A Dissertation
entitled
Inverse Problem of Lagrangian Mechanics
In Dimension Three
by
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Submitted to the Graduate Faculty as partial fulfillment of the requirements for the
Doctor of Philosophy Degree in Mathematics

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We discuss various aspects of the inverse problem of Lagrangian Mechanics. The problem consists of finding necessary and sufficient conditions for a system of second order ordinary differential equations to be the Euler-Lagrange equations of a regular Lagrangian function. The problem in dimension $n$ is studied, especially for systems with $n-1$ trivial equations and some necessary conditions are obtained. In dimension three, a case by case study of the dimension of the module of algebraic solutions is initiated. The problem in the linear connection case, that is, when the given data are the components of a symmetric connection, is also studied and some examples are constructed.
To my parents Bhagirath and Chitra Devi Bhattarai.
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<td>IP</td>
<td>Inverse Problem</td>
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<tr>
<td>ODE</td>
<td>Ordinary Differential equation</td>
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List of Symbols

\( d \) ............ the dimension of the module of solutions to the double hierarchy of the \( \Phi \) and \( \Psi \) conditions
Chapter 1

Introduction

1.1 The Helmholtz conditions

Consider a system of second order ordinary differential equations (ODE) of the form

\[ \ddot{x}^i = f^i(x^j, \dot{x}^j). \]  

(1.1)

To ease the working we shall denote \( \dot{x}^i \) by \( u^i \) throughout. Our problem is to determine if (1.1) are the Euler-Lagrange equations engendered by some unknown Lagrangian function \( L(x^j, u^j) \). Observe that the unknown function \( L \) is the solution of the following linear PDE system:

\[ f^k \frac{\partial L}{\partial u^i} \frac{\partial u^k}{\partial u^i} = \frac{\partial L}{\partial x^i} - u^k \frac{\partial L}{\partial x^k} \frac{\partial u^i}{\partial u^i}. \]  

(1.2)

This approach is taken by Muzsnay and Grifone [GM1, GM2] who use the techniques of Spencer cohomology. However, the majority of investigators follow Douglas [Dou], whose formulation of the problem, although ultimately equivalent, appears to be quite different. To explain Douglas’ approach, define the following \( n \times n \) matrix of functions:

\[ \Phi^i_j = \frac{1}{2} \frac{d}{dt} \left( \frac{\partial f^i}{\partial \omega} \right) - \frac{\partial f^i}{\partial x^j} - \frac{1}{4} \frac{\partial f^i}{\partial u^k} \frac{\partial f^k}{\partial u^j}. \]  

(1.3)
In Douglas’ approach the function $L$ is no longer the principal unknown but rather its Hessian $\frac{\partial L}{\partial u^i \partial u^j}$ which is denoted by $g_{ij}$. The notation is chosen so that if $L$ is homogeneous quadratic in the $u^i$ then $g_{ij}$ is just a classical metric tensor.

Starting from the PDE system (1.1) we can differentiate with respect to $x^i$ or $u^i$. After some symmetrization and rearrangement Douglas obtained the following conditions known universally as the Helmholtz conditions:

$$g\Phi = (g\Phi)^t \quad (1.4)$$

$$\frac{dg_{ij}}{dt} + \frac{1}{2} \frac{\partial f^k}{\partial u^j} g_{kj} + \frac{1}{2} \frac{\partial f^k}{\partial u^i} g_{ki} = 0. \quad (1.5)$$

$$\frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^j} = 0. \quad (1.6)$$

The surprising aspect of the Helmholtz conditions is that they are not only necessary but also sufficient for the existence of the Lagrangian $L$.

**Theorem 1.1.1.** (Douglas) Necessary and sufficient conditions for there to exist a Lagrangian so that (1.1) are its Euler-Lagrange equations are that there should exist a non-singular, symmetric matrix $g_{ij}$ depending on $(x^i, u^i)$ that satisfies (1.4), (1.5) and (1.6), known collectively as the Helmholtz conditions. To pass from the Hessian to the Lagrangian requires two integrations and the fact that appropriate linear and zeroth order terms may be added is a consequence of the Helmholtz conditions; the only ambiguity in passing from Hessian to Lagrangian is the trivial one of scaling by a constant and adding a total time derivative.

We consider the Helmholtz conditions as an algebro-differential system. Since there is a single matrix $\Phi$, one can always find non-degenerate solutions to (1.4), whatever the algebraic normal of $\Phi$ may be. In fact, (1.4) imposes at most $\binom{n}{2}$ conditions on the
components of $g$. As regards (1.5) it is possible to scale basis elements which are solutions to (1.4) by first integrals of (1.1) so as to satisfy (1.5). Upon integration, it is not constants of integration that enter but rather constants of motion of (1.1). To carry out the preceding step in practice, explicit first integrals of (1.1) are needed. Thus, there is no obstruction to solving (1.4) and (1.5) and at this stage it may not be possible to say if a Lagrangian $L$ exists. The final and most difficult step is to impose the so-called closure conditions (1.6).

The primary integrability conditions arising from the Helmholtz conditions are obtained as follows. Take the total time derivative of (1.4) and use (1.5) to eliminate the derivatives of $g_{ij}$. The process can be iterated and one obtains a hierarchy $\Phi$ of matrices defined recursively by

$$\Phi^{n+1} = \frac{d}{dt} \left( \Phi^n \right) - \frac{1}{2} \left[ \frac{\partial f}{\partial u}, \Phi^n \right]$$

and the multiplier $g$ is such that each $\Phi^n$ is self-adjoint relative to $g$. One thus obtains a sequence of algebraic conditions on the multiplier matrix $g_{ij}$. Some time we denote $\Phi^1$ by $\nabla \Phi$.

There is a second hierarchy of algebraic conditions that must be satisfied by $g$. Define functions $\Psi_{jk}^i$ by

$$\Psi_{jk}^i = \frac{1}{3} \left( \frac{\partial \Phi_{ij}^i}{\partial u^k} - \frac{\partial \Phi_{ik}^i}{\partial u^j} \right).$$

If we differentiate (1.4) with respect to $u^i$, skew-symmetrize and makes use of (1.6) we obtain

$$g_{mi} \Psi_{jk}^m + g_{mk} \Psi_{ij}^m + g_{mj} \Psi_{ki}^m = 0.$$  

Again we can take the time derivative of (1.9) and use (1.5) to eliminate the derivatives of $g_{ij}$. One obtains a sequence of tensors which satisfy cyclic conditions similar to
In the case of the canonical linear connection on a Lie group only the first set of conditions in each hierarchy is significant, the higher order conditions being satisfied identically.

We now assume that we have a basis of solutions to the double hierarchy of algebraic conditions. In the analytic category the solutions constitute a finite-dimensional module over the ring of first integrals. *If we cannot find a non-singular solution then we can be sure that no regular Lagrangian exists.*

### 1.2 Different approaches to the inverse problem

This Section discusses some earlier contributions to the solution of the inverse problem. Three important contributions from the 1980’s are the papers of Crampin [Cr], Henneaux and Shepley [HS] and Sarlet [Sa]. In [Cr] the fundamental algebro-differential system obtained by Douglas, known now somewhat misleadingly as the Helmholtz conditions, was recast into coordinate-free language. In [HS] the Kepler problem in dimension three was studied and a class of non-standard Lagrangians was obtained. In [Sa] it was shown how the Helmholtz conditions could be manipulated so as to derive some hidden, purely algebraic conditions. In the 1990’s investigations advanced on three fronts. In [AT] Anderson and Thompson presented an algorithm for solving the inverse problem in a concrete situation and it is that procedure that will be adopted here. Meanwhile Martinez, Sarlet and Crampin and others developed a powerful calculus associated to any second order ODE system. Finally, Muzsnay and Grifone took a different approach and completely by-passed the Helmholtz conditions [GM1, GM2]. They approached the inverse problem directly in terms of the Euler-Lagrange operator and used the techniques of the Frölicher-Nijenhuis theory of derivations and Spencer cohomology.

All of these different approaches have their advantages and disadvantages but it
is clear that none of them contains a “silver bullet” that will evade the fundamental difficulties inherent in the inverse problem, however, they are formulated. As a PDE system there is a bewildering collection of surprising integrability conditions [SCM], and it becomes very difficult to ascertain whether such conditions are really new or linear combinations, possibly with function coefficients, of previous conditions. Thus, it is very doubtful whether one can ever construct an effective procedure that could handle the inverse problem in all possible cases. This state of affairs is in marked contrast to the problem of constructing metrics rather than general Lagrangians. A deep underlying reason which explains the difference between the two situations is because the space of metrics is a finite-dimensional space whereas the space of Lagrangians may not be.

Now let us comment on the article of Henneaux [Henn]. Essentially Henneaux finds the first hierarchy corresponding to the self-adjointness of the Hessian matrix $g_{ij}$ relative to the higher order $\Phi$-tensors defined in equation (1.7). It is true that Henneaux works on a space of bivectors rather than two-forms but the content is essentially the same. He then argues that generically the first two such conditions will suffice to show the non-existence of a Lagrangian, a remark that we echo in our comments that follow Theorem 1.1.1. Similarly if there is a Lagrangian the first two conditions show the essential uniqueness of a Lagrangian. Of course for a particular system of second order ODE such observations are useful but it is important to understand that to date there is absolutely no Theorem that guarantees that they are the only conditions that need be considered. Indeed in the case of the canonical Lie group connection only the first condition in the first hierarchy is not empty and the Lagrangian, if one exists, is very far from being unique.
1.3 Outline of dissertation

In chapter two we define the primary invariant $d$ as the dimension of the module of solutions to the double hierarchy defined in Chapter one. Similarly we define $d_0$ but for variational Hessians. The main difficulty in the inverse problem is to find the hidden algebraic conditions which enable one to decide about the existence or nonexistence of a variational Hessian. After that we talk about systems of second order ODE for which the solution space of the algebraic conditions is of maximal dimension assuming that the matrix $\Phi$ is not a multiple of the identity. This maximum is $1 + \binom{n}{2}$ in dimension $n$. We identify many situations in which this maximum occurs; for example, Riemannian space of constant curvature, certain product systems, and systems with $n - 1$ trivial equations are such examples. We study various aspects of these examples.

In Chapter three we focus our study on system with $n - 1$ trivial equations. We obtain some new necessary conditions, solve the closure and time-derivative conditions. For the system with $n - 1$ trivial equations, $\Phi$ is always rank one and if it is nilpotent then Lagrangian must be linear in the velocity $u^1$. If the given system is independent of velocities then there is no regular Lagrangian unless right hand side only depends on $x^1$ os that the ODE system decomposes as a product.

In Chapter four we discuss the inverse problem for the case of a linear connection especially in dimension three. In the connection case we consider ODE’s for which the RHS’s are homogeneous quadratic in velocities. Even in the connection case, if we step up from dimension two to three, the number of arbitrary functions to be controlled jumps from six to eighteen. They give an indication of the big increase in complexity in stepping up from dimension two to three. However, if we put in a “real life system”, that is, one without arbitrary functions or constants, we should be able to execute the first necessary tests for variationality. We also give the normalization
of system with constant coefficients, study the curvature matrices and investigate various aspects of the problem in dimension three.

1.4 Prognosis

It is clear that we are nowhere near to having a complete solution to the inverse problem for linear connections in dimension three. The corresponding problem in dimension two has been solved almost completely, inasmuch as it is possible to solve the inverse problem in any real sense [AT, Tho3]. However, the solution for \( n = 2 \) relies heavily on Douglas’ classification for the inverse problem for general second order systems and such a classification is unavailable for \( n = 3 \). There is another important difficulty concerning the inverse problem which one can even see in Douglas’ paper on the general IP for \( n = 2 \); namely, it does not take long before one gets very far removed from concrete examples. In fact in terms of analyzing the IP starting from the Helmholtz conditions Douglas constructs a lot of alternants; essentially the idea is to construct compatibility conditions starting from two sets of equations, cross-differentiating and then hoping that the highest order terms will cancel out. A much more geometric interpretation of Douglas’ procedure is presented in [CSMBP]. Eventually one might hope that one can show that either a Hessian is singular or one obtains a “passive orthonomic” system in terms of Riquier theory [Dou] and one can claim existence of a Hessian and hence Lagrangian. The difficulty is that in practice these alternants become exceedingly complicated to write down and it is practically impossible to know if one is generating necessary or sufficient conditions. At that point we have to ask what is the point? Even the examples we do write down are pretty exotic from a physics or engineering point of view. In the future I would like to continue my investigation of the IP for linear connections when \( n = 3 \). One of the issues which I have not yet touched is if \( d = 4 \), in the terminology of Section (2.1), is
there always a Lagrangian? Many other such questions remain to be addressed but again most of them seem to be very difficult.
Chapter 2

Definition of the invariant $d$

2.1 The invariants $d$ and $d_0$

We shall define an integer $d$ to be the dimension of the module of solutions to the double hierarchy of the $\Phi$ and $\Psi$ conditions introduced in Chapter 1. We shall define a second integer $d_0$ to be the dimension of the module of variational Hessians. Clearly $d \geq d_0$ however, it is important to appreciate that strict inequality may occur; the reason is that many more algebraic conditions could restrain the possible variational Hessians. In fact it is precisely this issue which lies at the heart of the difficulty of trying to solve the inverse problem in general. To this day no one has succeeded in general in finding a set of algebraic conditions the solution of which determines whether a variational Hessian exists and determines the most general such Hessian. A related issue is that it is rarely possible to find the most general Lagrangian even if one has the most general variational Hessian: consider a free particle system for example.
2.2 Possible values of $d$

Clearly the integers $d$ and $d_0$ can vary between 0 and $\binom{n+1}{2}$. If $d = 0$ and hence $d_0 = 0$ there is no Lagrangian. A non trivial example of this case is given by

$$\ddot{x} = \dot{x}\dot{y}, \quad \ddot{y} = \dot{y}\dot{z}, \quad \ddot{z} = \dot{z}\dot{x}. \quad (2.1)$$

For this system, (1.4) imposes three, $\frac{1}{\Phi}$ one, $\frac{2}{\Phi}$ one and (1.9) imposes one condition on $g_{ij}$ thus implying that the sought after Hessian is zero.

