A Thesis

entitled

Reliability Assessment Using Bootstrapping and Identification of Point of Diminishing Returns

by

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Submitted to the Graduate Faculty as partial fulfillment of the requirements for the

Master of Science Degree in

Engineering

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May 2016
The safety of dynamic systems under random vibrations is usually quantified by using probabilistic methods. This is because of the inherent randomness associated with some parameters of a system. Simulation methods such as Monte Carlo Simulation (MCS), Separable Monte Carlo (SMC), and Importance Sampling (IS) are used to estimate the probability of failure of dynamic systems under random vibration. These methods are flawed or normally pose difficulties that prompt analyst to seek out other methods of simulation. In this research, bootstrapping was used to estimate the probability of failure and accuracy of dynamic systems under time-variant random vibration. MCS was used to compare the results obtained by bootstrapping. Increasing the degrees of freedom of a system results in an enormous increase in the total number of time histories of excitation created. This research focused on investigating the effect of increasing the degrees of freedom on the accuracy of the probability of failure of the systems under investigation. It was concluded that the points of diminishing returns of the standard deviation of the probability of failure is independent on the DOF of the systems under consideration.
I dedicate my thesis work to God Almighty. Without God, I will not be here. Also to my wonderful parents, Mr & Mrs. Fidelis Ugwumba, Thank you for the way you raised me and for all your love and support. To my elder sister, Precious Ugwumba, thank you for all you acts of love and kindness. I appreciate you a great deal. My other siblings, Victor Ugwumba, ThankGod Ugwumba, Emmanuel Ugwumba, Goodness Ugwumba, and God’sPower Ugwumba, thank you for all your love, encouragement, and support. To Marvelous Ugwumba, I miss you every day. There is so much I can say, but I will when I see you again.
Acknowledgements

I would like to express my gratitude to Dr. Nikolaidis for being a wonderful person and an advisor. Thank you for being a great mentor and believing in me; while this was a challenging experience, I learned a lot and had some fun. I would also like to thank my committee members, Professor Dr. Berhan, Professor Dr. Elahinia, and Professor Dr. Pourazady for their brilliant comments and suggestions.

My family were a consistent source of support all through my time in college, and words cannot express how grateful I am. Thank you for your unconditional love and patience. I would also like to thank my friends and roommate who helped make my journey enjoyable and memorable. To the Burson family, thank you for all your love and support.
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List of Abbreviations

$FI$ ................................................. Failure Indicator function

$MCS$ ............................................. Monte Carlo Simulation

$BS$ ............................................... Bootstraps

$SMC$ .............................................. Separable Monte Carlo simulation

$PDS$ ............................................ Power Spectral Density

$COV_{BS}$ ........................................ Coefficient of Variation for results obtained by bootstrapping

$COV_{MCS}$ ....................................... Coefficient of Variation for results obtained by $MCS$

$DOF$ ............................................. Degrees of freedom

$ESS$ .............................................. Equivalent Sample Size

$BSS$ ............................................. Bootstrapping Sample Size

$ISD$ .............................................. Importance Sampling Density
List of Symbols

\( \hat{p}_f^{avg}_{BS} \) ................................................. Average probability of failure obtained by bootstrapping

\( S(\omega_i) \) ..................................................... Pierson Moskowitz power spectral density evaluated for frequency \( \omega_i \)

\( A \) ................................................................. Parameter of Pierson Moskowitz power spectral density

\( B \) ................................................................. Parameter of Pierson Moskowitz power spectral density

\( H(\omega_i) \) ........................................................ Frequency response function evaluated for frequency \( \omega_i \)

\( |H(\omega_i)| \) .................................................... Magnitude of the frequency response function

\( K \) ................................................................. Stiffness matrix

\( M \) ................................................................. Mass matrix

\( \hat{p}_f \) ............................................................ Probability of failure estimate

\( l_i \) ................................................................. Failure indicator function

\( k \) ................................................................. Number of replications

\( \hat{p}_F_{i} \) ........................................................... \( i \)th element of the estimated probability of failure vector

\( f(t) \) ............................................................ Excitation time history

\( R \) ................................................................. Response

\( C \) ................................................................. Capacity or Damping matrix

\( G \) ................................................................. Limit state function

\( N \) ................................................................. Database sample size

\( D \) ................................................................. Dynamic stiffness matrix

\( \hat{p}_f^{MCS} \) ....................................................... Estimated probability of failure obtained using MCS

\( \hat{p}_f^{BS} \) ........................................................ Estimated probability of failure obtained using bootstrapping

\( \hat{p}_f^{SMC} \) ....................................................... Estimated probability of failure obtained using SMC

\( N_1, N_2, \ldots, N_n \) ........................................ Database size of all the random variables

\( \ddot{x}_1, \ddot{x}_2, \ddot{x}_3 \) ........................................ Acceleration of masses 1, 2, and 3

\( \dot{x}_1, \dot{x}_2, \dot{x}_3 \) ........................................... Velocity of masses 1, 2, and 3

\( x_1, x_2, x_3 \) .................................................. Displacement of masses 1, 2, and 3

\( m_1, m_2, m_3 \) ............................................... Masses 1, 2, and 3

\( k_1, k_2, k_3 \) ................................................... Stiffness

\( c_1, c_2, c_3 \) ................................................... Damping coefficients

\( X_1^i, X_2^i, \ldots, X^n_i \) ..................................... Random variables 1 to \( n \) of the Response of a system

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Random variables 1 to \( n \) of the capacity of a system

Samples of the random variables 1 to \( n \) of the response of a system

Response of the system

Element (mass) of a system

Degrees of freedom of a system

Average number of up-crossings

Coefficient of diffusion

Joint pdf of a process and its velocity

Joint pdf of a particle’s displacement

Force of excitation at masses 1, 2, and 3

Standard Deviation of the probability of failure obtained by Bootstrapping

Standard Deviation of the probability of failure obtained by \( MCS \)

Frequency

Frequency vector

Square root of the area under the PSD

Phase angle

Threshold lever

Phase shift
Chapter 1

Introduction

The reliability assessment of a dynamic system is often carried out analytically or by using simulation. Simulation methods such as standard Monte Carlo simulation (MCS), Separable Monte Carlo (SMC) simulation, Line Sampling (Schueller et al., 2003), the Auxiliary Domain Method (Kataygiantis et al., 2007), and Importance Sampling (IS) are a few of the many simulation methods used in reliability analysis. MCS is the most popular and simplest simulation method available. SMC, which is an extension of MCS, was developed by Haftka et al. (2010). It is a method that can increase the efficiency of standard Monte Carlo simulation by taking advantage of the fact that the performance function can be separable. In this thesis, the performance function is a function that is negative if the system fails and positive if the system survives. The expression for the performance function can be separated into functions than can be calculated independently.

1.1. Problem Definition & Significance

Engineers design, test, and service structures, materials, and processes so as to meet the everyday human needs. Testing is an integral part of the engineering process. Structures are subjected to loading that is time dependent in real life applications. The
reliability of a structure or mechanical system requires the knowledge of the forces that act on the system and the response of the system due to these forces. The response histories under some loading conditions can be calculated by using quasi-static analysis procedure. For others, calculation of response histories requires a complete dynamic analysis procedure. The response of structures whose parameters such as durations, frequencies, and amplitudes, are known or can be determined requires a deterministic dynamic analysis. In practice, we lack information due to the difficulty in determining the aforementioned properties of many dynamic systems. Some of this information can be obtained from past records or by observations of the system response to the applied loads. Examples are the frequency and magnitude of an earthquake and sea waves. Characteristics of such systems are determined using probabilistic methods. Deterministic analysis involves the use of a safety factor. In most cases, this analysis is very inefficient because it results into overdesign of a system and, therefore, leads to waste of resources. Probabilistic analysis produces more efficient designs than deterministic design because probabilistic design quantifies the risk of failure more accurately than deterministic design. On the other hand, probabilistic design is often more expensive than its deterministic counterpart because it requires analyzes of thousands or even millions of random realizations of a design.

1.2. Prior Work & Motivation

MCS is commonly used to estimate the effect of uncertainties that propagate from the input to the output variables. MCS does not allow resampling of data. The advantage of the method is that it produces uncorrelated values of the failure indicator function. In
this thesis, failure indicator function is a function that is equal to 1 if the system fails and 0 if the system survives. The method’s main drawback lies in the fact that it is too expensive computationally for estimating small probabilities, such as $10^{-3}$. The number of sample values required to achieve a given accuracy by MCS is proportional to the inverse of the probability of failure, therefore, a very large sample size is required to increase the accuracy in the estimation of the probability of failure. The simplest measure of accuracy is the standard deviation of the estimated probability of failure. The accuracy is improved by quadrupling the number of simulations/replications. The number of replications $N$ needed to estimate a probability of failure $p_f$ so that the standard deviation of the failure probability is $\sigma_{p_f}$ is $N = \frac{pf(1-p_f)}{\sigma_{p_f}^2}$. Therefore, calculating a probability of $10^{-3}$ with standard deviation $10^{-4}$ requires $10^5$ replications. Therefore, computation of the reliability of a system under random vibration is very expensive by MCS.

Naess (1990), Engelund et al. 1995, Shinozuka (1964), Rice (1944), Veneziano et. al. (1977), and Schall et al. (1991) are among the many researchers who have introduced analytical techniques for estimating the reliability of a dynamic system under random vibration. The merits of their method lie in the fact that these researchers derived explicit, closed form expressions for the probability of failure. On the other hand, these methods are less effective than simulation based methods because they rely on assumptions and approximations.

Majcher et al. (2015) used the total probability theorem to calculate the failure probability of a linear system with random parameters excited by stationary Gaussian processes. A space-filling design such as optimal symmetric Latin hypercube sampling
was first used to sample the input parameters space. A Kriging metamodel was created between input parameters and the output conditional probabilities, which enabled Majcher et al. (2015) to estimate the conditional probabilities of any set of input parameters. The total probability theorem was then applied to calculate the probability of failure of the system.

Jehan and Nikolaidis (2014) extended SMC method to evaluate the probability of failure of a dynamic system with multiple variables. The closed form equation

\[ \sigma_{pf} = \frac{1}{N^2} \sqrt{N^2 \sigma_I^2 + \frac{N^2(N-1)}{2} \sigma_1^2 \rho_1 + \frac{N^2(N-1)}{2} \sigma_2^2 \rho_2} \]

was developed to estimate the standard deviation of the probability of failure. \( \rho_1 \) and \( \rho_2 \) represent the correlation coefficient of values of the failure indicator function with the first and second random variable respectively, and \( N \) is the number of replications. An important advantage of the method lies in the fact that data can be resampled. However, resampling data introduces correlation between failure indicator functions determined from different time histories. The equation developed involves estimation of the correlation coefficient of pairs of failure indicator function values. The key idea of the proposed method is to calculate correlation coefficients between a pair of failure indicator function values separately for each number of common elements that may be shared by each time history. The method was used to estimate the reliability of a quarter car model and a wind turbine. A disadvantage of the method lies in the difficulty in estimating small correlation coefficients (for example \( 10^{-3} \)).

To compute small failure probabilities encountered in reliability analysis of engineering systems, Sui-Kui Au, and James L Beck (2003) proposed a new simulation
method called subset simulation. To demonstrate the efficiency of their method, the first excursion probability of a linear oscillator subjected to white noise excitation and a five-story nonlinear hysteretic shear building under uncertain seismic excitation were calculated. The key idea was to express the failure probability as a product of larger conditional failure probabilities by introducing intermediate failure events. With this method, the original problem of calculating a small failure probability is reduced to the computation of a sequence of conditional probabilities using simulation. The conditional probabilities are estimated efficiently by a Markov Chain Monte Carlo simulation technique based on the Metropolis algorithm [Metropolis et al., 1953]. One challenge with this method is making a proper choice of the conditional events.

L. Ren (2013) suggested a procedure for estimating the first excursion probability of failure of a nonlinear dynamic system subjected to Gaussian excitation. Their approach is based on the mean up-crossing rate and importance sampling method. Firstly, they found an equivalent linear system by using Poisson assumption and Rice formula. Linearizing the system is based on the principle that both systems have the same up-crossing rate for a specified threshold. Finally, they used importance sampling technique to estimate the failure probability of the linear system by first excursion. They applied their method to a duffing system and obtained accuracy better than that obtained from MCS.

C. Andrieu-Renaud (2004) in order to estimate the reliability of time variant systems, presented a method called PHI2, which is based on the outcrossing approach explained by Sudret et al. (2003). The method employs classical time-invariant reliability tools such as FORM and SORM. With this method, the probability of failure is related to
the mean number of outcrossing of the random process through the limit state surface. The computation of the outcrossing rate involves two successive analyzes. The first method uses simulation /FORM and the second evaluates the cumulative binomial distribution as firstly presented by Hagen and Tvedt (1991).

