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Chapter 1  Introduction and literature review

Contact mechanics, though classical, continues to be a subject of lively interest from many different perspectives. It generates a seemingly inexhaustible sequence of commercially important and challenging problems, possibly because of the physical effects that occur in the case of any contact between two surfaces, contact forces are the principal means by which loads are applied to a body and the resulting stress concentration is generally the most critical in the body.

Historically the development of the subject stems from the famous paper of Hertz (1) giving a restricted solution for the frictionless contact of two elastic bodies of ellipsoidal profile. Progress in contact mechanics in the second half of the 21st century has been largely associated with the removal of these restrictions. Important contributions, extensions, and applications have been developed by Ahmedi, keer, and Mura (2); Braber (3); Bental and Johnson (4); Boussinesq (5); Bryant and Keer (6); Carter (7); Cerruti (8); Conry and Seireg (9); Cooper (10); Dunders (11) ; Dyson(12); Gladwell (13); Goodman (14); Johnson(15); Kalker and Piotrowski(16); Keer, Dunders, and Tsai (17); Love (18); Mindlin (19); Lure (20); Paul and Hashemi (21); Poritsky (22); Sackfiled and Hills (23); Singh and Paul(24).

When two bodies come in contact, two cases can arise. In the first case; which will be considered in this research work; when the two elastic bodies are pressed together, the movement is so slow such that the dynamic effects can be neglected and it is assumed
that under the effect of the displacing force, one body is situated on the other in a state of equilibrium. In the second case dynamic effects are considered for which different solution methods are used which will not be investigated in this research work.

Although friction can often be ignored especially in the case of very smooth or lubricated bodies in contact, in many practical cases friction occurs and cannot be ignored. In solving problems involving friction substantial mathematical difficulties arise which complicates the solution to a far greater extent. The problems involving friction was investigated by many researchers. Valuable contributions were made by Catteneo(26) Who realized that the contact zone must be divided into regions of stick and slip in order to fulfil Coulomb in a pointwise manner, he solved the three dimensional contact problem for quadratic surface under an assumed tangential traction. The results was later experimentally confirmed and extended by Mindlin and Deresiewicz (27); Jager (28); and Ciavarella (29). Cattaneo and Mindlin’s solution related the stick zone radius to the radius of contact in the cases of cylinders and spheres in contact under symmetric normal and tangential tractions. These results were later studied by Goodman (1962), Hills, and Sackfield for dissimilar materials. The results was carried over for very limited three-dimensional contact problems with Ciavarella (29); Jager (28); Klarring (30); and later with Anderson (31).

For a long time no substantial progress was made in the field of three-dimensional contact. Most of the results on the subject of the three dimensional contact problems were obtained by Soviet scientists, many of which were translated into English because of its
scientific value. This was clear in the works of Dinnik (25); Belyaev, and Steirman who gave the solution of the contact problem in which the touching bodies is a paraboloid of higher degrees.

As it is impossible to account for every author who contributed to solving theoretical contact problem, it is recommended to refer to Gladwell (32) who provided a compendious treatment of the various contact geometries that had been treated including the invaluable survey of the rich Russian literature of the subject, and Johnson (33) who gives an excellent overview of contact problems that have been under considerations with good balance between mathematical thoroughness and engineering practicality. A common feature of this theoretical stage is that the geometry and the deformation of a body is usually assumed in such a way that available mathematical and mechanical tools can be used to obtain a closed form solution of the problem. This approach was very restrictive and only applies to a very special class of problems.

Due to the limitations of analytical solutions of contact, and the fact that in most practical cases suitable analytical solutions do not exist, the need for a straightforward numerical method is apparent. The contacting bodies may have complicated geometries and loading profile and the solution in its final form involves solving a system of algebraic equations instead of obtaining a closed form solution. Different numerical methods were employed; they have been categorized under diverse headings such as finite element methods, Boundary element methods, meshless methods, and variational methods.
Whenever large stress and strain gradients are present such as in contact problems the finite element method (FEM) is characteristically insufficient. The numerical treatment of contact constraints within the finite element techniques is based on two main strategies, i.e. penalty method (34, 35, 36, 37) and Lagrange multiplier method (34, 35, 38, 39).

The FE method was developed rapidly in recent years, many new mathematical theories and better optimization codes suggested the application of variational inequalities, which is a generalization of the principle of virtual work and include complicated boundary conditions. A very brief historical review of method in contact problems is as follows:

Wilson and Parson (40) solved frictionless contact problems using constant strain elements, Ohte (41) extended this method to problems involving friction. Chan and Tuba (42) presented an incremental method for both frictionless and frictional problems, followed by the work of Fredriksson (43); Gaerner (44); Okamoto and Nakazawa (45) who introduced the technique of using contact elements to determine contact stresses and deformations. The same technique was presented by Urzua et al. (46); and automated by Torstenfelt (47). Tseng and Olson (48) introduced a mixed finite element method in which both displacements and stresses were retained as variables, this method was then used by Ostachowiez (49) who also defined a gap element with variable stiffness for joining the contact surfaces.

The contact problems by finite element analysis is often conveniently undertaken with the use of gap elements, an iterative procedure continually checks the status of each gap element deleting and reinstating the element as required. Ideally the gap elements should have infinite stiffness to insure a non-intrusive contact between the two bodies in contact.
However, this is not possible in practice due to computation and numeric restriction. This parameter affects both accuracy and convergence behavior. A suitable value is problem dependent and commonly involves a trade off between accuracy and convergence efficiency.

There is a scope for error in this procedure. Contact can only occur between nodes and the nodes are a discrete distance apart. Thus, the most accurate estimate of the radius of the area of contact "r" would be the average radial coordinates of the last node that has contacted and the next one that is not in contact. Since the mean contact pressure involves a “1/r^2”, while the strain of the bodies in contact depends directly on r, any error in the estimation of the radius of contact is cubed. Therefore, a large number of nodes are to be used in order to minimize the effect of this error. In addition, the lack of the possibility of concentrating on a particular region of interest like the area of contact for example, lead us to conclude that the accuracy of the results is not guaranteed and is dependant on the experience of the analyst. Once friction is involved, all frictional effects are seen as additional constraints that an admissible solution should satisfy, and therefore introducing new equations with Lagrange Multipliers or with the Penalty method. This adds tremendously to the time and the complexity of the solution, and what remains missing in top is rigorous proof of numerical stability and uniqueness of this solution. In most common FE programs no error analysis or limits is provided so the user cannot guarantee the reliability of a contact solution involving friction. For some year’s general-purpose computer programs incorporating special contact elements have been available which added to the flexibility of FEM but still failed to get rid of most of the above-mentioned issues.
However, The finite element method must be acknowledged, as the most important and successful of the numerical techniques developed to date, especially in the field of linear static problems and the problems that involve complex geometries and loading profiles.

While progress in Finite element method was advancing rapidly, an alternative numerical technique was going through its early developing stages. The technique known as the Boundary element method (BEM), is based on Green’s theorem; The most striking feature of the BEM is that the dimensionality of the problem is reduced by one, which represents a considerable advantage over other methods, another advantage of the BEM is that no internal discretization of the domain is needed. However, although the coefficient matrix associated with the final set of equations is smaller than other techniques, it is fully populated and non-symmetric. The method also loses its appeal when a full field solution is required. The method has been exploited by many authors to solve contact problems [Anderson (50), Karami and Fenner (51), Karami (52), Tsuta and Yamaji (53); and Abdul-Mihsein et al (54)].

More recently many authors attempted to develop FEM-BEM hybrid methods to combine the advantages and trying to eliminate some of the disadvantages of both methods. Guyot, Kosier and Mourice (65) employed a coupled BEM and FEM to study friction contacts. And then later with same authors, where the total solution was started only in terms of unknown quantities on the contact surface, resulting in more economical computation compared to domain discretization methods such as the FEM.
In recent years in an attempt to reduce the time consumed of the mesh generation especially in complicated geometries, other methods referred to as ‘meshless methods’ were introduced. The concept was introduced in 1977, started gaining popularity after the publication of the diffuse element method by Nayroles et al. (55) and the element free Galerkin method by Belytschko et al. (56) and more recently by the work of Zhu (57); kim and Atluri (58); Lin and Atluri (59).

Other special methods; like the Variational methods; were introduced by Stampacchia (60) and followed by the work of Duvaut and Lions (61). Important contributions were made by Oden and kikuchi (62) who, along with others, undertook employing the method to broad class of contact problems, with especial interest in the finite element method.

The advantages of the linear static solution, and the disadvantages of the contact approach in FEM, together with the limitations of the theoretical theory of contact triggered this research work.

This research work proposes a numerical procedure to be used in conjunction with the finite element method for contact stresses based upon the point load superposition method developed by Paul et al (18, 21, 24, 63). The method combines the advantages of the finite element method and the boundary element method without the corresponding disadvantages. The procedure removes many of the Hertz analysis restrictions. The essence of the problem is to determine the normal and tangential forces distribution inside the contact area together with the shape and size of the contact area, which is not usually known before hand. This approach is based upon fundamental problem solutions
from the theory of elasticity. It employs boundary elements; within the contact surface; on the surfaces of the bodies in contact.

Distribution of traction in line for two-dimensional contact is determined by the superposition of overlapping triangular traction elements. Such piecewise linear distribution of tractions produces surface displacements that are smooth and continuous throughout the contact region, no internal discretization of the domain is needed and therefore no further approximations are imposed at internal points. The obtained solution provides the size and extent of the contact region together with the loading distribution. We can then separate the bodies and employ the well-developed and established linear static solution in the finite element method to find the stress distribution in the interior of the bodies as well as in the contact area.

The method was then extended to the three-dimensional problems employing the Bousinesq-Cerruti equations (5, 64). The analytical solutions are achieved using a set of overlapping conical elements with a circular base whose boundaries have been defined by an equally spaced cartesian grid built from the nodes at the centers of the loading elements with circular bases on the contact surface. The contact area is calculated in the same manner including its shape and size together with the loading distribution and the slip and stick zones. This sets the stage for the application of the finite element method to find the stresses throughout the two bodies.
Many restrictions were removed from Hertz solution, and many disadvantages were eliminated from traditional contact finite element analysis. The main highlights of this new method is:

♦ General non-conforming geometries and loading profiles can be solved.
♦ Frictional effects are included.
♦ Two and three-dimensional contact can be analyzed
♦ No gap elements are needed.
♦ Trial and error for gap element stiffness is eliminated.
♦ The extent and shape of contact is calculated using theory of elasticity, allowing the planning for the proper mesh.
♦ Concentrating on the area of interest when calculating the area of contact and the loading profile while having the ability of evaluating the stresses and deformations throughout the body.
♦ Avoiding the addition of any constraints or equations in the contacts problems involving friction, since the loading distribution and the contact area is calculated using the theoretical theory of contact.
♦ The finite element contact problem is reduced to a linear static problem, which is validated and known to have higher reliability and accuracy.
♦ Employing only the linear static solution in finite element Analysis allowing less modeling and solution time.

A number of example problems have been examined to verify the accuracy, sensitivity and convergence behavior of this method. The numerical results indicate excellent comparison with analytical solutions as well as previously verified numerical solutions.
The dissertation is divided into four main chapters, with this chapter being the introduction and literature review. The second chapter describes general geometrical consideration and constraints upon which the analysis was based. The chapter to follow presents the development of the numerical procedure for two-dimensional problems using linear overlapping triangles with examples and results. The last chapter reports the development of the semi-analytical three-dimensional contact technique using overlapping conical elements with example results and conclusion.
Before the problem of elasticity can be formulated, understanding of the geometry of the contacting surfaces is necessary. There are different formulations of contact problem such as the Classical and the Variational approach. In the Classical approach; which will be used in this research work; the contact problem is regarded as an ordinary problem of solid mechanics where the loads or displacements at the surface are prescribed. Some extra conditions are added which, at the end, the unknowns in the solution are determined. When two non-conforming bodies are brought in contact, they initially meet at a point or a line. Under the action of normal loading they deform in the vicinity of the first point or line to form an area (a strip or a batch) of contact that grows with the application of a larger load. If the bodies are subjected to tangential loading, the contact area shifts and becomes asymmetric about the original point or line of contact.

2.1 Elastic half plane and space.

Non-conforming elastic bodies in contact whose deformation is sufficiently small for the linear strain theory of elasticity to be applicable inevitably make contact over an area whose dimensions are small compared with the radii of curvature of the undeformed bodies. Since the loading distribution; and hence the contact stresses; are highly concentrated at the contact region and decreases rapidly away from the point of contact. Thus provided that the dimensions of the bodies are large compared to the dimensions of the contact area, the stresses in the contact region are not critically dependent upon the shape of the two bodies distant from the contact area or the precise way in which they are
supported. If these conditions are satisfied, then each body can be regarded as a semi-
infinite elastic solid bounded by a plane surface; or in other word; the body is said to be an elastic half space.

The half-space and half-plane must be regarded as geometric consequence of the consideration that the contact, effects may be regarded as local. In both, two and three-dimensional contact we arrive at a consistent theory since the contact load and displacements; and hence contact area and stresses; can be calculated completely. This assumption simplifies the boundary conditions and is widely used in the theory of elasticity. It is specially accurate for nonconforming contact analysis and therefore will be used in this research work.

2.2 Normal loading constraint

We shall now consider the deformation as a normal load $F$ is applied to two nonconforming bodies as shown in Figure (2.1).
When the two bodies are compressed, center points $C_1$ and $C_2$ move towards each other; along the Z-axis; by displacements $U_{z1}(0)$ and $U_{z2}(0)$ respectively. Consider two typical matching surface points $P_1$ and $P_2$ separated originally by a distance $h(x,y)$. Due to this normal loading these two point $P_1$ and $P_2$ on the surface of each body is displaced by $U_{z1}(x)$ and $U_{z2}(x)$ respectively. When no sliding occurs, points $P_1$ and $P_2$ are then assumed to be coincident within the contact surface then:

$$U_{z1}(x) + U_{z2}(x) = U_{z1}(0) + U_{z2}(0) - h(x) \quad \ldots \ldots \quad (2.1)$$

Rearranging

$$(U_{z1}(0) - U_{z1}(x)) + (U_{z2}(0) - U_{z2}(x)) = h(x)$$

For three-dimensional contact the loading is two-dimensional and the Normal constraint within the contact area can be expressed as

$$(U_{z1}(0,0) - U_{z1}(x,y)) + (U_{z2}(0,0) - U_{z2}(x,y)) = h(x, y)$$

If points $P_1$ and $P_2$ are outside the contact area then they do not touch and they follow the constraint:

$$U_{z1}(x) + U_{z2}(x) > U_{z1}(0) + U_{z2}(0) - h(x) \quad \ldots \ldots \quad (2.2)$$

Similarly in the case of three-dimensional contact:

$$U_{z1}(x, y) + U_{z2}(x, y) > U_{z1}(0,0) + U_{z2}(0,0) - h(x, y)$$

Where $h(x,y)$ is the original separation between the two bodies is equal to $|z_1| + |z_2|$ and can be calculated form the geometry of the surfaces.
2.3 Boundary conditions in contact problems with friction

2.3.1 Mechanics of friction

By nature, in contact problems friction always exist. Friction effects may, however, be neglected in many cases if the frictional forces are sufficiently small. On considering the class of contact problems where friction cannot be neglected, many friction laws were developed and are proven to hold for these type of problems. Examples of these laws are: linear, nonlinear, non-local friction laws, and of course the most famous; Coulomb law.

