I, Andoniaina Rarivoarimanana, hereby submit this original work as part of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences.

It is entitled:
Unbalanced Urn Models and Applications

Student's name: Andoniaina Rarivoarimanana

This work and its defense approved by:

Committee chair: Wlodzimierz Bryc, Ph.D.

Committee member: Joanna Mitro, Ph.D.

Committee member: Magda Peligrad, Ph.D.

University of Cincinnati
Abstract

Unbalanced Urn Models and Applications

We study the urn models with two colors where the number of balls added after each draw or each pair of draws depends on the color drawn (unbalanced). We show that the results for balanced urn models where the number of balls added after each draw remains constant extend to the unbalanced case. The average total number of balls with respect to discrete time \( n \) converges to the principal eigenvalue of the replacement matrix of the urn and the limiting fraction of balls of a given color is related to the eigenvector of the replacement matrix. Next, we consider the central limit theorem for the unbalanced urn models using the technique due to Mahmoud (2008) for balanced urn models. Then, we show that the generalized binomial distribution of Drezner and Farnum (1993) and a modified University Placement Test algorithm can be embedded into the generalized Pólya-Eggenberger and unbalanced urn models respectively.
Acknowledgements

I would like to thank my advisor Dr. Wlodek Bryc for his patience, kindness and willingness to spend countless hours with me throughout the course of my study. Thanks also to Dr. Joanna Mitro and Dr. Magda Peligrad for serving on my committee.
Contents

Abstract i
Acknowledgements iii
Contents iv
List of Figures vi

1 Introduction 1
   1.1 Preliminaries ................................................. 1
      1.1.1 Notation from Analysis ................................. 1
      1.1.2 Approximations of ratio gamma functions ............... 2
      1.1.3 Some Useful Inequalities ............................... 4
      1.1.4 Auxiliary definition and theorems from probability .... 7
   1.2 Urn Models .................................................. 8
      1.2.1 Notation .................................................. 8
      1.2.2 History of Generalized Pólya-Eggenberger Urn Models .... 9

2 Unbalanced Urn Models 14
   2.1 Limit Theorem for Simple Draw .............................. 15
   2.2 Limit Theorem for Urns Evolving by Two Draws .............. 38

3 Central Limit Theorem 49
   3.1 Martingale Central Limit Theorem ........................... 49
   3.2 Construction of Martingale ................................. 50
3.3 Main Result .................................................. 70

4 Application of balanced and unbalanced urn models ................. 73
   4.1 Drezner Generalized Binomial Distribution ...................... 73
      4.1.1 Definition ........................................... 73
      4.1.2 Main Results ......................................... 74
   4.2 Adaptive Tests ............................................. 78
      4.2.1 Introduction ......................................... 78
      4.2.2 Item Characteristic Curve .............................. 79
      4.2.3 Adaptive Testing Algorithms ........................... 81
      4.2.4 UC Math Placement Test Algorithm ................. 83
      4.2.5 UC Math Placement Test Scheme Modified .......... 87

Bibliography .......................................................... 92
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>$\mathbb{P}(Y = 1</td>
<td>x, \theta) = \frac{1}{1+e^{x-\theta}}$: Probability of correct response for Rasch model</td>
</tr>
<tr>
<td>4.2</td>
<td>Simulation of $\sqrt{n} (x_n - \theta)$ for UCMPT</td>
<td>87</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

1.1 Preliminaries

In this section we list some notations used throughout the dissertation. We denote by \( \Omega \) the sample space, \( \mathcal{F} \) the \( \sigma \)-algebra of events, and \( \mathbb{P} \) the probability measure. For a random variable \( X \) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) whose moment of order \( r > 0 \), \( \mathbb{E}(|X|^r) \) is finite, we denote by \( \|X\|_r = \left[ \mathbb{E}(|X|^r) \right]^{1/r} \). The notation \((a.s.)\) stands for almost surely. For \( \mu \in \mathbb{R} \) and \( \sigma > 0 \), \( \mathcal{N}(\mu, \sigma^2) \) denotes a random variable whose distribution is normal with mean \( \mu \) and variance \( \sigma^2 \). \( X_n \overset{c.c.}{\longrightarrow} X \), \( X_n \overset{a.s.}{\longrightarrow} X \), \( X_n \overset{p}{\longrightarrow} X \), \( X_n \overset{D}{\longrightarrow} X \), and \( X_n \rightarrow X \) in \( L^r \) denote \( X_n \) converges completely, almost surely (a.s.), in probability, in distribution, and in \( r \)-mean to \( X \) respectively.

1.1.1 Notation from Analysis

We use the standard Landau notation (see (Billingsley, 1986), Appendix A18, page 568). Suppose that \((x_n)_n\), \((y_n)_n\), and \((z_n)_n\) are real sequences and \( y_n > 0 \). Then

(i) \( x_n = \mathcal{O}(y_n) \) means \( \frac{x_n}{y_n} \) is bounded.

(ii) \( x_n \sim y_n \) means \( \frac{x_n}{y_n} \to 1 \) as \( n \to \infty \).

(iii) \( x_n = z_n + \mathcal{O}(y_n) \) means \( x_n = z_n + u_n \) for some real sequence \((u_n)_n\) satisfying \( u_n = \mathcal{O}(y_n) \).
1.1.2 Approximations of ratio gamma functions

Let $\Gamma(x)$ denote the gamma function defined by

$$
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0.
$$

The function $\Gamma$ is analytic on $(0, \infty)$ and for any $x > 0$,

$$
\Gamma(x + 1) = x\Gamma(x).
$$

Thus, for $x > 0$ and $n \in \mathbb{N},$

$$
\Gamma(x + n) = (x + n - 1)\Gamma(x + n - 1)
\quad = (x + n - 1)(x + n - 2)\Gamma(x + n - 2)
\quad = (x + n - 1)(x + k - 2) \cdots (x + n - (n - 1))(x + n - n)\Gamma(x + n - n)
\quad = \prod_{j=0}^{n-1} (x + j) \Gamma(x).
$$

So,

$$
(1.1) \quad \frac{\Gamma(n + x)}{\Gamma(x)} = \prod_{j=0}^{n-1} (j + x).
$$

For $x > 0$, $c_1 \geq 0$, and $c_2 \geq 0$, (Tricomi and Erdlyi, 1951), page 1

$$
(1.2) \quad \frac{\Gamma(x + c_1)}{\Gamma(x + c_2)} = x^{c_1-c_2} \left[ 1 + \frac{(c_1 - c_2)(c_1 + c_2 - 1)}{2x} + O(x^{-2}) \right]
$$

as $x \to \infty$. We have the following proposition:

**Proposition 1.1.** Let $c_1 > 0$, and $c_2 > 0$. Then,

(i)

$$
(1.3) \quad \lim_{n \to \infty} n^{c_2-c_1} \frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} = 1,
$$
(1.4) \[ \lim_{n \to \infty} n^{2(c_2-c_1)} \left[ \frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} \right]^2 = 1. \]

(ii) there exists a constant \( C_1 = C_1(c_1, c_2) > 0 \) such that

(1.5) \[ \frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} \leq C_1 n^{c_1-c_2} \quad \text{for all } n \geq 1. \]

Proof. (i) Substituting \( x \) by \( n, \ n \in \mathbb{N} \), in (1.2), we have

(1.6) \[ \frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} = n^{c_1-c_2} \left[ 1 + \frac{(c_1 - c_2)(c_1 + c_2 - 1)}{2n} + O\left(n^{-2}\right) \right] \]

as \( n \to \infty. \)

Multiplying both sides of (1.6) by \( n^{c_2-c_1} \), we get

\[ n^{c_2-c_1} \frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} = 1 + \frac{(c_1 - c_2)(c_1 + c_2 - 1)}{2n} + O\left(n^{-2}\right) \quad \text{as } n \to \infty. \]

Hence, \( \lim_{n \to \infty} n^{c_2-c_1} \frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} = 1. \)

\[ \lim_{n \to \infty} n^{2(c_2-c_1)} \left[ \frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} \right]^2 = \lim_{n \to \infty} \left[ n^{c_2-c_1} \frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} \right]^2 = 1^2 = 1. \]

To prove (ii), by (1.6), there exist a constant \( C'_1(c_1, c_2) > 0, \ N(C'_1) \in \mathbb{N} \) such that for all \( n \geq N, \)

\[ \left| \frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} - n^{c_1-c_2} \left[ 1 + \frac{(c_1 - c_2)(c_1 + c_2 - 1)}{2n} \right] \right| \leq C'_1 n^{c_1-c_2-2}. \]

Hence, for all \( n \geq N, \)

\[ \frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} \leq n^{c_1-c_2} \left[ 1 + \frac{(c_1 - c_2)(c_1 + c_2 - 1)}{2n} + C'_1 n^{-2} \right]. \]

Since each of the terms \( \frac{(c_1 - c_2)(c_1 + c_2 - 1)}{2n}, C_1 n^{-2} \) is bounded by a constant, we get

\[ \]
for all $n \geq N$,
\[
\frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} \leq C''_{1} n^{c_1-c_2},
\]
for some constant $C''(c_1, c_2) > 0$. It follows that for all $n \geq 1$,
\[
\frac{\Gamma(n + c_1)}{\Gamma(n + c_2)} \leq C_1 n^{c_1-c_2},
\]
for some constant $C_1(c_1, c_2) > 0$. 

### 1.1.3 Some Useful Inequalities

In this subsection, we state a number of mathematical inequalities which will be useful later. The following lemma is a version of results from (Gut, 2005), Lemma 3.1. page 559.

**Lemma 1.2.** Let $\gamma > 0$.

(i) For $n \geq 1$,

\[
\sum_{j=1}^{n} j^{\gamma-1} \leq \begin{cases} 
\frac{(n+1)^\gamma}{\gamma}, & \text{if } \gamma > 1 \\
n, & \text{if } \gamma = 1 \\
\frac{n^\gamma}{\gamma}, & \text{if } 0 < \gamma < 1,
\end{cases}
\]

and

\[
\lim_{n \to \infty} \frac{1}{n^\gamma} \sum_{j=1}^{n} j^{\gamma-1} = \frac{1}{\gamma}.
\]

(ii) For $n \geq 3$,

\[
\log n \leq \sum_{j=1}^{n} \frac{1}{j} \leq \log n + 1 \leq 2 \log n,
\]
\begin{align*}
\lim_{n \to \infty} \frac{1}{\log n} \sum_{j=1}^{n} \frac{1}{j} &= 1,
\intertext{(1.11)}
\sum_{j=2}^{n} \frac{1}{j} (\log j)^{\gamma - 1} &\leq C_1 (\log n)^\gamma,
\intertext{for some constant \(C_1 = C_1(\gamma) > 0,\) and}
\lim_{n \to \infty} \frac{1}{(\log n)^\gamma} \sum_{j=2}^{n} \frac{1}{j} (\log j)^{\gamma - 1} &= \frac{1}{\gamma}.
\end{align*}

We give a complete proof since these results are important for our thesis and Gut only gives a hint on how to prove them.

Proof. (i) Let \(n \in \mathbb{N}.\) Since \(x \mapsto x^{\gamma - 1}, x \geq 1,\) is increasing for \(\gamma > 1,\) constant for \(\gamma = 1,\) and decreasing for \(0 < \gamma < 1,\) we have the following three cases.

Case 1: \(\gamma > 1\)

\[
\sum_{j=1}^{n} j^{\gamma - 1} = \sum_{j=1}^{n} \int_{j}^{j+1} j^{\gamma - 1} dx \leq \sum_{j=1}^{n} \int_{j}^{j+1} x^{\gamma - 1} dx = \int_{1}^{n+1} x^{\gamma - 1} dx = \frac{(n+1)^\gamma - 1}{\gamma} \leq \frac{(n+1)^\gamma}{\gamma}.
\]

Case 2: \(\gamma = 1\)

\[
\sum_{j=1}^{n} j^{\gamma - 1} = \sum_{j=1}^{n} 1 = n \leq n.
\]

Case 3: \(0 < \gamma < 1\)

\[
\sum_{j=1}^{n} j^{\gamma - 1} = \sum_{j=1}^{n} \int_{j-1}^{j} j^{\gamma - 1} dx \leq \sum_{j=1}^{n} \int_{j-1}^{j} x^{\gamma - 1} dx = \int_{0}^{n} x^{\gamma - 1} dx = \frac{n^\gamma}{\gamma}.
\]

Since \(\frac{1}{n^\gamma} \sum_{j=1}^{n} j^{\gamma - 1} = \frac{1}{n} \sum_{j=1}^{n} \left( \frac{j}{n} \right)^{\gamma - 1}\) is the right-hand Riemann sum of the function \(x^{\gamma - 1}\)
with partition  \( \frac{1}{n} \) over the interval \([0, 1]\), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} j^{\gamma - 1} = \int_{0}^{1} x^{\gamma - 1} dx = \frac{1}{\gamma}.
\]

(ii) Since convergent sequences are bounded, the inequality (1.11) follows from (1.12).

To prove (1.12), since \( \lim_{n \to \infty} (\log n)^\gamma = \infty \), it suffices to give two-sided bounds for

\[
\sum_{j=J}^{n} \frac{(\log j)^{\gamma - 1}}{j}
\]

for some \( J \in \mathbb{N}, 2 \leq J \leq n \).

We choose \( J \in \mathbb{N} \) large enough so that the function \( x \mapsto \frac{(\log x)^{\gamma - 1}}{x} \) is decreasing for \( x \geq J \).

We have

\[
\sum_{j=J}^{n} \frac{(\log j)^{\gamma - 1}}{j} = \sum_{j=J}^{n-1} \frac{(\log j)^{\gamma - 1}}{j} + \frac{(\log n)^{\gamma - 1}}{n}
= \sum_{j=J}^{n-1} \frac{(\log j)^{\gamma - 1}}{j} \int_{j}^{j+1} \frac{1}{j} dx + \frac{(\log n)^{\gamma - 1}}{n}
\geq \sum_{j=J}^{n-1} \int_{j}^{j+1} \frac{(\log x)^{\gamma - 1}}{x} dx + \frac{(\log n)^{\gamma - 1}}{n}
= \int_{J}^{n} \frac{(\log x)^{\gamma - 1}}{x} dx + \frac{(\log n)^{\gamma - 1}}{n}
= \frac{(\log n)^{\gamma}}{\gamma} - \frac{\log J)^{\gamma}}{\gamma} + \frac{(\log n)^{\gamma - 1}}{n},
\]

and

\[
\sum_{j=J}^{n} \frac{(\log j)^{\gamma - 1}}{j} = \frac{(\log J)^{\gamma - 1}}{J} + \sum_{j=J+1}^{n} \frac{(\log j)^{\gamma - 1}}{j}
= \frac{(\log J)^{\gamma - 1}}{J} + \sum_{j=J+1}^{n} \frac{(\log j)^{\gamma - 1}}{j} \cdot \int_{j-1}^{j} dx
\leq \frac{(\log J)^{\gamma - 1}}{J} + \sum_{j=J+1}^{n} \int_{j-1}^{j} \frac{(\log x)^{\gamma - 1}}{x} dx = \frac{(\log J)^{\gamma - 1}}{J} + \int_{J}^{n} \frac{(\log x)^{\gamma - 1}}{x} dx
= \frac{(\log J)^{\gamma - 1}}{J} + \frac{(\log n)^{\gamma}}{\gamma} - \frac{(\log J)^{\gamma}}{\gamma}.
\]

Thus,

(1.13)

\[
\frac{(\log n)^{\gamma}}{\gamma} - \frac{(\log J)^{\gamma}}{\gamma} + \frac{(\log n)^{\gamma - 1}}{n} \leq \sum_{j=J}^{n} \frac{(\log j)^{\gamma - 1}}{j} \leq \frac{(\log n)^{\gamma}}{\gamma} - \frac{(\log J)^{\gamma}}{\gamma} + \frac{(\log J)^{\gamma - 1}}{J}.
\]
Since \( \sum_{j=2}^{J-1} \frac{(\log j)^{\gamma-1}}{j} \) is finite and \( \lim_{n \to \infty} (\log n)^{\gamma} = \infty \), we have

\[
\lim_{n \to \infty} \frac{1}{(\log n)^{\gamma}} \sum_{j=2}^{n} \frac{(\log j)^{\gamma-1}}{j} = \lim_{n \to \infty} \frac{1}{n^{\gamma}} \sum_{j=1}^{n} \frac{(\log j)^{\gamma-1}}{j} = \frac{1}{\gamma} \text{ by (1.13).}
\]

\[\square\]

### 1.1.4 Auxiliary definition and theorems from probability

We now state a definition and the theorems that will be used throughout this dissertation.

**Definition 1.1.1** ([Hsu and Robbins, 1947]). Let \( X_1, X_2, \cdots \) be random variables. \( X_n \) converges completely to 0 as \( n \to \infty \) if for every \( \epsilon > 0 \),

\[
\sum_{n=1}^{\infty} P(\{|X_n| > \epsilon\}) < \infty.
\]

**Remark 1.1.1.** The complete convergence implies the convergence almost surely by the Borel-Cantelli lemma.

**Theorem 1.3.** Let \( X, X_1, X_2, \cdots \) and \( Y, Y_1, Y_2, \cdots \) be real-valued random variables such that

\[ X_n \overset{a.s.}{\to} X \quad \text{and} \quad Y_n \overset{a.s.}{\to} Y \quad \text{as} \ n \to \infty, \]

and suppose that \( f : \mathbb{R}^2 \to \mathbb{R} \) is continuous. Then

\[ f(X_n, Y_n) \overset{a.s.}{\to} f(X, Y) \quad \text{as} \ n \to \infty. \]

**Theorem 1.4** (Cramér’s theorem or Slutsky’s theorem). Let \( X_1, X_2, \cdots \) and \( Y_1, Y_2, \cdots \) be sequences of random variables. Suppose that

\[ X_n \overset{P}{\to} X \quad \text{and} \quad Y_n \overset{P}{\to} c \quad \text{as} \ n \to \infty, \]
where $c$ is some constant. Then

(i) $X_n + Y_n \xrightarrow{D} X + c,$

(ii) $X_n - Y_n \xrightarrow{D} X - c,$

(iii) $X_n Y_n \xrightarrow{D} cX,$

(iv) $\frac{X_n}{Y_n} \xrightarrow{D} \frac{X}{c}$ for $c \neq 0$, as $n \to \infty$.

**Theorem 1.5** (The Cauchy-Schwarz inequality). Suppose that the random variables $X$ and $Y$ have finite variances. Then

$$E|XY| = \|XY\|_1 \leq \|X\|_2 \|Y\|_2 = \sqrt{E(X^2)E(Y^2)}.$$ 

**Theorem 1.6** (The Lebesgue dominated convergence theorem). Let $X$ and $X_1, X_2, \cdots$ be random variables. Suppose that $X_n \xrightarrow{a.s.} X$ as $n \to \infty$, and that for all $n \in \mathbb{N}$, $|X_n| \leq Y$ for some random variable $Y$ with $E(Y) < \infty$. Then

$$E(|X_n - X|) \to 0 \quad \text{and} \quad E(X_n) \to E(X) \quad \text{as} \quad n \to \infty.$$ 

### 1.2 Urn Models

#### 1.2.1 Notation

We consider an urn with two colors. Assume the urn initially contains a total of $t_0 > 0$ balls, of which $w_0$ are white and $b_0 = t_0 - w_0$ black. At each time epoch, a ball is drawn uniformly at random from the urn, its color is noted, and then it is put back in the urn. If a white ball is drawn, $a$ white balls and $b$ black balls are added in the urn; if a black ball is drawn, $c$ white balls and $d$ black balls are added in the urn. A negative value of $a, b, c, \text{or} \ d$ indicates that that balls are removed from the urn instead of being added.
This can be represented by the replacement matrix

\begin{equation}
A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{equation}

The rows indicate the color of the ball drawn whereas the columns represent the number of balls added (removed). For instance, the first row indicates a white ball is drawn and the first column the number of white balls added (removed) into the urn.

Let $W_n$, $B_n$, $T_n$, $\frac{W_n}{T_n}$ be the number of white balls, the number of black balls, the total number of balls, the proportion of white balls in the urn after $n$ draws, respectively.

Since we allow non-integer values for $a, b, c, d$, a more convenient definition is to define $(W_n, B_n)$ as an $\mathbb{R}_+^2$-valued Markov chain with transition probabilities

\[
\begin{align*}
\mathbb{P} [(W_n, B_n) = (W_{n-1}, B_{n-1}) + (a, b) | W_{n-1}, B_{n-1}] &= \frac{W_{n-1}}{T_{n-1}}, \\
\mathbb{P} [(W_n, B_n) = (W_{n-1}, B_{n-1}) + (c, d) | W_{n-1}, B_{n-1}] &= \frac{B_{n-1}}{T_{n-1}},
\end{align*}
\]

where $(W_0, B_0) = (w_0, b_0)$.

### 1.2.2 History of Generalized Pólya-Eggenberger Urn Models

In 1923, Eggenberger and Pólya (Eggenberger and Pólya, 1923) studied the urn model well-known in literature as the Pólya urn. They used it to derive the probability of the number of infected cases during an epidemic (Friedman, 1949, p.1). Since 1940, many authors have generalized this model in many different ways and by different methods. These models are commonly known as generalized Pólya-Eggenberger models.

In the Pólya urn, the added balls are always of the same color as the ball drawn. The corresponding replacement matrix is

\[
A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}
\]

where $a$ is a positive real number.
Theorem 1.7 (Eggenberger and Pólya, 1923). \( \frac{W_n}{T_n} \) converges a.s. to a random variable that has a beta distribution with parameters \( \frac{w_0}{a}, \frac{b_0}{a} \).

Bernstein (Bernstein, 1940b), Savkevitch (Savkevitch, 1940), and Friedman (Friedman, 1949) all independently generalized the Pólya urn model by adding \( b \) balls of the opposite color of the ball drawn. This urn model is known as Friedman’s urn. The corresponding replacement matrix is

\[
A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.
\]

(1.15)

Bernstein (Bernstein, 1940a) considered the following generalization that corresponds to the replacement matrix

\[
A = \begin{pmatrix} a & s-a \\ c & s-c \end{pmatrix}
\]

(1.16)

where \( s > 0 \).

Theorem 1.8 (Bernstein, 1940a). If \( (s-a)(a-c) \neq 0 \) and \( a-c \leq \frac{s}{2} \), then \( W_n \) is normally distributed in the limit.

D. Freedman (Freedman, 1965) used the method of moments, and Athreya and Karlin (Athreya and Karlin, 1968) developed a groundbreaking technique of embedding certain urn schemes into continuous time Markov branching processes to get the asymptotic behavior of Friedman’s urn.

Theorem 1.9 (Freedman, 1965). Assume \( b > 0 \) and \( a \neq b \) (the case \( a = b \) is trivial). Let \( \rho := \frac{a-b}{a+b} \) be the ratio of eigenvalues of the matrix \( A \) in (1.15). Then,

(i) \( \frac{W_n}{T_n} \xrightarrow{a.s.} \frac{1}{2} \)

(ii) If \( \rho > \frac{1}{2} \), then \( \frac{W_n - B_n}{n^\rho} \xrightarrow{a.s.} H \) and for any \( 0 < r < \infty \), \( \frac{W_n - B_n}{n^\rho} \to H \) in \( L^r \)

where \( H \) is a nondegenerate limiting random variable.
(iii) If \( \rho < \frac{1}{2} \), then \( \frac{W_n - B_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N} \left( 0, \frac{(a - b)^2}{1 - 2\rho} \right) \)

(iv) If \( \rho = \frac{1}{2} \), then \( \frac{W_n - B_n}{\sqrt{n \log n}} \xrightarrow{d} \mathcal{N} \left( 0, (a - b)^2 \right) \).

In 1985, Bagchi and Pal (Bagchi and Pal, 1985) generalized the Pólya-Eggenberger urn to a model that allowed removing of the balls with tenable conditions but the number of balls added or removed for each draw remains constant. Hence the total number of balls in the urn \( T_n \) is nonrandom. The corresponding replacement matrix is given by (1.14), where \( a, b, c, d \) and \( s \) are integer-valued and satisfy:

(i) \( a + b = c + d = s \geq 1 \) (we say the urn is balanced).

(ii) \( w_0 + b_0 \geq 1, w_0 \geq 0, \) and \( b_0 \geq 0 \)

(iii) \( a \neq c \)

(iv) \( b = s - a > 0 \) (i.e. \( s > a \)) and \( c > 0 \)

(v) If \( a < 0 \) then \( a \) divides \( c \) and \( a \) divides \( w_0 \). Similarly, if \( d < 0 \), then \( d \) divides \( b \) and \( d \) divides \( b_0 \).

**Theorem 1.10** (Bagchi and Pal, 1985). Let \( \rho := \frac{a - c}{s} \) be the ratio of the eigenvalues of the matrix \( A \) in (1.16).

(i) \( \frac{W_n}{T_n} \xrightarrow{p} \frac{c}{b + c} \) and \( \mathbb{E}(W_n) \sim \frac{c}{b + c} T_n \) as \( n \to \infty \).

