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Precoded Linear Dispersion Codes For Wireless MIMO Channels

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Abstract

A primary design objective in next generation wireless systems is to make efficient use of available bandwidth while keeping error rates low. Multiple Input, Multiple Output (MIMO) systems have shown promise toward fulfilling this objective. Space-time codes spread data symbols in time and space in order to create redundancy, which improves bit error rates (BER) but does not improve capacity. Spatial multiplexing techniques decompose the matrix channel into subchannels. This improves capacity but the resulting systems are not optimized with respect to BER.

Linear Dispersion Codes (LDCs) are linear codes which spread data symbols using dispersion matrices. These codes guarantee high spectral efficiency and have been shown to exhibit good error performance. LDC designs based on frame theory of wavelets explicitly optimize both capacity and error performance.

Conventional space-time codes such as Orthogonal Space-Time Block Codes (OSTBC) and spatial multiplexing systems assume that the transmitter has no knowledge of the channel. In some cases, however, feedback from receiver to transmitter can be established to convey channel state information (CSI). Such systems are called precoding systems.

In this dissertation, we propose Precoded Linear Dispersion Codes (P-LDCs), a family of precoders which assume that the receiver has perfect channel knowledge while the transmitter has statistical information about the transmit and receive correlation matrices. P-LDCs can be viewed as a unifying design strategy which utilizes the LDC structure to implement an optimal capacity-achieving transmit covariance strategy. Utilizing partial CSI, P-LDCs form a linear space-time precoder using dispersion matrices derived according to capacity
optimality criteria. P-LDC dispersion matrices are designed according to a two-fold objective: maximize spectral efficiency, then minimize pairwise codeword error probability for high signal to noise ratio (SNR).

Finally, we analyze frame-based LDCs and demonstrate that existing designs are subject to a capacity penalty under certain circumstances heretofore unexplored. We illustrate this capacity penalty and devise a structural constraint for the LDC precoder based on Grassmannian beamforming to reduce the possibility of this capacity penalty.
Acknowledgments

This dissertation is dedicated to my Lord and Savior, Jesus Christ, the source of my strength and eternal hope. Without Him I am nothing, and none of my academic accomplishments would have been possible. To Him I attribute all glory and thank Him above all for His love for me.

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“Now unto Him that is able to keep you from falling, and to present you faultless before the presence of His glory with exceeding joy, to the only wise God our Saviour, be glory and majesty, dominion and power, both now and ever. Amen.” - Jude 24-25
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Glossary of Acronyms

AOA  Angle of Arrival
AWGN  Additive White Gaussian Noise
BER  Bit Error Rate
BFGS  Broyden-Fletcher-Goldfarb-Shanno method
BLAST  Bell Labs Layered Space-Time Architecture
BS  Base Station
CDMA  Code Division Multiple Access
CSI  Channel State Information
DAST  Diagonal Algebraic Space-Time Code
DB  Decibel
D-BLAST  Diagonal Bell Labs Layered Space-Time Architecture
EGC  Equal Gain Combining
EM  Electromagnetic Field
ESPRIT  Estimation of Signal Parameters via Rotational Invariance Techniques
EVD  Eigenvalue Decomposition
FR  Full Rate
IEEE  Institute of Electrical and Electronics Engineers
IID  Independent, Identically Distributed
LDC  Linear Dispersion Code
LOS  Line Of Sight
LTAST  Linear Threaded Algebraic Space-Time Code
MAPS  Method of Alternating ProjectionS
MIMO  Multiple Input, Multiple Output
MISO  Multiple Input, Single Output
ML  Maximum Likelihood
MRC  Maximal Ratio Combining
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<tr>
<td>MS</td>
<td>Mobile Station</td>
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<tr>
<td>MSE</td>
<td>Mean Square Error</td>
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<td>MUSIC</td>
<td>Multiple Signal Classification</td>
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<td>OFDM</td>
<td>Orthogonal Frequency Division Multiplexing</td>
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<tr>
<td>OSTBC</td>
<td>Orthogonal Space-Time Block Code</td>
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<tr>
<td>PAM</td>
<td>Pulse Amplitude Modulation</td>
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<td>PDA</td>
<td>Personal Digital Assistants</td>
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<td>PEP</td>
<td>Pairwise Error Probability</td>
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<td>P-LDC</td>
<td>Precoded Linear Dispersion Code</td>
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<tr>
<td>PMF</td>
<td>Probability Mass Function</td>
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<tr>
<td>POCS</td>
<td>Projection Onto Convex Sets</td>
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<td>PSK</td>
<td>Phase Shift Keying</td>
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<td>QAM</td>
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<tr>
<td>QF</td>
<td>Quality Factor</td>
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<tr>
<td>QOSTBC</td>
<td>Quasi-Orthogonal Space-Time Block Code</td>
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<tr>
<td>RD</td>
<td>Rank Deficient</td>
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<tr>
<td>RF</td>
<td>Radio Frequency</td>
</tr>
<tr>
<td>RMS</td>
<td>Root Mean Square</td>
</tr>
<tr>
<td>RX</td>
<td>Receiver</td>
</tr>
<tr>
<td>SER</td>
<td>Symbol Error Rate</td>
</tr>
<tr>
<td>SFC</td>
<td>Space-Frequency Code</td>
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<tr>
<td>SIMO</td>
<td>Single Input, Multiple Output</td>
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<tr>
<td>SISO</td>
<td>Single Input, Single Output</td>
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<tr>
<td>SNR</td>
<td>Signal to Noise Ratio</td>
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<td>SQP</td>
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<td>STBC</td>
<td>Space-Time Block Code</td>
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<td>STC</td>
<td>Space-Time Code</td>
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<tr>
<td>STTC</td>
<td>Space-Time Trellis Code</td>
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<tr>
<td>Acronym</td>
<td>Description</td>
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<tr>
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<td>--------------------------------------------------</td>
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<tr>
<td>SVD</td>
<td>Singular Value Decomposition</td>
</tr>
<tr>
<td>TGn</td>
<td>IEEE 802.11 Task Group n</td>
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<tr>
<td>TAST</td>
<td>Threaded Algebraic Space-Time Code</td>
</tr>
<tr>
<td>TST</td>
<td>Threaded Space-Time Code</td>
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<td>TX</td>
<td>Transmitter</td>
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<td>V-BLAST</td>
<td>Vertical Bell Labs Layered Space-Time Architecture</td>
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<tr>
<td>WBE</td>
<td>Welch-Bound-Equality Sequence</td>
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<td>WI-FI</td>
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Chapter 1

Introduction

As the desire for on-demand wireless transfer of high speed data in small, handheld units has become more prevalent, the demands upon existing wireless technologies are becoming more stringent. Consumer demands have exceeded the technological ability of existing wireless infrastructures. In order to accommodate the desire for faster, more reliable communications, next generation wireless communication systems are required to have improved reliability, even in the presence of harsh channel conditions. Furthermore, these systems must achieve very high throughputs.

The need for more reliable, faster communications has been pushed to the forefront by the increasing popularity of handheld devices which engage in high speed wireless data transfer. Whereas previous generation commercial wireless devices, such as cellular phones, were primarily voice communication devices, the technology has changed rapidly with the development of readily available broadband access points for data transfer. Common office tools, such as Personal Digital Assistants (PDAs), commonly come equipped with wireless capability for accessing “Wi-Fi” (Wireless Fidelity) hot spots. These hot spots are used for all sorts of data transfer, from e-mail data to internet browsing to streaming audio and video. Even cellular phones are now equipped with software for internet browsing. Unfortunately,
consumer demands for high speed data transfer have rapidly surpassed the capabilities of existing technologies. Typical cellular phones cannot achieve data rates that satisfy most wireless consumers who are accustomed to the extremely high data rates of wired networked systems. This has necessitated new research into methods of supporting higher data rates over next generation wireless systems.

One of the most promising communication techniques for meeting these requirements is to use Multiple Input, Multiple Output (MIMO) systems. MIMO wireless systems utilize antenna arrays at both the transmitter and receiver to achieve exceptional reliability and high capacity [1, 2]. MIMO technology has become very popular in the research community since its inception in the mid 1990s, although it has experienced relatively little market penetration due to deployment issues. The focus of this work is to analyze existing methods of MIMO communication and to introduce a new technique of utilizing a MIMO system to achieve high data rates and low error rates. Our innovation builds on a body of MIMO research called precoding, and we present a novel space-time precoder known as a Precoded Linear Dispersion Code.

1.1 Early Attempts At System Optimization

As wireless communication technologies have gained wide acceptance over the last half century as an integral part of both consumer and military systems, the role of wireless communication systems has changed. In the formative years of wireless technology, radio waves were used to transmit low-bandwidth data such as voice information. System designers attempted to optimize the error rate of the system while simultaneously trying to deal with suboptimal antenna structures such as the low efficiency dipole [3]. As design engineers attempted to make wireless systems more efficient, they began investigating more efficient antenna design techniques with the objective of getting better directional gain and higher efficiencies from their antennas. Over the years, a host of antenna structures have been
devised in order to tailor an antenna’s physical parameters to a given application at a given frequency band. Among the advances resulting from such design objectives were the aperture antennas such as the conical horn for microwave frequencies, reflector antennas, and microstrip patch antennas. Patch antennas were a particularly interesting development in antenna theory, since it represented a major step in the miniaturization of antennas tailored for specific microwave frequencies. Patch antennas are a patch of copper on a circuit board, mounted above a ground plane with a substrate between the copper patch and the ground plane. Patch antennas’ small size has made them very attractive for designs where space is at a premium [4], since their dimensions are on the order of the wavelength of the microwave frequency, which is usually no more than a few millimeters. Furthermore, their characteristics and directivity can be changed easily by varying parameters such as the antenna’s substrate permeability.

Although physical antenna parameter manipulation enabled system designers to optimize communication links to a limited extent, it became clear as design constraints became even more demanding that simple antenna design techniques were inadequate to achieve the desired high data rates and low error rates. High quality factors (Q-factors) are insufficient to increase a wireless system’s data rate to the levels required to support high-speed broadband wireless data links. This prompted system designers to consider a combination of physical antenna design techniques in conjunction with signal processing. Antenna designers began co-locating groups of antennas, called antenna arrays, at the receiver or transmitter. By varying the relative phase shifts of the signals at each transmitter, designers were able to generate signals with special signal processing characteristics. The combination of physical antenna array design and signal processing techniques led to the innovation known as “smart antennas” [5].
1.2 Smart Antenna Systems

Smart antenna techniques became popular in the decade of the 1970s. In smart antenna systems, an antenna array is located at either the receiver or the transmitter (but not both). Such systems are either called Multiple-Input, Single Output (MISO), if there is a transmit array, or Single-Input, Multiple-Output (SIMO) if there is a receive array. Smart antenna systems can operate in one of two modes. In one application, the receiver utilizes multiple antennas to employ receive (or transmit) diversity, a concept in which the same signal is received (transmitted) multiple times with the objective that at least one of the transmissions occurs without error [6]. This is an inherently statistical process which depends upon the statistical nature of the wireless channel under consideration. In order to use diversity techniques, the antenna configuration must be designed to guarantee statistical independence of the antenna array elements. This requires appropriate channel modeling to establish the statistical variations of the channel amplitudes, such as whether they follow a Rayleigh or a Rice distribution [6]. Various methods of processing the data from the receive array, such as maximal ratio combining (MRC) and equal gain combining (EGC), have been developed to optimize metrics related to the error rate of the transmission.

Another possible use of smart antennas is to increase the system’s gain in a particular direction of space in order to receive (transmit) signals from only one small portion of the angular spectrum. In such designs, a concept known as beamforming is used to “steer” the antenna array’s focus in a single direction. In its simplest form, this requires precisely calculated phase shifts among the different antennas at the array, which upon proper characterization in a signal processing algorithm yields a reception (transmission) which occurs in a single direction. Many popular signal processing algorithms, such as MUltiple SIgnal Classification (MUSIC) [7] and Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) [8], were developed in conjunction with smart antennas. ESPRIT and MUSIC enable straightforward application of signal processing techniques to determine the angle of arrival (AOA) of an incident signal upon a receive antenna array.
The use of smart antennas in beamforming applications has been extended to rapidly
time-varying systems by utilizing adaptive algorithms. Using Wiener and Kalman filtering
techniques [9], it is possible to adapt the beamforming algorithms to vary with time and to
adapt to mobile users within a wireless system. Such systems can be made to adapt not only
to the location of users, but also to the number of users within the system. This enables the
smart antenna system to be flexible in that it can adapt to many different parameters in the
system while maintaining its functionality.

Diversity and beamforming are two very different concepts with very different objectives.
Diversity concepts are used to improve the reliability, or bit error rate (BER), of a wireless
link by introducing redundancy into the spatial domain, whereas beamforming serves to
focus transmissions in a single direction. Beamforming can function to conserve power, if
the system knows the intended direction of transmission, and it can also serve as a means
of multiuser communication, since a transmitter can distinguish between two distinct users
by their angular location. These dual uses of smart antennas have served to establish the
research field of antenna arrays as an active field in the wireless communications body of
research.

Despite their significant advantages, smart antennas also have certain disadvantages when
compared to conventional wireless techniques which do not utilize antenna arrays. The ne-
cessity of having multiple antennas requires multiple radio frequency (RF) or microwave
chains, one for each antenna [10]. This can lead to significantly increased implementation
costs which can prove prohibitive for many consumer applications. In an attempt to accom-
modate this problem, recent research has focused on antenna selection techniques, where a
small number of RF or microwave chains is used to switch among several antennas at the
array in order to optimize some system metric [11–14]. This has the effect of reducing the
number of RF or microwave chains in the system. Another issue in smart antenna systems is
the complexity of the wireless channel model. Whereas conventional wireless systems require
a single characterization between a transmit and receive antenna, a smart antenna channel
model must accommodate a *vector* channel between a single transmitter and a receive array (for SIMO) or between a transmit array and a single receiver (for MISO). Vector channels can present significant challenges in terms of channel estimation and/or prediction at the receiver, since the added complexity of the vector channel can require significant signal processing power.

Despite the significant advantages of smart antenna systems, they have not been able to provide system designers with the capability to meet the high data rates required to support broadband wireless data transmissions. The spectral efficiency of smart antenna systems is simply not adequate to support such high data rates [15], since the primary objective of smart antennas is either to improve BER or to improve antenna directionality. Neither of these design objectives by themselves will improve data rate substantially. As a consequence, wireless system designers needed another innovation in the field of antenna arrays to supplement the advances learned from smart antenna theory. This innovation required the simultaneous use of both a transmit and receive antenna array, a concept now known as a MIMO wireless communication system. MIMO systems can be viewed as a smart antenna array at both the transmitter and receiver.

### 1.3 MIMO Antenna Systems

Bell Labs developed one of the first MIMO systems. Their system, known as Bell Labs Layered Space-Time Architecture (BLAST), was a pioneer in MIMO technology because it proved the ability of multiple antenna architectures to achieve high capacity without increasing transmit power [1,16]. In the BLAST system, it was shown that, under certain assumptions regarding the channel, it is possible to decompose the MIMO wireless channel into a series of *subchannels*, each of which can be used for independent data modulation. This has been shown to dramatically increase the Shannon capacity of the wireless system [1,15]. Each subchannel can be handled independently, and separate data streams can
be transmitted on each substream. This concept, an example of which is called Vertical-BLAST (V-BLAST) because of the configuration of the signal processing algorithm required to decode it, is also known as spatial multiplexing and cannot be achieved using simple smart antennas [17].

A chief advantage of spatial multiplexing systems is that the capacity of an ideal MIMO configuration is substantially larger than that of MISO or SIMO systems. Provided that rather strict assumptions are satisfied in the MIMO channel, it has been shown that for a MIMO system with $N_T$ transmit antennas and $N_R$ receive antennas, the capacity is approximately equal to $\min(N_R, N_T)$ times larger than the capacity of a conventional single input, single output (SISO) system. Therefore, by using clever coding techniques, theoretically it is possible to achieve significantly higher data rates by using antenna arrays without expanding the system bandwidth. Such an advantage has been the driving force behind MIMO technology. MIMO systems enhance system throughput without expanding the required bandwidth and without requiring additional transmit power. As a result, MIMO systems are significantly more bandwidth and power efficient than their MISO, SIMO, and SISO counterparts [15].

In addition to providing the possibility of increased capacity, MIMO systems also provide tremendous flexibility for utilizing spatial diversity. Since MIMO systems have antenna arrays at both ends of a link, it is possible to utilize both transmit and receive diversity, in which the same signal is repeated multiple times at the transmitter and received multiple times at the receiver. MIMO systems have been shown to achieve diversity gains up to $N_R N_T$. Such diversity gains are achieved by employing spatial diversity at both ends of the link, in which multiple copies of the signal are used to improve the statistical likelihood of error-free reception.

As diversity techniques became popular, a new branch of MIMO research called space-time coding was developed as a means of uniting temporal and spatial diversity into a single coding scheme. The simplest of these schemes was developed by Alamouti [18]. In
this scheme, a two transmit, one receive system (referred to as a $2 \times 1$ system) is used to transmit a total of two symbols over two symbol periods. By manipulating the order in which the symbols appear on the transmit antennas as a function of time, it was shown that without loss of data rate relative to a SISO system, the BER could be significantly improved. Later research proved that this code could achieve even better BER performance if a second receive antenna was utilized. This code, known as the Alamouti space-time code, sparked a series of new space-time codes based on orthogonal coding theory [2, 19]. These codes devised novel ways of introducing temporal and spatial redundancy into a MIMO system in order to optimize conventional coding properties such as coding advantage to minimize error rate [20]. As space-time coding research has matured, more sophisticated space-time codes have been developed which deviate from orthogonal coding theory. Some of these codes are based on algebraic number theory [21] and other intricate details of coding theory and trellis coding (see, for example, [22–32]).

Just as smart antenna systems became more complex than point-to-point single antenna links, MIMO systems are more complex than smart antenna systems because of the introduction of the second antenna array. Whereas SISO channels are scalar, and smart antennas have vector channels, MIMO systems have matrix channels. There are a total of $N_T N_R$ channel coefficients, one for each path between each transmitter and receiver. Although the most optimistic assumption for MIMO channel models is that each channel coefficient is complex and independent, identically distributed (IID) from every other coefficient [15], this assumption is rarely valid because there are so many antennas co-located at both antenna arrays. As a consequence, significant research has been devoted to modeling MIMO channels in such a way as to capture realistic channel effects that occur due to fading and correlation among antenna elements. One way of generating test models for MIMO systems is to use ray tracing techniques, in which a complicated fading environment involving obstructions, distance, and even weather is either simulated in a computer program or is measured in an actual measurement campaign [33]. Other more recent efforts have attempted to generate statistical models better suited to numerical simulation. Among the more important of these
channel modeling efforts is the Kronecker channel model [34–36], which models independent fading environments at the transmitter and receiver. These models can be applied to the “ring of scatterers” model made popular in the wireless research [6]. While there have been some deficiencies reported with this model, including the well-known pinhole, or keyhole, effect [37,38], the Kronecker channel model is still the most widely used and accepted MIMO channel model.

1.3.1 Recent Research Thrust: A Unified Approach

An important observation regarding both BLAST and most space-time coding techniques is that they are designed with a single purpose in mind. For example, BLAST systems are intended to increase capacity, but little effort is expended in the design to minimize error rate. Therefore it can be said that spatial multiplexing systems such as BLAST are not optimized with respect to error performance. In most space-time coding (STC) systems, diversity advantage is the main benefit, not data rate or capacity improvement, and so most STC systems are not optimized with respect to multiplexing gain. As a result, most of the existing multiple antenna designs are inadequate because they fail to address a fundamental requirement of next generation wireless systems: simultaneous optimization of both capacity and error rate. One exception to this rule is the linear dispersion code (LDC), a code designed with both error rate and spectral efficiency as design parameters to be optimized [30–32,39,40].

LDCs are space-time codes designed to spread data symbols in both the temporal and spatial domains using non-orthogonal dispersion matrices. These dispersion matrices are designed with the objective of maximizing spectral efficiency. It has been shown that capacity-optimal dispersion matrices give rise to a family of non-unique STCs over which further optimization may occur. Early LDC designs did not search over the space of capacity-optimal codes in such a way as to guarantee any given performance parameters. Later LDC designs based upon the frame theory of wavelets [39], [41, pp.56–101] did optimize the capacity-
optimal family to improve diversity properties of the STC. The result of the frame-based LDC design is a STC which is capacity-optimal, or in some cases nearly capacity-optimal, and which possesses a quantifiable diversity and coding advantage, which helps predict the slope of the BER curve for high signal to noise ratio (SNR). As a consequence, LDCs represent a major breakthrough in the way STC is performed, since it bridged the gap between competing design extremes of high capacity and low error rate.

Although LDCs represent a significant achievement in STC design, LDCs are not without drawbacks. They require computational decoding schemes such as maximum likelihood (ML) decoding, or approximations thereof, such as sphere decoding [42]. These can be tremendously computational, especially if the number of symbol periods used by the code, $\tau$, is large. Furthermore, LDCs are derived from an implicit assumption that the transmitter has no knowledge of the wireless environment. In this sense, LDCs can be viewed as a “worst case design” since they are robustly designed for whatever channel characteristics may be present at any given time. In cases where the transmitter does have some knowledge of the wireless environment, LDC designs are suboptimal because they do not utilize any knowledge of the channel in their design.

### 1.3.2 Recent Research Thrust: Precoding

An increasingly popular research field in wireless communication theory is that of precoding. Precoding occurs whenever the transmitter is able to acquire some information about the wireless environment, whether it be through transmitter-based channel estimation or receiver-to-transmitter feedback through a system control channel. Precoding is a valuable tool in system design, since the presence of channel knowledge of any kind at the transmitter can significantly enhance spectral efficiency [43–47]. Precoding can also provide a means of optimizing symbol error rates (SER) and BER for constellation-specific designs [48, 49]. Precoding techniques can be classified according to the type of channel information the transmitter is assumed to have. If the transmitter is assumed to know the channel’s instantaneous
value with only a slight noise term in the estimate, the precoder is said to utilize channel mean information. If the transmitter is only able to characterize the covariance matrix at the transmitter and receiver, then the precoder is said to utilize channel covariance information.

Channel mean information assumes that the transmitter knows the channel with near perfect accuracy. The only error in the transmitter’s estimate is an estimation error term, which is usually a white noise term. This kind of channel knowledge is the most advantageous kind of information to have, since the transmitter can align its transmission to accommodate the peculiarities of any given channel realization. The problem with assuming channel mean information is that it is very costly to acquire. To achieve channel mean information means that the transmitter must either continually update its estimate of the channel, or it must have access to a very high bandwidth feedback channel from the receiver. While this may be possible in fixed wireless applications, if the channel mean varies rapidly then it is almost impossible to maintain an accurate estimate of the channel. As a result, channel mean information is not a very practical approach to precoding.

Channel covariance information assumes that the transmitter is able to acquire statistical information about the channel from the receiver. It is considered a form of “imperfect feedback” since the channel is not known with certainty [50,51]. There is a substantial difference between channel mean-based and channel covariance-based precoders. Channel mean precoders assume that the transmitter has access to instantaneous MIMO channel estimates which must be updated continually as the fading environment changes from one realization to another. Channel covariance precoders only assume that the transmitter knows how the channel realizations will change over time, and hence instantaneous channel information is not required. Because the transmitter does not need an instantaneous channel estimate, the bandwidth requirements on the feedback path from receiver to transmitter are not as restrictive as for channel mean precoders. Furthermore, in some cases the statistics of the fading environment stay relatively constant over a period of time, so that once the transmitter has acquired the statistical channel information it needs, that information does not need
updating for a fixed amount of time. However, since channel covariance precoders have less information than channel mean precoders, the potential performance benefits of covariance precoders are less than for mean precoders. The spectral efficiency of covariance-based systems will be less, and the BER will usually be greater. However, this disadvantage is offset by the significantly lowered feedback bandwidth requirements. Overall, this tradeoff usually makes channel covariance precoding the precoding mode of choice in systems with a MIMO feedback path and moderate to heavy user mobility.

1.4 Thesis Overview and Outline

In this thesis we focus on combining the unified STC concepts pioneered in linear dispersion codes with the precoding concepts which utilize partial channel state information at the transmitter. The objective of our work is to build upon the LDC theory to incorporate partial channel knowledge into the design of LDCs. The resulting codes, called Precoded Linear Dispersion Codes (P-LDCs), utilize channel covariance knowledge to modify the design of LDC parameters in order to increase the spectral efficiency of the system while maintaining reliable BER performance. Topics of interest to precoding systems, such as beamforming optimality and optimal power distribution, are derived and discussed.

P-LDCs are discussed in two very different contexts. In this work, we show that P-LDCs can be designed in order to guarantee global optimality across all SNRs, a type of design which we designate full rank precoding. Full rank precoders are designed using structural assumptions which guarantee that the precoder has the structural capability of carrying the full Shannon capacity of the channel. Under slightly modified assumptions, it is also possible to construct the P-LDC precoders with structural assumptions which result in the precoder’s inability to carry the full Shannon capacity for high SNR. We call these precoders rank deficient precoders. Rank deficient precoders suffer from capacity suboptimality but provide the system designer with additional flexibility in the design of the code. We quantify
the amount of capacity loss associated with rank deficiency. Further, we draw conclusions about the matrix distance relationships between the full rank and rank deficient precoders.

In addition to P-LDCs, we also present a design modification to the frame-based LDCs of [39]. We show through theoretical development and simulation that the frame-based LDCs of [39] can suffer from a previously undetected loss in capacity due to insufficient structural constraints on the form of the frame-based code. We propose a remedy to this structural deficiency by imposing a structural constraint on the precoder. This structural constraint is derived from the theory of Grassmannian frames [52]. We illustrate a novel application of a technique known as “alternating projections” in order to find a Grassmannian frame which imposes a structural constraint on the frame-based code. This alternating projection method is verified through numerical simulation, and we verify through ergodic capacity simulations that the structural constraint reduces the probability of experiencing the previously undetected problem of capacity loss.

The remainder of this dissertation is outlined as follows. Chapter 2 contains relevant background information which will provide the technical motivation for the work which follows. A detailed explanation of the capacity of MIMO systems, as well as the diversity capability of multielement arrays, is provided. A primer on basic space-time codes structure and a variety of their applications is presented. The structure of linear dispersion codes is explained and their relative advantages over other codes are discussed. A brief tutorial on precoding is also presented.

Chapter 3 presents a full rank, capacity-optimal P-LDC with BER optimization. The codes of Chapter 3 are assumed to achieve full rank for all values of SNR, and hence these codes are termed “ideal,” or “full-rank,” P-LDCs. It is shown that capacity-optimal P-LDCs are superior in spectral efficiency to capacity-optimal LDCs because of the presence of the additional channel information present at the transmitter. A detailed explanation of the effect of correlation at the transmitter and receiver is presented, and we illustrate that the dominant factor in the capacity performance of the P-LDCs is the transmit correlation
environment, not the receive correlation environment. We characterize the optimality range of beamforming for P-LDCs and prove that the region of optimality is no different than for other capacity-optimal precoders which do not spread the data in the time domain by a factor $\tau$. A method of BER enhancement based upon pairwise error probability between codewords is presented, and it is shown that such a modification to the capacity-optimal P-LDC structure can significantly improve BER.

Chapter 4 completes the P-LDC design technique of Chapter 3 by permitting the optimal precoders to become rank deficient. These precoders are prohibited from achieving full rank for all SNR values. As such, we illustrate these “non-ideal” precoders can achieve a significant portion of the capacity for a wide range of SNRs, but for high enough SNRs the non-ideal precoders will eventually suffer capacity loss. We explore the non-ideal precoding solutions in terms of matrix distance from the ideal precoding solutions of Chapter 3, and we derive useful relationships for analysis.

Chapter 5 presents a design modification to frame based LDCs of [39] which imposes additional structural constraints on the design of the LDC. We provide a brief tutorial on the frame theory of wavelets [41] and illustrate how to use this theory to improve on existing designs. These results are verified through simulation.

Finally, Chapter 6 presents a summary of the dissertation and suggests some areas for future research.
Chapter 2

Background

As MIMO technology has begun to filter down to consumer-end products, confusion has emerged as to what constitutes a MIMO system [53]. In January 2004 the Institute of Electrical and Electronics Engineers (IEEE) formed a new standardization task group called 802.11 Task Group n (TGn). The 802.11n standard will combine Orthogonal Frequency Division Multiplexing (OFDM) with MIMO technology, possibly in conjunction with Alamouti Codes [18], to specify a real data throughput of at least 100 Megabits/second. Since a standard is not likely to be adopted until 2007, care must be taken to define a MIMO system carefully, in terms of its analytical properties. This chapter presents the basic technical description of what constitutes a MIMO system and describes its major performance advantages.

MIMO systems utilize antenna arrays at both the transmitter and receiver of a communication system in order to improve performance. Through the use of diversity and/or spatial multiplexing techniques [1,6,16], it is possible to significantly improve the capacity and error rate of wireless systems by exploiting the properties of MIMO communication systems. The advantages of a MIMO configuration are not readily apparent until the matrix structure of the transmission channel is considered. Therefore, the theory of MIMO is
heavily dependent upon the theory of matrices, and many of the advantages of MIMO are best described in terms of matrix quantities. As a result, we will make considerable use of singular value decomposition (SVD) [9], eigenvalue decomposition (EVD) [54], and basic linear algebra concepts such as rank and determinant [55]. In what follows, we present a brief theoretical background on MIMO technology, its motivations, and its applications, and we shall stress this matrix relationship. We present only the information which is most relevant to our discussion in later chapters, and this brief MIMO primer should not be considered exhaustive.

2.1 Channel Model

Crucial to the performance of a MIMO channel is the statistical system used to characterize the interactions among the antennas at each array. The channel correlation properties depend heavily upon the scattering environment as well as the spacing among the antennas, the elevation of the antennas over the scattering environment [6], and the configuration of the antennas, whether they be arranged in a straight line, a circle, or some other geometric configuration. Models differ substantially depending upon whether the transmission occurs over indoor or outdoor terrains and whether the receiver or transmitter are mobile. One simulation technique for channel characterization which is very useful is ray tracing [56, 57], in which a complete scattering environment is simulated at the Electromagnetic Field (EM) level. Ray tracing models are especially useful for simulating urban environments where buildings are arranged in predictable configurations. These models are useful for measuring the effects of antenna correlation upon MIMO capacity in specific test cases. An example of a ray tracing environment is shown in Fig. 2.1.

Because ray tracing models simulate a specific environment, it is difficult to make constructive conclusions about the capacity of a general MIMO system from a specific ray tracing simulation. However, ray tracing is an important technique since most statistical channel
Figure 2.1: Example of a ray tracing environment with four buildings which cause scattering.
models are derived from long-term statistical averages of ray tracing simulations such as that of Fig. 2.1.

2.1.1 Channel Model: Ring of Scatterers Kronecker Model

The most common type of statistical channel model used for capacity analysis is a model which attempts to capture the effect of channel correlation in terms of a matrix quantity. The most popular such model was first proposed in 2000 [34] and has become known as the Kronecker channel model. The essence of the Kronecker channel model is that the correlation environments at the transmitter and receiver are assumed to be independent. In the specific model considered in this dissertation, the properties of ray tracing [58, 59] are utilized to characterize a statistically expected fading environment seen by a typical transmit or receive array. The ray tracing model assumed is called a “one ring model” and is shown in the Fig. 2.2.

In the model of Fig. 2.2, the important parameters are the angular spread of the scatterer ring with respect to the transmit array, $\Delta$, the distance $D$ between the base station (BS) and mobile station (MS), and the radius of the scattering ring, $R$. All scatterers located inside the radius $R$ are assumed to be located on the ring of scattering; all scatterers outside the ring are ignored. The details of the model are explained extensively in the MIMO channel
modeling literature [34], and those details are not presented here. However, specific features of this model need to be highlighted:

- Items which reflect off scatterers exactly once are considered; double reflections are ignored.
- All paths from transmitter to receiver are equally attenuated.
- The radius $R$ of the scatterer ring is determined by the root mean square (RMS) delay spread of the channel [6].
- Effective scatterers are located at angle $\theta$ with respect to the receiver, and $\theta$ is distributed uniformly on $[0, 2\pi)$.
- Each scatterer angle $\theta$ is associated with a phase shift, $\phi(\theta)$, which is drawn from a uniform distribution on $[0, 2\pi)$. The angle $\phi(\theta)$ represents the dielectric properties and the radial displacement from the scatterer ring of the actual scatterer.
- Each scatterer is designated as $S(\theta)$.

Assume there are $K$ scatterers, $S(\theta_k)$, for $k = 1, 2, \ldots, K$. The normalized complex path gain between transmit antenna $t$ and receive antenna $r$, $H_{r,t}$, has been shown to be [34]

$$H_{r,t} = \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \delta(\theta - \theta_k) \exp \left( -j \frac{2\pi}{\lambda} \cdot \left( D_{T_{A_r} \rightarrow S(\theta_k)} + D_{S(\theta_k) \rightarrow D_{R_{A_r}}} \right) + j\phi(\theta) \right) d\theta,$$

where $D_{a \rightarrow b}$ denotes the distance between object $a$ and $b$, and $\delta(\cdot)$ is the Dirac delta impulse function. Computing the correlation between different path gains yields terms of the form

$$\frac{1}{2\pi} \int_{0}^{2\pi} \exp (jx \cos \theta) d\theta = J_0(x),$$

where $J_0(x)$ is the zero order Bessel function of the first kind. By taking the correlation of (2.1), where the number and location of scatterers is unknown, it can be shown that the
correlation from one transmit antenna to two receive antennas is [34]

\[ E[H_{r_1,t}H_{r_2,t}^*] \rightarrow J_0 \left( \frac{2\pi}{\lambda} d^R(r_1, r_2) \right), \tag{2.2} \]

where \( d^R(r_1, r_2) \) represents the distance between receive antennas 1 and 2. In constructing (2.2), we assume both receive antennas are equidistant from the transmit array. In addition, the correlation between two similarly configured transmit antennas and one receive antenna is [34]

\[ E[H_{r,t_1}H_{r,t_2}^*] \rightarrow J_0 \left( \frac{2\pi}{\lambda} d^T(t_1, t_2) \right), \tag{2.3} \]

where \( d^T(t_1, t_2) \) represents the distance between transmit antennas 1 and 2. The expressions in (2.2) and (2.3) are useful in characterizing expected channel characteristics based upon averaged ray tracing simulations. Therefore the model we assume in this dissertation is actually a ray tracing simulation repeated many times to achieve a statistically expected channel correlation environment.

Regardless of the channel model used to characterize the interactions of the antennas at the transmitter and receiver, analysis of a MIMO system hinges upon characterization of the correlation of the MIMO channel, \( R_{MIMO} \). Letting \( h = \text{vec}(H) \), where \( \text{vec}(\cdot) \) denotes the vector-stacking operator, we define \( R_{MIMO} \) as [60]

\[ R_{MIMO} = E[hh^H]. \tag{2.4} \]

A chief benefit of using the Kronecker channel model proposed by [34] and expounded in (2.2) and (2.3) is that (2.4) has a very simplified structure for the Kronecker channel model. As explained in the channel modeling literature [34,47], the channel model in (2.2) and (2.3) has been shown to have the following impact on the entries of \( H \):

- All channel realizations \( H \) are independent, identically distributed (iid) in the time domain.
• Rows of \( \mathbf{H} \) are identically distributed. This means that the receive array is considered to be in the “far field” of the transmit array, and hence each receive antenna sees identical statistics with respect to the transmit array. Assume the \( i^{th} \) row of \( \mathbf{H} \) is \( \mathbf{h}_r \). The \( N_T \times N_T \) transmit correlation matrix is denoted \( \mathbf{R}_T = \mathbb{E}[\mathbf{h}_r \mathbf{h}_r^H] \).