The case $d = 0$ is to be regarded as the generic situation in that “almost all” systems of type (1.1) are not variational. Such a statement could be made precise by asserting that the space of variational systems form a variety in the space of all second order systems of type (1.1).

At the other extreme if $d = d_0 = \binom{n+1}{2}$, the so-called “case 1”, there is always a Lagrangian [AT]: the reason for the terminology “case 1” is that it generalizes the case of the same name for the special value $n = 2$. In fact case 1 occurs precisely when $\Phi = \lambda I$ for some function $\lambda$. If $d = 1$ and the Hessian is non-singular then the Lagrangian exists if and only if a certain one-form is closed and the Lagrangian is essentially unique [AT].

The following Lemma helps to understand the next level of maximality of $d$ after Case 1.

**Lemma 2.2.1.** If $\Phi \neq I$ then the maximal dimension solution of (1.4) is of dimension $1 + \binom{n}{2}$. Furthermore $\Phi$ is up to change of basis equivalent to one of the following two matrices where $\lambda \neq \mu$. 

10
\[
\begin{bmatrix}
\lambda & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda & 0 \\
0 & 0 & 0 & \cdots & 0 & \mu
\end{bmatrix},
\begin{bmatrix}
\lambda & 0 & 0 & \cdots & 0 & 1 \\
0 & \lambda & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda & 0 \\
0 & 0 & 0 & \cdots & 0 & \lambda
\end{bmatrix}
\]

**Proof.** We shall use induction. Observe first of all that for each of the two matrices above the solution space of (1.4) is \(1 + \binom{n}{2}\). Assume that the result holds in dimension \(n\) and consider \(\Phi\) as a given \((n + 1) \times (n + 1)\) matrix. We shall further assume in the first instance that \(\Phi\) has a real eigenvalue \(\alpha\). As such we may assume after change of basis that \(\Phi = \begin{bmatrix} \phi & x \\ 0 & \alpha \end{bmatrix}\) where \(\phi\) is \(n \times n\), \(x\) is a column \(n\)-vector and \(\alpha \in \mathbb{R}\).

Now let \(g\) be an \((n + 1) \times (n + 1)\) symmetric matrix that we write in block form as \(g = \begin{bmatrix} h & b \\ b^t & c \end{bmatrix}\) where \(h\) is \(n \times n\) symmetric, \(b\) is a column \(n\)-vector and \(c \in \mathbb{R}\). We find that \(g\Phi\) is given by \(\begin{bmatrix} h\phi & hx + ab \\ b^t\phi & b^tx + \alpha c \end{bmatrix}\). In order for \(g\Phi\) to be symmetric it remains to satisfy

\[
hx = (\phi^t - \alpha I)b. \quad (2.2)
\]

Now we consider some sub-cases. If \(\phi = \lambda I\) there are two possibilities according as \(\lambda = \alpha\) or \(\lambda \neq \alpha\). In the former case we must have that \(x \neq 0\) or else we have that \(\Phi = \alpha I\) and by change of basis we may reduce to the first kind of matrix given in the statement of the lemma. In the latter case we can reduce \(x\) to zero by change of basis and we obtain the second kind of matrix given in the statement of the lemma.

If \(\Phi \neq \lambda I\) we may assume by induction that \(\Phi\) is one of the two kinds of matrix given in the statement of the lemma. For the first kind, unless \(\lambda = \alpha\) then \(b_1 = b_2 = \cdots = b_{n-1} = 0\) and the dimension of the solution space to (1.4) is less than \(\binom{n+1}{2} + 1\).
However, in that case since \( h \) is block diagonal it must be that \( x \) has only its last entry non-zero; but then, since \( \lambda \neq \mu \), \( x \) can be removed entirely by change of basis and we have a matrix of the first kind given in the statement of the lemma after another change of basis that transposes \( e_n \) and \( e_{n+1} \).

Finally if \( \Phi \) is the second kind of matrix given in the statement of the lemma we must again have \( \lambda = \alpha \) in order for \( b_1 = b_2 = \ldots = b_n \) not to be determined. However, in that case it follows that only the first entry of \( x \) can be non-zero and then by change of basis \( \Phi \) can be reduced to a matrix of the second kind given in the statement of the lemma.

As a last remark it is easy to see that if \( \Phi \) has a complex eigenvalue then the dimension of the solution space to (1.4) is less than \( 1 + \binom{n}{2} \).

Thus according to the Lemma we cannot have \( 1 + \binom{n}{2} < d < \binom{n+1}{2} \). Notice that this condition is empty for \( n = 2 \) and so does not correspond to a special case in [Dou] which is concerned with the case \( n = 2 \).

**Lemma 2.2.2.** If \( d = 1 + \binom{n}{2} \) then \( \nabla \Phi = aI + b\Phi \) for some functions \( a, b \).

*Proof.* Assume first of all that in some basis \( \Phi = \begin{bmatrix} \lambda I_{n-1} & 0 \\ 0 & \mu \end{bmatrix} \). Then the most general symmetric matrix \( g \) that satisfies 1.4 is of the form \( g = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) where \( A \) is \((n-1) \times (n-1)\).

Now choose another matrix denoted by \( \nabla \Phi = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \) where \( \alpha \) is \((n-1) \times (n-1)\), \( \beta \) and \( \gamma \in \mathbb{R}^{n-1} \) and suppose that it too satisfies 1.4 relative to \( g \). Then we find that \( A\beta = b\gamma \) and \( A\alpha = \alpha' A \). Since \( A, b \) are arbitrary the first of these conditions implies that \( \beta = 0 \) and \( \gamma = 0 \) whereas the second that \( \alpha = cI_{n-1} \) for some \( c \in \mathbb{R} \). Thus \( \nabla \Phi \) has precisely the same block structure as does \( \Phi \). It follows that \( \nabla \Phi = aI + b\Phi \) for some functions \( a \) and \( b \).

The proof for the other Jordan normal form of \( \Phi \) is similar. \( \square \)

**Lemma 2.2.3.** If \( d = 1 + \binom{n}{2} \) then the conditions in the \( \Psi \)-hierarchy are identically satisfied by virtue of the \( \Phi \)-condition.
\textit{Proof.} If $\Phi$ is not a multiple of the identity then $d \leq 1 + \binom{n}{2}$. If the $\Psi$-condition imposes a new condition then we will have $d < 1 + \binom{n}{2}$.

If we consider the $\Phi$-condition only we see that it imposes at most $\binom{n}{2}$ conditions on the matrix $g$. It follows that the $\Phi$-condition itself is never enough to force $g$ to be singular. In principle we might expect that the $\nabla \Phi$-condition would impose another $\binom{n}{2}$ conditions on $g$. Of course there is no guarantee that all or even any of the conditions imposed by $\nabla \Phi$ are independent of the conditions imposed by $\nabla \Phi$. In the case $n = 2$ the $\Phi$-condition imposes at most a single condition and likewise for the higher order $\Phi$-conditions. It follows that it is only necessary to consider the first three terms of the $\Phi$-hierarchy. It is used in [Dou] to produce the fundamental $3 \times 3$ matrix. Of course for $n = 2$ the $\Psi$-conditions are vacuous.

\section{2.3 Systems with n-1 trivial equations}

We consider a system of ODE of the form

\begin{equation}
\ddot{x}^1 = 2f(x^i, \dot{x}^a), \quad \ddot{x}^a = 0 \quad (2 \leq a \leq n). \tag{2.3}
\end{equation}

We shall say that the system (2.3) that has $n - 1$ trivial equations. Clearly the $\Phi$-matrix for such a system is of rank one: it could be one of the two types that occur in Lemma 2.2.1 according as $\Phi^1_1$ is not or is zero assuming that $\Phi$ does not disappear completely. We shall investigate systems with $n - 1$ trivial equations systematically in Chapter 3.
2.4 Product systems

Another class of systems for which \( d = 1 + \binom{n}{2} \) can be described as follows. Take a system of type 1 in dimension \((n - 1)\) for which the coordinates are \((x^a, u^a)\) where \(1 \leq a \leq n - 1\) and add to it a single trivial geodesic equation

\[
\ddot{x}^n = 0.
\] (2.4)

Provided the system in dimension \((n - 1)\) is such that its \(\Phi\)-matrix is not zero the \(\Phi\)-matrix for the \(n\)-dimensional system will not be a multiple of the identity and yet clearly \(d = 1 + \binom{n}{2}\). However, this construction will not produce examples for which \(d = 1 + \binom{n}{2}\) in the case of a linear connection because the geodesic system belongs to case one if and only if the connection is flat which implies that the \(\Phi\)-matrix is zero.

2.5 Riemannian spaces of constant curvature

Among the simplest of all Riemannian metrics are the spaces of constant curvature. We discussed in Section 2.2 the so-called “case 1” where the matrix \(\Phi\) is a multiple of the identity. In the case of a linear connection it turns out that we are in case 1 if and only if the connection is flat. We discuss the inverse problem for linear connections in Chapter 4. It turns out that a space of constant curvature does not belong to case 1 unless it is of constant zero curvature. An important property of spaces of constant curvature is that the curvature tensor is parallel. It follows that the tensors \(\nabla \Phi\) and \(\nabla \Psi\) in the double hierarchy introduced in Chapter 1 are zero so it is only necessary to consider \(\Phi\) and \(\Psi\) themselves.

Spaces of constant curvature have been investigated in the context of the inverse problem in the case \(n = 2\) in [AT]. We refer to [Spi] for a discussion of spaces of constant curvature. Since our concerns are mainly of a local nature we shall quote
the following formulas: for a space of constant curvature $K$ there are local coordinates $(x^i)$ relative to which the components of the metric are given by

$$g_{ij} = \frac{\delta_{ij}}{F^2},$$

(2.5)

where

$$F = 1 + \frac{K}{4}(x^1)^2 + (x^2)^2 + ... (x^n)^2).$$

(2.6)

Furthermore the curvature tensor as type $(1,3)$ is given by

$$R^i_{jkl} = K(\delta^i_kg_{jl} - \delta^i_lg_{jk}).$$

(2.7)

Hence the only non-zero curvature components are given by

$$R^i_{jj} = -R^i_{jj} = \frac{K}{F^2}.$$ 

(2.8)

It turns out that $\Phi$ is given by

$$\Phi = \frac{K}{F^2} \begin{bmatrix}
(u^2)^2 + (u^3)^2 + ... + (u^n)^2 & -u^1u^2 & -u^1u^3 & ... & -u^1u^{n-1} & -u^1u^n \\
-u^1u^2 & (u^1)^2 + (u^3)^2 + ... + (u^{n-1})^2 + (u^n)^2 & -u^2u^3 & ... & -u^2u^{n-1} & -u^2u^n \\
... & ... & ... & ... & ... & ... \\
-u^1u^{n-1} & -u^2u^{n-1} & -u^3u^{n-1} & ... & * & -u^{n-1}u^n \\
-u^1u^n & -u^2u^n & -u^3u^n & ... & -u^{n-1}u^n & *
\end{bmatrix}.$$ 

(2.9)

Alternatively we may write

$$\Phi^i_j = \frac{K}{F^2} \left( \sum_{k=1}^n u^k u^j \delta^i_k - u^i u^j \right).$$

(2.10)

**Proposition 2.5.1.** The Jordan normal form of $\Phi$ as given by 2.9 is of the first type in Lemma 2.2.1 where $\lambda = \frac{K}{F^2} \sum_{k=1}^n u^k u^k$ and $\mu = 0$.