(ii) If \( \rho < \frac{1}{2} \), then \( \frac{W_n - \mathbb{E}(W_n)}{\sqrt{n}} \xrightarrow{d} \mathcal{N} \left( 0, \frac{b c \rho^2 s^2}{(1 - 2\rho)(b + c)^2} \right) \)

(iii) If \( \rho = \frac{1}{2} \), then \( \frac{W_n - \mathbb{E}(W_n)}{\sqrt{n \log n}} \xrightarrow{d} \mathcal{N} \left( 0, bc \right) \)

**Remark 1.2.1.** Their proof works for nonnegative real numbers \( a, b, c, d \). The condition \( a, b, c, d \) integers is only necessary for the assumption (v).

In 1989, Gouet (Gouet, 1989) used straightforward martingale arguments to improve the result (i) above to convergence a.s.. It also allows the case where \( (s - a)c = 0 \) and \( \max(s - a, c) > 0 \).
In 1993, Gouet (Gouet et al., 1993) proved a functional central limit theorem.

In 1996, Smythe (Smythe, 1996) generalized the assumptions of (Bagchi and Pal, 1985) by allowing the elements of (1.16) to be non-deterministic but the expected added numbers of balls remain constant, i.e. the matrix $E[A]$ has a constant row sum. He proved the following results:

**Theorem 1.11.** Let $\lambda_1, \lambda_2$ be the principal and the non-principal eigenvalues of the matrix $E[A]$, the expected value of the matrix $A$, $v = (v_1, v_2)$ the left eigenvector of $\lambda_1$. Denote by $X_n, Y_n$ the number of times a white (respectively a black) ball is drawn after $n$ draws.

(i) $\frac{T_n}{n} \xrightarrow{a.s.} \lambda_1$ as $n \to \infty$.

(ii) $E \left( \frac{T_n}{n} - \lambda_1 \right)^2 \leq \frac{C}{n}$ for a constant $C$.

(iii) $\frac{X_n}{n} \xrightarrow{p} v_1$, $\frac{Y_n}{n} \xrightarrow{p} v_2$, $\frac{W_n}{n} \xrightarrow{p} \lambda_1 v_1$, $\frac{B_n}{n} \xrightarrow{p} \lambda_1 v_2$.

(iv) If $\frac{\lambda_2}{\lambda_1} < \frac{1}{2}$, then $\left( \frac{W_n - n\lambda_1 v_1}{\sqrt{n}}, \frac{B_n - n\lambda_1 v_2}{\sqrt{n}} \right)$ is asymptotically bivariate.

Janson (Janson, 2006) studied the limit theorems for triangular replacement matrix. While our work in the next chapter was in progress, Renlund ((Renlund, 2010), (Renlund, 2011)) posted preprints on Arxiv in which he extended the assumption (i) of Bagchi and Pal ((Bagchi and Pal, 1985)) to **unbalanced case**, that is $a + b \neq c + d$.

**Theorem 1.12.** Let $a, b, c, d$ in (1.14) be nonnegative real numbers. Suppose that $\min\{a, b, c, d\} \geq 0$, $\max\{a, b, c, d\} > 0, a \neq d$ if $b = c = 0$. Define

\[(1.17) \quad f(x) := (c + d - a - b)x^2 + (a - 2c - d)x + c.\]

(i) $\frac{W_n}{T_n} \xrightarrow{a.s.} x^* \text{ where } x^* \text{ is a zero of } (1.17) \text{ in } [0, 1] \text{ and if two such points exists, it has the additional property that } f'(x^*) < 0.$

(ii) If $a + b = c + d$, $c \neq a < b + 2c$ and $bc > 0$ then

\[(1.18) \quad \sqrt{n} \left( \frac{W_n}{T_n} - \frac{c}{b + c} \right) \xrightarrow{d} \mathcal{N} \left( 0, \frac{bc(a - c)^2}{(a + b)(b + c)^2(b + 2c - a)} \right) \]
(iii) If \( a + b = c + d \), \( a = b + 2c \) and \( bc > 0 \) then

\[
(1.19) \quad \sqrt{\frac{n}{\ln n}} \left( \frac{W_n}{T_n} - \frac{c}{b+c} \right) \xrightarrow{D} \mathcal{N} \left( 0, \frac{bc}{4(b+c)^2} \right).
\]

**Remark 1.2.2.**

(i) These results were obtained as corollaries to more general results on stochastic approximation algorithms.

(ii) The results (ii) and (iii) are equivalent to those stated in Theorem 1.10 above.

(iii) For \( a + b \neq c + d \) (unbalanced case), Renlund \cite{Renlund, 2011} made a remark that the central limit theorem still holds but to write down the condition and the general formula is rather cumbersome.
Chapter 2

Unbalanced Urn Models

In this chapter, we study the limit theorem for the unbalanced urn model for both the simple draw and the two consecutive one ball draw. Throughout this chapter, we use the following assumptions:

Let $a, b, c, d$ be nonnegative real numbers, elements of the matrix $A$ in (1.14) satisfying $ad - bc \neq 0$ and one of the following conditions:

(A1) $bc > 0$

(A2) $b > 0$, $c = 0$, and $a < d$.

By symmetry, the condition (A2) is equivalent to (A2') $b = 0$, $c > 0$, and $a > d$ (by interchanging the two colors).

The eigenvalues of $A$ are

$$\lambda_1 = \frac{a + d + \sqrt{(a - d)^2 + 4bc}}{2},$$

$$\lambda_2 = \frac{a + d - \sqrt{(a - d)^2 + 4bc}}{2} = \frac{a + d - \sqrt{(a + d)^2 - 4(ad - bc)}}{2}.$$  

Note that

1. $\lambda_1$ and $\lambda_2$ are real,

2. $\lambda_1 \geq \lambda_2$,

3. $\lambda_1 = \lambda_2 (= a)$ if and only if $a = d$ and $bc = 0$ (i.e. $\min\{b, c\} = 0$),
4. \[
\begin{cases}
\lambda_2 > 0 & \text{if } ad - bc > 0 \\
\lambda_2 = 0 & \text{if } ad - bc = 0 \\
\lambda_2 < 0 & \text{if } ad - bc < 0.
\end{cases}
\]

Throughout this chapter, we order the eigenvalues of $A$ as above i.e. $\lambda_1 \geq \lambda_2$ and we define

\begin{equation}
(2.2) \quad m := \min\{a + b, c + d\}.
\end{equation}

### 2.1 Limit Theorem for Simple Draw

**Theorem 2.1.** Let $\alpha = (\alpha_1, \alpha_2)^\top$ be an eigenvector corresponding to the eigenvalue $\lambda_2$ of $A$. Then,

\begin{equation}
(2.3) \quad \frac{T_n}{n} \xrightarrow{a.s.} \lambda_1,
\end{equation}

\begin{equation}
(2.4) \quad \frac{W_n}{T_n} \xrightarrow{a.s.} \frac{\alpha_2}{\alpha_2 - \alpha_1} \quad \text{as } n \to \infty.
\end{equation}

**Lemma 2.2.** $\lambda_2 < m$, where $m$ is defined in (2.2).

**Proof.** If $\lambda_2 \leq 0$, then $\lambda_2 - m \leq 0 - m = -m < 0$ since $m > 0$.

Now, suppose $\lambda_2 > 0$. Recall for $x, y$ real numbers, $\min\{x, y\} = \frac{x + y - |x - y|}{2}$. We have

\[
\lambda_2 - m = \lambda_2 - \min\{a + b, c + d\} = \frac{a + d - \sqrt{(a - d)^2 + 4bc} - (a + b) + (c + d) - |(a + b) - (c + d)|}{2} = \frac{a + d - \sqrt{(a - d)^2 + 4bc} - (a + b) - (c + d) + |(a + b) - (c + d)|}{2} = \frac{1}{2} \left[ \sqrt{(a - d)^2 + 4bc} - a - b - c - d + |(a + b) - (c + d)| \right] = \frac{1}{2} \left[ |a + b - c - d| - (b + c) - \sqrt{(a - d)^2 + 4bc} \right].
\]

For $bc > 0$, we have by the triangle inequality $|a + b - c - d| \leq |a - d| + |b - c| <
\[ |a - d| + |b| + |c| = |a - d| + b + c \text{ and } \sqrt{(a - d)^2 + 4bc} > \sqrt{(a - d)^2} = |a - d|. \] Thus,

\[ \lambda_2 - m < \frac{1}{2} \left( |a - d| + b + c - (b + c) - |a - d| \right) \]

\[ = 0. \]

For \( b > 0, \ c = 0, \) and \( a < d, \) we have

\[ \lambda_2 - m = \frac{1}{2} (|a + b - d| - b - |a - d|) \]

\[ < \frac{1}{2} (|a - d| + b - |a - d|) \quad \text{by the triangle inequality} \]

\[ = 0. \]

\[ \square \]

We will also need the following well known property of linear recursions, which appears for example as Lemma 6.1 in (Freedman, 1965).

**Lemma 2.3.** If \( x_n, a_n, b_n \) are positive real numbers satisfying

\[ x_n \leq a_{n-1} x_{n-1} + b_{n-1} \text{ for } n \geq 1, \text{ then} \]

(i)

\[ (2.5) \]

\[ x_n \leq x_0 \prod_{j=0}^{n-1} a_j + \sum_{j=0}^{n-2} \left( \prod_{k=j+1}^{n-1} a_k \right) b_j + b_{n-1}. \]

(ii) If \( a_n = 1 + \frac{A}{Bn + C} \) with \( A > 0, \ B > 0, \ C > 0 \) and \( b_n \) is a positive constant, then there exists \( C_1 = C_1(A, B, C) > 0 \) such that

\[ x_n \leq \begin{cases} 
 C_1 n^{A/B} & \text{if } A > B, \\
 C_1 n \log n & \text{if } A = B, \\
 C_1 n & \text{if } A < B, 
\end{cases} \]

as \( n \to \infty. \)

**Proof.** (i) We proceed by induction.
The inequality (2.5) holds for \( n = 1 \).

\[
x_1 \leq a_0x_0 + b_0 = x_0 \prod_{j=0}^{1-1} a_j + \sum_{j=0}^{1-2} \left( \prod_{k=j+1}^{1-1} a_k \right) b_j + b_{1-1},
\]

where the middle term \( \sum_{j=0}^{1-2} \left( \prod_{k=j+1}^{1-1} a_k \right) b_j \) is defined to be 0 since the upper bound of summation, \( 1 - 2 = -1 \) is less than the lower bound of summation, 0.

Assume the inequality (2.5) holds for \( n \), we will show that it also holds for \( n + 1 \).

\[
x_{n+1} \leq a_n x_n + b_n \leq a_n \left[ x_0 \prod_{j=0}^{n-1} a_j + \sum_{j=0}^{n-2} \left( \prod_{k=j+1}^{n-1} a_k \right) b_j + b_{n-1} \right] + b_n
\]

\[
= x_0 \prod_{j=0}^{n} a_j + \sum_{j=0}^{n-2} \left( \prod_{k=j+1}^{n} a_k \right) b_j + a_n b_{n-1} + b_n
\]

\[
= x_0 \prod_{j=0}^{(n+1)-1} a_j + \sum_{j=0}^{(n+1)-2} \left( \prod_{k=j+1}^{n} a_k \right) b_j + b_n.
\]

Now, to prove (ii), we shall find upper bounds for \( \prod_{j=0}^{n-1} a_j \) and \( \sum_{j=0}^{n-2} \left( \prod_{k=j+1}^{1-1} a_k \right) \).

For any \( n \geq 2 \),

\[
\prod_{j=0}^{n-1} a_j = \prod_{j=0}^{n-1} \left( 1 + \frac{A}{Bj + C} \right)
\]

\[
= \prod_{j=0}^{n-1} \frac{Bj + C + A}{Bj + C}
\]

\[
= \prod_{j=0}^{n-1} \frac{j + \frac{A+C}{B}}{j + \frac{C}{B}}
\]

\[
= \frac{\Gamma \left( n + \frac{A+C}{B} \right)}{\Gamma \left( \frac{A+C}{B} \right)} \times \frac{\Gamma \left( \frac{C}{B} \right)}{\Gamma \left( n + \frac{C}{B} \right)} \text{ by (1.1)}
\]

\[
= \frac{\Gamma \left( n + \frac{A+C}{B} \right)}{\Gamma \left( n + \frac{C}{B} \right)} \Gamma \left( \frac{C}{B} \right).
\]
Using (1.5) (with \( c_1 = \frac{A+C}{B}, c_2 = \frac{C}{B} \)), we see that

\[
\prod_{j=0}^{n-1} a_j \leq C'_1 n^{A/B},
\]

for some constant \( C'_1 = C'_1(A,B,C) > 0 \).

Similarly, for any fixed \( j \leq n-2 \),

\[
\prod_{k=j+1}^{n-1} a_k = \left( \prod_{k=0}^{j-1} a_k \right) / \left( \prod_{k=0}^{j-1} a_k \right),
\]

\[
= \frac{\Gamma \left( n + \frac{A+C}{B} \right)}{\Gamma \left( n + \frac{C}{B} \right)} \times \frac{\Gamma \left( j + 1 + \frac{C}{B} \right)}{\Gamma \left( j + 1 + \frac{A+C}{B} \right)}.
\]

Thus,

\[
\sum_{j=0}^{n-2} \prod_{k=j+1}^{n-1} a_k = \frac{\Gamma \left( n + \frac{A+C}{B} \right)}{\Gamma \left( n + \frac{C}{B} \right)} \sum_{j=0}^{n-2} \Gamma \left( j + 1 + \frac{C}{B} \right)
\]

\[
= \frac{\Gamma \left( n + \frac{A+C}{B} \right)}{\Gamma \left( n + \frac{C}{B} \right)} \sum_{j=1}^{n-1} \Gamma \left( j + \frac{C}{B} \right).
\]

Using (1.5), we have

\[
\sum_{j=1}^{n} \frac{\Gamma \left( j + \frac{C}{B} \right)}{\Gamma \left( j + \frac{A+C}{B} \right)} \leq C_1 \sum_{j=1}^{n} j^{-A/B}, \quad \text{and}
\]

\[
\sum_{j=1}^{n} j^{-A/B} \leq \begin{cases} 
\sum_{j=1}^{\infty} j^{-A/B} = O(1) & \text{if } A/B > 1, \\
2 \log n & \text{if } A/B = 1, \quad \text{by (1.9), } n \geq 3 \\
n^{1-A/B} / (1 - A/B) & \text{if } A/B < 1 \quad \text{by (1.7)}.
\end{cases}
\]

Therefore, by (2.8), (2.9), and (1.5), we get

\[
\sum_{j=0}^{n-2} \prod_{k=j+1}^{n-1} a_k \leq C''_1 \times \begin{cases} 
(n^{A/B}) (1) = n^{A/B} & \text{if } A/B > 1, \\
(n \log n) = n \log n & \text{if } A/B = 1, \\
(n^{A/B}) \left( n^{1-A/B} \right) = n & \text{if } A/B < 1,
\end{cases}
\]

\[
C''_1 \times \begin{cases} 
(n^{A/B}) (1) = n^{A/B} & \text{if } A/B > 1, \\
(n \log n) = n \log n & \text{if } A/B = 1, \\
(n^{A/B}) \left( n^{1-A/B} \right) = n & \text{if } A/B < 1,
\end{cases}
\]
for some constant $C''_1 = C''_1(A, B, C) > 0$ as $n \to \infty$.

From (2.6) and (2.10), we have

$$x_n \leq \begin{cases} 
C'_1 n^{A/B} + C''_1 n^{A/B} & \text{if } A > B, \\
C'_1 n + C''_1 n \log n & \text{if } A = B, \\
C'_1 n^{A/B} + C''_1 n & \text{if } A < B,
\end{cases}$$

for some constant $C_1 = C_1(A, B, C) > 0$ as $n \to \infty$.

We will also need some information about the eigenvectors of the matrix $A$.

**Lemma 2.4.** Let $\alpha := (\alpha_1, \alpha_2)^\top$ be an eigenvector corresponding to the non-principal eigenvalue, $\lambda_2$ of $A$.

(i) If (A1) holds ($bc > 0$), then $\alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_1 \neq \alpha_2$, and $\alpha_2(\alpha_2 - \alpha_1) > 0$.

(ii) If (A2) holds ($b > 0, c = 0, a < d$), then $\alpha_1 \neq 0, \alpha_2 = 0$.

**Proof.** (i) Suppose $bc > 0$.

- To show $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, we proceed by contradiction.
  Without loss of generality, we suppose $\alpha_1 = 0$ and $\alpha_2 \neq 0$. Then $(0, 1)^\top$ would be an eigenvector corresponding to the eigenvalue $\lambda_2$. Thus,

$$\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}. $$

The above equation implies that $b = 0$ and $\lambda_2 = d$.

This is a contradiction since $b > 0$.

- To show $\alpha_1 \neq \alpha_2$, we also proceed by contradiction.
  Suppose $\alpha_1 = \alpha_2$. Then $u := (1, 1)^\top$ would be an eigenvector corresponding to the eigenvalue $\lambda_2$. Thus,

$$\begin{pmatrix} a + b \\ c + d \end{pmatrix} = Au = \lambda_2 u = \begin{pmatrix} \lambda_2 \\ \lambda_2 \end{pmatrix}. $$
We get \( \lambda_2 = a + b = c + d \).

But, \( a + b = c + d \) implies \( \lambda_1 = a + b = c + d \) and \( \lambda_2 = a - c \). Hence, \( a + b = a - c \) which implies \( b = -c < 0 \).

This is a contradiction since \( b > 0 \).

To show \( \alpha_2(\alpha_2 - \alpha_1) > 0 \), it suffices to show that \( \alpha_1 \alpha_2 < 0 \) (since \( \alpha_1 \alpha_2 < 0 \) implies \( \alpha_2(\alpha_2 - \alpha_1) = \alpha_2^2 - \alpha_1 \alpha_2 > -\alpha_1 \alpha_2 > 0 \))

Since \( A\alpha = \lambda_2 \alpha \), we have

\[
\begin{align*}
\alpha_1 + \alpha_2 &= \lambda_2 \\
\alpha_1 + d \alpha_2 &= \lambda_2 \alpha_2
\end{align*}
\]

(2.11)

\[
\begin{align*}
\alpha_1 &= \lambda_2 \alpha_1 - b \alpha_2 \\
d \alpha_2 &= \lambda_2 \alpha_2 - c \alpha_1
\end{align*}
\]

Multiplying the first equation by \( \alpha_2 \) and the second equation by \( \alpha_1 \), we get

\[
\begin{align*}
\alpha_1 \alpha_2 &= \lambda_2 \alpha_1 \alpha_2 - b \alpha_2^2 \\
d \alpha_1 \alpha_2 &= \lambda_2 \alpha_1 \alpha_2 - c \alpha_1^2
\end{align*}
\]

Adding both equations, we have

\[
(a + d) \alpha_1 \alpha_2 = 2 \lambda_2 \alpha_1 \alpha_2 - (b \alpha_2^2 + c \alpha_1^2)
\]

\[
(a + d - 2 \lambda_2) \alpha_1 \alpha_2 = -(b \alpha_2^2 + c \alpha_1^2)
\]

\[
(\lambda_1 + \lambda_2 - 2 \lambda_2) \alpha_1 \alpha_2 = -(b \alpha_2^2 + c \alpha_1^2) \quad \text{since } a + d = \text{Tr}(A) = \lambda_1 + \lambda_2
\]

\[
(\lambda_1 - \lambda_2) \alpha_1 \alpha_2 = -(b \alpha_2^2 + c \alpha_1^2)
\]

\[
\alpha_1 \alpha_2 = -\frac{b \alpha_2^2 + c \alpha_1^2}{\lambda_1 - \lambda_2}
\]

Since \( b > 0 \), \( c > 0 \), \( \alpha_1 \neq 0 \), \( \alpha_2 \neq 0 \), and \( \lambda_1 > \lambda_2 \), we have \( \alpha_1 \alpha_2 \) is negative.
(ii) Suppose $b > 0, c = 0, a < d$. Then, $\lambda_2 = a$. We have

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = a \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\begin{cases} a\alpha_1 + b\alpha_2 = a\alpha_1 \\ d\alpha_2 = a\alpha_2 \end{cases}$$

$$\begin{cases} b\alpha_2 = 0 \\ (d - a)\alpha_2 = 0 \end{cases}$$

Since $b \neq 0$ and $d - a \neq 0$, both equations imply $\alpha_2 = 0$. It follows that $\alpha_1 \neq 0$ since $(\alpha_1, \alpha_2)^\top$ is an eigenvector (a non-zero vector).

Recall from the introduction that $W_n, B_n, T_n$ denote the "number" of white, black, white and black balls in the urn after the $n^{th}$ draw, respectively. Recall that $w_0 \geq 0$, and $b_0 \geq 0$ are the initial number of white and black balls, respectively. Then, $t_0 := w_0 + b_0$ is the initial total number of balls, and clearly $T_n = W_n + B_n$.

Now, let $X_n$ denote the number of times a white ball is drawn after the $n^{th}$ draw and $Y_n$ for a black ball, $u := (1, 1)^\top$, and $\mathcal{F}_n$ be the sigma field $\sigma(X_1, X_2, \cdots, X_n)$. Then, we have

$$X_n + Y_n = n \tag{2.12}$$

$$T_n = (a + b)X_n + (c + d)Y_n + t_0 \tag{2.13}$$

$$\min\{a + b, c + d\}n + t_0 \leq T_n \leq \max\{a + b, c + d\}n + t_0 \tag{2.14}$$

$$\begin{pmatrix} W_n & B_n \end{pmatrix} = \begin{pmatrix} X_n & Y_n \end{pmatrix}A + \begin{pmatrix} w_0 & b_0 \end{pmatrix} \tag{2.15}$$

Define

$$Z_n := \alpha_1 X_n + \alpha_2 Y_n + \frac{\alpha_1 w_0 + \alpha_2 b_0}{\lambda_2} = \begin{pmatrix} X_n & Y_n \end{pmatrix} \alpha + \frac{(w_0 \ b_0)\alpha}{\lambda_2} \tag{2.16}$$
Heuristically, since $X_n + Y_n = n$, $\alpha_1$ and $\alpha_2$ have opposite signs, we expect the ratio $\frac{Z_n}{n}$ to be small as $n$ gets sufficiently large.

Consider

$$\Delta Z_n := Z_n - Z_{n-1} = \alpha_1 X_n + \alpha_2 Y_n - (\alpha_1 X_{n-1} + \alpha_2 Y_{n-1})$$

$$= \alpha_1 (X_n - X_{n-1}) + \alpha_2 (Y_n - Y_{n-1})$$

$$= \alpha_1 1\{X_n - X_{n-1} = 1\} + \alpha_2 1\{X_n - X_{n-1} = 0\}.$$

Note that for all $n \in \mathbb{N}$,

$$|\Delta Z_n| = |\alpha_1 1\{X_n - X_{n-1} = 1\} + \alpha_2 1\{X_n - X_{n-1} = 0\}|$$

$$\leq \max\{|\alpha_1|, |\alpha_2|\}. \tag{2.17}$$

We have the following lemmas:

**Lemma 2.5.** For all $n \in \mathbb{N}$,

$$\mathbb{E}[\Delta Z_n | \mathcal{F}_{n-1}] = \frac{\lambda_2}{T_{n-1}} Z_{n-1}. \tag{2.18}$$
Proof. For all \( n \in \mathbb{N} \), we have

\[
\mathbb{E} [\Delta Z_n | \mathcal{F}_{n-1}] = \alpha_1 \frac{W_{n-1}}{T_{n-1}} + \alpha_2 \frac{B_{n-1}}{T_{n-1}} \\
= \frac{\alpha_1 W_{n-1} + \alpha_2 B_{n-1}}{T_{n-1}} \\
= \frac{(W_{n-1} B_{n-1}) \alpha}{T_{n-1}} \\
= \frac{[(X_{n-1} Y_{n-1}) A + (w_0 \ b_0)] \alpha}{T_{n-1}} \quad \text{by } (2.15) \\
= \frac{(X_{n-1} Y_{n-1}) (A\alpha) + (w_0 \ b_0)\alpha}{T_{n-1}} \\
= \frac{(X_{n-1} Y_{n-1}) (\lambda_2 \alpha) + (w_0 \ b_0)\alpha}{T_{n-1}} \\
= \frac{\lambda_2}{T_{n-1}} [(X_{n-1} Y_{n-1}) \alpha + \frac{(w_0 \ b_0)\alpha}{\lambda_2}] \\
= \frac{\lambda_2}{T_{n-1}} Z_{n-1} \quad \text{by } (2.16).
\]

\[\square\]

Lemma 2.6. Recall \( m := \min\{a + b, c + d\} \) \[2.2\]. For any \( r \in \mathbb{N} \), there exists a constant

\( C_1 = C_1(a, b, c, d, t_0, r) > 0 \) such that

\[
\mathbb{E} \left( Z_{2r}^2 \right) \leq \begin{cases} 
C_1 n^{2r (\lambda_2 / m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
C_1 (n \log n)^r & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 n^r & \text{if } \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases}
\]

as \( n \to \infty \).

Proof. We proceed by induction with respect to \( r \).