Zio & Pedroni (2009) used the line sampling and subset simulation method to evaluate the probability of failure of a system. The underlying idea of the line sampling method is to employ lines instead of random points to probe the failure domain of high dimensional systems under analysis [Pradlwarter et al. (2005)]. Subset simulation is based on the fact that a small probability of failure can be expressed as a product of large conditional probabilities of some intermediate events. Their method was used to calculate the reliability of a cracked plate. They concluded that their method improved the efficiency of MCS in the estimation of the probability of failure.

Veit Bayer and Christian Bucher (1999), in order to reduce sample size, proposed the Importance sampling concept for reliability assessment of mechanical structures under random excitation. Importance sampling as a simulation method is a variance reduction technique widely used in reliability analysis. With this method, samples are generated from a simulation/sampling density instead of an original/true density. The sampling density is chosen so that a higher probability of failure is obtained compared to that obtained from the true density. The probability of failure estimated from the simulation density is biased, and a weighting function is introduced to correct for this bias. The weighting function is the ratio of the likelihood of the true density to the likelihood of the simulation density. The draw-back of Importance sampling is the difficulty in choosing an accurate simulation density. Choosing an Importance sampling
density (ISD) requires adequate knowledge of the system in the failure region. For a complex system, it becomes however, difficult to gain sufficient knowledge to construct a good ISD. Norouzi and Nikolaidis (2013) provided some guidelines for choosing a good ISD. The method does not work for a system with many random variables. This is because when the likelihood ratio is calculated, too many random variables lead to increased variability.

1.3. Objectives & Scope

The main objective of this research is to estimate the standard deviation of the probability of failure of a dynamic system under time-variant loading conditions by bootstrapping. A second objective is to determine the point of diminishing returns, which is the number of bootstraps beyond which the accuracy of the estimated failure probability does not increase appreciably with the bootstraps. In order to assess the effectiveness of the method, MCS is also employed to estimate probabilities of failure. The method involves the use of a closed-form equation for calculating the standard deviation of the probability of failure. To further demonstrate the merits of bootstrapping, Equivalent Sample Size (ESS) is calculated for different threshold levels using the closed form equation used for calculating the standard deviation of the probability of failure by MCS. ESS is defined as the number of sample values needed to perform MCS to obtain the same probability of failure and standard deviation of the probability of failure as that obtained using bootstrapping.
The third objective of this research is to investigate the change in the accuracy of bootstrapping with the number of degrees of freedom of a system. The system is set up so that, an additional mass accounts for an additional force to the system. A force time history is a superposition of 10 sinusoids. Each sinusoid is obtained by using a frequency value \( \omega \), which is drawn from a power spectral density (PSD). Each mass of the system requires its PSD from which frequency values can be drawn. In the simulations, sample values of the frequency \( \omega \) from different PSDs are kept separate. For a one degree of freedom system, a total number of \( \binom{N}{n} \) time histories of excitation can be generated using bootstrapping. \( N \) represents the frequency database size and \( n \) is the number of \( \omega \) values selected at random. The expression \( \binom{N}{n} \) stands for the number of distinct n-tuples than can be drawn from a pool of \( N \) objects. For a g-degree of freedom system under g simultaneous excitations, a total of \( \binom{N_1}{n_1} \binom{N_2}{n_2} \cdots \binom{N_g}{n_g} \) time histories can be generated. This is an enormous number of time histories that becomes available for use. This research focuses on investigating the advantage gained from the increase in the number of time histories.

1.4. Overview of Approach

The use of SMC in determining the reliability of dynamic systems requires reusing sample data and, therefore, the calculation of correlation coefficients. This calculation becomes difficult when small correlation coefficients, such as \( 10^{-3} \), are
involved. Because of this difficulty, the use of SMC for reliability analysis of dynamic systems becomes impractical. This difficulty encourages the use of simulation methods that does not require calculation of correlation. MCS does not pose the same difficulty as SMC, and it is a simpler method. This is because data is not reused in MCS. Therefore, failure indicator values are uncorrelated. On the other hand, MCS is flawed in the sense that it only allows for limited use of the data sample. Therefore, to improve the accuracy of the probability of failure, more data samples are needed making it computationally expensive.

The approach in this research employs bootstrapping (Efron 1979), which is an extension of SMC to estimate the reliability of a dynamic system. Both methods are based on the same idea of reusing of samples. Although, bootstrapping has an advantage over SMC and MCS in the sense that the samples of the harmonic response of a system are reused multiple times. Samples are drawn to create a time history of excitation and then the response of the system to the excitation is calculated. Failure analysis is performed, and a failure indicator value is obtained. Sampling and failure analysis is repeated for a set bootstrap value. The average value of the failure indicator function vector is calculated to obtain an estimate of the probability of failure. The analysis is replicated \( k \) times to obtain \( k \) probabilities of failure. The standard deviation of the probability of failure is estimated by calculating the standard deviation of the failure probability vector. The method uses data efficiently until a point of diminishing returns is reached. The point of diminishing returns occurs when the improvement in the accuracy of the probability of failure becomes negligible with the number of bootstraps.
1.5. Outline of the Thesis

This thesis is divided into five chapters, which are summarized below.

Chapter 1 introduces few of the methods of simulation available for estimating the probability of failure and the standard deviation of the probability of failure of dynamic systems. The problem investigated in this research is defined and its significance is explained. Prior work by other researchers is reviewed. The scope and objective of the research are explained. Finally, the chapter presents an overview of the method of approach.

Chapter 2 presents a literature survey of the estimation of the reliability analysis of dynamic systems. Uncertainty and variability are defined, and their sources are discussed. The concept and meaning of limit state function are reviewed. The methods of standard MCS and bootstrapping are explained. Bootstrapping and SMC methods are compared. Finally, the process of probabilistic analysis of a dynamic system is presented. A mathematical model involving the differential equation that governs the response of the system to given excitations is derived.

Chapter 3 presents the simulation steps and equations used for obtaining the standard deviation of the probability of failure of the dynamic systems under investigation. The matrix of efficiency used for comparing bootstrapping and MCS is also explained.

Chapter 4 presents the application of MCS and bootstrapping to a series of models of a building with different degrees of freedom. The results from the analysis are
presented and the accuracy and cost (the number of samples required to achieve same accuracy for both methods) of MCS and bootstrapping are made.

Chapter 5 proposes areas for future research and concludes.
Chapter 2

Assessment of Safety of Systems in Random Vibration

This chapter presents the concept of uncertainty and variability in random vibration and their sources. Limit state function is defined and explained. The simulation methods, steps of analysis, and algorithms used in this thesis are outlined in this chapter. The challenges of MCS and bootstrapping are also reviewed. The steps of the probabilistic analysis of a dynamic system are discussed and finally, the chapter presents historical developments in random vibration.

2.1 Introduction

Structural engineering involves the design, testing, and development of a system/structure. A structure is built to fulfill a well specified function under prescribed conditions of utilization. This process consists of some analysis for verification on a virtual product. As the parameters of the system are unknown in the initial design phase, a range of non-deterministic properties of the system has to be taken into account. The purpose of these analyzes is to determine the effects of loads and environmental factors on the structure and to improve the design for safety and durability. Under a loading condition, the response of a structure is calculated and compared to a set failure criteria.
To quantify the reliability of a dynamic system, a probabilistic approach has to be employed to account for the inherent randomness in the load and material properties of the system.

2.1.1 Uncertainties and Variabilities in Structural Analysis

Dirk Vandepitte & David Moens (2011) in their attempt to distinguish between variation, uncertainty, and error, defined variability as the variation that is inherent to the modeled physical system or the environment under consideration. Variability is described by a distributed quantity defined over a range of possible values. Variability can be referred to as irreducible uncertainty, referring to the fact that even when all information on the particular property is available, the quantity cannot be determined. They also defined uncertainty as a potential deficiency in any phase or activity of the modeling process that is due to lack of knowledge. Uncertainty is caused by incomplete information resulting from either vagueness or incongruity.

Much of the work involved with the engineering function deals with mathematical models of the engineer’s design and the physical phenomena encountered by them. These models aid the engineer in predicting the behavior of the product and, therefore, making the design efficient. The use of these models requires the engineer’s knowledge of the constants, parameters, and functional variables that enter the model. It is important to be aware of the uncertainties that is associated with these values. Therefore, the engineer’s design is associated with uncertainties and variabilities. Some uncertainties and variabilities may be ignored depending on their magnitude and likely consequences. If
only a small variation is expected around a large central value or if the effect on the
performance of a system is small, uncertainties and variations can be ignored. If this is
not the case, they are accounted for in design. The use of safety factors is one method for
dealing with significant uncertainties. With this method, the worst case scenario value of
the performance function is multiplied by a safety factor to arrive at a deterministic
design value. This process is typically very costly and results in the overdesign of a
structure or mechanical system. Simplicity is the only advantage of the safety factor
based design over its probabilistic counterpart.

The probabilistic design method yields safer and more economical designs than
deterministic design. However, probabilistic design requires a lot of information about
the system or structure, and usually involves extensive simulations.

2.1.2. Limit State Function

A system’s safety is modeled by the limit state function. Limit state function
defines the state of a structure; that is it determines if the structural capacity can no longer
fulfill the design requirements for first excursion failure or fatigue damage [Jehan &
Nikolaidis, 2014]. The limit state function is generally represented with letter $G$. It
assumes a positive value when the system is safe and a negative value when the system
fails.

The limit state function is a function of demand, $R$, and capacity, $C$, of a system.
For a system under random vibration, its response can be defined as load effects. The
applied loads can be due to the wind or a direct force applied to the system. The demand
R may be dependent on a set of random variables \(X_i^1, X_i^2, \ldots, X_i^n\). The capacity of the system can be defined as the amount of load a system can withstand before failure. In this research, the capacity of the system under investigation is defined as a threshold level of displacement. The Capacity \(C\) may also depend on other sets of random variables \(Y_j^1, Y_j^2, \ldots, Y_j^n\). The limit state function is expressed as

\[
G(R, C) = G[R(X_i^1, X_i^2, \ldots, X_i^n), C(Y_j^1, Y_j^2, \ldots, Y_j^n)]
\]  

(2.1)

The system fails if \(G < 0\), and survives if \(G \geq 0\).

The failure indicator function \((FI)\) is a vector of failure indicator values. A failure indicator value depends on the value of the limit state function. In the event of a system’s failure, \(G < 0\), the failure indicator function, \(I\), assumes the value of 0. Otherwise, the failure indicator function assumes the value of 1.

\[
G < 0, \quad I[G] = 1
\]

\[
G \geq 0, \quad I[G] = 0
\]

2.2. Simulation Methods

As mentioned in chapter 1, there are many simulation methods used for estimating the reliability of a dynamic system under random vibration. Such methods include MSC, SMC, importance sampling and bootstrapping. These methods have some similarities and in some cases are extensions of another method. It is also important to state that there are significant differences between the methods. The algorithms of these
methods will be presented in this section to point out their differences. Also, the challenges of these methods will be discussed.

2.2.1. MCS: An Overview

Monte Carlo simulation is the most popular simulation method used to estimate the reliability of a system. An advantage of MCS is its simplicity. However, MCS does not allow resampling of the values of a random variable. Each sample from a random variable $X^i$ is used just once in the course of the simulation. Therefore, in order to estimate accurately the reliability of a system, a lot of sample value calculations are required. The required number of new samples depends on the probability of failure, level of uncertainty and variation associated with the system in question.

Jehan & Nikolaidis (2014) described the sampling procedure for MCS with $n$ random variables with the aid of Table 2.1.

<table>
<thead>
<tr>
<th>Replications</th>
<th>$X_1^1$</th>
<th>$X_1^2$</th>
<th>$X_1^n$</th>
<th>$I_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$X_1^1$</td>
<td>$X_1^2$</td>
<td>$X_1^n$</td>
<td>$I_1$</td>
</tr>
<tr>
<td>2</td>
<td>$X_2^1$</td>
<td>$X_2^2$</td>
<td>$X_2^n$</td>
<td>$I_2$</td>
</tr>
<tr>
<td>...</td>
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<td>...</td>
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<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$N$</td>
<td>$X_N^1$</td>
<td>$X_N^2$</td>
<td>$X_N^n$</td>
<td>$I_N$</td>
</tr>
</tbody>
</table>
Table 2.1 illustrates a general reliability assessment procedure with \( n \) random variables. \( X_1^1, X_1^2, \ldots, X_1^n \) are random variables and \( I_i \) is the corresponding failure indicator function. From the table, sample values \( X_1^1, X_1^2, \ldots, X_1^n \) are used once in the limit state function to obtain a failure indicator value \( I_1 \). The same is true for \( X_2^1, X_2^2, \ldots, X_2^n \) and so forth.

