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ABSTRACT

Automotive NVH (Noise, Vibration and Harshness) refinement can be implemented by utilizing TPA (Transfer Path Analysis) for steady state and transient vibrations analysis in frequency domain and time domain, respectively. In frequency domain, a new spectral-based multi-substructure theory is formulated to compute the frequency responses of mechanical systems that can be subdivided into multiple inter-connected substructures. The proposed spectral-based approach employs the free substructure frequency response functions at the coupling, response and excitation coordinates of interest to construct the complete system model using a single efficient coupling step. Even though this proposed spectral-based approach is conceptually similar to the conventional transfer path analysis, it is more extensive because of the ability to analyze systems with arbitrary numbers of substructures and coupling coordinates. Hence, the methodology can be applied to treat complex multi-substructure mechanical structures commonly found in automotive and aerospace systems. In the present study, several lumped parameters mass-spring-damper systems are analyzed to validate the proposed theory. On the other hand, in time domain, a new time domain transfer path analysis is formulated to deal with a class of weakly nonlinear, transient structural dynamics problems. It combines the versatility of the frequency domain transfer path analysis for tracking vibration transmission between substructures and the generality of the time domain analysis for treating nonlinear transient dynamic response. It is suggested that the developed approach can be applied to analyze practical transient dynamic problems in
automotive systems. Also, the proposed time domain methods are applied to a lumped parameter mass-spring-damper dynamic system, and the results are shown to compare with the calculations results from the complete system method and a classical direct numerical integration routine.
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1 INTRODUCTION

Control of noise and vibration response in mechanical systems require an in-depth understanding of the underlying physics of vibratory energy transmissions. This need has led to a number of approaches that compute coupled system response from the dynamic characteristics of the substructure that make up the system. Some of the widely employed ones are discussed next. The conventional transfer path analysis (TPA) [1-3] that is designed for analyzing steady-state response of linear time-invariant vibro-acoustic system is primarily an experimental approach. It was initially formulated to analyze NVH (Noise, Vibration and Harshness) problems in vehicle systems by quantifying the vibratory energy transfer from one substructure to another along the critical structure-borne paths. The analysis is often applied to weakly coupled structures to identify the controlling parameters, components and structural paths that can be redesigned to improve response [4-12].

More recently, a relatively general spectral-based two-substructure theory was proposed by Zhen et al. [13-15] for computing the complete system characteristic from the frequency response functions (FRF) of selected points in the substructures. Unlike TPA, this approach accounts for the dynamic interactions between the connected substructures. In fact the TPA is a special case of Zhen’s substructuring analysis. Since the analysis is limited to the coupling of two substructures, it may be limited in utility when applied to systems that contain more than two substructures. For example, an assembled automotive system is typically made up of numerous substructures such as suspension, frame, body structure, passenger compartment and
powertrain, which makes the application of the two-substructure formulation cumbersome and prone to modeling errors. To overcome this limitation, Wang et al. [3] sub-divided the system into a cascade set of two-substructure pairs. This approach requires a multi-step calculation sequence that involves only two substructures within each step. Therefore, the approach yields the relationship between two connected substructures and it is quite feasible when the total number of substructures is small. However, as the number of substructure increases, the steps of applying two-substructure method increases as well and the analysis process may become lengthy, time consuming and low in efficiency. This renders the utility of the two-substructure formulation less attractive and more cumbersome for complex multi-substructure system. In addition, the multi-step approach does not reveal any valuable relationship between the substrucures except in the final coupling process because all the intermediate coupling processes only lead to partially assembled system. Accordingly, a more efficient physics-based multi-substructure theory that provides useful dynamic interactions of the connected substructures is needed. The focus of this current study is to address this gap by proposing a new one-step formulation for modeling the dynamics of complex system comprised of multiple inter-connected substructures. The proposed approach is expected to be more efficient and simpler to model digitally when the substructure size is large.

Secondly, those current TPA approaches that are performed in frequency domain inherently assume that the system behaves linearly. Also, the existing TPA approaches are mainly developed to analyze steady-state vibration response and not
formulated for transient dynamics. However, many vehicle NVH problems are dominated by nonlinear transient dynamic response such as clunk, shudder, and tip-in and tip-out events [16]. In the case of the clunk event [17], a sudden input at the gas pedal causes an undesirable shock vibration to transmit from the engine through the transmission and the rest of the powertrain system into the seat and floor where it can be felt by the passengers. Due to the intensity of the shock response as well as the nonlinear coupling stiffnesses and clearances in the components, the clunk response is highly transient and quite nonlinear. Typically, in the study of linear dynamic system, the superposition principle can be applied and system responses due to different excitations can be added linearly [18]. However, the superposition principle is not normally applicable to nonlinear dynamic systems, since the dynamic behavior is generally not proportional to excitation amplitude like in linear dynamic systems. Currently, there are three common methods for dealing with nonlinear structural dynamics: (1) geometrical method that employs the locus of the phase plane diagrams to qualitatively describe the motion of the system; (2) semi-analytical method that approximately formulates the nonlinear differential equation to obtain the characteristics of the motion; (3) numerical method that applies an integration algorithm to compute the dynamic response from the nonlinear differential equations [19]. The geometrical method is typically not accurate enough because it is a qualitative-based approach. The mathematical analysis in semi-analytical method is generally too complicated for treating equations of motion of higher-order dynamic system. In the third method on numerical integration, the computational effort needed
may become prohibitively impractical for large, complex mechanical systems. Hence, a more practical approach is warranted.

In chapter 2, the derivation of the proposed spectral-based multi-substructure theory will be presented first for three-substructure and four-substructure systems before generalizing it to N-substructure. To demonstrate the proposed concept, a lumped parameter, mass-spring-damper, multi-degrees-of-freedom mechanical vibrating system will be employed in the three-substructure and four-substructure analysis. To verify the proposed formulation, numerical studies involving multi-coupling multi-path mechanical system are performed. This work will be submitted as a journal paper and currently is in progress.

In chapter 3, a time-domain transfer path analysis method is proposed to address some of the shortcomings of the traditional frequency domain TPA approach. The formulation is derived by combining the spectral-based substructure method and a discrete, piecewise convolution theory. The spectral-based substructure method that was proposed by Zhen et al. [13-15] is a more comprehensive formulation that allows for the dynamic coupling between substructures. Unlike the classical TPA approaches, using the spectral-based substructure method as the base platform, a discrete, piecewise convolution is added to accommodate both weakly nonlinear and transient characteristics. This work has been submitted to the Noise Control Engineering Journal 2009 as a journal paper and recently has been accepted, and part of this work will be presented at the NOISE-CON 2010 (Wenwei Jiang and Teik C. Lim 2010).
2 GENERALIZED FREQUENCY-DOMAIN MULTI-SUBSTRUCTURE TRANSFER PATH ANALYSIS METHOD

2.1 Introduction

In this chapter, the proposed newer direct spectral-based multi-substructure method is introduced. Typically, in a two-substructure system, there are only two substructures and one set of coupling elements. On the other hand, in a multi-substructure system, say N-substructure, there are N substructures with at least N-1 set of coupling elements as shown in Figure 1. Moreover, for each substructure, a set of intra-substructure relationship exists between the coordinates contained in the same substructure, a second set of inter-substructure relationship exists between the coordinates of different substructures, and a third set of coupling relationship between the interfacial boundary coordinates. Taking into the account of all these substructure relationships, for the assembled N-substructure system, there are at least N-1 sets of relationships. This N-1 sets relationship matrix directly yields the system response in one computational sequence unlike the approach employed by Wang et al. [3] previously. This represents a significant advantage.

This chapter is organized as follows. Firstly, a one-step three-substructure formula is derived in Section 2.2. Secondly, in Section 2.3, a one-step four-substructure formula is derived. Thirdly, in Section 2.4, a generalized one-step N-substructure formula is derived. Fourthly, in Section 2.5, the validity of the proposed scheme is demonstrated by building a number of different types of lumped parameter mass-spring-damper multi-substructure dynamic systems. Finally, in
Section 2.6, the numerically results comparison is given.

Figure 1. An illustration of a type of N-substructure system.

### 2.2 Three-substructure System

For a multi-substructure mechanical system, the complete system response

\[
\{X_S\}
\]

can be described as
\[\{X_S\} = [H_{sub}]\{F_s\} + [H_F]\{F_c\}\]  
(1)

where \([H_{sub}]\) and \(\{F_s\}\) represent the transfer function vectors of each free substructure and their corresponding external force vectors, \([H_F]\) and \(\{F_c\}\) represent transfer function vectors of each set of coupling elements and their corresponding internal coupling reaction force vectors. Thus, for a three-substructure system shown in Figure 2, which consists of three substructures and two sets of coupling elements, the complete system response can be expressed as

\[\{X_S\} = [H_{sub}]\{F_s\} + [H_{c1}]\{F_{c1}\} + [H_{c2}]\{F_{c2}\}\]  
(2)

where the subscript \(c1\) and \(c2\) represent the coupling elements sets I and II, respectively. Equation (2) can be expanded as

\[
\begin{bmatrix}
\{X_S\}_{o(a)} \\
\{X_S\}_{c1(a)} \\
\{X_S\}_{c1(b)} \\
\{X_S\}_{c2(b)} \\
\{X_S\}_{c2(c)} \\
\{X_S\}_{o(c)} \\
\end{bmatrix} = \begin{bmatrix}
[H_s] & 0 & 0 \\
0 & [H_b] & 0 \\
0 & 0 & [H_c] \\
\end{bmatrix}
\begin{bmatrix}
\{F_s\}_{i(a)} \\
\{F_s\}_{c1(b)} \\
\{F_s\}_{c2(b)} \\
\{F_s\}_{c2(c)} \\
\{F_s\}_{i(c)} \\
\end{bmatrix}
\]

(3)

where the subscript \(i\) and \(o\) represent the input and output, \(a\), \(b\) and \(c\) represent the location of the corresponding component is in substructure A, B and C,
respectively. Also, the matrices \([H_A]\), \([H_B]\) and \([H_C]\) of the free substructure A, B and C are

\[
[H_A] = \begin{bmatrix}
[H_A]_{0(\alpha)\alpha(\alpha)} & [H_A]_{0(\alpha)\beta(\alpha)} \\
[H_A]_{\beta(\alpha)\alpha(\alpha)} & [H_A]_{\beta(\alpha)\beta(\alpha)}
\end{bmatrix}
\] (4a)

\[
[H_B] = \begin{bmatrix}
[H_B]_{\alpha(\beta)\alpha(\beta)} & [H_B]_{\beta(\beta)\beta(\beta)} \\
[H_B]_{\alpha(\beta)\alpha(\beta)} & [H_B]_{\alpha(\beta)\beta(\beta)}
\end{bmatrix}
\] (4b)

\[
[H_C] = \begin{bmatrix}
[H_C]_{\alpha(\gamma)\alpha(\gamma)} & [H_C]_{\beta(\gamma)\beta(\gamma)} \\
[H_C]_{\gamma(\gamma)\alpha(\gamma)} & [H_C]_{\gamma(\gamma)\beta(\gamma)}
\end{bmatrix}
\] (4c)

and the coefficients \(\alpha\), \(\beta\), \(\sigma\) and \(\tau\) are given by

\[\alpha = +1, \beta = -1 \quad \text{for} \quad x = a\]
\[\alpha = -1, \beta = +1 \quad \text{for} \quad x = b\]
\[\sigma = +1, \tau = -1 \quad \text{for} \quad y = b\]
\[\sigma = -1, \tau = +1 \quad \text{for} \quad y = c\]

(5)

As noted earlier, the vectors \(\{F_{c1(\alpha)}\}\) and \(\{F_{c2(\alpha)}\}\) are the internal coupling reaction forces of the coupling elements of sets I and II, which can be expressed as

\[
\{F_{c1(\alpha)}\} = [K_{c1}](\{x_S\}_{c1(\alpha)} - \{x_S\}_{c1(\alpha)}) \quad \text{and} \quad \{F_{c2(\alpha)}\} = -\{F_{c1(\beta)}\}
\] (6a)

\[
\{F_{c2(\alpha)}\} = [K_{c2}](\{x_S\}_{c2(\alpha)} - \{x_S\}_{c2(\alpha)}) \quad \text{and} \quad \{F_{c2(\alpha)}\} = -\{F_{c2(\beta)}\}
\] (6b)

From equation (3), \(\{X_{c1(\alpha)}\}\), \(\{X_{c1(\beta)}\}\), \(\{X_{c2(\beta)}\}\) and \(\{X_{c2(\gamma)}\}\) can be obtained as follows

\[
\{X_{S}\}_{c1(\alpha)} = \begin{bmatrix}
[H_A]_{0(\alpha)\alpha(\alpha)} & [H_A]_{0(\alpha)\beta(\alpha)} \\
[H_A]_{\beta(\alpha)\alpha(\alpha)} & [H_A]_{\beta(\alpha)\beta(\alpha)}
\end{bmatrix}\begin{bmatrix}
\{F_{c1(\alpha)}\} \\
\{F_{c1(\alpha)}\}
\end{bmatrix}
\] (7a)
Substituting equations (7a-d) into equations (6a) and (6b) yields

\[
\begin{bmatrix}
[H_B]_{k(b)l(b)}
\end{bmatrix}^T
\begin{bmatrix}
\{F_S\}_{c1(b)}
\end{bmatrix} =
\begin{bmatrix}
[H_B]_{k(b)l(b)}
\end{bmatrix}^T
\begin{bmatrix}
\{F_S\}_{c1(b)}
\end{bmatrix}
\]

