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I, Prasad Raman Joshi, hereby submit this as part of the requirements for the degree of:

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in:

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It is entitled:

An Elastic Contact Theory for Modeling Vibration

transmissibility through Rolling Element Bearings

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ABSTRACT

Rolling element bearings are a class of mechanism used to support and locate rotating shafts. Their construction and geometry is relatively complex. Even though they allow free rotation between the inner and outer raceways, translational vibration can still transmit through the bearings forming one of the major structure-borne paths in rotating machineries. Previous bearing load-deflection analysis is based on the Hertzian theory that describes the idealized contact between two bodies. The Hertzian contact theory also assumes one body as rigid and other mating pair as elastic. In the case of bearing analysis, the rolling elements are assumed elastic while the raceways are considered rigid, which tends to over-simplified the problem. In this thesis research, a more realistic contact model is introduced to improve the vibration response of rolling element-bearing systems as described below.

The primary objective of this research is to apply an elastic contact theory to formulate a new bearing stiffness matrix for use in predicting the vibration response of rotating systems and analyzing the vibration transmission through rolling element bearings. This proposed theory also accounts for the time-varying stiffness characteristics of the bearing due to orbital rotation of the rolling elements. Using the proposed bearing model, numerous parametric studies are performed to improve the understanding of the role of bearings in vibration transmission. The model is also employed to demonstrate the salient features. Obtrusive discrepancies are observed amongst the two theories for force preloading cases. Lumped parameter dynamic models based on the new contact-elastic theory are developed to study vibration transmission characteristics through single stage rotor and geared rotor systems. Also, the centrifugal
force, which is shown to have significant effect on the rotating system natural frequencies, is included in the new formulation. This effect is found more acute in angular contact ball bearings compared to other conventional designs.
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LIST OF SYMBOLS

- $a_i$: inner race groove curvature center
- $a_o$: outer race groove curvature center
- $d$: axial distance of rotor geometric center to the rotor mass center
- $d_b$: diameter of rolling element
- $d_e$: degree of interpenetration
- $d_i$: inner race diameter of bearing
- $d_m$: mean diameter of bearing
- $d_o$: outer race diameter of bearing
- $e$: eccentricity
- $e(t)$: kinematic transmission error
- $f(t)$: applied load vector
- $l_r$: length of distance from rotor/gear/pinion center to bearing geometry center
- $m$: mass of rolling element
- $m_e$: eccentric mass of rotor
- $n$: number of rolling elements in load deformation zone
- $r_c$: crown drop
- $r_j$: pitch radius of ball/roller
- $r_l$: radial clearance
- $x$: generalized displacement vector
- $y_1$: functions of contacting surface 1, profile function 1
- $y_2$: functions of contacting surface 2, profile function 2
\( w_1 \) elastic deformation of body 1
\( w_2 \) elastic deformation of body 2
\( t \) time
\( z \) gear ratio
\( A_0 \) unloaded center distance between inner race and outer raceway groove curvature center
\( A_j \) loaded center distance between inner race and outer raceway groove curvature center
\( C_m \) mesh damping coefficient
\( C_p \) multiplier constant to take care of positive values of pressure at field points in contact area
\( E_1 \) young’s modulus of elasticity of body 1
\( E_2 \) young’s modulus of elasticity of body 2
\( F_c \) Centrifugal force
\( F_{im} \) mean bearing forces in x, y, z direction where \((i = x, y, z)\)
\( K_a \) fluctuation of bearing stiffness coefficient about mean value
\( K_{mean} \) mean value of bearing stiffness coefficient over given time interval
\( K_m \) gear mess stiffness
\( L \) length of roller
\( M_{im} \) mean bearing moment about x and y axis where \((i = x, y)\)
\( Q_{ij} \) normal load at contact between rolling element and inner race for \( j^{th} \) rolling element
\( Q_{oj} \) normal load at contact between rolling element and inner race for \( j^{th} \) rolling element

\( Z \) total number of rolling element in bearing

\( \alpha_j \) loaded contact angle of \( j^{th} \) element

\( \alpha_0 \) unloaded contact angle

\( \beta \) Rayleigh damping proportionality constant

\( \beta_{im} \) mean angular misalignment about x and y axis, where \( (i = x, y) \)

\( \delta \) relative approach, defined as displacement of one point on one of the bodies relative to point on other body

\( \delta_{lm} \) mean radial displacement in x, y, and z direction, where \( (i = x, y) \)

\( \delta_{Bj(\psi_j)} \) resultant elastic deformation of \( j^{th} \) element for ball bearings

\( \delta_{Rj(\psi_j)} \) resultant elastic deformation of \( j^{th} \) element for roller bearings

\( \delta_{rij} \) effective \( j^{th} \) rolling element deformation in radial direction

\( \delta_{zj} \) effective \( j^{th} \) rolling element deformation in axial direction

\( \Delta_t \) time period

\( \Delta \) percentage deviation in frequency

\( \Psi \) angular distance of \( j^{th} \) rolling element from predefined X-axis

\( \Psi_T \) element-to-element angular distance

\( \lambda \) constant dependent on raceway control theory

\( \xi \) dimensionless parameter along \( j^{th} \) roller effective length

\( \Omega \) excitation frequency

\( \Omega_s \) mean rotating speed of shaft

\( \Omega_L \) mean rotating speed of load
\( \omega_m \) mean orbital speed of ball

Subscripts

1 pinion section
2 gear section
g gear
l left hand bearing
m mean
p pinion
r right hand bearing
s shaft
L load
R rotor
CHAPTER 1
INTRODUCTION

1.1 History

An archeological survey showed that Egyptians, ca. 2400 BC used lubricants like water in order to reduce effort required for transportation of very huge structures and stones [1]. The Assyrians, ca. 1100 BC used rollers under sledges in order to reduce manpower required for the movement of stones and structures. Dowson had conducted extensive research about history of ball and roller bearings and lubrication phenomena. Leonardo da Vinci great genius in tribology had carried out many experiments to study friction. Most recent record had revealed that, he sketched number of possible bearing configurations with balls and rollers.

1.2 Motivation

A bearing is a mechanical component that provides support and accurate positioning of an object while allowing it to rotate. Bearings are integral part of commonly used devices and high technological applications such as Internal Combustion engines, pumps, pulleys, motors, air craft engines, turbines, superchargers and many more. Bearings are further classified depending upon direction of loading as journal-bearings, which support load perpendicular to direction of rotation of axis and thrust bearings, which supports loads parallel to the axis of rotation. Depending upon nature of contact, bearings are also classified as sliding and rolling contact bearings. Although concept of rolling motion was known for thousands of years, practical use of rolling
element bearings did not occur until industrial revolution take place. The decisive steps
towards the development of ball and roller bearings were taken by the end of eighteenth
century when rolling element bearings are used for load carriages and heavy industrial
machineries. This development of bearings is further triggered with requirement of high-
speed bearings in aircraft engines. Now days there is tremendous competition among
various manufacturers served in favor of consumers, with low cost and highly reliable
bearings. Name rolling element bearings is used for all kind of bearings that can be used
to permit rotary motion of shaft relative to fixed housing. Some bearings are used to
permit translational motion of fixed structures relative to shaft in direction of motion of
shaft and some bearings permit rotational as well as translational motion between two
structures.

Rolling element bearings are more commonly used in machineries than
any other type of bearing for following reasons:

- Very low coefficient of friction usually ranges from 0.0005 to 0.008.
- Bearing deflections are less sensitive to load fluctuation than in any
  hydrodynamic bearings.
- Amount of lubricant require for operation is very small, importantly for many
  bearings lifetime supply of lubricant is provided.
- Can be used in transmission of radial as well as translational motion between two
  structures.
- Radial as well as thrust loads can be taken simultaneously. (e.g. angular contact
  bearings)
• Wide varieties of sizes are available. Hence they can be used in variety of application right from micro electro mechanical systems (MEMS) to huge channel tunneling application.

• Perform satisfactorily under various combinations of load, speed and operating temperature.

Although bearings appear to be very simple mechanisms, their actual construction and geometry is quite complex. As mentioned earlier, motivation for this research work comes from need to develop a superior bearing model, which will predict behavior of bearings under harsh operating conditions. For our research purpose we restrict our studies to angular contact ball bearings and cylindrical roller bearings. This pilot study examines a number of factors that affect rolling element bearing performance including effect of centrifugal forces and gyroscopic moment. Our main focus will be to develop elastic contact model that governs vibration transmissibility through bearings and predict dynamic response of system precisely and efficiently.

1.3 Specific Objectives of Project:

Some of the objectives of project are as follows:

• To develop elastic contact theory for studying vibration transmission characteristic of rolling element bearings.

• To analyze effects of contact angle variation and preloading condition variation on dominant stiffness coefficient for both ball and roller bearings

• Study effect of centrifugal force and gyroscopic moment on dominant stiffness coefficients.
• To study the effects of centrifugal force and gyroscopic moment on vibration response for single rotor and geared rotor systems.

1.4 Organization of thesis:

This document is organized in five chapters. This chapter serves as an introduction to the thesis work, includes history of bearings and specific objectives of the thesis. The remainder of this thesis will proceed as follows:

• Chapter 2 we will introduce the common approaches used prior to this research work. All previously employed theories in order to get stiffness matrix of bearing and their pros and cons are discussed and motivation to substantiate work to complete the thesis.

• Chapter 3 presents our solution methodology elastic linear time average (ELTA) theory, formulation of contact problem for bearing and algorithm to obtain coefficients of stiffness matrix using ELTA method.

• Chapter 4 deals with comparison of simulation results with previously employed mean linear time varying (MLTV) model [2] for various parametric studies. Qualitative discussion along with some analysis of physical systems shows effect of centrifugal force and gyroscopic moment on dynamic response of system.

• Finally in chapter 5 summarize research work and certain areas for future work are recommended.
CHAPTER 2
LITERATURE REVIEW

Major contributors in the field of bearing analysis are Jones [3], Harris [1], Palmgren [4] and Hammrock [5]. Firstly Lundberg and Palmgren [4] developed a theory to predict stress distribution at point of contact for normal loading. This theory was able to predict fatigue life of bearings to some extent with inclusion of empirical proportionality constant. Then Jones [3] developed a general method, to obtain all forces and elastic deformations analytically in a redundant system like ball and roller bearing. This theory was a successful attempt to improve precision of Lundberg and Palmgren [4] theory. Harris [1] and Gargiulo [6] developed a non-linear stiffness coefficient matrix. This derivation is based on force displacement relations, which are commonly used by designers to design rolling element bearings. The above formulation does not take into account effect of clearance. White [7] had modified the formulations developed by Harris [1] and Gargiulo [6] by adding effects of clearance and force on loaded contact angle. This model does not able to predict vibration transmission across the rolling element bearings.

In 1982 Rajab [8] tried to develop theory that will predict stiffness terms like $K_{b_r\theta}$, $K_{b_\theta\theta}$ that will couple radial and rotational degrees of freedom. But later Lim and Singh [9-13] had shown that the analysis done by Rajab [8] was erroneous and lead to inaccurate formulation of element stiffness matrix of order 2. In 1988 Young [14] had extended Rajab’s [8] formulation in order to incorporate stiffness terms along the axis of
bearing. Instead of integral terms he used discrete summation of stiffness terms over all loaded bearing elements. As this formulation was based on Rajab’s [8] formulation this also was erroneous.

Experimentally it is possible to measure stiffness in radial and axial direction only. Firstly Walford and Stone [15] performed measurement of stiffness and damping under oscillating condition. But measurement of damping values was somewhat inaccurate and the values are predicted on higher side. In 1987 Kraus et. al. [16] used modal analysis technique to determine bearing properties. They had studied effect of preload, speed $\Omega$ and bearing release on stiffness and damping. Their results indicate in general there is no effect of speed variation on stiffness, but damping decreases with increase in speed. Increase in bearing preload tends to increase stiffness and decrease damping. If bearing release increased it tends to increase preload and hence stiffness gets increased and so as damping. In order to predict vibration response of bearings more accurately Lim and Singh [9-13] developed one more model. That model considers more generalized formulation of load and displacement relationship based on Hertzian contact theory. This new theory produced a linear time invariant stiffness matrix of order five. Although bearing has six degrees of freedom it does not offer any resistance along its axis, so moment about its axis taken as zero. In all we have two radial, one axial and two rotational bending degrees of freedom for bearing. Their formulation was comparable with Jones [3] theory developed long back in 1960. In final paper among the series of 5 papers they had introduced effect of contact load distribution along length of roller. This effect usually predicts higher values of bending stiffness and cross-coupled terms associated with bending degrees of freedom. Lim and Singh [9-13] had done studies of
vibration transmissibility through rolling element bearings for cases like single stage rotor system and geared rotor system by using their linear time invariant stiffness formulation. Recently Houpert [17-18] proposed a novel analytical approach uniform for both ball and roller bearings. This model considers five relative race displacements in order to calculate loads in three directions $F_x$, $F_y$, $F_z$ and two tilting moments $M_x$ and $M_y$ about $X$ and $Y$ axis respectively. The advantage of this approach was it could be easily incorporated with any non-linear finite element analysis (FEA) package. Hernot et. al. [19] had developed similar kind of formulation to get stiffness matrix of order five. This formulation is suitable to obtain no-linear load displacement relationship.

