I, Prashant Patel, hereby submit this original work as part of the requirements for the degree of Master of Science in Mechanical Engineering.

It is entitled:
A Computational Study of Enhanced Heat Transfer in Low Reynolds Number Flows through Axially Twisted Ducts of Rectangular Cross Section

Student's name: Prashant Patel

This work and its defense approved by:

Committee chair: Raj Manglik, PhD
Committee member: Shaaban Abdallah, PhD
Committee member: Milind Jog, PhD
A Computational Study of Enhanced Heat Transfer in Low Reynolds Number Flows through Axially Twisted Ducts of Rectangular Cross Section

A Thesis Submitted to the

Graduate School

University of Cincinnati

in partial fulfillment of the

requirements for the degree of

MASTER OF SCIENCE (M.S.)

in the Department of Mechanical Engineering

of the School of Dynamic Systems

2011

By

Prashant Prakash Patel

B.E., University of Pune, India, 2005

Committee Co-Chairs:

Dr. R. M. Manglik

Dr. M. A. Jog
ABSTRACT

Low Reynolds number fully developed swirl flows through rectangular ducts that are helically twisted along their axis are computationally modeled. The twist ratio $\xi$ (= 180° twist pitch / hydraulic diameter) and rectangular flow cross-section aspect ratio $\alpha$ (= height / width) characterize the duct’s geometrical attributes. A parametric study delineates the influence of flow rate ($10 \leq \text{Re} \leq 1000$), type of fluid ($\text{Pr} = 5, 45$ and $100$) and duct geometry ($\xi = 3.0, 6.0, \text{and} 12.0; \alpha = 1.0, 0.75, \text{and} 0.5$) on swirl-flow generation, convection and wall friction behavior. The twisted duct-surface curvature is found to induce lateral fluid circulation, which results in the formation of axially helical swirl in the core of the duct. With increasing severity of duct twist ($\xi = 12.0 \rightarrow 3.0$) and flow Reynolds number, or decreasing cross-section aspect ratio ($\alpha = 1.0 \text{ or square} \rightarrow 0.5 \text{ or slender rectangle}$), the swirl structure breaks up into multiple peripheral vortices but with increased magnitude of the primary core secondary-flow cell. Consequently, in fixed pressure-gradient driven flows, the resultant lateral mixing of flow increases convection heat transfer significantly for twisted ducts relative to that in straight ducts of same cross section. Increasing swirl-induced mixing as $\alpha \rightarrow 0.5, \xi \rightarrow 3.0, \text{and} \text{Re} > \text{O}[100]$, characterized by pronounced core circulation accompanied with multiple peripheral vortices, is found to enhance the heat transfer coefficient by 2.6 to 14 times that in an equivalent straight duct. The larger benefits accrue in higher Pr liquids, and 2.4 to 13 times higher heat transfer rate can be accommodated on a fixed pumping power basis; alternatively, 50% to 90% reduction in heat exchanger surface area can be achieved on a fixed pressure drop and heat load basis.
ACKNOWLEDGEMENTS

I would like to express my sincere appreciation to my advisors Dr. Raj Manglik and Dr. Milind Jog for their support, guidance, and assistance throughout the research project. They have been true guide and mentor during different phases of my research. I would like to thank the committee member Dr. Shabaan Abdallah for agreeing to be on my thesis committee. I would also like to thank my friends and colleagues in the Thermal-Fluids and Thermal Processing Laboratory as well as student organization Hindu YUVA for their moral and material support during the tenure of research. I am especially grateful to my friend Advait Athavale, without his encouragement, I wouldn’t have enrolled in M.S.

Finally, I would like to thank the University of Cincinnati, for giving me this research opportunity and to make all the resources easily available from time to time.
TABLE OF CONTENTS

ABSTRACT i

ACKNOWLEDGMENTS ii

TABLE OF CONTENTS iii

LIST OF FIGURES v

NOMENCLATURE viii

1. INTRODUCTION 1

1.1 Heat transfer enhancements and twisted ducts 1

1.2 Swirl Flows 1

1.3 Aim and scope of study 5

2. LAMINAR FLOW IN TWISTED RECTANGULAR DUCT 6

2.1 Introduction 6

2.2 Mathematical formulation 6

2.3 Numerical Methodology 12

2.4 Results and discussions 14

2.5 Figures 21

3. HEAT TRANSFER IN TWISTED RECTANGULAR DUCT 30
3.1 Introduction

3.2 Mathematical formulation

3.3 Numerical Methodology

3.4 Results and discussions

3.5 Figures

4. CONCLUSION AND RECOMMENDATIONS

4.1 Conclusions

4.2 Recommendations

Bibliography

Appendix A. Mean Velocity and Friction Factor

Appendix B. Mean Temperature and Nusselt Number

Appendix C. Discritization of Governing Equations

Appendix D. Boundary Condition

Appendix E. Tables of Results
LIST OF FIGURES

1.1 Corrugated duct (Photo Courtesy: Brigg pipe systems)

1.2 Twisted insert (Photo Courtesy: Koch Heat Transfer Company)

1.3 Twisted duct (Photo Courtesy: Koch Heat Transfer Company)

2.1 Schematic (not to scale) geometrical description of the rectangular tube that is helically twisted about its axis.

2.2 Comparison of computed Fanning friction factor results with those reported by Masliyah and Nandakumar (Masliyah and Nandakumar, 1981) for square-duct twist ratios $\xi = 2.5$ and $5.0$.

2.3 The effect of flow cross section aspect ratio on the normalized axial velocity $\left( \frac{u_z}{u_{z,m}} \right)$ field in a twisted square and rectangular duct with $\xi = 3.0$ and $\text{Re} = 500$: (a) $\alpha = 1.0$, (b) $\alpha = 0.75$, and (c) $\alpha = 0.5$.

2.4 The effect of flow cross section aspect ratio on the stream function $\psi(x, y)$ distribution in a twisted square and rectangular duct with $\xi = 3.0$ and $\text{Re} = 500$: (a) $\alpha = 1.0$, (b) $\alpha = 0.75$, and (c) $\alpha = 0.5$.

2.5 Variation in swirl flow or secondary vortex structure with different Reynolds number flows in a twisted rectangular of cross-sectional aspect ratio $\alpha = 0.5$ and twist ratio $\xi = 3.0$: (a) $\text{Re} = 25$, (b) $\text{Re} = 100$, (c) $\text{Re} = 200$, and (d) $\text{Re} = 500$.

2.6 The effect of tube twist ratio on the normalized axial velocity $\left( \frac{u_z}{u_{z,m}} \right)$ field in the swirl-flow regime in a twisted square and rectangular duct with $\alpha = 0.5$ and $\text{Re} = 620$: (a) $\xi = 12.0$, (b) $\xi = 6.0$, and (c) $\xi = 3.0$. 
The effect of tube twist ratio on the stream function $\psi(x,y)$ distribution in the swirl-flow regime in a twisted square and rectangular duct with $\alpha = 0.5$ and $\text{Re} = 620$: (a) $\xi = 12.0$, (b) $\xi = 6.0$, and (c) $\xi = 3.0$.

Local wall shear stress distribution in steady flow through square and rectangular twisted ducts: (a) effect of cross-section aspect ratio $\alpha$, (b) effect of axial flow Re, and (c) effect of duct twist ratio $\xi$.

Variation of steady flow isothermal Fanning friction factors with Reynolds number in twisted rectangular ducts of different cross-section aspect ratio $\alpha$ and twist ratio $\xi$.

Description of the axially twisted rectangular duct geometry, and the two primary wall heating/cooling conditions.

The effect of flow cross section aspect ratio on the temperature $\left(\frac{T}{T_m}\right)$ distribution in a twisted square and rectangular ducts with $\text{H1}$ thermal condition, $\xi = 3.0$, $\text{Pr} = 5.0$, and $\text{Re} = 500$: (a) $\alpha = 1.0$, (b) $\alpha = 0.75$, and (c) $\alpha = 0.5$.

The effect of flow cross section aspect ratio on the temperature $\left(\frac{T}{T_m}\right)$ distribution in a twisted square and rectangular ducts with $\text{H1}$ thermal condition, $\xi = 3.0$, $\text{Pr} = 5.0$, and $\text{Re} = 500$: (a) $\alpha = 1.0$, (b) $\alpha = 0.75$, and (c) $\alpha = 0.5$.

The effect of axial twist ratio on the temperature $\left(\frac{T}{T_m}\right)$ distribution in the swirl-flow regime ($\text{Re} = 620$; $\text{Pr} = 5.0$) in a rectangular duct with $\alpha = 0.5$ and $\text{H}$ thermal condition: (a) $\xi = 12.0$, (b) $\xi = 6.0$, and (c) $\xi = 3.0$. 
3.5 The effect of axial twist ratio on the temperature \( \frac{T}{T_m} \) distribution in the swirl-flow regime \((\text{Re} = 620; \text{Pr} = 5.0)\) in a rectangular duct with \(\alpha = 0.5\) and H1 thermal condition: (a) \(\xi = 12.0\), (b) \(\xi = 6.0\), and (c) \(\xi = 3.0\).

3.6 Variation of Nu with Re, cross-section aspect ratio \(\alpha\), and twist ratio \(\xi\), in typical liquids flows in axially twisted rectangular ducts: (1) with T condition, and 2) with H1 condition.

3.7 Effect of Pr on the variation of \((\text{Nu}/\text{Nu}_{\text{st}})\) with Re in a axially twisted rectangular duct with \(\alpha = 0.5\) and \(\xi = 3.0\): (a) with T condition, and (b) with H1 condition.

3.8 Effect of twist ratio and cross-section aspect ratio of twisted rectangular ducts on enhanced heat transfer rate sustained in laminar liquid \((\text{Pr} = 5.0)\) flows with fixed pumping power and geometry constraints: (a) with T condition, and (b) with H1 condition.

3.9 Effect of liquid Pr and twist ratio on enhanced heat transfer rate sustained in laminar flows inside twisted rectangular ducts with \(\alpha = 0.5\), and fixed pumping power and geometry constraints: (a) with T condition, and (b) with H1 condition.

3.10 Effect of twist ratio and cross-section aspect ratio of twisted rectangular ducts on relative reduction in heat transfer surface area requirement in laminar liquid \((\text{Pr} = 5.0)\) flows with fixed heat load and pressure drop constraints: (a) with T condition, and (b) with H1 condition.

3.11 Effect of liquid Pr and twist ratio on relative reduction in heat transfer surface area requirement in laminar flows inside twisted rectangular ducts with \(\alpha = 0.5\), and fixed heat load and pressure drop constraints: (a) with T condition, and (b) with H1 condition.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_N, a_S, a_E, a_W, a_P$</td>
<td>coefficients in the general discretized equation</td>
<td></td>
</tr>
<tr>
<td>$A_N, A_S, A_E, A_W, A_P$</td>
<td>variables in power law scheme</td>
<td></td>
</tr>
<tr>
<td>$A$</td>
<td>heat transfer surface area, [m$^2$]</td>
<td></td>
</tr>
<tr>
<td>$A_c$</td>
<td>flow cross-section area, $(4ab)$ [m$^2$]</td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>half-length of smaller side of a rectangle, [m]</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>half-length of larger side of a rectangle, [m]</td>
<td></td>
</tr>
<tr>
<td>$c_p$</td>
<td>specific heat, [J/kg·K]</td>
<td></td>
</tr>
<tr>
<td>$D_e, D_w, D_n, D_s$</td>
<td>diffusive strengths</td>
<td></td>
</tr>
<tr>
<td>$d_h$</td>
<td>hydraulic diameter, $(4a/(1+\alpha))$, [m]</td>
<td></td>
</tr>
<tr>
<td>$F_e, F_w, F_n, F_s$</td>
<td>convective strengths</td>
<td></td>
</tr>
<tr>
<td>$f$</td>
<td>Fanning friction factor</td>
<td></td>
</tr>
<tr>
<td>$H$</td>
<td>axial pitch length for 180° twist of rectangular duct, [m]</td>
<td></td>
</tr>
<tr>
<td>$H_1$</td>
<td>uniform heat flux (UHF) with peripherally constant wall temperature condition</td>
<td></td>
</tr>
<tr>
<td>$h$</td>
<td>heat transfer coefficient, [W/m$^2$·K]</td>
<td></td>
</tr>
<tr>
<td>$k$</td>
<td>thermal conductivity of fluid, [W/m·K]</td>
<td></td>
</tr>
<tr>
<td>$J_x, J_y$</td>
<td>sum of convective and diffusive terms with respect to $x, y$ respectively</td>
<td></td>
</tr>
</tbody>
</table>
in general differential eqn