**Proof.** The matrix $\Phi$ is a linear combination of the identity and the rank one matrix
whose entries are \( u^i u^j \). The eigenvalues of the latter matrix are 0 with multiplicity \((n - 1)\) and \( \sum_{k=1}^{n} u^k u^k \) and it may be diagonalized since it is real symmetric. It follows that \( \Phi \) has eigenvalues \( \frac{K}{F^2} \sum_{k=1}^{n} u^k u^k \) with multiplicity \((n - 1)\) and 0 with multiplicity 1.

\[ \Phi \]

\[ \begin{bmatrix}
0 & 
\ldots &
0 & 
\ldots \\
\vdots &
\ddots &
\vdots \\
0 & 
\ldots &
(u^i)^2 &
\ldots &
-u^i u^j &
\ldots \\
\vdots &
\ddots &
\vdots \\
0 & 
\ldots &
-u^j u^i &
\ldots &
(u^i)^2 &
\ldots \\
\vdots &
\ddots &
\vdots \\
0 & 
\ldots &
0 & 
\ldots &
0
\end{bmatrix} \]

whose \((i,i),(i,j),(j,i),(j,j)\)-entries consist of \((u^i)^2, -u^i u^j, -u^j u^i, (u^i)^2\) where 1 \( \leq i < j \leq n\).

### 2.6 Lorentz metric

Consider the Lorentz metric

\[ 2dx dz + dy^2 + b(x, y, z) dz^2 \]  

where \( b(x, y, z) \) is a function of \((x, y, z)\). It geodesic equations are given by

\[ \ddot{x} = -2 b_x \dot{x} \dot{z} - 2 b_y \dot{y} \dot{z} - (2 b b_x + b_z) \dot{z}^2, \dot{y} = b_y \dot{z}^2, \dot{z} = b_x \dot{z}^2. \]
Generically for this geodesic system we find that $d = 2$. However, in the special case where $b$ is independent of $x$ the metric possesses a parallel null vector field. As such it turns out that $d = 4$.

**2.7 n=3**

Now we specialize our attention to the case $n = 3$. We can use the first two conditions in the $\Phi$ hierarchy to obtain the following $6 \times 6$ matrix denoted by $M$:

$$
\begin{bmatrix}
\phi_2^1 & -\phi_2^7 & 0 & \phi_3^2 - \phi_1^1 & -\phi_1^7 & \phi_2^3 \\
0 & \phi_3^3 & -\phi_3^7 & \phi_3^1 & \phi_3^2 - \phi_2^7 & -\phi_2^3 \\
-\phi_2^1 & 0 & \phi_2^7 & -\phi_2^3 & \phi_1^3 & \phi_1^7 - \phi_3^3 \\
\nabla \phi_2^1 & -\nabla \phi_2^7 & 0 & \nabla \phi_3^2 - \nabla \phi_1^1 & -\nabla \phi_1^7 & \nabla \phi_2^3 \\
0 & \nabla \phi_3^3 & -\nabla \phi_3^7 & \nabla \phi_3^1 & \nabla \phi_3^2 - \nabla \phi_2^7 & -\nabla \phi_2^3 \\
-\nabla \phi_1^1 & 0 & \nabla \phi_1^7 & -\nabla \phi_1^3 & \nabla \phi_1^2 & \nabla \phi_1^1 - \nabla \phi_3^3
\end{bmatrix} \quad (2.13)
$$

The $\Phi$ and $\nabla \Phi$ conditions can be paraphrased as stating the components of $g$, strung together as a vector in $\mathbb{R}^6$, must be in the null space of $M$. Immediately we get a necessary condition for a system to be variational, namely, $M$ must be singular.

According to the discussion in Section 2.2 the only possible values of $d$ are 0, 1, 2, 3, 4, 6. Furthermore the cases $d = 0, 1, 6$ are easily dealt with so we are left to focus on the values $d = 2, 3, 4$ as the difficult cases.
Chapter 3

Inverse problem for n-1 free particles

3.1 Phi-hierarchy

We start with a system of ODE of the form

\[\dddot{x}^i = 2f(x^i, \dot{x}^i), \quad \dddot{x}^i = 0 \quad (2 \leq i \leq n).\] (3.1)

We shall say that the system (3.1) that has \((n-1)\) trivial equations.

Let us consider the case \(n = 3\) and the following system:

\[\dddot{x} = a\dot{x}^2 + 2b\dot{x}\dot{y} + cy^2 + 2d\dot{y}\dot{z} + e\dot{z}^2 + 2f\dot{z}\dot{x}, \quad \dddot{y} = 0, \quad \dddot{z} = 0\] (3.2)

where \(a, b, c, d, e, f\) are functions of \(x, y, z\). Then it turns out for generic values of \(a, b, c, d, e, f\) invariants \(d\) and \(d_0\) introduced in Chapter 2 are three but there are certain special values where \(d = 4\). We shall see examples of this type later.

Let us return now to (3.1) and try to implement the Helmholtz conditions as explained in Chapter 2. As regards the solution to 1.4 there is the obvious solution consisting of an arbitrary symmetric lower right \((n-1) \times (n-1)\) block accounting
for \( \binom{n}{2} \) independent solutions. For the extra solution, consider the \( \Phi \) condition (1.4). In this case the \((i, j)\)-th entry of \( g\Phi \) is \( g_{i1}\Phi_j^1 \) and therefore

\[
g_{i1}\Phi_j^1 = g_{j1}\Phi_i^1 \tag{3.3}
\]

which entails that

\[
g_{i1} = \lambda \Phi_i^1 \tag{3.4}
\]

for some function \( \lambda \). The extra algebraic solution may be written as

\[
G = \lambda \begin{bmatrix}
\Phi_1^1 & \Phi_2^1 & \Phi_3^1 & \ldots & \Phi_n^1 \\
\Phi_2^1 & 0 & 0 & \ldots & 0 \\
\Phi_3^1 & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\Phi_n^1 & 0 & 0 & \ldots & 0
\end{bmatrix}.
\tag{3.5}
\]

If there is to be a Lagrangian for the system then this extra algebraic solution must remain non-zero and of course produce together with the lower right \( (n - 1) \times (n - 1) \) block a non-singular matrix.

We shall make the convention here that \( \Phi_j^1 \) will be denoted simply by \( \Phi_i \) since \( \Phi_j^i = 0 \) if \( i > 1 \). and likewise for the higher order \( \Phi \)'s in the \( \Phi \)-hierarchy. Consider next the \( \nabla \Phi \) equation. It is apparent that

\[
\nabla \Phi_j = \dot{\Phi}_j + \frac{\partial f}{\partial u} \Phi_1 - \frac{\partial f}{\partial u^1} \Phi_j. \tag{3.6}
\]

The \((i, j)\)-th entry of \( G\nabla \Phi \) is \( \Phi_i \nabla \Phi_j \) and therefore we must have that

\[
\Phi_i \nabla \Phi_j = \Phi_j \nabla \Phi_i \tag{3.7}
\]
which entails that

\[ \nabla \Phi = \nu \Phi \quad (3.8) \]

for some function \( \nu \). As result the entire \( \Phi \)-hierarchy is now satisfied. Condition (3.8) is a necessary condition for there to exist Lagrangian for (3.1).

### 3.2 Closure Conditions

In view of Section 3.1 we shall write the putative Hessian \( g_{ij} \) in the form

\[ g_{ij} = \lambda G_{ij} + h_{ij} \]

where \( \lambda \) is an unknown function and the first row and column of \( h_{ij} \) are zero. The closure conditions are

\[ \frac{\partial (\lambda \Phi_j)}{\partial u^k} = \frac{\partial (\lambda \Phi_k)}{\partial u^j} \quad (1 \leq j < k \leq n) \quad (3.9) \]

\[ \frac{\partial h_{ij}}{\partial u^k} = \frac{\partial h_{ik}}{\partial u^j} \quad (1 < j < k \leq n) \quad (3.10) \]

\[ \frac{\partial h_{ij}}{\partial u^1} = \frac{\partial (\lambda \Phi_i)}{\partial u^1} \quad (1 < i < j \leq n). \quad (3.11) \]

Suppose that the first of these closure conditions is satisfied. Then for some particular choice of \( \lambda \) shall write \( \lambda \Phi_i = \frac{\partial \psi}{\partial u^1 \partial u^i} \) for some function \( \psi \). The third closure condition (3.11) then gives

\[ \frac{\partial h_{ij}}{\partial u^1} = \frac{\partial^3 \psi}{\partial u^1 \partial u^i \partial u^j} \quad (3.12) \]

so that \( h_{ij} = \frac{\partial^2 \psi}{\partial u^1 \partial u^i} + \gamma_{ij} \) for \( (i, j \geq 2) \) where \( \gamma_{ij} \) are independent of \( u^1 \). Thus we may write

\[ g_{ij} = \frac{\partial^2 \psi}{\partial u^i \partial u^j} + \gamma_{ij}, \quad (1 \leq i, j \leq n) \quad (3.13) \]
The $\gamma_{ij}$ themselves form a Hessian in view of the second closure condition. Hence the $\gamma_{ij}$ may be “absorbed” into the $\frac{\partial^2 \psi}{\partial u_i \partial u_j}$ without changing the first row or column since the $\gamma_{ij}$ are independent of $u^1$ and the sum of Hessians is another Hessian. Hence we may assume without loss of generality that $\gamma_{ij} = 0$.

The one-form $\Phi = \Phi_i du^i$ must be integrable in the sense that it is exact after multiplication by a suitable integrating factor. We remark that the integrating factor $\lambda$ is very far from unique: in the first place it may be multiplied by a function of the $x^i$; secondly suppose that

$$\lambda \Phi = d\rho$$

where $\lambda$ is an integrating factor. Then also

$$\lambda H' \Phi = d(H(\rho))$$

where $H$ is a smooth function of $\rho$ with, nowhere vanishing derivative, so that $\lambda H'$ is also an integrating factor.

Note furthermore that the closure condition on the one-form $\Phi$ is empty for $n = 2$.

### 3.3 Time derivative conditions

The time derivative conditions for $g_{ij}$ can be broken down into two types:

$$\frac{d}{dt}(\lambda \Phi_i) + \lambda \Phi_1 \frac{\partial f}{\partial u^i} + \lambda \Phi_i \frac{\partial f}{\partial u^1} = 0, \quad (1 \leq i \leq n)$$

(3.16)

and

$$\frac{d}{dt}(\frac{\partial^2 \psi}{\partial u_i \partial u_j}) + \lambda \Phi_i f_{u^j} + \lambda \Phi_j f_{u^i} = 0, \quad (2 \leq i \leq j \leq n).$$

(3.17)
Proposition 3.3.1. Assuming that (3.8) holds and $\Phi_i \Phi_j \neq 0$ then the $i$th equation is equivalent to the $j$th equation in (3.16).

Proof. Condition (3.8) entails that $\Phi_i \nabla \Phi_j = \Phi_j \nabla \Phi_i$. Now we have

$$\nabla \Phi_i = \frac{d}{dt}(\Phi_i) + f_{u^i} \Phi - f_{u^j} \Phi_i$$

and hence we have the identity

$$\Phi_j \frac{d}{dt}(\Phi_i) - \Phi_i \frac{d}{dt}(\Phi_j) + \Phi_1 (f_{u^i} \Phi_j - f_{u^j} \Phi_i) = 0. \quad (3.18)$$

On the other hand if we multiply the left hand side of (3.16) by $\Phi_j$, interchange $i$ and $j$ and subtract we find precisely the equation above and hence the conclusion follows.

As regards (3.17) whatever value we find for $\lambda$ by solving the closure conditions, we may integrate: upon integration there will enter $\binom{n}{2}$ first integrals that are independent of $u^1$. In other words we will obtain the Hessian of a Lagrangian of the $(n - 1)$ free particle system that we know is present in (3.1) from the outset. The problem with (3.16) is that since $\lambda$ has, with a certain ambiguity, been determined from the closure conditions, there is no reason in general that (3.16) should be valid.

If $\Phi_i = 0$ where $i > 1$ then (3.16) implies that $\Phi_1 f_{u^i} = 0$. If $\Phi_1 \neq 0$ then $f_{u^i} = 0$. However, in that case $f_{x_i} = 0$. Thus our ODE system becomes indecomposable in the sense that it is a product of a system in $(n - 1)$ dimensions and a one-dimensional system. The inverse problem can then be studied for the $(n - 1)$ dimensional system and in the affirmative case we can obtain a Lagrangian for the $n$-dimensional system. Hence if $\Phi_1 \neq 0$ we shall assume that $\Phi_i \neq 0$ if $i > 1$.

If $\Phi_1 \neq 0$ then by integration we find that $\lambda \Phi_1 = Ae^F$ where $F$ satisfies $\dot{F} = -2f_{u^1}$ and $A$ is an arbitrary first integral of (3.1). On the other hand if $\Phi_1 = 0$ then $\Phi_i \neq 0$
for some $1 < i \leq n$ or else the Lagrangian that we are seeking will be degenerate; if
$\Phi_i \neq 0$ then $\lambda \Phi_i = Ae^{F}$ and again $A$ is an arbitrary first integral of (3.1).