In the present work only Coulomb’s law of external friction in static equilibrium is used. Coulomb’s law of friction is as follows

\[ |F_T| = |\mu F_N| \]

Where \( F_T \) is the frictional force, \( F_N \) is the normal load and \( \mu \) is the coefficient of friction. \( \mu \) is a function of the material of the two bodies in contact, their surfaces and the operational conditions. \( F_T \) is independent of the area of contact and the velocity of slip.

\[ |F_T| \leq |\mu F_N| \]

This holds for both resting and moving friction and in the case of equality, sliding contact occurs. Under the assumption that tangential loading is independent of the velocity of slip, and since Sliding is not studied in the research work presented, it is considered in a static sense and this allows us to use Coulomb friction laws in a local pointwise sense.

2.3.2 Stick and Slip Zones within the contact region

Catteneo (26) expressed the Tangential load to be equal:

\[ Q(x) = \frac{Q}{\pi (a^2 - x^2)^{5/2}} = \frac{\mu P}{\pi (a^2 - x^2)^{5/2}} \]
It is clear that these high tangential tractions at the edge of the contact area \((x=a)\) cannot be sustained, since they would require an infinite coefficient of friction. It was therefore concluded that, there must be a small relative motion referred as “Slip” over the part of the contact area and that this slip occurs at the edges of the contact area. While the rest of the contact region; where tangential tractions are lower than the limiting force of friction; will deform without slippage and is referred to as “Stick region”. As shown in Figure (2.2), we shall expect the stick zone to be in the center of the contact zone, and as \(Q\) approaches \(\mu P\), the radius of the stick zone shrinks to a point or a line. Any attempt to increase the tangential load in excess of that causes the two bodies to slide over each other.

![Figure (2.2)](image)

### 2.3.3 Tangential loading constraint

The second constraint in the loaded region depends upon the tangential loading and the friction conditions at the contact interface. Four cases might arise:

- The surfaces of the two bodies are perfectly smooth (frictionless) so that \(Q(x,y)=0\).
- The friction at the interface is sufficient to prevent any “Slip” between the two bodies and therefore the tangential displacement is constant.
- Partial slip occurs to limit the tangential tractions at the edges \(Q(x) \leq |\mu P(x)|\).
Sliding occurs which will not be considered in this research work.

From the above-mentioned cases, when applying a tangential force; whose magnitude is less than the force of limiting friction; of the two bodies in contact, it will induce frictional forces at the interface. This tangential loading causes an asymmetric deformation pattern that is limited by the friction coefficient.

![Figure (2.3)](image)

As shown in Figure (2.3) Consider two bodies pressed together with Normal force $F$, and consider points $P_1$ and $P_2$ on the two bodies that were coincident before the application of the tangential load $G$. After the application of the load, points $C_1$ and $C_2$ will displace tangentially; parallel to the x-axis; through displacements $U_{x1}(0)$ and $U_{x2}(0)$ respectively.

The slip between points $P_1$ and $P_2$ may be expressed as
\[ S_x = S_{x1} - S_{x2} = (U_{x1}(x) - U_{x1}(0)) - (U_{x2}(x) - U_{x2}(0)) \]

Similarly for three-dimensional contact \( S_x \) and \( S_y \) can be expressed as:

\[
\begin{align*}
S_x &= (U_{x1}(x, y) - U_{x1}(0, 0)) - (U_{x2}(x, y) - U_{x2}(0, 0)) \\
S_y &= (U_{y1}(x, y) - U_{y1}(0, 0)) - (U_{y2}(x, y) - U_{y2}(0, 0))
\end{align*}
\]

\[ \cdots \cdots \text{(2.3)} \]

In Stick Zone, where \( \mid T(x) \mid \leq \mid \mu F(x) \mid \), all surface points undergo the same tangential displacement. Therefore, if points \( P_1 \) and \( P_2 \) are in the stick region

\[ S_x = S_y = 0 \]

Equation (2.3) becomes:

\[
(U_{x1}(x) - U_{x1}(0)) - (U_{x2}(x) - U_{x2}(0)) = 0 \quad \cdots \cdots \text{(2.4)}
\]

For three-dimensional case:

And

\[
\begin{align*}
(U_{x1}(x, y) - U_{x1}(0, 0)) - (U_{x2}(x, y) - U_{x2}(0, 0)) &= 0 \\
(U_{y1}(x, y) - U_{y1}(0, 0)) - (U_{y2}(x, y) - U_{y2}(0, 0)) &= 0
\end{align*}
\]

Further if the two bodies have the same elastic moduli, since they are subjected to mutually equal and opposite surface traction, we can then say that:

\[ U_{x1}(x, y) = -U_{x2}(x, y) \quad \text{And} \quad U_{y1}(x, y) = -U_{y2}(x, y) \]

If points \( P_1 \) and \( P_2 \) are in the Slip region, The Tangential traction must always oppose the direction of slip, then Coulomb’s law relates tangential and normal loads:

\[ Q(x) = \mid \mu P(x) \mid \]

\[ \cdots \cdots \text{(2.5)} \]

For three-dimensional contact, friction is divided into two components \( \mu_x \) and \( \mu_y \) in the directions parallel to the \( X \) and \( Y \) direction respectively. In that case the tangential and normal loads within the slip zone is related by the following:
\[ Q_{xij}(x, y) = |\mu \cdot P_{ij}(x, y)| \]
\[ Q_{yij}(x, y) = |\mu \cdot P_{ij}(x, y)| \]
\[ Q_{xij}(x, y) = |\mu \cdot P_{ij}(x, y)| \]
\[ Q_{xij}(x, y) = |\mu \cdot P_{ij}(x, y)| \]

Where

\[ Q_{0ij}(x, y) = \sqrt{Q_{xij}^2(x, y) + Q_{yij}^2(x, y)} \]
\[ \mu_0(x, y) = \sqrt{\mu_x^2(x, y) + \mu_y^2(x, y)} \]
\[
\mu_x = \frac{Q_x}{Q_0} \mu_0 \quad \text{And} \quad \mu_y = \frac{Q_y}{Q_0} \mu_0
\]

Where \( ij \) represents any point in the contact region of coordinates \( X_i \) and \( Y_j \).

### 2.4 Equilibrium of forces in the contact region

Although the contact area is unknown, two conditions must be satisfied that is:

- The pressure distribution has to be compressive and in particular zero at the edge and of course outside the contact area,
- The contact pressure at all points increases monotonically with the normal force.
- The normal and tangential loads at all the points within the deformed contact region must satisfy equilibrium:

\[ F = \sum_{i,j=1}^{n} A_{ij} P_{ij} \]
\[ T = \sum_{i,j=1}^{n} A_{ij} Q_{ij} \]

Where \( A_{ij} \) is a constant depending upon the form and size of the pressure element. It is equal to area of the two-dimensional element and the volume of the three-dimensional element for two and three-dimensional contact respectively.
2.5 Principle of superposition and pressure elements overlapping

Due to the advancements in the computation and numerical techniques that involve solving complicated loading profiles. The importance of the principle of superposition emerges with modern computing facilities. Continuous distributions of tractions are replaced by a discrete set of elements. The boundary conditions are then satisfied at a discrete number of points to which the domain is divided and the solution in its final form involves solving a system of algebraic equations instead of obtaining a closed form solution. It is worth noting that, in the technological application of the theory of elasticity, good approximation of boundary conditions leads to a considerably accurate mathematical solution, and therefore it is vital to study the different kinds of loading distribution representation.

Figure (2.4)
Many representations were studied. The simplest representation of a loading distribution is an array of concentrated normal or tangential forces as shown in Figure (2.4.a). The difficulty lies in the infinite displacement that can occur at the point of application of these forces. This difficulty can be avoided by representing the load by adjacent columns of uniform loading acting on discrete segments of the surface, this leads to a stepwise distribution as shown in Figure (2.4.b). Although the surface displacement is now defined everywhere their gradients are infinite where there is a step change in traction. A piecewise linear distribution of traction as shown in Figure (2.4.c), on the other hand, produces surface displacements that are continuous and smooth throughout the domain of interest.

Such a distribution was used in the present work by superimposing triangular elements in the two-dimensional contact and conical elements in the three-dimensional contact. Then satisfying the boundary conditions described in the above sections the area of contact and the loading profiles can be found.
At first sight it might appear that to assume that two contacting bodies are perfectly elastic, homogeneous, isotropic, and in a plane state would be sufficient idealization to allow straightforward solution for a significant number of problems. Unfortunately, that is not so, although it is possible to make such assumptions, it is necessary to make further simplifying assumptions. As discussed in the previous chapter, one of the most common assumptions in the non-conformal problems is to assume that the area of contact is small relative to the dimensions of the two bodies in contact and then they can be modeled as an elastic half space. These problems can be efficiently treated employing the classical theory of elasticity. For us to be able to solve for the area of contact, and the loading associated with it, we shall start, therefore, with studying the stresses and deformations in an elastic half space. Making use of the boundary conditions resulted from the interaction between the two bodies in contact described in chapter (2) the solution can be obtained.

3.1 Loading of an elastic half space

Consider two cylinders in contact load, normally and tangentially, one dimensionally over a narrow strip (line loading) of width \((a+b)\). The loaded strip lies parallel to the \(Y\)-axis while the \(Z\)-axis is directed into the solid. Since the depth of the solid is large compared with the width of the loaded region, we shall assume a state of plane strain \((\varepsilon_y=0)\).
From the theory of elasticity, satisfying equilibrium, compatibility, Hook’s law, and the boundary conditions of the problem, the stress components are found as follows (33, 18 and 65):

\[
\sigma_r = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial \phi}{\partial \theta}
\]

\[
\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}
\]

\[
\tau_{r\theta} = \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial r} \right)
\]

Where: \(\phi(r, \theta)\) is a stress function.

We shall now proceed to discuss the solution to particular problems relevant to elastic contact theory.

3.1.1 **Concentrated normal and tangential force.**

Consider a concentrated load \(F\) per unit length distributed along the Y-axis acting in a direction normal to the surface as shown in Figure (3.1)
This problem was solved by Flamant using the stress function:

\[ \phi(r, \theta) = A r \theta \sin \theta \quad \text{......... (3.2)} \]

Using Equation (3.1) the radial and tangential stresses at point \( P \) anywhere in the half space can be expressed as (33,18,65):

\[ \sigma_r = 2A \frac{\cos \theta}{r} \quad \text{......... (3.3)} \]
\[ \sigma_\theta = \tau_{r\theta} = 0 \]

Where “\( A \)” is an arbitrary constant later found by equating the vertical component of the stress with the applied force.

Hence

\[ \sigma_r = -\frac{2F \cos \theta}{\pi r} \quad \text{......... (3.4)} \]
\[ \sigma_\theta = \tau_{r\theta} = 0 \]

Then the equivalent stress components in rectangular coordinates are:

\[ \sigma_x = -\frac{2F}{\pi} \frac{x^2 z}{(x^2 + z^2)^2} \quad \text{......... (3.5)} \]
\[ \sigma_z = -\frac{2F}{\pi} \frac{z^3}{(x^2 + z^2)^2} \]
\[ \tau_{xz} = -\frac{2F}{\pi} \frac{x z^2}{(x^2 + z^2)^2} \]
Using the stress strain relationships and assuming plane strain as mentioned before, the
displacement at any point “P” can be expressed as:

\[
\begin{align*}
    u_x &= -F \frac{(1-2\gamma)(1+\gamma)}{2E} \frac{x}{|x|} \\
    u_z &= 2F \frac{(1-\gamma^2)}{2E} \ln\left|\frac{1}{|x|}\right|
\end{align*}
\] ........ (3.6)

Where: “l” is the distance between point P and a point of zero displacement along the
Z-axis.

In a like manner, consider a concentrated load “T” per unit length distributed along the Y-axis acting in a direction tangential to the surface as shown in Figure (3.1), the stresses at
any point “P” are (33):

\[
\begin{align*}
    \sigma_x &= -\frac{2T}{\pi} \frac{x^3}{(x^2 + z^2)^2} \\
    \sigma_z &= -\frac{2T}{\pi} \frac{xz^2}{(x^2 + z^2)^2} \\
    \tau_{zx} &= -\frac{2T}{\pi} \frac{x^2z}{(x^2 + z^2)^2}
\end{align*}
\] ........ (3.7)

Therefore, the displacement at any point “P” can be expressed as:

\[
\begin{align*}
    u_x &= -2T \frac{(1-\gamma^2)}{2E} \ln|\frac{x}{|x|}| + c \\
    u_z &= T \frac{(1-2\gamma)(1+\gamma)}{2E} \left(\frac{x}{|x|}\right)
\end{align*}
\] ........ (3.8)

Where “c” is a constant determined from a zero displacement point.
3.1.2 Distributed normal and tangential loading

Consider an elastic half space of width \((a+b)\) loaded by an arbitrary distributed normal \(P(s)\) and tangential \(Q(s)\) loading as shown in Figure (3.2).

The stress at any point \(c(x,z)\) due to this loading can be obtained by integrating equations (3.5) and (3.8); in which “\(x\)” is replace by “\((x-s)\)” over the loaded region (33) and are given by:

\[
\sigma_x = -\frac{2}{\pi} \int_{-b}^{a} \left( \frac{zP(s)(x-s)^2}{((x-s)^2 + z^2)^2} + \frac{Q(s)(x-s)^3}{((x-s)^2 + z^2)^2} \right) ds
\]

\[
\sigma_z = -\frac{2}{\pi} \int_{-b}^{a} \left( \frac{z^3 P(s)}{((x-s)^2 + z^2)^2} + \frac{z^2 Q(s)(x-s)}{((x-s)^2 + z^2)^2} \right) ds \quad \text{(3.9)}
\]

\[
\tau_{xz} = -\frac{2}{\pi} \int_{-b}^{a} \left( \frac{z^2 P(s)(x-s)}{((x-s)^2 + z^2)^2} + \frac{zQ(s)(x-s)^2}{((x-s)^2 + z^2)^2} \right) ds
\]

Using the stress strain relationships and assuming plane strain as mentioned before, the displacement at any surface point \((z=0)\) can be expressed as (33):
\[ u_x = \frac{-(1-2\gamma)(1+\gamma)}{2E} \left[ \int_{-b}^{a} P(s)ds - \int_{a}^{x} P(s)ds \right] - \frac{2(1-\gamma^2)}{\pi E} \left[ \int_{-b}^{a} Q(s) \ln|x-s|ds \right] + C_1 \]
\[ u_z = -\frac{2(1-\gamma^2)}{\pi E} \int_{-b}^{a} P(s) \ln|x-s|ds + \frac{(1-2\gamma)(1+\gamma)}{2E} \left[ \int_{-b}^{x} Q(s)ds - \int_{a}^{x} Q(s)ds \right] + C_2 \]

(3.10)

Where \( C_1 \) and \( C_2 \) are constants to be determined at a reference point.

