1, 1\} \) and \( \frac{1}{\sqrt{2}} \{1, -1\} \), respectively.
EXAMPLE 5.1. Consider the following Lucas numbers where \( n = 2^4 \) and \( x = \{29, 18, 11, 7, 4, 3, 1, 2, -1, 3, -4, 7, -11, 18, -29, 47\} \). The Lucas numbers are beyond the scope of this paper refer to [11] and [7]. Use the Haar Wavelet and find the coefficients. Create a Module in Mathematica that transforms \( x \).

**SOLUTION.** **Step 1** first partition the given \( x \) so that:

\[
y = \{29, 18\} \quad \{11, 7\} \quad \{4, 3\} \quad \{1, 2\} \quad \{-1, 3\} \quad \{-4, 7\} \quad \{-11, 18\} \quad \{-29, 47\}
\]

**Step 2** compute \( a^k = \frac{1}{\sqrt{2}} y \{1, 1\} \) and \( d^k = \frac{1}{\sqrt{2}} y \{1, -1\} \) here \( k = 1 \). This results in the following

\[
a^1 = \left\{ \frac{47}{\sqrt{2}}, \frac{9\sqrt{2}}{\sqrt{2}}, \frac{7}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{7}{\sqrt{2}}, \frac{9\sqrt{2}}{\sqrt{2}} \right\}
\]

\[
d^1 = \left\{ \frac{11}{\sqrt{2}}, \frac{2\sqrt{2}}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -2\sqrt{2}, -\frac{11}{\sqrt{2}}, -\frac{29}{\sqrt{2}}, -\frac{38}{\sqrt{2}} \right\}
\]

**Step 3** Repetition of this process when \( a^k = \frac{1}{\sqrt{2}} a^{k-1} \{1, 1\} \) and \( d^k = \frac{1}{\sqrt{2}} a^{k-1} \{1, -1\} \) for \( k = 2, 3, \) and \( 4 \) gives:

\[
a^2 = \left\{ \frac{65}{2}, \frac{5}{2}, \frac{25}{2} \right\} \quad \text{and} \quad d^2 = \left\{ \frac{29}{2}, \frac{1}{2}, -\frac{11}{2} \right\}
\]

\[
a^3 = \left\{ \frac{75}{2\sqrt{2}}, \frac{15}{\sqrt{2}} \right\} \quad \text{and} \quad d^3 = \left\{ \frac{55}{2\sqrt{2}}, -\frac{5}{\sqrt{2}} \right\}
\]

\[
a^4 = \left\{ \frac{105}{4} \right\} \quad \text{and} \quad d^4 = \left\{ \frac{45}{4} \right\}
\]

Figure 9 shows the details of first level Haar wavelet transform. In example 5.1, the sequence had an even number of terms; if this is not the case, pad the sequence with zeros on both ends. Example 5.2 shows that the energy of the wavelet transform is conserved at every level.
5. DISCRETE WAVELET TRANSFORMS

\[
\begin{array}{ccccccc}
d_k & \mathbf{x} \cdot \phi_1^d & a_k & \mathbf{x} \cdot \psi_1^d & \psi_1^d & \psi_1^d, \ldots, N/2 & \phi_1^d & \phi_1^d, \ldots, N/2 \\
d_1 & \frac{11}{\sqrt{2}} & a_1 & \frac{47}{\sqrt{2}} & \psi_1^d & \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right\} & \phi_1^d & \left\{ \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right\} \\
d_2 & 2\sqrt{2} & a_2 & 9\sqrt{2} & \psi_2^d & \left\{ 0, 0, 0 \right\} & \phi_2^d & \left\{ 0, 0, 0 \right\} \\
d_3 & \frac{1}{\sqrt{2}} & a_3 & \frac{7}{\sqrt{2}} & \psi_3^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0, 0 \right\} & \phi_3^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0 \right\} \\
d_4 & -\frac{1}{\sqrt{2}} & a_4 & \frac{3}{\sqrt{2}} & \psi_4^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0 \right\} & \phi_4^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0 \right\} \\
d_5 & -2\sqrt{2} & a_5 & \sqrt{2} & \psi_5^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0 \right\} & \phi_5^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0 \right\} \\
d_6 & -\frac{11}{\sqrt{2}} & a_6 & \frac{3}{\sqrt{2}} & \psi_6^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0 \right\} & \phi_6^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0 \right\} \\
d_7 & -\frac{29}{\sqrt{2}} & a_7 & \frac{7}{\sqrt{2}} & \psi_7^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0 \right\} & \phi_7^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0 \right\} \\
d_8 & -38\sqrt{2} & a_8 & 9\sqrt{2} & \psi_8^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0 \right\} & \phi_8^d & \left\{ 0, 0, 0, 0, 0, 0, 0, 0 \right\}
\end{array}
\]

