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Semigroups and their Zero-Divisor Graphs

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Abstract

In this dissertation, we study the zero-divisor graphs of commutative semigroups with 0. The study of zero-divisor graphs was initiated by Istvan Beck in 1988, when he proposed a method for coloring a commutative ring by associating the ring to a simple graph, the vertices of which were defined to be the elements of the ring, with vertices \( x \) and \( y \) joined by an edge when \( xy = 0 \). In 1999 Anderson and Livingston changed this definition, restricting the set of vertices to the non-zero zero divisors of the ring, and from their paper work has proceeded in two directions. Specifically, Redmond investigated zero-divisor graphs of non-commutative rings, while DeMeyer, McKenzie, and Schneider looked at the zero-divisor graphs of commutative semigroups with 0. It is the second of these investigations on which we focus here. Starting with an overview of the essential information from graph theory, we quickly move to an investigation of the structure of semigroups with regard to their zero divisors, concluding that every semigroup can be partitioned into two subsemigroups, one of which is the set of zero divisors. Chapter III looks at the known results linking commutative semigroups and their zero-divisor graphs; in particular, we look at the results of DeMeyer and DeMeyer that determine a set of sufficient conditions for a given graph to be the zero-divisor graph of a commutative semigroup. Chapter IV focuses on extending these results, determining a larger set of graphs which must be the zero-divisor graph of a commutative semigroup. In Chapter V, we use these results to classify the connected graphs on six vertices as to whether or not each is the zero-divisor graph of a commutative semigroup. To accomplish this, we give specific examples of graphs that can be easily classified using the results of Chapter IV; however, we find that there are still some graphs to which the extended results do not apply, and for which we provide a method to classify them. The complete classification of the graphs on six vertices is given in Appendix 1. For graphs that are the zero-divisor graph of a commutative semigroup, we provide the Cayley table of a commutative semigroup; for those that are not, we provide a contradiction that prevents it from being such. In Chapter VI, we begin by noting the fact that every connected graph on three or four vertices is the graph of a commutative semigroup. In fact, most of the graphs are the zero-divisor graph of more than one commutative semigroup, and in the chapter we give methods for determining, up to isomorphism, all of the commutative zero-divisor semigroups for each of the graphs. The complete list of commutative zero-divisor semigroups for each of the graphs is given in Appendix 2. Finally, in Chapter VII, we extend the results of Redmond regarding ideal-based zero-divisor graphs of a commutative ring to the case of commutative semigroups, and close by commenting on a few properties that result from removing the assumption of commutativity from the semigroups.
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I. Introduction

In 1988, Istvan Beck began to investigate the possibility of coloring a commutative ring by associating to the ring a zero-divisor graph, defined as a simple graph, the vertices of which are the elements of the ring, with two distinct elements $x$ and $y$ being adjacent if and only if $xy = 0$ [4]. While the paper concentrated on the connection between the clique number and the chromatic number of the graph, the work inspired by Beck has concentrated on the interplay of rings and their zero-divisor graphs. Retaining the original definition, the next decade brought little progress. However, in 1999, Anderson and Livingston [3] changed the definition of the zero-divisor graph, defining the vertices of the graph to be the non-zero zero divisors of the commutative ring. This yielded a number of basic results on zero-divisor graphs of commutative rings, on which the work presented here is based. Subsequently, research has moved in several directions. For example, Anderson, Frazier, Lauve, and Livingston extended the results of [3] by looking at the clique number and planarity of zero-divisor graphs [2], while Akbari, Maimani, and Yassemi [1] investigated the properties necessary for a zero-divisor graph to be either planar or complete $r$-partite. Work was also done by Redmond [16], where he looked at some of the changes implied by basing the zero-divisor graph on a non-commutative ring. The results found in this present work are based primarily on the work of DeMeyer, McKenzie, and Schneider [9], where they based the zero-divisor graph on a commutative semigroup with 0. The results from their paper were extended in 2004 by DeMeyer and DeMeyer [8].
A natural question to ask is whether or not a given graph can be the zero-divisor graph of a commutative semigroup with 0. The answer turns out to be quite simple for connected graphs on three or four vertices, as all such graphs are the zero-divisor graph of a commutative semigroup with 0, and DeMeyer and DeMeyer [8] have provided a set of necessary and sufficient conditions for the graphs on five vertices. In this dissertation, we determine which of the graphs on six vertices can be the zero-divisor graph of a commutative semigroup with 0. To do this, we first prove that any commutative semigroup can be partitioned into two subsemigroups, one of which is the set of zero divisors. This allows us to restrict attention, without loss of generality, to commutative semigroups all of whose elements are zero divisors. We call such semigroups zero-divisor semigroups. We then use the necessary conditions stated in Theorem 1 of the DeMeyer and DeMeyer paper, extend the sufficient conditions stated in Theorem 3, and apply the results to as many of the graphs as we can. For the remainder of the graphs, we attempt to construct a Cayley table for a commutative semigroup with 0 for which the graph is the zero-divisor graph. To do this, we check for associativity by attempting to create a Light table for each of the non-zero elements of the hoped-for semigroup. In this manner, we either explicitly construct a semigroup for which the graph is the zero-divisor graph, or find a contradiction that prevents the graph from being such.

It turns out that if a graph can be the zero-divisor graph of a commutative semigroup with 0, then there infinitely many commutative semigroups with 0 having the graph as its zero-divisor graph. We determine, up to isomorphism, the zero-divisor semigroups that have each of the connected graphs on three or four vertices as their zero-divisor graph.
Furthermore, given a semigroup consisting of \( n \) non-zero zero divisors and its corresponding zero-divisor graph, we give a method for constructing commutative semigroups having an arbitrary number of elements \( \geq n + 1 \) and having the same zero-divisor graph. We also observe that this stands in stark contrast to the results obtained by Anderson and Livingston [3] in their work on zero-divisor graphs of commutative rings. In particular, we examine the multiplicative structure of the commutative rings on four and five elements, and compare them with the commutative semigroups found above, concluding that relatively few of the commutative semigroups give the multiplicative structure of the zero divisors of a commutative ring.

To conclude the paper, we present a few results currently unrelated to the above. First, we look at the results of Redmond [15], where he investigated the idea of basing the graph of a commutative ring on ideals rather than on zero divisors. While this was also investigated by Maimani, Pournaki, and Yassemi in [14], we extend the idea by basing the graph of a commutative semigroup on congruences, specifically the Rees congruence. Finally, we look at a few basic results that come from removing the assumption of commutativity from the semigroups.

To begin, we present the essential information from graph theory and semigroup theory used in the course of this paper, as well as several theorems that play a prominent role in our discussion.
II. Graph Theory and Semigroup Theory

In this chapter, we present the basic definitions from graph theory and semigroup theory essential to this paper. Unless otherwise noted, all definitions relating to graph theory are from Godsil and Royle [11], and all definitions relating to semigroup theory are from Howie [12]. The interested reader can find information from a purely graph-theoretic perspective in Diestel [7], while additional information about algebraic graph theory can be found in Biggs [5].

II.1 Graph Theory

A graph $X$ is a vertex set $V(X)$ and an edge set $E(X)$, where an edge is an unordered pair of distinct vertices of $X$. We will use $x$–$y$ to denote an edge, and the vertices that comprise an edge will be called adjacent vertices or neighbors. The complete set of neighbors for a vertex $x$ will be denoted $N(x)$, and the size of the set will be referred to as the valency of $x$. Note that $x$ is not an element of the set $N(x)$, so we will let $\overline{N(x)}$ denote $N(x) \cup \{x\}$. A vertex of valency 1 is called an end. If a vertex is one of the two vertices of an edge, the vertex is said to be incident with the edge. Note that with this definition of a graph, the possibility of an edge connecting a vertex to itself is not permitted, and neither is the possibility of two distinct vertices determining more than one edge. Such a graph is often referred to as a simple graph. However, since all of the graphs we consider will be of this type, we will omit the word “simple.”
A graph in which every pair of distinct vertices is adjacent is called a complete graph. We will denote the complete graph on \( n \) vertices by \( K_n \).

A subgraph \( Y \) of a graph \( X \) is a graph such that all of the vertices of \( Y \) are vertices of \( X \) and all of the edges of \( Y \) are edges of \( X \). The obvious subgraphs of a graph \( X \) are \( X \) itself, the null graph, consisting of no vertices and no edges, and the empty graph, consisting of one vertex from \( X \) and no edges. Other subgraphs include the induced subgraphs, which are obtained by removing some of the vertices of \( X \), along with any edges to which any of the removed vertices were incident. Of particular importance in graph theory is the concept of a clique, which is a complete subgraph.

A path of length \( r \) from vertex \( x \) to vertex \( y \) is a sequence of \( r + 1 \) distinct vertices, starting with \( x \) and ending with \( y \), such that consecutive vertices are adjacent. If for every pair of distinct vertices \( \{x, y\} \) there exists a path starting at \( x \) and ending at \( y \), then the graph is referred to as connected; otherwise, it is disconnected.

The distance between two vertices \( x \) and \( y \) is the length of the shortest path from \( x \) to \( y \), and will be denoted \( d(x, y) \). The diameter of a graph \( X \) is the maximum distance between two distinct vertices. This will be denoted \( \text{diam}(X) \).

A cycle is a connected (sub)graph where every vertex has exactly two neighbors, and an acyclic graph is a graph in which no subgraph is a cycle. The length of the shortest cycle of a graph \( X \) is known as the girth, and will be denoted \( \text{gr}(X) \). A connected, acyclic graph
is usually referred to as a tree. The core of a graph $X$ is the largest subgraph of $X$ in which every edge is the edge of a cycle in $X$ [8].

One particular type of graph is a **bipartite graph**, a graph $X$ whose vertex set $V(X)$ can be partitioned into two parts $V_1$ and $V_2$, such that every edge has one end in $V_1$ and one in $V_2$. A **complete bipartite graph** is a bipartite graph in which every vertex of $V_1$ is adjacent to every vertex of $V_2$.

![a bipartite graph](image1.png) ![a complete bipartite graph](image2.png)

Finally, we introduce the following definitions.

A graph on $n$ vertices such that $n - 1$ of the vertices are ends, all of which are adjacent only to the remaining vertex $x$, is called a **star graph with center** $x$. A graph which is the union of two star graphs whose centers $a$ and $b$ are connected by a single edge will be referred to as a **double-star graph with centers** $a$ and $b$, and the edge connecting the centers will be referred to as the **bridge of the double-star graph** (or just the **bridge** when the meaning is clear).

![a double-star graph with centers $a$ and $b$](image3.png)
A refinement of a graph $H$ is a graph $G$ such that the vertex sets of $G$ and $H$ are the same and every edge in $H$ is an edge in $G$. We extend this definition for a specific refinement in a double-star graph. A separably-refined double-star graph is one in which the removal of the bridge of the double-star graph causes the graph to become disconnected.

Note that in some graphs it will be possible for more than one pair of vertices to act as the centers of the double-star graph. For example, in the graph

vertices $a$ and $b$ could be the centers of the graph, as could the vertices $x$ and $y$, among others.
A polygon is a connected graph in which every vertex has valency 2. Polygons with a specific number of vertices will be called by the obvious names (a triangle is a polygon with three vertices, a quadrilateral a polygon with four vertices, etc.).

**II.2 Semigroup Theory**

**II.2.a Definitions**

A groupoid, denoted \((S, \cdot)\) is a set \(S\) with a binary operation \(\cdot\). A semigroup is a groupoid \((S, \cdot)\) in which the binary operation \(\cdot\) is associative. In this paper, the binary operation will be written multiplicatively. A commutative semigroup is a semigroup with the additional property that for any elements \(a\) and \(b\) in \(S\),

\[ ab = ba \]

We say that \(S\) is a semigroup with zero if \(S\) contains at least two elements, one of which is denoted \(0\), such that for all elements \(a\) in \(S\)

\[ a0 = 0a = 0. \]

The element \(0\) is referred to as the zero element of \(S\). Of particular note here is that this zero element is not an “additive identity”, as addition has not been defined on \(S\).

The zero element of a semigroup with zero is unique, since otherwise there would exist at least two elements, \(0\) and \(0'\), such that for all elements \(a\) in \(S\), both

\[ a0 = 0a = 0 \]
and

\[ a0' = 0'a = 0', \]

which would imply

\[ 0 = 00' = 0'. \]

If \( S \) is a semigroup, then a nonempty subset \( T \) of \( S \) is called a subsemigroup if for all \( x \) and \( y \) in \( T \), \( xy \) is an element of \( T \).

A non-empty subset \( A \) of a semigroup \( S \) is called a left ideal if \( SA \subseteq A \), and a right ideal if \( AS \subseteq A \). If \( A \) is both a right and left ideal, it is called a (two-sided) ideal.

**II.2.b Light’s Associativity Test**

Since the only requirements for a semigroup are closure and associativity, it is helpful to have a test for associativity. Given the Cayley table for a binary operation on a finite set, such a test exists. For the sake of completeness, we reproduce the test here, as found in [6]. As a note, the semigroups here are not necessarily commutative.

Let \((S, \cdot)\) be a finite groupoid. We can test to see if \( S \) is associative under \( \cdot \) as follows:

For each element \( s \) of \( S \),

1. Begin to create the \( s \)-table by copying the \( s \) row from the original \( \cdot \) - table onto the upper line of the \( s \)-table.
(2) Similarly, copy the $s$ column from the original $\,\cdot\,\,$-table into the left-hand column of the $s$-table.

(3) Copy the columns of the $\,\cdot\,\,$-table in the order specified by the upper line of the new $s$-table.

(4) If the rows of each of the new $s$-tables are just those of the original $\,\cdot\,\,$-table labeled by the left-hand column, then $S$ is associative under $\,\cdot\,$.

(5) If not, then the cells of the row that do not match those of the original $\,\cdot\,\,$-table indicate how the associativity fails. Specifically, and without loss of generality, let the first row of the original $\,\cdot\,\,$-table be the $a$-row and the second column of the original $\,\cdot\,\,$-table be the $b$-column. If the cell of the $s$-table that does not match is in row 1, column 2, then $(as)b \neq a(sb)$.

**Example:** Given the table

\[
\begin{array}{cccc}
  \cdot & a & b & c & d \\
  a & a & a & a & a \\
  b & a & b & b & b \\
  c & a & b & c & c \\
  d & a & b & c & d \\
\end{array}
\]

we illustrate the check for the $c$-table as described above.

(1) Begin to create the $c$-table by copying the $c$ row from the original $\,\cdot\,\,$-table onto the upper line of the $c$-table.
(2) Similarly, copy the \(c\) column from the original \(\textbullet\)-table into the left-hand column of the \(c\)-table.

\[
\begin{array}{cccc}
  c & a & b & c \\
\end{array}
\]

(3) Copy the columns of the \(\textbullet\)-table in the order specified by the upper line of the new \(c\)-table (i.e., fill the \(a\) column with the \(a\) column from the \(\textbullet\)-table, the \(b\) column with the \(b\) column from the \(\textbullet\)-table, and the two \(c\) columns with the \(c\) column from the \(\textbullet\)-table).

\[
\begin{array}{cccc}
  c & a & b & c \\
  a & a & a & a \\
  b & a & b & b \\
  c & a & b & c \\
  c & a & b & c \\
\end{array}
\]

(4) If the rows of the new \(c\)-table are just those of the original \(\textbullet\)-table labeled by the left-hand column, then we continue the process, checking the other element tables in the same way. If not, then the \(\textbullet\)-table is not associative.