From equations (3.10) we can conclude that if the loading distribution can be expressed as a function in terms of \( x \) and \( y \), then in principle the displacements due to this loading profile can be expressed explicitly. However, that is only true for simple loading profiles. This leads to the apparent importance of developing a numerical solution, through which simple and clearly defined loading profiles can be superimposed to form the desired actual loading profile.

### 3.2 Numerical procedure

The essence of the problem is to find the area of contact and to determine normal and tangential loading distribution over this area. Then to define this area of contact on the finite element model to be built accordingly with the knowledge of both the area of contact and loading distribution over such an area. The stresses and displacements are then calculated throughout the two bodies using the finite element method.

To achieve this goal, we shall assume that the loading distribution is divided into overlapping triangular pressure elements. We shall then divide an area around the point of contact that is guaranteed to be larger than the expected actual area of contact, into nodes at the bases of these triangular pressure elements. Employing equation (3.10) for such an element and satisfying the boundary conditions described in the previous chapter,
the displacements at each of these nodes can be found. The effect of each of these loading elements on all nodes can then be superimposed to find the general distribution. In the following sections, a detailed description of the method is provided.

3.2.1 Linear distribution of loading

Consider an element of width $2a$ as shown in Figure (3.3), assume a linear distribution for both the normal and tangential loads, in which the loading will increase from $P_0$ and $Q_0$ at $x=0$ to zero at $x=\pm a$ respectively.

![Figure (3.3)](image)

Where

$$
\begin{align*}
P(x) &= \frac{P_0}{a} (a - |x|) \\
Q(x) &= \frac{Q_0}{a} (a - |x|)
\end{align*}
$$

$|x| \leq a$  

\[ r_1^2 = (x - a)^2 + z^2, \quad \tan\theta_1 = z/(x - a) \]

\[ r_2^2 = (x + a)^2 + z^2, \quad \tan\theta_2 = z/(x + a) \]

\[ r^2 = x^2 + z^2, \quad \tan\theta_1 = z/x \]
Substituting Equation (3.11) in equation (3.9), integrate and simplify we obtain:

\[
\sigma_\theta = \frac{P_0}{\pi a} \left[ (x-a)\theta_1 + (x+a)\theta_2 - 2x\theta + 2z \ln \left( \frac{r_1 r_2}{r^2} \right) \right] + \frac{Q_0}{\pi a} \left[ 2x \ln \left( \frac{r_1 r_2}{r^2} \right) + 2a \ln \left( \frac{r_2}{r} \right) - 2z(\theta_1 + \theta_2 + \theta) \right]
\]

\[
\sigma_z = \frac{P_0}{\pi a} \left[ (x-a)\theta_1 + (x+a)\theta_2 - 2x\theta \right] - \frac{Q_0}{\pi a} \left[ 2z(\theta_1 + \theta_2 - 2\theta) \right]
\]

\[
\tau_x = -\frac{P_0}{\pi a} \left[ 2z(\theta_1 + \theta_2 - 2\theta) \right] - \frac{Q_0}{\pi a} \left[ (x-a)\theta_1 + (x+a)\theta_2 - 2x\theta + 2z \ln \left( \frac{r_1 r_2}{r^2} \right) \right]
\]

Similarly from Equations (3.11) and (3.10) we obtain:

\[
u_x = \alpha Q_0 \left[ (x+a)^2 \ln \left( \frac{x}{a} + 1 \right)^2 + (x-a)^2 \ln \left( \frac{x}{a} - 1 \right)^2 - 2x^2 \ln \left( \frac{x}{a} \right)^2 \right] + \beta P_0 + C_1
\]

\[
u_z = \alpha P_0 \left[ (x+a)^2 \ln \left( \frac{x}{a} + 1 \right)^2 + (x-a)^2 \ln \left( \frac{x}{a} - 1 \right)^2 - 2x^2 \ln \left( \frac{x}{a} \right)^2 \right] - \beta Q_0 + C_2 \tag{3.12}
\]

Where

\[
\alpha = -\frac{(1-\gamma^2)}{2\pi E a}
\]

\[
\beta = \begin{cases} 
+ \frac{a(1-2\gamma)(1+\gamma)}{2E} & \text{for } x < a \\
- \frac{x(a-|a/2|)(1-2\gamma)(1+\gamma)}{E a} & \text{for } |x| \leq a
\end{cases}
\]

\(C_1\) and \(C_2\) are constants

### 3.2.2 Numerical modeling of the contact problem

Divide the interface between the two bodies into \(n\) elements each of width “\(a\)”. Assume a triangular loading distribution on each of these elements as shown in Figure (3.4) each base of length “2\(a\)” and of height \(P_i\) and \(Q_i\) for normal and tangential pressure elements respectively.

Since the contact loads has to always be compressive. When the iteration gives some \(P_i\) with negative values, while \(P_{i+1}\) is the last positive value, with coordinates \(X_i\) and \(X_{i+1}\)
respectively, a new assumed contact width is needed for the next iteration. The new assumed contact width is found from an interpolation between these two adjacent values for a coordinate at which \( P=0 \).

By satisfying equilibrium, compatibility, together with the normal and tangential boundary conditions, and employing the principle of superposition the contact area and loading distribution can be found throughout the contact region.

From the figure (3.4) we observe that nodes are equally spaced in multiples of the element width “\( a \)”, that is the x-coordinate of each node “\( j \)” relative to any node “\( i \)” is

\[ x_k = k_{ij} a \]

By substituting in Equation (3.12), the displacement at node \( i \) due to loading at node \( j \) can me expressed as follows:

\[
\begin{align*}
    u_{ix} &= -\frac{(1-2\gamma)(1+\gamma)}{E} \eta_{ij} P_j + \frac{(1-\gamma^2)}{E} \xi_{ij} Q_j \\
    u_{ix} &= \frac{(1-2\gamma)(1+\gamma)}{E} \eta_{ij} Q_j + \frac{(1-\gamma^2)}{E} \xi_{ij} P_j
\end{align*}
\]

\[ \cdots \cdots (3.13) \]
Where

\[
\xi_{ij} = \begin{cases} 
-\frac{a}{2\pi} \left[ (k+1)^2 \ln(k+1)^2 + (k-1)^2 \ln(k-1)^2 - 2k^2 \ln k^2 \right] & |k| > 1 \\
0 & |k| = 0 \\
-\frac{2a}{\pi} \ln 4 & |k| = 1 
\end{cases}
\]

\[
\eta_{ij} = \begin{cases} 
a/2 & k > 0 \\
0 & k = 0 \\
-a/2 & k < 0 
\end{cases}
\]

\[k = i-j\]

Before applying the normal and tangential constraint it is convenient to put equation (3.13) in the format:

\[
\eta_{0j} = \eta_{0j} - \eta_{ij}, \quad \overline{\xi}_{ij} = \xi_{0j} - \xi_{ij}
\]

\[\eta_{0j} \text{ and } \xi_{0j} \text{ are the coefficients defined at the origin (x=0).}\]

By substituting equation (3.14) in Equation (2.1) to satisfy the normal boundary condition at the interface, and by the principle of superposition we obtain:

\[
\lambda_1 \sum_{j=-a}^{a} \eta_{ij} Q_j + \lambda_2 \sum_{j=-a}^{a} \overline{\xi}_{ij} P_j = h_i(x)
\]

\[i,j=-n,\ldots,0,\ldots,n\]
\[
\lambda_1 = \frac{(1-2\gamma_1)(1+\gamma_1)}{E_1} + \frac{(1-2\gamma_2)(1+\gamma_2)}{E_2} \\
\lambda_2 = \frac{(1-\gamma_1^2)}{E_1} + \frac{(1-\gamma_2^2)}{E_2}
\]

Where \(E_1, E_2, \gamma_1,\) and \(\gamma_2\) are the elastic modulii and the poisson’s ratios for the two bodies in contact respectively.

Substituting equation (3.14) into Equation (2.4) to satisfy the tangential boundary condition at the interface, and by the principle of superposition, we obtain for the stick region:

\[
\lambda_3 \sum_{j=0}^{m} \eta_{ij} P_j + \lambda_4 \sum_{j=-m}^{m} \xi_{ij} Q_j = 0 \\
\]

Where \(i=-m, \ldots, 0, \ldots, m\)

\[
\lambda_3 = -\frac{(1-2\gamma_1)(1+\gamma_1)}{E_1} + \frac{(1-2\gamma_2)(1+\gamma_2)}{E_2} \\
\lambda_4 = \frac{(1-\gamma_1^2)}{E_1} - \frac{(1-\gamma_2^2)}{E_2}
\]

For the slip region Equation (2.5) holds

\[
Q_i = \pm \mu P_i \quad i=-n, \ldots, -m, m, \ldots, n \quad \ldots \ldots (3.17)
\]

Then for the interface we have 2n surface elements with 2n+1 nodes and 4n+2 unknowns (\(P\) and \(Q\) at each node). Equations (3.14), (3.15), and (3.16) constitute 4n+2 equations. However, Equations (3.14) and (3.15) are degenerate at \(i=0\) since the coefficients \(\eta_{ij}\) and
\( \xi_{ij} \) are equal to zero at the origin, therefore, two more equations are needed and can be obtained from the global force summation:

\[
P = \sum A_i P_i \quad \text{…….. (3.18)}
\]
\[
Q = \sum A_i Q_i
\]

Where \( A_i \) is the area of a surface element and is equal to 0.5a.

The numerical procedure can then be summarized as:

1. Assume a contact region.
2. Divide the contact region into surface elements and nodes.
3. Solve equation (3.15), (3.16), and (3.18) for nodal loads.
4. Adjust the extent of the contact region by deleting the nodes with tensile normal force.
5. Adjust the stick and slip zones using equation (3.16) and (3.17).
6. Repeat step (1) through (5) until convergence is obtained.
7. Build a finite element model after defining the appropriate mesh on the contact area obtained from step (6).
8. Apply the normal and tangential loading over the contact area.
9. Use the linear static finite element solution to find the stresses and deformation throughout the body.

A flow chart of the numerical procedure is shown in Appendix A, while the computer code that was used to implement this method for the two-dimensional contact is shown in Appendix B.
3.3 Examples

A computer code was developed for the numerical method to find the area of contact and the loading distribution as described above (steps 1 through 6). As shown in Figure (3.5), two finite element models were built for each of the demonstrated examples. One for the second phase of the method (steps 7 through 9) as described in the previous section to find the stresses and deformation using the linear static finite element method. The other was modeled following the common practice of using gap elements and the nonlinear finite element solution for comparison purposes to illustrate the efficiency of the developed method. The results were then validated by analytical solution as available.

![Figure (3.5)](image-url)
3.3.1 **Normal contact of elastic cylinders - frictionless.**

Consider two cylinders of radii 150” and 100” respectively, elastic modulus of 30E6 Lb/in², and poison ratio of 0.3, pressed in contact by a force of 1000 pounds.

This example has been chosen to validate the equation formulation and the numerical approach using Hertz solution for two frictionless cylinders of similar materials.

![The effect of the increase of the total Normal load on the area of contact](image)

**Figure (3.6)**

Figure (3.6) demonstrates the stability of the method. The expected increase in the area of contact was observed when the pressing load increases. It was found that the area of contact increases with the increase in the pressing load. It follows the same trend as Hertz exact solution with very small error as shown.
From figure (3.7), comparing the solution time in both the traditional contact solution in the finite element method and the new method. It was shown that the solution time was reduced drastically in the developed method. As we increased the number of elements by four times to enhance accuracy, we concluded that, while the solution time in the traditional finite element method was doubled by nine times, it was only five times in the case of the hybrid method.

![Solution time comparison Between the Finite element method and developed method](image1)

**Figure (3.7)**

The effect of number of elements on accuracy (sensitivity Study)

![The effect of number of elements on accuracy (sensitivity Study)](image2)

**Figure (3.8)**
From figure (3.8), it was concluded that the higher the number of elements used, the less the error to the exact solution. The same number of 40 elements was used through the contact width in both methods to enhance accuracy and ensure comparison credibility between the two methods. While increasing the number of elements in the traditional finite element method more than forty elements were faced by convergence difficulties (due to the increase in total gap stiffness with the increase in the gap elements), the number of elements can still be increased in the newly developed hybrid method. Being the limit in the finite element method for this case, this number of elements was chosen only for comparison purposes between the two methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>Hertz</th>
<th>FE model</th>
<th>Numerical method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pmax (psi)</td>
<td>9351.35</td>
<td>9089.9</td>
<td>9351.9341</td>
</tr>
<tr>
<td>contact width (in)</td>
<td>0.136156</td>
<td>0.1333</td>
<td>0.136506796</td>
</tr>
<tr>
<td>Maximum Shear (psi)</td>
<td>2808</td>
<td>2267</td>
<td>2873</td>
</tr>
<tr>
<td>Pmax % error</td>
<td>N/a</td>
<td>2.80%</td>
<td>0.01%</td>
</tr>
<tr>
<td>Area % error</td>
<td>N/a</td>
<td>2.10%</td>
<td>0.26%</td>
</tr>
<tr>
<td>Max. Shear % error</td>
<td>N/a</td>
<td>19.27%</td>
<td>2.31%</td>
</tr>
</tbody>
</table>

**Table (3.1)**

As shown in table (3.1), and as mentioned above, although the same Finite element package, with the same number of elements was used in both the finite element model employing gap elements, and the hybrid numerical method. The results show major improvement over the finite element results. By calculating the area of contact, and the pressure distribution and then employing the finite element model to find stresses and
deformations. The solution for the loading distribution and the area of contact was accurate to less than half a percent error comparing to 2.5% in the case of the traditional finite element method. The major improvement occurred in the stress calculation, the error was about 2.3% down from a steep 19% in the FEM.

3.3.2 Normal contact of elastic cylinders with friction

To examine the effect of friction, consider two cylinders of radii 100” and 150”, elastic moduli of $30 \times 10^6$ and $103 \times 10^5$ Lb/in$^2$, and Poisson ratio of 0.3 and 0.33 respectively, pressed in contact by a force of 1000 pounds with coefficient of friction equal to 0.5.

Comparing with the Hertz solution for frictionless (all stick) solution. It was expected for the size of the area of contact under the same conditions to increase due to the slip zones while for the maximum pressure to decrease with the existence of friction. A sensitivity study was performed on the Finite element result in order to validate the results trend, it was concluded that with more and stiffer gap elements, higher accuracy is to be expected.
but with a major trade off with convergence efficiency and solution time. The trend of the results was directed towards the results obtained from our numerical solution. Which falls between the hertz solution and the results from the finite element model with higher element concentration.

The Finite element model and the load application is demonstrated in figure (3.10)
As shown in figure (3.9), the normal and tangential load distribution were found together with the width of contact, and slip zones where the tangential loading exceeded the limiting force of friction. The loading was then applied to a finite element method that was built with the knowledge of the contact width and the stick zone within which allowed proper planning. Note that, on the slip zone the tangential load applied to the finite element model is equal to “$\mu P$”, while in the stick zone, it is equal to the tangential loading directly from the computer code.

