I, Cheng Liu, hereby submit this original work as part of the requirements for the degree of:

Master of Science

in Mathematical Sciences

It is entitled:

Utility-based Futures Contract Pricing under Stochastic Interest Rate, Appreciation Rate and Dividend Yield

Student Signature: Cheng Liu

This work and its defense approved by:

Committee Chair: Srdjan Stojanovic, PhD

James Deddens, PhD

Jeesen Chen, PhD
Utility-based Futures Contract Pricing with Stochastic Interest Rate, Dividend yield and Appreciation Rate

Master Thesis
submitted to
the Graduate School of
The University of Cincinnati
in Partial Fulfillment
of the Requirements
for the Degree of

Master of Science

by
Cheng Liu
Department of Mathematics
The University of Cincinnati

August 2010
Committee Chair: Srdjan Stojanovic Ph.D.
Committee Member: James A Deddens Ph.D.
Committee Member: Jeesen Chen Ph.D.
Abstract

Futures contract is one of the oldest and simplest financial contracts, and the cost of carry model is no doubt the most popular pricing model for futures contracts. However, since it is derived for forward contracts and forward contracts price equals to futures contract only under some specific circumstances, this model has systematic pricing error and fails to capture some important properties of the futures contract, such as the dynamic interaction between the underlying and the futures contract.

Another model for futures contracts pricing is the general equilibrium model with stochastic interest rate and volatility. However, this model is based on some relatively strong assumptions and using logarithmic utility of wealth. In this paper, we implement a more general model to derive a closed-form pricing formula for the futures contracts pricing, with stochastic interest rate, dividend rate and appreciation rate. The model shows some different properties comparing with the classic cost of carry model and general equilibrium model.

In addition, we use the result of our model to set up a statistical analysis using Standard & Poor’s 500 index and the Standard & Poor’s 500 E-mini futures contracts’ historical data. The statistical analysis results shows that if one choose to use our utility-based pricing framework, then dividend yield rate plays a important role. Our result suggests a promising modification for the classic model.
# Table of Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Abstract</strong></td>
<td>ii</td>
</tr>
<tr>
<td><strong>Table of Contents</strong></td>
<td>iii</td>
</tr>
<tr>
<td><strong>Chapter</strong></td>
<td></td>
</tr>
<tr>
<td>1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 The Future Contracts</td>
<td>1</td>
</tr>
<tr>
<td>1.2 The Cost of Carry Model and Hemler’s Model</td>
<td>2</td>
</tr>
<tr>
<td>1.3 Utility-based Pricing Model</td>
<td>3</td>
</tr>
<tr>
<td>1.4 Standard&amp;Poor’s (S&amp;P) 500 E-mini Futures Contracts</td>
<td>3</td>
</tr>
<tr>
<td>2 Utility-based Pricing</td>
<td>5</td>
</tr>
<tr>
<td>2.1 The General Pricing and Hedging Methodology</td>
<td>5</td>
</tr>
<tr>
<td>2.2 Problem Setup</td>
<td>7</td>
</tr>
<tr>
<td>2.3 Solving the Risk Premium PDE</td>
<td>8</td>
</tr>
<tr>
<td>2.4 Solving the Pricing PDE</td>
<td>11</td>
</tr>
<tr>
<td>2.5 The Relation between the Proposed Model and Cost of Carry Model</td>
<td>15</td>
</tr>
<tr>
<td>2.6 The Relation between Utility-based Framework and General Equilibrium Framework</td>
<td>16</td>
</tr>
<tr>
<td>2.7 Coefficient of Dividend Rate</td>
<td>16</td>
</tr>
<tr>
<td>3 Empirical Results and Conclusion</td>
<td>19</td>
</tr>
<tr>
<td>3.1 Regression Setup</td>
<td>19</td>
</tr>
<tr>
<td>3.2 Data</td>
<td>20</td>
</tr>
<tr>
<td>3.3 Statistical Analysis</td>
<td>21</td>
</tr>
<tr>
<td>3.4 Conclusion</td>
<td>25</td>
</tr>
<tr>
<td><strong>References</strong></td>
<td>29</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

1.1 The Future Contracts

A futures contract is a standardized contract between two parties to buy or sell a specified asset of standardized quantity and quality at a specified future date at a price agreed today[9]. The contract are traded on the futures exchange. Futures contract’s underlying asset can be stock, bonds, rights or goods. One can take both long position and short position.

The price of the futures contract is determined by the market. The futures contract will be trading until the expiration date. The expiration date is also called delivery date or final settlement date. There are two forms of settlement, one is to make or take delivery of the underlying, the other is cash settlement[5].

A forward contract is very closely related to the futures contract. They are very similar to each other. However, it is in the specific details that these contracts differ[1][3]. First of all, futures contracts are exchange-traded and, therefore, are standardized contracts. Forward contracts, on the other hand, are private agreements between two parties and are not as rigid in their stated terms and conditions. Because forward contracts are private agreements, there is always a chance that a party may default on its side of the agreement. Futures contracts have clearing houses that guarantee the transactions, which lowers the probability of default. Secondly, the specific details concerning settlement and delivery are different. For forward contracts, settlement of the contract occurs at the end of the contract. Futures contracts are marked-to-market daily, which means that daily changes are settled day by day until the end of the contract. In addition, settlement for futures
contracts can occur over a range of dates. Forward contracts, on the other hand, only possess one settlement date. Lastly, because futures contracts are quite frequently employed by speculators, they are usually closed out prior to maturity and delivery usually never happens. On the other hand, forward contracts are mostly used by hedgers that want to eliminate the volatility of an asset’s price, and delivery of the asset or cash settlement will usually take place. It is shown that under stochastic interest rate, prices of futures contract and forward may differ[10].

1.2 The Cost of Carry Model and Hemler’s Model

The most popular model for pricing stock index futures contract is the cost of carry model, though it is derived for forward. This model expresses the futures contract price in terms of the underlying stock index value, the risk-free interest rate, and the dividend yield rate. The cost of carry model gives the following:

$$V = Se^{(r-D)\tau}$$

Its derivation relies on the no-arbitrage argument in which a trader replicates a futures contract position with spot positions in the stock and Treasury bill markets.

Despite its popularity, this model is restricted in many aspects. Hemler[4] derived a two factor equilibrium stock index futures pricing model by assuming interest rate and market volatility to be stochastic processes. Hemler’s model shows that if one accepts their general equilibrium model, then market volatility plays an important role in determining the stock index futures prices. However, in this model, dividend rate is taken as a deterministic, while its nature is stochastic indeed[7].

Hemler has shown that under their general equilibrium model with stochastic interest rate and market volatility, the futures contract price can be given as following:

$$V(\tau, S, v, r) = Sg_1(\tau)e^{rg_2(\tau) + vg_3(\tau)\tau}$$

where $g_1(t), g_2(t), g_3(t)$ are known.
1.3 Utility-based Pricing Model

This paper implements a utility-based neutral pricing framework with CRRA utility function developed by Stojanovic[12][6][11]. This relatively general pricing model contains a CRRA portfolio PDE (also called risk premium PDE) and a pricing PDE. Solving both of them leads to a “neutral derivative pricing” solution of the futures contracts. We first solve the CRRA portfolio PDE (also called risk premium PDE) and then plug in the risk premium solution into the pricing PDE to get a solution of the futures contract. Given that interest rate, dividend rate and appreciation rate, an explicit solution is not easy. However, we show that the pricing PDE can be solved by separation of variables, and this allows us to set up a regression analysis. We are also able to achieve an explicit solution of coefficient of dividend yields. In this way, we are able to compare our model with the cost of carry model and the Hemler’s model.