**Figure 9.** Details of the level 1 Haar wavelet transform.

Example 5.2. Use the largest \( a^k \) and all the \( d^k \) at each level found in the previous example to show that the energy of the transformation is conserved for every k-level transform. The energy of \( x \) is \( x \cdot x \) refer to Parseval’s relation 9 as a comparison.

**Solution.** In other words the energy is the square of the Euclidean norm of \( x \) or the sum of the squares of \( x \), denoted \( |x|^2 = 4935 \), and for vectors \( a^1 \) and \( d^1 \), the sums of the dot products \( a^1 \cdot a^1 \) and \( d^1 \cdot d^1 \). Therefore,

\[
\begin{align*}
a^1 \cdot a^1 + d^1 \cdot d^1 &= \left( \frac{47}{\sqrt{2}} \right)^2 + \left( 9\sqrt{2} \right)^2 + \left( \frac{7}{\sqrt{2}} \right)^2 + \left( \frac{3}{\sqrt{2}} \right)^2 + \sqrt{2}^2 + \left( \frac{3}{\sqrt{2}} \right)^2 \\
&+ \left( \frac{7}{\sqrt{2}} \right)^2 + \left( 9\sqrt{2} \right)^2 + \left( -\frac{11}{\sqrt{2}} \right)^2 + \left( -2\sqrt{2} \right)^2 + \left( -\frac{1}{\sqrt{2}} \right)^2 \\
&+ \left( \frac{1}{\sqrt{2}} \right)^2 + \left( 2\sqrt{2} \right)^2 + \left( \frac{11}{\sqrt{2}} \right)^2 + \left( \frac{29}{\sqrt{2}} \right)^2 + \left( 38\sqrt{2} \right)^2 \\
&= 4935,
\end{align*}
\]
and thus the energy for the first level Haar transform of the sequence is conserved. Similarly, the energy of the second level Haar transform,

\[ \mathbf{a}_2 \cdot \mathbf{a}_2 + \mathbf{d}_2 \cdot \mathbf{d}_2 + \mathbf{d}_1 \cdot \mathbf{d}_1 = \left( \frac{65}{2} \right)^2 + 5^2 + \left( \frac{5}{2} \right)^2 + \left( \frac{25}{2} \right)^2 + \left( \frac{-11}{\sqrt{2}} \right)^2 + \left( -2\sqrt{2} \right)^2 + \\
+ \left( -\frac{1}{\sqrt{2}} \right)^2 + \left( \frac{1}{\sqrt{2}} \right)^2 + \left( 2\sqrt{2} \right)^2 + \left( \frac{11}{\sqrt{2}} \right)^2 + \left( \frac{29}{\sqrt{2}} \right)^2 + \\
+ \left( 38\sqrt{2} \right)^2 + \left( -\frac{29}{\sqrt{2}} \right)^2 + (-2)^2 + \left( \frac{1}{2} \right)^2 + \left( \frac{11}{2} \right)^2 \]

\[ = 4935, \]