(7b)

\[
\begin{bmatrix}
[H_B]_{k(b)l(b)}
\end{bmatrix}^T
\begin{bmatrix}
\{F_S\}_{c2(b)}
\end{bmatrix} =
\begin{bmatrix}
[H_B]_{k(b)l(b)}
\end{bmatrix}^T
\begin{bmatrix}
\{F_S\}_{c2(b)}
\end{bmatrix}
\]

(7c)

\[
\begin{bmatrix}
[H_B]_{k(b)l(b)}
\end{bmatrix}^T
\begin{bmatrix}
\{F_S\}_{c2(c)}
\end{bmatrix} =
\begin{bmatrix}
[H_B]_{k(b)l(b)}
\end{bmatrix}^T
\begin{bmatrix}
\{F_S\}_{c2(c)}
\end{bmatrix}
\]

(7d)

Substituting equations (7a-d) into equations (6a) and (6b) yields

\[
\begin{bmatrix}
C_P
\end{bmatrix}^T
\begin{bmatrix}
\{F_c\}_{c1}\end{bmatrix} =
\begin{bmatrix}
-[H_A]_{k(a)l(a)}
\end{bmatrix}^T
\begin{bmatrix}
\{F_{c1}\}_{l(a)}
\end{bmatrix}
\]

(8a)

\[
\begin{bmatrix}
C_P
\end{bmatrix}^T
\begin{bmatrix}
\{F_c\}_{c2}\end{bmatrix} =
\begin{bmatrix}
-[H_A]_{k(a)l(a)}
\end{bmatrix}^T
\begin{bmatrix}
\{F_{c2}\}_{l(a)}
\end{bmatrix}
\]

(8b)

where \([C_P]\) and \([C_P]\) are coupling transfer function sets of coupling elements sets I and II, which are explicitly given by
\[
\begin{align*}
\begin{bmatrix} C_{11} \end{bmatrix} &= \left( [H_a]_{1(1)1(1)} + [H_b]_{1(1)1(1)} + K_{c1}^{-1} \right) \\
\begin{bmatrix} C_{12} \end{bmatrix} &= \left( [H_a]_{1(2)2(2)} + [H_b]_{1(2)2(2)} + K_{c2}^{-1} \right)
\end{align*}
\] (9a)

(9b)

Then, by solving equations (8a) and (8b) for \( \{F_{c1(a)}\} \) and \( \{F_{c2(b)}\} \), the final three-substructure model can be obtained by substituting \( \{F_{c1(a)}\} \) and \( \{F_{c2(b)}\} \) back into equation (3) and eliminating all the external excitation vectors \( \{F_S\} \) on both two sides of the equations. The same derivation approach is applied to a four-substructure system in the next section.

![Figure 2. A three-substructure lumped parameter mass-spring-damper system.](image)

2.3 Four-substructure Method

For a four-substructure system as shown in Figure 3, since there are three sets
of coupling elements, the coupled system equation of motion become

\[ \{ X_s \} = [H_{ab}] \{ F_s \} + [H_{cl}] \{ F_{c1(x)} \} + [H_{c2}] \{ F_{c2(x)} \} + [H_{c3}] \{ F_{c3(x)} \} \]  

(10)

which can be expanded as follow:

\[
\begin{bmatrix}
\{ X_{s1} \}_{a(a)} \\
\{ X_{s2} \}_{c1(a)} \\
\{ X_{s3} \}_{c1(b)} \\
\{ X_{s4} \}_{a(b)} \\
\{ X_{s5} \}_{c2(b)} \\
\{ X_{s6} \}_{c2(c)} \\
\{ X_{s7} \}_{a(c)} \\
\{ X_{s8} \}_{c3(c)} \\
\{ X_{s9} \}_{c3(d)} \\
\{ X_{s10} \}_{a(d)}
\end{bmatrix} =
\begin{bmatrix}
[H_A] & 0 & 0 & 0 \\
0 & [H_B] & 0 & 0 \\
0 & 0 & [H_C] & 0 \\
0 & 0 & 0 & [H_D]
\end{bmatrix}
\begin{bmatrix}
\{ F_{s1} \}_{a(a)} \\
\{ F_{s2} \}_{c1(a)} \\
\{ F_{s3} \}_{c1(b)} \\
\{ F_{s4} \}_{a(b)} \\
\{ F_{s5} \}_{c2(b)} \\
\{ F_{s6} \}_{c2(c)} \\
\{ F_{s7} \}_{a(c)} \\
\{ F_{s8} \}_{c3(c)} \\
\{ F_{s9} \}_{c3(d)} \\
\{ F_{s10} \}_{a(d)}
\end{bmatrix}
\]

(11)

where \([H_A]\), \([H_B]\), \([H_C]\) and \([H_D]\) matrices are

\[
[H_A] =
\begin{bmatrix}
[H_{A1}]_{a(a)c(a)} & [H_{A2}]_{a(a)c1(a)} \\
[H_{A3}]_{a(a)c1(a)} & [H_{A4}]_{a(a)c1(a)}
\end{bmatrix}
\]

(12a)

\[
[H_B] =
\begin{bmatrix}
[H_{B1}]_{c1(b)c1(b)} & [H_{B2}]_{c1(b)c2(b)} \\
[H_{B3}]_{c1(b)c2(b)} & [H_{B4}]_{c2(b)c2(b)} \\
[H_{B5}]_{c2(b)c2(b)} & [H_{B6}]_{c2(b)c2(b)}
\end{bmatrix}
\]

(12b)
\[ [H_C] = \begin{bmatrix}
[H_C]_{2(c)2(c)} & [H_C]_{2(c)l(c)} & [H_C]_{2(c)3(c)} \\
[H_C]_{l(c)2(c)} & [H_C]_{l(c)l(c)} & [H_C]_{l(c)3(c)} \\
[H_C]_{3(c)2(c)} & [H_C]_{3(c)l(c)} & [H_C]_{3(c)3(c)}
\end{bmatrix} \quad (12c) \]

\[ [H_D] = \begin{bmatrix}
[H_D]_{3(d)3(d)} & [H_D]_{3(d)l(d)} \\
[H_D]_{l(d)3(d)} & [H_D]_{l(d)l(d)}
\end{bmatrix} \quad (12d) \]

and where coefficients \( \alpha, \beta, \sigma, \tau, \chi \) and \( \gamma \) are given by

\[
\begin{align*}
\alpha &= +1, \beta = -1 \quad \text{for} \quad x = a \\
\alpha &= -1, \beta = +1 \quad \text{for} \quad x = b \\
\sigma &= +1, \tau = -1 \quad \text{for} \quad y = b \\
\sigma &= -1, \tau = +1 \quad \text{for} \quad y = c \\
\chi &= +1, \gamma = -1 \quad \text{for} \quad z = c \\
\chi &= -1, \gamma = +1 \quad \text{for} \quad z = d
\end{align*}
\quad (13) \]

Also, \( \{F_{c1(a)}\} \), \( \{F_{c2(a)}\} \) and \( \{F_{c3(a)}\} \) are the internal coupling reaction force vectors that can be expressed as

\[
\begin{align*}
\{F_{c1(a)}\} &= [K_{c1}](\{x_S\}_{c1(b)} - \{x_S\}_{c1(a)}) \quad ; \quad \{F_{c1(a)}\} = -\{F_{c1(b)}\} \quad (14a) \\
\{F_{c2(b)}\} &= [K_{c2}](\{x_S\}_{c2(c)} - \{x_S\}_{c2(b)}) \quad ; \quad \{F_{c2(b)}\} = -\{F_{c2(c)}\} \quad (14b) \\
\{F_{c3(c)}\} &= [K_{c3}](\{x_S\}_{c3(d)} - \{x_S\}_{c3(c)}) \quad ; \quad \{F_{c3(c)}\} = -\{F_{c3(d)}\} \quad (14c)
\end{align*}
\]

The remaining four-substructure model derivation procedure is identical to those of the three-substructure model. By substituting the six displacement vectors \( \{X_S\}_{c1(a)} \), \( \{X_S\}_{c1(b)} \), \( \{X_S\}_{c2(b)} \), \( \{X_S\}_{c2(c)} \), \( \{X_S\}_{c3(c)} \) and \( \{X_S\}_{c3(d)} \) from equation (11) into equations (14a-c) yield three equations with three unknown vectors \( \{F_{c1(a)}\} \), \( \{F_{c2(b)}\} \) and \( \{F_{c3(c)}\} \) given by
\[
\begin{bmatrix}
C_R
\end{bmatrix}^T
\begin{bmatrix}
\{F_{e1(a)}\}
\{F_{e2(b)}\}
\{F_{e3(c)}\}
\end{bmatrix}
=\begin{bmatrix}
-H_B\begin{bmatrix}(a)\end{bmatrix}
-H_B\begin{bmatrix}(b)\end{bmatrix}
-H_B\begin{bmatrix}(c)\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\{F_{e1(a)}\}
\{F_{e2(b)}\}
\{F_{e3(c)}\}
\end{bmatrix}
\]
\begin{bmatrix}
F_S\begin{bmatrix}(a)\end{bmatrix}
F_S\begin{bmatrix}(b)\end{bmatrix}
F_S\begin{bmatrix}(c)\end{bmatrix}
\end{bmatrix}
(15a)
\]

\[
\begin{bmatrix}
C_R
\end{bmatrix}^T
\begin{bmatrix}
\{F_{e1(a)}\}
\{F_{e2(b)}\}
\{F_{e3(c)}\}
\end{bmatrix}
=\begin{bmatrix}
-H_B\begin{bmatrix}(a)\end{bmatrix}
-H_B\begin{bmatrix}(b)\end{bmatrix}
-H_B\begin{bmatrix}(c)\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\{F_{e1(a)}\}
\{F_{e2(b)}\}
\{F_{e3(c)}\}
\end{bmatrix}
\begin{bmatrix}
F_S\begin{bmatrix}(a)\end{bmatrix}
F_S\begin{bmatrix}(b)\end{bmatrix}
F_S\begin{bmatrix}(c)\end{bmatrix}
\end{bmatrix}
(15b)
\]

\[
\begin{bmatrix}
C_R
\end{bmatrix}^T
\begin{bmatrix}
\{F_{e1(a)}\}
\{F_{e2(b)}\}
\{F_{e3(c)}\}
\end{bmatrix}
=\begin{bmatrix}
-H_B\begin{bmatrix}(a)\end{bmatrix}
-H_B\begin{bmatrix}(b)\end{bmatrix}
-H_B\begin{bmatrix}(c)\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\{F_{e1(a)}\}
\{F_{e2(b)}\}
\{F_{e3(c)}\}
\end{bmatrix}
\begin{bmatrix}
F_S\begin{bmatrix}(a)\end{bmatrix}
F_S\begin{bmatrix}(b)\end{bmatrix}
F_S\begin{bmatrix}(c)\end{bmatrix}
\end{bmatrix}
(15c)
\]

where the coupling transfer function sets \( [C_R] \), \( [C_R] \) and \( [C_R] \) are

\[
[C_R] = \left( [H_D]_{\begin{bmatrix} a \end{bmatrix}} + [H_B]_{\begin{bmatrix} a \end{bmatrix}} + K_{\begin{bmatrix} a \end{bmatrix}} \right)
\]
\[
[C_R] = \left( [H_B]_{\begin{bmatrix} b \end{bmatrix}} + [H_C]_{\begin{bmatrix} b \end{bmatrix}} + K_{\begin{bmatrix} b \end{bmatrix}} \right)
\]
\[
[C_R] = \left( [H_C]_{\begin{bmatrix} c \end{bmatrix}} + [H_D]_{\begin{bmatrix} c \end{bmatrix}} + K_{\begin{bmatrix} c \end{bmatrix}} \right)
\]

Thus, \( \{F_{e1(a)}\} \), \( \{F_{e2(b)}\} \) and \( \{F_{e3(c)}\} \) can be obtained by solving the above three equations (15a-c). Then, substituting \( \{F_{e1(a)}\} \), \( \{F_{e2(b)}\} \) and \( \{F_{e3(c)}\} \) back into equation (11) and eliminating all the external excitation vectors \( \{F_S\} \) on both two sides of the equation produces the final response formulation.
2.4 N-substructure Method

For a N-substructure system as shown in Figure 1, there are N-1 sets of coupling elements. Hence the coupled system equation of motion can be written as

\[
\{X_s\} = \left[H_{sub}\right]\{F_s\} + \left[H_{c(1)}\right]\{F_{c(1)}\} + \cdots + \left[H_{c(N-1)}\right]\{F_{c(N-1)}\}
\]

