A Dissertation
entitled
Two Problems in the Theory of
Toeplitz Operators on the Bergman Space

by
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Submitted as partial fulfillment of the requirements for the
Doctor of Philosophy Degree in Mathematics

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The University of Toledo
May 2009
An Abstract of

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In this thesis we deal with the zero product problem and the commuting problem for Toeplitz operators on the Bergman space over the unit disk of the complex plane.

For the zero product problem, we show that the zero product of two Toeplitz operators has only the trivial solution when one of the symbols has certain polar decomposition and the other is a general bounded symbol.

As for the commuting problem, we show that if the Fourier series of the bounded function $f$ is of the form $f(r\mathrm{e}^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r)$ where $N$ is a positive integer, and $T_f$ commutes with $T_{z+\bar{g}}$, where $g$ is a bounded analytic function on the open unit disk, then $T_f$ is a nontrivial linear combination of $T_{z+\bar{g}}$ and the identity operator $I$. Also, we describe all Toeplitz operators $T_f$ that commutes with $T_{z+\bar{z}}$, when the symbol $f$ is integrable, with respect to the Lebesgue area measure, on the unit disk.
Acknowledgments

I would like to thank my advisor Prof. Rao Nagisetty for all his help with this research. He has been an invaluable source of knowledge and insight. Without his guidance this project would not have been possible.

I would also like to thank Professors Željko Ćučković, Henry Wente, and Denis White for their willingness to be members of my Committee. They are all experts in their respective areas of research, and I appreciate them taking time to help with this project. I also would like to thank Prof. Issam Louhichi for carefully reading of my manuscript, and for the useful discussion I had with him when he was a visiting professor at University of Toledo.

I would like to thank Prof. Roshdi Khalil from University of Jordan, for his continuous encouragement.

Thanks also to my family and friends for their support and encouragement.

Finally I would like to thank my wife Rafeef for her constant support and patience. She has been so understanding of the time and effort it has taken to work my way through graduate school. Her cooperation and understanding have been a source of strength and encouragement to me.
To the soul of my father, Fawzi. To my mother, Ruqqaya.

To my great wife, Rafeef, and my daughter, Layan, and my son, Faris.
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Chapter 1

Hardy and Bergman Spaces

The theory of Bergman spaces draws much of its inspiration from the well established theory of Hardy spaces. Classical properties of Hardy spaces suggest analogous problems for Bergman spaces. We therefore begin this chapter with a brief overview of the Hardy space $H^2$.

1.1 Hardy Space

Let $\mathbb{C}$ be the complex plane, and let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1 \}$ be the open unit disk. The boundary of $\mathbb{D}$ is denoted by $\mathbb{T}$ which is the unit circle. Let $d\theta$ be the arc-length measure on $\mathbb{T}$. $L^2(\mathbb{T})$ shall denote the Banach space of Lebesgue measurable functions on $\mathbb{T}$ with

$$||f||_2 = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^2 d\theta \right\}^{\frac{1}{2}} < \infty.$$

$L^\infty(\mathbb{T})$ denotes the Banach space of bounded measurable functions $f$ on $\mathbb{T}$ with

$$||f||_\infty = \text{ess sup}\{ |f(\theta)|, \theta \in [0, 2\pi] \} < \infty.$$
Definition 1.1.1. A function $f : \mathbb{D} \to \mathbb{C}$ is called analytic on $\mathbb{D}$ if $f'(z)$ exists for all $z \in \mathbb{D}$.

Definition 1.1.2. The Hardy space $H^2(\mathbb{D})$ consists of analytic functions $f$ in $\mathbb{D}$ such that

$$||f||_2^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$ 

Theorem 1.1.1. Suppose $f \in H^2(\mathbb{D})$. Then

$$|f(z)| \leq \frac{||f||_2}{(1-|z|^2)^{1/2}}$$

for all $z \in \mathbb{D}$.

Proof. See [27, p. 253]

Corollary 1.1.2. Suppose $S$ is a compact subset of $\mathbb{D}$. Then there exists a positive constant $C$ such that

$$|f(z)| \leq C||f||_2$$

for all $f \in H^2(\mathbb{D})$ and all $z \in S$.

Using the above corollary, one can prove that the Hardy space $H^2(\mathbb{D})$ is Banach space.

Theorem 1.1.3. Suppose $f \in H^2(\mathbb{D})$. Then the limit

$$f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$$ (1.1)

exists for almost all $e^{i\theta} \in \mathbb{T}$. Furthermore, the limit function $f$ is in $L^2(\mathbb{T})$ with

$$||f||_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta. \quad (1.2)$$

Proof. See [13].
It follows from (1.2) that the relation (1.1) establishes an isometric isomorphism between $H^2(\mathbb{D})$ and a certain closed subspace of $L^2(\mathbb{T})$. Since the arc length measure $d\theta$ is finite, $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$. Given $f \in L^1(\mathbb{T})$, the Fourier coefficients of $f$ are given by

$$a_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)e^{-in\theta}d\theta, \quad n \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of all integers.

By (1.1), the Hardy space $H^2(\mathbb{D})$ can be identified with the subspace of $L^2(\mathbb{T})$ consisting of functions $f$ such that $a_n(f) = 0$ for all negative integers $n$.

**Definition 1.1.3.** The Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$, is defined to be the subspace of $L^2(\mathbb{T})$ consisting of all functions $f$ such that $a_n(f) = 0$ for all negative integers $n$.

**Definition 1.1.4.** Let $f \in L^1(\mathbb{T})$, the harmonic extension (or Poisson extension) of $f$ to $\mathbb{D}$, denoted by $\hat{f}(z)$, is defined by

$$\hat{f}(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)P_z(\theta)d\theta, \quad z \in \mathbb{D},$$

where $P_z(\theta) = \frac{1 - |z|^2}{1 - \bar{z}e^{i\theta}} = \Re\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right)$, is the Poisson kernel of $\mathbb{D}$.

The Poisson kernel is the real part of an analytic function, so $P_z(\theta)$ is harmonic in $z$ for fixed $\theta \in [0, 2\pi]$. It follows that $\hat{f}(z)$ is harmonic in $\mathbb{D}$. Moreover, if $a_n = a_n(f)$ are the Fourier coefficients of $f$, then

$$\hat{f}(z) = \sum_{n=0}^{\infty} a_nz^n + \sum_{n=1}^{\infty} a_{-n}\bar{z}^n.$$

This implies that for $f \in L^2(\mathbb{T})$ is in $H^2(\mathbb{T})$ if and only if $\hat{f}(z)$ is analytic in $\mathbb{D}$, in other words, $H^2(\mathbb{D})$ consists exactly of the Poisson extensions of functions in $H^2(\mathbb{T})$.

It will be convenient to say that a function $f \in L^2(\mathbb{T})$ (or possibly even in $L^1(\mathbb{T})$)
is analytic if all its Fourier coefficients with negative index vanish, i.e. \( a_n(f) = 0 \) for \( n = -1, -2, -3, \ldots \).

For any fixed integer \( n \), clearly the Fourier coefficient \( a_n(f) \) is a bounded linear functional on \( L^2(\mathbb{T}) \). According to the definition of the Hardy space 1.1.3, it follows that

\[
H^2(\mathbb{T}) = \bigcap_{n<0} \text{Ker}(a_n) = \{ f \in L^2(\mathbb{T}) : a_n(f) = 0 \}.
\]

Therefore, \( H^2(\mathbb{T}) \) is closed subspace of \( L^2(\mathbb{T}) \), and hence a Banach space. In fact, the space \( L^2(\mathbb{T}) \) is a Hilbert space with the inner product

\[
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt,
\]

for all \( f, g \in L^2(\mathbb{T}) \). Since \( H^2(\mathbb{T}) \) is a closed subspace of \( L^2 \), \( H^2(\mathbb{T}) \) is also a Hilbert space with the same inner product of \( L^2(\mathbb{T}) \).

### 1.2 Toeplitz Operators on the Hardy Space

Since \( H^2(\mathbb{T}) \) is a closed subspace of the Hilbert space \( L^2(\mathbb{T}) \), there exists an orthogonal projection from \( L^2(\mathbb{T}) \) onto \( H^2(\mathbb{T}) \), denoted by \( P_H \) and defined as

\[
P_H : H^2 \oplus (H^2)^\perp \rightarrow H^2
\]

\[
f = f_1 + f_2 \quad \mapsto \quad P_H(f) = P_H(f_1 + f_2) = f_1.
\]

**Definition 1.2.1.** Let \( \phi \in L^\infty(\mathbb{T}) \). The operator \( T_\phi : H^2(\mathbb{T}) \rightarrow H^2(\mathbb{T}) \), defined by \( T_\phi(f) = P_H(\phi f) \), is called the Toeplitz operator with symbol \( \phi \).

Now, one can easily check the following algebraic properties of Hardy space Toeplitz operators.
Theorem 1.2.1. If $\phi$ and $\psi$ are bounded on $\mathbb{T}$, $\lambda$ and $\beta$ are complex numbers. Then the Hardy space Toeplitz operators satisfy the following properties.

1. $T_{\lambda \phi + \beta \psi} = \lambda T\phi + \beta T\psi$.

2. $T_\phi^* = T_{\bar{\phi}}$.

3. $T_\phi$ is self-adjoint if and only if $\phi$ is real.

4. $T_\phi = 0$ if and only if $\phi = 0$.

5. If $\bar{\phi}$ or $\psi$ is analytic, then $T_{\bar{\phi}}T_\psi = T_{\bar{\phi}\psi}$

6. $T_1 = I$ is the identity operator.

Since $P_H$ is bounded on $L^2(\mathbb{T})$, it follows that $T_\phi$ is bounded whenever $\phi$ is bounded on $\mathbb{T}$.

Hardy space Toeplitz matrices were known for a while and extensive studies have been done on them. In the sixties, Brown and Halmos [9] introduced the operator theoretic approach to study Toeplitz matrices. The relationship between such matrices and the modern theory of Toeplitz operators on the Hardy space can be understood by the following.

Let $f \in L^\infty(\mathbb{T})$, and recall that the set \( \{e_n(\theta) = e^{in\theta}, \ n \geq 0\} \) is the standard orthonormal basis of the Hardy space $H^2(\mathbb{T})$. Since $T_f : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is a linear operator, it has the form

$$T_f e_n = \sum_{m=0}^{\infty} a_{m,n} e_m = \sum_{m=0}^{\infty} \langle T_f e_n, e_m \rangle e_m,$$

where $(a_{m,n})$ are the entries of the matrix of $T_f$. Since $f \in L^\infty(\mathbb{T}) \subseteq L^2(\mathbb{T})$, we have

$$f = \sum_{k=-\infty}^{\infty} b_k e_k,$$
where \( \{b_k\} \) is a sequence of complex numbers and \( \{e_n : n \in \mathbb{Z}\} \) is the orthonormal basis of \( L^2(\mathbb{T}) \). Therefore,

\[
T_f e_n = P_H (f e_n) = P_H \left( \sum_{k=-\infty}^{\infty} b_k e_k e_n \right) = \sum_{n+k \geq 0} b_k e_{k+n}.
\]

Thus,

\[
a_{m,n} = \langle T_f e_n, e_m \rangle = \sum_{n+k \geq 0} b_k \langle e_{k+n}, e_m \rangle = \sum_{n+k \geq 0} b_k \langle e_{k+n}, e_m \rangle = b_{m-n}
\]

for all nonnegative integers \( m \) and \( n \). i.e.

\[
a_{m+i,n+i} = b_{m-n}, \quad i \geq 0.
\]

Thus the matrix representation of the Toeplitz operator \( T_f \), in the standard orthonormal basis \( \{e_n : n \geq 0\} \) of \( H^2(\mathbb{T}) \), has the following form

\[
\begin{pmatrix}
a_{00} & a_{01} & a_{02} & a_{03} & \cdots \\
a_{10} & a_{11} & a_{12} & a_{13} & \cdots \\
a_{20} & a_{21} & a_{22} & a_{23} & \cdots \\
a_{30} & a_{31} & a_{32} & a_{33} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} =
\begin{pmatrix}
b_0 & b_{-1} & b_{-2} & b_{-3} & \cdots \\
b_1 & b_0 & b_{-1} & b_{-2} & \cdots \\
b_2 & b_1 & b_0 & b_{-1} & \cdots \\
b_3 & b_2 & b_1 & b_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The defining characteristic of a Toeplitz matrix is that the entries on each main diagonal are constant. This property actually characterizes Hardy space Toeplitz operators. In fact, for any linear operator \( T \) on \( H^2(\mathbb{T}) \) with the above matrix in the standard orthonormal basis of \( H^2(\mathbb{T}) \), there exists a function \( f \) of the form

\[
f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}
\]
such that $T = T_f$. Moreover, if $T$ is bounded then $f$ is bounded. In addition to the above criterion, we have the following characterization of Hardy space Toeplitz operators.