1.4 Standard&Poor’s (S&P) 500 E-mini Futures Contracts

In this paper, S&P 500 E-mini futures contracts’ historical data for 1997-2007 period is chosen to be analyzed. The Chicago Mercantile Exchange (CME) introduced the S&P 500 futures contracts back in spring of 1982. The S&P 500 futures market has now become today’s most actively traded equity futures market. The S&P 500 futures contract represents roughly 90% of all US stock index futures trading. The S&P 500 is comprised of the largest 500 listed stocks, therefore allowing you to easily and effectively buy or sell an extremely well diversified portfolio of stocks in one stock index futures contract. This allows you to make trading/investing decisions based on your overall outlook of the stock market.

There are four S&P 500 E-mini futures contracts released every year each on the third Friday of January, March, June, December. All of them expire on the third Friday in the sixth months with cash settlement. Trading day of each S&P 500 E-mini futures contract
is around 120 days. Theoretically the futures contract pricing should be equal to the price of underlying at the expiration date, otherwise one can take either long or short position to make arbitrage.
Chapter 2

Utility-based Pricing

2.1 The General Pricing and Hedging Methodology

An economy $E$ is a finite set of:

1) a cash, or money market account accruing interest at (possibly stochastic) short (interest) rate $r(t),$

2) factors, and

3) tradables.

Factors and tradables are denoted by $A(t)=\{A_1(t), A_2(t), ..., A_m(t)\}$ and $S(t)=\{S_1(t), S_2(t), ..., S_k(t)\}$, respectively, with typically nonempty intersection, and obeying Itô SDE dynamics

\[ dA(t) = b(t, A(t))dt + c(t, A(t)).dB(t) \]  
\[ dS(t) = S(t)(a_s(t, A(t)) - D(t, A(t)))dt + S(t)\sigma_s(t, A(t)).dB(t) \]

where $B(t)=\{B_1(t), B_2(t), ..., B_n(t)\}$ is a standard n-dimension Brownian motion, $b(t,A(t))$ is the m-vector of factor-drift, $c(t,A(t))$ is m x n factor-diffusion-matrix, $a_s(t, A(t))$ is the k-vector of (pre-dividend) appreciation rates for the tradables, $D(t, A(t))$ is the k-vector of dividend rates, $\sigma_s(t, A(t))$ is the volatility k x n-matrix. Functions $a_s(t, A(t))$, $\sigma_s(t, A(t))$, $b$, $c$ are called market coefficients. Multiplication "." is the usual matrix-vector or "dot" multiplication; multiplication " " is meant component-wise.

Consider a portfolio European contracts in the economy $E$, expiring at time $T \leq \infty$, with the terminal payoff equal to $v(A(T))=\{v_1(A(T)), v_2(A(T)), ..., v_l(A(T))\}$. As shown
in [12], in the case of the CRRA utility of wealth, for, say, \(0 < \gamma \neq 1\),

\[
\varphi_\gamma(X) = \frac{X^{1-\gamma}}{1-\gamma}
\]  

(2.3)

in case \(\gamma = 1\), we define as following:

\[
\varphi_1(X) = \log X
\]  

(2.4)

**Theorem 1** [12] Under the CRRA utility of wealth, \(g_\gamma(t, A)\) is a scalar function, characterized as an \(A\)-entire solution (see [12] for what is meant by \(A\)-entire solutions) of

\[
\frac{\partial g_\gamma(t, A)}{\partial t} + \left(b - \frac{(\gamma - 1)(a_s - r)}{\gamma} \cdot (\sigma_s \sigma_s^T)^{-1} \cdot c^T\right) \cdot \nabla g_\gamma(t, A) + \frac{1}{2} \nabla^2 g_\gamma(t, A) \cdot c \cdot c^T = -\mathbb{D}
\]  

(2.5)

Theorem 2 [12] Under the CRRA utility of wealth (with relative risk aversion \(\gamma > 0\)), the general Black-Scholes type system of (pricing) PDEs for a set of (tradable) contracts with prices \(V = \{V_1, ..., V_l\}\), paying dividends \(D = \{D_1, ..., D_l\}\), and with the terminal payoffs \(v = \{v_1, ..., v_l\}\), in any simple economy \(E\) with market coefficients \(a_s, \sigma_s, b, c\) and (possibly stochastic) interest rate \(r\), reads as

\[
\frac{\partial V(t, A)}{\partial t} + \nabla V(t, A) \cdot \left(b - (a_s - r) \cdot (\sigma_s \sigma_s^T)^{-1} \cdot c^T\right) + \nabla V(t, A) \cdot c \cdot \left(\sigma_s \sigma_s^T\right)^{-1} \cdot (a_s - r) = -\mathbb{D}
\]  

(2.7)
where \( g_\gamma(t, A) \) is the solution of PDE in **Theorem 1**. For \( t < T \leq \infty \), with terminal condition (if \( T < \infty \))

\[
V(T, A) = v(A)
\]  

(2.8)

### 2.2 Problem Setup

Consider a simple economy with stochastic interest rate \( r \), dividend yield rate \( D \) and appreciation rate \( a \), \( S \) is the stock index:

\[
dS = S(a - D)dt + S\sigma_1 dB_1(t)
\]  

(2.9)

\[
da = n(m - a)dt + \sigma_2 dB_2(t)
\]  

(2.10)

\[
dr = k(\theta - r)dt + \sigma_3 dB_3(t)
\]  

(2.11)

\[
dD = p(q - D)dt + \sigma_4 dB_4(t)
\]  

(2.12)

where

\[
dB_i dB_j = \rho_{i,j} dt
\]  

(2.13)

Thus the market coefficients are given as following:

\[
a_s = \{a\}; b = \{S(a - D), n(m - a), k(\theta - r), p(q - D)\};
\]

\[
c = \begin{pmatrix}
S\sigma_1 & 0 & 0 & 0 \\
\sigma_2\rho_{2,1} & \sigma_2\sqrt{1 - \rho_{2,1}^2} & 0 & 0 \\
\sigma_3\rho_{3,1} & \frac{\sigma_3(\rho_{2,1}\rho_{3,1} - \rho_{3,2})}{\sqrt{1 - \rho_{2,1}^2}} & \sigma_3\sqrt{\frac{2\rho_{2,1}\rho_{3,1}\rho_{3,2} - (\rho_{2,1}^2 + \rho_{3,1}^2 + \rho_{3,2}^2) + 1}{1 - \rho_{2,1}^2}} & 0 \\
\sigma_4\rho_{4,1} & \sigma_4 \mathbf{c}_{4,2} & \sigma_4 \mathbf{c}_{4,3} & \sigma_4 \mathbf{c}_{4,4}
\end{pmatrix};
\]

\[
\sigma_s = \begin{pmatrix}
\sigma_1 & 0 & 0 & 0
\end{pmatrix};
\]

(2.14)

where

\[
\mathbf{c}_{4,2} = \frac{(\rho_{4,2} - \rho_{2,1}\rho_{4,1})}{\sqrt{1 - \rho_{2,1}^2}}
\]
\[ c_{4,4} = \frac{\gamma_{1,1,1} \rho_{4,1} - \frac{(\gamma_{3,2} - \gamma_{2,2} \rho_{3,1})}{\gamma_{3,2} - \gamma_{2,2} \rho_{3,1}} \cdot \frac{(\gamma_{4,2} - \gamma_{2,2} \rho_{4,1})}{\gamma_{3,2} - \gamma_{2,2} \rho_{3,1}}}{\gamma_{3,2} - \gamma_{2,2} \rho_{3,1}} + 1}{\sqrt{\gamma_{3,2} - \gamma_{2,2} \rho_{3,1}^2} + 1} \] 