• First, we will show that the inequality \[2.19\] holds for \( r = 1 \).
Since $Z_n^2 = (Z_{n-1} + \Delta Z_n)^2 = Z_{n-1}^2 + 2Z_{n-1}\Delta Z_n + (\Delta Z_n)^2$, we have

$$
\mathbb{E} \left[ Z_n^2 | \mathcal{F}_{n-1} \right] = Z_{n-1}^2 + 2Z_{n-1}\mathbb{E} [\Delta Z_n | \mathcal{F}_{n-1}] + \mathbb{E} \left[ (\Delta Z_n)^2 | \mathcal{F}_{n-1} \right] - \mathbb{E} \left[ (\Delta Z_n)^2 | \mathcal{F}_{n-1} \right] 
$$

$$
= Z_{n-1}^2 + 2Z_{n-1} \left( \frac{\lambda_2}{T_{n-1}} Z_{n-1} \right) + \mathbb{E} \left[ (\Delta Z_n)^2 | \mathcal{F}_{n-1} \right] \quad \text{by (2.18)}
$$

(2.20) \quad = \left( 1 + \frac{2\lambda_2}{T_{n-1}} \right) Z_{n-1}^2 + \mathbb{E} \left[ (\Delta Z_n)^2 | \mathcal{F}_{n-1} \right].

Due to the first term of the recurrence of $\{\mathbb{E}[Z_n^2 | \mathcal{F}_{n-1}]\}_{n}$, we need to handle separately the cases $\lambda_2$ nonpositive and $\lambda_2$ positive.

Let $C_2 := \max\{|\alpha_1|, |\alpha_2|\}$ in (2.17).

**Case 1: $\lambda_2 \leq 0$**

From (2.17) and (2.20),

$$
\mathbb{E} \left[ Z_n^2 | \mathcal{F}_{n-1} \right] = \left( 1 + \frac{2\lambda_2}{T_{n-1}} \right) Z_{n-1}^2 + \mathbb{E} \left[ (\Delta Z_n)^2 | \mathcal{F}_{n-1} \right] \leq Z_{n-1}^2 + C_2^2.
$$

Thus,

(2.21) \quad $\mathbb{E}(Z_n^2) \leq \mathbb{E}(Z_{n-1}^2) + C_2^2.$

Using (2.21) recurrently, we get

(2.22) \quad $\mathbb{E}(Z_n^2) = \sum_{j=1}^{n} [\mathbb{E}(Z_j^2) - \mathbb{E}(Z_{j-1}^2)] \leq \sum_{j=1}^{n} C_2^2 = C_2^2 n.$

This shows that $\mathbb{E}(Z_n^2) \leq C_1 n$ where $C_1 = C_2^2$.

**Case 2: $\lambda_2 > 0$**
From (2.17) and (2.20), we have

\[ E \left[ Z_n^2 | F_{n-1} \right] = \left( 1 + \frac{2\lambda_2}{T_{n-1}} \right) Z_{n-1}^2 + E \left[ (\Delta Z_n)^2 | F_{n-1} \right] \]
\[ \leq \left( 1 + \frac{2\lambda_2}{\min\{a + b, c + d\}(n - 1) + t_0} \right) Z_{n-1}^2 + C_2^2 \quad \text{by (2.14)} \]
\[ = \left( 1 + \frac{2\lambda_2}{m(n - 1) + t_0} \right) Z_{n-1}^2 + C_2^2. \]

Thus,

\[ (2.23) \quad E \left( Z_n^2 \right) \leq \left( 1 + \frac{2\lambda_2}{m(n - 1) + t_0} \right) E \left( Z_{n-1}^2 \right) + C_2^2. \]

Note that \( \frac{\lambda_2}{m} < 1 \) by Lemma 2.2.

Applying Lemma 2.3 (ii) with \( A = 2\lambda_2, B = m, \) and \( C = t_0 \) in (2.23), we have

\[ (2.24) \quad E \left( Z_n^2 \right) \leq \begin{cases} 
C_1 n^{2\lambda_2/m} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
C_1 n \log n & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 n & \text{if } 0 < \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases} \]

for some constant \( C_1 = C(a, b, c, d, t_0) \) as \( n \to \infty. \)

This shows that \( E(Z_n^2) \) satisfies (2.19).

• For \( r \geq 2, \) assume the inequality (2.19) holds for \( l = 1, \ldots, r - 1, \) we will prove that it also holds for \( l = r. \)

\[ Z_n^{2r} = (Z_n^{r-1} + \Delta Z_n)^{2r} \]
\[ = \sum_{j=0}^{2r} \binom{2r}{j} Z_{n-1}^{2r-j}(\Delta Z_n)^j. \]
Taking the conditional expectation, we have

\[
\mathbb{E} \left[ Z_{n}^{2r} | \mathcal{F}_{n-1} \right] = \sum_{j=0}^{2r} \binom{2r}{j} Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_{n})^j | \mathcal{F}_{n-1} \right]
\]

\[
= Z_{n-1}^{2r} + 2r Z_{n-1}^{2r-1} \mathbb{E} [\Delta Z_{n} | \mathcal{F}_{n-1}] + \sum_{j=2}^{2r} \binom{2r}{j} Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_{n})^j | \mathcal{F}_{n-1} \right]
\]

\[
= Z_{n-1}^{2r} + 2r Z_{n-1}^{2r-1} \left( \lambda_2 \frac{Z_{n-1}}{T_{n-1}} \right) + \sum_{j=2}^{2r} \binom{2r}{j} Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_{n})^j | \mathcal{F}_{n-1} \right] \quad \text{by (2.18)}
\]

\[
= \left( 1 + \frac{(2r)\lambda_2}{T_{n-1}} \right) Z_{n-1}^{2r} + \sum_{j=2}^{2r} \binom{2r}{j} Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_{n})^j | \mathcal{F}_{n-1} \right] .
\]

So,

\[
\mathbb{E} \left[ Z_{n-1}^{2r} | \mathcal{F}_{n-1} \right] \leq \begin{cases} 
Z_{n-1}^{2r} + \sum_{j=2}^{2r} \binom{2r}{j} Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_{n})^j | \mathcal{F}_{n-1} \right] & \text{if } \lambda_2 \leq 0, \\
\left( 1 + \frac{(2r)\lambda_2}{m(n-1) + t_0} \right) Z_{n-1}^{2r} + \sum_{j=2}^{2r} \binom{2r}{j} Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_{n})^j | \mathcal{F}_{n-1} \right] & \text{by (2.14)}
\end{cases}
\]

if \( \lambda_2 > 0. \)

Thus,

\[
(2.25) \quad \mathbb{E} \left( Z_{n}^{2r} \right) \leq \begin{cases} 
\mathbb{E} \left( Z_{n-1}^{2r} \right) + \sum_{j=2}^{2r} \binom{2r}{j} \mathbb{E} \left\{ Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_{n})^j | \mathcal{F}_{n-1} \right] \right\} & \text{if } \lambda_2 \leq 0, \\
\left( 1 + \frac{(2r)\lambda_2}{m(n-1) + t_0} \right) \mathbb{E} \left( Z_{n-1}^{2r} \right) + \sum_{j=2}^{2r} \binom{2r}{j} \mathbb{E} \left\{ Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_{n})^j | \mathcal{F}_{n-1} \right] \right\} & \text{if } \lambda_2 > 0.
\end{cases}
\]
Next, we will find an upper bound for $\mathbb{E} \left\{ Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_n)^j | \mathcal{F}_{n-1} \right] \right\}$, $2 \leq j \leq 2r$.

$$
\left| \mathbb{E} \left\{ Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_n)^j | \mathcal{F}_{n-1} \right] \right\} \right| \leq \mathbb{E} \left\{ \left| Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_n)^j | \mathcal{F}_{n-1} \right] \right| \right\} \\
\leq \mathbb{E} \left\{ \left| Z_{n-1}^{2r-j} \right| \mathbb{E} \left[ |\Delta Z_n|^j | \mathcal{F}_{n-1} \right] \right\} \\
\leq C_j^2 \mathbb{E} \left| Z_{n-1}^{2r-j} \right| \text{ by (2.17).}
$$

Note that if $j$ is even, we can use directly the inequality (2.19) to bound $\mathbb{E} \left| Z_{n-1}^{2r-j} \right|$.

Let $j = 2l$ where $1 \leq l \leq r$.

$$
\mathbb{E} \left| Z_{n-1}^{2r-j} \right| = \mathbb{E} \left| Z_{n-1}^{2(r-l)} \right| \\
\leq \begin{cases} 
C_1 (n - 1)^{2(r-l)(\lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
C_1 \left[ (n - 1) \log(n - 1) \right]^{r-l} & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 (n - 1)^{r-l} & \text{if } \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases}
$$

for some constant $C_1 = C_1(a, b, c, d, t_0, r) > 0$ as $n \to \infty$.

Thus, for $j$ even and $2 \leq j \leq 2r$, we have

(2.26)

$$
\left| \mathbb{E} \left\{ Z_{n-1}^{2r-j} \mathbb{E} \left[ (\Delta Z_n)^j | \mathcal{F}_{n-1} \right] \right\} \right| \leq \begin{cases} 
C_1 (n - 1)^{(r-1)(2\lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
C_1 \left[ (n - 1) \log(n - 1) \right]^{r-1} & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 (n - 1)^{r-1} & \text{if } \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases}
$$

for some constant $C_1 = C_1(a, b, c, d, t_0, r)$ as $n \to \infty$.

Now, it remains to find an upper bound for $\mathbb{E} \left| Z_{n-1}^{2r-j} \right|$ for $j$ odd.
Let \( j = 2l + 1 \) where \( 1 \leq l \leq r - 1 \).

\[
\mathbb{E} \left| Z_{n-1}^{2r-j} \right| = \mathbb{E} \left| Z_{n-1}^{2r-(2l+1)} \right| = \mathbb{E} \left| Z_{n-1}^{r-l} Z_{n-1}^{r-l-1} \right| \leq \sqrt{\mathbb{E} \left[ Z_{n-1}^{2(r-l)} \right]} \sqrt{\mathbb{E} \left[ Z_{n-1}^{2(r-l-1)} \right]} \quad \text{by the Cauchy-Schwarz inequality}
\]

Thus, by (2.19)

\[
\mathbb{E} \left| Z_{n-1}^{2r-j} \right| \leq \begin{cases} 
C_1 (n - 1)^{2(r-l)(\lambda_2/m)} \times C_1 (n - 1)^{2(r-l-1)(\lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
\sqrt{C_1 \left[ (n - 1) \log(n - 1) \right]^{r-l} \times C_1 \left[ (n - 1) \log(n - 1) \right]^{r-l-1}} & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 (n - 1)^{r-l} \times C_1 (n - 1)^{r-l-1} & \text{if } \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases}
\]

where \( C_1 = \max \{ C_1(a, b, c, d, t_0, r - l) : 1 \leq l \leq r - 1 \} \).

\[
\mathbb{E} \left| Z_{n-1}^{2r-j} \right| \leq \begin{cases} 
C_1 (n - 1)^{2(r-l-1/2)(\lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
\sqrt{C_1 \left[ (n - 1) \log(n - 1) \right]^{r-l-1/2}} & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 (n - 1)^{r-l-1/2} & \text{if } \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases}
\]

as \( n \to \infty \).

So, for \( j \) odd and \( 2 \leq j \leq 2r \), we have

(2.27)

\[
|\mathbb{E} \{ Z_{n-1}^{2r-j} \mathbb{E} [(\Delta Z_n)^j | F_{n-1}] \}| \leq \begin{cases} 
C_1 (n - 1)^{2(r-1-1/2)(\lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
C_1 \left[ (n - 1) \log(n - 1) \right]^{r-1-1/2} & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 (n - 1)^{r-1-1/2} & \text{if } \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases}
\]
as \( n \to \infty \).

Due to the first term of the recurrence of \( \{E(Z_n^{2r})\}_n \) in (2.25), we need to handle separately the cases \( \lambda_2 \) nonpositive and \( \lambda_2 \) positive.

**Case 1: \( \lambda_2 \leq 0 \)**

From (2.25), (2.26), and (2.27), we have

\[
E(Z_n^{2r}) \leq E(Z_{n-1}^{2r}) + b_{n-1},
\]

where \( b_{n-1} \leq C_1(n - 1)^{r-1} \) for some \( C_1 = C_1(a, b, c, d, t_0, r) > 0 \).

Using this recurrently, we get

\[
E(Z_n^{2r}) = \sum_{j=1}^{n} \left[ E(Z_j^{2r}) - E(Z_{j-1}^{2r}) \right] \\
\leq C_1 \sum_{j=1}^{n} b_{j-1} \\
\leq C_1 \sum_{j=1}^{n} (j - 1)^{r-1} \\
\leq C_1 \frac{n^r}{r} \quad \text{by (1.7)},
\]

This shows that \( E(Z_n^{2r}) \leq C_1 n^r \) for some \( C_1 = C_1(a, b, c, d, t_0, r) > 0 \).

**Case 2: \( \lambda_2 > 0 \)**

Similarly, from (2.25), (2.26), and (2.27), we have

\[
E(Z_n^{2r}) \leq \left( 1 + \frac{(2r)\lambda_2}{m(n-1) + t_0} \right) E(Z_{n-1}^{2r}) + b_{n-1},
\]

where

\[
b_{n-1} = \begin{cases} 
C_1 (n - 1)^{(r-1)(2\lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
C_1 [(n - 1) \log(n - 1)]^{r-1} & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 (n - 1)^{r-1} & \text{if } 0 < \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases}
\]
for some $C_1 = C_1(a, b, c, d, r, t_0) > 0$ as $n \to \infty$.

Hence, an upper bound of $\mathbb{E}(Z_{r,n}^{2r})$ depends on the values of $\frac{\lambda_2}{m}$ and the upper bound of $\{b_n\}_n$.

Let $a_n = 1 + \frac{(2r)\lambda_2}{mn + t_0}$.

Using (2.6), (2.7) from the proof of Lemma 2.3 (ii) with $A = (2r)\lambda_2$, $B = m$, $C = t_0$, we get

\[(2.32) \quad \prod_{j=0}^{n-1} a_j \leq C_1 n^{2r(\lambda_2/m)} \quad \text{as } n \to \infty,\]

and for any fixed $j \leq n - 2$,

\[(2.33) \quad \prod_{k=j+1}^{n-1} a_k = \frac{\Gamma \left( n + \frac{(2r)\lambda_2 + t_0}{m} \right)}{\Gamma \left( n + \frac{t_0}{m} \right)} \frac{\Gamma \left( j + 1 + \frac{t_0}{m} \right)}{\Gamma \left( j + 1 + \frac{(2r)\lambda_2 + t_0}{m} \right)}.\]

Using (1.5), we have

\[(2.34) \quad \frac{\Gamma \left( n + \frac{(2r)\lambda_2 + t_0}{m} \right)}{\Gamma \left( n + \frac{t_0}{m} \right)} \leq C_4 n^{2r(\lambda_2/m)},\]
\[(2.35) \quad \frac{\Gamma \left( j + 1 + \frac{t_0}{m} \right)}{\Gamma \left( j + 1 + \frac{(2r)\lambda_2 + t_0}{m} \right)} \leq C_4 (j + 1)^{-2r(\lambda_2/m)}.\]

**Subcase 2.1: $\frac{1}{2} < \frac{\lambda_2}{m} < 1$**

From (2.33), (2.34), (2.35), and (2.31), we have

\[
\sum_{j=0}^{n-2} \left( \prod_{k=j+1}^{n-1} a_k \right) b_j \leq C_4 n^{2r(\lambda_2/m)} \sum_{j=0}^{n-2} (j + 1)^{-2r(\lambda_2/m)} j^{2(r-1)(\lambda_2/m)}
\]
for some $C_1 = C_1(a, b, c, d, t_0, r) > 0$ as $n \to \infty$.

$$\sum_{j=0}^{n-2} (j + 1)^{-2r(\lambda_2/m)} j^{2(r-1)(\lambda_2/m)} = \sum_{j=1}^{n-2} (j + 1)^{-2r(\lambda_2/m)} j^{2(r-1)(\lambda_2/m)}$$

$$\leq \sum_{j=1}^{n-2} j^{-2r(\lambda_2/m)} j^{2(r-1)(\lambda_2/m)} = \sum_{j=1}^{n-2} j^{-2r(\lambda_2/m)+2(r-1)(\lambda_2/m)} = \sum_{j=1}^{n-2} j^{-2\lambda_2/m}.$$}

Since the series $\sum_{j=1}^{\infty} j^{-2\lambda_2/m}$ converges for $\lambda_2/m > \frac{1}{2}$, we have

$$\sum_{j=0}^{n-2} (j + 1)^{-2r(\lambda_2/m)} j^{2(r-1)(\lambda_2/m)} = \mathcal{O}(1) \text{ as } n \to \infty.$$

Hence,

$$(2.36) \quad \sum_{j=0}^{n-2} \left( \prod_{k=j+1}^{n-1} a_k \right) b_j \leq C_1 n^{2r(\lambda_2/m)} \text{ as } n \to \infty.$$

By Lemma 2.3 (i), (2.32) and (2.36), we obtain

$$(2.37) \quad \mathbb{E} \left( Z_{n}^{2r} \right) \leq C_1 n^{2r(\lambda_2/m)} \text{ as } n \to \infty.$$

**Subcase 2.2:** $\frac{\lambda_2}{m} = \frac{1}{2}$

From (2.33), (2.34), (2.35), and (2.31), we have

$$\sum_{j=0}^{n-2} \left( \prod_{k=j+1}^{n-1} a_k \right) b_j \leq C_1 n^r \sum_{j=0}^{n-2} (j + 1)^{-r} (j \log j)^{r-1}$$
for some \( C_1 = C_1(a, b, c, d, r, t_0) > 0 \) as \( n \to \infty \).

\[
\sum_{j=0}^{n-2} (j + 1)^{-r} (j \log j)^{r-1} = \sum_{j=1}^{n-2} (j + 1)^{-r} (j \log j)^{r-1} \\
\leq \sum_{j=1}^{n-2} j^{-r} j^{-r-1} (\log j)^{r-1} = \sum_{j=1}^{n-2} j^{-r+r-1} (\log j)^{r-1} \\
= \sum_{j=2}^{n-2} j^{-1} (\log j)^{r-1} \leq C_1 (\log n)^r \text{ by (1.11)},
\]

for some constant \( C_1 = C_1(r) > 0 \).

Thus,

\[
\sum_{j=0}^{n-2} (j + 1)^{-r} (j \log j)^{r-1} = \mathcal{O} [(\log n)^r] \text{ as } n \to \infty.
\]

Hence,

\[
(2.38) \quad \sum_{j=0}^{n-2} \left( \prod_{k=j+1}^{n-1} a_k \right) b_j \leq C_1 n^r (\log n)^r,
\]

for some constants \( C_1 = C_1(a, b, c, d, t_0, r) > 0 \) as \( n \to \infty \).

By Lemma 2.3 (i), (2.32) and (2.38), we obtain

\[
(2.39) \quad \mathbb{E} (Z_n^{2r}) \leq C_1 (n \log n)^r,
\]

for some constant \( C_1 = C_1(a, b, c, d, t_0, r) > 0 \) as \( n \to \infty \).

**Subcase 2.3:** \( 0 < \frac{\lambda_2}{m} < \frac{1}{2} \)

From (2.33), (2.34), (2.35), and (2.31), we have

\[
\sum_{j=0}^{n-2} \left( \prod_{k=j+1}^{n-1} a_k \right) b_j \leq C_1 n^{2r(\lambda_2/m)} \sum_{j=0}^{n-2} (j + 1)^{-2r(\lambda_2/m)} j^{r-1}
\]
for some $C_1 = C_1(a, b, c, d, t, r) > 0$ as $n \to \infty$.

\[
\sum_{j=0}^{n-2} (j + 1)^{-2r(\lambda_2/m)} j^{r-1} = \sum_{j=1}^{n-2} (j + 1)^{-2r(\lambda_2/m)} j^{r-1}
\leq \sum_{j=1}^{n-2} j^{-2r(\lambda_2/m)} j^{r-1} = \sum_{j=1}^{n-2} j^{-2r(\lambda_2/m)+r-1}
\leq \frac{n^{-2r(\lambda_2/m)+r}}{-2r(\lambda_2/m)+r} \quad \text{by (1.7)}.
\]

Thus,

\[
\sum_{j=0}^{n-2} (j + 1)^{-2r(\lambda_2/m)} j^{r-1} = O \left[ n^{-2r(\lambda_2/m)+r} \right] \quad \text{as } n \to \infty.
\]

Hence,

\[
(2.40) \quad \sum_{j=0}^{n-2} \left( \prod_{k=j+1}^{n-1} a_k \right) b_j \leq C_1 n^r,
\]

for some constant $C_1 = C_1(a, b, c, d, t, r) > 0$ as $n \to \infty$.

By Lemma 2.3 (i), (2.32) and (2.40), we obtain

\[
(2.41) \quad \mathbb{E} \left( Z_{n}^{2r} \right) \leq C_1 n^r,
\]

for some constant $C_1 = C_1(a, b, c, d, t, r) > 0$ as $n \to \infty$.

We have proved (2.19) from (2.29), (2.30), (2.39), and (2.41).

We have the following proposition.

**Proposition 2.7.**

\[
\frac{Z_n}{n} \xrightarrow{c.c.} 0 \quad \text{as } n \to \infty.
\]

**Proof.** Let $\epsilon > 0$. We will show that the series $\sum_n \mathbb{P} \left( \left| \frac{Z_n}{n} \right| > \epsilon \right)$ converges.

Choose any $r_0 \in \mathbb{N}$ such that $r_0 \geq 2$ and $2r_0 \left( 1 - \frac{\lambda_2}{m} \right) > 1$ (This is possible since by
Lemma 2.2 \( \lambda_2 < m \). Hence, \( 2r \left( 1 - \frac{\lambda_2}{m} \right) \nearrow \infty \) as \( r \nearrow \infty \).

By Markov’s inequality,

\[
\sum_n P \left( \left| \frac{Z_n}{n} \right| > \epsilon \right) \leq \sum_n \frac{\mathbb{E}(Z_n^{2r_0})}{n^{2r_0}} = \frac{1}{\epsilon^{2r_0}} \sum_n \frac{\mathbb{E}(Z_n^{2r_0})}{n^{2r_0}}.
\]

By Lemma 2.6,

\[
\sum_n \frac{\mathbb{E}(Z_n^{2r_0})}{n^{2r_0}} \leq \begin{cases} 
\sum_n \frac{n^{2r_0(\lambda_2/m)}}{n^{2r_0}} = \sum_n \frac{1}{n^{2r_0(1-\lambda_2/m)}} < \infty \quad \text{for } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
\sum_n \frac{(n \log n)^{r_0}}{n^{2r_0}} = \sum_n \frac{1}{n^{r_0(\log n)^{-r_0}}} < \infty \quad \text{for } \frac{\lambda_2}{m} = \frac{1}{2}, \\
\sum_n \frac{n^{r_0}}{n^{2r_0}} = \sum_n \frac{1}{n^{r_0}} < \infty \quad \text{for } \frac{\lambda_2}{m} < \frac{1}{2}.
\end{cases}
\]

Hence, \( \frac{Z_n}{n} \xrightarrow{e.s.} 0 \) as \( n \to \infty \).

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. From (2.12) and (2.16), we have

\[
Z_n = \alpha_1 X_n + \alpha_2 Y_n + \frac{(w_0 - b_0)\alpha_2}{\lambda_2}
\]

\[
= \alpha_1 X_n + \alpha_2 (n - X_n) + \frac{(w_0 - b_0)\alpha_2}{\lambda_2}
\]

\[
= (\alpha_1 - \alpha_2)X_n + \alpha_2 n + \frac{(w_0 - b_0)\alpha_2}{\lambda_2}
\]

So, \( X_n = \frac{\alpha_2}{\alpha_2 - \alpha_1} n - \frac{Z_n}{\alpha_2 - \alpha_1} + \frac{(w_0 - b_0)\alpha}{\lambda_2(\alpha_2 - \alpha_1)} \)

(2.42)

\[
\frac{X_n}{n} = \frac{\alpha_2}{\alpha_2 - \alpha_1} - \frac{1}{\alpha_2 - \alpha_1} \frac{Z_n}{n} + \frac{1}{n} \frac{(w_0 - b_0)\alpha}{\lambda_2(\alpha_2 - \alpha_1)}.
\]

By Proposition 2.7

(2.43) \[
\frac{X_n}{n} \xrightarrow{a.s.} \frac{\alpha_2}{\alpha_2 - \alpha_1} \quad \text{as } n \to \infty.
\]
Thus,

\[(2.44) \quad \left( \frac{X_n}{n}, \frac{Y_n}{n} \right) \xrightarrow{a.s.} \left( \frac{\alpha_2}{\alpha_2 - \alpha_1}, \frac{-\alpha_1}{\alpha_2 - \alpha_1} \right) \text{ as } n \to \infty.\]

Using \((2.11), (2.13), (2.44), \) and \(1.3\) we have

\[
\frac{T_n}{n} = (a + b) \frac{X_n}{n} + (c + d) \frac{Y_n}{n} + \frac{t_0}{n} \\
\xrightarrow{a.s.} (a + b) \left( \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) + (c + d) \left( \frac{-\alpha_1}{\alpha_2 - \alpha_1} \right) \\
= \frac{(a + b)\alpha_2 - (c + d)\alpha_1}{\alpha_2 - \alpha_1} \\
= a\alpha_2 + b\alpha_2 - c\alpha_1 - d\alpha_1 \\
= a\alpha_2 + (d\alpha_2 - \lambda_2\alpha_1) + (d\alpha_2 - \lambda_2\alpha_2) - d\alpha_1 \\
= \frac{a(\alpha_2 - \alpha_1) - \lambda_2(\alpha_2 - \alpha_1) + d(\alpha_2 - \alpha_1)}{\alpha_2 - \alpha_1} \\
= a + d - \lambda_2 \\
(2.45) \\
= \lambda_1.
\]

Similarly, since \(W_n = aX_n + cY_n + w_0\), we have

\[
\frac{W_n}{n} = a \frac{X_n}{n} + c \frac{Y_n}{n} + \frac{w_0}{n} \\
\xrightarrow{a.s.} a \left( \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) + c \left( \frac{-\alpha_1}{\alpha_2 - \alpha_1} \right) \\
= \frac{a\alpha_2 - c\alpha_1}{\alpha_2 - \alpha_1} \\
= \frac{a\alpha_2 + (d\alpha_2 - \lambda_2\alpha_1)}{\alpha_2 - \alpha_1} \\
= \frac{(a + d - \lambda_2)\alpha_2}{\alpha_2 - \alpha_1} \\
(2.46) \\
= \lambda_1 \left( \frac{\alpha_2}{\alpha_2 - \alpha_1} \right).
\]

From \((2.46)\) and \((2.45)\), we obtain by Slutsky’s theorem, Theorem 1.4

\[
(2.47) \quad \frac{W_n}{T_n} \xrightarrow{a.s.} \frac{\alpha_2}{\alpha_2 - \alpha_1} \text{ as } n \to \infty.
\]
Corollary 2.8. As $n \to \infty$, for any $r > 0$,

(2.48) \[ \frac{Z_n}{T_n} \to 0 \quad \text{in } L^r \]

(2.49) \[ \frac{W_n}{T_n} \to \frac{\alpha_2}{\alpha_2 - \alpha_1} \quad \text{in } L^r \quad \text{and} \]

(2.50) \[ \mathbb{E}[(\Delta Z_n)^2 | \mathcal{F}_{n-1}] \to -\alpha_1 \alpha_2 \quad \text{in } L^r. \]

Proof. By (2.14) and Proposition 2.7, we have $\frac{Z_n}{T_n} \overset{a.s.}{\to} 0$ as $n \to \infty$.