It remains finally to satisfy (3.17) which may be rewritten using the following
formula

$$
\left[ \frac{\partial}{\partial u^i}, \frac{d}{dt} \right] = \frac{\partial}{\partial x^j} + 2f_{u^j} \frac{\partial}{\partial u^1}.
$$

(3.19)

Hence we obtain

$$
\frac{\partial}{\partial u^i} \left( \frac{d}{dt} \left( \frac{\partial \psi}{\partial u^i} \right) - \frac{\partial \psi}{\partial x^j} \right) + \psi_{x^j u^i} - \psi_{x^i u^j} + f_{u^j} \psi_{u^1} - f_{u^i} \psi_{u^1 u^j} = 0.
$$

(3.20)

In (3.20) the first group of terms are the derivatives of the Euler-Lagrange operator
applied to $\psi$. Thus we have the following necessary conditions

$$
\psi_{x^j u^i} - \psi_{x^i u^j} + f_{u^j} \psi_{u^1} - f_{u^i} \psi_{u^1 u^j} = 0.
$$

(3.21)

**Proposition 3.3.2.** If $\Phi_1 = 0$ then $\frac{d}{dt} \left( \frac{\Phi_j}{\Phi_1} \right) = 0$ if $\Phi_i \neq 0$ is a necessary condition for
there to exist a Lagrangian in the $n - 1$ trivial equations case.

*Proof.* From (3.7)

$$
\Phi_i \nabla \Phi_j - \Phi_j \nabla \Phi_i = 0.
$$

Using (3.6)

$$
\Phi_i \left( \frac{d}{dt} \Phi_j + \frac{\partial f}{\partial u^j} \Phi_1 - \frac{\partial f}{\partial u^1} \Phi_j \right) - \Phi_j \left( \frac{d}{dt} \Phi_i + \frac{\partial f}{\partial u^i} \Phi_1 - \frac{\partial f}{\partial u^1} \Phi_i \right) = 0.
$$

If $\Phi_1 = 0$ then

$$
\Phi_i \frac{d}{dt} \Phi_j - \Phi_j \frac{d}{dt} \Phi_i = 0.
$$
and hence
\[
\frac{d}{dt}(\frac{\Phi_j}{\Phi_i}) = 0.
\]

Thus in the case $\Phi_1 = 0$ the $\frac{\Phi_j}{\Phi_i}$ are first integrals whenever $\Phi_i \neq 0$.

### 3.4 Systems for which $\Phi$ is rank one nilpotent

In this section we focus on systems with $n - 1$ trivial equations for which $\Phi$ is rank one nilpotent. In this case we go back to the Euler-Lagrange expressions themselves. We note that $\Phi$ is rank one nilpotent iff $\Phi_1 = 0$. Since, if there is to be a regular Lagrangian, $g_{11}$ must be a non-zero multiple of $\Phi_1$ it follows that $\Phi$ is rank one nilpotent and variational only if $L$ is linear in $u^1$. Thus we shall write the Lagrangian in the form
\[
L = u^1 F_{x^1} + H(x^i, u^a)
\]
(3.22)

where $H(x^i, u^a)$ is a smooth function of $(x^i, u^a)$.

Now we go back to the Euler-Lagrange expressions themselves. Since $g_{11} = 0$ and $f_a = 0$ we find that $L$ must satisfy
\[
L_{x^1} - u^k L_{x^k u^1} = 0.
\]
(3.23)

Substituting 3.22 into 3.23 we find that $H_{x^1} = u^a F_{x^1 x^a}$ so that $H = u^a F_{x^a} + K(x^a, u^a)$ where $K(x^a, u^a)$ is a smooth function of $(x^a, u^a)$. Thus we have
\[
L = u^1 F_{x^1} + K(x^a, u^a).
\]
(3.24)

If we compute the Euler-Lagrange equations we find that the first one is satisfied and
the remaining \((n-1)\) are given by

\[
\frac{d}{dt}(u^i F_{x^i u^a} + K_{u^a}) = K_{x^a}. \tag{3.25}
\]

### 3.5 Systems with trivial equations and right hand side independent of \(u^i\)

**Theorem 3.5.1.** For the system

\[
\dot{u} = f(x,y,z), \quad \dot{v} = 0, \quad \dot{w} = 0 \tag{3.26}
\]

there is no regular Lagrangian unless \(f\) depends only on \(x\).

**Proof.** Since the right hand side of (3.26) is independent of \((u,v,w)\), we have \(\Phi_1^1 = -f_x, \Phi_2^1 = -f_y, \Phi_3^1 = -f_z\) and \(\nabla \Phi = \frac{d}{dt} \Phi\). The theory developed in Section 3.1 applies to system (3.26). From (3.8) we have

\[
\frac{d}{dt} f_x = \nu f_x, \quad \frac{d}{dt} f_y = \nu f_y, \quad \frac{d}{dt} f_z = \nu f_z. \tag{3.27}
\]

Eliminate \(\nu\) between the \(xy, yz\) and \(zx\) equations in 3.27, we get

\[
\begin{align*}
    f_x \frac{d}{dt} f_y &= f_y \frac{d}{dt} f_x, \\
    f_y \frac{d}{dt} f_z &= f_z \frac{d}{dt} f_y, \\
    f_z \frac{d}{dt} f_x &= f_x \frac{d}{dt} f_z.
\end{align*}
\]

Above equations are linear in \(u,v,w\), if we compare corresponding coefficients in each equation we get three system of partial differential equations (PDE) each system
containing three equation. For example, the system coming from first equation is

\[
\begin{align*}
  f_y f_{xx} - f_x f_{yx} &= 0, \\
  f_y f_{xy} - f_x f_{yy} &= 0, \\
  f_y f_{xz} - f_x f_{yz} &= 0.
\end{align*}
\]

The solution to the first system of PDE is \( f = f(x, z) \). The solution to the second system of PDE is \( f = f(x, y) \). The solution to the third system of PDE is \( f = f(x, y) \).

Hence the final solution is \( f = f(x) \) and there is obviously a Lagrangian for the decoupled system.

\[\square\]

### 3.6 Examples

1. The system

\[
\begin{align*}
  \dot{u}^i &= (u^1)^2 + (u^2)^2 + \ldots + (u^n)^2, & \dot{u}^i &= 0 & (2 \leq i \leq n)
\end{align*}
\]

is derivable from the Lagrangian

\[
L = e^{-2x^1(\frac{}{\sqrt{(u^1)^2 + \ldots + (u^n)^2}})} + (u^2)^2 + \ldots + (u^n)^2.
\]

2. The system

\[
\begin{align*}
  \dot{u}^1 &= \frac{(1 + (u^1)^2 + (u^2)^2 + \ldots + (u^n)^2)}{x^1}, & \dot{u}^i &= 0 & (2 \leq i \leq n)
\end{align*}
\]

is derivable from the Lagrangian

\[
L = x^1\sqrt{1 + (u^1)^2 + (u^2)^2 + \ldots + (u^n)^2}.\]

This example appears in [Dou] in the case \( n = 2 \).

3. Consider the system

\[
\begin{align*}
  \dot{u} &= au^2 + 2buv + cv^2 + 2dvw + ew^2 + 2fwu, & \dot{v} &= 0, & \dot{w} &= 0
\end{align*}
\]
where $a, b, c, d, e, f$ are functions of $x, y, z$; in other words we have the geodesics of a linear connection with two trivial equations. It turns out that

$$
\Phi = ((b_y - c_x + ac - b^2)v^2 + (f_z - e_x - f^2 + ae)w^2 + (a_z - f_z)wu
$$

$$
+ (a_y - b_x)uv + (f_y - 2bf + b_z - 2d_x + 2ad)vw)du
$$

$$
+ ((b_x - a_y)u^2 + (c_x - b_y - ac + b^2)uw + (c_z - d_y + bd - cf)v)w
$$

$$
+ (-e_y + be - df)v^2 + (b_z - 2f_y + d_x - ad + bf)wu)dv
$$

$$
((f_x - a_z)u^2 + (d_x - 2b_z + f_y + bf - ad)uv + (d_y - c_z - bd + cf)v^2
$$

$$
+ (e_y - be + df)v w + (e_x - f_z + f^2 - ae)wu)dw
$$

and that $\Phi \wedge d\Phi = 0$, that is, $\Phi$ is automatically integrable.

4. The system

$$
\dot{u} = 2f, \dot{v} = 0
$$

where $f$ satisfies $\Phi_1 = -2f_x - f^2 + \frac{df}{dt} = 0$ appears in [Dou]. However, it is not of interest from the point of view of linear connections because

$$
\Phi_1 = uvR^1_{112} + v^2R^1_{212}
$$

(3.28)

so if $\Phi_1 = 0$ then $R^1_{112} = R^1_{212} = 0$ and since also $R^2_{112} = R^2_{212} = 0$ the connection would be flat. From the closure conditions we find that

$$
(\lambda \Phi_1)_v = (\lambda \Phi_2)_u.
$$

(3.29)

Since $\Phi_1 = 0$ it follows that $\lambda \Phi_2$ is independent of $u$. There are two time-derivative equations coming from the components of $\Phi$. Since $\Phi_1 = 0$ the first
of them is an identity. The second one is

\[
\frac{d}{dt}(\lambda \Phi_2) + (\lambda \Phi_2) f_u = 0. \tag{3.30}
\]

Since \( \lambda \Phi_2 \) is independent of \( u \) it follows that \( \frac{d}{dt}(\ln(\lambda \Phi_2)) \) is linear in \( u \) and hence

\[
f_{uuu} = 0 \tag{3.31}
\]

is a necessary condition for there to be a Lagrangian for the given system. Douglas uses (3.29) to distinguish between his two subcases IIb 1′ and IIb1″: the two sub-cases are in fact distinguished precisely according as (3.29) is or is not satisfied. Systems which belong to sub-case IIb1″ are not variational whereas systems in sub-cases IIb1′ apparently are.

Let us investigate this claim further. The most general integrating factor of (3.29) is of the form

\[
e^F(x,y)G(v). \]

It follows that a Lagrangian must be of the form

\[
L = e^{F(x,y)}uG(v) + H(x, y, v). \tag{3.32}
\]

Assuming that \( v \) is a first integral, the first Euler-Lagrange equation is

\[
\frac{d}{dt}(e^F G) = uF_x e^F + H_x \tag{3.33}
\]

and hence

\[
vGe^F F_y = H_x. \tag{3.34}
\]

Hence we may introduce a function \( K = K(x, y, v) \) such that

\[
vGe^F = K_x, \ H = K_y. \tag{3.35}
\]
and the Lagrangian becomes

\[ L = \frac{uK_x}{v} + K_y. \]  

(3.36)

We find that the Euler-Lagrange equations are given by

\[
\dot{u} = \frac{u^2(K_{xx} - vK_{xxy}) + 2uv(K_{xy} - vK_{xyv}) + v^2(K_{yy} - vK_{yyv})}{vK_{xx} - K_x}, \\
\dot{v} = 0.
\]  

(3.37)

However, the system in (3.37) is not an arbitrary one that has a trivial equation. Indeed the coefficients of \( u^2 \) and \( 2uv \) terms are \( -(\ln(vK_{xx} - K_x))_x \) and \( -(\ln(vK_{xx} - K_x))_y \) so that the \( y \)-derivative of the first is the \( x \)-derivative of second. Our calculation seems to contradict Douglas’ conclusion [Dou], specifically Theorem 4 on page 80. However, it is not easy to read Douglas’ paper and the issue shall have to be revisited at a later stage.