\( L \)
length of flow channels or exchanger, [m]

\( L_x, L_y \)
coefficient of convective term with respect to \( x \), \( y \) respectively in general differential eqn

\( M_x, M_y \)
coefficient of diffusive term with respect to \( x \), \( y \) respectively in general differential eqn

\( \dot{m} \)
mass flow rate, [kg/s]

\( N \)
number of flow channels

\( \text{Nu} \)
Nusselt number, \((hd_h/k)\)

\( P_x, P_y, P_z \)
dimensionless pressure field components in fixed coordinate system

\( p_x, p_y, p_z \)
dimensionless pressure field components in rotating coordinates

\( \hat{P}, P \)
dimensionless pressure gradient vector

\( P_p \)
pumping power [W]

\( \Delta p \)
pressure drop [N/m²]

\( \text{Pr} \)
Prandtl number, \((\mu c_p/k)\)

\( P_w \)
wetted perimeter of flow duct, \((4a + 4b)\) [m]

\( Q \)
heat transfer rate [W]
\( q^\prime \) heat flux

\( Re \) Reynolds number, \( \left( \rho \tilde{U} \delta \mu \right) \) [-]

\( S_r \) source terms [-]

\( S_c \) constant source term

\( S_p \) dependant source term

\( S_{\varphi} \) source terms

\( \tilde{T} \) temperature vector [K]

\( T \) uniform wall temperature (UWT) condition

\( T_w \) dimensionless wall temperature

\( \Delta T_i \) approach or inlet temperature difference between two fluid streams, [K]

\( \tilde{U} \) velocity vector [m/s]

\( \tilde{U} \) dimensionless velocity vector

\( u_x, u_y, u_z \) dimensionless velocity components in rotating coordinates

\( \tilde{X}, \tilde{Y}, \tilde{Z} \) dimensional Cartesian coordinates in fixed space, [m]

\( x, y, z \) dimensionless rotating coordinate system

\( X, Y, Z \) dimensionless coordinates in fixed space
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nabla$</td>
<td>dimensionless covariant derivative</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>distance between two faces of the control volume</td>
</tr>
<tr>
<td>$\omega$</td>
<td>dimensionless vorticity</td>
</tr>
<tr>
<td>$\theta$</td>
<td>angular coordinate</td>
</tr>
<tr>
<td>$\theta'$</td>
<td>twist rate</td>
</tr>
<tr>
<td>$\nu$</td>
<td>fluid kinetic viscosity</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>duct flow cross section aspect ratio, $\left( \frac{2a}{2b} \right)$</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>heat exchanger effectiveness, $\left( \frac{Q}{Q_{\text{max}}} \right)$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>dimensionless function of twist ratio and aspect ratio</td>
</tr>
<tr>
<td>$\mu$</td>
<td>dynamic viscosity of fluid, $[\text{N} \cdot \text{s/m}^2]$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>fluid density, $[\text{kg/m}^3]$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>tube twist ratio, $\left( \frac{H}{d_h} \right)$</td>
</tr>
<tr>
<td>$\theta_z$</td>
<td>axial angle of rotation [rad]</td>
</tr>
<tr>
<td>$\theta'_z$</td>
<td>axial derivative of rotation angle [rad]</td>
</tr>
<tr>
<td>$\tau$</td>
<td>shear stress $[\text{N/m}^2]$</td>
</tr>
</tbody>
</table>
\( \omega_z \) 
dimensionless axial component of vorticity

\( \psi \) 
dimensionless stream function

\( \varphi \) 
general function

**Subscripts**

\( m \) 
bulk mean value

\( e, w, n, s \) 
boundaries of the control volume in various directions

\( E, W, N, S \) 
neighboring nodes

\( p \) 
midpoint of control volume

\( i \) 
angular nodes index

\( j \) 
radial nodes index

\( H1 \) 
pertaining to the H1 or UHF boundary condition

\( m \) 
mean or mixed-mean or bulk value

\( st \) 
referring to a straight, untwisted duct

\( T \) 
pertaining to the T or UWT boundary condition

\( w \) 
at duct wall conditions

\( st \) 
referring to a straight, untwisted duct
$x, y, z$ components in $x, y,$ and $z$ directions of rotating coordinates, respectively.

**Superscripts**

$\sim$ dimensional vectors

Lower case letters variables in rotating coordinate system

Upper case letters variables in fixed coordinate system
CHAPTER 1 INTRODUCTION

1.1 Heat Transfer Enhancement

Generally, design of any industrial process plants is influenced by need for optimum utilization of energy and hence effective usage of heat for economic design and operation. The need for effective utilization and recovery of heat has prompted the development of various heat transfer enhancement techniques for heat exchangers. Use of curved surfaces, extended surfaces and swirl flow devices are some of the heat transfer enhancement techniques. As mentioned in Bergles (1998), heat transfer enhancement techniques are classified as active or passive techniques. Active enhancement techniques are ones which need extra power to increase heat transfer. They include techniques such as generation of surface vibration, fluid vibration and electrostatic fields. Passive techniques require no external energy input, except for fluid movement. Consequently, passive techniques are often the preferred choice and they have seen wider applications. Passive techniques include the use of roughened surfaces, extended surfaces, coiled tubes, and swirl flow devices, among some others. Since swirl flow is involved in this study, it is discussed in more detail following section.

1.2 Swirl Flows

Swirl flow consists of regions of secondary recirculation within the cross-section which helps promote better mixing of fluid. This results in higher momentum transfer and increased convective heat transfer. Swirl flow can be generated using number of techniques such as corrugated surfaces, twisted tape inserts through ducts and twisted ducts

Corrugated surfaces:
The most common application using corrugated surface for enhanced heat transfer is Plate heat exchangers. The plate surface corrugations provide restriction to the cross-sectional flow, generating recirculation in the corrugation troughs. The intensity of resultant swirl flow depends on wall corrugation aspect ratio and wave-length as well type of corrugation. Spirally or helically corrugated ducts have also been found effective in heat transfer. Heat transfer enhancement using corrugated surface has been discussed in more detail by Metwally and Manglik (2004), Muley and Manglik (2005), Metwally and Manglik (2004). As mentioned in Bergles (1998), heat transfer enhancement using corrugated surfaces can be up to 400%.

![Corrugated duct](image.png)

**Figure 1.1** Corrugated duct (Photo Courtesy: Brigg pipe systems)

Twisted tape inserts:

The major advantage of twisted tape inserts over other swirl flow devices is that it does not require any significant change in configuration of existing shell and tube heat exchangers during retrofitting. Also it does not involve any significant increase in manufacturing costs. In ducts with twisted tape inserts, the helical nature of twisted surface help in generation of swirl flow. Depending on the tape material and roughness, there may be
increased shear losses due to the added surface area of the insert. However if properly fitted and provided proper thickness, the twisted tape may act as internal fins thus offsetting surface shear losses. Bergles (1998), Smithberg and Landis( 1964), Hong and Bergles(1976), Date (1974), Manglik and Bergles(1992, 2002), DuPlessis and Kröger (1984, 1987), Lin and Wang (2009), Date and Ray(2001) have shown that significant heat transfer improvement can be obtained with use of twisted tape inserts. The only possible disadvantage of the twisted tape insert is difficulty in using with non-uniform cross-sectional ducts.

![Twisted insert](image)

**Figure 1.2** Twisted insert (Photo Courtesy: Koch Heat Transfer Company)

Twisted ducts:

Todd (1977) was first one to provide hypothesis of the effects swirl flow by means of twist on straight ducts. Masliyah and Nandakumar (1981a, b) and Kim et al. (1988) studied the generation of swirls and their effect on heat transfer in twisted square-rectangular and elliptic ducts respectively. Twisted ducts may substitute the non-uniform cross-sectional ducts where inserting a twisted tape is difficult. Twisted surface of the duct essentially behaves in similar manner in generation of swirl flows as compared to twisted tape inserts.
Apparently, twisted ducts seem to have lesser shear surface area as compared to ducts with twisted inserts.

Figure 1.3 Twisted duct (Photo Courtesy: Koch Heat Transfer Company)

All of the swirl flow techniques discussed above involves increase shear area for flow which directly results in higher frictional losses and hence may require increased pumping power. So care has to be taken to optimize twist ratio to have superior performance than basic geometry.

Somehow, among all type of geometries, twisted rectangular geometries have been least studied. Barring Masliyah and Nandakumar (1981, a, b; 1982) there is no literature available on the topic. There is no significant work available detailing the effects of different aspect ratios on flow and heat transfer in twisted ducts. In order to successfully implement twisted rectangular ducts as heat transfer enhancement equipment in practical engineering applications, a comprehensive evaluation of different parameters vis-à-vis flow and heat transfer is must.
1.3 Aim and scope of study

The aim of this study is to evaluate twisted rectangular ducts for its suitability with respect to effective heat transfer by collecting friction factor and Nusselt number data, and characterizing their dependence on the aspect ratio of the twisted duct.

The scope includes investigation of the swirl flow behavior and the laminar convective heat transfer in twisted rectangular ducts. The fluid flow and thermal fields are simulated computationally. The Reynolds numbers, average friction factors and Nusselt numbers for different flow pressure drops, twisted geometries and Prandtl numbers are determined. The results are detailed in graphical as well as tabular form. The uniform wall temperature and uniform wall heat flux are considered for temperature boundary conditions. Twisted rectangular geometries of aspect ratio of 0.5, 0.75 and 1.0 and twisted ratios of 12, 6 and 3 are considered along with Prandtl number of 5, 45 and 100. Evaluation of performance is done in terms of overall increase in heat load (FG 2a criteria) and overall reduction in effective heat transfer area (VG1 criteria)
2.1 Introduction

In order to study feasibility of twisted duct in practical applications like heat exchangers, it is necessary to study flow behavior of fluid. Fluid flow influences various parameters in design of heat exchangers such as pressure drop, fouling at the walls, duct wall temperatures, vibration of ducts. Thus, nature of fluid flow has direct bearing on pumping power requirement, efficiency and durability of Heat exchangers. Hence it becomes necessary to study flow behavior inside twisted ducts for implementing them in industrial applications.

2.2 Mathematical Formulation

Steady-state, constant property flows with low Reynolds numbers \(10 \leq \text{Re} \leq 1000\) are considered. The duct geometry is depicted in Fig. 2.1, which has a rectangular cross section and whose surface is helically twisted about its straight axial center line. The severity of the helical twist is described by its pitch \(H\) that axially spans a \(180^\circ\) (or \(\pi\) radians) period. Its dimensionless representation is given by a twist ratio and that for the flow cross section by an aspect ratio, defined respectively as follows:

\[
\xi = \left(\frac{H}{d_h}\right), \quad \alpha = \left(\frac{2a}{2b}\right)
\] (2.1)

Furthermore, in order to render the governing equations dimensionless, the velocity vector \(\mathbf{U}\), the pressure vector \(\mathbf{P}\), and the differential operators are normalized by the length scale \(a\), the fluid viscosity \(\mu\), and its density \(\rho\) as per the following:
\[
\mathbf{U} = \left( \rho a \bar{U} / \mu \right), \quad \mathbf{P} = \left( \rho a^2 \bar{P} / \mu^2 \right), \quad \nabla = a \bar{\nabla}, \quad \text{and} \quad \nabla^2 = a^2 \bar{\nabla}^2 \tag{2.2a}
\]

The reference Cartesian coordinates are likewise given by

\[
X = \left( \tilde{X} / a \right), \quad Y = \left( \tilde{Y} / a \right) \quad \text{and} \quad Z = \left( \tilde{Z} / a \right) \tag{2.2b}
\]

Thus, the dimensionless forms of the continuity and momentum transport equations can be expressed, respectively, as follows:

\[
\nabla \mathbf{U} = 0, \quad \text{and} \quad \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla \mathbf{P} + \nabla^2 \mathbf{U} \tag{2.3a, b}
\]

Equations (3a) and (3b) describe the flow field in a fixed coordinate system \((X, Y, Z)\).