Note that

(2.51) \[ |Z_n| = \left| \alpha_1 X_n + \alpha_2 Y_n + \frac{w_0 \alpha_1 + b_0 \alpha_2}{\lambda_2} \right| \]

\[ \leq \max\{|\alpha_1|, |\alpha_2|\}(X_n + Y_n) + \left| \frac{w_0 \alpha_1 + b_0 \alpha_2}{\lambda_2} \right| \]

(2.52) \[ = \max\{|\alpha_1|, |\alpha_2|\} n \left| \frac{w_0 \alpha_1 + b_0 \alpha_2}{\lambda_2} \right|. \]

So, by (2.52) and (2.14), $\frac{Z_n}{T_n}$ is bounded by a non-zero constant.

Thus, by the Lebesgue dominated convergence theorem, Theorem 1.6

\[ \frac{Z_n}{T_n} \to 0 \quad \text{in } L^r \quad \text{as } n \to \infty. \]

Also, since $\frac{W_n}{T_n}$ is nonnegative and bounded by 1, we have by (2.47) and the Lebesgue dominated convergence theorem, Theorem 1.6

\[ \frac{W_n}{T_n} \to \frac{\alpha_2}{\alpha_2 - \alpha_1} \quad \text{in } L^r \quad \text{as } n \to \infty. \]

\[ \mathbb{P}(\Delta Z_n = \alpha_1 | \mathcal{F}_{n-1}) = \frac{W_n}{T_n}, \quad \mathbb{P}(\Delta Z_n = \alpha_2 | \mathcal{F}_{n-1}) = 1 - \frac{W_n}{T_n}. \]
So, we have

\[ \mathbb{E} \left[ (\Delta Z_n)^2 | F_{n-1} \right] = \alpha_1^2 \frac{W_{n-1}}{T_{n-1}} + \alpha_2^2 \left( 1 - \frac{W_{n-1}}{T_{n-1}} \right) \]
\[ = (\alpha_1^2 - \alpha_2^2) \frac{W_{n-1}}{T_{n-1}} + \alpha_2^2 \]
\[ = (\alpha_1^2 - \alpha_2^2) \left( \frac{W_{n-1}}{T_{n-1}} - \frac{\alpha_2}{\alpha_2 - \alpha_1} + \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) + \alpha_2^2 \]
\[ = (\alpha_1^2 - \alpha_2^2) \left( \frac{W_{n-1}}{T_{n-1}} - \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) + (\alpha_1^2 - \alpha_2^2) \frac{\alpha_2}{\alpha_2 - \alpha_1} + \alpha_2^2 \]
\[ = (\alpha_1^2 - \alpha_2^2) \left( \frac{W_{n-1}}{T_{n-1}} - \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) - (\alpha_1 + \alpha_2) \alpha_2 + \alpha_2^2 \]
\[ = (\alpha_1^2 - \alpha_2^2) \left( \frac{W_{n-1}}{T_{n-1}} - \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) - \alpha_1 \alpha_2. \]

Thus,

\[ \mathbb{E} \left[ (\Delta Z_n)^2 | F_{n-1} \right] - (-\alpha_1 \alpha_2) = (\alpha_1^2 - \alpha_2^2) \left( \frac{W_{n-1}}{T_{n-1}} - \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) . \]

Hence,

\[ \| \mathbb{E} \left[ (\Delta Z_n)^2 | F_{n-1} \right] - (-\alpha_1 \alpha_2) \|_r = |\alpha_1^2 - \alpha_2^2| \left\| \frac{W_{n-1}}{T_{n-1}} - \frac{\alpha_2}{\alpha_2 - \alpha_1} \right\|_r \]
\[ \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ by (2.49)}. \]

\[ \square \]

Remark 2.1.1. As a quick consequence of (2.14) and (2.45), we have the following bound easily while an elementary proof is complicated.

(2.53) \[ \min\{a + b, c + d\} \leq \lambda_1 \leq \max\{a + b, c + d\} . \]

To see this, \[ \min\{a + b, c + d\} + \frac{t_0}{n} \leq \frac{T_n}{n} \leq \max\{a+b,c+d\} + \frac{t_0}{n} \text{ by (2.14)} . \] Then, passing to the limit, we get (2.53) by (2.45).
2.2 Limit Theorem for Urns Evolving by Two Draws

We still consider an urn with balls of two colors, white $W$ and black $B$ but we draw two balls, one at a time, **consecutively**: at each step, we draw one ball from the urn uniformly at random and with replacement between the first and second draw. Based on the colors of the two drawn balls, additional white and black balls are added to the urn. The possible pairs of outcome are $\{W,W\}$, $\{W,B\} = \{B,W\}$, $\{B,B\}$. If the pair $\{W,W\}$ is drawn, we add $2a$ white balls and $2b$ black balls. Likewise, if the pair $\{B,B\}$ is drawn, we add $2c$ white balls and $2d$ black balls. Finally if the pair $\{W,B\}$ is drawn, we add $a + c$ white balls and $b + d$ black balls.

This can be represented by the rectangular replacement matrix

\[
\begin{pmatrix}
W & B \\
W & 2a & 2b \\
W & a + c & b + d \\
B & 2c & 2d
\end{pmatrix}
\]

Note that the total number of balls in the urn only changes each time two consecutive balls are drawn.

**Remark 2.2.1.** Another model where two balls are drawn **simultaneously** has been studied in Chapter 10 of [Mahmoud, 2008] and Section 3.2. of [Renlund, 2010].

Assume the urn initially contains a total of $t_0 > 0$ balls, of which $w_0 \geq 0$ are white and $b_0 = t_0 - w_0$ black.

Let $W_{2n}$ (resp. $B_{2n}$) be the number of white (resp. black) balls after $n$ pairs of draws and $T_{2n} := W_{2n} + B_{2n}$.

Let $X_{2n}$ (resp. $Y_{2n}$) denote the number of times a white (resp. black) ball is drawn after $n$ pairs of draws.

Let $X_{2n}^1$ (resp. $Y_{2n}^1$) denote the number of times a white (resp. black) ball is drawn at odd number of draws after $n$ pairs of draws, and $X_{2n}^2$ (resp. $Y_{2n}^2$) denote the number of times a white (resp. black) ball is drawn at even number of draws after $n$ pairs of draws.
After $n$ pairs of draws, we have

\begin{align*}
(2.55) & \quad X_{2n} = X_{2n}^1 + X_{2n}^2, \\
(2.56) & \quad Y_{2n} = Y_{2n}^1 + Y_{2n}^2, \\
(2.57) & \quad X_{2n}^1 + Y_{2n}^1 = n, \\
(2.58) & \quad X_{2n}^2 + Y_{2n}^2 = n.
\end{align*}

Since the ball drawn is put back into the urn after each draw, the proportions of balls in the urn between the two consecutive pairs of draw remain unchanged. Hence,

1. $X_{2n}^1$ and $X_{2n}^2$ are i.i.d.
2. $Y_{2n}^1$ and $Y_{2n}^2$ are i.i.d.
3. For every $n$, the $\mathbb{R}^2$-valued random variables $(X_{2n}^1, Y_{2n}^1)$ and $(X_{2n}^2, Y_{2n}^2)$ are independent and have the same distribution. But their distribution changes with respect to $n$ and these variables for different $n$ are dependent.

Thus, the replacement matrix (2.54) is equivalent to the square replacement matrix in simple draw

\begin{equation}
(2.59)
\begin{pmatrix}
1W & 1B & 2W & 2B \\
1W & a & b & 0 & 0 \\
1B & c & d & 0 & 0 \\
2W & 0 & 0 & a & b \\
2B & 0 & 0 & c & d
\end{pmatrix}
\end{equation}

Hence,

\begin{align*}
(2.60) & \quad W_{2n} = aX_{2n}^1 + cY_{2n}^1 + aX_{2n}^2 + cY_{2n}^2 + w_0, \\
(2.61) & \quad B_{2n} = bX_{2n}^1 + dY_{2n}^1 + bX_{2n}^2 + dY_{2n}^2 + b_0, \\
(2.62) & \quad T_{2n} = 2(a + b) \left( X_{2n}^1 + X_{2n}^2 \right) + 2(c + d) \left( Y_{2n}^1 + Y_{2n}^2 \right) + t_0.
\end{align*}
From (2.57), (2.58), and (2.62), we get

\[(2.63) \quad 2 \min\{a + b, c + d\} n + t_0 \leq T_{2n} \leq 2 \max\{a + b, c + d\} n + t_0.\]

Let \(X_{2n} = (X_{2n}^1, Y_{2n}^1, X_{2n}^2, Y_{2n}^2)\) and \(\mathcal{F}_{2n} = \sigma\left(\left(X_{2j}^1, Y_{2j}^1\right), j = 1, 2, \ldots, n\right)\) be the sigma field generated by the pairs of random variables \(X_{2j}^1, Y_{2j}^1\), \(j = 1, 2, \ldots, n\).

Note that from (2.57) and (2.58), the random variables \(Y_{2j}^1\) and \(Y_{2j}^2\), \(1 \leq j \leq n\) are \(\{\mathcal{F}_{2n}\}\)-measurable.

The equations (2.60) and (2.61) yield

\[(2.64) \quad (W_{2n} \quad B_{2n}) = \begin{pmatrix} X_{2n}^1 \ Y_{2n}^1 \\ X_{2n}^2 \ Y_{2n}^2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + (w_0 \quad b_0) = X_{2n} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + (w_0 \quad b_0).\]

Now, consider for \(n \geq 1\),

\[\Delta X_{2n} := X_{2n} - X_{2(n-1)} = (X_{2n}^1 - X_{2(n-1)}^1, Y_{2n}^1 - Y_{2(n-1)}^1, X_{2n}^2 - X_{2(n-1)}^2, Y_{2n}^2 - Y_{2(n-1)}^2).\]

We have

\[(2.65) \quad \mathbb{E} [\Delta X_{2n} | \mathcal{F}_{2(n-1)}] = \begin{pmatrix} W_{2(n-1)} & B_{2(n-1)} \\ T_{2(n-1)} & T_{2(n-1)} \end{pmatrix} \begin{pmatrix} W_{2(n-1)} & B_{2(n-1)} \end{pmatrix} = \frac{1}{T_{2(n-1)}} \begin{pmatrix} W_{2(n-1)} & B_{2(n-1)} \\ W_{2(n-1)} & B_{2(n-1)} \end{pmatrix}.\]
Using (2.64), we get

\begin{equation}
\mathbb{E} \left[ \Delta X_{2n} | \mathcal{F}_{2(n-1)} \right] = \frac{1}{T_{2(n-1)}} (X_{2(n-1)} R + (w_0 \ b_0 \ w_0 \ b_0)),
\end{equation}

where

\begin{equation}
R = \begin{pmatrix}
a & b & a & b \\
c & d & c & d \\
a & b & a & b \\
c & d & c & d
\end{pmatrix}.
\end{equation}

Note that

1. \( R = \begin{pmatrix} A & A \\ A & A \end{pmatrix} \) where \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is the replacement matrix (1.14) for the simple draw in Section 2.1.

2. \( R \) has real eigenvalues

\[
\tilde{\lambda}_1 = a + d + \sqrt{(a - d)^2 + 4bc} = a + d + \sqrt{(a + d)^2 - 4(ad - bc)},
\]

\[
\tilde{\lambda}_2 = a + d - \sqrt{(a - d)^2 + 4bc} = a + d - \sqrt{(a + d)^2 - 4(ad - bc)},
\]

\[
\tilde{\lambda}_3 = \tilde{\lambda}_4 = 0.
\]

3. \( \tilde{\lambda}_1 > \tilde{\lambda}_2 \) and \( \tilde{\lambda}_2 \neq 0 \) since \( ad - bc \neq 0 \) by assumption.

4. \( \tilde{\lambda}_1 = 2\lambda_1, \tilde{\lambda}_2 = 2\lambda_2 \) where \( \lambda_1 \) and \( \lambda_2 \) (see (2.1)) are the eigenvalues of the matrix \( A \).

This leads to the following proposition:

**Proposition 2.9.** The eigenvectors of the matrix \( R \) corresponding to the eigenvalue \( \tilde{\lambda}_2 \) are exactly of the form \((\alpha, \alpha)^\top\), where \( \alpha = (\alpha_1, \alpha_2)^\top \) is an eigenvector corresponding to the eigenvalue, \( \lambda_2 \) of \( A \) as in Theorem 2.1.
Proof. Let \( v = (v_1, v_2, v_3, v_4)^\top \) be an eigenvector of \( R \) corresponding to the eigenvalue \( \tilde{\lambda}_2 \).

First, we will show that \( v_1 = v_3 \) and \( v_2 = v_4 \).

Since \( R v = \tilde{\lambda}_2 v \), we have

\[
\begin{pmatrix}
A & A \\
A & A \\
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{\lambda}_2 \\
\tilde{\lambda}_2 \\
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
\end{pmatrix}.
\]

So,

\[
A \begin{pmatrix} v_1 \\ v_2 \\ v_4 \end{pmatrix} + A \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \tilde{\lambda}_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \text{and}
\]

\[
A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + A \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \tilde{\lambda}_2 \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}.
\]

Thus,

\[
\tilde{\lambda}_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \tilde{\lambda}_2 \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}.
\]

Since, \( \tilde{\lambda}_2 \neq 0 \), we must have \( v_1 = v_3 \) and \( v_2 = v_4 \).

Hence, \( v = (v_1, v_2, v_1, v_2)^\top \).

Now, we will show that \((v_1, v_2)^\top\) is an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda_2 \).
Substituting \(v_3\), and \(v_4\) by \(v_1\), and \(v_2\) in (2.68) respectively, we get

\[
2A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \tilde{\lambda}_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},
\]

\[
A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\tilde{\lambda}_2}{2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{since } \tilde{\lambda}_2 = 2\lambda_2.
\]

Hence, \((v_1, v_2)^\top\) is an eigenvector corresponding to the eigenvalue \(\lambda_2\) of \(A\).

**Theorem 2.10.** Let \(a, b, c, d\) be nonnegative real numbers in (2.54) satisfying \(ad - bc \neq 0\) and the assumptions (A1) or (A2). Denote by \(\lambda_1 > \lambda_2\) the eigenvalues of \(A\) and \(\alpha = (\alpha_1, \alpha_2)^\top\) an eigenvector of corresponding to the eigenvalue \(\lambda_2\) of \(A\). Then,

\[
\frac{T_{2n}}{2n} \xrightarrow{a.s.} \lambda_1 \quad \text{and} \quad \frac{W_{2n}}{T_{2n}} \xrightarrow{a.s.} \frac{\alpha_2}{\alpha_2 - \alpha_1}.
\]

Before we give a sketch proof of the theorem, we need the following lemmas:

**Lemma 2.11** ([Gut, 2005], Theorem 11.5 (i) page 253). Let \(a, b \in \mathbb{R}\), let \(X_1, X_2, \ldots\) and \(Y_1, Y_2, \ldots\) be two sequences of random variables.

Suppose that \(X_n\) and \(Y_n\) are independent for all \(n \geq 1\). If \(X_n + Y_n \xrightarrow{c.c.} a + b\) and \(Y_n \xrightarrow{p} b\) as \(n \to \infty\), then

\[X_n \xrightarrow{c.c.} a \quad \text{and} \quad Y_n \xrightarrow{c.c.} b \quad \text{as } n \to \infty.\]

**Lemma 2.12.** Let \(X_1, X_2, \ldots\) and \(Y_1, Y_2, \ldots\) be two sequences of random variables.

Suppose that \(X_n\) and \(Y_n\) are i.i.d. for all \(n \geq 1\). If \(X_n + Y_n \xrightarrow{c.c.} 0\), then

\[X_n \xrightarrow{p} 0 \quad \text{and} \quad Y_n \xrightarrow{p} 0 \quad \text{as } n \to \infty.\]

**Proof.** Since \(X_n + Y_n \xrightarrow{c.c.} 0\), we have \(X_n + Y_n \xrightarrow{a.s.} 0\), \(X_n + Y_n \xrightarrow{p} 0\) and \(X_n + Y_n \xrightarrow{D} 0\).
So,

\( (2.69) \quad \varphi_{X_n+Y_n}(t) \rightarrow \varphi_0(t) \quad \text{as } n \rightarrow \infty \quad \text{for all } t \in \mathbb{R}, \)

where \( \varphi_X \) denotes the characteristic function of the random variable \( X \).

\[
\begin{align*}
\varphi_{X_n+Y_n}(t) &= \mathbb{E} [e^{it(X_n+Y_n)}] \\
&= \mathbb{E} [e^{itX_n+itY_n}] \\
&= \mathbb{E} (e^{itX_n}e^{itY_n}) \\
&= \mathbb{E} (e^{itX_n}) \mathbb{E} (e^{itY_n}) \quad \text{(by independence)} \\
&= [\mathbb{E} (e^{itX_n})]^2 \quad (X_n \text{ and } Y_n \text{ have the same law}) \\
(2.70) \quad &= [\varphi_{X_n}(t)]^2 \\
(2.71) \quad \varphi_0(t) &= \mathbb{E} [e^{it(0)}] = 1.
\end{align*}
\]

The equations (2.69), (2.70), and (2.71) yields

\[
\varphi_{X_n}(t) \rightarrow \varphi_0(t) \quad \text{as } n \rightarrow \infty.
\]

Thus, \( X_n \xrightarrow{D} \delta(0) \) as \( n \rightarrow \infty \). Hence, \( X_n \xrightarrow{p} 0 \) as \( n \rightarrow \infty \). Finally, since \( X_n \) and \( Y_n \) have the same law, we also obtain \( Y_n \xrightarrow{p} 0 \) as \( n \rightarrow \infty \).

These two lemmas give us the following proposition.

**Proposition 2.13.** Under the same assumptions as Lemma 2.12 above, we have

\[
X_n \xrightarrow{c.c.} 0 \quad \text{and } Y_n \xrightarrow{c.c.} 0 \quad \text{as } n \rightarrow \infty.
\]

**Proof.** Lemma 2.12 implies \( Y_n \xrightarrow{p} 0 \) as \( n \rightarrow \infty \). So, the assumptions of Lemma 2.11 are satisfied with \( a = 0 \) and \( b = 0 \). Hence, the conclusion of Lemma 2.11 holds.

Now, we are ready to prove Theorem 2.10.
Sketch of proof. The proof is almost identical to that of Theorem 2.1. So, some details will be omitted.

First, we give a two-draw urn version of Lemma 2.5.

Let $v$ be an eigenvector corresponding to the eigenvalue $\tilde{\lambda}_2$ of the matrix $R$ (2.67). Define

$$Z_{2n} := X_{2n} v + \frac{(w_0 \ b_0 \ w_0 \ b_0)}{\tilde{\lambda}_2} v \quad \text{by (2.72)}$$

Then, for all $n \in \mathbb{N}$,

$$\mathbb{E}[\Delta Z_{2n}] = \mathbb{E}[\Delta X_{2n} v | \mathcal{F}_{2(n-1)}]$$

$$= \mathbb{E}[\Delta X_{2n} | \mathcal{F}_{2(n-1)}] v$$

$$= \frac{1}{T_{2(n-1)}} (X_{2(n-1)}R + (w_0 \ b_0 \ w_0 \ b_0)) v \quad \text{by (2.66)}$$

$$= \frac{1}{T_{2(n-1)}} X_{2(n-1)} \tilde{\lambda}_2 v + \frac{1}{T_{2(n-1)}} (w_0 \ b_0 \ w_0 \ b_0) v$$

$$= \frac{\tilde{\lambda}_2}{T_{2(n-1)}} \left[ X_{2(n-1)} v + \frac{(w_0 \ b_0 \ w_0 \ b_0)}{\tilde{\lambda}_2} v \right]$$

$$= \frac{\tilde{\lambda}_2}{T_{2(n-1)}} \quad \text{by (2.72)}.$$ 

Thus,

$$\mathbb{E}[Z_{2n}^2 | \mathcal{F}_{2(n-1)}] = \left( 1 + \frac{2\tilde{\lambda}_2}{T_{2(n-1)}} \right) Z_{2(n-1)}^2 + \mathbb{E}[\Delta Z_{2n}]^2 | \mathcal{F}_{2(n-1)}$$

and for all integers $r \geq 2$,

$$\mathbb{E}[Z_{2n}^2 | \mathcal{F}_{2(n-1)}] = \left( 1 + \frac{(2r)\tilde{\lambda}_2}{T_{2(n-1)}} \right) Z_{n-1}^{2r} + \sum_{j=2}^{2r} \binom{2r}{j} Z_{n-1}^{2r-j} \mathbb{E}[\Delta Z_{n}]^j | \mathcal{F}_{n-1}].$$

Recall $m := \min \{a + b, c + d\}$ defined in (2.2).

Since $|\Delta Z_{2n}| \leq \max \{|\alpha_1|, |\alpha_2|\}$ by Proposition 2.9, $\tilde{\lambda}_2 = 2\lambda_1$, and $T_{2n} \geq 2 \min \{a + b, c + d\}$,
\( n + t_0 = 2mn + t_0 \) by (2.63), we have for \( \lambda_2 > 0 \)

\[
1 + \frac{2r\lambda_2}{T_{2(n-1)}} \leq 1 + \frac{(2r)(2\lambda_2)}{2m(n-1) + t_0} = 1 + \frac{(2r)\lambda_2}{m(n-1) + t_0/2}
\]

The term \( 1 + \frac{(2r)\lambda_2}{m(n-1) + t_0/2} \) is similar to \( 1 + \frac{(2r)\lambda_2}{m(n-1) + t_0/2} \) used in the proof of Lemma 2.6 (see (2.25)).

Hence, for all \( r \in \mathbb{N} \), \( \{E(Z_{2n}^r)\}_n \) also satisfies the inequality (2.19).