(17)

which can be expanded as follows.
\[
\begin{bmatrix}
\{X\}_{i(m)} \\
\{X\}_{j(d)} \\
\vdots \\
\{X\}_{n(m)} \\
\end{bmatrix} = 
\begin{bmatrix}
[H_1] & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 \\
0 & 0 & [H_n] & 0 \\
0 & 0 & 0 & \ddots \\
0 & 0 & 0 & 0 & [H_N] \\
\end{bmatrix} + \begin{bmatrix}
0 \\
\vdots \\
\vdots \\
\vdots \\
0 \\
\end{bmatrix} + \cdots + 
\begin{bmatrix}
\{F\}_{e(i)} \\
\{F\}_{j(d)} \\
\vdots \\
\{F\}_{n(m)} \\
\end{bmatrix} + \cdots + 
\begin{bmatrix}
\{F\}_{e(N-1)} \\
\{F\}_{j(d)} \\
\vdots \\
\{F\}_{n(m)} \\
\end{bmatrix} \tag{18}
\]

where the subscript \(u\), \(m\) and \(d\) represent the top coupling elements set, middle non-coupling elements set and bottom coupling elements set of a substructure as can be seen in Figures 1 and 4.

where \([H_1]\), \([H_n]\) and \([H_N]\) matrices are
\[
[H_i] = \begin{bmatrix}
[H_{i}]_{o(m)(n)} & [H_{i}]_{a(m)c(d)} \\
[H_{i}]_{c(d)(n)} & [H_{i}]_{a(n)c(d)}
\end{bmatrix}
\] (19a)

\[
[H_n] = \begin{bmatrix}
[H_{n}]_{o(u)c(u)} & [H_{n}]_{a(u)(n)} & [H_{n}]_{a(u)c(d)} \\
[H_{n}]_{o(m)c(u)} & [H_{n}]_{a(m)(n)} & [H_{n}]_{a(m)c(d)} \\
[H_{n}]_{o(d)c(u)} & [H_{n}]_{a(d)(n)} & [H_{n}]_{a(d)c(d)}
\end{bmatrix}
\] (19b)

\[
[H_N] = \begin{bmatrix}
[H_{N}]_{o(u)c(u)} & [H_{N}]_{a(u)(n)} \\
[H_{N}]_{o(m)c(u)} & [H_{N}]_{a(m)(n)}
\end{bmatrix}
\] (19c)

Also, \( \{F_{e(i)}\} \), \( \{F_{e(n)}\} \) and \( \{F_{e(N)}\} \) are the internal coupling reaction force vectors of the coupling elements set I, n and N, 1<n<N, are given by

\[
\{F_{e(i)}\} = [K_{e}]\left(\{X_2\}_{e(u)} - \{X_1\}_{e(d)}\right)
\] (20a)

\[
\{F_{e(n)}\} = [K_{e(n)}]\left(\{X_{n+1}\}_{e(u)} - \{X_n\}_{e(d)}\right)
\] (20b)

\[
\{F_{e(N-1)}\} = [K_{e(N-1)}]\left(\{X_N\}_{e(u)} - \{X_{N-1}\}_{e(d)}\right)
\] (20c)

Now, suppose the internal coupling reaction force and external excitation vectors are related through a matrix \([R]\),

\[
\{F_e\} = [R]\{F_s\}
\] (21)

where the internal coupling reaction force vector \( \{F_e\} \) and \([R]\) can be expanded as

\[
\{F_e\} = \left\{\begin{array}{c}
\{F_{e(i)}\} \\
\{F_{e(n)}\} \\
\{F_{e(N)}\}
\end{array}\right\}^T
\] (22a)

\[
[R] = \begin{bmatrix}
[R_1] & \cdots & [R_n] & \cdots & [R_N]
\end{bmatrix}^T
\] (22b)

Thus, the relationship between each internal coupling reaction force and the external force vectors can be expressed as

\[
\{F_{e(i)}\} = [R_1]\{F_s\}
\] (23a)

\[
\{F_{e(n)}\} = [R_n]\{F_s\}
\] (23b)
\[
\{F_{e_{(N-1)}}\} = [R_{N-1}]\{F_S\} \quad (23c)
\]

Then, substituting \(\{F_{e_{(1)}}\}\), \(\{F_{e_{(n)}}\}\) and \(\{F_{e_{(N-1)}}\}\) in equations (23a-c) into equation (18), and eliminating all the external excitation \(\{F_S\}\) give rise to the system transfer function response matrix equation for the N-substructure system, which is explicitly given by

\[
[H_S] = \begin{bmatrix}
[H_1] & 0 & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 & 0 \\
0 & 0 & [H_n] & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & [H_{N}]
\end{bmatrix}
\]

\[
+ [R_1] + \cdots + [R_n] + \cdots + [R_{N-1}] = \begin{bmatrix}
0 & \cdots \\
\vdots & \ddots \\
[H_n]_{(i)\rightarrow(c)} \\
[H_{N-1}]_{(i)\rightarrow(c)} \\
0 & \cdots \\
\vdots & \ddots \\
[R_n] + \cdots + [R_{N-1}]
\end{bmatrix}
\]

\[
\begin{bmatrix}
H_{N-1} & \cdots & H_{N-1} \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
H_{N-1} & \cdots & H_{N-1}
\end{bmatrix} = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
[H_N]_{(i)\rightarrow(c)} \\
[H_N]_{(m)\rightarrow(c)} \\
[H_N]_{(n)\rightarrow(c)} \\
[H_N]_{(d)\rightarrow(c)} \\
[H_N]_{(i)\rightarrow(c)} \\
[H_N]_{(m)\rightarrow(c)} \\
[H_N]_{(n)\rightarrow(c)}
\end{bmatrix}
\]

Thus, by solving for \([R]\), the N-substructure model can be obtained.
From equations (8a), (8b), (15a), (15b) and (15c) in the three-substructure and four-substructure model, it can be generalized that for any arbitrary part of the N-substructure system comprising of two coupled substructures as depicted in Figure 4, the equation of motion of the coupling elements set can be written as

$$\begin{bmatrix} -[H_n]_{c(d)c(u)} & C_{P_c} \\ -[H_{n+1}]_{c(u)c(d)} \end{bmatrix} \begin{bmatrix} \{F_{c_1}\} \\ \{F_{c_2}\} \end{bmatrix} = \begin{bmatrix} -[H_n]_{c(d)c(u)} & \{F_n\}_{c(u)} \\ -[H_n]_{c(d)(m)} & \{F_n\}_{c(m)} \\ -[H_n]_{c(d)} & \{F_n\}_{c(d)} \\ [H_{n+1}]_{c(u)c(u)} & \{F_{n+1}\}_{c(u)} \\ [H_{n+1}]_{c(u)(m)} & \{F_{n+1}\}_{c(m)} \\ [H_{n+1}]_{c(u)c(d)} & \{F_{n+1}\}_{c(d)} \end{bmatrix} \begin{bmatrix} \{F_n\} \\ \{F_{n+1}\} \end{bmatrix}$$

(25)

Equation (25) can be rewritten in a more compact form as

$$[A] \{F_c\} = [B] \{F_S\}$$

(26)

where the internal coupling reaction force vector $\{F_c\}$ and external excitation vector $\{F_S\}$ are given by

$$\{F_c\} = [F_1] [F_2] [F_3] \cdots [F_{n-1}] [F_n] [F_{n+1}] [F_{n+2}] \cdots [F_{2n-1}] [F_{2n}] \begin{bmatrix} \{F_{c_1}\} \\ \{F_{c_2}\} \end{bmatrix}$$

(27a)

$$\{F_S\} = [F_{i(a)}] [F_{i(d)}] \cdots [F_{j(u)}] [F_{j(m)}] [F_{j(d)}] \cdots [F_{k(u)}] [F_{k(m)}] \begin{bmatrix} \{F_n\} \\ \{F_{n+1}\} \end{bmatrix}$$

(27b)

and where matrices $[A]$ and $[B]$ are

$$[A] = \begin{bmatrix} [C_n] & -[H_2]_{c(d)c(u)} & 0 & 0 & 0 & 0 & 0 & 0 \\ -[H_2]_{c(u)c(d)} & [C_n] & \vdots & 0 & 0 & 0 & 0 & 0 \\ 0 & -[H_2]_{c(d)c(u)} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots \\ \end{bmatrix}$$

(28a)
The coupling transfer function set \( \left[ C_{c_{\ell}} \right] \) is

\[
\left[ C_{c_{\ell}} \right] = \left( \left[ H_n \right]_{\ell(d)c(d)} + \left[ H_{n+1} \right]_{\ell(c)c(n)} + K_{c(n)}^{-1} \right)
\]  

(29)

Accordingly, the internal coupling reaction force vector \( \{ F_{c} \} \) can be calculated from

\[
\{ F_{c} \} = [A]^{-1} [B] \{ F_{S} \}
\]

(30)

which can also be written as

\[
\{ F_{c} \} = \frac{[A]^T}{|A|} [B] \{ F_{S} \} = |A|^{-1} [A]^T [B] \{ F_{S} \}
\]

(31)

where \([A]^T\) and \(|A|\) is the adjoint matrix and determinant of matrix \([A]\) respectively. Also, the relationship matrix \([R]\) between the internal coupling reaction force and external excitation vector can be expressed as

\[
[R] = \begin{bmatrix}
[R_1] & \cdots & [R_s] & \cdots & [R_N]
\end{bmatrix}^T = [A]^{-1} [B] = |A|^{-1} [A]^T [B]
\]

(32)

Finally, the internal coupling reaction force vector \( \{ F_{c} \} \) can be derived from
equation (30) or (31), and the N-substructure model can be obtained by substituting the corresponding row of $[R]$ from equation (32) into equation (24).

Figure 4. An arbitrary subset of the N-substructure system comprising of two coupled substructures

2.5. Lumped Parameter Systems

To validate the generality of the N-substructure formula, four different lumped parameter models are employed as depicted in Figures 5-8. Using these models, the system response functions computed from the spectral-based three-substructure model are compared to the exact results obtained from the complete system model. Figures 5-7 are basically three-substructure models while Figure 8 is a four-substructure system. The mathematical representations of these models are discussed next.
2.5.1 Three-substructure

As depicted in Figures 5-7, three different cases of three-substructure lumped parameter systems are considered in this study. Figure 5 shows a 7 degrees-of-freedom three-substructure system that is connected serially as $Sub(1) \rightarrow Sub(2) \rightarrow Sub(3)$. Utilizing equation (24), the system transfer function matrix $[H_s]$ of this three-substructure representation can be obtained by assuming $N$ equals to 3,

$$[H_s] = \begin{bmatrix} H_{S11} & H_{S12} & \cdots & H_{S17} \\ H_{S21} & H_{S22} & \cdots & H_{S27} \\ \vdots & \vdots & \ddots & \vdots \\ H_{S71} & H_{S72} & \cdots & H_{S77} \end{bmatrix}$$

$$= \begin{bmatrix} [H_1] \\ 0 \\ [H_2] \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ [H_1] \end{bmatrix}$$

$$= \begin{bmatrix} H_{12} \\ H_{12} \\ -H_{23} \\ -H_{24} \\ 0 \end{bmatrix} + [R_1] + \begin{bmatrix} H_{25} \\ H_{25} \\ H_{36} \\ H_{36} \end{bmatrix} [R_2]$$

$$= \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{12} \end{bmatrix}$$

$$= \begin{bmatrix} H_{23} & H_{24} & H_{25} \\ H_{24} & H_{24} & H_{25} \\ H_{25} & H_{25} & H_{25} \end{bmatrix}$$

where $[H_1]$, $[H_2]$ and $[H_3]$ matrices are

$$[H_1] = \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{12} \end{bmatrix}$$

$$[H_2] = \begin{bmatrix} H_{23} & H_{24} & H_{25} \\ H_{24} & H_{24} & H_{25} \\ H_{25} & H_{25} & H_{25} \end{bmatrix}$$

(33)

(34a)

(34b)
\[
[H_3] = \begin{bmatrix}
H_{36} & H_{397} \\
H_{36} & H_{377}
\end{bmatrix}
\]  

(34c)

The relationship matrix \([R]\) is

\[
[R] = \begin{bmatrix}[R_1] & [R_2] \end{bmatrix}^T = [A]^{-1} [B] = [A]^T [A] [B]
\]

(35)

where matrices \([A]\) and \([B]\) are

\[
[A] = \begin{bmatrix}
C_{R} & -H_{23} \\
-H_{23} & C_{P_5}
\end{bmatrix}
\]

(36a)

\[
[B] = \begin{bmatrix}
-H_{12} & 0 \\
-H_{12} & 0 \\
H_{23} & -H_{23} \\
H_{23} & -H_{24} \\
H_{23} & -H_{23} \\
0 & H_{36} \\
0 & H_{36}
\end{bmatrix}^T
\]

(36b)

and where the coupling transfer function sets \(C_{R}\) and \(C_{P_5}\) are

\[
C_{R} = H_{12} + H_{23} + (K_2 + j\omega C_2)^{-1}
\]

(37a)

\[
C_{P_5} = H_{23} + H_{36} + (K_5 + j\omega C_5)^{-1}
\]

(37b)
Similarly, as expected, the proposed theory can be applied to the three-substructure multi-path lumped parameter model in Figure 6 with 7 degrees-of-freedom, which is connected in parallel form: $Substructure(1) \rightarrow Substructure(2) \leftarrow Substructure(3)$. Since this sewed system also possesses three substructures and two sets of coupling elements, the corresponding three-substructure representation turns out to be the same as the first case as given by equation (33). Likewise, the expressions for matrices $[H_1]$ and the corresponding relationship matrix $[R]$ are the same as that in the system of the first case.
Figure 6. Case 2: Three-substructure, seven-degrees-of-freedom, lumped mass-spring-damper system.