For the system (3.37) we find that \( \Phi_1 \) is given by

\[
\Phi_1 = \frac{1}{(vF_v - F)^2} \left( u(v^2(K_y - K_{xy})K_{xx} + (K_{xx} - K_x)K_{xyv}) \right. \\
+ v(K_xK_{xy} - K_yK_{xx}) \bigg) + v^2((K_x - K_{xx})K_{yy} + K_{xy}(K_{xy} - K_y))
\]  

(3.38)

Thus in order for \( \Phi_1 \) to vanish we must have that

\[
v^2(K_y - K_{xy})K_{xxv} + (K_{xx} - K_x)K_{xyv} + v(K_xK_{xy} - K_yK_{xx}) = 0, \\
(K_x - K_{xx})K_{yy} + K_{xy}(K_{xy} - K_y) = 0.
\]  

(3.39, 3.40)

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We found several solutions to this system using Maple. One of them is given by

\[ L = \frac{uF(x,v)}{v} + v^2. \] (3.41)

where \( F \) is a function of \((x,v)\) only. The Euler-Lagrange equations are given by

\[
\begin{align*}
\dot{u} &= -u^2(\ln(vF_v - F))_x, \\
\dot{v} &= 0.
\end{align*}
\] (3.42)

The matrix \( \Phi \) is given by

\[
\Phi = \begin{bmatrix} 0 & \Phi_2 \\ 0 & 0 \end{bmatrix}
\] (3.43)

where

\[
\Phi_2 = -\frac{u^3v}{2(vF_v - F)^3}((vF_v - F)^2F_{xxx} - 3(v^2F_vF_{xx} - vFF_{xx}) \right.
\]

\[\left.-vF_xF_v + FF_x)F_{xxv} - v(vF_v - F)F_{uv}F_{xxv} + (vF_vF_{xx}
\]

\[-FF_{xx} + 3v^2(F_{xx})^2 - 6vF_xF_{xxv} + 3(F_x)^2F_{vv}).
\]
Chapter 4

Inverse Problem for Linear Connections

4.1 Helmholtz conditions for connections

Let us explain next how the general theory simplifies for the case of the geodesic equations associated to a linear connection. In the connection case the given system of ODE is

$$\ddot{x}^i = -\Gamma^i_{jk} u^j u^k \quad (4.1)$$

and the matrix $\Phi$ is of the form

$$\Phi^i_j = R^i_{kjl} u^k u^l \quad (4.2)$$

where $R^i_{kjl}$ are the components of the curvature tensor of the linear connection relative to a coordinate system $(x^i)$ is defined by

$$R^i_{hjk} = \frac{\partial \Gamma^i_{hj}}{\partial x^k} - \frac{\partial \Gamma^i_{hk}}{\partial x^j} + \Gamma^i_{mk} \Gamma^m_{hj} - \Gamma^i_{mj} \Gamma^m_{hk} \quad (4.3)$$
and the Ricci tensor by

\[ R_{ij} = R^k_{ijk}. \] (4.4)

The higher order \( \Phi \)-tensors in this case just correspond to covariant derivatives of the curvature tensor, for example,

\[ \nabla \Phi^i_j = R^i_{kjl} u^k u^l u^m. \] (4.5)

In particular if \( R \) is parallel then all the higher order \( \Phi \)-tensors vanish. Notice that in the connection case \( \Phi \) is singular because \( \Phi^i_j u^j = R^i_{kjl} u^k u^l u^j = 0 \) since \( R^i_{kjl} \) is skew-symmetric in \( j, l \).

Also for the case of a linear connection, one finds that

\[ \Psi^i_{jk} = R^i_{jkl} u^l. \] (4.6)

and again the higher order \( \Psi \)'s correspond to covariant derivatives of \( R \). Thus, for example,

\[ \nabla \Psi^i_{jk} = R^i_{jkl} u^l u^m. \] (4.7)

Again if \( R \) is parallel the higher order \( \Psi \)-tensors vanish.

The condition coming from (1.4) is

\[ (g_{mi} R^i_{pqj} - g_{ji} R^i_{pmq}) u^p u^q = 0, \] (4.8)

while the condition coming from (1.9) is

\[ (g_{mi} R^i_{pqj} + g_{qi} R^i_{pmj} + g_{ji} R^i_{pqm}) u^p = 0. \] (4.9)

If we contract \( u^q \) into (4.9) we find from (4.8) that
Thus, for the special case of a linear connection with parallel curvature tensor, we can use (4.9) and (4.10) as the first and only algebraic conditions in the double hierarchy.

Lemma 4.1.1. In the connection case in dimension three \( d \) cannot be one if there is a Lagrangian.

Proof. If a Lagrangian exists then \( g_{ij} \) is non-singular and there is a non-singular solution of \( \Phi \)-hierarchy . Our goal is to find one extra solution which is linearly independent to previous solution so that \( d \) is atleast two. Define

\[
G_{ij} = g_{ip}g_{jq}u^p u^q
\]  

Then

\[
G_{im} \Phi^m_j = G_{im} R^m_{kjl} u^k u^l
\]

\[
= g_{ip} g_{mq} R^m_{kjl} u^k u^l u^p u^q
\]

\[
= g_{ip} g_{mq} R^m_{kql} u^l u^p u^q
\]

Notice that last term is symmetric and skew-symmetric in \( q \) and \( l \) at the same time and therefore is zero. Therefore \( G_{ij} \) is solution of \( g \Phi \) equation. By the exact same argument and noticing \( \nabla \Phi^j = R^i_{hjk|m} u^h u^k u^m \) we can show that \( G_{ij} \) is solution of \( \Phi \)
hierarchy. Thus

\[ G_{im} \Psi_{jk}^m = G_{im} R_{hjk}^m u^h \]

\[ = g_{ip} g_{mq} R_{hjk}^m u^h u^p u^q \]

\[ = g_{ip} g_{mq} R_{hjk}^m u^h u^q u^p \]

\[ = 0 \]

if we contract \( u^k \) and make use of the \( g \Phi \) equation. Thus \( G_{m[i} \Psi_{jk]}^m = 0 \) and hence \( G_{ij} \) is solution of the \( g \Psi \) equation. Similarly we find that \( G_{ij} \) is solution of the \( (g \nabla \Psi) \) equation. The rank of \( G_{ij} \) is one but the rank of \( g_{ij} \) is three, so these two solution are linearly independent and hence \( d \geq 2 \).

Thus the only difficult and interesting cases when \( n = 3 \) are \( d = 2, 3, 4 \).

**Lemma 4.1.2.** If \( d = 1 + \binom{n}{2} \) in the connection case then \( \nabla \Phi = b \Phi \) for some function \( b \).

*Proof.* According to Lemma (2.2.2) we know that \( \nabla \Phi = a I + b \Phi \) for some functions \( a, b \). Since \( (u^i) \) is a common eigenvector for \( \Phi \) and \( \nabla \Phi \) it follows that \( a = 0 \).

4.2 \( d=4 \)

Let us begin the investigation of the case \( d = 4 \).

**Lemma 4.2.1.** In the connection case \( d = 4 \) only if the Jordan Normal form of \( \Phi \) is one of the following three forms:

\[
\begin{bmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

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Proof. Follows immediately from Lemma (2.2.1) and the fact Φ must be singular. □

We can display the components of the curvature by means of the following matrices:

\[
\begin{bmatrix}
R_{112}^1 & R_{212}^1 & R_{312}^1 \\
R_{112}^2 & R_{212}^2 & R_{312}^2 \\
R_{112}^3 & R_{212}^3 & R_{312}^3
\end{bmatrix}, \quad
\begin{bmatrix}
R_{123}^1 & R_{223}^1 & R_{323}^1 \\
R_{123}^2 & R_{223}^2 & R_{323}^2 \\
R_{123}^3 & R_{223}^3 & R_{323}^3
\end{bmatrix}, \quad
\begin{bmatrix}
R_{131}^1 & R_{231}^1 & R_{331}^1 \\
R_{131}^2 & R_{231}^2 & R_{331}^2 \\
R_{131}^3 & R_{231}^3 & R_{331}^3
\end{bmatrix}.
\]

The curvature components satisfy the following three Bianchi identities:

\[
R_{123}^1 + R_{312}^1 + R_{231}^1 = R_{123}^2 + R_{312}^2 + R_{231}^2 = R_{123}^3 + R_{312}^3 + R_{231}^3 = 0.
\]

Suppose that Φ has either of the first two forms given in Lemma (4.2.1). Then the matrix \( g \) must be of the form

\[
\begin{bmatrix}
\alpha & \beta & 0 \\
\beta & \gamma & 0 \\
0 & 0 & \delta
\end{bmatrix}.
\]

Since \( d = 4 \) the Ψ-condition (1.9) gives

\[
\alpha \Psi_{23} + \beta (\Psi_{23}^2 + \Psi_{31}^1) + \gamma \Psi_{31}^2 + \delta \Psi_{12}^3 = 0.
\]

It follows that we obtain the following conditions on the curvature components:

\[
R_{123}^1 = R_{223}^1 = R_{323}^1 = R_{131}^2 = R_{231}^2 = R_{331}^2 = R_{112}^3 = R_{212}^3 = R_{312}^3 = 0
\]

\[
R_{123}^2 + R_{131}^1 = R_{223}^1 + R_{231}^1 = R_{323}^1 + R_{331}^1 = 0.
\]
Now we impose the conditions (4.13), (4.15), (4.16) and (4.17) on the curvature components and solve so as to obtain where $a, b, c, d, e, f, g, h, j, k, m, n \in \mathbb{R}$.

\[
\begin{bmatrix}
e & f & -a \\
g & h & -b \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
b & -a & -d \\
c & j & k
\end{bmatrix},
\begin{bmatrix}
-b & a & d \\
m & -c & n
\end{bmatrix}.
\] (4.18)

As such we find that $\Phi$ is given by

\[
\begin{bmatrix}
euv + fv^2 - 2aw + buw - dw^2 \\
(aw - ea - fe)v \\
(cv - mu - nw)w
\end{bmatrix},
\begin{bmatrix}
(aw - ea - fe)u \\
(aw - ba + dw)u \\
(aw - ba + dw)v
\end{bmatrix}.
\] (4.19)

The eigenvalues are of (4.19) are

\[
0, fv^2 - dw^2 - gu^2 + (e - h)uv + 2bwu - 2avw,
\]

\[
mu^2 - dw^2 - jv^2 - 2cuv + (b + n)w - (a + k)vw.
\] (4.20)

In the first case of Lemma (4.2.1) we find that $m = -g, j = -f, k = a, h = 2c + e, n = b$ and so the curvature matrices are given by

\[
\begin{bmatrix}
e & f & -a \\
g & 2c + e & -b \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
b & -a & -d \\
c & -f & a
\end{bmatrix},
\begin{bmatrix}
-b & a & d \\
-m & -c & n
\end{bmatrix}.
\] (4.21)

In the second case of Lemma (4.2.1) we find that $d = 0, m = 0, j = 0, c = 0, n = -b, k = -a$ and so the curvature matrices are given by

\[
\begin{bmatrix}
e & f & -a \\
g & h & -b \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
b & -a & 0 \\
0 & 0 & -a
\end{bmatrix},
\begin{bmatrix}
-b & a & 0 \\
0 & 0 & -b
\end{bmatrix}.
\] (4.22)
Then the matrix $g$ must be of the form
\[
\begin{bmatrix}
0 & 0 & \gamma \\
0 & \delta & \epsilon \\
\gamma & \epsilon & \phi
\end{bmatrix}.
\] (4.23)

Since $d = 4$ the $\Psi$-condition (1.9) gives
\[
\alpha \Psi_{23}^1 + \beta (\Psi_{23}^2 + \Psi_{31}^1) + \gamma \Psi_{31}^2 + \delta \Psi_{12}^3 = 0.
\] (4.24)

We obtain the following conditions on the curvature components:
\[
R_{131}^2 = R_{231}^2 = R_{331}^2 = R_{112}^3 = R_{212}^3 = R_{312}^3 = 0 \quad (4.25)
\]
\[
R_{112}^2 + R_{131}^2 = R_{212}^3 + R_{231}^3 = R_{312}^3 + R_{331}^3 = 0 \quad (4.26)
\]
\[
R_{112}^1 + R_{123}^3 = R_{212}^1 + R_{223}^3 = R_{312}^1 + R_{332}^3 = 0. \quad (4.27)
\]

When we impose the conditions (4.13), (4.25), (4.26) and (4.27) on the curvature components we obtain
\[
\begin{bmatrix}
a & b & c \\
d & -a & e \\
0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
-(c + f) & g & h \\
-e & i & j \\
-a & -b & -c
\end{bmatrix}, \quad \begin{bmatrix}
k & f & m \\
0 & 0 & 0
\end{bmatrix}.
\] (4.28)

As such we find that $\Phi$ is given by
\[
\begin{bmatrix}
aw + bv^2 + (c - f)uw - kwv - mw^2 & -aw^2 - buv - (2c + f)uw + gvw + hw^2 & k w^2 + (c + 2f)uw + mw^2 - gw^2 - hwv \\
-(av - du - ew)v & -du^2 + auv + iuv + j w^2 & (eu - i v - j w)v \\
-(av - du - ew)w & -(aa + bv + cw)w & -da^2 + 2aw - euw + be^2 + cew
\end{bmatrix}.
\] (4.29)
The eigenvalues are of (4.29) are

\[ 0, jw^2 - du^2 + bv^2 + (c + i)vw - 2euw + 2auv, \]  
\[ mw^2 - du^2 + bv^2 + (c - f)vw - (e + k)uw + 2auv. \]  

In order for all the eigenvalues to be zero we must have that \( e = 0, f = c, b = 0, d = 0, j = 0, a = 0, m = 0, i = -c, k = 0 \) and so the curvature matrices are given by

\[
E_1 = \begin{bmatrix}
0 & 0 & c \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
E_2 = \begin{bmatrix}
-2c & g & h \\
0 & -c & 0 \\
0 & 0 & -c
\end{bmatrix},
E_3 = \begin{bmatrix}
0 & c & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]  

If we consider (4.31) since we are assuming that \( d = 4 \) we see that the covariant derivatives of the curvature components \( R_{ijklmn} \) must satisfy similar conditions as in (4.31): they will be linear combinations of the matrices \( E_1, E_2, E_3 \). Hence the Lie algebra of the holonomy group of the connection is a subalgebra of a solvable three-dimensional Lie algebra that has basis \( \{E_1, E_2, E_3\} \) and non-zero brackets of the form:

\[
[E_1, E_2] = 2cE_1 - cE_3, [E_2, E_3] = -cE_3, (c \in \mathbb{R}).
\]  

Proposition 4.2.1. A recurrent one-form is integrable relative to a symmetric connection.