Liew [2] had extended the idea of Lim and Singh [9-13] to formulate a new time varying rolling element bearing stiffness matrix. This formulation takes into account variation in stiffness of bearing due to rotational speed. This improved time averaged model has good match in results for vibration response with linear time varying one. Jang and Jeong [20-21] had developed a model deals with ball bearing vibration due to waviness in system of rotor support between two or more bearings. In their next paper centrifugal forces and gyroscopic moments of ball and waviness of rolling elements induced to get vibrations response. The results indicate centrifugal force and gyroscopic moment of ball play very important role in determination of principle frequencies and their harmonics.

All previous bearing models were based on Hertzian contact theory, which does not take into the account of the elasticity of both contacting surfaces. This assumption generally makes the model inadequate for bearings under more actual operating condition. In 1974 Singh and Paul [22] had developed a general method for numerical
analysis of frictionless non-conformable non-Hertzian contact between bodies of arbitrary shapes. The proposed elastic theory is based on numerical contact theory developed by Singh and Paul [22], modified suitably to predict dynamic behavior of rolling element bearings more precisely and accurately. Instead of assuming ellipsoidal pressure distribution this theory actually calculates pressure at field points, which will replicate pressure distribution in contact area. In general this thesis will be an extension of the work done by Liew [2], by taking into account elasticity of races and incorporating the effect of centrifugal force to produce more generalized rolling element bearing stiffness matrix.
CHAPTER 3
FORMULATION

3.1 Bearing Load – displacement relation:

The ball and roller bearing kinematics and coordinate system are shown in figure (1). These diagrams will indicate different degrees of freedom, any ball or roller bearing will have. These parameters are used to represent the bearing load – displacement relation. Resultant elastic deformation of ball or roller is dependent on angular position $\psi_j$.

![Figure 1. Ball Bearing Kinematics and Co-ordinate system [9]](image)
As shown in the figure (2) effective \( j \)th ball displacement can be obtained in terms of mean translational and mean rotational degrees of freedom as follows,

\[
(\delta)_{ij} = \delta_{xm} \cos \psi_j + \delta_{ym} \sin \psi_j - r_L \tag{3.1}
\]

\[
(\delta)_{ij} = \delta_z + r_j (\beta_{xm} \cos \psi_j + \beta_{ym} \sin \psi_j) \tag{3.2}
\]

where, \( r_l \) is radial clearance, \( \delta_{xm}, \delta_{ym}, \delta_{zm} \) are mean translational displacements and \( \beta_{xm}, \beta_{ym} \) are rotational mean misalignments. So in general radial and axial displacements can be represented in terms of five degrees of freedom for ball bearings.

Loaded center distance can be obtained as follows,

\[
(\delta^*)_{ij} = A_0 \cos \alpha_0 + (\delta)_{ij} \tag{3.3}
\]

\[
(\delta^*)_{ij} = A_0 \sin \alpha_0 + (\delta)_{ij} \tag{3.4}
\]

\[
A(\psi_j) = \sqrt{(\delta^*)_{ij}^2 + (\delta^*)_{ij}^2} \tag{3.5}
\]

We will only consider positive bearing displacements and therefore,

\[
\delta_B(\psi_j) = \begin{cases} 
A(\psi_j) - A_0, & \delta_{ij} > 0 \\
0, & \delta_{ij} \leq 0
\end{cases} \tag{3.6}
\]
where, $A$ and $A_0$ are loaded and unloaded center distances between inner $a_i$ and outer $a_0$ raceway groove curvature centers, and $a_0$ is the unloaded contact angle. This above equation (3.6) will give elastic displacement as function of position of rolling element. This value of deflection obtained is used in formulation of non-hertzian contact problem in order to calculate pressure distribution and contact area at contact point in bearings at different location of rolling element. At the same time, one can observe as an effect of preloading contact angle between inner race and ball changes in case of angular contact ball bearing. This change in contact angle is function of ball location, preloads and unloaded contact angle. This loaded contact angle can be given by following equation,

$$\tan(\alpha_j) = \frac{A_0 \sin a_0 + (\delta_{ij})}{A_0 \cos a_0 + (\delta_{ij})}$$

(3.7)

As shown in figure (3) same sort of relation can be obtained for roller element bearings. Initially, we represent elastic displacements of the roller bearing in radial and axial direction as a function of mean translational displacements and mean angular misalignments as follows,
\[(\delta)_{\psi j} = \delta_{xm} \cos \psi_j + \delta_{ym} \sin \psi_j - r_j \]  
(3.8)

\[(\delta)_{\psi j} = \delta_z + r_j \{\beta_{xm} \cos \psi_j + \beta_{ym} \sin \psi_j \} \]  
(3.9)

Elastic deformation of \(j^{th}\) roller can be represented as function of angular contact position of roller and characteristic dimensionless parameter \(\zeta\) along the length of roller.

\[V(\psi_j) = (\delta)_{r_j} \cos(\alpha_j) + (\delta)_{\psi_j} \sin(\alpha_j) - r_c \]  
(3.10)

\[W(\psi_j) = -\beta_{xm} \sin(\psi_j) + \beta_{ym} \cos(\psi_j) \]  
(3.11)

where \(r_j\) is the pitch radius, \(r_c\) is the crown drop, \(\delta_{xm}, \delta_{ym}, \delta_{zm}\) are the mean translational displacements and \(\beta_{xm}, \beta_{ym}\) are the mean rotational misalignments.

Then resultant elastic deformation of roller can be given as follows,

\[\delta_{R_i}(\psi_j, \zeta) = \begin{cases} 
V(\psi_j) + \zeta W(\psi_j), & \delta_{R_i} > 0 \\
0, & \delta_{R_i} \leq 0 
\end{cases} \quad -0.5 \leq \zeta \leq 0.5, \]  
(3.12)

### 3.2 Formulation of Contact Problem:

![Figure 4. Two bodies in contact with aligned principal Y-axis](image-url)
Two bodies denoted by 1 and 2 as shown in figure (4) are in contact. Co-ordinate axes system is as shown in figure. Plane X-Z is perpendicular to plane of paper, this is the plane that will contain contact area and Y-axis is pointing into bodies. During formulation of contact problem following assumptions are made:

- Both bodies are assumed to be frictionless.
- External load assumed to be compressive.
- Applied load can cause only rigid body translation and elastic deformation.
- Elliptical contact area at any contact point and symmetric about both axes.
- For this problem dimensions of contact region are assumed to be very small compared to radii of the indenting surfaces. This enables us to use elastic half space theory.
- Variation of stiffness coefficients about their mean value is not much, and hence elastic linear time-averaged theory is valid.

Hertz previously employed assumption of half space theory and he consider one body as rigid and one body elastic. Experimental results obtained by Fesier and Ollerton [23] had proved that half space assumption is fairly accurate only for Hertzian contacts where semi major diameter of contact ellipse is as large as half the smallest radius of curvature of either body. We had assumed elasticity of races as well as rolling element, hence this case will no longer be consider as Hertzian contact case. In case of non-Hertzian contact, it is reasonable to assume half space theory as long as surfaces are non-conformal. Then points in the contact region should satisfy following equation

$$d_1 + d_2 + f_1 + f_2 = \delta$$

(3.13)
where \( d_1, d_2 \) are given by Boussinseq solution, \( f_1, f_2 \) are the functions of contacting surfaces called profile functions. Generally, \( f_1, f_2 \) are from family of polynomials, for the problem of bearings they are assumed as

\[
f_1 = \frac{\left( x^2 + z^2 \right)^{\frac{1}{2}}}{R_1^2}
\]

\[
f_2 = \frac{\left( x^2 + z^2 \right)^{\frac{3}{2}}}{R_2^2}
\]

where \( R_1, R_2 \) are the radii of contacting surfaces 1 and 2 respectively. Boussinseq solution for normal point load on elastic half space will give us relation between elastic displacement and pressure distribution in contact area.

\[
d_i(x, z) = \frac{\left( 1 - \nu_i^2 \right)}{\pi E_i} \int_\Omega \frac{p(x', z') dx' dz'}{\left( x - x' \right)^2 + \left( z - z' \right)^2}^{\frac{3}{2}}
\]

where \( \nu_i, E_1, E_2 \) are Poisson’s ratio and Young’s modulus of the two bodies respectively. The symbol \( \Omega \) is the projection of the contact region on the tangent plane \((XZ)\). As stated in the assumption about the two bodies under compressive load, which naturally implies no tension is exerted upon each other, a physically meaningful solution at all points within the contact region will have the form,

\[
p(x, z) \geq 0
\]

So by solving above equations, the pressure distribution at the contact area can be obtained. The normal force through this contact area can be given as,

\[
F = \int_\Omega p dA = \sum_{i=1}^{n} p_i A_i
\]

The contacting surfaces can be described in functional form as,

\[
f(x, z) = y_1 + y_2 = f_1(x, z) + f_2(x, z)
\]
After combining all of the above equations, final equation will be of the form, which further can be solved to obtain pressure distribution inside the contact region,

\[ k \int p dA = k \int \frac{p(x', z') dx' dz'}{\left[(x-x')^2 + (z-z')^2\right]^{1/2}} = \delta - f(x, z) \quad (3.20) \]

where ‘k’ is called as elastic parameter and it is defined as follows,

\[ k = \frac{1 - \nu_1^2}{\pi E_1} + \frac{1 - \nu_2^2}{\pi E_2} \quad (3.21) \]

Contact problems like bearings will require priori knowledge of contact area and then, force that can be transmitted through bearing will be calculated. In any rolling element bearing number of balls or rollers are used, hence contact occur at number of points. This contact area is a function of ball or roller position in bearing. Contact area at any point of contact in bearing can be obtained by considering a hypothetical parameter \( d_e \), known as degree of interpenetration. The concept of interpenetration curve is defined as, the intersection of two undeformed surface when one body is moved with respect to other through arbitrary distance \( d_e \). So, for the problem of the bearing,

\[ d_e = C_p f(\delta_{ij}, \varphi_{ij}) \quad (3.22) \]

where \( C_p \) is constant multiplier, will take care of positive values of pressure at field points. Numerical solution is highly sensitive and largely depends upon value of interpenetration parameter. Hence, the choice of \( C_p \) is crucial; otherwise the problem will produce ill conditioned problems. In the case of rolling element bearing, the semi minor axis of ellipse can be correlated with \( d_e \) through following relation,

\[ a = \left[ \frac{d_e}{\left(\frac{1}{R_1^2} + \frac{1}{R_2^2}\right)} \right]^{1/2} \quad (3.23) \]
where $R_1, R_2$ are the radii of contacting surfaces 1 and 2 respectively.

In the case of bearings, elastic deformation is always given in terms of mean translational displacements and mean rotational displacements. Even if the force input conditions are given, the force input will be transformed into a corresponding displacement input conditions and then used to calculate the resultant elastic deformation. So for given choice of interpenetration and mean displacements by using above equations, the contact area can be computed.

In the assumption, any rolling element bearing will have an elliptical contact area as shown in figure (5) at all contact points, which are symmetric about their axes, so we can do analysis for only quarter of area.

![Figure 5. Elliptical contact area and its cellular division](image)

In order to get pressure distribution we are going to use “simply discretized numerical solution” method [22]. For any ball (or roller) location, the elliptical contact area is divided into a set of smaller cells. Usually, this division is done as 4, 9 and 33 cells; however finer mesh can be applied. It may be noted that the ratio of the amount of calculation involved and the accuracy obtained by using finer mesh is very small. At the same time, for some problems, the increase in mesh density may not outweigh the
improved accuracy. In most cases, the optimal values of mesh density are between 4 and 9. For convenience, the quarter of ellipse is divided into 4 cells for the analysis reported in this paper. We assume rectangular cells inside boundary and polygonal at outside boundary. For sake of uniformity and simplicity we assume field points as centroids \((x', z')\) of cells. With the help of following equations we can calculate pressure at those field points.

For field points where \(j \neq i\)

\[
kp_j \int_{A_j} \frac{dA}{r} dr = (\delta) - f_i
\]  
(3.24)

For field points where \(j = i\)

\[
k_p i \int_{A_i} \frac{dA}{r} dr = (\delta) - f_i
\]  
(3.25)

where \(r\) is distance between field points \(i\) and \(j\). So for the case where \(i = j\), there is going to be singularity in above equations. To overcome this problem, we will represent integral \(1/r\) in terms of elementary functions and by using numerical quadrature double integral scheme we will evaluate it.

\[
b_{ij} p_i = \delta - f_i
\]  
(3.26)

where

\(\delta\) = elastic deformation at point of contact, \(b_{ij}\) = coefficients obtained through evaluation of integral, \(p_i\) = pressure at field point, \(f_i\) = value of summation of profile function that is “initial separation between bodies”.