The third case, as shown in Figure 7, contains a three-substructure multi-path lumped parameter system with 9 degrees-of-freedom, which are connecting similarly as Substructure(1) → Substructure(2) → Substructure(3) → Substructure(1). Since the system has three sets of coupling elements, that is one more than cases 1 and 2, the corresponding three-substructure representation is given by
\[
[H_S] = 
\begin{bmatrix}
H_{S11} & H_{S12} & \cdots & H_{S19} \\
H_{S21} & H_{S22} & \cdots & H_{S29} \\
\vdots & \vdots & \ddots & \vdots \\
H_{S91} & H_{S92} & \cdots & H_{S99}
\end{bmatrix}
\]

\[
= 
\begin{bmatrix}
[H_1] & 0 & 0 \\
0 & [H_2] & 0 \\
0 & 0 & [H_3]
\end{bmatrix}
\]

\[
\begin{bmatrix}
H_{113} \\
H_{123} \\
H_{133} \\
-H_{2a4} \\
-H_{2a4} \\
0 \\
0 \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
0 \\
H_{2a6} \\
H_{2a6} \\
-H_{377} \\
-H_{377} \\
-H_{377}
\end{bmatrix}
+ 
\begin{bmatrix}
-H_{111} \\
-H_{121} \\
-H_{131} \\
0 \\
0 \\
H_{379} \\
H_{379} \\
H_{379}
\end{bmatrix}
\]

\[
\text{where } [H_1], [H_2] \text{ and } [H_3] \text{ matrices are}
\]

\[
[H_1] = 
\begin{bmatrix}
H_{111} & H_{112} & H_{113} \\
H_{121} & H_{122} & H_{123} \\
H_{131} & H_{132} & H_{133}
\end{bmatrix}
\]

\[
[H_2] = 
\begin{bmatrix}
H_{2a4} & H_{2a5} & H_{2a6} \\
H_{2a4} & H_{2a5} & H_{2a6} \\
H_{2a4} & H_{2a5} & H_{2a6}
\end{bmatrix}
\]

\[
[H_3] = 
\begin{bmatrix}
H_{377} & H_{379} & H_{379} \\
H_{377} & H_{379} & H_{379} \\
H_{377} & H_{379} & H_{379}
\end{bmatrix}
\]

The relationship matrix \([R]\) is

\[
[R] = [R_1 \ R_2 \ R_3]^T = [A]^{-1} [B] = [A]^{-1} [A]^T [B]
\]

\[
\text{where matrices } [A] \text{ and } [B] \text{ are}
\]
\[
\begin{bmatrix}
C_{p_1} & -H_{240} & -H_{330} \\
-H_{240} & C_{p_2} & -H_{337} \\
-H_{330} & -H_{337} & C_{p_3}
\end{bmatrix}
\]  
\hspace{1cm} (41a)

\[
\begin{bmatrix}
-H_{121} & 0 & H_{11} \\
-H_{122} & 0 & H_{12} \\
-H_{123} & 0 & H_{13} \\
H_{244} & -H_{244} & 0 \\
H_{245} & -H_{245} & 0 \\
H_{246} & -H_{246} & 0 \\
0 & H_{337} & -H_{337} \\
0 & H_{338} & -H_{338} \\
0 & H_{339} & -H_{339}
\end{bmatrix}
\]  
\hspace{1cm} (41b)

and where the coupling transfer function sets \( C_{p_1} \), \( C_{p_2} \) and \( C_{p_3} \) are

\[
C_{p_1} = H_{13} + H_{244} \left( K_3 + j\omega C_3 \right)^{-1}
\]  
\hspace{1cm} (42a)

\[
C_{p_2} = H_{244} + H_{337} \left( K_7 + j\omega C_7 \right)^{-1}
\]  
\hspace{1cm} (42b)

\[
C_{p_3} = H_{337} + H_{444} \left( K_{10} + j\omega C_{10} \right)^{-1}
\]  
\hspace{1cm} (42c)

By substituting \([R_1]\), \([R_2]\) and \([R_3]\) in equations (35) and (40) into equations (33) and (38), the corresponding three-substructure single-path seven-degrees-of-freedom and nine-degrees-of-freedom models can be obtained explicitly.
2.5.2 Four-substructure

The fourth case is a 10 degrees-of-freedom four-substructure lumped parameter system as shown in Figure 8. The relationship matrix between the internal coupling reaction force and external excitation can be obtained directly from equation (24) by setting N to be 4. The system transfer function matrix of this four-substructure system is then given by
\[
[H_S] = \begin{bmatrix}
H_{S11} & H_{S12} & \cdots & H_{S1,10} \\
H_{S21} & H_{S22} & \cdots & H_{S2,10} \\
\vdots & \vdots & \ddots & \vdots \\
H_{S,11} & H_{S,12} & \cdots & H_{S,1,10}
\end{bmatrix}
\]

\[
[H_1] = \begin{bmatrix}
0 & 0 & 0 \\
0 & [H_2] & 0 \\
0 & 0 & [H_3] \\
0 & 0 & 0 & [H_4]
\end{bmatrix}
\]

\[
\begin{bmatrix}
H_{1,12} \\
H_{1,22} \\
-H_{2,23} \\
-H_{2,24}
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
-H_{3,36} \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
-H_{3,36} \\
0
\end{bmatrix} = \begin{bmatrix}
[H_1] \\
[H_2] \\
[H_3] \\
[H_4]
\end{bmatrix}
\]

\[
(H_{43})
\]

where \([H_1], [H_2], [H_3] \) and \([H_4] \) matrices are

\[
[H_1] = \begin{bmatrix}
H_{11} & H_{12} \\
H_{12} & H_{122}
\end{bmatrix}
\]

\[
[H_2] = \begin{bmatrix}
H_{23} & H_{24} & H_{25} \\
H_{23} & H_{24} & H_{25} \\
H_{23} & H_{24} & H_{25}
\end{bmatrix}
\]

\[
[H_3] = \begin{bmatrix}
H_{36} & H_{37} & H_{38} \\
H_{36} & H_{37} & H_{38} \\
H_{36} & H_{37} & H_{38}
\end{bmatrix}
\]

\[
[H_4] = \begin{bmatrix}
H_{49} & H_{4,10} \\
H_{4,10} & H_{4,10}
\end{bmatrix}
\]

As there are three sets of coupling elements in this four-substructure system,
where matrices \( [A] \) and \( [B] \) are

\[
[A] = \begin{bmatrix}
C_{R_1} & -H_{2s3} & 0 \\
-H_{2s3} & C_{R_2} & -H_{3s6} \\
0 & -H_{3s6} & C_{R_3}
\end{bmatrix}
\] (46a)

\[
[B] = \begin{bmatrix}
-H_{1s1} & 0 & 0 \\
-H_{1s2} & 0 & 0 \\
-H_{2s3} & -H_{2s4} & 0 \\
-H_{2s4} & -H_{2s5} & 0 \\
0 & H_{3s6} & -H_{3s6} \\
0 & H_{3s7} & -H_{3s7} \\
0 & H_{3s8} & -H_{3s8} \\
0 & 0 & H_{4s9} \\
0 & 0 & H_{4s30}
\end{bmatrix}
\] (46b)

and where coupling transfer function sets \( C_{R_1} \), \( C_{R_2} \) and \( C_{R_3} \) are

\[
C_{R_1} = H_{1s2} + H_{2s3} + (K_2 + j\omega C_2)^{-1}
\] (47a)

\[
C_{R_2} = H_{2s5} + H_{3s6} + (K_5 + j\omega C_5)^{-1}
\] (47b)

\[
C_{R_3} = H_{3s8} + H_{4s9} + (K_8 + j\omega C_8)^{-1}
\] (47c)

Likewise, by substituting \( [R_1] \), \( [R_2] \) and \( [R_3] \) into equation (43), the corresponding four-substructure single-path 10 degrees-of-freedom model can be obtained explicitly.
2.5.3 Exact System Model

Next, in order to verify the validity of the above three-substructure and four-substructure models, their results will be compared with the exact results obtained by employing the complete system model and its transfer function matrix.
can be obtained by the direct forced response equation

$$\begin{align*}
H_S(\omega) &= (-\omega^2 [M_S] + j\omega [C_S] + [K_S])^{-1}
\end{align*}$$

(48)

The mass \([M_S]\), damping \([C_S]\), and stiffness \([K_S]\) matrices are expressed as \([M_{S1}], [C_{S1}], [K_{S1}], [M_{S2}], [C_{S2}], [K_{S2}], [M_{S3}], [C_{S3}], [K_{S3}], \) and \([M_{S4}], [C_{S4}], [K_{S4}]\) for the four dynamic systems shown in Figures 5-8 respectively.

For the first three-substructure system shown in Figure 5, the system matrices are

$$
[M_{S1}] = \text{diag}[M_1 \quad M_2 \quad \cdots \quad M_6 \quad M_7]
$$

(49a)

$$
[C_{S1}] = \begin{bmatrix}
C_1 & -C_1 & 0 & 0 & 0 & 0 & 0 \\
-C_1 & C_1 + C_2 & -C_2 & 0 & 0 & 0 & 0 \\
0 & -C_2 & C_2 + C_3 & -C_3 & 0 & 0 & 0 \\
0 & 0 & -C_3 & C_3 + C_4 & -C_4 & 0 & 0 \\
0 & 0 & 0 & -C_4 & C_4 + C_5 & -C_5 & 0 \\
0 & 0 & 0 & 0 & -C_5 & C_5 + C_6 & -C_6 \\
0 & 0 & 0 & 0 & 0 & -C_6 & C_6 + C_7
\end{bmatrix}
$$

(49b)

$$
[K_{S1}] = \begin{bmatrix}
K_1 & -K_1 & 0 & 0 & 0 & 0 & 0 \\
-K_1 & K_1 + K_2 & -K_2 & 0 & 0 & 0 & 0 \\
0 & -K_2 & K_2 + K_3 & -K_3 & 0 & 0 & 0 \\
0 & 0 & -K_3 & K_3 + K_4 & -K_4 & 0 & 0 \\
0 & 0 & 0 & -K_4 & K_4 + K_5 & -K_5 & 0 \\
0 & 0 & 0 & 0 & -K_5 & K_5 + K_6 & -K_6 \\
0 & 0 & 0 & 0 & 0 & -K_6 & K_6 + K_7
\end{bmatrix}
$$

(49c)

The second three-substructure system shown in Figure 6 possesses the following system matrices,

$$
[M_{S2}] = \text{diag}[M_1 \quad M_2 \quad \cdots \quad M_6 \quad M_7]
$$

(50a)
In the third three-substructure system shown in Figure 7, the system is defined by

$$[M_{S3}] = \text{diag} \begin{bmatrix} M_1 & M_2 & \cdots & M_8 & M_9 \end{bmatrix}$$  \hspace{1cm} (51a)

$$[C_{S3}] = \begin{bmatrix} C_1 + C_{10} & -C_i & 0 & 0 & 0 & 0 & 0 & 0 & -C_{10} \\ -C_i & C_1 + C_{10} & -C_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -C_i & C_1 + C_{10} & -C_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -C_i & C_1 + C_{10} & -C_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -C_i & C_1 + C_{10} & -C_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -C_i & C_1 + C_{10} & -C_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -C_i & C_1 + C_{10} & -C_i & 0 \\ -C_{10} & 0 & 0 & 0 & 0 & 0 & 0 & -C_{10} & C_{10} + C_{10} \\ -C_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -C_{10} \end{bmatrix}$$  \hspace{1cm} (51b)

$$[K_{S3}] = \begin{bmatrix} K_i + K_{10} & -K_i & 0 & 0 & 0 & 0 & 0 & 0 & -K_{10} \\ -K_i & K_i + K_{10} & -K_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -K_i & K_i + K_{10} & -K_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -K_i & K_i + K_{10} & -K_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -K_i & K_i + K_{10} & -K_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -K_i & K_i + K_{10} & -K_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -K_i & K_i + K_{10} & -K_i & 0 \\ -K_{10} & 0 & 0 & 0 & 0 & 0 & 0 & -K_{10} & K_{10} + K_{10} \\ -K_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K_{10} \end{bmatrix}$$  \hspace{1cm} (51c)

Finally, for the four-substructure system shown in Figure 8, the mass, damping and stiffness matrices are

$$[M_{S4}] = \text{diag} \begin{bmatrix} M_1 & M_2 & \cdots & M_9 & M_{10} \end{bmatrix}$$  \hspace{1cm} (52a)
The solutions to each of the lumped parameter models can be obtained by computing the left hand side of equation (48) at each frequency point of interest. The results are frequency response functions to be shown in the next section.