**Proposition 1.2.2.** A bounded linear operator $T$ on $H^2(\mathbb{T})$ is a Toeplitz operator if and only if $T^*_e e^{i\theta} T e^{i\theta} = T$.

**Proof.** If $T$ is a bounded Toeplitz operator, then $T = T_f$ for some $f \in L^\infty(\mathbb{T})$. Therefore,

$$T e^{-i\theta} T_f e^{i\theta} = T e^{-i\theta} f e^{i\theta} = T_f.$$ 

On the other hand, if $T^*_e e^{i\theta} T e^{i\theta} = T$, then for the standard basis $\{e_n : n \geq 0\}$ we have

$$\langle T e_n, e_m \rangle = \langle T^*_e e^{i\theta} T e^{i\theta} e_n, e_m \rangle = \langle T T e^{i\theta} e_n, T e^{i\theta} e_m \rangle = \langle T e_{n+1}, e_{m+1} \rangle.$$

This implies that the matrix of $T$ with respect to the basis $\{e_n : n \geq 0\}$ is a Toeplitz matrix, and hence $T$ is a Toeplitz operator. \qed

### 1.3 Bergman Space

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk of the complex plane $\mathbb{C}$. Let $dA = \frac{dxdy}{\pi} = \frac{rdrd\theta}{\pi}$ denote the normalized Lebesgue area measure on the unit disk $\mathbb{D}$, where $(r, \theta)$ are the polar coordinates in $\mathbb{C}$.

The complex space $L^2(\mathbb{D}, dA)$ is the Hilbert space of all square integrable functions on $\mathbb{D}$, with the inner product

$$\langle f, g \rangle = \int_\mathbb{D} f \overline{g} dA,$$

for all $f, g \in L^2(\mathbb{D}, dA)$.

**Definition 1.3.1.** The Bergman space, denoted by $L^2_a(\mathbb{D}, dA)$, consists of all functions
that are analytic in \( \mathbb{D} \). In other words,

\[
L^2_a(\mathbb{D}, dA) = H(\mathbb{D}) \cap L^2(\mathbb{D}, dA),
\]

where \( H(\mathbb{D}) \) is the space of all analytic functions on \( \mathbb{D} \).

The following theorem [14, p. 7], shows that functions in a Bergman space cannot grow too rapidly near the boundary.

**Theorem 1.3.1.** Point-evaluation is a bounded linear functional on the Bergman space \( L^2_a(\mathbb{D}, dA) \). More specifically, each function \( f \in L^2_a(\mathbb{D}, dA) \) has the property

\[
|f(z)| \leq \delta(z)^{-1} \| f \|_2, \quad \forall z \in \mathbb{D},
\]

where \( \delta(z) = \text{dist}(z, \mathbb{T}) \) is the distance from \( z \) to the boundary.

As a consequence, if \( C \) is a compact subset of \( \mathbb{D} \), there exists an element \( z_0 \in C \) such that \( \text{dist}(C, \mathbb{T}) = \text{dist}(z_0, \mathbb{T}) = \delta(z_0) \), therefore \( \delta(z_0) \leq \delta(z) \) for every \( z \in C \). Which implies that if \( f \in L^2_a(\mathbb{D}, dA) \), then there exists a constant \( M \) depending only on \( C \) such that

\[
|f(z)| \leq M \| f \|_2, \quad \forall z \in C.
\]

An immediate corollary is that norm convergence implies locally uniform convergence. In other words, if \( f_n \) and \( f \) are in \( L^2_a(\mathbb{D}, dA) \) and \( \| f_n - f \|_2 \to 0 \) as \( n \to \infty \), then \( f_n(z) \to f(z) \) uniformly on compact subsets of \( \mathbb{D} \). If \( f_n \in L^2_a(\mathbb{D}, dA) \) and \( \| f_n - f \|_2 \to 0 \), where \( f \in L^2(\mathbb{D}, dA) \), then there exists a subsequence \( \{ f_{n_k}(z) \} \) that converges to \( f(z) \) almost everywhere in \( \mathbb{D} \). But \( f_n \) is a Cauchy sequence in the \( L^2 \)-norm, and therefore a locally uniform Cauchy sequence. Hence it converges locally uniformly to a function \( h \) that is analytic in \( \mathbb{D} \). This implies \( f(z) = h(z) \) almost everywhere in \( \mathbb{D} \), and the limit function \( f \) can be identified with a function in \( L^2_a(\mathbb{D}, dA) \). Thus \( L^2_a(\mathbb{D}, dA) \) is a closed subspace of \( L^2(\mathbb{D}, dA) \). Since \( L^2(\mathbb{D}, dA) \) is a Hilbert space, then
the Bergman space $L^2_a(\mathbb{D}, dA)$ is also a Hilbert space with the same inner product of $L^2(\mathbb{D}, dA)$.

According to Theorem 1.3.1, each point-evaluation functional $\Lambda_z(f) = f(z)$ on $L^2_a(\mathbb{D}, dA)$ is bounded. So by Riesz representation theorem for Hilbert space, there exists a unique function $K_z \in L^2_a(\mathbb{D}, dA)$ such that $f(z) = \Lambda_z(f) = \langle f, K_z \rangle$ for every $f \in L^2_a(\mathbb{D}, dA)$. Thus,

$$f(z) = \langle f, K_z \rangle = \int_{\mathbb{D}} f(w)\overline{K_z(w)}dA(w).$$

The function $K(z, w) = \overline{K_z(w)}$ is known as the Bergman reproducing kernel, since it has the reproducing property

$$f(z) = \int_{\mathbb{D}} f(w)K(z, w)dA(w), \quad z \in \mathbb{D},$$

for all $f \in L^2_a(\mathbb{D}, dA)$. Besides the reproducing property, the Bergman kernel has the symmetry property, i.e. $K(z, w) = \overline{K(w, z)}$. This property shows that $K(z, w)$ is analytic in $z$ and co-analytic in $w$.

It should be emphasized that among functions in $L^2_a(\mathbb{D}, dA)$, the kernel function is uniquely determined by its reproducing property which is obtained from the Riesz representation theorem. In other words, if a function $l_z(w) = \overline{L(z, w)}$ belongs to $L^2_a(\mathbb{D}, dA)$ has the reproducing property, then $f(z) = \langle f, l_z \rangle = \langle f, K_z \rangle$. Thus $\langle f, l_z - K_z \rangle = 0$ for every $f \in L^2_a(\mathbb{D}, dA)$. This implies that $l_z - K_z = 0$, and therefore $K(z, w) = L(z, w)$ for all $z, w \in \mathbb{D}$.

Before we start calculating the kernel function of the unit disk $\mathbb{D}$, we need to review some facts and notations.

Since $L^2(\mathbb{D}, dA)$ is separable Hilbert space, $L^2_a(\mathbb{D}, dA)$ has the same property as well, which guarantees the existence of a countable orthonormal basis. Now, suppose
that \( \{ \phi_n \}_{n=0}^{\infty} \) is an orthonormal basis of \( L^2_a(\mathbb{D}, dA) \). Then

\[
\langle \phi_n, \phi_m \rangle = \delta_{nm} = \begin{cases} 
0, & n \neq m \\
1, & n = m
\end{cases}
\]

is the Kronecker delta. Furthermore, each function \( f \in L^2_a(\mathbb{D}, dA) \) has a unique expansion \( f = \sum_{n=0}^{\infty} a_n \phi_n \), convergent in norm and therefore uniformly convergent on compact subsets. Thus,

\[
f(z) = \sum_{n=0}^{\infty} a_n \phi_n(z), \quad z \in \mathbb{D}
\]

where \( a_n = \langle f, \phi_n \rangle \). By Parseval’s identity, \( \sum_{n=0}^{\infty} |a_n|^2 = ||f||^2 \). The following theorem asserts that the representation of the Bergman kernel of \( \mathbb{D} \) is independent of the choice of the orthonormal basis.

**Theorem 1.3.2.** The kernel function of \( \mathbb{D} \) has the representation

\[
K(z, w) = \sum_{n=0}^{\infty} \phi_n(z) \overline{\phi_n(w)},
\]

where the set \( \{ \phi_n \}_{n=0}^{\infty} \) is an arbitrary orthonormal basis of \( L^2_a(\mathbb{D}, dA) \).

**Proof.** Since \( K(z, w) \) is analytic in \( z \), it has a unique expansion \( K(z, w) = \sum_{n=0}^{\infty} c_n \phi_n(z) \) that converges uniformly on compact subsets of \( \mathbb{D} \), where \( c_n = \langle K(\cdot, w), \phi_n \rangle \). Therefore,

\[
\overline{c_n} = \langle K(\cdot, w), \phi_n \rangle = \langle \phi_n, K(\cdot, w) \rangle = \int_{\mathbb{D}} \overline{\phi_n(\eta)} K(\eta, w) dA(\eta) = \int_{\mathbb{D}} \phi_n(\eta) K(w, \eta) dA(\eta) = \phi_n(w).
\]
which implies \( c_n = \overline{\phi_n(w)} \) for all \( n \geq 0 \). Hence \( K(z, w) = \sum_{n=0}^{\infty} \phi_n(z) \overline{\phi_n(w)} \).

To calculate the Bergman kernel function of \( \mathbb{D} \), we need the following lemma.

**Lemma 1.3.3.** The set of functions \( \{ \psi_n(z) = \sqrt{n + 1} z^n \}_{n=0}^{\infty} \) form an orthonormal basis for \( L^2_a(\mathbb{D}, dA) \).

**Proof.** An easy calculation gives,

\[
\langle \psi_n, \psi_m \rangle = \int_D (\sqrt{n + 1} z^n) (\sqrt{m + 1} z^m) dA
= \sqrt{n + 1} \sqrt{m + 1} \int_0^{2\pi} \int_0^1 r^n e^{in\theta} r^m e^{-im\theta} \frac{1}{\pi} r dr d\theta
= \delta_{nm}
\]

To complete the proof, we need to show that they form a basis. So we have to verify that they span the whole space. Equivalently, we shall show that Parseval’s relation \( \sum_{n=0}^{\infty} |\langle f, \psi_n \rangle|^2 = ||f||_2^2 \) holds for every \( f \in L^2_a(\mathbb{D}, dA) \). Which is equivalent to the identity \( ||f||_2^2 = \sum_{n=0}^{\infty} |a_n|^2 \), for \( f(z) = \sum_{n=0}^{\infty} a_n z^n \). So it suffices to show that

\[
\langle f, \psi_n \rangle = \frac{a_n}{\sqrt{n + 1}}
\]

for \( n \geq 0 \). Let \( \mathbb{D}_\rho = \{ z \in \mathbb{C} : |z| \leq \rho < 1 \} \), it follows that

\[
\int_{\mathbb{D}_\rho} f(z) \overline{\psi_n(z)} dA(z) = \sqrt{n + 1} \int_{\mathbb{D}_\rho} a_n z^n \overline{z^n} dA = \frac{a_n}{\sqrt{n + 1}} \rho^{2n+2}
\]

for every \( n \geq 0 \) and every \( \rho < 1 \). Now letting \( \rho \to 1 \) completes the proof.

**Corollary 1.3.4.** The Bergman kernel function of \( \mathbb{D} \) is given by

\[
K(z, w) = \frac{1}{(1 - \bar{w}z)^2}.
\]

**Proof.** By Theorem 1.3.2, the Bergman kernel can be written as

\[
K(z, w) = \sum_{n=0}^{\infty} \psi_n(z) \overline{\psi_n(w)}.
\]
where the set of functions \( \{ \psi_n(z) = \sqrt{n + 1}z^n \}_{n \geq 0} \) is the orthonormal basis discussed in Lemma 1.3.3. Hence,

\[
K(z, w) = \sum_{n=0}^{\infty} (\sqrt{n + 1}z^n)(\sqrt{n + 1}w^n) = \sum_{n=0}^{\infty} (n + 1)(z\bar{w})^n = \frac{1}{(1 - \bar{w}z)^2}
\]

An easy calculation shows that

\[
\int_D \frac{1}{|1 - zw|^4} dA(w) = ||K_z||^2 = \frac{1}{(1 - |z|^2)^2}.
\]

We define the normalized reproducing kernel, denoted \( k_z \), by

\[
k_z(w) = \frac{K_z(w)}{||K_z||} = \frac{1 - |z|^2}{(1 - \bar{z}w)^2}.
\]

### 1.4 Toeplitz Operators on Bergman space

Since \( L^2_a(\mathbb{D}, dA) \) is closed subspace of the Hilbert space \( L^2(\mathbb{D}, dA) \), by the orthogonal projection theorem, \( L^2(\mathbb{D}, dA) \) has the following decomposition

\[
L^2(\mathbb{D}, dA) = L^2_a(\mathbb{D}, dA) \oplus (L^2_a(\mathbb{D}, dA))^\perp,
\]

where \( (L^2_a(\mathbb{D}, dA))^\perp \) is the orthogonal complement of the Bergman space. So there exists an orthogonal projection from \( L^2(\mathbb{D}, dA) \) onto \( L^2_a \), denoted by \( P \), and is defined by \( P(u + v) = u \) where \( u \in L^2_a(\mathbb{D}, dA) \), and \( v \in (L^2_a)^\perp \). Thus \( P(f) \in L^2_a(\mathbb{D}, dA) \) for all \( f \in L^2(\mathbb{D}, dA) \), and so we can use the reproducing kernel \( K(z, w) \) to give an explicit formula for \( Pf \) as follows

\[
P(f)(z) = \langle Pf, K_z \rangle = \langle f, PK_z \rangle = \langle f, K_z \rangle = \int_D \frac{f(w)}{(1 - \bar{w}z)^2} dA(w).
\]
Since \( (f - Pf) \) is orthogonal to \( K_z \in L^2_a(\mathbb{D}, dA) \), the projection operator \( P \) is known as the Bergman projection. The right-hand side of the above equation makes sense whenever \( f \in L^1(\mathbb{D}, dA) \), and so the domain of \( P \) can be extended to \( L^1(\mathbb{D}, dA) \). For \( f \in L^1(\mathbb{D}, dA) \) and \( z \in \mathbb{D} \), we define \( (Pf)(z) \) by the same formula as above. Since we can differentiate under the integral sign, clearly \( Pf \) is analytic on \( \mathbb{D} \) for each \( f \in L^1(\mathbb{D}, dA) \), and \( Pf = f \) for all \( f \in L^1_a(\mathbb{D}, dA) \).

