A Dissertation
entitled

Stability Analysis of Capillary Surfaces with Planar or Spherical Boundary in the Absence of Gravity

by

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We study stable capillary surfaces with planar or spherical boundary in the absence of gravity. If the boundary of the capillary surface is embedded in a plane, we prove that the only immersed stable capillary surface is the spherical cap. The second part of this dissertation treats the case when the capillary surface lies inside the unit ball in $\mathbb{R}^3$ with its boundary on the unit sphere. We construct a Killing vector field for the hyperbolic metric and use it to show that if the center of mass of the region bounded between the surface and the unit sphere is at the origin, the configuration cannot be stable. As a corollary of this approach we obtain a new proof of a theorem by Barbosa and do Carmo. We also provide a new proof of the stability of spherical caps on a plane or inside of the round ball, using exotic containers.
To Ralitza
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Chapter 1

Introduction

Capillary surfaces have been studied extensively in the past, most notably by T. Young, P. Laplace and C.F. Gauss. A comprehensive treatment of the theory of capillary surfaces can be found in R. Finn’s book [F]. The problem we address concerns a homogeneous liquid drop in contact with a smooth rigid boundary surface \( \Sigma \). We call the free surface of the drop \( \Omega \), the angle of contact \( \gamma \), and the wetted part of \( \Sigma \) we call \( \Sigma' \). The liquid drop occupies a connected region in space, \( T \), with a prescribed volume. The contact angle \( \gamma \) is measured relative to the interior of the liquid bounded by \( \Omega \) and \( \Sigma \). One can ask the following question: What is the shape of \( \Omega \) if the liquid drop is in equilibrium?

There are three energies associated with the above configuration. The first one is the free surface energy, which is proportional to the area of \( \Omega \) with coefficient equal to the surface tension. The second one is the wetting energy, which is a multiple of the area of \( \Sigma' \). The third is the gravitational energy. In this dissertation we assume the absence of gravity so the gravitational energy does not contribute. As it is well known, in order for the drop to be in equilibrium it must be a critical point for the potential energy functional \( E \). From the discussion above we obtain a formula for \( E \), namely \( E = \sigma \text{Area}(\Omega) - \sigma \tau \text{Area}(\Sigma') \), where \( \sigma \) is the surface tension and \( \tau \) is the
capillary constant. The constant \( \tau \) is a physical quantity that is predetermined and and in equilibrium equals \( \cos \gamma \). We must point out that the wetting ability and the surface tension of the liquid are the two physical phenomena that cause the drop to become stationary. The above configuration is said to be in a stationary state if the first variation of \( E \) is zero for any volume preserving perturbation. It is stable (strictly stable) if it is stationary and the second variation of \( E \) is non-negative (positive) for any non-trivial volume preserving perturbation. Precise definitions will be given in chapter two.

In this paper we will study the stability problem when the fixed boundary surface \( \Sigma \) is either a plane or a round sphere. In the latter case \( \Omega \) will lie inside the ball bounded by \( \Sigma \). One can study other configurations, e.g. when \( \Omega \) lies on the outside of the ball bounded by \( \Sigma \), or when \( \Sigma \) is a circular cylinder, or even when \( \Sigma \) is not smooth and has an edge. For example \( \Sigma \) could be a wedge or a cone.

The main tool for our study is the theory of calculus of variations (see [G-F] for a comprehensive introduction). We denote the first variation of \( E \) by \( \partial E \). If \( \partial E \) is zero subject to a volume constraint one finds that the angle of contact \( \gamma \) must be constant along \( \partial \Omega \subset \Sigma \), the mean curvature of \( \Omega \) must be constant, and \( \tau = \cos \gamma \). This computation is reviewed in the next chapter. If the case when gravity is present, the mean curvature of \( \Omega \) is proportional to the height. This discussion naturally leads to the study of constant mean curvature surfaces with boundary. If \( \Omega \) makes constant angle with \( \Sigma \) along \( \partial \Omega \), one can ask what are the possible shapes of \( \Omega \). This question is hard to resolve even if \( \Sigma \) has a simple geometry, e.g. a plane (See [E-B-M-R]), round sphere, or round cylinder. There are a few known examples for planar or spherical \( \Sigma \), including the spherical caps, right cylinders and Delaunay surfaces. One can generalize the problem and assume \( \Omega \) to be immersed, i.e. \( \Omega \) could have self intersections, which complicates the discussion even more. For this reason we may put an additional restriction on \( \Omega \) and study the same problem. The physical discussion
above suggests to assume that the second variation of the potential energy, $\partial^2 E$, is non negative, i.e $\Omega$ is stable. Assuming stability, in the case of $\Sigma$ being a plane or a sphere one can say much more.

The first part of this dissertation discusses the case of an immersed stable constant mean curvature surface $\Omega$ making a constant contact angle $\gamma$ with a plane $\Sigma$ (see Figure 1-1). It is known that if $\Omega$ is embedded, then it is a spherical cap (see [W 1]). We only assume that $\partial \Omega$ is embedded and that the surface $\Omega$ comes from above close to the boundary, but we allow $\Omega$ to be below $\Sigma$ away from $\partial \Omega$. Our main theorem shows that the only possible stable configuration of this type is the spherical cap. The spherical cap is weakly stable as is shown in [W]. For the proof of the main theorem we consider three cases. The first one is when $\Omega$ is of disk type with genus equal to 0. The proof of this case is known by a result of Nitsche (see [N], [F-M]) and does not assume stability. The second case is when $\Omega$ is of genus zero, but not of disk type. For this case we use an argument involving a Killing field, which is suggested in [R-S]. The paper of [R-S] has been our primary guide for considering two separate situations, when $\Sigma$ is a plane, and when $\Sigma$ is a round sphere. The third case is when the genus of $\Omega$ is positive. For this case we construct a perturbation that depends on the mean curvature of $\Omega$ and on the contact angle $\gamma$. A perturbation similar to this was used in [B-dC] to show that the round spheres are the only immersed stable constant mean curvature hypersurfaces in $\mathbb{R}^n$. The normal component of the constructed perturbation makes the second variation of the energy negative, while preserving the volume; therefore, $\Omega$ cannot be stable.

In the proof we use various theorems from analysis and differential geometry, most importantly the Gauss-Bonnet formula. One might say that we have almost a full understanding of this case. There are other interesting results related to the planar boundary case involving cylindrical surfaces (see [At] and [V]), which bifurcate in new surfaces that are non-symmetric and also unstable. An excellent source on bifurcation
theory is \(|S|\).

Figure 1-1: Immersed capillary surface "sitting" on a plane

In the second part of this thesis we give an alternative proof of the fact that the spherical caps are stable, using exotic containers. The exotic containers (see Figure 1-2) have been studied in their own right and are a fruitful subject of research in capillary theory. By construction, an exotic container bounds a continuum of non-congruent equilibrium surfaces. The surfaces all intersect the container at a fixed constant contact angle and all have the same volume. We show that the second variation of energy for a spherical cap in an exotic container is smaller then the second variation of energy for the case of a spherical or planar wetted area.

The last chapter of this dissertation discusses the case when \(\Sigma\) is a round sphere (see Figure 1-3). This case is studied in [R-S]. We prove however a new result here, which suggests again that the only stable immersed constant mean curvature surface in a ball is the spherical cap or the flat disk. We construct a one parameter family of
Figure 1-2: Exotic container

Figure 1-3: Immersed capillary surface in a ball
conformal vector fields on $\mathbb{R}^3$ from a series of two inversions with respect to spheres, which are actually Killing fields on the hyperbolic ball. The family depends on a fixed unit vector in $\mathbb{R}^3$. We then show that at least one member of this family makes the second variation of energy for $\Omega$ negative if $\Omega$ is not a spherical cap.

The main difficulty that arises in this approach is the fact that we cannot always control the volume constraint. In the special case when the center of mass of $T = \text{int}(\Omega \cup \Sigma')$ is located at the origin (and all known examples except the stable spherical caps have this feature), we get the desired result. After this we obtain some interesting results about geometric quantities using the main result in [L-Y]. Finally, as a simple consequence of our computations, we provide a new proof of the main theorem in [B-dC] for a two dimensional immersed compact closed surface in 3-space.
Chapter 2

Stability Analysis for Capillary Surfaces with Planar Boundary

In this chapter we prove a theorem that shows that there are no immersed stable capillary surfaces in $\mathbb{R}^3$ of genus $g > 0$, with planar boundary. It is known that an embedded stable capillary surface in half space must be a round spherical cap. We also state known facts for closed stable surfaces, disc type CMC-surfaces with planar boundary and discuss the non existence of stable capillary surfaces with planar boundary with genus zero and more then one connected boundary component.

The fact that leads to this discussion is that the only known stable CMC-surface with constant contact angle along the boundary, sitting on a plane in the absence of gravity is the spherical cap (these are the only embedded in half space). One gets spherical caps just by slicing the standard sphere $S^2$ by a plane. There is a whole family of them, which can be parametrized by the contact angle with the plane and by the volume of the cap. Of course for the spherical caps the mean curvature $H$ is a constant and so is the angle of contact with the planar boundary.

To show the non existence of other stable configurations of genus $g > 0$, we devise a special normal perturbation and show that it makes the second variation of energy
negative. This perturbation is related to the Mean Curvature Flow, and it is adjusted to keep the volume and the contact angle fixed. The case $g = 0$ is basically known, if one has only one boundary curve (see [N],[F-M]). Actually, for disc type immersion in this case we do not need the the stability condition, but the proof is quite different using complex differentials. For $g = 0$ and more boundary curves further investigation will be made, applying techniques from [R-S]. On the contrary, our main computation uses facts from Geometry, Integral formulae and some PDE’s.

2.1 Definitions and Classical Theorems

Here we define some basic notions from Differential geometry and Functional Analysis and list the theorems that are going to be used.

2.1.1 Preliminaries

First we define the notion of Mean Curvature. Let $x$ be an $C^2$-immersion of some abstract surface $D$ into $\mathbb{R}^3$ and $\Omega = x(D)$ be the surface represented by this immersion. A key object of the study of surfaces is the principal curvatures. To define them at point $p \in \Omega$ we use slicing with planes parallel to the normal vector $\xi$ of $\Omega$ at $p$. Thus, we get a family of curves made by the intersection with the vertical planes and the surface. The two principal curvatures $k_1$ and $k_2$ are the minimum and maximum possible curvatures of the profile curves. If they coincide we call this an umbilical point. At a non umbilical point the curves corresponding to $k_1$ and $k_2$ are known to be at $\frac{\pi}{2}$ angle within each other. A well know fact is that if a surface consists only of umbilical points, that it a part of plane or a sphere (See [Ho]).

Definition. Let $\Omega$ be given by a map $x : D \rightarrow \mathbb{R}^3$, with $\Omega = x(D)$, where $D$ is some abstract 2-dimensional manifold with boundary $\partial D$. A principal curvature vector at a point $p \in \Omega$ is one that is tangent to the profile curve whose curvature is a principle
curvature. There are two families of curves on a surface whose tangent lines at each point are in the direction of the principle vectors. The elements of this families are called curvature lines on the surface $\Omega$. This definition is valid only at a non-umbilical point ($k_1 \neq k_2$), since at an umbilical point any direction is a direction of principal curvature. For more details see [Ho].

Another useful fact is that each of the two principal vectors at a point of $\Omega$ represent two fields of line elements, if one varies the point continuously of the surface. Each of these line fields develops a singularity at an umbilical point and there is an index corresponding to the point. The sum of the indices at the isolated umbilical points is related to the genus of the surface and the Euler’s Characteristic via the Hopf-Poincare Theorem (see [Ho]). This proves to be useful in the Gauss-Bonnet Theorem which is stated in the next subsection without a proof, but will have a major use in the main theorem of this chapter. Next we define the main curvatures at a surface.

**Definition.** The **Gauss curvature** $K$ is defined to be the product of $k_1$ and $k_2$ and the **Mean Curvature** $H$ is defined to be the average of the principal Curvatures at each point on $\Omega$.

There are two quadratic expressions related to every surface, namely the First and The Second Fundamental Forms. If these two expressions are known and they if are consistent with each other (see Gauss-Codazzi-Mainardi equations in [O]), then the surface is determined up to a rigid motion in $\mathbb{R}^3$. In local coordinates $(u,v)$, they are defined as follows:

$$I = Edu^2 + 2Fduv + Gdv^2$$

$$II = Ldu^2 + 2Mduv + Ndv^2$$

where $E = x_u \cdot x_u$, $F = x_u \cdot x_v$, $E = x_v \cdot x_v$, and $L = x_{uu} \cdot \xi$, $M = x_{uv} \cdot \xi$, $N = x_{vv} \cdot \xi$. The following lemma is a classical result.
Lemma 2.1.1. Let $\Omega$ be a surface given by a patch $x = x(u,v)$ in local coordinates, like above. The Gauss Curvature $K$ and the Mean Curvature $H$ of $\Omega$ are given respectively by

$$2K = \frac{LN - M^2}{EG - F^2} \quad (2.1)$$

$$2H = \frac{GL + EN - 2FM}{EG - F^2} \quad (2.2)$$

where $E,F,G$ are the coefficients of the First Fundamental Form of $x$ and $L,M,N$ are the coefficients of the Second Fundamental Form of $x$.

Another way of describing $H$ is writing it in a divergence form if the surface is given by a graph $x = u, y = v$ and $z = f(u,v)$. Let

$$Tf = \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}$$

which represents a unit normal vector on the surface and $\nabla$ is the gradient operator. One can easily see by the above definition that $2H = \text{div} Tf$. If $x$ is a conformal map, i.e $E = G$ and $F = 0$ and if we take the scalar product of the divergence form equation for $H$ with $\nabla$, and rewrite it we arrive at the well known $H$-surface partial differential equation valid in conformal coordinates

$$\Delta x = 2H(x_u \wedge x_v) \quad (2.3)$$

where $\Delta$ is the standard vector Laplacian of the map $x$. We will make use of those formulae in the next sections.

2.1.2 Some Classical Theorems in Differential Geometry

Here we state some famous theorems, that will be used. These are classical results obtained in the $19^{th}$ century. The first two relate geometry with topology, and they
are well known and widely used in a lots of contexts. For proofs see [Ho], [O] or [Sp].

The first one is a formula about the invariance of the Euler’s Characteristic, and the second one gives a global integral formula for the Gauss Curvature.

**Theorem 2.1.2. (Euler’s Formula)**

For an immersed compact connected closed surface $M$ the Euler Characteristic $\chi(M)$ is equal to $2 - 2g - d$, where $g$ is the genus of the surface and $d$ is the number of boundary components.

Let $T$ be a triangulation of the surface $M$. The Euler characteristic $\chi(M)$ is defined to be the number of faces minus the number of edges plus the number of vertices in $T$. It is well known that that is an invariant of $M$, independent of the triangulation.

**Theorem 2.1.3. (Gauss-Bonnet Formula)**

For an immersed surface $M$ with Gauss Curvature $K$ and geodesic curvature $k_g$ of $\partial M$ in $M$, we have the following formula

$$\int \int_M KdS + \oint_{\partial M} k_g d\sigma = 2\pi \chi(M).$$

The next two classical theorems are also well known and we will use them extensively. The for proofs see [St].

Let $\Gamma$ be a curve parametrized by arc length on a surface $M$ and let $\kappa = dt/ds$ be the curvature vector of $\Gamma$, where $t$ is the unit tangent vector of $\Gamma$. We decompose this vector two components: normal and tangential to the surface, i.e $\kappa = \kappa_n + \kappa_g$.