By Proposition 2.7

\[
\frac{Z_{2n}^r}{n} \overset{c.c.}{\to} 0 \quad \text{as} \quad n \to \infty.
\]

It follows by (2.72) that

\[
\frac{X_{2n}^r v}{n} = \left( \frac{X_{2n}^1}{n} \frac{Y_{2n}^1}{n} \frac{X_{2n}^2}{n} \frac{Y_{2n}^2}{n} \right) v \overset{a.s.}{\to} 0 \quad \text{as} \quad n \to \infty,
\]

\[
\alpha_1 \frac{X_{2n}^1}{n} + \alpha_2 \frac{Y_{2n}^1}{n} + \alpha_1 \frac{X_{2n}^2}{n} + \alpha_2 \frac{Y_{2n}^2}{n} \overset{c.c.}{\to} 0 \quad \text{as} \quad n \to \infty.
\]

Since for every \( n \in \mathbb{N} \), the \( \mathbb{R}^2 \)-valued random variables \((X_{2n}^1, Y_{2n}^1)\) and \((X_{2n}^2, Y_{2n}^2)\) are independent and have the same distribution, the real-valued random variables \( \alpha_1 \frac{X_{2n}^1}{n} + \alpha_2 \frac{Y_{2n}^1}{n} \) and \( \alpha_1 \frac{X_{2n}^2}{n} + \alpha_2 \frac{Y_{2n}^2}{n} \) are independent for all \( n \in \mathbb{N} \). Thus, by Proposition 2.13,

\[
\alpha_1 \frac{X_{2n}^1}{n} + \alpha_2 \frac{Y_{2n}^1}{n} \overset{c.c.}{\to} 0 \quad \text{and} \quad \alpha_1 \frac{X_{2n}^2}{n} + \alpha_2 \frac{Y_{2n}^2}{n} \overset{c.c.}{\to} 0 \quad \text{as} \quad n \to \infty.
\]

Since \( X_{2n}^1 + Y_{2n}^1 = n \), \( X_{2n}^2 + Y_{2n}^2 = n \) (2.57), (2.58) and complete convergence implies almost sure convergence, we have

\[
(2.73) \quad \frac{X_{2n}^r}{n} \overset{a.s.}{\to} \left( \frac{\alpha_2}{\alpha_2 - \alpha_1}, \frac{-\alpha_1}{\alpha_2 - \alpha_1}, \frac{\alpha_2}{\alpha_2 - \alpha_1}, \frac{-\alpha_1}{\alpha_2 - \alpha_1} \right) \quad \text{as} \quad n \to \infty.
\]
From (2.63), \( T_{2n} = (a + b) \left( X_{2n}^1 + X_{2n}^2 \right) + (c + d) \left( Y_{2n}^1 + Y_{2n}^2 \right) + t_0 \). So,

\[
\frac{T_{2n}}{n} = \frac{(a + b)}{n} \left( X_{2n}^1 \frac{X_{2n}}{n} + X_{2n}^2 \frac{X_{2n}}{n} \right) + \frac{(c + d)}{n} \left( Y_{2n}^1 \frac{Y_{2n}}{n} + Y_{2n}^2 \frac{Y_{2n}}{n} \right) + \frac{t_0}{n}
\]

\[
\xrightarrow{a.s.} (a + b) \left( 2 \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) + (c + d) \left( \frac{-2\alpha_1}{\alpha_2 - \alpha_1} \right)
\]

\[
= 2 \left[ \frac{\alpha_2}{\alpha_2 - \alpha_1} \right] + (c + d) \left( \frac{-\alpha_1}{\alpha_2 - \alpha_1} \right)
\]

\[
= 2\lambda_1 \quad \text{by the same computation to get (2.45)}.
\]

Thus,

\[
(2.74) \quad \frac{T_{2n}}{2n} \xrightarrow{a.s.} \lambda_1 \quad \text{as } n \to \infty.
\]

Similarly, since \( W_{2n} = a \left( X_{2n}^1 + X_{2n}^2 \right) + c \left( Y_{2n}^1 + Y_{2n}^2 \right) + w_0 \left( 2.60 \right) \), we have

\[
\frac{W_{2n}}{n} = \frac{a}{n} \left( X_{2n}^1 \frac{X_{2n}}{n} + X_{2n}^2 \frac{X_{2n}}{n} \right) + \frac{c}{n} \left( Y_{2n}^1 \frac{Y_{2n}}{n} + Y_{2n}^2 \frac{Y_{2n}}{n} \right) + \frac{w_0}{n}
\]

\[
\xrightarrow{a.s.} a \left( 2 \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) + c \left( 2 \frac{-\alpha_1}{\alpha_2 - \alpha_1} \right)
\]

\[
= 2 \left[ a \left( 2 \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) + c \left( \frac{-\alpha_1}{\alpha_2 - \alpha_1} \right) \right]
\]

\[
= 2 \left( \lambda_1 \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) \quad \text{by the same computation to get (2.46)}.
\]

Thus,

\[
(2.75) \quad \frac{W_{2n}}{2n} \xrightarrow{a.s.} \lambda_1 \frac{\alpha_2}{\alpha_2 - \alpha_1}.
\]

(2.75) and (2.74) imply

\[
\frac{W_{2n}}{T_{2n}} \xrightarrow{a.s.} \frac{v_2}{v_2 - v_1}.
\]

From (2.74), (2.75), and Slutsky’s theorem, Theorem 1.4, we obtain

\[
\frac{W_{2n}}{T_{2n}} \xrightarrow{a.s.} \frac{\alpha_2}{\alpha_2 - \alpha_1}.
\]
Chapter 3

Central Limit Theorem

In this chapter, we give the central limit theorem for the unbalanced urn models for the simple draw. Also, we assume the assumption \((A1)\), i.e. \(bc > 0\).

3.1 Martingale Central Limit Theorem

We recall the definition of a martingale array and the central limit theorem for martingale array which appear in [Pollard, 1984], page 171 that we will use later.

Let \(\{X_{n,j}, 1 \leq j \leq n\}\) be any triangular array of random variables on \((\Omega, \mathcal{F}, \mathbb{P})\), and \(\{\mathcal{F}_{n,j}; 0 \leq j \leq n\}\) be any triangular array of sub-sigma algebras of \(\mathcal{F}\) such that for each \(n\) and \(1 \leq j \leq n\), \(\mathcal{F}_{n,j-1} \subset \mathcal{F}_{n,j}\).

**Definition 3.1.1.** We say the array \(\{X_{n,j}; 1 \leq j \leq n\}\) is a martingale difference array with respect to \(\mathcal{F}_{n,j}\) if

(i) Each \(X_{n,j}\) is \(\mathcal{F}_{n,j}\)-measurable and

(ii) \(E[X_{n,j}|\mathcal{F}_{n,j-1}] = 0\) for all \(n\) and \(1 \leq j \leq k_n\).

**Theorem 3.1.** Let \(\{X_{n,j}, \mathcal{F}_{n,j}; 1 \leq j \leq n\}\) be a martingale difference array satisfying

(i) for all \(\epsilon > 0\),

\[
\sum_{j=1}^{n} E \left[ (X_{n,j})^2 1_{\{|X_{n,j}| > \epsilon\}} | \mathcal{F}_{n,j-1} \right] \overset{p}{\to} 0 \quad \text{as } n \to \infty
\]

49
(ii) 

\[
\sum_{k=1}^{n} \mathbb{E} \left[ (X_{n,j})^2 | \mathcal{F}_{n,j-1} \right] \xrightarrow{p} \sigma^2 \quad \text{as } n \to \infty,
\]

where \( \sigma^2 \) is a positive constant.

Then, the martingale array \( \sum_{j=1}^{n} X_{n,j} \) converges in distribution to \( N(0, \sigma^2) \).

3.2 Construction of Martingale

In this section, we shall find a martingale associated with \( W_n \) and \( B_n \). Define

\[
Q_n := \alpha_1 W_n + \alpha_2 B_n = (W_n \ B_n) \alpha
\]

where \( \alpha = (\alpha_1, \alpha_2)^T \) is an eigenvector corresponding to the eigenvalue \( \lambda_2 \) of \( A \) in Chapter 2.

Note that from (2.15)

\[
Q_n = (X_n \ Y_n)A \alpha + (w_0 \ b_0)
\]

\[
= (X_n \ Y_n)(\lambda_2 \alpha) + (w_0 \ b_0)
\]

\[
= \lambda_2 \left[ (X_n \ Y_n)\alpha + \left( \frac{(w_0 \ b_0)}{\lambda_2} \right) \right]
\]

(3.4) 

\[
Q_n = \lambda_2 Z_n \quad \text{by (2.16)}
\]

(3.5) 

Thus, \( \Delta Q_n = \lambda_2 \Delta Z_n \).

So, by Lemma 2.5 we have

\[
\mathbb{E} \left[ \Delta Q_n | \mathcal{F}_{n-1} \right] = \lambda_2 \mathbb{E} \left[ \Delta Z_n | \mathcal{F}_{n-1} \right]
\]

\[
= \lambda_2 \left( \frac{\lambda_2}{T_{n-1}} Z_{n-1} \right)
\]

\[
= \frac{\lambda_2}{T_{n-1}} (\lambda_2 Z_{n-1})
\]

(3.6) 

\[
\mathbb{E} \left[ \Delta Q_n | \mathcal{F}_{n-1} \right] = \frac{\lambda_2}{T_{n-1}} Q_{n-1}.
\]
Let

\[(3.7) \quad \Delta M_n^Q := \Delta Q_n - \frac{\lambda_2}{T_{n-1}} Q_{n-1} \]

be the martingale differences associated with \(\{Q_n\}_n\).

We want to approximate the sequence of random variable \(\{T_{n-1}\}_{n\in\mathbb{N}}\) in the denominator by a deterministic sequence.

To do so, we define for \(n \geq 1\),

\[(3.8) \quad K_n := \lambda_1 n + t_0, \quad \text{and} \quad (3.9) \quad \Delta \tilde{M}_n^Q := \Delta Q_n - \frac{\lambda_2}{K_{n-1}} Q_{n-1}. \]

Note that the sequence \(\{\Delta \tilde{M}_n^Q\}_n\) is no longer a sequence of martingale differences.

Now, we will use the idea of [Mahmoud, 2008].

We shall seek deterministic constants \(\{\beta_{n,j}\}\) such that

\[(3.10) \quad Q_n + \epsilon_n = \sum_{j=1}^n \beta_{n,j} \Delta \tilde{M}_j^Q, \]

where \(\{\epsilon_n\}_n\) are deterministic error terms that will not affect the asymptotics.

We match the coefficients of the left-hand side and right-hand side of (3.10):

\[
Q_n + \epsilon_n = \beta_{n,n} \left( Q_n - Q_{n-1} - \frac{\lambda_2}{K_{n-1}} Q_{n-1} \right) + \beta_{n,n-1} \left( Q_{n-1} - Q_{n-2} - \frac{\lambda_2}{K_{n-2}} Q_{n-2} \right) \\
+ \beta_{n,n-2} \left( Q_{n-2} - Q_{n-3} - \frac{\lambda_2}{K_{n-3}} Q_{n-3} \right) \\
+ \beta_{n,n-3} \left( Q_{n-3} - Q_{n-4} - \frac{\lambda_2}{K_{n-4}} Q_{n-4} \right) \\
\vdots \\
+ \beta_{n,1} \left( Q_4 - Q_3 - \frac{\lambda_2}{K_3} Q_3 \right) + \beta_{n,2} \left( Q_3 - Q_2 - \frac{\lambda_2}{K_2} Q_2 \right) \\
+ \beta_{n,3} \left( Q_2 - Q_1 - \frac{\lambda_2}{K_1} Q_1 \right) + \beta_{n,1} \left( Q_1 - Q_0 - \frac{\lambda_2}{K_0} Q_0 \right)
\]
\[ Q_n + \epsilon_n = \beta_{n,n} Q_n + \left[ -\beta_{n,n} \left( 1 + \frac{\lambda_2}{K_{n-1}} \right) + \beta_{n,n-1} \right] Q_{n-1} \]
\[ + \left[ -\beta_{n,n-1} \left( 1 + \frac{\lambda_2}{K_{n-2}} \right) + \beta_{n,n-2} \right] Q_{n-2} \]
\[ + \left[ -\beta_{n,n-2} \left( 1 + \frac{\lambda_2}{K_{n-3}} \right) + \beta_{n,n-3} \right] Q_{n-3} \]
\[ \vdots \]
\[ + \left[ -\beta_{n,5} \left( 1 + \frac{\lambda_2}{K_4} \right) + \beta_{n,4} \right] Q_4 + \left[ -\beta_{n,4} \left( 1 + \frac{\lambda_2}{K_3} \right) + \beta_{n,3} \right] Q_3 \]
\[ + \left[ -\beta_{n,3} \left( 1 + \frac{\lambda_2}{K_2} \right) + \beta_{n,2} \right] Q_2 + \left[ -\beta_{n,2} \left( 1 + \frac{\lambda_2}{K_1} \right) + \beta_{n,1} \right] Q_1 \]
\[ + \beta_{n,1} \left( 1 + \frac{\lambda_2}{K_0} \right) Q_0. \]

So, we have \( \beta_{n,n} = 1 \)

\[ \beta_{n,n-1} = \beta_{n,n} \left( 1 + \frac{\lambda_2}{K_{n-1}} \right) = \left( 1 + \frac{\lambda_2}{K_{n-1}} \right) \]
\[ \beta_{n,n-2} = \beta_{n,n-1} \left( 1 + \frac{\lambda_2}{K_{n-2}} \right) = \left( 1 + \frac{\lambda_2}{K_{n-1}} \right) \left( 1 + \frac{\lambda_2}{K_{n-2}} \right) \]
\[ \beta_{n,n-3} = \beta_{n,n-2} \left( 1 + \frac{\lambda_2}{K_{n-3}} \right) = \left( 1 + \frac{\lambda_2}{K_{n-1}} \right) \left( 1 + \frac{\lambda_2}{K_{n-2}} \right) \left( 1 + \frac{\lambda_2}{K_{n-3}} \right) \]
\[ \vdots \]
\[ \beta_{n,j} = \prod_{l=j}^{n-1} \left( 1 + \frac{\lambda_2}{K_l} \right) \]
\[ \epsilon_n = \beta_{n,1} \left( 1 + \frac{\lambda_2}{K_0} \right) Q_0 = \prod_{l=1}^{n-1} \left( 1 + \frac{\lambda_2}{K_l} \right) \left( 1 + \frac{\lambda_2}{K_0} \right) Q_0. \]

\[ \beta_{n,j} = \prod_{l=j}^{n-1} \left( 1 + \frac{\lambda_2}{K_l} \right) = \prod_{l=j}^{n-1} \left( 1 + \frac{\lambda_2}{\lambda_1 l + t_0} \right) \]
Using (2.7) with $A = \lambda_2$, $B = \lambda_1$, and $C = t_0$, we get for any fixed $1 \leq j \leq n$,

$$\beta_{n,j} = \frac{\Gamma \left( n + \frac{\lambda_2 + t_0}{\lambda_1} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} \right)} \times \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda_2 + t_0}{\lambda_1} \right)},$$

and

$$\epsilon_n = Q_0 \prod_{l=0}^{n-1} \left( 1 + \frac{\lambda_2}{K_l} \right)$$

$$\beta_{n,j} = Q_0 \frac{\Gamma \left( n + \frac{\lambda_2 + t_0}{\lambda_1} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} \right)} \frac{\Gamma \left( \frac{t_0}{\lambda_1} \right)}{\Gamma \left( \frac{\lambda_2 + t_0}{\lambda_1} \right)}.$$

From (3.10),

$$Q_n = \sum_{j=1}^{n} \beta_{n,j} \Delta \tilde{M}_j^Q - \epsilon_n$$

$$= \sum_{j=1}^{n} \beta_{n,j} \left( \Delta \tilde{M}_j^Q - \Delta M_j^Q + \Delta M_j^Q \right) - \epsilon_n$$

$$= \sum_{j=1}^{n} \beta_{n,j} \left( \Delta \tilde{M}_j^Q - \Delta M_j^Q \right) + \sum_{j=1}^{n} \beta_{n,j} \Delta M_j^Q - \epsilon_n$$

$$Q_n = \sum_{j=1}^{n} \beta_{n,j} \Delta M_j^Q + \sum_{j=1}^{n} \beta_{n,j} \Delta \tilde{M}_j^Q - \epsilon_n.$$

$$\frac{Q_n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \beta_{n,j} \Delta M_j^Q - \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \beta_{n,j} \left( \Delta M_j^Q - \Delta \tilde{M}_j^Q \right) - \frac{\epsilon_n}{\sqrt{n}},$$

$$\frac{Q_n}{\sqrt{n \log n}} = \frac{1}{\sqrt{n \log n}} \sum_{j=1}^{n} \beta_{n,j} \Delta M_j^Q - \frac{1}{\sqrt{n \log n}} \sum_{j=1}^{n} \beta_{n,j} \left( \Delta M_j^Q - \Delta \tilde{M}_j^Q \right) - \frac{\epsilon_n}{\sqrt{n \log n}}.$$

Before we state the theorem, we need some lemmas.

The following lemma is a generalization of Cesàro averages which appears in [Gut, 2005], page 564 as Lemma 6.1.

**Lemma 3.2.** If $a_n$ are real numbers, $w_n$ are nonnegative numbers such that $a_n$ converges
to \( a \), \( a \in \mathbb{R} \) and \( B_n := \sum_{j=1}^{n} w_j \not\to \infty \) as \( n \to \infty \), then
\[
\frac{1}{B_n} \sum_{j=1}^{n} w_j a_j \to a \quad \text{as} \quad n \to \infty.
\]

Lemma 3.3. (i) If \( \frac{\lambda_2}{\lambda_1} < \frac{1}{2} \), then the martingale \( \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \beta_{n,j} \Delta M_j^Q \) converges in distribution to \( \mathcal{N} \left( 0, \frac{-\lambda_2^2 \alpha_1 \alpha_2}{1 - 2\lambda_2/\lambda_1} \right) \).

(ii) If \( \frac{\lambda_2}{\lambda_1} = \frac{1}{2} \), then the martingale \( \frac{1}{\sqrt{n \log n}} \sum_{j=1}^{n} \beta_{n,j} \Delta M_j^Q \) converges in distribution to \( \mathcal{N} \left( 0, -\lambda_2^2 \alpha_1 \alpha_2 \right) \).

Proof. We will apply Theorem 3.1 to the martingale difference array
\[
X_{n,j} = \begin{cases} 
\frac{\beta_{n,j} \Delta M_j^Q}{\sqrt{n}} & \text{for } \frac{\lambda_2}{\lambda_1} < \frac{1}{2} \\
\frac{\beta_{n,j} \Delta M_j^Q}{\sqrt{n \log n}} & \text{for } \frac{\lambda_2}{\lambda_1} = \frac{1}{2} 
\end{cases}
\]

We will show for both cases that all the assumptions of Theorem 3.1 are satisfied. First, from (3.7) and (3.5), we have for all \( j \in \mathbb{N} \),
\[
\Delta M_j^Q = \lambda_2 \left( \Delta Z_j - \frac{\lambda_2}{T_{j-1}} Z_{j-1} \right).
\]
So,
\[
|\Delta M_j^Q| \leq |\lambda_2| \left( |\Delta Z_j| + \left| \frac{Z_{j-1}}{T_{j-1}} \right| \right).
\]
Recall \( |\Delta Z_j| \leq \max\{\alpha_1, \alpha_2\} \) by (2.17) and \( \left| \frac{Z_{j-1}}{T_{j-1}} \right| \) is bounded by a non-zero constant by (2.52) and (2.14). Thus, \( \left\{ \Delta M_j^Q \right\}_j \) is uniformly bounded.

Let \( C_1' \) be the uniform bound of \( \left\{ \Delta M_j^Q \right\}_j \).

Now, we separate the two cases depending on the value of the ratio \( \lambda_2/\lambda_1 \).

**Case 1:** \( \frac{\lambda_2}{\lambda_1} < \frac{1}{2} \)
(i) First, we will show that the conditional Lindeberg condition (3.1) is satisfied.

\[
\max_{1 \leq j \leq n} |X_{n,j}| = \max_{1 \leq j \leq n} \frac{\beta_{n,j} |\Delta M_j^Q|}{\sqrt{n}} \\
\leq \frac{C_1}{\sqrt{n}} \max_{1 \leq j \leq n} \frac{\Gamma \left( n + \frac{\lambda_2 + t_0}{\lambda_1} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} \right)} \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda_2 + t_0}{\lambda_1} \right)} \text{ by (3.11)} \\
= C'_1 n^{-1/2} \frac{\Gamma \left( n + \frac{\lambda_2 + t_0}{\lambda_1} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} \right)} \max_{1 \leq j \leq n} \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda_2 + t_0}{\lambda_1} \right)} \text{ by (1.5)} \\
\leq C_1 n^{-1/2 + \lambda_2/\lambda_1} \text{ for some constant } C_1 > 0.
\]

Since \( C_1 n^{-1/2 + \lambda_2/\lambda_1} \to 0 \) as \( n \to \infty \), for any given \( \epsilon > 0 \) and \( n \geq \left( \frac{C_1}{\epsilon} \right)^{1/(1/2 - \lambda_2/\lambda_1)} \), the set \( \{ \max_{1 \leq j \leq n} |X_{n,j}| > \epsilon \} \) is empty. Hence, the conditional Lindeberg condition (3.1) is satisfied.

(ii) Now, we will show that

\[
\sum_{j=1}^{n} \mathbb{E} \left[ (X_{n,j})^2 | \mathcal{F}_{j-1} \right] \xrightarrow{p} \frac{-\lambda_2^2 \alpha_1 \alpha_2}{1 - 2\lambda_2/\lambda_1} \text{ as } n \to \infty.
\]

\[
\Delta M_j^Q = \lambda_2 \left( \Delta Z_j - \frac{\lambda_2}{T_{j-1}} Z_{j-1} \right) \\
(\Delta M_j^Q)^2 = \lambda_2^2 \left( \Delta Z_j - \frac{\lambda_2}{T_{j-1}} Z_{j-1} \right)^2 \\
= \lambda_2^2 \left\{ (\Delta Z_j)^2 - 2 \frac{\lambda_2}{T_{j-1}} Z_{j-1} \Delta Z_j + \frac{\lambda_2^2}{T_{j-1}^2} Z_{j-1}^2 \right\}.
\]
\[
\mathbb{E}\left[\left(\Delta M_{jQ}^2 \right)^2 | \mathcal{F}_{j-1}\right] = \lambda_2^2 \left\{ \mathbb{E}\left[ (\Delta Z_j)^2 | \mathcal{F}_{j-1}\right] - 2 \frac{\lambda_2}{T_{j-1}} Z_{j-1} \mathbb{E}[\Delta Z_j | \mathcal{F}_{j-1}] + \frac{\lambda_2^2}{T_{j-1}^2} Z_{j-1}^2 \right\}
\]

\[
= \lambda_2^2 \left\{ \mathbb{E}\left[ (\Delta Z_j)^2 | \mathcal{F}_{j-1}\right] - 2 \frac{\lambda_2}{T_{j-1}} Z_{j-1} \left( \frac{\lambda_2}{T_{j-1}} Z_{j-1} \right) + \frac{\lambda_2^2}{T_{j-1}^2} Z_{j-1}^2 \right\} \text{ by Lemma 2.5}
\]

\[
= \lambda_2^2 \left\{ \mathbb{E}\left[ (\Delta Z_j)^2 | \mathcal{F}_{j-1}\right] - \frac{\lambda_2^2}{T_{j-1}^2} Z_{j-1}^2 \right\}
\]

\[
= \lambda_2^2 \mathbb{E}\left[ (\Delta Z_j)^2 | \mathcal{F}_{j-1}\right] - \lambda_2^2 \left( \frac{Z_{j-1}}{T_{j-1}} \right)^2.
\]

\[
\sum_{j=1}^{n} \mathbb{E}\left[ (X_{n,j})^2 | \mathcal{F}_{j-1}\right] = \sum_{j=1}^{n} \mathbb{E}\left[ \left( \frac{\beta_{n,j} \Delta M_{jQ}}{\sqrt{n}} \right)^2 | \mathcal{F}_{j-1}\right]
\]

\[
= \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \mathbb{E}\left[ (\Delta M_{jQ})^2 | \mathcal{F}_{j-1}\right]
\]

\[
= \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \left\{ \lambda_2^2 \mathbb{E}\left[ (\Delta Z_j)^2 | \mathcal{F}_{j-1}\right] - \lambda_2^2 \left( \frac{Z_{j-1}}{T_{j-1}} \right)^2 \right\}.\]
\[
\sum_{j=1}^{n} \mathbb{E} \left[ (X_{n,j})^2 | F_{j-1} \right] - \left( -\frac{\lambda_2^2 \alpha_1 \alpha_2}{1 - 2\lambda_2/\lambda_1} \right)
\]
\[
= \lambda_2^2 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \mathbb{E} \left[ (\Delta Z_j)^2 | F_{j-1} \right] - \lambda_2^4 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \left( \frac{Z_{j-1}}{T_{j-1}} \right)^2
\]
\[
- \left( -\frac{\lambda_2^2 \alpha_1 \alpha_2}{1 - 2\lambda_2/\lambda_1} \right)
\]
\[
= \lambda_2^2 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \left\{ \mathbb{E} \left[ (\Delta Z_j)^2 | F_{j-1} \right] - (-\alpha_1 \alpha_2) \right\}
\]
\[
+ \lambda_2^2 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} (-\alpha_1 \alpha_2)
\]
\[
- \lambda_2^4 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \left( \frac{Z_{j-1}}{T_{j-1}} \right)^2
\]
\[
- \left( -\frac{\lambda_2^2 \alpha_1 \alpha_2}{1 - 2\lambda_2/\lambda_1} \right)
\]
\[
= \lambda_2^2 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \left\{ \mathbb{E} \left[ (\Delta Z_j)^2 | F_{j-1} \right] - (-\alpha_1 \alpha_2) \right\}
\]
\[
+ \lambda_2^2 (-\alpha_1 \alpha_2) \left( \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} - \frac{1}{1 - 2\lambda_2/\lambda_1} \right).
\]

Thus,
\[
\left\| \sum_{j=1}^{n} \mathbb{E} \left[ (X_{n,j})^2 | F_{j-1} \right] - \left( -\frac{\lambda_2^2 \alpha_1 \alpha_2}{1 - 2\lambda_2/\lambda_1} \right) \right\|_1
\]
\[
\leq \lambda_2^2 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \left\| \mathbb{E} \left[ (\Delta Z_j)^2 | F_{j-1} \right] - (-\alpha_1 \alpha_2) \right\|_1
\]
\[
+ \lambda_2^2 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \left\| \left( \frac{Z_{j-1}}{T_{j-1}} \right)^2 \right\|_1
\]
\[
+ \lambda_2^2 (-\alpha_1 \alpha_2) \left( \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} - \frac{1}{1 - 2\lambda_2/\lambda_1} \right).
\]

We will show that each term of the right-hand side of the equation above goes to 0 as \( n \to \infty \).
First, we will show that \( \lim_{n \to \infty} \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} = \frac{1}{1 - 2\lambda_2/\lambda_1} \).

\[
\sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} = \sum_{j=1}^{n} \frac{1}{n} \left[ \frac{\Gamma \left( n + \frac{\lambda_2 + t_0}{\lambda_1} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} \right)} \right] \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda_2 + t_0}{\lambda_1} \right)} \right]^2 
\]

\[
= \frac{1}{n} \left[ \frac{\Gamma \left( n + \frac{\lambda_2 + t_0}{\lambda_1} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} \right)} \right]^2 \sum_{j=1}^{n} \left[ \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda_2 + t_0}{\lambda_1} \right)} \right]^2 
\]

\[
= n^{-2\lambda_2/\lambda_1} \left[ \frac{\Gamma \left( n + \frac{\lambda_2 + t_0}{\lambda_1} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} \right)} \right]^2 \frac{1}{n^{1-2\lambda_2/\lambda_1}} \sum_{j=1}^{n} \left[ \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda_2 + t_0}{\lambda_1} \right)} \right]^2 
\]

Since \( \lim_{n \to \infty} n^{-2\lambda_2/\lambda_1} \left[ \frac{\Gamma \left( n + \frac{\lambda_2 + t_0}{\lambda_1} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} \right)} \right]^2 = 1 \) by (1.4) (with \( c_1 = \frac{\lambda_2 + t_0}{\lambda_1} \), \( c_2 = \frac{t_0}{\lambda_1} \)), we have

\[
(3.15) \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \sim \frac{1}{n^{1-2\lambda_2/\lambda_1}} \sum_{j=1}^{n} \left[ \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda_2 + t_0}{\lambda_1} \right)} \right]^2
\]

as \( n \to \infty \).

\[
(3.16) \frac{1}{n^{1-2\lambda_2/\lambda_1}} \sum_{j=1}^{n} \left[ \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda_2 + t_0}{\lambda_1} \right)} \right]^2 = \frac{B_n}{n^{1-2\lambda_2/\lambda_1}} \frac{1}{B_n} \sum_{j=1}^{n} w_j a_j
\]

where \( a_j = j^{2\lambda_2/\lambda_1} \left[ \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda_2 + t_0}{\lambda_1} \right)} \right]^2 \), \( w_j = j^{-2\lambda_2/\lambda_1} \), and \( B_n = \sum_{j=1}^{n} w_j \).