**Remark 1.4.1.** The Bergman projection satisfies the following properties.

1. \( P \) is a bounded projection from \( L^2(\mathbb{D}, dA) \) onto \( L^2_a(\mathbb{D}, dA) \).
2. \( P^* = P \), and \( P^2 = P \).
3. \( ||P|| = 1 \), where \( ||.|| \) is the operator norm.
4. Let \( n \) and \( m \) be a nonnegative integers. Then

\[
P(z^n\bar{z}^m) = \begin{cases} \frac{n-m+1}{n+1}z^{n-m}, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}
\]

**Definition 1.4.1.** For \( \phi \in L^\infty(\mathbb{D}) \), the Toeplitz operator \( T_\phi \) with symbol \( \phi \) is the operator from \( L^2_a(\mathbb{D}, dA) \) into \( L^2_a(\mathbb{D}, dA) \) defined by

\[
T_\phi(f) = P(\phi f) = \int_{\mathbb{D}} \frac{\phi(w)f(w)}{(1 - \bar{w}z)^2} dA(w), \quad f \in L^2_a(\mathbb{D}, dA).
\]

The study of Toeplitz operators on the Bergman space has begun in the seventies by G. McDonald and C. Sundberg in [24]. They proved that a Toeplitz operator \( T_\phi \) on the Bergman space \( L^2_a(\mathbb{D}, dA) \), where \( \phi \) is continuous on \( \mathbb{D} \), is compact if and only if \( \phi \) can be extended to a continuous function on the closed unit disk which vanishes on the boundary. Since then, mathematicians are interested in finding necessary and sufficient conditions for a symbol \( \phi \) such that the Toeplitz operator \( T_\phi \) is bounded,
compact or has finite rank. Also, they are interested in investigating when the product of two Toeplitz operators is a Toeplitz operator, when two Toeplitz operators commute, and so many other questions.

From the above definition, one can see that $T_\phi$ is bounded. In fact,

$$||T_\phi(f)||_2 = ||P(\phi f)||_2 \leq ||\phi||_\infty ||f||_2 < \infty, \quad f \in L^2_\phi(\mathbb{D}, dA).$$

Since the Bergman projection $P$ can be extended to $L^1(\mathbb{D}, dA)$, the Toeplitz operator $T_\phi$ with $\phi \in L^1(\mathbb{D}, dA)$ is well defined on $H^\infty$, the space of bounded analytic functions on $\mathbb{D}$. Hence, $T_\phi$ is always densely defined on $L^2_\phi(\mathbb{D}, dA)$. Since $P$ is not bounded on $L^1(\mathbb{D}, dA)$, $T_\phi$ can be unbounded for symbols in $L^1(\mathbb{D}, dA)$.

Even though we are interested in operators with bounded symbols, operators with unbounded symbols arise naturally. However, unbounded symbols can give rise to bounded Toeplitz operators on the Bergman space. For example, if $G \in L^1(\mathbb{D}, dA)$ and has compact support $K$ in $\mathbb{D}$, then one can define the Toeplitz operator $T_G$ as

$$T_G(f)(z) = \int_{\mathbb{D}} \frac{G(\xi)f(\xi)}{(1 - \xi z)^2} dA(\xi).$$

By taking the absolute value for both sides, we have

$$|T_G(f)(z)| \leq C \int_K |G| \sup_{K} |f| dA \leq C_1 ||f||_2 \int_K |G| dA.$$

The last inequality follows from the fact that for $f \in L^2_\phi(\mathbb{D}, dA)$, the $L^2$-norm dominates the sup norm on any compact set. In fact by Theorem 1.3.1,

$$|f(z)| \leq \pi^{-\frac{1}{2}} \delta(z)^{-1} ||f||_2,$$

for all $f \in L^2_\phi(\mathbb{D}, dA)$, where $\delta(z) = \text{dist}(z, \mathbb{T})$. This shows that the sup norm of $T_G f$
is dominated by a constant times the $L^2$-norm of $f$, and so $\|T_G f\|_2 \leq C \|f\|_2$.

Bergman space Toeplitz operators have some nice algebraic properties similar to Hardy space Toeplitz operators.

**Proposition 1.4.1.** Suppose that $\lambda$ and $\beta$ are complex numbers, $f$ and $g$ are bounded functions. Then Toeplitz operators have the following algebraic properties:

1. $T_{\lambda f + \beta g} = \lambda T_f + \beta T_g$.
2. $T_f^* = T_{\overline{f}}$.
3. If $T_f = 0$, then $f = 0$.
   Moreover, if $f \in H^\infty(\mathbb{D})$, then
4. $T_g T_f = T_{gf}$.
5. $T_{T_f} T_g = T_{T_g}$.

Proof. [27, p. 164]. \qed

### 1.5 Berezin transform

In this section we will introduce the Berezin transform, and some useful facts about it.

**Definition 1.5.1.** Suppose $T$ is a linear operator on $L^2_a(\mathbb{D}, dA)$ (not necessarily bounded) whose domain contains $H^\infty(\mathbb{D})$, then $T$ induces a function $B(T)(z)$ on $\mathbb{D}$ given by

$$B(T)(z) = \langle Tk_z, k_z \rangle, \quad z \in \mathbb{D},$$

where $k_z$ are the normalized reproducing kernels. The function $B(T)$ is called the Berezin transform of the operator $T$. 
If $T = T_{\phi}$ is a Toeplitz operator, with $\phi \in L^1(\mathbb{D}, dA)$, we will write $B(T_{\phi}) = B(\phi)$ since

$$B(T_{\phi})(z) = \langle T_{\phi}k_z, k_z \rangle = \langle \phi k_z, k_z \rangle.$$  

It is easy to see that

$$B(\phi)(z) = \int_{\mathbb{D}} \phi(w) |k_z(w)|^2 dA(w), \quad z \in \mathbb{D}.$$

Making a change of variable, we also have

$$B(\phi)(z) = \int_{\mathbb{D}} \phi \circ \varphi_z(w) dA(w), \quad z \in \mathbb{D},$$

where $\varphi_z(w) = \frac{z - w}{1 - \overline{z}w}$, $w \in \mathbb{D}$, is the Möbius transformation.

It follows from the definition of the Berezin transform that the mapping $\phi \to B(\phi)$ is linear and conjugation preserving. Also it is clear that if $\phi \geq 0$, then $B(\phi) \geq 0$. However, it is not multiplicative. i.e. $B(\phi \psi) \neq B(\phi)B(\psi)$.

Now, we give some basic properties of the Berezin transform whose proofs can be found in [27, ch. 6]. The first property asserts that the Berezin transform is Möbius invariant. We denote by $\text{Aut}(\mathbb{D})$ the Möbius group, which is the set of all analytic one to one maps from $\mathbb{D}$ onto $\mathbb{D}$.

**Proposition 1.5.1.** If $f \in L^1(\mathbb{D}, dA)$ and $\sigma \in \text{Aut}(\mathbb{D})$, then

$$B(f \circ \sigma)(z) = B(f) \circ \sigma(z), \quad z \in \mathbb{D}.$$

**Proposition 1.5.2.** The Berezin transform is injective on $L^1(\mathbb{D}, dA)$.

One remarkable and useful result about Berezin transform is that the only invariant functions under the Berezin transform are the harmonic ones. The sufficiency is easy and it was known for a while, see for instance [15]. The necessity was proved
independently by Engliš in [16] and by Ahern, Flores and Rudin in [1]. One can state this result, which is well exposed in [17], as follows.

**Theorem 1.5.3.** Suppose \( f \in L^1(\mathbb{D}, dA) \). Then \( B(f) = f \) if and only if \( f \) is harmonic on \( \mathbb{D} \).

Many problems in the study of Toeplitz operators rely heavily on the Berezin transform, and due to the above result, the Berezin transform was the main tool in the proof of many results.

**Corollary 1.5.4.** If \( \phi \in L^1(\mathbb{D}, dA) \) is harmonic, then \( T_\phi \) is bounded if and only if \( \phi \) is bounded.

**Proof.** If \( \phi \) is bounded then it is very clear that \( T_\phi \) is also bounded. Assume \( T_\phi \) is bounded. Then Theorem 1.5.3 implies

\[
|\phi(z)| = |B(\phi)(z)| = |\langle T_\phi k_z, k_z \rangle| \leq ||T_\phi||,
\]

for all \( z \in \mathbb{D} \), since \( ||k_z||_2 = 1 \). Hence \( \phi \) is bounded. \( \square \)
Chapter 2

On the Zero Product of Toeplitz Operators

This chapter is concerned with the zero product problem of Toeplitz operators, which can be stated as follows: "Assume the product of two Toeplitz operators is zero. Is it true that one of the symbols must be the zero function?". When it is the case we say that the zero product has the trivial solution. In Section 2.4, we give a partial answer to this problem. In fact, we shall prove that the zero product of two Bergman space Toeplitz operators has the trivial solution, if one of the symbols has a certain polar decomposition form, and the other is a general bounded symbol. For sake of completeness, we will also discuss throughout this chapter all former work and results that we believe are important in understanding the zero product problem for Toeplitz operators.

2.1 Products of Hardy Space Toeplitz Operators

In general, the set of all Toeplitz operators on the Hardy space \( H^2(\mathbb{T}) \) is not closed under multiplication. Consider, for instance, the unilateral shift in \( H^2(\mathbb{T}) \). It is a Toeplitz operator with symbol \( z = e^{i\theta} \). Its adjoint is also a Toeplitz operator...
with symbol $\bar{z} = e^{-i\theta}$. Moreover their product $T_z T_\bar{z}$ is again a Toeplitz operator, and we have

$$T_z^* T_\bar{z} = T_{|z|^2} = T_1 = I,$$

where $I$ is the identity operator on $H^2(\mathbb{T})$. Nevertheless, the product $T_z T_\bar{z}^*$ is not a Toeplitz operator! In fact, using Proposition 1.2.2, one can see that

$$T_z^* (T_z T_\bar{z}^*) T_z = (T_z^* T_z)(T_\bar{z}^* T_\bar{z}) = (I)(I) = I \neq T_z T_\bar{z}^*,$$

and therefore $T_z T_\bar{z}^*$ is not a Toeplitz operator.

It is natural to ask the following question: Under which conditions is the product of two Toeplitz operators a Toeplitz operator? In 1964, Brown and Halmos [9] solved this question for Toeplitz operators on $H^2(\mathbb{T})$, and gave necessary and sufficient conditions for the product of two bounded Toeplitz operators to be again a Toeplitz operator. The key idea of their proof is actually based on a nice property of the matrix representation of Toeplitz operators with respect to the orthonormal basis of $H^2(\mathbb{T})$. In other words, the entries of each diagonal in a Toeplitz matrix are the same. Their result is stated as follows.

**Theorem 2.1.1.** Let $\phi, \psi \in L^\infty(\mathbb{T})$. Then the product $T_\phi T_\psi$ is a Toeplitz operator if and only if either $\bar{\phi}$ is analytic or $\psi$ is analytic, and in both cases $T_\phi T_\psi = T_{\bar{\phi}\psi}$.

### 2.2 Products of Bergman Space Toeplitz Operators

Unfortunately, matrices of Toeplitz operators on $L^2_\alpha(\mathbb{D}, dA)$ do not satisfy the same nice property of those in $H^2(\mathbb{T})$. Therefore, there is no hope that the techniques used by Brown and Halmos in [9] can be applied to Bergman space Toeplitz operators.
Thus, function theory rather than matrix manipulations, is going to play a major role in obtaining analogous results to those of Brown and Halmos.