2.3 Solving the Risk Premium PDE

By Theorem 1, using the market coefficients in the risk premium PDE, we have the following:

\[
\begin{align*}
\frac{2\gamma_{1,1} \sqrt{1 - \rho_{3,1}^2}}{\sqrt{1 - \rho_{3,1}^2} (S^2 + \gamma_{1,1} (t, S, a, r, D)) + \gamma_{1,1} \rho_{1,1} (t, S, a, r, D) + \gamma_{1,1} \rho_{1,2} (t, S, a, r, D) + \gamma_{1,1} \rho_{2,1} (t, S, a, r, D)}
\end{align*}
\]
We try a solution in the form (see [8]):

\[
2\gamma \sigma_3^2 g_\gamma^{(0,0,2,0)}(t, S, a, r, \mathbb{D}) + 2mn \gamma g_\gamma^{(0,0,1,0)}(t, S, a, r, \mathbb{D}) - \\
2na \gamma g_\gamma^{(0,1,0,0)}(t, S, a, r, \mathbb{D}) + 2g_{\gamma}^{(0,0,0,1)}(t, S, a, r, \mathbb{D})
\]

\[
(k(\theta - r) + \sigma_2 \sigma_3 (\gamma \rho_{3,2} - (\gamma - 1)\rho_{2,1}\rho_{3,1}) g_{\gamma}^{(0,0,1,0)}(t, S, a, r, \mathbb{D})) - \\
2r S g_{\gamma}^{(0,1,0,0)}(t, S, a, r, \mathbb{D}) + 2Sa g_{\gamma}^{(0,1,0,0)}(t, S, a, r, \mathbb{D}) + \\
2r S g_{\gamma}^{(0,1,0,0)}(t, S, a, r, \mathbb{D}) - 2\mathbb{D} g_{\gamma}^{(0,1,0,0)}(t, S, a, r, \mathbb{D}) + 2g_{\gamma}^{(1,0,0,0)}(t, S, a, r, \mathbb{D})))
\]

\[
\sigma_1 + 2(\gamma - 1)(r - a) \sqrt{1 - \rho_{2,1}^2} (\sigma_1 \rho_{1,1} g_{\gamma}^{(0,0,0,1)}(t, S, a, r, \mathbb{D}) + \sigma_3 \rho_{3,1} g_{\gamma}^{(0,0,1,0)}(t, S, a, r, \mathbb{D}) + \\
\sigma_2 \rho_{2,1} g_{\gamma}^{(0,0,1,0)}(t, S, a, r, \mathbb{D})) = \frac{(\gamma - 1)}{\gamma} \left( \frac{(r - a)^2 + r \gamma}{\gamma} \right)
\]

(2.16)

With terminal condition

\[
g_{\gamma}(T, S, a, r, \mathbb{D}) = 0
\]

(2.17)

The PDE allows us to drop \( S \) and \( \mathbb{D} \) as following:

\[
\frac{1}{2\gamma \sigma_1^1 \sqrt{1 - \rho_{2,1}^2}} \left( 2(\gamma - 1)(r - a) \sqrt{1 - \rho_{2,1}^2} (\sigma_3 \rho_{3,1} g_{\gamma}^{(0,0,1)}(t, a, r) + \sigma_3 \rho_{2,1} g_{\gamma}^{(0,1,0)}(t, a, r)) + \sigma_1 \sqrt{1 - \rho_{2,1}^2}
\right)
\]

\[
\left( -\gamma \rho_{2,1}^2 g_{\gamma}^{(0,1,0)}(t, a, r)^2 \sigma_2^2 + \rho_{2,1}^2 g_{\gamma}^{(0,0,1)}(t, a, r)^2 \sigma_2^2 + \gamma g_{\gamma}^{(0,0,1)}(t, a, r)^2
\right)
\]

\[
+ \gamma \sigma_2^2 g_{\gamma}^{(0,0,0)}(t, a, r) + 2mn \gamma g_{\gamma}^{(0,0,0)}(t, a, r) - 2na \gamma g_{\gamma}^{(0,1,0)}(t, a, r) + \\
2g_{\gamma}^{(0,0,1)}(t, a, r) k(\theta - r) + \sigma_2 \sigma_3 (\gamma \rho_{3,2} - (\gamma - 1)\rho_{2,1}\rho_{3,1}) g_{\gamma}^{(0,1,0)}(t, a, r)
\]

\[
+ 2g_{\gamma}^{(1,0,0)}(t, a, r)
\right) = \frac{(\gamma - 1)}{\gamma^1} \left( \frac{(r - a)^2 + r \gamma}{\gamma} \right)
\]

(2.18)

With terminal condition

\[
g_{\gamma}(T, a, r) = 0
\]

(2.19)

We try a solution in the form (see [8]):

\[
g(t, S, a, r, \mathbb{D}) = \mathbb{H}_5(t)r^2 + a\mathbb{H}_3(t)r + \mathbb{H}_4(t)r + \mathbb{H}_0(t) + a\mathbb{H}_1(t) + a^2\mathbb{H}_2(t)
\]

(2.20)

The terminal conditions are
\[ H_i(T) = 0 \text{ for } i=1,2,3,4,5 \] (2.21)

By setting coefficients of \( a \) and \( r \) to be zero, we have the following ODE system for \( H_1(t) - H_5(t) \):

\[
\frac{1}{2} \sigma_2^2 H_3(t)^2 - \frac{1}{2} \sigma_2^2 \rho_{2,1} H_3(t)^2 + \frac{\sigma_2^2 \rho_{2,1} H_3(t)}{2 \gamma} + \frac{\sigma_2 \rho_{2,1} H_3(t)}{\gamma} + \frac{2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 \rho_5(t) H_3(t)}{\gamma} - 2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 H_3(t) H_5(t) - 2 \sigma_2 \sigma_3 \rho_{2,1} \rho_5 H_3(t) = 0
\] (2.22)

\[
\frac{2 \sigma_2^2 H_2(t)^2}{\gamma} + \frac{2 \sigma_2 \rho_{2,1} H_2(t)^2}{\gamma} - \frac{2 \sigma_2^2 \rho_{2,1} H_2(t)^2}{\gamma} - 2 n H_2(t) + \frac{2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 \rho_5(t) H_2(t)}{\gamma} + \frac{2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 \rho_5(t) H_2(t)}{\gamma} + \frac{2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 \rho_5(t) H_2(t)}{\gamma} + \frac{2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 \rho_5(t) H_2(t)}{\gamma} + \frac{2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 \rho_5(t) H_2(t)}{\gamma} + \frac{2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 \rho_5(t) H_2(t)}{\gamma} + \frac{2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 \rho_5(t) H_2(t)}{\gamma} + \frac{2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 \rho_5(t) H_2(t)}{\gamma} + \frac{2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 \rho_5(t) H_2(t)}{\gamma} + \frac{2 \sigma_2 \sigma_3 \rho_{2,1} \rho_3 \rho_5(t) H_2(t)}{\gamma} = 0
\] (2.23)

\[
\frac{\gamma - 1}{2 \gamma} + \frac{1}{\sqrt{1 - \rho^2_{2,1}}} (2 \sqrt{1 - \rho^2_{2,1}} (2 \sigma_2 \rho_{2,1} H_2(t) + \sigma_3 \rho_3 H_3(t)) - 2 (\gamma - 1) \sqrt{1 - \rho^2_{2,1}} (\sigma_2 \rho_{2,1} H_3(t) + 2 \sigma_3 \rho_3 H_5(t)) + \sigma_1 \sqrt{1 - \rho^2_{2,1}} (-4 \gamma \rho^2_{2,1} H_2(t) H_3(t)) \sigma^2_2 + 4 \gamma^2 \rho^2_{2,1} H_2(t) H_3(t) \sigma^2_2 + 4 \gamma H_2(t) H_3(t) \sigma^2_2 + 8 \sigma_3 (\gamma \rho_3 - (\gamma - 1) \rho_2 \rho_3) H_3(t) - k \gamma H_3(t) H_3(t) - 4 \sigma_3^2 (\gamma - (\gamma - 1) \rho^2_{3,1}) H_3(t) H_3(t) + 2 n H_3(t)) = 0
\] (2.24)