The effects of the helical twist of the rectangular surface can be incorporated in these equations by transforming them into a rotating coordinate system \((x, y, z)\), characterized by the axial angle of rotation \(\theta_z\) and its derivative \(\theta'_z\)

\[
\theta_z = \left( \pi Z a / H \right), \quad \text{and} \quad \theta'_z = \left( \partial \theta / \partial Z \right) = \left( \pi a / H \right) \tag{2.4}
\]

and functionally given by the following matrix:

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = \begin{bmatrix}
  \cos \theta_z & \sin \theta_z & 0 \\
  -\sin \theta_z & \cos \theta_z & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  X \\
  Y \\
  Z
\end{bmatrix} \tag{2.5}
\]

Similarly, the velocity vector \(\mathbf{U}\) can be transformed into the rotating coordinate system as,

\[
\begin{bmatrix}
  U_x \\
  U_y \\
  U_z
\end{bmatrix} = \begin{bmatrix}
  \cos \theta_z & -\sin \theta_z & 0 \\
  \sin \theta_z & \cos \theta_z & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  u_x \\
  u_y \\
  u_z
\end{bmatrix} \tag{2.6}
\]
and the derivatives of all scalar functions as,

\[
\begin{bmatrix}
\frac{\partial f}{\partial X} \\
\frac{\partial f}{\partial Y} \\
\frac{\partial f}{\partial Z}
\end{bmatrix} =
\begin{bmatrix}
\cos \theta_z & -\sin \theta_z & 0 \\
\sin \theta_z & \cos \theta_z & 0 \\
\theta'_y & \theta'_x & 1
\end{bmatrix}
\begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial z}
\end{bmatrix}
\]  

(2.7)

It should additionally be noted that because the flow is considered to be periodically fully developed, the scalar function derivatives \( \frac{\partial f}{\partial Z} \) and \( \frac{\partial f}{\partial z} \) are zero.

Using the transformations given by Eqs. (2.5) – (2.7), the continuity equation, expressed in the rotating coordinate system, is given by

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \Gamma \left( y \frac{\partial u_x}{\partial x} - x \frac{\partial u_y}{\partial y} \right) = 0
\]

(2.8)

where \( \Gamma \) represents the dimensionless twist ratio and flow cross section aspect ratio given by the following:

\[
\Gamma = \left( \pi a/H \right) = \left[ \pi (1+\alpha)/4\xi \right]
\]

(2.9)

Likewise, the \( x \)-, \( y \)-, and \( z \)-components, respectively, of the momentum transport equations can be expressed as

\[
\begin{align*}
&u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + \Gamma u_z \left( y \frac{\partial u_x}{\partial x} - x \frac{\partial u_y}{\partial y} \right) = -\nabla p_x + \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \\
&+ \Gamma^2 \left( y^2 \frac{\partial^2 u_x}{\partial x^2} + x^2 \frac{\partial^2 u_x}{\partial y^2} - 2xy \frac{\partial^2 u_x}{\partial x \partial y} - y \frac{\partial u_x}{\partial y} - x \frac{\partial u_x}{\partial x} - u_x + 2x \frac{\partial u_x}{\partial y} - 2y \frac{\partial u_x}{\partial x} \right)
\end{align*}
\]

(2.10a)
\[\begin{align*}
\frac{u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + \Gamma u_z \left( y \frac{\partial u_y}{\partial x} - x \frac{\partial u_x}{\partial y} - u_z \right) &= -\nabla p_x + \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \\
+ \Gamma^2 \left( y^2 \frac{\partial^2 u_x}{\partial x^2} + x^2 \frac{\partial^2 u_y}{\partial y^2} + 2xy \frac{\partial^2 u_y}{\partial x \partial y} - x \frac{\partial u_y}{\partial y} - x \frac{\partial u_y}{\partial y} - u_x + 2x \frac{\partial u_x}{\partial y} - 2y \frac{\partial u_y}{\partial x} \right) 
\end{align*}\]  

Furthermore, based on the usual definition of stream function, expressed in dimensionless form in rotating coordinates and given by

\[u_z = \frac{\partial \psi}{\partial y} - \Gamma u_y, \text{ and } u_y = -\frac{\partial \psi}{\partial x} + \Gamma x u_y\]  

the pressure gradients in Eqs. (2.9a) and (2.9b) can be eliminated by cross differentiation, and the vorticity transport equation obtained as

\[\begin{align*}
\frac{\partial \psi}{\partial y} \frac{\partial \omega_z}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega_y}{\partial y} + \Gamma \omega_z \left( y \frac{\partial u_y}{\partial x} - x \frac{\partial u_x}{\partial y} - u_z \right) + \Gamma \left( y \frac{\partial u_y}{\partial x} - x \frac{\partial u_y}{\partial y} + u_x \right) \frac{\partial u_x}{\partial x} \\
- \Gamma \left( y \frac{\partial u_y}{\partial x} - x \frac{\partial u_x}{\partial y} - u_z \right) \frac{\partial u_x}{\partial y} = \frac{\partial^2 \omega_x}{\partial x^2} + \frac{\partial^2 \omega_y}{\partial y^2} + \Gamma^2 \left( y^2 \frac{\partial^2 \omega_x}{\partial x^2} + x^2 \frac{\partial^2 \omega_y}{\partial y^2} + 2xy \frac{\partial^2 \omega_y}{\partial x \partial y} - x \frac{\partial u_y}{\partial y} - x \frac{\partial u_y}{\partial y} - u_x + 2x \frac{\partial u_x}{\partial y} - 2y \frac{\partial u_y}{\partial x} \right) 
\end{align*}\]  

Here the \(z\)-component of the dimensionless vorticity, defined as the curl of the velocity vector, is expressed in the rotating coordinate system as

\[\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\]
Also, by combining Eqs. (2.11) and (2.13), the stream function – vorticity equation is obtained as

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega_x + \Gamma u_z \left( x \frac{\partial u_z}{\partial x} + y \frac{\partial u_z}{\partial y} + 2u_z \right) \]  

(2.14)

Over the domain of the entire surface of the rectangular cross section of the flow duct, described by the limits \( x = \pm 1 \), and \( y = \pm (1/\alpha) \), the velocity, and concomitantly the stream function and pressure gradients, are subject to the no-slip boundary condition, or

\[ u_x, u_y, u_z = 0 ; \ \nabla p_x, \nabla p_y = 0 ; \text{ and } \psi = 0 \ @ (\pm 1, y) \text{ and } (x, \pm 1/\alpha) \]  

(2.15a)

The wall vorticity boundary condition is obtained from the stream function – vorticity equation and is described by

\[ \omega(\pm 1, y) = \left[ -\frac{\partial^2 \psi}{\partial x^2} + x \frac{\partial u_z}{\partial x} \right]_{(\pm 1, y)} ; \text{ and } \omega(x, \pm 1/\alpha) = \left[ -\frac{\partial^2 \psi}{\partial y^2} + y \frac{\partial u_z}{\partial y} \right]_{(x, \pm 1/\alpha)} \]  

(2.15b, c)

Furthermore, the axial periodicity condition for fully developed steady flow requires that

\[ u_z(x, y) \big|_{z+\Gamma} = u_z(x, y) \big|_z \]  

(2.15d)

Equations (2.8) – (2.15) thus provide the complete mathematical description of the flow field, with the attendant boundary and flow-periodicity conditions, for the computational solution.

Given the flow field, or the velocity distribution \((u_x, u_y, u_z)\), for the periodically fully developed flow in the twisted rectangular duct geometry, the macroscopic variables of design interest \((f \text{ and } Re)\) need to be calculated. Based on the conventional definition for the hydraulic diameter of the flow channel, given by
\[ d_h = \left( \frac{4A_c}{P_w} \right) = \left[ \frac{4a/(1+\alpha)}{1} \right] \]  

(2.16)

the mean axial flow Reynolds number, given by its usual hydraulic-diameter-based definition, can be re-expressed as

\[ \text{Re} = \left[ 4u_{z,m}/(1+\alpha) \right] \]  

(2.17)

where the mean axial velocity \( u_{z,m} \) obtained from the following integral:

\[ u_{z,m} = \frac{1}{A_c} \int_{A_c} u_z(x,y) dA_c \]  

(2.18)

The Fanning friction factor, based on the average wall shear stress, is defined as

\[ f = \left[ \frac{2\tau_w}{\rho \bar{U}_{z,m}^2} \right] \]  

(2.19)

Thus, by considering a force balance across a one-twist period of the flow cross section and simplifying it, the frictional loss can be expressed in the following form that is more commonly used in design practice:

\[ f \text{Re} = \frac{-8\nabla p_z}{(1+\alpha)^2 u_{z,m}} \]  

(2.20)

Additional details of the complete mathematical development of the flow-field equations as well as those for frictional loss can be found Appendix A, C and D.
2.3 Numerical Methodology

The periodically developed flow field in the axially twisted rectangular ducts is essentially described by the axial velocity $u_z$, vorticity $\omega_z$, and the stream function $\psi$, which are obtained by computationally solving Eqs.(2.10c), (2.12), and (2.14), respectively. The secondary velocities, $u_x$ and $u_y$, are obtained from Eqs. (2.11a) - (2.11b), and pressure distribution, $p_x$ and $p_y$, from Eqs (2.10a)-(2.10b). The requisite governing differential equations, however, can be restated in the following generalized form:

$$\frac{\partial}{\partial x} \left( L_x \phi \right) + \frac{\partial}{\partial y} \left( L_y \phi \right) - M_x \frac{\partial^2 \phi}{\partial x^2} - M_y \frac{\partial^2 \phi}{\partial y^2} = S_\phi$$

(2.21)

where $\phi$ represents $u_z$, $\psi$, and $\omega_z$, along with their concomitant collection of coefficient terms for $L_x$, $L_y$, $M_x$, and $M_y$, and the respective source term $S_\phi$. To obtain their discretized representations, the diffusion terms were expressed by central differencing and the convective terms by the power-law scheme (Patankar, 1980). Furthermore, the application of all the Dirichlet boundary conditions on $u$, $\psi$, and $p$ is straightforward. For the numerical application of the vorticity boundary conditions given by Eqs. (2.15b)-(2.15c), the second-order derivative of $\psi$ was evaluated by the Taylor-series expansion about the boundary node and into the flow field (Tannehill et al., 1984) to yield the discretized terms as,

$$\frac{\partial^2 \psi}{\partial x^2} = -2 \frac{\psi(x \pm 1 \Delta x, y)}{\Delta x^2}, \text{ and } \frac{\partial^2 \psi}{\partial y^2} = -2 \frac{\psi(x, y \pm 1 \Delta y)}{\Delta y^2}$$

(2.22)

The numerical results were obtained by solving the discretized governing equations using the Gauss-Seidel iterative method. Under-relaxation in $u_z$, $\psi$, and $\omega_z$, especially for higher
flow rates ($\text{Re} \rightarrow 1000$) and smaller $\xi$ was effected in the iterations; the under-relaxation factor varied from 0.2 to 1.0. Furthermore, to ensure stable convergence, several “inner loop” iterations were carried out to resolve the stream function, before updating iterating in an “outer loop” the axial velocity and vorticity. The solution convergence was established when the following constraint on the iterative residual $\varepsilon$ for vorticity was satisfied:

$$
\varepsilon = \max \left| \frac{\omega_z^{\text{new}} - \omega_z^{\text{old}}}{\omega_z^{\text{new}}} \right| \leq 10^{-6}
$$

(2.23)

For calculating the mean axial velocity from its distribution in the flow cross section, the second-order accurate Simpson’s rule was invoked for the numerical integration of Eq. (2.18) and the consequent determination of $(f\text{Re})$ from Eq. (2.19).

The computational domain was described grid that was selected post successive grid-refinement calculations for three test cases with $\xi = 3.0$, and $\alpha = 1.0$, 0.75, and 0.5, so as to render grid-independent solutions. For example, refining the grid for a square duct, $\alpha = 1.0$, with $\text{Re} = 500$ and $\xi = 3.0$, from $121 \times 121$ to $161 \times 161$ resulted in less than 1.4% change in $(f\text{Re})$. Similarly, for $\alpha = 0.75$, $\xi = 3.0$, and $\text{Re} = 500$, by increasing the grid from $85 \times 113$ to $121 \times 161$ the change in friction factor results was less than 1.4%. Thus, based on similar exercises for some other $\alpha$, $\xi$, and Re cases, mesh sizes of $121 \times 121$, $85 \times 113$, and $81 \times 161$, respectively, were used for $\alpha = 1.0$, 0.75, and 0.5, for all flow rates ($10 \leq \text{Re} \leq 1000$) and $\xi$.