To check the stability of the method in the cases involving friction, the effect of the coefficient of friction on the stick zone width was studied. As shown in figure the results showed agreement with the Catteneo slip theory. As we increase the coefficient of friction, the stick zone increases and the slip zone shrinks. At relatively high friction coefficient, the area of contact can be approximated to an all stick solution.
Due to the lack of exact solution for the frictional case and as discussed before, the solution was compared with Hetrz solution for frictionless (all stick) solution, and the Finite element method. As shown in table (3.2), relative to Hetrz solution, the size of the area of contact under the same conditions increased while for the maximum pressure to decrease due to the slip zones which arises from the existence of friction. While the sensitivity study performed on the Finite element results and discussed in figure (3.9) indicated that the results trend, with more and stiffer gap elements, was directed towards the results obtained from our numerical solution. Which indicates the higher accuracy of the newly developed hybrid method.

<table>
<thead>
<tr>
<th>Method</th>
<th>Hertz (Frictionless)</th>
<th>24 elements</th>
<th>32 elements</th>
<th>40 elements</th>
<th>Numerical solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pmax (psi)</td>
<td>6738</td>
<td>5895.4</td>
<td>5960.42</td>
<td>6372.9</td>
<td>6475</td>
</tr>
<tr>
<td>Width of contact (in)</td>
<td>0.18896</td>
<td>0.17499</td>
<td>0.1875</td>
<td>0.1931</td>
<td>0.202</td>
</tr>
<tr>
<td>Max Shear (psi)</td>
<td>2021</td>
<td>1744</td>
<td>1705</td>
<td>1692</td>
<td>1674</td>
</tr>
<tr>
<td>Solution Time (min)</td>
<td>N/A</td>
<td>35</td>
<td>58</td>
<td>99</td>
<td>17</td>
</tr>
</tbody>
</table>

Table (3.2)
The solution time however increased drastically with the increase of number of elements in the traditional finite element approach. Our method led to higher accuracy and required only a small fraction of solution time.

3.4 Conclusion

The execution of the method involves iterative procedure to determine the extent of the contact region. Therefore the preparation for the method is initially more laborious than the classical restricted Hertz method. In exchange, many restrictions are removed and it provides more accurate and comprehensive analysis than either the classical Hertz or the FEM techniques.

The burdensome nonlinear modeling, using gap elements, was avoided and both modeling and solution time was reduced, while accuracy was enhanced drastically. The ability to solve different loading geometric profiles and the clear vision of the load and contact area before building the finite element model reduces the modeling time and the effectiveness of the model.

Although these examples employ bodies with simple geometries (cylinders), the method outlined herein is not restricted to simple geometries. Indeed there are no restrictions on the geometry as long as the contacting surfaces are continuous and geometrically smooth.

The numerical solution works successfully with the two-dimensional contact problems. Although it is known not to be a simple extension, this success led us to believe that in
principle it can be then extended to include the three-dimensional problems by using a three-dimensional pressure element instead.

It is expected that the developed method will indeed provide greater accuracy in only a fraction of the time for the three-dimensional contact as it did in the case of the two-dimensional contact. Three-dimensional contact using the same technique will be covered in the chapter that follows.

For the two-dimensional contact future research work is recommended to include other contact cases, such as conformal contact, torsional, sliding, nonlinear material contact and dynamic contact.
Chapter 4  Three-Dimensional Contact

Although many contact problems can be dealt with using two-dimensional model approximation, there are still many cases where three-dimensional representation is crucial, this is clear when either the loading or the geometry is not symmetric. However, the solution of three-dimensional problems presents considerably greater difficulties than the solution of plane problems, and unfortunately it is not a simple extension to the two-dimensional case described in the previous chapter.

The initial premise is that it is supposed that the two elastic bodies occupy semi-infinite space (half space defined in chapter 2). The Classical approach to finding the displacements in an elastic half space is traced back to the Boussinesq (5) and Cerruti (8) who made use of the theory of potential. Here it is necessary to find a harmonic function whose value is given on both sides of a plane region of the outline of the area of contact. By making use of a properly selected system of coordinates which is selected in a way that this harmonic function satisfying Laplace’s equation expressed in terms of these coordinates, should be capable of being represented in the form of a product of three functions each of which depends on one variable.

Boussinesq and Cerrutti employed the theory of potential to find the components of displacements, they both obtained solutions to distributed normal and tangential loading over an area $S$ on an elastic half space. The displacement expressions; discussed in the sections to follow; are given in terms of integrals of the loading distributions over the area $S$. 
Hence, in theory, if the distributions of the loading within an area $S$ is known explicitly, the displacements at any point can be found by evaluating these integrals. Unfortunately, it is not possible to find an effective solution for any class of problems, therefore the need for a numerical technique is apparent.

### 4.1 Bousinesq and Cerruti Potential Functions

Consider an elastic half space as shown in Figure (4.1). The x-y plane is chosen to be on the surface while the z-axis into the body, point “$A(x,y,z)$” is a general point within the surface $S$. Assume a two dimensional loading distribution in the three directions; Normal.
pressure “\(P(x,y)\)” , and Tangential traction “\(Q(x,y)\)” and “\(Q_y(x,y)\)” ; acting on the Loaded area “\(S\)” of an elastic body.

Satisfying Laplace equation, the Potential functions can be expressed as:

\[
I_1 = \int \int_S Q_x(\xi, \eta) \Omega d\xi d\eta \\
G_1 = \int \int_S Q_y(\xi, \eta) \Omega d\xi d\eta \\
H_1 = \int \int_S P(\xi, \eta) \Omega d\xi d\eta
\]  

\[\text{......... (4.1)}\]

Where

\[
\Omega = z \ln(\omega + z) - \omega \\
\omega = \sqrt{(\xi - x)^2 + (\eta - y)^2 + z^2}
\]

In addition we define the potential functions:

\[
I = \frac{\partial I_1}{\partial z} \quad G = \frac{\partial G_1}{\partial z} \quad H = \frac{\partial H_1}{\partial z}
\]

\[
\psi_1 = \frac{\partial I_1}{\partial x} + \frac{\partial G_1}{\partial y} + \frac{\partial H_1}{\partial z} \\
\psi = \frac{\partial \psi_1}{\partial z} = \frac{\partial I}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}
\]  

\[\text{......... (4.2)}\]

Love (18) expressed the displacement components at any point “\(A(x,y,z)\)” in terms of the above functions as follows:
\[
\begin{align*}
    u_x &= \frac{1}{4\pi G} \left[ 2 \frac{\partial I}{\partial z} - \frac{\partial H}{\partial x} + 2\nu \frac{\partial \psi}{\partial x} - z \frac{\partial \psi}{\partial x} \right] \\
    u_y &= \frac{1}{4\pi G} \left[ 2 \frac{\partial G}{\partial z} - \frac{\partial H}{\partial y} + 2\nu \frac{\partial \psi}{\partial y} - z \frac{\partial \psi}{\partial y} \right] \\
    u_z &= \frac{1}{4\pi G} \left[ \frac{\partial H}{\partial z} + \left( (1-2\nu)\psi \right) - z \frac{\partial \psi}{\partial z} \right]
\end{align*}
\] ........ (4.3)

The equations quoted above provide a formal solution to the problem of deformation (hence stresses) in an elastic body with prescribed traction. With a defined area, and if the loading distribution within such a loaded area S at the surface (z=0) is known, then, in principle, the displacements at any point can be found. However, obtaining expressions in closed form, presents difficulty in any but the simple loading and geometric shapes, in addition, the normal and tangential tractions are fully coupled in Bousinesq and Cerruti integrals, and the loading distribution and the area of contact is not known before hand. Therefore, the need for the development of an analytical technique to overcome these difficulties in the classical approach was apparent.

Johnson described an alternative approach in which he employed the principle of superposition using the stresses and displacements produced by concentrated normal and tangential forces to find the stresses and deformation resulting from any distributed loading. So if the solution for a simple, well-defined loading profile (conical pressure element) over a simple area (circular cone base), can be achieved in a closed form using the above equations. Then satisfying the boundary conditions described in chapter (2) and following the same methodology used in chapter (3) for two-dimensional contact, an
efficient and convenient contact mechanics analysis tool for three-dimensional contact of non-conformal geometries can be achieved.

We shall proceed by deriving these integrals for a conical pressure element on a circular area, to pave the road for the numerical approach.

4.2 normal Loading

4.2.1 Concentrated normal force

Using the results from the previous section to find the displacements produced by a concentrated normal load “F”.

\[ Q_x = Q_y = 0 \]

\[ \iint P(\xi, \eta) d\xi d\eta = F \]

Therefore \( I = G = 0 \)

The Boussinesq potential functions \( \psi \) and \( \psi_1 \) can be reduced to:

\[ \psi_1 = \frac{\partial H_1}{\partial z} = H \]

\[ \psi = \frac{\partial \psi_1}{\partial z} = \frac{\partial H}{\partial z} \]

Substituting in equations (4.3)

\[ u_x = \frac{1}{4\pi G} \left[ -\frac{\partial H}{\partial x} + 2N \frac{\partial H}{\partial x} - z \frac{\partial^2 H}{\partial x \partial z} \right] \]

\[ u_y = \frac{1}{4\pi G} \left[ -\frac{\partial H}{\partial y} + 2N \frac{\partial H}{\partial y} - z \frac{\partial^2 H}{\partial y \partial z} \right] \]

\[ u_z = \frac{1}{4\pi G} \left[ \frac{\partial H}{\partial z} + (1 - 2N) \frac{\partial H}{\partial z} - z \frac{\partial^2 H}{\partial z^2} \right] \]
The displacements can therefore be expressed as:

\[
\begin{align*}
    u_x &= \frac{F}{4\pi G} \left[ \frac{xz}{\omega^3} - (1 - 2\nu) \frac{x}{\omega(\omega + z)} \right] \\
    u_y &= \frac{F}{4\pi G} \left[ \frac{yz}{\omega^3} - (1 - 2\nu) \frac{y}{\omega(\omega + z)} \right] \\
    u_z &= \frac{F}{4\pi G} \left[ \frac{z^2}{\omega^3} + 2 \frac{(1-\nu)}{\omega} \right]
\end{align*}
\] ........... (4.4)

For a point on the surface \( (z=0) \) the equations reduce to:

\[
\begin{align*}
    u_x &= \frac{F}{4\pi G} \left[ (1 - 2\nu) \frac{x}{x^2 + y^2} \right] \\
    u_y &= \frac{F}{4\pi G} \left[ (1 - 2\nu) \frac{y}{x^2 + y^2} \right] \\
    u_z &= \frac{F}{2\pi G} \left[ \frac{(1-\nu)}{\sqrt{x^2 + y^2}} \right]
\end{align*}
\] ........... (4.5)

### 4.2.2 Normal loading distribution

Using the results shown in equation (4.5) for concentrated load, the deflections produced by normal pressure \( P(\xi,\eta) \) distributed over area \( S \) can be found at any general point \( A(x,y,z) \) :

\[
\begin{align*}
    u_x &= -\frac{(1 - 2\nu)}{4\pi G} \int_S \frac{P(\xi,\eta)(x - \xi)}{\omega^2} d\xi d\eta \\
    u_y &= -\frac{(1 - 2\nu)}{4\pi G} \int_S \frac{P(\xi,\eta)(y - \eta)}{\omega^2} d\xi d\eta \\
    u_z &= \frac{(1-\nu)}{2\pi G} \int_S \frac{P(\xi,\eta)}{\omega} d\xi d\eta
\end{align*}
\] ........... (4.6)
Changing to Polar coordinates, equation (4.6) can be expressed as

\[ u_x = -(1 - 2\nu) \frac{1}{4\pi G} \iint_{S} P(S, \phi) \cos \phi \, dS \, d\phi \]

\[ u_y = -(1 - 2\nu) \frac{1}{4\pi G} \iint_{S} P(S, \phi) \sin \phi \, dS \, d\phi \]

\[ u_z = -(1 - \nu) \frac{1}{2\pi G} \iint_{S} P(S, \phi) \chi \, dS \, d\phi \]

\[ \ldots \ldots \ldots \ (4.7) \]

4.3 Tangential loading

4.3.1 Concentrated Tangential force

Assume concentrated tangential load acting in the x-direction the Normal Pressure and the tangential load in the y-direction are both taken to be zero.

\[ P = Q_y = 0 \]

\[ \iint_{S} Q_x(x, \eta) \chi \, d\xi \, d\eta = Q_x \]

Therefore

\[ G_1 = H_1 = G = H = 0 \]

\[ I_1 = \iint_{S} Q_x(x, \eta) \left[ z \ln(\omega + z) - \omega \right] \chi \, d\xi \, d\eta \]

Where

\[ \omega^2 = (\xi - x)^2 + (\eta - y)^2 + z^2 \]

The Boussinesq potential functions \( \psi \) and \( \psi_1 \) can be reduced to:

\[ \psi_1 = \frac{\partial I_1}{\partial x} \]

\[ \psi = \frac{\partial \psi_1}{\partial z} = \frac{\partial^2 I_1}{\partial x \partial z} \]

Substituting in equations (4.3)
The displacements can therefore be expressed as:

\[ u_x = \frac{Q_x}{4\pi G} \left[ \frac{1}{\omega} + \frac{x^2}{\omega^3} + (1 - 2\nu) \left( \frac{1}{\omega + z} + \frac{x^2}{\omega(\omega + z)^2} \right) \right] \]

\[ u_y = \frac{Q_x}{4\pi G} \left[ \frac{xy}{\omega^3} - (1 - 2\nu) \left( \frac{xy}{\omega(\omega + z)^2} \right) \right] \]

\[ u_z = \frac{Q_x}{4\pi G} \left[ \frac{xz}{\omega^3} + (1 - 2\nu) \left( \frac{x}{\omega(\omega + z)} \right) \right] \]

For a point on the surface \((z=0)\) the equations reduce to:

\[ u_x = \frac{Q_x}{2\pi G} \left[ (1 - \nu) \frac{1}{\omega} + \nu \frac{x^2}{\omega^3} \right] \]

\[ u_y = \frac{Q_x}{2\pi G} \left[ \nu \frac{xy}{\omega^3} \right] \]

\[ u_z = \frac{Q_x}{4\pi G} \left[ (1 - 2\nu) \left( \frac{x}{\omega^2} \right) \right] \]

Similarly, displacements due to a concentrated tangential load in the y-direction can be expressed as:
\begin{align*}
  u_x &= \frac{Q_y}{2\pi G} \left[ v \frac{x y}{\omega^3} \right] \\
  u_y &= \frac{Q_y}{2\pi G} \left[ (1-v) \frac{1}{\omega} + v \frac{y^2}{\omega^3} \right] \\
  u_z &= \frac{Q_y}{4\pi G} \left[ (1-2v) \left( \frac{y}{\omega^2} \right) \right]
\end{align*}

\text{………… (4.11)}

\section*{4.3.2 Tangential loading distribution}

Using the results shown in equation (4.9) for concentrated load, the deflections produced by tangential loading in the x-direction \(Q_x(\xi, \eta)\) distributed over area \(S\) can be found at any general point \(A(x,y,z)\) by superposition.

\begin{align*}
  u_x &= \frac{1}{2\pi G} \iiint_S Q_x(\xi, \eta) \left( \frac{1-v}{\omega} + v \frac{(x-\xi)^2}{\omega^3} \right) d\xi d\eta \\
  u_y &= \frac{1}{2\pi G} \iiint_S Q_x(\xi, \eta) v \frac{(x-\xi)(y-\eta)}{\omega^3} d\xi d\eta \\
  u_z &= \frac{1}{2\pi G} \iiint_S Q_x(\xi, \eta) v \frac{(x-\xi)}{\omega^2} d\xi d\eta
\end{align*}

\text{………… (4.12)}

Similarly the deflections produced by tangential loading in the y-direction \(Q_y(\xi, \eta)\) distributed over area S can be found at any general point \(A(x,y,z)\) by superposition.

\begin{align*}
  u_x &= \frac{1}{2\pi G} \iiint_S Q_y(\xi, \eta) v \frac{(x-\xi)(y-\eta)}{\omega^3} d\xi d\eta \\
  u_y &= \frac{1}{2\pi G} \iiint_S Q_y(\xi, \eta) \left( \frac{1-v}{\omega} + v \frac{(y-\eta)^2}{\omega^3} \right) d\xi d\eta \\
  u_z &= \frac{1}{2\pi G} \iiint_S Q_y(\xi, \eta) v \frac{(y-\eta)}{\omega^2} d\xi d\eta
\end{align*}

\text{………… (4.13)}
4.4 Loading over a circular region

Equation (4.7), (4.12), and (4.13) described in previous sections are the rudiments of developing solutions to distributed loading profile and would serve as the base of the subsequent sections. Solutions in closed form can be achieved for axi-symetric conical pressure distribution with a circular base.