• Columns of \( \mathbf{H} \) are identically distributed. Hence the transmit array is in the “far field” of the receive array, and each transmit antenna sees identical statistics with respect to the receive array. Denote the \( j^{th} \) column of \( \mathbf{H} \) as \( \mathbf{h}_c \). The \( N_R \times N_R \) receive correlation matrix is denoted as \( \mathbf{R}_R = \mathbb{E}[\mathbf{h}_c \mathbf{h}_c^H] \).

• Under the Kronecker channel structure, the MIMO correlation matrix \( \mathbf{R}_{\text{MIMO}} \) in (2.4) is given by

\[
\mathbf{R}_{\text{MIMO}} = \mathbf{R}_T \otimes \mathbf{R}_R ,
\]

where \( \otimes \) denotes the Kronecker tensor product [54].

These properties of the Kronecker channel model lead to a further characterization of the statistics of \( \mathbf{H} \). Decompose \( \mathbf{R}_R = (\mathbf{A}^H \mathbf{A})^T \) and \( \mathbf{R}_T = (\mathbf{B}^H \mathbf{B}) \), where \( \mathbf{A} \) and \( \mathbf{B} \) are non-unique matrices computed from any of a variety of matrix decomposition techniques. Define \( \mathbf{H}_w \) be a \( N_R \times N_T \) matrix of independent, identically distributed complex Gaussian random variables each with complex noise variance \( \mathcal{CN}(0,1) \). Then the Kronecker channel model has been shown to give the alternative statistical representation

\[
\mathbf{H} \sim \mathbf{A}^H \mathbf{H}_w \mathbf{B} ,
\]

where \( \sim \) indicates that each matrix entry on the right-hand-side of (2.6) is identically distributed with the corresponding matrix entry on the left-hand-side of (2.6). The representation (2.6) has several attractive properties which will be utilized in later chapters.

The channel matrix model for MIMO systems appears quite massive and very complex, especially for large dimensions \( N_R \) and \( N_T \). But the MIMO structure is quite powerful
because the matrix structure gives rise to a “pipeline” system for data flow. Whereas SISO systems consist of a single “pipe” because of the point-to-point nature of the link, MIMO systems have a total of $N_R N_T$ channel coefficients - a total of $N_R$ rows and $N_T$ columns. As will be explained in Section 2.2, the “pipes” of a MIMO matrix are called subchannels, and in order to optimize transmission it is necessary to characterize these transmission pipes. Some pipes can be modeled as being larger than others, hence they are more conducive to high speed, reliable data flow. Channel correlations $R_R$ and $R_T$ such as are given in (2.5) lead to narrowing of the pipes at their endpoints so as to restrict the overall data flow capability of the channel.

Impediments to transmission in a MIMO system can be classified into at least two categories [35]. First, it is possible for a MIMO matrix to have very large pipes for transmission, but have very correlated transmit and/or receive antennas. The effect upon the MIMO system is to reduce the data flow capability of the channel because the correlation restricts the size of the connection between the channel’s data pipes and the antenna arrays at each end of the link. Loosely speaking, this means that the channel itself has very favorable pipes for transmission, but the interface between the channel and the antennas restricts the flow at the endpoints. Secondly, it is possible for the channel itself to have only one dominant pipe and a series of secondary, smaller pipes which are so small as to be unsuitable for transmission. In this case, correlation among the transmit and receive antennas is not present, but the channel itself is of intrinsically poor quality. This case is called the “pinhole” channel and has been observed in certain rooftop diffraction scenarios [35,61,62]. Of these two categories of channels, the primary interest in this work is on the first scenario where the channel is of good intrinsic quality but is impeded by channel correlations at one or both ends of the link.

2.1.2 Weichselberger Channel Model

The aforementioned Kronecker channel model is by far the most widely accepted model for spatial correlation at the transmitter and receiver. One of the disadvantages of this
model, however, is that it assumes that the correlations at the transmitter and receiver are separable, that is, that the correlative effects at the transmitter and receiver are decoupled [36]. A limited amount of recent research has focused on a much less popular, more obscure model known as the Weichselberger model [63]. The Weichselberger model is more complex than the Kronecker model. Instead of computing the transmit and receive correlation matrices without regard to the statistical description of the transmitted signal using $R_R$ and $R_T$, it captures the statistics of the transmitted signal in conjunction with the statistics of the correlation environment. Therefore, the Weichselberger model computes two receive correlation matrices. Let the input/output system model be given by

$$y = Hx + n,$$

where $y$, $H$, $x$, and $n$, are the $N_R \times 1$ received vector, the $N_R \times N_T$ channel matrix, the $N_T \times 1$ transmitted signal vector, and the $N_R \times 1$ noise vector, respectively. The first receive correlation matrix is independent of the other link end:

$$R_R = E_{HH^H}$$

(2.7)

The second receive correlation matrix accommodates the statistics of the transmitted signal $x$, where $Q = E_{xx^H}$:

$$R_{R,Q} = E_{HQH^H}.$$

(2.8)

The Weichselberger model has been verified through experimental results [63, 64] and can model certain channel scenarios which the Kronecker model cannot [38].

### 2.1.3 Decisions About Channel Assumptions

Since the Weichselberger model considers two correlation matrices (2.7) and (2.8), this model is more extensive in its scope than the Kronecker model. However, this added com-
plexity has inhibited its widespread acceptance in the MIMO community because its analysis is more cumbersome. Because of this added complexity issue, all channel models discussed in this thesis will utilize the Kronecker model instead of the Weichselberger model.

For all future consideration in this work, the channel is considered to be constant for a fixed period of time $\tau$ over which transmission occurs. Channel matrices $H$ stay constant for a duration $\tau$ and change independently in time from one block of length $\tau$ to another block of length $\tau$. This is known as the block fading model and is quite useful for information theoretic analysis. Further, transmission is assumed to occur on frequency non-selective channels [6], so from a tapped delay line perspective the channel is modeled to have a single tap.

### 2.1.4 The Function of Multipath in MIMO Systems

Traditionally in wireless communication theory, multipath is a phenomenon whose influence is considered detrimental to the system. Quite often the effects of multiple path gains and phase shifts incident upon a receiver must be suppressed by signal processing algorithms. MIMO systems, however, are typically designed to exploit multipath diversity. As opposed to other systems, which need the effects of the channel matrix $H$ to be predictable, in MIMO systems the desire is that the channel matrix $H$ be “as random as possible”, i.e., the system performance is improved the better $H$ is approximated by a matrix of uncorrelated, complex Gaussian entries. Thus the assumptions about the MIMO channel correspond to a worst-case design scenario for many other system architectures.

In some ways, MIMO system design can be compared to code division multiple access (CDMA). In CDMA, different users are assigned codes, and the codes are designed to be orthogonal, or quasi-orthogonal, to each other to guarantee separability of the users. In MIMO systems under consideration herein, there is only one user but different substreams of data streaming from the different transmit antennas. The separability of the substreams
occurs as a result of the channel’s rich multipath environment. The richer the multipath, the more likely the MIMO channel is full rank, providing a greater degree of separability for each substream. How separable the channel is depends upon the SVD of the channel. When line of sight (LOS) is present, this reduces the effective channel rank and provides fewer substreams on which data transfer can occur. The result is a decrease in how efficiently the MIMO channel uses the system bandwidth [56, 65]. Therefore, MIMO systems which are attempting to utilize the transmit antenna arrays to send high data rates at low BER experience significantly degraded performance for strong LOS environments [66].

2.2 Capacity Benefits of MIMO

The initial motivation for MIMO was increased capacity through a matrix channel. The definition of capacity, or spectral efficiency, utilizes many key concepts from information theory [67]. We first present elementary capacity results for a SISO channel and then extend the results to MIMO channels.

2.2.1 SISO Capacity

Consider the simple input/output system shown in Fig. 2.3. The input to the system is a random variable, $X$, and the output of the system is $Y$. The “system” in the block is called the channel and represents a propagation environment through which the input signal
$X$ travels as well as an additive white Gaussian noise (AWGN) process. For the moment we consider the wireless environment to be a unity gain, all-pass filter and consider the noise only. Letting the joint probability mass function (PMF) of $X$ and $Y$ be $p(x, y)$ for $x \in X$ and $y \in Y$, the definition of joint entropy for random variables is [67, pp. 15]

$$H(X, Y) = - \sum_{x \in X} \sum_{y \in Y} p(x, y) \log p(x, y) . \quad (2.9)$$

From (2.9), we can define information in terms of entropy as

$$I(X; Y) = \sum_{x, y} p(x, y) \log \frac{p(x, y)}{p(x)p(y)} = H(X) - H(X|Y) , \quad (2.10)$$

where $p(X)$ and $p(Y)$ are the marginal PMFs of $X$ and $Y$, respectively. Since entropy is a measure of uncertainty, we can see from (2.10) that information can be described intuitively as the “residual uncertainty” between $X$ and $Y$ after the conditional entropy $H(X|Y)$ is removed from the uncertainty of $X$. In communication systems, the logarithm in (2.10) is in base 2 and hence the units of information are data rate per unit bandwidth, which translates into bits per second per Hertz, or (bps/Hz). Because of these units, we can see why capacity is considered a measure of spectral efficiency. If the data rate is kept constant but the bandwidth is reduced, the capacity increases; and, if the data rate increases over the same bandwidth the capacity increases.

An important problem in communication systems is to select an input distribution for $X$ which maximizes the mutual information $I(X; Y)$. The maximum over all $p(x)$ of the information defined in (2.10) is called capacity, and is defined as

$$C = \max_{p(x)} I(X; Y) . \quad (2.11)$$

For a real-valued Gaussian channel in which only AWGN is added and the wireless environment passes data through with unity gain, it has been shown [67, pp. 241] that the maximum
mutual information is achieved when $X$ follows a Gaussian distribution with power $P$ such that $X \sim \mathcal{N}(0, P)$. In such a case, if the noise power is $N$, the capacity has been shown to be [67]

$$
C = \frac{1}{2} \log \left(1 + \frac{P}{N}\right) = \frac{1}{2} \log_2 (1 + \text{SNR}) .
$$

The result in (2.12) is a fundamental result in communication systems.

An important result for communication systems is Shannon’s Coding Theorem [67, pp. 198-203]. This theorem guarantees that all data rates below $C$ in (2.12) are achievable, and that for any data rate $R < C$, it is possible to construct an error correcting code [68] such that the probability of detection error at the output is forced to zero. Therefore, the maximum data rate per unit bandwidth for a given system is $C$, and all data rates $R$ below $C$ are, theoretically at least, achievable. A voluminous field known as coding theory has emerged which has explored many techniques of constructing codes to achieve Shannon Capacity. These techniques range from block codes to convolutional codes and have expanded to recently developed turbo codes and trellis codes [20, 68–71]. Development of low complexity coding structures to achieve Shannon capacity for SISO, SIMO, MISO, and MIMO systems remains an open problem.

### 2.2.2 MIMO Capacity

The essential results for SISO capacity can be extended to the case of MIMO [15]. The MIMO system is characterized by $N_T$ transmit antennas and $N_R$ receive antennas. In Fig. 2.4, a basic MIMO system is illustrated. There are two contributions to the MIMO input/output expression. There is the channel, $H$, and an AWGN process, $n$. As opposed to the SISO case, here it is necessary to analyze the impact of the wireless channel $H$ upon the system capacity. The system equation for a MIMO system can be written in terms of an $N_T \times 1$ input vector $x$ which is launched into the channel and an $N_R \times 1$ receive vector $y$ incident upon the receive antenna array. The channel matrix $H$ is modeled as a matrix of random
variables [72] as described in Section 2.1 and is of dimension $N_R \times N_T$. Each component $[h_{ij}]$ of $H$ represents the propagation environment from the $j^{th}$ transmit antenna to the $i^{th}$ receive antenna. The vector $y$ is the $N_R \times 1$ receive vector and $n$ is a $N_R \times 1$ vector of AWGN at the receiver. Mathematically, we have

$$y = Hx + n.$$ (2.13)

Because of the random nature of $H$, a capacity expression for the MIMO configuration in (2.13) must involve a statistical expectation over all realizations of $H$. Denote the SNR of the system as $\rho$ and an identity matrix of dimension $n$ as $I_n$, and let $E[\cdot]$ denote statistical expectation. Much theoretical work has shown that the Shannon capacity expression for the case in (2.13) is [15]

$$C = E_H \left[ \log_2 \det \left( I_{N_R} + \frac{\rho}{N_T} HH^H \right) \right].$$ (2.14)

Extensive research into the characteristic behavior of MIMO systems has shown that under certain idealized channel conditions, MIMO systems can exhibit a capacity advantage over SISO systems equal to $\min(N_R, N_T)$ [1, 16]. This can be explained by examining the matrix nature of the channel $H$. Whereas the SISO channel has a single point-to-point link from transmitter to receiver, the MIMO configuration gives rise to a total of $N_R N_T$ channel
paths from transmitter to receiver. The channel matrix $\mathbf{H}$ can be decomposed using the singular value decomposition as follows:

$$\mathbf{H} = \mathbf{UDV}^H.$$  \hspace{1cm} (2.15)

In (2.15), $\mathbf{U}$ contains the left singular vectors, $\mathbf{V}$ contains the right singular vectors, and $\mathbf{D}$ contains the singular values on the diagonal. There are $\min(N_R, N_T)$ nonzero singular values in $\mathbf{D}$, and the matrix channel in (2.15) can be viewed as a series of $\min(N_R, N_T)$ subchannels over which communication can occur. Therefore, the chief difference between SISO and MIMO, in terms of capacity, is that the SISO channel contains only a single link from transmitter to receiver, whereas the MIMO channel is characterized by $\min(N_R, N_T)$ subchannels, each of which can theoretically support an independent communication substream. Notice that the presence of these subchannels permits additional data to be transmitted in a given unit of bandwidth \textit{without requiring additional transmit power}. The ideal capacity advantage of MIMO over SISO is proportional to $\min(N_R, N_T)$, and the proportionality constant has been characterized in a number of works [15,73,74]. An essential feature of MIMO systems is that they make it possible to increase the data rate per unit bandwidth and keep the power level the same.

An important illustration of this capacity concept was explored in research conducted by Bell Labs. Their work explored the use of layered coding schemes known as BLAST [16]. The BLAST scheme is illustrated in Fig. 2.5. The BLAST code transmits independent data streams on each of the subchannels of $\mathbf{H}$ in (2.15). The idea behind the original BLAST technique [16] is to rotate which data streams are placed on which subchannels.

The decoding must accommodate the rotation of the substreams through a process called nulling and canceling. The decoding process is illustrated in Fig. 2.6. Because of the diagonal structure of the encoding and decoding process evident in Figs. 2.5 and 2.6, this original BLAST structure has become known as Diagonal-BLAST, or D-BLAST, since encoding and decoding occurs along the diagonal. D-BLAST has been shown to be a useful technique of
Figure 2.5: Diagonal encoding process inherent in D-BLAST. The key feature is that $N_T$ substreams are encoded and transmitted across the $N_T$ antennas.
Figure 2.6: Diagonal decoding process inherent in D-BLAST. Nulling and canceling is the detection scheme of choice. Interference from previously detected symbols is subtracted off, and interference from undetected symbols is projected into the interference subspace.
approaching channel capacity in slowly time-varying MIMO channels, where the transmitter is moving at a slow enough speed that the channel varies very little from one block to another [75]. In a slow fading environment, it is possible that the MIMO channel can experience a deep fade on one or more channel paths, and this results in loss of data. The rotation of the data streams in the D-BLAST architecture protects any given substream from being subject to a deep fade for an extended period of time and this helps increase the capacity. This rotation of substreams can be viewed as a form of spatial encoding.

Alternative architectures known as Vertical-BLAST, or V-BLAST, have also been proposed which eliminate the rotation of substreams during the encoding and decoding process [17]. The V-BLAST techniques are better suited to fast fading channels in which the transmitter moves with enough speed that the channel varies quite rapidly. In fast fading environments, any deep fades are of very short duration. This enables the V-BLAST structure to eliminate the rotation of substreams, since all deep fades are modeled to occur with equal likelihood and with short time duration on all subchannels [6]. In addition to V-BLAST and D-BLAST, other space-time architectures have been developed which are suited to specific channel modeling situations. There is an abundance of literature on this topic, including the comprehensive survey article by Foschini, et al [76].

The BLAST architecture was designed using idealized channel assumptions. To achieve all the theoretical benefits of BLAST, the channel \( \mathbf{H} \) must consist of \( N_R N_T \) spatially independent, identically distributed (and hence uncorrelated), complex Gaussian random variables. Statistically, this means that the subchannels are completely independent and can carry data streams optimally. In practical systems, however, this channel assumption is unrealistic. In practical channels, correlation among the transmit antennas can decrease the achievable capacity of the MIMO system, and in some cases receive correlation is present which reduces the capacity potential even further. This correlative effect has been verified in several measurement campaigns [77, 78], and is modeled accurately in many cases by the Kronecker channel model. The effects of channel correlation, as well as multi-user interference, upon
MIMO capacity has been the subject of extensive research [35, 44, 79]. Some work has even focused on spectral manipulations using the discrete Fourier transform (DFT) to suppress the correlation properties in the channel and create an alternative representation known as a virtual channel [80, 81].

The important feature of BLAST codes is that they attempt to utilize all available subchannels for transmission. They are specifically designed to increase capacity. The use of all the available subchannels is known as spatial multiplexing and constitutes a major innovation in MIMO technology. However, a key drawback in BLAST design is that there is no explicit attempt to minimize error rate. The only design criterion relates to capacity. Therefore, we anticipate that the performance of BLAST systems is suboptimal in terms of error rate.

2.2.3 Methods of Quantifying Capacity

The information theoretic concept of capacity can be characterized in different ways. Existing MIMO research has focused on several ways of increasing capacity, but not all these methods are equivalent. Since “increasing MIMO capacity” can mean different things in different contexts, it is necessary to establish what constitutes an increase in channel capacity, and in what context that increase is meaningful. This description of channel capacity is divided into two subsections: Shannon capacity and channel outage capacity.

Shannon Capacity and Ergodic Capacity

The information theoretic metric stated in (2.14) is known as the Shannon Capacity Limit. Because of the statistical nature of the quantity (2.14), an approximation to this quantity is often achieved by taking a time average instead of a statistical average. This time-averaged quantity is known as the ergodic capacity and represents a measurable quantity in most MIMO systems, given enough time to acquire a reliable time-domain estimate.
of the statistical expectation. The statistical properties of ergodic capacity have been analyzed extensively in MIMO systems for both frequency-flat and frequency-selective fading channels [82]. It represents the absolute maximum data rate which can be supported per unit bandwidth in a given channel \( H \) and is given by the Shannon Coding Theorem [67]. Quantifying, analyzing, and designing codes according to the Shannon Limit is tantamount to designing to achieve an upper bound.

Information theoretic results dictate that MIMO systems designed for optimal ergodic capacity must utilize Gaussian codebooks for the input signal distribution. Sometimes in order to achieve this distribution, special coding structures must be developed in order to permit a Gaussian codebook to be used [39]. However, in practice, communication systems typically use complex-valued codebooks drawn from finite constellations, and as a result, systems designed according to an optimal ergodic capacity criterion fall short of the Shannon Capacity. It is possible to use block codes, convolutional codes, trellis codes, or adaptive modulation in conjunction with finite constellation designs in a MIMO system to force the input distribution to approach that of a complex Gaussian codebook [83–85].

The methods in the later chapters of this thesis utilize ergodic Shannon Capacity as the capacity metric of interest. By selecting ergodic Shannon Capacity, it is ensured that any codes input into the channel have the potential to reach the full capacity of the MIMO channel.

**Outage Capacity**

Outage capacity is a quantity derived to make it easier to measure the statistical quantity of (2.14) “on the fly”. Since the quantity in (2.14) is a statistical expectation, the quantity inside the expectation is clearly a random variable. Whereas Shannon capacity is defined to be the maximum value achieved by the statistical expectation in (2.14) over all distributions of \( H \), outage capacity looks at the quantity inside the expectation, that is, it analyzes the
The random quantity $I$ in (2.16) can be analyzed from a statistical perspective other than expectation, as in (2.14). For example, it is possible to analyze the probability that $I$ is less than a given rate $r$, that is, we can compute

$$C_{\text{outage}} = P[I \leq r].$$

The quantity in (2.17) is known as *outage capacity* [1] and represents the probability that a given realization of the channel $H$ cannot support a given data rate $r$ [86]. It abandons the long-term ergodic capacity perspective of Shannon Capacity and instead looks at the probability that an instantaneous channel realization cannot support a given data rate. From the Shannon capacity theory it is obvious that we must choose $r$ to be less than the Shannon limit given by (2.14). Typical outage capacities analyzed in the literature are 1% and 5% outages. The fact that (2.17) does not require a statistical average makes it very attractive for analysis. Much of the BLAST literature was derived using outage capacity analysis [1, 16], and outage capacity continues to be an area of active interest in MIMO communications research [87–90].

### 2.3 MIMO Reliability: Space-Time Coding

Whereas BLAST is designed to increase capacity, another branch of MIMO research has pursued the objective of optimizing error rates. This branch of MIMO is known as space-time coding (STC). STCs introduce redundancy into a transmission through the use of the spatial dimension of the MIMO arrays. In this way, STCs utilize some of the same coding principles as error control coding and error correction coding, except that STCs add another dimension, the spatial dimension, by manipulating the ordering of symbols on the antenna
arrays at the transmitter. Another coding possibility in the presence of frequency selective fading channels is to use the frequency domain instead of the time domain and to form a space-frequency code (SFC) [91,92]. Space-time codes (and space-frequency codes) can be divided into two broad categories: space-time block codes (STBC), and space-time trellis codes (STTC). In this section, we first summarize the performance criteria used to evaluate STCs and then we present a brief survey of the most commonly used strategies for designing STBCs and STTCs.

2.3.1 Performance Criteria

The key to understanding how MIMO systems can improve error performance is to interpret the multiple antenna system as a diversity system. Diversity is a principle in which the receiver is provided with multiple copies of a faded signal. If the probability that a given wireless link experiences a deep fade is given by $p$, and there are $L$ independent diversity branches (i.e., redundant copies of the signal), then the probability that all the $L$ branches are simultaneously in a deep fade is given by $p^L$ [6]. Therefore, by providing multiple independent copies of a signal, it is possible to offset the effects of deep fading by utilizing redundancy. Examples of diversity include [6]:

- **Pattern, or Angle, Diversity.** By using MIMO antenna arrays to beamform in non-overlapping directions, it is possible to use spatially correlated antennas to transmit and receive independently faded signals by transmitting and receiving in different spatial directions.

- **Polarization Diversity.** By utilizing transmit antennas that have one substream sent on a vertically polarized antenna and another substream sent on a horizontally polarized antenna, independent copies of a signal can be received at a receive antenna [93,94]. It is possible to achieve orthogonal polarizations off a single antenna, and hence some benefits of MIMO can be realized on SISO systems by clever manipulation...
of polarization properties.

- **Field Diversity.** Electromagnetic Field diversity utilizes the uncorrelated nature of the magnetic $H$ field and the electric $E$ field.

- **Frequency Diversity.** Communication systems achieve frequency diversity when redundant data are sent in frequency bands that are uncorrelated.

- **Multipath Diversity.** In CDMA systems with multipath, in which multiple copies of a signal are received at different delays due to the propagation environment, it is possible to resolve the different paths in such a way as to utilize the delayed copies of the signal to increase the SNR, and hence decrease the BER.

- **Time Diversity.** By re-sending the same data sequentially in time such that the multiple transmission occurs after any deep fades in the channel have passed, time diversity can be achieved.

- **Spatial Diversity.** By spacing antennas far enough apart to ensure they are uncorrelated, different transmit antennas can send data streams across independent subchannels with a specified redundancy factor.

MIMO STC systems utilize time and spatial diversity to provide diversity branches among independent transmit and receive antennas.

The performance criteria for diversity systems are explained extensively in the literature [2] and hence only a thumbnail sketch is provided here. Assume that a codeword $c$ is transmitted and spans $\tau$ symbol intervals across $N_T$ transmit antennas, and that a codeword $e$ is received, where

\[
\begin{align*}
\mathbf{c} &= c_1^1 c_1^2 \ldots c_1^{N_T} c_2^1 c_2^2 \ldots c_2^{N_T} \ldots c_\tau^1 c_\tau^2 \ldots c_\tau^{N_T} \\
\mathbf{e} &= e_1^1 e_1^2 \ldots e_1^{N_T} e_2^1 e_2^2 \ldots e_2^{N_T} \ldots e_\tau^1 e_\tau^2 \ldots e_\tau^{N_T}.
\end{align*}
\]
In (2.18), $c^j_i$ represents the transmission at time index $i$ and antenna $j$, and $e^j_i$ is the receiver’s estimate of the transmission from antenna $j$ at time index $i$. If we desire to characterize the probability of error in a system with transmitted and estimated codewords as in (2.18), we need to compute $P(c \neq e)$. An exact value for this expression is difficult to find [95], and consequently it is necessary to approximate this error probability. Two assumptions can make the analysis simpler. First, assume a *union bound* on the probability of error. Denote the entire alphabet of potential transmitted codewords $c = \{c_k\}_{k=1}^K$, where $K$ is the alphabet size. Further, we condition on a particular transmitted codeword, $c_m$. Denote the linear distance between vectors $a$ and $b$ as $a \parallel b$. The union bound says that

$$P(e \neq c_m) \leq \sum_{i \neq m} P(c_i \parallel e \leq c_m \parallel e) \quad (2.19)$$

By using the union bound we can approximate the probability of error by an upper bound of summed terms. A second useful assumption is that we choose only the dominant term(s) in (2.19). The dominant terms are found by making an assumption that the signal to noise ratio is high. It is easy to prove that for sufficiently high SNR, the dominant term(s) in (2.19) are those which involve the constellation point(s) $c_i$ which are closest in distance to $c_m$, i.e., the dominant terms are the “closest neighbors”. Therefore, if the symbol $c_i$ is closest to $c_m$, then the union bound (2.19) is approximated by

$$P(e \neq c_m) \approx P(c_m \rightarrow c_i) \quad (2.20)$$

which represents the probability that codeword $c_m$ is incorrectly detected as $c_i$. The criterion stated in (2.20) is known as the Pairwise Error Probability (PEP) criterion and is often used in asymptotic BER analysis.

To develop a design criterion for STCs so that the expression in (2.20) is optimized, some additional mathematical manipulation is necessary. The PEP criterion in (2.20) can
be re-stated in terms of maximum likelihood decision metrics \( m \), such that [96]

\[
m(e, c_m; \text{CSI}) = \log p(e|c_m, \text{CSI}) ,
\]

(2.21)

where \( p(e|c_m, \text{CSI}) \) is the probability density function of the error conditioned on a transmitted codeword \( c_m \) and transmit channel state information (CSI). In addition to these ML metrics, we need an additional tool from communication theory known as the Chernoff Bound [95]. The Chernoff Bound provides a simple means of approximating the \( Q(\cdot) \) function characteristic of Gaussian BER curves. Denoting a Gaussian random variable as \( X \) and the moment generating function by \( \Phi_X(s) \), we can say that [97, pp. 249]

\[
P(X > 0) \leq \Phi_X(s) ,
\]

(2.22)

where \( s \) is an unknown Chernoff parameter. Denote the power associated with the \( n^{th} \) vector element \( c_n \) as \( p_n \), and the noise variance as \( \sigma_N^2 \). We can utilize the ML metrics of (2.21) and the Chernoff Bound of (2.22), to determine that, for a code length \( N_T \tau \) and Chernoff parameters \( s \),

\[
P(c \rightarrow e) \leq \prod_{n=1}^{N_T \tau} \exp \left( -s p_n |c_n - e_n|^2 (1 - 2s \sigma_N^2) \right) .
\]

(2.23)

The expression (2.23) requires solving for the optimal Chernoff parameter, \( s_{opt} \). By taking the derivative of the product (2.23) with respect to \( s \) and setting the expression equal to zero, we find that \( s_{opt} = \frac{1}{4\sigma_N^2} \). We then can simplify (2.23) to

\[
P(c \rightarrow e) \leq \prod_{n=1}^{N_T \tau} \exp \left( -p_n |c_n - e_n|^2 \right) .
\]

(2.24)

Since the product of exponentials is the exponential of the sum of the exponent terms, we can define a Euclidean distance metric \( d^2(c, e) \), such that

\[
d^2(c, e) = \sum_{n=1}^{N_T \tau} p_n^2 |c_n - e_n|^2 .
\]
Combining terms, and letting $\frac{E_s}{N_0}$ be the system SNR, it can be shown that

$$P(c \rightarrow e) \leq \exp \left( \frac{d^2(c,e)E_s}{4N_0} \right).$$

(2.25)

The expression in (2.25) is a key result for all STC designs.

It is possible to characterize (2.25) using specially defined matrices. Define the matrices $A$ and $B$ as follows:

$$A = \{A_{pq} : A_{pq} = \sum_{t=1}^{T} (c^t_p - e^t_p)(c^t_q - e^t_q)^* \},$$

(2.26)

$$B(c,e) = \begin{pmatrix}
  e^1_1 - c^1_1 & e^1_2 - c^1_2 & \ldots & \ldots & e^1_r - c^1_r \\
  e^2_1 - c^1_1 & e^2_2 - c^1_2 & \ldots & \ldots & e^2_r - c^1_r \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  e^{N_T}_1 - c^{N_T}_1 & e^{N_T}_2 - c^{N_T}_2 & \ldots & \ldots & e^{N_T}_r - c^{N_T}_r
\end{pmatrix}.$$  

(2.27)

From the matrices $A$ and $B$ defined in (2.26) and (2.27), we define the rank of $A$ as $r$ and the eigenvalues of $A$ as $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$. It can be shown that [2]

$$P(c \rightarrow e) \leq \left( \prod_{i=1}^{r} \lambda_i \right)^{-N_R} (E_s/4N_0)^{-rN_R}.$$  

(2.28)

Notice in (2.28) that there are two isolated terms: a term relating to the coding advantage, $(\lambda_1\lambda_2\ldots\lambda_r)^{\frac{1}{2}}$, and a diversity advantage term of order $rN_r$. By making these observations from (2.28), the following two fundamental lemmas for STC design have been used extensively in the literature [2]:

**Lemma 1. Rank Criterion for STC Design.** Since the maximum rank $r$ of $A$ is $N_T$, the maximum diversity advantage of any STC is $N_R N_T$. To achieve this maximum, the matrix $B(c,e)$ must be full rank for any codeword pair $c$ and $e$. If $B(c,e)$ has a minimum
rank r over any codeword pair, then the diversity advantage is \( rN_R \).

**Lemma 2. Determinant Criterion for STC Design.** If a diversity advantage of \( rN_R \) has been achieved, then the minimum of \( r^{th} \) roots of the sum of determinants of all \( r \times r \) principal cofactors of \( A(c,e) \) taken over all possible codeword pairs \( c \) and \( e \) specifies the coding advantage. If the system achieves full diversity, then the minimum of the determinant of \( A(c,e) \) must be maximized over all codeword pairs \( c \) and \( e \).

The criteria for STC design given in Lemmas 1 and 2 have set the standard for diversity-achieving STC design. Although they are only approximate design techniques based upon a union bound and high SNR assumptions, in general they have been widely accepted as the most efficient, simplest way of designing STCs with improved BER performance and manageable complexity.

### 2.3.2 Space-Time Block Codes

STBCs are the most popular STCs because they are the simplest to encode and decode. STBCs are self-contained within a given coding block, so that decoding over a STBC of length \( \tau \) requires an observed received sequence of length \( \tau \) only. The encoding process occurs without memory from one block to the next. We present a summary of the most basic, and widely accepted, STBCs: Alamouti Codes, Orthogonal STBC and Quasi-Orthogonal STBCs, number theoretic codes, algebraic STCs, universal STCs, and linear dispersion codes.

**Alamouti STCs**

One of the first STCs, and still one of the most popular, is known as the Alamouti space-time code [18]. Originally a transmit diversity scheme, the Alamouti code is the simplest of a class of STCs known as Orthogonal Space-Time Block Codes (OSTBC). Alamouti codes borrow heavily from the theory of MRC [6]. In MRC schemes, which were originally designed
for receive diversity arrays (SIMO systems), the receiver estimates the path coefficients between the transmitter and the receive array. We designate these vectorized coefficients as $h = [h_1, h_2]^T$. If a symbol $s$ is transmitted, and there is a 2-element receive array, then there are two signals, $r_1$ and $r_2$, received at the antenna array:

$$r_1 = h_1s + n_1$$
$$r_2 = h_2s + n_2$$

where $n_1$ and $n_2$ are AWGN noise. The MRC receiver combines the received signals, $r_1$ and $r_2$, such that the soft estimate, $\hat{s}$, becomes

$$\hat{s} = h_1^*r_1 + h_2^*r_2 .$$

(2.29)

An ML decision is then made upon the soft decision metric in (2.29). The metric presented in (2.29) can be shown to maximize the SNR at the receiver, which results in optimal BER performance [6].

Whereas MRC is a diversity combining technique [6] for signals at a multiple antenna receiver, in MIMO systems there are multiple antennas at both ends of the link. Because of the multiple transmit antennas, it is necessary to utilize transmit diversity to spread the transmitted signals across the multiple transmit antennas. In the Alamouti code, there are two antennas at both the transmitter and receiver. Alamouti STCs control which symbols appear on which antenna in a given time index [18].

Alamouti codes span exactly $\tau = 2$ symbol periods, and exactly $Q = 2$ symbols of any real or complex constellation are transmitted during those 2 symbol periods. Therefore, the average normalized data rate of the Alamouti STC is 1 symbol/period. From this we see that Alamouti codes do not share the high data rates of BLAST, since no attempt is made to utilize the independent subchannels to send independent data streams.

The two symbols transmitted during a given Alamouti code are denoted $\{s_1, s_2\}$. During
the first symbol period, the transmitter sends the symbols $s_1$ and $s_2$ from antennas 1 and 2, respectively. During the second period, it sends $-s_2^*$ and $s_1^*$. Graphically, we can envision the encoding process as taking place in two dimensions, the time and space dimensions, as shown in Table 2.1:

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>Antenna 1</th>
<th>Antenna 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$s_2$</td>
<td></td>
</tr>
<tr>
<td>$-s_2^*$</td>
<td>$s_1^*$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.1: Two Dimensional Coding of Alamouti STC in Space and Time

When viewed as a matrix, the Alamouti STC in Table 2.1 is an orthogonal matrix. This justifies calling the Alamouti STC an OSTBC.