So, in all we will have ‘n’ equation and ‘n’ unknowns. These simultaneous equations are solved to get pressures at field points, which will give good representation of pressure distribution inside contact area. Then force normal to contact area will be
obtained by multiplication of cell area and pressure at field point of cell. Since analysis is done only for quarter of ellipse, thus total force can be obtained as,

$$F_j = 4 \times p \times A'$$

(3.27)

where $p$ is a row vector consist of pressure at field points and $A'$ is a column vector consist of areas of different cells. This normal force is then used to get components of forces in $X$, $Y$, $Z$ direction and moments about $X$ and $Y$-axes.

$$
\begin{bmatrix}
F_{xm} \\
F_{ym} \\
F_{zm} \\
M_{xm} \\
M_{ym}
\end{bmatrix}
= \sum_{j}^{z} F_j \begin{bmatrix}
\cos(\alpha_j)cos(\psi_j(t)) \\
\cos(\alpha_j)\sin(\psi_j(t)) \\
\sin(\alpha_j) \\
r_j \sin(\alpha_j)\cos(\psi_j(t)) \\
r_j \sin(\alpha_j)\cos(\psi_j(t))
\end{bmatrix}
$$

(3.28)

Stiffness can be calculated by the method of small perturbation, then in-line linear stiffness in $X$ direction $K_{xx}$ can be obtained as follows,

$$K_{xx} = \frac{\partial F_x}{\partial \delta_x} \equiv \frac{\Delta F_x}{\Delta \delta_x} \equiv \frac{F_x(\delta'_x) - F_x(\delta_x)}{\delta'_x - \delta_x}$$

(3.29)

And cross-coupled stiffness $K_{xy}$ can be obtained as,

$$K_{xy} = \frac{\partial F_y}{\partial \delta_x} \equiv \frac{\Delta F_y}{\Delta \delta_x} \equiv \frac{F_y(\delta'_x) - F_y(\delta_x)}{\delta'_x - \delta_x}$$

(3.30)

where, $\delta'_x$ is mean elastic deformation in $X$ direction after very small perturbation in $\delta_x$.

Similarly in line linear stiffness $K_{yy}$ in $Y$ direction can be obtained as,

$$K_{yy} = \frac{\partial F_y}{\partial \delta_y} \equiv \frac{\Delta F_y}{\Delta \delta_y} \equiv \frac{F_y(\delta'_y) - F_y(\delta_y)}{\delta'_y - \delta_y}$$

(3.31)

where, $\delta'_y$ is mean elastic deformation in $Y$ direction after very small perturbation in $\delta_y$.

In line linear stiffness $K_{zz}$ in $Z$ direction can be given as,
\[ K = \begin{bmatrix}
K_{xx} & K_{xy} & K_{xz} & K_{x\delta x} & K_{x\delta y} \\
K_{yx} & K_{yy} & K_{yz} & K_{y\delta x} & K_{y\delta y} \\
K_{zx} & K_{zy} & K_{zz} & K_{z\delta x} & K_{z\delta y} \\
K_{x\delta x} & K_{x\delta y} & K_{z\delta x} & K_{x\delta x} & K_{x\delta y} \\
K_{y\delta x} & K_{y\delta y} & K_{z\delta x} & K_{y\delta x} & K_{y\delta y}
\end{bmatrix} \]  

(3.32)

where, \( \delta_z' \) is mean elastic deformation in Z direction after very small perturbation in \( \delta_z \).

If we consider angular misalignments of shaft then in line angular stiffness of bearing \( M_{xx} \) can be given as,

\[ M_{xx} = \frac{\partial M_x}{\partial \beta_x} = \frac{\Delta M_x}{\Delta \beta_x} = \frac{M_x(\beta_x') - M_x(\beta_x)}{\beta_x' - \beta_x} \]  

(3.33)

where \( \beta_x' \) is mean rotational misalignment about X axis after very small perturbation in \( \beta_x \). Cross-coupled angular stiffness \( M_{xy} \) can be given as,

\[ M_{xy} = \frac{\partial M_y}{\partial \beta_x} = \frac{\Delta M_y}{\Delta \beta_x} = \frac{M_y(\beta_y') - M_y(\beta_y)}{\beta_y' - \beta_y} \]  

(3.34)

In line angular stiffness \( M_{yy} \) in Y direction is given as,

\[ M_{yy} = \frac{\partial M_y}{\partial \beta_y} = \frac{\Delta M_y}{\Delta \beta_y} = \frac{M_y(\beta_y') - M_y(\beta_y)}{\beta_y' - \beta_y} \]  

(3.35)

where \( \beta_y' \) is mean rotational misalignment about Y axis after very small perturbation in \( \beta_y \). In the same fashion remaining terms of the stiffness matrix are obtained. Stiffness matrix is assumed to be symmetric hence; \( K_{xy} = K_{yx} \) this equality is valid.

In case of rolling element bearings number of loaded element varies from \( n \) to \( n+1 \) after certain time period \( \Delta t \). This variation of balls will cause variation in stiffness from point to point. This variation in stiffness is considered while calculating mean
stiffness coefficients in our model. This phenomenon is explained with the help of following figure (6),

![Figure 6. Variation in number of rolling elements as function of time](image)

At time $t = 0$, we can see there are five elements in bearing deformation zone. But after small interval of time, say $\Delta t$, there will be only four elements in bearing deformation zone. Hence, we can say that values of stiffness coefficients of bearing at time $t = 0$ will be larger than values of stiffness coefficients after period $\Delta t$. This variation in stiffness will actually introduce dynamic characteristics in bearing. This effect will be considered during calculation, and will result into mean stiffness coefficients matrix for bearing. These stiffness coefficients will be calculated through simple program. Flow chart is attached in appendix (A). Algorithm of this program can be explained below.

Initially, we get input data required to calculate all bearing parameters. Set the counter for number of balls and resolution of points between two ball locations. In the
case of the rolling element bearings rolling, the elements are separated by angle \( \psi_T \) given by following equation (3.37),

\[
\psi_T = \frac{360}{Z}
\]  

(3.37)

where \( Z \) = total number of rolling elements in bearing.

By using these parameters we can get the value of deformation in actual loaded contact zone along with dimensions of contact area. Our basic assumption is this area will be elliptical, so we get minor axis of this ellipse from parametric relation with elastic deformation. Then pressure in the contact zone will be obtained through numerical contact model explained earlier. This will give load normal to contact area and will be resolved to get forces in three directions \( X \), \( Y \), and \( Z \) and moments about \( X \) and \( Y \)-axis respectively. Then we will calculate stiffness coefficients for each ball location with the use of method of perturbation and then, add them together to get complete bearing stiffness coefficients for one complete revolution of bearing. Then, to increase counter for resolution to vary ball location of ball by small angle corresponding to small time interval \( \Delta t \), repeat the same procedure till one ball reaches the exact location of other ball, since there onwards we will get same stiffness values again. Then, this procedure is repeated for various locations of balls then all these coefficients are summed together and then average of these will give us elastic linear time averaged (ELTA) stiffness matrix.

\[
K_{ELTA} = \frac{\sum_{i=1}^{Z} \sum_{j=1}^{n} K_{ij}}{n}
\]  

(3.38)

where \( i \) = counter for number of balls \( (1,2,...Z) \), \( j \) = counter for time resolution \( (1,2,...n) \).

So in this way we will calculate complete dynamic stiffness matrix for bearing.
3.3 Effect of Centrifugal Force and Gyroscopic Moment:

In the above formulation of stiffness matrix, we did not consider effect of centrifugal forces and gyroscopic moment. In the case of low speed like (0 – 5000 rpm), the effect of centrifugal forces and gyroscopic moment are negligibly small and so they are not considered while doing the analysis. But if speed of operation is increased to 10000 rpm, the effect of centrifugal force and gyroscopic moment becomes significantly high. In high speed angular contact ball bearings this effect will tend to increase magnitude of contact angle at inner raceway and will decrease the same at outer raceway. This effect will significantly change the deflection versus load characteristics of ball bearing system. Hence stiffness matrix coefficients of bearings will definitely get modified and this modifies vibration characteristics of bearing supported systems. In high-speed roller bearings centrifugal force and gyroscopic moment affect loading conditions at inner race and outer race. This effect will tend to increase loading at outer race causing larger deformations at outer race. In the case of many high-speed bearings centrifugal forces and gyroscopic moment affect lubrication characteristics. Hence, this modifies frictional effects, but in most of the applications analysis is done by neglecting frictional effects. In the case of ball bearings, in order to include the above effects, we need to get the change in contact angle at inner race. This can be obtained as follows,

![Figure 7. Ball loading at any angular position [2]](image-url)
From the figure (7) we can write equations of equilibrium as follows,

\[ Q_{ij} \sin(\alpha_{ij}) - Q_{oj} \sin(\alpha_{oj}) - \frac{2M_{ij}}{D} \cos(\alpha_{ij}) = 0 \]  \hspace{1cm} (3.39)

\[ Q_{ij} \cos(\alpha_{ij}) - Q_{oj} \cos(\alpha_{oj}) + \frac{2M_{ij}}{D} \sin(\alpha_{ij}) + F_c = 0 \]  \hspace{1cm} (3.40)

In above equations \( F_c \) is the value of centrifugal force and \( M_{ij} \) is the gyroscopic moment of ball, they can be obtained by following formulae,

\[ F_c = \frac{1}{2} md_m \omega_m^2 \]  \hspace{1cm} (3.41)

\[ M_{ij} = I_p \omega_b \omega_m \sin \xi \]  \hspace{1cm} (3.42)

where, \( m \) is mass of ball, \( d_m \) is mean diameter, \( I_p \) is polar moment of inertia, \( \omega_b \) is the spinning speed of ball, \( \omega_m \) is orbital speed of ball and \( \xi \) is the angle between spinning axis of ball and centerline of bearing.

If we solve the above set of equations then we will get solution for \( \alpha_{ij} \) and \( \alpha_{oj} \). This \( \alpha_{ij} \) will be new contact angle at inner race due to effect of centrifugal force and gyroscopic moment, with the use of this we can get modified stiffness matrix. This modified stiffness matrix can be used to get vibration response of different systems; this is studied in next section. Following are different systems analyzed in order to study the vibration response through rolling element bearings.
3.4 Single Stage Rotor System:

Above schematic (8) shows single-stage rotor system consist of rotor mounted on shaft supported between two identical rolling element bearings, prime mover, load, casing and couplings. In this arrangement prime mover rotates at angular speed $\Omega_s$, assuming no loss between couplings load also rotates at the same speed $\Omega_s$. Then vibration response of this system is analyzed using the lumped parameter technique. The equation of motion of the system is written as follows,

$$[M]x + [C]x + [K]x = f(t)$$  \hspace{1cm} (3.43)

where, $[M]$ is lumped mass matrix, $[K]$ is Stiffness matrix, $[C] = \beta[K]$ is viscous damping matrix, $x$ is assumed to be generalized displacement vector and $f(t)$ is excitation force will be a function of time and frequency. Analysis of this system is done in two
ways, using stiffness and damping matrix as a function of time i.e linear time varying (LTV) analysis and elastic linear time averaged (ELTA) technique analysis.

In LTV technique stiffness matrix will be function of time hence new equation of motion should be used,

$$\begin{align*}
[M]\ddot{x}+[C(t)]\dot{x}+[K(t)]x &= f(t) \\
\end{align*}$$

(3.44)

$[K(t)]$ and $[C(t)]$ represent time dependent stiffness and damping matrix. This stiffness matrix is time varying stiffness matrix of only rolling element bearing, since shaft, casing, coupling and rotor are assumed to be rigid. So, bearing stiffness matrix will be obtained by equation (3.36). In ELTA technique $[K]$ matrix will be average of different stiffness matrices at different interval of time over one complete revolution of bearings at given frequency. Hence, the stiffness matrix will be no longer a function of time, so $[K]$ and $[C]$ both will be time invariant in nature but still having ability to predict dynamic characteristics of bearing systems accurately. Lumped mass matrix $[M]$ can be represented as,

$$[M] = diag[Mass \mid Inertia]$$

(3.45)

\[
[M] = \begin{bmatrix}
M & 0 & 0 & 0 \\
0 & M & 0 & 0 \\
0 & 0 & M & 0 \\
0 & 0 & 0 & I_{xx} \\
0 & 0 & 0 & I_{yy}
\end{bmatrix}
\]

(3.46)

where, $M$ is lumped mass combinations of mass of two bearings, casing, rotor, shafts and couplings. And mass moment inertia’s $I_{xx}$ and $I_{yy}$ are strictly mass moment inertias of the rotor about $X$ and $Y$-axes respectively.
In the single-stage rotor system shaft, the shafting system is supported between two identical rolling element bearings. These are designated as left hand bearing and right hand bearing, since both will have different stiffness matrix. Element stiffness matrix will be developed for left hand bearing, and for the right hand bearing, it will be obtained by using transformation matrix. This transformation matrix will depend largely on configuration of bearing, and it is different for back-to-back and face-to-face configurations,

$$[K]_i = T_i^{-1}[K]_i T_i, \quad (i = l \text{ or } r) \quad (3.47)$$

where $T_i$ for back-to-back configuration can be given as follows,

$$T_r = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad (3.48)$$

Combination these two stiffness matrices of bearings will give us global stiffness matrix for single-stage rotor system. The procedure is explained in detail in Appendix [B].