For example this means that $\kappa_n = k_n N$ where $N$ is unit normal to the surface and $k_n$ is the normal curvature. Also, $\kappa = kn$, where $n$ is the normal vector to $\Gamma$ and $k$ is the curvature of $\Gamma$. The vectors $\kappa_n$ and $\kappa_g$ are called normal curvature vector and geodesic curvature vector, respectively.
Theorem 2.1.4. (Meusnier’s Theorem)

Let $\Gamma_1, \Gamma_2$ be curves on a surface $M$ passing through some given point $p$ and having the same tangent vector at $p$. Then, $\Gamma_1$ and $\Gamma_2$ have the same normal curvature vector $\kappa_n$ at $p$ and their osculating circles form a sphere. Furthermore one has

$$k \cos \phi = k_n$$

where $\phi$ is the acute angle between $\kappa$ and $\kappa_n$.

Earlier we defined the principle curvatures $k_1$ and $k_2$. Let $t$ be any tangent vector to the surface $M$ at $p$. The unit normal vector $N$ to $M$ and $t$ determine a plane that intersects $M$ at a certain curve. Call $k_t$ the normal curvature of that curve and $\alpha$ the angle between $t$ and the principle curvature direction determined by $k_1$. The next theorem provides a relationship between the principle curvatures, the angle $\alpha$ and $k_t$.

Theorem 2.1.5. (Euler’s Theorem)

$$k_t = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha.$$ 

Although there are many more classical theorems in the differential geometry of surfaces we will stop with the list here and if needed state other important theorems in the next sections.

2.1.3 Theorems on Constant Mean Curvature Surfaces

There are several important theorems dedicated to the study of constant mean curvature surfaces. in [Ho], H. Hopf did an extensive study on the subject and proved two major theorems. One of them was due to A. Alexandrov and the other to Hopf himself. For completeness, in this subsection we state these results and hint the ideas behind them.
Theorem 2.1.6. *(Alexandrov’s Theorem)*

An embedded closed compact constant mean curvature in $\mathbb{R}^3$, must be the round sphere.

The proof he gave is rather involved and uses facts from Elliptic Partial Differential Equations together with some ideas from geometry. Basically, the idea is as follows: one use a family of moving parallel planes and when they start intersecting the surface, a reflection is done. Once the reflected portion touches the surface from the inside, a touching principal is used. The touching principal states that if two solutions of the same elliptic partial differential equation touch at a point they must coincide in a whole neighborhood. Since the mean curvature equation is just like this we have, that the sphere (which we already know is a solution) is the only possibility. That follows from the fact that the sphere is the only closed compact surface with symmetries in any given direction. For the original translated paper of Alexandrov see [A]. The other main theorem of Hopf’s discussion is the following.

Theorem 2.1.7. *(Hopf’s Theorem)*

An immersed closed compact constant mean curvature surface with of genus 0 in $\mathbb{R}^3$, must be the round sphere.

He basically gave two proofs. In the first one he constructs a function out of the coefficients of the Second Fundamental Form and uses some complex analysis theory. He showed that if one uses complex coordinates $\omega$ and $\overline{\omega}$ and introduce the function $\Phi(\omega, \overline{\omega}) = L - N - 2iM$, then $\Phi$ is holomorphic on a constant mean curvature surface. The proof after that is done using holomorphic quadratic differentials and Riemann surfaces. The second one uses lines of curvature. This ideas will be used in some of the next sections. Interestingly, the two theorems look very much alike, but the proofs are completely different. Hopf also made a conjecture about the higher genus case for immersions, which was proved to be false by H. Wente in [W 2].


2.2 Capillary Effect

In this section we define stability and capillarity in terms of the energy and the volume for a given configuration. An extensive source on the subject is [F]. From now, until the end of the chapter \( \Omega \) will be an oriented compact surface immersed in \( \mathbb{R}^3 \) with nonempty planar boundary in the \( x, y \)-plane. Let it be given by a \( C^{2,\alpha} \)-immersion \( x(u, v) : D \to \mathbb{R}^3 \), with \( x(D) = \Omega \) and \( x(\partial D) = \partial \Omega \). Also, we assume that \( \partial \Omega \) is a finite collection of non-intersecting simple closed curves, i.e. \( \partial \Omega \) is embedded in \( \mathbb{R}^3 \). We label the boundary \( \partial \Omega \) by \( \Gamma \), and the regions in \( \mathbb{R}^2 \) bounded by \( \Gamma \) we name \( \Sigma' \). The boundary \( \Gamma \) is also oriented and it is assumed that \( \Omega \) comes from above near the boundary. We denote the areas on the surface and the wetted area by \( |\Omega| \) and \( |\Sigma'| \), respectively. We denote the angle of contact between the surface and the wetted by \( \gamma \). The surface area of \( \Omega \) is given by

\[
|\Omega| = \int \int_D dS
\]

where \( dS \) is the surface element on \( \Omega = x(D) \). We will assume that \( \Omega \) is extendable in a neighborhood of \( \Gamma \) so we can compute tangent vectors, normal vectors, etc. Now, here is a key definition.

**Definition.** An immersed surface is called capillary if it has constant mean curvature and makes constant contact angle with the walls along its boundary.

Next to define is our main object of interest - the energy.

**Definition.** The energy function of the above configuration after dividing by the surface tension \( \sigma \), is given by

\[
E = |\Omega| - \tau|\Sigma'|
\]

with \(-1 < \tau < 1\) being some predetermined constant.
Our main goal in such problems is to minimize the energy subject to the natural constraints that arise. To do that we should look among all the nearby surfaces that are admissible. We get them if we apply a perturbation to the original surface. Thus, we need to define what is an admissible variation, for example see [R-S].

Definition. Admissible variation of \( x \) is a differentiable map \( \Phi : (-\epsilon, \epsilon) \times D \to \mathbb{R}^3 \), such that \( \Phi(t, p) = \Phi(p), p \in D \), is an immersion and \( \Phi_0 = x \).

Again as we assumed before the surface \( \Omega \) can be extended across its boundary. That will allow us to keep the boundary planar after applying an admissible variation.

Definition. The volume functional \( V : (-\epsilon, \epsilon) \to \mathbb{R} \) is defined by

\[
V(t) = \frac{1}{3} \int \int_D (\Phi_t \cdot \xi_t) dS_t
\]

where \( \xi_t \) and \( dS_t \) are the unit outward normal and the surface element on \( \Phi_t(D) \).

The corresponding variational field is \( Y(p) = \frac{\partial \Phi}{\partial t}(p)|_{t=0} \) and we denote its normal part by \( \phi \). Now we need to write the first and the second variation formulae related to \( \phi \). We need to set the first variation equal to 0, subject to the volume constraint \( V'(t) = 0 \), and to investigate the second variation. We also have a volume constraint, because the variation \( \phi \) must preserve the volume. This means that we should introduce a Lagrange multiplier \( \lambda \) and compute the First and Second Variation for the expression \( E + \lambda V \). For proofs of the first and second variation formulae subject to a volume constraint one may check [W] and [R-S], respectively.

Theorem 2.2.1. (First Variation Formula)

Let \( d\sigma \) be the line element on the boundary \( \Gamma \) and \( dS \) be the surface element on \( \Omega \). The first variation formula for the energy of \( x \) in the direction of \( \phi \), subject to a volume
constraint implies that

\[ \partial(E)[\phi] \equiv \left. \frac{d}{dt} E[\phi] \right|_{t=0} = -2 \int \int_D H \phi dS + \oint_{\partial D} (-\tau \csc \gamma + \cot \gamma) \phi d\sigma \]  

(2.5)

\[ \partial(V)[\phi] \equiv \left. \frac{d}{dt} V[\phi] \right|_{t=0} = \int \int_D \phi dS \equiv 0. \]  

(2.6)

Formula (2.6) represents the rate of change of the volume, so if we want constant volume it must be zero. It follows from (2.5) and (2.6) that \( H \) and \( \gamma \) must be constants in order to have an extremal of \( E \), subject to the volume being stationary. The latter follows from the observation that the constant \( \tau \) must be equal to \( \cos \gamma \) in order to have the boundary integral equal to zero. This means that \( \Omega \) must be a capillary surface.

**Definition.** A capillary surface is called stable if the second variation is non-negative for all admissible perturbations with normal components \( \phi \neq 0 \) and strictly stable if the second variation is positive for all admissible perturbations.

The next theorem gives a formula for the second variation. This will be our main object of interest. We will assume that the configuration is stable and will choose a special \( \phi \) and manipulate that formula to get a contradiction, unless we have a spherical cap.

**Theorem 2.2.2. (Second Variation Formula)**

Following the above notation the formula for the Second Variation of \( E \) is:

\[ \partial^2(E)[\phi] \equiv \left. \frac{d^2}{dt^2} E[\phi] \right|_{t=0} = \int \int_D [||\nabla \phi||^2 - (k_1^2 + k_2^2) \phi^2] dS + \oint_{\partial D} p \phi^2 d\sigma \]  

(2.7)

where \( \nabla \phi \) is the surface gradient of \( \phi \), \( k_1 \) and \( k_2 \) are the principal curvatures, and 
\( p = K_{\Omega} \cot \gamma + K_{\Sigma} \csc \gamma \). Here \( K_{\Omega} \) and \( K_{\Sigma} \) are the signed normal curvatures of \( \Omega \) and \( \Sigma \) with respect to the boundary. Of course, condition (2.6) should be fulfilled (for the proof see [W] or [R-S]).
In our case $\Sigma$ is planar so $K_\Sigma = 0$ and if we take a vertical slice and consider the profile curve, $K_\Omega$ will be its curvature. If the profile curve bends towards the boundary the sign of that normal curvature is taken to be positive. In the proof of our main theorem in this chapter, we will rewrite this formula in a more convenient way using Green’s Identities.

2.3 The Sphere and the Spherical Cap are Stable

2.3.1 Stability for closed surfaces

Here we state the uniqueness of immersed stable closed constant mean curvatures in $\mathbb{R}^n$. J. Barbosa and M. do Carmo first proved in [B-dC] that the only possibility is the round sphere. They used a particular variation field, with first-order change of volume being zero and which makes the second variation negative unless the surface in question is a sphere. Later, H. Wente gave a shorter proof in [W 3], based on the
construction of a specific one-parameter family of immersions. In the next chapter we will provide a new proof based on a construction of a conformal vector field. These are some of the ideas that we borrowed in our main theorem proof in section (2.5).

**Theorem 2.3.1. (Barbosa and do Carmo)**

Let $M$ be compact oriented $n$-manifold and let $x : M \to \mathbb{R}^{n+1}$ be an immersion with non-zero constant mean curvature $H$. Then $x$ is stable if and only if the surface represented by it is a round sphere $S^n \subset \mathbb{R}^{n+1}$.

The key ingredient of their proof is the vector field $1 + H(x \cdot \xi)$, which is a part of the vector field we are going to use in our main theorem.

Now we move to the case of spherical caps, namely parts of spheres with planar boundary.

### 2.3.2 The Spherical caps

The spherical caps, being capillary surfaces with planar boundary are stable. This means that the Second Variation corresponding to any small admissible displacement is nonnegative, and is zero only for horizontal shifts and for trivial perturbations that do not change the shape of the spherical cap. For this purpose we will use the construction of the so called *Exotic containers* to provide a new proof for the stability of spherical caps. Next chapter will be fully dedicated to this discussion.

### 2.4 The Genus 0 Case

Here we investigate the case when the genus of a capillary surface with planar boundary is zero. There are major results based on Complex Analysis and also on Geometry. One approach is Hopf’s idea to construct a function on the surface which is holomorphic if and only if the surface is of constant mean curvature and the use of complex differentials. Another idea is to use lines of curvature. The second method
uses the fact that, if one has constant contact angle and the boundary consists of curvature lines on the wall, then the boundary curves are curvature lines on the surface. In our case this holds since the boundary is planar and any curve in the plane is a curvature line. Of course, an excellent reference is Hopf’s book [Ho] on Differential Geometry. Another method devised for spheres in [R-S] could be applied.

2.4.1 Disc type Surfaces of Genus 0

The case of immersed constant mean curvature surface with planar or spherical boundary and making constant contact angle with the boundary is resolved by Nitsche and Finn-McCuan (see [N],[F-M]). The first author proves a theorem which states that a immersed disk type surface in a ball which makes constant angle with the boundary sphere is a flat disk or a round spherical cap. He proved the theorem for right angle but pointed out that the idea works for any angle. It was reproved in [R-S] too. Finn and McCuan had similar results for surfaces with planar boundary.

In this section we also state a classical result (for proof see [Sp, vol.3]).

Theorem 2.4.1. (Terquem-Joachimsthal Theorem)

Let $M$ and $N$ be two embedded surface is $\mathbb{R}^3$ with nonempty intersection - a common boundary, which we call $C$. If $C$ consists of curvature lines in $M$ and the contact angle between $M$ and $N$ is constant along $C$, then $C$ consists of curvature lines in $N$ also.

This theorem proves to be very useful in our discussion, since the capillary wall for us is planar and the angle of contact is constant. It is easy to show that on a plane all lines are lines of curvature. Now we move to the next subsection where other methods will be used. We need to investigate the genus 0 case but with more than one boundary curves.
2.4.2 General Genus 0 case

The ideas for this discussion come from [R-S]. Here the surface $\Omega$ has genus zero but there could be possibly more then one boundary curves. We assume that the planar boundary $\Gamma = \partial \Omega$ is embedded but the surface itself could be immersed. Let $\Gamma$ belongs to the plane $z = 0$. We adapt the method used in [R-S] for our purposes. As before $\Omega$ is given by mapping $x : D \rightarrow \Omega$. Let $p_0 \in \Omega$ be a point such that the euclidean distance to the the plane containing $\Gamma$ is maximal. Obviously there is at least one point with that property. Let $\xi$ be the unit normal to the surface. From our setup it follows that $\xi(p_0)$ is parallel to the $z$-axis. Later in this chapter, we will see that the mean curvature of $\Omega$ must be negative, so the vector $\xi(p_0)$ will point out in the positive $z$-direction. Denote the Killing field induced by rotations around the line directed by $\xi(p_0) = k$ with $X$, i.e $X = p \wedge \xi(p_0)$, where $p$ is a point in $\mathbb{R}^3$ and $\wedge$ is the usual wedge product in $\mathbb{R}^3$. Consider the function $\phi(p) = \langle X(x(p)), \xi(p) \rangle$. Because the rotational invariance around $\xi(p)$, it follows that $\phi(p)$ is a Jacobi field on the surface. Using the notation from the previous sections one has

$$\Delta \phi + (k_1^2 + k_2^2) \phi = 0$$

with $\phi_\nu + p\phi = 0$ on $\Gamma$. Also $\phi(p_0) = 0$ and $\nabla \phi(p_0) = 0$. Therefore the second variation of energy in the direction of $\phi$ is zero and the volume constraint holds. As is [R-S], using the Gauss-Bonnet Theorem one can show that there are at least three nodal regions of $\phi$, i.e. $\Omega - \phi^{-1}(0)$ has at least three connected components. Let $\Omega_i, i = 1, 2, 3$ are the nodal regions of $\phi$ and $\phi_i = \phi$ on $\Omega_i$ and zero elsewhere. Now construct $\tilde{\phi} = \sum_{i=1}^{3} c_i \phi_i$ with $c_1, c_2, c_3$ - constants. Then one can adjust the constants using the volume constraint to get a smaller number for the second variation so it would make it negative. This shows that there are no surfaces of the assumed type with two or more connected boundary components. Thus, we can conclude that the
spherical caps are the only immersed stable CMC-surfaces with planar embedded boundary, having genus \( g = 0 \) and constant contact angle along the boundary with the plane \( \Sigma \).