2.2.1.1. MCS Algorithm

For \( i = 1:1:N \):
- Sample a value from each random variable without replacement
- Use the sampled values in the limit state function
- Check for failure. The system fails if \( G < 0 \) and the system survives if \( G \geq 0 \)
- Record \( I[G] = 1 \) if the system fails and \( I[G] = 0 \) if the system survives
- Check if \( i = N \)
  - No
  - Yes

Estimate the fail probability and the standard deviation of the probability of failure of the system
The method involves the following steps

a. Generate a database of sample size of \( N \) for each of the random variables \( X^1_i \), \( X^2_i \), ..., \( X^n_i \). \( N \) represents the number of replications.

b. Select a value from each of the random variables without replacement.

c. Calculate the value of the limit state function \( G \).

d. Check for failure. The system fails if \( G < 0 \) and it survives if \( G \geq 0 \).

e. Update the failure indicator function with \( I[G] = 1 \) if the system fails and \( I[G] = 0 \) if the system survives.

f. Repeat steps b to e until the number of set replications is reached.

g. Obtain an estimate of the probability of failure \( \overline{p}_{MCS} \) and the standard deviation \( \sigma_{\overline{p}_{MCS}} \) of the system using equations 2.2 and 2.3 respectively.

\[
\overline{p}_{MCS} = \frac{1}{N} \sum_{i=1}^{N} I_i
\]  

(2.2)

\[
\sigma_{\overline{p}_{MCS}} = \sqrt{\frac{\overline{p}_{MCS}(1-\overline{p}_{MCS})}{N}}
\]  

(2.3)

2.2.1.2. Challenges of MCS

As stated earlier, MCS is the simplest simulation method available. However, the method is flawed in the sense that samples are not reused. This limits the accuracy of the estimated reliability of the system. The accuracy of the method can only be improved by increasing the number of samples of each of the random variable. The calculations of
these sample values are expensive, which makes the method computationally expensive when highly accurate estimates are required.

### 2.2.2. Bootstrapping: An Overview

A reliability assessment problem can be classified into four categories: Problems involving 1) systems whose both properties and applied loads are time invariant, 2) time invariant systems subjected to time variant loads, 3) time variant systems subjected to time-invariant loads, and 4) time variant systems under time-variant loads. In this thesis bootstrapping is used for estimating the reliability of time-invariant randomly vibrating systems. The method involves sampling with replacement values of all the random variables associated with the system. With bootstrapping, a closed form equation for estimating the standard deviation of the probability of failure of the system is not available. Resampling of sample values introduces correlation of the failure indicator values. Even though the failure indicator values are correlated, correlation coefficients are not calculated. The method becomes invalid when sample values are sampled multiple times. In this analysis, a new set of phase angles uniformly distributed from 0 to $2\pi$ were used to reduce the effect of correlation.

In bootstrapping, the probability of failure $\hat{p}_{f BS}$ is estimated numerous times using the values of the of the failure indicator function ($FI$). $\hat{p}_{f BS}$ is obtained by calculating the average value of $FI$. The standard deviation of the probability of failure $\sigma_{\hat{p}_{f BS}}$ is obtained by estimating the standard deviation of the $\hat{p}_{f BS}$ vector.
Jehan & Nikolaidis (2014) described the sampling procedure for bootstrapping with \( n \) random variables with the aid of the table 2-2. From the table, sample values from the random variables \( X_i^1, X_i^2, \ldots, X_i^n \) are sampled at random and are reused. The samples values are used in the limit state function to obtain a failure indicator values \( I_1 \) to \( I_k \). Symbol \( k \) represents the number of replications. Reusing sample values enables the analyst to generate an enormous number of distinct values of the failure indicator function.

Table 2.2 Sampling procedure for bootstrapping with \( n \) random variables

<table>
<thead>
<tr>
<th>Sample</th>
<th>( X^1 )</th>
<th>( X^2 )</th>
<th>( \ldots )</th>
<th>( X^n )</th>
<th>( I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( X_1^1 )</td>
<td>( X_1^2 )</td>
<td>( \ldots )</td>
<td>( X_1^n )</td>
<td>( I_1 )</td>
</tr>
<tr>
<td>2</td>
<td>( X_2^1 )</td>
<td>( X_2^2 )</td>
<td>( \ldots )</td>
<td>( X_2^n )</td>
<td>( I_2 )</td>
</tr>
<tr>
<td>\ldots</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
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<tr>
<td>\ldots</td>
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<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>\ldots</td>
<td>( X_{N_1}^1 )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>\ldots</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
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<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>\ldots</td>
<td>( X_{N_2}^2 )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>\ldots</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( N_1 \times N_2 \times \ldots \times N_n )</td>
<td>( \ldots )</td>
<td>( X_{N_n}^n )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( I_{N_1 \times N_2 \times \ldots \times N_n} )</td>
</tr>
</tbody>
</table>

Table 2.2 illustrates a general reliability assessment procedure with \( n \) random variables.

\( X_i^1, X_i^2, \ldots, X_i^n \) for bootstrapping.
2.2.2.1. Bootstrapping Algorithm

For $i = 1:1:k$

For $j = 1:1:BS$

Sample a value from each random variable with replacement

Use the sampled values in the limit state function

Check for failure. The system fails if $G < 0$ and the system survives if $G \geq 0$

Record $I[G] = 1$ if the system fails and $I[G] = 0$ if the system survives

Check if $j = BS$

Estimate the fail probability of the system

Check if $i = k$

Estimate the failure probability and the standard deviation of the probability of failure of the system
The method involves the following steps.

a. Generate a database of sample size $N$ for each of the random variables $X_1^1, X_1^2, \ldots, X_1^n$.

b. Select at random a value from each database with replacement.

c. Calculate the value of the limit state function $G$.

d. Check for failure. The system fails if $G < 0$, it survives if $G \geq 0$.

e. Update the failure indicator function with $I(G) = 1$ if the system fails and $I(G) = 0$ if the system survives.

f. Check the bootstrap count. If the count equals the set bootstrap value, estimate of the probability of failure $\hat{p}_{f_{BS}}$.

g. Repeat steps b – f for a set number of replication (for example 1000 replications). A thousand estimates of $\hat{p}_{f_{BS}}$ are obtained.

h. Check the replication count. If the count equals the set replication value, estimate the standard deviation of the probability of failure $\sigma_{\hat{p}_{f_{BS}}}$ by estimating the standard deviation of the $\hat{p}_{f_{BS}}$ vector.

2.2.2.2. Challenges of Bootstrapping

Bootstrapping is an efficient method of simulation for estimating the reliability of a dynamic system under random vibration. The method reuses sample values. The reuse of sample values results to having correlated failure indicator values. The method becomes invalid when failure indicator function values are correlated.
2.2.2.3. Differences Between Bootstrapping and SMC

SMC as a method of simulation is an extension of MCS. The major difference between these methods is that samples are reused in SMC. For a dynamic system with \( n \) random variables, each of the sample values of one of the random variables is sampled with every sample value from the other \( n-1 \) random variables. Table 2-3 describes the sampling procedure of SMC [Jehan & Nikolaidis, 2014]. If \( N_1, N_2, \ldots, N_n \) represent the size of each of the random variables \( X_1^i, X_2^i, \ldots, X_n^i \), a total of \( N_1^* N_2^* \ldots, N_n^* \) sampling is possible. Sample values from the random variables are always reused in SMC. If \( N \) is the size of each of the random variable, a sample value is resampled \( N(n-1) \) times. The failure indicator values obtained in SMC are all correlated. This prompts the calculation of correlation coefficients. With bootstrapping, samples are drawn at random and with replacement. Therefore, it is not known what sample value from a particular random variable is resampled and how many times. An enormous number of the limit state values are possible. Correlation of the failure indicator function obtained with bootstrapping is ignored.
Table 2.3 illustrates a general reliability assessment procedure with \( n \) random variables.

\[ X_i^1, X_i^2, \ldots, X_i^n \] for SMC.
2.2.2.4. SMC Algorithm

\[ \text{For } i = 1:1:N1 \]

- Sample a value from each random variable with replacement

\[ \text{For } j = 1:1:Nn \]

- Sample a value from each random variable with replacement

- Use the sampled values in the limit state function

- Check for failure. The system fails if \( G < 0, I[G] = 1 \) and the system survives if \( G \geq 0, I[G] = 0 \)

- Check if \( j = Nn \)

  - No
  - Yes

- Estimate the fail probability of the system

  - No
  - Check if \( i = N1 \)

    - Yes

- Estimate the failure probability and the standard deviation of the probability of failure of the system

\( N1, \ldots, Nn \) represents the database size of the random variables.

The method involves the following steps
a. Generate a database of sample size $N$ for each of the random variables $X_1^n, X_2^n, \ldots, X_i^n$.

b. Select at random a value from the first database without replacement.

c. Calculate the value of the limit state function $G$.

d. Check for failure. The system fails if $G < 0$, it survives if $G \geq 0$.

e. Update the failure indicator function with $I(G) = 1$ if the system fails and $I(G) = 0$ if the system survives.

f. Using the same sampled value with all the sample values from the other random variables, calculate limit state function values. A total of $N^{n-1}$ $FI$ values are obtained.

g. Repeat steps b – f for the rest of the sample values from the first random variable.

h. In the end, a total of $N^n$ $FI$ values are obtained.

i. Estimate of the probability of failure $\hat{p}_{SMC}$ of the system using equation (2.4) or (2.5).

\[ \hat{p}_{SMC} = \frac{1}{N^n} \sum_{i=1}^{N^n} I_i \] (2.4)

\[ \hat{p}_{SMC} = \frac{1}{N_1+N_2+\ldots+N_n} \sum_{i=1}^{N_1+N_2+\ldots+N_n} I_i \] (2.5)

Equation (2.4) is used for the case when all the random variables are of the same sample size.

Equation (2.5) is used for the case when the random variables are of the different sample size.
j. Calculate the standard deviation of the probability of failure $\sigma_{\bar{f}_{SMC}}$ using equation (2.6) derived by Jehan & Nikolaidis (2014).

$$
\sigma_{\bar{f}_{SMC}} = \frac{1}{N^2} \sqrt{N^2 \sigma_I^2 + \frac{N^2(N-1)}{2} \sigma_I^2 \rho_1 + \frac{N^2(N-1)}{2} \sigma_I^2 \rho_2}
$$

(2.6)

2.3. Probabilistic Analysis of Dynamic Systems Under Random Vibrations

The probabilistic analysis of a dynamic system under random vibration loading condition involves the following steps:

- Construct the probabilistic models of the excitation.
- Calculate the system’s response due to the excitation/applied load
- Perform failure analysis

2.3.1. Modeling of Excitation

It is important to model correctly the excitation of a dynamic system under random vibration. If the excitation is not modeled correctly, the analysis of the system becomes erroneous. For simplicity, the forces applied on most systems are modeled as time invariant (constant) or time variant, periodic. When the excitation is modeled to be time dependent, it is usually thought to be periodic. This assumption simplifies the analysis of the system. In reality, the load applied on a dynamic system is not periodic. For example, consider the load applied on a car due to the bumps on the road. The bumps have different depths and occur at very random intervals. Another example is the force on
the wings of an airplane due to the wind. This cannot be modeled as a constant amplitude force because wind speed differs during taxing, flight and landing of an airplane.

In order to model correctly the excitation on a dynamic system, sine waves are superimposed, and each sine wave is created with its own frequency ($\omega_i$) and amplitude.

Figure (2.1) depicts the time history of excitation obtained from superimposing $n$ sinusoids created with $n$ frequencies($\omega_i$).

Figure 2-1 Time history of excitation created by Fourier Transform

The excitation of a random variable is modeled as a signal that is represented by a power spectral density (PSD), which describes how the power of a signal or time series is distributed over the different frequencies. The two parameter Pearson- Moskowitz Spectrum [Pearson-Moskowitz, 1964] is a widely used power spectral density to describe the energy distribution in a dynamic system. The PSD $S(\omega)$ of Pearson- Moskowitz is defined in equation (2.7).

$$S(\omega) = \frac{A}{\omega^5} e^{\left(-\frac{B}{\omega^4}\right)}$$  \hspace{1cm} (2.7)

In the equation above, $A$ and $B$ are parameters estimated from measured time histories of the random process. The time histories of excitation can be obtained using the
Shinozuka’s method [Shinozuka et al. 1972]. The method is based on a superposition of harmonic function shown below:

\[ f(t) = \sum_{i=0}^{n} \sqrt{\frac{2}{n}} \sigma \cos(\omega_i t - \varphi_i) \]  

(2.8)

where \( i = 1,2,\ldots, n \). \( n \) is the number of harmonics superimposed.