4.4.1 Normal Axisymmetric loading

Assume a conical pressure distribution in the form of \( P = P_0 \left(1 - \frac{r^2}{a^2}\right)^\frac{Y}{2} \) applied to a circular base (area \( S \)) of radius “a”.

4.4.1.1 Within the loaded circle

Figure (4.2)
Where

\[ t^2 = r^2 + 2rS \cos \phi + S^2 \]
\[ a^2 - t^2 = (a^2 - r^2) - 2rS \cos \phi - S^2 \]
\[ S_{1,2} = -r \cos \phi \pm \sqrt{r^2 \cos^2 \phi + (a^2 - r^2)} \]
\[ P(S,\phi) = \frac{P_0}{a} \left( (a^2 - r^2) - 2r \cos \phi - S^2 \right)^{0.5} \]

As shown in figure (4.2), the pressure “P” at “C(S,\phi)”, acting on a surface element “SdSd\phi” can be regarded as concentrated force. From equations (4.7) the displacements at any internal surface point “B(x,y)” that lies within the cone base (the loaded circle) can be expressed as:

\[ u_x = -\frac{(1-\nu)}{4\pi G} \int_0^{2\pi} \int_0^{S_1} \frac{P_0}{a} \left( (a^2 - r^2) - 2r \cos \phi - S^2 \right)^{0.5} \cos \phi dSd\phi \]
\[ u_y = -\frac{(1-\nu)}{4\pi G} \int_0^{2\pi} \int_0^{S_1} \frac{P_0}{a} \left( (a^2 - r^2) - 2r \cos \phi - S^2 \right)^{0.5} \sin \phi dSd\phi \]
\[ u_z = \frac{(1-\nu)}{2\pi G} \int_0^{2\pi} \int_0^{S_1} \frac{P_0}{a} \left( (a^2 - r^2) - 2r \cos \phi - S^2 \right)^{0.5} dSd\phi \]

Integrating equations (4.15) throughout the loaded circle, the limits of the integration are from 0 to 2\pi and form 0 to S_1; which is the +ve root of “AS^2+BS+C=0” as shown in equation (4.14). The expression for the displacements can be found in close form and can be expressed as:
\[ u_x = -\frac{(1-2\nu)(1+\nu)}{2\pi E} P_0 a^2 \left( 1 - \left( \frac{a^2 - x^2 - y^2}{a^2} \right)^{1.5} \right) \frac{x}{x^2 + y^2} \]
\[ u_y = -\frac{(1-2\nu)(1+\nu)}{2\pi E} P_0 a^2 \left( 1 - \left( \frac{a^2 - x^2 - y^2}{a^2} \right)^{1.5} \right) \frac{y}{x^2 + y^2} \]
\[ u_z = \frac{(1-\nu^2)}{4E} \pi P_0 \left( 2a^2 - r^2 \right) \]

4.4.1.2 Outside the loaded circle

Where

\[ 0 = (a^2 - r^2) + 2rS \cos \phi - S^2 \]
\[ S_{1,2} = -2r \cos \phi \pm \sqrt{r^2 \cos^2 \phi - (a^2 - r^2)} \]
\[ P(S,\phi) = \frac{P_0}{a} \left( (a^2 - r^2) + 2rS \cos \phi - S^2 \right)^{0.5} \]
\[ \phi_1 = \sin^{-1} \left( \frac{a}{r} \right) \]
As shown in figure (4.3), for an external point “B(x,y)” on the surface outside the loaded circle (the cone base), the limits of the integration in equation (4.7) are ±φ₁, the displacement at any point outside the loaded circle can then be expressed as:

\[
\begin{align*}
  u_x &= -\frac{(1-2\nu)}{4\pi G} \int_{-\phi_1}^{\phi_1} \int_{S_1} P_0 \left( a^2 - r^2 + 2rS \cos \phi - S^2 \right)^{0.5} \cos \phi dSd\phi \\
  u_y &= -\frac{(1-2\nu)}{4\pi G} \int_{-\phi_1}^{\phi_1} \int_{S_1} P_0 \left( a^2 - r^2 + 2rS \cos \phi - S^2 \right)^{0.5} \sin \phi dSd\phi \\
  u_z &= -\frac{(1-\nu)}{2\pi G} \int_{-\phi_1}^{\phi_1} \int_{S_1} P_0 \left( a^2 - r^2 + 2rS \cos \phi - S^2 \right)^{0.5} dSd\phi
\end{align*}
\] ……… (4.18)

Integrating equations (4.18) throughout the loaded circle, the limits of the integration are from “-φ₁” to “φ₁” and form ‘S₂’ to ‘S₁’; which are the roots of “AS₁²+BS+C=0” as shown in equation (4.1); the expression for the displacements can be found in close form and can be expressed as:

\[
\begin{align*}
  u_x &= -\frac{(1-2\nu)(1+\nu)}{3E} P_0 a^2 \left( \frac{x}{x^2+y^2} \right) \\
  u_y &= -\frac{(1-2\nu)(1+\nu)}{3E} P_0 a^2 \left( \frac{y}{x^2+y^2} \right) \\
  u_z &= \frac{(1-\nu)}{E} \frac{P_0}{2a} \left( 2a^2 - r^2 \right) \sin^{-1} \left( \frac{a}{r} \right) + r^2 \left( \frac{a}{r} \right) \left( 1 - \frac{a^2}{r^2} \right)^{0.5}
\end{align*}
\] ……… (4.19)

4.4.2 **Tangential Axisymmetric loading**

Assume a Tangential loading distribution in the form of 

\[
\begin{align*}
  Q_x &= Q_{x_0} \left( 1 - r^2/a^2 \right)^{0.5} \\
  Q_y &= Q_{y_0} \left( 1 - r^2/a^2 \right)^{0.5}
\end{align*}
\]
acting parallel to the “OX” and the “OY” direction respectively and applied to a circular region (area S) or a cone base of radius “a” as shown in figure (4.4).
4.4.2.1 Within the loaded circle

From figure (4.4)  

\[ S^2 = (\xi - x)^2 + (\eta - y)^2 \]

\[ (\xi^2 + \eta^2) = (x + SC\cos\phi)^2 + (y + SS\sin\phi)^2 \]

Reorganizing and introducing \( \alpha \) and \( \beta \) as:

\[ \alpha^2 = a^2 - x^2 - y^2 \quad \text{and} \quad \beta = x\cos\phi + y\sin\phi \]

Therefore \( Q_x \) can be put in the form:

\[ Q_x (S, \phi) = \frac{Q_{x0}}{a} (\alpha^2 + 2\beta S - S^2)^{0.5} \]
As shown in figure (4.4), the load “\(Q_x\)” at “\(C(S,\phi)\)”, acting on a surface element “\(SdSd\phi\)” can be regarded as concentrated force. From equations (4.12) the displacements at any internal surface point “\(B(x,y)\)” that lies within the cone base (the loaded circle) can be expressed as:

\[
\begin{align*}
\frac{u_x}{2\pi G} & = \frac{1}{2\pi G} \int_0^{2\pi} \int_0^a \frac{Q_0}{a} \left( \alpha^2 - 2\beta S - S^2 \right)^{0.5} (1-v) + v\cos^2\phi dSd\phi \\
\frac{u_y}{4\pi G} & = \frac{\nu}{4\pi G} \int_0^{2\pi} \int_0^a \frac{Q_0}{a} \left( \alpha^2 - 2\beta S - S^2 \right)^{0.5} \sin\phi\cos\phi dSd\phi \\
\frac{u_z}{4\pi G} & = \frac{(1-2\nu)}{4\pi G} \int_0^{2\pi} \int_0^a \frac{Q_0}{a} \left( \alpha^2 - 2\beta S - S^2 \right)^{0.5} \cos\phi dSd\phi
\end{align*}
\]

……… (4.20)

Where the limit “\(S_1\)” is given by point “\(D\)” that is lying on the boundary of the base circle shown in figure (4.4), for which \( S_1 = -\beta + \sqrt{\alpha^2 + \beta^2} \).

Integrating equation (4.20), the surface displacement at any internal point due to the tangential load “\(Q_x(S,\phi)\)” parallel to “\(OX\)” can be expressed as:

\[
\begin{align*}
\frac{u_x}{16Ea} & = \frac{(1+\nu)}{16Ea} \pi Q_0 \left( 4(2-v)a^2 - (4-3\nu)x^2 - (4-v)y^2 \right) \\
\frac{u_y}{16Ea} & = \frac{\nu(1+v)}{16Ea} \pi Q_0 (2\nu xy)
\end{align*}
\]

……… (4.21)

Similarly, as shown in figure (4.4), the load ”\(Q_y\)” at “\(C(S,\phi)\)”, acting on a surface element “\(SdSd\phi\)” can be regarded as concentrated force. From equations (4.13) the displacements due to “\(Q_y\)” at any internal surface point ‘\(B(x,y)\)” that lies within the cone base (the loaded circle) can be expressed as:
\[
\begin{align*}
    u_x &= \frac{v}{4\pi G} \int_0^{2\pi} \int_0^a \frac{Q_{y_0}}{a} (\alpha^2 - 2\beta S - S^2)^{0.5} \sin\phi \cos\phi \, dS \, d\phi \\
    u_y &= \frac{1}{2\pi G} \int_0^{2\pi} \int_0^a \frac{Q_{y_0}}{a} (\alpha^2 - 2\beta S - S^2)^{0.5} (1 - \nu + \nu \cos^2 \phi) \, dS \, d\phi \\
    u_z &= \frac{(1 - 2\nu)}{4\pi G} \int_0^{2\pi} \int_0^a \frac{Q_{y_0}}{a} (\alpha^2 - 2\beta S - S^2)^{0.5} \cos\phi \, dS \, d\phi
\end{align*}
\]

Where the limit “\(S_1\)” is given by point “\(D\)” that is lying on the boundary of the base circle shown in figure (4.4), for which \(S_1 = -\beta + \sqrt{\alpha^2 + \beta^2}\).

Integrating equation (4.20), the surface displacement at any internal point due to the tangential load “\(Q_y(S,\phi)\)” parallel to “\(OY\)” is as follow:

\[
\begin{align*}
    u_x &= \frac{v(1 + \nu)}{16Ea} \pi Q_{y_0} (2\nu xy) \\
    u_y &= \frac{(1 + \nu)}{16Ea} \pi Q_{y_0} (4(2 - \nu) a^2 - (4 - 3\nu) y^2 - (4 - \nu) x^2)
\end{align*}
\]

It should be noted that the normal displacement “\(u_z\)” resulting from the tangential loading is not zero but negligibly small and has a negligible effect on the overall normal displacement relative to the normal displacement resulting from normal loading as will be shown in subsequent sections.

**4.4.2.2 Outside the loaded circle**

As shown in figure (4.4), the load “\(Q_x\)” at “\(C(S,\phi)\)”, acting on a surface element “\(SdSd\phi\)” can be regarded as concentrated force. As shown in the previous section, starting from equations (4.12), and integrating with the limits of \(\phi = \pm \sin^{-1}(a/r)\) the
displacements due to "\( Q_x \)" at any external surface point "\( B'(x,y) \)" that outside the cone base (the loaded circle) can be expressed as:

\[
u_x = \frac{(1 + \nu)Q_x}{4Ea} \left[ (2-\nu) \left( 2a^2 - r^2 \right) \sin^{-1} \left( \frac{a}{r} \right) + ar \left( 1 - a^2/r^2 \right)^{1.5} \right] + 0.5v \left( r^2 \sin^{-1} \left( \frac{a}{r} \right) + \left( 2a^2 - r^2 \right) \left( 1 - a^2/r^2 \right)^{0.5} \left( \frac{a}{r} \right) (x^2 - y^2) \right)
\] .......... (4.24)

\[
u_y = \frac{v(1 + \nu)Q_x}{4Ea} \left( r^2 \sin^{-1} \left( \frac{a}{r} \right) + \left( 2a^2 - r^2 \right) \left( 1 - a^2/r^2 \right)^{0.5} \left( \frac{a}{r} \right) (xy) \right)
\]

Similarly, from equations (4.13) the displacements due to "\( Q_y \)" at any external surface point "\( B'(x,y) \)" that outside the cone base (the loaded circle) can be expressed as:

\[
u_x = \frac{v(1 + \nu)Q_y}{4Ea} \left( r^2 \sin^{-1} \left( \frac{a}{r} \right) + \left( 2a^2 - r^2 \right) \left( 1 - a^2/r^2 \right)^{0.5} \left( \frac{a}{r} \right) (xy) \right)
\]

\[
u_y = \frac{(1 + \nu)Q_y}{4Ea} \left[ (2-\nu) \left( 2a^2 - r^2 \right) \sin^{-1} \left( \frac{a}{r} \right) + ar \left( 1 - a^2/r^2 \right)^{1.5} \right] + 0.5v \left( r^2 \sin^{-1} \left( \frac{a}{r} \right) + \left( 2a^2 - r^2 \right) \left( 1 - a^2/r^2 \right)^{0.5} \left( \frac{a}{r} \right) (y^2 - x^2) \right)
\] .......... (4.25)

4.5 **Numerical approach**

Many Non-Hertzian contact problems do not permit analytical solution in closed form, especially in three-dimensional problems with friction that involve partial slip. This has led to the development of the newly developed hybrid numerical approach described in this section.
As mentioned in the previous chapter for the two-dimensional contact problems, the essence of the problem is to find the area of contact together with the normal and tangential traction distribution that are coupled in general. Satisfying the normal and tangential boundary conditions (described in chapter two) at the interface both inside and outside the contact area. The extent of the contact area and the loading profiles can be found. In case that friction is involved, the slip and stick regions are to be defined as well. In this developed hybrid elasticity and finite element method approach for three-dimensional contact problems, the problem was treated as a static problem, where deriving the Boussinesq and Cerruti equations for a simple shaped conical loading profiles and employing the principle of super position, the stick and slip regions can be defined precisely together with the traction distribution. Then taking advantage of the efficiency and robustness of the commercially established and proven linear static solution of the finite element method, the stresses and displacements throughout the body can be found.