### 2.5 The Main Theorem

Here we prove our main result of this chapter

**Main Theorem.** There exists no stable capillary surface with planar boundary, that is immersed in \( \mathbb{R}^3 \) and having genus \( g > 0 \).

We assume that the boundary of \( \Omega \) is embedded, i.e. it consists of finite number of simple closed curves. Also we assume that the surface can be extended across its boundary. This way we can assure that the boundary stays planar after a normal perturbation. Also we assume that \( \Omega \) comes from above to \( \Sigma \). We will construct a special normal perturbation, for which the Second Variation is negative and the volume is preserved. First we need to rewrite (2.7) using Green’s 1st Identity. We also assume that our mappings are \( C^{2,\alpha} \) (in fact capillary surfaces are analytic by standard regularity theory), so we can compute derivatives at the boundary and we can extend the surface around the boundary \( \Gamma \). The variation that we use, does not necessarily keep the boundary planar. That is why we extend the surface across the boundary so after the perturbation the new surface has planar boundary, i.e. \( \partial \Phi_t(D) \) belongs to the same plane as \( \Omega = x(D) \). This is the way it has been done in [W].

**Theorem 2.5.1. (Green’s 1st Identity)**

Let \( \phi \) and \( \psi \) be scalar functions defined on some surface \( \Omega \) in \( \mathbb{R}^2 \) with \( \phi \in C^2(\overline{\Omega}), \psi \in C^1(\overline{\Omega}) \) (For our latter purposes \( \phi \) will be \( C^\infty \)). Then the following formula holds

\[
\int \int_{\Omega} \nabla \phi \cdot \nabla \psi dS = - \int \int_{\Omega} \psi \Delta \phi dS + \int_{\partial \Omega} \psi \phi \nu d\sigma \tag{2.8}
\]
where $\phi_\nu$ is the outward normal derivative.

Here $\nabla$ and $\Delta$ are the surface gradient and Laplacian operators respectively. This is an application of the Divergence theorem. For proof of the theorem see [E]. If we substitute $\phi$ for $\psi$ in (2.8) and apply it to (2.7) we get the following

$$
\int \int_D |\nabla^2 \phi| - (k_1^2 + k_2^2)\phi^2 dS = \int \int_D \phi [-\Delta \phi - (k_1^2 + k_2^2)\phi] dS + \oint_{\partial D} \phi \phi_\nu d\sigma \quad (2.9)
$$

and for $\partial^2 E$ we obtain the following

$$
\partial^2 E = \int \int_D ( -L\phi ) \phi dS + \oint_{\partial D} ( \phi_\nu + p\phi ) \phi d\sigma \quad (2.10)
$$

where $L\phi = \Delta \phi + (k_1^2 + k_2^2)\phi$, $\partial V \equiv \int \int_D \phi dS = 0$ and $p = K_\Omega \cot \gamma + K_\Sigma \csc \gamma$. The operator $L$ is called the Jacobi operator.

We can try a Rayleigh quotient approach, see [E]. Note, that one can normalize $\phi$ and $\Omega$ by requiring $\int \int_\Omega \phi^2 ds = 1$. Let $\phi_1$ be the eigenfunction, corresponding to the lowest eigenvalue $\lambda_1$ in the spectrum of $L\phi$ with boundary condition $\phi_\nu + p\phi = 0$. Now if we substitute $\phi_1$ into (2.10) we get $\lambda_1$ back. We do not know the sign of $\lambda_1$, so we cannot deduce anything from this observation. So we want to introduce certain perturbation $\Phi$ with normal part $\phi$, to make the Second Variation negative. From now until the end of the section, we assume that $\Omega$ is an immersed stable capillary surface, extendable over the boundary, with planar boundary and genus $g$ bigger than zero. We may choose that the planar boundary belongs the $z = 0$ plane and close to the boundary it is above the plane. This means that $H < 0$ for the embedded case since one can use the maximum principle for elliptic partial differential equations for the mean curvature equation to see that. For the immersed case $H$ is also negative. This fact follows from the proof of theorem (2.5.5) below.

Let $\Phi$ be the perturbation that sends $x \rightarrow x + t\xi + Htx + ctk + O(t^2)$. Here $t \in [-\epsilon, \epsilon]$, $k = (0,0,1)$ is the unit vertical vector, $c$ is a constant, $\xi$ - the outward
unit normal on the surface and $H$ is the mean curvature of the surface. The normal part of it is $\phi = \xi \cdot \frac{\partial \Phi}{\partial t}(p)|_{t=0}$, where $p \in \Omega$. When we compute this quantity, we get $\phi = 1 + H(x \cdot \xi) + c(k \cdot \xi)$, with $c$ to be determined from (2.6). The variation $1 + H(x \cdot \xi)$ is the one used in [R-S]. Remember that $\gamma$ is the constant angle of contact of $\Omega$ with the planar boundary.

Lemma 2.5.2. Condition (2.6) implies that $c = -\cos \gamma$, i.e

$$\phi = 1 + H(x \cdot \xi) - \cos \gamma (k \cdot \xi)$$

in order to keep the volume fixed.

Proof. One needs to adjust $c$ in $\phi = 1 + H(x \cdot \xi) - c(k \cdot \xi)$ to get the integral of $\phi$ over the surface $\Omega$ to be zero.

$$0 = \int \int_D \phi dS = \int \int_D (1 + H(x \cdot \xi) - c(k \cdot \xi))dS = |\Omega| + H \int \int_D (x \cdot \xi)dS + c \int \int_D (k \cdot \xi)dS$$

where $|\Omega|$ is the area of the surface $\Omega$. The quantity $\int \int_D (k \cdot \xi)dS$ is easily computed by the Divergence theorem. We know for the embedded case that

$$\int \int_D (k \cdot \xi)dS + \int \int_{\Sigma'} (k \cdot \xi)dS = \int \int_T \text{div} k dV = 0$$

since $k$ is a constant vector. Here $\Sigma'$ is the wetted part bounded by $\Gamma$, $T$ is the solid bounded by $\Omega$ and $\Sigma$, $\xi$ is unit outward normal to $\partial T = \Omega \cup \Sigma'$ and $dV$ is the volume element in $\mathbb{R}^3$. On $\Sigma'$ it is true that the unit vector $k$ is equal to $-\xi$ so we have

$$\int \int_D (k \cdot \xi)dS = -\int \int_{\Sigma'} (k \cdot \xi)dS = \int \int_{\Sigma'} dS = |\Sigma'|.$$

For the immersed case there is no actually a solid $T$, but one can still apply the divergence theorem. In this case $\Omega \cup \Sigma'$ separate $\mathbb{R}^3$ into finite number of connected
regions with one of them unbounded. On the bounded regions one can use the divergence theorem and the calculation will be the same as in the embedded case since \( \text{div} \mathbf{k} = 0 \) everywhere on \( \mathbb{R}^3 \). Thus, we get again

\[
\int \int_D (\mathbf{k} \cdot \xi) dS = |\Sigma'|.
\]

Let’s compute the other integral in the volume constraint, i.e. \( \int \int_D H(x \cdot \xi) dS \). Assuming conformal coordinates and using (2.3) and (2.8) and we get that

\[
\int \int_D H(x \cdot \xi) dS = \frac{1}{2} \int \int_D (x \cdot \Delta x) dS = -\frac{1}{2} \int \int_D |\nabla x|^2 dS + \frac{1}{2} \oint_{\partial D} (x \cdot x_\nu) d\sigma.
\]

Here \( \Delta x \) and \( \nabla x \) are the vector surface Laplacian and the vector surface gradient of \( x \). In conformal coordinates the square of the surface gradient of \( x \) is

\[
|\nabla x|^2 = \frac{1}{E}((x_u \cdot x_u) + (x_v \cdot x_v)) = \frac{1}{E}(E + E) = 2
\]

so

\[
-\frac{1}{2} \int \int_D |\nabla x|^2 dS = -\frac{1}{2} \int \int_D 2dS = -|\Omega|.
\]

Also if \( \mathbf{n} \) is the unit normal of \( \Gamma \) in \( \Sigma \) we have \( x_\nu = (\cos \gamma)\mathbf{n} - (\sin \gamma)\mathbf{k} \). Therefore \( (x \cdot x_\nu) = \cos \gamma (x \cdot \mathbf{n}) \). It follows that

\[
\frac{1}{2} \oint_{\partial D} (x \cdot x_\nu) d\sigma = \frac{\cos \gamma}{2} \oint_{\partial D} (x \cdot \mathbf{n}) d\sigma = \cos \gamma |\Sigma'|.
\]

The proof of the last equality will be given in Lemma 2.5.4 (part one), which is an application of the divergence theorem. Let us summarize the results that we have so far

\[
0 = |\Omega| + H \int \int_D (x \cdot \xi) dS + c \int \int_D (k \cdot \xi) dS
\]
\[ c \iint_D (k \cdot \xi) dS = c|\Sigma'| \]
\[ \iint_D H(x \cdot \xi) dS = -|\Omega| + \frac{1}{2} \oint_{\partial D} (x \cdot x_\nu) d\sigma \]
\[ \frac{1}{2} \oint_{\partial D} (x \cdot x_\nu) d\sigma = \cos \gamma |\Sigma'|. \]

Altogether these four facts give us

\[ 0 = |\Omega| - |\Omega| + \frac{1}{2} \oint_{\partial D} (x \cdot x_\nu) d\sigma + c|\Sigma'| = \cos \gamma |\Sigma'| + c|\Sigma'|. \]

This implies that \( c = -\cos \gamma \), therefore \( \phi = 1 + H(x \cdot \xi) - \cos \gamma (k \cdot \xi) \) and \( \iint_D \phi dS = 0. \)

Now, we will compute the boundary term for this particular \( \phi \). For a sketch see Figure 2-2. The claim is that \( \phi_\nu + p\phi = 0 \). Another useful fact is that \( \Gamma \) is a line of curvature for both the plane \( \Sigma \) and the surface \( \Omega \) (see the Terquem-Joachimsthal Theorem in the previous section). On \( \Gamma \) we have

\[ \phi_\nu + p\phi = (1 + H(x \cdot \xi) - \cos \gamma (k \cdot \xi))_\nu + K_\Omega (\cot \gamma) \phi \]
\[ = H \frac{\partial}{\partial \nu}(x \cdot \xi) - \cos \gamma \frac{\partial}{\partial \nu}(k \cdot \xi) + K_\Omega (\cot \gamma) \phi. \]

Now we compute the normal derivative and taking into account the fact that \( (x_\nu \cdot \xi) = 0 \) and \( k \) is constant vector. Since \( \Gamma \) is a line of curvature we get that \( \frac{\partial}{\partial \nu}(k \cdot \xi) = -K_\Omega (k \cdot x_\nu) \) and \( \frac{\partial}{\partial \nu}(x \cdot \xi) = -K_\Omega (x \cdot x_\nu) \). Substituting in in the boundary expression from above we have the following

\[ \phi_\nu + p\phi = -K_\Omega H(x \cdot x_\nu) + (\cos \gamma) K_\Omega (k \cdot x_\nu) + K_\Omega (\cot \gamma) \phi. \quad (2.11) \]

We use some more relations to rewrite this expression. Here \( n \) is the unit normal...
vector of $\Gamma$ in the plane $\Sigma$.

\[
(x \cdot x_\nu) = \cos \gamma (x \cdot n) \quad (x \cdot \xi) = \sin \gamma (x \cdot n) \quad (k \cdot x_\nu) = -\sin \gamma.
\]

Using this facts (2.11) becomes

\[
K_\Omega [-H (\cos \gamma)(x \cdot n) - \cos \gamma \sin \gamma + \cot \gamma (1 + H (x \cdot \xi) - \cos \gamma (k \cdot \xi))]
\]

\[
= K_\Omega [-H (\cos \gamma)(x \cdot n) - \cos \gamma \sin \gamma + (\cot \gamma) H \sin \gamma (x \cdot n) + \cot \gamma - \cot \gamma \cos \gamma \cos \gamma]
\]

since on $\Gamma$ we have $k \cdot \xi = \cos \gamma$. The first and the third term in the above expression cancel and we are left with

\[
K_\Omega [-\cos \gamma \sin \gamma + \cot \gamma - \cot \gamma \cos^2 \gamma]
\]

\[
= K_\Omega \cot \gamma [1 - \sin^2 \gamma - \cos^2 \gamma] = 0.
\]

This shows that the expression $\phi_\nu + p\phi \equiv 0$ on $\Gamma$ which means that

\[\text{Figure 2-2: } k, n, \xi \text{ and } x_\nu\]
\[ \int_{\partial D} (\phi_{\nu} + p\phi)\phi d\sigma = 0. \]

Therefore in the second variation formula \((2.7)\) the boundary term is zero and it transforms into

\[ \partial^2 E = \int \int_{D} (-L\phi)\phi dS. \quad (2.12) \]

Lemma 2.5.3.

\[ \partial^2 E = \int \int_{D} (-L\phi)\phi dS = - \int \int_{D} \frac{(k_1 - k_2)^2}{2} dS - \int_{\partial D} K_\Omega(\cos \gamma)[H(x \cdot n) + \sin \gamma]d\sigma. \quad (2.13) \]

Proof. First we have that for \(\phi = 1 + H(x \cdot \xi) - \cos \gamma(k \cdot \xi)\)

\[ (L\phi)\phi = (L\phi)(1 + H(x \cdot \xi) - \cos \gamma(k \cdot \xi)) = (L\phi) + (L\phi)(H(x \cdot \xi) - \cos \gamma(k \cdot \xi)) \]

so

\[ \int \int_{D} (-L\phi)\phi dS = - \int \int_{D} (L\phi)dS - \int \int_{D} (L\phi)(H(x \cdot \xi) - \cos \gamma(k \cdot \xi))dS. \]

Now, let’s compute \(L\phi\)

\[ L\phi = L_1 + L(H(x \cdot \xi)) - (\cos \gamma)L(k \cdot \xi) = k_1^2 + k_2^2 + HL(x \cdot \xi). \]

Here \(L_1 = \Delta 1 + (k_1^2 + k_2^2)1 = k_1^2 + k_2^2\) and \(L(k \cdot \xi) = 0\) (see [B-dC], page 347 Proposition (2.24)). Again from [B-dC], page 348 Lemma (3.6) it follows that \(L(x \cdot \xi) = -2H\).