2.6 Numerical Results

To determine the accuracy of the N-substructure model, results of three cases of three-substructure systems and a four-substructure system are compared with the exact results obtained from the complete system. Also, for the sake of brevity, only the response that relate to mass 1 will be shown and it may be noted that the response trends for the other coordinates are similar.

First of all, for the first \( Sub(1) \rightarrow Sub(2) \rightarrow Sub(3) \) type three-substructure case defined in Table 1 and Figure 5, the comparison results are given in Figure 9 for both amplitude and phase function. In this figure, seven different plots for \( H_{S11}, H_{S12}, H_{S13}, H_{S14}, H_{S15}, H_{S16} \) and \( H_{S17} \) are shown. Here \( H_{Sij} \) referes to the response of mass 1

\[
[C_{S4}] = \begin{bmatrix}
C_1 & -C_1 & 0 & 0 & 0 & 0 \\
-C_1 & C_1 + C_2 & -C_2 & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & -C_8 & C_8 + C_9 & -C_9 \\
0 & 0 & 0 & 0 & -C_9 & C_9
\end{bmatrix} \quad (52b)
\]

\[
[K_{S4}] = \begin{bmatrix}
K_1 & -K_1 & 0 & 0 & 0 & 0 \\
-K_1 & K_1 + K_2 & -K_2 & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & -K_8 & K_8 + K_9 & -K_9 \\
0 & 0 & 0 & 0 & -K_9 & K_9
\end{bmatrix} \quad (52c)
\]
due to an external excitation at coordinate j. In all plots, the results of the three-substructure model match precisely to the exact response.

Table 1. Parameters for the first three-substructure system.

<table>
<thead>
<tr>
<th>Model</th>
<th>Mass(Kg)</th>
<th>Stiffness(N/m)</th>
<th>Damping(Ns/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substructure A</td>
<td>$M_1=4$</td>
<td>$K_1=90000$</td>
<td>$C_1=5$</td>
</tr>
<tr>
<td></td>
<td>$M_2=10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Damped Elastic Connector I</td>
<td>$K_2=50000$</td>
<td></td>
<td>$C_2=10$</td>
</tr>
<tr>
<td>Substructure B</td>
<td>$M_3=10$</td>
<td>$K_3=12000$</td>
<td>$C_3=110$</td>
</tr>
<tr>
<td></td>
<td>$M_4=7$</td>
<td>$K_4=60000$</td>
<td>$C_4=12$</td>
</tr>
<tr>
<td></td>
<td>$M_5=9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Damped Elastic Connector II</td>
<td></td>
<td>$K_5=80000$</td>
<td>$C_5=30$</td>
</tr>
<tr>
<td>Substructure C</td>
<td>$M_6=6$</td>
<td>$K_6=100000$</td>
<td>$C_6=6$</td>
</tr>
<tr>
<td></td>
<td>$M_7=15$</td>
<td>$K_7=20000$</td>
<td>$C_7=6$</td>
</tr>
</tbody>
</table>
The comparison results of the second Sub(1) → Sub(2) ← Sub(3) type three-substructure case defined in Table 2 and Figure 6 are shown in Figure 10 for the same set of $H_{S11}, H_{S12}, H_{S13}, H_{S14}, H_{S15}, H_{S16}$ and $H_{S17}$ response functions.

Table 2. Parameters for the second three-substructure system.

<table>
<thead>
<tr>
<th>Model</th>
<th>Mass(Kg)</th>
<th>Stiffness(N/m)</th>
<th>Damping(Ns/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substructure A</td>
<td>$M_1=4$</td>
<td>$K_1=90000$</td>
<td>$C_1=5$</td>
</tr>
<tr>
<td></td>
<td>$M_2=10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Damped Elastic Connector I</td>
<td></td>
<td>$K_2=50000$</td>
<td>$C_2=10$</td>
</tr>
<tr>
<td>Substructure B</td>
<td>$M_3=10$</td>
<td>$K_3=12000$</td>
<td>$C_3=110$</td>
</tr>
<tr>
<td></td>
<td>$M_4=7$</td>
<td>$K_4=60000$</td>
<td>$C_4=12$</td>
</tr>
<tr>
<td></td>
<td>$M_5=9$</td>
<td>$K_5=80000$</td>
<td>$C_5=30$</td>
</tr>
<tr>
<td>Damped Elastic Connector II</td>
<td></td>
<td>$K_6=100000$</td>
<td>$C_6=6$</td>
</tr>
<tr>
<td>Substructure C</td>
<td>$M_6=6$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$M_7=15$</td>
<td>$K_7=20000$</td>
<td>$C_7=6$</td>
</tr>
</tbody>
</table>

Figure 9. Comparison results for $H_{S11}, H_{S12}, H_{S13}, H_{S14}, H_{S15}, H_{S16}$ and $H_{S17}$. (Keys: solid line -----, result of the first 3-substructure model; dotted line -·-·-·, exact result of complete system model).
Figure 10. Comparison results for $H_{S11}$, $H_{S12}$, $H_{S13}$, $H_{S14}$, $H_{S15}$, $H_{S16}$ and $H_{S17}$. (Keys: solid line ——, result of the second 3-substructure model; dotted line ——, exact result of complete system model).

Table 3. Parameters for the third three-substructure system.

<table>
<thead>
<tr>
<th>Model</th>
<th>Mass (Kg)</th>
<th>Stiffness (N/m)</th>
<th>Damping (Ns/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substructure A</td>
<td>$M_1=4$</td>
<td>$K_1=90000$</td>
<td>$C_1=5$</td>
</tr>
<tr>
<td></td>
<td>$M_2=10$</td>
<td>$K_2=50000$</td>
<td>$C_2=10$</td>
</tr>
<tr>
<td></td>
<td>$M_3=10$</td>
<td></td>
<td>$C_3=110$</td>
</tr>
<tr>
<td>Damped Elastic Connector I</td>
<td></td>
<td>$K_3=12000$</td>
<td></td>
</tr>
<tr>
<td>Substructure B</td>
<td>$M_4=7$</td>
<td>$K_4=60000$</td>
<td>$C_4=12$</td>
</tr>
<tr>
<td></td>
<td>$M_5=9$</td>
<td>$K_5=80000$</td>
<td>$C_5=30$</td>
</tr>
<tr>
<td></td>
<td>$M_6=6$</td>
<td>$K_6=100000$</td>
<td>$C_6=6$</td>
</tr>
<tr>
<td>Damped Elastic Connector II</td>
<td></td>
<td>$K_7=20000$</td>
<td></td>
</tr>
<tr>
<td>Substructure C</td>
<td>$M_7=15$</td>
<td></td>
<td>$C_7=6$</td>
</tr>
<tr>
<td></td>
<td>$M_8=4$</td>
<td>$K_8=50000$</td>
<td>$C_8=50$</td>
</tr>
<tr>
<td></td>
<td>$M_9=7$</td>
<td>$K_9=15000$</td>
<td>$C_9=30$</td>
</tr>
<tr>
<td>Damped Elastic Connector III</td>
<td></td>
<td>$K_{10}=120000$</td>
<td></td>
</tr>
</tbody>
</table>

Similarly, Figure 11 shows the comparison results of the third circular Sub (1) → Sub (2) → Sub (3) → Sub (1) type three-substructure case defined in Table
3 and Figure 7.

![Graphs of Amplitude and Phase for HS11(w), HS12(w), and HS13(w)]
Figure 11. Comparison results for \(H_{S11}, H_{S12}, H_{S13}, H_{S14}, H_{S15}, H_{S16}, H_{S17}, H_{S18}\) and \(H_{S19}\). (Keys: solid line , result of the third 3-substructure model; dotted line , exact result of complete system method).

### 2.6.2 Four-substructure Model

**Table 4. Parameters for four-substructure system.**

<table>
<thead>
<tr>
<th>Model</th>
<th>Mass(Kg)</th>
<th>Stiffness(N/m)</th>
<th>Damping(Ns/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substructure A</td>
<td>(M_1=4)</td>
<td>(K_1=90000)</td>
<td>(C_1=5)</td>
</tr>
<tr>
<td></td>
<td>(M_2=10)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Damped Elastic Connector I</td>
<td>(M_3=10)</td>
<td>(K_2=50000)</td>
<td>(C_2=10)</td>
</tr>
<tr>
<td>Substructure B</td>
<td>(M_4=7)</td>
<td>(K_3=12000)</td>
<td>(C_3=110)</td>
</tr>
<tr>
<td></td>
<td>(M_5=9)</td>
<td>(K_4=60000)</td>
<td>(C_4=12)</td>
</tr>
<tr>
<td>Damped Elastic Connector II</td>
<td>(M_6=6)</td>
<td>(K_5=80000)</td>
<td>(C_5=30)</td>
</tr>
<tr>
<td>Substructure C</td>
<td>(M_7=15)</td>
<td>(K_6=100000)</td>
<td>(C_6=6)</td>
</tr>
<tr>
<td></td>
<td>(M_8=4)</td>
<td>(K_7=20000)</td>
<td>(C_7=6)</td>
</tr>
<tr>
<td>Damped Elastic Connector III</td>
<td>(M_9=7)</td>
<td>(K_8=50000)</td>
<td>(C_8=50)</td>
</tr>
<tr>
<td>Substructure D</td>
<td>(M_9=7)</td>
<td>(K_9=15000)</td>
<td>(C_9=30)</td>
</tr>
<tr>
<td></td>
<td>(M_{10}=12)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Finally, for the four-substructure case defined in Table 4 and Figure 8, the comparison results are shown in Figure 12 for $H_{S11}, H_{S12}, H_{S13}, H_{S14}, H_{S15}, H_{S16}, H_{S17}, H_{S18}, H_{S19}$ and $H_{S1,10}$ response functions. Like Figure 9, all the three cases show excellent correlation between the multi-substructure predictions and the exact results.
2.7. Conclusion

In this chapter, a new one-step spectral-based multi-substructure approach has been formulated which can accurately represent systems comprising of any number of substructures. In the derivations, detailed formulations of three-substructure,
four-substructure and N-substructure systems have been presented. Predictions from
the multi-substructure model match the exact results of the complete system very well.
The newly proposed spectral-based direct multi-substructure approach is expected to
be sufficiently versatile for analyzing complex mechanical systems made up from
many inter-connected substructures. Also, the single-step calculation sequence is
expected to improve the calculation efficiency and ease digital programming. Finally,
the proposed multi-substructure method may be more suitable in revealing the
physical relationship among the substructures. Further study is underway to
demonstrate these key salient features.
3 TIME-DOMAIN TWO-SUBSTRUCTURE TRANSFER PATH ANALYSIS

METHOD

3.1 Introduction

In this chapter, a time-domain transfer path analysis method that combine the spectral-based two substructure method proposed by Zhen et al. [13-15] and a discrete, piecewise convolution theory is introduced. The discrete, piecewise convolution divides the nonlinear motion process into sufficient number of time segments. Each segment is assumed to be quasi-linear. The total nonlinear response is the sum of all the response of all temporal segments. To demonstrate the nonlinear features of the proposed approach, numerical studies employing a lumped parameter mass-spring-damper dynamic model being subjected to an impulse excitation are conducted, and comparison of temporal response from the 2-substructure single-path model to direct numerical integration of the full, un-simplified, nonlinear differential equations of motion are performed. The analysis also examines some of the limitations of the proposed time-domain TPA method especially when the system is highly nonlinear and very lightly damped. Several cases were studied to illustrate the effects of damping level, degree of nonlinearity and excitation magnitude.

The chapter is organized as follows. Firstly, the spectral-based two substructure method proposed by Zhen et al. [13-15] is introduced in Section 3.2. Secondly, in Section 3.3, idea of time domain transfer function is introduced. Thirdly, in Section 3.4, a lumped parameter mass-spring-damper dynamic model being subjected to an impulse excitation is built in two cases: (1) linear coupling; (2)
nonlinear coupling. Finally, in Section 3.5, numerically results comparison is given.