Since the Alamouti code uses a $2 \times 2$ channel matrix, it is necessary to represent a total of 4 path gains from transmitter to receiver. These channel coefficients can be detailed as follows. Denote $h_1$ as the path gain from transmitter 1 (TX1) to receiver 1 (RX1), $h_2$ as the path gain from TX1 to RX2, $h_3$ as the path gain from TX2 to RX1, and $h_4$ as the path gain from TX2 to RX2. All these complex-valued coefficients are assumed invariant with respect to time, so the channel remains constant over the duration of the code. This is referred to as the “block fading” model, since the channel is constant across the entire code block. With these definitions, the received signals can be described mathematically. The transmitter has two symbols to transmit during the Alamouti code, $s_1$ and $s_2$. Defining $r_1$ as the received signal at time index 1 at RX1, $r_2$ as the signal at RX1 at time index 2, $r_3$ as the received signal at RX2 at time index 1, and $r_4$ as the signal at RX2 at time index 2, and defining four independent AWGN noise values as $n_1$, $n_2$, $n_3$, and $n_4$, we can describe the received signals mathematically as [18]

$$
\begin{align*}
    r_1 &= h_1 s_1 + h_2 s_2 + n_1 \\
    r_2 &= -h_1 s_2^* + h_2 s_1^* + n_2 \\
    r_3 &= h_3 s_1 + h_4 s_2 + n_3 \\
    r_4 &= -h_3 s_2^* + h_4 s_1^* + n_4
\end{align*}
$$

(2.30)
The Alamouti code utilizes the following combining scheme, which is technically *not* a maximal ratio combiner but resembles it very closely. The soft decision estimates are $\hat{s}_0$ and $\hat{s}_1$, which are sent directly to the maximum likelihood decoder. The soft decision estimates are

$$
\hat{s}_0 = h_1^* r_1 + h_2 r_2^* + h_3^* r_3 + h_4 r_4^*
$$

$$
\hat{s}_1 = h_2^* r_1 - h_1 r_2^* + h_3^* r_3 - h_4 r_4^*
$$

(2.31)

It can be shown that the signals in (2.31) are equivalent to the signals in a four-branch MRC problem, which means the diversity order of the transmit diversity Alamouti code is equal to that of a 1-transmit, 4-receive MRC system. This means that the slope of the BER curve for high SNR of the two cases is identical. However, the Alamouti code suffers a 3 decibel (dB) penalty with respect to the MRC system because each transmit antenna sends data with only half the total system power, as opposed to the single transmit, 4 receive system, in which the transmitter transmits data off the single antenna with the full power of the system.

**Generalized OSTBCs and Quasi-Orthogonal STBCs**

The Alamouti codes were the first in a series of investigations into OSTBC. Further investigations were conducted to determine for what antenna configurations OSTBC could be constructed [19]. It has been shown that OSTBC exist for only a very select few configurations of antennas. For example, a square OSTBC exist for real-valued constellations such as Pulse Amplitude Modulation (PAM) only if there are 2, 4, or 8 transmit and receive antennas. Permitting mismatch among the transmit and receive antenna dimensions, it has been shown [19] that non-square generalized OSTBC exist for any number of transmit antennas. These generalized OSTBC are generated using Hurwitz-Radon coding theory [98, 99]. However, for complex constellations such as $m$-ary phase shift keying (PSK), the Alamouti code is the only OSTBC which achieves full diversity (i.e., maximum negative slope for the BER curve at high SNR) and full rate (i.e., an average of 1 symbol per 1 symbol period).
Expanding beyond a $2 \times 2$ transmit-receive configuration requires that the average data rate be decreased to $\frac{3}{4}$ or below. This is a major disadvantage of OSTBC, since in the process of enhancing BER the data rate has been decreased below that of SISO systems.

Since the promise of MIMO includes improved data rate and improved error performance, the development of OSTBC was insufficient for many applications which needed full data rate STCs. As a consequence, the field of Quasi-Orthogonal STCs (QOSTBC) was developed [100]. These space-time codes were developed for the specific cases where more than two transmit antennas were needed, full rate (1 symbol/sec) was required of the code, and full diversity was not a strict requirement. Several such QOSTBCs have been designed and characterized [100–102]. In these cases the slope of the BER curves at high SNRs is less optimal than in the OSTBC case because the design trade-off involved the diversity order of the system. However, recent research has focused on improving the BER performance of QOSTBCs by constellation manipulation and rotation [103–105].

**Number Theory Codes, Algebraic STCs, Universal STCs, and Others**

OSTBC, QOSTBC, and Alamouti codes were among the first STCs to be developed. These codes are at once both easy to construct and theoretically elegant because of their simple linear construction. Unfortunately, this simplicity has been proven to yield systems that do not always achieve the best possible performance. For example, it is possible to utilize number theory [21] to build codes that are more complex but which outperform OSTBC in terms of coding gain, even for small numbers of antennas [26]. The added complexity is derived from the fact that complicated number theoretic notation is used in the construction of the code. These number theoretic codes achieve a spectral efficiency of 1 symbol/sec/Hz and outperform OSTBC BER curves for many system architectures, such as Alamouti codes for the case where $N_T = 2$.

Number theoretic coding was further investigated when algebraic approximation was
introduced into the coding scheme [106]. Two codes have been developed which utilize this complicated theory to construct codes which achieve spectral efficiencies higher than 1 symbol/sec/Hz and which achieve good BER performance [107]. In these codes, an approximation of an irrational number by a rational number is performed, a process known as Diophantine Approximation. The Diophantine process in these codes uses continued fractions to approximate the irrational numbers $\sqrt{2}$ and $\sqrt{3}$ [108]. A continued fraction is any number $x$ of the form

$$x = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \ldots}}},$$

where $a_0$, $a_1$, $\ldots$, and $b_1$, $b_2$, $\ldots$, are integers. By starting with PSK constellations and then applying the algebraic number theory in conjunction with algebraic approximation, it is possible to construct codes of high rate and good reliability.

Whereas most of the aforementioned STCs attempt to optimize either capacity or BER, another line of STC research has focused on bridging the gap between these two design extremes. The basic principle behind such efforts is that a MIMO system has a certain capability of improving two different performance metrics, and to accept performance gains in only one of the two metrics is to accept a suboptimal system design. The tradeoffs and design difficulties inherent in designing codes which achieve improvement in both metrics have been explored recently in pioneering work by Zheng and Tse [109]. In order to design a system which achieves both high capacity and low error rate, the Threaded Space-Time (TST) Architecture [110] has been proposed. TST codes subsume as a special case both the D-BLAST and V-BLAST architectures of Section 2.2.2. TST codes formalize a general definition of a layer and a thread within a STC. In a system with $N_T$ transmit antennas and a code spanning $\tau$ symbol intervals, a given transmission from a specific antenna at a specific time index $t$ is called a layer, or a thread. TST codes can be substantially different based upon the way the layers are assigned as a function of space and time. Unlike the BLAST techniques, each layer is treated as an independent algebraic space-time code and is not designed for a particular receiver. Algebraic STCs are codes designed according to
binary design criteria for PSK-modulated STCs and are guaranteed to achieve full spatial diversity [111]. The TST codes were designed in order to achieve a target data rate by utilizing enough threads to achieve that rate, and the algebraic coding on each thread is designed to maximize the overall spatial and temporal diversity of the transmission [110].

The TST framework [110] has been extended to the case of frequency selective channels [92]. When utilizing the time, space, and frequency domains to encode data, this is referred to as space-time-frequency coding. In the context of TST coding, two alternative designs have been proposed. One alternative is to design a STBC using the TST framework with the specific objective of exploiting the multipath inherent in the frequency selective channel. In this way, the STC is utilizing the frequency selectivity in conjunction with an ML receiver. Within the time domain, a serially concatenated code is used which further enhances the performance. A second design choice is to use OFDM to convert the frequency selective MIMO channel into a series of frequency flat subchannels, after which a space frequency code based upon the same algebraic coding structure of [110] is utilized to exploit the channel's diversity.

The algebraic STC concept was developed even more fully by the innovation of Linear Threaded Algebraic Space-Time Codes (L-TAST), a subset of the TST structure [22]. L-TAST codes are linear over the field of complex numbers. L-TAST were designed with the specific intent that the tradeoff between data rate and diversity advantage could be analyzed and optimized based upon several different design criteria. L-TAST codes have been proven to achieve the optimal peak-to-average power ratio (PAR) for Quadrature Amplitude Modulation (QAM) and PSK input constellations when there is only one receive antenna. L-TAST codes were followed by Diagonal Algebraic STCs (DASTs) [27, 112]. DAST codes are block codes which rely upon the Hadamard transform to send a rotated version of the information-carrying symbols to achieve a data rate of 1 symbol/sec/Hz and full transmit diversity over quasi-static and fast fading channels [112]. Since the Hadamard transform is meaningful only for numbers of transmit antennas that are multiples of 4, the methodology
can be adapted to any number of transmit antennas $N_T$ by rotating the symbol vector and transmitting it over an $N_T \times N_T$ space-time code matrix. Details of the implementation of the DAST technique, including optimal combining at the receiver, optimal and suboptimal decoding methods, and capacity considerations, are elaborated in [27]. Although DAST codes are not proficient at bridging the gap between high data rate and low BER, they are of considerable theoretical interest because they achieve full diversity advantage for any number of transmit antennas, a significant improvement over OSTBC.

The culmination of the threaded STC framework was the Universal space-time code [24]. The Universal STC utilizes all the theory of the TAST, DAST, and numerical coding techniques with Diophantine approximation, to construct full-rate and full-diversity coherent STCs for any number of transmit and receive antennas. The basic concept of Universal codes is to decompose a MIMO channel into a series of threads, and treat each thread as a SISO channel over which algebraic coding is performed. The threads are differentiated from each other by the use of Diophantine approximation. Universal STCs have also been extended to form differential STCs by the use of the Cayley Transform when the MIMO channel is noncoherent. The Cayley transform is a unitary matrix $V$ constructed from the input data set $A$ by the relation [113]

$$V = (I + jA)^{-1} (I - jA) .$$

Use of the Cayley transform enables Universal STCs to span the entire spectrum of channel types, from coherent to noncoherent. Universal STCs are the culmination of a great amount of complicated algebraic number-theoretic research leveraged to accomplish what simple OSTBC could not: high rate, full diversity STCs.

Linear Dispersion Codes

Whereas TAST, DAST, and Universal codes rely upon complicated algebraic number theory to encode data to achieve high data rate and full diversity, another branch of coding,
linear dispersion coding (LDC), has emerged based upon a very simple coding framework [30, 114]. Linear dispersion codes are not designed to achieve a certain data rate, but rather to achieve optimality in terms of MIMO capacity [30, 39]. This distinguishes them from a many of the aforementioned STBCs which make no guarantees about Shannon’s capacity, and in fact are guaranteed to be capacity-suboptimal for high SNR. The development of LDCs has progressed through several stages, and the state of the art in LDC coding relies upon frame theory of wavelets [41] to simultaneously guarantee near-capacity-optimality and high diversity advantage. A tight frame is a “tall” matrix $A$ of dimension $m \times n$, with $m \geq n$, such that $A^H A = I_n$, with $I_n$ as the $n \times n$ identity matrix [39].

Frame-based LDCs can transmit $M$ data symbols, $\{\alpha_m\}_{m=0}^{M-1}$, over the duration of the block code, where $M$ is a system parameter under the control of the designer. The design of LDCs centers upon the selection of a group of dispersion matrices, $\{M_m\}_{m=0}^{M-1}$. LDCs span $\tau$ symbol periods, so if there are $N_T$ transmit antennas in the system, the overall $N_T \times \tau$ transmitted codeword $S(\alpha_0, \alpha_1, \ldots, \alpha_{M-1})$ is a function of these dispersion matrices. The structure of a frame-based LDC is [39]

$$S(\alpha_0, \alpha_1, \ldots, \alpha_{M-1}) = \sum_{m=0}^{M-1} \alpha_m M_m .$$

The simplicity of the expression (2.32) is the driving force behind LDCs. They are simple to encode, provided that an optimization routine for the dispersion matrices $\{M_m\}_{m=0}^{M-1}$ is selected.

The method of selecting the dispersion matrices requires a complete characterization of the input/output relationship. Because the LDC spans $\tau$ symbol periods, we define the $N_R \times \tau$ matrix $Y$ of received signals at the receiver. Denoting the $N_R \times N_T$ channel matrix $H$ and the $N_R \times \tau$ noise matrix $N$, we can write the input/output expression as

$$Y = HS(\alpha_0, \alpha_1, \ldots, \alpha_{M-1}) + N .$$

(2.33)
The expression (2.33) can be simplified by stacking all the time domain elements into a single vector. Let the vector operator \( \text{vec}(\cdot) \) of a matrix \( \mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N] \) be defined as

\[
\text{vec}(\mathbf{A}) = [\mathbf{a}_1^T, \mathbf{a}_2^T, \ldots, \mathbf{a}_N^T]^T.
\]

Using this definition, we define \( \mathbf{y} = \text{vec}(\mathbf{Y}), \mathbf{s} = [\alpha_0, \alpha_1, \ldots, \alpha_{M-1}]^T, \mathbf{H} = \mathbf{I}_\tau \otimes \mathbf{H}, \) and \( \mathbf{n} = \text{vec}(-\mathbf{N}). \) Finally, denote the dispersion super-matrix \( \mathbf{X} = [\text{vec}(\mathbf{M}_0), \text{vec}(\mathbf{M}_1), \ldots, \text{vec}(\mathbf{M}_{M-1})] \) of dimension \( N_T \tau \times M. \) Using these definitions, (2.33) simplifies to

\[
\mathbf{y} = \mathbf{H} \mathbf{X} \mathbf{s} + \mathbf{n} \quad (2.34)
\]

The frame-based structure of LDCs ensures a simple design technique for the dispersion matrices. By designing \( \mathbf{X} \), we simultaneously design the dispersion matrices \( \{\mathbf{M}_m\}_{m=0}^{M-1} \), so it suffices only to specify a design technique for \( \mathbf{X} \). In the absence of channel correlations, it has been shown that the optimal structure for the matrix \( \mathbf{X} \) must *whiten* the input, that is, it must preserve the uncorrelated nature of the channel. As such, the capacity optimal design is to select \[15\]

\[
\mathbf{X} \mathbf{X}^H = \frac{1}{N_T} \mathbf{I}_{N_T \tau} \quad (2.35)
\]

Unfortunately for many systems, the system configuration cannot support this optimal design. For example, in many cases it is necessary to choose \( M < N_T \tau \) because of a need to simplify detection or reduce memory constraints. In such cases it is impossible to choose a matrix \( \mathbf{X} \) such that (2.35) is satisfied due to basic linear algebra. A linear subspace of dimension \( M \) is spanned by an orthonormal basis of \( M \) vectors, but in order to satisfy (2.35) it would require \( N_T \tau > M \) orthonormal basis vectors. When \( M < N_T \tau \), the matrix \( \mathbf{X} \) is a “tall” matrix and can achieve only a portion of the true channel capacity. It has been shown that for tall systems, a selection for \( \mathbf{X} \) which achieves significant, but suboptimal, capacity
is the frame based structure in which

\[ \mathbf{X}^H \mathbf{X} = \frac{\tau}{M} \mathbf{I}_M. \]  

(2.36)

Essentially, (2.36) is a compromise solution to the optimal solution in (2.35). Whereas (2.35) requires the selection of \( N_T \tau \) orthonormal vectors of dimension \( M \times 1 \), (2.36) selects \( M \) orthonormal vectors of dimension \( N_T \tau \times 1 \). The resulting system architecture has been studied extensively in the literature [31,39,40].

An important feature of the frame-based LDCs of (2.36) is that \( \mathbf{X} \) is not unique. Recent research has attempted to exploit this non-uniqueness to further improve system error rates [31]. A design for \( \mathbf{X} \) using (2.36) can be subjected to further constraints on \( \mathbf{X} \) in order to guarantee certain levels of diversity, thus improving error performance for high SNR values. To impose such constraints, it is necessary to utilize the rank and determinant criteria of Section 2.3.1. In order to perform this optimization, it is necessary to select a constellation for the codeword \( \mathbf{s} \) and compute all possible error vectors \( \mathbf{e} \) over a length of time \( \tau \). Further, let us define \( \mathbf{X}_t = [M_{0,t}, M_{1,t}, \ldots, M_{M-1,t}] \), where \( M_{i,t} \) is the \( t \)th column of the matrix \( M_i \). If we compute the codeword error matrix

\[ \mathbf{R}_S = \sum_{t=0}^{\tau-1} \mathbf{X}_t^* \mathbf{e}^* \mathbf{e}^T \mathbf{X}_t^T \]

then it is possible to utilize the rank and determinant criteria in conjunction with \( \mathbf{R}_S \) as follows [31,39]:

- Confining the optimization to the class of frame-based codes of (2.36), maximize the diversity order of the code by maximizing the minimum rank of \( \mathbf{R}_S \) over all codeword pairs \( \mathbf{c} \) and \( \mathbf{e} \).

- Maximize the coding advantage by maximizing the minimum product of the nonzero eigenvalues of \( \mathbf{R}_S \).
Justification for this design is presented in the STC literature [39].

Recent innovations have focused on improving LDC designs. Alternative LDC structures have been proposed which resemble OSTBCs, which can simplify system architecture [115]. Alternative designs have been proposed which reduce the complexity of the ML decoder in order to reduce the number of computations required for symbol detection [116]. The simple time-domain LDC structure has been extended to frequency selective channels while maintaining system capacity and overall code simplicity [117]. Further advances in LDC theory have dealt with optimizing a pseudo-orthogonal LDC such that a Mean Square Error (MSE) metric is minimized [118]. Other recent advances have continued to make LDC research a vibrant field [119–121].

While the frame-based LDC does not guarantee true capacity optimality, and it also does not guarantee full MIMO diversity, it does achieve a useful balance between capacity and performance, and it utilizes a simple design technique. A slight complication of LDC design is that in the diversity optimization procedure, a specific constellation is selected for $s$. If no further coding is performed on the input data stream, then the capacity of the overall system becomes restricted by the selection of the constellation size. Thus in order to achieve high capacity, the input data sequence $s$ must undergo further coding not considered here. However, no restriction is placed on the input vector sequence $s$ that it be manipulated by algebraic number theory. Further, LDCs address the important issue of Shannon capacity, a topic not addressed by more complicated STCs such as universal STCs. Because of the relative simplicity of LDCs, they have carved their own niche in the STC literature.

A Final Comment on STBC

The STBCs discussed herein are not an exhaustive list of codes. Indeed, STBC research has produced literally hundreds of STBCs in recent years, many with very specific design objectives as it pertains to system parameters such as number of transmit and receive an-
tennas, required SNR, and diversity-rate tradeoff. No two codes are exactly alike, and each one has a certain advantage based upon its construction. However, the codes presented here reflect the most important STBCs since they reflect the broad research trends and the most important technical advances in the field.

2.3.3 Space-Time Trellis Codes

A companion theory to STBC is space-time trellis coding (STTC). Trellis codes have the property of combining the concepts of convolutional coding and constellation expansion [20, 70, 71]. It has been shown that trellis codes can achieve high spectral efficiencies and excellent error rate properties because of set partitioning, or constellation partitioning. The basic idea of trellis coding is as follows: First, one bit of redundancy is added to every \( m \) source bits, which expands the constellation from \( 2^m \) to \( 2^{m+1} \) signals. The resulting larger constellation is then repeatedly partitioned into a series of smaller constituent constellations, each of which possesses a greater minimum distance (and hence more desirable BER properties) than the larger constellation from which it was derived. This process is illustrated in Fig. 2.7. A convolutional encoder is then used to encode the source bits according to a very specific procedure outlined in [70, 71]. The designation “trellis coded modulation” refers to the trellis state diagram resulting from the convolutional coding technique. An example convolutional encoder and the corresponding trellis diagram is illustrated in Figs. 2.8 and 2.9.

Trellis coding continues to be an active research field for STC [2]. The theory of these codes has been carefully explored and the implications upon PSK design have been detailed [111]. STTC can be studied from a rich coding theoretic perspective, and the coding gain they provide far exceeds that of STBC. However, widespread implementation has been inhibited due to higher encoding complexity, and the ML decoding is much more complicated. Similar to STBC, the theory of STTC has focused on optimizing the diversity and coding gains as well as the capacity. Additional work has been performed on optimizing the distance spectrum of the code, which characterizes the likelihood of important pairwise error
Figure 2.7: Trellis codes partition larger constellations into smaller sub-constellations.

Figure 2.8: Eight-State Ungerboeck Convolutional Decoder [20].
• Signals in the same, lowest partition in the partition tree are assigned to parallel transitions.
• Signals in the preceding transition are assigned to transitions that start or stop in the same state.
• All signals are used equally often.

Figure 2.9: Trellis Diagram for the Eight-State Code of Fig. 2.8.
events [122]. Recent research has suggested that STTC may not be as effective as a serial concatenation of a STBC and a traditional convolutional code [123].

2.4 Precoding for MIMO

Previous sections have focused on capacity benefits and diversity advantages available to a MIMO system designer. These designs all assumed that the channel was modeled as having no correlation at the transmitter or receiver. Practical multiple antenna systems, as explained in Section 2.1, typically are characterized by correlation at the transmitter and/or receiver. When this correlation is present, the system performance degrades because the design assumptions are violated. Channel capacity decreases [44], and the effectiveness of STCs is also reduced. An excellent reference on the capacity of precoded systems is the thesis of Syed A. Jafar [124].

Precoding refers to a method of preprocessing a transmit signal in order to compensate for channel impairments. Intuitively, if the channel distorts a transmission, the function of a precoder is to pre-distort the transmission so that the serial combination of precoder and channel cancel each other out. The method of precoding used in a system is heavily dependent upon the channel model used in the design. Because the Kronecker channel model explained in Section 2.1.1 is by far the most widely accepted way of modeling channels statistically, we adopt the model of Section 2.1.1 and use the notation therein.

2.4.1 Channel Mean Precoding

Precoding is costly to include in a design. In order to precode, the transmitter must have access to some knowledge about the channel’s structure, and this can be expensive to acquire in terms of system resources. The simplest type of precoder, known as a mean feedback precoder, requires that the transmitter always know the instantaneous channel
realization, or channel mean $H$ [43, 125, 126]. In mean feedback models, the transmitter assumes the channel matrix $H$ is a random matrix with a given mean $\hat{H}$ corrupted by a noise term. Usually, this channel mean information is acquired through a feedback path from the receiver to transmitter, where the receiver is assumed to have estimated the channel. The noise term which corrupts the transmitter’s channel mean estimate reflects the uncertainty in the channel estimate. This uncertainty can arise due to channel estimation errors, errors induced by the feedback channel, and channel variations due to the feedback delay [48]. Often, the receiver will send a quantized version of the channel mean. Methods of intelligently generating quantized channel estimates has been the focus of much research [127, 128]. The quantization inherent in many such schemes is another source of feedback error.

Research on channel mean feedback has shown that in order to optimize both error rate and capacity in the presence of channel mean information at the transmitter, the optimal strategy is to transmit data along the strongest eigenmodes of the channel matrix $H$ [43, 48, 129]. The only difference between capacity-optimal and error-rate optimal coding strategies is the power allocated among the transmission eigenmodes. These power allocations must be computed via numerical methods and vary as a function of design objective.

Although it provides an interesting theoretical construct, channel mean precoding is highly impractical in all but a few cases. The feedback of an entire channel matrix $H$ requires that a total of $N_TN_R$ complex channel coefficients be transmitted from receiver to transmitter. If the channel is fast fading, this feedback must occur once for each coherence time of the channel [6]. In addition, the feedback channel quality must be monitored and estimated periodically to ensure reliability. Even quantized feedback techniques are insufficient to make mean precoding a viable option in most systems. Therefore, mean precoding is best suited to very slowly time-varying channels, typically stationary indoor channels, in which neither the channel matrix $H$ nor the feedback channel exhibit much change over time.
2.4.2 Channel Covariance Precoding

Another type of precoding is known as covariance feedback [43]. This feedback system makes direct use of the correlation terms in the Kronecker channel model of (2.5) and (2.6). Instead of the receiver sending back an estimate of the channel mean to the transmitter, it only sends back the channel correlation matrices $R_R$ and $R_T$. In this way, the transmitter models the MIMO channel as being a complex Gaussian matrix with zero mean and covariance matrix given by $R_{MIMO}$.

Covariance feedback significantly reduces the volume of information on the feedback channel. Since channel estimates are not required at the transmitter, the channel can be rapidly time varying and this does not require additional feedback. As long as the statistics of the channel are assumed to be constant over a lengthy time duration, a realistic assumption in some wireless channels, the transmitter can withstand rapid variations in $H$. Therefore, the coherence time of the channel is not a concern as in the case of channel mean precoding. A quantity of greater interest is the statistical coherence time, the time over which the channel statistics are assumed constant.

Covariance feedback has been the subject of extensive research. Its impact upon channel capacity has been thoroughly investigated [45, 46, 130, 131]. Recent research has proven that in order to achieve capacity and error rate optimality in the presence of channel covariance knowledge at the transmitter, the precoder must transmit data along the strongest eigenvectors of the correlation matrix $R_T$ [49]. The directions of transmission are independent of the receive correlation environment $R_R$, but the amount of power allocated to each eigenvector is a function of $R_R$. The optimal power allocation is determined through numerical optimization [130, 131].

Because it imposes less strict requirements on the channel’s time variation, channel covariance feedback has increased in popularity as a plausible method of improving the capacity and error rates of systems characterized by correlated channels. By preconditioning the
transmitted signal, the capacity of the wireless system can be improved significantly and the BERs can be minimized. Because the transmitter does not need frequent updates of the very slowly time varying channel statistics, the feedback channel does not need to transmit $N_T N_R$ complex channel coefficient estimates every channel coherence time. This reduced feedback volume helps conserve system resources and makes the precoding scheme viable.

\section{2.5 Perspective on This Work}

The research contained in this thesis utilizes the concepts of each of the previous sections. Our major objective is to construct a new class of space-time precoders for correlated channels which achieves both high capacity and low error rates. We utilize the Kronecker channel statistics to model the transmit and receive correlation matrices, $R_T$ and $R_R$. After searching extensively through the many STBC design options we utilized the space-time linear dispersive coding framework of section 2.3.2 with modifications to accommodate precoding for the correlations present in the channel. Using the capacity results in the literature [130], we derive new capacity expressions relevant to our new coding structure and verify optimality. Further, we impose spatial diversity restrictions upon our capacity-optimal codes in order to improve error performance.
Chapter 3

Ideal Precoded Linear Dispersion Codes

Conventional space-time codes such as OSTBC and spatial multiplexing systems such as BLAST assume that the transmitter has no knowledge of the channel. In some slowly time-varying systems, however, it is conceivable that a feedback path from receiver to transmitter can be established to convey information about the channel’s state. Such systems are called precoding systems. Recent MIMO research has focused on precoding techniques in which the transmitter has either full or partial channel state information (CSI). These research efforts have focused either on optimizing capacity [47, 131], or have attempted to minimize BER [48, 49]. Little research has addressed both of these objectives simultaneously in the case of partial transmit CSI.

In this chapter, we propose Precoded Linear Dispersion Codes (P-LDCs), a family of precoders which assume that the receiver has perfect channel knowledge while the transmitter has statistical information about the transmit and receive correlation matrices. P-LDCs can be viewed as a unifying design strategy which utilizes the LDC structure of [39] to implement the optimal capacity-achieving transmit covariance strategy of [131]. Utilizing
partial CSI, P-LDCs form a linear space-time precoder using dispersion matrices derived according to capacity optimality criteria specified in [131]. P-LDC dispersion matrices are designed according to a two-fold objective: maximize spectral efficiency, and then minimize pairwise codeword error probability for high signal to noise ratio (SNR). In this chapter, we derive an expression for the capacity-optimal precoder and demonstrate that the directions of transmission are independent of the receive correlation matrix but are dependent upon the transmit correlation matrix. We further derive a formula for the numerical computation of the optimal transmission powers of the precoder. Because of the time spreading factor, our results reveal a repetitive pattern in how the transmit power is distributed as a function of time and space. We also discuss “τ-Beamforming,” a modified version of beamforming for systems which span multiple symbol periods. This τ-beamforming is characterized in terms of optimality regions as a function of the transmit and receive correlation environment. We verify that the presence of the time spreading factor does not change the condition of optimality for beamforming derived in [131]. Because of the structure of the linear code, we demonstrate that the capacity-optimal precoders are not unique and hence can be optimized according to an asymptotic error performance criterion.

The performance advantage gained by P-LDCs is that it is possible to introduce redundancy using the time and spatial domains, all while utilizing some transmit channel knowledge to facilitate optimal transmission. As a consequence of the construction of the code, it is possible to ensure efficient use of the existing spectrum while guaranteeing the SER performance is predictable for high SNR.

This work builds on the dispersive coding framework for transmitters with no channel knowledge [39]. Independent work, developed in parallel in [132], has derived time-dispersive codes for channel models utilizing the virtual channel model [81], which assumes a linear antenna array. The P-LDCs designed herein can be generalized to arbitrary array geometries. We also provide explicit capacity derivations, a detailed explanation of the nature of τ-Beamforming for P-LDCs, and the relationship between transmit and receive correlations
and dispersion matrix design, topics not addressed in [132] or elsewhere.

This chapter is organized as follows. In Section 3.1, the channel model under consideration is detailed and the linear form of the code construction for P-LDCs is explained. In Section 3.2, we present the design of P-LDCs according to a capacity criterion, in which we derive optimal directions of transmission, powers associated with each transmission direction, and conditions for optimality for \( \tau \)-Beamforming. In Section 3.3, we utilize the non-uniqueness of the P-LDC of Section 3.2 to present a design modification for optimal error performance at high SNR. In Section 3.4, we evaluate the performance of P-LDCs using numerical simulations.

Throughout this work, we use \( \mathbb{E}_H(\cdot) \) to denote statistical expectation over all possible channels \( H \) and \( \text{tr}(\cdot) \) to indicate the trace of a matrix. Vectors are designated by boldface lowercase letters and matrices by boldface uppercase letters. The transpose operation is designated \((\cdot)^T\), and the conjugate transpose operation is \((\cdot)^H\). The operator \( \text{vec}(A) \) stacks the columns of a matrix \( A \) into a single column vector.

### 3.1 System Model

The P-LDC precoder design uses a well-established statistical channel model in conjunction with linear dispersion matrices to form the resulting code. The assumptions behind the channel model and the underlying precoder structure are outlined below.

#### 3.1.1 Channel Model

In much of the space-time coding literature, the matrix channel is modeled as a matrix of independent, identically distributed (iid) complex Gaussian random variables. The deficiencies of this model in real-world propagation scenarios marked by transmit and receive correlations have been well documented [38], because they fail to accommodate practical
propagation scenarios in which the transmitter, or receiver, or both, experience some degree of spatial correlation [34,36].

We assume a propagation environment in which there are correlations among the antennas [34]. The correlations at the receive array are assumed to be independent of those at the transmit array. We assume there are \( N_R \) receive and \( N_T \) transmit antennas in a frequency nonselective MIMO system, resulting in an \( N_R \times N_T \) matrix \( \mathbf{H} \). Define the \( N_T \times N_T \) transmit correlation matrix \( \mathbf{R}_T \) and the \( N_R \times N_R \) receive correlation matrix \( \mathbf{R}_R \). Elementary linear algebra permits us to say that for the Hermitian, positive semidefinite matrices \( \mathbf{R}_R \) and \( \mathbf{R}_T \), we have the relations

\[
\mathbf{R}_R = (M^H M)^T \quad \text{and} \quad \mathbf{R}_T = (N^H N)^T
\]

for matrices \( M \) and \( N \) [34]. Using such assumptions, and utilizing eigendecomposition, we can make the following statements about the channel model, where we use \( \sim \) to denote identical statistical distribution, and \( \mathbf{H}_w \) is a matrix of iid complex Gaussian entries distributed according to \( \mathcal{CN}(0, 1) \):

\[
\mathbf{H} \sim M^H \mathbf{H}_w N \quad \text{(3.1)}
\]

\[
\mathbf{R}_R = V \Gamma V^H \quad \text{(3.2)}
\]

\[
\mathbf{R}_T = U \Lambda U^H \quad \text{(3.3)}
\]

where \( U \) is the eigenvector matrix of \( \mathbf{R}_T \), and \( V \) is the eigenvector matrix for \( \mathbf{R}_R \). The diagonal matrices \( \Lambda \) and \( \Gamma \) contain the eigenvalues of \( \mathbf{R}_T \) and \( \mathbf{R}_R \) in descending order, i.e., \( \lambda_1 \geq \lambda_2 \geq \ldots \lambda_{N_T} \) and \( \gamma_1 \geq \gamma_2 \geq \ldots \gamma_{N_R} \). Under these assumptions, the rows of \( \mathbf{H} \) are identically distributed, as are the columns. Assuming (3.1), this gives rise to the Kronecker channel model [36], in which \( \text{cov}(\text{vec}(\mathbf{H})) = \mathbf{R}_T \otimes \mathbf{R}_R \), where \( \otimes \) denotes the Kronecker tensor product and \( \text{cov}(\cdot) \) denotes the covariance of the argument.

We assume communication occurs in the downlink. It is assumed that the receiver knows the channel \( \mathbf{H} \) perfectly, but the transmitter only knows \( \mathbf{R}_R \) and \( \mathbf{R}_T \). This constitutes the covariance channel model, which models realistic propagation environments in which the transmitter cannot track instantaneous channel variations [43]. This model is particularly
useful in cases where the channel may vary slowly, or even rapidly when the variations occur with statistical predictability as a result of the spatial correlation environment.

### 3.1.2 Code Construction Model

Assume the space-time precoder spans $\tau$ symbol periods. The transmitter generates $M > \tau$ complex-valued symbols, $\{\alpha_m\}_{m=0}^{M-1}$, to be transmitted over the duration of the code. The P-LDC encodes these symbols into an $N_T \times \tau$ codeword matrix $S(\alpha_0, \alpha_1, \ldots, \alpha_{M-1})$. Define $Y$ as the $N_R \times \tau$ collection of vectors collected at the receive antenna array of dimension $N_R \times 1$, $\mathcal{E}_S/\sigma^2$ as the transmit SNR, and $N$ as an $N_R \times \tau$ matrix of iid, circularly symmetric complex Gaussian random variables with variance $\sigma^2$. Then,

$$ Y = \sqrt{\mathcal{E}_S}HS(\alpha_0, \alpha_1, \ldots, \alpha_{M-1}) + N \quad (3.4) $$

In (3.4), the symbols $\{\alpha_m\}_{m=0}^{M-1}$ are drawn from a complex constellation and each complex symbol modulates a single, complex-valued dispersion matrix. This formulation has certain limitations in that certain types of coding schemes are not possible, such as the Alamouti scheme. However, it has been shown that modulation schemes such as (3.4) do not suffer significant performance penalties relative to “separate modulation” schemes, in which the real part and imaginary parts of a symbol each modulate a separate dispersion matrix [39].

The key feature of P-LDCs, shared with LDCs of [39], is the linear form of $S$ utilizing a collection of dispersion matrices $\{M_0, M_1, \ldots, M_{M-1}\}$ as follows:

$$ S(\alpha_0, \alpha_1, \ldots, \alpha_{M-1}) = \sum_{m=0}^{M-1} \alpha_m M_m \quad (3.5) $$

Within (3.4), there are several degrees of flexibility provided by the dispersion code structure, since $\tau$, $M$, $N_R$, and $N_T$ are all parameters under the control of the system designer.