Taking this into account we have

\[ L\phi = k_1^2 + k_2^2 - 2H^2 = k_1^2 + k_2^2 - \frac{(k_1 + k_2)^2}{2} = \frac{(k_1 - k_2)^2}{2}. \]
Getting back to the integral of \((−Lφ)φ\) we obtain

\[
\int \int_D (−Lφ)φdS = −\int \int_D \frac{(k_1 − k_2)^2}{2}dS − \int \int_D \frac{(k_1 − k_2)^2}{2}(H(x \cdot ξ) − \cos γ(k \cdot ξ))dS.
\]

Let us set \(ψ = H(x \cdot ξ) − \cos γ(k \cdot ξ)\), thus we need to compute \(\int \int_D (Lφ)ψdS\). From (2.8) it follows that

\[
\int \int_D (Lφ)ψdS = \int \int_D (Lψ)φdS + \oint_{∂D} (φνψ − ψνφ)dσ.
\]

To get this, we apply (2.8) to \(Δφψ\) and to \(Δψφ\) and subtract them from each other. In fact this is called Green’s second Identity. To get the formula for the operator \(L\) instead of \(Δ\) one just adds to both sides \((k_1^2 + k_2^2)φψ\). We know from the previous calculations that \(Lψ = −2H^2\) so

\[
\int \int_D (Lφ)ψdS = −2H^2 \int \int_D φdS + \oint_{∂D} (φνψ − ψνφ)dσ
\]

but we know \(\int \int_Ω φdS = ∂V = 0\). Using \(ψ = φ − 1\) the above formula reduces to

\[
\int \int_D (Lφ)ψdS = \oint_{∂D} (φν(φ − 1) − (φ − 1)νφ)dσ = −\oint_{∂D} φνdσ.
\]

From (2.11) we know that on \(Γ\), \(φν = −KΩH(x \cdot x_ν) + (\cos γ)KΩ(k \cdot x_ν)\) and we also know that on the boundary \((x \cdot x_ν) = \cos γ(x \cdot n), (k \cdot x_ν) = −\sin γ\) therefore

\[
φν = −KΩH(\cos γ)(x \cdot n) − (\cos γ)KΩ\sin γ
\]

and

\[
\int \int_D (Lφ)ψdS = \oint_{∂D} KΩ[H(\cos γ)(x \cdot n) + (\cos γ)KΩ\sin γ]dσ.
\]
Now substituting back \((L\phi)\psi\) in the formula

\[
- \int \int_D (L\phi)\phi dS = - \int \int_D \frac{(k_1 - k_2)^2}{2} dS - \int \int_D (L\phi)\psi dS
\]

we get

\[
- \int \int_D (L\phi)\phi dS = - \int \int_D \frac{(k_1 - k_2)^2}{2} dS - \oint_{\partial D} \left[ K_\Omega H(\cos \gamma)(x \cdot n) + (\cos \gamma)K_\Omega \sin \gamma \right] d\sigma.
\]

thus we have (2.13) which was the statement of the lemma.

To continue we have to rewrite (2.13). From Meusnier’s Theorem and from Euler’s Theorem in subsection (1.1.2) we know that on \(\Gamma\), \(2H = K_\Omega + k_2 = K_\Omega + (\sin \gamma)k_G\), where \(k_G\) is the curvature of the boundary. We have the following expression for \(K_\Omega\)

\[
K_\Omega = 2H - k_G \sin \gamma
\]

and therefore (2.13) becomes

\[
\partial^2 E = - \int \int_D \frac{(k_1 - k_2)^2}{2} dS - \oint_{\partial D} \cos \gamma[2H - k_G \sin \gamma][H(x \cdot n) + \sin \gamma] d\sigma
\]

\[
= - \int \int_D \frac{(k_1 - k_2)^2}{2} dS
\]

\[
- \oint_{\partial D} \cos \gamma[2H^2(x \cdot n) + 2H \sin \gamma - (\sin \gamma)k_G H(x \cdot n) - (\sin^2 \gamma)k_G] d\sigma.
\]

Lemma 2.5.4. Let \(\Sigma'\) is the region bounded by \(\Gamma\), and \(|\Sigma'|\) its area, and \(|\Gamma|\) the length of the boundary. Note the boundary may consist of several curves, so \(\Sigma'\) may not be connected. Let \(d\) be the number of boundary curves, i.e the number of components of \(\Gamma\). Of course \(\Sigma'\) belongs to the planar wall \(\Sigma\). We have the the following three formulae:

\[
\oint_{\partial D} (x \cdot n) d\sigma = 2|\Sigma'|
\]  

(2.14)
\[ \oint_{\partial D} k_{\Gamma}(x \cdot n) d\sigma = -|\Gamma| \] (2.15)

\[ \left| \oint_{\partial D} k_{\Gamma} d\sigma \right| \leq 2\pi d. \] (2.16)

Proof. Let \( x_1, x_2 \) be the coordinates in \( \Sigma \) which would mean that \( x = \langle x_1, x_2, 0 \rangle \).

Formula (2.14) follows immediately from a two dimensional version of the Divergence theorem and since \( \Gamma \) has no self intersections one has

\[ \int \int_{\Sigma'} \left( \text{div} x \right) dx_1 dx_2 = 2 \int \int_{\Sigma'} dx_1 dx_2 = 2|\Sigma'|. \]

For (2.15) let us use arc length. Let \( T = x' \) be the unit tangent vector of \( \Gamma \). We know that \( T' = k_{\Gamma} n \). Using integration by parts and the fact that \( \Gamma \) is a union of simple closed curves we get

\[ \oint_{\partial D} k_{\Gamma}(x \cdot n) d\sigma = \oint_{\partial D} (x \cdot T') d\sigma = -\oint_{\partial D} (x' \cdot T) d\sigma = -\oint_{\partial D} (T \cdot T) d\sigma = -\oint_{\partial D} d\sigma = -|\Gamma|. \]

The curve \( \Gamma \) has \( d \) components. Let us just focus on one of them - \( \Gamma_1 \). We will show that on \( \Gamma_1 \) the integral of the curvature \( k_{\Gamma} \) is \( \pm 2\pi \). Inequality (2.16) will then follow since \( \Gamma \) is embedded. Call \( \alpha \) the angle of inclination of \( \Gamma_1 \). Using arc length and standard definition from calculus we know that \( k_{\Gamma_1} = \alpha' \). Using the Fundamental Theorem of Calculus the integral of \( k_{\Gamma_1} \) will be the change of \( \alpha \) if the curve \( \Gamma_1 \) is traced once. But for simple closed curve this change is equal to \( \pm 2\pi \). The sign depends on the induced orientation of the component \( \Gamma_1 \). So we get

\[ \oint_{\partial D_1} \alpha' d\sigma = \pm 2\pi \]

and therefore

\[ \left| \oint_{\partial D} k_{\Gamma} d\sigma \right| \leq 2\pi d. \]
Here $x(\partial D_1) = \Gamma_1$. 

We also need another fact that may be found written in a slightly more general form in [E-B-M-R]. We have all the ingredients to prove it now.

**Theorem 2.5.5.** *(Balancing Formula)*

*In the above notation one has*

$$\sin \gamma |\Gamma| = -2H |\Sigma'|.$$  \hspace{1cm} (2.17)

**Proof.** Choose conformal coordinates. First of all, from the proof of lemma (2.5.2) we saw that

$$\int \int_D (k \cdot \xi) dS = |\Sigma'|$$

and also we know from (2.3) that for the surface Laplacian in conformal coordinates we have $\Delta X = 2H \xi$. When $H \neq 0$ (for us $H$ is negative), we use this fact and *Green's First Identity* to get

$$\int \int_D (k \cdot \xi) dS = \frac{1}{2H} \int \int_D (k \cdot \Delta x) dS = \frac{1}{2H} \int_{\partial D} (k \cdot x_\nu) d\sigma.$$ 

This equality holds since $k$ is a constant vector, therefore its surface gradient is zero.

From a previous discussion of the result that the boundary term in $\partial^2 E$ is zero, we know that on $\Gamma$, $(k \cdot x_\nu) = -\sin \gamma$. Combining that fact with the above expressions for the integral of $(k \cdot \xi)$ over $\Omega$ we arrive at

$$|\Sigma'| = \int \int_D (k \cdot \xi) dS = \frac{1}{2H} \int_{\partial D} (k \cdot x_\nu) d\sigma = -\frac{1}{2H} \sin \gamma |\Gamma|. $$

Taking the first and the last expressions in the above line and multiplying by $-2H$ we get

$$-2H |\Sigma'| = \sin \gamma |\Gamma|$$
which was the result to prove. Note that this is an indication that the mean curvature $H$ of $\Omega$ must be negative for the immersed case, since all the other quantities in (2.17) are positive.

Note that from this formula it follows that

$$|H| = \frac{\sin \gamma |\Gamma|}{2|\Sigma'|} \leq \frac{|\Gamma|}{2|\Sigma'|}.$$ 

This is a special case of a well known result by E. Heinz on nonexistence of surfaces of constant mean curvature, which puts a bound on $H$ see [H]. Now, if we use (2.14) and (2.15) we can rewrite the Second variation once more.

$$\partial^2 E = -\int_D \frac{(k_1 - k_2)^2}{2} dS + \cos \gamma \left[ -4H^2|\Sigma'| - 2H|\Gamma| \sin \gamma - (\sin \gamma)H|\Gamma| + \sin^2 \gamma \oint_{\partial D} k_T d\sigma \right]$$

and if we use (2.17) we arrive at

$$\partial^2 E = -\int_D \frac{(k_1 - k_2)^2}{2} dS + \cos \gamma \left[ 2H(\sin \gamma)|\Gamma| - 2H(\sin \gamma)|\Gamma| - H(\sin \gamma)|\Gamma| + \sin^2 \gamma \oint_{\partial D} k_T d\sigma \right].$$

After the obvious cancellation we obtain

$$\partial^2 E = -\int_D \frac{(k_1 - k_2)^2}{2} dS + \cos \gamma \left[ -(\sin \gamma)H|\Gamma| + \sin^2 \gamma \oint_{\partial D} k_T d\sigma \right]$$

and by using (2.17) again, we get

$$\partial^2 E = -\int_D \frac{(k_1 - k_2)^2}{2} dS + \cos \gamma \left[ 2H^2|\Sigma'| + \sin^2 \gamma \oint_{\partial D} k_T d\sigma \right].$$

We can easily convince ourselves that this last expression is zero if $\Omega$ is the standard spherical cap. On a spherical cap all points are umbilical, i.e. $k_1 = k_2$ everywhere so the first integral is zero. Furthermore we can see that $H^2|\Sigma'| = \pi \sin^2 \gamma$ no matter
what the scaling and for a spherical cap $\oint_{\partial D} k_\Gamma d\sigma = -2\pi$ since we chose to work with
the outward unit normal and for us $k_\Gamma \leq 0$. This means that the second expression
is also zero so the whole variation is zero. Thus, on a spherical cap this particular
variation just "lifts" the cap but does not change its geometry.

One can express the first integral in the above formula in another way.

\[
\frac{(k_1 - k_2)^2}{2} = \frac{k_1^2 + k_2^2 + 2k_1k_2 - 4k_1k_2}{2} = \frac{(k_1 + k_2)^2 - 4k_1k_2}{2} = \frac{(2H)^2 - 4K}{2} = 2H^2 - 2K.
\]

The integral of the square of $H$ minus the integral of $K$ is a well known quantity and it
goes in the literature by the name of Willmore Energy. There is a famous conjecture
related to it, which one can see in [B]. Continuing with the second variation and using
the previous formula we get

\[
\partial^2 E = -2 \int_D H^2dS + 2 \int_D KdS + \cos \gamma [2H^2|\Sigma'| + \sin^2 \gamma \oint_{\partial D} k_\Gamma d\sigma].
\]

Next, we apply the Gauss-Bonnet Formula, see (2.4).

\[
\partial^2 E = -2 \int_D H^2dS + 4\pi \chi(\Omega) - 2 \oint_{\partial D} k_g d\sigma + \cos \gamma [2H^2|\Sigma'| + \sin^2 \gamma \oint_{\partial D} k_\Gamma d\sigma]
\]

and we use again Meusnier’s Theorem and Euler’s Theorem in subsection
(1.1.2), to get that, $k_g = \pm (\cos \gamma)k_\Gamma$ on each $\Gamma_i$. From (2.16) it follows that

\[
\left| \oint_{\partial D} k_g d\sigma \right| \leq | \cos \gamma | 2\pi d.
\]

Also we know that if $\Omega$ has genus $g$ and $d$ boundary curves, that $\chi(\Omega) = 2 - 2g - d$.
This follows from the fact that one can attach flat discs to the surface at the boundary
to make it closed and reattaching the disks will decrease $\chi(\Omega)$ exactly with $d$. Taking all this into account we have the following

$$\partial^2 E = -2H^2 \left[ \int_D dS - \cos \gamma |\Sigma'| + 4\pi \chi(\Omega) - 2 \int_{\partial D} k_g d\sigma + \cos \gamma \sin^2 \gamma \int_{\partial D} k_\Gamma d\sigma \right]$$

$$\leq -2H^2 [\Omega] - \cos \gamma |\Sigma'| + 4\pi (2 - 2g - d) + 4\pi d |\cos \gamma| + |\cos \gamma|(\sin^2 \gamma) 2\pi d$$

$$= -2H^2 [\Omega] - \cos \gamma |\Sigma'| + 4\pi (2 - 2g) - 2\pi d [2 - 2|\cos \gamma| - |\cos \gamma| \sin^2 \gamma].$$

The first term is always negative because $|\Omega| > |\Sigma'|$ since $\Sigma'$ is a planar surface spanning $\Gamma$, so

$$\partial^2 E < 4\pi (2 - 2g) - 2\pi d [2 - 2|\cos \gamma| - |\cos \gamma| \sin^2 \gamma].$$

For us $\gamma \in (0, \pi)$ and there are two cases: $\gamma \leq \pi/2$ or $\gamma > \pi/2$.

(Case I) If $\gamma \leq \pi/2$, we have $2 - 2|\cos \gamma| - |\cos \gamma| \sin^2 \gamma = 2 - \cos \gamma(2 + \sin^2 \gamma)$ and if $\gamma = 0$ this quantity is zero. Taking the derivative of this expression with respect to $\gamma$ one gets

$$[2 - \cos \gamma(2 + \sin^2 \gamma)]_\gamma = \sin \gamma (2 + \sin^2 \gamma) - 2 \cos \gamma \sin \gamma \cos \gamma$$

$$= \sin \gamma (2 + \sin^2 \gamma - 2 \cos^2 \gamma)$$

$$= 3 \sin \gamma \sin^2 \gamma$$

$$\geq 0$$

whenever $0 \leq \gamma \leq \pi$. So at zero this quantity is zero and after that its derivative is nonnegative, so it is nondecreasing. This means that it is always nonnegative and it is positive on $(0, \pi/2]$. In view of that we have once again that

$$\partial^2 E < 4\pi (2 - 2g)$$

so if the genus $g$ of $\Omega$ is positive we get a that the second variation of energy is
negative, i.e. $\Omega$ is unstable.

*(Case II)* If $\gamma > \pi/2$ we have a similar calculation. The quantity

$$2 - 2|\cos \gamma| - |\cos \gamma| \sin^2 \gamma = 2 + 2 \cos \gamma + \cos \gamma \sin^2 \gamma$$

is bigger than zero on $(\pi/2, \pi)$, since at $\pi/2$ it is equal to 2, its derivative is decreasing on $(\pi/2, \pi)$ and at $\pi$ it is equal to 0. So in this case again

$$\partial^2 E < 4\pi(2 - 2g)$$

and if $g > 0$ we get that $\Omega$ is unstable.

One should point out that the discussion above fully resolves the case for an immersed stable capillary surface with planar embedded boundary and that is the only boundary for which the stability is completely understood.

Another thing to mention is a different proof of the case $\gamma \geq \pi/2$ and one boundary curve, using the *Isoperimetric Inequality* (for proof of it see [O]).