In [2], P. Ahern and Ž. Čučković studied the analogous of the Brown-Halmos theorem for Toeplitz operators on $L^2_a(\mathbb{D}, dA)$. They proved that, with some extra hypothesis on the two symbols, the theorem of Brown-Halmos holds on $L^2_a(\mathbb{D}, dA)$. However, they also provided examples to show that it fails for general symbols, even for symbols continuous up to the boundary. The exciting techniques used in this paper are very useful and completely new, which makes it significantly a rich paper. Their result is stated in the following theorem.

**Theorem 2.2.1.** If $f$ and $g$ are bounded harmonic functions, and $h$ is bounded $C^2$ function with bounded invariant Laplacian, then $T_FT_g = T_h$ holds if and only if either $\overline{f}$ or $g$ is analytic, and in either case $fg = h$.

Recall that the invariant Laplacian, often denoted by $\tilde{\Delta}$, is equal to $(1 - |z|^2)^2 \Delta$, where $\Delta = \frac{\partial^2}{\partial z \partial \bar{z}}$ is the Laplacian.

**Definition 2.2.1.** A function $F \in L^1(\mathbb{D}, dA)$ is said to be a nearly bounded function if there exists an $\rho < 1$ such that $F$ is integrable over the compact disk $\{|z| \leq \rho\}$, and is bounded on the corona $\{z \in \mathbb{D} : \rho < |z| < 1\}$.

If such $\rho$ exists, then by linearity of the Toeplitz operator about its symbol, $T_F$ can be written as $T_F = T_{F\chi_{\{|z| \leq \rho\}}} + T_{F\chi_{\{|\rho| < |z| < 1\}}}$. Since $F\chi_{\{|z| \leq \rho\}}$ is an integrable function with compact support, $T_{F\chi_{\{|z| \leq \rho\}}}$ is not only bounded but also compact. On the other hand, $T_{F\chi_{\{|\rho| < |z| < 1\}}}$ is clearly a bounded operator since its symbol $F\chi_{\{|\rho| < |z| < 1\}}$ is bounded. Therefore $T_F$ is bounded as sum of two bounded operators.

Nearly bounded functions will be useful in the construction of counter-examples to Brown-Halmos theorem obtained by Ahern and Ž. Čučković in [2].

Contrary to the case of Toeplitz operators on $H^2(\mathbb{T})$, for those defined on $L^2_a(\mathbb{D}, dA)$ there exist functions $\phi$, $\psi \in L^1(\mathbb{D}, dA)$ such that $T_\phi$ and $T_\psi$ are bounded and their
product $T_\phi T_\psi$ is a Toeplitz operator but neither $\bar{\phi}$ is analytic nor $\psi$ is analytic. Nevertheless, it is possible to obtain a theorem of Brown-Halmos type by requiring symbols to satisfy certain conditions.

If $\phi$ is a bounded harmonic function on $\mathbb{D}$, then there exist two analytic functions $\phi_1$ and $\phi_2$ such that

$$\phi = \phi_1 + \phi_2.$$  

This decomposition is unique if we require $\phi_2(0) = 0$. Of course, $\phi_1$ and $\phi_2$ are not necessarily bounded but they are certainly Bloch functions. We recall that the Bloch space consists of all analytic functions $f$ on $\mathbb{D}$ such that

$$\sup\{(1 - |z|^2)|f'(z)| : z \in \mathbb{D}\} < +\infty.$$  

In fact, $\phi$ can be written as $\phi = u + iv$, where $u = \Re \phi$ and $v = \Im \phi$ are real harmonic functions. Since any real harmonic function is the real part of an analytic function, there exist unique analytic functions $f$ and $g$ on $\mathbb{D}$ satisfying

$$u = \Re(f) \text{ and } f(0) = 0,$$

and

$$v = \Re(g) \text{ and } g(0) = 0.$$  

Thus

$$\phi = \Re(f) + i\Re(g) = \frac{1}{2}(f + \bar{f}) + i\frac{1}{2}(g + \bar{g}) = \frac{1}{2}(f + ig) + \frac{1}{2}(f - ig).$$  

If we denote $\phi_1 = \frac{1}{2}(f + ig)$ and $\phi_2 = \frac{1}{2}(f - ig)$, we have $\phi = \phi_1 + \phi_2$. Since the
Bergman projection $P$ is bounded from $L^\infty(\mathbb{D})$ onto the Bloch space (see [5, p. 13], or [27, p. 102]), therefore $P(\phi) = \phi_1$ and $P(\overline{\phi}) = \phi_2$ are Bloch functions.

Let us consider two bounded harmonic symbols $\phi = \phi_1 + \overline{\phi}_2$ and $\psi = \psi_1 + \overline{\psi}_2$ on $\mathbb{D}$. Then, using Proposition 1.4.1, the product $T_\phi T_\psi$ is equal to

$$T_\phi T_\psi = T_{(\phi_1 + \overline{\phi}_2)} T_{(\psi_1 + \overline{\psi}_2)} = T_{\phi_1 \psi_1} + T_{\phi_1 \overline{\psi}_2} + T_{\overline{\phi}_2 \psi_1} + T_{\overline{\phi}_2 \overline{\psi}_2}.$$  

(2.1)

So if $T_{\phi_1} T_{\overline{\psi}_2}$ is a Toeplitz operator, then $T_\phi T_\psi$ will be also a Toeplitz operator. This is the essence of [3] in which Ahern characterized, using the Berezin transform, all analytic bounded functions $\phi$ and $\psi$ such that the product $T_\phi T_\overline{\psi}$ is a Toeplitz operator. Consequently, Ahern gave necessary and sufficient conditions for the product of two Toeplitz operators with bounded harmonic symbols to be a Toeplitz operator. Ahern’s result depends on the following proposition in [2].

**Proposition 2.2.2.** Let $\phi$ and $\psi$ be two analytic functions in $L^1(\mathbb{D}, dA)$ such that the Toeplitz operators $T_\phi$ and $T_\psi$ are bounded on $L^2_n(\mathbb{D}, dA)$, and let $u \in L^1(\mathbb{D}, dA)$ be a nearly bounded function. Then the following statements are equivalent

(i) $T_\phi T_{\overline{\psi}} = T_u$.

(ii) $\phi \overline{\psi} = B(u)$, where $B$ is the Berezin transform.

(iii) For all $(z, w) \in \mathbb{D} \times \mathbb{D}$ we have

$$\phi(z) \overline{\psi(w)} = (1 - zw)^2 \int_{\mathbb{D}} \frac{u(\xi)}{(1 - \xi z)^2(1 - \xi w)^2} dA(\xi).$$

Now assume that one of the two symbols $\phi$ or $\psi$ is constant. Then $\phi \overline{\psi}$ is a harmonic function and according to Theorem 1.5.3, harmonic functions are invariant under the Berezin transform $B$, so $\phi \overline{\psi} = B(\phi \overline{\psi})$. Thus if $T_\phi T_{\overline{\psi}} = T_u$, then (ii) of Proposition 2.2.2 implies $B(\phi \overline{\psi}) = B(u)$. Since $B$ is injective, we have $\phi \overline{\psi} = u$. We
will refer to this case as the trivial case. In fact, in the decompositions $\phi = \phi_1 + \overline{\phi}_2$ and $\psi = \psi_1 + \overline{\psi}_2$, if $\phi_1$ or $\psi_2$ is constant, then it is clear that $\overline{\phi}$ is analytic or $\psi$ is analytic. Therefore the product $T_\phi T_\psi$ is the Toeplitz operator $T_{\phi_\psi}$. Under certain conditions this trivial case will be the only situation in which we have $T_\phi T_\psi = T_{\phi_\psi}$.

A similar result to this was given by Proposition 2 in [2], and it involves the invariant Laplacian. The invariant laplacian has the nice property that it commutes with the Berezin transform i.e. $\tilde{\Delta}(Bu) = B(\tilde{\Delta}u)$. This fact is crucial in the proof of the following proposition.

**Proposition 2.2.3.** If $\phi$ and $\psi$ are analytic in $\mathbb{D}$ and if $\phi \overline{\psi} = B(u)$ for some function $u \in L^1(\mathbb{D}, dA) \cap C^2(\mathbb{D})$ such that $\tilde{\Delta}u$ is bounded on $\mathbb{D}$, then either $\phi$ is constant or $\psi$ is constant.

Here it is important to emphasize that the conditions in Proposition 2.2.3 are sufficient but not necessary. In other words there exists a function $u \in L^1(\mathbb{D}, dA)$ such that $B(u) = \phi \overline{\psi}$ where $\phi$ and $\psi$ are both analytic but neither $\phi$, nor $\psi$ is constant. The following two examples, obtained in [2, p. 208] and [3, p. 208], illustrate this situation. In addition, they show that the condition of ”$\tilde{\Delta}u$ being bounded” is not necessary.

**Lemma 2.2.4.**

1. $z \overline{z} = B(u_1)(z)$, where $u_1(z) = 1 + \log |z|^2$.

2. $z \overline{z}^2 = B(u_2)(z)$, where $u_2(z) = 2 \overline{z} - \frac{1}{z}$.

It is clear that neither $u_1$, nor $u_2$ belongs to $C^2(\mathbb{D})$. Moreover, $u_1$ and $u_2$ are not bounded but they are nearly bounded.

It follows from Proposition 2.2.2 that

$$T_z T_{\overline{z}} = T_{1 + \log |z|^2}$$
and

\[ T_z T_{z^2} = T_{z^2 \frac{1}{2}}. \]

Note that in both examples the function \( \phi \) (which is equal to \( z \)) and the function \( \psi \) (which is equal to \( z \) in the first example, and \( z^2 \) in the second example) are analytic polynomials and that the degree of their product is less than or equal to 3. Moreover, according to Proposition 1.5.1, for every \( \varphi \in Aut(\mathbb{D}) \) we have

\[ \varphi \overline{\varphi} = B(u_1 \circ \varphi), \]

and

\[ \varphi \overline{\varphi}^2 = B(u_2 \circ \varphi). \]

This is equivalent, according to Proposition 2.2.2, to

\[ T_\varphi T_{\overline{\varphi}} = T_{u_1 \circ \varphi}, \]

and

\[ T_\varphi T_{\overline{\varphi}^2} = T_{u_2 \circ \varphi}. \]

The following theorem [3, p. 207] characterizes all analytic functions \( \phi \) and \( \psi \) such that \( \overline{\phi \psi} = B(u) \) for some function \( u \in L^1(\mathbb{D}, dA) \). It tells us that both \( \phi \) and \( \psi \) are analytic polynomial functions of a certain Möbius transformation (automorphism of the disk). Moreover, the degree of the product of the two polynomials does not exceed 3.

**Theorem 2.2.5.** If \( \phi \) and \( \psi \) are two non-constant analytic functions in \( \mathbb{D} \) and \( \overline{\phi \psi} = B(u) \) where \( u \in L^1(\mathbb{D}, dA) \) then there are non-constant polynomials \( p \) and \( q \) with \( \deg(pq) \leq 3 \) and there exists \( z_0 \in \mathbb{D} \) such as \( \phi = p \circ \varphi_{z_0} \) and \( \psi = q \circ \varphi_{z_0} \).

Actually, this theorem is a generalization of Proposition 2.2.3, since the condition
on the invariant Laplacian of $u$ is removed. Indeed, if we look at the two examples of Lemma 2.2.4, it is clear that $u_1$ and $u_2$ are not in $C^2(\mathbb{D})$, and therefore their Laplacians are not even defined.

As an application of the previous theorem, the following corollary [3, p. 215] characterizes all bounded harmonic functions $\phi$ and $\psi$ for which the product $T_{\phi}T_{\psi}$ is a Toeplitz operator.

**Corollary 2.2.6.** If $\phi = \phi_1 + \overline{\phi}_2$ and $\psi = \psi_1 + \overline{\psi}_2$ are bounded harmonic functions, where $\phi_1$, $\phi_2$, $\psi_1$, and $\psi_2$ are analytic in $\mathbb{D}$ and neither $\phi_1$ nor $\psi_2$ is constant. Then the following statements are equivalent

i) There exits a function $h \in L^1(\mathbb{D}, dA)$, where $T_h$ is bounded on $L^2_a(\mathbb{D}, dA)$, such that $T_{\phi_1}T_{\psi_2} = T_h$.

ii) There are non-constant analytic polynomials $p$ and $q$ with $\deg(pq) \leq 3$ and $z_0 \in \mathbb{D}$ such that $\phi_1 = p \circ \varphi_{z_0}$, $\psi_2 = q \circ \varphi_{z_0}$ and $h$ must be of the form

$$h = u \circ \varphi_{z_0} + \overline{\varphi}_2 \psi_1 + \phi_1 \psi_1 + \phi_2 \psi_2,$$

where the function $u$ is defined by $pq = B(u)$.