We represent (3.4) in vector form by vertically stacking the time dimensions of $Y$, $N$, and
To that end, define the $N_R \tau \times 1$ vector $y = \text{vec}(Y)$, the $N_R \tau \times 1$ vector $n = \text{vec}(N)$, the $N_R \tau \times N_T \tau$ matrix $\mathcal{H} = I_\tau \otimes H$, and the $M \times 1$ vector $s = [\alpha_0, \alpha_1, \ldots, \alpha_{M-1}]^T$. We assume $\mathbb{E}[ss^H] = I_M$. Also define

$$\mathcal{X} = [\text{vec}(M_0), \text{vec}(M_1), \ldots, \text{vec}(M_{M-1})]. \quad (3.6)$$

These definitions lead to the alternative system representation

$$y = \sqrt{\mathcal{E}_S} \mathcal{H} \mathcal{X} s + n. \quad (3.7)$$

The objective of this work is to design dispersion matrices $[M_0, M_1, \ldots, M_{M-1}]$, or equivalently $\mathcal{X}$, such that the system capacity is maximized and an error performance criterion is satisfied. For reasons which we clarify later, we select $M = N_T \tau$. This yields a square matrix $\mathcal{X}$ and guarantees that the design can achieve capacity optimality for all SNR values.

### 3.1.3 Decoding

Decoding of the P-LDC is accomplished by use of the maximum likelihood (ML) decoding scheme, which is optimal under the assumption of equally probable complex Gaussian symbols. The estimated symbols $\hat{s}$ are given by

$$\hat{s} = \arg \min \| y - \sqrt{\mathcal{E}_S} \mathcal{H} \mathcal{X} s \|^2_2. \quad (3.8)$$

As with conventional LDCs, decoding of P-LDCs using the ML rule is quite complex, but the storage requirements are reduced because the code is specified by the set of dispersion matrices. Decoding complexity can be reduced by utilizing sphere decoding techniques [133].
3.2 Design of Capacity-Optimal P-LDCs

The capacity-optimal design proposed in this section is a combination of two widely accepted concepts: the linear dispersion code of [39], and the capacity-optimal precoding strategy based on channel covariance knowledge in [131]. We combine these two concepts into a capacity-optimal design based upon the assumption of a Gaussian codebook. The Gaussian codebook assumption of this section is then relaxed in the next section in order to optimize error rates for practical complex constellations.

The design of capacity-optimal P-LDCs assumes block fading, that is, both the statistics of the channel and the channel itself remain constant for the duration of the code. From one channel realization to the next, the statistics remain constant but the channel is permitted to vary as a function of time.

3.2.1 Mathematical Description of the Capacity Problem

Standard analysis of (3.7) yields the following expression for the ergodic capacity [15]:

\[ C = \max_{X : \text{tr}(XX^H) = \tau} \frac{1}{\tau} C(XX^H), \tag{3.9} \]

where

\[ C(XX^H) = E_H \left[ \log \det \left( I_{N_rT} + \frac{\mathcal{E}_S}{\sigma^2} \mathcal{H} XX^H \mathcal{H}^H \right) \right]. \tag{3.10} \]

In order to design an optimal precoder \( X \), a derivation of the capacity-optimal product \( XX^H \) must be performed, which in turn yields the complete set of dispersion matrices \( \{M_m\}_{m=0}^{M-1} \).

For simplicity, define \( Q = XX^H \), such that (3.10) becomes

\[ C(Q) = E_H \left[ \log \det \left( I_{N_rT} + \frac{\mathcal{E}_S}{\sigma^2} \mathcal{H} Q \mathcal{H}^H \right) \right], \tag{3.11} \]
where we restrict $Q$ to be Hermitian and non-negative definite. It also must satisfy a power constraint, $\text{tr} (Q) = \tau$. Notice that $Q$ is the product of two square matrices. This is a result of our selection of $M = N_T \tau$. Had we selected $M < N_T \tau$, the matrix product $XX^H$ would have had dimensions of $N_T \tau \times N_T \tau$ but would have had a maximum rank of $M$. Our selection of $M = N_T \tau$ avoids the possibility of $Q$ not being able to achieve the full rank $N_T \tau$.

Finding an optimal solution to (3.11) requires modification of similar capacity calculations for codes which span only one time index (i.e., $\tau = 1$). Existing capacity calculations assume that each row and column of $H$ is identically distributed. In the model (3.10), each entry in $H$ is not a random variable, which is a result of the sparseness caused by the Kronecker tensor product. For example, if we denote the first $N_R$ rows of $H$ as $\{\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_{N_R}\}$, we see that $\hat{h}_i = [h_i, 0, 0, \ldots, 0]$, where $h_i$ is the $i$th row of $H$. We compute the covariance as

$$E \left[ \hat{h}_i^T \hat{h}_i^* \right] = \begin{bmatrix} R_T & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}.$$  

The next $N_R$ rows of $H$ are $\hat{h}_i = [0_{(1 \times N_T)}, h_i-N_R, 0, 0, \ldots, 0]$, and computing the covariance yields

$$E \left[ \hat{h}_i^T \hat{h}_i^* \right] = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & R_T & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}.$$  

Clearly, not all of the rows of $H$ are identically distributed because of the variability of the location of $R_T$ on the diagonal. Similar calculations show that while the columns of $H$ are identically distributed, the columns of $H$ are not. This motivates the capacity discussion which follows.

Let $W = I_r \otimes H_w$, $\Sigma_R = I_r \otimes R_R$, and $\Sigma_T = I_r \otimes R_T$. These matrices are of dimension
\[(N_{R\tau} \times N_{T\tau}), (N_{R\tau} \times N_{R\tau}),\) and \((N_{T\tau} \times N_{T\tau}),\) respectively, and are diagonally time-copied versions of the smaller, constituent matrices. It can be shown that \([34]\)

\[
\mathcal{H} \sim \Sigma_{R}^{\frac{1}{2}} W \Sigma_{T}^{\frac{1}{2}}. \tag{3.12}
\]

Here we stress that \(\sim\) indicates that the identical statistical distribution of both sides of (3.12) follows the Kronecker block structure of the channel matrix \(\mathcal{H}\).

### 3.2.2 Optimal Solution

Substitution of (3.12) into (3.11) permits us to express \(C(Q)\) as

\[
C(Q) = E_{\mathcal{H}} \left[ \log \det \left( I_{N_{R\tau}} + \frac{E_{S}}{\sigma^2} \Sigma_{R}^{\frac{1}{2}} W \Sigma_{T}^{\frac{1}{2}} Q \Sigma_{T}^{\frac{1}{2}} W^{H} \Sigma_{R}^{\frac{1}{2}} \right) \right]. \tag{3.13}
\]

To gain further insight into this expression, we must utilize the eigendecompositions of \(R_{T}\) and \(R_{R}\) given in (3.2) and (3.3). Define the Kronecker tensor products of the eigenvector and eigenvalue matrices as follows: \(\hat{U} = I_{\tau} \otimes \hat{U}, \hat{V} = I_{\tau} \otimes \hat{V}, \hat{\Gamma} = I_{\tau} \otimes \hat{\Gamma},\) and \(\hat{\Lambda} = I_{\tau} \otimes \Lambda.\) Clearly \(\hat{U}\) and \(\hat{V}\) retain their unitarity from \(U\) and \(V\) because of the time blocking on the diagonal. We can then express

\[
\Sigma_{R} = \hat{V} \hat{\Gamma} \hat{V}^{H} \tag{3.14}
\]

\[
\Sigma_{T} = \hat{U} \hat{\Lambda} \hat{U}^{H}. \tag{3.15}
\]

From the above development, the following theorem describes the solution to (3.9) and (3.11). This method of proof utilizes a general capacity framework developed for \(\tau = 1\) MIMO space-time codes ([46, 131]), and we generalize it to the multiple time index codes used in this work.

**Theorem 1.** A capacity-optimal precoder \(Q\) for the system described in (3.9) and (3.11) is given by \(Q = \hat{U} \Xi \hat{U}^{H},\) where \(\hat{U}\) is implicitly defined as the eigenvector matrix in (3.15) and contains the time-copied eigenvectors of the transmit correlation matrix, \(R_{T}\). \(\Xi\) is a diagonal
power distribution matrix.

Proof. We follow the steps directly from [131] and [46], reproducing them here in abbreviated form to emphasize that the time-blocking structure does not alter their results. Substituting (3.14) and (3.15) into (3.13), we obtain

\[
C(Q) = E_H \left[ \log \det \left( I_{NR\tau} + \frac{\mathcal{E}_S}{\sigma^2} \tilde{V} \tilde{\Gamma}^{\frac{1}{2}} \tilde{V}^H \mathbf{W} \tilde{\Lambda}^{\frac{1}{2}} \tilde{U}^H \mathbf{Q} \tilde{U} \tilde{\Lambda}^{\frac{1}{2}} \tilde{U}^H \mathbf{W}^H \tilde{V} \tilde{\Gamma}^{\frac{1}{2}} \tilde{V}^H \right) \right].
\]

Since the \( \log \det (\cdot) \) function is invariant with respect to multiplication by a unitary matrix \( \tilde{V} \), we can simplify as follows:

\[
C(Q) = E_H \left[ \log \det \left( I_{NR\tau} + \frac{\mathcal{E}_S}{\sigma^2} \tilde{\Gamma}^{\frac{1}{2}} \tilde{V}^H \mathbf{W} \tilde{\Lambda}^{\frac{1}{2}} \tilde{U}^H \mathbf{Q} \tilde{U} \tilde{\Lambda}^{\frac{1}{2}} \tilde{U}^H \mathbf{W}^H \tilde{V} \tilde{\Gamma}^{\frac{1}{2}} \right) \right].
\]

We make the substitution \( \tilde{Q} = \tilde{U}^H \mathbf{Q} \tilde{U} \). Consistent with [46] and [131], our objective is to show that \( \tilde{Q} = \tilde{U}^H \mathbf{Q} \tilde{U} \) is a diagonal matrix, which implies that the eigenvectors of \( \mathbf{Q} \) are identical to the eigenvectors \( \tilde{U} \) of the transmit correlation matrix, which will conclude the proof.

Multiplication by unitary matrices does not affect the trace operator, so \( \text{tr}(\tilde{Q}) = \text{tr}(\mathbf{Q}) \).

The matrix product \( \tilde{V}^H \mathbf{W} \tilde{U} \) has identical characteristics as \( \mathbf{W} \) because of the block-unitarity of \( \tilde{U} \) and \( \tilde{V} \) and the block structure of \( \mathbf{W} \). Then we have

\[
C = \max_{\mathbf{Q}: \text{tr}(\mathbf{Q}) = \tau} E_H \left[ \log \det \left( I_{NR\tau} + \frac{\mathcal{E}_S}{\sigma^2} \tilde{\Gamma}^{\frac{1}{2}} \tilde{W} \tilde{\Lambda}^{\frac{1}{2}} \tilde{Q} \tilde{\Lambda}^{\frac{1}{2}} \tilde{Q}^H \tilde{W}^H \tilde{\Gamma}^{\frac{1}{2}} \right) \right].
\]

As stated already, our objective is to show that for an optimal precoder \( \mathbf{Q}_{opt} \), the matrix product \( \tilde{Q} = \tilde{U}^H \mathbf{Q}_{opt} \tilde{U} \) is diagonal. Because \( \tilde{\Lambda}^{\frac{1}{2}} \) is also diagonal, this is equivalent to showing that the matrix product \( \tilde{\Lambda}^{\frac{1}{2}} \tilde{Q} \tilde{\Lambda}^{\frac{1}{2}} = \tilde{\Lambda}^{\frac{1}{2}} \tilde{U}^H \mathbf{Q}_{opt} \tilde{U} \tilde{\Lambda}^{\frac{1}{2}} \) is diagonal.

Denote the optimal precoding matrix \( \mathbf{Q}_{opt} \), and designate the optimal matrix product \( \tilde{Q} = \tilde{\Lambda}^{\frac{1}{2}} \tilde{U}^H \mathbf{Q}_{opt} \tilde{U} \tilde{\Lambda}^{\frac{1}{2}} \). Clearly the power constraint \( \text{trace}(\tilde{\Lambda}^{-\frac{1}{2}} \tilde{Q} \tilde{\Lambda}^{-\frac{1}{2}}) \leq \tau \) must be satisfied.
Substituting into (3.16), we obtain

$$C = E_H \left[ \log \det \left( I_{N_R \tau} + \frac{\xi_S}{\sigma^2} \tilde{\Gamma}_1^2 W \tilde{Q} W H \tilde{\Gamma}_1^2 \right) \right] .$$  (3.17)

Eigendecomposing \( \hat{Q} \) into eigenvectors \( Z \) and eigenvalues \( D_Z \), and simplifying the \( \det(\cdot) \) function which is invariant with respect to multiplication by unitary matrices, we have

$$C = E_H \left[ \log \det \left( I_{N_R \tau} + \frac{\xi_S}{\sigma^2} \tilde{\Gamma}_1^2 W D_Z W H \tilde{\Gamma}_1^2 \right) \right] .$$  (3.18)

The power constraint \( \text{trace}(\tilde{\Lambda}^{-\frac{1}{2}} \hat{Q} \tilde{\Lambda}^{-\frac{1}{2}}) \leq \tau \) in (3.17) becomes \( \text{trace}(\tilde{\Lambda}^{-\frac{1}{2}} D_Z \tilde{\Lambda}^{-\frac{1}{2}}) \leq \tau \), and previous work in [46] establishes that this new trace constraint is satisfied.

Because the optimal capacity-achieving expression in (3.18) is a diagonal matrix \( D_Z \), this proves that the quantity \( \tilde{\Lambda}^{\frac{1}{2}} \tilde{Q} \tilde{\Lambda}^{\frac{1}{2}} = \hat{\Lambda}^{\frac{1}{2}} \tilde{U} H Q_{opt} \tilde{U} \hat{\Lambda}^{\frac{1}{2}} \) is diagonal, which implies that the eigenvectors \( U_{opt} \) of \( Q_{opt} \) are aligned with the eigenvectors \( \hat{U} \). This concludes the proof.

\[\square\]

The key result of Theorem 1 is that the capacity-optimal transmit strategy is to transmit independent complex Gaussians along the eigenvectors of \( \hat{U} \). Because of the time-blocked nature of the problem, the block-diagonal structure of \( \hat{U} \) leads to a block-diagonal structure in its eigenvectors. This means that each eigenvector contains a set of zeros along with one of the eigenvectors of the transmit covariance matrix \( R_T \).

To interpret the results of Theorem 1, consider the fact that the \( N_T \tau \) eigenvectors of \( \hat{U} \) are actually \( \tau \) copies of \( N_T \) eigenvectors of \( R_T \), with the remaining elements of the eigenvectors being zeros. As a consequence, the capacity-optimal transmission scheme utilizes the same \( N_T \) eigenvectors \( \tau \) times. Heuristically, the P-LDC coding scheme computes an optimal way of transmitting data across \( N_T \) eigenvectors a total of \( \tau \) times.
Power Constraints

The trace constraint in (3.9) for P-LDCs is less restrictive than the constraints imposed on conventional LDCs, which assume no channel knowledge at the transmitter [39]. Although in [30] and [39] a “total power constraint” is mentioned, the focus of most of their designs was on cases where restrictions were placed on the individual dispersion matrices. The trace constraint in (3.9) is equivalent to $\text{Tr} \left( \sum_{m=0}^{M-1} M_m M_m^H \right) = \tau$. While constraints on individual dispersion matrices are useful when the transmitter is operating without CSI because they force power to be distributed equally across all symbols and transmit antennas in all time indices, this is not optimal when the transmitter can orient its transmissions along the geometry of the channel. In fact, such a scheme is obviously suboptimal in the case of covariance knowledge at the transmitter, since it is advantageous to use as much information about the channel as possible to ensure spectrally efficient communication.

The power constraint (3.9) reflects a total power constraint over the entire length of the code. Hence we do not restrict instantaneous transmit power but rather the average transmit power.

Suboptimality Due to Rank Deficiency

A key assumption in the derivation of Theorem 1 was that the precoder $Q$ could attain the optimal system capacity. However, recall that in order for the capacity optimal solution to be achievable, $Q$ must be of sufficient rank to support the power distribution $\Xi$, which has a maximum rank of $N_T \tau$. Since $Q = \mathbf{X} \mathbf{X}^H$, the rank of $Q$ is a function of the dimensions of $\mathbf{X}$, which are a function of the parameters of the system. Since the dimensions of $\mathbf{X}$ are $N_T \tau \times M$, the maximum rank obtained by $Q$ is $\min(N_T \tau, M)$. If $M < N_T \tau$, the system parameters prevent the system from achieving capacity optimality for some matrices $\Xi$ due to rank deficiency.

In this chapter, we emphasize that in order to circumvent the possibility of a rank-deficient
precoder $X X^H$, we have forced the P-LDC design to achieve capacity optimality by selecting $M = N_T \tau$, which is equivalent to forcing the system to engage in spatial multiplexing. Codes which permit rank deficiency are discussed in Chapter 4.

Characterization of the Precoder

In (3.7), the precoder of the MIMO system was $X$. We can characterize $X$ easily from the expression $XX^H = \hat{U} \Xi \hat{U}^H$.

**Theorem 2.** The capacity-achieving precoder $X$ is given by $X = \hat{U} \Xi^{1/2} \Psi$, where $\Psi$ is a unitary matrix of dimension $M \times M$.

**Proof.** Proof of this theorem can be accomplished by simple substitution of the quantity $X = \hat{U} \Xi^{1/2} \Psi$ into the product $XX^H$ to reveal that $XX^H = \hat{U} \Xi \hat{U}^H$.

From Theorem 2, we see that the optimal precoding expression $XX^H$ gives rise to a non-unique precoder $X$ because of the presence of the unitary matrix $\Psi$. Because of this non-uniqueness, it is possible to search over the family of optimal precoders $X$ in order to find a capacity-optimal precoder which performs best according to an error performance criterion.

### 3.2.3 Description of Optimal Power Distribution

The capacity-optimal precoder $X$ is described in Theorem 2 in terms of the geometry of the channel, but no method is given for computing the optimal power distribution $\Xi$. No analytical closed-form expressions exist for the optimal power distribution, and these coefficients must be computed through numerical simulation. Efficient methods such as gradient descent algorithms can be used to compute $\Xi$ [43]. As will be shown later, the optimal power distribution resembles a water-filling solution in the sense that stronger subchannels get allocated more transmit power, while weaker subchannels get allocated less power.
It can be shown that the optimal power distribution \( \Xi = \{ \xi_1, \xi_2, \ldots, \xi_{N_T} \} \) can be derived from the following optimization problem, where \( \{ \lambda_i \} \) are the eigenvalues of \( \Sigma_T \) [31]:

\[
\Xi = \max_{\sum_{i=1}^{N_T} \xi_i = 1} E_H \left[ \log \det \left( I_{N_R} + \sum_{i=1}^{N_T} \frac{E_S}{\sigma^2} \xi_i \lambda_i \widetilde{W}_i \widetilde{W}_i^H \right) \right],
\]

where \( \widetilde{W} \) is defined as

\[
\widetilde{W} = \begin{pmatrix}
\sqrt{\gamma_1} w_1 \\
\sqrt{\gamma_2} w_2 \\
\vdots \\
\sqrt{\gamma_{N_R}} w_{N_R}
\end{pmatrix}.
\]

Here, \( \widetilde{W}_i \) is the \( i^{th} \) column of \( \widetilde{W} \), \( \{ \gamma_i \}_{i=1}^{N_T} \) are the eigenvalues of \( \Sigma_R \), and \( \{ w_i \}_{i=1}^{N_T} \) are the rows of the Gaussian matrix \( W \).

The optimal power coefficients given by (3.19) are a function of the receive correlation matrix through \( \widetilde{W} \). However, the optimal transmission directions are \( \hat{U} \), which are independent of the receive correlation matrix. Thus we see that in the P-LDC case, the directions of transmission are unaffected by the receive correlation environment, so that the geometry used for transmission is a function only of the transmit environment. This is consistent with the assumptions of the Kronecker channel model, which assumes the transmit and receive correlation environments are independent. However, the power allocation across the optimal geometry is strongly dependent upon the receive correlation environment. These observations are consistent with the results for capacity-optimal codes characterized by \( \tau = 1 \) [31].

Because the optimal power coefficients are a function of the receive environment as well as the signal to noise ratio present in the system, the nature of the P-LDC solution presented in (3.19) is much different than that of conventional LDCs [39]. LDCs are designed under the assumption of no CSI at the transmitter, which means that one capacity-optimal design is sufficient for all SNRs [39]. P-LDCs, in contrast, specify an optimal precoder \( X \) for each SNR value, since the solution to (3.19), which resembles a water-fill, will be different for
each SNR. As a consequence, the computational complexity of our P-LDC design will far exceed that of LDCs because a capacity-optimal precoder must be computed for each SNR value. In practice, this issue can be alleviated somewhat by partitioning the SNR space into regions over which the precoder is assumed to be constant. However, complexity is still high because the channel statistics can be expected to change with the same rate as the SNR, so obtaining periodic updates of the channel statistics is a necessity.

Considering the time multiplicity factor $\tau$ which is present in the P-LDC structure, it is necessary to determine whether there is a multiplicity present in the optimal power coefficients, $\{\xi_i\}_{i=1}^{N_T\tau}$. The result is in the following theorem.

**Theorem 3.** There are at most $N_T$ unique power coefficients in the power allocation matrix $\Xi$, and each of these coefficients is repeated $\tau$ times. In cases where there are $k < N_T$ unique power coefficients, then $(N_T - k)$ of the power coefficients are zeros. In this case, there are $k$ unique power coefficients repeated $\tau$ times each, and there are $(N_T - k)\tau$ power coefficients that are zero.

**Proof.** Consider the optimization in (3.19). Inserting the power constraint, we can express (3.19) in terms of a Lagrange multiplier as follows:

$$J(\Xi) = E_H \left[ \log \det \left( I_{N_R\tau} + \sum_{i=1}^{N_T\tau} \frac{\mathcal{E}_S}{\sigma^2} \xi_i \lambda_i \widetilde{W}_i \widetilde{W}_i^H \right) \right] + \beta \left( \sum_{i=1}^{N_T\tau} \xi_i - \tau \right),$$

where $\beta$ is the Lagrange multiplier. Without loss of generality, we analyze the cost function with respect to $\xi_1$, and using the matrix derivative property $\frac{\partial \det(Y)}{\partial x} = \det(Y) \text{tr} \left( Y^{-1} \frac{\partial Y}{\partial x} \right)$, we obtain

$$\frac{\partial J(\Xi)}{\partial \xi_1} = E_H \left[ \text{tr} \left\{ \left( I_{N_R\tau} + \sum_{i=1}^{N_T\tau} \frac{\mathcal{E}_S}{\sigma^2} \xi_i \lambda_i \widetilde{W}_i \widetilde{W}_i^H \right)^{-1} \left( \frac{\mathcal{E}_S}{\sigma^2} \lambda_1 \widetilde{W}_1 \widetilde{W}_1^H \right) \right\} \right] + \beta.$$
Now applying the partial derivative to the cost function with respect to $\xi_{N_T+1}$, we obtain

$$\frac{\partial J(\Xi)}{\partial \xi_{N_T+1}} = E_H \left[ \text{tr} \left\{ \left( I_{N_R} + \sum_{i=1}^{N_T} \frac{\mathcal{E}_S}{\sigma^2} \xi_i \lambda_i \tilde{\mathbf{W}}_i \tilde{\mathbf{W}}_i^H \right)^{-1} \left( \frac{\mathcal{E}_S}{\sigma^2} \lambda_{N_T+1} \tilde{\mathbf{W}}_{N_T+1} \tilde{\mathbf{W}}_{N_T+1}^H \right) \right\} \right] + \beta.$$  \hspace{1cm} (3.20)

Note that $\lambda_1 = \lambda_{N_T+1}$ in the above development. By expanding these expressions and utilizing the placement of the zeros in $\tilde{\mathbf{W}}_1$ and $\tilde{\mathbf{W}}_{N_T+1}$, and by setting the derivatives in these expressions equal to zero and equating them, we can determine that

$$E_H \left[ \text{tr} \left\{ \left( I_{N_R} + \sum_{i=1}^{N_T} \frac{\mathcal{E}_S}{\sigma^2} \xi_i \lambda_i \tilde{\mathbf{W}}_i \tilde{\mathbf{W}}_i^H \right)^{-1} \left( \frac{\mathcal{E}_S}{\sigma^2} \lambda_1 \tilde{\mathbf{W}}_1 \tilde{\mathbf{W}}_1^H \right) \right\} \right] = E_H \left[ \text{tr} \left\{ \left( I_{N_R} + \sum_{i=1}^{N_T} \frac{\mathcal{E}_S}{\sigma^2} \xi_i + \sum_{i=1}^{N_T} \xi_i \lambda_i \tilde{\mathbf{W}}_i \tilde{\mathbf{W}}_i^H \right)^{-1} \left( \frac{\mathcal{E}_S}{\sigma^2} \lambda_1 \tilde{\mathbf{W}}_1 \tilde{\mathbf{W}}_1^H \right) \right\} \right]. \hspace{1cm} (3.21)$$

Notice that the upper limit in the summation has been reduced in (3.21) by utilizing the time-blocked nature of the system and condensing it. For any given realization of $H$, note that the instantaneous value of $\tilde{\mathbf{W}}_i$ in (3.21) is equal on both sides of the equality. That is, despite the fact that the two sides of (3.21) are equal in distribution, the instantaneous values of $\tilde{\mathbf{W}}_i$ are equal for any given code realization. This is a direct result of the Kronecker tensor product structure in (3.7), which causes the time blocking structure which leads to the repetition of the values of $\tilde{\mathbf{W}}_i$. Because the instantaneous values of $\tilde{\mathbf{W}}_i$ are equal, we can set the arguments of the expectations in (3.21) equal for any given code realization, which necessitates the selection of $\xi_i = \xi_{i+N_T}$ for $i = 1, 2, \ldots, N_T$. Repeated application gives us the general result that $\xi_i = \xi_{i+tN_T}$ for $t = 1, 2, \ldots, \tau - 1$ and $i = 1, 2, \ldots, N_T$.

By extending the P-LDC over multiple time indices, the code is able to utilize the strongest transmission modes $\tau$ times, leading to a multiplicity of $\tau$ in the power distribution. It is easy to show that the P-LDC transmits with unity power in each time index, summing to a total power of $\tau$ over the duration of the code.

Combining the results of Theorems 1 and 3, the optimal precoder $Q_{opt}$ has a block-
diagonal form. As a consequence, we see that it is possible to choose $\Psi = I$ in order to define a precoder $\mathbf{X}$ which also possesses a time-blocked structure. While this is certainly a feasible solution, a time-blocked $\mathbf{X}$ corresponds to a dispersion matrix set, $\{M_m\}_{m=0}^{M-1}$, wherein each dispersion matrix $M_m$ has only one nonzero column. Having only one nonzero column for a given dispersion matrix implies that the symbol modulating that dispersion matrix is spread into only one time index, which is not optimal in terms of error rate and defeats the purpose of the space-time coding structure. Therefore, it is necessary to find a better matrix selection $\Psi$ in order to optimize error performance.

3.2.4 $\tau$-Beamforming

Because of Theorem 3, each transmit power $\xi_i$ is repeated $\tau$ times. Consequently, it is impossible to achieve a traditional beamformer with only one active beam. Therefore, P-LDC codes can only achieve “$\tau$-Beamforming”, a state in which $\tau$ of the transmit powers are nonzero. This corresponds to the case in which $R_T$ has only one dominant eigenmode, and $\Sigma_T$ uses that eigenmode $\tau$ times over the $\tau$ indices of the code.

We analyze over what range of SNR $\tau$-Beamforming is optimal. We prove that the introduction of the code latency $\tau$ and the repetition of the beams does not change the region of optimality from the $\tau = 1$ case given in [131]. We summarize our results in the following theorem.

**Theorem 4.** $\tau$-Beamforming is optimal provided that

$$
\frac{E_S}{\sigma^2} \lambda_2 \leq \frac{1 - E_H \left[ \frac{1}{1 + \lambda_1 \|W_1\|^2} \right]}{\sum_{i=1}^{N_R} \gamma_i - \frac{E_S}{\sigma^2} \lambda_1 \sum_{k=1}^{N_R} \gamma_k \beta_k},
$$

(3.22)

where $\beta_k = E_H \left[ \frac{\gamma_k \|W_k\|^2}{1 + \frac{E_S}{\sigma^2} \lambda_1 \sum_{k=1}^{N_R} \gamma_k \|W_k\|^2} \right]$. This result is identical to the $\tau = 1$ case presented in [131].
**Proof.** The steps of this proof parallel those presented in [131] with modification due to the time factor \( \tau \). We assume only \( 2\tau \) eigenmodes are used by the precoder, and extension to the more general case is straightforward. We prove this theorem by assigning power \((1 - p)\) to the \( \tau \) strongest eigenmodes of the channel and power \( p \) to the second strongest set of \( \tau \) eigenmodes. The strongest eigenmodes of the channel correspond to the \((1 + N_Tk)\)th eigenvalues of \( \Xi \), and the second strongest eigenmodes of the channel correspond to the \((2 + N_Tk)\)th eigenvalues of \( \Xi \), for \( k = 1, 2, \ldots, \tau - 1 \). Then the capacity of the system can be expressed as

\[
C(p) = E_H \left[ \log \det \left( I_{N_R\tau} + \frac{E_S}{\sigma^2} (1 - p) \lambda_1 \sum_{k=0}^{\tau-1} \tilde{W}_{1+N_Tk} \tilde{W}_{1+N_Tk}^H + p \lambda_2 \sum_{k=0}^{\tau-1} \tilde{W}_{2+N_Tk} \tilde{W}_{2+N_Tk}^H \right) \right].
\]

(3.23)

Because of the equality of the nonzero elements of \( \tilde{W}_{i+jN_T} \) and \( \tilde{W}_{i+kN_T} \), for \( k \neq j \), we can simplify (3.23) by defining \( x_1 \) to be the first (nonzero) \( N_T \) elements of \( \tilde{W}_1 \) and \( x_2 \) to be the first set of (nonzero) \( N_T \) elements of \( \tilde{W}_2 \). Then (3.23) can be shown to simplify as follows:

\[
C(p) = E_H \left[ \log \det \left( I_{N_R\tau} + \frac{E_S}{\sigma^2} (1 - p) \lambda_1 (I_{\tau} \otimes x_1 x_1^H) + p \lambda_2 (I_{\tau} \otimes x_2 x_2^H) \right) \right].
\]

(3.24)

We define \( B(p) = (1 - p) \lambda_1 (I_{\tau} \otimes x_1 x_1^H) + p \lambda_2 (I_{\tau} \otimes x_2 x_2^H) \). By construction, \( B(p) \) has exactly 2 unique nonzero eigenvalues, each of which is repeated \( \tau \) times. Denoting these two eigenvalues as \( a_1(B) \) and \( a_{\tau+1}(B) \), and by using rules from matrix algebra, we see that

\[
\text{tr}(B) = \tau a_1(B) + \tau a_{\tau+1}(B) = (1 - p) \lambda_1 \tau x_1^H x_1 + p \lambda_2 \tau x_2^H x_2 = (1 - p) \lambda_1 \tau \|x_1\|^2 + p \lambda_2 \tau \|x_2\|^2.
\]

Since the determinant of a matrix equals the product of its eigenvalues, (3.24) can be simplified. Expressing the functional relationship between the eigenvalues of \( B \) and \( p \), we denote
\( a_1(B) = a_1(p) \) and obtain the following:

\[
C(p) = E_H \left[ \log \left( \left( 1 + \frac{E_S}{\sigma^2} a_1(p) \right)^\tau \left( 1 + \frac{E_S \text{tr}(B) - \tau a_1(p)}{\tau} \right)^\tau \right) \right]
\]

\[
= E_H \left[ \log \left( 1 + \frac{E_S}{\sigma^2} a_1(p) \right)^\tau \right] + E_H \left[ \log \left( 1 + \frac{E_S}{\sigma^2} \left( (1 - p)\lambda_1\|x_1\|^2 + p\lambda_2\|x_2\|^2 - a_1(p) \right) \right)^\tau \right], \quad (3.25)
\]

where, again, the exponent \( \tau \) is the time duration of the code in symbol periods. The exponent in the argument of the logarithm becomes a multiplier out front. Taking the derivative of \( C(p) \) with respect to \( p \) and setting it equal to zero reveals that \( \tau \) becomes a common factor which cancels out of the equation. The remaining steps of the proof are identical to similar proofs regarding optimality of beamforming for \( \tau = 1 \) [131].

### 3.3 Design Modification for Error Performance

As already discussed, precoders designed according to Theorem 2 are not unique. The presence of the unitary matrix \( \Psi \) gives rise to an infinite number of design possibilities for any given precoder. As shown in Fig. 3.1, the performance of a given precoder in terms of BER is highly dependent upon what criterion is used to select \( \Psi \). In Fig. 3.1, two codes are generated using two different values for \( \Psi \): one \( \Psi \) is randomly generated, and one \( \Psi \) is generated according to an optimality criterion derived later in this section. Fig. 3.1 illustrates clearly that the two codes have very different error performance despite the fact that they both are capacity-optimal.

In this section, we derive an optimality criterion to select the unitary matrix \( \Psi \). In doing so, we will make use of a popular space-time code design technique which attempts to maximize the diversity and coding gain of the system at high SNR.
Figure 3.1: Comparison of two P-LDC codes, one optimized with respect to pairwise error probability (PEP) and one unoptimized. The codes are both capacity optimal.
### 3.3.1 Derivation of Optimal Solution

Assuming that a symbol set \(\{\alpha_m\}_{m=0}^{M-1}\) is transmitted and that a symbol set \(\{\hat{\alpha}_m\}_{m=0}^{M-1}\) is received and decoded, the most general error metric is to compute the probability that \(S \neq \hat{S}\), where \(S = \sum_{m=0}^{M-1} M_m \alpha_m\) and \(\hat{S} = \sum_{m=0}^{M-1} M_m \hat{\alpha}_m\). However, this error metric is quite difficult to use in a general solution for a P-LDC design. Consequently, we utilize a union bound to upper bound the error metric, and we further utilize a Chernoff bound to derive a pairwise error probability (PEP) metric. Such assumptions are common in the precoding literature because for high SNR, the PEP becomes the dominating term in the error probability metric [134].

Using the PEP criterion, we modify the P-LDC error design criterion to minimize the following metric:

\[
P(S \rightarrow \hat{S}) \leq \exp\left(\frac{-E_{sd^2}(S,\hat{S})}{4\sigma^2}\right), \tag{3.26}
\]

where \(d^2(S,\hat{S})\) is the squared Frobenius norm of the distance between a correct codeword \(S\) and an incorrectly detected codeword \(\hat{S}\).

A detailed analysis of the metric in (3.26) has been performed by many authors for the case of statistical channel knowledge when orthogonal space-time codes have been used [134,135]. As with the analysis for OSTBC, our development assumes the Kronecker channel model, which implies that the correlation matrix of the MIMO channel \(H\) can be expressed as \(R_{\text{MIMO}} = R_T \otimes R_R\), which is assumed to be full rank and hence invertible. While our results are similar to those obtained for OSTBC, there are some differences which need to be underscored because of the non-orthogonal nature of the P-LDC code construction.