Additionally, the comparison of present numerical results with the limited results for the cases with $\alpha = 1.0$, and $\xi = 5$ and 2.5 given by Masliyah and Nandakumar (1981) is noteworthy. As seen in Fig. 2, while the difference between the two sets of results for $\xi = 5.0$ and low Re is negligible, thereby further validating the numerical simulations of the
present work, there is considerable deviation at higher Re and when $\xi = 2.5$. The numerical solutions in Masliyah and Nandakumar (1981) were obtained with a rather coarse grid ($21 \times 21$), which is apparently inadequate for capturing the complete swirl behavior, particularly in small $\alpha$ and $\xi$ ducts in higher Re flows. Also, as shown in the companion paper (Masliyah and Nandakumar, 1981), Masliyah and Nandakumar have reported rather anomalous heat transfer results, again perhaps because of the coarse grid and other numerical issues, which show a reduction in heat transfer that is contrary to the expected enhancement due to increased momentum transport.

2.4 Results and Discussion

Both the flow cross-sectional area of the rectangular duct and the axial helical twist, described respectively by $\alpha$ and $\xi$, have a substantially marked influence on the flow behavior. The effects of $\alpha$ on the axial velocity field and stream function distribution are graphically depicted in Figs. 2.3 and 2.4, respectively. The variations in the normalized axial velocity $(u_z/u_{z,m})$ distribution seen in the contour plots of Fig. 3, for the case with a twist ratio of $\xi = 3.0$ and flow rates with $Re = 500$, clearly indicate the growth and intensification of swirl flow with decreasing cross-sectional area of the duct ($\alpha = 1.0$, square duct $\rightarrow \alpha = 0.5$, flat rectangular duct). The accentuation of helical swirl in the mid-section of the duct is reflected in the increasing peak velocity with decreasing cross-section area and aspect ratio $\alpha$. As the core axial flow intensifies, the spatial coverage of the central region secondary circulation tightens and shrinks to scale with the narrower width of the flatter ($\alpha = 0.5$) rectangular duct. This promotes the generation of additional pairs of counter-rotating
vortices, albeit much smaller in size and magnitude, at each of the top and bottom large-side end regions of the duct.

The change in the structure of fluid recirculation in the twisted tube due to a reduction in the cross-sectional aspect ratio is also seen in the dimensionless stream function $\psi(x,y)$ plots given in Fig. 2.4. In the case of a square duct, the sharp edges of the geometry constrain the local flow field because of viscous effects in a manner that gives rise to the development of rather small corner recirculation cells seen in Fig. 4(a). With increasing flattening of the duct ($\alpha = 1.0 \rightarrow 0.75 \rightarrow 0.5$) and the consequent core-flow acceleration, which increases the magnitude of the central-region swirl but shrinks its size, the recirculation in the corner regions of the duct begin to grow both spatially and in strength. This is seen from the development of a pair of non-symmetric counter-rotating vortex cells at either of the narrower-ends of the rectangular duct that increase substantially in size in the smaller $\alpha$ rectangular duct. Also, the outer cells are diagonally symmetrical on either end of the central vortex, and such multiple-cell fluid circulation lends to cross-stream mixing of the axial flow in the narrower end regions of the channel and thus to enhanced momentum transport (Manglik and Bergles, 2002; Manglik, Maramraju and Bergles, 2001; Manglik and You, 2002).

As seen from the vector plots of the $x$-$y$ velocity components in Fig. 2.5, the swirl flow structure evolves with increasing flow rate in a duct with a fixed helical twist and cross-sectional aspect ratio, $\xi = 3.0$ and $\alpha = 0.5$ in this case. When $Re = 25$ (Fig. 2.5a), there is simply a gentle axially helical stirring of the flow as it follows the contours of the outer surface twist of the duct. This stirring intensifies with higher core flow acceleration and a
tighter counter-clockwise swirl is formed when $Re = 100$ (Fig. 2.5b). The fluid circulation has a flat elliptical profile that is skewed $N - W$, or in the direction of the duct twist, leaving relatively stagnant region in $N - E$ and $S - W$ corners of the duct. A vortex with clockwise rotation sets up in each of these corners as $Re$ increases to 200, as seen in Fig. 2.5(c), and the central swirl or stirring intensifies to become more circular in spatial coverage. The core circulation accentuates even more when $Re = 500$ so as to occupy a virtual circular region of diameter $2a$ in the center of the channel (Fig. 2.5d). The corner vortices also increase in magnitude, while a second much smaller vortex cell is also formed in the $N$-$W$ and $S$-$E$ regions. The pair of dissimilar counter-rotating cells that is diametrically symmetrical in the narrower ends of the duct, along with a near-circular helical core circulation essentially characterizes the fully-developed swirl-flow structure.

A similar spatial evolution of the swirl structure occurs with change in the helical twist ratio of the rectangular duct. This is evident from the axial velocity and stream function distributions depicted, respectively, in Figs. 2.6 and 2.7 for $Re = 620$ in a duct with $\alpha = 0.5$. As seen from the variations in the axial flow field in Fig. 2.6, the peak core velocity increases and the skewed elliptical distribution in the channel cross section changes to a near circular profile as $\zeta$ decreases from $12.0 \rightarrow 6.0 \rightarrow 3.0$. With $\zeta = 3.0$ the core circulation is confined to diameter equal to the narrow width ($2a$) of the duct, and secondary smaller peaks in the axial velocity are formed in the opposite narrow corner regions to give a diametrically symmetrical multi-humped axial flow profile. It is around the secondary corner peaks and the adjacent space that the two counter-rotating vortex cells are formed at either end of the duct. This behavior is mimicked by the stream function distributions in Fig. 2.7, again for $Re = 620$, $\alpha = 0.5$, and twist ratios of $\xi = 12.0$, 6.0, and 3.0. With increasing severity of duct
twist, or decreasing $\xi$, the higher magnitude and circular spread of the core flow circulation along with the generation of a pair of re-circulating cells at opposite ends of the duct are clearly evident.

That the duct-twist induced swirl flow increases momentum transport is reflected in the variation in the local wall shear stress around the periphery of the rectangular duct. The effects of the channel’s cross-sectional aspect ratio, flow Reynolds number, and twist ratio, respectively, on the normalized local wall shear stress ($\tau_{w}/\tau_{w,m}$) are depicted in Figs. 2.8(a) – 2.8(c). In the swirl regime with $Re = 500$ and $\xi = 3.0$ (Fig. 2.8a), as the aspect ratio of the flow cross section decreases ($\alpha \to 0.5$), not only does the local peak shear stress increases but additional, albeit smaller, peaks are obtained in the distribution along the longer wall of the duct (wall 2). The latter are centered on the lateral location of the increased wall-velocity gradients due to the formation of dissimilar vortices in the diagonally opposite narrow ends of the rectangular duct. Correspondingly, the maximum local $\tau_{w}$ on wall 2 also increases as a result of greater core flow velocity; the peak stress along wall 1 (smaller side of rectangle), however, diminishes due to some fluid stagnation caused by the presence of the pair of unequal re-circulation cells. A virtually similar behavior is exhibited when the flow Re increases in a duct of fixed aspect ratio and twist ($\alpha = 0.5, \xi = 3.0$; Fig. 2.8b), and when the twist ratio decreases but Re remains constant in the swirl regime ($\alpha = 0.5, Re = 620$; Fig. 2.8c). The relative magnitudes of the respective maximum local and/or average wall shear stress are of course different in each case, and in the latter case of decreasing $\xi$, as seen from Fig. 2.8(c), the enhanced swirl due to the increasing severity of helical twist of the
rectangular duct is evident in the triple-humped distribution of shear stress along wall 2 that has a significantly higher maximum \( \left( \tau_{w} / \tau_{w,m} \right) \) in the core flow region.

The increased wall shear stress lends to greater pressure loss for the same flow rate, and the concomitant friction factor behavior is presented in Fig. 2.9. These results describe the effects of \( \xi \) and \( \alpha \) on the variation in \( (f \, \text{Re}) \), normalized with the frictional loss \( (f \, \text{Re})_{st} \) in an untwisted duct of the same aspect ratio, with the flow rate. It is evident that in a flow channel with fixed \( \alpha \), the frictional loss increases significantly with increasing severity of duct twist (or decreasing \( \xi = 12 \rightarrow 6 \rightarrow 3 \)) and Reynolds number. Higher flow friction is similarly incurred when the cross-sectional aspect ratio decreases (\( \alpha = 1.0 \rightarrow 0.75 \rightarrow 0.5 \)) and the duct twist ratio remains the same. This essentially reflects the tighter and stronger core-region swirl with higher peak axial velocity, accompanied with the development of smaller vortex cells in the narrow ends of flatter rectangular ducts (see Figs. 2.3, 2.4, 2.6, and 2.7). The onset and growth of such secondary flow behavior with increasing axial flow rate is also distinguishable in the friction factor results from upward inflection in the plots of \( (f \, \text{Re})/(f \, \text{Re})_{st} \) versus Re given in Fig. 2.9. As \( \text{Re} \rightarrow O[100] \), the increasing strength of the core swirl coupled with the growth of corner vortices enhance the momentum transport, and this magnifies further with smaller \( \alpha \) and \( \xi \). With low flow rates or as \( \text{Re} \rightarrow O[10] \), on the other hand and regardless of \( \alpha \) and \( \xi \), the magnitude of \( (f \, \text{Re}) \) remains constant but higher than \( (f \, \text{Re})_{st} \). The higher wall-shear stress in this case primarily results from the relatively larger effective flow length (or surface area or residence time) due to the tighter helical twist of the duct (or decreasing \( \xi \)) and flow acceleration due to duct contraction (or decreasing \( \alpha \)).
In effect, the following two distinct regimes represent the interplay of $\alpha$, $\xi$, and Re on the frictional loss: (i) a viscous flow regime with a gentle stirring of the fluid and constant ($f \text{Re}$), and (ii) a swirl flow regime with multiple circulation cells, characterized by stronger core circulation and growing self-sustained vortices in corners or narrower ends of the duct, where ($f \text{Re}$) increases with flow rate. This behavior, though similar to other situations where secondary flows are generated by a geometrical attribute of the channel (Manglik and You, 2002; Metwally and Manglik, 2004; Manglik and Ranganathan, 1997, Manglik and Bergles, 2002), is both qualitatively and quantitatively different. A case in point is provided by the comparison with the friction loss obtained in an equivalent circular tube (same $d_h$) fitted with a twisted-tape insert of negligible thickness and $\xi = 3.0$. Even though its ($f \text{Re}$) characteristic curve is similar to those of the rectangular ducts, in the latter the quantitative influence of $\xi$ varies with $\alpha$. In the case of twisted-tape inserts, swirl production in the partitioned tube is primarily an outcome of $\xi$ and is characterized by cross-stream mixing due to a pair of counter-rotating vortex cells in the core of the flow (Manglik and Bergles, 2002; Manglik, Maramraju and Bergles, 2001; Manglik and Ranganthan, 1997, Manglik and Bergles, 1993). In rectangular ducts, however, the core flow is merely helically stirred, albeit with higher axial velocity, and re-circulating vortex cells appear only in the small narrow regions of flatter ducts. Also, as seen in Fig. 2.9, there is no general Re value that demarcates the transition from stirred-viscous to swirl-flow regimes; different envelopes of Re for different $\alpha$ and $\xi$ combinations are discernible. The effect of twist is thus in effect modulated by the cross-sectional aspect ratio – as $\alpha$ decreases, swirl effects increase with decreasing $\xi$, and vice versa. A completely different scaling from that for twisted-tape inserts.
is therefore needed to develop a correlating relationship, and addressing this task, which is not in the scope of the present study, could benefit from some experimental inputs.
2.5 FIGURES

Figure 2.1 Schematic (not to scale) geometrical description of the rectangular tube that is helically twisted about its axis.
Figure 2.2 Comparison of computed Fanning friction factor results with those reported by Masliyah and Nandakumar (1981a) for square-duct twist ratios $\xi = 2.5$ and 5.0.
Figure 2.3  The effect of flow cross section aspect ratio on the normalized axial velocity \( \left( \frac{u_z}{u_{z,m}} \right) \) field in a twisted square and rectangular duct with \( \xi = 3.0 \) and \( \text{Re} = 500 \): (a) \( \alpha = 1.0 \), (b) \( \alpha = 0.75 \), and (c) \( \alpha = 0.5 \).
Figure 2.4 The effect of flow cross section aspect ratio on the stream function $\psi(x, y)$ distribution in a twisted square and rectangular duct with $\xi = 3.0$ and $Re = 500$: (a) $\alpha = 1.0$, (b) $\alpha = 0.75$, and (c) $\alpha = 0.5$. 
Figure 2.5 Variation in swirl flow or secondary vortex structure with different Reynolds number flows in a twisted rectangular of cross-sectional aspect ratio $\alpha = 0.5$ and twist ratio $\zeta = 3.0$: (a) $Re = 25$, (b) $Re = 100$, (c) $Re = 200$, and (d) $Re = 500$. 
Figure 2.6 The effect of tube twist ratio on the normalized axial velocity \( \left( \frac{u_z}{u_{z,m}} \right) \) field in the swirl-flow regime in a twisted square and rectangular duct with \( \alpha = 0.5 \) and \( \text{Re} = 620 \): (a) \( \xi = 12.0 \), (b) \( \xi = 6.0 \), and (c) \( \xi = 3.0 \).
Figure 2.7 The effect of tube twist ratio on the stream function $\psi(x, y)$ distribution in the swirl-flow regime in a twisted square and rectangular duct with $\alpha = 0.5$ and Re = 620: (a) $\xi = 12.0$, (b) $\xi = 6.0$, and (c) $\xi = 3.0$. 
Figure 2.8 Local wall shear stress distribution in steady flow through square and rectangular twisted ducts: (a) effect of cross-section aspect ratio $\alpha$, (b) effect of axial flow $Re$, and (c) effect of duct twist ratio $\xi$. 
Figure 2.9 Variation of steady flow isothermal Fanning friction factors with Reynolds number in twisted rectangular ducts of different cross-section aspect ratio $\alpha$ and twist ratio $\xi$. 
CHAPTER 3 HEAT TRANSFER IN TWISTED RECTANGULAR DUCT