We have

\[
\lim_{n \to \infty} a_n = j^{2\lambda_2/\lambda_1} \left[ \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda_2 + t_0}{\lambda_1} \right)} \right]^2 = 1 \text{ by (1.5)},
\]

\[
B_n = \sum_{j=1}^{n} j^{-2\lambda_2/\lambda_1} \to \infty \text{ as } n \to \infty \text{ since } 2\lambda_2/\lambda_1 < 1.
\]
By Lemma 3.2

\begin{equation}
\lim_{n \to \infty} \frac{1}{B_n} \sum_{j=1}^{n} w_j a_j = 1,
\end{equation}

and by (1.8) (with \( \gamma = 1 - 2\lambda_2/\lambda_1 > 0 \)),

\begin{equation}
\lim_{n \to \infty} \frac{B_n}{n^{1-2\lambda_2/\lambda_1}} = \frac{1}{1 - 2\lambda_2/\lambda_1}.
\end{equation}

Hence, (3.15), (3.16), (3.17), and (3.18) yield

\begin{equation}
\lim_{n \to \infty} \frac{\sum_{j=1}^{n} (\beta_{nj})^2}{n} = \frac{1}{1 - 2\lambda_2/\lambda_1}.
\end{equation}

Now, we will show that

\[
\lim_{n \to \infty} \frac{\sum_{j=1}^{n} (\beta_{nj})^2}{n} \left\| \mathbb{E} \left[ (\Delta Z_j)^2 | F_{j-1} \right] - (-\alpha_1 \alpha_2) \right\|_1 = 0
\]

and

\[
\lim_{n \to \infty} \frac{\sum_{j=1}^{n} (\beta_{nj})^2}{n} \left\| \left( \frac{Z_{j-1}}{T_{j-1}} \right)^2 \right\|_1 = 0.
\]

Similarly, from (3.15), and (3.16), we have

\[
\sum_{j=1}^{n} (\beta_{nj})^2 \left\| \mathbb{E} \left[ (\Delta Z_j)^2 | F_{j-1} \right] - (-\alpha_1 \alpha_2) \right\|_1
\sim \frac{B_n}{n^{1-2\lambda_2/\lambda_1}} \frac{1}{B_n} \sum_{j=1}^{n} w_j a_j \quad \text{as } n \to \infty.
\]

where \( a_j = j^{2\lambda_2/\lambda_1} \left[ \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda + t_0}{\lambda_1} \right)} \right]^2 \mathbb{E} \left[ (\Delta Z_j)^2 | F_{j-1} \right] - (-\alpha_1 \alpha_2) \right\|_1, \ w_j = j^{-2\lambda_2/\lambda_1}, \) and \( B_n = \sum_{j=1}^{n} w_j. \)

Since \( \lim_{n \to \infty} j^{2\lambda_2/\lambda_1} \left[ \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{\lambda + t_0}{\lambda_1} \right)} \right]^2 = 1, \) and

\[
\lim_{n \to \infty} \left\| \mathbb{E} \left[ (\Delta Z_j)^2 | F_{j-1} \right] - (-\alpha_1 \alpha_2) \right\|_1 = 0 \quad \text{by (2.50),}
\]

59
\[
\lim_{n \to \infty} a_n = 1 \times 0 = 0.
\]
Hence, by Lemma 3.2 and (3.18),

\[
(3.20) \quad \lim_{n \to \infty} \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \| \mathbb{E} \left[ (\Delta Z_j)^2 | \mathcal{F}_{j-1} \right] - (-\alpha_1 \alpha_2) \|_1 = \frac{1}{1 - 2\lambda_2/\lambda_1} \times 0 = 0.
\]

Also, since \( \lim_{n \to \infty} \left\| \frac{Z_{j-1}}{T_{j-1}} \right\|_1^2 = \left\| \frac{Z_{j-1}}{T_{j-1}} \right\|_2^2 = 0 \) by (2.48), by substituting the term \( \| \mathbb{E} \left[ (\Delta Z_j)^2 | \mathcal{F}_{j-1} \right] - (-\alpha_1 \alpha_2) \|_1 \) in the proof above, we get

\[
(3.21) \quad \lim_{n \to \infty} \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n} \left\| \frac{Z_{j-1}}{T_{j-1}} \right\|_1^2 = 0.
\]

\[
\left\| \sum_{j=1}^{n} \mathbb{E} \left[ (X_{n,j})^2 | \mathcal{F}_{j-1} \right] - \left( -\frac{\lambda_2^2 \alpha_1 \alpha_2}{1 - 2\lambda_2/\lambda_1} \right) \right\|_1 \longrightarrow 0 \quad \text{as} \quad n \to \infty \quad \text{follows from (3.19), (3.20), (3.21)}.
\]

**Case 2:** \( \frac{\lambda_2}{\lambda_1} = \frac{1}{2} \)

We substitute \( \lambda_2/\lambda_1 \) and the denominator \( n \) in **Case 1** above by \( 1/2 \) and \( n \log n \) respectively. We get

\[
\max_{1 \leq j \leq n} |X_{n,j}| = \max_{1 \leq j \leq n} \frac{\beta_{n,j}|\Delta M_j^Q|}{\sqrt{n \log n}} \\
\leq C_1 (\log n)^{-1/2} \longrightarrow 0 \quad \text{as} \quad n \to \infty.
\]

Thus, for any given \( \epsilon > 0 \) and \( n \geq \exp \left( \frac{C_1}{\epsilon} \right)^2 \), the set \( \{ \max_{1 \leq j \leq n} |X_{n,j}| > \epsilon \} \) is empty. Hence, the *conditional Lindeberg condition* (3.1) is satisfied.

Now, it remains to show that \( \sum_{j=1}^{n} \mathbb{E} \left[ (X_{n,j})^2 | \mathcal{F}_{j-1} \right] \overset{p}{\longrightarrow} -\lambda_2^2 \alpha_1 \alpha_2 \) as \( n \to \infty \).
Similarly, we get

\[
\sum_{j=1}^{n} \mathbb{E} \left[ (X_{n,j})^2 | \mathcal{F}_{j-1} \right] = \sum_{j=1}^{n} \mathbb{E} \left[ \left( \frac{\beta_{n,j} \Delta M_{j}^Q}{\sqrt{n \log n}} \right)^2 | \mathcal{F}_{j-1} \right]
\]

\[
= \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} \mathbb{E} \left[ (\Delta M_{j}^Q)^2 | \mathcal{F}_{j-1} \right]
\]

\[
= \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} \left\{ \lambda_2^2 \mathbb{E} \left[ (\Delta Z_{j})^2 | \mathcal{F}_{j-1} \right] - \frac{Z_{j-1}}{T_{j-1}} \right\}.
\]

\[
\sum_{j=1}^{n} \mathbb{E} \left[ (X_{n,j})^2 | \mathcal{F}_{j-1} \right] - (-\lambda_2^2 \alpha_1 \alpha_2)
\]

\[
= \frac{\lambda_2^2}{n \log n} \sum_{j=1}^{n} (\beta_{n,j})^2 \mathbb{E} \left[ (\Delta Z_{j})^2 | \mathcal{F}_{j-1} \right] - \frac{Z_{j-1}}{n \log n} \left( \frac{Z_{j-1}}{T_{j-1}} \right)^2
\]

\[
- (-\lambda_2^2 \alpha_1 \alpha_2)
\]

\[
= \lambda_2^2 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} \left\{ \mathbb{E} \left[ (\Delta Z_{j})^2 | \mathcal{F}_{j-1} \right] - (-\alpha_1 \alpha_2) \right\}
\]

\[
+ \lambda_2^2 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} (-\alpha_1 \alpha_2)
\]

\[
- \lambda_2^4 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} \left( \frac{Z_{j-1}}{T_{j-1}} \right)^2
\]

\[
- (-\lambda_2^2 \alpha_1 \alpha_2)
\]

\[
= \lambda_2^2 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} \left\{ \mathbb{E} \left[ (\Delta Z_{j})^2 | \mathcal{F}_{j-1} \right] - (-\alpha_1 \alpha_2) \right\} - \lambda_2^4 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} \left( \frac{Z_{j-1}}{T_{j-1}} \right)^2
\]

\[
+ \lambda_2^2 (-\alpha_1 \alpha_2) \left( \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} - 1 \right).
\]
Thus,

\[
\left\| \sum_{j=1}^{n} \mathbb{E} \left[ (X_{n,j})^2 | \mathcal{F}_{j-1} \right] - (-\lambda_2^2 \alpha_1 \alpha_2) \right\|_1 \\
\leq \lambda_2^2 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} \left\| \mathbb{E} \left[ (\Delta Z_j)^2 | \mathcal{F}_{j-1} \right] - (-\alpha_1 \alpha_2) \right\|_1 \\
+ \lambda_2^2 \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} \left\| \left( \frac{Z_{j-1}}{T_{j-1}} \right)^2 \right\|_1 \\
+ \lambda_2^2 (-\alpha_1 \alpha_2) \left| \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} - 1 \right|.
\]

Similarly, from (3.15) and (3.16)

\[
\sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} \approx \frac{1}{\log n} \sum_{j=1}^{n} \left[ \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{t_0}{\lambda_1} + \frac{1}{2} \right)} \right]^2 \text{ as } n \to \infty,
\]

and

\[
\frac{1}{\log n} \sum_{j=1}^{n} \left[ \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{t_0}{\lambda_1} + \frac{1}{2} \right)} \right]^2 = B_n \frac{1}{\log n} \sum_{j=1}^{n} w_j a_j
\]

where \( a_j = \left[ \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{t_0}{\lambda_1} + \frac{1}{2} \right)} \right]^2 \), \( w_j = \frac{1}{j} \), and \( B_n = \sum_{j=1}^{n} \frac{1}{j} \).

Since \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \left( \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_2} + \frac{1}{2} \right)} \right)^2 = 1 \) by (1.5),

and \( \lim_{n \to \infty} \frac{B_n}{\log n} = \lim_{n \to \infty} \frac{1}{\log n} \sum_{j=1}^{n} \frac{1}{j} = 1 \) by (1.10),

we have

\[
\lim_{n \to \infty} \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} = 1.
\]
Using the same argument as in the proof of Case 1 above, we can show that

\[
\lim_{n \to \infty} \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} \left\| \mathbb{E} \left[ (\Delta Z_j)^2 | \mathcal{F}_{j-1} \right] - (-\alpha_1 \alpha_2) \right\|_1 = 0,
\]

and \[
\lim_{n \to \infty} \sum_{j=1}^{n} \frac{(\beta_{n,j})^2}{n \log n} \left\| \left( \frac{Z_{j-1}}{T_{j-1}} \right)^2 \right\|_1 = 0.
\]

\[ \square \]

**Lemma 3.4.** If \( a + b \neq c + d \), then there exists a constant \( C_1 = C_1(a, b, c, d, w_0, b_0) > 0 \) such that

\[
\left\| \frac{T_n - K_n}{n} \right\|_{2} \leq \begin{cases} 
C_1 n^{-(1 - \lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
C_1 n^{-1/2}(\log n)^{1/2} & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 n^{-1/2} & \text{if } \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases}
\]

as \( n \to \infty \).

**Proof.**

\[
\frac{T_n - K_n}{n} = \frac{T_n - (\lambda_1 n + t_0)}{n} = \frac{T_n - \lambda_1 - t_0}{n} = \frac{(a + b)X_n + (c + d)Y_n + t_0}{n} - \frac{\lambda_1 - t_0}{n} = \frac{(a + b)X_n + (c + d)[n - X_n]}{n} - \lambda_1 \quad \text{by (2.12)}
\]

\[
= [(a + b) - (c + d)] \frac{X_n}{n} + (c + d) - \lambda_1
\]

\[
= \left[ (a + b) - (c + d) \right] \left[ \frac{\alpha_2}{\alpha_2 - \alpha_1} - \frac{1}{\alpha_2 - \alpha_1} \frac{Z_n}{n} + \frac{1}{n \lambda_2 (\alpha_2 - \alpha_1)} \frac{(w_0 b_0) \alpha}{\alpha_1} \right] + (c + d) - \lambda_1 \quad \text{by (2.42)}
\]

\[
= (a + b) \left( \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) + (c + d) \left( \frac{-\alpha_1}{\alpha_2 - \alpha_1} \right) - \lambda_1
\]

\[
- \left[ (a + b) - (c + d) \right] \frac{Z_n}{n \lambda_2} + \frac{1}{n} \left[ (a + b) - (c + d) \right] (w_0 b_0) \alpha \quad \text{by (2.45)}
\]

\[
= - \left[ (a + b) - (c + d) \right] \frac{Z_n}{\alpha_2 - \alpha_1} + \frac{1}{n} \left[ (a + b) - (c + d) \right] (w_0 b_0) \alpha.
\]

Let \( C_3 = \left| \frac{(a + b) - (c + d)}{\alpha_2 - \alpha_1} \right|, C_4 = \left| \frac{[(a + b) - (c + d)](w_0 b_0) \alpha}{\lambda_2} \right| \).
Note that $C_3 > 0$ since $a + b \neq c + d$.

Hence, we have

\begin{equation}
\left\| \frac{T_n - K_n}{n} \right\|_2 \leq C_3 \left\| \frac{Z_n}{n} \right\|_2 + \frac{C_4}{n}.
\end{equation}

(3.22)

By Lemma 2.6 (with $r = 1$), we have

\[
\left\| \frac{Z_n}{n} \right\|_2 = \frac{1}{n} \left\| Z_n \right\|_2
\]

\[
\leq \begin{cases}
\frac{1}{n} \times \sqrt{C_1' n^{2\lambda_2/m}} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
\frac{1}{n} \times \sqrt{C_1' n \log n} & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
\frac{1}{n} \times \sqrt{C_1'} & \text{if } \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases}
\]

for some $C_1' = C_1'(a, b, c, d) > 0$ as $n \to \infty$. Hence,

\begin{equation}
\left\| \frac{Z_n}{n} \right\|_2 \leq \begin{cases}
C_1'' n^{-(1-\lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
C_1'' n^{-1/2} (\log n)^{1/2} & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1'' n^{-1/2} & \text{if } \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases}
\end{equation}

(3.23)

where $C_1'' = \sqrt{C_1'}$ as $n \to \infty$.

The lemma follows from (3.22) and (3.23). \hfill \Box

**Lemma 3.5.** If $a + b \neq c + d$, then there exists a constant $C_1 = C_1(a, b, c, d, w_0, b_0) > 0$ such that

\[
\left\| \Delta M_n^Q - \Delta \tilde{M}_n^Q \right\|_1 \leq \begin{cases}
C_1 n^{-2(1-2\lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
C_1 n^{-1} \log n & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 n^{-1} & \text{if } \frac{\lambda_2}{m} < \frac{1}{2},
\end{cases}
\]

as $n \to \infty$.  

64
Proof.

\[
\Delta M^Q_n - \Delta \tilde{M}^Q_n = \left( Q_n - Q_{n-1} - \frac{\lambda_2}{T_{n-1}} Q_{n-1} \right) - \left( Q_n - Q_{n-1} - \frac{\lambda_2}{K_{n-1}} Q_{n-1} \right) \\
= \lambda_2 \left( \frac{1}{K_{n-1}} - \frac{1}{T_{j-1}} \right) Q_{n-1} \\
= \lambda_2 \left( \frac{T_{n-1} - K_{n-1}}{T_{n-1} K_{n-1}} \right) (\lambda_2 Z_{n-1}) \text{ by (3.4)} \\
= \lambda_2 \left( \frac{T_{n-1} - K_{n-1}}{n - 1} \right) \left( \frac{Z_{n-1}}{n - 1} \right) \left( \frac{(n - 1)^2}{T_{n-1} K_{n-1}} \right) .
\]

From (2.14), we have for all \( n \geq 1, \)

\[
[m(n-1) + t_0] [\lambda_1 (n-1) + t_0] \leq T_{j-1} K_{n-1} \\
\leq \left[ \max\{a + b, c + d\} (n-1) + t_0 \right] \left[ \lambda_1 (n-1) + t_0 \right] \\
\leq \left[ \frac{1}{\max\{a + b, c + d\} (n-1) + t_0} \right] \left[ \lambda_1 (n-1) + t_0 \right] \leq \frac{1}{T_{n-1} K_{n-1}} \\
\leq \left[ \frac{1}{m(n-1) + t_0} \right] \left[ \lambda_1 (n-1) + t_0 \right] \leq \frac{1}{T_{n-1} K_{n-1}} \\
\leq \frac{(n - 1)^2}{(n - 1)^2} \leq \frac{1}{T_{n-1} K_{n-1}} \\
\leq \frac{(n - 1)^2}{m(n-1) + t_0} \left[ \lambda_1 (n-1) + t_0 \right].
\]

This shows that for all \( n \geq 2, \) \( \frac{(n - 1)^2}{T_{n-1} K_{n-1}} \) is bounded away from zero and bounded by a non-zero constant.

Let \( C'_1 = C'_1(a, b, c, d, w_0, b_0) \) be the uniform bound of \( \left\{ \frac{(n - 1)^2}{T_{n-1} K_{n-1}} \right\}_{j \geq 2} \).

Thus, for \( n \geq 2, \)

\[
\|\Delta M^Q_n - \Delta \tilde{M}^Q_n\|_1 = \lambda_2^2 \left\| \left( \frac{T_{n-1} - K_{n-1}}{n - 1} \right) \left( \frac{Z_{n-1}}{n - 1} \right) \left( \frac{(n - 1)^2}{T_{n-1} K_{n-1}} \right) \right\|_1 \\
\leq \lambda_2^2 C'_1 \left\| \left( \frac{T_{n-1} - K_{n-1}}{n - 1} \right) \left( \frac{Z_{n-1}}{n - 1} \right) \right\|_1 \\
\leq \lambda_2^2 C'_1 \left\| \frac{T_{n-1} - K_{n-1}}{n - 1} \right\|_2 \left\| \frac{Z_{n-1}}{n - 1} \right\|_2 \text{ by the Cauchy-Schwarz inequality.}
\]
By Lemma 3.4 and (3.23), we have

\[ \| \Delta M^Q_n - \Delta \tilde{M}^Q_n \|_1 \leq \begin{cases} 
C_1 n^{-2(1-\lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
C_1 n^{-1} \log n & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 n^{-1} & \text{if } \frac{\lambda_2}{m} < \frac{1}{2}, 
\end{cases} \]

for some constant \( C_1 > 0 \) as \( n \to \infty \).

**Lemma 3.6.** Assume \( a + b \neq c + d \).

(i) If \( \frac{\lambda_2}{\lambda_1} < \frac{1}{2} \) and \( \frac{\lambda_2}{m} < \frac{3}{4} \), then

\[ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \beta_{n,j} (\Delta M^Q_j - \Delta \tilde{M}^Q_j) \xrightarrow{p} 0. \]

(ii) If \( \frac{\lambda_2}{\lambda_1} = \frac{1}{2} \) and \( \frac{\lambda_2}{m} < \frac{3}{4} \), then

\[ \frac{1}{n \log n} \sum_{j=1}^{n} \beta_{n,j} (\Delta M^Q_j - \Delta \tilde{M}^Q_j) \xrightarrow{p} 0. \]

**Proof.** (i) Let \( \frac{\lambda_2}{\lambda_1} < \frac{1}{2} \) and \( \frac{\lambda_2}{m} < \frac{3}{4} \). By Markov’s inequality, it suffices to show that

\[ (3.24) \quad \mathbb{E} \left| \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \beta_{n,j} (\Delta M^Q_j - \Delta \tilde{M}^Q_j) \right| \longrightarrow 0 \quad \text{as } n \to \infty. \]
Since \( \lim_{n \to \infty} n^{-\lambda_2/\lambda_1} \frac{\Gamma\left(n + \frac{\lambda_2 + t_0}{\lambda_1}\right)}{\Gamma\left(n + \frac{t_0}{\lambda_1}\right)} = 1 \) by (1.3),

\[
\frac{1}{n^{1/2}} \frac{\Gamma\left(n + \frac{\lambda_2 + t_0}{\lambda_1}\right)}{\Gamma\left(n + \frac{t_0}{\lambda_1}\right)} \sum_{j=1}^{n} \frac{\Gamma\left(j + \frac{t_0}{\lambda_1}\right)}{\Gamma\left(j + \frac{\lambda_2 + t_0}{\lambda_1}\right)} \left\| \Delta M_j^Q - \Delta \tilde{M}_j^Q \right\|_1 
\sim \frac{1}{n^{1/2-\lambda_2/\lambda_1}} \sum_{j=1}^{n} \frac{\Gamma\left(j + \frac{t_0}{\lambda_1}\right)}{\Gamma\left(j + \frac{\lambda_2 + t_0}{\lambda_1}\right)} \left\| \Delta M_j^Q - \Delta \tilde{M}_j^Q \right\|_1 \quad \text{as } n \to \infty.
\]

(3.25)

\[
\frac{1}{n^{1/2-\lambda_2/\lambda_1}} \sum_{j=1}^{n} \frac{\Gamma\left(j + \frac{t_0}{\lambda_1}\right)}{\Gamma\left(j + \frac{\lambda_2 + t_0}{\lambda_1}\right)} \left\| \Delta M_j^Q - \Delta \tilde{M}_j^Q \right\|_1 = \frac{B_n}{n^{1/2-\lambda_2/\lambda_1}} \frac{1}{B_n} \sum_{j=1}^{n} w_j a_j
\]

where \( a_j = j^{\lambda_2/\lambda_1+1/2} \frac{\Gamma\left(j + \frac{\lambda_2 + t_0}{\lambda_1}\right)}{\Gamma\left(j + \frac{t_0}{\lambda_1}\right)} \left\| \Delta M_j^Q - \Delta \tilde{M}_j^Q \right\|_1, \) \( w_j = j^{-(\lambda_2/\lambda_1+1/2)}, \) and \( B_n = \sum_{j=1}^{n} w_j. \)

Since \( \lim_{n \to \infty} n^{\lambda_2/\lambda_1} \frac{\Gamma\left(n + \frac{t_0}{\lambda_1}\right)}{\Gamma\left(n + \frac{\lambda_2 + t_0}{\lambda_1}\right)} = 1 \) by (1.3),

(3.26)

\[
a_n \sim n^{1/2} \left\| \Delta M_n^Q - \Delta \tilde{M}_n^Q \right\|_1 \quad \text{as } n \to \infty.
\]

By Lemma 3.5,

\[
\left\| \Delta M_n^Q - \Delta \tilde{M}_n^Q \right\|_1 \leq \begin{cases} 
C_1 n^{1/2} n^{-2(1-\lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < \frac{3}{4}, \\
C_1 n^{1/2} \log n & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 n^{1/2} n^{-1} & \text{if } \frac{\lambda_2}{m} < \frac{1}{2}.
\end{cases}
\]

67
for some constant \( C_1 > 0 \) as \( n \to \infty \). So,

\[
(3.28) \quad n^{1/2} \left\| \Delta M_n^Q - \Delta \tilde{M}_n^Q \right\|_1 \leq \begin{cases} 
C_1 n^{-3/2 + 2\lambda_2/m} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < \frac{3}{4}, \\
C_1 n^{-1/2} \log n & \text{if } \frac{\lambda_2}{m} = \frac{1}{2}, \\
C_1 n^{-1/2} & \text{if } \frac{\lambda_2}{m} = \frac{1}{2},
\end{cases}
\]

as \( n \to \infty \).