**Theorem 2.5.6. (Isoperimetric Inequality)**

*Among all the simple closed curves in the plane having fixed length $L$ and enclosing area $A$, the circle bounds the most area. Moreover, we always have*

$$L^2 \geq 4\pi A$$

*(2.20)*

*with equality holding only in the case of a circle.*

For one boundary curve with the chosen orientation (using the outward normals),
we have \( \oint_{\partial D} k_\Gamma d\sigma = -2\pi \). Substituting in (2.19) and using (2.17) we get

\[
\partial^2 E = - \int \int_D \frac{(k_1 - k_2)^2}{2} dS + \cos \gamma \left[ \frac{2 \sin^2 \gamma |\Gamma|^2 |\Sigma'|}{4 |\Sigma|^2} - 2 \pi \sin^2 \gamma \right]
\]

\[
= - \int \int_D \frac{(k_1 - k_2)^2}{2} dS + 2 \cos \gamma \sin^2 \gamma \left[ \frac{|\Gamma|^2 - 4 \pi |\Sigma'|}{4 |\Sigma'|} \right].
\]

Now we apply (2.20) to the second part of the above expression and use the fact that on \([\pi/2, \pi)\), \(\cos(\gamma) \leq 0\), to get that \(\partial^2 E \leq 0\) and it is zero only when \(\Omega\) is totally umbilical and \(\Gamma\) is a circle, i.e. we have a spherical cap. This is a contradiction with the assumption that \(\Omega\) is not spherical and stable. Notice, that this estimation does not depend on the genus of \(\Omega\). Actually this method can be generalized for boundary with \(s\) component curves that do not enclose each other. If the boundary curves are \(\{\Gamma_i\}_{i=1}^s\) and the regions \(\Sigma_i\) that they bound have mutually empty intersections, we can use the Isoperimetric Inequality on each \(\Gamma_i\). As before we assume that each \(\Gamma_i\) is a simple closed curve in \(\Sigma\) and since \(|\Gamma| = \sum_{i=1}^s |\Gamma_i|\) we have by the Cauchy inequality and by (2.20) that

\[
|\Gamma|^2 \geq \sum_{i=1}^s |\Gamma_i|^2 \geq 4\pi \sum_{i=1}^s |\Sigma_i|.
\]

Here \(x(\partial D_i) = \Gamma_i\). Now by (2.16)

\[
\oint_{\partial D_i} k_\Gamma d\sigma = -2\pi
\]

with the sign depending on the orientation that is induced by the outward normal of \(\Omega\) and by \(\bigcup_{i=1}^s \Sigma_i = \Sigma'\), we can do the same estimate like above and deduce a contradiction with the stability of \(\Omega\), when \(\gamma \in [\pi/2, \pi)\).

The ideas and results that we applied in this chapter are all well known and broadly used in the past, but they have not been used in this way and entirety before. To summarize the main result, we assume that \(\Omega\) is an immersed capillary surface with planar boundary and positive genus. Then we construct a perturbation with normal
component that makes the second variation of the energy negative, which implies that there are no stable configurations of the assumed type. Therefore in the absence of gravity, the spherical cap is the only immersed stable capillary surface with boundary embedded in a plane.
Chapter 3

Exotic Containers and Stability of the Spherical Caps

The main goal of this chapter is to show the known result that spherical caps “sitting” inside of a ball or on a plane are weakly stable and the only perturbations which do not increase the energy are shifts in 3-space. We provide an alternative proof using Exotic containers. Again no gravity is assumed. For us, an Exotic container is a rotational container which can be partially filled with liquid in a continuum of different ways (having the same volume), each one in equilibrium, such that each member of the family of interfaces makes the same constant contact angle \( \gamma \in (0, \pi) \) with the container and they are all mutually non-congruent. Of coarse, the interfaces are all spherical except one which is planar.

We use a construction carried out by Finn (see [F 1]) for the container. Since it is rotationally symmetric, the problem reduces to finding the equation for the profile curve. We pick coordinates \( r \) and \( z \) in the right half plane. Following Finn, the curve will be of the form \( r = r(z) \). Also let \( w = \sqrt{1+r^2} \). The differential equation for the profile curve is derived from the constant volume condition and the same constant
contact angle $\gamma$. We have the following formula obtained in [F 1]

$$rr'' + w^2(2w \sin \gamma + r' \sin \gamma \cos \gamma + 1 + \sin^2 \gamma) = 0. \tag{3.1}$$

From (3.1) one can easily see that the curvature of the profile curve $K_e$ is always negative

$$K_e = \frac{r''}{w^3} = -\frac{2w \sin \gamma + r' \sin \gamma \cos \gamma + 1 + \sin^2 \gamma}{rw} \tag{3.2}$$

since the numerator in (3.2) is

$$\sin(\gamma)[2\sqrt{1 + r'^2} + r' \cos \gamma] + 1 + \sin^2 \gamma \geq \sin \gamma |r'| + 1 + \sin^2 \gamma > 1.$$

The profile curve has a minimum point, so equation (3.1) represents the curve on both sides of that minimum. To represent the curve around the minimum point one can invert $z$ as a function of $r$ and get a similar equation again with negative curvature.

Now the idea is to draw a tangent sphere, with center at the $z$-axis to every point of the profile curve (see figure 3-1). This will mean that the continuum of interfaces still makes the same contact angle with the tangent spheres. One can easily obtain formula for the curvature $K_\Sigma$ of the tangent spheres, namely $|K_\Sigma| = \frac{1}{rw}$. Now, from (3.2) and the estimate for the numerator it follows that

$$|K_e| = \frac{2w \sin \gamma + r' \sin \gamma \cos \gamma + 1 + \sin^2 \gamma}{rw} \geq \frac{1}{rw} = |K_\Sigma| \tag{3.3}$$

and since $K_e$ is negative it follows that

$$K_\Sigma > K_e$$

at every point of the profile curve. Now let us compare the second variation of energy for a spherical cap in the cases when it ”sits” in a sphere and when it is in the Exotic

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Figure 3-1: Exotic container and the tangent sphere

container. If a spherical cap is inside of the container it will be a member of the continuum of equilibrium configurations, i.e. the first variation of energy $\partial E$ is 0 if we use the container as boundary. Let us recall the formula for $\partial^2 E$ for a normal variation $\phi$ (see formula (2.7)): For a spherical or planar boundary $\Sigma$ we have

$$
\left. \partial^2 E[\phi] \right|_\Sigma = \int \int_D \left[ |\nabla \phi|^2 - (k_1^2 + k_2^2)\phi^2 \right] dS + \oint_{\partial D} \left[ K_\Omega \cot \gamma + K_\Sigma \csc \gamma \right] \phi^2 d\sigma
$$

and for the exotic container we have

$$
\left. \partial^2 E[\phi] \right|_{\epsilon} = \int \int_D \left[ |\nabla \phi|^2 - (k_1^2 + k_2^2)\phi^2 \right] dS + \oint_{\partial D} \left[ K_\Omega \cot \gamma + K_{\epsilon} \csc \gamma \right] \phi^2 d\sigma
$$

for a fixed angle $\gamma$, fixed volume and a spherical cap $\Omega$. The only difference between the two formulae above is the term containing the normal curvatures in the boundary
integral. Therefore if $\phi$ is not identically zero on $\partial D$, one gets

$$\partial^2(E)[\phi]_{\Sigma} - \partial^2(E)[\phi]_e = \oint_{\partial D} [K_{\Sigma} - K_e](\csc \gamma) \phi^2 d\sigma > 0$$

which implies that

$$\partial^2(E)[\phi]_{\Sigma} > \partial^2(E)[\phi]_e.$$

We now have two cases.

The family of equilibrium spherical caps generates a vector field whose normal perturbation is symmetric, $\phi = \phi(r)$. The volume constraint is satisfied for this function. It is known that for the exotic container this particular $\phi$ provides the infimum for the second variation of energy among all admissible symmetric functions and $\partial^2(E)[\phi]_e = 0$. Therefore $\partial^2(E)[\phi]_{\Sigma} > 0$.

Next we have the case when $\phi$ is not symmetric, (for example $\phi$ could be a rotation about the center of $\Sigma$). The volume constraint is satisfied now. We need to solve the Jacobi equation

$$L\phi = \lambda \phi$$

$$\phi_\nu + p\phi = 0$$

by separating variables, i.e. $\phi = A(r)B(\theta)$. After solving we obtain that the smallest eigenvalue $\lambda_1$ is zero and the eigenspace for $\lambda_1$ is 2-dimensional. In chapter two was pointed out that $\partial^2(E)[\phi]_{\Sigma} = \lambda_1$, so we get the weak stability of the spherical cap $\Omega$ if $\Sigma$ is a plane or a sphere. This also shows that $\Omega$ is unstable in the exotic container.

The above results can be summarized in the following theorem.

**Theorem 3.0.7.** Let $\phi \neq 0$ at $\partial D$. We have $\partial^2(E)[\phi]_{\Sigma} > \partial^2(E)[\phi]_e$. Moreover

1) If $\phi$ is a rotation about the center of $\Sigma$, then $\partial^2(E)[\phi]_{\Sigma} = 0 \Rightarrow \partial^2(E)[\phi]_e < 0$.

2) If $\phi$ is symmetric, then $\partial^2(E)[\phi]_e \geq 0 \Rightarrow \partial^2(E)[\phi]_{\Sigma} > 0$. 

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Chapter 4

Discussion on the Stability of Capillary Surfaces in a Ball

In this chapter we will investigate the stability of a capillary surface with a spherical boundary, that lies entirely in a ball with the same sphere as a boundary. This was the main problem discussed in [R-S] and again this paper will be a guideline for us. There are two main results in [R-S]. The first one is that the stable surfaces described above must be spherical caps or flat discs when their genus is zero. The second one is that if the capillary surface is also minimal ($H = 0$), then its genus is at most one and the surface has at most three connected boundary components. Some of their results hold if the ambient space is not Euclidean - could be using either hyperbolic or spherical metric.

Our approach in this problem is similar to the one we used in Chapter 2, when we had a planar boundary instead of a spherical one. We construct a specific vector field and exhibit a normal perturbation from it. That vector field was hinted in [R-S]. It has some very nice properties which we will show, e.g it is conformal and tangent to the bounding sphere. It actually comes from a linear fractional transformation in the complex plane which is rotated about an axis to create a there dimensional vector
field on the unit ball. We will provide some new results on the above problem and give an alternative proof of a theorem of Barbosa and do Carmo which was stated in chapter 2, in the section on closed surfaces.

4.1 Preliminaries

The setup is absolutely the same as in chapter 2 section 2 - just the boundary is spherical. One has a surface Ω that lies entirely in a unit ball $B$ whose boundary of $\Omega$ will lie in $\partial B = S^2$. Again let $\Omega$ be given by a $C^{2,\alpha}$-immersion (in fact $C^\infty$) $\bar{x}(u, v) : D \to \Omega \subset B$, where $D$ is some abstract 2-dimensional manifold (see [Ho]).

We call the boundary $\partial \Omega$, $\Gamma$, and the regions in $S^2$ bounded by $\Gamma$ we label $\Sigma'$. We denote the areas on the surface and the wetted area by $|\Omega|$ and $|\Sigma'|$, respectively. We denote the angle of contact between the surface and the wetted area by $\gamma$. We will assume that $\Omega$ is extendable in a neighborhood of $\Gamma$ so we can compute tangent vectors, normal vectors, etc. We will put the origin of $\mathbb{R}^3$ in the center of the unit ball $B$.

Again like in section (2.2) we compute the first and the second variation of the energy $= |\Omega| - \cos \gamma |\Sigma'|$, in a direction of a function $\phi$, subject to stationary volume. The function $\phi$ is the normal component of a perturbation. If a surface is a critical point for the first variation and makes the second variation positive for all admissible (preserving the volume to the 1$^{st}$ order) normal variations $\phi$, we call it a strictly stable capillary surface. The first, second variation and the volume constraint are the same as in (2.2). We will exhibit a normal variation $\phi$ via a conformal vector field $F$ which we will define in the next section. After that we will do the main calculations for the second variation formula using that particular $\phi$. In fact we will construct a family of normal variations for every unit vector in $\mathbb{R}^3$. 
4.2 A Construction of a Conformal Vector Field on $\mathbb{R}^3$

In this section we will construct a vector field on the unit ball in $\mathbb{R}^3$. It will be conformal, i.e. it is the infinitesimal generator of a family of conformal maps. Also, it will be tangent to the unit sphere $S^2$. A vector field $F$ is conformal if it satisfies $F = X'(t)|_{t=0}$ with $X(0) = Id$, where $X$ is a family of conformal maps from $\mathbb{R}^3$ into $\mathbb{R}^3$, parametrized by $t$. The family $X = <x(u,v,w,t), y(u,v,w,t), z(u,v,w,t)>$ parametrized by $t$, is conformal with respect to the standard scalar product if $|X_u|^2 = |X_v|^2 = |X_w|^2$ and $(X_u \cdot X_v) = (X_u \cdot X_w) = (X_w \cdot X_v) = 0$ for all $t$. Let first construct the vector field $F$ and then show that it is conformal. The idea comes from the linear fractional transformations in complex analysis. The construction of such a vector field is discussed in [R-S] and it is used there to prove the second main result of the paper. Here we will obtain an explicit formula for it.

Let $z$ be the complex coordinate in $\mathbb{R}^2$. A linear fractional transformation is a function of the form

$$w = \frac{az + b}{cz + d}$$

where $a, b, c, d$ are complex numbers with $ad - bc \neq 0$. It is well known (see [Al]) that the linear fractional transformations, being meromorphic functions are conformal mappings. Let $t$ be a real number. It is also known from [Al] that the most general linear fractional transformation that maps the unit disc into itself is given by

$$w = e^{i\theta} \frac{z - \alpha}{1 - z\bar{\alpha}}$$

with $|\alpha| < 1$. We define

$$w_t = \frac{z + t}{1 + tz}$$

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to be a family of transformations depending on \( t \in (-1, 1) \). Notice that \( w_t \) is of hyperbolic type (see [Al]), i.e. it has two fixed points, namely \( z = \pm 1 \). The family \( w_t \) maps the unit disc \( B_2 \subset \mathbb{R}^2 \) into itself. To see that we compute

\[
|w_t|^2 = w_t \bar{w}_t = \frac{z + t}{1 + tz} \cdot \frac{\bar{z} + t}{1 + t\bar{z}} = \frac{|z|^2 + t^2 + t(z + \bar{z})}{1 + t^2|z|^2 + t(z + \bar{z})}.
\]

From this we see that either \( |w_t| = |z| = 1 \) or \( |w_t| > 1, |z| > 1 \) or \( |w_t| < 1, |z| < 1 \), which shows that \( w_t : B_2 \to B_2 \). To get a vector field on \( B_2 \) from the family of transformations defined above we differentiate \( w_t \) with respect to \( t \) and set \( t \) equal to zero.

\[
\frac{d}{dt} w_t \bigg|_{t=0} = \frac{(1 + tz) - (z + t)z}{(1 + tz)^2} \bigg|_{t=0} = \frac{1 - z^2}{(1 + tz)^2} \bigg|_{t=0} = 1 - z^2.
\]

Let \( x, y \) be coordinates in \( R^2 \) so that \( z = x + iy \). Rewriting \( \frac{d}{dt} w_t \bigg|_{t=0} \) we get

\[
\frac{d}{dt} w_t \bigg|_{t=0} = 1 - z^2 = 1 - (x + iy)^2 = 1 - x^2 + y^2 - 2ixy
\]

so the vector field that we get on \( B_2 \) (actually on \( \mathbb{R}^2 \)) will be \( <1 - x^2 + y^2, -2xy>^T \). This vector field fixes two points, namely \( z = \pm 1 \). Rotating about the \( x \)-axis we get a vector field on \( \mathbb{R}^3 \). From now on we call it will it \( F \) and after the rotation we get

\[
F = <1 - x^2 + y^2 + z^2, -2xy, -2xz>^T.
\]

It generates a flow on \( \mathbb{B}^3 \) with fixed points (\( \pm 1, 0, 0 \)). Now we show \( F \) is conformal. To do so we will obtain a condition on the differential of \( F \). We state again the conditions on the family \( X = <x(u, v, w, t), y(u, v, w, t), z(u, v, w, t)>^T \).