\( \sigma \) represent the square root of the energy of the PSD. The energy is defined as the area under the power spectral density.

\( \omega_i \) = random frequencies extracted from the power spectral density.

\( \varphi_i \) = the phase angles of the harmonics. The phase angle follows a uniform distribution in the range \([0,2\pi]\).

Equations (2.9) are an example on how the Shinozuka equation is superimposed using \( n \) frequency values. Each equation \( f^k(t) \) represents a time history of excitation. \( k \) represents the number of time histories created. Samples of time histories of excitation are shown in the figure (2.3) [Jehan & Nikolaidis, 2014].

\[ f^1(t) = \sqrt{\frac{2}{n}} \sigma \cos(\omega_1^1 t - \varphi_1^1) + \sqrt{\frac{2}{n}} \sigma \cos(\omega_2^1 t - \varphi_2^1) + \ldots, + \sqrt{\frac{2}{n}} \sigma \cos(\omega_n^1 t - \varphi_n^1) \]

\[ f^2(t) = \sqrt{\frac{2}{n}} \sigma \cos(\omega_1^2 t - \varphi_1^2) + \sqrt{\frac{2}{n}} \sigma \cos(\omega_2^2 t - \varphi_2^2) + \ldots, + \sqrt{\frac{2}{n}} \sigma \cos(\omega_n^2 t - \varphi_n^2) \]

\vdots

\[ f^k(t) = \sqrt{\frac{2}{n}} \sigma \cos(\omega_1^k t - \varphi_1^k) + \sqrt{\frac{2}{n}} \sigma \cos(\omega_2^k t - \varphi_2^k) + \ldots, + \sqrt{\frac{2}{n}} \sigma \cos(\omega_n^k t - \varphi_n^k) \]

(2.9)
2.3.2. Calculation of System’s Response

The system responds to the applied excitation. A good knowledge and understanding of the system’s behavior to the excitation is important in order to model its response. The calculation of the response of the system involves creating the mathematical models that correctly fits the system’s behavior under random excitation. In this process, it is assumed that the system is linear. The mathematical model created involves a number of differential equations. The number of differential equations depends on the number of degrees of freedom of the system. The nature of the equation of motion for a general case is given in equation (2.10). An example on how the differential
equations are obtained is discussed next using a 3 DOF system. The system is composed of three masses $m_1, m_2,$ and $m_3,$ stiffness $k_1, k_2,$ and $k_3,$ and damping coefficients $c_1, c_2,$ and $c_3$ all connected in series. Therefore, the equations obtained is for a system in series. Figure 2-3 below shows the 3 DOF system.

$$M\ddot{X} + C\dot{X} + KX = F$$  \hspace{1cm} (2.10)

where $M$ is the mass matrix, $C$ is the damping matrix, $K$ is the stiffness matrix, and $F$ is the excitation/force matrix.
Using Newton's second law,

\[ \sum F = m\ddot{x} \quad (2.11) \]

The following equations are obtained from the free body diagram above.

\[ m_1 \ddot{x}_1 = -c_1 \dot{x}_1 - k_1 x_1 + f_1(t) + c_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) \quad (2.12) \]

\[ m_2 \ddot{x}_2 = f_2(t) - c_2 (\dot{x}_2 - \dot{x}_1) - k_2 (x_2 - x_1) + c_3 (\dot{x}_3 - \dot{x}_2) + k_3 (x_3 - x_2) \quad (2.13) \]

\[ m_2 \ddot{x}_2 = f_3(t) - c_3 (\dot{x}_3 - \dot{x}_2) - k_3 (x_3 - x_2) \quad (2.14) \]

The equations (2.11-13) are manipulated and put in matrix form to obtain equation (2.14)
\[
\begin{bmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3 
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\ddot{x}_3 
\end{bmatrix}
+ \begin{bmatrix}
c_1 + c_2 & -c_2 & 0 \\
-c_2 & c_2 + c_3 & -c_3 \\
0 & -c_3 & c_3 
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 
\end{bmatrix}
+ \begin{bmatrix}
k_1 + k_2 & -k_2 & 0 \\
-k_2 & k_2 + k_3 & -k_3 \\
0 & -k_3 & k_3 
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 
\end{bmatrix}
= \begin{bmatrix}
f_1 \\
f_2 \\
f_3 
\end{bmatrix}
\]

\[ (2.15) \]

\[
M = \begin{bmatrix}
m_1 & 0 & 0 \\
0 & m_2 & 0 \\
0 & 0 & m_3 
\end{bmatrix}, \quad C = \begin{bmatrix}
c_1 + c_2 & -c_2 & 0 \\
-c_2 & c_2 + c_3 & -c_3 \\
0 & -c_3 & c_3 
\end{bmatrix}, \quad K = \begin{bmatrix}
k_1 + k_2 & -k_2 & 0 \\
-k_2 & k_2 + k_3 & -k_3 \\
0 & -k_3 & k_3 
\end{bmatrix},
\]

\[
F = \begin{bmatrix}
f_1 \\
f_2 \\
f_3 
\end{bmatrix}
\]

\[
M \text{ represents the mass matrix of the system, } C \text{ represents the damping matrix of the system, } K \text{ represents the stiffness matrix of the system, } F \text{ represents the force vector of the system. The response } x(t) \text{ of the system is given by equation (2.16)}
\]

\[
X(\omega) = [K - \omega^2 M + j\omega C]^{-1}F(\omega) \quad (2.16)
\]

\[
D = [K - \omega^2 M + j\omega C]
\]

\[
H(\omega) = [K - \omega^2 M + j\omega C]^{-1} \quad (2.17)
\]

\[
D \text{ represents the dynamic stiffness matrix and } H(\omega) \text{ represents the frequency response function of the system. For a system with no damper, } C = 0 \text{ and the frequency response function becomes:}
\]

\[
H(\omega) = [K - \omega^2 M]^{-1} \quad (2.18)
\]

Using response equation obtained above equation (2.17) and the Shinozuka equation for excitation equation (2.8), the response equation becomes:

\[
x(t) = \sqrt{\frac{2}{\pi}} \sigma \sum_{i=0}^{n} |H(\omega_i)| \cos(\omega_i t - \varphi_i + \theta_i) \quad (2.19)
\]

where \(|H(\omega_i)|\) is the magnitude of the frequency function and \(\theta_i\) is the phase shift of the transfer function.
Equation (2.19) is the response of a mass due to the excitation on the same mass. For a 3-DOF system, nine separate responses are calculated. A general equation for the response of a dynamic system is given in Nikolaidis & Norouzi (2015).

\[ x^{(k)}_p(t) = \left[ \sum_{g=1}^L \sum_{n=1}^N \sqrt{\frac{2}{N}} \sigma_g \left| H_{pg}(\omega_{gn}) \right| \cos(\omega_{gn} t - \varphi_{gn} + \angle H_{pg}(\omega_{gn})) \right]^{(k-1)} \quad (2.20) \]

\[ \theta_i = \angle H_{pg}(\omega_{gn}) \]

Equation (2.19) represents the response of the \( p \)th element, \( L \) is the number of degrees of freedom, and \( k \) is the number of replications.

### 2.3.3. Failure Analysis

A system under dynamic load can fail either due to fatigue or first excursion [Nikolaidis 2004]. Fatigue is a phenomenon that results in the sudden fracture of a component after a period of cyclic loading. The process involves the initiation and growth of crack. Failure by first excursion occurs when the maximum response of the dynamic system exceeds an allowable threshold value \( \alpha \). An example of failure by first excursion is the failure of the wing of an aircraft when flying in gusty weather. The wings fail when its maximum displacement exceeds an allowable threshold value.

![First excursion failure](image)
First Excursion Failure

The problem of finding the probability of failure by first excursion during a given period is equivalent to the problem of finding the probability distribution of the maximum of a random process vector that contains infinite random variables. Methods for the estimation of the probability of failure by first excursion are presented by Nigam (1983). The estimate of the probability of failure due to first excursion is the same as that calculated by determining the number of outcrossing during a period. In general, the average up-crossing rate (average number of up-crossings of a level $\alpha$ per unit time) at time $t$ time is given by Rice (1944, 1945) in equation (2.21).

$$v^+(\alpha, t) = \int_0^\infty \dot{x} f_{x(t), x'(t)}(\alpha, \dot{x}, t) d\dot{x}$$ \hspace{1cm} (2.21)

$f_{x(t), x'(t)}(\alpha, \dot{x}, t)$ is the joint pdf of the process and its velocity.

The system under investigation fails when the response, $R$, of the system exceeds the capacity, $C$. The capacity in this research is the set threshold level $\alpha$. The maximum displacement over a period is calculated and then compared to the threshold level $\alpha$. If the limit state function $G$ is less than 0, the system fails. Otherwise, the system survives.

2.2.4. Brief Historical Development of Random Vibration Analysis

In 1905, Albert Einstein wrote the first paper on random. His paper was on the movement of small particles suspended in a stationary liquid. Before that, in 1827, a Scottish botanist named, Robert Brown speculated that particle motions were due to
molecular-kinetic motion after observing the motion of particles of pollen in a fluid suspension. He speculated that motion was caused by unobserved particles impacting the particles of the pollen. He became the pioneer of the Brownian motion. At this time, an equation for the Brownian motion had not been derived. In 1905, Einstein through a non-direct approach developed a mathematical theory to describe the Brownian motion. A direct approach will be to develop equations of motion that govern the motion of the particles in Brownian motion. Instead, he argued that if the molecular-kinematic theory of heat is applicable in describing the motion of the Brownian particle, then pressures on Brownian particles can be established. These pressures would cause the diffusion of small particles in a suspension. He developed a diffusion equation governing the probability density function of a collection of Brownian particles [Thomas Paez L (2012)]. The equation developed by Einstein is for a single, one-dimensional component of motion and is given below.

$$\frac{\partial f}{\partial t} = d \frac{\partial^2 f}{\partial x^2} \quad (2.22)$$

$f(x, t)$ is the PDF of particle displacement, $x$ is particle displacement, $t$ is time and $d$ is the coefficient of diffusion.

The problem Einstein solved was equated to a simple mechanical system with a mass tied to the ground with a damper. His work is thought of as the first solution to a random vibration analysis.

Advancements were made independently by Simoluchowski (1926) and Furth (1917). Instead of a simple system as that used by Einstein, they used a single degree of freedom system that displays oscillating response. The system composes of a mass,
spring, and a damper. Simoluchowski was first to write the Fokker-Planck equation for a SDOF system. Planck (1927) and Fokker (1913) started with the discrete space time framework of the random walk and offered arguments regarding the relative and limiting values of various parameters in the model. They found that a limiting form of the random walk model is the partial differential equation developed by Einstein to characterize the Brownian motion.

Ornstein (1919) overcame this limitation and formed the bases of what is used presently. His method focuses on analyzing the random vibration of a system directly based on the equation of motion that governs the system. The governing equation motion is given by

\[ m\ddot{X} + c\dot{X} + kX = W \]  

(2.23)

Symbol \( m \) is the mass of the system, \( c \) is the damping coefficient of the system, \( k \) is the spring constant, and \( W \) is the excitation. \( X(t) \) and \( W(t) \) are capitalized because they are formally considered to be random processes. The initial conditions on equation (2.23) involves the joint probability distribution of displacement and velocity. In a later paper, Uhlenbeck and Ornstein (1930) described the excitation and response in terms of their averages. The use of averages was an important innovation and it provided a good foundation for most random vibration analysis done today.

S. O. Rice (1944, 1945) introduced ideas on how to develop probabilistic measures of some important characteristics of the response of a system. He did this by developing equations that govern the rate of crossings of a threshold level \( \alpha \) by a random process, the probability distribution of the envelope of a stationary, narrowband Gaussian
random process, and the probability distribution of the maxima of a Gaussian random process.