During our analysis, it is assumed that damping is proportional to stiffness. So, time dependent viscous damping matrix can be given by the following equation:

$$[C] = \beta[K] \quad (3.49)$$

where $\beta$ is proportionality constant. In single stage rotor main cause of forcing function is mass imbalance. Hence, the applied force vector will be function of $\Omega_s$ and $t$, and can be given by following formula,
where $m_e$ is the eccentric mass, $e$ is eccentricity, $\Omega_s$ is the shaft speed and $d$ is axial distance from the rotor geometric center to the rotor mass center. Then, $x$ is considered as a generalized displacement vector and is given by,

$$x(t) = \begin{bmatrix} X \\ Y \\ Z \\ \theta_x \\ \theta_y \end{bmatrix}$$

(3.51)

The response of single-stage geared rotor system can be obtained by solving the following a set of governing equations of motion for generalized linear vibrating systems [24]. In the case of forced periodic excitation, the steady-state response vector of system can be given explicitly as,

$$\{x(t)\} = \left[ -\Omega^2[M] + i\Omega[C] + [K]^{-1} \right] \{f(t)\}$$

(3.52)
3.5 Geared Rotor System:

The schematic of the geared rotor system is shown in Figure 9. This system consist of a pair of spur gear mounted on two shafts namely, driver shaft and driven shaft, prime mover, load, couplings, casing and four identical rolling element bearings. Prime mover rotates at speed of $\Omega_s$, but load rotates at some different speed $\Omega_L$ depending on gearing ratio.

$$\Omega_L = (z)\Omega_s$$  \hspace{1cm} (3.53)

where $z$ is gear ratio, ($z =$ teeth of pinion/teeth of gear)

Vibration response of geared rotor system is analyzed using lumped mass technique. Lumped mass model of this geared rotor system possess 12 degrees of freedom (DOF). This is because each gear possesses three translational $\delta x, \delta y, \delta z$, and three rotational degrees of freedom $\beta x, \beta y, \beta z$ respectively. Governing equation of
motion of this system is same as equation (3.43); only difference is in the order and values of $[M]$, $[K]$ and $[C]$ matrices. Mass and moment of inertia of prime mover and load are neglected; hence lumped mass matrix $[M]$ for this system is given as,

$$[M] = [[M_p]] [[M_G]]$$  \hspace{1cm} (3.54) 

where, $[M_p]$ and $[M_G]$ are mass matrices of pinion and gear respectively and they are given as follows,

$$[M_i] = \begin{bmatrix} M_x & 0 & 0 & 0 & 0 \\ 0 & M_y & 0 & 0 & 0 \\ 0 & 0 & M_z & 0 & 0 \\ 0 & 0 & 0 & I_x & 0 \\ 0 & 0 & 0 & 0 & I_y \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (\text{where, } i = p \text{ or } G)$$  \hspace{1cm} (3.55) 

We focus our analysis to study vibration transmission through bearings only, and therefore the rest of the elements of the system like, shafts, couplings, gears, prime mover and loads are assumed to be rigid. In bearing formulation we had considered elasticity of balls and races of bearing. Stiffness matrix of complete system is combined stiffness matrices of all four rolling element bearings. Complete formulation of this matrix is explained in detail in appendix [B]. The same stiffness matrix can be used to get vibration response for ELTA as well as linear time varying (LTV) case. In LTV case, angular distance is function of time hence at each interval of time stiffness matrix will get modified and the current value of this matrix is used to compute generalized displacement vector. In stiffness matrix subscripts $l$ and $r$ are used for left hand bearing and right hand bearing respectively. Subscripts $1$ and $2$ are used to indicate pinion and gear sections respectively. Stiffness matrix of left hand or right hand bearing is obtained through formulation of stiffness matrix and depending upon bearing mounting
configuration used, stiffness matrix of the other bearing is obtained by using transformation matrix.

\[ [K(i)]_r = T_r^{-1} [K(i)]_l T_r \quad (i = l \text{ or } r) \quad (3.56) \]

\[ T_r = \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0 & 0 \\
    0 & 0 & -1 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & -1 \\
    0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad (3.57) \]

Proportional damping matrix is given as,

\[ [C(i)] = \beta [K(i)] \quad (3.58) \]

where \( \beta \) is Rayleigh proportionality constant. For the system under consideration, the generalized displacement vector can be given as,

\[ x(t) = \begin{bmatrix}
    X_1 \\
    Y_1 \\
    Z_1 \\
    \theta_{X1} \\
    \theta_{Y1} \\
    \theta_{Z1} \\
    X_2 \\
    Y_2 \\
    Z_2 \\
    \theta_{X2} \\
    \theta_{Y2} \\
    \theta_{Z2}
\end{bmatrix} \quad (3.59) \]
In geared rotor system, the main cause of forcing function is kinematical transmission error. Hence applied force vector will be function of time dependent error $e(t)$, and can be given by following formula,

$$\{f(t)\} = \begin{bmatrix} 0 \\ -K_m e(t) - C_m \dot{e}(t) \\ 0 \\ 0 \\ 0 \\ -K_m R_1 e(t) + C_m R_1 \dot{e}(t) \\ 0 \\ K_m e(t) + C_m \dot{e}(t) \\ 0 \\ 0 \\ 0 \\ K_m R_2 e(t) + C_m R_2 \dot{e}(t) \end{bmatrix} \quad (3.60)$$

where $K_m$ is the gear mesh stiffness and $C_m$ is the mesh damping stiffness. $R_1$ and $R_2$ are the radii of gear and pinion, respectively. Then response of geared rotor system is obtained through equation (3.43).
CHAPTER 4
RESULTS AND DISCUSSION

4.1 Comparison of proposed elastic theory and existing theory:

This chapter mainly comprises of comparison of rolling element bearing stiffness coefficients of the proposed elastic time averaged theory and existing analytical mean linear time-averaged theory Liew [19] and dynamic response analysis of various systems. From hereafter results obtained through proposed, elastic linear time averaged will theory be labeled as ELTA, linear time varying by LTV and existing mean linear time-varying theory by Liew [19], will be labeled as MLTV. For analysis purpose certain design parameters are used for angular contact ball bearing and roller bearing, which are listed in table 1. These parameters set consist of young’s modulus of elasticity of rolling element $E_1$ and that of inner race is $E_2$, Poisson’s ratio $\mu$, number of rolling elements $Z$, radial clearance $r_L$, crown drop $r_c$, rolling element diameter $d_b$, unloaded contact angle $\alpha_0$, unloaded center distance between inner and outer raceway groove curvature centers $A_0$ and radii of inner groove curvature center for ball and pitch radius for roller $r_j$. Discussion of the results will begin with validation of our assumption of using time averaged stiffness coefficients.

![Figure 10. Variation of stiffness as a function of time](image-url)
Above figure (10) indicates at any point of time stiffness of bearing can be given as,

\[ K_b = K_{\text{mean}} \pm K_a \]  

(3.61)

where \( K_{\text{mean}} \) is average value of stiffness over given interval of time, \( K_a \) is amplitude of variation of stiffness about mean and \( K_b \) is instantaneous value of stiffness.

**Table 1**

*Design Parameters for Ball and Roller Bearings*

<table>
<thead>
<tr>
<th>Parameter Description</th>
<th>Ball bearing</th>
<th>Roller Bearing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus for element, ( E_1 ) (N/mm(^2))</td>
<td>2.1x10(^{11})</td>
<td>2.1x10(^{11})</td>
</tr>
<tr>
<td>Young’s modulus for inner race, ( E_2 ) (N/mm(^2))</td>
<td>2.1x10(^{11})</td>
<td>2.1x10(^{11})</td>
</tr>
<tr>
<td>Poisson’s ratio, ( \mu )</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Number of rolling elements, ( Z )</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>Radial clearance, ( r_L ) (mm)</td>
<td>0.00005</td>
<td>0.00175</td>
</tr>
<tr>
<td>Crown drop, ( r_c ) (mm)</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>Roller length, ( L ) (mm)</td>
<td>-</td>
<td>10</td>
</tr>
<tr>
<td>Rolling element diameter, ( d_b ) (mm)</td>
<td>9.525</td>
<td>9.525</td>
</tr>
<tr>
<td>Unloaded contact angle, ( \alpha_0 ) (degree)</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>Unloaded center distance, ( A_0 ) (mm)</td>
<td>0.05</td>
<td>-</td>
</tr>
<tr>
<td>Inner race groove curvature radius, ( r_j ) (mm)</td>
<td>19.7625</td>
<td>19.7625</td>
</tr>
</tbody>
</table>

During the formulation we had assumed that the variation of stiffness about their mean value is not much, hence we can use average value of the stiffness for all practical
purposes. The rolling element bearing stiffness coefficients as a function of normalized element-to-element angular distance \( \frac{\Psi}{\Psi_T} \) are given in figures (11-16).

\[
\frac{\Psi}{\Psi_T} = \frac{\Psi}{360/Z}
\]

where \( Z = \) number of rolling elements.

Figures (11-13) indicate variation of stiffness for angular contact ball bearing for \( (\alpha_0 = 30^0) \) and figures (14-16) indicates variation of stiffness for taper roller bearing for taper angle \( (\alpha_0 = 30^0) \). Almost for all stiffness coefficients variation is observed as some function of either sine or cosine. As expected, the variation of stiffness is observed to be symmetric about the ratio \( \frac{\Psi}{\Psi_T} = 0.5 \). Dominant stiffness terms like \( K_{xx}, K_{yy}, K_{zz}, K_{\theta x\theta x}, K_{\theta y\theta y}, K_{x\theta y}, K_{z\theta y}, K_{xz} \) and \( K_{y\theta x} \) will have some non zero average value, but other cross coupling terms will have average value equal to zero. Figures (11-12) and figures (14-15) indicate variation of stiffness as a function \( \frac{\Psi}{\Psi_T} \) for number of rolling elements \( Z = 9 \), but more parametric analysis shows that if we increase number of elements then the spikes observed in variation of stiffness will be smoothen out. The results are as shown in figure 13 for ball bearing and in figure 16 for roller bearing for \( Z = 12 \). For all parameters remain same if we increase number of elements from 9 to 12 then at any instant of time load is shared by more number of balls, this will smoothen out distribution of load and hence the variation in stiffness is more uniform. At the same time stiffness of bearing is higher if we increase number of elements by keeping all parameters same. In general assumption of mean stiffness is valid since variation of stiffness coefficients is within \( (0.1-1\%) \) for most of the non-zero stiffness coefficients.
Normalized element to element angular distance ($\Psi/\Psi_T$)

Figure 11. Variation of stiffness coefficients as a function of normalized element-to-element angular distance for $\delta x = 0.025$ mm, for Angular contact Ball Bearing. ($Z = 9$)
Figure 12. Variation of stiffness coefficients as a function of normalized element-to-element angular distance for $\delta x = 0.025 \text{ mm}$ and $\beta y = 0.005 \text{ rad}$, for Angular contact Ball Bearing. ($Z = 9$)
Normalized element to element angular distance ($\Psi/\Psi_T$)

Figure 13. Variation of stiffness coefficients as a function of normalized element-to-element angular distance for $\delta x = 0.025 \text{ mm}$, for Angular contact Ball Bearing. ($Z = 12$)
Normalized element to element angular distance ($\Psi/\Psi_T$)

Figure 14. Variation of stiffness coefficients as a function of normalized element-to-element angular distance for $\delta x = 0.025 \, mm$, for Roller Bearing. ($Z = 9$)
Normalized element to element angular distance ($\Psi/\Psi_T$)

Figure 15. Variation of stiffness coefficients as a function of normalized element-to-element angular distance for $\delta x = 0.025 \text{ mm}$ and $\beta y = 0.005 \text{ rad}$, for Roller bearing. ($Z = 9$)
Normalized element to element angular distance \( (\Psi / \Psi_T) \)

Figure 16. Variation of stiffness coefficients as a function of normalized element-to-element angular distance for \( \delta x = 0.025 \text{ mm} \), for Roller bearing. \((Z = 12)\)
Figure 17 shows the variation of stiffness as a function of unloaded contact angle for constant mean radial displacement for angular contact ball bearings. All diagonal elements like $K_{xx}$, $K_{yy}$, $K_{zz}$, $K_{\theta x\theta x}$, and $K_{\theta y\theta y}$ shows difference in values between ELTA and MLTV. Values of stiffness obtained by ELTA method are smaller in magnitude than MLTV because ELTA method uses elastic contact theory, which assumes elasticity of both inner race and rolling element on contrary to rigid race formulation used in MLTV method. Figure 18 indicates the variation of stiffness as a function of unloaded contact angle for constant mean radial displacement and constant mean angular misalignment for angular contact ball bearings. Variation of stiffness is very small for given range of unloaded contact angle hence almost straight lines are observed in both methods.