### 2.3 Quasihomogeneous Toeplitz operators

We have seen in the previous section that the question of when the product of two Toeplitz operators with harmonic symbols is a Toeplitz operator was partially solved by Ahern and Ž. Čučković in [2], then completely solved by Ahern in [3]. The same question was treated by Issam Louhichi, Elizabeth Strouse and Lova Zakariasy in [20] but for a different class of symbols. In particular, they studied the product of Toeplitz operators with the so-called **quasihomogeneous symbols**. In this section


we shall introduce quasihomogeneous Toeplitz operators, and summarize some useful facts about them, that will be needed later for our main results.

**Definition 2.3.1.** A function $f$ is said to be quasihomogeneous of degree $p \in \mathbb{Z}$, if there exists a radial function $\phi$ such that for every $z = re^{i\theta} \in \mathbb{D}$, $f$ can be written as

$$f(re^{i\theta}) = e^{ip\theta}\phi(r).$$

The associated Toeplitz operator $T_f$ is also called quasihomogeneous Toeplitz operator of degree $p$.

Recall that a function $\phi$ is called radial if for every $z \in \mathbb{D}$, we have

$$\phi(z) = \phi(|z|).$$

From now on, we identify radial functions in $L^1(\mathbb{D}, dA)$ with functions in $L^1([0, 1], rdr)$. The main reason to consider such symbols is that any function in $L^2(\mathbb{D}, dA)$ is the sum of quasihomogeneous functions. To see this, let $\mathcal{R}$ be the space of all square integrable radial functions on $[0, 1]$ with respect to the measure $rdr$. Since the set of polynomials in $(z, \bar{z})$ is dense in $L^2(\mathbb{D}, dA)$ and for any two integers $k_1 \neq k_2$, $e^{ik_1 \theta}\mathcal{R}$ is orthogonal to $e^{ik_2 \theta}\mathcal{R}$, it follows that $L^2(\mathbb{D}, dA)$ has the following decomposition

$$L^2(\mathbb{D}, dA) = \bigoplus_{k \in \mathbb{Z}} e^{ik\theta}\mathcal{R}.$$

Thus any function $f \in L^2(\mathbb{D}, dA)$ can be written as

$$f(re^{i\theta}) = \sum_{k=-\infty}^{\infty} e^{ik\theta} f_k(r), \quad f_k \in \mathcal{R}. $$
Moreover, if \( f \in L^\infty(D) \) then for every \( r \in [0, 1) \) and every \( k \in \mathbb{Z} \), we have

\[
|f_k(r)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(re^{i\theta}) e^{-ik\theta} d\theta \right| \leq \sup_{z \in D} |f(z)|,
\]

and therefore the function \( f_k \) is bounded on \( D \) for every \( k \in \mathbb{Z} \).

Although this kind of decomposition does not occur for functions in \( L^1(D, dA) \), having some results on the product of Toeplitz operators with quasihomogeneous symbols could give rise to more general symbols.

Let \( p \in \mathbb{N} \), and let \( \phi \in L^1([0, 1], r dr) \) be a radial function such that \( T_\phi \) is bounded on \( L^2_a(D, dA) \). Now, applying the Toeplitz operator \( T_{e^{-ip\theta} \phi} \) on elements of the orthogonal basis of \( L^2_a(D, dA) \), namely \( \{\xi^n : n \geq 0\} \), gives us

\[
T_{e^{-ip\theta} \phi}(\xi^n)(z) = \int_D \frac{e^{-ip\theta} \phi(w) w^n}{(1 - z\bar{w})^2} dA(w)
= \int_D e^{-ip\theta} \phi(w) w^n \sum_{k=0}^{+\infty} (k + 1) z^k \bar{w}^k dA(w)
= \int_0^1 \int_0^{2\pi} \phi(r) r^n e^{i(n-p)\theta} \sum_{k=0}^{+\infty} (k + 1) r^k e^{-ik\theta} \frac{d\theta}{\pi} r dr
= \int_0^1 \phi(r) \sum_{k=0}^{+\infty} (k + 1) \int_0^{2\pi} e^{i(n-p-k)\theta} \frac{d\theta}{\pi} r^n z^k r dr.
\]

If \( n \leq p - 1 \), then there will be no \( k \geq 0 \) such that \( n = p + k \). But if \( n \geq p \), then there exists \( k \geq 0 \) such that \( n = k + p \). Moreover

\[
\int_0^{2\pi} e^{i(n-p-k)\theta} \frac{d\theta}{\pi} = \begin{cases} 0 & \text{if } k + p \neq n, \\ 2 & \text{if } k + p = n. \end{cases}
\]
Thus

\[
T_{e^{-\phi}}(\xi^n)(z) = \begin{cases} 
0 & \text{if } n \leq p - 1, \\
2(n - p + 1) \int_0^1 \phi(r) r^{2n-p+1} \, dr \, z^{n-p} & \text{if } n \geq p.
\end{cases}
\]

Similarly, we can show that for any integer \( n \geq 0 \)

\[
T_{e^{\phi}}(\xi^n)(z) = 2(n + p + 1) \int_0^1 \phi(r) r^{2n+p+1} \, dr \, z^{n+p}.
\]

The Mellin transform of a function \( \phi \) in \( L^1([0,1], rdr) \) is defined by

\[
\hat{\phi}(z) = \int_0^{+\infty} \phi(r) r^{z-1} dr.
\]

Since we are applying the Mellin transform to functions in \( L^1([0,1], rdr) \), we shall consider these functions to be zero on \((1, +\infty)\). It is clear that for a function \( \phi \in L^1([0,1], rdr) \), the map \( z \mapsto \hat{\phi}(z) \) is bounded on the half-plane \( \{ z : \Re z \geq 2 \} \) and analytic on the half-plane \( \{ z : \Re z > 2 \} \). In summary, we have the following lemma which we will use often.

**Lemma 2.3.1.** If \( p \) is a positive integer and \( \phi \) is a radial function in \( L^1(\mathbb{D}, dA) \) such that \( T_\phi \) is bounded Toeplitz operator, then for every integer \( n \in \mathbb{N} \) we have:

\[
T_{e^{-\phi}}(\xi^n)(z) = 2(n + p + 1) \hat{\phi}(2n + p + 2) z^{n+p},
\]

and

\[
T_{e^{\phi}}(\xi^n)(z) = \begin{cases} 
0 & \text{if } n < p, \\
2(n - p + 1) \hat{\phi}(2n - p + 2) z^{n-p} & \text{if } n \geq p.
\end{cases}
\]

**Remark 2.3.1.** (i) By the above lemma, one can see that a quasihomogeneous Toeplitz operator acts on any element of the orthogonal basis of \( L^2_\phi(\mathbb{D}, dA) \) as
a shift operator with holomorphic weight. This property has been emphasized by Louhichi and Rao in [19]. They named such an operator by Holomorphic Weighted Shift Operator and denoted it by HWS. This new notion appears to be of a great help when dealing with products of quasihomogeneous Toeplitz operators.

(ii) The matrix of the quasihomogeneous Toeplitz operator with respect to the orthonormal basis of the Bergman space $L^2_a(\mathbb{D}, dA)$, is a matrix whose elements are zero except those on the diagonal. For example, if $p \in \mathbb{N}$ and $\phi \in L^1(\mathbb{D}, dA)$ is radial such that $T_\phi$ is bounded and if $A = (a_{ij})$ is the Toeplitz matrix of $T_{e^{i\theta}\phi}$ with respect to the orthonormal basis $\{\sqrt{k+1}z^k\}_{k \geq 0}$ then

$$a_{ij} = \begin{cases} 
0 & \text{if } i - j \neq p \\
2\sqrt{j+1}(j+p+1)\hat{\phi}(2j+p+2) & \text{if } i - j = p 
\end{cases}$$

We shall often use the following classical theorem [25, p. 102], which infer that the Mellin transform is injective.

**Theorem 2.3.2.** Suppose $\phi$ is a bounded analytic function on the right half-plane $\{z \in \mathbb{C} : \Re z > 0\}$. If $\phi$ is zero on the following distinct points $z_1, z_2, \ldots$, where

1. $\inf |z_n| > 0$;
2. $\sum_{n \geq 1} \Re \left( \frac{1}{z_n} \right) = \infty$;

then $\phi$ is identically zero on $\{z \in \mathbb{C} : \Re z > 0\}$.

As a consequence of Theorem 2.3.2, the following corollary says that a function is uniquely determined by the values of its Mellin transform on an arithmetic sequence.

**Corollary 2.3.3.** Suppose $\{n_k\}_{k \geq 0}$ is a sequence of positive integers greater than or
equal to 2 and satisfies the condition

$$\sum_{k \geq 0} \frac{1}{n_k} = +\infty.$$  

If $\phi \in L^1([0,1], rdr)$ is such that $\hat{\phi}(n_k) = 0$ for every $k \geq 0$, then $\phi \equiv 0$.

**Proof.** If $\hat{\phi}(n_k) = \int_0^1 \phi(r)r^{n_k-1}dr = 0$ for every $k \geq 0$, then by the previous theorem $\hat{\phi}(z) = 0$ for every $z \in \{z \in \mathbb{C} : \Re z > 2\}$, and therefore $\phi$ is zero. \qed

### 2.4 On the Zero Product Problem

In this section, we shall give a partial answer to the following question (Q) Under which conditions on the symbols $f$ and $g$ does the zero product $T_fT_g = 0$ have a trivial solution, i.e. $f = 0$ or $g = 0$?

Brown and Halmos in [9] answered this question for Hardy space Toeplitz operators. In fact, their result is a direct consequence of Theorem 2.1.1.

**Corollary 2.4.1.** Let $\phi, \psi \in L^{\infty}(\mathbb{T})$. Then $T_\phi T_\psi = 0$ if and only if $\phi = 0$ or $\psi = 0$.

In other words, among the class of Toeplitz operators on $H^2(\mathbb{T})$ there are no zero divisors.

**Proof.** If $T_\phi = 0$ or $T_\psi = 0$, then clearly we get $T_\phi T_\psi = 0$. Now, suppose $T_\phi T_\psi = 0$, then the product $T_\phi T_\psi$ is a Toeplitz operator, since its matrix is a Toeplitz matrix.

So by Theorem 2.1.1, either $\tilde{\phi}$ is analytic or $\psi$ is analytic, and in any case we have $T_\phi T_\psi = T_{\phi \psi} = 0$. Therefore, $\phi \psi = 0$. If $\psi$ is analytic and not identically zero (thus $\psi \in H^2(\mathbb{T})$) then the set of all zeros of $\psi$ is of measure zero and in this case $\phi \psi = 0$ implies that the function $\phi = 0$ almost everywhere on $\mathbb{T}$ and thus $T_\phi = 0$.

If $\tilde{\phi}$ is analytic, then by taking the adjoint of both sides of $T_\phi T_\psi = 0$, we have
$T_{\overline{\psi}}T_{\phi} = T_{\overline{\psi}\phi} = 0$, which implies that $\overline{\psi}\phi = 0$ and in the case if $\phi$ is not identically zero, then clearly $\psi = 0$. \hfill \Box$

In the Bergman space, things are different and the question (Q) has no complete answer yet.

In [2], Ahern and Čučković considered bounded harmonic symbols on $\mathbb{D}$ and proved that a theorem of Brown-Halmos type holds only in the trivial case (see Theorem 2.2.1). They were able to answer the zero product problem when both symbols are bounded harmonic functions.

**Corollary 2.4.2.** If $f$ and $g$ are bounded harmonic functions and $T_fT_g = 0$ then either $f = 0$ or $g = 0$.

*Proof.* By Theorem 2.2.1, we obtain $T_fT_g = T_{fg} = 0$. Thus $fg = 0$ in $\mathbb{D}$. Since $f$ and $g$ are both harmonic, at least one of them must be zero. \hfill \Box

Since the question (Q) is solved for Toeplitz operators with bounded harmonic symbols, one can ask what would be the answer to the zero product problem of two Toeplitz operators with general symbols that are not harmonic in $\mathbb{D}$? In [12], Čučković conjectured that, if the product of two Toeplitz operators is zero and if one of the symbols is nonzero harmonic then the other symbol must be zero. The idea of Čučković was to combine both the Berezin and the Mellin transformations to study the zero product problem. He obtained the following result.

**Theorem 2.4.3.** Let $f \in L^\infty(\mathbb{D})$ and let $g$ be the function defined by $g(z) = z^m - \overline{z}^n$, where $m, n \in \mathbb{N}$. If $T_fT_g = 0$ then $f = 0$.