It is important to state that all of the advancements made in the development of random vibration would not have been possible without the work of Norbert Weiner (1930). He developed the idea of spectral density, although, he attributed the fundamental idea to Schuster (1906). Weiner’s definition of spectral density was based on the idea of autocorrelation function. Autocorrelation function of a random process is the average value over all time of the product of the process at time $t$ times the process at time $t + \tau$ [Thomas Paez L (2012)]. Prior to Weiner, the ideal white noise was a random process that possesses a spectral density that is constant over all frequencies from $-\infty$ to $\infty$. Such a process has spectral density with infinite area and therefore an infinite mean signal. For practical purposes, this does not exist. Consequently, the ideal white noise was used in modeling applications that were simple.
Chapter 3

Method of Approach

Bootstrapping is more efficient than MCS for reliability analysis because it maximizes the potential of the available data. However, the method becomes invalid when failure indicator functions are correlated. If the correlation is neglected, correlated failure indicator function values are generated, which results in an erroneous estimate of the standard deviation of the probability of failure. To avoid correlation of the failure indicator functions, the method by Rubinstein & Kroese (2001) was employed. Their method introduces random phase angles in calculating the excitation and response of the system. MCS method that does not reuse sample values. Therefore, the problem of correlation is not encountered with MCS. The method’s biggest disadvantage is its inefficiency in estimating very small probabilities of failure (for example $10^{-3}$). The author uses MCS for validation: the same analysis was performed using MCS and the results obtained were compared to those obtained from bootstrapping regarding accuracy and computational cost.
3.1. Outline

This chapter presents the simulation methods used for failure analysis of the dynamic system under random vibration in chapter 2. In this research, Monte Carlo simulation and bootstrapping are the methods used to estimate the reliability of the system under investigation. The simulation steps in calculating the probability of failure and the standard deviation of this probability will be described for the two methods. Also, the matrix of efficiency employed in this research is explained in detail. The matrix of efficiency is a tool used to quantify the efficiency of bootstrapping compared to MCS. Given a probability of failure $\hat{p}_{f_{avgBS}}$ and its standard deviation $\sigma_{\hat{p}_{f_{avgBS}}}$ from bootstrapping, equation (2.3) is used to calculate the sample size needed to achieve the same accuracy through MCS.

3.2. Bootstrapping

In reliability analysis of a system under random vibration, a power spectral density function is used to model the energy of the system. The Pierson-Moskowitz (1964) power spectral density is used to model the energy of excitation of the system in this research. The equation for the spectrum is:

$$ S(\omega) = \frac{A}{\omega^5} e^{-\frac{\omega}{\omega_d}} $$  \hspace{1cm} (3.1)

With defined values for $A$ and $B$, frequency values were drawn from the probability density function (PDF) of the power spectral density (PSD). The PDF is the PSD
normalized by the area under the PSD. The frequency database contains \( N \) frequency values

\[
ω_{\text{dat}} = \begin{bmatrix}
ω_1 \\
ω_2 \\
\vdots \\
ω_N
\end{bmatrix}
\]

From the database, ten frequency values are picked at random with replacement and used to create a time history of excitation using the Shinozuka method for superimposing sinusoids:

\[
f(t) = \sum_{i=1}^{n} \sqrt{\frac{Z}{n}} \sigma \cos(ω_i t - ϕ_i)
\]

(3.2)

In the above equation, \( n \) is the number of frequencies \( ω_i \) sampled at random with replacement. Symbol \( σ \) represents the square root of the area under the PSD function and \( ϕ_i \) represents the phase angles of the time history.

Ten frequencies were found to be adequate for representing the power spectral density of the excitation. The phase angles, \( ϕ_i \), follow a uniform distribution ranging from \([0, 2\pi]\).

The phase angles effectively reduce the effect of correlation. In the case where a frequency value is drawn twice, it is unlikely that the sinusoids will both have the same sampled phase angle. Since the phase angle will most likely be different, the time history of the excitation will be different even though the same frequency value was sampled.

Note that the phase angle does not alter the spectrum of the time history obtained because the frequency values were sampled from a PDF that is proportional to the PSD.

After the time history of excitation is calculated, the response of the system is obtained by using the equation that relates the response \( x(t) \) to the excitation \( f(t) \):
\[ x(t) = \frac{\sqrt{2}}{\sqrt{N}} \sigma \sum_{i=1}^{N} |H(\omega_i)| \cos [\omega_i t - \varphi_i] \] (3.3)

\( H(\omega_i) \) is the frequency response function and \( |H(\omega_i)| \) is its magnitude. The frequency response function is obtained using equation (3.4):

\[ H(\omega) = [K - \omega^2 M]^{-1} \] (3.4)

where \( K \) and \( M \) represent the stiffness and mass matrices of the system, respectively.

Three models of buildings with one, two, and eight stories are considered. Damping is assumed negligible. Therefore, the damping matrix \( C \) is neglected when calculating the frequency response function in equation (3.4).

After the time history of the response is calculated, failure analysis is performed to determine if the system fails due to first excursion within a period \( T \). The maximum response of the system during period \( T \) is calculated and compared to a set threshold level \( \alpha \). The system fails if the maximum displacement exceeds \( \alpha \), and it survives otherwise.

This process produces one failure indicator function value. If the system fails, the failure indicator function assumes the value of 1, and the value of 0 if it survives.

The process of calculating excitation, response, and failure analysis are repeated for a set number of bootstraps. The number of failure indicator function values equals to the number of bootstraps. The probability of failure of the system is obtained by estimating the mean of the failure indicator function using equation (3.5)

\[ \bar{p}_f = \frac{1}{BS} \sum_{i=1}^{BS} I_i \] (3.5)

Symbol \( BS \) represents the number of bootstraps.
The steps followed to obtain $\hat{p_f}$ are repeated $k$ times. $k$ represents the number of replications and is equal to 1000. Each of the 1000 $\hat{p_f}$ values is stored in a vector. The average probability of failure of the system is obtained using the equation (3.6)

$$\hat{p_f}_{avg_{BS}} = \frac{1}{k} \sum_{i=1}^{k} \hat{p_F}_i$$

(3.6)

$\hat{p_F}_i$ represents the $i$th entry of vector of failure probability $\hat{p_f}$.

Note that the estimates of the failure probability in vector $\hat{p_f}$ could be correlated because they could share some frequencies. The standard deviation of the probability of failure is estimated using equation (3.7):

$$\sigma_{\hat{p_f}_{BS}} = \sqrt{\frac{\sum_{i=1}^{k} (\hat{p_F}_i - \hat{p_f}_{avg})^2}{k-1}}$$

(3.7)

The behavior of the standard deviation of the probability of failure $\sigma_{\hat{p_f}_{BS}}$ with the number of bootstraps is investigated by repeating the analysis for different numbers of bootstraps.

### 3.3. MCS

In reliability analysis of a system under random vibration, power spectral density function is used to model the distribution of the energy of the system on the range of frequencies. The Pierson-Moskowitz (1964) power spectral density is used to model the energy of the system in this research. The spectrum is shown in (3.1)

Frequency values are drawn from the probability density function (PDF) of the power spectral density (PSD). The PDF is the PSD normalized by the area under the PSD. The frequency database contains $N$ frequency values:
\[ \omega_{dat} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_N \end{bmatrix} \]

From the database, ten frequency values are picked at random without replacement and are used to create a time history of excitation using Shinozuka method (Equation 3.2). In the above equation, \( n \) is the number of frequencies \( \omega_i \) sampled at random without replacement.

The probability of obtaining two exactly identical time histories of excitation is negligible. It is important to state that the phase angle does not alter the spectrum of the time history obtained because the frequency values were sampled from a PDF that is proportional to the PSD.

After the time history of excitation is calculated, the response of the system is obtained by using the equation (3.3).

The system under investigation is a series of buildings and were assumed undamped. After the time history of response is calculated, failure analysis is performed to determine if the system fails or survives. First excursion failure analysis was used. The response function was maximized over a period, and the maximum response of the system was obtained and then compared to a set threshold level \( \alpha \). The system fails if the maximum displacement is greater than \( \alpha \) and it survives otherwise. This process produces one failure indicator function value. If the system fails, the failure indicator function assumes the value 1, and the value 0 otherwise. The processes of calculating the excitation, response, and failure analysis are repeated for all frequencies in the database.

A number of failure indicator values equal to \( k = \frac{N}{10} \) is obtained, where \( N \) is the database size.
size. The probability of failure of the system is obtained by estimating the mean of the failure indicator function using equation (3.8)

\[ \hat{p}_f = \frac{1}{k} \sum_{i=1}^{k} I_i \]  

(3.8)

The standard deviation of the probability of failure is estimated using equation (3.9)

\[ \sigma_{pf} = \sqrt{\frac{\hat{p}_f (1 - \hat{p}_f)}{k}} \]  

(3.9)

3.4. Metrics of Efficiency: Equivalent Sample Size (ESS)

Bootstrapping is a more efficient simulation method than MCS. It is cost-effective because it uses more information in a database with sample values of the random variables. The concept of equivalent sample size was used to quantify the improvement in efficiency from using bootstrapping. ESS was calculated using the results obtained at the point of diminishing returns. The Equivalent Sample Size (ESS) is the number of samples values needed to estimate the probability of failure with the same accuracy (measured by the standard deviation of the probability of failure) as that obtained from bootstrapping.

Equation (3.9) is solved for \( k \) to obtain equation (3.10)

\[ k = \frac{\hat{p}_f (1 - \hat{p}_f)}{\sigma_{pf}^2} \]  

(3.10)

Given \( \sigma_{pf} \) and \( \hat{p}_f \) obtained from bootstrapping, \( k \) can be calculated using equation (3.10), and it represents the number of replications required for estimating the probability of failure with a prescribed standard deviation \( \sigma_{\hat{p}_f_{avgBS}} \). For each replication, 10 frequency values are used to calculate the time history of excitation. Therefore, the total number of
sample values needed to create $k$ failure indicator function values to obtain the same accuracy as bootstrapping by using MCS is given by equation (3.11):

$$ESS = 10 \times k$$

(3.11)
Chapter 4

Demonstration of Reliability Analysis Using Bootstrapping and MCS

This chapter demonstrates and compares bootstrapping and MCS in estimating the probability of failure and the standard deviation of the probability of failure of dynamic systems. The above two methods are applied to estimate the probability of failure and the standard deviation of the probability of failure of a series of models of buildings. Different threshold levels are used in the analysis, therefore, the probabilities of failure considered range in magnitude from $10^{-2}$ to $10^{-5}$. Plots are created to verify the behavior of $\sigma_{\bar{p}_{avgBS}}$ with the number of bootstraps ($BS$). Similarly, plots are created for the probability of failure, $\bar{p}_{avgBS}$. From the plots, points of diminishing returns for each level are identified. Finally, the equivalent sample size, ESS, is calculated to demonstrate the cost effectiveness of bootstrapping.

4.1. Chapter Overview

This section focuses on the reliability of systems under random vibration. A series of multistory buildings are considered. The reliability of a one degree of freedom system (one story building) is assessed. Then the reliability analysis system is repeated for two
and eight story buildings. The systems involve 20, 40, and, 160 random variables respectively. The first excursion probability of failure and its standard deviation are estimated using bootstrapping and MCS.

4.1.1. One Story Building

First the study considers a one degree of freedom system building with a mass of the floor equal to $m_1$. The mass is attached to the ground by a spring with stiffness $k_1$. The study assumes that damping is negligible (figure 4-1). The system involves 20 random variables: 10 random frequencies and 10 random phase angles. It is modeled with the following properties:

Sprung mass, $m_1 = 2000 \text{ kg}$ and stiffness, $k_1 = 35000 \frac{KN}{m}$

Excitation $y_1(t)$

Response $x_1(t)$

Figure 4-1 One degree of freedom system
The building is thought to be stationed in a windy area. The wind is modeled as a random process, and its energy is modeled using the Pierson-Moskowitz power spectral density \( S(\omega) \), which is shown in equation (4.1)

\[
S(\omega) = \frac{A}{\omega^5} e^{-\frac{B}{\omega^4}}
\]  \hspace{1cm} (4.1)

\( A \) and \( B \) are parameters of the PSD; \( A = 17000 \ m^2\ sec^{-4} \) and \( B = 390 \ sec^{-4} \).

Figure 4-2 shows the PSD of the wind. The energy of excitation is concentrated in a relatively narrow range.

![Figure 4-2 PSD of the wind](image)

The probability distribution function (PDF) of the wind is obtained by normalizing the PSD with the area under the PSD as shown below.

\[
f_s(\omega) = \frac{S(\omega)}{\text{Area}}
\]  \hspace{1cm} (4.2)

The area is obtained by integrating the area under the PSD. The PDF is shown in figure 4-3.
4.1.1.1. **Bootstrapping: Steps of Analysis for One Story Building**

To estimate the probability of failure and its standard deviation, numerous time histories of excitation of the building are created, the response of the building to each excitation is calculated, and the failure probability due to first excursion is calculated. The process consists of ten steps:

1. Create a database of 10,000 frequencies by drawing these frequencies from PSD of the wind.
2. Select at random with replacement ten frequency values $\omega_i$ ($i = 1, \ldots, 10$) from database. Draw 10 phase angles $\varphi_i$ ($i = 1, \ldots, 10$) from a uniform PDF from 0 to $2\pi$. 

![Figure 4-3 PDF of the wind](image)
3. Generate a time history of excitation on the mass by using equation (4.3):

\[ y_1(t) = \sqrt{2/N} \sigma_1 \sum_{i=1}^{N} \cos (\omega_i t - \phi_i) \]  

(4.3)

Figure 4-4 shows a sample time history of excitation obtained using equation (4.3).