is conserved. The energy of the third level Haar transform

\[ \mathbf{a}_3 \cdot \mathbf{a}_3 + \mathbf{d}_3 \cdot \mathbf{d}_3 + \cdots + \mathbf{d}_1 \cdot \mathbf{d}_1 = \left( \frac{75}{2\sqrt{2}} \right)^2 + \left( \frac{15}{\sqrt{2}} \right)^2 + \left( -\frac{11}{\sqrt{2}} \right)^2 + \left( -2\sqrt{2} \right)^2 + \left( -\frac{1}{\sqrt{2}} \right)^2 + \\
+ \left( \frac{1}{\sqrt{2}} \right)^2 + \left( 2\sqrt{2} \right)^2 + \left( \frac{11}{\sqrt{2}} \right)^2 + \left( \frac{29}{\sqrt{2}} \right)^2 + \left( 38\sqrt{2} \right)^2 + \\
+ \left( -\frac{29}{2} \right)^2 + (-2)^2 + \left( \frac{1}{2} \right)^2 + \left( \frac{11}{2} \right)^2 + \left( -\frac{55}{2\sqrt{2}} \right)^2 + \left( 5\sqrt{2} \right)^2 \]

\[ = 4935, \]

is conserved. Even the energy of the fourth level Haar transform is

\[ \mathbf{a}_4 \cdot \mathbf{a}_4 + \mathbf{d}_4 \cdot \mathbf{d}_4 + \cdots + \mathbf{d}_1 \cdot \mathbf{d}_1 = \left( \frac{105}{4} \right)^2 + \left( -\frac{11}{\sqrt{2}} \right)^2 + \left( -2\sqrt{2} \right)^2 + \left( -\frac{1}{\sqrt{2}} \right)^2 + \left( \frac{1}{\sqrt{2}} \right)^2 + \\
+ \left( 2\sqrt{2} \right)^2 + \left( \frac{11}{\sqrt{2}} \right)^2 + \left( \frac{29}{\sqrt{2}} \right)^2 + \left( 38\sqrt{2} \right)^2 + \left( -\frac{29}{2} \right)^2 + \\
+ (-2)^2 + \left( \frac{1}{2} \right)^2 + \left( \frac{11}{2} \right)^2 + \left( -\frac{55}{2\sqrt{2}} \right)^2 + \left( 5\sqrt{2} \right)^2 + \left( -\frac{45}{4} \right)^2 \]

\[ = 4935. \]

Thus the energy of the sequence is conserved through the wavelet decomposition.
5.7. 1D Inverse Haar Transform. The inverse of the L-1 mapping transforms \(a^1\) and \(d^1\) back into \(x\) where

\[
x = a^1 + d^1.
\]

(5.58)

Similarly, the inverse L-\(k\) transform is the sums of the \(k\) running differences and the \(k^{th}\) running average. Hence,

\[
x = a^k + d^k + d^{k-1} + \cdots + d^1
\]

(5.59)

gives the original signal from the decomposed signal.

To show that the relation in equation 5.59 occurs, the following examples use the \(a^k\) and \(d^k\) from example 5.1 and performs the inverse wavelet transform to retrieve the original signal \(x\).

**Example 5.3.** Start with the fourth level Haar transform and perform the one dimensional inverse wavelet transform to recover the original signal.

**Solution.** To retrieve the original solution from the L-4 Haar transform one needs 5.57 and the \(d^k\)’s from 5.54 - 5.56. To retrieve \(a^{k-1}\) manually one performs the inverse transform by joining

\[
\frac{1}{\sqrt{2}}\{1, -1\} \cdot \{a^k, d^k\} \quad \text{and} \quad \frac{1}{\sqrt{2}}\{1, 1\} \cdot \{a^k, d^k\}.
\]

(5.60)