Here, the \(a\) row of the \(c\)-table is the same as the \(a\) row of the \(\textbullet\)-table, the \(b\) row of the \(c\)-table is the same as the \(b\) row of the \(\textbullet\)-table, and both \(c\) rows of the \(c\)-table
are the same as the $c$ row of the $\bullet$-table. Since this is true for all of the remaining tables (the $a$-table, the $b$-table, and the $d$-table, shown below), the $\bullet$-table is associative.

Example: Given the table

$$
\begin{array}{c|cccc}
\cdot & a & b & c & d \\
\hline
a & a & a & a & a \\
b & a & a & a & a \\
c & a & b & b & b \\
d & a & b & c & c \\
d & a & b & c & d \\
\end{array}
$$

(1) Begin to create the $a$-table by copying the $a$ row from the original $\bullet$-table onto the upper line of the $a$-table.

$$
\begin{array}{c|cc}
\cdot & a & b \\
\hline
a & a & c \\
b & b & c \\
c & a & b \\
\end{array}
$$

(2) Similarly, copy the $a$ column from the original $\bullet$-table into the left-hand column of the $a$-table.
Copy the columns of the •-table in the order specified by the upper line of the new a-table.

If the rows of the new a-table are just those of the original •-table labeled by the left-hand column, then we continue by checking the b-table, followed by checking the c-table if we find no contradictions in the b-table. If the rows of the new a-table are just those of the original •-table labeled by the left-hand column, then we stop; this table is not associative.

The a-row of the a-table is not the same as the a-row of the •-table. Specifically, the problem is in the second cell, which should be c. This tells us that \((aa)b \neq a(ab)\). To check this, we see \((aa)b = ab = c\), but \(a(ab) = ac = a\).

There are other contradictions in the a-table, but we can stop the process here; the operation defined by the original table is not associative.
III. Fundamental Concepts and Important Theorems

We begin this chapter by defining the zero-divisor graph of a commutative semigroup with 0, along with other concepts linking graph theory to semigroup theory. We then present several theorems, both original and from other sources, which will play an essential role in our discussion.

III.1 Zero-Divisor Graphs of Commutative Semigroups with 0

We assume throughout that $S$ is a commutative semigroup with 0, and unless explicitly stated otherwise, it is implied that when we speak of a commutative semigroup it is actually a commutative semigroup with 0. The zero-divisor graph of $S$, denoted $\Gamma(S)$, is a graph with vertex set $V(\Gamma(S))$ the nonzero zero divisors of $S$, and edge set $E(\Gamma(S)) = \{(x, y) \mid x, y \in S, x \neq y \text{ and } xy = 0\}$. Note that, by definition, $\Gamma(S)$ is a simple graph.

III.2 The Theorems of DeMeyer and DeMeyer

Theorems 1-3 of DeMeyer and DeMeyer [8] will play a crucial role in classifying the graphs on six vertices as to whether or not each can be the zero-divisor graph of a commutative semigroup. We present them here, complete with proofs. To do this, we will need the following lemma of DeMeyer, McKenzie, and Schneider [9].

**Lemma III.1.** If $a \rightarrow x \rightarrow b$ is a path in a zero-divisor graph $\Gamma(S)$, then either $\{0, x\}$ is an ideal in $S$ or $a \rightarrow x \rightarrow b$ is contained in a cycle of length $\leq 4$.

**Proof.** Let $ann(a) = \{y \in S \mid ya = 0\}$. Either $ann(a) \cap ann(b) = \{0, x\}$, or there is a $c \notin \{0, x\}$ with $ac = bc = 0$. In the first case, $\{0, x\}$ is an ideal, since if there exists a $y$ in...
such that \( xy \notin \{0, x\} \), then \( xy = z \) for some \( z \) in \( S\{0, x\} \). But then \( z \in \text{ann}(a) \cap \text{ann}(b) \), since \( az = axy = 0 \) and \( zb = xyb = 0 \). This contradicts the fact that

\[
\text{ann}(a) \cap \text{ann}(b) = \{0, x\}.
\]

In the second case, \( a-x-b-c-a \) is a cycle of length \( \leq 4 \).

An important corollary to this lemma is the following:

**Corollary III.1.** If \( a, x \in V(\Gamma(S)) \) and \( x \) is adjacent to an end \( a \), then \( \{0, x\} \) is an ideal in \( S \).

**Proof.** If \( \Gamma(S) \) has only two vertices, then this is trivial. Otherwise, since \( a \) is an end and \( \Gamma(S) \) is connected, we know there exists a vertex \( b \) in \( \Gamma(S) \) such that \( x-b \) is an edge of \( \Gamma(S) \). Then \( a-x-b \) is a path in \( \Gamma(S) \) not contained in a cycle, and therefore \( \{0, x\} \) is an ideal in \( S \).

**Theorem III.1 (DeMeyer and DeMeyer [8]).** If \( \Gamma(S) \) is the zero-divisor graph of \( S \), a commutative semigroup, then \( \Gamma(S) \) satisfies all of the following conditions.

1. \( \Gamma(S) \) is connected.
2. \( \text{diam}(\Gamma(S)) \leq 3 \)
3. If \( \Gamma(S) \) contains a cycle, then
   
   (i) the core of \( \Gamma(S) \) is a union of triangles and quadrilaterals, and
   
   (ii) any vertex not in the core of \( \Gamma(S) \) is an end.
4. For each pair \( x, y \) of nonadjacent vertices of \( \Gamma(S) \), there is a vertex \( z \) with
   
   \( N(x) \cup N(y) \subset \overline{N(z)} \). We will refer to this as the neighbor condition.
Proof. For (1) and (2), let \( x, y \) be vertices in \( \Gamma(S) \). Then there exist \( z \neq 0 \) and \( w \neq 0 \) such that \( xz = yw = 0 \). If \( xy = 0 \), then \( x\text{–}y \) is a path of length 1 connecting \( x \) and \( y \). If \( xy \neq 0 \) and \( wz = 0 \), then \( x\text{–}z\text{–}w\text{–}y \) is a path of length 3 connecting \( x \) and \( y \). Finally, if \( xy \neq 0 \) and \( wz \neq 0 \), then \( x\text{–}wz\text{–}y \) is a path of length 2 connecting \( x \) and \( y \). Therefore, any two vertices can be connected by a path of length \( \leq 3 \).

For (3), let \( a \) be a vertex in the core of \( \Gamma(S) \) and assume that \( a \) is not in any triangle or quadrilateral in \( \Gamma(S) \). Then \( a \) is part of a cycle \( a\text{–}b\text{–}c\text{–}d\text{–}e\text{–}\cdots\text{–}a \), and by Lemma III.1, \( \{0, a\} \) is an ideal in \( S \). Then \( ad = ac = a \). Since \( cd = 0 \), we have \( 0 = a(cd) = (ac)d = (a)d \), a contradiction. Therefore, the core of \( \Gamma(S) \) must consist exclusively of triangles and quadrilaterals.

To show that any vertex not in the core of \( \Gamma(S) \) is an end of \( \Gamma(S) \), we begin by noting that since \( \Gamma(S) \) has a cycle, we know that \( \Gamma(S) \) has at least three vertices. Since \( diam(\Gamma(S)) \leq 3 \), one of the following must be true for any vertex \( a \) in \( \Gamma(S) \):

(i) \( a \) is in the core,

(ii) \( a \) is an end of \( \Gamma(S) \),

(iii) \( a\text{–}x\text{–}b \) is a path in \( \Gamma(S) \), with \( b \) in the core, or

(iv) \( a\text{–}x\text{–}y\text{–}b \) is a path in \( \Gamma(S) \) with \( b \) in the core.

To finish the proof of statement (3), we begin with (iv). Let \( a\text{–}x\text{–}y\text{–}b \) be a path in \( \Gamma(S) \), with \( b \) in the core and assume neither \( x \) nor \( y \) are in the core. Then \( b \) is a vertex of a triangle or quadrilateral. If \( c \) is a vertex of the triangle or quadrilateral such that \( b\text{–}c \) is an edge of \( \Gamma(S) \), then \( a\text{–}x\text{–}y\text{–}b\text{–}c \) is the shortest path connecting \( a \) and \( c \). This contradicts the fact that \( diam(\Gamma(S)) \leq 3 \). Therefore, at least one of \( x \) and \( y \) must be in the core. In other words, this must reduce to case (iii).
So assume that \(a\rightarrow x\rightarrow b\) is a path with \(b\) in the core. The question to be resolved here is whether or not \(x\) is in the core of \(\Gamma(S)\), since it is clearly not an end. So, assume that \(x\) is not in the core of \(\Gamma(S)\). By Lemma III.1, \(\{0, x\}\) is an ideal in \(S\) and \(x\rightarrow b\rightarrow c\rightarrow d\rightarrow b\) or \(x\rightarrow b\rightarrow c\rightarrow d\rightarrow e\rightarrow b\) is a sequence of connected vertices in \(\Gamma(S)\). Since \(x\) is not in the core, \(cx \neq 0\), so \(cx = x\). Then \(b\rightarrow c\rightarrow d\) is part of a cycle of length \(\leq 4\) in the core of \(\Gamma(S)\), and \(dx = d(cx) = (dc)x = 0\), which would make \(x\) a vertex in the cycle \(x\rightarrow b\rightarrow c\rightarrow d\rightarrow x\), contradicting the fact that \(x\) is not in the core of \(\Gamma(S)\). Therefore, \(x\) is in the core of \(\Gamma(S)\), and (3) is proven.

Finally, for (4), let \(x\) and \(y\) be nonadjacent vertices of \(\Gamma(S)\). Then \(xy = z\) for some \(z \neq 0\) in \(S\). If \(a \in N(x) \cup N(y)\), then either \(ax = 0\) or \(ay = 0\). Either way, \(az = a(xy) = 0\), so \(a \in \overline{N(z)}\). Therefore, \(N(x) \cup N(y) \subset \overline{N(z)}\).

**Theorem III.2 (DeMeyer and DeMeyer [8]).** If \(\Gamma\) is a graph satisfying conditions (1)-(4) of Theorem III.1 and there are five or fewer vertices in \(\Gamma\), then \(\Gamma\) is the graph of a semigroup.

**Proof.** The proof is given by enumerating all of the graphs with fewer than six vertices meeting conditions (1)-(4) and writing down a semigroup for each graph.

As a note, all of the connected graphs on three or four vertices meet the conditions of Theorem III.1, and therefore each is the graph of some commutative semigroup.

**Theorem III.3 (Demeyer and DeMeyer [8]).** The following graphs are the graphs of semigroups.

1. A complete graph or a complete graph together with an end.
A complete bipartite graph or a complete bipartite graph together with an end.

(3) A refinement of a star graph.

(4) A graph which has at least one end and has diameter $\leq 2$.

(5) A double-star graph.

**Proof.** For (1), a complete graph $\Gamma$ is the graph of the null semigroup $S$, in which for any two elements $a$ and $b$ in $S$, $ab = 0$. If $x$ is an additional vertex adjacent only to $a$ in $\Gamma$, then beginning with $S$, define a semigroup $T$ with elements $S \cup \{x\}$ and the following relations: $x^2 = x$, $xa = 0$, and $xb = b$ for all $b \neq a$ in $S$. We can use Light’s associativity test to verify that $T$ is, in fact, a semigroup, and from there it is straightforward to draw the graph of $\Gamma(T)$ and verify that this is the given complete graph with an end.

For (2), let $p$ and $q$ be any natural numbers, and let $\{a_i\}_{i=1}^p$ and $\{b_j\}_{j=1}^q$ be the vertices of the complete bipartite graph. Define a semigroup $S$, with elements the vertices of the graph, as follows.

(i) for every ordered pair $\{i, j\}$, $a_ib_j = 0$,

(ii) $a_i^2 = 0$

(iii) for every ordered pair $\{i, k\}$ with $i, k \neq 1$ and $i \leq k$, $a_ia_k = a_i$, and

(iv) for every ordered pair $\{j, l\}$ with $j \leq l$, $b_jb_l = b_l$.

The table determined by these properties is shown below:
Using Light’s associativity test, the table is easily verified to be the Cayley table of a commutative semigroup for which the complete bipartite graph is the zero-divisor graph.

For part (3), let $\Gamma$ be a refinement of a star graph with center $z$ and end vertices $\{x_i\}_{i=1}^p$ for some natural number $p \geq 2$. Define a semigroup $S$, with elements the vertices of the graph, as follows.

(i) $z^2 = 0$

(ii) for all $i$, $x_i^2 = z$ and $x_i z = 0$

(iii) if $x_i$ and $x_j$ are adjacent in $\Gamma$, $x_i x_j = 0$; otherwise, $x_i x_j = z$

Any triple product $a(bc) = (ab)c = 0$, so $S$ is a semigroup, and $\Gamma(S)$ is the zero-divisor graph.

For (4), let $\Gamma$ be a graph of diameter $\leq 2$ with an end vertex $x$. Let $z$ be adjacent to $x$.

Then because $diam(\Gamma) \leq 2$, all other vertices of $\Gamma$ are connected to $z$ by an edge, $z$ is the center of a star graph which spans $\Gamma$, and $\Gamma$ is the graph of a semigroup by (3).
Finally, for (5), let $x$ and $y$ be the centers of a double-star graph, with $\{a_i\}_{i=1}^p$ the ends adjacent to $x$ and $\{b_j\}_{j=1}^q \cup \{c\}$ the ends adjacent to $y$ for some natural numbers $p$ and $q$.

By Light’s associativity test, the following table is associative, and $\Gamma$ is the zero-divisor graph of $S$.

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### III.3 Other Important Theorems

**Corollary III.2.** A polygon $\Gamma$ with five or more vertices is never the zero-divisor graph of a commutative semigroup.

**Corollary III.3.** A polygon $\Gamma$ with five or more vertices and any number of additional ends is never the graph of a commutative semigroup.

**Proof.** Both corollaries follow immediately from Theorem III.1(3).
III.3.a The Subsemigroups $Z_0(S)$ and $S \setminus Z_0(S)$

**Theorem III.4.** Let $S$ be a commutative semigroup. If $Z(S)$ is the set of non-zero zero divisors of $S$, then $Z(S) \cup \{0\}$ forms a subsemigroup.