3.1 Introduction

Effective heat transfer is hallmark of design of any type of heat exchanger. Since heat transfer has direct bearing on energy costs of any process, it becomes critical factor in cost-benefit analysis of investment. With energy cost continuously rising amid decreasing supply, it becomes more important to develop new techniques for effective heat transfer. Heat exchangers with twisted duct can be one of the ways to increase heat transfer. One of the advantages of twisted ducts is that on account of its geometry, has benefit of increased heat transfer area for same effective length as compared to straight duct. But this cannot be the only criteria for using twisted ducts as effective heat transfer technique. As seen in previous chapter twist greatly influences velocity profile and hence pumping power requirement of flow. Hence it must be assumed that twist also affects temperature distribution and hence heat transfer across duct. To what extent twist influences heat transfer is studied in following sections in order to justify industrial applications of twisted duct.

3.2. Mathematical Formulation

Incompressible, steady laminar flows of constant property viscous liquids inside the twisted rectangular ducts are considered. For the given periodically-developed flow field along with the geometrical attributes of the flow channel described in Fig. 3.1, the governing thermal energy equation, expressed in primitive variables, is given by the following:

\[ \rho c_p \mathbf{U} \cdot \nabla \mathbf{T} = k \nabla^2 \mathbf{T} \]  

(3.1)

By defining a non-dimensional temperature as
\[ T = \frac{T_w - T}{a \text{Re Pr} \left( \frac{dT_m}{dZ} \right)} \]  

(3.2)

where the Reynolds number is given by

\[ \text{Re} = \frac{4 \rho u_{z,m} a}{\mu (1 + \alpha)} = \frac{4 u_{z,m}}{(1 + \alpha)} \]  

(3.3)

the energy equation can be transformed and restated in dimensionless form, and in terms of

the rotating coordinate system described in the companion chapter 2, as follows:

\[ u_x \frac{\partial T}{\partial x} + u_y \frac{\partial T}{\partial y} + \Gamma u_z \left( y \frac{\partial T}{\partial x} + x \frac{\partial T}{\partial y} \right) \]

\[-\frac{1}{\text{Pr}} \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \Gamma^2 \left( y^2 \frac{\partial^2 T}{\partial x^2} + x^2 \frac{\partial^2 T}{\partial y^2} - 2xy \frac{\partial^2 T}{\partial x \partial y} - y \frac{\partial T}{\partial x} - x \frac{\partial T}{\partial y} \right) \right] = S_r \]

(3.4a)

Two fundamental and commonly applied (Shah and London, 1978) thermal boundary

conditions at the duct walls are considered. As indicated in Fig. 3.1, these conditions and

their typical physical manifestations are the following: (1) uniform wall temperature (UWT

or \( T \)), usually encountered in steam or high heat-capacity-rate stream heating/cooling in a

heat exchanger, and (2) uniform wall heat flux (UHF or \( H1 \)), which represents

heating/cooling in counter flow of equal heat-capacity-rate streams in a heat exchanger. The

corresponding source term in Eq. (3.4a) in each case is given by

\[
S_r = \begin{cases} 
\frac{(u_{z,m} T / \text{Re Pr} T_m)}{(u_{z,m} / \text{Re Pr})} 
& \text{for } T \\
\frac{(u_{z,m} / \text{Re Pr})}{(u_{z,m} T / \text{Re Pr} T_m)} 
& \text{for } H1 
\end{cases} 
\]

(3.4b)
For the two thermal boundary condition cases considered in this analysis, namely $T$ and $H1$, to which Eq. (3.4) is subjected, the dimensionless representation of temperature around the peripheral wall domain of the rectangular channel can be expressed as

$$T = 0 \text{ at } (\pm 1, y) \text{ and } (x, \pm 1/\alpha)$$

Likewise, the constraint of periodically fully-developed flow condition requires that

$$\left.\left(\frac{T}{T_m}\right)\right|_{z} = \left.\left(\frac{T}{T_m}\right)\right|_{z=\Gamma}$$

where, it may be recalled from chapter 2, $\Gamma$ is the dimensionless $180^\circ$ axial twist pitch. In essence, it also represents both the twist ratio $\xi$ and duct cross section aspect ratio $\alpha$ as given by

$$\Gamma = (\pi a/H) = \left[\pi (1 + \alpha)/4\xi\right]$$

The average or bulk-mean temperature is obtained from its usual definition as

$$T_m = \frac{1}{u_{c,m}A_c} \int u_c(x, y)T(x, y)dA_c$$

Finally, based on the energy balance between wall heat transfer and enthalpy change of the flowing fluid, the average Nusselt number can be determined from

$$Nu = \left[1/(1 + \alpha)T_m\right]$$

Extended details of the coordinate transformations, non-dimensionalization, and the development of preceding expressions are given in Appendix B,C and D.
3.3 Numerical Methodology

The schemes for discretization and computational solution are essentially the same as those employed for the fluid flow problem in the chapter 2. The governing equation that describes the temperature field in the periodically developed flow inside the axially twisted rectangular ducts was discretized using central differencing for the diffusion terms, and the power-law scheme (Patankar, 1978) for the convective terms. Also, the dimensionless representation of the two thermal boundary conditions ($\mathbf{\mathbf{T}}$ and $\mathbf{H1}$), given by Eq. (3.5a), constitutes a straightforward Dirichlet boundary condition. Numerical solutions of the discretized governing equation were obtained by the Gauss-Seidel method (Tannehill, Anderson and Pletcher, 1984), and for convergence the results were constrained by the following iterative residual condition:

$$\varepsilon = \max \left| \frac{\omega^\text{new}_z - \omega^\text{old}_z}{\omega^\text{new}_z} \right| \leq 10^{-6} \quad (3.9)$$

The mixing-cup or bulk-mean temperature, given by the integration in Eq. (3.7), was determined from the temperature and velocity distributions by using the second-order accurate Simpson’s rule (Tannehill, Anderson and Pletcher, 1984). More details about the discretization, numerical methodology, and its implementation are given in Appendix C and D.

Because the temperature field solutions require the axial velocity distribution as an input, as dictated by the governing differential equation (3.4), the grid that described the computational domain was the same as that for the flow field. Grids with $121 \times 121$, $85 \times 113$, and $81 \times 161$ nodes, respectively, were used for $\alpha = 1.0$, $0.75$, and $0.5$, for all flow rates ($10 \leq \frac{\rho}{\rho_0} \leq 100$).
Re ≤ 1000) and ξ. However, in order to re-confirm grid-independent solutions for temperature distribution, a sample mesh-refinement calculation was carried out for the test case of α = 1.0, ξ = 3.0, Pr = 5, and with Re = 500. Refining the grid from 121×121 to 131×131 in this case, after obtaining the requisite velocity solution, yielded less than 1.0% residual difference in the temperature field. It may be noted that this contrasts significantly from the rather coarse grid of 21×21 nodes used by Masliyah and Nandakumar (Masliyah and Nandakumar, 1981), which has perhaps been the cause of the anomalies in their results (Masliyah and Nandakumar, 1981, Nandakumar and Masliyah, 1983).

3.4. Results and Discussion

3.4.1. Heat transfer characteristics As can be surmised from the governing energy equation, Eq. (3.4), the forced convective temperature field is not only dependent upon the fluid flow conditions (Re, Pr, and wall heating/cooling), but also on the axial twist and cross-sectional aspect ratio of the duct. With growth of the core helical circulation and swirl as α decreases, the peak temperature also increases and much sharper wall gradients are obtained. This is evident from the isotherms plotted in Fig. 3.2 for Pr = 5.0 liquid flows with constant Re = 500 in twisted rectangular ducts with ξ = 3.0 and the T wall condition. Moreover, the production of additional pairs of vortices in the narrow ends of a flatter duct (α = 0.5) and the attendant cross-stream mixing lends to greater thermal mixing as well as mitigation of corner-region thermal stratification seen in larger α ducts. This results in a somewhat more uniform temperature field, which, of course, enhances the convection heat transfer. The behavior with the H1 heating/cooling condition is likewise similar (see Fig. 3.3), though the temperature distribution tends to be much more uniform in this case as compared to that with
the T boundary condition. The relatively smaller difference between the core-region and peripheral peak temperatures with the H1 condition is clearly indicative of this.

The influence of the variation in twist ratio, and the consequent swirl generation, on the temperature field in a rectangular cross-section duct with $\alpha = 0.5$, is depicted in Figs. 3.4 and 3.5, respectively, for the T and H1 wall thermal conditions. The isotherms, or $(T/T_m)$ contour plots, clearly indicate the reduction of corner-region thermal stratification as the magnitude of core circulation increases with smaller $\xi$ values, and the onset of multiple vortices in narrower portions of the duct. As the swirl structure changes with increasing severity of axial twist of the duct, so does the temperature distribution evolve into a tighter skewed elliptical spiral. The peak core temperature $(T/T_m)$ first increases as $\xi$ reduces from 12.0 to 6.0, and then decreases as $\xi = 3.0$. The latter is again a reflection of a more uniform and flatter temperature distribution, albeit with multiple peaks, due to the swirl-induced cross-stream mixing and mimicking the spatial flow distribution. As a consequence, the bulk temperature is much higher. The wall-gradients are, nevertheless, successively higher in each case, thereby reflecting the enhanced heat transfer with decreasing $\xi$. Again, the temperature distribution across the flow cross section with the H1 thermal boundary tends to be more uniform (relatively less difference between the peak temperatures in the multi-humped profiles) in reference to that observed with the T condition.

Given the parametric modulation of the temperature field by $\xi$, $\alpha$, and Re, the concomitant Nusselt number behavior is presented in Fig. 6 for both the T and H1 boundary conditions. Typical liquid flows ($Pr = 5.0$, ~ water) are represented, and Nu is seen to be higher than that in the equivalent straight duct ($Nu/Nu_{st} > 1$) in all cases ($0.5 \leq \alpha \leq 1.0$, and
$3.0 \leq \xi \leq 12.0$) as Re increases. With higher flow rates ($\text{Re} \to 1000$) and tighter duct twist (smaller $\xi$), the more pronounced helical swirl promotes greater convective transport and hence higher Nu. This is especially the case in ducts with a flatter rectangular cross section ($\alpha = 0.5$), where the higher thermal mixing in the flow yields substantially higher Nu ($\sim 2.5$ times that in a straight duct, or $\text{Nu}_{st}$). In fact swirl sets in with rather low flow rates ($\text{Re} \sim 50$) when $\xi = 3.0$ in the duct with $\alpha = 0.5$. In a square duct ($\alpha = 1.0$), however, these effects are much less and any significant swirl-induced enhancement is manifest only at relatively higher flow rate ($\text{Re} \sim 400$) when the twist ratio is large ($\xi = 12.0$). Such thermal performance essentially mirrors the flow behavior outlined in the companion paper (Patel et al, 2012), where two enhancement regimes have been identified. At low flow rates ($\text{Re} \to 10$), the higher ($\text{Nu}/\text{Nu}_{st}$) is primarily due to the increased effective residence time or flow length. When swirl sets in, generally in the range $50 < \text{Re} < 300$, depending upon both $\alpha$ and $\xi$, the Nusselt number increases sharply as a result of better fluid mixing effected by the secondary circulation in the flow cross section.