Since

\[
a^4 = \left\{ \frac{105}{4} \right\} \quad \text{and} \quad d^4 = \left\{ \frac{45}{4} \right\}
\]
then find that

$$a^3 = \left\{ \frac{1}{\sqrt{2}} \{1, -1\} \cdot \{a^4, d^4\}, \frac{1}{\sqrt{2}} \{1, 1\} \cdot \{a^4, d^4\} \right\}$$

$$= \left\{ \frac{1}{\sqrt{2}} \{1, -1\} \cdot \left\{ \frac{105}{4}, \frac{45}{4} \right\}, \frac{1}{\sqrt{2}} \{1, 1\} \cdot \left\{ \frac{105}{4}, \frac{45}{4} \right\} \right\}$$

$$= \left\{ \frac{75}{2\sqrt{2}}, 15 \right\}. $$

Next using the result from above and $d^3$ to find $a^2$

$$c^2 = \left\{ \frac{1}{\sqrt{2}} \{1, -1\} \cdot \{a^3, d^3\}, \frac{1}{\sqrt{2}} \{1, 1\} \cdot \{a^3, d^3\} \right\}$$

$$= \left\{ \frac{1}{\sqrt{2}} \{1, -1\} \cdot \left\{ \frac{75}{2\sqrt{2}}, \frac{15}{2\sqrt{2}}, -\frac{55}{2\sqrt{2}}, \frac{5\sqrt{2}}{2} \right\}, \frac{1}{\sqrt{2}} \{1, 1\} \cdot \left\{ \frac{75}{2\sqrt{2}}, \frac{15}{2\sqrt{2}}, -\frac{55}{2\sqrt{2}}, \frac{5\sqrt{2}}{2} \right\} \right\}$$

$$= \left( \frac{65}{2}, \frac{5}{2} \right) \left( \frac{5}{2}, \frac{25}{2} \right).$$

such that

$$a^2 = \{c_{1,1}^2, c_{2,1}^2, c_{1,2}^2, c_{2,2}^2\}. $$

Furthermore, the previous result and $d^2$ gives $a^1$

$$c^1 = \left\{ \frac{1}{\sqrt{2}} \{1, -1\} \cdot \{a^2, d^2\}, \frac{1}{\sqrt{2}} \{1, 1\} \cdot \{a^2, d^2\} \right\}$$

$$= \left\{ \frac{1}{\sqrt{2}} \{1, -1\} \cdot \left\{ \frac{29}{2}, 2, -\frac{1}{2}, -\frac{11}{2}, \frac{65}{2}, 5, \frac{5}{2}, \frac{25}{2} \right\}, \frac{1}{\sqrt{2}} \{1, 1\} \cdot \left\{ \frac{29}{2}, 2, -\frac{1}{2}, -\frac{11}{2}, \frac{65}{2}, 5, \frac{5}{2}, \frac{25}{2} \right\} \right\}$$

$$= \left( \frac{47}{\sqrt{2}}, \frac{7}{\sqrt{2}} \right) \left( \frac{7}{\sqrt{2}}, \frac{9\sqrt{2}}{\sqrt{2}} \right).$$

where

$$a^1 = \{c_{1,1}^1, c_{2,1}^1, c_{1,2}^1, c_{2,2}^1, c_{1,3}^1, c_{2,3}^1, c_{1,4}^1, c_{2,4}^1\}. $$
Thus, to retrieve the original signal, perform the inverse transform on the result from above and \( d^1 \) to find \( x \)

\[
c^0 = \left\{ \frac{1}{\sqrt{2}} \left\{ 1, -1 \right\} \cdot \left\{ a^1, d^1 \right\}, \frac{1}{\sqrt{2}} \left\{ 1, 1 \right\} \cdot \left\{ a^1, d^1 \right\} \right\}
\]

\[
= \left\{ \frac{1}{\sqrt{2}} \left\{ 1, -1 \right\} \cdot \left\{ \frac{47}{\sqrt{2}}, \frac{9\sqrt{2}}{\sqrt{2}}, \frac{7}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{7}{\sqrt{2}}, \frac{9\sqrt{2}}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{2\sqrt{2}}{\sqrt{2}}, \frac{11}{\sqrt{2}}, \frac{29}{\sqrt{2}}, \frac{38\sqrt{2}}{\sqrt{2}} \right\}, \right. \\
\left. \frac{1}{\sqrt{2}} \left\{ 1, 1 \right\} \cdot \left\{ \frac{47}{\sqrt{2}}, \frac{9\sqrt{2}}{\sqrt{2}}, \frac{7}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{7}{\sqrt{2}}, \frac{9\sqrt{2}}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{2\sqrt{2}}{\sqrt{2}}, \frac{11}{\sqrt{2}}, \frac{29}{\sqrt{2}}, \frac{38\sqrt{2}}{\sqrt{2}} \right\} \right\}
\]