Proof. Another way to express that the one-form \( \alpha \) is integrable is to say that there
is a second one-form $\theta$ such that

$$\nabla_X \alpha = \langle X, \theta \rangle \alpha. \quad (4.33)$$

If $Y$ is another vector field we have

$$X \langle Y, \alpha \rangle = \langle \nabla_X Y, \alpha \rangle + \langle X, \theta \rangle \langle Y, \alpha \rangle. \quad (4.34)$$

If we interchange $X$ and $Y$ and use the symmetry of the connection we find that

$$d\alpha(X, Y) = \langle X, \theta \rangle \langle Y, \alpha \rangle - \langle Y, \theta \rangle \langle X, \alpha \rangle, \quad (4.35)$$

in other words $d\alpha = \theta \wedge \alpha$.

So coming back to (4.31) we can assert that the one-forms $\alpha$ and $\beta$ are integrable so that locally, by the Frobenius theorem, we may assume that $\alpha = dy$ and $\beta = dz$.

Notice that if we extend $(y, z)$ to a coordinate system $(x, y, z)$ then the vector field is automatically recurrent since it is in the common kernel of $\alpha$ and $\beta$.

At the level of the geodesics we find that they are of the form

$$\dot{u} = au^2 + 2bu + cv^2 + 2dw + ew^2 + 2fwu, \quad \dot{v} = gv^2, \quad \dot{w} = hw^2 \quad (4.36)$$

where $a, b, c, d, e, f, g, h$ are functions of $x, y, z$.

If we now calculate $\Phi$ we find that it is of the form

$$\left[ \begin{array}{ccc} \Phi_1^1 & \Phi_1^2 & \Phi_1^3 \\ \alpha v^2 & -v(\alpha u + \beta w) & \beta v^2 \\ \delta w^2 & \gamma w^2 & -w(\delta u + \gamma v) \end{array} \right] \quad (4.37)$$

where $\alpha = R_{212}^2 = -g_x, \beta = -R_{223}^2 = -g_z, \gamma = R_{323}^3 = -h_y, \delta = -R_{331}^3 = -h_x$.

However, since $\alpha, \beta, \gamma, \delta$ are functions of $x, y, z$ only, we see that the second and third
rows in (4.37) can be proportional if and only if one of them, say the the third row, is zero. In this case \( h \) is a function of \( y \) and \( z \) only: we may make a coordinate change so that the equations in (4.44) become trivial. Thus:

**Proposition 4.2.2.** If a three-dimensional system of geodesics is such that its \( \Phi \)-matrix is rank one nilpotent and in the form given in Lemma 4.2.1 then the system is of the form 4.44 in which also it may be assumed that \( h = 0 \).

We will close this Section by remarking that in practice it is virtually impossible to put \( \Phi \) into Jordan form using a transformation that is linear in the velocities \( u, v, w \). Speaking more abstractly, such a transformation preserves the tangent bundle of the space whose local coordinates are \((x, y, z)\). However, in order to put \( \Phi \) into Jordan form it will be necessary to use transformations that are not lifted from the basis. As such the curvature components are no longer well defined.

### 4.3 Further investigation of the \( \Phi \) rank one nilpotent case

**Lemma 4.3.1.** Suppose that \( \Phi \) has the third form given in Lemma 4.2.1. Then if \( d = 4 \) it may assumed that the coefficient matrices of \( v^2 \) and \( w^2 \) in \( \Phi \) are zero.

**Proof.** The matrix \( \Phi \) is a linear combination of six matrices with coefficients homogeneous quadratic in \( u, v, w \). If \( \Phi \) is rank one nilpotent so too are these six matrices. The three matrices whose coefficients are \( u^2, v^2, w^2 \) each have a column of zeroes: if each of them is rank one nilpotent we may assume that they are of the form

\[
\begin{bmatrix}
0 & ak & a \\
0 & -ck & -c^2k \\
0 & c & ck
\end{bmatrix},
\begin{bmatrix}
dm & 0 & dm^2 \\
e & 0 & em \\
-d & 0 & -dm
\end{bmatrix},
\begin{bmatrix}
fp & fp^2 & 0 \\
-f & -fp & 0 \\
g & gp & -b
\end{bmatrix}.
\]  
(4.38)
where $a, c, d, e, f, g, k, m, p \in \mathbb{R}$. Now consider the first three rows of (2.13) and use the same pattern for each of the three rank one nilpotent matrices in (4.38). We obtain the following $9 \times 6$ matrix

$$
\begin{bmatrix}
  a & 0 & 0 & -ck & 0 & c \\
  0 & -ck^2 & -c & ak & 2ck & -a \\
  -ak & 0 & 0 & ck^2 & 0 & -ck \\
  0 & -e & 0 & -dm & d & 0 \\
  0 & em & 0 & dm^2 & -dm & 0 \\
  -d^2m & 0 & -d & -em & e & 2dm \\
  fp^2 & f & 0 & -2fp & -g & gp \\
  0 & 0 & -gp & 0 & fp & -fp^2 \\
  0 & 0 & g & 0 & -f & fp
\end{bmatrix}.
$$

(4.39)

If we are to have the invariant $d = 4$ then the (4.39) matrix must have rank two. We see that the first and third, fourth and fifth and eighth and ninth, respectively, rows are proportional. Hence we may discard rows three, five and eight as regards determining its rank. As such, examining the resulting $6 \times 6$ matrix and the disposition of zero entries, we see that (4.39) has rank two only if, without loss of generality,

$$
d = e = f = g = m = p = 0.
$$

(4.40)

Now we use these values in $\Phi$ and impose the remaining conditions that insure that $\Phi$ is rank one nilpotent. We obtain several cases. Here is one of them with the
following values for the curvature matrices

\[
R_{ij}^{12} = \begin{bmatrix}
0 & 0 & 0 \\
\lambda & 0 & 0 \\
\mu & 0 & 0 \\
\end{bmatrix},
R_{ij}^{23} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix},
R_{ij}^{31} = \begin{bmatrix}
0 & 0 & 0 \\
-\nu & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\] (4.41)

where \(\lambda^2 + \mu \nu = 0\). With the curvature matrices of the form (4.41) the matrix \(\Phi\) is given by

\[
\Phi = \begin{bmatrix}
0 & 0 & 0 \\
u(\lambda v + \mu w) & -\lambda u^2 & -\mu u^2 \\
u(\nu v - \lambda w) & -\nu u^2 & \lambda u^2 \\
\end{bmatrix}
\] (4.42)

and the a basis for the solution space to (1.4) is given by

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
\lambda & 0 & 0 \\
\lambda & 0 & 0 \\
\end{bmatrix},
\begin{bmatrix}
0 & \lambda & \mu \\
\lambda & 0 & 0 \\
\mu & 0 & 0 \\
\end{bmatrix},
\begin{bmatrix}
0 & 0 & -(\lambda v + \mu w) \\
0 & 0 & \lambda u \\
-(\lambda v + \mu w) & \lambda u & 2\mu u \\
\end{bmatrix}
\] (4.43)

With the solution space given by (4.43) we find that the \(\Psi\)-condition (1.9) is identically satisfied. If we consider the curvature matrices that appear in (4.39) we see that they span a one-dimensional Lie algebra; furthermore, the covariant derivatives of the curvature matrices must satisfy similar conditions given by (4.39). Hence the holonomy group of the connection is one-dimensional and because of their structure it must have a parallel one-form and two vector fields in its kernel. At the level of the geodesics we find that they are of the form

\[
\dot{u} = 0, \dot{v} = b(x, y, z)v^2, \dot{w} = c(x, y, z)w^2
\] (4.44)

where \(b, c\) are functions of \(x, y, z\). The condition \(\lambda^2 + \mu \nu = 0\) following (4.41) translates
We can satisfy this equation by setting \( b = a_z, c = -a_y \) and then we obtain a \( \Phi \) which is rank one nilpotent on substituting \( a \) and its derivatives into (4.42) and (4.45).

## 4.4 Normalization in the constant coefficient case

Consider an ODE system of the form

\[
\dot{u}_1 = \alpha u_1^2 + 2\beta^a u_1 u_a + \gamma^{bc} u_b u_c, \quad \dot{u}_a = 0,
\]

(4.46)

where as usual \( 2 \leq a, b, c \leq n \) and \( \dot{x}_1 = u_1, \dot{x}_a = u_a \) the summation convention applies and \( \alpha, \beta^a, \gamma^{bc} \) are constants.

Let us assume first of all that \( \alpha \neq 0 \). Then define

\[
\bar{x}_1 = \alpha x_1, \quad \bar{x}_a = x_a.
\]

(4.47)

If we drop the bars the effect is to reduce \( \alpha \) in (4.46) to unity. Next define

\[
\bar{x}_1 = x_1 + \beta^a x_a, \quad \bar{x}_a = x_a.
\]

(4.48)

The effect is to reduce the \( \beta_a \)'s in (4.46) to zero; the \( \gamma^{bc} \) also change but since they are arbitrary we shall continue to call them \( \gamma^{bc} \) and again we shall drop the bars. Finally we have complete latitude to make a linear change in the \( n - 1 \)-dimensional subspace spanned by the \( u_a \)'s. As such, the \( \gamma^{bc} \) transform as a quadratic form and may be diagonalized with diagonal entries \( \pm 1 \).

Now consider the case \( \alpha = 0 \). We shall consider just the case where \( n = 3 \). Then

\[
2buv + cv^2 + 2dvw + ew^2 + 2fwu = 2u(bv + fw) + cv^2 + 2dvw + ew^2
\]

may be replaced.
by $2uv + cv^2 + 2dvw + ew^2$ unless $b = f = 0$ in which case we easily reduce to the case $a = b = d = f = 0, c = \pm 1, e = \pm 1$. Otherwise $2uv + cv^2 + 2dvw + ew^2$ can be reduced to $2buv \pm v^2 \pm w^2$ by completing the squared term in $w$ and rescaling $v$. Now completing the squared term in $v$ leads to $-b^2 u^2 + (bu + v)^2 + w^2$ or $b^2 u^2 - (bu - v)^2 \pm w^2$ and replacing $bu \pm v$ by $v$ the expression is reduced to a sum of squares. We saw at the beginning of the argument how it could be assumed that if $\alpha \neq 0$ it can be reduced to unity. So in conclusion we can reduce the expression to $\epsilon u^2 \pm v^2 \pm w^2$ where $\epsilon = 0, 1$.

As regards the case $\epsilon = 0$ it produces a flat connection so the inverse problem is not interesting. If we consider the first kind of normalized system

$$\ddot{x} = x^2 + y^2 + \dot{z}^2, \quad \dot{y} = 0, \quad \ddot{z} = 0$$ (4.49)

and make the orthogonal change of variables

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix}$$ (4.50)

we obtain, on dropping the bars,

$$\ddot{x} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2, \quad \ddot{y} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2, \quad \ddot{z} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2.$$ (4.51)

A Lagrangian for (4.49) is

$$\mathcal{L} = \frac{e^{-2t}(\dot{x}^2 - \dot{y}^2 - \dot{z}^2)}{\sqrt{\dot{y}^2 + \dot{z}^2}} + \dot{y} \dot{z}$$ (4.52)
whereas a Lagrangian for (4.51) is

\[ \mathcal{L} = \frac{e^{-2(x+y+z)}[\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 4(\dot{x}\dot{y} + \dot{y}\dot{z} + \dot{z}\dot{x})]}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - \dot{x}\dot{y} - \dot{y}\dot{z} - \dot{z}\dot{x}}} + \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - \dot{x}\dot{y} - \dot{y}\dot{z} - \dot{z}\dot{x}. \] (4.53)

Incidentally this Lagrangian can be extended to \( n \) dimensions. A Lagrangian for the second kind of normalized system

\[ \ddot{x} = \dot{x}^2 - \dot{y}^2 - \dot{z}^2, \dot{y} = 0, \dot{z} = 0 \] (4.54)

is

\[ \mathcal{L} = \frac{e^{-2x}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{\sqrt{\dot{y}^2 + \dot{z}^2}} + \dot{y} \dot{z} \] (4.55)

But so far we have not been able to find a Lagrangian for the last kind of normalized system

\[ \ddot{x} = \dot{x}^2 + \dot{y}^2 - \dot{z}^2, \dot{y} = 0, \dot{z} = 0. \]

4.5 Examples of existence of Lagrangian for \( d=4 \)

Again we are considering the three-dimensional connection case with two trivial equations and for which \( d = 4 \) and hence for which a Lagrangian exists:

1. Constant connection coefficients: examples of system with two trivial equation with constant coefficient are (4.49) and (4.54)with Lagrangian (4.52) and (4.55) respectively.