\[
\frac{1}{\gamma} (\sigma_1 (2 \gamma - (\gamma - 1) \rho^2_{2,1}) H_4(t) \sigma^2_2 + \sigma_3 (\gamma \rho_3 - (\gamma - 1) \rho_2 \rho_3) H_4(t) \sigma^2_2 + \gamma H_3(t) H_4(t) \sigma^2_2 + 2 (\gamma - (\gamma - 1) \rho^2_{2,1}) H_2(t) \sigma^2_2 + \sigma_3 (\gamma \rho_3 - (\gamma - 1) \rho_2 \rho_3) H_3(t) \sigma_2 - n \gamma) H_1(t) + k \gamma \theta H_3(t) + 2 H_2(t) (m n \gamma + \sigma_2 \rho_{2,1} \rho_3 \rho_5(t) H_4(t) + \gamma H_4(t)) - (\gamma - 1) (\sigma_2 \rho_{2,1} H_1(t) + \sigma_3 \rho_3 H_4(t))) = 0
\] (2.25)
\(( -\gamma^2 + 2k\theta H_5(t)\gamma - 2\sigma_2\sigma_3\rho_2,1\rho_3,1H_1(t)H_5(t)\gamma + 2\sigma_2\sigma_3\rho_3,2H_1(t)H_5(t)\gamma + H'_5(t)\gamma + \gamma + \\
\(( \gamma - (\gamma - 1)\rho_2^2,1)H_1(t)\sigma_2^2 + \sigma_3(\gamma\rho_3,2 - (\gamma - 1)\rho_2,1\rho_3,1)H_4(t)\sigma_2 + mn\gamma)H_3(t) + \\
(2\sigma_3^2(\gamma - (\gamma - 1)\rho_2^2,1)H_5(t) - k\gamma)H_4(t) + 2\sigma_2\sigma_3\rho_2,1\rho_3,1H_1(t)H_5(t)\) \\
\sigma_1 + (\gamma - 1)(\sigma_2\rho_2,1H_1(t) + \sigma_3\rho_3,1H_4(t))) = 0 \tag{2.26}
\)

This ODE system (2.22-2.26) has five unknown functions and five equations. Given appropriate condition, the risk premium PDE can be solved. For example, it can be solved numerically.

### 2.4 Solving the Pricing PDE

We shall assume that the risk premium was solved. The next step is solving the pricing PDE. By **Theorem 2**, we have the following pricing PDE. \(V(t, S, a, D)\) is the price of the futures contract. It is a function of time \(t\), the underlying spot \(S\), the appreciation rate \(a\) and the dividend yield rate \(D\). Notice that the right hand side is \(-rV(t, S, a, D)\), which is the implicit “dividend” introduced by the resettlement feature of futures contracts[2][3]:

\[-rV(t, S, a, r, D) + \\
\sigma_2\sqrt{1 - \rho_2^2} \left( \sigma_4c_{4,2} V^{(0,0,0,0,1)}(t, S, a, r, D) + \sigma_2\sqrt{1 - \rho_2^2} V^{(0,0,1,0,0)}(t, S, a, r, D) - \\
\frac{\sigma_3(\rho_2,1\rho_3,1 - \rho_2^2,1)\sqrt{1 - \rho_2^2}}{\sqrt{1 - \rho_2^2}}(H_1(t) + 2aH_2(t) + rH_3(t)) + \\
\left( \sigma_3 \frac{-\rho_2^2,1 + 2\rho_3,1\rho_2,1 - \rho_3,1 - \rho_2^2,1 + 1}{1 - \rho_2^2,1} \right) \left( \sigma_4c_{4,3} V^{(0,0,0,1)}(t, S, a, r, D) + \\
\sigma_3 \frac{-\rho_2^2,1 + 2\rho_3,1\rho_2,1 - \rho_3,1 - \rho_2^2,1 + 1}{1 - \rho_2^2,1} V^{(0,0,0,1,0)}(t, S, a, r, D) - \\
\frac{1}{\sqrt{1 - \rho_2^2}} \left( \sigma_3(\rho_2,1\rho_3,1 - \rho_3,2) \left( \sigma_4c_{4,2} V^{(0,0,0,0,1)}(t, S, a, r, D) + \sigma_2\sqrt{1 - \rho_2^2} V^{(0,0,0,0,0,1,0)}(t, S, a, r, D) \right) - \\
\frac{1}{\sqrt{1 - \rho_2^2}} (\sigma_3(\rho_2,1\rho_3,1 - \rho_3,2) \left( \sigma_4c_{4,2} V^{(0,0,0,0,0,1,0)}(t, S, a, r, D) - \frac{\sigma_3(\rho_2,1\rho_3,1 - \rho_3,2)\sqrt{1 - \rho_2^2}}{\sqrt{1 - \rho_2^2}} \right) \right) \right) \\
(aH_3(t) + H_4(t) + 2rH_5(t)) + \left( p(q - D) - \frac{(a - r)\sigma_4\rho_4,1}{\sigma_1} \right) V^{(0,0,0,0,1)}(t, S, a, r, D) + \\
\left( k(\theta - r) - \frac{(a - r)\sigma_3\rho_3,1}{\sigma_1} \right) V^{(0,0,0,0,1,0)}(t, S, a, r, D) + \\
\left( n(m - a) - \frac{(a - r)\sigma_2\rho_2,1}{\sigma_1} \right) V^{(0,0,1,0,0)}(t, S, a, r, D) + \frac{n(m - a) - \sigma_2\rho_2,1}{\sigma_1} V^{(0,0,0,0,0,0)}(t, S, a, r, D) + \frac{n(m - a) - \sigma_2\rho_2,1}{\sigma_1} V^{(0,0,1,0,0,0)}(t, S, a, r, D) + \]

\]
\[
(S(a - D) - S(a - r))V^{(0,1,0,0,0)}(t, S, a, r, \hat{D}) + \\
\frac{1}{2} \left( \sigma^2 V^{(0,0,0,2)}(t, S, a, r, \hat{D})c_{4,2}^2 + \sigma_4 c_{4,2} \left( \sigma_4 c_{4,2} V^{(0,0,0,2)}(t, S, a, r, \hat{D}) + \sigma_2 \sqrt{1 - \rho_{21}^2} \right) \right) - \\
\frac{1}{1 - \rho_{21}^2} \sigma V^{(0,0,0,2)}(t, S, a, r, \hat{D}) - \frac{\sigma_3 \rho_2 \rho_3 \rho_1}{1 - \rho_{21}^2} \left( \sigma_4 c_{4,2} V^{(0,0,0,1)}(t, S, a, r, \hat{D}) + \sigma_2 \sqrt{1 - \rho_{21}^2} V^{(0,0,2,0,0)}(t, S, a, r, \hat{D}) - \\
\frac{\sigma_3 \rho_2 \rho_3 \rho_1 \rho_2 - \rho_{21}^2}{1 - \rho_{21}^2} \right) + \sigma_4 c_{4,3} \left( \sigma_4 c_{4,3} V^{(0,0,0,2)}(t, S, a, r, \hat{D}) \right) \right) + \\
\sigma_3 \sqrt{\frac{-\rho_{21}^2 + 2 \rho_3 \rho_1 \rho_2 - \rho_{21}^2 - \rho_{21}^2 + 1}{1 - \rho_{21}^2} V^{(0,0,0,1)}(t, S, a, r, \hat{D})} \right) + \\
\sigma_4 \left( \sigma_4 \rho_{11} V^{(0,0,0,2)}(t, S, a, r, \hat{D}) + \sigma_3 \rho_{31} V^{(0,0,0,1)}(t, S, a, r, \hat{D}) + \\
\sigma_2 \rho_{21} V^{(0,0,1,0,1)}(t, S, a, r, \hat{D}) + S \sigma_1 V^{(0,1,0,0,0)}(t, S, a, r, \hat{D}) \right) + \\
\sigma_3 \rho_{31} \left( \sigma_4 \rho_{11} V^{(0,0,0,1)}(t, S, a, r, \hat{D}) + \sigma_3 \rho_{31} V^{(0,0,0,2)}(t, S, a, r, \hat{D}) + \\
\sigma_2 \rho_{21} V^{(0,0,1,0,0)}(t, S, a, r, \hat{D}) + S \sigma_1 V^{(0,1,1,0,0)}(t, S, a, r, \hat{D}) \right) + \\
\sigma_2 \rho_{21} \left( \sigma_4 \rho_{11} V^{(0,0,1,0,1)}(t, S, a, r, \hat{D}) + \sigma_3 \rho_{31} V^{(0,0,1,0,0)}(t, S, a, r, \hat{D}) + \\
\sigma_2 \rho_{21} V^{(0,0,2,0,0)}(t, S, a, r, \hat{D}) + S \sigma_1 V^{(0,1,0,0,0)}(t, S, a, r, \hat{D}) \right) + \\
S \sigma_1 \left( \sigma_4 \rho_{11} V^{(0,1,1,0,1)}(t, S, a, r, \hat{D}) + \sigma_3 \rho_{31} V^{(0,1,1,0,0)}(t, S, a, r, \hat{D}) + \sigma_2 \rho_{21} \right) \right) \right) + \\
V^{(1,0,0,0,0)}(t, S, a, r, \hat{D}) = -r V(t, S, a, r, \hat{D}) (2.27)
\]