Since each term of the right-hand side of (3.28) goes to 0 as \( n \to \infty \), we have \( \lim_{n \to \infty} a_n = 0 \) by (3.27).

\[ B_n = \sum_{j=1}^{n} j^{-(\lambda_2/\lambda_1 + 1/2)} \nearrow \infty \text{ as } n \to \infty \text{ since } \lambda_2/\lambda_1 + 1/2 < 1. \]

By Lemma (3.2),

\[
(3.29) \quad \lim_{n \to \infty} \frac{1}{B_n} \sum_{j=1}^{n} w_j a_j = 0,
\]

and by (1.8) (with \( \gamma = 1/2 - \lambda_2/\lambda_1 > 0 \)),

\[
(3.30) \quad \lim_{n \to \infty} \frac{B_n}{n^{1/2 - \lambda_2/\lambda_1}} = \frac{1}{1/2 - \lambda_2/\lambda_1}.
\]

Hence, (3.25), (3.26), (3.29), and (3.30) yield (3.24).

To prove (ii), we substitute \( \lambda_2/\lambda_1 \) and the denominator \( \sqrt{n} \) in the proof of (i) above by \( 1/2 \) and \( \sqrt{n \log n} \) respectively. We get

\[
\mathbb{E} \left| \frac{1}{\sqrt{n \log n}} \sum_{j=1}^{n} \beta_{n,j} (\Delta M_j^Q - \Delta \tilde{M}_j^Q) \right|
\leq \frac{1}{(n \log n)^{1/2}} \frac{\Gamma \left( n + \frac{t_0}{X_1} + \frac{1}{2} \right)}{\Gamma \left( n + \frac{X_1}{X_1} + \frac{1}{2} \right)} \sum_{j=1}^{n} \frac{\Gamma \left( j + \frac{t_0}{X_1} \right)}{\Gamma \left( j + \frac{X_1}{X_1} + \frac{1}{2} \right)} \left\| \Delta M_j^Q - \Delta \tilde{M}_j^Q \right\|_1,
\]

68
and

\[
\frac{1}{(n \log n)^{1/2}} \sum_{j=1}^{n} \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} + \frac{1}{2} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} + \frac{1}{2} \right)} \left\| \Delta M_j^Q - \Delta \tilde{M}_j^Q \right\|_1 \\
\sim \frac{1}{(\log n)^{1/2}} \sum_{j=1}^{n} \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{t_0}{\lambda_1} + \frac{1}{2} \right)} \left\| \Delta M_j^Q - \Delta \tilde{M}_j^Q \right\|_1 \quad \text{as } n \to \infty.
\]

\[
\frac{1}{(\log n)^{1/2}} \sum_{j=1}^{n} \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{t_0}{\lambda_1} + \frac{1}{2} \right)} \left\| \Delta M_j^Q - \Delta \tilde{M}_j^Q \right\|_1 = \frac{B_n}{(\log n)^{1/2}} \frac{1}{n} \sum_{j=1}^{n} w_j a_j
\]

where \( a_j = j \left( \log j \right)^{1/2} \frac{\Gamma \left( j + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( j + \frac{t_0}{\lambda_1} + \frac{1}{2} \right)} \left\| \Delta M_j^Q - \Delta \tilde{M}_j^Q \right\|_1, \) \( w_j = j^{-1} \left( \log j \right)^{-1/2}, \) and \( B_n = \sum_{j=1}^{n} w_j. \)

Since \( \lim_{n \to \infty} n^{1/2} \frac{\Gamma \left( n + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} + \frac{1}{2} \right)} = 1 \) by (1.3),

\[
a_n \sim (n \log n)^{1/2} \left\| \Delta M_n^Q - \Delta \tilde{M}_n^Q \right\|_1 \quad \text{as } n \to \infty.
\]

By assumption, \( \frac{\lambda_2}{\lambda_1} = \frac{1}{2}, \) and \( \lambda_1 \geq m \) by (2.53), we have \( \frac{\lambda_2}{m} \geq \frac{\lambda_2}{\lambda_1} = \frac{1}{2}. \) Thus, \( \frac{1}{2} \leq \frac{\lambda_2}{m} < \frac{3}{4} \)

since \( \frac{\lambda_2}{m} < \frac{3}{4} \) by assumption.

Hence, by Lemma 3.5

\[
(n \log n)^{1/2} \left\| \Delta M_n^Q - \Delta \tilde{M}_n^Q \right\|_1 \leq \begin{cases} 
(n \log n)^{1/2} C_1 n^{-2(1-\lambda_2/m)} & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
(n \log n)^{1/2} C_1 n^{-1} \log n & \text{if } \frac{\lambda_2}{m} = \frac{1}{2},
\end{cases}
\]

for some constant \( C_1 > 0 \) as \( n \to \infty. \) So,

\[
(n \log n)^{1/2} \left\| \Delta M_n^Q - \Delta \tilde{M}_n^Q \right\|_1 \leq \begin{cases} 
C_1 n^{-3/2+2\lambda_2/m} \left( \log n \right)^2 & \text{if } \frac{1}{2} < \frac{\lambda_2}{m} < 1, \\
C_1 n^{-1/2} \left( \log n \right)^{3/2} & \text{if } \frac{\lambda_2}{m} = \frac{1}{2},
\end{cases}
\]

as \( n \to \infty. \)
Since both \( \lim_{n \to \infty} C_1 n^{-3/2+2\lambda_2/\lambda_1} (\log n)^2 = 0 \) and \( C_1 n^{-1/2} (\log n)^{3/2} = 0 \), we have \( \lim_{n \to \infty} a_n = 0 \).

So, by Lemma 3.2 \( \lim_{n \to \infty} \frac{1}{B_n} \sum_{j=1}^{n} w_j a_j = 0 \).

Now, it remains to show that \( \lim_{n \to \infty} \frac{B_n}{(\log n)^{1/2}} \) exists and is finite.

By (1.12) (with \( \gamma = 1/2 \)),

\[
\lim_{n \to \infty} \frac{B_n}{(\log n)^{1/2}} = \lim_{n \to \infty} \frac{1}{(\log n)^{1/2}} \sum_{j=1}^{n} j^{-1}(\log j)^{-1/2} = \frac{1}{1/2} = 2,
\]

which is finite. \( \Box \)

### 3.3 Main Result

In this section, we state the main result of this chapter 3, the central limit theorem for the unbalanced urn models for the case of simple draw.

**Theorem 3.7.** (i) If \( \frac{\lambda_2}{\lambda_1} < \frac{1}{2} \) and \( \frac{\lambda_2}{m} < \frac{3}{4} \), then \( \frac{Q_n}{\sqrt{n}} \) converges in distribution to \( N \left( 0, \frac{-\lambda_2^2\alpha_1\alpha_2}{1 - 2\lambda_2/\lambda_1} \right) \).

(ii) If \( \frac{\lambda_2}{\lambda_1} = \frac{1}{2} \) and \( \frac{\lambda_2}{m} < \frac{3}{4} \), then \( \frac{Q_n}{\sqrt{n \log n}} \) converges in distribution to \( N \left( 0, -\lambda_2^2\alpha_1\alpha_2 \right) \).

**Proof.** Recall (3.12),

\[
\epsilon_n = Q_0 \frac{\Gamma \left( \frac{t_0}{\lambda_1} \right) \Gamma \left( n + \frac{\lambda_2 + t_0}{\lambda_1} \right) \Gamma \left( n + \frac{t_0}{\lambda_1} \right)}{\Gamma \left( \lambda_2 + t_0 \right) \Gamma \left( n + \lambda_2 + t_0 \right)}.
\]

So,

\[
\frac{\epsilon_n}{\sqrt{n}} = Q_0 \frac{\Gamma \left( \frac{t_0}{\lambda_1} \right) 1}{\Gamma \left( \lambda_2 + t_0 \right)} \frac{\Gamma \left( n + \frac{\lambda_2 + t_0}{\lambda_1} \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} \right)} \frac{\Gamma \left( n + \lambda_2 + t_0 \right)}{n^{-\lambda_2/\lambda_1} \Gamma \left( 1/2 - \frac{\lambda_2}{\lambda_1} \right)},
\]

and

\[
\frac{\epsilon_n}{\sqrt{n \log n}} = Q_0 \frac{\Gamma \left( \frac{t_0}{\lambda_1} \right) 1}{\Gamma \left( \lambda_2 + t_0 + \frac{1}{2} \right) \Gamma \left( n + \frac{t_0}{\lambda_1} + \frac{1}{2} \right)} \frac{\Gamma \left( n + \lambda_2 + t_0 + \frac{1}{2} \right)}{(\log n)^{1/2} n^{-1/2}} \frac{\Gamma \left( 1/2 \right)}{\Gamma \left( n + \frac{t_0}{\lambda_1} + \frac{1}{2} \right)}.
\]
\[
\lim_{n \to \infty} \frac{\epsilon_n}{\sqrt{n}} = 0 \quad \text{for} \quad \frac{\lambda_2}{\lambda_1} < \frac{1}{2}, \quad \text{and} \\
\lim_{n \to \infty} \frac{\epsilon_n}{\sqrt{n \log n}} = 0 \quad \text{for} \quad \frac{\lambda_2}{\lambda_1} = \frac{1}{2}.
\]

Theorem 3.7 follows from applying Slutsky’s theorem, Theorem 1.4 to Lemma 3.6, Lemma 3.3, and (3.31).

Corollary 3.8.
(i) If \( \frac{\lambda_2}{\lambda_1} < \frac{1}{2} \) and \( \frac{\lambda_2}{m} < \frac{3}{4} \), then \( \sqrt{n} \left( \frac{W_n}{T_n} - \alpha_2 - \alpha_1 \right) \) converges in distribution to \( N \left( 0, \frac{1}{\alpha_2 - \alpha_1} \right) \).

(ii) If \( \frac{\lambda_2}{\lambda_1} = \frac{1}{2} \) and \( \frac{\lambda_2}{m} < \frac{3}{4} \), then \( \sqrt{n \log n} \left( \frac{W_n}{T_n} - \alpha_2 - \alpha_1 \right) \) converges in distribution to \( N \left( 0, \frac{1}{4(\alpha_2 - \alpha_1)^2} \right) \).

Proof.

\[
\frac{W_n}{T_n} - \frac{\alpha_2}{\alpha_2 - \alpha_1} = \frac{(\alpha_2 - \alpha_1)W_n - \alpha_2 T_n}{T_n(\alpha_2 - \alpha_1)} = \frac{(\alpha_2 - \alpha_1)W_n - \alpha_2(W_n + B_n)}{T_n(\alpha_2 - \alpha_1)} = -\frac{1}{\alpha_2 - \alpha_1} \frac{1}{T_n} \frac{Q_n}{\sqrt{n}}.
\]

Since \( \frac{T_n}{n} \overset{a.s.}{\to} \lambda_1 \) by (2.45), the corollary (i) follows from Slutsky’s theorem, Theorem 1.4, and Theorem 3.7 (i).

Similarly,

\[
\sqrt{\frac{n}{\log n}} \left( \frac{W_n}{T_n} - \frac{\alpha_2}{\alpha_2 - \alpha_1} \right) = -\frac{1}{\alpha_2 - \alpha_1} \frac{1}{\frac{T_n}{n}} \frac{Q_n}{\sqrt{n \log n}}.
\]

Hence, the corollary (ii) follows from Slutsky’s theorem, Theorem 1.4, Theorem 3.7 (ii), and \( \frac{\lambda_2}{\lambda_1} = \frac{1}{2} \).

Corollary 3.8 is not new but more explicit. Renlund (Renlund, 2011) obtained this result via stochastic approximation method without the assumption \( \frac{\lambda_2}{m} < \frac{3}{4} \). However,
he did not write down the general formula since it is rather cumbersome.

To illustrate the results of Corollary 3.8 we will use the example given in Renlund, 2011.

**Example 3.3.1.** Let \( A \) be the replacement matrix

\[
\begin{pmatrix}
4 & 5 \\
3 & 2 \\
\end{pmatrix}
\]

So, \( m = \min\{a + b, c + d\} = \min\{9, 5\} = 5. \)

\( A \) has eigenvalues \( \lambda_1 = 7, \lambda_2 = -1 \) with eigenvector \( \alpha = (\alpha_1, \alpha_2)^T = (-1, 1). \)

Since \( \lambda_2 < 0 \), the assumptions \( \frac{\lambda_2}{\lambda_1} < \frac{1}{2} \) and \( \frac{\lambda_2}{\lambda_1} < \frac{3}{4} \) are easily satisfied.

\[
\frac{\alpha_2}{\alpha_2 - \alpha_1} = \frac{1}{1 - (-1)} = \frac{1}{2}
\]

\[
\frac{-(\lambda_2/\lambda_1)^2\alpha_1\alpha_2}{(1 - 2\alpha_2/\lambda_1)(\alpha_2 - \alpha_1)^2} = \frac{-(1/7)^2(-1)(1)}{(1 - 2(-1/7))(1 - (-1))^2} = \frac{1}{252}.
\]

Thus,

\[
\sqrt{n} \left( \frac{W_n}{T_n} - \frac{1}{2} \right) \text{ converges in distribution to } \mathcal{N}(0, 1/252).
\]
Chapter 4

Application of balanced and unbalanced urn models

In this last chapter, we give an application of the balanced and the unbalanced urn models.

4.1 Drezner Generalized Binomial Distribution

4.1.1 Definition

Drezner and Farnum (Drezner and Farnum, 1993) define the following Markov chain: Let $0 < p < 1$ and $0 \leq \theta_n < 1$ for all $n \in \mathbb{N}$. Let $X_1$ be a Bernoulli random variable with parameter $p$. For $n \geq 2$,

$$\mathbb{P}(X_n = X_{n-1} + \delta | X_{n-1}) = \begin{cases} (1 - \theta_{n-1})p + \theta_{n-1}\left(\frac{X_{n-1}}{n-1}\right) & \text{if } \delta = 1; \\ (1 - \theta_{n-1})(1 - p) + \theta_{n-1}\left(1 - \frac{X_{n-1}}{n-1}\right) & \text{if } \delta = 0; \\ 0 & \text{otherwise.} \end{cases}$$

where $X_n$ denotes the number of successes in $n$ trials.

Remark 4.1.1. When $\theta_n = 0$ for all $n \in \mathbb{N}$, $\{X_n\}_n$ has exactly the Binomial distribution with parameters $n$ and $p$. 

73
4.1.2 Main Results

Let $B(x, y)$ denote the beta function defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x > 0, y > 0.$$ 

It is well-known that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x > 0, y > 0,$$

where $\Gamma$ is the gamma function.

**Theorem 4.1.** *(Drezner and Farnum, 1993)* For all $n \geq 1$,

\begin{enumerate}[(i)]
  \item $\mathbb{E}(X_n) = np$,
  \item If $\theta_n = \theta$ constant,
    
    $$\text{Var}(X_n) = \begin{cases} 
      \frac{np(1-p)}{1-2\theta} \left[ 1 - \frac{1}{n B(n,2\theta)} \right] & \text{if } \theta \neq 0, \frac{1}{2}; \\
      np(1-p) & \text{if } \theta = 0; \\
      np(1-p) \sum_{j=1}^{n} \frac{1}{j} & \text{if } \theta = \frac{1}{2}.
    \end{cases}$$
\end{enumerate}

From now on, we assume for all $n \in \mathbb{N}$, $\theta_n = \theta$ constant.

Heyde *(Heyde, 2004)* proved the central limit theorem for $\{X_n\}$ when $\theta \leq \frac{1}{2}$ using the Martingale Convergence Theorem. Two years later, unaware of Heyde’s paper, Drezner *(Drezner, 2006)* proved a weak version of the same result under additional technical assumptions.

**Theorem 4.2.** *(Heyde, 2004)*

\begin{enumerate}[(i)]
  \item If $\theta < \frac{1}{2}$, then $\frac{X_n - np}{\sqrt{n}}$ converges in distribution to $\mathcal{N} \left( 0, \frac{p(1-p)}{1-2\theta} \right)$
  \item If $\theta = \frac{1}{2}$, then $\frac{X_n - np}{\sqrt{n \log n}}$ converges in distribution to $\mathcal{N} \left( 0, p(1-p) \right)$
  \item If $\theta > \frac{1}{2}$, then $\frac{X_n - np}{n^\theta} \xrightarrow{a.s.} X$, where $X$ is a proper random variable such that $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) = \frac{p(1-p)}{(2\theta-1)(2\theta)}$, and $\mathbb{E}(X^3) \neq 0$ if $p \neq \frac{1}{2}$.
\end{enumerate}
Now, we will show that the results of Theorem 4.2 (i), (ii) can be derived from Theorem 1.10 after we associate a balanced urn process with two colors to the Markov chain \( \{X_n\}_n \).

**Proposition 4.3.** Let \( 0 < p < 1 \) and \( 0 \leq \theta < 1 \). Define \( W_0 = p, B_0 = 1 - p \) and for all \( n \geq 1 \),

\[
W_n = n(1 - \theta)p + \theta X_n, \quad \text{and} \quad B_n = n(1 - \theta)(1 - p) + \theta (n - X_n).
\]

Then, \( (W_n, B_n) \) is a balanced urn process with two colors whose replacement matrix is given by

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (1 - \theta)p + \theta & (1 - \theta)(1 - p) \\ (1 - \theta)p & (1 - \theta)(1 - p) + \theta \end{pmatrix}.
\]

**Proof.** First, we need to show that \( (W_n, B_n) \) is an \( \mathbb{R}_+^2 \)-Markov chain with transition probabilities

\[
\mathbb{P} [(W_n, B_n) = (W_{n-1}, B_{n-1}) + (a, b) | W_{n-1}, B_{n-1}] = \frac{W_{n-1}}{T_{n-1}},
\]

\[
\mathbb{P} [(W_n, B_n) = (W_{n-1}, B_{n-1}) + (c, d) | W_{n-1}, B_{n-1}] = \frac{B_{n-1}}{T_{n-1}},
\]

where \( T_n := W_n + B_n \) and \( a, b, c, \) and \( d \) are the entries of the matrix \( A \) in (4.2).

Note that \( T_0 = W_0 + B_0 = 1 \) and for all \( n \geq 1 \),

\[
T_n = n(1 - \theta)p + \theta X_n + n(1 - \theta)(1 - p) + \theta (n - X_n)
\]

\[
= n(1 - \theta) + \theta n = n.
\]

We have,

\[
W_n = n(1 - \theta)p + (1 - \theta)p + \theta (X_{n-1} + \delta)
\]

\[
= n(1 - \theta)p + \theta X_{n-1} + (1 - \theta)p + \theta \delta
\]

\[
= W_{n-1} + (1 - \theta)p + \theta \delta.
\]
Similarly,

\[ B_n = n(1 - \theta)(1 - p) + (1 - \theta)(1 - p) + \theta(n - 1 + 1 - (X_{n-1} + \delta)) \]
\[ = n(1 - \theta)(1 - p) + \theta(n - 1 - X_{n-1}) + (1 - \theta)(1 - p) + \theta(1 - \delta) \]
\[ = B_{n-1} + (1 - \theta)(1 - p) + \theta(1 - \delta). \]

Thus,

\[
(4.4) \quad W_n = \begin{cases} 
W_{n-1} + (1 - \theta)p + \theta & \text{if } \delta = 1, \\
W_{n-1} + (1 - \theta)p & \text{if } \delta = 0,
\end{cases}
\]

and

\[
(4.5) \quad B_n = \begin{cases} 
B_{n-1} + (1 - \theta)(1 - p) & \text{if } \delta = 1, \\
B_{n-1} + (1 - \theta)(1 - p) + \theta & \text{if } \delta = 0.
\end{cases}
\]

Hence, from (4.3), (4.4), and (4.5), it is left to show that for all \( n \geq 1 \),

\[
\begin{align*}
\frac{W_{n-1}}{T_{n-1}} &= \mathbb{P}(X_n = X_{n-1} + \delta|X_{n-1}) \quad \text{for } \delta = 1, \\
\frac{B_{n-1}}{T_{n-1}} &= \mathbb{P}(X_n = X_{n-1} + \delta|X_{n-1}) \quad \text{for } \delta = 0,
\end{align*}
\]

where \( X_0 := 0 \).

\[
\frac{W_0}{T_0} = \frac{p}{1} = p = \mathbb{P}(X_1 = X_0 + \delta|X_0) \quad \text{for } \delta = 1, \\
\frac{B_0}{T_0} = \frac{1 - p}{1} = 1 - p = \mathbb{P}(X_1 = X_0 + \delta|X_0) \quad \text{for } \delta = 0,
\]

since \( X_1 \) is a Bernoulli random variable with parameter \( p \).
For $n \geq 2$,

\[
\frac{W_{n-1}}{T_{n-1}} = \frac{(n-1)(1-\theta)p + \theta X_{n-1}}{n-1}
\]

\[
= (1-\theta)p + \theta \left( \frac{X_{n-1}}{n-1} \right)
\]

\[
= \mathbb{P}(X_n = X_{n-1} + \delta | X_{n-1}) \text{ for } \delta = 1 \text{ by (4.1)},
\]

\[
\frac{B_{n-1}}{T_{n-1}} = \frac{(n-1)(1-\theta)(1-p) + \theta(n-1 - X_{n-1})}{n-1}
\]

\[
= (1-\theta)(1-p) + \theta \left( 1 - \frac{X_{n-1}}{n-1} \right)
\]

\[
= \mathbb{P}(X_n = X_{n-1} + \delta | X_{n-1}) \text{ for } \delta = 0 \text{ by (4.1)}.
\]

- Now, we will show that the matrix $A$ has a constant row sum.

\[
a + b = (1-\theta)p + \theta + (1-\theta)(1-p) = (1-\theta)(p + (1-p)) + \theta
\]

\[
= (1-\theta)(1) + \theta = 1,
\]

\[
c + d = (1-\theta)p + (1-\theta)(1-p) + \theta = (1-\theta)(p + (1-p)) + \theta
\]

\[
= (1-\theta)(1) + \theta = 1.
\]

\[\square\]

Now, we can apply the results of Theorem [1.10] $A$ has eigenvalues $\lambda_1 = a + b = c + d = 1 = s$ and $\lambda_2 = a - c = (1-\theta)p + \theta - (1-\theta)p = \theta$.

So, $\rho = \frac{\lambda_2}{\lambda_1} = \frac{\theta}{1} = \theta$.

(i) If $\theta < \frac{1}{2}$, then $\frac{W_n - \mathbb{E}(W_n)}{\sqrt{n}}$ converges in distribution to $\mathcal{N}(0, \sigma^2)$ where $\sigma^2 = \frac{bc\rho^2 s}{(1-2\rho)(b+c)^2} = \frac{p(1-p)\theta^2}{1-2\theta}$

(ii) If $\theta = \frac{1}{2}$, then $\frac{W_n - \mathbb{E}(W_n)}{\sqrt{n \log n}}$ converges in distribution to $\mathcal{N}(0, \sigma^2)$ where $\sigma^2 = bc = (1-\theta)^2p(1-p) = (1-1/2)^2p(1-p) = p(1-p)/4$. 

77
\[ W_n = n(1 - \theta)p + \theta X_n \]
\[ \mathbb{E}(W_n) = n(1 - \theta)p + \theta \mathbb{E}(X_n) \]
\[ W_n - \mathbb{E}(W_n) = \theta (X_n - \mathbb{E}(X_n)) \]
\[ = \theta (X_n - np) \text{ by Theorem 4.1 (i)}. \]

So,
\[ \frac{X_n - np}{\sqrt{n}} = \frac{1}{\theta} \frac{W_n - \mathbb{E}(W_n)}{\sqrt{n}}, \]
and similarly for \( \theta = \frac{1}{2} \),
\[ \frac{X_n - np}{\sqrt{n \log n}} = \frac{1}{\theta} \frac{W_n - \mathbb{E}(W_n)}{\sqrt{n \log n}} \]
\[ = 2 \frac{W_n - \mathbb{E}(W_n)}{\sqrt{n \log n}}. \]

We get the results from (i) and (ii) above, and Slutsky’s theorem, Theorem 1.4.

### 4.2 Adaptive Tests

#### 4.2.1 Introduction

First, we will give the definitions of a conventional test and an adaptive test.

**Definition 4.2.1.** A test is said to be conventional if the questions administered to all the examinees are the same.

**Definition 4.2.2.** A test is said to be adaptive if the questions presented to each examinee are adjusted depending on the examinee’s responses to the preceding questions.

The basic rule in adaptive testing is that when an examinee answers a question correctly (incorrectly), then the next question administered to her should be more difficult (easier) (Lord, 1971a).

Examples of adaptive test are the Graduate Management Admission Test (GMAT),
the Graduate Record Examinations (GRE), the Test of English as a Foreign Language (TOEFL), the Microsoft Certification exams.

A considerable amount of research have been done in adaptive testing both theoretically and practically. We recommend the interested reader to consult (van der Linden et al., 2000) and (Wainer et al., 2000). Here, we will review only the relevant literature that is connected to the University of Cincinnati Mathematics Placement Test that we will describe below.

### 4.2.2 Item Characteristic Curve

The characteristic curve of an item (ICC) represents the probability of correct response to the question as a function of the ability level, $\theta$, $\theta \in \mathbb{R}$, of the examinee.