\[
= \begin{pmatrix}
29 & 11 & 4 & 1 & -1 & -4 & -11 & -29 \\
18 & 7 & 3 & 2 & 3 & 7 & 18 & 47
\end{pmatrix}.
\]

where

\[
x = \left\{ c^0_{1,1}, c^0_{2,1}, c^0_{1,2}, c^0_{2,2}, c^0_{1,3}, c^0_{2,3}, c^0_{1,4}, c^0_{2,4}, c^0_{1,5}, c^0_{2,5}, c^0_{1,6}, c^0_{2,6}, c^0_{1,7}, c^0_{2,7}, c^0_{1,8}, c^0_{2,8} \right\}.
\]

In other words, transpose and flatten the matrix \( c^0 \) to retrieve the original signal.

Understanding the mathematics behind a one dimensional wavelet transform, multiresolution analysis, and subband coding leads to the next section on wavelet transforms and image processing in two dimensions.
6. Wavelet Transforms and Image Processing

This chapter will examine the two dimensional wavelet transform, discuss topics relevant to multiresolution analysis, apply the Daubechies Wavelet with and without convolution in conjunction with the Laplacian on an image and show how the different levels of decomposition are used to depict images. Since analyzing images is the goal, the two dimensional discrete wavelet is defined in the next section.

$$x(s, t) = a \varphi(i + 1, j, k)$$

Figure 10. The Original Image

Figure 11 represents the image decomposition after one and two passes through a two dimensional discrete wavelet transform.

6.1. Wavelets in Two Dimensions. In two dimensions, the father scaling function 5.1 is a separable function defined as

$$\varphi(s, t) = \varphi(s) \varphi(t)$$

that has the scaled and translated basis function

$$\varphi_{i,j,k}(s,t) = 2^{i/2} \varphi(2^i s - j, 2^i t - k)$$
where $i$ is the level of resolution. The mother wavelet \ref{eq:5.4} in two dimensions is given as

\begin{equation}
\psi^t(s, t) \quad \text{for} \quad t = \{H, V, D\}
\end{equation}

where $H$ stands for horizontal, $V$ stands for vertical, and $D$ stands for diagonal.

The mother wavelet has three basis functions

\begin{equation}
\psi^H(s, t) = \psi(s) \varphi(t)
\end{equation}

\begin{equation}
\psi^V(s, t) = \varphi(s) \psi(t)
\end{equation}

\begin{equation}
\psi^D(s, t) = \psi(s) \psi(t)
\end{equation}

with scaled and translated versions

\begin{equation}
\psi^H_{i,j,k}(s, t) = 2^{i/2} \psi^H(2^i s - j, 2^i t - k)
\end{equation}

\begin{equation}
\psi^V_{i,j,k}(s, t) = 2^{i/2} \psi^V(2^i s - j, 2^i t - k)
\end{equation}
\[ \psi_{i,j,k}^D(s,t) = 2^{i/2} \psi^D(2^i s - j, 2^i t - k). \]

Notice that the wavelet basis functions are also separable. This next part defines the two dimensional DWT since images are discrete functions.

The two dimensional DWT for an \( J \times K \) image, \( x(s,t) \), implements the separable, one dimensional scaling and wavelet functions first on the rows and then the columns or vice versa. This gives the following approximation coefficients for an arbitrary scale \( i_0 \)

\[ a_{\varphi}(i_0, j, k) = \frac{1}{\sqrt{JK}} \sum_{s=0}^{J-1} \sum_{t=0}^{K-1} x(s,t) \varphi_{i_0,j,k}(s,t) \]  

and the detail coefficients are given as

\[ d_{\psi}^l(i, j, k) = \frac{1}{\sqrt{JK}} \sum_{s=0}^{J-1} \sum_{t=0}^{K-1} x(s,t) \psi_{i_0,j, k}^l(s,t) \text{ for } l = \{H,V,D\}. \]

Figure 12 visualizes one pass of an image through the analysis portion of the 2D FWT.