2. Function coefficient but a specific function:

\[ \dot{x} = e^x(v^2 + w^2), \quad \dot{y} = 0, \quad \dot{z} = 0 \]
for which a Lagrangian is

\[ \mathcal{L} = \frac{u^2}{2\sqrt{v^2 + w^2}} + e^x\sqrt{v^2 + w^2} + (v^2 + w^2). \]

3. Arbitrary function coefficient:

\[ \ddot{x} = b'(x)vw, \quad \ddot{y} = 0, \quad \ddot{z} = 0 \]

for which a Lagrangian is

\[ \mathcal{L} = \frac{u^2}{2\sqrt{vw}} + b(x)\sqrt{vw} + vw. \]

4.6 \( \Phi \) nilpotent in the connection case

We now investigate the possibility that \( \Phi \) is rank one nilpotent in the connection case. From the previous section we know that there must be two trivial equations:

\[ \dot{u} = au^2 + 2buv + cv^2 + 2dvw + ew^2 + 2fwu, \quad \dot{v} = 0, \quad \dot{w} = 0. \]

We note that the curvature components are given by, again suppressing the upper index 1,

\[ R_{112} = a_y - b_x, \quad R_{212} = b_y - c_x + ac - b^2, \quad R_{231} = d_x - b_z + bf - ad, \]
\[ R_{123} = b_z - f_y, \quad R_{223} = c_z - d_y + bd - cf, \quad R_{312} = f_y - d_x + ad - bf, \]
\[ R_{131} = f_x - a_z, \quad R_{331} = e_x - f_z + f^2 - ae, \quad R_{323} = d_z - e_y + be - df, \]
and that the components of $\Phi$ are given by

$$
\Phi_1 = (a_y - b_x)uw + (a_z - f_x)uw + (b_y - c_x + ac - b^2)v^2 + 2(f_y - e_x + ad - bf)vw \\
+ (f_x - d_y + ae - f^2)w^2
$$

$$
\Phi_2 = (b_x - a_y)u^2 + (c_x - b_y + b^2 - ac)uv + (b_z - 2f_y + d_x + bf - ad)uw \\
+ (c_y - d_x + bd - cf)vw + (d_z - e_y + be - df)vw
$$

$$
\Phi_3 = (f_x - a_z)u^2 + (d_x - 2b_z + f_y + bf - ad)uv + (e_x - f_z + f^2 - ae)uw \\
+ (d_y - c_z + cf - bc)v^2 + (e_y - d_z + df - be)vw.
$$

Thus, if $\Phi$ is nilpotent, we have that $R_{112} = R_{113} = R_{212} = R_{313} = R_{312} - R_{231} = 0$.

Now the one-form $\Phi$ is given by $\Phi = w^2(vR_{223} + wR_{323} + \frac{3w}{2}R_{233})d(\frac{v}{w})$ where $d$ denotes the “fibre” exterior derivative, that is, relative to the coordinates $u, v, w$. Thus an integrating factor for $\Phi$ is $\frac{1}{w^2(vR_{223} + wR_{323} + \frac{3w}{2}R_{233})} F'(\frac{v}{w})$; it is not the most general integrating factor which is in fact $\frac{F'}{M(x,y,z)w^2(vR_{223} + wR_{323} + \frac{3w}{2}R_{233})}$ where $F$ is a smooth function of $\frac{v}{w}$ and $M(x,y,z)$ is a smooth function of $x, y, z$.

Now we resume from (3.29) and in view of the form of the integrating factor just described we can assert that the Lagrangian is of the form

$$
\mathcal{L} = (uM_x + vM_y + wM_z)F(\frac{v}{w}) + K(y, z, v, w)
$$

(4.56)

where $F, K, M$ are smooth functions of $\frac{v}{w}$ and $y, z, v, w$ and $x, y, z$, respectively.

We may assume that $F$ is not constant or else $\mathcal{L}$ is gauge-equivalent to a degenerate Lagrangian. We can analyze the Euler-Lagrange equations as follows: the first Euler-Lagrange equation tells us that $\frac{v}{w}$ is a first integral. As regards the second and third Euler-Lagrange equations, if we multiply the second by $\frac{v}{w}$ and add to the third we find that in order for the second and third Euler-Lagrange equations to be consistent
that
\[ \frac{v}{dt} \frac{dK_v}{dt} + w \frac{dK_w}{dt} = vK_y + wK_z. \]  
(4.57)

Using (4.57) and the fact that \( \frac{v}{w} \) is a first integral we can now solve for \( \dot{v} \) and \( \dot{w} \) to find that
\[
\dot{v} = v \frac{(vK_y + wK_z - v^2K_{yv} - vw(K_{yw} + K_{zw}) - w^2K_{zw})}{v^2K_{vv} + 2vwK_{vw} + w^2K_{ww}} \]  
(4.58)
\[
\dot{w} = w \frac{(vK_y + wK_z - v^2K_{yv} - vw(K_{yw} + K_{zw}) - w^2K_{zw})}{v^2K_{vv} + 2vwK_{vw} + w^2K_{ww}}. \]

Hence in order to have two trivial equations \( K \) must satisfy the PDE
\[
vK_y + wK_z - v^2K_{yv} - vw(K_{yw} + K_{zw}) - w^2K_{zw} = 0. \]  
(4.59)

The second Euler-Lagrange equation is
\[
\dot{u}M_x + u^2M_{xx} + 2uvM_{xy} + 2uwM_{xz} + v^2M_{yy} + 2vwM_{yz} + w^2M_{zz} + \frac{w(\frac{dK_w}{dt} - K_z)}{F'} = 0. \]  
(4.60)

In order to have the geodesics of a linear connection it is necessary that there exist functions \( A(x, y, z) \) and \( B(x, y, z) \) such that
\[
\frac{dK_v}{dt} - K_y = (Av + Bw)F'. \]  
(4.61)

but recall that \( F \) is a function of \( \frac{v}{w} \) only. Thus (4.57) and (4.61) are necessary and sufficient for the Euler-Lagrange equations to be the geodesics of a linear connection together with the understanding that \( F \) is a function of \( \frac{v}{w} \) only. Alternatively we can substitute (4.61) into (4.57) so as obtain
\[
w(\frac{dK_w}{dt} - K_z) = v(Av + Bw)F'. \]  
(4.62)
Then we assert that (4.61) and (4.62) are necessary and sufficient for Lagrangian (4.56) to engender the geodesics of a linear connection subject to the condition that $F$ is a function of $\frac{v}{w}$ only. Suppose then that (4.60) and two trivial equations are the indeed the geodesics of a linear connection. If $\frac{dK_v}{dt} - K_y = 0$, so that in (4.61) we have $A = B = 0$ then also from (4.62) we shall have $w(\frac{dK_w}{dt} - K_z) = 0$ so that $K$ would be a Lagrangian in $(y, z, v, w)$-space. Furthermore the connection in dimension three in that case turns out to be flat.

Assuming that (4.60) is homogeneous quadratic we can calculate the curvature of the associated linear connection. More generally since $K$ and $F$ depend only on $y, z, v, w$ we can simply add terms $Ev^2, 2Hvw, Gw^2$ into the terms (4.60) that depend only on $M$. The only non-zero curvature terms in such a connection are $R_{221}, R_{231}, R_{223}, R_{312}, R_{331}, R_{323}$, recalling that we suppress the upper index when there are $n - 1$, in this case two, trivial equations. Furthermore, we find that

$$R_{212} = \frac{EM_{xx}}{M_x}, R_{313} = \frac{GM_{xx}}{M_x}, R_{213} - R_{312} = 0.$$ (4.63)

Now for a general connection that has two trivial equations we have

$$\Phi_1 = uvR_{112} + uwR_{113} + R_{212}v^2 + vw(R_{213} + R_{312}) + w^2R_{313} = 0.$$ (4.64)

Thus, in order to have $\Phi$ nilpotent it is necessary that besides $R_{112}, R_{113}$ which are already zero, that $R_{212} = R_{213} + R_{312} = R_{313} = 0$. The only way to satisfy these conditions is to have $E = G = H = 0$, giving a flat connection, or else $M_{xx} = 0$.

Thus the question of whether we can find a connection where two of the geodesics are trivial and for which $\Phi$ is rank one nilpotent hangs on the delicate issue of whether we can find a functions $A, B, F, K$ such that (4.61) and (4.62) for which also $F$ is a function of $\frac{v}{w}$ only. We shall see some affirmative examples in the next Section.
4.7 \( \Phi \) nilpotent in the constant coefficient case

We know that \( \Phi^i_j = R^i_{kjl} u^k u^l \) and Ricci tensor \( R_{ij} = R^k_{ijk} \). In the connection case we know that \( \Phi^i_j \) is singular. If \( \Phi^i_j \) is nilpotent its trace is zero which is equivalent to the Ricci tensor being skew-symmetric. Apparently not much is known about connections whose Ricci tensor is skew-symmetric see however, [Der]. For \( n = 3 \) in order for \( \Phi^i_j \) to be nilpotent it is necessary and sufficient that as well as Ricci being skew-symmetric that the trace of \( \Phi^2 \) should be zero which gives \( R^i_{(kjl} R^p_{mni)j} = 0 \) where the round parentheses denote the symmetric part of the tensor.

**Theorem 4.7.1.** If \( \Phi \) is nilpotent in the connection case with two trivial equation and constant coefficients then the connection is flat.

*Proof.* In this case the only non-zero curvature components are

\[
\begin{align*}
R^1_{221} & = b^2 - ac, \\
R^1_{232} & = cf - bd, \\
R^1_{331} & = f^2 - ae,
\end{align*}
\]

\[
\begin{align*}
R^1_{231} & = bf - ad, \\
R^1_{321} & = bf - ad, \\
R^1_{332} & = df - be.
\end{align*}
\]

Because we have two trivial equations, the second and third rows of \( \Phi \) are zero. By hypothesis \( \Phi \) is nilpotent which gives \( \Phi^1_1 = 0 \) and hence

\[
(ac - b^2)v^2 + 2(ad - bf)vw + (ae - f^2)w^2 = 0.
\]

Each of the three coefficients in the above equation must be zero, which forces \( R^i_{kjl} \equiv 0 \). So the connection is flat. \( \square \)

If we have function coefficient then above remark is not true. A class of examples
of non flat connections in the case of two trivial equation are the following:

\[
\ddot{x} = c(y, z)\ddot{y}^2 + 2d(y, z)\dot{y}\dot{z} + e(y, z)\dot{z}^2 - \frac{2\dot{z}\dot{x}}{z}
\]

\[
\ddot{y} = 0
\]

\[
\ddot{z} = 0
\]

One of the simplest example is

\[
\ddot{x} = \dot{y}\dot{z} - \frac{2\dot{z}\dot{x}}{z}
\]

\[
\ddot{y} = 0
\]

\[
\ddot{z} = 0
\]

For this particular example

\[
\Phi = \begin{bmatrix}
0 & \dot{z}^2 & -\frac{\dot{z}\dot{x}}{z} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and the non-zero curvature component is \( R_{323}^1 = \frac{1}{2z} \).

### 4.8 Structure of curvature in the \( \Phi \) rank one nilpotent case

For the sake of simplicity, we rewrite the curvature components (4.12) in terms of the new notation

\[
\begin{bmatrix}
a & d & i \\
g & b & e \\
f & h & c
\end{bmatrix},
\begin{bmatrix}
C & F & H \\
I & A & D \\
E & G & B
\end{bmatrix},
\begin{bmatrix}
\beta & \epsilon & \sigma \\
\tau & \gamma & \psi \\
\delta & \rho & \alpha
\end{bmatrix}
\]

(4.65)
Lemma 4.8.1. In the connection case, if $\Phi$ is rank one nilpotent then the curvature matrices are also rank one.