Let $Y$ denote an examinee’s response, with values 1 if the response is correct and 0 if the response is wrong to a question whose difficulty is $x$, $x \in \mathbb{R}$.

Rasch (Rasch, 1960) proposed the following logistic function:

$$P(Y = 1|x, \theta) = \frac{1}{1 + e^{x-\theta}}.$$

![Figure 4.1](image)

Figure 4.1: $P(Y = 1|x, \theta) = \frac{1}{1 + e^{x-\theta}}$: Probability of correct response for Rasch model.
(i) the examinee has a 50% chance of correct response to a question of difficulty that matches her ability.

(ii) Her chance of responding correctly to questions of difficulty higher than her ability decreases rapidly to 0.

(iii) Her chance of responding correctly to questions of difficulty lower than her ability increases rapidly to 1.

Some generalizations of the Rasch model have been considered (see for e.g. (Chang et al., 2009)):

- The two-parameter logistic (2-PL) model, whose ICC is given by

\[
\Pr(Y = 1 | x, \theta) = \frac{1}{1 + e^{a(x - \theta)}},
\]

where \(a\) is an additional item parameter called the discriminating power.

- The three-parameter logistic (3-PL) model, whose ICC is given by

\[
\Pr(Y = 1 | x, \theta) = c + (1 - c) \frac{1}{1 + e^{a(x - \theta)}},
\]

where \(a > 0\), \(0 \leq c < 1\) are additional item parameters called the discriminating power, the degree of guessing respectively.

- Lord (Lord, 1971a) considered another form of the item characteristic curve (ICC) by substituting the logistic function in (4.8) to a cumulative distribution function of a standard normal random variable.

\[
c + (1 - c)\Phi [a(x - \theta)],
\]

where \(\Phi\) denotes the cumulative distribution of the standard normal random variable.
The problem is that we do not know in advance the true ability, \( \theta \), of the examinee.

Lord (Lord, 1971a) noted that if an examinee answers all \( n \) questions in a test correctly, then we are not able to pinpoint her ability level. For example, we cannot tell how she compares with another examinee who also answers all the questions correctly. A similar conclusion applies when the examinee does not know the answer to any of the questions in the test. The main focus of the next subsection is how to estimate \( \theta \).

4.2.3 Adaptive Testing Algorithms

In this subsection, we recall the definition of stochastic approximation and and some adaptive testing algorithms.

**Definition 4.2.3** ([Benaim, 1999]). Let \( \{X_n\}_{n \in \mathbb{N}} \) be a stochastic process in the Euclidean space \( \mathbb{R}^m \) and adapted to a filtration \( \{\mathcal{F}_n\} \).

We say \( \{X_n\} \) is a stochastic approximation algorithm if it satisfies

\[
X_{n+1} - X_n = \gamma_{n+1} V_{n+1},
\]

where \( V_{n+1} \) is a random variable and \( \gamma_n > 0 \) is a "small" step size.

Suppose \( n \) questions with difficulties \( x_1, \ldots, x_n, n = 1, 2, \ldots \) have been selected and administered, and the responses of the examinee are \( Y_1, \ldots, Y_n \). The selection of the \((n + 1)\)th question, whose difficulty will be \( x_{n+1} \), will be based on the previous questions, \( x_1, \ldots, x_n \) and the responses \( Y_1, \ldots, Y_n \).

Lord (Lord, 1971a, Lord, 1971b) used the Robbins-Monro stochastic approximation procedure (Robbins and Monro, 1951). For all \( n \geq 1 \),

\[
x_{n+1} - x_n = \frac{\gamma}{n} (Y_n - \alpha),
\]

where \( \alpha \), and \( \gamma \) are positive real numbers. He stated the following theorem:

**Theorem 4.4.** If the ICC follows \([4.9]\) and \( \alpha = \frac{1+c}{2} \), then \( x_n \xrightarrow{p} \theta \) as \( n \to \infty \).
From now on, we assume the Rasch model \([4.6]\). Define for \(t \in \mathbb{R}, f(t) := \frac{1}{1 + e^{t}}\). \(f\) is monotone decreasing on \((-\infty, \infty)\) and has an inverse.

Lord (Lord, 1980) proposed another adaptive testing procedure based on maximum likelihood estimators consisting of the following steps (Chang et al., 2009):

**Step 1:** Initialization. Specify the difficulty level, \(x_1\) of the initial question. If the examinee’s first response is correct (i.e. \(Y_1 = 1\)), then choose the succeeding questions with increasing difficulties \(x_1 \leq x_2 \leq \ldots \leq x_{n_0}\), where \(n_0 := \inf \{n \in \mathbb{N} : Y_n = 0\}\) is the first time a wrong response occurs. On the other hand, if the examinee’s response is wrong (i.e. \(Y_1 = 0\)), then select the succeeding questions with decreasing difficulties \(x_1 \geq x_2 \geq \ldots \geq x_{n_0}\), where \(n_0 := \inf \{n : Y_n = 1\}\) is the first time a correct response occurs.

**Step 2:** Estimation. For each \(n \geq n_0\), maximize the likelihood function

\[
(4.12) \quad L_n(\theta) := \prod_{j=1}^{n} [f(x_j - \theta)^{Y_j} [1 - f(x_j - \theta)]^{1-Y_j}],
\]

So, \(\log L_n(\theta) = \sum_{j=1}^{n} \{Y_j f(x_j - \theta) + (1 - Y_j) \log [1 - f(x_j - \theta)]\}\).

\[
\frac{d}{d\theta} \log L_n(\theta) = \sum_{j=1}^{n} \left[ Y_j \frac{-f'(x_j - \theta)}{f(x_j - \theta)} + (1 - Y_j) \frac{f'(x_j - \theta)}{1 - f(x_j - \theta)} \right]
\]

\[
= \sum_{j=1}^{n} \left\{ \frac{(-Y_j [1 - f(x_j - \theta)] + (1 - Y_j) f(x_j - \theta)) f'(x_j - \theta)}{f(x_j - \theta) [1 - f(x_j - \theta)]} \right\}.\]

Since \(f'(x) = -\frac{e^x}{(1+e^x)^2}\) and \(f(x) [1 - f(x)] = \frac{1}{1+e^x} \left( 1 - \frac{1}{1+e^x} \right) = \frac{e^x}{(1+e^x)^2}\), we have

\[
\frac{d}{d\theta} \log L_n(\theta) = \sum_{j=1}^{n} \left[ -Y_j + f(x_j - \theta) \right] (-1)
\]

\[
= \sum_{j=1}^{n} [Y_j - f(x_j - \theta)].
\]
Since the response sequence \( \{Y_1, \ldots, Y_n\} \) contains both 0 and 1, the maximum likelihood estimating (MLE) equation

\[
\sum_{j=1}^{n} [Y_j - f(x_j - \theta)] = 0
\]

has a unique solution.

So, define \( x_{n+1} \), the \((n + 1)^{st}\) estimate of \( \theta \), to be that solution. It means

\[
\sum_{j=1}^{n} [Y_j - f(x_j - x_{n+1})] = 0.
\]

We have the following theorem:

**Theorem 4.5** ([Ying and Wu, 1997], [Chang et al., 2009]). Let \( (x_n)_{n \in \mathbb{N}} \) be the sequences of difficulty specified by Steps 1–2. Then

(i) \( x_n \xrightarrow{a.s.} \theta \) as \( n \to \infty \) and

(ii) \( \sqrt{n}(x_n - \theta) \) converges in distribution to \( \mathcal{N}(0, 4) \).

**4.2.4 UC Math Placement Test Algorithm**

In this subsection, we give a description of and conjecture for the University of Cincinnati Mathematics Placement Test (UCMPT) Algorithm. The design of the UCMPT algorithm is based on the book written by Wright and Stone on tests designs [Wright and Stone, 1979]. The UCMPT algorithm is a modification of the maximum likelihood estimate (MLE).

**Step 1:** same as described in subsection 4.2.3

**Step 2:** Estimation. For each \( n \geq n_0 \), substitute \( x_1, \ldots, x_n \) of the MLE equation (4.13) by \( \bar{x}_n \), their average \( (\bar{x}_n = (x_1 + \ldots + x_n) / n) \). Thus, we have

\[
\sum_{j=1}^{n} Y_j - f(\bar{x}_n - \theta) = 0.
\]
Since the response sequence \( \{Y_1, \ldots, Y_n\} \) contains both 0 and 1, the equation (4.15) has a \textit{unique} solution.

So, define \( x_{n+1} \), the \((n + 1)\)th estimate of \( \theta \), to be that solution. It means

\[
(4.16) \quad \sum_{j=1}^{n} [Y_j - f(\bar{x}_n - x_{n+1})] = 0.
\]

From now on, define \( S_1 := Y_1 \) and for \( n \geq 2 \), \( S_n := Y_1 + \ldots + Y_n \).

The equation (4.16) is equivalent to

\[
S_n - nf(\bar{x}_n - x_{n+1}) = 0,
\]

\[
f(\bar{x}_n - x_{n+1}) = \frac{S_n}{n},
\]

\[
\bar{x}_n - x_{n+1} = f^{-1}\left(\frac{S_n}{n}\right)
\]

\[
= \log\left(\frac{1 - \frac{S_n}{n}}{\frac{S_n}{n}}\right)
\]

\[
\bar{x}_n - x_{n+1} = \log\left(\frac{n - S_n}{S_n}\right).
\]

We get for all \( n \geq n_0 \),

\[
(4.17) \quad x_{n+1} = \bar{x}_n + \log\left(\frac{S_n}{n - S_n}\right).
\]

Thus, for all \( n \geq n_0 + 1 \),

\[
(4.18) \quad nx_{n+1} = x_1 + \ldots + x_{n-1} + x_n + n \log\left(\frac{S_n}{n - S_n}\right), \text{ and}
\]

\[
(4.19) \quad (n - 1)x_n = x_1 + \ldots + x_{n-1} + (n - 1) \log\left(\frac{S_{n-1}}{n - 1 - S_{n-1}}\right).
\]

Subtracting (4.19) from (4.18) yields

\[
n(x_{n+1} - x_n) = n \log\left(\frac{S_n}{n - S_n}\right) - (n - 1) \log\left(\frac{S_{n-1}}{n - 1 - S_{n-1}}\right)
\]

\[
x_{n+1} - x_n = \log\left(\frac{S_n}{n - S_n}\right) - \frac{n - 1}{n} \log\left(\frac{S_{n-1}}{n - 1 - S_{n-1}}\right).
\]
Hence, we have the following recursion

\[(4.20) \quad x_{n+1} - x_n = \log \left( \frac{S_n}{n - S_n} \right) - \frac{n - 1}{n} \log \left( \frac{S_{n-1}}{n - 1 - S_{n-1}} \right), \quad n \geq n_0 + 1. \]

**Remark 4.2.1.** Since \( \log \left( \frac{S_n}{n - S_n} \right) \) and \( \log \left( \frac{S_{n-1}}{n - 1 - S_{n-1}} \right) \) are undefined for \( n = 1, \ldots, n_0 + 1 \), if we want to use the recursion (4.20) from the beginning, we can skip Step 1 and use the following recursion for \( n \geq 1 \),

\[(4.21) \quad x_{n+1} - x_n = \log \left( \frac{S_n + \gamma}{n - S_n + \gamma} \right) - \frac{n - 1}{n} \log \left( \frac{S_{n-1} + \gamma}{n - 1 - S_{n-1} + \gamma} \right), \]

where \( \gamma \) is a fixed positive number (for example \( \gamma = 1 \)).

Let \( h(t) := \log \left( \frac{t}{1 - t} \right) \) for \( 0 < t < 1 \) so that (4.20) becomes

\[x_{n+1} - x_n = h \left( \frac{S_n}{n} \right) - \frac{n - 1}{n} h \left( \frac{S_{n-1}}{n - 1} \right), \quad n \geq n_0 + 1.\]

For \( 0 < t < 1 \),

\[h(1 - t) = -h(t), \quad h'(t) = \frac{1}{t} + \frac{1}{1 - t}, \quad h''(t) = -\frac{1}{t^2} + \frac{1}{(1 - t)^2}.\]

So, \( h(1/2) = 0, \quad h'(1/2) = 4, \quad \text{and} \quad h''(1/2) = 0.\)

Expanding \( h \) into the Taylor series at \( t = 1/2 \), we get

\[x_{n+1} - x_n = +4 \left( \frac{S_n}{n} - \frac{1}{2} \right) + \frac{1}{2} h''(\frac{S_n}{n} - \frac{1}{2}) \left( \frac{S_n}{n} - \frac{1}{2} \right)^2 - \frac{n - 1}{n} \left[ 4 \left( \frac{S_{n-1}}{n - 1} - \frac{1}{2} \right) + \frac{1}{2} h''(\frac{S_{n-1}}{n - 1} - \frac{1}{2}) \left( \frac{S_{n-1}}{n - 1} - \frac{1}{2} \right)^2 \right],\]

where \( S_n^*, S_{n-1}^* \) are random variables depending on \( \frac{S_n}{n} \) and between \( \frac{S_n}{n} \) and 1/2, depending on \( \frac{S_{n-1}}{n-1} \) and between \( \frac{S_{n-1}}{n-1} \) and 1/2 respectively.
After simplification,

\[
x_{n+1} - x_n = \frac{4}{n} \left( Y_n - \frac{1}{2} \right) + \frac{1}{2} n h''(S_n^*) \left( \frac{S_n}{n} - \frac{1}{2} \right)^2
- \frac{1}{2} n h''(S_{n-1}^*) \left( \frac{S_{n-1}}{n-1} - \frac{1}{2} \right)^2.
\]

(4.22)

**Remark 4.2.2.** If we drop the last two terms of the right-hand side of (4.22), then the UCMPT algorithm becomes the Robbins-Monro procedure

\[
x_{n+1} - x_n = \frac{4}{n} \left( Y_n - \frac{1}{2} \right).
\]

(4.23)

We have the following theorem as a direct application of Theorem 1 (Sacks et al., 1958).

**Theorem 4.6.** If \( \{x_n\} \) satisfies (4.23), then

(i) \( x_n \xrightarrow{a.s.} \theta \) as \( n \to \infty \) and

(ii) \( \sqrt{n} (x_n - \theta) \) converges in distribution to \( \mathcal{N}(0, 4) \).

**Conjecture:** Our simulations suggests that the conclusion of Theorem 4.6 remains true if \( \{x_n\} \) satisfies the UCMPT algorithm (4.22).

We chose \( n = 10^7 \) and ran 10,000 simulations. We got the following statistics:

- mean = 0.006887
- median = 0.029701
- variance = 4.07130472
- skewness = -0.0069269
- kurtosis = -0.01466.

The continuous curve in figure 4.2 represents the probability density function of \( \mathcal{N}(0, 4) \).
4.2.5 UC Math Placement Test Scheme Modified

Since \( \lim_{n \to \infty} \frac{n - 1}{n} = 1 \), let us substitute the term \( \frac{n - 1}{n} \) by 1 in (4.21). So, for \( n \geq 1 \),

\[
(4.24) \quad x_{n+1} = x_n + \log \left( \frac{S_n + \gamma}{n - S_n + \gamma} \right) - \log \left( \frac{S_{n-1} + \gamma}{n-1 - S_{n-1} + \gamma} \right).
\]

Iterating (4.24) yields

\[
(4.25) \quad x_{n+1} = x_1 + \log \left( \frac{S_n + \gamma}{n - S_n + \gamma} \right), \quad n \geq 1.
\]

We have the following theorem:

**Theorem 4.7.** Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence of random variables satisfying (4.25). Then

(i) \( x_n \xrightarrow{a.s.} \frac{x_1 + \theta}{2} \) as \( n \to \infty \) and

(ii) \( \sqrt{n} \left( x_n - \frac{x_1 + \theta}{2} \right) \) converges in distribution to \( \mathcal{N} \left( 0, \frac{1 + e^{(x_1-\theta)/2}}{3 e^{(x_1-\theta)/2}} \right) \).

**Remark 4.2.3.** Theorem 4.7 (i) shows that the sequence of \( (x_n)_n \) converges to a different limit. Hence, the term \( \frac{1}{n} \) in (4.21) is not negligible.

Before we prove the theorem, we need the following lemma.
Lemma 4.8 (The Delta Method). Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of random variables satisfying
\[
\sqrt{n} (X_n - \mu) \text{ converges in distribution to } \mathcal{N}(0, \sigma^2),
\]
where \(\mu\) and \(\sigma^2\) are finite valued constants, \(\sigma^2 > 0\).
If \(g\) is a real-valued, differentiable function on an interval, \(I, \mu \in I, \text{ and } g'(\mu) \neq 0\), then
\[
\sqrt{n} (g(X_n) - g(\mu)) \text{ converges in distribution to } \mathcal{N}(0, \sigma^2 [g'(\mu)]^2).
\]

Now, we are ready to prove the Theorem 4.7 above.

Proof. Without loss of generality, we may assume \(\theta = 0\) by subtracting \(\theta\) on both sides of (4.25).

(i) We shall construct an urn process \((W_n, B_n)\) from \(\{S_n\}_n\).

Recall \(S_n = Y_1 + \cdots + Y_n\) where
\[
\mathbb{P}(Y_j = 1|x_1, \ldots, x_j) = \mathbb{P}(Y_j = 1|x_j) = \frac{1}{1 + e^{x_j}}, \quad 1 \leq j \leq n.
\]
Let \(\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)\). Note that \(x_{n+1}\) is \(\mathcal{F}_n\)-measurable.
It is easy to see that for all \(n \geq 1\), \(\{S_n\}_n\) is a Markov chain with transition probabilities
\[
\begin{align*}
\mathbb{P}(S_n = S_{n-1} + 1|\mathcal{F}_{n-1}) &= \mathbb{P}(Y_n = 1|x_n), \\
\mathbb{P}(S_n = S_{n-1}|\mathcal{F}_{n-1}) &= \mathbb{P}(Y_n = 0|x_n),
\end{align*}
\]
where \(S_0 := 0\).

\[
\begin{align*}
\mathbb{P}(Y_n = 1|x_n) &= \frac{1}{1 + e^{x_n}} \\
&= \frac{1}{1 + e^{x_1} \left(\frac{S_{n-1} + \gamma}{n - 1 - S_{n-1} + \gamma}\right)} \\
&= \frac{n - 1 - S_{n-1} + \gamma}{n - 1 - S_{n-1} + \gamma + e^{x_1} (S_{n-1} + \gamma)}
\end{align*}
\]
(4.26)

and
\[
\begin{align*}
\mathbb{P}(Y_n = 0|x_n) &= \mathbb{P}(S_n = S_{n-1}|x_n) \\
&= 1 - \frac{1}{1 + e^{x_n}} \\
&= \frac{e^{x_1} (S_{n-1} + \gamma)}{n - 1 - S_{n-1} + \gamma + e^{x_1} (S_{n-1} + \gamma)}
\end{align*}
\]
(4.27)
Now, let \( \{(W_n, B_n)\}_n \) be an \( \mathbb{R}^2_n \)-Markov chain with replacement matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and transition probabilities

\[
\begin{align*}
\mathbb{P}[(W_n, B_n) = (W_{n-1}, B_{n-1}) + (a, b)] &= \frac{W_{n-1}}{T_{n-1}}, \\
\mathbb{P}[(W_n, B_n) = (W_{n-1}, B_{n-1}) + (c, d)] &= \frac{B_{n-1}}{T_{n-1}},
\end{align*}
\]

where \( a, b, c, d \) are to be determined, \( T_n := W_n + B_n \), and \((W_0, B_0) = (\gamma, e^{x_1}\gamma)\).

To determine \( a, b, c, d \), we set the transition probabilities of \( \{(W_n, B_n)\}_n \) to be equal to those of \( \{S_n\}_n \).

From (4.26) and (4.27), we get

\[
(4.28) \quad \begin{cases} 
W_{n-1} &= \frac{n - 1 - S_{n-1} + \gamma}{n - 1 - S_{n-1} + \gamma + e^{x_1}(S_{n-1} + \gamma)}, \\
B_{n-1} &= \frac{e^{x_1}(S_{n-1} + \gamma)}{n - 1 - S_{n-1} + \gamma + e^{x_1}(S_{n-1} + \gamma)}.
\end{cases}
\]

So for \( n \in \mathbb{N} \cup \{0\} \),

\[
\begin{align*}
W_n &= n - S_n + \gamma \\
B_n &= e^{x_1}(S_n + \gamma) \\
T_n &= n - S_n + \gamma + e^{x_1}(S_n + \gamma).
\end{align*}
\]

We have for \( \delta \in \{0, 1\} \),

\[
\begin{align*}
W_n &= n - (S_{n-1} + \delta) + \gamma \\
&= n - 1 - S_{n-1} + 1 - \delta \\
&= W_{n-1} + 1 - \delta.
\end{align*}
\]
and

\[
B_n = e^{x_1} (S_n + \gamma) \\
= e^{x_1} (S_{n-1} + \delta + \gamma) \\
= e^{x_1} (S_{n-1} + \gamma) + e^{x_1} \delta \\
= B_{n-1} + e^{x_1} \delta.
\]

Thus,

\[
(W_n, B_n) = \begin{cases} 
(W_{n-1}, B_{n-1}) + (0, e^{x_1}) & \text{if } \delta = 1, \\
(W_{n-1}, B_{n-1}) + (1, 0) & \text{if } \delta = 0.
\end{cases}
\]

Hence, the replacement matrix \( A \) of \( \{(W_n, B_n)\}_n \) is given by

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
0 & e^{x_1} \\
1 & 0
\end{pmatrix}
\]

The matrix (4.29) has eigenvalues \( \lambda_1 = e^{x_1/2}, \lambda_2 = e^{-x_1/2} \) and eigenvector corresponding to \( \lambda_2, \alpha = (\alpha_1, \alpha_2)^T \) where \( \alpha_1 = -e^{x_1/2} \) and \( \alpha_2 = 1. \)

By Theorem 2.1

\[
W_n \xrightarrow{T_n} \alpha_2 = \frac{1}{1 - (-e^{x_1/2})} = \frac{1}{1 + e^{x_1/2}}.
\]

Since the function \( x \mapsto \frac{1}{1 + e^x} \) is monotone on \( (-\infty, \infty) \), (4.26), (4.28) and (4.30) yield \( x_n \xrightarrow{n} \frac{x_1}{2} \) as \( n \to \infty. \)

(ii) Since \( \lambda_2 = -e^{x_1/2} < 0, \frac{\lambda_2}{\lambda_1} < \frac{1}{2}. \) By Corollary 3.8 (i),

\[
\sqrt{n} \left( \frac{W_n}{T_n} - \frac{1}{1 + e^{x_1/2}} \right) \text{ converges in distribution to } \mathcal{N} \left( 0, \sigma^2 \right)
\]
where $\sigma^2 = \frac{-(\lambda_2/\lambda_1)^2 \alpha_1 \alpha_2}{(1 - 2\lambda_2/\lambda_1)(\alpha_2 - \alpha_1)^2}$
\begin{align*}
&= -\left(-e^{x_{1/2}}/e^{x_{1/2}}\right)^2 \left(-e^{x_{1/2}}\right) (1) \\
&= \frac{1 - 2\left(-e^{x_{1/2}}/e^{x_{1/2}}\right) \left[1 - \left(-e^{x_{1/2}}\right)^2\right]}{(1 - 2(-1))(1 + e^{x_{1/2}})^2} \\
&= \frac{e^{x_{1/2}}}{3(1 + e^{x_{1/2}})^2}.
\end{align*}
(4.31)

Now, let $g(x) := \log \left(\frac{1-x}{x}\right)$ for all $x \in (0, 1)$, the inverse function of $x \mapsto \frac{1}{1+e^x}$.

$g$ is differentiable on $(0, 1)$ and for all $x \in (0, 1)$, $g'(x) = \frac{-1}{1-x} - \frac{1}{x} = -\frac{1}{x(1-x)} \neq 0$. By the Delta Method, Lemma 4.8 we have

$$
\sqrt{n} (x_n - x_{1/2}) \text{ converges in distribution to } \mathcal{N}(0, \sigma_g^2)
$$

where $\sigma_g^2 = \sigma^2 g' \left(\frac{1}{1+e^{x_{1/2}}}\right)$
\begin{align*}
&= \frac{e^{x_{1/2}}}{3(1 + e^{x_{1/2}})^2} \left[-\frac{1}{1+e^{x_{1/2}}} \left(1 - \frac{1}{1+e^{x_{1/2}}}\right)\right]^2 \text{ by (4.31)} \\
&= \frac{e^{x_{1/2}}}{3(1 + e^{x_{1/2}})^2} \left[-\frac{1}{1+e^{x_{1/2}}} \frac{1}{1+e^{x_{1/2}}}\right]^2 \\
&= \frac{e^{x_{1/2}}}{3(1 + e^{x_{1/2}})^2} \left(1 + e^{x_{1/2}}\right)^4 \\
&= \frac{1 + e^{x_{1/2}}}{3e^{x_{1/2}}^2}.
\end{align*}
Bibliography

Athreya, K. B. and Karlin, S. (1968). Embedding of urn schemes into continuous time
markov branching processes and related limit theorems. The Annals of Mathematical

Eggenberger urn model, with an application to computer data structures. SIAM

probabilites XXXIII, pages 1–68. Springer.

Bernstein, S. (1940a). Nouvelles applications des grandeurs aléatoires
SSSR], 4:137–150.


response theory models with applications to computerized adaptive tests. The Annals