3.2 Two-substructure System

Suppose a mechanical system can be divided into two substructures. For example, a vehicle system is made up of its body and passenger compartment as one substructure that is coupled to the powertrain and frame as the other substructure. Given this 2-substructure system, it is possible to express the system frequency response function in terms of the individual substructure dynamic characteristics. To demonstrate this mathematically, we consider a 4-degree-of-freedom, 2-substructure, single coupling path lumped parameter system as shown in Figure 13. This system comprises of three main parts: (1) substructure A; (2) substructure B; (3) a pair of dynamic stiffness and damping that connect both substructures. The corresponding mathematical model is described below by applying the approach suggested by Zhen et al. [13-15]

The transfer function relationship denoted by \([H_A(\omega)]\) between an arbitrary input excitation coordinate vector \(\{F_A(\omega)\}\) and an arbitrary output response coordinate vector \(\{X_A(\omega)\}\) in substructure A can be described as

\[
\{X_A(\omega)\} = [H_A(\omega)]\{F_A(\omega)\}
\]

where the frequency response function (FRF) matrix of substructure A can be expressed as

\[
[H_A(\omega)] = \begin{bmatrix}
H_{A11}(\omega) & H_{A12}(\omega) \\
H_{A21}(\omega) & H_{A22}(\omega)
\end{bmatrix} = (-\omega^2 [M_A] + j\omega [C_A] + [K_A])^{-1}
\]

In the above equation, the mass \([M_A]\), damping \([C_A]\) and stiffness \([K_A]\) matrices
are

\[ M_A = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \]  
(55a)

\[ C_A = \begin{bmatrix} C_1 & -C_1 \\ -C_1 & C_1 \end{bmatrix} \]  
(55b)

\[ K_A = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \]  
(55c)

Accordingly, equation (53) can be rewritten as

\[
\begin{bmatrix} X_{A1}(\omega) \\ X_{A2}(\omega) \end{bmatrix} = \begin{bmatrix} H_{A11}(\omega) & H_{A12}(\omega) \\ H_{A21}(\omega) & H_{A22}(\omega) \end{bmatrix} \begin{bmatrix} F_{A1}(\omega) \\ F_{A2}(\omega) \end{bmatrix}
\]  
(56)

where \( X_{A1}(\omega) \) and \( X_{A2}(\omega) \) are the response functions of the two coordinates in substructure A, \( F_{A1}(\omega) \) and \( F_{A2}(\omega) \) are the possible applied excitation functions in substructure A, and \( H_{A11}(\omega) \), \( H_{A12}(\omega) \), \( H_{A21}(\omega) \), \( H_{A22}(\omega) \) are the transfer functions corresponding to the two coordinates of interest in substructure A. For a more complex substructure with significantly larger number of coordinates, only those of interest are tracked and not all coordinates have to be represented in this matrix. As shown in Figure 13, \( X_{A2}(\omega) \) is the coupling coordinate.

Similarly, the transfer function relationship denoted by \( [H_B(\omega)] \) between an input excitation coordinate vector \( \{F_B(\omega)\} \) and an output response coordinate vector \( \{X_B(\omega)\} \) in substructure B can be formulated using the same approach for substructure A. In this case, the mass \( [M_B] \), damping \( [C_B] \) and stiffness \( [K_B] \) matrices of substructure B version of equation (54) are given by

\[ M_B = \begin{bmatrix} M_3 & 0 \\ 0 & M_4 \end{bmatrix} \]  
(57a)
\[ C_B = \begin{bmatrix} C_3 & -C_3 \\ -C_3 & C_3 + C_4 \end{bmatrix} \]  

(57b)

\[ K_B = \begin{bmatrix} K_3 & -K_3 \\ -K_3 & K_4 \end{bmatrix} \]  

(57c)

Also, the substructure B version of equation (56) can be obtained by replacing subscripts A with B, 1 with 3 and 2 with 4 to yield

\[
\begin{bmatrix}
X_{B3}(\omega) \\
X_{B4}(\omega)
\end{bmatrix} =
\begin{bmatrix}
H_{B33}(\omega) & H_{B34}(\omega) \\
H_{B43}(\omega) & H_{B44}(\omega)
\end{bmatrix}
\begin{bmatrix}
F_{B3}(\omega) \\
F_{B4}(\omega)
\end{bmatrix}
\]

(58)

In this case, the coupling coordinate is \( X_{B3}(\omega) \).

Employing the same process used for substructures A and B, the relationship between the external excitation vector \( \{F_S(\omega)\} \) and response vector \( \{X_S(\omega)\} \) in a complete system can be expressed as follows:

\[
\{X_S(\omega)\} = [H_S(\omega)] \{F_S(\omega)\}
\]

(59)

which can be expanded into

\[
\begin{bmatrix}
X_{S1}(\omega) \\
X_{S2}(\omega) \\
X_{S3}(\omega) \\
X_{S4}(\omega)
\end{bmatrix} =
\begin{bmatrix}
H_{S11}(\omega) & H_{S12}(\omega) & H_{S13}(\omega) & H_{S14}(\omega) \\
H_{S21}(\omega) & H_{S22}(\omega) & H_{S23}(\omega) & H_{S24}(\omega) \\
H_{S31}(\omega) & H_{S32}(\omega) & H_{S33}(\omega) & H_{S34}(\omega) \\
H_{S41}(\omega) & H_{S42}(\omega) & H_{S43}(\omega) & H_{S44}(\omega)
\end{bmatrix}
\begin{bmatrix}
F_{S1}(\omega) \\
F_{S2}(\omega) \\
F_{S3}(\omega) \\
F_{S4}(\omega)
\end{bmatrix}
\]

(60)

By applying the superposition method since each free substructure is assume to behave linearly, the above relationship can also be re-expressed as

\[
\{X_S(\omega)\} = \left[H_{Sub}(\omega)\right] \{F_S(\omega)\} + \left[H_C(\omega)\right] \{F_{C(S)}(\omega)\}
\]

(61)

where \( H_{Sub}(\omega) \) is the transfer function matrix that only comprises of the two sets of transfer function vectors of free substructures A and B, \( H_C(\omega) \) is the transfer function matrix related to the coupling part, and \( \{F_{C(S)}(\omega)\} \) is the internal reaction force vector of the coupling spring and damping. These terms are explicitly given by
$$\begin{bmatrix} H_{A12}(\omega) & H_{A12}(\omega) & 0 \\ H_{A21}(\omega) & H_{A22}(\omega) & \end{bmatrix}$$ (62a)

$$\begin{bmatrix} \alpha H_{A12}(\omega) \\ \alpha H_{A22}(\omega) \\ H_B33(\omega) \end{bmatrix} = \begin{cases} \alpha = 1, \beta = -1 & \text{for } x = A \\ \alpha = -1, \beta = 1 & \text{for } x = B \end{cases}$$ (62b)

$$F_{C(x)}(\omega) = \begin{cases} (K_x + j\omega C_2)(X_3(\omega) - X_2(\omega)) & \text{for } x = A \\ (K_x + j\omega C_2)(X_2(\omega) - X_3(\omega)) & \text{for } x = B \end{cases}$$ (62c)

where \(X_2\) and \(X_3\) are the displacements of masses 2 and 3 respectively.

Using equations (60-61) to substitute for \(X_2\) and \(X_3\) in the above equation and rearranging to yield another expression for \(F_{C(A)}(\omega)\) given by

$$F_{C(A)}(\omega) = \left( H_{A22}(\omega) + H_{B33}(\omega) + (K_x + j\omega C_2)^{-1} \right)^{-1} \begin{bmatrix} -H_{A21}(\omega) \\ -H_{A22}(\omega) & H_{B33}(\omega) \\ H_{B34}(\omega) \end{bmatrix} \{F_x(\omega)\}$$ (63)

Since equation (59) is equivalent to equation (61), by substituting \(F_{C(A)}(\omega)\) in equation (63) back into equation (61) and eliminating the external excitation vector \(\{F_x(\omega)\}\) in both equations (59) and (61), the \([H_S]\) matrix can be further expressed in terms of the substructure transfer functions and coupling stiffness and damping parameters:

$$\begin{bmatrix} H_{S11}(\omega) & H_{S12}(\omega) & H_{S13}(\omega) & H_{S14}(\omega) \\ H_{S21}(\omega) & H_{S22}(\omega) & H_{S23}(\omega) & H_{S24}(\omega) \\ H_{S31}(\omega) & H_{S32}(\omega) & H_{S33}(\omega) & H_{S34}(\omega) \\ H_{S41}(\omega) & H_{S42}(\omega) & H_{S43}(\omega) & H_{S44}(\omega) \end{bmatrix}$$
The above formulation essentially provides a way to compute the coupled system response using the dynamic characteristics of the free substructures for the lumped parameter system shown in Figure 13.

\[
\begin{bmatrix}
H_{411}(\omega) & H_{412}(\omega) & 0 \\
H_{421}(\omega) & H_{422}(\omega) & 0 \\
0 & H_{433}(\omega) & H_{434}(\omega) \\
0 & H_{443}(\omega) & H_{444}(\omega)
\end{bmatrix}
\]

\[
\begin{bmatrix}
H_{A21}(\omega) \\
H_{A22}(\omega) \\
-H_{B33}(\omega) \\
-H_{B43}(\omega)
\end{bmatrix}
\begin{bmatrix}
H_{A21}(\omega) \\
H_{A22}(\omega) \\
-H_{B33}(\omega) \\
-H_{B43}(\omega)
\end{bmatrix}^T
\]  

\[
[C] = \left( H_{422}(\omega) + H_{B33}(\omega) + (K_2 + j\omega C_2)^{-1}\right)^{-1} \tag{64b}
\]

Figure 13. A 4-degree-of-freedom, 2-substructure, single-path lumped mass-spring-damper system.
3.3 Time Domain Transfer Function

A time-domain transfer function (TTF) or also generally known as the impulse response function \( h(t) \) can be defined as the inverse Fourier transform of a frequency domain transfer function that is often referred to as a frequency response function (FRF). Accordingly, the time-domain response \( x(t) \) is simply the convolution of TTF and the time-domain excitation function \( f(t) \) as depicted in Figure 14 where \( \delta(t) \) represents the unit impulse and \( \tau \) a real-valued number. The response of a linear time-invariant system due to \( \delta(t) \) is \( h(t) \). Hence, given an excitation \( \delta(t-\tau) \), the response is then \( h(t-\tau) \). Applying the theory of superposition, an excitation input of the form \( \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau \) will yield the corresponding response given by \( \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau \). Finally, if \( \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau \) equals to \( f(t) \) whose corresponding output is \( x(t) \) as noted above, then \( \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau \) is equal to \( x(t) \), which is the fundamental basis for the convolution integration applied in this study.

Considering the fundamental theory of convolution, the relationship between the excitation and response in time domain for a single-degree-of-freedom mechanical system can be expressed as

\[
x(t) = \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau = h(t) \otimes f(t)
\]  

(65)

that can be expanded as follows for a multiple degrees of freedom mechanical system:

\[
x_{m}(t) = \int_{-\infty}^{+\infty} f_{n}(\tau)h_{mn}(t-\tau)d\tau = h_{mn}(t) \otimes f_{n}(t)
\]  

(66)

In the above equations (65) and (66), \( \otimes \) represents the convolution as described earlier, \( m \) is the response coordinate and \( n \) corresponds to the excitation coordinate.
Note that the impulse response function \( h(t) \) is analogous to the frequency domain transfer function \( H(\omega) \) but operates in time domain.

\[
\begin{align*}
\delta(t) & \quad h(t) \\
\delta(t-\tau) & \quad h(t-\tau) \\
f(\tau)\delta(t-\tau) & \quad f(\tau)h(t-\tau) \\
\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau & \quad \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \\
f(t) & \quad x(t)
\end{align*}
\]

**Figure 14.** Fundamental theory of time-domain convolution employed in this study.

Since the above convolution integration assumes linear behavior, it is theoretically not valid when the system is characteristically nonlinear. In this case, the convolution cannot be described as an integral. To overcome this limitation, a discrete form based on piecewise quasi-linear principle is applied. This piecewise discrete convolution method divides the nonlinear process into a number of time steps that is sufficiently small, such that within each time step, the system is assumed to be approximately linear. The discrete form of equation (65) is given by

\[
x(t_i) = \sum_{\tau=-\infty}^{\infty} f(\tau)h(t_i-\tau) \tag{67}
\]

where \( t_i \) represents the \( i \)-th time step.
For multi-degree-of-freedom systems, the above equation can be expanded into the following form:

\[
x_m(t_i) = \sum_{n=1}^{\infty} \sum_{\tau=-\infty}^{\infty} f_n(\tau) h_{mn}(t_i - \tau) \rightarrow h_{mn}(t_i) = F^{-1}(H_{Smn,1}(\omega))
\]

\[
x_m(t_2) = \sum_{n=1}^{\infty} \sum_{\tau=-\infty}^{\infty} f_n(\tau) h_{mn}(t_2 - \tau) \rightarrow h_{mn}(t_2) = F^{-1}(H_{Smn,2}(\omega))
\]

\[
x_m(t_i) = \sum_{n=1}^{\infty} \sum_{\tau=-\infty}^{\infty} f_n(\tau) h_{mn}(t_i - \tau) \rightarrow h_{mn}(t_i) = F^{-1}(H_{Smn,i}(\omega))
\]

where \( F^{-1} \) represents inverse fast fourier transform (IFFT), \( H_S \) represents the complete system transfer function matrix obtained from equations (64a) and (64b), \( t_1, t_2, \) and \( t_i \) represent the first, second and the \( i \)-th time step, and \( m, n, i \) represent the row, column and the \( i \)-th step, respectively.