In [4], Ahern and Čučković answered the zero product problem for two Toeplitz operators in the case where one of the symbols is quasihomogeneous and the other is an integrable function.
Theorem 2.4.4. Let $p$ be a positive integer. Let $f$ and $\psi$, where $\psi$ is radial, be in $L^1(\mathbb{D}, dA)$ such that $T_f$ and $T_\phi$ are bounded. If $T_f T_{e^{ip\theta} \psi} = 0$, then either $f = 0$ or $\psi = 0$.

Remark 2.4.1. (i) If $p \in \mathbb{N}$ and $T_f T_{e^{-ip\theta} \psi} = 0$, then compose both sides from the right by $T_{z^p}$ to obtain $T_f T_{r^p\psi} = 0$. But the symbol $r^p\psi$ is radial function, in particular it is quasihomogeneous of degree 0, and therefore by the previous theorem $f = 0$ or $r^p\psi = 0$ i.e. $\psi = 0$.

(ii) If $T_{e^{ip\theta} \psi} T_f = 0$ then by taking the adjoint of both sides we have $T^*_f T_{e^{-ip\theta} \psi} = 0$, and so the same reasoning as in (i) is still valid.

Now, we are ready to present our contribution toward solving the zero product problem. Our result is an improvement of Theorem 2 in [12, p. 237]. The technique used in our proof is completely new and different from the technique used by Ćučković. More precisely, we use a recent result, obtained by D. Luecking in [23], about finite rank Toeplitz operators on $L^2_a(\mathbb{D}, dA)$, which can be stated as follows.

"The only finite rank Toeplitz operator is the zero operator".

We state our result [22] as follows.

Theorem 2.4.5. Let $f \in L^\infty(\mathbb{D})$ and let $g \in L^\infty(\mathbb{D})$ such that $g(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} g_k(r)$, where $N$ is a positive integer. Assume $n_0 \geq 0$ to be the smallest integer such that $\hat{g}_N(2n + N + 2) \neq 0$ for all $n \geq n_0$. If $T_f T_g = 0$, then $f = 0$.

Proof. For all $n \geq 0$, we have

$$T_g(z^n) = \sum_{k=-\infty}^{N} T_{e^{ik\theta} g_k(r)}(z^n).$$
According to Lemma 2.3.1, $T_{e^{ik\theta}g_k(r)}(z^n) = 0$ for all $k \leq -n - 1$. Therefore

\[
T_g(z^n) = \sum_{k=-n}^{N} T_{e^{ik\theta}g_k(r)}(z^n)
\]

\[
= \sum_{k=-n}^{N} 2(n + k + 1)\hat{g}_k(2n + k + 2)z^{n+k}
\]

\[
= 2(n + N + 1)\hat{g}_N(2n + N + 2)z^{n+N} + \sum_{k=-n}^{N-1} 2(n + k + 1)\hat{g}_k(2n + k + 2)z^{n+k}.
\]

Replacing $n$ by $n_0$, we have

\[
T_g(z^{n_0}) = 2(n_0 + N + 1)\hat{g}_N(2n_0 + N + 2)z^{n_0+N} + \sum_{k=-n_0}^{N-1} 2(n_0 + k + 1)\hat{g}_k(2n_0 + k + 2)z^{n_0+k},
\]

By hypothesis $\hat{g}_N(2n_0 + N + 2) \neq 0$. Thus

\[
z^{n_0+N} = \frac{1}{\lambda_{n_0}} \left[ \sum_{k=-n_0}^{N-1} 2(n_0 + k + 1)\hat{g}_k(2n_0 + k + 2)z^{n_0+k} - T_g(z^{n_0}) \right],
\]

where $\lambda_{n_0} = 2(n_0 + N + 1)\hat{g}_N(2n_0 + N + 2)$. This clearly implies

\[
z^{n_0+N} \in \text{Span}\{T_g(z^{n_0}), 1, z, \ldots, z^{n_0+N-1}\}. \tag{2.2}
\]

Redoing the same argument for $n = n_0 + 1$, one can see that

\[
T_g(z^{n_0+1}) = 2(n_0 + N + 2)\hat{g}_N(2n_0 + N + 4)z^{n_0+N+1} + \sum_{k=-n_0-1}^{N-1} 2(n_0 + k + 2)\hat{g}_k(2n_0 + k + 4)z^{n_0+k+1}.
\]

Since $\hat{g}_N(2n_0 + 2 + N + 2) \neq 0$, we can write $z^{n_0+N+1}$ as a linear combination of
$T_g(z^{n_0+1})$, 1, $z, \ldots, z^{n_0+N}$, which means that

$$z^{n_0+N+1} \in \text{Span}\{T_g(z^{n_0+1}), 1, z, \ldots, z^{n_0+N}\}.$$  

Therefore, (2.2) implies

$$z^{n_0+N+1} \in \text{Span}\{T_g(z^{n_0+1}), T_g(z^{n_0}), 1, z, \ldots, z^{n_0+N-1}\}.$$  

In fact, using the same method, we prove that for all $l \geq 0$

$$z^{n_0+N+l} \in \text{Span}\{T_g(z^{n_0+1}), \ldots, T_g(z^{n_0}), 1, z, \ldots, z^{n_0+N-1}\},$$

and so

$$T_f(z^{n_0+N+l}) \in \text{Span}\{T_fT_g(z^{n_0+1}), \ldots, T_fT_g(z^{n_0}), T_f(1), T_f(z), \ldots, T_f(z^{n_0+N-1})\}.$$  

But $T_fT_g(z^n) = 0$ for all $n \geq 0$. So

$$T_f(z^{n_0+N+l}) \in \text{Span}\{T_f(1), T_f(z), \ldots, T_f(z^{n_0+N-1})\}, \text{ for all } l \geq 0.$$  

Therefore the rank of $T_f$ is at most equal to $n_0 + N$. Hence, using Luecking’s result, $T_f$ must be zero and so $f = 0$. \hfill \square

**Remark 2.4.2.** Let $f \in L^\infty(\mathbb{D})$ and let $g(z) = Q(z) + \overline{h}(z)$, where $Q$ is an analytic polynomial and $h$ is any bounded analytic function. If $T_fT_g = 0$, then $f = 0$. In fact, since the Mellin transform of any monomial in $z$ never equals to zero, the hypothesis “$\hat{g}N(2n + N + 2) \neq 0$ for all $n \geq n_0$” of Theorem 2.4.5 is obviously satisfied, and therefore $f$ must be zero. Actually this result supports Čučković’s conjecture in [12, p. 235] that we mentioned early in this section.
Chapter 3

Commuting Toeplitz operators on the Bergman space

The commuting problem of Toeplitz operators is one of the most challenging problems in the theory of Toeplitz operators. This problem can be stated as follows

*If two Toeplitz operators commute, what can we say about their symbols?*

Brown and Halmos [9] gave a complete answer to this question for Toeplitz operators on $H^2(\mathbb{T})$. In fact, they characterize the commutant of a given Toeplitz operator on $H^2(\mathbb{T})$. We recall that the commutant of a Toeplitz operator is the set of all Toeplitz operators that commute with it. Their result is stated as follows.

**Theorem 3.0.6.** Let $\phi, \psi$ be two bounded functions in $\mathbb{T}$. Then $T_\phi T_\psi = T_\psi T_\phi$ if and only if

(i) both $\phi$ and $\psi$ are analytic, or

(ii) both $\bar{\phi}$ and $\bar{\psi}$ are analytic, or

(iii) there exist $\alpha, \beta \in \mathbb{C}$ not both zeros such that $\alpha \phi + \beta \psi$ is constant on $\mathbb{T}$.
In other words, Toeplitz operators on $H^2(\mathbb{T})$ commute only in the trivial case. In fact, if both symbols $\phi$ and $\psi$ are analytic (respectively $\bar{\phi}$ and $\bar{\psi}$ are analytic), then their associated Toeplitz operators (respectively their adjoints $T^*_\phi$ and $T^*_\psi$) are just the operators of multiplication by $\phi$ and $\psi$ respectively. Since any two multiplication operators commute, $T_\phi$ will commute with $T_\psi$ (respectively since their adjoints commute, then the operators themselves will also commute). On the other hand it is well known that any operator commutes with itself and the identity. So if $\phi$ and $\psi$ satisfy (iii), i.e. one symbol is a linear combination of the other and one, then by linearity of Toeplitz operators in their symbols, $T_\phi$ and $T_\psi$ commute. In the proof of Theorem 3.0.6, Brown and Halmos used the well-known property of Toeplitz matrices on $H^2(\mathbb{T})$, namely the entries at each diagonal are equal.

As we mentioned before, in the Bergman space $L^2_a(D,dA)$ things are different, and in spite of all the work that has been done over the last decades, the problem of commuting Toeplitz operators is still not resolved. In this chapter, we start by discussing former results, that have been obtained so far, regarding this question. In the last section (3.4) of this chapter we will give our results in this context.

### 3.1 Commuting Toeplitz operators with harmonic symbols

In [6], Axler and Čučković studied commuting Toeplitz operators, on the Bergman space $L^2_a(\mathbb{D},dA)$, with bounded harmonic symbols. They obtained a result similar to Theorem 3.0.6. In fact, they show that on the Bergman space, two Toeplitz operators with bounded harmonic symbols commute only in the trivial cases. However, their proof is completely different from the proof provided by Brown and Halmos for commuting Toeplitz operators on $H^2(\mathbb{T})$. In fact, the property of Toeplitz matrices (the entries of each diagonal are the same) is no longer true on $L^2_a(\mathbb{D},dA)$, and the
only Toeplitz operators on $L^2_a(\mathbb{D}, dA)$, that satisfy such property are the identity and the zero operators. The result of Axler and Ćučković can be stated as follows.

**Theorem 3.1.1.** Suppose that $f$ and $g$ are bounded harmonic functions on $\mathbb{D}$. Then

$$T_f T_g = T_g T_f$$

if and only if

(i) $f$ and $g$ are both analytic on $\mathbb{D}$, or

(ii) $\overline{f}$ and $\overline{g}$ are both analytic on $\mathbb{D}$, or

(iii) there exist constants $a, b \in \mathbb{C}$, not both zero, such that $af + bg$ is constant on $\mathbb{D}$.

The main tool in the proof of the above theorem is a characterization of harmonic functions by a conformally invariant mean value property.

The following characterization of normal Toeplitz operators is a direct consequence of the previous theorem. An operator is called normal if it commutes with its adjoint.

**Corollary 3.1.2.** Suppose $\phi$ is a bounded harmonic function on $\mathbb{D}$. Then the operator $T_\phi$ is normal, i.e. $T_\phi T_\phi^* = T_\phi^* T_\phi$, if and only if the set $\phi(\mathbb{D})$ lies on some line in the complex plane $\mathbb{C}$.

### 3.2 Commutants of analytic Toeplitz operators

A Toeplitz operator $T_\phi$ is called analytic, if the symbol $\phi$ is analytic. In this case $T_\phi$ is just the multiplication operator by $\phi$. Since any two multiplication operators commute, if $T_\phi$ and $T_\psi$ are analytic on $L^2_a(\mathbb{D}, dA)$, then clearly $T_\phi T_\psi = T_\psi T_\phi$. We will refer to this case as a trivial case.
In [10], Čučković characterized the commutants of all powers of the shift operator $T_z$ on $L^2_a(\mathbb{D}, dA)$. It turns out that for every $n \geq 1$, the commutant of $T_{z^n}$ is the the set of all analytic Toeplitz operators. Later in [7], Axler, Čučković and Rao generalized this result by replacing the unit disk $\mathbb{D}$ with any bounded open domain $\Omega$ in $\mathbb{C}$, and the symbol $\phi(z) = z^n$ with an arbitrary nonconstant bounded analytic function on $\Omega$. The proof of their theorem depends on the the following approximation theorem due to Bishop [8].

**Theorem 3.2.1.** Let $\phi$ be a nonconstant bounded analytic function on an open bounded domain $\Omega$. Then the norm closed subalgebra of $L^\infty(\Omega)$ generated by $\bar{\phi}$ and bounded analytic functions on $\Omega$ contains $C(\bar{\Omega})$

Here we will state the result of [7] and its proof.

**Theorem 3.2.2.** If $\phi$ is a nonconstant bounded analytic function on $\Omega$ and $\psi$ is a bounded measurable function on $\Omega$ such that $T_\phi$ and $T_\psi$ commute, then $\psi$ is analytic.