That the duct-surface helical-curvature-induced convective heat transfer is further enhanced in higher Prandtl number fluid flows is amply highlighted by the ($\text{Nu}/\text{Nu}_{st}$) plots in Fig. 3.7. Three different fluids are represented, namely, $\text{Pr} = 5$ (~ water), 45 (~ ethylene glycol), and 100 (~ oils and lubricants). Up to 14 times higher Nu is obtained in $\text{Pr} = 100$ flows in a twisted duct with $\xi = 3.0$ and aspect ratio $\alpha = 0.5$, relative to that in the equivalent straight duct. This reaffirms the observations in previous studies with twisted-tape inserts and coiled tubes (Manglik, 2003; Manglik and Bergles, 2002; Yerra et al, 2006) that swirl flows impart larger thermal benefits in higher Pr liquids. An interesting feature of Fig. 3.7 is that the relative heat transfer performance ($\text{Nu}/\text{Nu}_{st}$) with the H1 wall thermal condition is
less than that with the $T$ condition in high Prandtl number flows, and vice versa in low Pr flows. In the former case of high Pr liquids, the swirl generation and the associated cross-stream mixing of the fluid is substantially greater when $Re \to 1000$, and as a consequence the effect of wall heating/cooling condition (whether $H1$ or $T$) diminishes. The difference between $Nu_{H1}$ and $Nu_T$ in an $\alpha = 0.5$ duct, with $\xi = 3.0$ and $Pr = 45$, is only $+5.2\%$, and this diminishes to $+2.3\%$ in $Pr = 100$ liquid flow Appendix E. This is akin to what is typically seen in turbulent transport inside tubes (Bhatti and Shah, 1987; Petukhov and Popav, 1963).

However, in straight ducts with $0.5 \leq \alpha \leq 1.0$, $Nu_{st,H1}$ is about 1.21-to-1.22 times higher than $Nu_{st,T}$, and hence $(Nu/Nu_{st})_{H1} < (Nu/Nu_{st})_{T}$ in Fig. 7 for $Pr = 45$ and 100, as $Re \to 1000$ and when the flow field is characterized by a well-established and more intense swirl. The secondary-flow generation behavior, on the other hand, is much less pronounced in $Pr \leq 5$ fluid flows, even at higher Reynolds number, and $Nu_{H1} > Nu_T$.

3.4.2. Enhanced performance evaluation The relative thermal-hydrodynamic performance of any enhancement technique needs to be appropriately evaluated in order to extract any applications-based benefits for its usage in a heat exchanger design. Several performance evaluation factors or figures of merit have been proposed (Bergles, 1998; Manglik, 2003; Webb and Kim 2005) to carry out such an optimization exercise. These criteria essentially require comparing and contrasting the convection heat transfer and friction factor behavior with and without the enhancement. To this end, in the present analysis, the Nusselt number and friction factor characteristics of axially twisted rectangular ducts are assessed relative to those that would normally accrue in equivalent straight rectangular ducts. The fully-developed laminar flow isothermal Fanning friction factors and Nusselt numbers for the latter case, with the two fundamental thermal boundary conditions
(UWT or T, and UHF or H1), are dependent upon the duct’s aspect ratio $\alpha$ and they can be predicted (Shah and London, 1978) by the following:

$$f_{Re} = 24.0 \left(1.0 - 1.3553\alpha + 1.9467\alpha^2 - 1.7012\alpha^3 + 0.9564\alpha^4 - 0.2537\alpha^5\right)$$ (3.10)

$$Nu_T = 7.541 \left(1.0 - 2.610\alpha + 4.970\alpha^2 - 5.119\alpha^3 + 2.702\alpha^4 - 0.548\alpha^5\right)$$ (3.11)

$$Nu_{H1} = 8.235 \left(1.0 - 2.0421\alpha + 3.0853\alpha^2 - 2.4765\alpha^3 + 1.0578\alpha^4 - 0.1861\alpha^5\right)$$ (3.12)

In evaluating the trade-off between higher heat transfer coefficient and the accompanying friction penalty associated with enhancement techniques, one goal in engineering practice is often to accommodate a higher heat transfer rate $Q$ with the constraint of fixed pumping power $P_p$ (Bergles, 1998; Manglik, 2003; Yerra et al, 2006). Additionally, the heat exchanger geometry ($N$ and $L$), and approach temperature difference $\Delta T_i$, are considered to be the same as in the case without enhancement. The heat transfer rate $Q$ in any heat exchanger can be quantified on the basis of its effectiveness $\varepsilon$ as,

$$Q = \varepsilon \left(\dot{m}c_p\right)\Delta T_i \equiv hA\Delta T_i = \left(\pi kNL\Delta T_i\right)\text{Nu}$$ (3.13)

and the pumping power $P_p$ required to overcome frictional pressure drop is given by

$$P_p = \left(\dot{m}/\rho\right)\Delta p = \left(\pi\mu^2L/2\rho^2d_h^2\right)\left(f_{Re}^3\right)$$ (3.14)

On the basis of Eq. (3.13), the ratio of heat transfer rate for the twisted duct $Q$ and that for the straight duct $Q_s$ can then be expressed as,

$$\left(Q/Q_s\right) = \left(Nu/Nu_s\right)_{N,L,d_h,\Delta T_i}$$ (3.15a)
Furthermore, for the specified operating conditions in a duct of hydraulic diameter $d_h$, the constraint of equal pumping power requires that

$$
\left(f \text{ Re}^3\right) = \left(f \text{ Re}^3\right)_{st}
$$

This mandates that Re in the twisted duct must be lower for a given value of Re$_{st}$ in an equivalent straight duct, because the former has comparatively much higher friction factors.

As is evident from the plots in Figs. 3.8 and 3.9, depending upon $\alpha$, $\xi$, Re, and Pr, up to 13 times higher heat transfer rate $Q$ can be accommodated in a heat exchanger with axially twisted rectangular ducts. Moreover, the enhanced performance is seen to be the highest with smaller $\alpha (\rightarrow 0.5)$ and $\xi (\rightarrow 3.0)$, and increasingly so with higher flow rates ($\text{Re} \rightarrow 1000$). This is essentially brought about by fully developed swirl in the flow filed and the concomitant cross-stream mixing. It may be recalled that the transition for this condition ranges from $\text{Re} \sim 50$ ($\alpha = 0.5$, $\xi = 3.0$) to $\text{Re} \sim 300$ ($\alpha = 1.0$, $\xi = 12.0$). At lower flow rates ($\text{Re} \rightarrow 10$), however, the enhancement is relatively small; about 1.1 times with $\alpha = 0.5$ and negligible with $\alpha = 1.0$. The rather small improvement in this regime is not surprising, as in the absence of swirl effects the helical twisting of the duct does not provide substantive length or surface area increase. Furthermore, the mitigation of wall thermal boundary condition ($T$ or $H1$) effects due to extensive swirl-induced flow mixing at higher Reynolds number in larger Prandtl number liquids and flatter ducts with tighter axial twist is also evident in the results of Fig. 3.9. As a consequence, particularly with in higher Pr liquid flows, the use of twisted ducts in exchangers where the $T$ condition would prevail would accommodate relatively higher heat transfer rates for the same flow rates and pumping power requirements.
Another commonly adopted criterion for the design of new high-performance and compact exchangers is to evaluate the effective reduction in the needed surface area \( A \) for fixed heat duty \( Q \) and pressure drop \( \Delta p \) specifications (Bergles, 1998; Manglik, 2003). Reduced surface area requirements, but without compromising the thermal performance, essentially translate to smaller heat exchangers with attendant materials and cost savings. To carry out such an analysis, the following expression for pressure drop in duct flows is considered:

\[
\Delta p = f \left( 4L/d_h \right) \left( \rho u_{\text{z,m}}^2 / 2 \right) = \left( 2\mu^2 L / \rho d_h^3 \right) \left( f \text{Re}^2 \right)
\]  

(3.16)

Consequently, from the imposition of the condition that \( \Delta p \) should be the same in both the twisted and straight rectangular duct, which also have equal cross-section area and length, it then follows that

\[
\left( f \text{Re}^2 \right) = \left( f \text{Re}^2 \right)_{\text{st}}
\]

(3.17a)

This once again indicates that for the comparative evaluation, \( \text{Re} \) in a twisted duct is less than that in the equivalent straight duct because of the higher \( f \) in the former case. For fixed \( Q \), \( \Delta T \), \( \Delta p \), and \( d_h \), the ratio of respective surface area requirements can therefore be expressed as,

\[
\left( A/A_{\text{st}} \right) = \left( \text{Nu}_{\text{st}} / \text{Nu} \right)_{Q,\Delta T,\Delta p, d_h}
\]

(3.17b)

The evaluation of reduced heat transfer surface area needs in a typical exchanger constructed with axially twisted rectangular ducts, relative to that with straight rectangular ducts, for different conditions is depicted in Figs. 3.10 and 3.11. Variations in \( (A/A_{\text{st}}) \) with
Re along with those in the geometrical attributes $\alpha$ and $\xi$ of the flow channels are graphed. It is evident from these plots that surface-area-reduction ratio is much higher for ducts with smaller $\alpha$ and/or smaller $\xi$. For instance, up to 52.5\% area reduction is obtained for heat exchange with Pr = 5 liquids because of the higher swirl-enhanced convective transport in ducts with $\alpha = 0.5$, $\xi = 3.0$, and as $Re \to 1000$. With more viscous liquids (Pr = 100) there is even greater enhancement benefit and the required surface area in such a case is only ~10\% of that needed with straight rectangular channel. Once again, especially with $\alpha = 0.5$, $\xi = 3.0$, Pr $\geq 45$, and Re $> 100$, the diminishing effect of wall heating/cooling conditions, T or H1, is evident and the relatively larger enhancement benefits accrue with the T condition in more viscous (high Pr) liquid flows. With low flow rates (Re $\to 10$), even with a tight twist of $\xi = 3.0$, area reduction of only about 7.5\% - 10\% is possible for $0.5 \leq \alpha \leq 1.0$; with $\xi = 12.0$, on the other hand, and at the low flow rates, there is negligible enhancement advantage, regardless of the duct aspect ratio.

In general, the optimal enhancement benefits are obtainable primarily in the fully-developed swirl flow region (Re $> O[100]$), irrespective of the rectangular duct’s cross-sectional aspect ratio $\alpha$ and axial twist $\xi$. Both the design objectives of increased heat load and smaller surface area requirements can be accommodated based on their respective fixed pumping power and heat load/pressure drop constraints. Additional figures of merit for different design needs (Bergles and Manglik, 2011; Bergles et al 1974) can be similarly evaluated. Moreover, it may be noted here that in practice such flow channels would likely be configured as plate-fin type compact heat exchanger cores (Shah and Sekulic,2003; Kays and London,1998; Hesselgreaves, 2001), with possibly the use of twisted rectangular plate fins sandwiched between partition plates.
3.5 FIGURES