With terminal condition \( V(T, S, a, r, \hat{D}) = S \). Try solution in form of
\[
V(t, S, a, r, \hat{D}) = S g_1(t) e^{r g_2(t) + a g_3(t) - D g_4(t)}
\]

Plugging it into equation 2.27, then we have:
\[
\frac{1}{2} e^{r g_2(t) + a g_3(t) - D g_4(t)} S \left( \sigma^2 c_{4,2}^2 g_1(t) g_4(t)^2 + 2 \left( p(q - D) + \frac{(r - a) \sigma_4 \rho_{11}}{\sigma_1} \right) g_1(t) g_4(t) + \\
\sigma_4 c_{4,3} g_1(t) \left( \frac{\rho_{21}^2 - 2 \rho_3 \rho_1 \rho_2 + \rho_{21}^2 + \rho_{21}^2 - 1}{\rho_{21}^2} \right) g_2(t) + \sigma_4 c_{4,3} g_4(t) \right) g_4(t) + \\
\right)
\]

(2.28)
\[
\sigma_4 \rho_4 g_1(t) \left( \sigma_1 + \sigma_3 \rho_3 g_2(t) + \sigma_2 \rho_2 g_3(t) + \sigma_4 \rho_4 g_4(t) \right) g_4(t) - \\
+ 2(r - \mathbb{D}) g_1(t) + 2 \left( k(\theta - r) + \frac{(r-a)\sigma_3 \rho_3}{\sigma_1} \right) g_1(t) g_2(t) + \\
2 \left( n(m - a) + \frac{(r-a)\sigma_2 \rho_2}{\sigma_1} \right) g_1(t) g_3(t) + \\
\sigma_4 c_{4,2} g_1(t) (\sigma_3 \rho_3 - \rho_3,1) g_2(t) - \sigma_2 \rho_2 g_3(t) + \sigma_4 c_{4,2} \sqrt{1 - \rho_2^2} g_4(t)
\]

\[
+ \sqrt{\frac{\rho_2^2 - 2\rho_4 \rho_3 \rho_2 + \rho_2^2 \rho_3^2 + \rho_2^2 - 1}{\rho_2^2 - 1}} \sigma_3 g_1(t) g_2(t)
\]

\[
\left( \sqrt{\frac{\rho_2^2 - 2\rho_4 \rho_3 \rho_2 + \rho_2^2 \rho_3^2 + \rho_2^2 - 1}{\rho_2^2 - 1}} \sigma_3 g_2(t) + \sigma_4 c_{4,3} g_4(t) \right) + \\
\sigma_1 g_1(t) \left( \sigma_3 \rho_3 g_2(t) + \sigma_2 \rho_2 g_3(t) + \sigma_4 \rho_4 g_4(t) \right) + \\
\sigma_3 \rho_3 g_1(t) g_2(t) \left( \sigma_1 + \sigma_3 \rho_3 g_2(t) + \sigma_2 \rho_2 g_3(t) + \sigma_4 \rho_4 g_4(t) \right) + \\
\sigma_2 \rho_2 g_1(t) g_3(t) \left( \sigma_1 + \sigma_3 \rho_3 g_2(t) + \sigma_2 \rho_2 g_3(t) + \sigma_4 \rho_4 g_4(t) \right) - \\
\sigma_2 g_1(t) g_3(t) \left( \sigma_3 \left( \rho_2,1 - \rho_3,2 \right) g_2(t) + \left( \rho_2^2 - 1 \right) \sigma_2 g_3(t) - \sigma_4 c_{4,2} \sqrt{1 - \rho_2^2} g_4(t) \right) - \\
\frac{1}{\rho_2^2 - 1} \sigma_3 \left( \rho_2,1 \rho_3,1 - \rho_3,2 \right) g_1(t) g_2(t)
\]

\[
\left( \sigma_3 \left( \rho_2,1 \rho_3,1 - \rho_3,2 \right) g_2(t) + \left( \rho_2^2 - 1 \right) \sigma_2 g_3(t) - \sigma_4 c_{4,2} \sqrt{1 - \rho_2^2} g_4(t) \right) - \\
2 \sigma_2 g_1(t) \left( \sigma_3 \left( \rho_2,1 \rho_3,1 - \rho_3,2 \right) g_2(t) + \left( \rho_2^2 - 1 \right) \sigma_2 g_3(t) - \sigma_4 c_{4,2} \sqrt{1 - \rho_2^2} g_4(t) \right)
\]

\[
\left( H_1(t) + 2aH_2(t) + rH_3(t) \right) - \frac{1}{\rho_2^2 - 1} 2 \sigma_3 g_1(t) \left( \left( \rho_2^2 - 1 \right) \left( \rho_3^2 - 1 \right) \sigma_3 g_2(t) + \\
\left( \rho_2^2 - 1 \right) \sigma_2 \left( \rho_2,1 \rho_3,1 - \rho_3,2 \right) g_3(t) + \left( \sigma_4 c_{4,2} \sqrt{1 - \rho_2^2} \rho_3,2 \rho_2,1 \rho_3,1 \right) - \\
c_{4,3} \left( \rho_2^2 - 1 \right) \sqrt{\frac{\rho_2^2 - 2\rho_4 \rho_3 \rho_2 + \rho_2^2 \rho_3^2 + \rho_2^2 - 1}{\rho_2^2 - 1}} \sigma_4 c_{4,3} \right)
\]

\[
\left( aH_3(t) + H_4(t) + 2rH_5(t) \right) + 2g_1(t) + 2g_1(t) \left( rg_2'(t) + ag_3'(t) + Dg_4'(t) \right) = 0
\]

(2.29)

With terminal condition

\[
g_1(T) = 1, g_2(T) = 0, g_3(T) = 0, g_4(T) = 0
\]

(2.30)

Setting all coefficients of \( r, a \) and \( \mathbb{D} \) and constant terms to be zero, we have the following system ODE's:

\[
\left( \sigma_1 \left( \rho_2^2 \left( g_2(t) \left( 2\rho_2^2 - \rho_3^2 \rho_2,1 \right) H_5(t) - \sigma_2 \sigma_3 \rho_3,2 H_3(t) + k \right) - \sigma_4 c_{4,2} \sqrt{1 - \rho_2^2} \right) \right) g_4(t) H_5(t) - 2 \sigma_3 \sigma_4 c_{4,3} \left( \frac{1}{\rho_2^2 - 1} \left( \rho_2^2 - 2\rho_3 \rho_2 \rho_2,1 + \rho_2^2 + \rho_2^2 - 1 \right) \right) g_4(t) H_5(t) - \\
2 \sigma_3 \sigma_2 \rho_3,2 g_3(t) H_5(t) - 2 \sigma_2^2 g_3(t) H_3(t) - 1 + \\
\]
\[ g_2(t) \left( -2\sigma_3^2 (\rho_{3,1}^2 - 1) H_5(t) + \sigma_2 \sigma_3 \rho_{3,2} H_3(t) - k \right) - \sigma_3 \rho_{3,1} \rho_{2,1} \]
\[ = \left( 2\sigma_4 c_{1,2} \sqrt{1 - \rho_{2,1}^2 g_4(t) H_5(t)} + \sigma_2 (g_2(t) H_3(t) + 2g_3(t) H_5(t)) \right) + \]
\[ \sigma_2 \sigma_4 c_{1,2} \sqrt{1 - \rho_{2,1}^2 g_4(t) H_3(t)} + 2\sigma_3 \sigma_4 c_{4,3} \sqrt{\frac{\rho_{2,1}^2 - 2\rho_{3,1} \rho_{3,2} \rho_{2,1} + \rho_{3,1}^2 + \rho_{3,2}^2 - 1}{\rho_{2,1}^2 - 1}} g_4(t) H_5(t) + \]
\[ 2\sigma_3 \sigma_4 c_{4,2} \sqrt{1 - \rho_{2,1}^2 \rho_{3,2} g_4(t) H_5(t)} + \sigma_2 \rho_{2,1}^2 g_3(t) H_3(t) + \]
\[ \sigma_2 \sigma_3 \rho_{3,1} \rho_{3,1}^2 (g_2(t) H_3(t) + 2g_3(t) H_5(t)) + \]
\[ 2\sigma_2 \sigma_3 \rho_{3,2} g_3(t) H_5(t) + (g_2') (t) + \sigma_2^2 g_3(t) H_3(t) + 1 \]
\[ = (\rho_{2,1}^2 - 1) (\sigma_3 \rho_{3,1} g_2(t) + \sigma_2 \rho_{2,1} g_3(t) + \sigma_4 \rho_{4,1} g_4(t)) = (g_2') (t) \]
\[ pg_4(t) - g_4'(t) + 1 = 0 \]  \hspace{1cm} \text{(2.34)}

Solving the above ODE system with terminal condition (2.30) will yield a closed-form solution of the futures contract pricing formula. It has the following form:

\[ V(t, S, a, r, D) := Sg_1(t)e^{rg_2(t)+ag_3(t)+Dg_4(t)} \]  \hspace{1cm} \text{(2.35)}

which can also be rewritten as following:

\[ \log(V_S) := \log g_1(t) + g_2(t)r + g_3(t)a + g_4(t)D \]  \hspace{1cm} \text{(2.36)}

### 2.5 The Relation between the Proposed Model and Cost of Carry Model

In the proposed model, interest rates are stochastic, thus stock index futures and forward prices need not be equal\cite{10}\cite{3}. The futures prices are given by equation 2.36, while the corresponding forward price is easily determined by setting all market coefficients other than \( \sigma_1 \) to be zero. Then we have the following PDE:

\[ S^2\sigma_1^2V^{(0,2,0,0,0)}(t, S, a, r, D) + 2S(r - D)V^{(1,0,0,0,0)}(t, S, a, r, D) + 2V^{(1,0,0,0,0)}(t, S, a, r, D) = 0 \]  \hspace{1cm} \text{(2.37)}

with terminal condition \( V(T, S, a, r, D) = S \). It is solved by

\[ V = Se^{(r-D)(T-t)} \]  \hspace{1cm} \text{(2.38)}

which is the cost of carry model.
2.6 The Relation between Utility-based Framework and General Equilibrium Framework

In Hemler’s paper, they are assuming stochastic volatility and interest rate. The market coefficients are defined as following:

\[ a_s = \{v+r\} \]
\[ b = \{S(r+v-\rho), \alpha-\beta r-\gamma v, \delta-(\beta-\gamma)v\} \]
\[ c = \begin{pmatrix} S\sqrt{v} & 0 & 0 \\ 0 & \eta\sqrt{r+v} & -\xi\sqrt{v} \\ 0 & 0 & \xi\sqrt{v} \end{pmatrix} \]
\[ \sigma_s = \begin{pmatrix} \sqrt{v} & 0 & 0 \end{pmatrix} \]

Here, \( v \) is volatility of underlying. We should notice that they are assuming volatility and interest rate are not correlated with underlying. Another thing need to be mentioned is that the utility function they are using is logarithmic utility, which is a special case in the utility function we are using (\( \gamma = 1 \)).

\[
(g_{\gamma})^{(0,0,1,0)}(t, S, r, v) \left( (\gamma^2 \eta(r + v) + v \xi^2) V^{(0,0,1,0)}(t, S, r, v) - v \xi^2 V^{(0,0,0,1)}(t, S, r, v) \right) \\
+ v \xi^2 \left( V^{(0,0,0,1)}(t, S, r, v) - V^{(0,0,1,0)}(t, S, r, v) \right) (g_{\gamma})^{(0,0,0,1)}(t, S, r, v) \\
+ \frac{1}{2} \left( v S^2 v^{(0,2,0,0)}(t, S, r, v) + (\gamma^2 \eta(r + v) + v \xi^2) V^{(0,0,2,0)}(t, S, r, v) + V \xi^2 V^{(0,0,0,2)}(t, S, r, v) \right) \\
\left( -2v \xi^2 V^{(0,1,1,1)}(t, S, r, v) \right) + (-r \beta - \gamma \alpha) V^{(0,0,1,0)}(t, S, r, v) + (v(\gamma - \beta) + \delta) V^{(0,0,0,1)}(t, S, r, v) + \\
S(r - \rho) V^{(0,1,0,0)}(t, S, r, v) + V^{(1,0,0,0)}(t, S, r, v) = 0
\] (2.40)

Compare this PDE with the pricing PDE in Hemler’s model, we notice that Hemler’s model with logarithmic utility is a special case that \( g(t, S, r, v) = 0 \), which means risk premium terms are not included.

2.7 Coefficient of Dividend Rate

In cost of carry model and Hemler’s general equilibrium model, the coefficient of \( \mathbb{D} \) is


\[ u^{Hemler}(t) = -(T - t) \]

(2.41)

while in our model, we have the following:

\[ u(t) = -\frac{e^{-\rho T}(e^{\rho(T-\tau)} + e^{\rho T})}{\rho} \]

(2.42)

in figure 1 we have plots of \( u(t) \) and \( u^{Hemler}(t) \). We would expect our model perform different from the cost of carry model or Hemler’s model when \( T - t \) is near 0.
Figure 1. Plot of coefficients of dividend of cost of carry model/Hemler’s model and our model. Red line is plot of \( u^{Hemler}(t) = -(T - t) \). Others are plots of \( u(t) = -e^{-pt}(e^{ptp(T - t)} + e^{pT}) \) with different p values.
Chapter 3

Empirical Results and Conclusion

3.1 Regression Setup

To test our model, we denote $L_\tau$ to be $\log\left(\frac{V_t}{S_t}\right)$, where $V_t$ is the spot price of the futures contract, $S_t$ is the spot price of the underlying. From the above results, let $\tau=T-t$ ($T$ is the expiration date of the futures contract), then we have:

$$L_\tau = \alpha_\tau + \beta_\tau r + \lambda_\tau a + \kappa_\tau D + \epsilon_\tau$$ (3.1)

where the regression coefficients $\alpha_\tau, \beta_\tau, \lambda_\tau, \kappa_\tau$ are related to $g_1, g_2, g_3$ and $g_4$. Residual term $\epsilon_\tau$ is included to reflect the possible measure error in the $L_\tau$ terms. If we assume the following holds at any time $t>0$

Assumption: all market $(n, m, k, \theta, p, q, \sigma_i, \rho_{i,j})$ coefficients are non-zero constants.

Then, we know that $\alpha_\tau, \beta_\tau, \lambda_\tau, \kappa_\tau$ are dependent on $\tau$ only. So for all S&P 500 futures contracts, if $\tau$ is the same, they share the same $\alpha_\tau, \beta_\tau, \lambda_\tau, \kappa_\tau$. So We have:

$$L_{\tau,i} = \sum_{i=1}^{6} D_i \alpha_{\tau,i} + \sum_{i=1}^{6} D_i \beta_{\tau,i} r + \sum_{i=1}^{6} D_i \lambda_{\tau,i} a + \sum_{i=1}^{6} D_i \kappa_{\tau,i} D + \sum_{i=1}^{6} D_i \epsilon_{\tau,i}$$ (3.2)

$i$ is the number of months left to maturity, $D_i$ is a dummy variable. $D_i=1$ if and only if the dependent variable corresponds to an i-month to maturity contract, 0 otherwise.