**Proof.** Since $\psi$ is bounded, $\psi \in L^2(\Omega, dA) = L^2_a(\Omega, dA) \oplus (L^2_a(\Omega, dA))^\perp$, where $(L^2_a(\Omega, dA))^\perp$ is the orthogonal complement of the Bergman space $L^2_a(\Omega, dA)$. Hence $\psi = f + u$ with $f \in L^2_a(\Omega, dA)$ and $u \in L^2(\Omega, dA) \ominus L^2_a(\Omega, dA)$. If $n$ is a nonnegative integer, then

$$T_{\phi^n}T_\psi(1) = \phi^n P(f + u) = \phi^n f,$$

and

$$T_\psi T_{\phi^n}(1) = P(f \phi^n + u \phi^n) = f \phi^n + P(u \phi^n).$$

Since $T_\phi T_\psi = T_\psi T_\phi$, $T_{\phi^n}T_\psi = T_\psi T_{\phi^n}$ for all $n \geq 0$. Therefore, the equations above must be equal, and so $P(u \phi^n) = 0$ for all $n \geq 0$. Hence if $h$ is a function in $L^2_a(\Omega, dA)$,
we have

\[ 0 = \langle h, u\phi^n \rangle = \int_{\Omega} \bar{u}h\bar{\phi}^n \, dA, \text{ for } n \geq 0. \]

Because the equation above holds for an arbitrary bounded analytic function \( h \) on \( \Omega \) and every nonnegative integer \( n \), Bishop’s result implies that

\[ \int_{\Omega} \bar{u}w \, dA = 0 \]

for every \( w \) in \( C(\Omega) \) the set of all continuous functions on the closure of \( \Omega \). Since \( C(\Omega) \) is dense in \( L^2(\Omega, dA) \), we have

\[ \int_{\Omega} |u|^2 \, dA = 0, \]

and so \( u = 0 \). Hence \( \psi = f \) and therefore \( \psi \) is analytic.

\[ \Box \]

### 3.3 Commuting Toeplitz operators with quasi-homogeneous symbols

In this section, we are interested in the commutant of a quasihomogeneous Toeplitz operator. The commutativity of such operators was first studied by Čučković and Rao in [11]. They characterized all functions \( \psi \) bounded on \( \mathbb{D} \) such that \( T_\psi \) and \( T_{e^{i\theta}r^m} \) commute, with \( p \) and \( m \) are two positive integers. They obtained the following result.

**Theorem 3.3.1.** Let \( \psi(re^{i\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} \psi_k(r) \) and \( \phi(re^{i\theta}) = e^{i\delta \theta} r^m \) be two functions in \( L^\infty(\mathbb{D}) \), where \( \psi_k \in \mathcal{R}, m \in \mathbb{N}, \) and \( \delta \in \mathbb{N} \). Then \( T_\phi \) and \( T_\psi \) commute if and only
if for every \( k \in \mathbb{Z} \), there exist constants \( a_0(k), a(k), b(k), c(k), \) and \( d(k) \) in \( \mathbb{R} \) such that:

\[
\hat{\psi}_k(z) = a_0(k) \frac{\Gamma(z/2\delta + a(k))\Gamma(z/2\delta + b(k))}{\Gamma(z/2\delta + c(k))\Gamma(z/2\delta + d(k))},
\]

where \( \Gamma \) is the Gamma function. Moreover, the value of the constants are \( a(k) = k/2\delta \), \( b(k) = (m + \delta - k)/2\delta \), \( c(k) = (2\delta - k)/2\delta \), and \( d(k) = (k + \delta + m)/2\delta \).

In the proof of their theorem, Čučković and Rao showed for a bounded function \( \psi(re^{ik\theta}) = \sum_{k \in \mathbb{Z}} e^{ik\theta} \psi_k(r) \), if \( T_\psi \) commutes with a given quasihomogeneous Toeplitz operator, then the later commutes with each \( T_{e^{ik\theta}\psi_k} \), \( k \in \mathbb{Z} \). Later on, Louhichi and Zakariasy [21] used this crucial fact to characterize all bounded functions \( \psi \) such that \( T_\psi \) commutes with \( T_{e^{ip\theta}\phi} \). In fact, by looking separately at the quasihomogeneous terms of negative and positive degrees in the polar decomposition of the symbol \( \psi \), they were able to show first that two non-trivial quasihomogeneous Toeplitz operators with degrees of opposite signs can never commute. Second they proved that a nontrivial Toeplitz operator with radial symbol cannot commute with nonzero quasihomogeneous Toeplitz operator of positive degree. Third they proved that, for a given quasihomogeneous Toeplitz operator \( T_{e^{ip\theta}\phi} \) of positive degree and for a fixed but arbitrary positive integer \( s \), if there exists a radial function \( \psi \) such that \( T_{e^{ip\theta}\phi}T_{e^{is\theta}\psi} = T_{e^{is\theta}\psi}T_{e^{ip\theta}\phi} \), then \( \psi \) is unique up to multiplication by a constant. Finally, by putting together all these results on quasihomogeneous Toeplitz operators, they obtained the following theorem.

**Theorem 3.3.2.** Let \( \phi \) be a nonzero bounded radial function, \( p \) be a positive integer and \( \psi(re^{ik\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} \psi_k(r) \in L^\infty(\mathbb{D}) \). Then

a) \( T_\psi \) commutes with \( T_{e^{ip\theta}\phi} \) if and only if \( T_{e^{ik\theta}\psi_k} \) commutes with \( T_{e^{ip\theta}\phi} \) for all \( k \in \mathbb{Z} \).

b) If there exists \( k \in \mathbb{Z}_- \) and a bounded radial function \( \psi_k \) such that

\[
T_{e^{ip\theta}\phi}T_{e^{ik\theta}\psi_k} = T_{e^{ik\theta}\psi_k}T_{e^{ip\theta}\phi}
\]
then \( \psi_k \) must be equal to zero.

c) If there exists \( k \in \mathbb{Z}_+ \) and a bounded radial function \( \psi_k \) such that

\[
T_{e^{ip\theta}\phi}T_{e^{ik\theta}\psi_k} = T_{e^{ik\theta}\psi_k}T_{e^{ip\theta}\phi}
\]

then \( \psi_k \) is unique up to a constant factor. In particular \( \psi_0 \) is a constant.

The above theorem is true for \( p < 0 \) also, which can be seen by taking the adjoints.

In [18], Louhichi continued investigating the commutants of quasihomogeneous Toeplitz operators and obtained a relationship between commutativity, roots and powers of such operators. In fact, Louhichi introduced a new notion namely the \( T\)-\( p \)th root of a quasihomogeneous Toeplitz operator of degree \( p \) which played an important role in interpreting and extending the results of [11].

**Definition 3.3.1.** Let \( \phi \) be a nonzero bounded radial function and \( p \) be a positive integer. \( T_{e^{i\theta}\psi} \) is said to be the \( T\)-\( p \)th root of the Toeplitz operator \( T_{e^{ip\theta}\phi} \) if and only if there exists a nonzero bounded radial function \( \psi \) such that

\[
T_{e^{ip\theta}\phi} = (T_{e^{i\theta}\psi})^p.
\]

Using this notion of \( T\)-\( p \)th root, Louhichi obtained the following result.

**Theorem 3.3.3.** Let \( \phi \) be a nonzero bounded radial function and \( p \) be a positive integer. Assume that \( T_{e^{ip\theta}\phi} \) has a \( T\)-\( p \)th root \( T_{e^{i\theta}\psi} \). Suppose that

\[
f(re^{i\theta}) = \sum_{k=-\infty}^{+\infty} e^{ik\theta} f_k(r) \in L^\infty(\mathbb{D})
\]

is such that

\[
T_{e^{ip\theta}\phi}T_f = T_fT_{e^{ip\theta}\phi}.
\]

Then
(i) \( f_k = 0 \) for every \( k < 0 \).

(ii) If \( k \geq 0 \) and if \((T_{e^{i\theta}})^k\) is a Toeplitz operator, then either \( T_{e^{i\theta}} f_k = c (T_{e^{i\theta}})^k \), where \( c \) is a constant, or \( f_k = 0 \).

(iii) If \( k \geq 0 \) and if \((T_{e^{i\theta}})^k\) is not a Toeplitz operator, then \( f_k = 0 \).

### 3.4 Commutants of Toeplitz operators with bounded harmonic symbols

In [7], Axler, Čučković and Rao proved that non-trivial Toeplitz operators with analytic symbols commute only with analytic Toeplitz operators. Moreover, they asked the following question.

(Q) Suppose \( \phi \) is a bounded harmonic function on \( \mathbb{D} \), that is neither analytic nor conjugate analytic. If \( f \) is a bounded measurable function on \( \mathbb{D} \) such that \( T_{\phi} \) and \( T_f \) commute, must \( f \) be of the form \( a\phi + b \) for some constants \( a, b \)?

In this section, we give a partial answer to this question. In fact, we characterize all operators \( T_f \) that commute with \( T_{z+g(z)} \), in the case when the symbols \( f \) have the form \( f(re^{i\theta}) = \sum_{k=-\infty}^{N} c^{ik\theta} f_k(r) \), where \( N \) is a positive integer, and \( g(z) \) is a bounded analytic function on \( \mathbb{D} \).

We shall use often the following lemma [18, p. 1469] in the proof of our main results.

**Lemma 3.4.1.** Let \( F \) and \( G \) be two nonzero bounded analytic functions on the right-half plane \( \Pi = \{ z \in \mathbb{C} : \Re z > 2 \} \). If there exist \( p \in \mathbb{N} \) such that:

\[
F(z)G(z + p) = F(z + p)G(z),
\]  

(3.1)
then

\[ F = cG, \text{ where } c \text{ is a constant.} \]

\textbf{Proof.} Substituting in (3.1), \( z \) by \( z_n = z + np \) for \( n = 0, \ldots, k - 1 \), we obtain the following \( k \) equations:

\[
\begin{align*}
F(z)G(z + p) &= F(z + p)G(z) \\
F(z + p)G(z + 2p) &= F(z + 2p)G(z + p) \\
&\quad \vdots \quad \vdots \\
F(z + (k - 1)p)G(z + kp) &= F(z + kp)G(z + (k - 1)p).
\end{align*}
\]

Multiplying the above \( k \) equations side by side and dividing the obtained equation by the repeated factors on both sides of the equation, imply that

\[ F(z)G(z + kp) = F(z + kp)G(z). \quad (3.2) \]

Since \( G \) is nonzero by hypothesis then there exists a point \( z_0 \in \Pi \) such that \( G(z_0) \neq 0 \).

Let \( E := \{k \in \mathbb{N} : G(z_0 + kp) = 0\} \). If

\[
\sum_{k \in E} \Re\left(\frac{1}{|z_0 + kp|}\right) = \infty,
\]

then, according to Theorem 2.3.2, \( G \) is zero which contradicts the hypothesis. Thus

\[
\sum_{k \in E^c} \Re\left(\frac{1}{|z_0 + kp|}\right) = \infty,
\]

where \( E^c \) is the complement of \( E \) in \( \mathbb{N} \). For every \( k \in E^c \), equation (3.2) implies that:

\[
\frac{F(z_0 + kp)}{G(z_0 + kp)} = \frac{F(z_0)}{G(z_0)}.
\]
If we assume \( c = \frac{F(z_0)}{G(z_0)} \), then \((F - cG)(z_0 + kp) = 0\) for every \( k \in E^c \) and so by Theorem 2.3.2, \( F - cG = 0 \) in \( \Pi \).

A simple calculation gives the following property, for the Bergman projection, which is used often in our proofs.

\[
P(z^n z^m) = \begin{cases} 
\frac{n-m+1}{n+1} z^{n-m}, & \text{for } n \geq m \\
0, & \text{for } n < m
\end{cases} \quad (3.3)
\]

Now, we are ready to state and prove the first main result of this section.

**Proposition 3.4.2.** Let \( f(re^{i\theta}) = \sum_{k=-\infty}^{N} f_k(r)e^{ik\theta} \in L^\infty(D) \), where \( N \) is a positive integer, and let \( g(z) = \sum_{n=0}^{\infty} a_n z^n \) be a bounded analytic function in \( D \). If \( T_f \) commutes with \( T_{z+g(z)} \), then \( N = 1 \).

**Proof.** We proceed by contradiction, so let us assume that \( N \geq 3 \). Since for every \( n \geq 0 \)

\[
T_f T_{z+g(z)}(z^n) = T_{z+g(z)} T_f(z^n). \quad (3.4)
\]

On the left side of (3.4), the monomial in \( z \) of highest degree is \( z^{n+N+1} \), and it comes only from \( T_{fN e^{iN\theta}} T_z(z^n) \). On the other hand, the monomial in \( z \) with highest degree on the right side of (3.4), also is \( z^{n+N+1} \) and it comes from \( T_z T_{fN e^{iN\theta}}(z^n) \) only. Therefore, \( T_{fN e^{iN\theta}} T_z(z^n) = T_z T_{fN e^{iN\theta}}(z^n) \) for every \( n \geq 0 \). According to Theorem 3.2.2, \( f_N e^{iN\theta} \) is analytic and so must have the form \( f_N e^{iN\theta} = c_N z^N \) where \( c_N \) is a constant. Similarly, the monomial \( z^{n+N} \) appears on both sides of (3.4) only from \( T_{f_N^{-1} e^{i(N-1)\theta}} T_z(z^n) \), and \( T_z T_{f_N^{-1} e^{i(N-1)\theta}}(z^n) \), which implies

\[
T_{f_N^{-1} e^{i(N-1)\theta}} T_z(z^n) = T_z T_{f_N^{-1} e^{i(N-1)\theta}}(z^n).
\]

Therefore by Theorem 3.2.2, \( f_{N-1}(r) = c_{N-1} r^{N-1} \).