Variation of diagonal coefficients of element stiffness matrix like $K_{xx}$, $K_{yy}$, $K_{zz}$, $K_{\theta x\theta x}$, and $K_{\theta y\theta y}$ as a function of unloaded contact angle for roller bearings for constant mean radial displacement is shown in figure 19. All diagonal stiffness terms obtained by ELTA method show same trend in variation as MLTV, but as expected values of stiffness coefficients for ELTA are on lower side than MLTV. Figure 19 shows stiffness coefficients $K_{xx}$, $K_{yy}$ values have maximum difference for small values of unloaded contact angle but $K_{zz}$, $K_{\theta x\theta x}$ and $K_{\theta y\theta y}$ have maximum difference for larger values of unloaded contact angle. This is because stiffness calculation scheme in radial direction dominated by cosine function and in axial direction it is dominated by sine function of unloaded contact angle. Figure 20 shows variation of stiffness for constant mean radial displacement as well as constant mean angular misalignment for roller bearings. The diagonal stiffness coefficients $K_{xx}$, $K_{yy}$, $K_{zz}$ and $K_{\theta x\theta x}$ follows same trend as MLTV, but
Unloaded contact angle (degrees)

Figure 17. Comparison of stiffness coefficients as a function of unloaded contact angle between ELTA (-) and MLTV (--) for $\delta x = 0.025 \, mm$, for Angular contact Ball Bearing.

for stiffness coefficient $K_{\theta x\theta y}$, ELTA method shows much smooth variation compared with MLTV.

Till now we had considered all stiffness variation against unloaded contact angle for displacement as preload, but in real life condition it is impractical to measure actual preload in terms of individual displacement components so usually force preload conditions are used for analysis. Figures (21-22) indicate variation of diagonal stiffness
coefficients for various force input condition for angular contact ball bearings. As shown in figure 21 for preload $F_x = 1000N$ and $M_y = 1000Nm$ MLTV model shows that bearing system will become unstable for unloaded contact angle around 73 degrees, but ELTA model shows smooth variation of stiffness till 90 degrees. Similar results are obtained for preloading condition of $F_y = 1000N$ and $M_x = 1000Nm$ as shown in figure 22, where...
Unloaded contact angle (degrees)
Figure 19. Comparison of stiffness coefficients as a function of unloaded contact angle between ELTA (-) AND MLTV (--) for \( \delta x = 0.025 \text{ mm} \), for Roller Bearing.

MLTV model shows instability at much lower angle around 42 degrees, but even for this case ELTA model hold same trend of stiffness variation till 90 degrees.

Figures (23-24) indicate variation of diagonal stiffness coefficients for various force input condition for roller bearings. As shown in figure 23, for \( F_z = 1000N \) MLTV method shows abrupt spikes in variation of stiffness compared with smooth variation
Unloaded contact angle (degrees)

Figure 20. Comparison of stiffness coefficients as a function of unloaded contact angle between ELTA (-) and MLTV (--) for \( \delta y = 0.025 \, mm \) and \( \beta x = 0.005 \, rad \), for Roller Bearing.

observed in ELTV method. MLTV model shows that bearing system will become unstable for unloaded contact angle of 90 degrees, but ELTA model shows smooth variation of stiffness till 90 degrees. Figure 24 shows for preload \( F_y = 1000N \) MLTV shows random variation of stiffness values for unloaded contact angles from 0 to 89 degrees.
Figure 21. Comparison of stiffness coefficients as a function of unloaded contact angle between ELTA (-) and MLTV (--) for force and moment input $F_x = 1000N$ and $M_y = 1000Nm$, for Angular contact Ball Bearing.

degrees, at 90 degree of unloaded contact angle abrupt spike is observed resulting into stiffness value to be infinite, but ELTA model shows smooth variation in stiffness for given range of unloaded contact angle and some practical value of stiffness for bearing
Unloaded contact angle (degrees)

Figure 22. Comparison of stiffness coefficients as a function of unloaded contact angle between ELTA (-) and MLTV (--) for force and moment input $F_y = 1000N$ and $M_x = 1000Nm$, for Angular contact Ball Bearing.

system for unloaded contact angle equal to 90 degrees. So in general, ELTA model produces much better results than previous MLTV method for force input condition.

More parametric studies are carried out to check effectiveness of ELTA method.

Variation of the rolling element bearing stiffness coefficients for a constant unloaded
Figure 23. Comparison of stiffness coefficients as a function of unloaded contact angle between ELTA (-) and MLTV (--) for force and moment input $F_z = 1000N$ for Roller Bearing.

Unloaded contact angle (degrees)

contact angle ($\alpha_0 = 30^\circ$) as a function of constant mean radial displacement and constant mean angular misalignment are shown in figures from (25-28). Figure 25 shows the variation of diagonal stiffness coefficient as a function of preload in $X$ direction for angular contact ball bearing. Stiffness coefficient obtained from ELTA method shows lower values having same trend in variation as MLTV method. Figure 26 shows the
Unloaded contact angle (degrees)

Figure 24. Comparison of stiffness coefficients as a function of unloaded contact angle between ELTA (-) and MLTV (--) for force and moment input $F_y = 1000N$ for Roller Bearing.

variation of stiffness coefficients for constant mean radial displacement in $X$ direction as a function of constant mean angular misalignment about $X$ axis for angular contact ball bearing. Here we can observe diagonal stiffness coefficients $K_{xx}$ and $K_{yy}$ shows larger difference in magnitude between two methods compared with other diagonal stiffness
Preload in x-direction, $\delta_x (mm)$

Figure 25. Comparison of stiffness coefficients as a function of preload in x-direction between ELTA (-) and MLTV (--) for Angular contact Ball Bearing.

coefficients. In case of roller bearing, figure 27 shows the variation of all diagonal terms of stiffness matrix as a function of constant mean radial displacement in $X$ direction. Two methods show higher difference in magnitude of stiffness terms. Figure 28 indicates the variation of $K_{xx}$, $K_{yy}$, $K_{zz}$, and $K_{\theta_x\theta_x}$ for constant mean radial displacement in $X$ direction as a function of constant mean angular misalignment about $X$ axis for roller bearing. ELTA
Preload in x-direction, $\beta_x (\text{rad})$

Figure 26. Comparison of stiffness coefficients as a function of angular misalignment about x-axis between ELTA (-) and MLTV (--) for Angular contact Ball Bearing.

method shows smooth variation in stiffness coefficients compared to sudden drop in values of stiffness at $\beta_x = 0.001$ rad for MLTV method. In this figure variation of $K_{\theta_x \theta_y}$ is not shown since MLTV method gives infinite value of stiffness at $\beta_x = 0.001$ rad, on contrary ELTA method shows some finite value of stiffness. ELTA method shows higher values of stiffness for variation of $K_{zz}$ compared to MLTV method.
Preload in x-direction, δx (mm)

Figure 27. Comparison of stiffness coefficients as a function of preload in x-direction between ELTA (-) and MLTV (--) for Roller Bearing.
Preload in x-direction, $\beta_x \ (\text{rad})$

Figure 28. Comparison of stiffness coefficients as a function of angular misalignment about x-axis between ELTA (-) and MLTV (--) for Roller Bearing.

In case of angular contact ball bearings, the centrifugal forces and gyroscopic moment will increase the contact angle at inner race and will decrease the same at outer race hence it will change the load displacement characteristics of bearing and hence significant change in stiffness of bearing is observed. Effect of centrifugal forces and gyroscopic moment on diagonal stiffness coefficients as a function of unloaded contact
Unloaded contact angle (degrees)

Figure 29. Comparison of stiffness coefficients as a function of unloaded contact angle between with (--) and without (-) centrifugal force consideration for $\delta_x = 0.025$ mm for Angular contact Ball Bearing.

angle for constant mean radial displacement for angular contact ball bearings is shown in figure 29. All diagonal elements show higher values of stiffness if effect of centrifugal force and gyroscopic moment is considered. It should be noted that at unloaded contact angle equal to zero, centrifugal force and gyroscopic moment has no effect on stiffness matrix. Figure 30 shows the variation of diagonal stiffness coefficients as a function of
Unloaded contact angle (degrees)

Figure 30. Comparison of stiffness coefficients as a function of unloaded contact angle between with (--) and without (-) centrifugal force consideration for $\delta_y = 0.025$ mm for Angular contact Ball Bearing.

unload contact angle for constant mean radial displacement for angular contact ball bearing. Figure 30 shows similar type of variation of stiffness coefficients as observed in figure 29. Figure 31 shows the effect of centrifugal force and gyroscopic moment on the variation diagonal stiffness coefficients for constant mean radial displacement in $X$ direction as a function of unloaded contact angle for roller bearings. Even after
Unloaded contact angle (degrees)

Figure 31. Comparison of stiffness coefficients as a function of unloaded contact angle between without (-) and with (--) centrifugal force consideration for $\delta_x = 0.025$ mm for Roller Bearing.

consideration of centrifugal force and gyroscopic moment, for roller bearings very little variation on stiffness coefficients is observed. Comparison of stiffness as a function of unloaded contact angle for constant mean radial displacement in $Y$ direction and constant mean angular displacement about $X$-axis for roller bearings is shown in figure 32. Even for this case variation of stiffness is very small. Although centrifugal force and
Unloaded contact angle (degrees)

Figure 32. Comparison of stiffness coefficients as a function of unloaded contact angle between without (-) and with (--) centrifugal force consideration for $\delta_y = 0.025$ mm and $\beta_x = 0.005$ rad for Roller Bearing.

Gyroscopic moment changes loading at inner race and outer race but, in case of roller bearings contact angle, which is actually taper angle, remains unchanged.
4.2 Case study 1: Single-stage rotor system

Single stage rotor system shown in figure (8) is used for vibration response analysis. This system consist of a rigid rotor mounted on shaft, shaft supported between two identical rolling element bearings, a load, a prime mover and couplings. Except bearing all other elements of the system are considered to be rigid. Any lumped parameter model of bearing possesses five degrees of freedom, three in translation and two in rotation. Primary source of excitation will be mass imbalance of rotor.

Table 2

Design Parameters Single-Stage Rotor System

<table>
<thead>
<tr>
<th>Parameter Description</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of rotor, $M_R$ (Kg)</td>
<td>0.86</td>
</tr>
<tr>
<td>Mass moment of Inertia, $I_{xR}, I_{yR}$ (Kg m$^2$)</td>
<td>$7.534 \times 10^{-3}$</td>
</tr>
<tr>
<td>Rotor- bearing geometry center length, $l_r$ (m)</td>
<td>$8 \times 10^{-2}$</td>
</tr>
<tr>
<td>$d^*$, (mm), $e^{**}$, (mm)</td>
<td>$1 \times 10^{-3}$</td>
</tr>
<tr>
<td>Mean radial displacements, $\delta_{xm}$ (m)</td>
<td>$0.025 \times 10^{-3}$</td>
</tr>
<tr>
<td>Mean radial displacements, $\delta_{ym}$ (m)</td>
<td>$0.025 \times 10^{-3}$</td>
</tr>
<tr>
<td>Mean angular misalignment, $\beta_{ym}$ (m)</td>
<td>$0.025 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

* Axial distance between rotor geometric center and mass center.
** Eccentricity of rotor.

Figure 33 shows the comparison of the dynamic response due to mass imbalance of the system for mean constant radial displacement in $X$ and $Y$ direction between ELTA and LTV method for angular contact ball bearings. Close match for angular displacement
amplitude about $X$ and $Y$-axis, but very small amount of deviation is observed for radial and axial displacement amplitude. Comparison of the dynamic response of this system due to mass imbalance for a constant mean radial displacement and constant mean angular misalignment between ELTA and LTV method are shown in figure 34 for angular contact ball bearings. As shown in figure 34, very close match is observed for radial displacement amplitude in $X$, $Y$ and $Z$ direction. For angular displacement amplitude about $X$ and $Y$-axis close match is observed in natural frequency, but some deviation is observed between two response spectra.

Figure 35 indicates the comparison of the dynamic response due to mass imbalance of the system for mean constant radial displacement in $X$ and $Y$ direction between ELTA and LTV method for roller bearings. As shown in figure 35, both dynamic response spectra match closely for displacement amplitude in radial direction and angular displacement amplitude about $X$ and $Y$-axis. Comparison of the dynamic response of this system due to mass imbalance for a constant mean radial displacement and constant mean angular misalignment between ELTA and LTV method are shown in figure 36 for roller bearings. Radial displacement amplitude shows close match between response spectra and natural frequencies of the system. Angular displacement amplitude about $X$ and $Y$-axis shows good match at resonance frequencies of the system but significant variation in response spectra.