**Proof:** Let $Z_0 = Z(S) \cup \{0\}$, and let $a, b \in Z(S)$. We only need to show that $Z_0$ is closed, since it inherits associativity from $S$. If $ab = 0$, then $ab \in Z_0$. Otherwise, since $a \in Z(S)$, there exists some $c \in Z(S)$ such that $ca = 0$. This implies that $0 = (ca)b = c(ab)$, and therefore $ab \in Z_0$. ■

**Theorem III.5.** Let $S$ be a commutative semigroup. With $Z_0$ as above, $S \setminus Z_0$ is a subsemigroup.

**Proof:** Let $a, b \in S \setminus Z_0$. Again here, we only need to show that $S \setminus Z_0$ is closed. By our assumption, we know that $ab \neq 0$, because if $ab = 0$, we would have $a, b \in Z(S)$, a contradiction. If there existed some $c \in S$ such that $c(ab) = 0$, then we would have $0 = c(ab) = (ca)b$, which would mean that $b \in Z(S)$, again a contradiction. Therefore, $ab \in S \setminus Z_0$. ■

Clearly, $Z_0 \cup S \setminus Z_0 = S$ and $Z_0 \cap S \setminus Z_0 = \emptyset$. Therefore, without loss of generality, we can and will assume that all of the nonzero elements of the semigroups with which we are working are zero divisors of the semigroup.
It is interesting to note that the properties shown in the previous two theorems are not shared by commutative rings, for there is no reason why a subset that is closed with respect to multiplication should be closed with respect to addition. In fact, in a finite ring, any set closed under addition must include zero, so the set of elements that are not zero divisors can never form a subring. Likewise, in general the set of zero divisors will not form a subring. For example, in $\mathbb{Z}_6$, $Z_0 = \{0, 2, 3, 4\}$. But, since $2 + 3 = 5 \notin Z_0$, $Z_0$ is not a subring of $\mathbb{Z}_6$. 
IV. Extensions of the Results of DeMeyer and DeMeyer

We now begin to classify the graphs on six vertices as to whether or not they can be the zero-divisor graph of a commutative semigroup. To this end, we first extend the results of Theorem 3 of DeMeyer and DeMeyer [8], creating large categories of graphs that we will be able to immediately classify. We then classify the remaining graphs “by hand,” either explicitly finding a commutative semigroup for which the given graph is the zero-divisor graph, or explicitly finding a contradiction that prevents the graph from being the zero-divisor graph of any commutative semigroup. Since the graphs on five vertices were completely classified by DeMeyer and DeMeyer [8], we assume throughout that the graphs in question have at least six vertices.

For completeness, we restate Theorem 3 of DeMeyer and DeMeyer [8] here.

**Theorem III.3.** Each of the following is the graph of a semigroup.

1. a complete graph or a complete graph together with an end
2. a complete bipartite graph or a complete bipartite graph together with an end
3. a refinement of a star graph
4. a graph which has at least one end and has diameter \( \leq 2 \)
5. a double-star graph

Note that in a complete bipartite graph with vertex sets \( A \) and \( B \) such that \( a - b \) is an edge of the graph for all \( a \in A, b \in B \), if \( |A| = 1 \) or \( |B| = 1 \), then the graph is a star graph. Thus,
when discussing bipartite graphs, we implicitly assume that each of the vertex sets
contains more than one element.

An immediate extension of (1) is the following.

**Corollary IV.1.** A complete graph together with any number of ends all of which
emanate from the same vertex of a complete graph is the graph of a commutative
semigroup.

**Proof.** This is just the refinement of a star graph, with the center being the vertex from
which the ends emanate. So, by part (3) above, each of these is the graph of a semigroup.

■

Corollary IV.1 is actually a special case of the following more general theorem.

**Theorem IV.1.** A complete graph together with any number of ends, each of which
emanates from one of two vertices, is the graph of a commutative semigroup.

**Proof.** For any positive integers \( p \) and \( q \), let \( \{a_i\}_{i=1}^p \) and \( \{b_j\}_{j=1}^q \) be the ends, and let \( w, x, y_1, y_2, \ldots, y_{n-3}, \) and \( z \) be the vertices of the complete graph (on \( n \geq 3 \) vertices), with each
\( a_i \) adjacent to \( w \) and each \( b_j \) adjacent to \( x \). Using Light’s associativity test, the following
multiplication table is easily seen to give a commutative semigroup for which the graph is
the zero-divisor graph. ■
When we investigate the case where the ends emanate from three distinct vertices, we need to look at two separate cases: (1) the underlying complete graph has exactly three vertices; and (2) the underlying complete graph has more than three vertices.

**Theorem IV.2.** *The complete graph on three vertices, from each of which emanates at least one end, is the graph of a commutative semigroup.*

**Proof.** Let \( x, y, \) and \( z \) be the vertices of the complete graph on three vertices. Also, for some positive integers \( p, q, \) and \( r, \) let \( \{a_i\}_{i=1}^p, \{b_j\}_{j=1}^q, \) and \( \{c_k\}_{k=1}^r, \) be the ends of the graph, where \( a_i-x, b_j-y, \) and \( c_k-z \) are edges of the graph. Using Light’s associativity test, the following multiplication table is easily seen to give a commutative semigroup for which the graph is the zero-divisor graph.
Theorem IV.3. The complete graph on four or more vertices together with any number of ends emanating from at least three different vertices is never the graph of a commutative semigroup.

Proof. We begin with the case where the ends emanate from exactly three different vertices.

For some positive integer \( n \), let \( w, x, y \), and \( \{z_h\}_{h=1}^n \) be the vertices of the complete graph on at least four vertices. Also, let \( A, B, \) and \( C \) be the sets of ends adjacent to vertices \( w, x, \) and \( y \), respectively. For all \( a \in A \), \( b \in B \), and \( c \in C \), we have \( ax = x, ay = y, bw = w \).
by = y, cw = w, and cx = x by Corollary III.1. Letting \( a' \in A \) and \( b' \in B \), we attempt to find a value for \( ab \).

We know that \( a' \neq 0 \)

If \( ab = a' \), then \( 0 = a0 = a(bx) = (ab)x = a'x \), a contradiction.

If \( ab = b' \), then \( 0 = 0b = (wa)b = w(ab) = wb' \), a contradiction.

If \( ab = c \), then \( 0 = a0 = a(bx) = (ab)x = cx \), a contradiction.

If \( ab = w \), then \( 0 = wy = (ab)y = a(by) = ay \), a contradiction.

If \( ab = x \), then \( 0 = xy = (ab)y = a(by) = ay \), a contradiction.

If \( ab = z_h \), then \( 0 = z_hy = (ab)y = a(by) = ay \), a contradiction.

Therefore, \( ab = y \).

Similarly, \( ac = x \) and \( bc = w \).

Letting \( z, z' \in \{z_h\}_{h=1}^n \), we now attempt to find a value for \( az \).

We know that \( az \neq 0 \).

If \( az = a' \), then \( 0 = a0 = a(zy) = (az)y = a'y \), a contradiction.

If \( az = b \), then \( 0 = a0 = a(zy) = (az)y = by \), a contradiction.

If \( az = c \), then \( 0 = a0 = a(zx) = (az)x = cx \), a contradiction.

If \( az = w \), then \( 0 = yz = (ba)z = b(az) = bw \), a contradiction.

If \( az = x \), then \( 0 = xz = (ca)z = c(az) = cx \), a contradiction.

If \( az = y \), then \( 0 = yz = (ba)z = b(az) = by \), a contradiction.

If \( az = z' \), then \( 0 = yz = (ba)z = b(az) = bz' \), a contradiction.

With no possible value for \( az \), the first case is complete.

We now examine the case where there are ends emanating from at least four different vertices.
We start with a complete graph together with any number of ends emanating from at least four different vertices. For some positive integer $n$, let $w, x, y,$ and $\{z_h\}_{h=1}^n$, be the vertices of the complete subgraph that are adjacent to at least one end, and let $V$ be the set of vertices of the complete subgraph that are not adjacent to an end. Also, let $A, B, C,$ and $D_h, 1 \leq h \leq n$, be the sets of ends adjacent to vertices $w, x, y,$ and $z_h$ respectively. For all $a \in A, b \in B, c \in C,$ and $d_h \in D_h$, we have $ax = x, ay = y, a z_h = z_h, bw = w, by = y, b z_h = z_h, cw = w, cx = x, c z_h = z_h, d_h w = w, d_h x = x,$ and $d_h y = y$ by Corollary III.1. Letting $v \in V$, we attempt to find a value for $ab$.

We know that $ab \neq 0$, and the contradictions that prevent $ab = a, ab = b, ab = c, ab = w,$ and $ab = x$ are identical to the above.

If $ab = d_h$, then $0 = a0 = a(bx) = (ab)x = d_h x$, a contradiction.

If $ab = y$, then $0 = y z_h = (ab) z_h = a(b z_h) = a z_h$, a contradiction.

If $ab = z_h$, then $0 = z_h y = (ab) y = a(by) = a y$, a contradiction.

If $ab = v$, then $0 = vy = (ab) y = a(by) = a y$, a contradiction.

With no possible value for $ab$, the proof is complete. ■

We now look to extend result (2) of Theorem III.3.

**Theorem IV.4.** A complete bipartite graph together with any number of ends emanating from the same vertex is the graph of a commutative semigroup.

**Proof.** For any natural numbers $p \geq 2$ and $q \geq 2$, let $\{a_i\}_{i=1}^p$ and $\{b_j\}_{j=1}^q$ be the vertices of the complete bipartite graph such that $a_i$ is adjacent to $b_j$ for every ordered pair $(i, j)$ where $1 \leq i \leq p, 1 \leq j \leq q$. Also, for some natural number $r$, and without loss of
generality, let \( \{x_k\}_{k=1}^\Gamma \) be ends of the graph, all emanating from \( a_1 \). Using Light’s associativity test, the following table is easily verified to be a commutative semigroup for which the graph is the zero-divisor graph. ■

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This turns out to be the best we can do, as is shown by the following.

**Theorem IV.5.** A complete bipartite graph together with two or more ends emanating from at least two distinct vertices is never the graph of a commutative semigroup.

**Proof.** For any natural numbers \( p \geq 2 \) and \( q \geq 2 \), let \( \{a_i\}_{i=1}^p \) and \( \{b_j\}_{j=1}^q \) be the vertices of the complete bipartite graph such that \( a_i \) is adjacent to \( b_j \) for every ordered pair \((i, j)\), where \( 1 \leq i \leq p \), \( 1 \leq j \leq q \). If \( a_1 \) is adjacent to an end, then \( \{0, a_1\} \) is an ideal. Likewise, if \( a_2 \) is adjacent to an end, then \( \{0, a_2\} \) is an ideal. However, since \( a_1a_2 \neq 0 \), this would imply \( a_1a_2 = a_1 \) and \( a_1a_2 = a_1 \), which is impossible since \( a_1 \) and \( a_2 \) are distinct vertices.
Therefore, at most one of the vertices in \( \{a_i\}_{i=1}^p \) can be adjacent to an end. Likewise, at most one of the vertices in \( \{b_j\}_{j=1}^q \) can be adjacent to an end. Therefore, the only way we can have ends emanating from more than one vertex is if they emanate from exactly two vertices, with one in \( \{a_i\}_{i=1}^p \) and the other in \( \{b_j\}_{j=1}^q \). Without loss of generality, let \( x - a_1 \) and \( y - b_1 \) be edges of the graph, with \( x \) and \( y \) being ends. We attempt to determine the value of \( xy \).

We know that \( xy \neq 0 \).

If \( xy = x \), then \( xb_1 = xyb_1 = x0 = 0 \). Then \( a_1 - x - b_1 \) is a path in \( \Gamma \), contradicting the given that \( x \) is an end. Similarly, \( xy \neq y \).

If \( xy = a_1 \), then \( xyb_2 = a_1b_2 = 0 \). Since \( a_1 \) is the only vertex adjacent to \( x \), this implies that \( yb_2 = a_1 \) or \( yb_2 = x \). If \( yb_2 = a_1 \), then \( 0 = y0 = y(b_2a_2) = (yb_2)a_2 = a_1a_2 \), a contradiction. If \( yb_2 = x \), then \( 0 = y0 = y(b_2a_2) = (yb_2)a_2 = xa_2 \). With \( x \) adjacent to \( a_1 \), this contradicts the given that \( x \) is an end. Similarly, we see \( xy \neq b_1 \).

If \( xy = a_i \) for any \( 2 \leq i \leq p \), then \( 0 = 0y = (a_i)x y = a_1(xy) = a_1a_i \), a contradiction.

Similarly, we see \( xy \neq b_j \).

Therefore, we have no value for \( xy \) for this case. ■

There is no obvious extension for part (3) of Theorem III.3.

Part (4) is as good as we can hope for, because

1. the graph
has diameter = 2 and no ends, but is not the graph of any commutative semigroup by Corollary III.2; and

(2) the graph

has diameter = 3 but is not the graph of any commutative semigroup by Theorem IV.5.

In looking at part (5) of Theorem III.3, we take a cue from part (3) and see if the refinements of a double-star graph can be the graph of a commutative semigroup. The answer is an unsatisfying “maybe”, as we see in the following two examples.

**Example 1.** The graph

is a refinement of the double-star graph
and is the graph of a commutative semigroup, by Theorem III.2.

**Example 2.**  The graph

![Diagram of a graph](image)

is a refinement of the double-star graph

![Diagram of a double-star graph](image)

and is not the graph of a commutative semigroup. This can be shown using Theorem III.2, since for this graph \(a\) and \(y\) are not adjacent, and there is no vertex \(m\) with

\[
N(a) \cup N(y) \subseteq \overline{N(m)}.
\]

We can, however, show the following.

**Theorem IV.6.**  A separably-refined double-star graph is never the graph of a commutative semigroup.

**Proof.**  Let \(x\) and \(y\) be the centers of a double-star graph, and for any natural numbers \(p\) and \(q\), let \(\{a_i\}_{i=1}^p\) and \(\{b_j\}_{j=1}^q\) be the ends of the graph, with \(a_i\) adjacent to \(x\) and \(b_j\) adjacent to \(y\) for each \(a_i\) and \(b_j\). Without loss of generality, let the edge \(a_1 \rightarrow a_2\) be in the refinement.
of the double-star graph. Note that since this graph is separably-refined, none of the $a_i$ are adjacent to $y$, none of the $b_j$ are adjacent to $x$, and none of the $a_i$ are adjacent to any of the $b_j$. We attempt to find a vertex $m$ such that $N(a_i) \cup N(y) \subseteq \overline{N(m)}$. Notice that since $N(y) = \{x, b_1, b_2, ..., b_q\}$, and since $y$ is the only vertex adjacent to all of these vertices, the only vertex $m$ with $N(y) \subset \overline{N(m)}$ is $m = y$. However, since $a_i \notin N(y)$, vertices $a_i$ and $y$ fail to meet the neighbor condition. Hence, this graph is not the graph of a semigroup by Theorem III.2. ■
V. The Graphs on Six Vertices

With the results of the preceding chapter in hand, we now determine whether or not each of the graphs on six vertices can be the graph of a commutative semigroup. Since the graph of a commutative semigroup must be connected, we focus on only the connected graphs on six vertices. Up to isomorphism, there are 112 such graphs, as shown by Leck [13].

We begin by finding all of the graphs with at least one vertex of valency 5. All of these graphs are, by definition, either a star graph or the refinement of a star graph, and therefore are the graph of a commutative semigroup by Theorem III.3(3). Note that this includes the complete graph on six vertices and the complete graph on five vertices together with an end, so this set includes the graphs that meet the conditions of Theorem III.3(1). There are 34 such graphs:
One of the remaining graphs is a complete graph on three vertices with three ends, each of which emanates from a different vertex, and therefore is the graph of a commutative semigroup by Theorem IV.2.

Drawing the graph as

makes the situation clear.