\[
|X_u|^2 = |X_v|^2 = |X_w|^2
\]
\[(X_u \cdot X_v) = (X_u \cdot X_w) = (X_w \cdot X_v) = 0\]

with \(X(u, v, w, 0) = (u, v, w)\) and \(F = (F_1, F_2, F_3)^T = X'(t)|_{t=0}\). Here \(F\) is given with respect to the \((u, v, w)\) coordinates. Differentiating the first condition with respect to \(t\) and setting \(t = 0\), one gets

\[
2\left(\frac{dX_u}{dt}(0) \cdot X_u(0)\right) = 2\left(\frac{dX_v}{dt}(0) \cdot X_v(0)\right) = 2\left(\frac{dX_w}{dt}(0) \cdot X_w(0)\right)
\]

and from \(x(u, v, w, 0) = u, y(u, v, w, 0) = v, z(u, v, w, 0) = w\) it follows that \(X_u(0) = <1, 0, 0>^T, X_v(0) = <0, 1, 0>^T, X_w(0) = <0, 0, 1>^T\) and

\[
\frac{dx_u}{dt}(0) = \frac{dy_v}{dt}(0) = \frac{dz_w}{dt}(0)
\]

but

\[
\frac{dx_u}{dt}(0) = \frac{dF_1}{du}, \frac{dy_v}{dt}(0) = \frac{dF_2}{dv}, \frac{dz_w}{dt}(0) = \frac{dF_3}{dw}
\]

therefore

\[
\frac{dF_1}{du} = \frac{dF_2}{dv} = \frac{dF_3}{dw}
\]

Differentiating the second condition on \(X\) with respect to \(t\), setting \(t = 0\) and using \(X_u(0) = <1, 0, 0>^T, X_v(0) = <0, 1, 0>^T, X_w(0) = <0, 0, 1>^T\) we get

\[
0 = \left(\frac{dX_u}{dt}(0) \cdot X_v(0)\right) + \left(\frac{dX_v}{dt}(0) \cdot X_u(0)\right) = \frac{dy_u}{dt}(0) + \frac{dx_v}{dt}(0) = \frac{dF_2}{du} + \frac{dF_1}{dv}
\]

\[
0 = \left(\frac{dX_u}{dt}(0) \cdot X_w(0)\right) + \left(\frac{dX_w}{dt}(0) \cdot X_u(0)\right) = \frac{dz_u}{dt}(0) + \frac{dx_w}{dt}(0) = \frac{dF_3}{du} + \frac{dF_1}{dw}
\]

\[
0 = \left(\frac{dX_w}{dt}(0) \cdot X_v(0)\right) + \left(\frac{dX_v}{dt}(0) \cdot X_w(0)\right) = \frac{dy_w}{dt}(0) + \frac{dz_v}{dt}(0) = \frac{dF_2}{dw} + \frac{dF_3}{dv}
\]

If one integrates these conditions, he will get the conditions on \(X\). To summarize, a vector field \(F = (F_1(u, v, w), F_3(u, v, w), F_3(u, v, w))^T\) is conformal if and only if

\[
\frac{dF_1}{du} = \frac{dF_2}{dv} = \frac{dF_3}{dw}
\]
Thus, we can rewrite the above calculations in a matrix form by stating the following theorem

**Theorem 4.2.1.** Let \( F = \langle F_1(u, v, w), F_2(u, v, w), F_3(u, v, w) \rangle^T \) be a vector field on \( \mathbb{R}^3 \), and let \( DF = \begin{bmatrix} DF_1 & DF_2 & DF_3 \\ \frac{DF_1}{du} & \frac{DF_2}{dv} & \frac{DF_3}{dw} \\ \frac{DF_1}{du} & \frac{DF_2}{dv} & \frac{DF_3}{dw} \end{bmatrix} \) be the differential of \( F \). Then, \( F \) is conformal

if and only if

\[
DF + DF^T = \lambda(u, v, w)Id
\]

where \( DF^T \) is the transpose matrix of \( DF \), \( \lambda(u, v, w) \) is a scalar function and \( Id \) is the identity matrix in \( \mathbb{R}^3 \).

Now if we write the \( F \) we constructed above in the \((u, v, w)\) coordinates we get

\[
F = \langle F_1, F_2, F_3 \rangle^T = \langle 1 - u^2 + v^2 + w^2, -2uv, -2uw \rangle^T
\]

and a simple differentiating shows that the conditions derived above are fulfilled. Thus \( F \) is conformal.

One must point out that there are other ways to show that \( F \) is conformal. One can use *Liouville’s theorem for conformal mappings* (see [Bl]) and argue that the family of mappings \( X_t \) corresponds to two inversions with respect to spheres. Another way is to directly integrate \( F \), i.e solve the autonomous system

\[
\frac{dx}{dt} = 1 - x^2 + y^2 + z^2, \quad \frac{dy}{dt} = -2xy, \quad \frac{dz}{dt} = -2xz
\]

with initial conditions

\[
x(0) = u, y(0) = v, z(0) = w
\]
and check that the solution is a conformal family of mappings parametrized by $t$. It can be done as follows. From the second and the third equation in the above system one can express $z$ in terms of $y$ and then solve the first and the second equation using the integrating factor $1/y^2$. Another fact to mention is that the vector field $F$ is actually Killing vector field for the hyperbolic 3-space. It is conformal in $\mathbb{R}^3$ so it must be Killing in $\mathbb{H}^3$ (see [Sp]), assuming that the integrated vector field maps the unit ball into itself. Also one can see that the Lie group of all conformal maps of the unit ball in $\mathbb{R}^3$ into itself, i.e. all isometric maps of the unit ball in $\mathbb{H}^3$ into itself is six dimensional. Later in the chapter we will construct a family of conformal vector fields depending on a unit vector (the vector field $F$ is a member of that family) and this space is 3-dimensional. Another space of conformal vector fields comes from the rotations in $\mathbb{R}^3$ and it is also 3-dimensional. One can check directly that $F$ is a Killing field for the hyperbolic metric. A Killing field is the infinitesimal generator of one parameter Lie group of isometries with respect to a given metric. The condition that a vector field $F$ is Killing is that the Lie derivative of the metric in the direction of $F$ is zero. For details on the above statements and for definition of Lie derivative of a tensor one can see [L]. The metric in the hyperbolic space $\mathbb{H}^3$ in coordinates $x, y$ and $z$ is given by

$$g_{ij} = \frac{4 \delta_{ij}}{(1 - x_1^2 - x_2^2 - x_3^2)^2} = \frac{4 \delta_{ij}}{\omega^2}$$

where $i$ and $j$ are between one and three. From [L] we get the formula for the Lie derivative of the metric tensor in the direction of $F = <F_1, F_2, F_3>$$$
\mathcal{L}_F g_{ij} = (\nabla g_{ij} \cdot F) + g_{nj} \frac{d F_n}{dx_i} + g_{in} \frac{d F_n}{dx_j}.$$

Here we use Einstein’s summation notation (summing over repeated dummy indices). Replacing $x_1, x_2, x_3$ with $x, y, z$, taking into account that $g_{ij}$ is a scalar multiple of $\delta_{ij}$
and using the formula for $F$ we compute

$$\mathcal{L}_F g_{11} = (\nabla g_{11} \cdot F) + 2g_{11} \frac{dF_1}{dx}$$

$$= \frac{16x(1 - x^2 + y^2 + z^2)}{\omega^2} + \frac{16y(-2xy)}{\omega^2} + \frac{16z(-2xz)}{\omega^2} + \frac{4(-2x)}{\omega^2}$$

$$= \frac{16}{\omega^2}(x - x^3 + xy^2 + xz^2 - 2xy^2 - 2xz^2 - x(1 - x^2 - y^2 - z^2)) = 0.$$

Using the fact metric $g_{ij}$ is conformal to the flat one, i.e. $g_{11} = g_{22} = g_{33} > 0$ and $g_{ij} = 0$ if $i \neq j$ and $\frac{dF_1}{dx} = \frac{dF_2}{dy} = \frac{dF_3}{dz} = -2x$, we obtain

$$\mathcal{L}_F g_{22} = (\nabla g_{22} \cdot F) + 2g_{22} \frac{dF_2}{dy} = (\nabla g_{11} \cdot F) + 2g_{11} \frac{dF_1}{dx} = \mathcal{L}_F g_{11} = 0$$

$$\mathcal{L}_F g_{33} = (\nabla g_{33} \cdot F) + 2g_{33} \frac{dF_3}{dz} = (\nabla g_{11} \cdot F) + 2g_{11} \frac{dF_1}{dx} = \mathcal{L}_F g_{11} = 0.$$

Finally, if $i \neq j$ and again using $x_1, x_2, x_3$, we have

$$\mathcal{L}_F g_{ij} = g_{jj} \frac{dF_j}{dx_j} + g_{ii} \frac{dF_i}{dx_i} = g_{ii} (\frac{dF_j}{dx_j} + \frac{dF_i}{dx_i}) = 0.$$

The last equality follows from theorem (4.2.1) or simply by differentiating the components of $F$. We therefore showed that $\mathcal{L}_F g_{ij} = 0$ for all $i, j$, thus $F$ is Killing with respect to the hyperbolic metric.

Now we proceed with showing two important results about the vector field $F$. Let us write again the formula (2.10) from chapter 2 for the second variation of energy and the volume constraint (2.6).

$$\partial^2 \mathcal{E} = \int \int_D (-L \phi) dS + \int_{\partial D} (\phi_{\nu} + p \phi) d\sigma$$

$$\partial \mathcal{V} [\phi] = \left. \frac{d}{dt} V[\phi] \right|_{t=0} = \int \int \phi dS$$

where $\phi$ is the normal component of the vector field $F$, i.e. $\phi = (F \cdot \xi)$, with $\xi$ being
the unit outward normal vector to $\Omega$. Again as before $\gamma$ is the contact angle between $\Omega$ and the wetted area $\Sigma$. The vector field $F$ is conformal as we saw from the previous discussion. Our next lemma is of general interest and will imply that for such an $F$ the boundary integral in the volume condition $\text{(2.10)}$ is zero. For proof see [W].

**Lemma 4.2.2.**

$$\partial \gamma = \phi \nu + p \phi$$

and $\phi \nu + p \phi = 0$ on $\Gamma \subseteq S^2$ since $\phi$ is the normal component of the conformal vector field $F$.

Since $F$ is angle preserving, the boundary integral in the second variation formula is zero for $\phi = (F \cdot \xi)$. Therefore for that particular $\phi$ the second variation of energy is just

$$\partial^2 E = - \int \int_D (L\phi)\phi dS.$$  

We shall now rewrite $\phi$ in another form that will be useful later in this chapter. Remember that $\Omega$ was given by the mapping $\bar{x}(u, v) = < x(u, v), y(u, v), z(u, v) >^T$ from some abstract 2-dimensional surface $D$ into the unit ball $B \subset \mathbb{R}^3$. We compute

$$\phi = (F \cdot \xi) = (< 1 - x^2 + y^2 + z^2, -2xy, -2xz > \cdot \xi) = (< 1 + |\bar{x}|^2 - 2x^2, -2xy, -2xz > \cdot \xi)$$

therefore from the above line one finds

$$\phi = (1 + |\bar{x}|^2)(i \cdot \xi) - 2x(\bar{x} \cdot \xi) = (i \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}]) \quad (4.1)$$

where $i$ is the unit vector in the $x$ direction. This also shows that $F$ is a member of one parameter family of conformal fields parametrized by a unit vector in $\mathbb{R}^3$. Finally in this chapter we will show another property of $F$ which happens to be very useful.

**Lemma 4.2.3.** *The vector field $F$ is tangent to the unit sphere $S^2$.***
Proof. The unit sphere $S^2$ is given by a position vector $< x, y, x >^T$ where $1 - x^2 - y^2 - z^2 = 0$. To show that $F$ is tangent to the unit sphere we need to show that $(F \cdot < x, y, z >^T) = 0$ on $S^2$.

$$(F \cdot < x, y, z >^T) = (1 - x^2 + y^2 + z^2, -2xy, -2xz) \cdot < x, y, z >$$

$$= x - x^3 + xy^2 + xz^2 - 2xy^2 - 2xz^2 = x(1 - x^2 - y^2 - z^2) = 0.$$  \qed

4.3 The Main Results

In this section we will exhibit some new results concerning the stability of capillary surfaces in a ball and a new proof of a theorem by Barbosa and do Carmo (see theorem (2.3.1)). Our considerations will be mainly based on the use of the function $\phi = (F \cdot \xi)$, described above as the normal component of the conformal vector field $F$.

4.3.1 Capillary Surfaces in a Ball

In the section (4.2) we obtained a function $\phi$ which is the normal component of a perturbation. That perturbation came from a conformal vector field $F$ on $\mathbb{R}^3$, tangent to the unit sphere. We showed that the boundary integral in the second variation of energy for the normal component $\phi = (F \cdot \xi)$ is zero. Now we continue investigating this function. First we start with calculation that will indicate the main difficulty of our approach.

Lemma 4.3.1. The volume constraint (2.6) may not be satisfied by the function $\phi$. In fact

$$\partial V \equiv \int \int_D \phi dS = -6x_0 \text{Vol}(T)$$
where $\text{Vol}(T)$ is the volume of the solid region $T$ bounded between $\Omega$ and the wetted area $\Sigma$ and $x_0$ is the $x$-component of the center of mass of $T$.

Proof. Let the solid $T \subset \mathbb{R}^3$ be represented by a position vector $\bar{x} = <x, y, z>$. The center of mass of $T$ is defined to be the point $\bar{x}_0 = <x_0, y_0, z_0>$ with equation

$$\bar{x}_0 = \frac{\int \int T \bar{x} dxdydz}{\text{Vol}(T)}$$

where the volume of $T$ is given by

$$\text{Vol}(T) = \frac{1}{3} \int \int \partial T (x \cdot \xi) dS = \frac{1}{3} \int \int D (x \cdot \xi) dS + \frac{1}{3} \int \int \Sigma' (x \cdot x) dS = \frac{1}{3} \int \int D (x \cdot \xi) dS + \frac{1}{3} |\Sigma'|.$$ 

For the immersed case the volumes some of regions of $T$ might be counted with weights.

Having defined that, we prove the lemma using lemma (4.2.3) and the divergence theorem. From lemma (4.2.3) we know that $F$ is tangent to the unit sphere. Applying the divergence theorem one has

$$\int \int \phi dS = \int \int_D (F \cdot \xi) dS + \int \int \Sigma' (F \cdot x) dS = \int \int T \text{div} F dxdydz$$

but $\int \int \Sigma' (F \cdot x) dS$ is zero since $\Sigma \subseteq S^2$. The vector field $F$ is tangent to the unit sphere and the $\xi = \bar{x}$ on $\Sigma'$, so $\phi = (F \cdot \xi) = 0$ on $\Sigma$. The divergence $\text{div} F$ equals to $-6x$, therefore

$$\int \int_D \phi dS = \int \int_T \text{div} F dxdydz = -6 \int \int_T x dxdydz = -6x_0 \text{Vol}(T). \quad \text{(4.2)}$$

The previous result shows that $\phi$ does not always fix the volume form. It does
when $x_0 = 0$ or more generally when $x_0 = 0$. We can always rotate the coordinate axes to get that $x_0 = 0$ but this will lead to further complications as will see in the next pages. So for now we will leave formula (4.2) as it stands.

The next key calculation will give us formula for $L\phi$ on $\Omega$. Remember

$$L\phi = \Delta \phi + (k_1^2 + k_2^2)\phi$$

where $\Delta$ is the surface Laplacian on $\Omega$ and $k_1, k_2$ are the principle curvatures of the surface. In conformal coordinates $\Delta$ is just the standard flat Laplacian, divided by $E = (\bar{x}_u \cdot \bar{x}_u)$. Again, as usual we denote the mean curvature of $\Omega$ by $H$.