Centrifugal force and gyroscopic moment significantly changes principal frequencies of single-stage rotor system, which are observed from comparison between dynamic response of system with and without centrifugal force and gyroscopic moment.
Figure 33. Comparison of Dynamic response of single stage rotor system due to dynamic unbalance force between ELTA (*) and LTV (--) for a constant radial deformation $\delta_x = 0.025$ mm and $\delta_y = 0.025$ mm for Angular contact Ball Bearing: (1) radial displacement amplitude in x- direction; (2) radial displacement amplitude in y- direction; (3) radial displacement amplitude in z- direction; (4) angular displacement amplitude about x- axis.
Figure 34. Comparison of Dynamic response of single stage rotor system due to dynamic unbalance force between ELTA (*) and LTV (--) for a constant radial deformation $\delta_y = 0.025$ mm and $\beta_x = 0.025$ mm for Angular contact Ball Bearing: (1) radial displacement amplitude in x-direction; (2) radial displacement amplitude in y-direction; (3) radial displacement amplitude in z-direction; (4) angular displacement amplitude about x-axis.
Figure 35. Comparison of Dynamic response of single stage rotor system due to dynamic unbalance force between ELTA (*) and LTV (--) for a constant radial deformation $\delta_x = 0.025 \text{ mm}$ and $\delta_y = 0.025 \text{ mm}$ for Roller Bearing: (1) radial displacement amplitude in x-direction; (2) radial displacement amplitude in y-direction; (3) angular displacement amplitude about x-axis; (4) angular displacement amplitude about y-axis.
Figure 36. Comparison of Dynamic response of single stage rotor system due to dynamic unbalance force between ELTA (*) and LTV (--) for a constant radial deformation $\delta_x = 0.025$ mm and $\beta_x = 0.025$ mm for Roller Bearing: (1) radial displacement amplitude in x-direction; (2) radial displacement amplitude in y-direction; (3) angular displacement amplitude about x-axis; (4) angular displacement amplitude about y-axis.
Table 3 and 4 indicates the comparison of natural frequencies with and without centrifugal force and gyroscopic moment and percentage deviation between the same for single-stage rotor system.

The deviation $\Delta$ in frequency is represented in percentage form in following table,

$$
\Delta(\%) = \left( \frac{f_{ELTA\ with\ CF} - f_{ELTA\ without\ CF}}{f_{ELTA\ without\ CF}} \right) \times 100
$$

(3.60)

Table 3

*Comparison of Principal Natural Frequencies in Hz for the Single-Stage Rotor System for Angular Contact Ball Bearing*

<table>
<thead>
<tr>
<th>Mode</th>
<th>Constant mean radial displacement $\delta_x = 0.025$ (mm)</th>
<th>Constant mean radial displacement and mean angular misalignment $\delta_y = 0.025$ (mm), $\beta_x = 0.005$ (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f_{ELTA\ without\ CF}$</td>
<td>$f_{ELTA\ with\ CF}$</td>
</tr>
<tr>
<td>1</td>
<td>1485.8</td>
<td>1491</td>
</tr>
<tr>
<td>2</td>
<td>2147.6</td>
<td>2163.4</td>
</tr>
<tr>
<td>3</td>
<td>2488.5</td>
<td>2506.8</td>
</tr>
<tr>
<td>4</td>
<td>3143.7</td>
<td>3144</td>
</tr>
<tr>
<td>5</td>
<td>3225</td>
<td>3227.6</td>
</tr>
<tr>
<td>Mode</td>
<td>$\delta_x = 0.025$ (mm)</td>
<td>$\delta_y = 0.025$ (mm)</td>
</tr>
<tr>
<td>------</td>
<td>------------------------</td>
<td>------------------------</td>
</tr>
<tr>
<td></td>
<td>$f_{ELTAwithoutCF}$</td>
<td>$f_{ELTAwithCF}$</td>
</tr>
<tr>
<td>1</td>
<td>3072.8</td>
<td>3065.9</td>
</tr>
<tr>
<td>2</td>
<td>3559.1</td>
<td>3559.1</td>
</tr>
<tr>
<td>3</td>
<td>4505</td>
<td>4505</td>
</tr>
<tr>
<td>4</td>
<td>5197.2</td>
<td>5173.6</td>
</tr>
</tbody>
</table>

Figure 37 shows the comparison of the dynamic response of system due to mass imbalance for a constant mean radial displacement in $X$ direction for angular contact ball bearings. As shown in figure, response of system in radial $X$ direction is almost unaffected even if we consider centrifugal force and gyroscopic moment, but for response in radial $Y$, $Z$ direction and angular displacement about $X$ and $Y$-axis will get shifted rightwards. Appreciable frequency shift of 0.3 to 0.7 % is observed for first 3 lower modes of vibration, but very little shift in frequency is observed for higher modes of vibration. Dynamic response of the system due to mass imbalance for a constant mean radial displacement in $Y$ direction and for constant mean angular misalignment about $X$-axis for angular contact ball bearings is shown in figure 38. In figure 38, some effect of centrifugal force and gyroscopic moment of ball is observed on radial displacement amplitude in $X$, $Y$, $Z$ direction as well as angular displacement amplitude about $X$ and $Y$-
Figure 39 shows the comparison of dynamic response of the system with and without centrifugal force due to mass imbalance for a constant mean radial displacement $X$ direction for roller bearings. Although loading on outer race increases, significant changes in stiffness values are not observed for roller bearing. First and fourth mode of vibration shows little change in frequency of $0.2258\%$ and $0.4535\%$ respectively, but second and third mode of vibration will remain unaffected by the effect of centrifugal force and gyroscopic moment showing frequency shift of $0.0011\%$ and $0.0009\%$.

Comparison of the dynamic response of the system with and without centrifugal force due to mass imbalance for a constant mean radial displacement $Y$ direction for roller bearings is shown in figure 40. Frequency shift of $0.2261\%$ is observed for first and third mode of vibration and second and fourth modes will show no difference in frequencies.
Figure 37. Comparison of Dynamic response of single stage rotor system due to dynamic unbalance force, without (-) and with (--) centrifugal force consideration for a constant radial deformation $\delta_x = 0.025$ mm for Angular contact Ball Bearing: (1) radial displacement amplitude in x-direction; (2) radial displacement amplitude in y-direction; (3) radial displacement amplitude in z-direction; (4) angular displacement amplitude about x-axis.
Figure 38. Comparison of Dynamic response of single stage rotor system due to dynamic unbalance force, without (-) and with (--) centrifugal force consideration for a constant radial deformation $\delta_y = 0.025$ mm and angular misalignment $\beta_x = 0.005$ (rad) for Angular contact Ball Bearing: (1) radial displacement amplitude in x-direction; (2) radial displacement amplitude in y-direction; (3) radial displacement amplitude in z-direction; (4) angular displacement amplitude about x-axis.
Figure 39. Comparison of Dynamic response of single stage rotor system due to dynamic unbalance force, without (-) and with (--) centrifugal force consideration for a constant radial deformation $\delta_x = 0.025$ mm for Roller Bearing: (1) radial displacement amplitude in x- direction; (2) radial displacement amplitude in y- direction; (3) radial displacement amplitude in z- direction; (4) angular displacement amplitude about x- axis.
Figure 40. Comparison of Dynamic response of single stage rotor system due to dynamic unbalance force, without (−) and with (--) centrifugal force consideration for a constant radial deformation $\delta_y = 0.025$ mm for Roller Bearing: (1) radial displacement amplitude in x- direction; (2) radial displacement amplitude in y- direction; (3) radial displacement amplitude in z- direction; (4) angular displacement amplitude about x- axis.

4.3 Case study 2: Geared Rotor System

Schematic of geared rotor system shown in figure (9) is used for the analysis.

Simplest possible model of geared rotor system consist of pair of rigid shafts, pair of rigid
spur gears, four identical rolling element bearings, set of couplings, prime mover and load. Main objective of this study is to observe vibration transmission through bearings hence, all other elements of the system assumed to be rigid. Lumped parameter model of this system exhibits 12 DOF.

Table 5

**Design Parameters Geared Rotor System**

<table>
<thead>
<tr>
<th>Parameter Description</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of Gear and Pinion, $M_G, M_p$ (Kg)</td>
<td>0.5</td>
</tr>
<tr>
<td>Mass moment of Inertia of Gear and Pinion, $I_{yp}, I_{yg}$ (Kg m$^2$)</td>
<td>$1.5 \times 10^{-4}$</td>
</tr>
<tr>
<td>Mass moment of Inertia of Gear and Pinion, $I_{zp}, I_{zg}$ (Kg m$^2$)</td>
<td>$3 \times 10^{-4}$</td>
</tr>
<tr>
<td>Rotor- bearing geometry center length, $l_r$ (mm)</td>
<td>25</td>
</tr>
<tr>
<td>Gear and Pinion pitch radius, $R_G, R_p$ (mm)</td>
<td>34.6</td>
</tr>
<tr>
<td>Gear mesh stiffness, $K_m$ (N/m)</td>
<td>$1 \times 10^8$</td>
</tr>
<tr>
<td>Gear and Pinion Pressure angle (degree)</td>
<td>20°</td>
</tr>
<tr>
<td>Number of teeth on pinion and gear</td>
<td>28</td>
</tr>
<tr>
<td>Amplitude of kinematic transmission error, $e(t)$ (inch)</td>
<td>$1 \times 10^{-6}$</td>
</tr>
<tr>
<td>Mean radial displacements, $\delta_{xm}$ (mm)</td>
<td>0.025</td>
</tr>
<tr>
<td>Mean radial displacements, $\delta_{ym}$ (mm)</td>
<td>0.025</td>
</tr>
<tr>
<td>Mean angular misalignment, $\beta_{ym}$ (rad)</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Figure 41 shows the comparison of the dynamic response due to kinematic transmission error of the geared rotor system for mean constant radial displacement in $X$. 

90
and $Y$ direction between ELTA and LTV method for angular contact ball bearings. Response of the system in radial $Y$ direction and about $Z$ axis for pinion and gear are studied, since $Y$ axis is assumed to be parallel to gear mesh line of action and maximum response is observed in this direction. Excellent match for radial displacement amplitude in $Y$ direction and angular displacement amplitude about $Z$-axis for both pinion and gear is observed. Comparison of the dynamic response of the system due to kinematic transmission error for a constant mean radial displacement in $Y$ direction and constant mean angular misalignment about the $Z$-axis between ELTA and LTV method are shown in Figure 42 for angular contact ball bearings. As shown in figure 42, very close match is observed between two response spectra for radial displacement amplitude in $Y$ direction and angular displacement amplitude about $Z$-axis for both pinion and gear.

Figure 43 indicates the comparison of the dynamic response due to kinematic transmission error of the system for mean constant radial displacement in $Y$ and $Z$ direction between ELTA and LTV method for roller bearings. As shown in figure 43, both dynamic response spectra match closely for radial displacement amplitude in $Y$ direction and angular displacement amplitude about $Z$-axis for pinion as well as gear. Comparison of the dynamic response of this system due to kinematic transmission error for a constant mean radial displacement in $X$ direction and constant mean angular misalignment about $X$-axis between ELTA and LTV method are shown in figure 44 for roller bearings. Figure 44 shows, the dynamic response of the system by ELTA model is deviated slightly from LTV model for radial displacement amplitude in $X$ and $Y$ direction.
Figure 41. Comparison of Dynamic response of Geared rotor system due to dynamic unbalance force between ELTA (*) and LTV (--), for a constant radial deformation $\delta_x = 0.025$ mm and $\delta_y = 0.025$ mm for Angular contact Ball Bearing: (1) radial displacement amplitude in y- direction for gear; (2) radial displacement amplitude in y- direction for pinion; (3) angular displacement amplitude about z- axis for gear; (4) angular displacement amplitude about z- axis for pinion.
Figure 42. Comparison of Dynamic response of Geared rotor system due to dynamic unbalance force between ELTA (*) and LTV (--) for a constant radial deformation $\delta_y = 0.025$ mm and $\beta_x = 0.025$ mm for Angular contact Ball Bearing: (1) radial displacement amplitude in y- direction for gear; (2) radial displacement amplitude in y- direction for pinion; (3) angular displacement amplitude about z- axis for gear; (4) angular displacement amplitude about z- axis for pinion.
Figure 43. Dynamic response of Geared rotor system due to dynamic unbalance force between ELTA (*) and LTV (--) for a constant radial deformation $\delta_y = 0.025$ mm $\delta_z = 0.025$ mm for Roller Bearing: (1) radial displacement amplitude in y- direction for gear; (2) radial displacement amplitude in y- direction for pinion; (3) angular displacement amplitude about z- axis for gear; (4) angular displacement amplitude about z- axis for pinion.
Figure 44. Dynamic response of Geared rotor system due to dynamic unbalance force between ELTA (*) and LTV (--) for a constant radial deformation $\delta_x = 0.025$ mm $\beta_x = 0.025$ mm for Roller Bearing: (1) radial displacement amplitude in y- direction for gear; (2) radial displacement amplitude in y- direction for pinion; (3) angular displacement amplitude about z- axis for gear; (4) angular displacement amplitude about z- axis for pinion.
Table 6