Two more of the graphs are complete bipartite, and two others are complete bipartite with one end. Therefore, by Theorem III.3(2), each of these is the graph of a commutative semigroup.
The first two graphs are clearly complete bipartite. The third graph is isomorphic to

\[ \text{Graph A} \]

while the fourth is isomorphic to

\[ \text{Graph B} \]

Two others are double-star graphs, which means they meet the condition set in Theorem III.3(5), and therefore each is the graph of a commutative semigroup.

\[ \text{Graph C} \]

To emphasize the centers of the graphs, we note that the first graph above is isomorphic to

\[ \text{Graph D} \]

while the second is isomorphic to

\[ \text{Graph E} \]
An additional graph is a complete bipartite graph with two ends emanating from the same vertex, and therefore, by Theorem IV.4, it is the graph of a commutative semigroup.

Finally, two more of the graphs are complete graphs with ends emanating from at most two different vertices, and therefore each is the graph of a commutative semigroup by Corollary IV.1 and Theorem IV.1.

To show the ends more clearly, we draw these graphs as
On the negative side, we have nine graphs with diameter > 3, and therefore they are not the graph of a commutative semigroup by Theorem III.1(2). The graphs are shown below, with a path of length 4 that causes the diameter to be greater than 3 emphasized.

An additional 25 of the graphs do not meet the neighbor condition of Theorem III.1(4), and therefore none is the graph of a commutative semigroup. Two possible vertices that do not meet the neighbor condition are highlighted in each of the graphs below.
Of the remaining graphs, four are separably-refined double-star graphs, and therefore none is the graph of a commutative semigroup by Theorem IV.6. The centers of the double-star graphs have been highlighted in the drawings below.

One of the remaining graphs is a pentagon with one end, and another is a hexagon, so by Corollaries III.2 and III.3, neither is the graph of a commutative semigroup.

The first graph is isomorphic to
and the second is isomorphic to

Finally, one graph is complete bipartite with two ends emanating from distinct vertices, and therefore is not the graph of a commutative semigroup by Theorem IV.5.

This is more easily seen drawing the graph as

For the remaining 27 graphs, we must either determine a commutative semigroup for which the given graph is the zero-divisor graph, or find a contradiction that prevents the graph from being the graph of a commutative semigroup. It turns out that there are 24
graphs for which we can find a semigroup, and three for which we find a contradiction, as classified below.

24 graphs of commutative semigroups

Three graphs that are not the graph of a commutative semigroup

As an example, we now show the process used to find a contradiction that appears in attempting to determine a commutative semigroup for the graph.
First, we begin to make a Cayley table for the commutative semigroup, filling in the zeros as appropriate.

\[
\begin{array}{c|cccc}
\cdot & a & b & c & x & y & z \\
\hline
a & 0 & 0 & & & & \\
b & & 0 & 0 & & & \\
c & 0 & & 0 & & & \\
x & 0 & & & 0 & & \\
y & 0 & & & 0 & & \\
z & 0 & 0 & 0 & 0 & & \\
\end{array}
\]

To fill in the remaining cells, we first attempt to limit the possibilities by looking for contradictions. Starting with \(ab\), we note that:

\[ab \neq 0\]

If \(ab = a\), then \(0 = a0 = a(by) = (ab)y = ay\), a contradiction.

If \(ab = b\), then \(0 = 0b = (xa)b = x(ab) = xb\), a contradiction.

If \(ab = c\), then \(0 = a0 = a(by) = (ab)y = cy\), a contradiction.

If \(ab = x\), then \(0 = a0 = a(by) = (ab)y = xy\), a contradiction.

If \(ab = y\), then \(0 = 0b = (xa)b = x(ab) = xy\), a contradiction.

Therefore, the only option is \(ab = z\).
Similar calculations yield $ay = z$ and $az = z$.

Next, we look at the possibilities for $bc$.

$bc \neq 0$.

If $bc = a$, then $0 = 0c = (yb)c = y(bc) = ya$, a contradiction.

If $bc = b$, then $0 = b0 = b(ca) = (bc)a = ba$, a contradiction.

If $bc = c$, then $0 = 0c = (yb)c = y(bc) = yc$, a contradiction.

If $bc = x$, then $0 = 0c = (yb)c = y(bc) = yx$, a contradiction.

If $bc = y$, then $0 = b0 = b(ca) = (bc)a = ya$, a contradiction.

If $bc = z$, then $0 = b0 = b(ca) = (bc)a = za$, a contradiction.

Since there are no options for $bc$, this cannot be the zero-divisor graph of a commutative semigroup.

As an example of a graph for which we can find a commutative semigroup, we now look at the following graph:

We begin the same way as before, beginning to construct a Cayley table by filling in the known zeros:
By Corollary III.1, we know that $bc = b$.

Starting in the $a$-row of the table, we try to find a value for $ay$.

We know that $ay \neq 0$.

If $ay = b$, then $0 = 0y = (ca)y = c(ay) = cb$, a contradiction.

If $ay = c$, then $0 = 0y = (xa)y = x(ay) = xc$, a contradiction.

If $ay = x$, then $0 = 0y = (ca)y = c(ay) = cx$, a contradiction.

If $ay = y$, then $0 = 0y = (xa)y = x(ay) = xy$, a contradiction.

Therefore, $ay \in \{a, z\}$.

Similar calculations yield $az \in \{a, z\}$.

For $c^2$, we have the following. As a reminder, we have already established that $bc = b$ and $xc = x$.

If $c^2 = 0$, then $0 = b0 = bc^2 = (bc)c = bc$, a contradiction.

If $c^2 = a$, then $0 = xa = xc^2 = (xc)c = xc$, a contradiction.

If $c^2 = b$, then $0 = xb = xc^2 = (xc)c = xc$, a contradiction.
If \( c^2 = x \), then \( 0 = bx = bc^2 = (bc)c = bc \), a contradiction.

If \( c^2 = y \), then \( 0 = by = bc^2 = (bc)c = bc \), a contradiction to the above.

If \( c^2 = z \), then \( 0 = bz = bc^2 = (bc)c = bc \), a contradiction to the above.

Therefore, \( c^2 = c \).

Next, we try to find a value for \( cx \).

We know that \( cx \neq 0 \).

If \( cx = a \), then \( 0 = 0x = (zc)x = z(cx) = za \), a contradiction.

If \( cx = b \), then \( 0 = by = (cx)y = c(xy) \), which implies that \( xy = a \) or \( xy = z \).

If \( xy = a \), then \( 0 = x0 = x(yz) = (xy)z = az \), a contradiction.

If \( xy = z \), then \( 0 = x0 = x(ya) = (xy)a = za \), a contradiction.

If \( cx = c \), then \( 0 = c0 = c(xb) = (cx)b = cb \), a contradiction.

If \( cx = y \), then \( 0 = 0x = (ac)x = a(cx) = ay \), a contradiction.

If \( cx = z \), then \( 0 = 0x = (ac)x = a(cx) = az \), a contradiction.

Therefore, \( cx = x \).

Similar calculations yield \( cy = x \).

We now attempt to find a value for \( y^2 \). Using the fact obtained above that \( cy = x \), we first note that \( cy^2 = (cy)y = xy \). From there, we have the following:

If \( y^2 = 0 \), then \( 0 = c0 = cy^2 = xy \), a contradiction.

If \( y^2 = a \), then \( 0 = ca = cy^2 = xy \), a contradiction.
If $y^2 = b$, then $0 = ab = ay^2 = (ay)y$. From above, we know that $ay$ is either $a$ or $z$. But this would mean that either $0 = (ay)y = ay$, or $0 = (ay)y = zy$, both of which are contradictions.

If $y^2 = c$, then $0 = 0y = (by)y = by^2 = bc$, a contradiction.

If $y^2 = x$, then $0 = xz = y^2z = y(yz)$. Since our current assumption is that $y^2 = x$, $0 = y(yz)$ implies that $yz = b$. But this would mean that $0 = y0 = y(ze) = (yz)e = bc$, a contradiction.

If $y^2 = z$, then $0 = cz = cy^2 = xy$, a contradiction.

Therefore, $y^2 = y$.

Using this result, we can now find a value for $xy$.

We know that $xy \neq 0$.

If $xy = a$, then $0 = (zx)y = z(xy) = za$, a contradiction.

If $xy = b$, then $0 = by = (xy)y = x(y^2) = xy$, contradicting the assumption that $xy = b$.

If $xy = c$, then $0 = x0 = x(yb) = (xy)b = cb$, a contradiction.

If $xy = y$, then $0 = 0y = (ax)y = a(xy) = ay$, a contradiction.

If $xy = z$, then $0 = 0y = (ax)y = a(xy) = az$, a contradiction.

Therefore, $xy = x$.

Finally, with this result and $cy = x$ from above, we note that $x^2 = (x)x = (cy)x = c(yx) = cx = x$. Filling in all of these results, we have
From here, we look at the Light tables for each of the elements.
We already know from above that $ay \in \{a, z\}$ and $az \in \{a, z\}$. Looking at the Light table for $a$, and comparing it with the Cayley table, we see that the possibilities for $a^2$ are $0$, $a$, and $z$. The original Cayley table tells us that $ya \neq 0$, $yz \neq 0$, $za \neq 0$, and $zy \neq 0$. Looking at the Light tables for $y$ and $z$, we see that $ya \in \{a, z\}$, $yz \in \{a, z\}$, $za \in \{a, z\}$, $zy \in \{a, z\}$, and $z^2 \in \{0, a, z\}$.

None of these options are in conflict with one another (for example, the options dictated by the Light table for $yz$ are the same as those dictated by the Light table for $zy$), so we’ll take it one cell at a time in the Cayley table and the Light tables for $a$, $y$, and $z$. Since the choices for $a^2$ will only affect the Light table for $a$, and $z^2$ will only affect the Light table for $z$, we begin by filling in $ay = a$. 

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We already know from above that $ay \in \{a, z\}$ and $az \in \{a, z\}$. Looking at the Light table for $a$, and comparing it with the Cayley table, we see that the possibilities for $a^2$ are $0$, $a$, and $z$. The original Cayley table tells us that $ya \neq 0$, $yz \neq 0$, $za \neq 0$, and $zy \neq 0$. Looking at the Light tables for $y$ and $z$, we see that $ya \in \{a, z\}$, $yz \in \{a, z\}$, $za \in \{a, z\}$, $zy \in \{a, z\}$, and $z^2 \in \{0, a, z\}$. None of these options are in conflict with one another (for example, the options dictated by the Light table for $yz$ are the same as those dictated by the Light table for $zy$), so we’ll take it one cell at a time in the Cayley table and the Light tables for $a$, $y$, and $z$. Since the choices for $a^2$ will only affect the Light table for $a$, and $z^2$ will only affect the Light table for $z$, we begin by filling in $ay = a$. 

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Next, we fill in $az = a$. 

\[
\begin{array}{cccccc}
  y & a & 0 & x & x & y \\
  a & 0 & 0 & 0 & a & a \\
  a & 0 & 0 & 0 & 0 & 0 \\
 x & 0 & 0 & x & x & x & 0 \\
 x & 0 & 0 & x & x & x & 0 \\
 y & a & 0 & x & x & y & 0 & 0 & 0 \\
 z & a & 0 & 0 & 0 & 0 & a & a \\
\end{array}
\]
Then, we fill in \( yz = a \).

\[
\begin{array}{cccccccc}
\cdot & a & b & c & x & y & z & a \\
\hline
a & 0 & 0 & 0 & a & a & a & 0 \\
b & 0 & b & 0 & 0 & 0 & 0 & 0 \\
c & 0 & b & c & x & x & 0 & 0 \\
x & 0 & 0 & x & x & x & 0 & 0 \\
y & a & 0 & x & x & y & a & a \\
z & a & 0 & 0 & 0 & a & a & a \\
\hline
y & a & 0 & x & x & y & a & a \\
z & a & 0 & 0 & 0 & a & a & a \\
\end{array}
\]

At this point, we are forced to make the choice \( a^2 = a \) from the column of the Light table for \( y \) relating to \( yz \). Also, looking at the column of the Light table for \( z \) relating to \( z^2 \), we see that we can narrow the choices for \( z^2 \) to \( a \) and \( z \). Making the choice \( z^2 = a \), we have
This leaves us to make a choice for $b^2$. Based on the Light table for $b$, we can choose either $b^2 = 0$ or $b^2 = b$. Since neither of these creates a contradiction, we choose $b^2 = b$.

This completes the process, and yields the following Cayley table of a commutative semigroup for which the graph is the zero-divisor graph.
It is worth mentioning that there are several places in the above construction where choices were made between several options. Choosing differently does, in some instances, create a different commutative semigroup whose zero-divisor graph is, nonetheless, the one given. We will see this explicitly in Chapter VI, where we determine all of the possible commutative semigroups for each of the graphs on three and four vertices.

There are instances where determining whether or not a given graph is the graph of a commutative semigroup takes a bit more work and insight. As an example of this, we attempt to determine a commutative semigroup for which the graph

is the zero-divisor graph.
As usual, we attempt to construct a Cayley table for the hoped-for commutative
semigroup by first filling in the known zeros.

\[
\begin{array}{c|cccccc}
\cdot & a & b & c & x & y & z \\
\hline
a & 0 & 0 & 0 & 0 & & \\
b & & 0 & 0 & & & \\
c & 0 & & 0 & 0 & 0 & \\
x & 0 & 0 & 0 & 0 & & \\
y & 0 & 0 & 0 & 0 & & \\
z & 0 & 0 & & & & \\
\end{array}
\]

Next, we try to find a value for \(ab\).

We know that \(ab \neq 0\).

If \(ab = b\), then \(0 = 0b = (za)b = z(ab) = zb\), a contradiction.

If \(ab = x\), then \(0 = 0b = (za)b = z(ab) = zx\), a contradiction.

If \(ab = y\), then \(0 = 0b = (za)b = z(ab) = zy\), a contradiction.

If \(ab = z\), then \(0 = 0b = (ya)b = y(ab) = yz\), a contradiction.

There are no obvious contradictions for \(ab = a\) or \(ab = c\).

Similarly, if we look for a value for \(bc\), we find that there are two options: either \(bc = a\) or \(bc = c\).

So, we have four cases to consider: (1) \(ab = bc = a\); (2) \(ab = bc = c\); (3) \(ab = a\) and \(bc = c\); and (4) \(ab = c\) and \(bc = a\).

**Case 1.** Assume that \(ab = a\) and \(bc = a\), and attempt to find a value for \(b^2\).

If \(b^2 = 0\), then \(0 = a0 = ab^2 = (ab)b = ab\), a contradiction.
If $b^2 = a$, then $0 = ca = cb^2 = (cb)b = ab$, a contradiction.

If $b^2 = c$, then $0 = ac = ab^2 = (ab)b = ab$, a contradiction.

If $b^2 = x$, then $0 = cx = cb^2 = (cb)b = ab$, a contradiction.

If $b^2 = y$, then $0 = cy = cb^2 = (cb)b = ab = 0$, a contradiction.

If $b^2 = z$, then $0 = 0b = (yb)b = yb^2 = yz$, a contradiction.

Therefore, $b^2 = b$, and similarly, $z^2 = z$.

However, if we then try to find a value for $bz$, we find the following.

We know that $bz \neq 0$.

If $bz = a$, then $0 = az = (bz)z = b(z^2) = bz$, a contradiction.