**Theorem 4.3.2.** Let $\phi = (F \cdot \xi) = (i \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])$ (see (4.1)). Then

$$L\phi = 4(i \cdot [\xi + H\bar{x}])$$ (4.3)

where $L\phi = \Delta \phi + (k_1^2 + k_2^2)\phi$.

**Proof.** As we mentioned before the surface Laplacian has rather simple form in conformal coordinates and one can always introduce conformal coordinates locally. For an arbitrary function $f$ on a manifold $M$, it is defined to be (see [B])

$$\Delta f = \frac{1}{\sqrt{(det(g)}} \sum_{j,k} \frac{\partial}{\partial x_j} \left( g^{jk} \sqrt{(det(g)} \frac{\partial f}{\partial x_k} \right)$$

where $g$ is the metric. In two dimensions and in conformal coordinates we have that $det(g) = E^2$, $g^{11} = g^{22} = 1/E$ and $g^{12} = g^{21} = 0$. We will rename the variables $x_1, x_2$ in the formula for the Laplacian and call them $u$ and $v$ respectively. Assuming conformal coordinates, we get the following formula for the surface Laplacian in two dimensions

$$\Delta f = \frac{f_{uu} + f_{vv}}{E}.$$
Now let us start computing $L\phi$. According to the above discussion for the surface Laplacian of $\phi$ we get $\Delta \phi = (\phi_{uu} + \phi_{vv})/E$. Using formula (4.1) and the fact that $(\bar{x}_u \cdot \xi) = 0$ we get

$$\phi_u = (i \cdot [(1 + |\bar{x}|^2)_u \xi + (1 + |\bar{x}|^2) \xi_u - 2(\bar{x}_u \cdot \xi)\bar{x} - 2(\bar{x} \cdot \xi)\bar{x}_u])$$

$$= (i \cdot [(1 + |\bar{x}|^2)_u \xi + (1 + |\bar{x}|^2) \xi_u - 2(\bar{x} \cdot \xi)\bar{x}_u])$$

and

$$\phi_{uu} = (i \cdot [(1 + |\bar{x}|^2)_{uu} \xi + 2(1 + |\bar{x}|^2)_u \xi_u + (1 + |\bar{x}|^2) \xi_{uu}$$

$$- 2(\bar{x}_u \cdot \xi)\bar{x} - 2(\bar{x} \cdot \xi)\bar{x}_u - 2(\bar{x} \cdot \xi)\bar{x}_u - 2(\bar{x} \cdot \xi)\bar{x}_{uu}])$$

$$= (i \cdot [(1 + |\bar{x}|^2)_{uu} \xi + 2(1 + |\bar{x}|^2)_u \xi_u + (1 + |\bar{x}|^2) \xi_{uu}$$

$$+ 2L \bar{x} - 2(\bar{x} \cdot \xi_{uu})\bar{x} - 4(\bar{x} \cdot \xi_u)\bar{x}_u - 2(\bar{x} \cdot \xi)\bar{x}_{uu}])$$

where $L$ is one of the coefficients of the second fundamental form of $\bar{x}$ and should not be confused with the Jacobi operator. We can obtain similar formula for $\phi_{vv}$, replacing $u$ by $v$ and $L$ by $N$. Adding the equations for $\phi_{uu}$ and $\phi_{vv}$ and using $E\Delta \phi = \phi_{uu} + \phi_{vv}$ we get

$$E\Delta \phi = (i \cdot [E\Delta(1 + |\bar{x}|^2) \xi + 2(1 + |\bar{x}|^2)_u \xi_u + 2(1 + |\bar{x}|^2)_v \xi_v + (1 + |\bar{x}|^2) E\Delta \xi$$

$$+ 2(L + N) \bar{x} - 2(\bar{x} \cdot E\Delta \xi)\bar{x} - 4(\bar{x} \cdot \xi_u)\bar{x}_u - 4(\bar{x} \cdot \xi_v)\bar{x}_v - 2(\bar{x} \cdot \xi) E\Delta \bar{x}])$$

(4.4)

We want to show that some of the terms in (4.4) add up to zero. Set

$$A = 2(1 + |\bar{x}|^2)_u \xi_u + 2(1 + |\bar{x}|^2)_v \xi_v - 4(\bar{x} \cdot \xi_u)\bar{x}_u - 4(\bar{x} \cdot \xi_v)\bar{x}_v$$

$$= 4[(\bar{x} \cdot \bar{x}_u)\xi_u + (\bar{x} \cdot \bar{x}_v)\xi_v - (\bar{x} \cdot \xi_u)\bar{x}_u - (\bar{x} \cdot \xi_v)\bar{x}_v].$$

One can find in [O] the Fundamental acceleration formulae, which give expressions
for the second derivatives of $\bar{x}$ and the first derivatives $\xi$ in the frame $(\bar{x}_u, \bar{x}_v, \xi)$. In conformal coordinates the first derivatives of $\xi$ are

$$\xi_u = -\frac{L\bar{x}_u + M\bar{x}_v}{E}$$

$$\xi_v = -\frac{M\bar{x}_u + N\bar{x}_v}{E}$$

where $L, M, N$ are the coefficients of the second fundamental form of $\bar{x}$. Applying them to $A$ we get

$$A = -\frac{4}{E}[(\bar{x} \cdot \bar{x}_u)(L\bar{x}_u + M\bar{x}_v) + (\bar{x} \cdot \bar{x}_v)(M\bar{x}_u + N\bar{x}_v)
- (\bar{x} \cdot (L\bar{x}_u + M\bar{x}_v))\bar{x}_u
- (\bar{x} \cdot (M\bar{x}_u + N\bar{x}_v))\bar{x}_v]
= -\frac{4}{E}[L(\bar{x} \cdot \bar{x}_u)\bar{x}_u + M(\bar{x} \cdot \bar{x}_v)\bar{x}_v + M(\bar{x} \cdot \bar{x}_v)\bar{x}_u + N(\bar{x} \cdot \bar{x}_u)\bar{x}_v
- L(\bar{x} \cdot \bar{x}_u)\bar{x}_u
- M(\bar{x} \cdot \bar{x}_v)\bar{x}_u
- M(\bar{x} \cdot \bar{x}_u)\bar{x}_v
- N(\bar{x} \cdot \bar{x}_v)\bar{x}_v] = 0.$$

Remember that in conformal coordinates $\Delta \bar{x} = 2H\xi$ and $H = (L + N)/2E$. In view of this and the fact that $A = 0$, equation (4.4) transforms into

$$E\Delta \phi = (i \cdot [E\Delta (1 + |\bar{x}|^2)\xi + (1 + |\bar{x}|^2)E\Delta \xi + 2(L + N)\bar{x} - 2(\bar{x} \cdot E\Delta \xi)\bar{x} - 2(\bar{x} \cdot \xi)E\Delta \bar{x})
= (i \cdot [E\Delta (1 + |\bar{x}|^2) - 4H(\bar{x} \cdot \xi))\xi + (1 + |\bar{x}|^2)E\Delta \xi + 4HE\bar{x} - 2(\bar{x} \cdot E\Delta \xi)\bar{x})].$$

Our next claim is that the quantity $B$

$$B = E(\Delta (1 + |\bar{x}|^2) - 4H(\bar{x} \cdot \xi)) = 4E.$$

First

$$E\Delta (1 + |\bar{x}|^2) = (\bar{x} \cdot \bar{x})_{uu} + (\bar{x} \cdot \bar{x})_{vv} = 2[(\bar{x}_u \cdot \bar{x}_u) + (\bar{x}_v \cdot \bar{x}_v) + (\bar{x} \cdot \bar{x}_{uu}) + (\bar{x} \cdot \bar{x}_{vv})].$$
therefore

\[ B = 2[(\bar{x}_u \cdot \bar{x}_u) + (\bar{x}_v \cdot \bar{x}_v) + (\bar{x}_w \cdot \bar{x}_w) - 4HE(\bar{x} \cdot \xi)] \]

\[ = 2E + 2E + 2(\bar{x} \cdot E \Delta \bar{x}) - 4HE(\bar{x} \cdot \xi) = 4E + 2(\bar{x} \cdot E2H\xi) - 4HE(\bar{x} \cdot \xi) = 4E. \]

Now from (4.5) we get

\[ \Delta \phi = \frac{(i \cdot [4E\xi + (1 + |\bar{x}|^2)E\Delta \xi + 4HE\bar{x} - 2(\bar{x} \cdot E\Delta \xi)\bar{x}])}{E} \]

\[ = (i \cdot [4\xi + (1 + |\bar{x}|^2)\Delta \xi + 4H\bar{x} - 2(\bar{x} \cdot \Delta \xi)\bar{x}]). \] (4.6)

Having obtained formula this expression for \( \Delta \phi \), we are in a position to compute \( L\phi \).

From (4.1) and (4.6) we get

\[ L\phi = \Delta \phi + (k_1^2 + k_2^2)\phi \]

\[ = (i \cdot [4\xi + (1 + |\bar{x}|^2)\Delta \xi + 4H\bar{x} - 2(\bar{x} \cdot \Delta \xi)\bar{x} + (k_1^2 + k_2^2)((1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}]). \]

Combining the coefficients of \((1 + |\bar{x}|^2)\) and \(\bar{x}\) respectively, we get

\[ L\phi = \left( i \cdot \left[ (\Delta \xi + (k_1^2 + k_2^2)\xi)(1 + |\bar{x}|^2) - 2((\bar{x} \cdot \Delta \xi) + (k_1^2 + k_2^2)(\bar{x} \cdot \xi))\bar{x} + 4(\xi + H\bar{x}) \right] \right) \]

\[ = \left( i \cdot \left[ (1 + |\bar{x}|^2)L\xi - 2(\bar{x} \cdot (\Delta \xi + (k_1^2 + k_2^2)\xi))\bar{x} + 4(\xi + H\bar{x}) \right] \right) \]

\[ = \left( i \cdot \left[ (1 + |\bar{x}|^2)L\xi - 2(\bar{x} \cdot L\xi)\bar{x} + 4(\xi + H\bar{x}) \right] \right). \] (4.7)

But \( L\xi = 0 \) just like in the proof of lemma (2.5.3). This is true because

\[ \xi = (i \cdot \xi), (j \cdot \xi), (k \cdot \xi)^T \]

so we can apply [B-dC], page 347 Proposition (2.24) to each component of \( \xi \). This
last observation reduces (4.7) to

\[ L\phi = (i \cdot 4[\xi + H\vec{x}]) \]

which was the statement of this theorem.

Let us write the expression that we have for \( \partial^2 E \) as of now

\[
\partial^2 E = -\int \int_D (L\phi)\phi dS = -4\int \int_D (i \cdot [(1 + |\vec{x}|^2)\xi - 2(\vec{x} \cdot \xi)\vec{x}]) (i \cdot [\xi + H\vec{x}]) dS. \tag{4.8}
\]

One can replace \( i \) in formula (4.2) by any constant vector, e.g. \( j \) or \( k \). All the results that we proved so far will be valid. Without loss of generality let us replace \( i \) by \( j \). The corresponding vector field, call it \( G \), will be \( G = < -2xy, 1 + x^2 - y^2 + z^2, -2yz > \). \( G \) also generates a flow on the unit ball with two fixed points, namely \((0, \pm1, 0)\). It will be conformal again due to the same arguments and an analogue of lemma (4.2.2) will hold, i.e. \( G \) will be tangent to \( S^3 \). The volume constraint of the corresponding normal component function \( \eta = (G \cdot \xi) \) will be \(-6y_0Vol(T)\) and \( L\eta = 4(j \cdot [\xi + H\vec{x}]) \).

This suggests that we can build a \( 3 \times 3 \) matrix \( Q \) from expressions similar to (4.8).

Let \( p, q \) be constant vectors in \( \mathbb{R}^3 \). We call

\[
Q(p, q) = -4\int \int_D (p \cdot [(1 + |\vec{x}|^2)\xi - 2(\vec{x} \cdot \xi)\vec{x}]) (q \cdot [\xi + H\vec{x}]) dS.
\]

The entries in the matrix \( Q = \{ Q \}_{ij}, 1 \leq i, j \leq 3 \) are defined so that the index 1 corresponds to the vector \( i \), the index 2 to \( j \) and the index 3 to \( k \). For example \( Q_{12} = Q(i, j), Q_{32} = Q(k, j), Q_{11} = Q(i, i) \). From the formula for \( Q_{ij} \) it follows easily that the matrix \( Q \) is symmetric. Standard linear algebra allows us to diagonalize it with respect to the standard Euclidean coordinates as needed. We will prove that the trace of \( Q, trQ = Q_{11} + Q_{22} + Q_{33} \) is non positive but first we need an auxiliary lemma.
Lemma 4.3.3.
\[ \Delta |\bar{x}|^2 = 4(1 + H(\bar{x} \cdot \xi)). \]

*Proof.* Assuming conformal coordinates \(u, v\) and differentiating one obtains

\[ \Delta |\bar{x}|^2 = \Delta(\bar{x} \cdot \bar{x}) = (\nabla \cdot 2(\bar{x} \cdot \nabla \bar{x})) = 2(\bar{x} \cdot \Delta \bar{x}) + 2|\nabla \bar{x}|^2. \]

Here \(\nabla\) is the surface gradient and the second term is given by

\[
|\nabla \bar{x}|^2 = \left( \frac{<x_u, x_v>}{\sqrt{E}} \cdot \frac{<x_u, x_v>^T}{\sqrt{E}} \right) + \left( \frac{<y_u, y_v>}{\sqrt{E}} \cdot \frac{<y_u, y_v>^T}{\sqrt{E}} \right) + \left( \frac{<z_u, z_v>}{\sqrt{E}} \cdot \frac{<z_u, z_v>^T}{\sqrt{E}} \right) \\
= \frac{x_u^2 + x_v^2 + y_u^2 + y_v^2 + z_u^2 + z_v^2}{E} = \frac{E + E}{E} = 2.
\]

Using and the relation \(\Delta \bar{x} = 2H\xi\) one gets

\[ \Delta |\bar{x}|^2 = \Delta(\bar{x} \cdot \bar{x}) = 2(\bar{x} \cdot \Delta \bar{x}) + 2|\nabla \bar{x}|^2 = 4H(\bar{x} \cdot \xi) + 4 \]

which was the formula to prove. \(\square\)

Theorem 4.3.4. Let \(Q\) be the matrix defined above. We claim that the trace of \(Q\),

\[ trQ \leq 0. \]

*Proof.* In the above notation we have

\[ Q_{11} = -4 \int_D (i \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])(i \cdot [\xi + H\bar{x}])dS \]

\[ Q_{22} = -4 \int_D (j \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])(j \cdot [\xi + H\bar{x}])dS \]

\[ Q_{33} = -4 \int_D (k \cdot [(1 + |\bar{x}|^2)\xi - 2(\bar{x} \cdot \xi)\bar{x}])(k \cdot [\xi + H\bar{x}])dS. \]