![Figure 4-4 Time history of excitation](image)

4. Calculate the response time history of mass one due to the force at one by using equation (4.4):

\[ x_1(t) = \sqrt{2/N} \sigma_1 \sum_{i=1}^{N} |H_1(\omega_i)| \cos [\omega_i t - \phi_i + \angle H_1(\omega_i)] \]  

(4.4)

Figure 4-5 shows a sample path of the response time history obtained using equation (4.4)
In the above equations, $\sigma_1$ is the area under the PSD and $H_1$ is the frequency response function:

$$H_1(\omega_i) = [K - \omega^2M]^{-1} \quad (4.5)$$

5. Obtain the maximum displacement of the response time history by maximizing the response function. The response function was maximized using a Matlab built-in function.

6. Compare the maximum displacement to different threshold levels $\alpha$ to check for failure. For this purpose, calculate the failure indicator function, which assumes a value of zero if the system survives and a value of one if it fails.

$$I[G \geq 0] = 0 \text{ system survives}$$

$$I[G < 0] = 1 \text{ system fails}$$

7. Repeat steps 1-7 50 times (initial bootstrap value) and calculate a probability of failure ($\hat{p}_{f_{BS}}$) by taking the mean of the failure indicator function or using equation (4.6)

$$\hat{p}_{f_{BS}} = \frac{1}{50} \sum_{i=1}^{50} I_i \quad (4.6)$$
8. Repeat steps 1-8 a thousand times to obtain a thousand values of \(\hat{p}_{f_{BS}}\) and store them in a vector.

9. Estimate the average probability of failure \(\hat{p}_{f_{avgBS}}\) using equation (4.7).

\[
\hat{p}_{f_{avgBS}} = \frac{1}{1000} \sum_{i=1}^{1000} \hat{p}_{f_{BS_i}}
\]  

(4.7)

10. Estimate the standard deviation of the probability of failure \(\sigma_{\hat{p}_{f_{avgBS}}}\) by using equation (4.8)

\[
\sigma_{\hat{p}_{f_{avgBS}}} = \sqrt{\frac{\sum_{i=1}^{1000} (\hat{p}_{f_{BS_i}} - \hat{p}_{f_{avgBS}})^2}{1000-1}}
\]  

(4.8)

11. Repeat steps 1-10 for different bootstrap values to check for convergence.

4.1.1.1.1. Presentation of Results From Bootstrapping: One Story Building

The average probability of failure and its standard deviation at different bootstrap values obtained from simulation using bootstrapping is shown in Tables 4.1 and 4.2.

<table>
<thead>
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<th>BS</th>
<th>(\hat{p}<em>{f</em>{avgBS}})</th>
<th>(\sigma_{\hat{p}<em>{f</em>{avgBS}}})</th>
<th>(\hat{p}<em>{f</em>{avgBS}})</th>
<th>(\sigma_{\hat{p}<em>{f</em>{avgBS}}})</th>
<th>(\hat{p}<em>{f</em>{avgBS}})</th>
<th>(\sigma_{\hat{p}<em>{f</em>{avgBS}}})</th>
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<td>2.60E-03</td>
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<tr>
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Table 4.1. Average probability of failure and its standard deviation for different bootstrap value and threshold levels $\alpha$

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<th>$\hat{p}_{\text{avg}}$</th>
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</tr>
</tbody>
</table>

Table 4.2. Average probability of failure and its standard deviation for different bootstrap value and threshold levels $\alpha$ (continued)

Figures 4.6-15 show the results in Tables 4.1 and 4.2. It is observed that the estimator of the probability of failure converges for higher probabilities of failure after 40,000 bootstraps.
Figure 4-6 Average probability of failure versus number of bootstraps for level 0.12.

Figure 4-7 Average probability of failure versus number of bootstraps for level 0.2.
Figure 4-8 Average probability of failure versus number of bootstraps for level 0.25.

Figure 4-9 Average probability of failure versus number of bootstraps for level 0.265.
Figure 4-10 Average probability of failure versus number of bootstraps for level 0.28.

Figure 4-11 Standard deviation of the probability of failure versus number of bootstraps for level 0.12.
Figure 4-12 Standard deviation of the probability of failure versus number of bootstraps for level 0.2.

Figure 4-13 Standard deviation of the probability of failure versus number of bootstraps for level 0.25.
Figure 4-14 Standard deviation of the probability of failure versus number of bootstraps for level 0.265.

Figure 4-15 Standard deviation of the probability of failure versus number of bootstraps for level 0.28.
4.1.1.2. Validation of Results Using MCS: Steps of Analysis of One Story Building

The steps below are used to calculate the probability of failure and its standard deviation using MCS.

1. Create a database of 10,000 frequencies using the PSD of the wind. Store the frequencies in a database.
2. Create a database of 10,000 phase angles. The phase angles follows a uniform distribution in the range [0 - 2π].
3. Select at random without replacement ten frequency values $\omega_i (i = 1, \ldots, 10)$ from the database and 10 phase angles $\varphi_i (i = 1, \ldots, 10)$.
4. Calculate a time history of excitation of the mass using equation (4.3)
   The time history of excitation obtained is similar to that shown in figure 4-4
5. Calculate the response time history of mass one due to the force at one by equation (4.4).
6. Find the maximum value of each response time history.
7. Compare the maximum displacement to different threshold levels $\alpha$ to check for failure. The failure indicator function assumes a value of 0 if the system survives and a value of 1 otherwise.
   $\mathbb{I}[G \geq 0] = 0$ system survives
   $\mathbb{I}[G < 0] = 1$ system fails
8. Repeat steps 1-7 until you use all values in the database.
9. Estimate the probability of failure using equation (4.9)
\[ \hat{p}_{MCS} = \frac{1}{k} \sum_{i=1}^{k} l_i \] 

(4.9)

10. Estimate the standard deviation of the probability of failure \( (\sigma_{\hat{p}_{MCS}}) \) by using equation (4.10)

\[ \sigma_{\hat{p}_{MCS}} = \sqrt{\frac{\hat{p}_{MCS}(1-\hat{p}_{MCS})}{k}} \] 

(4.10)

4.1.1.2.1. Results From MCS: One Story Building

The probability of failure and its standard deviation at different threshold levels obtained from MCS is shown in Table 4.3.

<table>
<thead>
<tr>
<th>Level</th>
<th>0.12</th>
<th>0.2</th>
<th>0.25</th>
<th>0.265</th>
<th>0.28</th>
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<tr>
<td>( \hat{p}_{MCS} )</td>
<td>7.0E-03</td>
<td>4.0E-03</td>
<td>3.0E-03</td>
<td>2.0E-03</td>
<td>2.0E-03</td>
</tr>
<tr>
<td>( \sigma_{\hat{p}_{MCS}} )</td>
<td>2.64E-03</td>
<td>2.0E-03</td>
<td>1.73E-03</td>
<td>1.41E-03</td>
<td>1.41E-03</td>
</tr>
</tbody>
</table>

Table 4.3 Probability of failure and its standard deviation at different threshold levels \( \alpha \) using MCS.

The probabilities of failure obtained from MCS and corresponding standard deviations are compared with those obtained from bootstrapping as shown in figures 4-16-17.
Figure 4-16 Probability of failure from MCS and bootstrapping for all levels.

Figure 4-17 Standard deviation of probability of failure from MCS (horizontal lines) and bootstrapping (curves) for all levels.
Figure 4-16 does not show the true nature of the plots of the standard deviations from bootstrapping. This is because the results from MCS are larger than those obtained by bootstrapping. The superimposed plots were created to show how the probabilities of failure vary for the different threshold levels.

4.1.1.3. Equivalent Sample Size and Coefficient of Variation (COV): One Story Building

The equivalent sample size for a failure probability is the number of MCS needed to estimate this probability with same accuracy as with bootstrapping. For example, if the value of ESS for a given failure probability is 10, then ten replications of MCS are required to estimate the same probability with the same accuracy using MCS as using one bootstrap. In other words, one bootstrap is equivalent to 10 MCS. The equivalent sample size was calculated using equation (3.13). As stated in section 3.3, this was done to demonstrate the cost effectiveness of bootstrapping compared to MCS. Table 4.4 shows the results obtained for the equivalent sample size.

<table>
<thead>
<tr>
<th>Level</th>
<th>Bootstraps</th>
<th>$\hat{p}_{avg, BS}$</th>
<th>$\sigma \hat{p}_{avg, BS}$</th>
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<th>BSS</th>
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<td>405758</td>
<td>10,000</td>
</tr>
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</table>

Table 4.4 Calculated equivalent sample size for the one story building.
The equivalent sample space was calculated using the results at the point of diminishing returns. Figure 4-18 shows the variation of ESS with the threshold level. The plot was made to investigate the behavior of ESS with the threshold level.

Figure 4-18 ESS & BSS (Bootstrapping Sample Size) for different threshold levels

To determine whether or not the estimated probability of failure and its standard deviation were good, the coefficient of variation of the estimates were calculated. For bootstrapping, $COV$ was calculated at the point of diminishing returns. Table 4.5 shows the result obtained.

<table>
<thead>
<tr>
<th>Level</th>
<th>bootstraps</th>
<th>$\hat{p}_{avgs}$</th>
<th>$\sigma_{\hat{p}_{BS}}$</th>
<th>$COV_{BS}$</th>
<th>$\hat{p}_{MCS}$</th>
<th>$\sigma_{MCS}$</th>
<th>$COV_{MCS}$</th>
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<td>2.64E-03</td>
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<td>1.41E-03</td>
<td>7.05E-01</td>
</tr>
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</table>

Table 4.5 Bootstrapping & MCS results for the one story building- $COV$
4.1.1.4. Observations: One Story Building

The average probability of failure, $\widehat{p}_{avgBS}$, fluctuates for lower bootstrap values and stabilizes with the number of bootstraps. For lower $\alpha$ values (high probability of failure), the average probability of failure oscillates at higher values and stabilizes at a lower values of the probability of failure. The standard deviation of the probability of failure, $\sigma_{\widehat{p}_{avgBS}}$, is observed to decrease with the number of bootstraps. Initially, $\sigma_{\widehat{p}_{avgBS}}$ decreases very fast until a point of diminishing returns is reached. Beyond the point of diminishing returns, $\sigma_{\widehat{p}_{avgBS}}$, reduces slowly. The point of diminishing returns occurs at a bootstrap value of 40,000. As should be the case, it is observed from Tables 4.1 & 4.2 that quadrupling bootstraps values reduces $\sigma_{\widehat{p}_{avgBS}}$ into a half.

The probability of failure obtained from MCS decreases with the threshold level. The decrease is insignificant compared to that obtained from bootstrapping as it is only by a magnitude of 3.5 (ratio of the highest probability of failure to the lowest). The same can be said for the standard deviation of the probabilities of failure.

The equivalent sample size, ESS, varies within a small range (380,000 – 430,000). The $COV$ obtained for bootstrapping range from 3 – 86% and that from MCS ranges from 38 – 70%. For both methods, $COV$ increased as the probability of failure reduced.
4.1.2. Two Story Building

The second example is a two degrees of freedom system building that is modeled as a system with floor masses $m_1$ and $m_2$ attached by a spring with stiffness $k_2$. Mass $m_1$ is attached to the ground with a spring of stiffness $k_1$. The study assumes that damping is negligible (figure 4-19). The system involves 40 random variables: 20 random frequencies and 20 random phase angles. It is modeled with the following properties:

Sprung mass, $m_1 = 2000 \text{ kg}$ and un-sprung mass, $m_2 = 1700 \text{ kg}$, and Stiffness, $k_1 = k_2 = 35000 \frac{\text{KN}}{\text{m}}$.

![Two degrees of freedom system diagram](image)

**Figure 4-19 Two degrees of freedom system**

Similar to the one story building example, the wind's energy is modeled using the Pierson-Moskowitz power spectral density $S(\omega)$, which is shown in equation (4.1). The
excitation of each mass is created with frequencies from different PSDs with the parameters listed below:

Parameters of mass \( m_1 \) PSD; \( A_1 = 17000 \, m^2 \text{sec}^{-4} \) and \( B_1 = 390 \, \text{sec}^{-4} \).

Parameters of mass \( m_2 \) PSD; \( A_2 = 17500 \, m^2 \text{sec}^{-4} \) and \( B_2 = 350 \, \text{sec}^{-4} \).

The probability density function of the PSD is obtained in a similar way as that explained in section 4.1.1.