Our analysis parallels that of [134] and [135]. We construct a matrix \(R_S = \sum_{t=0}^{T-1} X_t^* e e^T X_t^T\), where \(e = s - \hat{s}\) and \(X_t = [M_{0,t}, M_{1,t}, \ldots, M_{M-1,t}]\). In the definition of \(X_t\), we use the subscript \(t\) to denote the \(t\)-th column of the matrix \(M_t\). Due to the time-blocking structure of the matrix channel \(H\), for the covariance channel knowledge case we derive a performance
metric of
\[
\max_{\mathbf{s}} \left( - \log \det \left[ \frac{\mathcal{E}_S}{\sigma^2} (\mathbf{I}_{N_R} \otimes \mathbf{R}_S) + \mathbf{R}_{\text{MIMO,}\mathcal{H}}^{-1} \right] \right),
\]
where \(\mathbf{R}_{\text{MIMO,}\mathcal{H}}\) is the correlation matrix of the time-blocked MIMO channel matrix \(\mathcal{H}\).

The expression in (3.27) simplifies greatly for orthogonal space-time block codes (OSTBC), since the data symbols are chosen to be orthogonal to each other. In the case of P-LDCs, however, no such orthogonality is guaranteed. As a result, we are forced to expand the expression in (3.27) using the P-LDC code expansion formulas in terms of dispersion matrices (3.5). Recognizing that our objective is to minimize the quantity in (3.27) for any pair of codeword matrices, we refine our expression of the codeword difference matrix to be
\[
\mathbf{R}_S = \left( \sum_{m=0}^{M-1} \mathbf{M}_m^* (\alpha_{mk} - \alpha_{ml})^* \right) \left( \sum_{m=0}^{M-1} \mathbf{M}_m^T (\alpha_{mk} - \alpha_{ml}) \right).
\]
Making this substitution in (3.27), and using Kronecker tensor product identities [136], we can write
\[
\max_{\mathbf{s}} \left( - \log \det \left[ \frac{\mathcal{E}_S}{\sigma^2} (\mathbf{I}_{N_R} \otimes \mathbf{R}_S) + \mathbf{R}_{T}^{-1} \otimes \mathbf{R}_{R}^{-1} \right] \right).
\]
Applying a succession of Kronecker matrix properties and the eigenvalue decompositions in (3.15), we can write
\[
\max_{\mathbf{s}} \left( - \log \det \left[ \frac{\mathcal{E}_S}{\sigma^2} (\mathbf{I}_{N_R} \otimes \mathbf{R}_S) + \mathbf{\Lambda}^{-1} \otimes \mathbf{\Gamma}^{-1} \right] \right).
\]
It is noteworthy that \(\mathbf{R}_S\) as given in (3.28) is different in structure from many expressions derived elsewhere in the precoding literature. Other research has focused on the specific case where \(\mathbf{R}_S\) is designed for OSTBC, whereas in (3.28) we must accommodate a non-orthogonal structure dictated by the dispersion matrices, \(\{\mathbf{M}_m\}_{m=0}^{M-1}\).

Our design objective is to maximize the metric in (3.28) for high SNR. Notice that in (3.28), the term that dominates for high SNR is the term that is multiplied by \(\frac{\mathcal{E}_S}{\sigma^2}\), and the
term containing the correlation matrix eigenvalues becomes negligible. Therefore, for high SNR, we see that the correlation environment plays a diminished role in the asymptotic BER behavior of the system.

With some manipulation, we can derive an upper bound on the probability of decoding a codeword $S$ incorrectly as $\hat{S}$ to be

$$
P(S \rightarrow \hat{S}) \leq \frac{1}{\left(\frac{\mathcal{E}_b}{4\sigma^2}\right)^{N_R \text{Rank}(R_S)} \prod_{i=1}^{\text{Rank}(R_S)} s_i^{\frac{N_R}{N_R}}} ,
$$

where $s_i$ are the eigenvalues of the matrix $R_S$.

In (3.29), the two terms in the denominator are related to the diversity advantage and coding advantage of the space-time code, respectively. The diversity advantage is given by the smallest product $N_R \text{Rank}(R_S)$ over all possible $R_S$ and the coding advantage is given by the smallest product $\prod_{i=1}^{\text{Rank}(R_S)} s_i^{N_R}$. Hence, the objective of our selection of $\Psi$ should be according to space-time coding guidelines originally set forth for space-time codes with no transmit CSI, called the rank and determinant design criteria [2, 19]. These criteria involve choosing $\Psi$ such that the diversity and coding advantage are maximized over all codeword error possibilities. In such an optimization procedure, it is usually advantageous to target a desired diversity advantage and then to maximize the coding advantage for that targeted diversity. Mathematical procedures for accomplishing such a search are discussed in the space-time coding literature [39].

It is noteworthy that the BER design modification for P-LDCs, while similar to that of LDCs [39], is different in the sense that a unique optimization must be performed for each SNR. With LDCs, there was no transmit CSI, which gave rise to a capacity-optimal design which was independent of SNR. With P-LDCs, however, the capacity-optimal solution is a function of SNR because of the water-filling principle. Consequently, each SNR value requires its own unique optimization. Because this optimization must be performed over the entire range of both high and low SNR, this is problematic from an analysis point of view.
The diversity and coding advantage design principles outlined herein assume high SNR. As a result, the design technique we present here will not be as effective for low SNR values as for high SNR values. For example, by targeting a certain diversity advantage, we are determining the slope of the BER curve for high SNR. Our design will not be very revealing for low SNR because for low SNR the BER curve will not have achieved its asymptotic slope.

Design methods which improve upon the design criteria for the low-SNR region are an open question. However, our simulation results will reveal that, even with this design caveat, our P-LDCs achieve some BER improvement for all values of SNR when compared to unoptimized P-LDCs.

3.3.2 Techniques for Generating Optimal Solutions

There are many techniques that can be used to optimize the selection of $\Psi$. Among these methods are gradient descent methods, which yield local optimizations in the region of the initial selection of $\Psi$. Design modifications intended to reduce computational complexity involving the use of Householder reflections are sometimes used [39].

A simple method of selecting an optimal $\Psi$ is to generate a large number of random “candidate” matrices and to choose the one which satisfies the rank and determinant criteria most satisfactorily. Such a method has been shown to yield good results given a large enough sample alphabet size [39]. We choose to implement our search using this method for two reasons. First, it is extremely simple to implement. Second, due to the very high computational burden incurred by the need to select an optimal $\Psi$ for each SNR value, the random technique provides us with an easy way of reducing the computational burden by eliminating the need for complicated optimization procedures.
3.3.3 Characteristics of Optimal Solutions

In the attempt to minimize the metric presented in (3.29), we have shown that the rank and determinant criteria must be observed. These criteria must be re-evaluated at each SNR, since the optimal solution is a function of the SNR. Of the two terms in the denominator of (3.29), we defined the coding gain as the minimum product of

$$\text{Rank}(R_S) \prod_{i=1}^{s_i} s_i^{N_R},$$

where $s_i$ are the eigenvalues of $R_S$. Notice that the number of terms in this product is a function of the rank of $R_S$, which in turn is a function of the capacity-optimal solution $X$.

As the SNR increases, the number of beams utilized in the capacity-optimal solution will increase. As the number of beams increases, the rank of $R_S$ will increase. The coding gain achievable by the P-LDC is a function of the product of the nonzero eigenvalues of $R_S$. However, we show in Section 3.4 that it is possible for there to be an abrupt change in the maximum product of the nonzero eigenvalues at the transition SNRs where a new capacity-optimal beam is added to the system. For example, at the transition SNR value where the capacity optimal solution transitions from having one active beam to two active beams, it is possible to transition from a single eigenvalue of $R_S$ which is on the order of 1 to a double eigenvalue $R_S$ with one eigenvalue on the order of 1 and the other eigenvalue very small, even down to the order of magnitude $10^{-2}$. In cases such as these, the maximum coding gain achieved by the P-LDC decreases significantly because the product of the nonzero eigenvalues is affected by the introduction of a smaller eigenvalue of $R_S$.

Because of this transition effect, it is possible to see abrupt disruptions in BER plots where the capacity-optimal solution requires an additional beam to achieve optimality. The locations of these transition points always occur at SNR values where a new beam is introduced to the capacity optimal solution, resulting in increased rank of $X$. Quantifying the shift in the BER curve is very difficult, since the method of generating the codes requires
the creation of many random candidate $\mathcal{X}$ realizations, and the optimal solution is itself a function of the optimization problem in (3.19), which does not have a closed-form solution. Therefore, the best we can conclude is that these transition points can lead to an abnormal appearance of the BER plots, and the degree of severity of this abnormality is a function of the relative strength of the new eigenvalue of $R_S$ in comparison to the other eigenvalues of $R_S$.

3.4 Performance Evaluation

In this section, we present numerical simulations illustrating the performance of the P-LDC code design. We present results illustrating the capacity optimality of our solution, the optimality range of beamforming for our P-LDC design, the BER performance of our optimized code, and the effect of the size of the random candidate $\Psi$ alphabet upon BER performance. Throughout this presentation we discuss the effects of both transmit and receive correlation upon both capacity and error performance.

The correlation matrices $R_R$ and $R_T$ are generated using the well-known ring of scattering model [34]. The transmit antennas experience correlations given by $J_0 \left( \Delta \frac{2\pi}{\mu} d_y \right)$, where $J_0 (\cdot)$ is the zero-order Bessel function of the first kind, $\mu$ is the wavelength, $\Delta$ is the transmit angle spread, and $d_y$ is the antenna spacing at the transmitter assuming a linear array. The receive antennas experience correlations given by $J_0 \left( \frac{2\pi}{\mu} d_x \right)$, where $d_x$ is the distance between antennas at the receive array. Note that these models can be adjusted to accommodate non-linear arrays.

3.4.1 Capacity Optimality of P-LDCs

The Precoded Linear Dispersion Codes as designed herein are guaranteed to optimize capacity. In Fig. 3.2, we compare the spectral efficiency of the P-LDC design technique to the LDC frame-based designs [39]. In all cases, we set $\tau = 2$ and $M = N_T\tau$. For this figure, we
assumed perfect antenna decorrelation at the receiver and a transmit correlation characterized by an angular spread of 1.5° and an antenna spacing of one wavelength. From Fig. 3.2, it is apparent that even for MIMO systems with only two transmit and receive antennas, the capacity gains achieved by utilizing partial channel state information are substantial. Notice that as the dimensionality of the channel is increased to three and then to four, the amount of improvement gained by utilizing statistical transmit CSI becomes even more sizable. For a 4 × 4 system, the gap between the LDC and P-LDC designs is nearly 2 bps/Hz.

The result observed in Fig. 3.2 demonstrates that having statistical transmit CSI can
Figure 3.3: Comparison of spectral efficiency for a 2 × 2 system for a variety of receive correlation environments. It is observed that even extreme receive correlation environments still have the potential for very large spectral efficiencies.

significantly improve performance. As compared to conventional LDC designs, in which the transmitter operates blindly without any knowledge of the eigenstructure of the channel, P-LDCs utilize transmit CSI to maximize spectral efficiency. Although the P-LDC design is considerably more computational in nature than LDCs, the performance gains are clear from Fig. 3.2. Spectral efficiency can be dramatically increased when using P-LDCs.

When analyzing spectral efficiency in a P-LDC system, it is instructive to investigate the effect of receive correlation upon the overall capacity of the system. In Fig. 3.3, we compare the spectral efficiencies of five combinations of transmit and receive correlation environments. These five cases are described below:
Case 1. No transmit or receive correlation.

Case 2. No receive correlation, but the transmit antenna spacing is one wavelength and the transmit angular spread is 15°.

Case 3. No receive correlation, but the transmit antenna spacing is one wavelength and the transmit angular spread is 1.5°.

Case 4. The receive antenna separation distance is $\frac{1}{4}$ wavelength, the transmit antenna spacing is one wavelength, and the transmit angular spread is 15°.

Case 5. The receive antenna separation distance is $\frac{1}{4}$ wavelength, the transmit antenna spacing is one wavelength, and the transmit angular spread is 1.5°.

From Fig. 3.3, we see that in the absence of receive correlation, which corresponds to cases 1-3, the spectral efficiency of the P-LDC is not significantly affected by an angular spread at the transmitter of 15°, but it is affected substantially at high SNR for a transmit angular spread of 1.5°. Consistent with the antenna correlation model [34], we expect the transmit correlative effect to increase with decreasing angular spread, which has been shown to decrease spectral efficiency [47]. The results in Fig. 3.3 confirm this expectation.

In Fig. 3.3, we observe from cases 4-5 that a receive correlation environment corresponding to an antenna spacing of a quarter wavelength does not significantly affect spectral efficiency. We have previously asserted that the capacity-optimal transmission directions are unaffected by receive correlation, but the optimal power distribution is. From Fig. 3.3, it is apparent that the dominant factor in determining spectral efficiency is the transmit correlation environment.

An apparent feature of Fig. 3.3 is that for low SNR, the spectral efficiency associated with cases 3 and 5 exceeds the spectral efficiency of cases 1, 2, and 4. This indicates that extremely harsh channel conditions - conditions in which antennas are very closely spaced, or the angular spread is very small - can actually lead to higher spectral efficiencies than
Figure 3.4: Comparison of spectral efficiency for $3 \times 3$ and $4 \times 4$ systems in the absence of receive correlation for two different transmit environments. The P-LDCs maintain high spectral efficiency even in harsh transmission environments.

more ideal channels for low SNR. The reason for this phenomenon is that, when the channel environment is harsh, there is usually one dominant eigendirection which is preferred in the system. At low SNR, capacity-optimal transmission schemes such as P-LDCs converge to a beamformer, and therefore the capacity-optimal designs will transmit all their energy along the dominant eigendirection. The more harsh the environment, the more dominant the eigendirection, which translates into higher capacity at low SNR. However, as SNR becomes large, the more ideal channel conditions achieve higher spectral efficiencies, as expected [47].

While the results in Fig. 3.3 consider only a $2 \times 2$ system, in Fig. 3.4 we present results for
higher dimensioned systems. This figure does not consider receive correlation. Notice that the spectral efficiency performance gap between the more harsh transmission environments becomes more noticeable as the number of antennas increases from 2 in Fig. 3.3 to 3 and 4 in Fig. 3.4. The reason for the increased gap in performance for higher dimensioned systems is that as more antennas are added to a system, more eigendirections become available for achieving capacity. However, in a large-dimensioned system with a very harsh transmission environment, the optimal transmission strategy is to beamform even for higher SNR values. For example, in generating Fig. 3.4, our results indicate that for the $4 \times 4$ case, beamforming is optimal for the angular spread of $1.5^\circ$ up through 6 dB, whereas for the less harsh angular spread of $15^\circ$, beamforming becomes suboptimal for SNR values less than 0 dB. Since beamforming is optimal for the more harsh environment over a larger range of SNR, more system resources are being wasted. For a $2 \times 2$ system, beamforming only disregards one of the MIMO channel eigendirections, whereas for a $4 \times 4$ system, beamforming disregards three of the four MIMO channel eigendirections. More system resources are wasted in beamforming as the dimensionality increases, hence the widening performance gap evident in Fig. 3.4.

3.4.2 Optimality Range of $\tau$-Beamforming

We have proven that the optimality range of $\tau$-Beamforming is identical to the optimality range of traditional beamforming for systems that do not introduce the code latency $\tau$. In Fig. 3.5, we present the optimality range of $\tau$-beamforming in the absence of receive correlation as a function of the number of antennas in the system. Consistent with previous work on optimality of beamforming, we normalize the axes to be the product of the eigenvalues of the transmit correlation matrix and the SNR [47]. As expected, with increased channel dimensionality we see the range of optimality of $\tau$-beamforming decreases.

In Fig. 3.6, we present comparative results for the optimality range of $\tau$-beamforming as a function of the receive correlation environment. Figs. 3.5 and 3.6 are a graphical representation of (3.22). For both the $3 \times 3$ and $4 \times 4$ cases, we observe that introducing
Figure 3.5: Optimality range of beamforming as a function of MIMO channel dimensionality for the more harsh transmit correlation case 3. We observe that larger channel dimensions have a smaller range of optimality than smaller dimensioned systems.
receive correlation into a system with a harsh transmit correlation environment reduces the optimality range of $\tau$-Beamforming. However, the amount of reduction is not significant. Furthermore, we also conducted simulations to test the optimality range of $\tau$-Beamforming in the presence of receive correlation and less harsh transmit channel conditions with an angular spread of $15^\circ$, and we discovered that the region of optimality was unaffected by the introduction of receive correlation. From these results and from Fig. 3.6, we can conclude that the region of optimality, when normalized as we have shown here, is not very sensitive to the receive correlation environment.
3.4.3 Error Performance of P-LDCs

The P-LDC design technique is a dual-design approach. First, guarantee capacity optimality, and second, search within the family of capacity-optimal codes for a PEP-optimal code. We illustrated in Fig. 3.1 that it is possible to generate two capacity-optimal P-LDCs with vastly different error performance curves. Using the PEP optimality criteria, we generated P-LDCs and modified them according to the rank and determinant criteria. We selected a simple Binary Phase Shift Keying (BPSK) modulation scheme and have plotted the performance curves in Fig. 3.7. For reference in our BER plots, we have included the performance of a frame-based LDC for comparison. Comparison with a frame-based LDC is a relevant comparison because such LDCs are conventional, non-precoded LDCs which experience a similar type of error performance enhancement as P-LDCs. This provides an “apples to apples” comparison between the precoded and non-precoded cases. In Fig. 3.7, we assume perfect receiver decorrelation and a transmit correlation characterized by an antenna spacing of one wavelength and an angular spread of $7.5^\circ$. In all cases, we set $\tau = 2$ and $M = N_T \tau$.

From Fig. 3.7, it is clear that for all channel dimensions, the P-LDCs outperform the LDCs. The performance gap increases as the dimensionality of the channel increases. For example, in a $2 \times 2$ P-LDC system, to achieve a BER of $10^{-3}$ requires an SNR of approximately 11 dB whereas for the LDC an SNR of 12 dB is required. However, for the $4 \times 4$ case, the P-LDC requires an SNR of almost 8 dB to achieve $10^{-3}$, whereas the LDC requires an SNR of 10 dB.

From Fig. 3.7, it is apparent that the P-LDC plots are not perfectly smooth as one would expect. There are two causes for these discontinuities. First, this effect occurs because a new code is derived for each SNR value, and each individual code is optimized in isolation from every other code with respect to the rank and determinant criteria. Conventional LDCs require only a single code which is effective for all SNR values, so only one PEP optimization is necessary. In the P-LDC framework, however, there is no guarantee that the code derived

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Figure 3.7: BER performance of P-LDC design for various channel configurations compared with LDC designs. There is significant improvement in terms of BER for the capacity-optimal designs.
for a certain SNR value will have identical coding gain properties as a code derived at a
different SNR. As a consequence, the BER plots are discontinuous, although they follow a
clear trend for high SNR. The discontinuities are difficult to eliminate because each SNR
needs a different P-LDC code to ensure capacity optimality is achieved.

The second cause of the discontinuities is the method by which the codes were generated.
When performing the PEP optimization, it is possible to define a cost function and perform
a complicated gradient-descent optimization to ensure local optimality of the solution [39].
However, we chose not to implement this method of generating the PEP-optimal codes
because the optimization routine would have to be repeated for each SNR, which becomes
computationally prohibitive. The alternative we chose to implement was to generate $L$
random realizations of the unitary matrix $\Psi$ and to choose the matrix which yielded the
best coding and diversity gains. The “smoothness” of the BER curve is a function of the
size $L$ of the candidate alphabet from which we chose $\Psi$. Although to obtain a truly optimal
solution requires that we choose $L$ to be on the order of $10^6$, practically it is impossible to
conduct a simulation for such a large value of $L$ at each SNR value. As a consequence, we
compromised on the size of $L$ and selected $L = 10^4$, a reduction in the alphabet size by
a factor of 100. Therefore, some of the discontinuous nature of the BER curve in Fig. 3.7
is a result of an insufficient alphabet size $L$. This insufficient alphabet size is a necessary
compromise to ensure feasibility of the P-LDC design technique.

The effect of the selection of the candidate alphabet size $L$ is evident in Fig. 3.8 for BPSK
modulation for a $3 \times 3$ system with no receive correlation and an angular spread of $1.5^\circ$. Here,
we show that the amount of performance improvement gained by increasing the alphabet
size from 100 to 5000 is negligible. From Fig. 3.8, it is evident that a candidate alphabet
size of $L = 10$ is too small to obtain reliable BER simulation data, but the performance
gap between the $L = 100$ and $L = 5000$ curves is quite small. From this we can infer that
using an alphabet size on the order of $10^2$ is sufficient to get a PEP-solution that is relatively
close to the optimal solution. Further refinement by setting $L$ on the order of $10^3$ or larger
Figure 3.8: BER performance of P-LDCs as a function of candidate alphabet size $L$. For comparably small values of $L$, good, albeit suboptimal, error performance is achieved.
achieves fine tuning of the solution but provides little tangible performance gain. Since P-LDCs require so much computation, we consider it an acceptable compromise to settle for a slightly suboptimal PEP solution provided that we can choose a candidate alphabet size on the order of $10^2$ to minimize computational complexity.

One peculiarity of the PEP-optimal solution is that it is possible to see irregularities in the BER curves at SNR values where a new transmission beam becomes necessary to achieve capacity optimality. This effect is illustrated in Fig. 3.9 for a $3 \times 3$ system with no receive correlation and an angular spread of $1.5^\circ$ using BPSK. The capacity-optimal solution switches from having one active beam to having two active beams between the SNR values 12 and 13. Notice that the BER increases suddenly for SNR=13 dB, resulting in a discontinuity in the BER curve. This produces two piecewise continuous portions of the BER curve.

It is tempting to try and compensate for the discontinuity in Fig. 3.9 by artificially forcing the second beam to receive no transmit power. This results in a system with the same number of active beams as for lower SNR values. However, because the beam is forced to be inactive, the resulting code is no longer capacity optimal, since the power allocation to the beams is determined from a capacity optimization technique. We restrict our discussion to purely capacity-optimal codes here and leave analysis of modified schemes to compensate for the effect seen in Fig. 3.9 for future investigation.
Figure 3.9: Example of the BER suddenly increasing as a result of an additional beam becoming active in the capacity optimal solution.
Chapter 4

Non-Ideal Precoding Structures

In the previous chapter, we constructed Precoded Linear Dispersion Codes under the assumption that the number of symbols involved in the transmission, $M$, was equal to the product of the number of transmit antennas, $N_T$, and the time spread of the code, $\tau$, that is, $M = N_T \tau$. This permitted the precoder, $\mathbf{X}$, to achieve the rank of the optimal precoder for all values of SNR, which simplified the analysis. In practical systems, however, flexibility is often a key design goal. In many cases, the need for optimal capacity for all SNR values may not be necessary, and in those cases we can relax the requirement that $M$ be equal to $N_T \tau$. Additionally, other system concerns such as decoding complexity, memory, or code latency restrictions may induce the design selection $M < N_T \tau$. When $M < N_T \tau$, care must be taken to understand the consequences to the precoding system in terms of capacity and error rate. We investigate these consequences to system design in this chapter.

This chapter completes the P-LDC design procedure by enabling the system designer to treat the quantity $M$ as a variable design parameter. We will show in this chapter that P-LDCs with $M < N_T \tau$ can achieve optimal capacity for certain low SNR values, but for high SNR values they cannot achieve true capacity optimality. We illustrate this capacity loss and characterize the suboptimal P-LDCs. Moreover, since precoders designed under the
assumption that $M < N_T \tau$ are unable to achieve optimal capacity for all SNRs, we propose several classifications of approximate P-LDCs whose capacity performance come close to the optimal case when $M = N_T \tau$. We use these approximate P-LDCs to illustrate some key distance relationships between the $M = N_T \tau$ precoder and the $M < N_T \tau$ precoder.

This chapter is organized as follows. In Section 4.1, we present the model for the capacity suboptimal case in which $M < N_T \tau$. We discuss the reasons for the suboptimality in Subsection 4.1.2 as well as the characteristics of the system model which lead to two disjoint SNR regions, one in which the P-LDC is optimal in terms of capacity and one in which it is suboptimal. In Subsection 4.1.3, we provide an upper and a lower bound on the ergodic capacity of our rank deficient precoded LDC, and in subsection 4.1.4 we provide a BER optimization procedure for the family of rank deficient precoders. Subsection 4.1.5 provides simulation results illustrating the amount of capacity loss due to the rank deficiency. Section 4.2 presents a variety of matrix distance calculations which illuminate the relationship between the full rank precoders of Chapter 3 and Chapter 4. We make some concluding remarks in Section 4.3.

4.1 P-LDC Dimension Mismatch Model

In Chapter 3, we presented the solution to the optimization problem stated in (3.9) and (3.10). In deriving this solution, we made the assumption that $M = N_T \tau$. As a consequence of this assumption, the matrix $Q = xx^H$ could achieve the maximum rank achieved by the optimal solution derived in Theorem 1.

Sometimes in practical systems, the decoding latency can become a significant factor in system design. Recall that ML decoding is assumed in the decoding of P-LDCs, and as a consequence the decoding complexity grows linearly with the number of symbols to decode, $M$. This motivates the system designer to keep $M$ as small as possible to minimize decoding complexity. This leads us to the discussion of the design choice $M < N_T \tau$. In such a system,
the matrix product $Q = \mathbf{X}\mathbf{X}^H$ cannot achieve the maximum possible rank of the optimal solution in Theorems 1 and 2 because of the tall nature of $\mathbf{X}$. This phenomenon is discussed below.

### 4.1.1 Capacity Suboptimality

Recall the description of the capacity problem stated in Section 3.2.1 in (3.9), (3.10), and (3.11), which we reproduce here for convenience of presentation:

$$
C = \max_{\mathbf{X} : \text{Tr}(\mathbf{X}\mathbf{X}^H) = \tau} \frac{1}{\tau} C(\mathbf{X}\mathbf{X}^H),
$$

where

$$
C(\mathbf{X}\mathbf{X}^H) = E_H \left[ \log \det \left( I_{NR}\tau + \frac{\xi_S}{\sigma^2} \mathbf{H}\mathbf{X}\mathbf{X}^H\mathbf{H}^H \right) \right].
$$

Recalling that $\mathbf{X}\mathbf{X}^H = Q$, we defined $\mathbf{X}$ to be an $N_T\tau \times M$ precoding matrix defined in (3.6), which we reproduce here:

$$
\mathbf{X} = \left[ \text{vec} (\mathbf{M}_0), \text{vec} (\mathbf{M}_1), \ldots, \text{vec} (\mathbf{M}_{M-1}) \right],
$$

where the P-LDC codeword is constructed from the basis matrices as

$$
S(\alpha_0, \alpha_1, \ldots, \alpha_{M-1}) = \sum_{m=0}^{M-1} \alpha_m \mathbf{M}_m.
$$

From the above definitions, we have

$$
C(Q) = E_H \left[ \log \det \left( I_{NR}\tau + \frac{\xi_S}{\sigma^2} \mathbf{H}Q\mathbf{H}^H \right) \right].
$$

Our ability to define $\mathbf{X}$ with arbitrary dimension as we have done above is permitted by the definition of $\mathbf{X}$ given in (4.3). Recall that $\mathbf{X}$ is a collection of the vectorized forms of
the dispersion matrices, \( \{M_m\}_{m=0}^{M-1} \). The vertical dimension of \( \mathbf{X} \) must be \( N_T \tau \) because it interfaces with the equivalent time-delayed channel, \( \mathcal{H} \). However, the horizontal dimension of \( \mathbf{X} \) is a function of how many dispersion matrices were selected at the outset of the design, which is \( M \). In turn, the number of dispersion matrices selected is a direct result of the number of symbols which need to be transmitted across the channel. This chapter assumes \( M \) is a design variable.

By selecting \( \mathbf{X} \) such that its dimensions are \( N_T \tau \times M \), where \( M < N_T \tau \), we will construct a precoder \( \mathbf{Q} = \mathbf{X} \mathbf{X}^H \) which has dimensions of \( N_T \tau \times N_T \tau \) but which is of rank \( M < N_T \tau \) because of the inherited rank-deficiency of \( \mathbf{X} \). In what follows, we refer to the case in which \( M < N_T \tau \) as the rank deficient precoding case. Our derivation of the optimal rank deficient precoder \( \mathbf{X} \) will proceed in two steps: first, solve for the optimal rank-deficient precoder \( \mathbf{Q} \), and second, solve for the optimal precoder \( \mathbf{X} \) such that \( \mathbf{X} \mathbf{X}^H = \mathbf{Q} \). The first result concerning capacity optimality for the rank-deficient \( \mathbf{Q} \) case is contained in the following theorem. In this result, our assumptions concerning the channel model are unchanged from Section 3.1.2, that is, we assume the input-output system expression of the system is given by (3.7). We further assume that \( M < N_T \tau \). Consistent with previous derivations, we let the transmit and receive correlation matrices be designated according to the Kronecker channel model as \( \mathbf{R}_R = \mathbf{V} \Gamma \mathbf{V}^H \) and \( \mathbf{R}_T = \mathbf{U} \Lambda \mathbf{U}^H \). The time-blocked notation of (3.14)-(3.15) from Section 3.2.2 is continued in this discussion, that is, \( \Sigma_R = \tilde{\mathbf{V}} \tilde{\Gamma} \tilde{\mathbf{V}}^H \) and \( \Sigma_T = \tilde{\mathbf{U}} \tilde{\Lambda} \tilde{\mathbf{U}}^H \). Our solution to the capacity problem in (4.1)-(4.4) for the rank deficient case is as follows.

**Theorem 5.** Assume \( M < N_T \tau \). The form of the capacity-optimal rank-deficient precoder \( \mathbf{Q} \) for the system described in (4.1)-(4.4) is identical to the case in which \( \mathbf{Q} \) is of dimension \( N_T \tau \times N_T \tau \). The optimal solution is given by \( \mathbf{Q} = \hat{\mathbf{U}} \Xi \hat{\mathbf{U}}^H \), where \( \hat{\mathbf{U}} \) is defined in (3.15) and contains the time-copied eigenvectors of the transmit correlation matrix, \( \mathbf{R}_T \). The power distribution matrix \( \Xi \) must be computed through numerical optimization.

**Proof.** The proof of the \( N_T \tau \times N_T \tau \) case presented in (3.16)-(3.18) did not make any assumptions about the dimensions of \( \mathbf{X} \), and hence the results apply for the general case in
which $M < N_T \tau$ as well. Therefore, the form of the solution is unaltered.

Because of Theorem 5, it is apparent that the form of the optimal solution for the rank-deficient precoder is unchanged from the case in which the precoder $Q$ supports the full rank of the channel. However, because the rank of the precoder $Q$ is less than $N_T \tau$, it is apparent that the rank of the matrix product $\hat{U} \Xi \hat{U}^H$ is less than $N_T \tau$. Since the form of the solution is the same as the case $M = N_T \tau$ and the eigenvector matrix $\hat{U}$ is unaffected by the rank deficiency, as indicated by Theorem 5, then we anticipate that the rank deficiency must affect the power distribution matrix, $\Xi$, such that only $M < N_T \tau$ of the diagonal entries are nonzero. This result is stated in the following theorem.

**Theorem 6.** Assume $M < N_T \tau$. The optimal power distribution matrix, $\Xi$, is computed by numerically solving the following optimization problem:

$$\Xi = \max_{\sum_{i=1}^{M} \xi_i = \tau} E_H \left[ \log \det \left( I_{N_R \tau} + \sum_{i=1}^{M} \frac{\xi_i \lambda_i \hat{W}_i \hat{W}_i^H}{\sigma^2} \right) \right],$$

where $\hat{W}$ is defined as

$$\hat{W} = \begin{pmatrix}
\sqrt{\xi_1} w_1 \\
\sqrt{\xi_2} w_2 \\
\vdots \\
\sqrt{\xi_{N_R \tau}} w_{N_R \tau}
\end{pmatrix}.$$

The form of the solution in (4.5) is identical to that of (3.19) with the exception that the sum constraint is applied over $M$ eigenvalues instead of $N_T \tau$ eigenvalues, and that the summation in the argument of the $\log \det (\cdot)$ function is over $M$ terms only.

**Proof.** See [131].

The most straightforward way of obtaining the capacity optimal design for the rank deficient precoder is to solve the optimization problem (4.5) directly by numerical optimization.
techniques. One method of solving this numerical problem directly is to utilize the `fmincon` function in Matlab. The function `fmincon` uses a Sequential Quadratic Programming (SQP) algorithm to solve a quadratic programming problem at each iteration. An estimate of the Hessian of the Lagrangian is updated using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method [137, pp. 268-271]. We do not elaborate on this method, since adequate documentation on SQP algorithms can be found in any standard linear programming textbook [137]. In utilizing SQP algorithms, it is important to enforce two constraints: first, that the total power in the system is fixed at some value (we use \( \tau \) for simplicity); and second, that the rank of the optimal power distribution cannot exceed \( M \).

### 4.1.2 Discussion of Suboptimal Solution

The solution to the optimization problem for the rank deficient precoder is different from the full rank precoder because the rank of the power distribution matrix \( \Xi \) can be no greater than \( M \) instead of \( N_T\tau \). As a consequence, at least \((N_T\tau - M)\) diagonal entries in \( \Xi \) are zero, and possibly more are zero depending on the SNR and the waterfilled solution at that SNR. Because of the similarity between the solutions for the full rank and rank deficient cases, it is important to discuss their similarities and differences.

Recall that the full rank solution resembles a water-filled solution. That is, depending upon the SNR, the largest eigenvalues of the time-copied transmit correlation matrix, \( \Sigma_T \), are used as subchannels for transmission across the channel, and the amount of power applied to those subchannels depends upon the optimization problem (3.19). The number of active subchannels is a function of the signal strength relative to the noise power. For low signal power, the optimal solution reduces to a simple \( \tau \)-Beamformer, where only \( N_T \) of the subchannels are active. For very high signal power, or alternatively, very low noise power, the optimal solution applies power equally to all subchannels, and the number of active subchannels becomes \( N_T\tau \). For intermediate SNRs, the traditional water-filling principles apply. These principles are illustrated in Fig. 4.1. As is apparent from Fig. 4.1, the strongest eigen-
Figure 4.1: Traditional water-filling problem where strongest subchannels, represented graphically by deeper “water wells,” are filled first, and the weakest subchannels are filled last.

vectors (corresponding to the “deepest wells” in the water-fill) receive most of the power, whereas the weaker subchannels (the “shallow wells”) get less power, or no power at all.

For the rank-deficient precoder, we permit at most $M < N_T \tau$ subchannels to be active. From the form of the optimization problem in (4.5) and from simulation data we see that the solution still has the form of a water-filled solution. Therefore, for low SNR, the optimal power distribution is again a $\tau$-Beamformer. For high SNR, however, the precoder does not have enough rank to access all $N_T \tau$ subchannels in the system, and it is restricted to only $M$ active subchannels. Because of the water-filling nature of the optimal solution (4.5), the $M$ largest eigenvalues of the transmit correlation matrix are selected and used for transmission in the channel. The optimal power distribution applied to these subchannels resembles a waterfill among these $M$ subchannels. This waterfill is illustrated in Fig. 4.2 for the case $M = 3$. Notice that the water-fill region is smaller because of the smaller number of “wells” in which the power can be distributed (i.e., the number of permissible active subchannels is smaller).