The two dimensional DWT can also utilize digital filters and downsampling to implement the one dimensional FWT on the rows, and then on the columns of the output from the first two-band filter bank. After convolution on the rows, the columns are downsampled to produce the approximation and detail subimages, which contain the vertical information, and have horizontal resolutions reduced by a factor of two. These subimages are each sent through another set of highpass and lowpass filters that convoles the columns and then downsamples the rows. As seen in figure 12, the approximation subimage, \( a_{\varphi}(i, j, k) \), went through two lowpass filters and downsamplers, which reduce its resolution by a factor of four. The horizontal detail subimage, \( d_{\psi}^H(i, j, k) \), was sent through the lowpass filter that retains the vertical approximations and then the highpass filter on the columns that retains
6. WAVELET TRANSFORMS AND IMAGE PROCESSING

\[ a_\phi(i + 1, j, k) \ast h_\phi(−k) \downarrow 2 \ast h_\phi(−j) \downarrow 2 a_\phi(i, j, k) \]

\[ a_\psi(i + 1, j, k) \ast h_\psi(−k) \downarrow 2 \ast h_\psi(−j) \downarrow 2 d_V \psi(i, j, k) \]

\[ a_\psi(i, j, k) \ast h_\psi(−j) \downarrow 2 d_H \psi(i, j, k) \]

\[ a_\psi(i, j, k) \ast h_\psi(−j) \downarrow 2 d_D \psi(i, j, k) \]

**Figure 12. Two Dimensional Analysis Filter Bank**

the horizontal details. The vertical detail subimage, \( d_V \psi(i, j, k) \), went through the highpass filter on the rows that retains the high vertical frequencies, and then the lowpass filter on the columns, which filters out the horizontal details, keeping only the low horizontal frequencies. Whereas, the diagonal subimage, \( d_D \psi(i, j, k) \), passed through both high pass filters, thus retaining the details in the vertical and then the horizontal direction. Since all of these subimages were downsampled by 2 in both the horizontal and vertical directions, all of them have been reduced by a factor of four.

Furthermore, given equations 6.7, 6.8-6.11 the two dimensional inverse DWT is

\[
x(s, t) = \frac{1}{\sqrt{JK}} \sum_j \sum_k a_\phi(i_0, j, k) \varphi_{i_0, j, k}(s, t) + \frac{1}{\sqrt{JK}} \sum_l \sum_{i=i_0}^{\infty} \sum_j \sum_k d_l \psi(i, j, k) \psi_{l, j, k}(s, t)
\]

for \( l = \{H, V, D\} \).
The original image can be reproduced through the synthesis filter bank shown in figure 13.

\[
a_{\phi}(i+1, j, k) + a_{\psi}(i, j, k) + h_{\phi}(k) \uparrow 2 + h_{\psi}(j) \uparrow 2
\]

\[
\ast h_{\phi}(k) \uparrow 2 + h_{\psi}(j) \uparrow 2 \ast dV_{\psi}(i, j, k) + \ast h_{\psi}(j) \uparrow 2 \ast dH_{\psi}(i, j, k) + \ast h_{\psi}(j) \uparrow 2 \ast dD_{\psi}(i, j, k)
\]

Figure 13. Two Dimensional Synthesis Filter Bank

6.2. Transforming from the Spatial Domain to the scale-translation Domain. When an image is transformed via a DWT from the spatial domain to the scale-translation domain, the transformation process is referred to as a forward DWT. The forward DWT process can be performed with Mathematica’s DiscreteWaveletTransform command. Peaks in the data, or in other words, most of the information, are contained in the low frequencies, whereas noise in the data is typically located within high frequencies. This allows easy modification of an image to remove the noise.