**Comparison of Principal Natural Frequencies in Hz for the Geared Rotor System for Angular Contact Ball Bearing**

<table>
<thead>
<tr>
<th>Mode</th>
<th>Constant mean radial displacement $\delta_x = 0.025$ (mm)</th>
<th>Constant mean radial displacement and mean angular misalignment $\delta_y = 0.025$ (mm), $\beta_x = 0.005$ (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f_{ELTA\text{withoutCF}}$</td>
<td>$f_{ELTA\text{withCF}}$</td>
</tr>
<tr>
<td>1</td>
<td>1448.8</td>
<td>1460.5</td>
</tr>
<tr>
<td>2</td>
<td>1448.8</td>
<td>1460.5</td>
</tr>
<tr>
<td>3</td>
<td>2293.8</td>
<td>2286.1</td>
</tr>
<tr>
<td>4</td>
<td>2631.3</td>
<td>2649.4</td>
</tr>
<tr>
<td>5</td>
<td>4122.9</td>
<td>4123.4</td>
</tr>
<tr>
<td>6</td>
<td>4122.9</td>
<td>4123.4</td>
</tr>
<tr>
<td>7</td>
<td>4538</td>
<td>4549.6</td>
</tr>
<tr>
<td>8</td>
<td>5159.9</td>
<td>5198</td>
</tr>
<tr>
<td>9</td>
<td>5159.9</td>
<td>5198</td>
</tr>
<tr>
<td>10</td>
<td>5421.6</td>
<td>5428.9</td>
</tr>
<tr>
<td>11</td>
<td>7044.5</td>
<td>7046.6</td>
</tr>
<tr>
<td>12</td>
<td>7044.5</td>
<td>7046.6</td>
</tr>
</tbody>
</table>

Table 6 and 7 indicates the comparison of natural frequencies with and without centrifugal force and percentage deviation between the same for geared rotor system.
Table 7

Comparison of Principal Natural Frequencies in Hz for the Geared Rotor System for Roller Bearing

<table>
<thead>
<tr>
<th>Mode</th>
<th>Constant mean radial displacement $\delta_x = 0.025$ (mm)</th>
<th>Constant mean radial displacement and mean angular misalignment $\delta_x = 0.025$ (mm), $\beta_y = 0.005$ (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f_{ELTA\text{without}CF}$</td>
<td>$f_{ELTA\text{with}CF}$</td>
</tr>
<tr>
<td>1</td>
<td>1867.8</td>
<td>1867.8</td>
</tr>
<tr>
<td>2</td>
<td>2865.1</td>
<td>2865.1</td>
</tr>
<tr>
<td>3</td>
<td>2865.1</td>
<td>2865.1</td>
</tr>
<tr>
<td>4</td>
<td>3532.6</td>
<td>3532.6</td>
</tr>
<tr>
<td>5</td>
<td>5313.2</td>
<td>5313.2</td>
</tr>
<tr>
<td>6</td>
<td>5908.2</td>
<td>5908.2</td>
</tr>
<tr>
<td>7</td>
<td>5908.2</td>
<td>5908.2</td>
</tr>
<tr>
<td>8</td>
<td>6058.2</td>
<td>6058.2</td>
</tr>
<tr>
<td>9</td>
<td>7405.8</td>
<td>7405.8</td>
</tr>
<tr>
<td>10</td>
<td>7405.8</td>
<td>7405.8</td>
</tr>
<tr>
<td>11</td>
<td>10724</td>
<td>10724</td>
</tr>
<tr>
<td>12</td>
<td>10724</td>
<td>10724</td>
</tr>
</tbody>
</table>

Centrifugal forces play key role in determination vibration frequencies of geared rotor system. Figure 45 shows the comparison of the dynamic response of system with and without centrifugal forces due to kinematic transmission error for a constant mean
radial displacement in $X$ direction for angular contact ball bearings. Because of symmetric nature of arrangement repeated roots are observed for modes 1, 5, 8 and 11. Except for third mode of vibration all modes of vibration shows increase in value of natural frequency from 0.01 % to 0.8 %. Third mode of vibration shows decrease in value of natural frequency by 0.3378 %. Dynamic response of the system due to kinematic transmission error for a constant mean radial displacement in $Y$ direction and for constant mean angular misalignment about $X$-axis for angular contact ball bearings is shown in figure 46. Repeated roots are observed at modes 4 and 11. Frequency shift of more than 1 % in natural frequencies is observed for modes 4, 5, 7 and 8. Mode 1, 11 and 12 shows drop in value of natural frequency by very small amount.

Figure 47 shows the comparison of dynamic response of the system with and without centrifugal force due to kinematic transmission error for a constant mean radial displacement $X$ direction for roller bearings. Although loading on outer race increases, significant changes in stiffness values are not observed for roller bearing; hence no change is observed in values of natural frequencies of geared rotor system. Comparison of the dynamic response of the system with and without centrifugal force due to kinematic transmission error for a constant mean radial displacement $X$ direction and constant mean angular misalignment about $Y$-axis for roller bearings is shown in figure 48. Natural frequencies remain same even if we consider effect of centrifugal force and gyroscopic moment of roller.

So in general, we can conclude that natural frequencies of the system consisting of angular contact ball bearings will get affected by the centrifugal forces and gyroscopic moment of ball compared to rolling element bearings.
Figure 45. Comparison of Dynamic response of Geared rotor system due to dynamic unbalance force, without (-) and with (--) centrifugal force consideration for a constant radial deformation $\delta_x = 0.025$ mm for Angular contact Ball Bearing: (1) radial displacement amplitude in y- direction for gear; (2) radial displacement amplitude in y-direction for pinion; (3) angular displacement amplitude about z-axis for gear; (4) angular displacement amplitude about z-axis for pinion.
Figure 46. Comparison of Dynamic response of Geared rotor system due to dynamic unbalance force, without (-) and with (--) centrifugal force consideration for a constant radial deformation $\delta_y = 0.025$ mm for Angular contact Ball Bearing: (1) radial displacement amplitude in y-direction for gear; (2) radial displacement amplitude in y-direction for pinion; (3) angular displacement amplitude about z-axis for gear; (4) angular displacement amplitude about z-axis for pinion.
Figure 47. Comparison of Dynamic response of Geared rotor system due to dynamic unbalance force, without (-) and with (--) centrifugal force consideration for a constant radial deformation $\delta_x = 0.025$ mm for Roller Bearing: (1) radial displacement amplitude in $y$-direction for gear; (2) radial displacement amplitude in $y$-direction for pinion; (3) angular displacement amplitude about $z$-axis for gear; (4) angular displacement amplitude about $z$-axis for pinion.
Figure 48. Comparison of Dynamic response of Geared rotor system due to dynamic unbalance force, without (-) and with (--) centrifugal force consideration for a constant radial deformation $\delta_x = 0.025$ mm and angular misalignment $\beta_y = 0.005$ for Roller Bearing: (1) radial displacement amplitude in y- direction for gear; (2) radial displacement amplitude in y- direction for pinion; (3) angular displacement amplitude about z- axis for gear; (4) angular displacement amplitude about z- axis for pinion.
CHAPTER 5
PARAMETRIC STUDIES

Various parametric studies are carried out to study the force transmission through angular contact bearings. Figure (49) shows the total dynamic load and its components at the various ball locations for constant mean radial displacement in $X$ direction and constant mean axial displacement in $Z$ direction as a function of number of balls. For this complex loading condition we can easily see that initially only few balls were loaded and as number of ball increases load are taken by more number of balls. Load distribution will be much smoother if we increase number of balls and balls will not see much variation in dynamic loading during rotation of bearings. Figure (50) shows the total dynamic load and its components at the various ball locations for constant mean radial displacement in $Y$ direction and for constant mean angular misalignment about $Y$-axis as a function of radial clearance. For complex loading condition we don’t much variation in dynamic force for clearance 0, 0.0001 and 0.001 mm this is because of less variation in extent of loading zone but force transmission capacity of bearing reduces. For clearance of 0.01 mm force transmission capacity as well as extent of loading zone reduces.

Figure (51) shows the total dynamic load and its components at the various roller locations for constant mean radial displacement in $X$ direction and constant mean axial displacement in $Z$ direction as a function of number of roller. For this complex loading condition we can easily see that initially only few rollers were loaded and as number of roller increases load are taken by more number of rollers. Load distribution will be much smoother if we increase number of rollers and rollers will not see much variation in
Figure 49. Variation of forces transmitted through bearing at various angular locations for $\delta x = 0.025 \text{ mm}$ and $\delta z = 0.025 \text{ mm}$, for Angular contact Ball Bearing ($Z = 9 (-)$, $11 (-\cdot-)$ and $13 (-\cdot-)$).

dynamic loading during rotation of bearings. Figure (52) shows the total dynamic load and its components at the various roller locations for constant mean radial displacement in $Y$ direction and for constant mean angular misalignment about $Y$-axis as a function of radial clearance. For complex loading condition we don’t much variation in dynamic force for clearance 0, 0.0001 and 0.001 mm this is because of less variation in extent of
Hence, from the point of view of design we can tune the stiffness of system by selecting appropriate values of number of rolling elements and radial clearance. If system demands higher natural frequency and smoother load distribution then it is desirable to
increase number of rolling element. In order to improve performance of bearing system zero radial clearance is desirable but practically it is impossible to have zero radial clearance, hence keep radial clearance as minimum as possible.

Figure 51. Variation of forces transmitted through bearing at various angular locations for $\delta x = 0.025 \, mm$ and $\delta z = 0.025 \, mm$, for Roller Bearing ($Z = 9 (-)$, 11 (--) and 13 (-----)).
Figure 52. Variation of forces transmitted through bearing at various angular locations for $\delta_y = 0.025 \, \text{mm}$ and $\beta_y = 0.025 \, \text{mm}$, for Roller Bearing (clearance = 0 (−), 0.0001 (−−), 0.001 (− · −) and 0.01mm (·)).
CHAPTER 6
CONCLUSION AND FUTURE WORK

6.1 Conclusion

This research presents an elastic contact theory by considering higher order profile functions of contacting surfaces, and taking into the account of the elasticity of the ball as well as raceway structure of the bearing. Various parametric studies revealed that, the elements of the coefficient stiffness matrix obtained from this theory tend to be lower than those computed by the mean linear time-averaged theory (MLTV) [9]. The ELTA theory appears more accurate compared to the MLTV [9] theory especially for bearing load input conditions as the variation of the stiffness coefficients for the entire range of unloaded contact angle was captured more accurately. The ELTA theory also blends time variation characteristics like MLTV [9] method; hence an improved elastic model is proposed as an essence of this research. Taking into the account of the centrifugal forces and gyroscopic moments of the ball further widens the applicability of this theory. Dynamic model of geared rotor system is developed on the basis of this theory to investigate the vibration transmission characteristics through angular contact ball bearings. Deviation in natural frequencies of geared rotor system is observed after considering centrifugal force and gyroscopic moment of the ball. Various parametric studies will be helpful for bearing designer to select the appropriate bearing parameters. The elastic linear time-averaged theory is a systematic approach for predicting vibration transmission characteristics through bearings of a system efficiently without performing large-scale computation such as the finite element method. Further research is required to
incorporate various tribological issues like lubrication film thickness during formulation of the contact problem bearings.

### 6.2 Future Work:

We have illustrated a comprehensive approach to formulate stiffness matrix of rolling element bearings for predicting vibration transmission characteristics. Following are some of the recommendations for future work:

- Since the research was extensively oriented around dry contact bearings it does not take into account lubrication film thickness and their effect on stiffness matrix. More detail analysis is required to formulate stiffness matrix, which will account for above facts and replicate real life bearing compliance properties.

- Experimental validation of the stiffness values and principal frequencies for various case studies.

- Formulations of bearing contact problem with more mesh density, based on functional regularization method [22].

- Application of elastic linear time averaged theory for more complex machinery systems.

- Implementation of this model into non-linear finite element package to describe a new bearing element.