To achieve this goal, as shown in figure (4.5), an array of overlapping hyperbolic cones was erected on an equally spaced Cartesian mesh. The centers of the circular bases of these cones, each coincides with a mesh point, therefore, the distance between nodes is equal to the radius of the pressure conical element “a”. The grid is defined only on the contact surface and spread to an initially assumed contact area that is guaranteed to be larger than the actual contact area.
Combining equations (4.16), (4.21) and (4.23), the displacements of any point \( ij \) falls inside the circular base, due to a conical normal loading in the form

\[
\text{Figure (4.5)}
\]
\[ P(x, y) = P_{ij} \left(1 - r_{ij}^2 / a^2\right)^{\frac{1}{2}}, \] and conical tangential loading in the form
\[ Q_x(x, y) = Q_{xij} \left(1 - r_{ij}^2 / a^2\right)^{\frac{1}{2}}, \text{ and } Q_y(x, y) = Q_{yij} \left(1 - r_{ij}^2 / a^2\right)^{\frac{1}{2}} \]
acting over this circular base of radius “a” can be expressed as:

\[
\begin{align*}
  u_{xij} &= -\frac{(1-2\nu)(1+\nu)}{2\pi E} P_{ij} a^2 \left(1 - \left(\frac{a^2-x^2-y^2}{a^2}\right)^{1.5}\right) \left(\frac{x}{x^2+y^2}\right) + \\
  &+ \frac{(1+\nu)}{16Ea} \pi Q_{xij} \left(4(2-\nu)a^2 - (4-3\nu)x^2 - (4-\nu)y^2\right) + \frac{\nu(1+\nu)}{16Ea} \pi Q_{yij} \left(2\nu xy\right)
\end{align*}
\]

\[
\begin{align*}
  u_{yij} &= -\frac{(1-2\nu)(1+\nu)}{2\pi E} P_{ij} a^2 \left(1 - \left(\frac{a^2-x^2-y^2}{a^2}\right)^{1.5}\right) \left(\frac{y}{x^2+y^2}\right) + \\
  &+ \frac{\nu(1+\nu)}{16Ea} \pi Q_{xij} \left(2\nu xy\right) + \frac{(1+\nu)}{16Ea} \pi Q_{yij} \left(4(2-\nu)a^2 - (4-3\nu)x^2 - (4-\nu)y^2\right)
\end{align*}
\]

\[
\begin{align*}
  u_{zij} &= \frac{(1-\nu^2)}{4E} \frac{\pi P_{ij}}{a} \left(2a^2 - x^2 - y^2\right)
\end{align*}
\]

Although the displacements outside the cone base were evaluated as shown in previous sections. It was found that, employing Saint-Venant’s principle leads to satisfactory results. If for a point outside the cone base, this cone was replaced by a statically equivalent load.

Saint-Venant’s principle states that, ”if some distribution of forces acting on a portion of the surface of a body is replaced by a different distribution of forces acting on the same portion of the body, then the effects of the two different distributions on the parts of the body sufficiently far removed from the region of application of the forces are essentially the same, provided that the two distributions of forces are statically equivalent”.

Combining equations (4.5), (4.10) and (4.11), the displacements of any point ‘kl” falls outside side the circular base, due to a conical normal and tangential pressure element as
described above, acting over a circular base of radius “a” away from point ‘kl’ can be expressed as:

\[
\begin{align*}
& u_{xkl} = -\frac{(1-2\nu)(1+\nu)}{3E} P_{yj} a^2 \left( \frac{x}{x^2 + y^2} \right) + \\
& \frac{(1+\nu)}{3E} \pi Q_{x,yj} a^2 \left( \frac{1-\nu}{\sqrt{x^2 + y^2}} + \nu \frac{x^2}{(x^2 + y^2)^{1.5}} \right) + \frac{\nu (1+\nu)}{3E} \pi Q_{y,yj} a^2 \left( \frac{xy}{(x^2 + y^2)^{1.5}} \right) \\
& u_{xkl} = -\frac{(1-2\nu)(1+\nu)}{3E} P_{yj} a^2 \left( \frac{y}{x^2 + y^2} \right) + \\
& \frac{(1+\nu)}{3E} \pi Q_{x,yj} a^2 \left( \frac{1-\nu}{\sqrt{x^2 + y^2}} + \nu \frac{y^2}{(x^2 + y^2)^{1.5}} \right) + \frac{\nu (1+\nu)}{3E} \pi Q_{y,yj} a^2 \left( \frac{xy}{(x^2 + y^2)^{1.5}} \right) \\
& u_{xkl} = -\frac{(1-2\nu)(1+\nu)}{3E} P_{yj} a^2 \left( \frac{1}{\sqrt{x^2 + y^2}} \right) + \frac{(1-2\nu)(1+\nu)}{3E} \pi Q_{x,yj} a^2 \left( \frac{x}{x^2 + y^2} \right) + \\
& \frac{(1-2\nu)(1+\nu)}{3E} \pi Q_{y,yj} a^2 \left( \frac{y}{x^2 + y^2} \right) \\
\end{align*}
\]

Figure (4.6)
From figure (4.6) we can conclude:

\[
\Delta X = aK_x \\
K_x = k - i \\
\Delta Y = aK_x \\
M_y = l - j
\]

\[
R = a\sqrt{k_x^2 + M_y^2}
\]

From equation (2.1) the normal constraints is

\[
U_{x1}(x, y) + U_{x2}(x, y) = U_{x1}(0,0) + U_{x2}(0,0) - h(x, y)
\]

Figure (4.6) demonstrates the grid of nodes created at the centers of conical pressure elements. By satisfying the boundary conditions and superimposing the effect of the loading at all the nodes \((X_k, Y_l)\) on a point \((X_i, Y_j)\), the total displacement at this point can be calculated.

Substituting equation (4.26), (4.27) and (4.28) into the right hand side of equation (2.1), the right hand side of equation (2.1) is

\[
(U_{x1}(x, y) + U_{x2}(x, y))_{ij} = \sum_k \sum_l z_{ijkl} B_z P_{kl} + z_{ijkl} C_z Q_{X,kl} + z_{ijkl} D_z Q_{Y,kl}
\]

Where:

\[
z_{ijkl}, \quad z_{ijkl}, \quad z_{ijkl}
\]

are the normal constraint coefficients of point \(ij\) due to normal load \(P_{kl}\) and tangential load \(Q_{X,kl}\) and \(Q_{Y,kl}\) at any point \(kl\) respectively, and are equal to:

\[
z_{ijkl} = \begin{cases} 
2 - K_x^2 - M_y^2 & \text{for } R \leq a \\
\frac{1}{\sqrt{K_x^2 + M_y^2}} & \text{for } R \geq a
\end{cases}
\]
\[
\varepsilon_{ijkl} = \begin{cases} 
0 & \text{for } R \leq a \\
\frac{K_x}{K_x^2 + M_y^2} & \text{for } R \geq a
\end{cases}
\]

\[
\xi_{ijkl} = \begin{cases} 
0 & \text{for } R \leq a \\
\frac{M_y}{K_x^2 + M_y^2} & \text{for } R \geq a
\end{cases}
\]

\[B_z = C_z \]

\[D_z = \begin{cases} 
a & \text{for } R \leq a \\
\frac{(1-2\nu_1)(1+\nu_1)a}{2E_1} + \frac{(1-2\nu_2)(1+\nu_2)a}{2E_2} & \text{for } R \geq a
\end{cases}
\]

\[B_z, \ C_z, \ \text{and } D_z \] are normal constraint constants associated with normal load \(P\), and tangential load \(Q_x\), and \(Q_y\) respectively. They are dependent on the materials of the two bodies in contact and are equal:

\[
B_z = \left\{ \begin{array}{ll} 
\frac{\left(1-v_1^2\right)a}{4E_1} + \frac{\left(1-v_2^2\right)a}{4E_2} & \text{for } R \leq a \\
\frac{(1-v_1)a}{E_1} + \frac{(1-v_2)a}{E_2} & \text{for } R \geq a
\end{array} \right.
\]

Substituting equation (4.29) into equation (2.1). The normal constraint of a point at \((X_i, Y_j)\) due to normal and tangential loading at \((X_k, Y_l)\) can be expressed in the form:

\[
h(x, y)_{ij} = \sum_k \sum_l \bar{h}_{ijkl} B_z P_{kl} + \bar{\xi}_{ijkl} C_z Q_{X,kl} + \bar{\xi}_{ijkl} D_z Q_{Y,kl} \]

Where,

\[h(x, y)_{ij}\] is the initial separation, at point “ij”, described in chapter (2) and is dependant on the geometries of the two bodies.
\( \bar{\eta}_{ijkl}, \bar{\xi}_{ijkl}, \text{and } \bar{\zeta}_{ijkl} \) are the normal coefficients relative to the reference point that was taken as the origin for convenience and are equal to:

\[
\bar{\eta}_{ijkl} = \eta_{00kl} - \eta_{ijkl} \\
\bar{\xi}_{ijkl} = \xi_{00kl} - \xi_{ijkl} \\
\bar{\zeta}_{ijkl} = \zeta_{00kl} - \zeta_{ijkl}
\]

Within the stick zone, as shown in equation (2.4) the tangential constraint is:

\[
U_{x1}(x, y) + U_{x2}(x, y) = U_{x1}(0, 0) + U_{x2}(0, 0) \\
U_{y1}(x, y) + U_{y2}(x, y) = U_{y1}(0, 0) + U_{y2}(0, 0)
\]

Substituting equation (4.26), (4.27) and (4.28) into the right hand side of equation (2.4). The tangential displacement of a point at \((X_k, Y_l)\) due to normal and tangential loading at \((X_i, Y_j)\) can be expressed in the form:

\[
(U_{x1}(x, y) + U_{x2}(x, y))_{ij} = \sum_k \sum_l \eta_{ijkl} B_{x kl} + \xi_{ijkl} C_{x kl} + \zeta_{ijkl} D_{x kl}
\]

\[
(U_{y1}(x, y) + U_{y2}(x, y))_{ij} = \sum_k \sum_l \eta_{ijkl} B_{y kl} + \xi_{ijkl} C_{y kl} + \zeta_{ijkl} D_{y kl}
\]

\[\ldots (4.31)\]

Where:

"\( \eta_{ijkl}, \xi_{ijkl}, \text{and } \zeta_{ijkl} \)" are the ‘X and Y’ tangential constraint coefficients of point \(ij\) due to normal load \(P_{kl}\) and tangential load \(Q_{X,kl}\) and \(Q_{Y,kl}\) at any point \(kl\) respectively, and are equal to:

\[
x_{ijkl} = \begin{cases} 
\frac{K_x}{K_x^2 + M_y^2} & \text{for } R \leq a \\
\left(1 - (1 - K_x^2 - M_y^2)^{1.5\ldots} \right) \frac{K_x}{K_x^2 + M_y^2} & \text{for } R \geq a
\end{cases}
\]
\[ x_{ijkl} = \begin{cases} 
\left( \frac{1 + v_1}{16E_1} \left( 4(2 - v_1) - (4 - 3v_1)K_x^2 + (4 - v_1)M_y^2 \right) + \frac{1 + v_2}{16E_2} \left( 4(2 - v_2) - (4 - 3v_2)K_x^2 + (4 - v_2)M_y^2 \right) \right), & R \leq a \\
\left( \frac{1 + v_1}{3E_1} \left( 1 - v_1 \right) + \frac{1 - v_2}{3E_2} \left( 1 - v_2 \right) \right), & R \geq a \\
\left( \frac{1 - v_1}{K_x^2 + M_y^2} \left( K_x^2 + M_y^2 \right)^{1.5} \right), & R \geq a \\
\left( \frac{1 - v_2}{K_x^2 + M_y^2} \left( K_x^2 + M_y^2 \right)^{1.5} \right), & R \geq a 
\end{cases} \]

\[ \zeta_{ijkl} = \begin{cases} 
M_xK_x, & R \leq a \\
K_xM_y, & R \leq a \\
K_x^2 + M_y^2, & R \geq a 
\end{cases} \]

\( B_x, C_x, \) and \( D_x \) are normal constraint constants associated with normal load \( P, \) and tangential load \( Q_x, \) and \( Q_y \) respectively. They are dependent on the materials of the two bodies in contact and are equal:

\[ B_x = \begin{cases} 
- a \left( \frac{(1 + v_1)(1 - 2v_1)}{2\pi E_1} + \frac{(1 + v_2)(1 - 2v_2)}{2\pi E_2} \right), & R \leq a \\
- a \left( \frac{(1 + v_1)(1 - 2v_1)}{3E_1} + \frac{(1 + v_2)(1 - 2v_2)}{3E_2} \right), & R \geq a 
\end{cases} \]

\[ C_x = \begin{cases} 
\pi a, & R \leq a \\
2\pi a, & R \geq a 
\end{cases} \]

\[ D_x = \begin{cases} 
\pi a \left( \frac{(1 + v_1)\nu_1}{16E_1} + \frac{(1 + v_2)\nu_2}{16E_2} \right), & R \leq a \\
\pi a \left( \frac{(1 + v_1)\nu_1}{3E_1} + \frac{(1 + v_2)\nu_2}{3E_2} \right), & R \geq a 
\end{cases} \]
Similarly the tangential constraints coefficients in y-direction can be expressed as:

\[
y\eta_{ijkl} = \begin{cases} 
\frac{M_y}{K_x^2 + M_y^2} & \text{for } R \leq a \\
\left(1-(1-K_x^2-M_y^2)^{1.5}\right) \frac{M_y}{K_x^2 + M_y^2} & \text{for } R \geq a
\end{cases}
\]

\[
y\xi_{ijkl} = \begin{cases} 
\frac{M_y K_x}{K_x M_y} & \text{for } R \leq a \\
\frac{M_y K_x}{K_x + M_y^2} & \text{for } R \geq a
\end{cases}
\]

\[
y\zeta_{ijkl} = \begin{cases} 
\left\{ \frac{1}{16E_1} \left(4(2-v_1)-(4-3v_1)M_y^2+(4-v_1)K_x^2\right) \right\} & \text{for } R \leq a \\
\left\{ \frac{1}{16E_2} \left(4(2-v_2)-(4-3v_2)M_y^2+(4-v_2)K_x^2\right) \right\} & \text{for } R \geq a
\end{cases}
\]

\[
B_y, C_x, \text{ and } D_x \text{ are normal constraint constants associated with normal load } P, \text{ and tangential load } Q_x, \text{ and } Q_y \text{ respectively. They are dependent on the materials of the two bodies in contact and are equal:}
\]

\[
B_y = \begin{cases} 
-a \left( \frac{(1+v_1)(1-2v_1)}{2\pi E_1} + \frac{(1+v_2)(1-2v_2)}{2\pi E_2} \right) & \text{for } R \leq a \\
-a \left( \frac{(1+v_1)(1-2v_1)}{3E_1} + \frac{(1+v_2)(1-2v_2)}{3E_2} \right) & \text{for } R \geq a
\end{cases}
\]
Substituting equation (4.30) into equation (2.4). The normal constraint of a point that falls in the stick region at \((X_i, Y_j)\) due to normal and tangential loading at \((X_k, Y_l)\) can be expressed in the form:

\[ 0 = \sum_k \sum_l \text{Re} \overline{n}_{ijkl} B_x P_{kl} + \overline{\xi}_{ijkl} C_x Q_{X,kl} + \overline{\zeta}_{ijkl} D_x Q_{Y,kl} \]

\[ 0 = \sum_k \sum_l \text{Im} \overline{n}_{ijkl} B_y P_{kl} + \overline{\eta}_{ijkl} C_y Q_{X,kl} + \overline{\nu}_{ijkl} D_y Q_{Y,kl} \]

\[ \ldots \ldots \quad (4.32) \]

\( \overline{n}_{ijkl}, \overline{\eta}_{ijkl}, \text{and } \overline{\mu}_{ijkl} \) are the tangential coefficients relative to the reference point that was taken as the origin for convenience and are equal to:

\[ \text{Re} \overline{n}_{ijkl} = x_{0000} - x_{ijkl} \]
\[ \text{Im} \overline{n}_{ijkl} = y_{0000} - y_{ijkl} \]
\[ \text{Re} \overline{\xi}_{ijkl} = x_{0000} - x_{ijkl} \]
\[ \text{Im} \overline{\xi}_{ijkl} = y_{0000} - y_{ijkl} \]
\[ \text{Re} \overline{\zeta}_{ijkl} = x_{0000} - x_{ijkl} \]
\[ \text{Im} \overline{\zeta}_{ijkl} = y_{0000} - y_{ijkl} \]

The tangential coefficients in y-direction can be expressed in the same manner.