For example, utilize the Haar wavelet to move from lower resolution to the next higher resolution. In figure 14 there are three sets of detail coefficients and one set of approximation coefficients associated with each refinement level. By
default, Mathematica returns only the detail coefficients at each level. However, it is useful to analyze the approximation coefficients, which will be shown in another section on transforming to the spatial domain. Before the next example, there is an introduction to the Laplacian convolution filter.

The Laplacian is useful in image processing for edge detection. The Laplacian, denoted $L(x, y)$, is the sum of the second partial derivatives of the pixel intensity of an image, denoted $I(x,y)$, and is given by the following
\[ L(x, t) = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} \]

6.2.1. The Laplacian Filter. The Laplacian filter involves the process of convolution between a kernel and the image. This filter is used to detect the peaks, or the rapid intensity changes, of an image by an isotropic measurement of its second spatial derivative. In other words, measurement of its second spatial derivative that do not have directional variation. To implement the Laplacian on a discrete set of pixels, find a discrete convolution kernel to approximate the second partial derivatives defined above.

Figure 15 is an image after convolution with the Laplacian filter that implements a kernel with positive peaks.
Next is a visualization of the Daubechies wavelet transform alone and in combination with a Laplacian convolution filter. Figure 6.2.1 shows the original image and its transform into the wavelet domain.
The highest frequency in each refinement is located in the lower right corner of the pictures, and notice the details captured with each refinement level. Realize that the original photo is the input for the Daubechies wavelet transform without convolution. However, the input for the transform including convolution with the Laplacian is the image convolved with the Laplacian before it is transformed.

Notice the detail of the robot mouth is less visible when the Daubechies wavelet is used alone. Thus, wavelets in conjunction with other methods may produce the best result. The choice of refinement levels was to save on computation time and allow the image to be analyzed subjectively.
6.3. Transforming from the scale-translation Domain to the Spatial Domain. After an image is manipulated in the scale-translation domain, the image is transformed from the scale-translation domain to the spatial domain: this transformation process is referred to as an inverse DWT. The inverse DWT process can be performed with Mathematica’s `InverseWaveletTransform` command. In the following, the Daubechies wavelet is utilized to perform the DWT, which sharpens the coefficients, and then the inverse DWT is performed. Notice the sharper outlines on the arms of the robot in figure 6.3.
Recall figure 15, which is the image convolved with the Laplacian filter. Although the wavelet image plots in figure 6.2.1 are similar, the inverse DWT produces very different results, as shown in figure 16, which does not resemble the input.

![Image](image.png)

**Figure 16.** Inverse DWT on an Image Convolved with the Laplacian

Now, let's use the approximation and detail coefficients from different refinement levels in figure 14, and the inverse DWT to reconstruct the image. The top left image in figure 17 invoked the `Automatic` setting in Mathematica and truncated 52.1% of the coefficients. The top right image was reconstructed using
Figure 17. Image Reconstruction via the IDWT

the coefficients from the first refinement level which truncated 51.8% coefficients.

Whereas, the images in the bottom row were reconstructed from the second and third refinement levels, and truncated 87.5% and 96.7% coefficients, respectively.
7. Conclusion

This thesis encompassed wavelet transformations through multiresolution analysis on digital signals. It discussed the fundamentals of digital signal processing and discrete time signals, the technique of convolution, and the properties of discrete time and linear time invariant systems. Following these fundamentals, the mathematical foundation of the Laplace and Fourier transforms, along with their properties, that relate to the continuous wavelet transform, were analyzed. Next, the discrete transforms were brought to an understanding through the z-transform, the DFT, and the DCT. Furthermore, the sampling process, separability, truncation, and the measurements, such as the energy relation and power spectrum, were introduced. And finally, derivation of the discrete wavelet transform with the underlying theory of subband coding through filter banks, multiresolution analysis, and the relation to the fast wavelet transform, were considered when performing wavelet transforms on images. With future research, wavelet coefficients could be used to produce a three dimensional representation of a two dimensional figure.
Bibliography


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