If $bz = b$, then $0 = b0 = b(zc) = (bz)c = bc$, a contradiction.

If $bz = c$, then $0 = cz = (bz)z = b(z^2) = bz$, a contradiction.

If $bz = x$, then $0 = bx = b(bz) = (b^2)z = bz$, a contradiction.

If $bz = y$, then $(b^2)z = by = 0 \Rightarrow (bz) = 0$, a contradiction.

If $bz = z$, then $0 = 0z = (yb)z = y(bz) = yz$, a contradiction.

Therefore, there is no possible value for $bz$ in this case.

Case 2. Similarly, $ab = c$ and $bc = c$ ends with no possible value for $bz$.

Case 3. Assume that $ab = a$ and $bc = c$, and attempt to find a value for $b^2$.

If $b^2 = 0$, then $0 = a0 = ab^2 = (ab)b = ab$, a contradiction.

If $b^2 = a$, then $0 = ca = cb^2 = (cb)b = cb$, a contradiction.

If $b^2 = c$, then $0 = ac = ab^2 = (ab)b = ab$, a contradiction.

If $b^2 = x$, then $0 = cx = cb^2 = (cb)b = cb$, a contradiction.
If \(b^2 = y\), then \(0 = cy = cb^2 = (cb)b = cb\), a contradiction.

If \(b^2 = z\), then \(0 = 0b = (yb)b = yb^2 = yz\), a contradiction.

Therefore, \(b^2 = b\), and similarly, \(z^2 = z\).

The remainder of this case is identical to the conclusion to Case 1, ending with no possible value for \(b_2\).

**Case 4.** Assume that \(ab = c\) and \(bc = a\), and attempt to find a value for \(b^2\).

If \(b^2 = 0\), then \(0 = a0 = ab^2 = (ab)b = cb\), a contradiction.

If \(b^2 = a\), then \(0 = ca = cb^2 = (cb)b = ab\), a contradiction.

If \(b^2 = c\), then \(0 = ac = ab^2 = (ab)b = cb\), a contradiction.

If \(b^2 = x\), then \(0 = cx = cb^2 = (cb)b = ab\), a contradiction.

If \(b^2 = y\), then \(0 = cy = cb^2 = (cb)b = ab\), a contradiction.

If \(b^2 = z\), then \(0 = 0b = (yb)b = yb^2 = yz\), a contradiction.

Therefore, \(b^2 = b\), and similarly, \(z^2 = z\).

The remainder of this case is identical to the conclusion to Case 1, ending with no possible value for \(b_2\).

We are out of options for \(ab\) and \(bc\), and therefore this is not the zero-divisor graph of any commutative semigroup.

The results for all 112 graphs are provided in Appendix 1.
VI. Semigroups with Three or Four Non-Zero Zero Divisors

In the previous chapter, we concentrated on determining whether a given graph on six vertices is the zero-divisor graph of any commutative semigroup. Looking at Theorem 1 of DeMeyer and DeMeyer [8], the question is easily answered for graphs on three or four vertices, as every connected graph on three or four vertices is the zero-divisor graph of a commutative semigroup. This being the case, we look at each of the connected graphs on three or four vertices and determine (up to isomorphism) all of the commutative semigroups that have the given graph as their zero-divisor graph, thereby providing a complete list, up to isomorphism, of all zero-divisor semigroups of order four or five.

There are essentially two methods for determining the semigroups. We will demonstrate the first by determining the commutative semigroups for the complete graph on three vertices, and the second by determining the commutative semigroups for one of the graphs on four vertices. As in the previous chapter, we assume that the semigroups contain only zero-divisors.

Before proceeding, we provide a useful lemma.

**Lemma VI.1.** Let $S$ be a commutative semigroup and $x, y$ be distinct elements of $S$ such that $xy = 0$. Then

(a) if $x^2 = x$, then $y^2 \neq x$, and

(b) if $x^2 = y$, then $y^2 = 0$. 

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Proof. For part (a), if $y^2 = x$, then $0 = 0y = (xy)y = x(y^2) = x^2 = x$, a contradiction.

Therefore, $y^2 \neq x$.

For part (b), if $x^2 = y$, then $0 = x0 = x(xy) = (x^2)y = y^2$, and therefore $y^2 = 0$, as claimed.

VI.1 The Graphs on Three Vertices

There are two connected graphs on three vertices, yielding a total of 20 isomorphism types of commutative zero-divisor semigroups of order 4. We demonstrate the first method for determining the semigroups by finding, up to isomorphism, all of the commutative semigroups for the complete graph on three vertices. Looking at the graph, we begin by filling in the zeros of the Cayley table indicated by the edges of the graph:

\[
\begin{array}{ccc}
\cdot & a & b & c \\
a & 0 & 0 & \\
b & 0 & 0 & \\
c & 0 & 0 & \\
\end{array}
\]

We then list all of the possible combinations for the cells on the main diagonal, up to symmetry:
Using Lemma VI.1, we can remove several of the options. Specifically,

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{contradiction} & \\
\hline
0 & a & b & \(c^2 = b \Rightarrow b^2 = 0\) \\
\hline
0 & c & b & \(c^2 = b \Rightarrow b^2 = 0\) \\
\hline
a & 0 & a & \(c^2 = a \Rightarrow a^2 = 0\) \\
\hline
a & a & a & \(b^2 = a \Rightarrow a^2 = 0\) \\
\hline
a & a & b & \(b^2 = a \Rightarrow a^2 = 0\) \\
\hline
a & b & a & \(c^2 = a \Rightarrow a^2 = 0\) \\
\hline
b & a & a & \(a^2 = b \Rightarrow b^2 = 0\) \\
\hline
a & c & b & \(b^2 = c \Rightarrow c^2 = 0\) \\
\hline
b & c & a & \(a^2 = b \Rightarrow b^2 = 0\) \\
\hline
\end{tabular}
\end{center}

For the remaining options, we form the Light tables and check for associativity. For example, if \(a^2 = a\), \(b^2 = 0\) and \(c^2 = b\), then the corresponding Cayley table would be
and the Light table for each of the non-zero elements would be

\[
\begin{array}{|c|ccc|}
\hline
\cdot & a & b & c \\
\hline
a & a & 0 & 0 \\
b & 0 & 0 & 0 \\
c & 0 & 0 & b \\
\hline
\end{array}
\]

Finding no contradictions between the rows and columns of the Light tables and the Cayley table, this Cayley table gives a commutative semigroup whose zero-divisor graph is the complete graph on three vertices.

Similarly, the other options found in Table 1 and not ruled out in Table 2 create no contradiction between the corresponding Light and Cayley tables, and each gives a commutative semigroup whose zero-divisor graph is the complete graph on three vertices. We then need to check whether or not any of the semigroups are isomorphic. However, by the descriptions we gave for \(a^2\), \(b^2\), and \(c^2\) in each case, it is easily verified that none of the semigroups are isomorphic, and therefore we have seven non-isomorphic commutative semigroups, each of which has the complete graph on three vertices as its zero-divisor graph.

The only other connected graph on three vertices is
Using the method shown below for graphs on four vertices, we find 13 non-isomorphic commutative semigroups, each having this graph as its zero-divisor graph. The Cayley tables for these semigroups are given in Appendix 2A.

### VI.2 The Graphs on Four Vertices

There are six connected graphs on four vertices, yielding a total of 131 isomorphism types of commutative semigroups of order 5. We demonstrate the second method for determining the semigroups by finding, up to isomorphism, the commutative semigroups that have the graph

![Graph](image)

as their zero-divisor graph.
There are two difficulties facing us as we begin: (1) determining which of the possible Cayley tables actually represent an associative operation; and (2) determining which of the resulting semigroups are isomorphic.

We start by filling in the zeros of the Cayley tables and the corresponding Light tables for the graph:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We then create a list of possibilities for each of the empty cells of the table. Beginning with \( a^2 \), we find the following.

If \( a^2 = b \), then \( 0 = a0 = a(ad) = a^2d = bd \), a contradiction. So, \( a^2 \neq b \).

Using the symmetry of the graph, this also implies \( a^2 \neq d \).

Therefore, \( a^2 \in \{0, a, c\} \).

The symmetry of the graph also implies \( b^2 \in \{0, b, d\} \), \( c^2 \in \{0, a, c\} \), and \( d^2 \in \{0, b, d\} \).

Looking next at \( ac \):

If \( ac = b \), then \( 0 = cd = a(cd) = (ac)d = bd \), a contradiction. So, \( ac \neq b \).

Using the symmetry of the graph, this also implies \( ac \neq d \).

Therefore, \( ac \in \{a, c\} \).
The symmetry of the graph also implies $bd \in \{b,d\}$.

From here, we check the combinations of the above possibilities. To do this, we fill in the cells of the Cayley table one at a time, fill in the corresponding cells in the Light tables, and check for contradictions. Beginning with $a^2 = 0$, we have

\[
\begin{array}{c|cccc}
\cdot & a & b & c & d \\
\hline
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
a & 0 & 0 & c & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
a & 0 & c & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

This choice does not create a contradiction, so we continue in a similar fashion, attempting to determine a value for $ac$. Filling in the tables for $ac = c$, we have

\[
\begin{array}{c|cccc}
\cdot & a & b & c & d \\
\hline
a & 0 & 0 & c & 0 \\
b & 0 & 0 & 0 & 0 \\
c & c & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
a & 0 & 0 & c & 0 \\
b & 0 & 0 & 0 & 0 \\
c & c & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
a & 0 & c & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & c & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
a & 0 & c & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & c & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

This creates a contradiction in the third column of the $a$-table, which must have a $c$ in the first cell to match with the $c$-column of the Cayley table, contradicting the 0 required by the choice of $a^2 = 0$. The specific contradiction implied by this is
If \( a^2 = 0 \) and \( ac = c \), then \( 0 = 0c = a^2c = (ac)c = a(ac) = ac \).

Therefore, if \( a^2 = 0 \), then the only option for \( ac \) is \( ac = a \). Filling in the Cayley and Light tables, we have

\[
\begin{array}{cccc}
\cdot & a & b & c & d \\
\hline
a & 0 & 0 & a & 0 \\
b & 0 & a & 0 & 0 \\
c & a & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
a & 0 & 0 & a & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
a & 0 & 0 & a & 0 \\
b & 0 & 0 & 0 & 0 \\
c & a & 0 & c & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

At this point, we notice that the top of the \( c^2 \) column of the \( c \)-table is an \( a \). Looking at the Cayley table, the only column with an \( a \) in its top cell is \( c \), and therefore we are forced to conclude that \( c^2 = c \). Filling this in, we get

\[
\begin{array}{cccc}
\cdot & a & b & c & d \\
\hline
a & 0 & 0 & a & 0 \\
b & 0 & a & 0 & 0 \\
c & a & 0 & c & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
a & 0 & 0 & a & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
a & 0 & 0 & a & 0 \\
b & 0 & 0 & 0 & 0 \\
c & a & 0 & c & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]

Next, we choose \( b^2 = 0 \), and fill in the tables accordingly.
Seeing no contradictions, we continue, attempting to find a value for $bd$. Using the
symmetry of the graph, the choice of $b^2 = 0$ implies $bd = b$ in the same way that $a^2 = 0$
implied $ac = a$. Filling in the tables, we have

$$
\begin{array}{cccc}
\cdot & a & b & c & d \\
 a & 0 & 0 & a & 0 \\
 b & 0 & 0 & 0 & 0 \\
 c & a & 0 & c & 0 \\
 d & 0 & 0 & 0 & 0 \\
\end{array}
$$

$$
\begin{array}{cccc}
 a & 0 & 0 & a & 0 \\
 b & 0 & 0 & 0 & b \\
 c & a & 0 & c & 0 \\
 d & 0 & b & 0 \\
\end{array}
$$

This leaves $d^2$ as the only open cell in the Cayley table. Noticing that according to the $d^2$
column of the $d$-table, the second cell must be $b$, we look at the Cayley table and
conclude that the only option for $d^2$ is $d$. We could also have come to this conclusion
based on the symmetry of the graph, with $d^2 = d$ following from $c^2 = c$. Filling this in, we
have
Therefore, we have

\[
\begin{array}{c|cccc}
\cdot & a & b & c & d \\
\hline
a & 0 & 0 & a & 0 \\
b & 0 & 0 & b & 0 \\
c & a & 0 & c & 0 \\
d & 0 & b & 0 & d \\
\end{array}
\]

\[
\begin{array}{c|cccc}
a & 0 & 0 & a & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & 0 & a & 0 \\
d & 0 & 0 & b & 0 \\
\end{array}
\]

as one possible commutative semigroup for which the graph

\[
\begin{array}{c}
a \\
b \\
c \\
d \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \\
\end{array}
\]

is the zero-divisor graph, and indeed the only option for which \(a^2 = b^2 = 0\).

However, \(b^2 = 0\) was a choice, not a requirement. It is possible that \(b^2 = b\) or \(b^2 = d\). So our next step is to return to that point in the process and fill in the tables with \(b^2 = b\).
We continue in this way, filling in the tables and looking for contradictions. If no contradictions exist, we have found a new commutative semigroup for which the given graph is the zero-divisor graph.

We give the complete set of calculations based on the choice $a^2 = 0$. The numbers to the left of the calculations correspond to the Cayley tables that follow the calculations.

If $a^2 = 0$, then

$ac = a$ implies $c^2 = c$ (because $a = ac = (a)c = (ac)c = a(c^2)$, but since $aa = 0, ab = 0, ac = a$, and $ad = 0$, the only option for $c^2$ is $c$); then

if $b^2 = 0$, then

(1) $bd = b$ implies $d^2 = d$ (because $b = bd = (b)d = (bd)d = b(d^2)$, but since $ba = 0, bb = 0, bc = 0, and bd = b$, the only option for $d^2$ is $d$).

$bd \neq d$, since if $bd = d$, then $0 = 0d = (b^2)d = b(bd) = bd$, a contradiction.

else if $b^2 = b$, then

if $bd = b$, then

$d^2 \neq 0$, since if $d^2 = 0$, then $b = bd = (b)d = (bd)d = bd^2 = b0 = 0$, a contradiction.

(2) $d^2 = b$ creates no contradictions.

(3) $d^2 = d$ creates no contradictions.

else if $bd = d$, then

(4) $d^2 = 0$ creates no contradictions.

(5) $d^2 = b$ creates no contradictions.
(6) \[ d^2 = d \] creates no contradictions.

else if \( b^2 = d \), then

(7) \[ bd = b \] implies \( d^2 = d \) (because \( b = bd = (b)d = (bd)d = b(d^2) \), but since \( ba = 0 \),

\[ bb = 0, \ bc = 0, \text{ and } bd = b, \] the only option for \( d^2 \) is \( d \))

(8) \[ bd = d \] implies \( d^2 = d \), since \( d = bd = b(d) = b(bd) = (b^2)d = dd = d^2 \)

\[ ac \neq c, \] since if \( ac = c \), then \( 0 = 0c = a^2c = a(ac) = ac, \) a contradiction.