For the slip zone equation (2.6) Holds

\[ Q_{yij}(x, y) = |\mu_y P_y(x, y)| \]
\[ Q_{xij}(x, y) = |\mu_x P_y(x, y)| \]
\[ Q_{yij}(x, y) = |\mu_y P_x(x, y)| \]

\[ \ldots \ldots \quad (4.33) \]
Where

\[
Q_{aij}(x, y) = \sqrt{Q_{xij}(x, y)^2 + Q_{yij}(x, y)^2}
\]

\[
\mu_0(x, y) = \sqrt{\mu_x^2(x, y) + \mu_y^2(x, y)}
\]

\[
\mu_x = \frac{Q_x}{Q_0} \mu_0 \quad \text{And} \quad \mu_y = \frac{Q_y}{Q_0} \mu_0
\]

Then for the interface as shown in figure (4.6), the domain is divided into \(2nx2n\) nodes. Therefore, we have \((2n+1)^2\) surface elements with \((2n+1)^2\) nodes and \(3(2n+1)^2\) unknowns \((P, Q_x \text{ and } Q_y \text{ at each node})\). Equations (4.31), (4.32), and (4.33) constitute \(3(2n+1)^2\) equations. However, Equations (4.31) and (4.32) are degenerate at “\(i=0\)” and “\(j=0\)” since the coefficients are equal to zero at the origin, therefore, three more equations are needed and can be obtained from the global force summation:

\[
F = \sum_{i,j=-n}^{n} A_{ij} P_{ij}
\]

\[
T_x = \sum_{i,j=-n}^{n} A_{ij} Q_{xij}
\]

\[
T_y = \sum_{i,j=-n}^{n} A_{ij} Q_{yij}
\]

where:

\(A_{ij}\) is the characteristic volume of the conical element with circular base of radius “\(a\)” and is equal to \(2\pi a^2/3\).

\(F, T_x, \text{ and } T_y\) are the total Normal and tangential loads applied to the two contacting bodies.
4.6 Examples

Following the methodology described in the previous chapter (refer to the flow chart described in Appendix A) and using the equations derived in the previous section (equations 4.30, 4.32, 4.33, and 4.34) a computer code was developed for the three-dimensional contact problems with friction. The area of contact was found together with the loading distribution and the stick and slip zones.

Two Finite element models were built. The first being the second phase of the method using linear static finite element solution to find the stresses and displacements. The second model was built employing gap elements for comparison and validation purposes. The results were then compared with other methods and the exact solution to compare accuracy and efficiency of the method.

4.6.1 Contact of elastic spheres without friction

Consider two elastic spheres of radii 100 mm and 150 mm, elastic modulii of 206840 and 193050 N/mm² respectively, and poison ratio of 0.27 pressed in contact by a force of 1000 N without friction. This example was chosen since there exists an exact solution for that case which enables the validation the methodology and the solution formulation.

As shown in figure (4.7), two sets of nodes were allocated, that is, for each row of nodes the last boundary node that falls inside the contact area (last node with compressive load) and the first adjacent node outside the contact area. Two different methods were used to find the exact area of contact employing these sets of nodes. By interpolation between
these two nodes, the position of the points of zero pressure can be found, and therefore the contact region can be defined by a piecewise linear function, or a higher order function that satisfies the locus of all these locations. Due to the symmetry in this problem, it was expected that the area of contact is a circle, the least square method was used to define the best circle fit between the two sets of nodes shown. The two methods led to very similar results with the first being more general.

The least square equation used is as follows:

$$ r_i = \sqrt{x_i^2 + y_i^2} + \frac{a}{2} $$

$$ r = \frac{1}{n} \sum_{i=1}^{n} r_i $$

Figure (4.7)
Where \( r \) is the estimated radius of the circle, \( a \) is the radius of the cone, \( x_i \) and \( y_i \) are the coordinates of node \( \text{“i”} \).

<table>
<thead>
<tr>
<th>Method</th>
<th>Hertz</th>
<th>FE model</th>
<th>Numerical method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_{\text{max}} ) (N/mm(^2))</td>
<td>854.3278</td>
<td>909.9371</td>
<td>854.1688</td>
</tr>
<tr>
<td>contact radius (mm)</td>
<td>0.7475813</td>
<td>0.69998</td>
<td>0.748339</td>
</tr>
<tr>
<td>Maximum Stress (N)</td>
<td>264.8416</td>
<td>336.34</td>
<td>277.606</td>
</tr>
<tr>
<td>( P_{\text{max}} ) % error</td>
<td>N/a</td>
<td>6.55%</td>
<td>0.1861%</td>
</tr>
<tr>
<td>Area % error</td>
<td>N/a</td>
<td>4.75%</td>
<td>0.3378%</td>
</tr>
<tr>
<td>Max. Stress % error</td>
<td>N/a</td>
<td>27.6%</td>
<td>4.82%</td>
</tr>
<tr>
<td>Solution Time</td>
<td>N/a</td>
<td>4 (hrs)</td>
<td>38 (min)</td>
</tr>
</tbody>
</table>

*Table (4.1)*

As shown in Table (4.1) the numerical results are extremely close to those of the classical Hertz solutions. Indeed the area of contact and the max pressure are virtually identical whereas in the case of the finite element method the errors are much higher but still acceptable in practicality. For the stresses the method error was about 4.8% down from 27.6% in the case of the finite element method that shows the enhancement of the hybrid method over the traditional finite element method. Although not quoted above the modeling time was four times higher in the case of the FEM, while the solution time was about 38 minutes in case of the Hybrid method down from four hours in the FEM.

The number of elements used was the same for comparison purposes and was limited by the finite element method rather than the hybrid method, therefore even higher accuracy is to be expected if the number of elements was increased however due to the low errors resulted this seemed unnecessary for the comparative purpose of this problem.
4.6.2 **Contact of elastic spheres with friction**

Consider two elastic spheres each of radii 2 mm, elastic modulii of 432000 and 72000 N/mm$^2$ respectively, and poison ratio of 0.32, with coefficient of friction of 0.1, pressed in contact by a force of 10 N.

This problem was solved in the past with two different approaches. An analytical approach by Spence (64), and more recently, a numerical approach by Guyot and Kosior (66).

<table>
<thead>
<tr>
<th>Method</th>
<th>Hertz</th>
<th>Spence</th>
<th>Guyot</th>
<th>FEM</th>
<th>Hybrid Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pmax (N/mm$^2$)</td>
<td>2092.5</td>
<td>N/A</td>
<td>2156.6</td>
<td>2386.5</td>
<td>2149.331</td>
</tr>
<tr>
<td>Contact Radius (mm)</td>
<td>0.0478</td>
<td>N/A</td>
<td>0.0475</td>
<td>0.04318</td>
<td>0.047664</td>
</tr>
<tr>
<td>Stick / Contact radius</td>
<td>N/A</td>
<td>0.530</td>
<td>0.690</td>
<td>0.512</td>
<td>0.596886</td>
</tr>
<tr>
<td>Solution Time</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>7 (hrs)</td>
<td>51 min</td>
</tr>
</tbody>
</table>

**Table (4.2)**

As shown in Table (4.2), the results from the newly developed method were compared with the results provided from the above-mentioned approaches in addition to the results from the finite element method using gap elements. It should be noted here that the same element length was used except in the case of the finite element model where convergence problems hindered the efforts of raising the number of element more than 14 elements through the radius of contact.
(b) Tangential loading distribution

Max. Pressure at the center of the area of contact

Figure (4.8)
The normal and tangential distribution was calculated, together with the radius of contact “a” and the stick zone “c”. Figure (4.8.a) shows a 3D contour plot of the normal pressure distribution in which the height of the contour indicates its value, the pressure falls from its maximum value at the center to zero at the edges of the contact area.

As shown in figure (4.8.b), The contour plot of the tangential loading distribution over the area of contact is divided into two zones. The slip annulus, where the radius is larger than the stick radius “c” but less than the radius of contact, the tangential traction follows the Coulomb’s law of friction (quoted in Chapter 2). The stick zone, in which the tangential traction is; less than the limiting force of friction, and does not scale with the normal pressure, but rather locally satisfying the equilibrium and boundary conditions. Note that the tangential traction falls to zero at the point of symmetry at the center of the contact region.

Figure (4.9)
As shown in figure (4.9), following the same methodology discussed in the previous example, a finite element model of the two spheres in contact was built and then another model for only one of the spheres under the normal and tangential loads calculated from the numerical scheme shown in figure (4.8.a,b).

The results for the newly developed method showed better agreement with other approaches than in the case of the finite element solution with a major saving in the solution and modeling time. It should be noted here that the finite element model had very limited flexibility to the increase in the number of elements, which was faced by convergence problems and a steep increase in solution time.
4.7 Conclusion

The examples described in this chapter demonstrate the fidelity of this hybrid method in the case of three-dimensional contact as it did in the case of the two-dimensional contact. Comparing to the traditional finite element approach in contact problems, the new method showed higher accuracy relative to exact solution and other numerical and analytical methods, in a fraction of the solution and modeling time. It was clear that the limit on increasing the number of elements to enhance accuracy was imposed by the traditional finite element approach rather than the hybrid method and therefore even more elements can be used to enhance the accuracy without the fear of convergence problems.

This approach combined a lot of the features and flexibility in the finite element method while showing a great deal of enhancement on the accuracy, solution time, and the efficiency of the results.

Using iterative analytical solution in calculating the area of contact and the loading distribution offers many advantages:

- Eliminating displacement interpolation errors.
- Smaller domain is utilized; this reduces computation cost of each iteration and therefore, enhances the solution efficiency.
- Higher accuracy in the calculation of the area of contact and the loading distribution.
- General non-conforming geometries and loading profiles can be solved.
• Frictional effects are included.

Obtaining the area of contact and the loading profiles using the iterative solution based on the theory of elasticity and not the finite element method has many benefits that was clear from the numerical examples presented. These benefits are:

• No gap elements are needed and therefore the non-linearity of the solution is eliminated.

• Trial and error for gap element stiffness is eliminated.

• Ability to use more elements in the area of interest without hindering convergence behavior and with a slight increase in solution time.

• Avoiding the addition of any constraints or equations in the contacts problems involving friction, which leads to a smaller stiffness matrix and therefore less solution time. The finite element part of the method does not involve friction even if the friction is involved in the problem but instead the friction is accounted for in the calculation of the contact area and the loading distribution.

• The finite element contact problem is reduced to a linear static problem, which is validated and known to have higher reliability and accuracy than the nonlinear large displacement finite element solution.

• Employing only the linear static solution in finite element Analysis allowing less modeling and solution time. The extent and shape of contact is calculated using theory of elasticity, allowing the planning for the proper mesh.
Since crack propagation is governed by tensile stresses, just knowing the skin stresses
might not be helpful, especially that max stress might occur elsewhere in the body.
Concentrating on the area of interest when calculating the area of contact and the loading
profile while having the ability of evaluating the stresses and deformations throughout the
body is an important enhancement over both the FEM and the BEM and enhances the
ability to predict crack occurrences.

The Basic theory was presented for both the two and three-dimensional contact problems,
together with the algorithm and numerical examples that showed accuracy, efficiency and
robustness of the method.
Future research work is recommended, employing different friction laws and including
sliding and dynamic effects. Different interpolation schemes for the area of contact can
be compared for accuracy, and the use of higher order conical elements could offer an
ever better enhancement in the overall accuracy of the method.
More research is needed to include the non-conformal three-dimensional in dynamic,
rolling, and sliding contact analysis, the conformal contact, and the contact involving
non-linear material and thermal boundary conditions.
References


50. T. Anderson,” The second generation Boundary element contact problem”, In proc. of the fourth international seminar on recent advances in boundary element methods, Southampton, 1982


53. T. Tsuta, and S. Ymagi, Boundary element analysis of contact thermoelastoplastic problems with creep and the numerical technique”, In proc. of the fifth international seminar on recent advances in boundary element methods, Springer Verlag, Berlin, 1983.