Figure 3.1. Description of the axially twisted rectangular duct geometry, and the two primary wall heating/cooling conditions.
Figure 3.2 The effect of flow cross section aspect ratio on the temperature \( \left( \frac{T}{T_m} \right) \) distribution in a twisted square and rectangular ducts with \( T \) thermal condition, \( \xi = 3.0, \text{ Pr } = 5.0, \text{ and } \text{Re } = 500 \): (a) \( \alpha = 1.0 \), (b) \( \alpha = 0.75 \), and (c) \( \alpha = 0.5 \).
Figure 3.3 The effect of flow cross section aspect ratio on the temperature \( \left( \frac{T}{T_m} \right) \) distribution in a twisted square and rectangular ducts with H1 thermal condition, \( \xi = 3.0 \), Pr = 5.0, and Re = 500: (a) \( \alpha = 1.0 \), (b) \( \alpha = 0.75 \), and (c) \( \alpha = 0.5 \).
Figure 3.4 The effect of axial twist ratio on the temperature \( \left( T/T_m \right) \) distribution in the swirl-flow regime (Re = 620; Pr = 5.0) in a rectangular duct with \( \alpha = 0.5 \) and T thermal condition: (a) \( \zeta = 12.0 \), (b) \( \zeta = 6.0 \), and (c) \( \zeta = 3.0 \).
Figure 3.5 The effect of axial twist ratio on the temperature \( T/T_m \) distribution in the swirl-flow regime (Re = 620; Pr = 5.0) in a rectangular duct with \( \alpha = 0.5 \) and H1 thermal condition: (a) \( \xi = 12.0 \), (b) \( \xi = 6.0 \), and (c) \( \xi = 3.0 \).
Figure 3.6.1  Variation of Nu with Re, cross-section aspect ratio $\alpha$, and twist ratio $\xi$, in typical liquids (Pr = 5,45,100) flows in axially twisted rectangular ducts with T condition.
Figure 3.6.2 Variation of Nu with Re, cross-section aspect ratio $\alpha$, and twist ratio $\xi$, in typical liquids ($Pr = 5, 45, 100$) flows in axially twisted rectangular ducts with H1 condition.
Figure 3.7 Effect of \( \text{Pr} \) on the variation of \( \left( \frac{\text{Nu}}{\text{Nu}_{st}} \right) \) with \( \text{Re} \) in a axially twisted rectangular duct with \( \alpha = 0.5 \) and \( \zeta = 3.0 \): (a) with T condition, and (b) with H1 condition.
Figure 3.8 Effect of twist ratio and cross-section aspect ratio of twisted rectangular ducts on enhanced heat transfer rate sustained in laminar liquid ($Pr = 5.0$) flows with fixed pumping power and geometry constraints: (a) with $T$ condition, and (b) with $H1$ condition.
Figure 3.9 Effect of liquid Pr and twist ratio on enhanced heat transfer rate sustained in laminar flows inside twisted rectangular ducts with $\alpha = 0.5$, and fixed pumping power and geometry constraints: (a) with $T$ condition, and (b) with $H1$ condition.
Figure 3.10 Effect of twist ratio and cross-section aspect ratio of twisted rectangular ducts on relative reduction in heat transfer surface area requirement in laminar liquid ($Pr = 5.0$) flows with fixed heat load and pressure drop constraints: (a) with $T$ condition, and (b) with $H1$ condition.
Figure 3.11 Effect of liquid Pr and twist ratio on relative reduction in heat transfer surface area requirement in laminar flows inside twisted rectangular ducts with $\alpha = 0.5$, and fixed heat load and pressure drop constraints: (a) with T condition, and (b) with H1 condition.
4.1 Conclusion

The effects of cross-sectional geometry and axially helical twist pitch on the low Reynolds number \((10 \leq \text{Re} \leq 1000)\) flow behavior in twisted rectangular ducts have been computationally investigated. Numerical solutions for incompressible, single-phase, periodically developed, constant property, forced convection in square-to-rectangular ducts \((1.0 \geq \alpha \geq 0.5)\) with helical twist ratios of \(12 \geq \xi \geq 3\) are presented. To study thermal behavior, Prandtl numbers 4, 45 and 100 are considered.

Both the reduction in cross-sectional area or \(\alpha\) and severity of axially helical twist of the tube or \(\xi\) are seen to have a compound influence on the flow field. The curvature of edge sections of the rectangular duct, especially that with smaller \(\alpha\), tends to produce secondary circulation in the corner ends of the channel and substantially alter the axial flow. Two different regimes characterize the consequent flow field, where at low \(\text{Re}\), viscous effects dominate and the duct twist only lends to a gentle helical stirring of the axial flow. At higher \(\text{Re}\) and with flattening of the rectangular duct, or decreasing \(\alpha\), and increasing severity of the duct twist, or decreasing \(\xi\), on the other hand, the core swirl increases and multiple vortex cells are generated in the narrower ends of flatter rectangular ducts. The enhanced momentum transport in this swirl-flow regime is incurred due to the cross-stream mixing in the end regions and significantly higher core acceleration, which leads to substantially higher heat transfer. In the viscous regime where the flow is stirred but generally undisturbed, the relatively higher heat transfer, compared to that in an equivalent straight untwisted duct, results mainly due to increased flow length due to the reduction in \(\xi\) of the duct geometry.
Depending upon the fluid Pr, the Nusselt number increases by as much as 2.5 to 14 times that in equivalent straight ducts. The larger convective transport is achieved in more viscous liquids (Pr → 100), essentially due to the markedly stronger momentum transport in the swirl regime. Also, the effect of the type of wall heating or cooling, represented by the $T$ and $H1$ conditions, diminishes drastically as swirl intensifies (Re → 1000). With low flow rates (Re → 10), the increase in Nu is rather small and is primarily due the higher effective flow length (or thermal residence time).

The enhanced momentum transfer due to swirl flow also leads to substantially higher wall shear stress and friction factors. In the viscous regime, the relatively higher ($f$ Re), compared to that in an equivalent straight untwisted duct, is on account of increased flow length and peak axial velocity due to the reduction in $\alpha$ and/or $\xi$ of the duct geometry.

The enhanced performance of twisted rectangular ducts is further evaluated on the basis of two popularly employed metrics: (a) increased heat transfer rate for fixed pumping power, and (b) reduced heat transfer surface area for fixed heat load and pressure drop, relative to straight equivalent channels. In the former case, up to 1.1 to 13 times higher $Q$ can be accommodated depending upon the type of liquid, its flow rate, and duct geometry. Whereas as much as 52.5% to 90% less surface area is required in a new heat exchanger, when designed based on the latter figure of merit. The greater enhancement benefits, in both cases, are achieved in ducts with smaller aspect ratio ($\alpha = 0.5$) and/or twist ratio ($\xi = 3.0$), particularly in liquid flows with high Pr and Re. Also, the effects of wall heating/cooling condition (represented by the $T$ and $H1$ boundary conditions) diminishes with increasing
swirl development, which becomes more pronounced as $\alpha \to 0.5$, $\xi \to 3.0$, $Pr \geq 45$, and $Re > 100$.

4.2 Recommendations

In present study, computational results were produced but to put twisted tubes to use for practical purpose, some sort of empirical correlations needs to be produced. Also, three twist ratios and aspect ratios are considered and results obtained are only for $Re$ upto 1000 and $Pr$ of 5, 45 and 100. Sample size of these parameters needs to be increased to further understand the behavior of flow through twisted tubes.


APPENDIX A: MEAN VELOCITY AND FRICTION FACTOR

The mean flow velocity is defined as

\[ u_{z,m} = \frac{1}{A_c} \int_{c} u_c(x, y) \, dA_c = \frac{1}{4\alpha} \int_{c}^{\alpha} \int_{1-\alpha}^{1} u_c(x, y) \, dx \, dy \]  

A.1

The Reynolds number is defined as

\[ \text{Re} = \frac{d_c \bar{u}_{z,m}}{v} = 4\alpha \frac{\bar{u}_{z,m}}{(1+\alpha)v} = \left[4u_{z,m}/(1+\alpha)\right] \]  

A.2

The Fanning friction factor, based on the average wall shear stress, is defined as

\[ f = \left[2\bar{\tau}_w/\rho \bar{U}_{z,m}^2\right] \]  

A.3

From a force balance over an elemental duct, the average wall shear stress is

\[ \bar{\tau}_w = \frac{A_x}{P_w} \left( \frac{\partial \bar{P}}{\partial Z} \right) = \frac{a}{1+\alpha} \left( \frac{\partial \bar{P}}{\partial Z} \right) \]  

A.4

Therefore, the fanning friction factor is reduced to,

\[ f = 2 \frac{a}{1+\alpha} \left[ 1/\rho \bar{U}_{z,m}^2 \right] = 2 \frac{-P_z}{(1+\alpha)u_{z,m}^2} \]  

A.5

From Eqs. A.2 and A.5, we get

\[ f \, \text{Re} = \frac{-8P_z}{(1+\alpha)^2 u_{z,m}^2} \]  

A.6
APPENDIX B: MEAN TEMPERATURE AND NUSSELT NUMBER

The mean temperature is defined as

$$
\bar{T}_m = \frac{1}{A_z u_{z,m}} \int \bar{T} u_z (\tilde{x}, \tilde{y}) dA_z
$$

Dimensionless temperature is given by,

$$
T = \frac{\bar{T}_w - \bar{T}}{a \text{Re} \text{Pr}(d\bar{T}_m/d\tilde{Z})}
$$

Expressing eqn. B.2 in dimensionless form gets,

$$
\bar{T}_w - a \text{Re} \text{Pr}(d\bar{T}_m/d\tilde{Z}) T_m = \frac{1}{A_z u_{z,m}} \int \bar{T}_w u_z dA_z - a \text{Re} \text{Pr}(d\bar{T}_m/d\tilde{Z}) \frac{1}{A_z u_{z,m}} \int T u_z dA_z
$$

Where,

$$
\frac{1}{A_z u_{z,m}} \int \bar{T}_w u_z dA_z = \bar{T}_w
$$

Substituting eqn. B.4 in eqn. B.3 gives dimensionless mean temperature as

$$
T_m = \frac{1}{A_z u_{z,m}} \int T u_z dA_z = \frac{1}{4\alpha u_{z,m} - 1 - \alpha} \int \alpha T u_z (x, y) dx dy
$$

After the temperature distribution is determined, the Nusselt number can be evaluated in terms of the mean temperature. From an energy balance over an elemental duct,

$$
q'' P_h d\tilde{Z} = \dot{m} C_p d\bar{T}_m
$$

Where $P_h$ is heated perimeter

$$
q'' = \frac{\dot{m} C_p}{P_h} d\bar{T}_m = h (T_w - \bar{T}_m)
$$
Also,

\[ \dot{m} = \rho \tilde{u}_{z,m} A_c \]

We know that,

\[ \text{Nu} = \frac{h d_h}{k} = \frac{h}{k} \frac{4a}{1 + \alpha} \]

Rearranging eqn.9 using Eqs. B.7 and B.8

\[ \text{Nu} = \frac{\rho \tilde{u}_{z,m} A_c C_p}{(T_w - \tilde{T}_m) P_h} \frac{d\tilde{T}_m}{dZ} 1 + \alpha = \frac{\text{Re Pr} C_p}{(T_w - \tilde{T}_m)} \frac{d\tilde{T}_m}{dZ} \frac{a}{1 + \alpha} \]

B.10

Where in

\[ \frac{T_w - T_m}{a \text{ Re Pr}(dT_m/d\tilde{Z})} = T_m \]

So final expression for Nusselt number is reduced to,

\[ \text{Nu} = \frac{1}{1 + \alpha} T_m \]

B.11
APPENDIX C. DISCRETIZATION OF GOVERNING EQUATIONS

The general form requisite differential equations is

\[
\frac{\partial (L_x \phi)}{\partial x} + \frac{\partial (L_y \phi)}{\partial y} - M_x \frac{\partial^2 \phi}{\partial x^2} - M_y \frac{\partial^2 \phi}{\partial y^2} = b
\]

Where \( \phi, L_x, L_y, M_x, M_y \) and \( b \)

<table>
<thead>
<tr>
<th>( \varphi )</th>
<th>( L_x )</th>
<th>( L_y )</th>
<th>( M_x )</th>
<th>( M_y )</th>
<th>( b = S_c + S_p \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( S_c = \omega_z - \Gamma u_z \left{ x \frac{\partial u_z}{\partial x} + y \frac{\partial u_z}{\partial y} + 2u_z \right}, S_p = 0 )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( \frac{\partial \psi}{\partial y} + \Gamma^2 x )</td>
<td>-( \frac{\partial \psi}{\partial x} + \Gamma^2 y )</td>
<td>1+( \Gamma^2 y^2 )</td>
<td>1+( \Gamma^2 x^2 )</td>
<td>( S_c = \Gamma \left{ y \frac{\partial u_z}{\partial x} - x \frac{\partial u_z}{\partial y} + u_z \right}, S_p = \Gamma \left{ y \frac{\partial u_z}{\partial x} - x \frac{\partial u_z}{\partial y} \right} )</td>
</tr>
<tr>
<td>( u_z )</td>
<td>( \frac{\partial \psi}{\partial y} + \Gamma^2 x )</td>
<td>-( \frac{\partial \psi}{\partial x} + \Gamma^2 y )</td>
<td>1+( \Gamma^2 y^2 )</td>
<td>1+( \Gamma^2 x^2 )</td>
<td>( S_c = -P_z - \Gamma \left{ yP_z - xP_y \right} - \Gamma^2 \left{ 2xy \frac{\partial^2 u_z}{\partial x \partial y} \right} )</td>
</tr>
<tr>
<td>( T )</td>
<td>( Pr \frac{\partial \psi}{\partial y} + \Gamma^2 x )</td>
<td>-( Pr \frac{\partial \psi}{\partial x} + \Gamma^2 y )</td>
<td>1+( \Gamma^2 y^2 )</td>
<td>1+( \Gamma^2 x^2 )</td>
<td>( S_c = \frac{Tu_z}{ReT_m} - \Gamma^2 \left{ 2xy \frac{\partial^2 T}{\partial x \partial y} \right} ) for UWT</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( S_c = \frac{u_z}{Re} - \Gamma^2 \left{ 2xy \frac{\partial^2 T}{\partial x \partial y} \right} ) for UHF</td>
</tr>
</tbody>
</table>
Introducing a term total flux $J$, as sum of convective and diffusive terms, by defining

$$J_x = L_x - M_x \frac{\partial \phi}{\partial x}$$

$$J_y = L_y - M_y \frac{\partial \phi}{\partial y}$$

This implies

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = S_c + S_p \phi$$

Integrating eqn.C.3 over the control volume, we have

$$\int_{w}^{e} \int_{n}^{s} \frac{\partial J_x}{\partial x} \partial x \partial y + \int_{w}^{e} \int_{n}^{s} \frac{\partial J_y}{\partial y} \partial x \partial y = \int_{w}^{e} \int_{n}^{s} \left( S_c + S_p \phi \right) \partial x \partial y$$

$$\left( J_{xe} - J_{wn} \right) \Delta y + \left( J_{en} - J_{ws} \right) \Delta x = \left( S_c + S_p \phi \right) \Delta x \Delta y$$