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Foiling the integrands in each $Q_{ii}$ we get

\[Q_{11} = -4 \int \int_D [(1 + |\bar{x}|^2)(i \cdot \xi)^2 + (H(1 + |\bar{x}|^2) - 2(\bar{x} \cdot \xi))(i \cdot \bar{x})(i \cdot \xi) - 2H(\xi \cdot \bar{x})(i \cdot \bar{x})^2]dS\]

\[Q_{22} = -4 \int \int_D [(1 + |\bar{x}|^2)(j \cdot \xi)^2 + (H(1 + |\bar{x}|^2) - 2(\bar{x} \cdot \xi))(j \cdot \bar{x})(j \cdot \xi) - 2H(\xi \cdot \bar{x})(j \cdot \bar{x})^2]dS\]

\[Q_{33} = -4 \int \int_D [(1 + |\bar{x}|^2)(k \cdot \xi)^2 + (H(1 + |\bar{x}|^2) - 2(\bar{x} \cdot \xi))(k \cdot \bar{x})(k \cdot \xi) - 2H(\xi \cdot \bar{x})(k \cdot \bar{x})^2]dS.\]

Observing that

\[(i \cdot \xi)^2 + (j \cdot \xi)^2 + (k \cdot \xi)^2 = |\xi|^2 = 1\]

\[(i \cdot \bar{x})(i \cdot \xi) + (j \cdot \bar{x})(j \cdot \xi) + (k \cdot \bar{x})(k \cdot \xi) = (\bar{x} \cdot \xi)\]

\[(i \cdot \bar{x})^2 + (j \cdot \bar{x})^2 + (k \cdot \bar{x})^2 = |\bar{x}|^2\]

we compute the the trace of $Q$ by summing up $Q_{ii}$ and simplifying to get

\[trQ = \sum_{i=1}^{3} Q_{ii}\]

\[= -4 \int \int_D [(1 + |\bar{x}|^2) + (H(1 + |\bar{x}|^2) - 2(\bar{x} \cdot \xi))(\bar{x} \cdot \xi) - 2H(\xi \cdot \bar{x})|\bar{x}|^2]dS\]

\[= -4 \int \int_D [H(1 + |\bar{x}|^2) - 2|\bar{x}|^2)(\bar{x} \cdot \xi) + 1 + |\bar{x}|^2 - 2(\bar{x} \cdot \xi)^2]dS\]

\[= -4 \int \int_D [H(1 - |\bar{x}|^2)(\bar{x} \cdot \xi) + 1 + |\bar{x}|^2 - 2(\bar{x} \cdot \xi)^2]dS.\] (4.9)

Next, we estimate the integrand in (4.9). From Cauchy-Schwarz inequality it follows that $(\bar{x} \cdot \xi)^2 \leq |\bar{x}|^2|\xi|^2 = |\bar{x}|^2$, therefore

\[1 + |\bar{x}|^2 - 2(\bar{x} \cdot \xi)^2 \geq 1 + |\bar{x}|^2 - 2|\bar{x}|^2 = 1 - |\bar{x}|^2.\]

Notice that $1 - |\bar{x}|^2 \geq 0$ since $\bar{x}$ represents the surface $\Omega$ which lies entirely in the
unit ball $B$. From the previous inequality and (4.9) we have

$$-trQ = 4 \int \int_D [H(1 + |\bar{x}|^2 - 2|\bar{x}|^2)(\bar{x} \cdot \xi) + 1 + |\bar{x}|^2 - 2(\bar{x} \cdot \xi)^2]dS$$

$$\geq 4 \int \int_D [H(1 - |\bar{x}|^2)(\bar{x} \cdot \xi) + 1 - |\bar{x}|^2]dS$$

$$= 4 \int \int_D [(1 - |\bar{x}|^2)(1 + H(\bar{x} \cdot \xi))]dS.$$

Applying the result from lemma (4.3.3) we obtain the following inequality

$$-trQ \geq \int \int_D (1 - |\bar{x}|^2)\Delta|\bar{x}|^2dS.$$  (4.10)

Finally we apply Green’s First Identity to (4.10) and also the fact that $1 - |\bar{x}|^2 = 0$ on $\partial \Omega \subseteq S^2$.

$$-trQ \geq \int \int_D (1 - |\bar{x}|^2)\Delta|\bar{x}|^2dS$$

$$= \oint_{\partial D} (1 - |\bar{x}|^2)\frac{\partial}{\partial \nu}(\Delta|\bar{x}|^2)d\sigma - \int \int_D (\nabla(1 - |\bar{x}|^2) \cdot \nabla|\bar{x}|^2)dS$$

$$= -\int \int_D (\nabla(1 - |\bar{x}|^2) \cdot \nabla|\bar{x}|^2)dS$$

$$= \int \int_D (\nabla|\bar{x}|^2 \cdot \nabla|\bar{x}|^2)dS$$

$$= \int \int_D |\nabla|\bar{x}|^2|^2dS \geq 0$$

or in other words

$$trQ \leq 0.$$

It also follows that if $\bar{x}$ is not a constant vector, then $trQ < 0$ and $Q$ has at least one negative eigenvalue. If $Q$ is assumed diagonal (we can do that since it is symmetric), then at least one of $Q_{11}, Q_{22}, Q_{33}$ is negative provided $\bar{x}$ is not a constant vector. If the center of mass of the solid $T$ is at the origin, from lemma (4.3.1) we
know that \( \partial \text{Vol}(T) = 0 \) for \( \phi = (i \cdot [(1 + |\vec{x}|^2)\xi - 2(\vec{x} \cdot \xi)\vec{x}]) \) and it will be also true if we replace \( i \) by \( j \) or \( k \). The next result is a corollary of theorem (4.3.4) and of the above discussion.

**Theorem 4.3.5.** If the center of mass of \( T \) (the solid region bounded by \( \Sigma' \) and \( \Omega \)) is at the origin, then \( \Omega \) is unstable.

*Proof.* We just mentioned that for all of the following functions

\[
\phi_1 = (i \cdot [(1 + |\vec{x}|^2)\xi - 2(\vec{x} \cdot \xi)\vec{x}])
\]

\[
\phi_2 = (j \cdot [(1 + |\vec{x}|^2)\xi - 2(\vec{x} \cdot \xi)\vec{x}])
\]

\[
\phi_3 = (k \cdot [(1 + |\vec{x}|^2)\xi - 2(\vec{x} \cdot \xi)\vec{x}])
\]

the volume constraint \( \partial \text{Vol}(t) = 0 \) is satisfied, when the center of mass of \( T \) is at the origin. We can rotate the coordinate axes so that \( Q \) is diagonal and that will preserve the origin. We know at least one of \( Q_{11}, Q_{22}, Q_{33} \) is negative. Without loss of generality say it is \( Q_{11} \). But \( Q_{11} \) is the second variation of the energy functional in the direction of \( \phi_1 \). So the perturbation with normal component \( \phi_1 \) will have negative second variation of energy and will preserve volume. This shows that the surface \( \Omega \) is unstable. \( \square \)

We should remark that of all known examples on the above configurations only the spherical cap does not bound a body with center of mass at the origin.

We continue this chapter with several other results, which are also corollaries of the preceding discussion. Again we assume we rotated the axes so that the matrix \( Q \) is diagonal and \( \vec{x} \) is not degenerate so \( \text{tr}Q < 0 \). From this we know that at least one of the \( Q_{ii} \) is negative. Now, if two of them happen to be negative we can show easily that \( \Omega \) is unstable. Take \( \psi = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 \), where \( c_1, c_2, c_3 \) are arbitrary real constants, which are not all equal to zero. The matrix \( Q \) being diagonal immediately
implies that

\[- \int \int_D (L\psi) \psi dS = c_1^2 Q_{11} + c_2^2 Q_{22} + c_3^2 Q_{33}\]

and

\[\int \int_D \psi dS = (-6 Vol(T))(c_1 x_0 + c_2 y_0 + c_3 z_0)\]

where \(\bar{x}_0 = <x_0, y_0, z_0>\) is the center of mass of the solid \(T\). Without loss of generality assume that \(Q_{11}\) and \(Q_{22}\) are negative and pick \(c_3 = 0\). Then

\[- \int \int_D (L\psi) \psi dS = c_1^2 Q_{11} + c_2^2 Q_{22} < 0\]

\[\int \int_D \psi dS = (-6 Vol(T))(c_1 x_0 + c_2 y_0).\]

We can always pick \(c_1, c_2\) not both zero, so that the quantity \(c_1 x_0 + c_2 y_0\) is zero. This shows that \(\Omega\) is unstable, because the perturbation with normal component \(\psi = c_1 \phi_1 + c_2 \phi_2\) has a negative second variation of energy and satisfies the volume constraint.

The center of mass for a spherical cap inside of the unit ball is never at the origin. One sees that simply by observation. So if we suspect that all capillary surfaces in a ball besides the spherical caps are unstable, we can make the following conjecture.

For a non spherical capillary surface \(\Omega\) in a unit ball the center of mass of the solid \(T\) is at the origin or the matrix \(Q\) has at least two negative eigenvalues or both. This would imply \(\Omega\) is unstable.

In conclusion of this chapter we will provide some additional calculations. As we saw in lemma (4.3.1), formula \(4.2\)

\[\int \int_D \phi dS = \int \int_D (F \cdot \xi) dS = \int \int_T \text{div} F dxdydz = -6 \int \int_T x dxdydz = -6x_0 Vol(T).\]
We get another formula for the volume using Green’s First Identity.

\[
2H \int \int_D \phi dS = \int \int_D (F \cdot 2H \zeta) dS = \int \int_D (F \cdot \Delta \bar{x}) dS \\
= \oint_{\partial D} (F \cdot \bar{x}_\nu) d\sigma - \int \int_D (\nabla F \cdot \nabla \bar{x}) dS.
\]

Assuming conformal coordinates we get that

\[
\nabla F \cdot \nabla \bar{x} = (\nabla (1 - x^2 + y^2 + z^2) \cdot \nabla x) + (\nabla (-2xy) \cdot \nabla y) + (\nabla (-2xz) \cdot \nabla z) \\
= \begin{pmatrix}
-2xx_u + 2yy_u + 2zz_u - 2xx_v + 2yy_v + 2zz_v
\end{pmatrix} \begin{pmatrix}
x_u, x_v
\end{pmatrix}^T \\
+ \begin{pmatrix}
-2yy_u + 2xu_y, -2xy_v - 2xv_y
\end{pmatrix} \begin{pmatrix}
y_u, y_v
\end{pmatrix}^T \\
+ \begin{pmatrix}
-2zz_u + 2xz_u, -2xz_v - 2xz_v
\end{pmatrix} \begin{pmatrix}
z_u, z_v
\end{pmatrix}^T \\
= \begin{pmatrix}
-2(x^2_u + y^2_u + z^2_u) - 2x(x^2_v + y^2_v + z^2_v)
\end{pmatrix} \\
= \begin{pmatrix}
-2xE - 2xE
\end{pmatrix} = -4x.
\]

From this computation one gets

\[
2H \int \int_D \phi dS = \oint_{\partial D} (F \cdot \bar{x}_\nu) d\sigma + \int \int_D 4xdS. \quad (4.11)
\]

Assume the the contact angle $\gamma$ is $\pi/2$. Therefore $\bar{x} = \bar{x}_\nu$ and the line integral in (4.11) disappears since $(F \cdot \bar{x}) = 0$ on $\partial \Omega = \Gamma$. From this fact and (4.2), (4.11) we see that

\[
H(-6Vol(T)x_0) = 2|\Omega|x^*
\]

where $x^* = <x^*, y^*, z^*>$ is the center of mass of the free surface $\Omega$ itself. The previous formula will be true if we repeat the calculation with $\phi_2$ and $\phi_3$. Therefore

\[
H(-6Vol(T)x_0) = 2|\Omega|x^*
\]
which means that $\bar{x}_0$ and $\bar{x}^*$ lie on a line passing through the origin. Moreover if $H = 0$ we get $\bar{x}^* = 0$. Also $\bar{x}_0 = \bar{0}$ if and only if $\bar{x}^* = \bar{0}$, provided that $H \neq 0$.

In [L-Y] is shown that the Willmore Energy $W = \int \int_D (H^2 - K) dS$ is invariant under conformal transformations. In view of that and the fact that $F$ is conformal vector filed we see that in the direction of $\phi$

$$\partial W = \partial \int \int_D (H^2 - K) dS = 0. \quad (4.12)$$

One can see using the results in [W] and the Gauss-Bonnet formula that the following equation holds.

**Lemma 4.3.6.**

$$\partial W = H \int \int_D L \phi dS - 2H^3 \int \int_D \phi dS + (\cot \gamma) H^2 \int_{\partial D} \phi d\sigma + \partial \oint_{\partial D} k_g d\sigma.$$ 

We also know from [R-S] that $k_g = (\cos \gamma) \bar{k}_g + \sin(\gamma)$ where $\bar{k}_g$ is the geodesic curvature of $\Gamma$ in $S^2$. So when $\gamma = \pi/2$ we have that $k_g = 1$ and

$$0 = \partial W = H \int \int_D L \phi dS - 2H^3 \int \int_D \phi dS + \partial \oint_{\partial D} d\sigma$$

therefore using theorem (4.3.2) we have

$$2H^3 \int \int_D \phi dS = H \int \int_D L \phi dS + \partial \oint_{\partial D} d\sigma = 4H \int \int_D (\mathbf{i} \cdot [\xi + H \bar{x}]) dS + \partial \oint_{\partial D} d\sigma = 4H^2 \int \int_D x dS + 4H \int \int_D (\mathbf{i} \cdot \xi) dS + \partial \oint_{\partial D} d\sigma.$$
Combining the previous line with (4.11) we get

\[ 4H^2 \int_D x dS = 4H^2 \int_D x dS + 4H \int_D (\mathbf{i} \cdot \mathbf{\xi}) dS + \partial \int_{\partial D} d\sigma \]

which provide us with a nice formula for \( \partial \int_{\partial D} d\sigma \), in the direction of \( \phi \), when \( \gamma = \pi/2 \)

\[ \partial \int_{\partial D} d\sigma = -4H \int_D (\mathbf{i} \cdot \mathbf{\xi}) dS. \]

Notice that the right hand side does not depend on \( \phi \) at all. This ends the discussion in this section.

### 4.3.2 Alternative Proof of a Theorem by Barbosa and do Carmo

Finally we will provide a new proof of the fact the the only closed compact surface, immersed in \( \mathbb{R}^3 \) and having non-zero constant mean curvature is the round sphere. It is an easy corollary of the discussion in the previous section. Let \( \Omega \) be a closed surface, which is again given by a map \( x \) from an abstract 2-dimensional closed surface \( D \) into the flat 3-space. Since there is no boundary the energy is given simply by the area, i.e

\[ E = |\Omega| \]

and the second variation of the energy functional \( E \) in a direction of \( \phi \) is given by

\[ \partial^2 E = -\int \int_D (L\phi) \phi dS = -\int \int_D (\Delta \phi + (k_1^2 + k_2^2) \phi) \phi dS \]

subject to the volume constraint

\[ \partial Vol(int(\Omega)) = \int \int_D \phi dS = 0. \]
Of course, as before the CMC-surfaces are critical points for the energy $E$. The surface $\Omega$ has no boundary in this case and we can assume that the center of mass of $T = \text{int}(\Omega)$ is at the origin, so that the volume constraint is satisfied according to lemma (4.3.1). We can now use the same vector fields that we used in the previous section and apply theorems (4.3.4) and (4.3.5) to get that the trace of the matrix $Q$ constructed in section 4.3.1 is negative if $\Omega$ is non spherical. As mentioned $Q$ is symmetric, so we can diagonalize it by a rotation. Of course that will preserve the trace and the volume constraint will be fulfilled. Therefore either $Q_{11}$ or $Q_{22}$ or $Q_{33}$ we be negative. Without loss of generality assume that $Q_{11}$ is negative. Now we will have that $\phi = (F \cdot \xi)$ makes $\partial^2 E$ negative. Thus, if $\Omega$ is stable, it must be the round sphere.
References