For Hemler’s general equilibrium model, denote that $L'_{\tau}$ to be $\log\left(\frac{V^D_{e\tau}}{S_t}\right)$, then
\[ L'_\tau = \alpha'_\tau + \beta'_\tau r_t + \chi'_\tau V + \epsilon'_\tau \]  

(3.3)

where \( V \) is the volatility of the underlying. The regression coefficients \( \alpha'_\tau, \beta'_\tau, \chi'_\tau \) are given in Hemler’s paper [4]. Then we have the same set up as equation 3.2:

\[ L'_{\tau,i} = \sum_{i=1}^{6} D_i \alpha'_{\tau,i} + \sum_{i=1}^{6} D_i \beta'_{\tau,i} r_t + \sum_{i=1}^{6} D_i \chi'_{\tau,i} V + \sum_{i=1}^{6} D_i \epsilon'_{\tau,i} \]  

(3.4)

where \( i \) is months left to expiration. \( D_i = 1 \) if and only if the dependent variable corresponds to an \( i \)-month to maturity contract, 0 otherwise.

For the cost of carry model, the setup is identical to the general equilibrium model except there are restrictions on the coefficients:

\[
\begin{align*}
\alpha'_{\tau,i} & = 0 \\
\beta'_{\tau,i} & = \tau \\
\chi'_{\tau,i} & = 0 
\end{align*}
\]  

(3.5)

where \( i = 1, 2, 3, 4, 5, 6 \).

This setup allows us to perform statistical analysis to compare these three models. We will only look at Hemler’s general equilibrium model and our model.

### 3.2 Data

The futures prices’ quotations are from Normans History Data, S&P500 index quotations are from the Mathematica financial data server. S&P 500 E-mini futures contracts have been trading since 1997. However the trading volume was very slim at the beginning. As a result, we use data from 2001 to 2007 to estimate regression coefficients. Dividend data is from http://www.econ.yale.edu/ shiller/data.htm. We take most recently reported dividends and divide by the index prices. However it may be a biased estimation since sometimes the recently reported dividends may lag by a quarter or two.
To measure the volatility of the S&P 500 index, we use the procedure of French, Schwes and Stambaugh (1987). That is, we estimate the monthly volatility as the sum of the squared daily returns plus twice the sum of the product of adjacent returns over the following 30 days. The volatility estimates are annualized by multiplying them by 12. For the risk-free interest rate, we use the 6-month Treasury-bill discount yield quotations as published on www.ustreas.gov.

Moreover, we use monthly observations because one month is the smallest time interval for which we can obtain relatively exact dividend yield data. All observations correspond to the last business day for each month. All futures contracts’ prices are for the nearby contract, which implies that the corresponding times to expiration are approximately one to six months. Typically trading period of a S&P 500 E-mini contract is around 120 days, there are approximately 20 trading days each month. We treat $\tau$, which is time to expiration date, as following: we take the last day of the first trading month as an observation in our data set, and we say it is roughly 6-month to maturity. The last day of the second trading month is another observation in our data set, we say it is roughly 5-month to maturity etc. This not very accurate since there may be several days difference. It would be a good assumption until there are only 1 or 2 months left. We would expect inaccurate regression results near expiration since the first order derivative of risk premium function increase significantly toward expiration, a few days difference may lead to significant error terms.

### 3.3 Statistical Analysis

Observations are grouped according to time to expiration. Linear regression analysis is performed using R version 2.11.0. Table 3.1 and 3.2 shows summary of the linear regression results of the general Ito SDE model. It shows that interest rate is the most significant regressor for all $\tau$’s. Significance of dividend vanishes toward maturity, while appreciation rate is only significant around 4-month to maturity. Figure 3.1 shows a plot of residues of the regression. Residues are uniformly distributed around zero untill close to maturity.
Table 3.3-3.5 shows summary statistics for the pricing errors of the general equilibrium and general Ito SDE models for S&P 500 stock index futures contracts’ prices. Root of mean squared errors shows that general Ito SDE model performs better when the contracts are far away from maturity, while general equilibrium model does slightly better around maturity. The residues of general equilibrium model shows significant correlation with appreciation rate and dividend yield rate of the underlying, while the residues of the general Ito SDE model shows significant correlation with volatility.
Table 3.1: Summary 1 of Regression Analysis of our model (Significant codes: \(0^{***} 0.001^{**} 0.01^* 0.05 \cdot 0.1 \ 1\))

<p>| Coefficients | months to maturity | Estimate  | Std. Error | t value | Pr(&gt;|t|) |
|--------------|--------------------|-----------|------------|---------|---------|
| interception | 1                  | 2.650e-04 | 4.391e-04  | 0.604   | 0.5533  |
|              | 2                  | -0.0002900| 0.0009335  | -0.311  | 0.7595  |
|              | 3                  | -0.0003234| 0.0009403  | -0.344  | 0.73465 |
|              | 4                  | 0.0004983 | 0.0008386  | 0.594   | 0.559   |
|              | 5                  | 0.0017213 | 0.0009704  | 1.774   | 0.0921  |
|              | 6                  | 0.0036364 | 0.0011551  | 3.148   | 0.00529 ** |
| interest     | 1                  | 5.080e-02 | 2.258e-03  | 22.494  | 3.72e-15 *** |
|              | 2                  | 0.1287791 | 0.0049812  | 25.853  | 2.87e-16 *** |
|              | 3                  | 0.2260032 | 0.0058149  | 38.866  | &lt; 2e-16 *** |
|              | 4                  | 0.3097560 | 0.0054515  | 56.820  | &lt; 2e-16 *** |
|              | 5                  | 0.4006332 | 0.0062558  | 64.042  | &lt; 2e-16 *** |
|              | 6                  | 0.5039948 | 0.0084369  | 59.737  | &lt; 2e-16 *** |
| appreciation | 1                  | 8.598e-05 | 1.206e-03  | 0.071   | 0.9439  |
|              | 2                  | 0.0018580 | 0.0017526  | 1.060   | 0.3024  |
|              | 3                  | -0.0026103| 0.0019881  | -1.313  | 0.20483 |
|              | 4                  | -0.0041941| 0.0024960  | -1.680  | 0.109   |
|              | 5                  | -0.0010874| 0.0022699  | -0.479  | 0.6374  |
|              | 6                  | 0.0006408 | 0.0029246  | 0.219   | 0.82891 |
| dividend     | 1                  | -6.922e-02| 2.684e-02  | -2.579  | 0.0184 * |
|              | 2                  | -0.1305290| 0.0580863  | -2.247  | 0.0367 * |
|              | 3                  | -0.2038071| 0.0592674  | -3.439  | 0.00275 ** |
|              | 4                  | -0.3098980| 0.0525179  | -5.901  | 1.11e-05 *** |
|              | 5                  | -0.5012067| 0.0601784  | -8.329  | 9.17e-08 *** |
|              | 6                  | -0.6919059| 0.0716584  | -9.656  | 9.23e-09 *** |</p>
<table>
<thead>
<tr>
<th>Months to Maturity</th>
<th>Degrees of freedom</th>
<th>Residual standard error</th>
<th>Adjusted R-squared</th>
<th>F-statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19</td>
<td>0.0001481</td>
<td>0.9624</td>
<td>188.9</td>
<td>2.552e-14</td>
</tr>
<tr>
<td>2</td>
<td>19</td>
<td>0.0003193</td>
<td>0.9726</td>
<td>260.9</td>
<td>1.300e-15</td>
</tr>
<tr>
<td>3</td>
<td>19</td>
<td>0.0003524</td>
<td>0.9883</td>
<td>618.3</td>
<td>&lt; 2.2e-16</td>
</tr>
<tr>
<td>4</td>
<td>19</td>
<td>0.0003413</td>
<td>0.9943</td>
<td>1275</td>
<td>&lt; 2.2e-16</td>
</tr>
<tr>
<td>5</td>
<td>19</td>
<td>0.0004079</td>
<td>0.9949</td>
<td>1420</td>
<td>&lt; 2.2e-16</td>
</tr>
<tr>
<td>6</td>
<td>19</td>
<td>0.0005313</td>
<td>0.9942</td>
<td>1260</td>
<td>&lt; 2.2e-16</td>
</tr>
</tbody>
</table>
3.4 Conclusion