Because the water-filled solution is a function of the system SNR, the full-rank precoder
Figure 4.2: Restricted water-filling problem where only $M$ of $N_T \tau$ subchannels are available for use. The restricted water-fill can be viewed as a “dammed” water-fill problem where the first $M$ subchannels receive all the power. The total power - the area of the gray regions in this Figure and in Fig. 4.1 - are equal. The weakest $N_T \tau - M$ subchannels are not used.

and the rank-deficient precoder are equivalent for a region of low SNR. As already discussed, both systems degenerate into a $\tau$-beamformer for low SNR. As SNR is increased, both systems begin to add power into more subchannels, that is, the signal power “spills” into higher water wells. The order in which these subchannels are made active, and the SNR values for which these transitions occur, are the same for both cases. The solutions diverge when the SNR becomes large enough that the number of required subchannels to make the system capacity optimal exceeds the maximum rank of the precoder, $M$. We thus see that there are two regions of interest for the rank-deficient precoder: a low-SNR region in which it achieves capacity optimality, and a high-SNR region in which it does not achieve full-rank capacity optimality.
4.1.3 Bounds on the Capacity Suboptimality

Precedents exist for computing the bounds on capacity loss for a rank-deficient precoder. These bounds were introduced in the context of frame-based LDCs [31, 39]. In the case of LDCs designed for uncorrelated channels using frame theory [39], quantifying the amount of suboptimality due to a rank deficiency in the precoder \( Q = \mathbf{X} \mathbf{X}^H \) was accomplished by utilizing the frame-based structure of the code. For LDC codes, the capacity-optimal design for the full rank precoder was derived to be \( \mathbf{X} \mathbf{X}^H = \frac{1}{N_T} \mathbf{I}_{N_T \tau} \), where \( \mathbf{I}_{N_T \tau} \) is the \( N_T \tau \times N_T \tau \) identity matrix. In order to compensate for the rank deficient case, frame-based codes select \( \mathbf{X} \) such that

\[
\mathbf{X}^H \mathbf{X} = \frac{\tau}{M} \mathbf{I}_M ,
\]

with \( \mathbf{I}_M \) being the \( M \times M \) identity matrix. The rationale for this design is detailed elsewhere [39].

The simple structure in (4.6) for LDCs makes it possible to analyze an upper and lower bound on the frame-based capacity using the Poincaré Separation Theorem [54, pp. 190]. The Poincaré Separation Theorem utilizes the orthonormal nature of the columns of \( \mathbf{X} \) to draw conclusions about the eigenstructure of the system, which enables a simple derivation of loose upper and lower capacity bounds. Unfortunately, P-LDCs do not share the frame-based structure of their LDC predecessors. In the case of P-LDCs, we do not make the design choice that \( \mathbf{X}^H \mathbf{X} = \frac{\tau}{M} \mathbf{I}_M \), since this choice was predicated upon the assumption of no transmit channel knowledge. No such simplifying structure exists for P-LDCs, which precludes the use of the Poincaré Separation Theorem. Consequently, we must find a different set of bounds to see how “good” the rank-deficient codes are in terms of achieved capacity.

For P-LDCs, a suitable lower bound is the LDC code design which assumes \( \mathbf{X}^H \mathbf{X} = \frac{\tau}{M} \mathbf{I}_M \). Since the LDC was designed in the absence of knowledge of the covariance environment at the transmitter, we know that the P-LDC must outperform the LDC in terms of capacity, since conditioning on additional \textit{a priori} information will increase the expectation in (4.2).
It is therefore instructive to compare the P-LDC performance with rank-deficient precoders to the equivalent LDC design to obtain a lower bound.

For upper bound determination, we recognize that
\[
\max_{\mathbf{X} : \text{Tr}(\mathbf{XX}^H) = \tau} \frac{1}{\tau} C(\mathbf{XX}^H) \leq \max_{\mathbf{X} : \text{Tr}(\mathbf{XX}^H) = \tau} \frac{1}{\tau} C(\mathbf{XX}^H), \quad (4.7)
\]
since adding constraints to an optimization problem cannot increase the value of the global maximum. As a result, we may use our full rank precoding results to upper bound our capacity results for the rank deficient precoder.

### 4.1.4 BER Optimization of $\mathbf{X}$

Once $\mathbf{Q}$ is determined by solving the optimization problem in (4.1)-(4.4), the optimal rank deficient precoder is of the following structure, where $\Psi$ is a unitary matrix of dimension $N_T \tau \times N_T \tau$:
\[
\mathbf{X} = \hat{\mathbf{U}} \Xi^{1/2} \Psi.
\]  \( (4.8) \)

The expression (4.8) can be simplified. Since $\Xi$ has at most $M$ nonzero diagonal entries, we can simplify the precoder expression for $\mathbf{X}$ by defining the $N_T \tau \times M$ matrix $\hat{\mathbf{U}}_1$ to contain the $M$ eigenvectors corresponding to the largest eigenmodes of the channel. Define $\Xi_1$ to be a $M \times M$ diagonal matrix containing the waterfilled power distribution of the rank-deficient optimization problem. Finally, define an arbitrary $M \times M$ unitary matrix $\Psi_1$. Then we can rewrite (4.8) as
\[
\mathbf{X} = \hat{\mathbf{U}}_1 \Xi_1^{1/2} \Psi_1.
\]  \( (4.9) \)

It is apparent from (4.9) that the rank-deficient precoder differs from the full rank precoder based upon the size of the dimension of the unitary matrix, $\Psi_1$, since the largest dimension achieved by $\Psi_1$ is $M \times M$. 

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In order to determine the matrix $\Psi_1$, the design modification procedures for BER enhancement in Section 3.3 are repeated for the rank deficient precoder. The rank and determinant criteria, as discussed in (3.26)-(3.29), are used to select $\Psi_1$ so as to optimize the coding gain and diversity advantage of the code for high SNR. The design procedure does not change due to the rank deficiency and is not repeated here.

### 4.1.5 Simulation Results

In the discussion which follows, we implement several system configurations to illustrate the loss of capacity due to rank deficiency. We quantify these results in terms of two important system metrics: spectral efficiency and BER.

**Capacity Results**

The rank deficiency incurred by permitting $M < N_T \tau$ has consequences for the capacity performance of a given MIMO system. As we show through simulation, the degree to which the spectral efficiency is diminished is a function of several parameters, including transmit correlation properties, the number of substreams $M$, and the number of antennas present in the system. Because the effects upon performance are functions of so many variables, it is difficult to draw decisive general conclusions about the impact of rank deficiency upon capacity.

Consider the capacity plot in Fig. 4.3. The simulation environment adopted in this figure is a 2-transmit, 2-receive MIMO system with a variety of angular spreads $\Delta$ ranging from 20 degrees down to 2 degrees. In this $2 \times 2$ system characterized by $\tau = 2$, there are a total of $N_T \tau = 4$ subchannels available for transmission. In Fig. 4.3, we compare precoders which use all available subchannels (designated Full Rate, or FR, such that $M = 4$), with rank deficient precoders (designated RD, such that $M = 3$). For the correlation environment in which $\Delta = 20^\circ$, a noticeable difference in spectral efficiency is evident for high SNR.
Figure 4.3: Comparison of full rank (FR) and rank deficient P-LDC precoders for a variety of transmit correlation environments in a 2 × 2 MIMO system. Full rank precoders have $M = 4$, whereas the rank deficient precoders herein have $M = 3$. 
However, for the harsh environment in which $\Delta = 2^\circ$, the full rank and rank deficient precoding curves are identical. This behavior can be explained by results from information theory. The reason for this performance is that there are only two unique subchannels, each of which has a redundancy of 2. Since the capacity-optimal solution resembles a water-fill, the strongest eigenmodes receive the greatest power first. In the harshly correlated environment, there is a significant difference in strength between the strongest eigenmode and the next eigenmode. Therefore, in the harsh environment, there is a very “deep” well into which power gets poured, and in Fig. 4.3 there is not enough signal power for there to be any “pouring” of the power into the weaker subchannel. Therefore, in the harsh environment the water-fill only attempts to utilize 2 subchannels for the SNR values shown, whereas both the RD and FR precoders attempt to utilize greater than 2 subchannels, so both precoders yield identical spectral efficiency.

In Fig. 4.4, simulation results for a $3 \times 3$ system are presented. The system of Fig. 4.4 uses $\tau = 2$ and therefore a total of $N_T \tau = 6$ subchannels are available for transmission. The RD precoder is selected to utilize $M = 4$ of these subchannels, whereas the FR precoder utilizes all $M = 6$ subchannels. Two different correlation environments are illustrated - one with a transmit antenna separation distance of one wavelength ($1 \lambda$) and an angular spread of $20^\circ$, and one with a transmit antenna separation distance of 1.5 wavelengths ($1.5 \lambda$) and an angular spread of $15^\circ$. The key result of Fig. 4.4 is that neither correlation environment experiences much loss of capacity due to a rank deficiency in the precoder. This is explained by the fact that the correlation environments of Fig. 4.4 have two dominant subchannels, each with a multiplicity of 2, so that most of the power of the water-fill gets placed into the strongest $M = 4$ subchannels - the same subchannels utilized by the rank-deficient precoder.

The results of Fig. 4.4 indicate that for $3 \times 3$ systems with relatively low correlation, the difference in capacity between the rank deficient precoder and the full rank precoder is minimal. In the specific case of Fig. 4.4, a total of 6 subchannels are available and 4 of these are utilized by the RD precoder. In some ways, the RD precoder of Fig. 4.4 is “barely
Figure 4.4: Comparison of FR and RD precoders for 3 × 3 MIMO configuration with a variety of transmit correlation environments. Little difference between FR and RD precoders is observed.
Figure 4.5: Comparison of FR and RD precoders for 3×3 MIMO configuration. Progressively higher degrees of correlation at the transmitter are illustrated.

rank deficient,” since it only discards the two weakest subchannels. The four strongest subchannels are still available for use, which makes capacity performance of the RD strong in this case. However, if we repeated the simulation of Fig. 4.4 with fewer subchannels available for the RD precoder in a similar correlation environment, we anticipate that a more noticeable capacity gap would exist between the RD and FR precoders. We illustrate this result in Fig. 4.5.

In Fig. 4.5 we present more capacity results for RD precoders in a 3 × 3 system with similar physical environmental parameters as in Fig. 4.4. The RD precoders in this plot utilize only 3 subchannels, instead of 4 as in Fig. 4.4. In Fig. 4.5, notice that for angular spreads of 10° and 15°, a noticeable difference in capacity exists between the RD and FR
precoders. By comparison, we see that eliminating just one additional subchannel for a RD precoder can have significant consequences for the RD precoder in terms of spectral efficiency. Notice that the gap in performance is greatest for $15^\circ$, and smaller for $10^\circ$.

Finally, Fig. 4.5 reveals that for an angular spread of $2^\circ$, the RD and FR precoders have identical capacity performance. This is explained by the fact that for highly correlated environments, the optimal transmitter design is a beamformer, which utilizes only one beam. In the context of P-LDCs, this means a $\tau$-Beamformer, which in our case requires $\tau = 2$ subchannels to be used. Since the RD precoder uses $M = 3 > \tau$ subchannels, the RD precoder is able to achieve true capacity optimality in this case, resulting in a coincident curve with the FR case.
For completeness of our results, we also present capacity curves for the $4 \times 4$ case in Fig. 4.6. In this case, the RD precoders utilize only 4 subchannels out of an available $N_T \tau = 8$. From Fig. 4.6, even for reasonably mild correlation environments, the difference in capacity between the RD and FR precoders is quite noticeable. Again, for the harshly correlated environment with $\Delta = 5^\circ$, the RD and FR precoders are equivalent because only 4 subchannels are utilized in the FR precoder for the range of SNRs shown in Fig. 4.6.

In Sec. 4.1.3, the use of LDCs as lower bounds for P-LDCs in terms of capacity performance was discussed. We present some simulation results here because such lower bounds provide some useful insights into the behavior of systems utilizing RD precoders. In Fig. 4.7, a $3 \times 3$ system for $\tau = 2$ is presented with an angular spread of $15^\circ$, a transmit antenna spac-
ing of 1.5\( \lambda \), and no receive correlation. The RD precoder assumes only \( M = 3 \) of \( N_T = 6 \) subchannels are in use. In such a case, we have already discussed that a significant drop in capacity occurs from the FR to the RD precoder. However, it is interesting to note that there is not much difference between a “blind” LDC using three subchannels and the RD-P-LDC. From this we see that it is possible to incur a substantial penalty in terms of capacity by utilizing a RD precoder. In the system of Fig. 4.7, by discarding 3 subchannels the system has lost so much information about the transmission environment that a blindly designed LDC would provide nearly equivalent performance. Further reduction of the number of active subchannels to \( M = 2 \) would force the RD-P-LDC capacity curve even closer to the LDC capacity curve.

Figure 4.8: Lower bound LDCs presented for comparison against RD-P-LDCs in a harsh correlation environment.
Fig. 4.8 presents similar results as Fig. 4.7 for a $3 \times 3$ system, but with a more harsh correlation environment. In Fig. 4.8, we assume a transmit antenna spacing of $1\lambda$ (as opposed to $1.5\lambda$ for Fig. 4.7), and an angular spread of $10^\circ$ (as opposed to $15^\circ$ for Fig. 4.7). As before, the RD precoder uses only $M = 3$ of 6 total subchannels. From this figure, we see that in the more harsh environment, sacrificing 3 subchannels in the RD precoder is less dramatic in terms of reducing capacity relative to the FR precoder, and the RD precoder still has a significant 1.7 bps/Hz advantage over a blindly designed $3 \times 3$ LDC.

The difference between the effects of Figs. 4.7 and 4.8 can be explained by recognizing the value of the statistical channel information being discarded due to rank deficiency. In Fig. 4.7, a mild correlation environment is assumed, so the water-filled subchannels are distributed more evenly than in a harsh environment. In such a case where the discarded subchannels are likely to be in use, we expect significant performance loss due to rank deficiency. It turns out that orthogonally designed dispersion matrices [39] are little worse than the RD precoders of Fig. 4.7. However, in Fig. 4.8, a more harsh environment is assumed, which means the water-filled subchannels are more disproportionate in assigning power to stronger subchannels. Since in the harsh environment the subchannels discarded due to rank deficiency are not heavily needed, the RD precoder significantly outperforms the LDC. Essentially, the RD precoder of Fig. 4.8 has nearly all the information it needs to achieve capacity, and the performance is significantly better than the LDC code.

BER Results

When discussing the effects of rank deficiency on the BER of a system when compared to a full rank precoded system, it is important to specify in what way the comparison is made. For example, consider the BER plot of Fig. 4.9, in which we compare the BER of a FR and RD precoder for BPSK modulation in a $3 \times 3$ system with an angular spread of $10^\circ$ and a transmit antenna spacing of $\lambda$. The RD precoder utilizes only $M = 3$ subchannels, whereas the FR precoder utilizes $M = N_T \tau = 6$ subchannels.
Figure 4.9: BER comparison for $3 \times 3$ system with $M = 3$ for RD precoder and $M = 6$ for FR precoder. BPSK modulation is assumed for both cases. SNR is considered per block.
An obvious conclusion to draw from Fig. 4.9 is that the BER performance of the RD precoder is much better than that of the FR precoder. However, it must be mentioned that the effective data rate of the two systems is quite different. The FR precoder sends a total of 6 bits over $\tau = 2$ symbol periods, resulting in a data rate of 3 bps/Hz. The RD precoder only sends 3 bits over $\tau = 2$ symbol periods, resulting in a data rate of 1.5 bps/Hz. Therefore, we see that by keeping the modulation constant and reducing the number of subchannels being utilized, the RD precoder has a distinct advantage over its FR counterpart, since it is required to send only half the data of the FR precoder. Therefore, the BER improvement observed in Fig. 4.9 is not surprising.
For comparison, we present Fig. 4.10, which provides a more “fair” comparison between the RD and FR precoders, assuming we desire to keep the data rate constant. To effect this fair comparison, we utilize the exact physical parameters as in Fig. 4.9, but this time a different modulation is selected for the RD case in order to keep the overall data rate constant in the two systems. To this end, we select BPSK for the $M = 6$ substreams of the FR precoder, and we select 4-QAM for the $M = 3$ substreams of the RD precoder. The effective data rate of both systems is 3 bps/Hz.

In Fig. 4.10, we see that if the data rate is kept constant between the FR and RD precoders, the FR precoder substantially outperforms the RD precoder. In fact, a full 3.5 dB separates the two curves. This result is not surprising, however. The FR precoder has more dimensions over which to spread the data in both space and time, whereas the RD precoder has only half the dimensions of the FR precoder. Moreover, the FR precoder uses a constellation with a larger minimum distance than the RD precoder. In order to handle the same amount of data, the RD precoder is at a significant disadvantage in terms of available diversity branches.

A potential problem with comparing FR and RD precoders in terms of BER is the necessity of keeping the data rates constant. If we desire to achieve a target data rate, there are many issues to address, such as what constellation to use on which subchannel in order to minimize the overall error rate. In our example in Fig. 4.10, we assumed the same constellation was used on all subchannels, but this is not a necessity. This can be made a problem of adaptive modulation, and the result is that it is possible to utilize different constellations on different subchannels. While this is certainly possible in terms of the existing adaptive modulation theory [49], it presents other problems which complicate system design. For example, when selecting different constellations for different subchannels, this affects the BER optimization procedure discussed in Sec. 3.3, which is specific to a given constellation. Hence selection of different constellations for different subchannels presents the need for an iterative design, where an initial constellation choice is guessed for design of the optimal
BER criterion (Sec. 4.10), then a new optimal constellation choice must be made. This process must be repeated until an equilibrium state is achieved where successive iterations yield identical results. Because of this design complication, the results we present here are based upon the simplifying assumption that all subchannels utilize the same constellation, regardless of whether this is optimal in terms of BER.

4.2 Characterizing the Rank Deficient Solution

The solution to the capacity optimization problem described in (4.1)-(4.4) is unique; that is, given the optimal eigendirections \( \hat{U} \) corresponding to the strongest eigenmodes of the correlation environment, there is a unique set of coefficients, \( \Xi \), which optimizes the capacity. The optimization expressions which describe the rank deficient solution do not reveal much useful information about the relationship between the full rank, capacity-optimal precoder and the rank deficient, capacity optimal precoder. In this section, we focus on some basic approximation techniques which reveal useful information about the structural relationship between the full rank and rank deficient solutions.

This section focuses on matrix distance relationships between the full rank and rank deficient precoders. We illustrate that there is a key relationship between these two classes of precoders which relates to a Frobenius norm difference matrix. By using these Frobenius norm distance metrics and numerical optimization techniques, we illustrate some properties of the rank deficient precoder which are intuitively satisfying in their relationship to the structure of the channel.

4.2.1 Minimum Frobenius Norm Approximation

Instead of solving (4.1)-(4.4) directly through numerical optimization, it is possible to derive a minimum distance approximate solution. Assume that the full rank precoder, which
we designate $Q_{\text{opt}}$, has been solved using numerical techniques (such as an SQP algorithm).

Assume that we desire to derive an estimate, $Q_{\text{est}}$, of rank $t < N_T\tau$, such that the following criterion is satisfied:

$$Q_{\text{est}} = \min_{Q_{\text{rank}(Q)=t}} \|Q_{\text{opt}} - Q\|$$ \hspace{1cm} (4.10)

From (4.10), we see that $Q_{\text{est}}$ represents a minimum distance estimate between the full rank capacity optimal code, $Q_{\text{opt}}$, and the rank-deficient estimate, $Q_{\text{est}}$. What this method attempts to do is to approximate a large dimensioned subspace by a smaller dimensioned subspace. Methods of such computations are called low-rank matrix approximation [138].

In solving (4.10), we must address three issues. First, we must determine what norm to use in the minimization. Secondly, we must solve for the optimal solution. Thirdly, once we find a solution, we must ascertain whether that solution will be valid for other norms. We address these issues in order below.

**Determination of Which Norm to Use**

A logical choice for the norm to use is the Frobenius norm, which measures Euclidean distance in the matrix space. The Frobenius norm of an $m \times n$ matrix $A$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{i,j}|^2} ,$$

where $a_{i,j}$ denotes the element in $i^{th}$ row and the $j^{th}$ column of $A$. Using the Frobenius norm in (4.10), $Q_{\text{est}}$ represents the matrix in $M$-dimensional space that is closest, in the Euclidean distance sense, to the optimal precoder $Q_{\text{opt}}$, whose rank is a function of the water-fill and can be as large as $N_T\tau$. 
Optimal Solution

The solution to (4.10) can be derived by using a well-known result stated in the Eckart-Young Lemma from matrix algebra. We state the Eckart-Young Lemma below [54, pp. 427-450].

**Eckart-Young Lemma.** Let an \( m \times n \) matrix \( A \) have rank \( k \) and let its SVD be given by \( A = VD^H \). Assume the singular values \( \{\sigma_1, \sigma_2, \ldots, \sigma_k\} \) are contained in \( D \) and that the singular values are arranged in non-increasing order on the diagonal. The \( m \times n \) matrix \( A_1 \) of rank \( k_1 < k \) which solves the problem

\[
A_1 = \min_{B: \text{rank}(B)=k_1} \|B - A\|_F^2
\]

is given by

\[
A_1 = VD_1W^H,
\]

where the first \( k_1 \) entries of \( D_1 \) are \( \{\sigma_1, \sigma_2, \ldots, \sigma_{k_1}\} \), and the remaining \( n-k_1 \) diagonal entries of \( D_1 \) are zero.

Because of the Eckart-Young Lemma, we propose the following theorem to solve (4.10).

**Theorem 7.** Minimum Distance Rank Deficient Precoder. Assume the eigenvectors \( \hat{U} \) and eigenvalues \( \tilde{\Lambda} \) of the transmit correlation matrix, \( \Sigma_T \), are known at the transmitter. Solve for the full dimensional capacity optimal precoder, \( Q_{\text{opt}} \), for the \( M = N_T \tau \) case using techniques of Chapter 3. The rank of this precoder is a function of the SNR in the system and is designated as \( \rho \). The suboptimal precoder \( Q_{\text{est}} \) of rank \( \rho \leq N_T \tau \) which solves (4.10) is defined as

\[
Q_{\text{est}} = \hat{U} \Xi_1 \hat{U}^H, \tag{4.11}
\]

where the nonzero entries on the diagonal of \( \Xi_1 \) are the \( \rho \) largest power distribution values from \( \Xi_{\text{opt}} \).
Generalization of Minimum Norm Approximation

We refer to the solution (4.11) as an SVD-truncated solution. This solution (4.11) can be shown to minimize the distance metric in (4.10) for any unitarily invariant norm. A unitarily invariant norm \( \| \cdot \| \) is one in which pre-multiplication and/or post-multiplication by a unitary matrix does not change the value of the norm. Mathematically, we have:

\[
\| A \| = \| UV \|
\]

for a \( n \times n \) matrix \( A \) and unitary matrices \( U \) and \( V \). It has been shown [54, pp. 450] that the minimum distance solution proposed by the Eckart-Young Lemma is valid for all unitarily invariant norms. Therefore, the Theorem (4.11) is valid for any unitarily invariant norm.

Among the unitarily invariant norms for which (4.11) applies are the matrix spectral norm \( \| \cdot \|_2 \) and the trace norm, also known as the Ky-Fan norm \( \| \cdot \|_{\Sigma} \) [54, pp. 445], [139]. Additionally, the family of \( l_p \) norms, when applied to the singular values of a matrix, generate unitarily invariant norms known as the Schatten \( p \)-norms for which (4.11) applies [54, pp. 441].

Power Loss

A key feature of (4.11) is that \( Q_{est} \) suffers power loss relative to \( Q_{opt} \). Because some of the diagonal components of \( \Xi \) have been zeroed in the formation of \( \Xi_1 \), the total power contained in \( Q_{est} \) is less than that of \( Q_{opt} \). Assuming high SNR, there are \( M \) active subchannels in the rank-deficient precoder and \( N_{T,T} \) active subchannels in the full rank precoder. Denote the power distribution values for the rank deficient precoder as \( \{ \mu_i \}_{i=1}^M \) and for the full rank precoder as \( \{ \gamma_i \}_{i=1}^{N_{T,T}} \). Then we have the power in the estimated system, \( P_{est} \), to be

\[
P_{est} = \sum_{i=1}^{M} \mu_i < P_{opt} = \sum_{i=1}^{N_{T,T}} \gamma_i \tag{4.12}
\]
Because of the power loss inherent in (4.12), there are at least two observations we can make in regard to system performance. First, the capacity performance of (4.11) will not be as good for high SNR as that obtained by solving (4.1)-(4.4) directly through numerical optimization. This is true because the solution (4.11) was not derived from a capacity optimality criterion, whereas (4.1)-(4.4) were. Further, the BER performance of the codes derived by (4.11) will not be as promising as that obtained through numerical optimization, since the precoder uses less average transmit power. However, the advantage of this code is that the amount of capacity loss and BER increase is not that large for some cases, and in situations where power consumption is a design consideration, the approximation (4.11) can provide a useful alternative to numerical optimization.
Key Result

By utilizing the minimum distance Frobenius norm metric discussed in this Section, we have shown how to approximate, in the sense of any unitarily invariant norm, the full-rank precoder by truncating the smallest eigenvalues. As shown in Fig. 4.11, the optimal rank deficient precoder is not the same as the minimum Frobenius norm estimate, since at high SNR there is significant deviation from the optimal spectral efficiency for a $3 \times 3$ system. This result is not unexpected, since significant loss of power is possible due to the truncation of channel modes and the resulting loss of power. The precoder in Fig. 4.11 is outperformed by the rank deficient solution because it is operating at a significant power disadvantage.

4.2.2 Power-Reallocated SVD Truncation

A logical corollary to the SVD truncation method is to eliminate the power loss by adding the power from the eliminated eigendirections back in to the overall power distribution. For simplicity, assume high SNR, so that the number of active subchannels is $M$. The power-reallocated SVD truncation technique is described as follows.

Theorem 8. Power-Reallocated Algorithm. Compute the estimation in (4.11) of Theorem 7, and let $P_{\text{diff}} = P_{\text{opt}} - P_{\text{est}}$. Add the power $\frac{P_{\text{diff}}}{M}$ back into each of the $M$ subchannels so that the total power in the system is $P_{\text{opt}}$.

Since the optimization problem in (4.1)-(4.4) does not have a closed-form solution, the relationship between this power-reallocated SVD truncation technique and the truly capacity-optimal solution for the rank-deficient case is not immediately evident. However, as we explain here, these two techniques are identical. Recall that the full-rank capacity-optimal solution gave a water-filled precoder whose rank was determined by the quality of the channel and could be as large as $N_T \tau$. This water-filled solution filled the subchannels of the channel in order of their quality. In the case of the full-rank precoding case, the water-fill could extend to all the available subchannels as in Fig. 4.1. In the case of the rank-deficient case, the
maximum rank of the precoder is clipped at $M$. As a result, the remaining $N_T \tau - M$ subchannels are restricted, and any power that would have flowed into the remaining $N_T \tau - M$ subchannels is “trapped” in the first $M$ subchannels, as illustrated in Fig. 4.2. Until the SNR becomes high enough that the $M^{th}$ subchannel is filled, the full-rank and rank-deficient precoders are equivalent; the differences arise only when the SNR becomes large enough that added subchannels beyond the initial $M$ subchannels are needed.

As illustrated in Fig. 4.12, we see that the rank-deficient capacity-optimal precoder coincides with the power-reallocated SVD truncation procedure described above. The power-reallocated SVD truncation procedure first computes the “closest” low-rank estimate of the full-rank precoder by finding the strongest eigenmodes of the channel and computing the optimal power allocation if all subchannels were permitted to be used. Then, the “leftover” power is re-distributed evenly across all the active subchannels so that all the power is used.
Relation to LDCs Based on Frame Theory

The power-reallocated SVD truncation procedure has a specific relationship to frame-based LDCs not previously explored [31, 39, 40] and it highlights some key relationships, and an inherent design flaw, in frame-based LDCs. Frame-based LDCs are designed under the assumption of no channel correlation. The basic input/output model for frame-based LDCs is the same as for P-LDCs, and as such, the optimal precoder $\mathbf{X}\mathbf{X}^H$ is of dimension $N_T\tau \times N_T\tau$ but has a rank that is limited by $\mathbf{X}$, which is of dimension $N_T\tau \times M$, where $M \leq N_T\tau$. From prior work on LDCs [39], the optimal solution for a full rank precoder where $M = N_T\tau$ is

$$\mathbf{X}\mathbf{X}^H = \frac{1}{N_T} \mathbf{I}_{N_T\tau}.$$ (4.13)

For cases in which $M < N_T\tau$, it is impossible for (4.13) to hold because $\mathbf{X}$ is of insufficient size to support a full rank precoder. As a consequence, frame theory proposes an alternative design criterion in which [39]

$$\mathbf{X}^H \mathbf{X} = \frac{\tau}{M} \mathbf{I}_M.$$ (4.14)

An important aspect of (4.13) is to recognize that the water-filling problem for the uncorrelated channel becomes a flat water-fill in which each subchannel is of equal quality and receives equal power. In a full-rank precoder given by (4.13), this is accomplished by choosing $\mathbf{X}$ such that it is comprised of $N_T\tau = M$ orthonormal row vectors, or equivalently, $M = N_T\tau$ orthonormal column vectors, and it is scaled such that the total power in the system is $\tau$.

In the rank-deficient case (4.14), however, it is impossible to choose $N_T\tau$ orthonormal row vectors of dimension $1 \times M$ since $M < N_T\tau$. As a result, the frame design instead chooses $M$ orthonormal column vectors of dimension $N_T\tau \times 1$.

The relationship between (4.13) and (4.14) is as follows. Consider the waterfill for the full-rank precoder given in Fig. 4.13. Notice that there are $M = N_T\tau$ equally weighted subchannels, and each channel is represented by an orthonormal vector. This is true because the optimal precoder for this case is a scaled identity (4.13), whose eigendecomposition is
Figure 4.13: Water-filling concept for equally weighted subchannels in the frame-based LDC.

given by

\[ I_{N_T \tau} = \text{CSC}^H , \]

where \(\text{C}\) is a \(N_T \tau \times N_T \tau\) unitary matrix and \(S\) is an identity matrix. For the rank-deficient case, depicted in Fig. 4.14, the power is “trapped” in only \(M\) subchannels. Following the procedure of the adjusted SVD truncation technique, we find the minimum distance estimate of the full-rank precoder by choosing \(M\) eigenvectors from (4.13) and placing them in \(\mathbf{X}\). Denote \(\mathbf{X} = \sqrt{\frac{\tau}{N_T}} \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M \}\). Clearly, because of the rank deficiency, we have \(\mathbf{X}^H \mathbf{X} = \frac{1}{N_T} \sum_{i=1}^{M} \mathbf{u}_i \mathbf{u}_i^H \neq I_{N_T \tau}\). Finally, we take the leftover power that has not been used and reallocate it among the active subchannels. This leads to a definition of \(\mathbf{X} = \sqrt{\frac{\tau}{M}} \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M \}\). As such, we obtain \(\mathbf{X}^H \mathbf{X} = \frac{\tau}{M} \sum_{i=1}^{M} \mathbf{u}_i \mathbf{u}_i^H\).

The minimum distance estimation with power re-allocation described above gives us the final solution

\[ \mathbf{X}^H \mathbf{X} = \frac{\tau}{M} \sum_{i=1}^{M} \mathbf{u}_i \mathbf{u}_i^H . \]  

By utilizing the orthonormality of the columns of \(\mathbf{X}\), it is easy to see that

\[ \mathbf{X}^H \mathbf{X} = \frac{\tau}{M} I_M , \]

which is the definition of a frame-based code [39]. Thus we see that frame-based LDCs are
Figure 4.14: Water-filling concept for equally weighted subchannels in the frame-based LDC with a rank limitation of $M$ placed on the precoder.

From the above development we see that the frame-based LDCs are identical to the power-adjusted minimum distance approximators for uncorrelated channels. In particular, the frame-based LDCs first compute the optimal precoder for full rank $M = N_T \tau$, which is an identity matrix, and then they compute estimates which find

$$
\min_{A: A^H A = \frac{1}{N_T} I_M} \| I_{N_T \tau} - A A^H \|_F .
$$

(4.17)

The final step in the LDC frame-based design is to re-distribute the power so that the total power in the system is $\tau$.

In the SVD truncation technique, as well as the technique of re-allocating the power due to eigenmode truncation, we have given specific attention to the Frobenius norm approximation between a given $XX^H$ and the full rank optimal precoder. In the frame-based structure of LDCs, we have shown the equivalency between the frame design technique and the power reallocated SVD truncation procedure. However, in Heath’s LDC solution for the case of
no channel correlations [39], he has made no structural considerations regarding how well \(XX^H\) approximates the full rank precoder. A useful issue to address is how to construct tight frame-based LDCs such that the expression (4.17) is satisfied and that the basic structure of \(XX^H\) is “close” in some sense to \(I_{N_T}\). Because of the design flaw that no structural constraint is imposed on the quantity \(XX^H\) in frame-based LDC design, we anticipate that it is possible to obtain poor capacity performance for certain selections of \(X\). This is an issue we address in a later chapter.

4.2.3 Minimum Distance with Modified Constraints

We can utilize the extensive literature on low-rank matrix approximation to describe another method of approximating the full-rank matrix \(Q_{opt}\) with a lower rank matrix \(Q_{est}\). We utilize the general framework of structured low rank approximation [140], which seeks to solve the following problem. Given a matrix \(A\) of dimension \(m \times n\), and integer \(k\) such that \(1 \leq k \leq \text{rank}(k)\), a class of matrices \(\Omega\), and a norm \(\| \cdot \|\), find an estimator matrix \(B_1\) of rank \(k\) such that

\[
\|A - B_1\| = \min_{B \in \Omega, \text{rank}(B) = k} \|A - B\|.
\]

The difference between (4.18) and (4.10) is the presence of the constraint that \(B_1 \in \Omega\). The subspace \(\Omega\) represents an additional constraint on the estimator matrix \(B_1\). In some applications, \(\Omega\) can represent a structural constraint, such that \(\Omega\) is the set of Hankel, or Toeplitz, matrices. In such problems it is possible to exploit certain structural properties of \(\Omega\), since in this case \(\Omega\) is an affine subspace. In our P-LDC application, the constraint \(\Omega\) is not structural but rather it is a power constraint. We define \(\Omega\) to be the set of all matrices \(A\) such that \(\text{trace}(AA^H) = \tau\).

Methods of solving (4.18) for a variety of assumptions regarding \(\Omega\) have been discussed thoroughly in the mathematics literature [140–144]. Theoretically, it is possible to use “lift and project” techniques to solve (4.18) when \(\Omega\) is an affine subspace [140]. For the case
where Ω is a power constraint, exotic minimization routines are unnecessary, and we can resort to more traditional SQP algorithms, such as are used by Matlab’s fmincon function.