4.1.2.1. **Bootstrapping: Steps of Analysis for a Two Story Building**

The steps used to estimate the probability of failure and its standard deviation of the two-story building is similar to that used in section 4.1.1.1. One of the difference is that two different the time history of excitation are created for the two masses. A general expression for the equation used for calculating the time history of excitation is given by equation (4.11).

\[
y_p(t) = \sqrt{\frac{2}{N}} \sigma_p \sum_{i=1}^{N} \cos (\omega_i t - \varphi_i)
\]  
(4.11)

Another difference is the number of times histories of response calculated. A total of 4-time histories of the response of the two masses due to the two-time histories of excitation is calculated. Equation (4.12) shows a general expression.

\[
x_p(t) = \sqrt{\frac{2}{N}} \sigma_g \sum_{g=1}^{G} \sum_{i=1}^{N} |H_{pg}(\omega_{gi})| \cos [\omega_{gi} t - \varphi_{gi} + \angle H_{pg}(\omega_{gi})]
\]  
(4.12)

The symbol \( p \) represents the \( p \text{th} \) element of the system. In this case, \( p = 2 \).
4.1.2.1.1. Presentation of Results From bootstrapping: Two Story Building

The average probability of failure and its standard deviation at different bootstrap values obtained from simulation using bootstrapping is shown in Tables 4.6 and 4.7.

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<td>$\hat{p}<em>{avg</em>{BS}}$</td>
<td>$\sigma_{\hat{p}<em>{avg</em>{BS}}}$</td>
<td>$\hat{p}<em>{avg</em>{BS}}$</td>
<td>$\sigma_{\hat{p}<em>{avg</em>{BS}}}$</td>
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Table 4.6 Average probability of failure and its standard deviation at different bootstrap values and threshold levels $\alpha$. 
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Table 4.7 Average probability of failure and its standard deviation at different bootstrap values and threshold levels \( \alpha \) (continued).

Plots of the average probability of failure and its standard deviation with the number of bootstraps are created to show their behavior as the number of bootstraps is increased. The plots are shown in figures 4-19-28.
Figure 4-20 Average probability of failure versus number of bootstraps for level 0.3

Figure 4-21 Average probability of failure versus number of bootstraps for level 0.35
Figure 4-22 Average probability of failure versus number of bootstraps for level 0.417

Figure 4-23 Average probability of failure versus number of bootstraps for level 0.44
Figure 4-24 Average probability of failure versus number of bootstraps for level 0.5

Figure 4-25 Standard deviation of the probability of failure versus number of bootstraps for level 0.3
Figure 4-26 Standard deviation of the probability of failure versus number of bootstraps for level 0.35

Figure 4-27 Standard deviation of the probability of failure versus number of bootstraps for level 0.417
Figure 4-28 Standard deviation of the probability of failure versus number of bootstraps for level 0.44

Figure 4-29 Standard deviation of the probability of failure versus number of bootstraps for level 0.5
4.1.2.2. Validation of Result Using MCS: Steps of Analysis of Two Story Building

The steps used to estimate the probability of failure and its standard deviation of the two-story building is similar to that used in section 4.1.1.3. One of the difference is the fact that two different the time history of excitation are created for the two masses. A general expression for the equation used for calculating the time history of excitation is given by equation (4.11). Another difference is the number of times histories of response calculated. A total of 4-time histories of the response of the two masses due to the two-time histories of excitation are calculated. Equation (4.12) shows a general expression.

4.1.2.2.1. Results From MCS: Two Story Building

The probability of failure and its standard deviation at different threshold levels obtained from MCS is shown in Table 4.8.

<table>
<thead>
<tr>
<th>Level</th>
<th>0.3</th>
<th>0.35</th>
<th>0.417</th>
<th>0.44</th>
<th>0.5</th>
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<td>3.0E-03</td>
<td>3.0E-03</td>
<td>3.0E-03</td>
</tr>
<tr>
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<td>1.73E-03</td>
<td>1.73E-03</td>
<td>1.73E-03</td>
<td>1.73E-03</td>
</tr>
</tbody>
</table>

Table 4.8 Probability of failure and its standard deviation at different threshold levels $\alpha$ using MCS.

The probabilities of failure obtained from MCS and corresponding standard deviations are compared with those obtained from bootstrapping as shown in figures 4.29-30.
Figure 4.30 Probability of failure from MCS and bootstrapping for all levels.

Figure 4.31 Standard deviation of probability of failure from MCS (horizontal lines) and bootstrapping (curves) for all levels.
Figure 4.30 does not show the true nature of the plots of the probability of failure from bootstrapping. This is because the results from MCS are larger than those obtained by bootstrapping. The superimposed plots were created to show how the probability of failures varies for the different threshold level.

4.1.2.3. Equivalent Sample Size and Coefficient of Variation (COV): Two Story Building

The equivalent sample size was calculated using equation (3.13). As stated in section 3.3, this was done to demonstrate the cost effectiveness of bootstrapping compared to MCS. Table 4.9 shows the results obtained for the equivalent sample size.

<table>
<thead>
<tr>
<th>Level</th>
<th>Bootstraps</th>
<th>( \hat{p}_{avg BS} )</th>
<th>( \sigma_{\hat{p}_{avg BS}} )</th>
<th>ESS</th>
<th>BS</th>
</tr>
</thead>
<tbody>
<tr>
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<td>383,175</td>
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<td>7.51E-03</td>
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<td>10,000</td>
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<td>10,000</td>
</tr>
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</table>

Table 4.9 Calculated equivalent sample size for the two story building.

The equivalent sample space was calculated using the results at the point of diminishing returns. Figure 4.32 the variation of ESS with the threshold level. The plot was used to investigate the behavior of ESS with the threshold level.
To determine whether or not the estimated probability of failure and its standard deviation were good, the coefficient of variation of the estimates were calculated. For bootstrapping, $COV$ was calculated at the point of diminishing returns.

Tables 4.10 shows the result obtained.

**Table 4.10 Bootstrapping & MCS results for the two story building- $COV$**

<table>
<thead>
<tr>
<th>Level</th>
<th>bootstraps</th>
<th>$\hat{p}_{avgBS}$</th>
<th>$σ_{\hat{p}_{BS}}$</th>
<th>$COV_{BS}$</th>
<th>$\hat{p}_{MCS}$</th>
<th>$σ_{MCS}$</th>
<th>$COV_{MCS}$</th>
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</tr>
<tr>
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<td>4.41E-04</td>
<td>5.88E-02</td>
<td>3.0E-03</td>
<td>1.73E-03</td>
<td>5.76E-01</td>
</tr>
<tr>
<td>0.417</td>
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<td>2.30E-04</td>
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<td>1.73E-03</td>
<td>5.76E-01</td>
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<td>5.76E-01</td>
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<tr>
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<td>1.73E-03</td>
<td>5.76E-01</td>
</tr>
</tbody>
</table>

**4.1.2.4. Observations: Two Story Building**

The same phenomena were observed for the average probability of failure, $\hat{p}_{avgBS}$. It fluctuates for lower bootstrap values and stabilizes with the number of
The standard deviation of the probability of failure, $\sigma_{\hat{P}_{avgBS}}$, is observed to decrease with the number of bootstraps. Initially, $\sigma_{\hat{P}_{avgBS}}$ decreases very fast until a point of diminishing returns is reached. Beyond the point of diminishing returns, $\sigma_{\hat{P}_{avgBS}}$ reduces slowly. The point of diminishing returns also occurs at a bootstrap value of 40,000. As should be the case, it is observed from Tables 4.6 & 7 that quadrupling bootstraps values cuts $\sigma_{\hat{P}_{avgBS}}$ into a half.

The probability of failure obtained from MCS does not decrease with the threshold level. $\hat{P}_{MCS}$ is constant for all the threshold levels. This could be because the threshold levels used are not appropriate for the system. Consequently, the standard deviation of the probabilities of failure remained constant.

The equivalent sample size, ESS, varies within a small range (380,000 – 440,000). The COV obtained for bootstrapping range from 5 – 155% and that from MCS ranges from 57 – 57%. For both methods, COV increased as the probability of failure reduced.

### 4.1.3. Eight Story Building

The system under consideration is an eight degree of system building. It has floor masses $m_1$ to $m_8$ attached by a spring with stiffness $k_1$ to $k_8$. Mass $m_1$ is attached to the ground with spring of stiffness $k_1$. The system is assumed to be undamped (figure 4-32). It involves 160 random variables: 80 random frequencies and 80 random phase angles. It is modeled with the following properties:
Sprung mass, \(m_1 = 2000 \text{ kg}\)

Un-sprung mass, \(m_2 = 2000 \text{ kg}, m_3 = 2000 \text{ kg}, m_4 = 2000 \text{ kg}, m_5 = 2000 \text{ kg}, m_6 = 2000 \text{ kg}, m_7 = 2000 \text{ kg}, m_8 = 1700 \text{ kg}\)

Stiffness, \(k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = k_7 = k_8 = 35000 \frac{\text{KN}}{m}\)

Figure 4-33 Eight degrees of freedom system

Similar to the previous example, the wind’s energy is modeled using the Pierson-Moskowitz power spectral density \(S(\omega)\), which is shown in equation (4.1). The excitation of each mass is created with frequencies from different PSDs with the parameters listed below:

Parameters of mass \(m_1\) PSD; \(A_1 = 17000 \text{ m}^2\text{sec}^{-4}\) and \(B_1 = 390 \text{ sec}^{-4}\).

Parameters of mass \(m_2\) PSD; \(A_2 = 17500 \text{ m}^2\text{sec}^{-4}\) and \(B_2 = 350 \text{ sec}^{-4}\).

Parameters of mass \(m_1\) PSD; \(A_3 = 17000 \text{ m}^2\text{sec}^{-4}\) and \(B_3 = 390 \text{ sec}^{-4}\).

Parameters of mass \(m_2\) PSD; \(A_4 = 15000 \text{ m}^2\text{sec}^{-4}\) and \(B_4 = 395 \text{ sec}^{-4}\).

Parameters of mass \(m_1\) PSD; \(A_5 = 18000 \text{ m}^2\text{sec}^{-4}\) and \(B_5 = 400 \text{ sec}^{-4}\).
Parameters of mass $m_2$ PSD; $A_6 = 17500 \, m^2 sec^{-4}$ and $B_6 = 390 \, sec^{-4}$.

Parameters of mass $m_1$ PSD; $A_7 = 18000 \, m^2 sec^{-4}$ and $B_7 = 395 \, sec^{-4}$.

Parameters of mass $m_2$ PSD; $A_8 = 18500 \, m^2 sec^{-4}$ and $B_8 = 380 \, sec^{-4}$.

The probability density function of the PSD is obtained in a similar way as that explained in section 4.1.1.

4.1.3.1. **Bootstrapping: Steps of Analysis for an Eight Story Building**

The steps used to estimate the probability of failure and its standard deviation of the eight-story building is similar to that used in section 4.1.2.1. Instead of two-time histories of excitation, eight of them are created for the eight masses. The same general expression used in equation (4.11) is used. The number of time histories of response calculated increases to 64. The same general expression shown in equation (4.12) is used. In this case $p = 8$.

4.1.3.1.1. **Presentation of Results From bootstrapping: Eight Story Building**

The average probability of failure and its standard deviation at different bootstrap values obtained from simulation using bootstrapping is shown in Tables 4.11 & 4.12.

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<td>50,000</td>
<td>4.74E-04</td>
<td>9.70E-05</td>
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<td>60,000</td>
<td>4.75E-04</td>
<td>8.90E-05</td>
<td>4.77E-05</td>
<td>2.80E-05</td>
<td></td>
<td></td>
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<td>70,000</td>
<td>4.73E-04</td>
<td>8.15E-05</td>
<td>4.80E-05</td>
<td>2.54E-05</td>
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Table 4.11 Average probability of failure and its standard deviation at different bootstrap values and threshold levels \( \alpha \).
Table 4.12 Average probability of failure and its standard deviation at different bootstrap values and threshold levels $\alpha$ (continued).

Plots of the average probability of failure and its standard deviation with the number of bootstraps are created to show their behavior as the number of bootstraps is increased. The plots are shown in figures 4-34-43.

Figure 4-34 Average probability of failure versus number of bootstraps for level 0.5
Figure 4-35 Average probability of failure versus number of bootstraps for level 0.57

Figure 4-36 Average probability of failure versus number of bootstraps for level 0.69
Figure 4-37 Average probability of failure versus number of bootstraps for level 0.72

Figure 4-38 Average probability of failure versus number of bootstraps for level 0.78
Figure 4-39 Standard deviation of the probability of failure versus number of bootstraps for level 0.5

Figure 4-40 Standard deviation of the probability of failure versus number of bootstraps for level 0.57
Figure 4-41 Standard deviation of the probability of failure versus number of bootstraps for level 0.69

Figure 4-42 Standard deviation of the probability of failure versus number of bootstraps for level 0.72
4.1.3.2. Validation of Results Using MCS: Steps of Analysis Eight Story Building

The steps used to estimate the probability of failure and its standard deviation of the eight-story building are similar to that used in section 4.1.1.3. The differences outlined in section 4.1.3.1 apply to the MCS analysis. The same general expressions are used to calculate the time histories of excitation and response and the same number of time histories of excitation and response are calculated.