In this paper, we derived a closed-form pricing formula under utility-based neutral pricing framework with stochastic interest rate, dividend yield rate and appreciation rate. Our model shows some different properties comparing to the cost of carry model and Hemler’s general equilibrium mode. One of the most important difference is the coefficient of dividend

\[ V(t, S, r, V) = S f_1(t) e^{-D \tau} e^{f_2(t) r + f_3(t) V} \]  

while our model give the following (solution of \( g_4(t) \) in equation 2.34):

\[ V(t, S, \alpha, r, D) = S g_1(t) e^{-\frac{e^{-p(T - \tau)} + e^{p(T)}}{p} D} e^{g_2(t) r + g_3(t) \alpha} \]

cost of carry model and Hemler’s model suggest coefficient of dividend to be linear over time. In figure 1 we can see that if \( p \) is small enough (slightly bigger than 0), then there is not much difference between these two models. However, if \( p \) is significant bigger than 0, then the difference in the coefficients between cost of carry or Hemler’s and our model are significant, especially when \( T - t \) (or say \( \tau \)) is close to \( T \). Thus we are expecting our model to perform better then Hemler’s model while \( t \) is small, or equivalent \( \tau \) is close to \( T \). The regression analysis favors our model over Hemler’s model when \( \tau \) is close to \( T \).

Another thing to mention is that the risk premium function \( g_\gamma(t) \) change rapidly around maturity, thus slightly difference in \( \tau \) may lead to significant difference in risk premium, which may lead error in pricing. This is also the major error source for regression analysis.

In a word, if one choose to use our model for futures contracts pricing, then dividend rate plays differently from the classic models. Another thing to mention is that the appreciation rate of the underlying may play a roll at a certain time during trading period, but appreciation rate is not necessarily a significant factor all the time.
Table 3.3: Summary Statistics for the Pricing Errors of Hemler’s General Equilibrium model, Cost of Carry model and our model for the Nearby S&P 500 Stock Index Futures Price during 2001 to 2007 period (In sample test)

<table>
<thead>
<tr>
<th>Statistic</th>
<th>τ</th>
<th>Hemler’s</th>
<th>Our’s</th>
<th>Cost of Carry</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE</td>
<td>1</td>
<td>0.0001334010</td>
<td>0.0001346395</td>
<td>0.000278327</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0002792087</td>
<td>0.0002901738</td>
<td>0.000608197</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.0002741506</td>
<td>0.0003203327i</td>
<td>0.000344238</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.0003106786</td>
<td>0.000310168</td>
<td>0.000536762</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.0003993633</td>
<td>0.0003707002</td>
<td>0.000415553</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.0005853394</td>
<td>0.0004828952</td>
<td>0.000973419</td>
</tr>
<tr>
<td>Mean</td>
<td>1</td>
<td>8.983613e-21</td>
<td>5.156428e-21</td>
<td>0.000163373</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6.165157e-21</td>
<td>1.031401e-20mile</td>
<td>0.000471612</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4.418151e-21</td>
<td>3.103869e-21</td>
<td>-0.000327291</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>1.443581e-20</td>
<td>8.982462e-21</td>
<td>-0.000420643</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>3.80014e-20</td>
<td>1.594171e-20</td>
<td>-0.000730986</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>1.162829e-20</td>
<td>1.178020e-20</td>
<td>-0.000732205</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>1</td>
<td>0.0001363992</td>
<td>0.0001376655</td>
<td>0.000230398</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0002854838</td>
<td>0.0002966954</td>
<td>0.000392671</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.0002803120</td>
<td>0.0003275321</td>
<td>0.00035038</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.000317661</td>
<td>0.0003171389</td>
<td>0.000340919</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.0004083388</td>
<td>0.0003790316</td>
<td>0.000418267</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.0005984948</td>
<td>0.0004937481</td>
<td>0.000655836</td>
</tr>
</tbody>
</table>
Table 3.4: Summary Statistics for the Pricing Errors of Hemler’s General Equilibrium model and our Model for the Nearby S&P 500 Stock Index Futures Price during 2001 to 2007 period (Continued 1)

<table>
<thead>
<tr>
<th>Statistic</th>
<th>τ</th>
<th>Hemler’s Model</th>
<th>Our’s Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correlation with r</td>
<td>1</td>
<td>-1.256763e-16</td>
<td>-1.209564e-16</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-2.490972e-17</td>
<td>-9.038907e-18</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-3.093421e-17</td>
<td>-1.182969e-16</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-1.354594e-16</td>
<td>-6.792836e-17</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>-6.256334e-17</td>
<td>-1.011249e-16</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>-1.486070e-16</td>
<td>-1.177240e-16</td>
</tr>
<tr>
<td>Correlation with volatility</td>
<td>1</td>
<td>1.812078e-17</td>
<td>0.1434773</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6.084157e-18</td>
<td>0.4289663</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-3.192473e-17</td>
<td>0.3948113</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-7.537416e-17</td>
<td>0.3160925</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>3.272655e-18</td>
<td>0.2741166</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>-4.345541e-17</td>
<td>0.08567282</td>
</tr>
<tr>
<td>Correlation with D</td>
<td>1</td>
<td>-0.06310871</td>
<td>-7.835632e-16</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.01883481</td>
<td>-1.5331e-16</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.1787333</td>
<td>-9.006562e-16</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-0.08795285</td>
<td>-1.112024e-16</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>-0.4381255</td>
<td>-4.066447e-16</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>-0.5447503</td>
<td>-3.949187e-16</td>
</tr>
<tr>
<td>Correlation with a</td>
<td>1</td>
<td>0.04371798</td>
<td>3.061485e-18</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.3963232</td>
<td>5.49705e-17</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.1186376</td>
<td>1.565708e-17</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-0.3295837</td>
<td>3.956392e-17</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>-0.1515805</td>
<td>1.926321e-17</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.1304383</td>
<td>5.333212e-18</td>
</tr>
</tbody>
</table>
Table 3.5: Summary Statistics for the Pricing Errors of Hemler’s model and our Model for the Nearby S&P 500 Stock Index Futures Price during 2001 to 2007 period (Continued 2)

<table>
<thead>
<tr>
<th>Statistic</th>
<th>$\tau$</th>
<th>Hemler’s Model</th>
<th>Our’s Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st order autocorrelation</td>
<td>1</td>
<td>0.088</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.094</td>
<td>0.028</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-0.152</td>
<td>-0.010</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-0.067</td>
<td>0.046</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>-0.135</td>
<td>-0.153</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>-0.048</td>
<td>-0.102</td>
</tr>
<tr>
<td>2nd order autocorrelation</td>
<td>1</td>
<td>0.287</td>
<td>0.358</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.134</td>
<td>-0.044</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-0.028</td>
<td>-0.067</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-0.164</td>
<td>0.081</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>-0.141</td>
<td>-0.127</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>-0.019</td>
<td>-0.288</td>
</tr>
<tr>
<td>3rd order autocorrelation</td>
<td>1</td>
<td>-0.014</td>
<td>0.089</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.052</td>
<td>0.063</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-0.102</td>
<td>-0.040</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-0.027</td>
<td>0.140</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>-0.048</td>
<td>-0.016</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.097</td>
<td>-0.054</td>
</tr>
</tbody>
</table>
References