Proof. In theory, there are 27 components of curvature. By using Bianchi identity

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0 \quad (4.66)$$

we could reduce it to 24 components. If $\Phi^i_j$ is nilpotent then Ricci tensor is skew-symmetric. Ricci is skew imposes 6 more linear conditions on $R^i_{klj}$, so in principle we can reduce the number of curvature component down to 18. Then by assumption we write second row of $\Phi$ is $\lambda$ times first and third row of $\Phi$ is $\mu$ times first row. Then pick independent equations (equating corresponding coeff. of $u, v, w$ in corresponding entries) and solve for curvature component in term of $H, F$ and $\epsilon$.

$$a = 2\lambda \mu F - \mu^2 H - 3\mu \epsilon$$
$$\beta = 2\lambda^2 F - \lambda \mu H - 3\lambda \epsilon$$
$$g = \lambda a = \mu \beta$$
$$b = \lambda G = \lambda \mu F$$
$$f = \mu a$$
$$A = \lambda F$$
$$B = \mu H$$
$$e = \lambda (\mu H - \lambda F + \epsilon)$$
$$E = \mu (\lambda F - \mu H - 2\epsilon)$$
$$\tau = \lambda \beta$$
$$\alpha = \lambda \mu H$$
$$\rho = \lambda^2 H.$$

With these solution the curvature matrices are given by:

$$\begin{bmatrix}
    a & \mu F & \epsilon - \lambda F + \mu H \\
    \lambda a & \lambda \mu F & \lambda (\epsilon - \lambda F + \mu H) \\
    \mu a & \mu^2 F & \mu (\epsilon - \lambda F + \mu H)
\end{bmatrix},
\begin{bmatrix}
    \lambda F - \mu H - 2\epsilon & F & H \\
    \lambda (\lambda F - \mu H - 2\epsilon) & \lambda F & \lambda H \\
    \mu (\lambda F - \mu H - 2\epsilon) & \mu F & \mu H
\end{bmatrix},
\begin{bmatrix}
    \beta & \epsilon & \lambda H \\
    \lambda \beta & \lambda \epsilon & \lambda^2 H \\
    \mu \beta & \mu \epsilon & \mu \lambda H
\end{bmatrix}.$$
4.9  $d=3$

Let us begin the investigation of case $d = 3$. This case can happen in one of three ways and applies generally to the inverse problem and not just for linear connections:

1. $\Phi$ imposes three conditions on $g$ and then $\Psi$ and $\nabla \Phi$ conditions are identities.

2. $\Phi$ imposes two conditions, $\Psi$ condition is an identity and then $\nabla \Phi$ imposes one new condition, $\nabla^2 \Psi$ condition is identity.

3. $\Phi$ imposes two conditions, $\Psi$ imposes one condition (it can only impose one) and then $\nabla \Phi, \nabla \Psi$ conditions are identities.

An example where $\Phi$ imposes three conditions on $g$ and the rest of the algebraic hierarchy are identities

$$\ddot{x} = (au + v)w, \ddot{y} = (av - u)w, \ddot{z} = 0 \quad (4.67)$$

and a Lagrangian for (4.67) is

$$\mathcal{L} = \frac{e^{-az}}{2w}[(v^2 - w^2) \cos z + 2uv \sin z] + f(w) \quad (4.68)$$

The geodesics (4.67) come from the canonical symmetric connection of a Lie group whose Lie algebra is numbered as 3.7 in [PSWZ]. It is defined as one-half the Lie bracket on either left or right-invariant vector fields and extended to arbitrary vector fields by using the Leibnitz rule. It turns out that the curvature tensor for such a connection is parallel so that as far as the inverse problem is concerned only the first
terms in the double hierarchy of algebraic conditions are of significance. For more details about the canonical symmetric connection in the context of the inverse problem, the reader is referred to [ST].

In (4.67), if $a = 0$ then $d = 4$ (Euclidean group of plane) and if $a \neq 0$ then $d = 3$ (Lie Group) and the $\Phi$-matrix is given by

$$
\Phi = \frac{1}{4} \begin{bmatrix}
1 - a^2 & -2a & 0 \\
2a & 1 - a^2 & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

Another example of system where $\Phi$ equation impose three conditions and the rest of the conditions in the algebraic hierarchies is identity is given by

$$
\ddot{x} = a(x)u^2, \ddot{y} = b(x)v^2, \ddot{z} = c(x)w^2.
$$

In (4.70), the basis of algebraic solution for $g$ is

$$
\begin{bmatrix}
0 & 0 & w \\
0 & 0 & 0 \\
w & 0 & -u
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & w & 0 \\
w & -u & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

Example of a system where $\Phi$ imposes two conditions, $\nabla \Phi$ imposes one new condition and rest of the algebraic hierarchy is identity

$$
\ddot{x} = vw, \ddot{y} = wu, \ddot{z} = uv.
$$
In 4.72, the basis of algebraic solution for \( g \) is

\[
\begin{bmatrix}
  v^2 & -uv & 0 \\
  -uv & u^2 & 0 \\
  0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
  0 & 0 & 0 \\
  0 & v^2 & -vw \\
  0 & -vw & v^2
\end{bmatrix}, \quad
\begin{bmatrix}
  w^2 & 0 & -wu \\
  0 & 0 & 0 \\
  -wu & 0 & u^2
\end{bmatrix}
\]

(4.73)

But solution (4.73) is singular, so there is no regular Lagrangian for the system (4.72).

### 4.10 \( d=2 \)

An example of existence of Lagrangian when \( d = 2 \) is

\[ \mathcal{L} = e^{f(x,y,z)}(u^2 + v^2 + w^2) \]  

(4.74)

which is a conformally flat metric. A class of examples when the geodesics are non-trivial and \( d = 2 \) is given by

\[
\ddot{x} = -\frac{f_x(x, y, z)u^2}{p}, \quad \ddot{y} = -\frac{f_y(x, y, z)v^2}{q}, \quad \ddot{z} = -\frac{f_z(x, y, z)w^2}{r}
\]

(4.75)

and a Lagrangian for (4.75) is

\[ \mathcal{L} = e^{f(x,y,z)}u^p v^q w^r. \]

(4.76)

Another example where \( d = 2 \) is the Lorentz metric (2.11) in Chapter 2 when \( b_x \neq 0 \). In that case \( \Phi \) imposes three conditions, \( \Psi \) none and \( \nabla \Phi \) one. If \( b_x = 0 \) then \( d = 2 \).

An example of non-existence of Lagrangians is given by

\[
\ddot{x} = uw, \quad \ddot{y} = uv, \quad \ddot{z} = 0.
\]

(4.77)
In (4.77) $\Phi$ imposes three conditions and $\nabla \Phi$ imposes one condition and the basis of algebraic solution space for $g$ is

$$
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
w & 0 & -u \\
0 & 0 & 0 \\
-u & 0 & 0 \\
\end{bmatrix}.
$$

(4.78)

However, solution (4.78) is singular, so there is no regular Lagrangian for the system (4.77). So the system is not variational. Another similar example of non-existence is given by

$$
\ddot{x} = uv, \quad \dot{y} = 0, \quad \ddot{z} = uw.
$$

(4.79)

### 4.11 Levi-Civita connections

A Levi-Civita connection is a symmetric linear connection which is engendered by a pseudo-Riemannian metric. By its very definition the geodesics of such a connection are variational with a Lagrangian $\mathcal{L}$ being provided by the standard kinetic energy Lagrangian. Any smooth function $F(\mathcal{L})$ is also a Lagrangian subject only to the non-degeneracy condition as is discussed in [AT]. In this section we consider several special classes of Levi-Civita connections.

We consider first of all Riemannian metrics that are (conformally flat), that is to say, the metric $g_{ij}$ is given by

$$
g_{ij} = e^{2h(x^i)} \delta_{ij}
$$

(4.80)

where $\delta_{ij}$ is the Euclidean metric and $h(x^i)$ is a function of the coordinates $(x^i)$. For
n = 3 the geodesic equations are given by

\[
\begin{align*}
\dot{u} &= -h_x u^2 - 2h_y uv - 2h_z uw + h_y v^2 + h_z w^2 \\
\dot{v} &= -h_y v^2 - 2h_z vw - 2h_x vu + h_z w^2 + h_x u^2 \\
\dot{w} &= -h_z w^2 - 2h_x uw - 2h_y vw + h_x u^2 + h_y v^2.
\end{align*}
\]

Furthermore the only non-zero curvature components as a type (1,3) tensor, are, allowing for symmetries, given by

\[
\begin{align*}
R^{1}_{212} &= -(h_{xx} + h_{yy} + (h_z)^2) = -c \\
R^{2}_{323} &= -(h_{yy} + h_{zz} + (h_x)^2) = -a \\
R^{3}_{131} &= -(h_{zz} + h_{xx} + (h_y)^2) = -b \\
R^{1}_{231} &= h_{yz} - h_y h_z = \lambda \\
R^{2}_{312} &= h_{zx} - h_z h_x = \mu \\
R^{3}_{123} &= h_{xy} - h_x h_y = \nu.
\end{align*}
\]

(The quantities \(a, b, c, \lambda, \mu, \nu\) will be used soon.) The same pattern holds in higher dimensions where the only non-zero curvature components are when there are two pairs of repeated indices and three distinct indices with one of them being repeated.

Here is an interesting property of conformally flat metrics.

**Proposition 4.11.1.** For a conformally flat metric the \(\Phi\)-matrix is symmetric.

**Proof.** This fact is obvious since the conformally flat metric \(g\) itself is a solution of (1.9). \(\square\)

We remark that the property of \(\Phi\) being symmetric is not invariant under an
arbitrary change of basis, in a possibly non-holonomic frame; however, it will be so under a conformally orthogonal change. We notice also that all the matrices in the \( \Phi \)-hierarchy will also be symmetric.

We next investigate when in the case \( n = 3 \) we can have \( d = 4 \). We begin by noting that

\[
\Phi = \begin{pmatrix}
-cv^2 - 2 \lambda vw - b w^2 & cuv + \lambda uw + \mu vw - \nu w^2 & bwu + \nu vw + \lambda uw - \mu v^2 \\
cuv + \lambda uw + \mu vw - \nu w^2 & -aw^2 - 2 \mu wu - cu^2 & avw + \mu uv + \nu wu - \lambda u^2 \\
 bwu + \nu vw + \lambda uw - \mu v^2 & avw + \mu uv + \nu wu - \lambda u^2 & -bu^2 - 2 \nu uv - au^2
\end{pmatrix}
\]  
(4.83)

where the quantities \( a, b, c, \lambda, \mu, \nu \) are defined in (4.82). We know that one eigenvalue of \( \Phi \) is zero and because \( \Phi \) is symmetric the eigenvalues are real. If we start with the characteristic polynomial of \( \det(xI - \Phi) \) and divide by \( x \) we obtain a quadratic equation for \( x \). Its discriminant is a degree four polynomial homogeneous in \( u, v, w \) of which three terms are

\[
((b - c)^2 + 4\lambda^2)u^4 + ((c - a)^2 + 4\mu^2)v^4 + ((a - b)^2 + 4\nu^2)w^4.
\]  
(4.84)

Hence the characteristic polynomial of \( \Phi \) has repeated roots only if

\[
a = b = c, \lambda = \mu = \nu = 0.
\]  
(4.85)

On the other hand (4.85) are precisely the conditions to obtain a space of constant curvature.

If the space is not of constant curvature then \( d = 2 \) or \( d = 3 \). In both these cases \( \Phi \) is diagonalizable, possibly by a transformation that is not linear in the \( u, v, w \), with distinct real roots. In the case \( d = 3 \) then \( \nabla \Phi \) must be simultaneously diagonalizable and hence must commute with \( \Phi \).

In fact this situation can be generalized. We quote the fact that for \( n = 3 \) a Riemannian metric may be diagonalized in coordinates, that is, not just in a non-
holonomic frame: see [dTY] and references therein. Thus we can write the metric as

\[ ds^2 = e^{2\alpha(x,y,z)}dx^2 + e^{2\beta(x,y,z)}dy^2 + e^{2\gamma(x,y,z)}dz^2. \]  

(4.86)

For the metric in (4.86) it turns out that the only non-zero components of the curvature are, just like for conformally flat spaces which is a special case, when there are two pairs of repeated indices and three distinct indices with one of them being repeated. This time \( \Phi \) is not necessarily symmetric since \( R_{121}^1 = e^{2\alpha} R_{1221} \) and \( R_{1221}^2 = e^{2\beta} R_{1221} \) where \( R_{1221} \) is a component of the fully covariant curvature tensor. Nonetheless we obtain exactly the same condition and conclusion as in (4.84) where now

\[ a = R_{323}^2, \quad b = R_{131}^3, \quad c = R_{212}^1, \quad \lambda = R_{231}^1, \quad \mu = R_{312}^2, \quad \nu = R_{123}^3. \]  

(4.87)

It should be noted that the reduction to diagonal form is not essential here but if it is not done it becomes impossible for Maple to do the calculation, which was the starting point for the investigation.

We summarize the entire situation by means of the following Theorem.

**Theorem 4.11.1.** A Riemannian metric in dimension three has \( d = 4 \) if and only if it is a space of constant non-zero curvature. The metric has \( d = 3 \) if and only if it is not a space of constant non-zero curvature, \( \Phi \) and \( \nabla \Phi \) commute and the \( \Psi \)-condition is an identity by virtue of the \( \Phi \)-condition. Finally the metric has \( d = 2 \) if and only if it is not a space of constant non-zero curvature and either \( \Phi \) and \( \nabla \Phi \) do not commute or the \( \Psi \)-condition is not an identity by virtue of the \( \Phi \)-condition.

In spite of 4.11.1 we do not know an example of a metric for which \( d = 3 \).
References


[Der] A Derdzinski *Connections with skew-symmetric Ricci tensor on surfaces*