Appendix A  Flow chart for the structure of the developed method

1. Read Input Data
2. Assume, all stick, contact area
3. Solve For Normal and Tangential distribution
4. Compute New contact area
5. Check contact area for convergence
   - Not converged
     - Adjust Contact area
     - Not converged
       - Check contact area for convergence
         - Not converged
           - Adjust Contact area
           - Not converged
             - Check contact area for convergence
               - Not converged
                 - Adjust Contact area
                 - Not converged
                   - Check contact area for convergence
                     - Not converged
                       - Adjust Contact area
                       - Not converged
                         - Check contact area for convergence
                           - Not converged
                             - Adjust Contact area
                             - Not converged
                               - Check contact area for convergence
                                 - Not converged
                                   - Adjust Contact area
                                   - Not converged
                                     - Check contact area for convergence
                                       - Not converged
                                         - Adjust Contact area
                                         - Not converged
                                           - Check contact area for convergence
                                             - Not converged
                                               - Adjust Contact area
                                               - Not converged
                                                 - Check contact area for convergence
                                                   - Not converged
                                                     - Adjust Contact area
                                                     - Not converged
                                                       - Check contact area for convergence
                                                         - Not converged
                                                           - Adjust Contact area
                                                             - Normal Loading converged?
                                                               - Y
                                                                - Divide area into stick and slip
                                                                  - Y
                                                                    - Build FE model with calculated contact area
                                                                      - Shape and extent Known
                                                                        - Apply calculated Loading distribution
                                                                                   - Solve for stresses and displacements throughout the two bodies
                                                                                     - N
                                                                                       - Adjust Slip zone
                                                                                         - X
                                                                                             - Build FE model with calculated contact area
                                                                                               - Shape and extent Known
                                                                                                 - Apply calculated Loading distribution
                                                                                                      - Solve for stresses and displacements throughout the two bodies
                                                                                                        - N
                                                                                                            - Adjust Slip zone
                                                                                                               - X
                                                                                                                   - Build FE model with calculated contact area
                                                                                                                     - Shape and extent Known
                                                                                                                                   - Apply calculated Loading distribution
                                                                                                                                 - Solve for stresses and displacements throughout the two bodies
                                                                                                                                               - N
                                                                                                                                                   - Adjust Slip zone
                                                                                                                                                      - X
                                                                                                                                                                         - Build FE model with calculated contact area
                                                                                                                                                                             - Shape and extent Known
                                                                                                                                                                                               - Apply calculated Loading distribution
                                                                                                                                                                                               - Solve for stresses and displacements throughout the two bodies
                                                                                                                                                                                                                     - N
                                                                                                                                                                                                                       - Adjust Slip zone
                                                                                                                                                                                                                                      - X
                                                                                                                                                                                                                                                   - Build FE model with calculated contact area
                                                                                                                                                                                                                                                            - Shape and extent Known
                                                                                                                                                                                                                                                                 - Apply calculated Loading distribution
                                                                                                                                                                                                                                                                                     - Solve for stresses and displacements throughout the two bodies
                                                                                                                                                                                                                                                                                                                                                      - N
                                                                                                                                                                                                                                                                                                                                                              - Adjust Slip zone
                                                                                                                                                                                                                                                                                                                                                                                  - X
                                                                                                                                                                                                                                                                                                                                                                                                          - Build FE model with calculated contact area
                                                                                                                                                                                                                                                                                                                                                                                                           - Shape and extent Known
                                                                                                                                                                                                                                                                                                                                                                                                 - Apply calculated Loading distribution
                                                                                                                                                                                                                                                                                                                                                                                                                     - Solve for stresses and displacements throughout the two bodies
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  - N
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              - Adjust Slip zone
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  - X
Appendix B

Two-dimensional contact program

PROGRAM TWODIM
*

DOUBLE PRECISION E1, E2, R1, R2, NU1, NU2, A1, A2, A3, EPS
DOUBLE PRECISION C1, C2, D1, D2, B1, X, MU, W, WRIGHT, WLEFT
INTEGER I, J, L, M, N, L1, L2, LL, MN, MN2, MN3
INTEGER LLL, MN1, M2, N2, L3, LFLAG, RFLAG, MM, NN, ITER
REAL PI, K, F, T
DOUBLE PRECISION P, BB, X1, X2, WLST, WRST, ASUM
DOUBLE PRECISION C, CC, D, DD, Q, QL, QR
DOUBLE PRECISION A

DIMENSION P(100), Q(100), QL(100), QR(100)
DIMENSION C(100,100), CC(100,100), D(100,100), DD(100,100)
DIMENSION A(98,99)

PI = 3.141593

OPEN (UNIT=8, FILE='RESULTS.DAT', STATUS='NEW')
*

PRINT*, 'ENTER CYLINDER I RADIUS R1 ='
READ*, R1
PRINT*, 'ENTER CYLINDER II RADIUS R2 ='
READ*, R2
PRINT*, 'ENTER MODULUS OF ELASTICITY FOR CYLINDER I '
READ*, E1
PRINT*, 'ENTER MODULUS OF ELASTICITY FOR CYLINDER II '
READ*, E2
PRINT*, 'ENTER POISSON RATIO FOR CYLINDER I '
READ*, NU1
PRINT*, 'ENTER POISSON RATIO FOR CYLINDER II '
READ*, NU2
PRINT*, 'ENTER THE COEFFICIENT OF FRICTION '
READ*, MU
PRINT*, 'ENTER THE CONTACT NORMAL FORCE /UNIT LENGTH '
READ*, F
PRINT*, 'ENTER THE TANGENTIAL FORCE /UNIT LENGTH '
READ*, T
PRINT*, 'ENTER THE NUMBER OF ELEMENTS '
READ*, N
*

* ASSUME AREA OF CONTACT AS (150% OF HERTZ)
*

E3 = (E1 * E2) / ((E2 * (1 - NU1 ** 2)) + (E1 * (1 - NU2 ** 2)))
R3 = (R1 * R2) / (R1 + R2)
*

A IS THE HALF WIDTH (HALF THE CONTACT AREA OF WIDTH 2A)
*

A3 = SQRT(((4 * F * R3) / (PI * E3))
WLEFT = 1.2 * A3
WRIGHT = WLEFT
M=N
MM=M
NN=N
ITER=0

*
$105 \ \text{ASUM=WRIGHT+WLEFT}$

\[ W = \frac{\text{ASUM}}{N+M} \]

* \[ C1 = \left( \frac{(1-\nu1^2)}{E1} + \frac{(1-\nu2^2)}{E2} \right) \]

\[ D1 = \left( -1 \right) \left( \frac{(1-2\nu1)(1+\nu1)}{E1} + \frac{(1-2\nu2)(1+\nu2)}{E2} \right) \]

\[ C2 = \left( \frac{1}{E1} \right) + \left( \frac{(1-\nu2^2)}{E2} \right) \]

\[ D2 = \left( \frac{(1-2\nu1)(1+\nu1)}{E1} + \frac{(1-2\nu2)(1+\nu2)}{E2} \right) \]

* \[ MN = M+N+1 \]

DO 10 I=1,MN,1
DO 20 J=1,MN,1
K=J-I
IF(ABS(K).GT.1) THEN
A1=((k+1)**2)* LOG((k+1)**2);
A2=((k-1)**2)* LOG((k-1)**2);
A3=((k)**2)* LOG((k)**2);
C(I,J) = \left( -W/(2*PI) \right) \left( A1+A2-(2*A3) \right) ;
ELSEIF(ABS(K).EQ.0) THEN
C(I,J)=0.0
ELSEIF(ABS(K).EQ.1) THEN
*C1=(((1-NU1**2)/E1)+((1-NU2**2)/E2))
D1=((-1*(((1-2*NU1)*(1+NU1))/E1))+(((1-2*NU2)*(1+NU2))/E2))
C2=(-1*(1-NU1**2)/E1)+((1-NU2**2)/E2)
D2=((1-2*NU1)*(1+NU1))/E1)+(((1-2*NU2)*(1+NU2))/E2)

*ln4=1.386294361
C(I,J)=(-2*W*1.386294361)/(PI)
ENDIF
IF(K.GT.0) THEN
D(I,J)=-W/2
ELSEIF(K.LT.0) THEN
D(I,J)=W/2
ENDIF
20 CONTINUE
10 CONTINUE

DO 30 I=1,MN,1
DO 40 J=1,MN,1
CC(I,J)=C(M+1,J)-C(I,J)
DD(I,J)=D(M+1,J)-D(I,J)
40 CONTINUE
30 CONTINUE

* \[ MN2 = (2*(M+N+1)) + 1 \]

DO 50 I=1,MN,1
L1=I-M-1
L=I+MN
DO 60 J=1,MN,1
CC(I,J)=C1*CC(I,J)
A(I,2*J)=D1*DD(I,J)
BB=(sqrt((R1**2)-(L1*w)**2))+( sqrt((R2**2)-(L1*w)**2));
A(I,MN2)=R1+R2-BB;
* \[ BB=(sqrt((R1**2)-(L1*w)**2))+( sqrt((R2**2)-(L1+w)**2)); \]

* Region 1 in the matrix, Notice that P is odd and Q is even

* \[ A(I,(2*J)-1)=C1*CC(I,J) \]
A(I,2*J)=D1*DD(I,J)
BB=(sqrt((R1**2)-(L1+w)**2))+( sqrt((R2**2)-(L1+w)**2));
A(I,MN2)=R1+R2-BB;

* Region 3 in the matrix.... FROM MN TO 2MN

* STICK REGION

IF (L1.GE.-MM.AND.L1.LE.NN) THEN
A(L,(2*J)-1)=D2*DD(I,J)
A(L,2*J)=C2*CC(I,J)
A(L,MN2)=0.0
ENDIF

* SLIP REGION
* Q(I)=MU*P(I)
* 
IF(L1.LT.-MM.OR.L1.GT.NN) THEN
   IF (J.EQ.I) THEN
      A(L,(2*J)-1)=MU
      A(L,2*J)=-1
      A(L,MN2)=0.0
   ELSE
      A(L,J)=0.0
      A(L,MN2)=0.0
   ENDIF
ENDIF

* Region 2 in the matrix at zero
*
IF (I.EQ.M+1) THEN
   A(I,(2*J)-1)=W
   A(I,(2*J))=0.0
   A(I,MN2)=F
   A(L,(2*J)-1)=0.0
   A(L,(2*J))=W
   A(L,MN2)=T
ENDIF

60 CONTINUE
50 CONTINUE

DO 111 I=1,90,1
   WRITE (8,*) I
   WRITE (8,*) (A(I,J),J=1,91)
111 CONTINUE

* Solve system of equations
EPS=1.E-30
MN3=2*MN
CALL GAUSS(MN3,1,A,EPS)

DO 120 I=1,MN,1
   P(I)=A((2*I)-1,MN2)
   Q(I)=A((2*I),MN2)
120 CONTINUE
PRINT*, P(1), P(7)

* Adjust the contact area
IF (P(MN).LT.0) THEN
   DO 70 I=MN,M+1,-1
      IF (P(I).LT.0.0.AND.P(I-1).GT.0.0) THEN
         X1=W*(ABS(P(I))/(ABS(P(I-1))+ABS(P(I))))
         WRIGHT=WRIGHT-((MN-I)*W)-X1
         RFLAG=2
      ENDIF
   70 CONTINUE
ELSEIF (P(MN).GE.0) THEN
   DO 77 I=M+N+1,M+1 ,-1
      QR(I)=MU*P(I)
77 CONTINUE
ELSE
   DO 78 I=1,90,1
      QR(I)=P(I)
78 CONTINUE
ENDIF
IF (ABS(Q(I)).GE.ABS(QR(I))) THEN
  NN=N-(M+N+1-I)
  WRST=NN*W
  RFLAG=2
ELSE
  RFLAG=1
ENDIF

77  CONTINUE
ELSE
  RFLAG=1
ENDIF

IF (P(1).LT.0) THEN
  DO 80 I=1,M+1,1
    IF (P(I).LT.0.0.AND.P(I+1).GT.0.0) THEN
      X2=W*(ABS(P(I))/(ABS(P(I+1))+ABS(P(I))))
      WLEFT=WLEFT-(I*W)+(W-X2)
      LFLAG=2
    ENDIF
  80 CONTINUE
ELSEIF (P(1).GT.0.001) THEN
  * WLEFT=WLEFT+W
  * LFLAG=3
ELSEIF (P(1).GE.0) THEN
  DO 88 I=1,M+1,1
    QL(I)=MU*P(I)
    IF (ABS(Q(I)).GE.ABS(QL(I))) THEN
      MM=M+1-I
      WLST=MM*W
      LFLAG=2
    ELSE
      LFLAG=1
    ENDIF
  88 CONTINUE
ELSE
  LFLAG=1
ENDIF

*Check if the slip turn out to be bigger
IF (WLST.GT.WLEFT) THEN
  WLEFT=WLST
ELSEIF (WRST.GT.WRIGHT) THEN
  WRIGHT=WRST
ENDIF

******** SET NEW ELEMENT WIDTH******
ASUM=WRIGHT+WLEFT
W=(ASUM)/(N+M)
* MM=(WLST/W)
* NN=(WRST/W)

WRITE (8,*) WRIGHT, WLEFT, ASUM, W, MN3
WRITE (8,*) MM, NN, WLST, WRST
DO 110 I=1,MN,1
  WRITE (8,*) I, P(I), Q(I)
110  CONTINUE
ITER=ITER+1
PRINT*, ITER
IF (LFLAG.NE.1.OR.RFLAG.NE.1) THEN
SUBROUTINE GAUSS(N,M,A,DELT)

C       ***********************************************************
C       * FUNCTION: THIS SUBROUTINE COMPUTES THE SOLUTIONS FOR M
C       *           SYSTEMS WITH N EQUATIONS AND N UNKNOWNS USING
C       *           GAUSSIAN ELIMINATION
C       * USAGE:
C       *       CALL SEQUENCE: CALL GUASS(N,M,A,DELT)
C       * PARAMETERS:
C       *      INPUT:  N=NUMBER OF EQUATIONS AND UNKNOWNS
C       *              M=NUMBER OF SYSTEMS(RIGHT HAND SIDE VECTORS)
C       *              A=N BY M+N ARRAY OF COEFFICIENTS AUGMENTED
C       *                WITH EACH RIGHT SIDE VECTOR
C       *              DELT=MACHINE ZERO(TOLERANCE)
C       *      OUTPUT:
C       *              A(1,N+J),...,A(N,N+J)
C       *              =SOLUTION OF THE J-TH SYSTEM(J=1,...,M)
C       ***********************************************************

double precision A,DELT
DIMENSION A(N,M+N)
IF(N.GT.1) THEN
   DO 1 K=1,N-1
      U=ABS(A(K,K))
      MM=K+1
      IN=K
      DO 2 I=MM,N
         IF(ABS(A(I,K)).GT.U) THEN
            U=ABS(A(I,K))
            IN=I
         END IF
      2 CONTINUE
      IF(K.NE.IN) THEN
         DO 3 J=K,M+N
            X=A(K,J)
            A(K,J)=A(IN,J)
            A(IN,J)=X
         3 CONTINUE
      END IF
   1 CONTINUE
   DO 2 I=MM,N
      IF(ABS(A(I,K)).GT.U) THEN
         U=ABS(A(I,K))
         IN=I
      END IF
   2 CONTINUE
   IF(K.NE.IN) THEN
      DO 3 J=K,M+N
         X=A(K,J)
         A(K,J)=A(IN,J)
         A(IN,J)=X
      3 CONTINUE
   END IF
   IF(U.LT.DELT) THEN
      WRITE(6,4)
   END IF
END

GOTO 105
ELSE
   WRITE (8,*) CONVERGED
ENDIF
END
**FORWARD ELIMINATION STEP**

DO 5 I=MM,N
    DO 5 J=MM,M+N
        A(I,J)=A(I,J)-A(I,K)*A(K,J)/A(K,K)
      5 CONTINUE

**BACK SUBSTITUTION**

DO 6 K=1,M
    A(N,K+N)=A(N,K+N)/A(N,N)
    DO 6 IE=1,N-1
        I=N-IE
        IX=I+1
        DO 7 J=IX,N
            A(I,K+N)=A(I,K+N)-A(J,K+N)*A(I,J)
        7 CONTINUE
    A(I,K+N)=A(I,K+N)/A(I,I)
  6 CONTINUE

ELSE IF(ABS(A(1,1)).LT.DELT) THEN
    WRITE(6,4)
    RETURN
END IF

DO 8 J=1,M
    A(1,N+J)=A(1,N+J)/A(1,1)
  8 CONTINUE
RETURN
END