- Analyzing contact between rolling element and race as 2-D or 3-D contact problem to get displacement load distribution relationship.
REFERENCES


APPENDIX A

THE FLOW CHART OF MEAN LINEAR STIFFNESS CALCULATION

- Enter Bearing Input Parameters Initialize Kxx, Kyy, etc = 0

  Set counter ‘k’ for number of Balls (1,2,...z)
  Initialize Kx1, Ky1 etc = 0

  Set counter ‘g’ for various ball positions (1,2,…n)

  Calculate elastic deformation at contact between inner race and ball

  Determine contact area

  Calculate load at each ball position

  Calculate force components in x, y, z directions and moment about x, and y axis

  Calculate stiffness at each ball location at each degree of freedom

  \[ Kx1 = Kx1 + Kx \]

  \[ g \geq n \]

  yes

  \[ Kxx = Kxx + Kx \]

  no

  \[ k \geq z \]

  yes

  \[ Kbxx = Kxx/g \]
APPENDIX B

STIFFNESS MATRIX FOR LUMPED PARAMETER MODEL

Single-stage rotor system:

\[
[K] = \begin{bmatrix}
[K]_{11} & [K]_{12} & [K]_{13} & [K]_{14} & [K]_{15} \\
[K]_{21} & [K]_{22} & [K]_{23} & [K]_{24} & [K]_{25} \\
[K]_{31} & [K]_{32} & [K]_{33} & [K]_{34} & [K]_{35} \\
[K]_{41} & [K]_{42} & [K]_{43} & [K]_{44} & [K]_{45} \\
[K]_{51} & [K]_{52} & [K]_{53} & [K]_{54} & [K]_{55}
\end{bmatrix}
\]

\[
[K]_{11} = [K_{xx}]_l + [K_{xx}]_r, \quad [K]_{12} = [K_{xy}]_l + [K_{xy}]_r, \quad [K]_{13} = [K_{xz}]_l + [K_{xz}]_r,
\]

\[
[K]_{14} = [K_{x\theta\theta}]_l + [K_{x\theta\theta}]_r + \{[K_{xy}]_l - [K_{xy}]_r\}/r,
\]

\[
[K]_{15} = [K_{x\phi\phi}]_l + [K_{x\phi\phi}]_r + \{[K_{xx}]_l - [K_{xx}]_r\}/r,
\]

\[
[K]_{22} = [K_{yy}]_l + [K_{yy}]_r, \quad [K]_{23} = [K_{yz}]_l + [K_{yz}]_r,
\]

\[
[K]_{24} = [K_{y\theta\theta}]_l + [K_{y\theta\theta}]_r + \{[K_{yy}]_l - [K_{yy}]_r\}/r,
\]

\[
[K]_{25} = [K_{y\phi\phi}]_l + [K_{y\phi\phi}]_r + \{[K_{yx}]_l - [K_{yx}]_r\}/r,
\]

\[
[K]_{33} = [K_{zz}]_l + [K_{zz}]_r, \quad [K]_{34} = [K_{z\theta\theta}]_l + [K_{z\theta\theta}]_r + \{[K_{zy}]_l - [K_{zy}]_r\}/r,
\]

\[
[K]_{35} = [K_{z\phi\phi}]_l + [K_{z\phi\phi}]_r + \{[K_{xz}]_l - [K_{xz}]_r\}/r,
\]

\[
[K]_{44} = [K_{\theta\theta\theta}]_l + [K_{\theta\theta\theta}]_r + \{[K_{y\theta}]_l + [K_{y\theta}]_r\}/r^2 + \{[K_{\theta\phi}]_l + [K_{\theta\phi}]_r\}/r,
\]

\[
[K]_{45} = [K_{\theta\phi\phi}]_l + [K_{\theta\phi\phi}]_r + \{[K_{\theta\phi\phi}]_l + [K_{\theta\phi\phi}]_r\}/r,
\]

\[
[K]_{55} = [K_{\phi\phi\phi}]_l + [K_{\phi\phi\phi}]_r + \{[K_{\phi\phi\phi}]_l + [K_{\phi\phi\phi}]_r\}/r^2 + \{[K_{\phi\phi\phi}]_l + [K_{\phi\phi\phi}]_r\}/r,
\]
Geared Rotor System:

\[
[K] = \begin{bmatrix}
[K]_{11} & [K]_{12} & [K]_{13} & [K]_{14} & [K]_{15} & [K]_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\
[K]_{21} & [K]_{22} & [K]_{23} & [K]_{24} & [K]_{25} & [K]_{26} & 0 & -K_m & 0 & 0 & 0 & -K_m R_2 \\
[K]_{31} & [K]_{32} & [K]_{33} & [K]_{34} & [K]_{35} & [K]_{36} & 0 & 0 & 0 & 0 & 0 & 0 \\
[K]_{41} & [K]_{42} & [K]_{43} & [K]_{44} & [K]_{45} & [K]_{46} & 0 & 0 & 0 & 0 & 0 & 0 \\
[K]_{51} & [K]_{52} & [K]_{53} & [K]_{54} & [K]_{55} & [K]_{56} & 0 & 0 & 0 & 0 & 0 & 0 \\
[K]_{61} & [K]_{62} & [K]_{63} & [K]_{64} & [K]_{65} & [K]_{66} & 0 & K_m R_1 & 0 & 0 & 0 & K_m R_2 R_1 \\
0 & 0 & 0 & 0 & 0 & 0 & [K]_{17} & [K]_{18} & [K]_{19} & [K]_{20} & [K]_{21} & [K]_{22} \\
0 & -K_m & 0 & 0 & 0 & K_m R_1 & [K]_{37} & [K]_{38} & [K]_{39} & [K]_{40} & [K]_{41} & [K]_{42} \\
0 & 0 & 0 & 0 & 0 & 0 & [K]_{57} & [K]_{58} & [K]_{59} & [K]_{60} & [K]_{61} & [K]_{62} \\
0 & 0 & 0 & 0 & 0 & 0 & [K]_{107} & [K]_{108} & [K]_{109} & [K]_{110} & [K]_{111} & [K]_{112} \\
0 & 0 & 0 & 0 & 0 & 0 & [K]_{117} & [K]_{118} & [K]_{119} & [K]_{120} & [K]_{121} & [K]_{122} \\
0 & -K_m R_2 & 0 & 0 & 0 & -K_m R_2 R_1 & [K]_{127} & [K]_{128} & [K]_{129} & [K]_{130} & [K]_{131} & [K]_{132}
\end{bmatrix}
\]

\[
[K]_{11} = [K_{xx}]_1 + [K_{xx}]_1, \quad [K]_{12} = [K_{xy}]_1 + [K_{xy}]_1, \quad [K]_{13} = [K_{xz}]_1 + [K_{xz}]_1,
\]

\[
[K]_{14} = [K_{x\theta z}]_1 + [K_{x\theta z}]_1 + [K_{xy}]_1 - [K_{xy}]_1 I_r,
\]

\[
[K]_{15} = [K_{x\theta y}]_1 + [K_{x\theta y}]_1 + [K_{xx}]_1 - [K_{xx}]_1 I_r,
\]

\[
[K]_{16} = [K_{x\theta z}]_1 + [K_{x\theta z}]_1,
\]

\[
[K]_{22} = [K_{yy}]_1 + [K_{yy}]_1 + [K_{xx}]_1 + [K_{xx}]_1,
\]

\[
[K]_{23} = [K_{yz}]_1 + [K_{yz}]_1,
\]

\[
[K]_{24} = [K_{y\theta y}]_1 + [K_{y\theta y}]_1 + [K_{yy}]_1 - [K_{yy}]_1 I_r,
\]

\[
[K]_{25} = [K_{y\theta x}]_1 + [K_{y\theta x}]_1 + [K_{yy}]_1 - [K_{yy}]_1 I_r,
\]

\[
[K]_{33} = [K_{zz}]_1 + [K_{zz}]_1,
\]

\[
[K]_{34} = [K_{z\theta z}]_1 + [K_{z\theta z}]_1 + [K_{zz}]_1 - [K_{zz}]_1 I_r,
\]

\[
[K]_{35} = [K_{z\theta y}]_1 + [K_{z\theta y}]_1 + [K_{zz}]_1 - [K_{zz}]_1 I_r,
\]

\[
[K]_{36} = [K_{z\theta x}]_1 + [K_{z\theta x}]_1,
\]

\[
[K]_{44} = [K_{\theta \theta z}]_1 + [K_{\theta \theta z}]_1 + [K_{yy}]_1 + [K_{yy}]_1 I_r^2 + [K_{\theta \theta y}]_1 + [K_{\theta \theta y}]_1 I_r,
\]

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\[
K_{45} = \left[ K_{\theta_1\theta_1} \right] _2 + \left[ K_{\theta_1\theta_2} \right] _{12} + \left\{ \left[ K_{\theta_1\alpha} \right] _{12} + \left[ K_{\theta_1\alpha} \right] _{21} \right\} I_r, \quad \left[ K_{46} = \left[ K_{\theta_1\theta_1} \right] _1 + \left[ K_{\theta_1\theta_1} \right] _{12}, \right.
\]

\[
K_{55} = \left[ K_{\theta_1\theta_2} \right] _1 + \left[ K_{\theta_1\alpha} \right] _{12} + \left\{ \left[ K_{\theta_1\alpha} \right] _{12} + \left[ K_{\theta_1\alpha} \right] _{21} \right\} I_r^2 + \left\{ \left[ K_{\theta_1\alpha} \right] _{12} + \left[ K_{\theta_1\alpha} \right] _{21} \right\} I_r,
\]

\[
K_{56} = \left[ K_{\theta_1\theta_2} \right] _1 + \left[ K_{\theta_1\theta_2} \right] _{12}, \quad \left[ K_{56} = \left[ K_{\theta_1\theta_1} \right] _1 + \left[ K_{\theta_1\theta_1} \right] _{12} + K_m R^2_1, \right.
\]

\[
K_{77} = \left[ K_{xx} \right] _2 + \left[ K_{xx} \right] _{22}, \quad \left[ K_{78} = \left[ K_{xy} \right] _2 + \left[ K_{xy} \right] _{22}, \quad \left[ K_{79} = \left[ K_{xz} \right] _2 + \left[ K_{xz} \right] _{22}, \right.
\]

\[
K_{710} = \left[ K_{xt} \right] _2 + \left[ K_{xt} \right] _{22} + \left\{ \left[ K_{xt} \right] _{22} - \left[ K_{xt} \right] _{21} \right\} I_r, \quad \left[ K_{712} = \left[ K_{xt} \right] _2 + \left[ K_{xt} \right] _{22}, \right.
\]

\[
K_{s8} = \left[ K_{zy} \right] _2 + \left[ K_{zy} \right] _{22} + K_m, \quad \left[ K_{s9} = \left[ K_{yz} \right] _2 + \left[ K_{yz} \right] _{22}, \right.
\]

\[
K_{810} = \left[ K_{yt} \right] _2 + \left[ K_{yt} \right] _{22} + \left\{ \left[ K_{yt} \right] _{22} - \left[ K_{yt} \right] _{21} \right\} I_r, \quad \left[ K_{812} = \left[ K_{yt} \right] _2 + \left[ K_{yt} \right] _{22} + K_m R^2_1, \right.
\]

\[
K_{99} = \left[ K_{zz} \right] _2 + \left[ K_{zz} \right] _{22}, \quad \left[ K_{910} = \left[ K_{zt} \right] _2 + \left[ K_{zt} \right] _{22} + \left\{ \left[ K_{zt} \right] _{22} - \left[ K_{zt} \right] _{21} \right\} I_r, \quad \left[ K_{912} = \left[ K_{zt} \right] _2 + \left[ K_{zt} \right] _{22}, \right.
\]

\[
K_{j10} = \left[ K_{\alpha_1\alpha_1} \right] _2 + \left[ K_{\alpha_1\alpha_2} \right] _{22} + \left\{ \left[ K_{\alpha_1\alpha} \right] _{22} - \left[ K_{\alpha_1\alpha} \right] _{21} \right\} I_r, \quad \left[ K_{j12} = \left[ K_{\alpha_1\alpha} \right] _2 + \left[ K_{\alpha_1\alpha} \right] _{22}, \right.
\]

\[
K_{j010} = \left[ K_{\alpha_1\alpha_1} \right] _2 + \left[ K_{\alpha_1\alpha_2} \right] _{22} + \left\{ \left[ K_{\alpha_1\alpha} \right] _{22} + \left[ K_{\alpha_1\alpha} \right] _{21} \right\} I_r^2 + \left\{ \left[ K_{\alpha_1\alpha} \right] _{22} + \left[ K_{\alpha_1\alpha} \right] _{21} \right\} I_r, \quad \left[ K_{j012} = \left[ K_{\alpha_1\alpha} \right] _2 + \left[ K_{\alpha_1\alpha} \right] _{22}, \right.
\]

\[
K_{j011} = \left[ K_{\alpha_1\alpha_2} \right] _2 + \left[ K_{\alpha_1\alpha_2} \right] _{22} + \left\{ \left[ K_{\alpha_1\alpha} \right] _{22} + \left[ K_{\alpha_1\alpha} \right] _{21} \right\} I_r^2 + \left\{ \left[ K_{\alpha_1\alpha} \right] _{22} + \left[ K_{\alpha_1\alpha} \right] _{21} \right\} I_r, \quad \left[ K_{j012} = \left[ K_{\alpha_1\alpha} \right] _2 + \left[ K_{\alpha_1\alpha} \right] _{22}, \right.
\]

\[
K_{i111} = \left[ K_{\alpha_1\alpha_1} \right] _2 + \left[ K_{\alpha_1\alpha_2} \right] _{22} + \left\{ \left[ K_{\alpha_1\alpha} \right] _{22} + \left[ K_{\alpha_1\alpha} \right] _{21} \right\} I_r^2 + \left\{ \left[ K_{\alpha_1\alpha} \right] _{22} + \left[ K_{\alpha_1\alpha} \right] _{21} \right\} I_r, \quad \left[ K_{i112} = \left[ K_{\alpha_1\alpha} \right] _2 + \left[ K_{\alpha_1\alpha} \right] _{22} + K_m R^2_1, \right.
\]