<table>
<thead>
<tr>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
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<tbody>
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<td>0</td>
</tr>
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<td>0</td>
<td>0</td>
<td>b</td>
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<td>b</td>
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<td>c</td>
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<td>b</td>
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<td>c</td>
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<td>0</td>
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<td>d</td>
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<tr>
<td>c</td>
<td>a</td>
<td>b</td>
<td>0</td>
<td>c</td>
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<td>b</td>
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</tr>
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<td>0</td>
<td>a</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
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<table>
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<td>0</td>
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<tr>
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<td>0</td>
<td>0</td>
<td>b</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>0</td>
<td>b</td>
<td>d</td>
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</tbody>
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<table>
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<td>0</td>
<td>a</td>
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<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>0</td>
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<tr>
<td>d</td>
<td>d</td>
<td>0</td>
<td>d</td>
<td>0</td>
</tr>
</tbody>
</table>

If we examine the tables more closely, we realize that applying the permutation \((b \ d)\) to the elements of Table 1 produces Table 4, and therefore the two tables determine isomorphic semigroups. The same is true for Tables 2 and 8, Tables 3 and 6, and Tables
5 and 7. Of course, we could have seen this prior to making the calculations by using the symmetry of the graph, noticing, for example, that the calculations that produced Tables 1 and 4 do nothing more than switch the roles of \( b \) and \( d \). Removing the unnecessary tables and renumbering, we have

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0 c</td>
<td>0 d</td>
<td>3</td>
<td>a 0</td>
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<td>2</td>
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<td>4</td>
<td>a 0</td>
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<td>0 b</td>
<td>0 b</td>
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</table>

Keeping the symmetry of the graph in mind, we now change the choice of \( a^2 = 0 \) to \( a^2 = a \) and, proceeding in a similar fashion, get the following results. The numbers to the left of the calculations correspond to the tables that follow, giving the actual table determined by the calculation or one to which the table determined by the calculation is isomorphic.

If \( a^2 = a \) then

if \( ac = a \) then

\[ c^2 \neq 0, \text{ since if } c^2 = 0, \text{ then } 0 = a0 = ac^2 = (ac)c = ac = a, \text{ a contradiction.} \]

if \( c^2 = a \) then

if \( b^2 = 0 \) then

(2) \( bd = b \) implies \( d^2 = d \), as above.

\( bd \neq d \), as above.
else if \( b^2 = b \), then

\[
\text{if } bd = b, \text{ then }
\]

\[
d^2 \neq 0, \text{ as above.}
\]

(5) \( d^2 = b \) creates no contradictions.

(6) \( d^2 = d \) creates no contradiction.

else if \( bd = d \), then

(2) \( d^2 = 0 \) creates no contradictions.

(7) \( d^2 = b \) creates no contradictions.

(6) \( d^2 = d \) creates no contradictions.

else if \( b^2 = d \), then

(7) \( bd = b \implies d^2 = d \), as above.

(5) \( bd = d \implies d^2 = d \), as above.

else if \( c^2 = c \), then

\[
\text{if } b^2 = 0, \text{ then }
\]

(3) \( bd = b \implies d^2 = d \), as above.

\[
bd \neq d, \text{ as above.}
\]

else if \( b^2 = b \), then

\[
\text{if } bd = b, \text{ then }
\]

\[
d^2 \neq 0, \text{ as above.}
\]

(6) \( d^2 = b \) creates no contradictions.

(8) \( d^2 = d \) creates no contradictions.

else if \( bd = d \), then

(3) \( d^2 = 0 \) creates no contradictions.

(9) \( d^2 = b \) creates no contradictions.
(8) \[ d^2 = d \] creates no contradictions.

else if \( b^2 = d \), then

(9) \[ bd = b \] implies \( d^2 = d \), as above.

(6) \[ bd = d \] implies \( d^2 = d \), as above.

else if \( ac = c \), then

if \( c^2 = 0 \), then

if \( b^2 = 0 \), then

(1) \[ bd = b \] implies \( d^2 = d \), as above.

\[ bd \neq d \], as above.

else if \( b^2 = b \), then

if \( bd = b \), then

\[ d^2 \neq 0 \], as above.

(2) \[ d^2 = b \] creates no contradictions.

(3) \[ d^2 = d \] creates no contradictions.

else if \( bd = d \), then

(1) \[ d^2 = 0 \] creates no contradictions.

(4) \[ d^2 = b \] creates no contradictions.

(3) \[ d^2 = d \] creates no contradictions.

else if \( b^2 = d \), then

(4) \[ bd = b \] implies \( d^2 = d \), as above.

(2) \[ bd = d \] implies \( d^2 = d \), as above.

else if \( c^2 = a \), then

if \( b^2 = 0 \), then

(4) \[ bd = b \] implies \( d^2 = d \), as above.
\[bd \neq d, \text{ as above.}\]

else if \(b^2 = b\), then

\[\text{if } bd = b, \text{ then}\]

\[d^2 \neq 0, \text{ as above.}\]

\((7)\)

\[d^2 = b \text{ creates no contradictions.}\]

\((9)\)

\[d^2 = d \text{ creates no contradictions.}\]

\[\]

else if \(bd = d\), then

\((4)\)

\[d^2 = 0 \text{ creates no contradictions.}\]

\((10)\)

\[d^2 = b \text{ creates no contradictions.}\]

\((9)\)

\[d^2 = d \text{ creates no contradictions.}\]

\[\]

else if \(b^2 = d\), then

\((10)\)

\[bd = b \implies d^2 = d, \text{ as above.}\]

\((7)\)

\[bd = d \implies d^2 = d, \text{ as above.}\]

else if \(c^2 = c\), then

\[\text{if } b^2 = 0, \text{ then}\]

\((3)\)

\[bd = b \implies d^2 = d, \text{ as above.}\]

\[bd \neq d, \text{ as above.}\]

else if \(b^2 = b\), then

\[\text{if } bd = b, \text{ then}\]

\[d^2 \neq 0, \text{ as above.}\]

\((6)\)

\[d^2 = b \text{ creates no contradictions.}\]

\((8)\)

\[d^2 = d \text{ creates no contradictions.}\]

else if \(bd = d\), then

\((3)\)

\[d^2 = 0 \text{ creates no contradictions.}\]
\[(9) \quad d^2 = b \text{ creates no contradictions.}\]

\[(8) \quad d^2 = d \text{ creates no contradictions.}\]

else if \(b^2 = d\), then

\[(9) \quad bd = b \implies d^2 = d, \text{ as above.}\]

\[(6) \quad bd = d \implies d^2 = d, \text{ as above.}\]

We now change the choice of \(a^2 = a\) to \(a^2 = c\). However, using the symmetry of the graph, we see that this change will produce no new semigroups, because

(a) all of the options created by \(a^2 = c\) will be isomorphic to the semigroups above with \(c^2 = a\); and

(b) the options created by \(a^2 = c\) and \(c^2 = a\) would correspond to the options created by \(b^2 = d\) and \(d^2 = b\) above, and none of these produce a semigroup.

Therefore, the Cayley tables for the additional semigroups determined by the above calculations are the following.

<table>
<thead>
<tr>
<th>5</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tr>
<td>a</td>
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<td>c</td>
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<td>d</td>
<td>0</td>
<td>d</td>
<td>0</td>
<td>b</td>
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</tbody>
</table>
This concludes the calculations for the commutative semigroups for the graph

\[
\begin{array}{cccc}
  a & & d \\
  b & & c \\
\end{array}
\]

The complete list of commutative semigroups (up to isomorphism) for all of the connected graphs on four vertices is determined in a similar way, and the results are found in Appendix 2B.

A few observations:

First, it would seem reasonable to expect that the more non-zero cells we have, the more non-isomorphic commutative semigroups we would have for a given graph. This turns out not to be the case, as there is exactly one commutative semigroup (up to isomorphism) for which the graph

\[
\begin{array}{ccc}
  a & & d \\
  b & & c \\
\end{array}
\]

is the zero-divisor graph.

Second, while each of the connected graphs on four vertices has at least one commutative semigroup for which it is the zero-divisor graph, this is not the case for commutative
rings. If we let $R$ be a commutative ring with 1, then the definitions for the zero-divisor graph of the ring are as expected, with the zero-divisor graph of $R$, denoted as $\Gamma(R)$, as an undirected (and, by definition, simple) graph with vertex set $V(\Gamma(R)) = \{ \text{the non-zero zero-divisors of } R \}$ and edge set $E(\Gamma(S)) = \{ (x, y) \text{ such that } x \neq y \text{ and } xy = 0 \}$. With these definitions, only three of the six connected graphs on four vertices are the zero-divisor graphs of a commutative ring, and even then the number of non-isomorphic rings is limited. Specifically, it was shown by Anderson and Livingston [3] that the star graph on four vertices and the quadrilateral graph each are the zero-divisor graph of exactly one commutative ring ($\mathbb{Z}_2 \times \mathbb{F}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$, respectively), and the complete graph on four vertices is the zero-divisor graph of exactly two non-isomorphic commutative rings ($\mathbb{Z}_{25}$ and $\mathbb{Z}_5[\mathbb{X}]/(\mathbb{X}^2)$). We see the same thing with the connected graphs on three vertices, as the star graph on three vertices is the zero-divisor graph of exactly three non-isomorphic commutative rings ($\mathbb{Z}_6$, $\mathbb{Z}_8$, and $\mathbb{Z}_2[\mathbb{X}]/(\mathbb{X}^3)$), and the complete graph on three vertices is the zero-divisor graph of exactly two non-isomorphic commutative rings ($\mathbb{Z}_2[\mathbb{X}, \mathbb{Y}]/(\mathbb{X}^2, \mathbb{X}\mathbb{Y}, \mathbb{Y}^2)$ and $\mathbb{F}_4[\mathbb{X}]/(\mathbb{X}^2)$).

What this means is that the multiplicative structure of the set of non-zero zero divisors of a commutative ring is severely limited. For example, of the 36 possible commutative semigroups that have the star graph on four vertices as their zero-divisor graph, only one,

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</table>
gives the underlying multiplicative structure of the set of zero divisors of a commutative ring. In general, this is because the multiplicative structure of the ring as a whole is much more complicated, since the ring will also have a number of elements that are not zero divisors. For the remaining 35, it is not possible for the semigroup to be embedded as the zero-divisor subsemigroup of the multiplicative semigroup of any commutative ring.

Similarly, of the 10 commutative semigroups for the quadrilateral graph, only one,

\[
\begin{array}{cccc}
\cdot & a & b & c & d \\
\hline
a & \text{a} & 0 & c & 0 \\
b & 0 & b & 0 & d \\
c & c & 0 & a & 0 \\
d & 0 & d & 0 & b
\end{array}
\]

gives the underlying multiplicative structure of the set of non-zero zero divisors of a commutative ring, and for the remaining 9 it is not possible for the semigroup to be embedded as the zero-divisor subsemigroup of the multiplicative semigroup of any commutative ring. For the complete graph on four vertices, even through there are two non-isomorphic commutative rings that have the graph as their zero-divisor graph, the set of non-zero zero divisors of each ring has the same underlying multiplicative structure, namely

\[
\begin{array}{cccc}
\cdot & a & b & c & d \\
\hline
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0
\end{array}
\]

For the star graph on three vertices, only two of the 13 possible semigroups,
give the multiplicative structure of the set of non-zero zero divisors of a commutative ring, since
both $\mathbb{Z}_8$ and $\mathbb{Z}_2[X]/(X^3)$ have the semigroup on the left as the underlying multiplicative structure
for the set of zero divisors, while the semigroup on the right gives the underlying multiplicative
structure for the set of zero divisors of $\mathbb{Z}_6$. As for the complete graph on three vertices, of the
seven possible semigroups that have the graph as their zero-divisor graph, only one,

\[
\begin{array}{c|ccc}
\cdot & a & b & c \\
\hline
a & 0 & 0 & 0 \\
b & 0 & 0 & 0 \\
c & 0 & 0 & 0 \\
\end{array}
\quad \text{and} \quad 
\begin{array}{c|ccc}
\cdot & a & b & c \\
\hline
a & c & 0 & a \\
b & 0 & b & 0 \\
c & a & 0 & c \\
\end{array}
\]

gives the multiplicative structure of the set of zero divisors of a commutative ring, since it is the
multiplicative structure of both $\mathbb{Z}_2[X,Y]/(X^2, XY, Y^2)$ and $\mathbb{F}_4[X]/(X^2)$.

Finally, if we lift the restriction that all of the elements of the semigroup must be zero divisors,
we see that there are actually infinitely many semigroups having the above graphs as their zero-
divisor graphs. For example, we can use Light’s Associativity Test to verify that each of the
following Cayley tables represents a semigroup, and it is obvious that each has the complete
graph on four vertices as its zero-divisor graph:

\[
\begin{array}{c|cccc}
\cdot & a & b & c & d \\
\hline
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 \\
\end{array}
\]
In fact, once we have found the Cayley table of a semigroup whose zero-divisor graph is a given graph, we can produce infinitely many semigroups that are not isomorphic to the given semigroup, but whose zero-divisor graph is isomorphic to the given graph. One method for producing such semigroups is found in the pattern of the second and third tables above.

Following this pattern, another possible semigroup would be

\[
\begin{array}{cccccc}
\cdot & a & b & c & d & e \\
a & 0 & 0 & 0 & 0 & a \\
b & 0 & 0 & 0 & 0 & b \\
c & 0 & 0 & 0 & 0 & c \\
d & 0 & 0 & 0 & 0 & d \\
e & a & b & c & d & e \\
f & a & b & c & d & e \\
g & a & b & c & d & e \\
h & a & b & c & d & e \\
\end{array}
\]

In Theorems III.4 and III.5, we showed that any semigroup \(S\) is the union of two subsemigroups \(Z_0\) and \(S \setminus Z_0\), where \(Z_0\) is the set of zero divisors of \(S\). Conversely, we can generalize the above and show that given any semigroup \(Z_n\) consisting of 0 and \(n\) additional zero divisors, and a semigroup \(T\) not containing 0, we can create a semigroup \(S = Z_n \cup T\) by defining multiplication
as follows: if \( a \in Z_a \) and \( x \in T \), then \( ax = a \). To verify that \( S \) is a semigroup, let \( a, b, c \in Z_a \) and \( x, y, z \in T \), where \( ab = c \) and \( xy = z \). Then

\[
(ab)x = cx = c = ab = a(bx)
\]

and

\[
(ax)y = ay = a = az = a(xy).
\]

In fact, we can always construct a commutative semigroup consisting of a set of zero divisors, \( Z \), of arbitrary cardinality by simply defining \( ab = 0 \) for any two elements \( a, b \in Z \). Since there exist commutative semigroups of arbitrary order and not containing 0 (such as \( T \) above), we can therefore construct a semigroup \( S \) consisting of arbitrarily many non-zero zero divisors and arbitrarily many elements that are not zero divisors.

As a concluding remark, this presents another distinction between the case of commutative semigroups and that of commutative rings. It was shown in Ganesan [10] that any commutative ring having \( n \) non-zero zero divisors does not contain more than \((n + 1)^2\) elements, a restriction that clearly does not hold in the case of commutative semigroups.
VII. Other Results

In this chapter, we present various other results obtained in the course of our research.