The results of the discrete time-domain response and TTF can be computed by using equations (68a), (68b) and (68c), and sorted sequentially according to each time step as given below:

\[
x_m(t_i) = [x_m(t_1), x_m(t_2), \ldots, x_m(t_i)]
\]

\[
h_{mn}(t_i) = [h_{mn}(t_1), h_{mn}(t_2), \ldots, h_{mn}(t_i)]^T
\]

where \( x_m(t_i) \) is the response value at time step \( t_i \), \( x_m(t) \) is the aggregate nonlinear time-domain response vector, \( h_{mn}(t_i) \) is the TTF at time step \( t_i \), and \( h_{mn}(t) \) is the aggregate TTF vector. A set of typical results are illustrated in Figures 15a and 15b for \( x_2(t) \) and \( h_{21}(t) \) respectively. Note that the main difference between the response vector \( x_2(t) \) and the TTF vector \( h_{21}(t) \) is that \( x_2(t) \) is comprised of a group of single point value, but \( h_{21}(t) \) consists of a family of impulse response curves where each curve represents the TTF in each time step shown in equations
(68a), (68b) and (16c). Thus, as shown in Figure 15a, each discrete point represents the actual response of $x_2(t)$. However, in Figure 15b, each of the curves represents the impulse response function $h_{21}(t)$ at time $t$. An example of discrete convolution in Matlab code is shown in appendix.

![Figure 15a. Example of $X_2(t)$ response for $\varepsilon=0$.](image1)

![Figure 15b. Example of $h_{21}(t)$ response for $\varepsilon=0.01$.](image2)
3.4 Lumped Parameter System

To develop the transfer path analysis in time domain, a 4-degree-of-freedom, 2-substructure single-path lumped parameter system subjected to an impulse excitation is employed as depicted in Figure 13. Using this system, both linear and nonlinear cases are studied. The linear case is mainly used to validate the fundamental theory by comparing the results of two-substructure model to the response predicted using the exact formulation of the complete lumped parameter system. On the other hand, in the nonlinear case, because a closed form solution does not exist, the results of two-substructure model are compared to the predicted response obtained from direct application of numerical integration on the raw differential equations of motion. The analyses of both cases are described further in the next section.

3.4.1 Linear Coupling Case

First, we consider the coupling terms in the lumped mass-spring-damper system shown in Figure 13 as linear elements of \( K_2 \) and \( C_2 \). An impulse form of external force \( f_i(t) \) shown in Figure 16 is applied to mass 1. The impulsive force is about 0.02 second long, and the peak value reaches \( 10^3 \) N. For the complete system, the response matrix of the four masses are derived exactly by processing the convolution of TTF matrix and the time-domain force vector. Since there is only one excitation force \( f_i(t) \), the relationship between the input and output coordinates is

\[
\begin{bmatrix}
  x_1(t) \\
  x_2(t) \\
  x_3(t) \\
  x_4(t)
\end{bmatrix} = \begin{bmatrix}
  h_{11}(t) & h_{12}(t) & h_{13}(t) & h_{14}(t) \\
  h_{21}(t) & h_{22}(t) & h_{23}(t) & h_{24}(t) \\
  h_{31}(t) & h_{32}(t) & h_{33}(t) & h_{34}(t) \\
  h_{41}(t) & h_{42}(t) & h_{43}(t) & h_{44}(t)
\end{bmatrix} \otimes \begin{bmatrix}
  f_i(t) \\
  0 \\
  0 \\
  0
\end{bmatrix}
\]  

(70)
The TTF matrix with elements \( h_{ij}(t), i,j=1,2,3,4 \), can be substituted by the inverse Fourier Transform of the FRF yielding

\[
\begin{bmatrix}
  x_1(t) \\
  x_2(t) \\
  x_3(t) \\
  x_4(t)
\end{bmatrix} = \begin{bmatrix}
  h_{11}(t) \otimes f_1(t) \\
  h_{21}(t) \otimes f_1(t) \\
  h_{31}(t) \otimes f_1(t) \\
  h_{41}(t) \otimes f_1(t)
\end{bmatrix} = \begin{bmatrix}
  F^{-1}(H_{11}(\omega)) \otimes f_1(t) \\
  F^{-1}(H_{21}(\omega)) \otimes f_1(t) \\
  F^{-1}(H_{31}(\omega)) \otimes f_1(t) \\
  F^{-1}(H_{41}(\omega)) \otimes f_1(t)
\end{bmatrix}
\]

(71)

where \( H_{11}(\omega), H_{21}(\omega), H_{31}(\omega) \) and \( H_{41}(\omega) \) are extracted from the complete system FRF matrix \( [H(\omega)] \) that can be expressed as

\[
[H(\omega)] = \begin{bmatrix}
  H_{11}(\omega) & H_{12}(\omega) & H_{13}(\omega) & H_{14}(\omega) \\
  H_{21}(\omega) & H_{22}(\omega) & H_{23}(\omega) & H_{24}(\omega) \\
  H_{31}(\omega) & H_{32}(\omega) & H_{33}(\omega) & H_{34}(\omega) \\
  H_{41}(\omega) & H_{42}(\omega) & H_{43}(\omega) & H_{44}(\omega)
\end{bmatrix} = (-\omega^2[M] + j\omega[C] + [K])^{-1}
\]

(72)

where the mass \([M]\), damping \([C]\), and stiffness \([K]\) matrices are

\[
[M] = \begin{bmatrix}
  M_1 & 0 & 0 & 0 \\
  0 & M_2 & 0 & 0 \\
  0 & 0 & M_3 & 0 \\
  0 & 0 & 0 & M_4
\end{bmatrix}
\]

(73a)

\[
[C] = \begin{bmatrix}
  C_1 & -C_1 & 0 & 0 \\
  -C_1 & C_1 + C_2 & -C_2 & 0 \\
  0 & -C_2 & C_2 + C_3 & -C_3 \\
  0 & 0 & -C_3 & C_3 + C_4
\end{bmatrix}
\]

(73b)

\[
[K] = \begin{bmatrix}
  K_1 & -K_1 & 0 & 0 \\
  -K_1 & K_1 + K_2 & -K_2 & 0 \\
  0 & -K_2 & K_2 + K_3 & -K_3 \\
  0 & 0 & -K_3 & K_3 + K_4
\end{bmatrix}
\]

(73c)

In the 2-substructure setup, the response matrix of the four masses can also be obtained by the convolution of the TTF matrix and the time-domain force vector similar to the complete system. The primary difference here is that the TTF matrix in the 2-substructure case is derived from the frequency response functions that are
calculated from the 2-substructure theory discussed earlier.

Similar to complete system method shown in equation (70), the response matrix is derived from the convolution of TTF matrix and the time-domain force vector. Simplifying the above equation (70) and replacing \( h(t) \) with \( H_s \) yields

\[
\begin{bmatrix}
    x_1(t) \\
    x_2(t) \\
    x_3(t) \\
    x_4(t)
\end{bmatrix} = \begin{bmatrix}
    h_{11}(t) \otimes f_1(t) \\
    h_{21}(t) \otimes f_1(t) \\
    h_{31}(t) \otimes f_1(t) \\
    h_{41}(t) \otimes f_1(t)
\end{bmatrix} = \begin{bmatrix}
    F^{-1}(H_{s11}(\omega)) \otimes f_1(t) \\
    F^{-1}(H_{s21}(\omega)) \otimes f_1(t) \\
    F^{-1}(H_{s31}(\omega)) \otimes f_1(t) \\
    F^{-1}(H_{s41}(\omega)) \otimes f_1(t)
\end{bmatrix}
\] (74)

Note that the above \( H_s \) terms can be computed from the dynamic characteristics of free substructures A and B using equation (64).

![Figure 16. Applied impulse force \( F_1(t) \).](image)

3.4.2. Nonlinear Coupling Case

Next, consider the coupling spring \( K_2 \) in the lumped parameter system of Figure 13 to be nonlinear. Mathematically, it can be represented generically as
\[ K_2 = (1 + \varepsilon (x_2 - x_3)^2)k_2 \]  

(75)

where \( k_2 \) is the stiffness value of the previous linearly coupled system and \( \varepsilon \) is a small non-linear coefficient used to express the degree of nonlinearity in the coupling. Hence, the main difference between the linear and non-linear systems is the coupling stiffness \( K_2 \). Note that this form of non-linear coupling may represent a wide variety of joining elements such as u-joints and dampers in automotive systems. The substructure themselves will remain principally linear internally as assumed earlier, which enables the fundamental transfer function representation be employed. Therefore, the mechanical system is comprised of two linearly behaving substructures coupled by a nonlinear stiffness element.

![Diagram](image-url)

*Figure 17. Piecewise discrete convolution algorithm.*

To compute the response of the nonlinear problem, the proposed piecewise discrete convolution that has been introduced above is applied to approximate the nonlinear response. The summary flowchart of the computational process is shown in
Figure 17. Within each time step, the system is assumed to be quasi-linear. $K_{2,j}$ can be computed from equation (75) and $H_{5,j}$ can be obtained by substituting $K_{2,j}$ into equation (64). Thus, the system response can be calculated in the same way as the above linear coupling case. By relying on this iterative process, the dynamic response can be computed by sorting each time step results sequentially as described in equation (69). To verify the accuracy, the predicted results will be compared to the calculations from the direct numerical integration method. The numerical algorithm is based on an explicit Runge-Kutta technique that is applied to perform the direct time-domain integration of the raw differential equations of motion. It is essentially a one-step solver for computing $x(t_i)$, and only requires the solution at the immediate preceding time point $x(t_{i-1})$ [17].

3.5. Numerical Results

3.5.1 Linear Response

To validate the basic formulation, the linear system is first analyzed. Here, the exact results from the complete system are compared to the response predicted from the 2-substructure method. The analysis is applied using the set of parameters given in Table 5.

<table>
<thead>
<tr>
<th>Model</th>
<th>Mass(Kg)</th>
<th>Stiffness(N/m)</th>
<th>Damping(Ns/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substructure A</td>
<td>$M_1=4$</td>
<td>$K_1=90000$</td>
<td>$C_1=5$</td>
</tr>
<tr>
<td>$M_2=10$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Damped Elastic Connector</td>
<td>$K_2=50000$</td>
<td>$C_2=10$</td>
<td></td>
</tr>
<tr>
<td>Substructure B</td>
<td>$M_3=10$</td>
<td>$K_3=12000$</td>
<td>$C_3=110$</td>
</tr>
<tr>
<td>$M_4=7$</td>
<td></td>
<td>$K_4=60000$</td>
<td>$C_4=12$</td>
</tr>
</tbody>
</table>

Figures 18-21 present the comparison results of displacement responses $x_i$,.
$x_2$, $x_3$ and $x_4$ from the exact solution of the complete system and 2-substructure prediction model. As shown in these figures, the results match exactly, which confirms that the proposed 2-substructure formulation coupled with the time-domain transfer path analysis is able to predict the impulse response of a linear system precisely.

Figure 18. Comparison of $X_1(t)$ of exact result of complete system and prediction from the two-substructure system in Table 5. (Keys: solid line , result of 2-substructures method; dotted line ( ), exact result of complete system).
Figure 19. Comparison of $X_2(t)$ of exact result of complete system and prediction from the two-substructure system in Table 5. (Keys: solid line ——, result of 2-substructures method; dotted line (••••••), exact result of complete system method).

Figure 20. Comparison of $X_3(t)$ of exact result of complete system and prediction from the two-substructure system in Table 5. (Keys: solid line ——, result of 2-substructures method; dotted line (••••••), exact result of complete system method).
3.5.2 Nonlinear Response

Having verified that the proposed time-domain 2-substructure model works exactly in the linear coupling case, we apply the same approach to the nonlinear problem where the coupling is nonlinear. To determine the accuracy of the approach, the results from the direct numerical integration method are employed as the basis for comparison to the 2-substructure time-domain transfer path analysis calculation.

As discussed above, to compare the results of both approaches, a set of computational parameters are assigned in Tables 6, 7 and 8. Also, an external impulse force applied on mass 1 is the same as the force on the linear system shown in Figure 16. To illustrate the salient features and limitation of the proposed time-domain 2-substructure approach, the following three cases are examined: (a) light damping; (b)
high damping; and (c) lower excitation amplitude. For the sake of brevity only the
response of mass 3 will be shown. It may be noted that the response trends for the
other lumped masses are similar.

Table 6. Parameters for lightly damped nonlinear case.