Through numerical simulation, it turns out that adding the power constraint as indicated in (4.18) yields a solution which is identical to the rank deficient capacity optimal precoder. In principle, then, we can state that the minimum distance estimate of the full rank precoder, constrained to have full power τ and a rank equal to M, is equivalent to the optimal rank deficient precoder.

4.3 Concluding Remarks

We emphasize that the rank deficient codes of this chapter, while unable to support the Shannon capacity of the channel for all SNRs, give the system designer added flexibility since the number of subchannels, M, is flexible. As illustrated in this chapter through extensive simulations, for a wide range of system architectures and SNRs, the loss of capacity is not great given that M is “close” in value to $N_T \tau$. Therefore, for these cases we deem it an acceptable design tradeoff to exchange a small amount of capacity in return for additional design flexibility.

There are additional opportunities for future research on this topic. In particular, an analysis of which types of precoders - partial rank or full rank - can achieve a better BER for a given number of subchannels would be useful. Implementation of an adaptive modulation technique which can achieve a target BER for a given number of active subchannels would provide additional insight into the partial rank design technique. Such an adaptive design would have to optimize which subchannels would be modulated with given constellations in order to achieve a target data rate at a given BER.
Chapter 5

Structurally Constrained
Frame-Based Linear Dispersion Codes

Previous chapters have focused on the case in which partial knowledge about the channel correlation environment was accessible to the transmitter. Based upon this transmit covariance knowledge, precoded linear dispersion codes were designed which utilized the covariance information to pre-align transmissions in a MIMO channel along the modes of the channel statistics in order to optimize performance metrics. In this chapter, we analyze a related scenario in which no channel correlation information is present at the transmitter. This may indicate that there is no correlation present, or that this information is not accessible due to the lack of a feedback channel from receiver to transmitter.

In the absence of channel covariance knowledge, the design of P-LDCs degenerates into the design of linear dispersion codes. LDCs have been designed for a variety of scenarios [32, 39, 114, 116, 117]. Among the key contributions in the field of LDCs has been the work of Hassibi, which focused on ergodic capacity optimality [32], and Heath, which focused on joint capacity and error rate optimality [39, 40]. Our overriding objective is to optimize both capacity and error performance metrics, hence we will base our analysis on the works of
In this chapter, we illustrate an inherent design flaw in the LDC designs of Heath. These designs are based upon the frame theory of wavelets [41]. We show that while this frame-based structure is an important starting point for capacity-optimal design, the designs of [31,39,40] contain an inherent design flaw which makes it possible for existing frame-based LDCs to suffer significant capacity losses due to poor precoder selection. To address this potential capacity loss, we design an improved LDC which does not experience such drastic capacity losses. We impose an additional constraint upon the capacity optimal design which dictates that the precoder satisfy a structural matrix constraint in addition to a frame constraint. We prove that this additional constraint is equivalent to computing a nearly Grassmannian or equiangular frame, or a packing in a Grassmannian manifold [52]. To derive the Grassmannian frame, we apply a search technique known as alternating projections to search through the space of tight frames to find a specific frame which satisfies the Grassmannian constraint. The performance of the resulting code is dependent upon the convergence of the alternating projection algorithm. Although this search technique has been applied to a related problem of finding equiangular tight frames [145], searching for Grassmannian tight frames of arbitrary dimension via alternating projection is a novel contribution of this chapter. We verify the performance of our structurally constrained LDC through numerical simulation.

5.1 Motivation For Additional Frame-Based LDC Constraints

In order to illustrate the need for additional constraints in existing LDC designs, consider a wireless MIMO channel whose input/output relationship is determined by (2.33) and (2.34). LDCs utilize a set of dispersion matrices, \{M_0, M_1, \ldots, M_{M-1}\}, which form a space-time transmission given by (2.32). We also adopt the super-matrix notation of Chapter 2 which
utilizes \( \mathbf{X} \) instead of explicit use of the dispersion matrices. We consider the design problem in which it is desired to optimize the ergodic capacity. Mathematically, we must compute a precoder \( \mathbf{X} \) such that the following relationship is satisfied:

\[
C = \max_{\text{tr}(\mathbf{X}\mathbf{X}^H) \leq \tau} \frac{1}{\tau} E_{\mathbf{H}} \log \det \left( \mathbf{I}_{N_{RT}} + \frac{E_{S}}{N_0} \mathbf{H} \mathbf{X} \mathbf{X}^H \mathbf{H}^H \right)
\]

(5.1)

In a design scenario in which (5.1) is the objective and no channel knowledge is available to the transmitter, it has been proven that the optimal precoder \( \mathbf{X} \) satisfies the relationship [15,39]

\[
\mathbf{X}\mathbf{X}^H = \frac{1}{N_T} \mathbf{I}_{N_T \tau} .
\]

(5.2)

The relationship in (5.2) shows that the function of the matrix precoder product \( \mathbf{X}\mathbf{X}^H \) is to whiten the input to the channel so that the argument of the log det (\( \cdot \)) function in (5.1) is \( \left( \mathbf{I}_{N_{RT}} + \frac{E_{S}}{N_0 N_T} \mathbf{H} \mathbf{H}^H \right) \).

In order for the LDC precoder \( \mathbf{X} \) to satisfy the relationship in (5.2), the matrix product must equal an identity matrix of dimensions \( N_T \tau \times N_T \tau \). This requires that the precoding matrix \( \mathbf{X} \) be of rank \( N_T \tau \). Recall that the dimensions of \( \mathbf{X} \) are \( N_T \tau \times M \), where \( M \) is the number of substreams to be transmitted across the channel. We see that in order for (5.2) to hold, we must have \( M = N_T \tau \). In many cases, the decoding complexity of choosing \( M = N_T \tau \) is too high, or the required memory to effect decoding is restrictive. As a consequence, in many designs it is not feasible to choose \( M = N_T \tau \) but instead a choice of \( M < N_T \tau \) is necessitated. In these cases, it is not possible to achieve true capacity optimality. This case is comparable to the P-LDC design case discussed in Chapter 4, but without the channel covariance knowledge at the transmitter.

In a compromise design, Heath proposed an alternative solution based on the frame theory of wavelets [39]. A frame is defined as any tall matrix \( \mathbf{A} \) of dimension \( m \times n \), with \( m > n \), such that \( \mathbf{A}^H \mathbf{A} = \mathbf{I}_n \). To approximate a “nearly capacity optimal” solution whose
performance approaches that of the solution to (5.1), frame-based LDCs assume that

\[ \mathbf{X}^H \mathbf{X} = \frac{\tau}{M} \mathbf{I}_M. \]  

(5.3)

The intuitive link between (5.3) and (5.2) is that (5.2) requires the selection of \( N_T \tau \) orthonormal column vectors of dimension \( N_T \tau \times 1 \). To satisfy (5.3), the best we can do is choose \( N_T \tau \) orthonormal row vectors of dimension \( 1 \times M \). Both solutions require the selection of an orthonormal basis.

Frame-based codes designed according to (5.3) have been shown to have strong capacity performance on average. Results reported in [39, Fig. 3] claim that capacity performance is predictable as a function of SNR and is “nearly optimal”. However, we point out several important aspects of this design which are not favorable. First, the design in (5.3) makes no explicit attempt to approximate the truly capacity optimal solution given in (5.2). No guarantee is made that the frames achieve capacity optimality, and in the following discussion we will demonstrate they do not. Secondly, since these codes are not guaranteed to be capacity optimal, there is variability among different selections of frames \( \mathbf{X} \) in terms of the spectral efficiency of the MIMO channel. Therefore, it is possible to select two different precoders, \( \mathbf{X}_1 \) and \( \mathbf{X}_2 \), such that (5.3) is satisfied but different capacities are achieved. We refer to this effect as “frame variability”. Thirdly, we demonstrate that the frame variability is significantly understated by the results in [39], where it is claimed that the frame variability is a negligible factor in frame-based LDC capacity performance. Indeed, we will show that frame variability can degrade spectral efficiency significantly depending upon system configuration.

5.1.1 No Guarantee of Capacity Optimality

Frame-based LDCs make no guarantee of capacity optimality, since the design criterion (5.3) is not based upon an optimization but rather upon the seemingly arbitrary selection
\( \mathcal{X}^H \mathcal{X} = \frac{\tau}{M} \mathbf{I}_M \). However, this frame-based selection is not truly arbitrary. Using the theory of Sections 4.2.2 and 4.2.3, we have already shown that the closest approximation of a matrix, in the Frobenius norm sense, subject to a trace power constraint and a maximum rank constraint, is given by the result in Theorem 8. The highlights of our observations are as follows. Let a rank \( \rho \) matrix \( \mathbf{A} \) have singular value decomposition \( \mathbf{U} \mathbf{D} \mathbf{V}^H \). We assume the singular values, and hence the associated singular vectors, are arranged in nonincreasing order on the diagonal. We make the following claims:

- The rank \( \rho_1 < \rho \) matrix \( \mathbf{A}_1 \) which minimizes the distance \( \| \mathbf{A} - \mathbf{A}_1 \|_F \) is given as
  \( \mathbf{A}_1 = \mathbf{U} \mathbf{D}_1 \mathbf{V}^H \), where \( \mathbf{D}_1 \) contains the \( \rho_1 \) largest singular values of \( \mathbf{A} \) on the diagonal, and the remaining diagonal elements are zero (SVD truncation).

- The rank \( \rho_1 < \rho \) matrix \( \mathbf{A}_2 \) which minimizes the distance \( \| \mathbf{A} - \mathbf{A}_2 \|_F \), subject to the constraint that the power in \( \mathbf{A}_2 \) is equal to the power in \( \mathbf{A}_1 \), can be solved by utilizing a two step process. First, compute the minimum distance estimate via SVD truncation. Secondly, compute the power difference between the SVD truncated estimate and \( \mathbf{A} \) and add this power in equally among all the active subchannels.

The frame-based code of (5.3) satisfies a minimum distance metric in relation to the optimal precoding vector. Recall that the full rank optimal code for LDCs is a scaled \( N_{T\tau} \times N_{T\tau} \) identity matrix, \( \frac{1}{N_{T\tau}} \mathbf{I}_{N_{T\tau}} \). In the rank deficient case \( M < N_{T\tau} \), frames assume \( \mathcal{X}^H \mathcal{X} = \frac{\tau}{M} \mathbf{I}_M \). We endeavor to compute a rank \( M < N_{T\tau} \) approximation \( \mathbf{A}_1 \) to the \( N_{T\tau} \times N_{T\tau} \) identity matrix such that \( \| \frac{1}{N_{T\tau}} \mathbf{I}_{N_{T\tau}} - \mathbf{A}_1 \|_F \) is minimized, subject to an equal power constraint. From the above discussion, we follow a two-part process. First, decompose the identity matrix into its singular values, such that \( \frac{1}{N_{T\tau}} \mathbf{I}_{N_{T\tau}} = \frac{1}{N_{T\tau}} \mathbf{D} \mathbf{U}^H \), where \( \mathbf{D} \) is an identity matrix and \( \mathbf{U} \) is an arbitrary unitary matrix. We then select \( M \) of the singular vectors in \( \mathbf{U} \) and set the remaining \( N_{T\tau} - M \) singular values in \( \mathbf{D} \) to zero. Denote the \( N_{T\tau} \times M \) matrix of utilized singular vectors as \( \mathbf{U}_1 \). After adding in the power lost due to SVD truncation, we obtain
\( \mathbf{x} \mathbf{x}^H = \frac{\tau}{M} \mathbf{U}_1^H \mathbf{U}_1 \). By reversing the order, we see that

\[
\mathbf{x}^H \mathbf{x} = \frac{\tau}{M} \mathbf{U}_1^H \mathbf{U}_1 = \frac{\tau}{M} \mathbf{I}_M .
\]

(5.4)

We can conclude that frame-based LDCs, as proposed in [39], satisfy an equal-power, rank-deficient minimum distance relationship to the full rank, capacity optimal precoder. This motivates the design (5.3) and establishes that frame-based designs are not arbitrary but are intended to satisfy a minimum distance constraint. This relationship has not been pointed out heretofore. Still, this is no guarantee of capacity optimality, since a Frobenius norm relationship does not guarantee spectral efficiency.

### 5.1.2 Frame Variability

From the previous discussion, (5.4) reveals that the matrix product \( \mathbf{U}_1^H \mathbf{U}_1 \) yields an identity matrix. However, the matrix product \( \mathbf{U}_1 \mathbf{U}_1^H \) is non-unique. Provided that \( \mathbf{U}_1 \) satisfies (5.4), any selection of \( M \) orthonormal vectors of dimension \( N_T \tau \times 1 \) will suffice to form a tight frame, but this non-uniqueness of \( \mathbf{U}_1 \) can lead to significant performance variation in terms of spectral efficiency, since the capacity expression in (5.1) depends upon \( \mathbf{x} \mathbf{x}^H \), not \( \mathbf{x}^H \mathbf{x} \). We refer to this phenomenon as frame variability.

Previous authors have claimed that frame variability has a minimal impact upon LDC capacity [39, Fig. 3]. Existing research has made this claim [39] after conducting 100 Monte Carlo simulations choosing random realizations for \( \mathbf{U}_1 \), and resulting capacity plots have shown that the capacity is relatively stable as a function of \( \mathbf{U}_1 \). However, these simulations are hardly conclusive, since 100 Monte Carlo simulations do not provide a large sample size. Furthermore, selecting random realizations for \( \mathbf{U}_1 \) without hand-selecting several simple test cases is poor design practice. We will show that previously reported capacity results were unduly optimistic because the randomness of the selection of \( \mathbf{U}_1 \) made the probability of choosing a low-capacity frame-based code unlikely, but not statistically negligible. Provided
that we can show explicitly that a simple test case can severely restrict the capacity of the frame-based LDC structure in [39] for certain architectures, we can validate our claim that existing frame-based LDCs contain an inherent design flaw.

For Fig. 5.1, we consider a simple $2 \times 2$ MIMO channel with no transmit or receive correlation. Assume the LDC will span $\tau = 3$ symbol periods and that we wish to transmit $M = 2$ substreams. In Fig. 5.1, we show the result of 50 Monte Carlo simulations revealing the capacity variation if the frame codes are selected randomly. The solid lines represent the frame variability over 50 random realizations. Notice that the frame variability only causes a 0.3 bps/Hz variability at high SNR. However, as a simple test case, let us now select $\mathbf{X}$...
such that

$$\mathbf{X} = \sqrt{\frac{3}{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This test case is composed of two scaled elementary vectors, and the frame is a scaled, truncated identity matrix. Direct substitution shows that $\mathbf{X}^H \mathbf{X} = \frac{r}{M} \mathbf{I}_M$, and hence $\mathbf{X}$ is the simplest frame which satisfies (5.3). This elementary test case produces a spectral efficiency plot in Fig. 5.1 which is noticeably lower than the randomly generated frames. At high SNR, the elementary test case has a spectral efficiency which is 0.5 bps/Hz less than the upper bound of the randomly generated frames. For a $2 \times 2$ system, this constitutes a significant capacity loss.

If we utilize the same $2 \times 2$ channel but this time select $M = 3$, we show in Fig. 5.2 that the effect of the frame variability is still noticeable, even as the matrix product $\mathbf{X}^H \mathbf{X}$ is closer to being the ideal scaled identity matrix. We use the following elementary test matrix:

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Fig. 5.2 clearly shows that a capacity loss on the order of 0.4 bps/Hz occurs when using the scaled elementary vectors as compared to the best case of the randomly generated frames.

The effect of frame variability can be seen even more clearly for larger dimensioned
Figure 5.2: Illustration of Frame Variability for $2 \times 2$ system, with $M = 3$, $\tau = 3$. 
Figure 5.3: Illustration of Frame Variability for $3 \times 3$ system, with $M = 3$, $\tau = 3$.

Consider Fig. 5.3, in which a $3 \times 3$ system is simulated using $\tau = 3$ and $M = 3$. As before, we plot the results of 50 Monte Carlo simulations on the same plot with the results of an elementary vector system, where we define

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^H.$$

As is evident from Fig. 5.3, a capacity loss of almost 1 bps/Hz can occur due to frame variability when compared to the best performing randomly selected frame LDCs. We comment that the loss of 1 bps/Hz in a $3 \times 3$ MIMO system is quite significant, since it represents a
non-negligible loss of the multiplexing advantage gained through the use of MIMO.

A similar definition of a frame comprised of scaled elementary vectors for a $3 \times 3$ system with $\tau = 3$ and $M = 4$ reveals in Fig. 5.4 that the loss of capacity due to frame deficiency is noticeable even for larger values of $M$. For high SNR, a loss of 0.5 bps/Hz is observed due to frame variability when compared to the best performing randomly selected frame-based code. This is important to observe, since as $M$ increases in value the matrix product $\mathbf{x}\mathbf{x}^H$ gets closer, in the Frobenius norm sense, to the optimal scaled identity matrix. Thus despite the reduced distance, frame variability can still significantly reduce system capacity.

As a final justification for the insufficiency of the frame constraint to yield predictably high spectral efficiency, consider Figs. 5.5 and 5.6. These plots reveal that frame variability
Figure 5.5: Illustration of Frame Variability for $5 \times 5$ system, with $M = 5$, $\tau = 3$. 
is a problem for larger antenna configurations than just the $2 \times 2$ and $3 \times 3$ cases we have been discussing. Fig. 5.5 presents simulation results for a $5 \times 5$ system with $\tau = 3$ and $M = 5$. If we establish $\mathbf{x}$ to be a scaled $15 \times 5$ tall matrix of scaled elementary vectors, we see that the lower bound reference curve lags the randomly generated frame-based LDCs by 1.5 bps/Hz. Similarly, Fig. 5.6 presents simulation results for a $7 \times 7$ system with $\tau = 3$ and $M = 8$. For this case, frame variability reduces the spectral efficiency of the code by 2 bps/Hz for high SNR. It appears that the trend is for the reduction in spectral efficiency to grow with the increasing MIMO system dimension.
A Final Comment

In the above discussion, we have shown that it is possible to generate by hand a degenerate case in which the capacity is significantly reduced by utilizing elementary column vectors in the formation of $\mathbf{X}$. A legitimate counter-argument is that in many cases, the probability of Heath’s method generating such an $\mathbf{X}$ in [39] in extremely remote, especially for larger dimensions of $N_{\tau T}$ and $M$. However, we maintain that there is a continuum of rank-deficient solutions lower bounded by the elementary vector case. It is easy to generate an initial sample $\mathbf{X}$ consisting of elementary column vectors, and then add an arbitrarily scaled noise matrix. After adjusting the resultant $\mathbf{X}$ using Gram-Schmidt methods to guarantee orthogonality and proper scaling, it can be shown that there are many solutions whose performance curves reside “close” to the elementary vector lower bound. The degree of proximity between the elementary lower bound and the noise-added $\mathbf{X}$ is a function of the strength of the noise added. We note that while the probability that Heath’s method will result in an $\mathbf{X}$ consisting of pure elementary vectors is remote, the probability that Heath’s method will result in an $\mathbf{X}$ with a dominant entry in each column is greater. From our simulations we have observed that the key factor in determining how close an $\mathbf{X}$ is to the lower bound depends on whether each column of $\mathbf{X}$ has a single, dominant entry, while the remaining entries are small in norm.

5.1.3 Proposed Remedy

The objective of the foregoing simulations is to establish that frame variability is a problem which needs to be addressed in the frame-based LDC literature. Although previously published simulation results have not reported the conclusions reached in this section, the fact that poorly performing frame-based LDCs can be generated by hand so easily indicates that the designs of [31,39,40] need to be modified to reduce the amount of frame variability.

The method proposed in this chapter to remedy the frame variability is to add an addi-
tional constraint. The only constraint in [39] is that $X^H X = \frac{\tau}{M} I_M$. As already discussed, this places no restrictions on $XX^H$. Observe that when the frames constructed of elementary vectors are utilized, the form of $XX^H$ is that of a square $N_T \times N_T$ matrix, of which $M$ diagonal entries are $\frac{\tau}{M}$, and the remaining entries are all zeros. All the off-diagonal elements are zeros. In contrast, notice that the ideal matrix product is $\frac{1}{N_T} I_{N_T}$, which has identical entries on all the diagonal elements and zeros on the off-diagonal.

We propose to modify the frame-based LDC design of [39] in the following way. First, we determine that the frame relationship $X^H X = \frac{\tau}{M} I_M$ must hold true to preserve the minimum distance properties. Secondly, we impose the structural constraint that all the diagonal elements of $XX^H$ must be identical and nonzero, and the off-diagonal entries of $XX^H$ must be minimized. In this way, we guarantee that in addition to obtaining a matrix product $XX^H$ which is minimally distant from $\frac{1}{N_T} I_{N_T}$ in the Frobenius norm sense, we also guarantee that the structure of $XX^H$ resembles the structure of $\frac{1}{N_T} I_{N_T}$.

We therefore endeavor to design a precoding LDC matrix $X$ such that the following properties are satisfied by $X$:

- $X^H X = \frac{\tau}{M} I_M$
- $[XX^H]_{ii} = a$, for all $i = 1, 2, \ldots, N_T$, where $a \neq 0$ and $a$ is real
- $[XX^H]_{ij} = b_{ij}$, for $i \neq j$, where $b_{ij}$ is minimized

A tall matrix $X$ which satisfies the above requirements is known as a Grassmannian Frame [52]. Computation of Grassmannian frames has been shown to be very complicated [145]. In the following sections, we present the basics of Grassmannian Frames and Packings and we propose a method of computing equiangular frames and Grassmannian frames using a procedure known as alternating projections.

Once the optimal equiangular or Grassmannian frame, $X$, has been computed, BER minimization can still occur through use of the methods of Section 3.3. Since these methods
are unaltered as a result of $XX^H$ being fixed instead of being random, we do not present
the BER optimization again in this chapter.

5.2 Theory of Grassmannian Frames and Packings in
Grassmannian Manifolds

To establish the theoretical framework for imposing an additional structural constraint
to the LDC as suggested in Section 5.1.3, we first need to lay the basic groundwork for the
analysis of frame systems. To do this, we present some basics of matrix frame theory and
some of the elementary theory on matrix manifolds from which frame theory is derived. We
present only a thumbnail sketch of the tools needed to solve our problem, and for a more
detailed literature survey the reader is referred to [52,145–150].

5.2.1 Preliminaries

Consider the finite dimensional Hilbert Space $\mathbb{C}^n$ defined by the $n \times 1$ vector space of
complex numbers. As a Hilbert Space, $\mathbb{C}^n$ consists of all $n \times 1$ vectors of complex numbers.
By definition of finite dimensional Hilbert Spaces [41], any two column vectors $x$ and $y$ in
$\mathbb{C}^n$ have an inner product given by

\[
\langle x, y \rangle = x^T y^* .
\]

The norm of a vector in the Hilbert Space $x$ is given as

\[
\|x\| = \sqrt{\langle x, x \rangle} .
\]

Although we have restricted our attention to Hilbert vector spaces, other types of Hilbert
spaces are also possible, including spaces of functions.
Consider a collection of vectors in the Hilbert space \(\mathbb{C}^n\), \(\{x_k\}_{k \in H}\), where \(H\) is a countable index set. The set \(\{x_k\}_{k \in H}\) is defined to be a frame for \(\mathbb{C}^n\) if there exist positive constants, or frame bounds, \(A\) and \(B\), such that

\[
A\|x\|_2^2 \leq \sum_{k \in H} |\langle x, x_k \rangle|^2 \leq B\|x\|_2^2 ,
\]

(5.5)

for every \(x\) in \(\mathbb{C}^n\). As opposed to orthonormal bases, which utilize the minimum number of linearly independent vectors to span a given subspace, frames are useful for overcomplete representations of subspaces. A tight frame is a frame for which the two frame bounds \(A\) and \(B\) are equal, that is,

\[
\sum_{k \in H} |\langle x, x_k \rangle|^2 = A\|x\|_2^2 .
\]

(5.6)

for all \(x \in \mathbb{C}^n\).

We will demonstrate by means of a simple example that frames do not constitute an orthonormal basis but rather an overcomplete representation of the subspace spanned by an orthonormal basis. Since tight frames form an overcomplete representation of a subspace, we can identify \(A\) as the redundancy ratio, which indicates the degree to which the frame set is overcomplete. If the redundancy ratio \(A\) is 1, then the frame set is equal to an orthonormal set. In general, we have \(A \geq 1\).

A Simple Example of Frames

We reproduce a simple example of frames first presented in [41, pp. 56-57]. Consider the 2-dimensional real vector space, and let us denote three vectors in that space as \(e_1 = (0, 1)\), \(e_2 = (-\sqrt{3}/2, -1/2)\), and \(e_3 = (\sqrt{3}/2, -1/2)\). Denote any random vector \(x\) as \((x_1, x_2)\). Performing the mathematical operation in (5.5), we compute

\[
\sum_{j=1}^{3} |\langle x, e_j \rangle|^2 = \frac{3}{2} \left[ |x_1|^2 + |x_2|^2 \right] = \frac{3}{2}\|x\|^2 .
\]
Thus these three vectors constitute a tight frame with a frame bound $A = B = \frac{3}{2}$. Therefore, the redundancy of the frame is $\frac{3}{2}$. These three vectors are shown in Fig. 5.7. The sense in which tight frames permit an overcomplete representation of a signal can be explained as follows. Assume that, for all $\mathbf{x} \in \mathbb{R}^2$, we have (5.6) satisfied. Then, through polarization identities, we can say that
$$
\mathbf{x} = A^{-1} \sum_j <\mathbf{x}, \mathbf{x}_j> \mathbf{x}_j .
$$

This is very similar to the representation of an arbitrary vector $\mathbf{x}$ by an orthonormal basis, but here the frame set $\{\mathbf{x}_j\}$ is an overcomplete set, that is, there are more frame vectors than the minimum needed to span the subspace.

### 5.2.2 Manifolds and Packings in Spaces

We can utilize the concepts of Section 5.2.1 to define two related quantities, the Stiefel Manifold and the Grassmann Manifold. The definitions of these quantities differ slightly in notation from that of the literature in order to accommodate the dimensions of the equivalent
channel $\mathcal{HX}$ to be considered shortly. The *Stiefel Manifold* $S(a, b)$ for $a \geq b$ is defined as the set of all unitary $a \times b$ matrices [151]. Mathematically,

$$
S(\tau, M) = \{ X \in \mathbb{C}^{a \times b} : X^H X = I_b \}.
$$

(5.8)

In (5.8), $S(a, b)$ is a “tall” matrix whose columns are orthonormal. It is easily shown that Stiefel Manifold matrices consist of $b$ column vectors which comprise a tight frame according to the definition of (5.6). Therefore, Stiefel Manifolds and tight frames are very closely related, since the columns of a Stiefel Manifold form a tight frame. For the remainder of this paper, we use the terms “Stiefel Manifold” and “tight frame” interchangeably.

The *Gram Matrix* of a frame is defined as the outer product of the frame $X$, such that

$$
G = XX^H,
$$

(5.9)

where the outer product $XX^H$ is an $a \times a$ matrix of rank $b$. We remark that $XX^H \neq I_a$, whereas $X^H X = I_b$. The diagonal elements of a Gram matrix are the squared norms of the frame vectors, and the off-diagonal elements of $G$ are the inner products among the frame vectors [52].

Given a Stiefel Manifold, a *Grassmannian Manifold* can be defined by identifying an equivalence relation on the Stiefel Manifold. Two elements $P$ and $Q$ in $S(a, b)$ are equivalent if the $b$-dimensional column vectors span the same subspace, or equivalently, if $P = QU$ for some unitary matrix $U$. The Grassmannian manifold $G(a, b)$ is the quotient space of $S(a, b)$ with respect to this equivalence relation. It can be shown that $G(a, b)$ is the set of all $b$-dimensional subspaces of $\mathbb{C}^a$.

The Grassmannian Manifold is not equivalent to a Stiefel Manifold. The Stiefel Manifold consists of $b$ orthonormal column vectors of dimension $a \times 1$, where $a > b$. In matrix notation, Stiefel Manifolds take the form of a tall, unitary matrix (a tight frame). In contrast, Grassmannian Manifolds can be represented by matrices of any dimension, such as a square.
matrix of dimension $a \times a$. For example, a square matrix $Y$ of dimension $a \times a$ but with rank $b$ has columns (and rows) which comprise a Grassmannian manifold, since the columns of $Y$ form a $b$-dimensional subspace of $\mathbb{C}^a$.

The terms defined above - frames, tight frames, Stiefel Manifolds, and Grassmannian Manifolds - are useful in solving packing problems. Packing problems involve an optimal placement of a finite set of elements into a space in such a way that a distance metric is maximized. For example, define a finite set of complex vectors $X$ of dimension $N$. The complex vector space $\mathbb{C}^N$ with a given distance function $d(\cdot)$ defined on the space $\mathbb{C}^N$ has a packing function $p(\cdot)$ given by [145]

$$p(X) = \min_{m \neq n} d(x_m, x_n) . \tag{5.10}$$

The type of distance function $d(\cdot)$ is a choice of the designer. Many alternatives exist, including geodesic, chordal, and spectral distance measures [150]. Our work focuses on spectral distance measures.

An optimal packing of the type shown in (5.10) can be defined by choosing a set of complex vectors $X$ with cardinality $b$. Then the optimal packing for these $b$ points is given by

$$\max_{|X|=b} p(X) , \tag{5.11}$$

where $|\cdot|$ is the cardinality of the argument. Intuitively, we are maximizing the minimum distance between two points on a sphere.

One packing problem of particular interest for our application is that of sphere packings of all one-dimensional subspaces of a vector space [52]. The sphere packing problem is defined mathematically as follows.
Sphere Packing Problem

The sphere packing problem is the problem of packing points on the sphere in $\mathbb{C}^N$ to maximize the minimum distance among the points. If a point $P$ is in the packing, then the point $-P$ is also in the packing. This is known as a spherical antipodal code [152]. Since this problem is a point packing problem, this is equivalent to choosing the Grassmannian Manifold $G(N,1)$, which defines an antipodal point pair on a sphere, and minimizing the distance among $M$ points in $G(N,1)$ in order to minimize a distance metric.

5.2.3 Important Classifications of Frames Relevant to LDCs

Within frame theory, there are several classifications of frames which are important to examine. We begin by examining Welch-Bound Inequality Sequences.

Welch-Bound Inequality Sequences

A frame given by
\[ X \in \mathbb{C}^{a \times b} : X^H X = I_b , \]  
with $a > b$, is also known as a Welch-Bound Equality Sequence (WBE) [153]. Matrices $X$ of the form (5.12) have been characterized by the lower bound on the sum of the squared cross correlations of the columns of $X$. This lower bound, first derived by Welch in [154], is called the Welch Bound and states the following:
\[ \text{tr} (X^H X)^2 \geq \frac{a^2}{b} . \]  
(5.13)

Welch showed that any matrix $X$ of the form (5.12) satisfies the lower bound (5.13) with equality, and hence $X$ is called a Welch-Bound Equality Sequence. WBEs are unit-norm tight frames. Since WBEs satisfy (5.13) with equality, we see that WBEs minimize the total
squared correlation present in the system.

Equiangular Tight Frames

A subclass of WBEs is known as equiangular tight frames. An equiangular tight frame is a tall, unit-norm tight frame of the form (5.12) wherein each row of $X$ has an identical inner product. Mathematically, if we designate the $a$ rows of $X$ as $\{x_1, x_2, \ldots, x_a\}$, then we have

$$|<x_k, x_m>| = c, \quad k \neq m$$

(5.14)

where $c$ is some positive constant and $k \in \{1, 2, \ldots, a\}$ and $m \in \{1, 2, \ldots, a\}$. Equiangular tight frames can be viewed as having a constant interference among all $a$ rows of the frame $X$. Utilizing the terminology of Section 5.2.2, we can say that the Gram Matrix of an equiangular tight frame is given by

$$\text{abs}(XX^H) = \begin{bmatrix}
1 & c & c & \ldots & c \\
c & 1 & c & \ldots & c \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
c & c & c & \ldots & 1
\end{bmatrix},$$

(5.15)

where $\text{abs}(XX^H)$ denotes the magnitude of each matrix entry in $XX^H$. Each of the $b$ vectors in the $a \times 1$-dimensional subspace which satisfy (5.15) are equally spaced in the $a \times 1$-dimensional subspace. The frames illustrated in Fig. 5.7 are equiangular, since the three vectors are all spaced at $\frac{2\pi}{3}$ radians with respect to each other.

This interference invariance property is particularly useful in CDMA signature sequence design. CDMA signatures are used as “keys” to differentiate among different users in a multiple access communication system. The chief property desired for these signatures is that any two users’ signatures have zero cross correlation, and any given signature has unit norm. In practical systems where the length of the signature sequence is less than the
number of users in the system, there are limitations on how many ideal CDMA signatures can be designed. When there are too many users to guarantee zero cross correlation among all signatures, equiangular tight frames can be used to design signatures with unit norm and known cross correlation. This use of equiangular frames has been explored in [155] and [156].

Establishing the existence of equiangular tight frames is problematic. It has been shown that for real-valued equiangular frames, it is necessary that the dimensions of the matrix $X$ must satisfy $a > b$ and $a \leq \frac{1}{2}b(b + 1)$, and for complex-valued $X$ the dimensions must satisfy $a \leq b^2$ [52]. However, these are necessary conditions only, and while other necessary conditions have been computed [157], no sufficient conditions have been derived. As a result, it is very difficult to determine with certainty whether equiangular tight frames exist for a given set of dimensions $a$ and $b$ of $X$.

An interesting subproblem in (5.14) is to generate a tight frame with minimal cross correlation $c$ among the rows of $X$. From [52, 155], it is shown that minimizing the cross correlation in (5.14) is equivalent to maximizing the packing radius of $a$ lines in a $(b - 1)$-dimensional space such that (5.15) is satisfied. This is equivalent to the sphere packing problem referenced in Section 5.2.2. In minimizing $c$, it is necessary to maximize a spectral distance constraint. Intuitively, this is equivalent to maximizing the distance among $a$ points located on a unit sphere in the space $\mathbb{C}^b$. More detailed definitions, including descriptions of the principal angles between any given two vectors, is given in [145, pp.149-150] and [150]. For our purposes, it is sufficient to state the general result that the maximal packing radius, $\rho$, for this problem is given by

$$\rho^2 \leq \frac{(b - 1)a}{b(a - 1)}.$$  (5.16)

In addition to (5.16), it has been shown that this maximal packing radius of $\rho$ translates into a minimum correlation $c$ among the rows of $X$ given by

$$c = \sqrt{1 - \rho^2} \geq \sqrt{\frac{a - b}{b(a - 1)}}.$$  (5.17)
Equiangular tight frames are a subclass of Grassmannian frames, which are defined shortly. The key result regarding equiangular tight frames, obtained in many places in the mathematics literature (see for example [52, 146, 155]), is that equiangular tight frames given by (5.15) satisfy (5.17) with equality, that is, equiangular tight frames achieve the minimum correlation possible among the rows of $X$. Frame matrices $X$ designed to satisfy (5.16) have been shown to be optimal packings in Grassmannian manifolds [52].