VII.1 Zero-Divisor Graphs Based on Ideals

Redmond [15] extended the idea of a zero-divisor graph of a commutative ring with 1, basing the graph on an ideal of the ring. Letting \( I \) be an ideal of \( R \) and \( x, y \) elements of \( R \setminus I \), Redmond defined the vertex set to be \( V(\Gamma_I) = \{ x \in R \setminus \{0\} : \exists y \in R \setminus I \text{ with } xy \in I \} \) and the edge set to be \( E(\Gamma) = \{(x, y) : x, y \in R/I, x \neq y, xy \in I \} \), and commented on various properties of these graphs.

Since the role of ideals in ring theory is filled by congruences in the study of semigroups, we combine the above ideas and explore the properties of a congruence-based zero-divisor graph. As before, \( S \) is taken to be a commutative semigroup with 0 throughout, and the operation is written multiplicatively. Also, as in the preceding chapters, we assume that the semigroup with which we are working contains only zero divisors.

VII.1.a. Congruences

In this section we present the basic definitions and results, as found in Howie [12], necessary to the study of congruences in semigroups.
**Definition VII.1.** A binary relation $\rho$ on a semigroup $S$ is called *compatible* if $(x, y), (w, z) \in \rho$ implies $(xw, yz) \in \rho$.

**Definition VII.2.** A compatible equivalence relation $\rho$ on a semigroup $S$ is called a *congruence*.

For the sake of clarity, we restate the definitions of *right ideal*, *left ideal*, and *(two-sided)* ideal for semigroups.

**Definition VII.3.** A *left ideal* is a nonempty subset $A$ of $S$ such that $SA \subseteq A$. A *right ideal* is defined analogously. A *(two-sided)* ideal is a nonempty subset $A$ of $S$ such that $A$ is both a left and right ideal. The above sets are identical when $S$ is commutative. Since here $S$ is assumed to be commutative, we will simply refer to the subset $A$ as an ideal.

**Theorem VII.1.** Let $1_s = \{(x, x) \mid x \in S\}$, and $\rho_I = (I \times I) \cup 1_s$, where $I$ is an ideal of $S$. Then $\rho_I$ is a congruence on $S$.

**Proof.** First, notice that $(x, y) \in \rho_I$ if and only if either $x = y$ and $x \in S$, or $x, y \in I$. Then, $x \in S$ implies $(x, x) \in \rho_I$, $(x, y) \in \rho_I$ implies $(y, x) \in \rho_I$, and $(x, y), (y, z) \in \rho_I$ implies $(x, z) \in \rho_I$ are obvious. As for compatibility, there are three cases to check:

**Case 1.** $(x, x)(y, y)$, where $x, y \in S$.

$(x, x)(y, y) = (xy, xy)$, and since $xy \in S$, we have $(xy, xy) \in \rho_I$.

**Case 2.** $(x, x)(w, z)$, where $x \in S$ and $w, z \in I$.

$(x, x)(w, z) = (xw, xz)$, and since $I$ is an ideal, we have $xw, xz \in I$. Therefore,
(xw, xz) ∈ ρ₁.

Case 3. (x, y)(w, z), where and x, y, w, z ∈ I.

(x, y)(w, z) = (xw, yz), and since I is an ideal, we have xw, yz ∈ I. Therefore,

(xw, yz) ∈ ρ₁. ■

Definition VII.4. A congruence of the type ρ₁ in Theorem 1 is called a Rees congruence.

Definition VII.5. Let ρ be a relation on S. Then ρ(x) := {y ∈ S | (x, y) ∈ ρ}, and S/ρ is the set of subsets ρ(x) of S.

Observe that when ρ is a congruence, S/ρ will be a partition of S.

For the remainder of this section, ρ will indicate a congruence unless otherwise stated.

Define a binary operation on S/ρ in a natural way, namely, ρ(x) ρ(y) = ρ(xy). This operation is well defined, since if ρ(x) = ρ(w) and ρ(y) = ρ(z), then (x, w), (y, z) ∈ ρ implies (xy, wz) ∈ ρ, and therefore ρ(xy) = ρ(wz).

Theorem VII.2. With multiplication defined as ρ(x) ρ(y) = ρ(xy), S/ρ is a semigroup.

Proof. Let x, y, z ∈ S. Then (ρ(x) ρ(y)) ρ(z) = ρ(xy) ρ(z) = ρ((xy)z) = ρ(x(yz)) = ρ(xy) ρ(yz) = ρ(x) (ρ(y) ρ(z)), and hence S/ρ is a semigroup. ■
Of particular interest for us is the semigroup $S/\rho_I$, which is commonly denoted as $S/I$. Since the semigroups we are using are commutative, our $S/I$ are actually commutative semigroups.

**VII.1.b. $\Gamma_I(S)$**

We now define the graph $\Gamma_I(S)$ and present some basic properties of these graphs. The results are completely analogous to those found in Redmond [15], with only minor changes required to change from rings to semigroups.

**Definition VII.6.** Let $S$ be a commutative semigroup, $I$ an ideal of $S$, and $x$, $y$ distinct elements of $S \setminus I$. Then the graph $\Gamma_I(S)$ is an undirected graph with vertex set $V(\Gamma_I) =$ \{ $x \in S \setminus I : \exists y \in S \setminus I$ with $xy \in I$ \} and edge set $E(\Gamma_I) =$ \{ $(x, y) : x, y \in S/I$ and $x y \in I$ \}. We begin with a few observations analogous to those found in [15].

**Theorem VII.3.** If $I = (0)$, then $\Gamma_I(S) = \Gamma(S)$.

**Proof.** If $x$ and $y$ are distinct elements of $SV$ such that $x, y \in V(\Gamma_I)$ and $(x, y) \in E(\Gamma_I)$, then since $I = (0)$, we have $xy = 0$. Therefore $x, y \in V(\Gamma)$ and $(x, y) \in E(\Gamma)$. In other words, $\Gamma_I(S)$ is a subgraph of $\Gamma(S)$.

On the other hand, if $x$ and $y$ are distinct elements of $SV$ such that $x, y \in V(\Gamma)$ and $(x, y) \in E(\Gamma)$, then $xy = 0 \in I$, and since $I = (0)$, we have that $x$ and $y$ are distinct elements of $SV$. Hence,
In other words, $\Gamma(S)$ is a subgraph of $\Gamma_I(S)$, and therefore, $\Gamma_I(S) = \Gamma(S)$. □

**Definition VII.7.** A **prime ideal** of $S$ is an ideal $I$ such that $xy \in I \Rightarrow x \in I$ or $y \in I$.

**Theorem VII.4.** Let $I$ be a nonzero ideal of $S$. Then $\Gamma_I(S) = \emptyset$ if and only if $I$ is a prime ideal of $S$.

**Proof.** If $I$ is a prime ideal, then $xy \in I \Rightarrow x \in I$ or $y \in I$. However, for $x$ and $y$ to be vertices of $\Gamma_I(S)$, we need both of them to be in $S \setminus I$. Therefore, $\Gamma_I(S) = \emptyset$.

On the other hand, assume that $\Gamma_I(S) = \emptyset$. Then, if $x \in S \setminus I$ and $xy \in I$ for some $y \in S$, it must be true that $y \in I$, since otherwise $(x, y) \in E(\Gamma_I)$ and $\Gamma_I(S)$ would not be empty. Hence, $I$ is a prime ideal of $S$. □

**Theorem VII.5.** Let $I$ be an ideal of $S$. Then $\Gamma_I(S)$ is connected with diam$(\Gamma_I(S)) \leq 3$.

**Proof.** Assume that $x$ and $y$ are distinct vertices of $\Gamma_I(S)$.

**Case 1.** $xy \in I$. Then $x - y$ is a path in $\Gamma_I(S)$.

**Case 2.** $xy \notin I, x^2 \in I, y^2 \in I$. Then $x - xy - y$ is a path in $\Gamma_I(S)$. 

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Case 3. \( xy \notin I, x^2 \notin I, y^2 \notin I \). Since \( x \in V(\Gamma_i) \), \( \exists a \in S \setminus I \) such that \( xa \in I \). If \( ay \in I \), then \( x - a - y \) is a path in \( \Gamma_i(S) \). If \( ay \notin I \), then \( x - ay - y \) is a path in \( \Gamma_i(S) \).

Case 4. \( xy \notin I, x^2 \in I, y^2 \notin I \). The proof of this is similar to Case 3.

Case 5. \( xy \notin I, x^2 \notin I, y^2 \notin I \). Since \( x, y \in \Gamma_i(S) \), \( \exists a, b \in S \setminus I \) (where \( a, b \notin \{x, y\} \) is implied by the condition for this case) such that \( xa, by \in I \). If \( a = b \), then \( x - a - y \) is a path in \( \Gamma_i(S) \). So assume \( a \neq b \). If \( ab \in I \), then \( x - a - b - y \) is a path in \( \Gamma_i(S) \), and if \( ab \notin I \), then \( x - ab - y \) is a path in \( \Gamma_i(S) \).

Therefore, \( \Gamma_i(S) \) is connected and \( diam(\Gamma_i(S)) \leq 3 \). ■

We now reach a point of departure from the situation which arises in the setting of rings, and therefore from the results obtained by Redmond [15]. As shown there, if \( I \) is an ideal of a ring \( R \), then \( \Gamma(R/I) \) is a subgraph of \( \Gamma_i(R) \). However, in the case of semigroups, \( S/I \) indicates the congruence classes under the Rees congruence. Note that \( S/I = I \cup \{x\} \mid x \in S \setminus I \}; in other words, in \( S/I \), \( \rho(x) = \{x\} \) if \( x \in S \setminus I \), and \( \rho(x) = I \) if \( x \in I \). For our purposes, the important differences between \( S/I \) and \( R/I \) are: (1) the cosets formed by \( S/I \) need not have the same number of elements; and (2) at most one of the cosets has more than one element, namely, the coset \( \{I\} \). It is these differences that lead to the following.
Theorem VII.6. Let $I$ be an ideal of $S$. Then $\Gamma_I(S) = \Gamma(S/I)$.

Proof. We begin by showing that $S/I$ is a semigroup. We already know that $S/I$ is a semigroup from Theorem VII.2, so we only need to find the element $\rho(x)$ of $S/I$ such that 

$$\rho(x)\rho(y) = \rho(y)\rho(x) = \rho(x) \forall \rho(y) \in S/I,$$

which, if it exists, is known to be unique. We claim that $I$ is this element. We know that if $x$ and $y$ are any two elements of $I$, then $\rho(x) = \rho(y)$. So choose $x$ as the representative element of $I$. Since $x \in I$ implies $xz \in I$ for all $z \in S$, we have 

$$\rho(z)\rho(x) = \rho(xz) = \rho(x).$$

Similarly, $\rho(z)\rho(x) = \rho(x)$. Thus $I$ is the zero element of $S/I$.

Now let us denote $\Gamma_I(S)$ as $\Gamma_I$ and $\Gamma(S/I)$ as $\Gamma$. We show that $\Gamma_I = \Gamma$. First, let $x$ and $y$ be distinct elements of $S \setminus I$ such that $x, y \in V(\Gamma_I)$ and $(x, y) \in E(\Gamma_I)$. Then $xy \in I$. Since $I$ is the zero element of $S/I$, this means that $xy = 0$ if we look at the calculation in terms of $S/I$. Hence, $x, y \in V(\Gamma)$ and $(x, y) \in E(\Gamma)$. In other words, $\Gamma_I(S)$ is a subgraph of $\Gamma(S/I)$. On the other hand, if $x$ and $y$ are distinct elements of $(S/I)\setminus 0$ such that $x, y \in V(\Gamma)$ and $(x, y) \in E(\Gamma)$, then $xy = 0$. But in $S/I$, $I$ is the zero element, so what we really have is that $x$ and $y$ are distinct elements of $S \setminus I$ and $xy \in I$. As such, $x, y \in V(\Gamma_I)$ and $(x, y) \in E(\Gamma_I)$. In other words, $\Gamma(S/I)$ is a subgraph of $\Gamma_I(S)$, and therefore, $\Gamma_I(S) = \Gamma(S/I)$. ■

Since $S/I$ is a commutative semigroup, $\Gamma(S/I)$ has the same properties as any zero-divisor graph based on a commutative semigroup. In particular, the following property, with which we end this section, holds.
Theorem VII.7. Let $S$ be a commutative semigroup. Then we can construct $T$, a commutative semigroup of arbitrarily large cardinality $|T| > |S|$, and an ideal $I$ of $T$, such that $\Gamma(T/I) = \Gamma(S)$.

Proof. Given $S$, let $X = \{x_j : j \in J\}$ for some index set $J$, and let $T = S \cup X$. Fix an element $x_1$ of $X$, and let $T$ have the following properties.

1. The operation on $S \subset T$ is the same as the operation on $S$.
2. $\forall x_i, x_j \in T \setminus S, \ x_i x_j = x_i$
3. $\forall s \in S, \ sx_i = x_i s = 0$

We claim $T$ is the desired semigroup, with $X \cup \{0\}$ as the ideal. To check that $T$ is a semigroup, we first note that the operation on $S$ and $X$ is clearly associative. Since $T$ is also clearly commutative, we only need to check the following cases to confirm associativity on all of $T$. For both cases, $s_1, s_2 \in S$ and $x_i, x_j \in X$.

Case 1: Since $s_1 s_2 \in S$, $(s_1 s_2) x_i = s_3 x_i = 0$ and $s_1 (s_2 x_i) = s_4 0 = 0$, so

$$(s_1 s_2) x_i = s_1 (s_2 x_i).$$

Case 2: $(s_1 x_i) x_j = 0 x_j = 0$ and $s_1 (x_i x_j) = s_4 x_i = 0$, so

$$(s_1 x_i) x_j = s_1 (x_i x_j).$$
Now, let \( I = X \cup \{0\} \). Then \( T/I = \left( \bigcup_{s \in S} \{s\} \right) \cup I \). Since \( \forall s_i, s_j \in S, s_i s_j \in I \) if and only if \( s_i s_j = 0 \), we have \( V(\Gamma(T/I)) = \{ s_i \in T \setminus I \mid \exists s_j \in T \setminus I \text{ such that } s_i s_j = 0 \} \) and
\[
E(\Gamma(T/I)) = \{ (s_i, s_j) \mid s_i, s_j \in T \setminus I, s_i s_j = 0 \},
\]
where \( s_i \neq s_j \). This is precisely the vertex and edge set of \( \Gamma(S) \), and therefore \( \Gamma(T/I) = \Gamma(S) \). ■

**VII.2 Noncommutative Semigroups with 0**

In [16], Redmond explored several ways in which one can define the zero-divisor graph of a noncommutative ring and presented results based on these different definitions. Here, we explore one of the possibilities as it relates to noncommutative semigroups.

Let \( S \) be a noncommutative semigroup with 0. \( Z_L(S) \) will denote the set of left zero divisors of \( S \) (i.e. the elements \( a \) in \( Z_L(S) \) such that there exists a \( b \) in \( S \setminus \{0\} \) with \( ab = 0 \)). \( Z_R(S) \) will denote the set of right zero divisors of \( S \), defined analogously.

As we have done previously, \( \Gamma(S) \) will denote the zero-divisor graph of \( S \). This graph, however, is a directed graph, depending on the way in which two elements multiply to equal zero. Our convention will be the following: if \( a \) and \( b \) are elements of \( S \) and \( ab = 0 \), then the directed edge connecting \( a \) to \( b \) will point toward \( b \). This will be denoted as \( a \rightarrow b \).