<table>
<thead>
<tr>
<th>Model</th>
<th>Mass(Kg)</th>
<th>Stiffness(N/m)</th>
<th>Damping(Ns/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substructure A</td>
<td>$M_1=4$</td>
<td>$K_1=90000$</td>
<td>$C_1=5$</td>
</tr>
<tr>
<td></td>
<td>$M_2=10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Damped Elastic</td>
<td></td>
<td>$K_2=50000(1+\varepsilon(x_2-x_3)^2)$</td>
<td>$C_2=10$</td>
</tr>
<tr>
<td>Connector</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Substructure B</td>
<td>$M_3=10$</td>
<td>$K_3=12000$</td>
<td>$C_3=110$</td>
</tr>
<tr>
<td></td>
<td>$M_4=7$</td>
<td>$K_4=60000$</td>
<td>$C_4=12$</td>
</tr>
</tbody>
</table>

In this section, as noted in Table 6, the light damping level is assigned to the
coupling elements. To illustrate the nonlinearity of coupling stiffness $K_2$ clearly, a
relationship between the elastic spring force and the displacement difference of $x_2$
and $x_3$ is presented in Figure 22 in which the nonlinear coefficient $\varepsilon$ is set to be 0.1.
As depicted in Figure 22, the relationship does not exhibit as a straight line, but a
curved one, where the curvature represents the degree of nonlinearity of the coupling
stiffness $K_2$. In Figures 23 and 24, when the non-linear coefficient $\varepsilon$ equals to zero or
a small value 0.0001, results of 2-substructure model matches almost exactly with that
of numerical integration. In Figures 25, 26, 27 and 28, when $\varepsilon$ equals to 0.001, 0.01,
0.1 and 0.5, respectively, visible deviation between the two results can be seen. In fact,
the deviation increases as $\varepsilon$ increases. The comparison of results from these two
methods indicates that in light damping case, when the non-linearity of coupling is
small, the numerical studies subjected to an impulse excitation show excellent
temporal response correlation between the function predicted from the 2-substructure
model and direct numerical integration results. Even though as the nonlinearity of the
coupling increases leading to greater deviation between the two calculations, the results still match reasonably well especially at the beginning high amplitude part. Results deviation of the 2-substructure model and direct numerical integration are depicted in Figure 29.

Figure 22. Relationship between $F_2(t)$ and $X_2(t)$-$X_3(t)$ in light damping nonlinear case when $\varepsilon=0.1$. 

Figure 23. Comparison of results when $\varepsilon=0$ for light damping nonlinear case.
(Keys: solid line , result of 2-substructure model; dashed line – – – , result of solving ordinary differential equation numerically).

Figure 24. Comparison of results when $\varepsilon=0.0001$ for light damping nonlinear case.
(Keys: solid line , result of 2-substructure model; dashed line – – – , result of solving ordinary differential equation numerically).
Figure 25. Comparison of results when $\varepsilon=0.001$ for light damping nonlinear case. (Keys: solid line - - - - - , result of 2-substructure model; dashed line - - - ., result of solving ordinary differential equation numerically).

Figure 26. Comparison of results when $\varepsilon=0.01$ for light damping nonlinear case. (Keys: solid line - - - - - , result of 2-substructure model; dashed line - - - ., result of solving ordinary differential equation numerically).
Figure 27. Comparison of results when $\varepsilon=0.1$ for light damping nonlinear case. (Keys: solid line — , result of 2-substructure model; dashed line --- , result of solving ordinary differential equation numerically).

Figure 28. Comparison of results when $\varepsilon=0.5$ for light damping nonlinear case. (Keys: solid line — , result of 2-substructure model; dashed line --- , result of solving ordinary differential equation numerically).
Figure 29. Results deviation of 2-substructure model and solving ordinary differential equation numerically when (a) $\varepsilon=0$, (b) $\varepsilon=0.0001$, (c) $\varepsilon=0.001$, (d) $\varepsilon=0.01$, (e) $\varepsilon=0.1$ and (f) $\varepsilon=0.5$, for light damping nonlinear case.
Table 7. Parameters for highly damped nonlinear case.

<table>
<thead>
<tr>
<th>Model</th>
<th>Mass(Kg)</th>
<th>Stiffness(N/m)</th>
<th>Damping(Ns/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substructure A</td>
<td>$M_1=4$</td>
<td>$K_1=90000$</td>
<td>$C_1=50$</td>
</tr>
<tr>
<td></td>
<td>$M_2=10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Damped Elastic Connector</td>
<td></td>
<td>$K_2=50000(1+\varepsilon(x_2-x_3)^2)$</td>
<td>$C_2=100$</td>
</tr>
<tr>
<td>Substructure B</td>
<td>$M_3=10$</td>
<td>$K_3=12000$</td>
<td>$C_3=110$</td>
</tr>
<tr>
<td></td>
<td>$M_4=7$</td>
<td>$K_4=60000$</td>
<td>$C_4=120$</td>
</tr>
</tbody>
</table>

Next, as noted in Table 7, higher damping level is assigned to the coupling elements. Similar to the above light damping case, the relationship between the force acting on $K_2$ and the displacements difference of $x_2$ and $x_3$ in Figure 29 does not exhibit as a straight line but a curved one, where the nonlinear coefficient $\varepsilon$ is set to be 0.1 and the curvature represents the degree of nonlinearity of coupling stiffness $K_2$. It is found that although the curvature is not as large as that in the light damping case, it still shows a significant level of nonlinearity. As depicted in Figures 30a, 30b and 30c, when $\varepsilon$ equals to 0, 0.0001 and 0.001 respectively, the results of the two methods match exactly. However, as depicted in Figures 30d, 30e and 30f, when $\varepsilon$ equals to 0.01, 0.1 and 0.5, respectively, visible deviation between these two approaches can be seen. Similarly to the above, the comparison of these results indicates that in higher damping case, when the non-linearity of coupling is small, the correlation between the 2-substructure model and direct numerical integration results are excellent. Like in the light damping case, even though as the nonlinearity in the coupling increases leading to more severe deviation between the two approaches, the two results still match quite well especially at the beginning high amplitude part. However, the observed deviation is not as severe as in the light damping case.
Figure 30. Relationship between $F_2(t)$ and $X_2(t)-X_3(t)$ for higher damping nonlinear case when $\varepsilon=0.1$
Figure 31. Comparison of results when (a) $\varepsilon=0$, (b) $\varepsilon=0.0001$, (c) $\varepsilon=0.001$, (d) $\varepsilon=0.01$, (e) $\varepsilon=0.1$ and (f) $\varepsilon=0.5$, for higher damping nonlinear case. (Keys: solid line , result of 2-substructure model; dashed line , result of solving ordinary differential equation numerically).

Next, the response of the parameters shown in Table 8 and Figure 31 with the lower excitation amplitude level impulse force (i.e. peak value of the force decreases to 10 N) is examined. Similar to the previous two cases, a plot of the relationship between the force acting on $K_2$ and the displacement difference of $x_2$ and $x_3$ is illustrated in Figure 32, where the nonlinear coefficient $\varepsilon$ is set to be 0.1. With the lower external force, the nonlinearity of the coupling stiffness $K_2$ diminishes significantly and acts nearly as a straight line. Accordingly, as depicted in Figures 33a, 21b, 21c, 21d, 21e and 21f, when $\varepsilon$ equals to 0, 0.0001, 0.001, 0.01, 0.1 and 0.5, respectively, all the results match exactly.

Table 8. Parameters for lightly damped nonlinear case with reduced force amplitude.

<table>
<thead>
<tr>
<th>Model</th>
<th>Mass(Kg)</th>
<th>Stiffness(N/m)</th>
<th>Damping(Ns/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substructure A</td>
<td>$M_1=4$</td>
<td>$K_1=90000$</td>
<td>$C_1=5$</td>
</tr>
<tr>
<td></td>
<td>$M_2=10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Damped Elastic Connector</td>
<td>$K_2=50000(1+\varepsilon(x_2-x_3)^2)$</td>
<td></td>
<td>$C_2=10$</td>
</tr>
<tr>
<td>Substructure B</td>
<td>$M_3=10$</td>
<td>$K_3=12000$</td>
<td>$C_3=110$</td>
</tr>
<tr>
<td></td>
<td>$M_4=7$</td>
<td>$K_4=60000$</td>
<td>$C_4=12$</td>
</tr>
</tbody>
</table>
Figure 32. Lower excitation amplitude impulse force $F_1(t)$.

Figure 33. Relationship between $F_2(t)$ and $X_2(t)-X_3(t)$ for the nonlinear case of lower impulse force amplitude when $\varepsilon=0.1$. 
Figure 34. Comparison of results when (a) $\varepsilon=0$, (b) $\varepsilon=0.0001$, (c) $\varepsilon=0.001$, (d) $\varepsilon=0.01$, (e) $\varepsilon=0.1$ and (f) $\varepsilon=0.5$, for the nonlinear case of lower impulse force amplitude. (Keys: solid line , result of 2-substructure model; dashed line , result of solving ordinary differential equation numerically).

3.6 Conclusion

In this study, a new time-domain transfer path analysis (TTF) method has been formulated by combining the spectral-based substructure method and discrete piecewise convolution scheme based on the classical convolution integral, and applied
successfully to study a lumped mass-spring-damper 4-degree-of-freedom system. When the system is linear and weakly non-linear, the results are exactly the same as the known system response. When nonlinearity is significant, visible deviation between the results of TTF 2-substructure method and the exact system response can be observed even though the overall match is still quite reasonable especially at the beginning high amplitude part. The source of error may be from the process of IFFT and convolution. Research effort is in progress to expand the study to multi-substructure and multi-path problem, as well as more practical automotive shock response problems involving nonlinear coupling. Also, the scope of nonlinearity the discrete piecewise convolution can handle will be examined in the future study. Since the proposed method is developed for continuous polynomial type nonlinearity, extension to nonlinear functions that are discontinuous is left for future work.
4 CONCLUSION AND RECOMMENDATIONS FOR FURTHER WORK

The objectives of this research are to expand the spectral-based substructure method from only two-substructure to a generalized N-substructure and formulate a newer time domain substructure method to deal with the transient problems of a mechanical system with nonlinear coupling.

The generalized frequency domain N-substructure method is a direct one-step method. It is summarized from the derivation of three-substructure formula and four-substructure formula. The comparison results show excellent agreement between the multi-substructure method and the complete system method. The proposed spectral-based direct multi-substructure approach is expected to be sufficiently versatile for analyzing complex mechanical systems made up from many inter-connected substructures. Also, the single-step calculation sequence is expected to improve the calculation efficiency and ease digital programming. Finally, the proposed multi-substructure method may be more suitable in revealing the physical relationship among the substructures. From another point of view, the generalized N-substructure method provides more choices for the substructure division where N could be from 2 to infinite. For instance, for a 9-substructure system, it can be divided to a variety of assembly subsystems, like 3-3-3, 2-3-4, 4-5 and so on. The appropriate division will be chosen for the sake of calculation convenience.

The time domain substructure transfer path analysis method is developed since the frequency domain TPA does not work well for nonlinear transient problems. It is derived by combining the spectral-based two substructure method and a discrete,
piecewise convolution theory. When the system is linear and weakly non-linear, the results are excellent. When nonlinearity is significant, although visible deviation of the results can be observed, the overall match is still quite reasonable especially at the beginning high amplitude part.

The proposed frequency-domain generalized N-substructure method and the time-domain substructure method optimize the utilization and expand the field of the application of substructure method in transfer path analysis. Future work will be: (1) applied the generalized frequency domain N-substructure method for more complicated multi-substructure mechanical system to demonstrate its key salient features; (2) expand the time-domain substructure method in the multi-substructure mechanical system with multi-nonlinear-coupling elements sets.
REFERENCES


APPENDIX

Part I: Parameters
Note: “ht” represents the impulse response function (TTF) and “ft” represents the external excitation.
clear all;
close all;
clc;
N=1024;
T=0.1;
delt=T/N;
t1=0:delt:(6*N-1)*delt;
t2=0:delt:(3*N-1)*delt;
Lt1=length(t1);
Lt2=length(t2);
leng=Lt1+Lt2-1;
ht=10*exp(-20*t1).*sin(20*pi*t1);
htt=[ht,zeros(1,(leng-Lt1))];
ft=5*sin(20*pi*t2);
ftt=[ft,zeros(1,(leng-Lt2))];

Part II: MATLAB Conv Command Method
Note: “conv” is the most common matlab command to compute the convolution in one step. “FT” below is the convolution results of “ht” and “ft”.
FT=conv(ht,ft);
figure;
plot(FT,'g--');

Part III: FFT method
Note: “fft” represents fast fourier transform, which can transform the signal from time-domain to frequency-domain and “ifft” represents the inverse fast fourier transform, which can transform the signal from frequency-domain back to time domain In fact, the convolution result of two signals in time domain equals to the inverse fourier transform of the product of the two signals in frequency domain.
H=fft(htt);
F=fft(ftt);
figure;
plot(htt);
figure;
plot(ftt);
x=H.*F;
x=ifft(X);
figure;
plot(x,'b');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Part IV: Discrete Convolution method
Note: Discrete convolution method can be utilized for nonlinear convolution since it calculates the results step by step. However, this discrete convolution method has no advantage by comparing with the above two methods in linear system since it solves for the convolution results analytically.

y=zeros(1,leng);
for i=1:leng
    temp=0;
    for j=1:leng
        if i>=j
            temp=temp+htt(j).*ftt(i+1-j);
        else
            break;
        end
    end
    y(i)=temp;
end
figure;
plot(y,'r--');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
End