**Grassmannian Frames**

Grassmannian frames are a type of unit-norm tight frame in which the maximum cross correlation in (5.14) is minimized. Mathematically, Grassmannian frames $X$ are characterized by the following properties:

- The diagonal elements of the Gram Matrix $XX^H$ are unity, i.e., $[XX^H]_{ii} = 1$.
- The maximum of the off-diagonal elements of the Gram Matrix are minimized, i.e.,

  $$
  \max_{i,j} [XX^H]_{ij} \leq c_{\max},
  $$

  for $i \neq j$ where $c_{\max}$ is the lower bound given by (5.17). Grassmannian frames involve a max-min design procedure.

It is important to note that Grassmannian frames ensure that the maximum cross correlation among rows of $X$ is minimized. However, in general, rows of $X$ are not guaranteed to have equal cross correlation, that is, for general Grassmannian frames we have

$$
| \langle x_k, x_m \rangle | \neq | \langle x_k, x_n \rangle |
$$

for $m \neq n$. 

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5.3 Application of Equiangular Tight Frames to LDCs

In the context of LDCs, we have shown that the matrix constraint placed on $\mathbf{X}^H\mathbf{X}$ is given by (5.4), which determines a tight-frame structure for $\mathbf{X}$. However, as we have shown in Section 5.1.1, the frame constraint gives rise to frame variability in terms of calculating MIMO capacity, that is, there are an infinite number of $\mathbf{X}$ matrices which satisfy (5.4), and this non-uniqueness gives rise to a continuum of capacity values. Some $\mathbf{X}$ matrices yield severely deficient capacity, as demonstrated in Figs. 5.1-5.6. Furthermore, we have illustrated that the matrix products $\mathbf{X}\mathbf{X}^H$ which give severe rank deficiency can share a characteristic elementary structure shown in Section 5.1.2.

We propose to reduce the possibility of severely rank-deficient codes by imposing a structural constraint on the matrix product $\mathbf{X}\mathbf{X}^H$. We design $\mathbf{X}$ to be an equiangular tight frame such that the Gram Matrix, $\mathbf{X}\mathbf{X}^H$, is either of the form (5.15) or closely approximates that form. As such, the cross-correlations among the rows of $\mathbf{X}$ are minimized, and the angles among the $N_T\tau \times 1$ vectors in the $M$-dimensional subspace are maximized. This design choice precludes the possibility that $\mathbf{X}$ be composed of elementary vectors, which led to the most extreme rank deficiency explored in Section 5.1.2. However, we will show that the method used to generate the frames which approximate the form of (5.15) can experience convergence issues which may affect the ability of the code to reduce frame variability. Furthermore, frame variability cannot be eliminated because we cannot guarantee the uniqueness of an equiangular frame even when one exists. However, despite this non-uniqueness, we illustrate that equiangular frames can reduce frame variability significantly.

Mathematical intuition suggests that this equiangular design choice will improve spectral efficiency over the worst case capacity for several reasons. First, it removes the possibility of incurring a severe capacity penalty due to a gross structural deficiency in the precoder. Further, equiangular frames achieve an optimal subspace packing of $N_T\tau$ points on a sphere in $M$-dimensional space. This means that the rows of $\mathbf{X}$ provide an optimally spaced redundant
means of spanning an $M$-dimensional space.

In this section, we will use a mathematical technique known as the Method of Alternating Projections to obtain an approximation of an equiangular tight frame for use as an LDC. We will then evaluate the capacity of this equiangular frame-based LDC through simulation and compare it to the worst-case designs considered in Section 5.1.2.

5.3.1 The Method of Alternating Projections

The Method of Alternating Projections (MAPS) is a well-known technique also known as Projection Onto Convex Sets (POCS). In the context considered here, however, the class of tight frames does not constitute a convex set, so the designation MAPS is used instead to differentiate the methods of analysis due to the non-convexity. In this section, we provide only a brief sketch of the MAPS technique, since the details of implementation are not of concern to us here. For detailed discussions and analyses of MAPS, see [158, 159]. The presentation here is based upon that of Tropp [52].

MAPS relies on the use of two sets of matrices, $A$ and $B$, with identical dimensions. We assume $A$ contains matrices with a spectral constraint, that is, the eigenvalues of the set $A$ are restricted in some way. The set $B$ contains matrices with a structural constraint, that is, diagonal elements of the matrices of $B$ are restricted and/or the off-diagonal elements are restricted in value. The Method of Alternating Projections algorithm can be described as follows:

INPUT:
- An arbitrary matrix $A_0$ with appropriate dimensions
- The number of iterations, $D$

OUTPUT:
- A matrix $A_F$ in $A$ and a matrix $B_F$ in $B$. 

PROCEDURE:

1. Initialize $d = 0$.

2. Find a matrix $B_d$ in $\mathcal{B}$ such that

$$B_d \in \min_{B \in \mathcal{B}} \| B - A_d \|_F$$

3. Find a matrix $A_{d+1}$ in $\mathcal{A}$ such that

$$A_{d+1} \in \min_{A \in \mathcal{A}} \| A - B_d \|_F$$

4. Increment $d$ by one.

5. Repeat Steps 2-4 until $d = D$.

6. Let $A_F = A_d$ and $B_F = B_d$.

In this algorithm, steps 2 and 3 involve projection of one subspace onto another through a minimum distance calculation.

5.3.2 Convergence

It has been shown [52] that the method of alternating projections never increases the distance between successive iterates. This is obvious because each pass through the algorithm minimizes the distance between successive approximations. However, establishing strong convergence has proven difficult because of the non-convexity of the set of unit norm tight frames. We briefly present some elementary convergence results derived in the literature to establish the validity of the MAPS method for solving our problem. Since this analysis is not a novel contribution of this dissertation, we do not include detailed proofs of convergence but rather we simply highlight the key results. This presentation is based on [52].
The spaces $\mathcal{A}$ and $\mathcal{B}$ form a product space $\mathcal{A} \times \mathcal{B}$ which generates a set of all points $(A, B)$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Alternating projection utilizes a continuous function $f$ such that some distance metric on $(A, B)$ is minimized. Hence, $f : \mathcal{A} \times \mathcal{B} \to \mathbb{R}_+$. The function $f$ induces two mappings given by

$$M_A(B) = \min_{A \in \mathcal{A}} f(A, B),$$

$$M_B(A) = \min_{B \in \mathcal{B}} f(A, B).$$

The method of alternating projections generates a sequence of points $(A_d, B_d)$ via the rules

$$B_d \in M_B(A_d),$$

$$A_{d+1} \in M_A(B_d).$$

Because of this, it has been shown that if $\mathcal{A}$ and $\mathcal{B}$ are compact, then the MAPS algorithm is characterized by the following:

- The iterative sequence has at least one, and possibly many, limit points (also known as accumulation points). These accumulation points are termination points of the algorithm.

- Each accumulation point of the sequence lies in the product space $\mathcal{A} \times \mathcal{B}$.

- Each accumulation point $(A_F, B_F)$ satisfies

$$f(A_F, B_F) = \lim_{d} f(A_d, B_d).$$

Research in [52] has established the required compactness necessary for the algorithm to converge to an accumulation point. Convergence is geometric. However, no guarantees are made that the accumulation point reached by the algorithm is a Grassmannian frame or an equiangular tight frame. This is one of the weaknesses of the algorithm.
5.3.3 Identifying The Constraints of The Problem

In the context of the problem we are solving in this section, we determine that it is easiest to work with the Gram Matrices, $\mathbf{X}^H\mathbf{X}$, of the LDC matrix $\mathbf{X}$. Recognizing that we wish to impose the frame constraint (5.4), which states that $\mathbf{X}^H\mathbf{X} = \frac{\tau}{M}\mathbf{I}_M$, it is easy to see that all the eigenvalues of $\mathbf{X}^H\mathbf{X}$ are equal to $\frac{\tau}{M}$. From linear algebra, the nonzero eigenvalues of $\mathbf{XX}^H$ are also $\frac{\tau}{M}$. Therefore, we can impose a spectral constraint on the set $A$ which requires that the $N_T\tau$ eigenvalues of $\mathbf{XX}^H$ are $\{\frac{\tau}{M}, \frac{\tau}{M}, \ldots, \frac{\tau}{M}, 0, \ldots, 0\}$. However, in order to orient this discussion in terms of unit-norm tight frames as discussed in [52], we force the eigenvalues of $\mathbf{XX}^H$ to be $\{\frac{N_T\tau}{M}, \frac{N_T\tau}{M}, \ldots, \frac{N_T\tau}{M}, 0, \ldots, 0\}$. After the algorithm terminates, we re-scale the Gram Matrix such that $\mathbf{XX}^H$ has the desired eigenvalue set, $\{\frac{\tau}{M}, \frac{\tau}{M}, \ldots, \frac{\tau}{M}, 0, \ldots, 0\}$. The structural constraint imposed on set $B$ is that the frame be a Grassmannian frame with the maximum cross correlation among rows of $\mathbf{X}$ minimized. A matrix which belongs to the intersection of sets $A$ and $B$ is an equiangular frame of the form given by (5.15). Both sets $A$ and $B$ are of dimensions $N_T\tau \times N_T\tau$.

In summary, we make the following definitions:

- $A = \left\{ \mathbf{A} \in \mathbb{C}^{N_T\tau \times N_T\tau} : \lambda_\mathbf{A} = \begin{bmatrix} N_T\tau & N_T\tau & \ldots & N_T\tau \\ \frac{M}{M} & \frac{M}{M} & \ldots & \frac{M}{M} \\ \frac{N_T\tau - M}{M} & \frac{N_T\tau - M}{M} & \ldots & \frac{N_T\tau - M}{M} \end{bmatrix} \right\}$, where $\lambda_\mathbf{A}$ are the eigenvalues of the matrix $\mathbf{A}$.

- $B = \left\{ \mathbf{B} \in \mathbb{C}^{N_T\tau \times N_T\tau} : \mathbf{B} = \mathbf{B}^H, \text{diag}(\mathbf{B}) = 1, \max_{m \neq n} |b_{mn}| \leq c = \sqrt{\frac{N_T\tau - M}{M(N_T\tau - 1)}} \right\}$, where the constant $c$ is derived from (5.17) with the dimensional substitution $a = N_T\tau$ and $b = M$.

The Projection of Step 2

The projection of Step 2 requires the computation of the Gram Matrix of a Grassmannian tight frame closest in Frobenius norm to $\mathbf{A}_d$. Results in [52, Prop. 2] reveal that the matrix
\( B_d \) closest to \( A_d \), with maximum correlation \( c \) given by
\[
c = \sqrt{\frac{N_T \tau - M}{M(N_T \tau - 1)}} ,
\] (5.18)
can be computed as
\[
[B]_{ij} = b_{ij} = \begin{cases} a_{ij}, & |a_{ij}| \leq c \\ c e^{j \arg(a_{ij})}, & \text{otherwise} \end{cases} .
\] (5.19)

Proof can be reconstructed from the arguments of [52, 146].

The Projection of Step 3

The projection of Step 3 requires the computation of the closest matrix, in Frobenius norm distance, in set \( \mathcal{A} \) to a matrix in set \( \mathcal{B} \). Perform an eigendecomposition on the matrix \( B_d \in \mathcal{B} \) such that \( B = U \Lambda U^H \) with the diagonal entries of \( \Lambda \) arranged in nonincreasing order. Let \( U_d \) be the \( N_T \tau \times M \) matrix formed from the first \( M < N_T \tau \) columns of \( U \). Based upon [52, Thm. 3], the closest matrix \( A \in \mathcal{A} \) to a matrix \( B_d \in \mathcal{B} \) is \( N_T \tau M U_d U_d^H \). A complete proof is given in [52]. This closest matrix is unique if the \( M \)th eigenvalue of \( \Lambda \) is strictly greater than the \( (M + 1) \)st eigenvalue of \( \Lambda \).

After the algorithm terminates, the matrix \( \mathcal{X} \) must be re-scaled to reflect that the desired eigenvalue set of \( \mathcal{X} \mathcal{X}^H \) is \( \{ \frac{\tau}{M}, \frac{\tau}{M}, \ldots, \frac{\tau}{M}, 0, \ldots, 0 \} \).

5.3.4 Novelty, Dimensionality, and Simulation Dimensions

The purpose of the alternating projection system explained in Section 5.3.3 is to compute a Gram Matrix, \( \mathcal{X} \mathcal{X}^H \), such that the frame \( \mathcal{X} \) is equiangular and has the property that \( \mathcal{X}^H \mathcal{X} = \frac{\tau}{M} I_M \). As already mentioned, necessary and sufficient conditions for the dimensions of \( \mathcal{X} \) do not exist for establishing \textit{a priori} whether such a frame exists. Loose necessary conditions for complex frames have been established, such as \( N_T \tau \leq M^2 \) [146]. However,
in the context of a typical MIMO communication system, we cannot always guarantee that such a mathematical constraint will be satisfied.

Existing literature on alternating projections has focused on computing true equiangular frames or “nearly” equiangular frames which satisfy $N_T \tau \leq M^2$ [52,156]. We will not restrict ourselves to the case $N_T \tau \leq M^2$ in executing our simulations in order to test the ability of the alternating projection procedure to find a matrix “close” to an equiangular matrix, even though for $N_T \tau > M^2$ a true equiangular matrix cannot be found. In this way, we test whether it is possible to derive precoders $\mathbf{X}$ with desirable structural properties regardless of the dimensions of the system. In particular, we test the alternating projection algorithm to see if it is possible to compromise and obtain a nearly Grassmannian frame instead of an equiangular frame for the case in which $N_T \tau \geq M^2$. To our knowledge, this extension represents a novel application of the alternating projection algorithm, although the proof of convergence of the algorithm to a fixed point has already been computed for all cases of $M$ and $N_T \tau$ [52, Thm. 4].

Once the termination point of the Alternating Projection algorithm is reached, we have a Gram Matrix, $\mathbf{X} \mathbf{X}^H$. From this Gram Matrix, it is a simple matter of performing eigen-decomposition or a rank-revealing QR decomposition on $\mathbf{X} \mathbf{X}^H$ in order to find the optimal frame, $\mathbf{X}$. This $\mathbf{X}$ becomes the optimal precoder to be used in our formulation of the LDC.

We point out that the inability of the MAPS algorithm to always converge to an equiangular frame is problematic in terms of reducing the frame variability illustrated in Section 5.1.2. We will show that when MAPS converges to an equiangular frame, the frame variability is reduced significantly, but when MAPS does not converge, the performance improvement is compromised. MAPS may not converge for several reasons - the dimensions of the system may make it impossible to satisfy the Welch Bound, or perhaps necessary conditions for existence of an equiangular frame are not satisfied.

Additionally, we remark that even when MAPS converges to an equiangular frame, there is no guarantee that it will always converge to the same equiangular frame, since equiangular
frames are not unique. Therefore, we will see that a residual frame variability exists even when MAPS converges to an equiangular frame, and this frame variability results from the fact that different equiangular frames result in slightly different capacities, since the matrix product $XX^H$ is different. However, we show that the amount of frame variability is slight compared to the unmodified Heath case [39].

5.4 Simulation Results

Our simulations focus on establishing the theoretical predictions of Section 5.3 in two ways. First, we establish the usefulness of applying the equiangular tight frame theory to the LDC problem and we determine that use of equiangular tight frames can reduce the frame variability of Section 5.1.2, subject to the convergence of the MAPS algorithm. As such, we show that we can prevent capacity loss due to deficient structure of the matrix $XX^H$ for certain dimensions of precoders $X$. Second, we establish the usefulness of the method of alternating projections as a means of computing “quasi-Grassmannian frames” which do not meet the strict definition of Grassmannian frames but come “close” in a peak-overshoot sense which will be discussed below.

5.4.1 Reduction of Frame Variability

In Section 5.1.2 we demonstrated through six MIMO configurations in Figs. 5.1-5.6 that it is possible to utilize the frame-based LDC structure of [39] and see significant variation in the achieved capacity of the space-time code. This variability was due to the structure of the matrix product, $XX^H$. In the discussion of Section 5.1.2, we showed that by selecting $X$ to be a collection of elementary vectors, it was possible to reduce the capacity so significantly as to lose a substantial part of the benefit of the MIMO multiplexing structure. In Sections 5.1.3 and 5.3, we proposed to modify the LDC structure of [39] by forcing the matrix product
\(XX^H\) to have a specific structure specified by an equiangular tight frame constraint, the theory of which was expanded in Section 5.2. This structure preserved the original tight frame constraint proposed in [39] but removed the possibility of \(X\) being an elementary vector set. In this section, we establish the validity of the proposed remedy of Section 5.1.3 by analyzing several MIMO configurations.

We begin with a case where the MAPS algorithm converges to an equiangular frame. We select \(N_T = 2\), \(\tau = 2\), and \(M = 2\). We first present the result of the equiangular code resulting from the convergence of MAPS. In Fig. 5.8, it is clear that the use of the equiangular frame produces a frame variability that is very small. However, it is also clear that frame variability has not been eliminated. This is a result of the fact that MAPS converged to several different complex matrices over the course of 30 simulations, all of which satisfied the equiangular tight frame constraint. As a consequence, there is not a unique equiangular frame for these system dimensions, and hence frame variability cannot be eliminated entirely. Still, Fig. 5.8 illustrates that the amount of variability is very small.

We compare Fig. 5.8 with Fig. 5.9, in which 30 random frame realizations are generated according to [39]. Notice in Fig. 5.9 that the amount of frame variability is significantly increased because \(XX^H\) is not structurally constrained. Additionally, we remark that the lower bound, generated with elementary vectors, demonstrates an even more significant capacity variability. All of the equiangular frames prevent this degenerate case.

Now reconsider Fig. 5.1. Notice there is noticeable, but not large, frame variability in this figure for a system configuration of \(M = 2\), \(N_T = N_R = 2\), and \(\tau = 3\). For high SNR, as much as 0.3 bps/Hz is lost over 50 random realizations of \(X\) due to poor structure of the matrix product \(XX^H\). The worst-case elementary vector code suffers a full 0.5 bps/Hz penalty due to poor code structure. By applying the theory of Section 5.2 and using the MAPS algorithm, we attempted to derive an equiangular frame. However, we discovered that for this set of system parameters, MAPS does not converge to an equiangular frame, and as a result, it is possible to generate many different “nearly equiangular” frames which are close
Figure 5.8: Drastic reduction in frame variability due to use of equiangular frames. Here, $N_T = \tau = 2$, $M = 2$. 

\[ \text{Spectral Efficiency - Equiangular Frames} \]
Figure 5.9: Increased frame variability utilizing randomly generated tight frames, with elementary degenerate case as reference. $N_T = \tau = 2$, $M = 2$. 
Figure 5.10: Frame variability resulting from a set of 50 “nearly equiangular” tight frames output by MAPS. A comparison of this figure and Fig. 5.1 shows little improvement, although the added structural constraint precludes the possibility of the degenerate case, whereas Heath’s method does not. $N_T = N_R = 2$, $M = 3$, $\tau = 3$.

in structure to an equiangular frame but do not satisfy the definition exactly. Because MAPS does not converge to an equiangular frame, there are many possible non-unique convergence matrices for MAPS, depending on the initial condition fed to the algorithm.

Because of this non-uniqueness, there still appears to be frame variability in Fig. 5.10. In fact, there appears to be little difference between Figs. 5.1 and 5.10. However, we point out one significant difference. Fig. 5.1 was generated using only 50 random LDCs; as the plot suggests, even over 50 simulations it is possible to obtain a code which is relatively close to the lower bound generated by the elementary vectors. Over 5000 or 5 million such iterations,
the frame variability would be much more spread over the range indicated in Fig. 5.1. In
Fig. 5.10, on the other hand, despite the non-uniqueness of the solution generated by MAPS,
the similarity in structure between all the Gram Matrices generated by MAPS and the ideal
identity matrix prevents the possibility that the frame variability will extend as far as in
Heath’s case. As a result, we can conclude that even when MAPS does not converge to an
equiangular frame, the frame variability is reduced.

It is instructive to reflect on what the equiangular frames and nearly equiangular frames
have, and have not, achieved, to understand the novelty of the technique presented in this
chapter. We re-iterate the following observations:

- The use of an equiangular frame fixes the matrix product, $\mathbf{XX}^H$, to be a predictable,
  fixed matrix. This occurs only when MAPS converges to an equiangular frame, which
  only occurs for certain MIMO system configurations.

- Even when MAPS converges to an equiangular frame, the equiangular frame is not
  guaranteed to be unique, so frame variability cannot be eliminated entirely. However,
  when MAPS converges to equiangular frames, the frame variability has been shown to
  be drastically reduced.

- When MAPS does not converge to an equiangular frame, the resulting code is “close” to
  equiangular in structure, and identical in terms of spectral properties (eigenvalues). As
  a consequence, this technique can reduce frame variability even when an equiangular
  frame does not exist, albeit the amount of reduction is dependent upon the system
  parameters and cannot be reliably predicted.

- The equiangular frame achieves very good capacity performance when compared against
  randomly generated precoders $\mathbf{X}$. However, since no explicit optimization of (5.1) is
  performed, true capacity optimality is not guaranteed.
5.4.2 Quasi-Grassmannian Frames

In Section 5.3.4, we pointed out that equiangular tight frames only exist for certain configurations of $N_T$, $\tau$, and $M$. Utilizing the terminology of [146], it can be shown that a necessary condition for the existence of a complex-valued equiangular tight frame is that

$$N_T\tau \leq M^2.$$  \hspace{1cm} (5.20)

This necessary condition is problematic for LDC design for two reasons. First, this is a necessary condition only, and as illustrated in [156], there is no guarantee that an equiangular tight frame exists even when the conditions of (5.20) are met. Furthermore, the constraint of (5.20) is an artificial one in terms of MIMO configuration, since in practical systems it is possible to violate (5.20) by certain choices of $N_T$, $\tau$, and $M$. For example, by choosing $M = 2$, $N_T = 3$, and $\tau = 2$, we have clearly violated (5.20) with valid choices of LDC parameters. This motivates an investigation into whether the MAPS method of Section 5.3.1 can locate a matrix product $XX^H$ which is “Quasi-Grassmannian,” or close to a Grassmannian frame in structure. The results of our investigation are below.

We conducted simulations using MAPS to see if it was possible to generate “nearly equiangular,” or at the very least “nearly Grassmannian,” frames, for a variety of combinations of $N_T$, $\tau$, and $M$. We summarize our findings in the following table, with novel results designated with an asterisk (*):

As is obvious from Table 5.1, there are several results which can be considered novel. Previous research efforts have focused on finding close approximations to equiangular frames (see, for example, [156, Table I]), whereas in Table 5.1 we have not restricted our results to approximations of equiangular frames. For example, for a configuration of $N_T\tau = 6$ and $M = 2$, it is impossible to even approximate an equiangular frame because no such frame exists due to (5.20). However, we investigated to see what kind of solution the MAPS
algorithm converged to. Our result is given below:

\[
\mathbf{x} = \begin{bmatrix}
-0.2599 - j0.7181 & 0.5909 - j0.2599 \\
0.5909 + j0.2599 & 0.2599 - j0.7181 \\
-0.3675 - j0.0899 & 0.9257 \\
0.3403 - j0.2098 & 0.8205 - j0.4087 \\
0.9257 & 0.3675 - j0.0899 \\
-0.8205 - j0.4087 & 0.3403 + j0.2098
\end{bmatrix}.
\]  

(5.21)

From (5.20), it is apparent that the Welch Bound of (5.13) cannot be met, since we have \( M = 2 < N_{\tau} \tau = 6 \). However, the minimum correlation possible, if the Welch Bound were to be met, is given by \( c = 0.6325 \). Compare this to the outer product matrix \( \mathbf{x}\mathbf{x}^H \) given below:

\[
\text{abs}(\mathbf{x}\mathbf{x}^H) = \frac{1}{\sqrt{2}} \begin{bmatrix}
\sqrt{2} & 0 & 1 & 1 & 1 & 1 \\
0 & \sqrt{2} & 1 & 1 & 1 & 1 \\
1 & 1 & \sqrt{2} & 1 & 0 & 1 \\
1 & 1 & 1 & \sqrt{2} & 1 & 0 \\
1 & 1 & 0 & 1 & \sqrt{2} & 1 \\
1 & 1 & 1 & 0 & 1 & \sqrt{2}
\end{bmatrix}.
\]  

(5.22)

As is evident from (5.22), the frame in (5.21) bears no resemblance to an equiangular frame,
since each off-diagonal entry in $\mathbf{X}\mathbf{X}^H$ does not have equal magnitude. However, the maximum of the off-diagonal magnitudes in (5.22) is given by $c_{\min} = 0.7071$, which is about 12% greater than the absolute minimum dictated by the Welch Bound. This means that $\mathbf{X}$ given by (5.21) is "close" to being a Grassmannian frame. Therefore, in the case $M = 2$ and $N_T\tau = 6$, we have determined that it is possible to use MAPS to find frames which approach being Grassmannian with a peak overshoot of the Welch Bound by 12%.

As a final example of the usefulness of the MAPS technique for finding nearly Grassmannian frames, we consider the case $M = 2$ and $N_T\tau = 5$. Consider the following frame:

$$
\mathbf{X} = \begin{bmatrix}
0.9095 & 0.2080 + j0.3240 \\
0.8313 - j0.3615 & -0.3977 + j0.0631 \\
-0.3852 + j0.0866 & 0.9099 \\
-0.3955 + j0.0698 & -0.6040 - j0.6702 \\
0.1510 + j0.7149 & 0.6806 + j0.2910
\end{bmatrix}.
$$

(5.23)

The outer product matrix $\mathbf{X}\mathbf{X}^H$ of the frame in (5.23) is similar to (5.22) because the form is not that of an equiangular frame. Again, however, the output of the MAPS algorithm yields a matrix in (5.23) which is almost a frame (the inner product matrix $\mathbf{X}^H\mathbf{X}$ is almost a scaled identity), and the outer product has maximum cross correlation among rows of value $c = 0.7187$, whereas the Welch Bound dictates that a WBE sequence would have a maximum cross correlation of $c = 0.6124$. This constitutes a 17.4% overshoot.

The other results from Table 5.1 are redundant with those presented above. The key result we wish to stress is that it is possible to use the MAPS algorithm to generate quasi-equiequangular, or at least nearly Grassmannian, frames, even for cases where the dimensions preclude exact computation of a true equiangular frame. This has potential application for CDMA signature design, as suggested in [156]. In CDMA signature design, it is customary to have more users in the system than available codes to differentiate among users. Because of this, it is problematic to design good signature sequences such that each signature has equal
norm and different users do not interfere with each other. Signatures designed according to WBE sequence design guidelines, or “nearly Grassmannian” guidelines, place a limit on inter-user interference due to signature cross correlation. Because the dimensions of a given CDMA system may not meet the necessary restrictions to ensure existence of a WBE sequence, we have illustrated that the MAPS technique can be used to find frames which are almost Grassmannian, which is a reasonable compromise given that no sequence exists which satisfies the Welch Bound with equality.
Chapter 6

Conclusions and Future Research

The future of MIMO research is quite promising. Because MIMO antenna systems provide the promise of simultaneous data rate and error rate enhancement without the need for additional transmit power, MIMO will continue to be the focus of intense research in the coming years. In particular, more work will focus on bridging the gap between the extremes of increased capacity and reduced data rate through improved space-time code design. As processing speeds increase, the implementation restrictions for more complicated STC designs utilizing trellis codes and turbo codes will ease. Furthermore, the popularity of precoding will increase since it provides the transmitter with a means of orienting its transmission along the geometry of the wireless channel, which greatly improves all aspects of system performance. The insertion of precoding techniques into new wireless standards is dependent upon reliable means of quantizing the precoding vectors in order to reduce the volume of feedback needed to effect reliable transmission.

This dissertation makes a contribution to the state of the art in MIMO wireless system design by helping to bridge the gap between the competing design extremes of increased spectral efficiency and decreased error rate. Through the use of an established LDC precoding structure, we introduce a new space-time code which utilizes channel state information
to further increase capacity and reduce errors. We note restrictions on the dimensions of the system and explore all the ramifications of choosing the MIMO system parameters as design variables. Furthermore, we investigate ways to utilize mathematical theory on matrix structural constraints in order to guarantee that an existing LDC design delivers the capacity promised by the MIMO multiplexing structure. We outline the contributions of this thesis in a chapter-by-chapter summary below.

6.1 Chapter 2 Summary

The content of Chapter 2 provides an introduction to the basics of MIMO technology and reflects the result of well over a year and a half of personal, independent study. While it makes no claim to be comprehensive, it can serve as a primer to MIMO terminology for any student or professional interested in learning the basics of the motivations and advantages of MIMO. This chapter can serve as a “launching point” for someone desiring to begin a research project in MIMO.

Chapter 2 presents the basic explanation behind the capacity benefits of MIMO and explains the decomposition of a MIMO channel into independent subchannels. It illustrates the concept behind BLAST and explains why transmission across independent subchannels can increase the data rate without expanding the system bandwidth. It also presents the basic argument about why MIMO can be viewed as a combination of spatial and temporal diversity techniques to facilitate excellent BER. A summary of a broad class of space time codes is presented, with special emphasis on the most popular STC, the Alamouti structure, which is being implemented in next-generation MIMO standards. Chapter 2 was written in an effort to be readable by a general audience with specific technical expertise in wireless communications. Furthermore, this chapter is well referenced, which provides the reader with the opportunity to learn more about any given topic of interest.
6.2 Chapter 3 Summary and Novelty

In Chapter 3, two concepts were united in an attempt to improve both capacity and error rate of wireless systems. The linear dispersive STC structure was utilized in combination with channel covariance knowledge at the transmitter to form a new, precoded STC. By forcing the number of symbols, $M$, to be equal to the product of the number of transmitters and the time spread of the code, $N_T\tau$, we completely characterized this new STC. The novel contributions of this chapter are:

- A novel combination of the Kronecker-based covariance channel knowledge model and the LDC structure of [39] to form a new STC with improved ergodic capacity and error rate in comparison to non-precoded LDCs. This new code is both capacity optimal and SER-optimal for high SNR.
- A capacity analysis of this precoded LDC (P-LDC) which illustrates that the P-LDC is capacity-optimal.
- A description of “$\tau$-Beamforming” and illustration that the region of optimality for $\tau$-Beamforming is identical to that of standard beamforming.
- Identification and characterization of an irregularity in the BER plot of the P-LDC due to the need re-compute a new optimal precoder for each SNR value.
- Complete characterization of the new code through simulation.

6.3 Chapter 4 Summary and Novelty

Chapter 4 presents an extension of the ideas of Chapter 3. Whereas in Chapter 3 the design was restricted to have $M = N_T\tau$, reflecting a desire that the precoder should always be able to deliver the full capacity of the channel, Chapter 4 relaxed the design restrictions
so that the parameters $M$, $N_T$, and $\tau$ could be under the control of the system designer. We demonstrated that choosing $M < N_T \tau$ had consequences for the system capacity, and we demonstrated the nature of the capacity loss. In cases where $M < N_T \tau$, the precoder $\mathcal{X}$ becomes a tall matrix, which leads to a rank deficiency in the matrix product $\mathcal{X} \mathcal{X}^H$ for high enough SNR. The novel contributions of this chapter are:

- Relaxation of the design constraint in Chapter 3 that $M = N_T \tau$ and re-formulating the P-LDC code in terms of a tall matrix precoder, $\mathcal{X}$.
- Derivation of a new capacity formula reflecting suboptimal capacity if $M < N_T \tau$.
- Demonstration through simulation of the amount of capacity loss due to the rank deficiency in the precoder $\mathcal{X}$.
- Characterization of the distance relationships between the truly capacity-optimal solution of Chapter 3 and the suboptimal solution of Chapter 4.

### 6.4 Chapter 5 Summary and Novelty

Chapter 5 again focused on the LDC structure of [39], but this time in the absence of precoding. In this chapter, we identified a design flaw in the frame-based LDC structure of [39] which permits poorly structured precoders $\mathcal{X}$ to generate severely capacity-deficient precoders. As noted in the literature, frame-based LDCs are not capacity-optimal. We demonstrated in this chapter that the frame-based design of [39] satisfied a minimum distance Frobenius norm constraint between a rank-deficient code and the truly optimal code. However, we also demonstrated through extensive simulation that it is possible to observe considerable variation in the capacity performance of frame-based codes, a phenomenon which we term “frame variability.” We utilized the theory of Grassmannian frames and equiangular frames to impose an additional constraint on the frame-based codes of [39] in order to reduce the probability that the code would experience capacity-deficiency due to
poor structuring of the precoder $\mathbf{X}$. Furthermore, we utilized the method of alternating projections to solve for “nearly Grassmannian” frames, even when true equiangular frames are guaranteed not to exist. The novel contributions of this chapter are:

- Identification and illustration through simulation of a design flaw inherent in [39] which permits randomly generated LDC codes $\mathbf{X}$ to unexpectedly yield poor capacity performance.

- Identification of the cause of this design flaw, which is a structural problem in the formation of the frame matrix $\mathbf{X}$. The degenerate case of this design flaw occurs when $\mathbf{X}$ is comprised of all elementary vectors.

- Identification of a method to reduce the probability of the design flaw of [39] by utilizing the theory of equiangular frames. By imposing additional constraints on the frame-based $\mathbf{X}$, it was shown in this chapter that:
  
  - Frame variability can be reduced significantly, depending upon convergence properties of the MAPS technique.
  - Nearly capacity-optimal behavior occurs when MAPS converges to true equiangular frames in the LDC coding structure.
  - Use of true equiangular frames reduces the randomness inherent in randomly generated frames.

- Extension of the MAPS algorithm to solve for “nearly Grassmannian” frames, even when equiangular frames do not exist.

### 6.5 Directions For Future Research

The material in this thesis can serve to launch future research in some of the following ways. In Chapters 3 and 4, a method of uniting LDC and precoding theory was presented
in terms of the Kronecker-based covariance channel model. It is possible to also use channel mean knowledge at the transmitter to pre-align the transmissions along the geometry of the channel. Therefore, one possible extension of this work is to apply LDC theory in conjunction with channel mean transmitter channel knowledge. Further, the assumptions in these two chapters were that the channel had correlation properties at the transmitter and receiver which were decoupled. This is not always the case, as shown in Chapter 2. It is an open question whether the P-LDC structure could be extended to channels whose transmit and receive correlations are not decoupled.

The material in Chapter 5 deals with computation of Grassmannian frames and equiangular frames. The potential applications of equiangular frames and Grassmannian frames in the mathematical arena are too numerous to mention here. From a communication perspective, Grassmannian frames have been identified as having at least two significant areas of application which are worthy of further exploration. First, when designing precoders in MIMO systems, it is often desirable to quantize the precoding vectors in order to reduce the volume of information in the feedback channel. When performing such quantization, it is necessary to form an alphabet of prospective precoding vectors which is optimal in some way. In the terminology of this paper, Grassmannian frames can be viewed as a way of optimally placing a set of $N_T \tau$ points on an $M$-dimensional unit sphere. Because of this interpretation, Grassmannian frames have been identified as a potential way of designing quantized vector alphabets. A second application for equiangular and Grassmannian frames is in CDMA signature design. CDMA signatures need to be as nearly orthonormal to each other as possible in order to prevent interference among users. Since Grassmannian frames seek to find a frame with minimal cross correlation, and since equiangular frames share that goal but seek to force the interference among any two signatures to be constant, Grassmannian frames have been identified as a possible tool in the design of CDMA signatures.
Bibliography