Several of the fundamental concepts with which we have been working change when we are working with directed graphs. First, for any two vertices \( x \) and \( y \), if \( xy = 0 \) and \( yx = 0 \), we will need two edges connecting \( x \) and \( y \), one pointing in each direction. Therefore, the graphs with
which we are working will no longer be simple. Resulting from this is the fact that a directed
graph can have a cycle of length 2. Finally, for a directed graph to be connected, there must be a
directed path connecting any two vertices. In other words, the edges connecting the vertices
must all be pointing in the same direction.

With the above notation and definitions, we now present a few basic results. These are
essentially analogous to those found by Redmond [16], with only minor changes required to
change from rings to semigroups.

**Theorem VII.7.** $\Gamma(S)$ is connected if and only if $Z_L(S) = Z_R(S)$. Moreover, if $\Gamma(S)$ is connected,
then $\text{diam}(\Gamma(S)) \leq 3$.

**Proof.** Assume $Z_L(S) = Z_R(S)$, and let $x, y \in V(\Gamma(S))$, where $x \neq y$.

If $xy = 0$, then $x \to y$ is a path in $\Gamma(S)$.

If $xy \neq 0$, $x^2 = 0$ and $y^2 = 0$, then $x \to xy \to y$ is a path in $\Gamma(S)$.

If $xy \neq 0$, $x^2 = 0$ and $y^2 \neq 0$, then $\exists b \in S \setminus \{0, x, y\}$ such that $by = 0$.

If $xb = 0$, then $x \to b \to y$ is a path in $\Gamma(S)$.

If $xb \neq 0$, then $x \to xb \to y$ is a path in $\Gamma(S)$.

If $xy \neq 0$, $x^2 \neq 0$ and $y^2 = 0$, then $\exists a \in S \setminus \{0, x, y\}$ such that $xa = 0$.

If $ay = 0$, then $x \to a \to y$ is a path in $\Gamma(S)$.

If $ay \neq 0$, then $x \to ay \to y$ is a path in $\Gamma(S)$.

If $xy \neq 0$, $x^2 \neq 0$ and $y^2 \neq 0$, then $\exists a, b \in S \setminus \{0, x, y\}$ such that $xa = by = 0$. 
If $ab = 0$, then $x \rightarrow a \rightarrow b \rightarrow y$ is a path in $\Gamma(S)$.

If $ab \neq 0$, then $x \rightarrow ab \rightarrow y$ is a path in $\Gamma(S)$.

The converse is true by definition since, in order for a directed graph to be connected, each vertex must have both at least one edge pointing into the vertex and at least one edge pointing out of the vertex. Therefore, every element corresponding to the vertices of the graph must be in both $Z_L(S)$ and $Z_R(S)$.

Looking at the paths above, we have also shown that $diam(\Gamma(S)) \leq 3$. ■

**Theorem VII.8.** Let $S$ be a finite noncommutative semigroup, with no nonzero nilpotent elements and at least one nonzero zero divisor. Then $S$ has a nonzero two-sided zero divisor.

**Proof.** Let $a$ be a nonzero zero divisor of $S$. Without loss of generality, assume $a$ is a left zero divisor.

If $S = \{0, a\}$, then $a^2 = 0$, contradicting the fact that $S$ has no nonzero nilpotent elements. So $|S| \geq 3$.

If $S = \{0, a, b\}$, then $ab = 0$. Look at $ba$.

If $ba = 0$, then both $a$ and $b$ are nonzero two-sided zero divisors.

If $ba = a$, then $aba = a^2$, which implies $0 = a^2$, a contradiction.

If $ba = b$, then $bab = b^2$, which implies $0 = b^2$, a contradiction.

If $|S| > 3$ and $a, b$ are elements of $S$ such that $ab = 0$, then consider $ba$. 
If $ba = 0$, $ba = a$, or $ba = b$, then we have the same results as above.

If $ba = c$ for some $c$ in $S\setminus\{0, a, b\}$, then $aba = ac$ which implies $0 = ac$. Similarly, $bab = cb$, which implies $0 = cb$. So, $c$ is a nonzero two-sided zero divisor. ■

**Theorem VII.9.** Let $S$ be a noncommutative semigroup. If $\Gamma(S)$ contains a cycle, then the girth of the graph, denoted $gr(\Gamma)$, is at most 3.

**Proof.** If there exists an element $x$ in $S$ such that $x^2 = 0$, then $gr(\Gamma(S)) = 1$.

If there exist distinct elements $x$, $y$ in $S$ such that $xy = yx = 0$, then $gr(\Gamma(S)) \leq 2$. So assume no such elements exist in $S$, and note that for a cycle to exist in $\Gamma(S)$, all edges of the cycle must go the same direction. Also, assume $x_0 \to x_1 \to \ldots \to x_{n-1} \to x_n \to x_0$ is a cycle of shortest length in $\Gamma(S)$, and that $n \geq 3$ (i.e., $gr(\Gamma(S)) > 3$). If $x_1x_n = 0$, then $x_0 \to x_1 \to x_n \to x_0$ is a cycle in $G(S)$, and $gr(G) \leq 3$, contradicting the assumption that $x_0 \to x_1 \to \ldots \to x_{n-1} \to x_n \to x_0$ is a cycle of shortest length in $\Gamma(S)$.

If $x_1x_n \neq 0$, then $x_0 \to x_1x_n \to x_0$ is a cycle in $\Gamma(S)$, and $gr(\Gamma(S)) \leq 2$, contradicting the assumption that $x_0 \to x_1 \to \ldots \to x_{n-1} \to x_n \to x_0$ is a cycle of shortest length in $\Gamma(S)$.

Thus, $gr(\Gamma(S)) \leq 3$. ■
Bibliography


Appendix 1: Complete Classification of the Graphs on Six Vertices

A: The graphs on six vertices that are the zero-divisor graph of a commutative semigroup

A star graph:

Refinements of a star graph:

Double-star graphs:
Complete bipartite graphs:

Complete bipartite graph with two ends emanating from the same vertex:

Complete bipartite graphs with one end:

Complete graph with two ends emanating from different vertices:

Complete graph with three ends emanating from two vertices:

Complete graph with ends emanating from three different vertices:
The following 24 graphs are each the graph of a commutative semigroup, and each is presented with the Cayley table of a commutative semigroup for which the graph is its zero-divisor graph.

\[
\begin{array}{cccccc}
\bullet & a & b & c & x & y & z \\
\hline
a & a & a & a & 0 & 0 & a \\
b & a & b & b & 0 & 0 & b \\
c & a & b & c & x & 0 & c \\
x & 0 & 0 & x & x & 0 & 0 \\
y & 0 & 0 & 0 & y & y & y \\
z & a & b & c & 0 & y & z \\
\end{array}
\]

\[
\begin{array}{cccccc}
\bullet & a & b & c & x & y & z \\
\hline
a & 0 & a & a & 0 & 0 & 0 \\
b & a & b & b & 0 & 0 & a \\
c & a & b & b & 0 & x & a \\
x & 0 & 0 & 0 & 0 & x & 0 \\
y & 0 & 0 & x & x & y & x \\
z & 0 & a & a & 0 & x & 0 \\
\end{array}
\]
\begin{array}{|c|c|c|c|c|c|}
\hline
& a & b & c & x & y & z \\
\hline
a & x & x & 0 & 0 & 0 & 0 \\
b & x & 0 & 0 & x & b & x \\
c & 0 & x & 0 & x & c & x \\
x & 0 & 0 & 0 & 0 & x & 0 \\
y & c & b & x & 0 & 0 & 0 \\
z & x & x & 0 & 0 & 0 & 0 \\
\hline
\end{array}

\begin{array}{|c|c|c|c|c|c|}
\hline
& a & b & c & x & y & z \\
\hline
a & 0 & a & a & 0 & 0 & 0 \\
b & a & b & b & 0 & 0 & a \\
c & a & b & c & 0 & 0 & a \\
x & 0 & 0 & 0 & 0 & x & 0 \\
y & 0 & 0 & 0 & x & y & x \\
z & 0 & a & a & 0 & x & 0 \\
\hline
\end{array}

\begin{array}{|c|c|c|c|c|c|}
\hline
& a & b & c & x & y & z \\
\hline
a & a & a & a & 0 & 0 & 0 \\
b & a & a & a & 0 & 0 & x \\
c & a & a & a & 0 & 0 & x \\
x & 0 & 0 & 0 & 0 & x & 0 \\
y & 0 & 0 & 0 & 0 & x & x \\
z & 0 & x & x & x & x & z \\
\hline
\end{array}
\[
\begin{array}{ccccccc}
\cdot & a & b & c & x & y & z \\
a & 0 & a & a & 0 & 0 & 0 \\
b & a & b & b & 0 & 0 & a \\
c & a & b & c & x & 0 & a \\
x & 0 & 0 & x & x & 0 & 0 \\
y & 0 & 0 & 0 & y & y & y \\
z & 0 & a & a & 0 & y & y \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\cdot & a & b & c & x & y & z \\
a & a & 0 & a & 0 & a & 0 \\
b & b & b & z & b & 0 & a \\
c & b & z & b & 0 & a & x \\
x & 0 & 0 & 0 & 0 & x & 0 \\
y & x & 0 & 0 & x & 0 & y \\
z & z & z & z & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\cdot & a & b & c & x & y & z \\
a & 0 & a & a & 0 & 0 & 0 \\
b & a & b & b & 0 & 0 & z \\
c & a & b & b & 0 & 0 & z \\
x & 0 & 0 & 0 & x & x & 0 \\
y & 0 & 0 & 0 & x & x & 0 \\
z & 0 & z & z & 0 & 0 & 0 \\
\end{array}
\]
\begin{array}{llllll}
\hline
a & b & c & x & y & z \\
\hline
a & a & a & 0 & 0 & 0 \\
b & a & b & a & x & 0 \\
c & a & a & c & 0 & y \\
x & 0 & x & 0 & x & 0 \\
y & 0 & 0 & y & 0 & y \\
z & 0 & 0 & z & 0 & 0 \\
\hline
\end{array}

\begin{array}{llllll}
\hline
a & b & c & x & y & z \\
\hline
a & a & a & 0 & 0 & 0 \\
b & a & a & a & 0 & x \\
c & a & a & c & 0 & 0 \\
x & 0 & 0 & 0 & 0 & x \\
y & 0 & x & 0 & x & y \\
z & 0 & 0 & z & 0 & 0 \\
\hline
\end{array}

\begin{array}{llllll}
\hline
a & b & c & x & y & z \\
\hline
a & a & x & x & x & 0 \\
b & x & b & z & 0 & y \\
c & x & z & c & 0 & 0 \\
x & x & 0 & 0 & 0 & 0 \\
y & 0 & y & 0 & 0 & y \\
z & 0 & 0 & z & 0 & 0 \\
\hline
\end{array}
\[
\begin{array}{cccccccc}
\cdot & a & b & c & x & y & z \\
 a & a & a & 0 & 0 & 0 & 0 \\
b & a & a & x & 0 & 0 & 0 \\
c & 0 & x & c & x & 0 & x \\
x & 0 & 0 & x & 0 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & y & y \\
z & 0 & 0 & x & 0 & y & y \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\cdot & a & b & c & x & y & z \\
 a & 0 & a & 0 & 0 & 0 & 0 \\
b & a & b & a & 0 & 0 & a \\
c & 0 & a & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & 0 & x & 0 \\
y & 0 & 0 & 0 & x & y & x \\
z & 0 & a & 0 & 0 & x & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\cdot & a & b & c & x & y & z \\
 a & 0 & a & 0 & 0 & 0 & 0 \\
b & a & b & a & 0 & a & a \\
c & 0 & a & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & x & x & 0 \\
y & 0 & a & 0 & x & x & 0 \\
z & 0 & a & 0 & 0 & 0 & z \\
\end{array}
\]
\begin{itemize}
\item \begin{tabular}{|c|c|c|c|c|c|}
\hline
 & a & b & c & x & y \\
\hline
a & 0 & a & 0 & 0 & 0 \\
b & a & b & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 & c \\
x & 0 & 0 & 0 & x & x \\
y & 0 & 0 & 0 & x & x \\
z & 0 & 0 & c & 0 & c \\
\hline
\end{tabular}
\end{itemize}

\begin{itemize}
\item \begin{tabular}{|c|c|c|c|c|c|}
\hline
 & a & b & c & x & y \\
\hline
a & 0 & a & 0 & 0 & 0 \\
b & a & b & 0 & 0 & 0 \\
c & 0 & 0 & c & 0 & 0 \\
x & 0 & 0 & 0 & x & 0 \\
y & 0 & 0 & 0 & y & y \\
z & 0 & 0 & c & x & y \\
\hline
\end{tabular}
\end{itemize}

\begin{itemize}
\item \begin{tabular}{|c|c|c|c|c|c|}
\hline
 & a & b & c & x & y \\
\hline
a & a & a & 0 & 0 & 0 \\
b & a & b & 0 & x & 0 \\
c & 0 & 0 & c & 0 & 0 \\
x & 0 & x & 0 & x & 0 \\
y & 0 & 0 & 0 & y & y \\
z & 0 & 0 & c & 0 & y \\
\hline
\end{tabular}
\end{itemize}
B: The graphs on six vertices that are not the zero-divisor graph of a commutative semigroup

Graphs with diameter > 3:

Separably-refined doubly stars:
Complete bipartite with two ends emanating from different vertices:

Graphs that violate Theorem III.1(4):

A hexagon:

A pentagon with one end:
None of the following three graphs are the graph of a commutative semigroup, and each is presented with a contradiction that prevents it from being such.

\[ az = b, \text{ but then } a(zx) \neq (az)x \]

\[ ab = x \text{ and } cz = y, \text{ but then } bc \text{ does not have a value} \]

\[ b^2 = b, z^2 = z, ab \in \{a, c\}, \text{ and } bc \in \{a, c\}, \text{ but then } bz \text{ does not have a value} \]
Appendix 2: Complete List of Commutative Semigroups for Each of the Connected Graphs on Three or Four Vertices

A: The Graphs on Three Vertices

13 semigroups
7 semigroups

\[
\begin{array}{ccc}
\cdot & a & b & c \\
a & 0 & 0 & 0 \\
b & 0 & 0 & 0 \\
c & 0 & 0 & 0 \\
\end{array}
\quad \begin{array}{ccc}
\cdot & a & b & c \\
a & 0 & 0 & 0 \\
b & 0 & a & 0 \\
c & 0 & 0 & a \\
\end{array}
\quad \begin{array}{ccc}
\cdot & a & b & c \\
a & a & 0 & 0 \\
b & 0 & b & 0 \\
c & 0 & 0 & 0 \\
\end{array}
\]

A: The Graphs on Four Vertices

1 semigroup

\[
\begin{array}{cccc}
\cdot & a & b & c & d \\
a & a & 0 & a & 0 \\
b & b & b & 0 & d \\
c & c & d & a & 0 \\
d & 0 & d & 0 & 0 \\
\end{array}
\]
12 semigroups

\[
\begin{array}{c|cccc}
\cdot & a & b & c & d \\
\hline
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & a \\
\end{array}
\begin{array}{c|cccc}
\cdot & a & b & c & d \\
\hline
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & d \\
c & 0 & 0 & a & 0 \\
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10 semigroups

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