4.1.3.2.1. Results From MCS: Eight Story Building

The probability of failure and its standard deviation at different threshold levels obtained from simulation using MCS is shown in Table 4.13.
Levels | 0.5 | 0.57 | 0.69 | 0.72 | 0.78  
|-------|-----|------|------|------|------|  
\(p_{f_{MCS}}\) | 2.0E-03 | 2.0E-03 | 2.0E-03 | 1.0E-03 | 1.0E-03 |  
\(\sigma_{p_{f_{MCS}}}\) | 1.41E-03 | 1.41E-03 | 1.41E-03 | 1.0E-03 | 1.0E-03 |  

Table 4.13 Probability of failure and its standard deviation at different threshold levels \(\alpha\) using MCS.

The probabilities of failure obtained from MCS and corresponding standard deviations are compared with those obtained from bootstrapping as shown in figures 4.43-44.

Figure 4-44 Probability of failure from MCS and bootstrapping for all levels.
Figure 4-45 Standard deviation of the probability of failure from MCS (horizontal) and bootstrapping (curves) for all levels.

Figure 4-29 does not show the true nature of the plots of the standard deviations from bootstrapping because their unstable behavior at lower bootstraps is not well shown. This is because the results from MCS are larger than those obtained by bootstrapping. The superimposed plots were created to show how the probability of failures varies for the different threshold level.

4.1.3.4. Equivalent Sample Size and Coefficient of Variation (COV): Eight Story Building

The equivalent sample space was calculated using equation (3.13). As stated in section 3.3, this was done to demonstrate the cost effectiveness of bootstrapping compared to MCS. Table 4.14 shows the results obtained for the equivalent sample size.
The equivalent sample space was calculated using the results at the point of diminishing returns. Figure 4-31 shows the variation of ESS with the threshold level. The plot was used to investigate the behavior of ESS with the threshold level.

![Figure 4.46 ESS & BSS (Bootstrapping Sample Size) versus threshold levels](image)

To determine whether or not the estimated probability of failure and its standard deviation were good, the coefficient of variation of the estimates were calculated. For bootstrapping, \( COV \) was calculated at the point of diminishing returns. Tables 4.15 shows the result obtained.
Table 4.15 Bootstrapping & MCS results for the eight story building- \( COV \)

<table>
<thead>
<tr>
<th>Level</th>
<th>bootstraps</th>
<th>( \hat{p}_{avgBS} )</th>
<th>( \sigma_{\hat{p}_{BS}} )</th>
<th>( COV_{BS} )</th>
<th>( \hat{p}_{MCS} )</th>
<th>( \sigma_{MCS} )</th>
<th>( COV_{MCS} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>40,000</td>
<td>1.01E-02</td>
<td>4.94E-04</td>
<td>4.90E-02</td>
<td>2.00E-03</td>
<td>1.41E-03</td>
<td>7.06E-01</td>
</tr>
<tr>
<td>0.57</td>
<td>40,000</td>
<td>6.65E-03</td>
<td>4.10E-04</td>
<td>6.17E-02</td>
<td>2.00E-03</td>
<td>1.41E-03</td>
<td>7.06E-01</td>
</tr>
<tr>
<td>0.69</td>
<td>40,000</td>
<td>2.45E-03</td>
<td>2.46E-04</td>
<td>1.01E-01</td>
<td>2.00E-03</td>
<td>1.41E-03</td>
<td>7.06E-01</td>
</tr>
<tr>
<td>0.72</td>
<td>40,000</td>
<td>4.73E-04</td>
<td>1.09E-04</td>
<td>2.30E-01</td>
<td>1.00E-03</td>
<td>9.99E-04</td>
<td>9.99E-01</td>
</tr>
<tr>
<td>0.78</td>
<td>40,000</td>
<td>4.80E-05</td>
<td>3.34E-05</td>
<td>6.96E-01</td>
<td>1.00E-03</td>
<td>9.99E-04</td>
<td>9.99E-01</td>
</tr>
</tbody>
</table>

### 4.1.2.5. Observations: Eight Story Building

For the eight degrees of freedom system, the same phenomena were observed for the average probability of failure, \( \hat{p}_{avgBS} \). It fluctuates for lower bootstrap values and stabilizes with the number of bootstraps. The standard deviation of the probability of failure, \( \sigma_{\hat{p}_{avgBS}} \), is observed to decrease with the number of bootstraps. Initially, \( \sigma_{\hat{p}_{avgBS}} \) decreases very fast until a point of diminishing returns is reached. Beyond the point of diminishing returns, the \( \sigma_{\hat{p}_{avgBS}} \), reduces slowly. Beyond this point, more bootstraps results to no appreciable gain. The point of diminishing returns also occurs at a bootstrap value of 40000. As should be the case, it is observed from Tables 4.11 & 4.12 that quadrupling bootstraps values cuts \( \sigma_{\hat{p}_{avgBS}} \) into a half.

The probability of failure obtained from MCS starts out constant and then increases at a higher threshold level of 0.72 and stays constant. This could be because the threshold levels used are not appropriate for the system. The same observation was made for the standard deviation of the probabilities of failure.
The equivalent sample space, ESS, varies within a small range (390,000 – 430,000). The $COV$ obtained for bootstrapping range from 5 – 70% and that from MCS ranges from 70 – 100%. For both methods, $COV$ increased as the probability of failure reduced.

4.2. Discussion

For all of the systems investigated in this research, the probabilities of failure of the systems fluctuate at lower bootstrap values and stabilizes with the number of bootstraps. Bootstrapping method estimated probabilities of failure in the magnitude $10^{-2} – 10^{-5}$, while MCS estimated probability in the magnitude $10^{-3}$. This shows the inefficiency of MCS in calculating small probabilities of failure.

At low bootstrap values, poor accuracy of the probabilities of failure is obtained. As the number of bootstraps in increased, this accuracy of the estimates improved significantly for lower threshold values (probability of failure in the range $10^{-2} – 10^{-3}$). To improve the accuracy of the probabilities of failure at higher threshold levels (probabilities of failure in the range $10^{-4} – 10^{-5}$), more bootstraps are required. Probability of failure in the magnitude of $10^{-3}$ was the focus of this research. For probabilities of failure in the range $10^{-2} – 10^{-3}$, the accuracy of the probabilities of failure was good. This was determined by calculating the coefficient of variation $COV$, which is the ratio of the standard deviation with the estimated mean. An estimated probability of failure is good when the $COV \leq 10\%$. The accuracy of the estimated probabilities of failure using MCS was not good because the $COV$ calculated for all the degrees of freedom and threshold levels was at least 38%. 

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Increasing the number of degrees of freedom of the systems increases the number of time histories possible. One of the objectives of this research was to investigate how the accuracy of the probability of failure changes with an increase in the number of degrees of freedom. It is known that the accuracy increases with the number of bootstraps, but a point of diminishing return is reached when more bootstraps are picked, and no appreciable gain is achieved. For all of the examples in this research and in cases where the accuracy of the probabilities of failure is high, the point of diminishing returns was observed to occur at the same number of bootstraps (40,000). Therefore, the point of diminishing returns is independent of the number of degrees of freedom.

Equivalent sample sizes were calculated using the results at the initial point of diminishing returns. The equivalent size calculated for the one-story building is in the range of 380,000 – 430,000. This means that the number of samples required to perform MCS to obtain the same accuracy of the probability of failure is 38 – 43 times that used for bootstrapping (10,000 sample values were used). For the two-story and eight-story building, ESS is in the range of 380,000 – 440,000 and 390,000 – 430,000 respectively. The equivalent sample sizes for the three examples do not vary significantly. For lower threshold values (high probability of failure in the magnitude $10^{-2} - 10^{-3}$), the calculated value for the COV shows that the estimated probabilities of failure and their standard deviations at the point of diminishing returns were accurate because their COV is less than 10%. For higher threshold values (lower probability of failure in the magnitude $10^{-4} - 10^{-5}$), the estimated probabilities of failure and their standard deviations are not good estimates because their COV > 10%. The estimates can be improved by using higher bootstraps. This research focused on higher probabilities of
failure in the magnitude $10^{-2} - 10^{-3}$. For MCS, the calculated $COV$ for all the threshold levels implies that the estimated probabilities of failure and their standard deviations are not accurate. This is true for all the degrees of freedom. The least $COV$ for all the threshold values is 38%. This can only be improved by increasing the database size, which results to more expense. To achieve the same accuracy from bootstrapping by using MCS, the equivalent sample size gives an estimate of the number of samples required to achieve this. The results from ESS shows that hundreds of thousands more sample values are required to match the results from bootstrapping by using MCS.
Chapter 5

Conclusion

This chapter discusses the conclusions obtained in this thesis. The probability of failure and standard deviation of the probability of failure obtained using bootstrapping, and MCS were discussed, and the coefficients of variation (COV) for the different threshold levels were calculated to validate the estimates of the probabilities of failure. The use of ESS enabled the author to assess the efficiency of bootstrapping and compare its efficiency to that of MCS.

In this thesis, the probability of failure and the standard deviation of the probability of failure of systems under random loading conditions were estimated using bootstrapping and MCS. The change in the accuracy of bootstrapping with the number of degrees of freedom of a system was investigated. Using bootstrapping, the standard deviation of the probability of failure of any system reaches a point of diminishing returns, which is the number of bootstraps beyond which the accuracy of the estimated failure probability does not increase appreciably with the bootstraps. The objective was to investigate whether or not the point of diminishing returns increases with the degrees of freedom of the system. Both methods were compared in terms of their accuracy efficiency by calculating the equivalent sample size. Random vibration analysis of a one-story building was performed and then increased to two and then eight stories.
As expected, bootstrapping method produced more accurate estimates of the probability of failure of the buildings under random excitation. Tables 4.5, 4.10 & 4-15 summarizes the results obtained for the different degrees of freedom. For bootstrapping, results at the point of diminishing returns are used. The COV of the results obtained for probabilities of failure in the magnitude $10^{-2} \& 10^{-3}$ using bootstrapping suggests that the probability of failure and standard deviation of the probability of failure obtained are accurate ($COV \leq 10\%$). For lower probabilities of failure in the magnitude $10^{-4} \& 10^{-5}$, the results obtained were inaccurate as suggested by the $COV$ obtained. For all the probabilities of failure, the results from MCS were inaccurate since their $COV$ was higher than 10\%.

Also, it was observed that the point of diminishing returns was independent of the number of degrees of freedom of the system. For all the different DOF and threshold levels $\alpha$, the point of diminishing returns was observed to be at 40,000 bootstraps. It can be concluded that the point of diminishing returns is independent of the number of degree of freedom since the point of diminishing returns was reached at the same value of bootstrap for all the systems investigated. At this point, the coefficient of variation that is the ratio of standard deviation to the mean, suggests that the $\sigma_{pf_{BS}}$ obtained is a good estimate for lower threshold levels $\alpha$. For higher $\alpha$ values (very low probability in the magnitude $10^{-4} \& 10^{-5}$), the number of bootstraps needs to be increased to obtain better estimate for $\sigma_{pf_{BS}}$.

To make comparisons between bootstrapping and MCS, equivalent sample size, ESS is calculated. The equivalent sample size was calculated using the results obtained at the point of diminishing return. Table 4.4, 4.9, & 4.14 summarizes the results obtained.
for ESS. Bootstrapping analysis was performed with a database size of 10,000. The results from Tables 4.4, 4.9, & 4.14 clearly show that the equivalent sample size needed to obtain the same accuracy as that obtained from bootstrapping is in the range of 38 – 44 times the database size of bootstrapping. This shows the cost effectiveness of bootstrapping.

An investigation of the effect of the number of frequencies picked at random to calculate the time histories of excitation is a possible area of further research.
References


Appendix A

Derivation of Equivalent Sample Size Equation (Section 3.4)

The standard deviation of the probability of failure using MCS, is given by the equation:

\[ \sigma_{\hat{p}_f} = \sqrt{\frac{\hat{p}_f (1-\hat{p}_f)}{k}} \]  \hspace{1cm} (A.1)

\( k \) is the number of replications and \( \hat{p}_f \) is the estimated probability of failure.

Given a standard deviation and estimated probability of failure \( \sigma_{\hat{p}_f} \) and \( \hat{p}_f \) respectively, the number of replications can be calculated by solving for \( k \).

\( k \) is given by the equation below:

\[ k = \frac{\hat{p}_f (1-\hat{p}_f)}{\sigma_{\hat{p}_f}^2} \]  \hspace{1cm} (A.2)

Each replication uses 10 sample values of frequency. Therefore, the equivalent sample size is given by the equation below:

\[ ESS = k \times 10 \]  \hspace{1cm} (A.3)