Back substituting eqn.C.3 in eqn.C.5,
\[
\left\{ L_x - M_x \frac{\partial \phi}{\partial x} \right\}_x - \left\{ L_x - M_x \frac{\partial \phi}{\partial x} \right\}_w \right\} \Delta y + \left\{ L_y - M_y \frac{\partial \phi}{\partial y} \right\}_n - \left\{ L_y - M_y \frac{\partial \phi}{\partial y} \right\}_s \right\} \Delta x = \left( S_c + S_p \phi_p \right) \Delta x \Delta y
\]

C.6

Where diffusive terms are expressed as,

\[
\left( \frac{\partial \phi}{\partial x} \right)_e = \frac{\left( \phi_e - \phi_p \right)}{\Delta x}, \quad \left( \frac{\partial \phi}{\partial x} \right)_w = \frac{\left( \phi_p - \phi_w \right)}{\Delta x}
\]

\[
\left( \frac{\partial \phi}{\partial y} \right)_n = \frac{\left( \phi_n - \phi_p \right)}{\Delta y},\quad \left( \frac{\partial \phi}{\partial y} \right)_s = \frac{\left( \phi_p - \phi_s \right)}{\Delta y}
\]

C.7

Let,

\[
D_e = M_e \Delta y / \Delta x \quad F_e = L_e \Delta y
\]

\[
D_w = M_w \Delta y / \Delta x \quad F_w = L_w \Delta y
\]

\[
D_n = M_n \Delta x / \Delta y \quad F_n = L_n \Delta x
\]

\[
D_s = M_s \Delta x / \Delta y \quad F_s = L_s \Delta x
\]

\[
\phi_e = \frac{\left( \phi_E - \phi_p \right)}{\Delta x}, \quad \phi_w = \frac{\left( \phi_p - \phi_W \right)}{\Delta x}, \quad \phi_n = \frac{\left( \phi_n - \phi_p \right)}{\Delta y}, \quad \phi_s = \frac{\left( \phi_p - \phi_s \right)}{\Delta y}
\]

C.8

Rearranging eqn.C6 using Eqs. C.7-C.9, we have

\[
\left( D_e + D_w + D_n + D_s + \frac{F_e}{2} - \frac{F_w}{2} + \frac{F_n}{2} - \frac{F_s}{2} \right) \phi_p =
\]

\[
\left( D_e - \frac{F_e}{2} \right) \phi_E + \left( D_n + \frac{F_n}{2} \right) \phi_n + \left( D_s + \frac{F_s}{2} \right) \phi_s + \left( D_s + \frac{F_s}{2} \right) \phi_s + \left( S_c + S_p \phi_p \right) \Delta x \Delta y
\]

C.9

This can be generalized as

\[
a_p \phi_p = a_E \phi_E + a_W \phi_W + a_N \phi_N + a_S \phi_s + S_c \Delta x \Delta y
\]

C.10
Where in,

\[ a_E = \left( D_e - \frac{F_e}{2} \right) \]
\[ a_s = \left( D_w + \frac{F_w}{2} \right) \]
\[ a_N = \left( D_n - \frac{F_n}{2} \right) \]
\[ a_s = \left( D_s + \frac{F_s}{2} \right) \]
\[ a_p = \left( a_E + a_w + a_N + a_s + F_e - F_w + F_n - F_s - S_p \phi_p \Delta x \Delta y \right) \]

C.11

\[ a_E = D_e A_e + \max(-F_e, 0) \]
\[ a_w = D_w A_w + \max(F_w, 0) \]
\[ a_N = D_n A_n + \max(-F_n, 0) \]
\[ a_s = D_s A_s + \max(F_s, 0) \]

C.12

By Power law scheme,

\[ A_e = \max(0, (1 - 0.1 \text{abs}(F_e / D_e))^5) \]
\[ A_w = \max(0, (1 - 0.1 \text{abs}(F_w / D_w))^5) \]
\[ A_n = \max(0, (1 - 0.1 \text{abs}(F_n / D_n))^5) \]
\[ A_s = \max(0, (1 - 0.1 \text{abs}(F_s / D_s))^5) \]
APPENDIX D. IMPLEMENTATION OF BOUNDARY CONDITIONS

Boundary Conditions

<table>
<thead>
<tr>
<th></th>
<th>( x = \pm 1 ), ( y )</th>
<th>( x, y = \pm 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \psi )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( \frac{\partial^2 \psi}{\partial x^2} + \left{ x \frac{\partial v_z}{\partial x} \right} )</td>
<td>( -\frac{\partial^2 \psi}{\partial y^2} + \left{ y \frac{\partial v_z}{\partial y} \right} )</td>
</tr>
<tr>
<td>( u_z )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( T )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

All dependant variables are zero at the boundaries except for vorticity. Vorticity boundary condition is expressed in terms of Taylor series.
APPENDIX E. TABULATION OF RESULTS

Table 1 $\alpha = 0.5$ and $\xi = 12$.

<table>
<thead>
<tr>
<th>$P_z$</th>
<th>$Re$</th>
<th>$fRe$</th>
<th>$Nu$ (UWT)</th>
<th>$Nu$ (UHF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>60.26128</td>
<td>15.73395</td>
<td>3.552648</td>
<td>3.811016</td>
</tr>
<tr>
<td>200</td>
<td>120.0914</td>
<td>15.9493</td>
<td>3.618996</td>
<td>4.190189</td>
</tr>
<tr>
<td>1500</td>
<td>719.4918</td>
<td>20.8622</td>
<td>5.331812</td>
<td>15.1256</td>
</tr>
</tbody>
</table>
Table 2 $\alpha = 0.5$ and $\xi = 6$.

<table>
<thead>
<tr>
<th>$P_z$</th>
<th>Re</th>
<th>$f_{Re}$</th>
<th>Nu (UWT)</th>
<th>Nu (UHF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>58.01641</td>
<td>16.34276</td>
<td>3.586289</td>
<td>3.839314</td>
</tr>
<tr>
<td>200</td>
<td>112.8833</td>
<td>16.79873</td>
<td>3.706067</td>
<td>4.640209</td>
</tr>
<tr>
<td>1500</td>
<td>615.6909</td>
<td>23.09961</td>
<td>5.864152</td>
<td>19.11509</td>
</tr>
<tr>
<td>2500</td>
<td>938.8315</td>
<td>25.24809</td>
<td>7.330012</td>
<td>26.42728</td>
</tr>
<tr>
<td>2800</td>
<td>1030.693</td>
<td>25.75757</td>
<td>7.760616</td>
<td>28.21212</td>
</tr>
</tbody>
</table>
Table 3 $\alpha =0.5$ and $\xi = 3$.

<table>
<thead>
<tr>
<th>$P_z$</th>
<th>$Re$</th>
<th>$f/Re$</th>
<th>Nu (UWT)</th>
<th>$f/Re$</th>
<th>Nu (UHF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>49.97094</td>
<td>18.97399</td>
<td>3.799155</td>
<td>4.600208</td>
<td>6.01713</td>
</tr>
<tr>
<td>1200</td>
<td>415.8489</td>
<td>27.36037</td>
<td>6.20971</td>
<td>22.62067</td>
<td>35.20816</td>
</tr>
<tr>
<td>1600</td>
<td>527.8588</td>
<td>28.73945</td>
<td>7.181201</td>
<td>27.17531</td>
<td>41.62165</td>
</tr>
<tr>
<td>1800</td>
<td>581.7906</td>
<td>29.33472</td>
<td>7.663657</td>
<td>29.13932</td>
<td>44.35875</td>
</tr>
<tr>
<td>2000</td>
<td>634.4742</td>
<td>29.88768</td>
<td>8.137303</td>
<td>30.9379</td>
<td>46.85949</td>
</tr>
</tbody>
</table>
Table 4 $\alpha =0.75$ and $\zeta = 12$.

<table>
<thead>
<tr>
<th>$P_z$</th>
<th>$Re$</th>
<th>$j/Re$</th>
<th>$Nu$ (UWT)</th>
<th>$Nu$ (UHF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1500</td>
<td>595.5325</td>
<td>15.03909</td>
<td>3.338285</td>
<td>7.283737</td>
</tr>
</tbody>
</table>
Table 5 $\alpha =0.75$ and $\zeta = 6$.

<table>
<thead>
<tr>
<th>$P_z$</th>
<th>Re</th>
<th>$f/Re$</th>
<th>Nu (UWT)</th>
<th>Nu (UHF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>120.2893</td>
<td>14.89122</td>
<td>3.178931</td>
<td>4.703201</td>
</tr>
</tbody>
</table>
Table 6 $\alpha =0.75$ and $\xi = 3$.

<table>
<thead>
<tr>
<th>$P_z$</th>
<th>Re</th>
<th>$f_{Re}$</th>
<th>Nu (UWT)</th>
<th>Nu (UHF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>38.31814</td>
<td>15.5823</td>
<td>3.379787</td>
<td>4.232341</td>
</tr>
<tr>
<td>200</td>
<td>75.62116</td>
<td>15.79147</td>
<td>3.397588</td>
<td>6.664792</td>
</tr>
<tr>
<td>1500</td>
<td>492.7061</td>
<td>18.17771</td>
<td>5.123929</td>
<td>16.82349</td>
</tr>
<tr>
<td>2500</td>
<td>758.5787</td>
<td>19.67774</td>
<td>6.630729</td>
<td>23.4743</td>
</tr>
<tr>
<td>2800</td>
<td>831.2546</td>
<td>20.11221</td>
<td>7.074453</td>
<td>25.10681</td>
</tr>
</tbody>
</table>
Table 7 $\alpha = 1.0$ and $\xi = 12$.

<table>
<thead>
<tr>
<th>$P_z$</th>
<th>Re</th>
<th>$f/Re$</th>
<th>5</th>
<th>45</th>
<th>100</th>
<th>5</th>
<th>45</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Nu (UWT)</td>
<td></td>
<td></td>
<td>Nu (UHT)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>2.802257</td>
<td>14.27421</td>
<td>2.964623</td>
<td>2.965709</td>
<td>2.96854</td>
<td>3.578001</td>
<td>3.578663</td>
<td>3.580336</td>
</tr>
<tr>
<td>500</td>
<td>140.0157</td>
<td>14.28411</td>
<td>2.979959</td>
<td>3.537426</td>
<td>4.634228</td>
<td>3.593785</td>
<td>4.121347</td>
<td>5.164657</td>
</tr>
</tbody>
</table>
Table 8 $\alpha =1.0$ and $\xi =6$.

<table>
<thead>
<tr>
<th>$P_z$</th>
<th>$Re$</th>
<th>$Re_f$</th>
<th>$Nu$ (UWT)</th>
<th>$Nu$ (UHF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4000</td>
<td>1043.676</td>
<td>15.33043</td>
<td>4.658821</td>
<td>15.39125</td>
</tr>
</tbody>
</table>
Table 9 $\alpha = 1.0$ and $\xi = 3$.

<table>
<thead>
<tr>
<th>$P_z$</th>
<th>Re</th>
<th>$f/Re$</th>
<th>Nu (UWT)</th>
<th>Nu (UHF)</th>
<th>Nu (UWT)</th>
<th>Nu (UHF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>261.0926</td>
<td>15.32024</td>
<td>3.802721</td>
<td>10.00148</td>
<td>15.0704</td>
<td>4.450289</td>
</tr>
<tr>
<td>5000</td>
<td>1126.708</td>
<td>17.75082</td>
<td>7.845098</td>
<td>29.77281</td>
<td>45.2782</td>
<td>8.587777</td>
</tr>
</tbody>
</table>