Now the terms in \( z \) of degree \( n + N - 1 \), on both left and right sides of (3.4), come
only from \( T_{c_N z^n} T_{\frac{1}{n+2}}(z^n) + T_{\frac{T}{n+2}} T_{c_N z^n}(z^n) \) and \( T_{\frac{1}{n+2}} T_{c_N z^n}(z^n) + T_{z} T_{\frac{T}{n+2}} f_{n+2}(z^n) \) respectively. This implies that

\[
T_{c_N z^n} T_{\frac{1}{n+2}}(z^n) + T_{\frac{T}{n+2}} T_{c_N z^n}(z^n) = T_{\frac{1}{n+2}} T_{c_N z^n}(z^n) + T_{z} T_{\frac{T}{n+2}} f_{n+2}(z^n), \quad \text{for all } n \geq 1,
\]

which is, using Lemma 2.3.1, equivalent to

\[
2(n+N)\hat{f}_{N-2}(2n+N+2) - 2(n+N-1)\hat{f}_{N-2}(2n+N) = c_N\hat{a}_1 \left( \frac{n+N}{n+N+1} - \frac{n}{n+1} \right),
\]

for all \( n \geq 1 \). Thus for all \( z \) such that \( \Re z \geq 1 \), Corollary 2.3.3 implies that

\[
2(z+N)\hat{f}_{N-2}(2z+N+2) - 2(z+N-1)\hat{f}_{N-2}(2z+N) = c_N\hat{a}_1 \left( \frac{z+N}{z+N+1} - \frac{z}{z+1} \right).
\]

(3.5)

Let \( F \) and \( G \) be two functions defined by

\[
F(z) = 2(z+N-1)\hat{f}_{N-2}(2z+N)
\]

and

\[
G(z) = \frac{z+N-1}{z+N} + \frac{z+N-2}{z+N-1} + \ldots + \frac{z}{z+1}.
\]

It is easy to see that, both \( F \) and \( G \) are bounded and analytic in \( \{ z : \Re z > 1 \} \), and that equation (3.5) is equivalent to

\[
F(z+1) - F(z) = c_N\hat{a}_1 \left( G(z+1) - G(z) \right), \quad \text{for } \Re z \geq 1.
\]

Hence the function \( F - c_N\hat{a}_1 G \) is 1-periodic, and by Lemma 3.4.1, we conclude that \( F(z) - c_N\hat{a}_1 G(z) = c \) where \( c \) is a constant. Therefore

\[
\hat{f}_{N-2}(2z+N) = \frac{c}{2z+2N-2} + \frac{c_N\hat{a}_1}{2z+2N-2} \sum_{j=1}^{N} \frac{z+N-j}{z+N+1-j}.
\]

(3.6)
By rewriting equation (3.6) as sum of partial fractions, we obtain

\[
\widehat{f}_{N-2}(2z + N) = \frac{c}{2z + 2N - 2} + \sum_{j=1}^{N} \frac{c_Na_1}{z + N + 1 - j} \tag{3.7}
\]

\[
= \frac{c_Na_1}{2z + 2N} + \frac{c + c_Na_1}{2z + 2N - 2} - \frac{2c_Na_1}{(2z + 2N - 2)^2} \tag{3.8}
\]

\[-c_Na_1 \sum_{j=3}^{N} \frac{1}{j - 2} \left( \frac{1}{2z + 2N + 2 - 2j} - \frac{j - 1}{2z + 2N - 2} \right). \tag{3.9}
\]

Since \( \widehat{r}^m(z) = \frac{1}{z + m} \) and \( \widehat{r}^m \ln r(z) = -\frac{1}{(z + m)^2} \) for any integer \( m \), the above equality becomes

\[
\widehat{f}_{N-2}(2z + N) = c_Na_1 \widehat{r}^N(2z + N) + (c + c_Na_1)\widehat{r}^{N-2}(2z + N) + 2c_Na_1 \widehat{r}^{N-2} \ln r(2z + N)
\]

\[-c_Na_1 \sum_{j=3}^{N} \frac{1}{j - 2} (r^{N+2-2j}(2z + N) - (j - 1)r^{N-2}(2z + N)),
\]

and so Corollary 2.3.3 implies

\[
f_{N-2}(r) = c_Na_1 r^N + (c + c_Na_1) r^{N-2} + 2c_Na_1 r^{N-2} \ln r - c_Na_1 \sum_{j=3}^{N} \frac{1}{j - 2} (r^{N+2-2j} - (j - 1)r^{N-2}).
\]

Since by hypothesis \( f \) is bounded, for each \( k \leq N \), \( f_k \) also must be bounded. Now in the expression of \( f_{N-2} \), the term \( r^{N-2} \) is bounded if and only if \( N \geq 2 \). On the other hand, the terms \( r^{N+2-2j} \) in the sum are bounded if and only if \( N - 2j \geq -2 \) for all \( 3 \leq j \leq N \). In particular for \( j = N \), we have \( N \leq 2 \). Hence \( N = 2 \), which contradicts our assumption "\( N \geq 3 \)". Therefore \( N \leq 2 \). But, by taking \( N = 2 \) in Equation (3.7) and then using Corollary 2.3.3 again, we obtain that

\[
f_0(r) = c_2a_1 r^2 + (c + c_2a_1) + 2c_2a_1 \ln r.
\]

Since the function \( \ln r \), and so \( f_0 \), is unbounded in \((0, 1)\), \( N \) can not equal 2. Therefore \( N \leq 1 \). \( \square \)
**Remark 3.4.1.** Taking $N = 1$ in Equation (3.6) of the previous proof, implies

$$\hat{f}_{-1}(2z + 1) = \frac{c_1 \hat{a}_1}{2z + 2} + \frac{c}{2z} = c_1 \hat{a}_1 \hat{r}(2z + 1) + c r^{-1}(2z + 1).$$

Clear $f_{-1}(r) = c_1 \hat{a}_1 r + c r^{-1}$ is bounded if and only if $c = 0$. Therefore

$$T_{e^{-i\theta} f_{-1}} = c_1 \hat{a}_1 T_{\hat{z}}.$$

The following theorem is a partial answer to the question (Q) mentioned at the beginning of this section.

**Theorem 3.4.3.** Assume $f(re^{i\theta}) = \sum_{k=-\infty}^{N} f_k(r)e^{ik\theta} \in L^\infty(\mathbb{D})$ and $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is a bounded analytic function in $\mathbb{D}$. If $T_f$ commutes with $T_{\frac{z}{z + g(z)}}$, then there exist two constants $c_0$ and $c_1$ such that $T_f = c_1 T_{\frac{z}{z + g(z)}} + c_0 I$, where $I$ is the identity operator on $L^2_a(\mathbb{D}, dA)$.

**Proof.** By Proposition 3.4.2, we have

$$N \leq 1, \ T_{e^{i\theta} f_1} = c_1 T_{\hat{z}}, \text{ and } T_{e^{-i\theta} f_{-1}} = c_1 \hat{a}_1 T_{\hat{z}},$$

where $c_1$ is a constant. When subtracting $c_1 T_{\frac{z}{z + g(z)}}$ from $T_f$, we obtain

$$T_f - c_1 T_{\frac{z}{z + g(z)}} = T_{f_0 - c_1 \hat{a}_0} + \sum_{k \leq -1} T_{e^{ik\theta}(f_k - c_1 \overline{a}_{-k} r^{-k})}.$$  

Moreover, the new operator $T_f - c_1 T_{\frac{z}{z + g(z)}}$ still commutes with $T_{\frac{z}{z + g(z)}}$. Thus, for any element in the orthogonal basis $\{z^n : n \geq 1\}$ of $L^2_a(\mathbb{D}, dA)$, we have

$$\left[ T_{f_0 - c_1 \hat{a}_0} + \sum_{k \leq -1} T_{e^{i\theta}(f_k - c_1 \overline{a}_{-k} r^{-k})} \right] T_{\frac{z}{z + g(z)}}(z^n) = T_{\frac{z}{z + g(z)}} \left[ T_{f_0 - c_1 \hat{a}_0} + \sum_{k \leq -1} T_{e^{i\theta}(f_k - c_1 \overline{a}_{-k} r^{-k})} \right](z^n).$$  

(3.10)
On the left side of (3.10), the monomial in $z$ of highest degree is $z^{n+1}$, and it comes only from $T_{f_0-c_1a_0}T_z(z^n)$. On the right side of (3.10), the monomial of highest degree in $z$ is also $z^{n+1}$, and it comes from the product $T_zT_{f_0-c_1a_0}(z^n)$. Therefore $T_{f_0-c_1a_0}T_z(z^n) = T_zT_{f_0-c_1a_0}(z^n)$ for every $n \geq 1$. Since $T_z$ is an analytic Toeplitz operator, Theorem 3.2.2 implies that $f_0 - c_1\overline{a_0}$ is analytic symbol. But a quasihomogeneous symbol of degree 0 is analytic if and only if it is constant. Hence, there exist a constant $c_0$ such that $f_0 = c_0 + c_1\overline{a_0}$.

Similarly, since the operator $T_f - c_1T_{z+g(z)} - c_0 I = \sum_{k=-\infty}^{-1} T_{e^{ik\theta}(f_k-c_1\overline{a}_r)}$ commutes with $T_{z+g(z)}$, and using the same argument as before, we show that $T_{e^{-i\theta}(f_1-c_1\overline{a}_r)}$ commutes with $T_z$. However, a quasihomogeneous Toeplitz operator of negative degree cannot commute with a nonzero analytic Toeplitz operator unless it is zero. So we must have $f_1 - c_1\overline{a}_r = 0$, i.e. $f_1 = c_1\overline{a}_r$. Redoing the same argument for all $k \leq -2$, we prove that $f_k = c_1\overline{a}_r e^{-k}$. Now, when we reconstitute the symbol $f$, we observe that

$$f(re^{i\theta}) = c_1 e^{i\theta}r + (c_1\overline{a}_0 + c_0) + c_1 \sum_{k=-\infty}^{-1} \overline{a}_k e^{ik\theta} r^{-k}$$

$$= c_1 z + c_1 \sum_{k=0}^{\infty} \overline{a}_k e^{-ik\theta} r^{-k} + c_0$$

$$= c_1(z + g(z)) + c_0.$$

Hence, $T_f = c_1T_{z+g(z)} + c_0 I$, which completes the proof.

In other words, Theorem 3.4.3 says that in the commutant of $T_{z+g(z)}$, the only Toeplitz operators with bounded symbols are polynomials in $T_{z+g(z)}$ of degree at most one. Since the boundedness of the symbol is not necessary to insure the boundedness of the associated Toeplitz operator, what else would the commutant of $T_{z+g(z)}$ contain besides linear combinations of itself and the identity operator? It is well known that, for any bounded harmonic function $\phi$, $T_\phi$ commutes not only with any linear
combination of itself and the identity, but also with any polynomial $Q(T_\phi)$ in the set of bounded operators on $L^2_a(\mathbb{D}, dA)$. Clearly not any polynomial in $T_\phi$ will be a Toeplitz. Among this set of all polynomials in $T_\phi$, we are interested only in those which are themselves bounded Toeplitz operators.

In [22], we shall prove, under some conditions, that the commutant of $T_{z+\bar{z}}$ is reduced to the set of polynomials in $T_{z+\bar{z}}$ that are Toeplitz operators. Our choice for the harmonic symbol $z + \bar{z}$ is motivated by the fact that the commutant of both the shift $T_z$ and its adjoint $T_{\bar{z}}$ has been already well studied and completely characterized (the commutant of $T_z$ is the set of all analytic Toeplitz operators and the commutant of $T_{\bar{z}}$ is the set of all anti-analytic Topelitz operators). However, in the current literature we did not find any work that treats the commutant of either $T_{z+\bar{z}}$ or any other bounded harmonic Toeplitz operator. We are firmly convinced (even though the general case is not proved yet) that the commutant of any bounded harmonic Toeplitz operator consists only of polynomials of itself that are themselves Toeplitz operators.

The following lemma is simple but crucial in our attempt to characterize the commutant of $T_{z+\bar{z}}$.

**Lemma 3.4.4.** The products $T_{z+\bar{z}}^2$ and $T_{z+\bar{z}}^3$ are both Toeplitz operators. Moreover

$$T_{z+\bar{z}}^2 = T_z^2 + T_{1+\ln|z|^2} + T_{|z|^2} + T_{\bar{z}^2},$$

and

$$T_{z+\bar{z}}^3 = T_z^3 + T_{z(1+\ln|z|^2)} + T_{2z^2} + T_{2\bar{z}^2} + T_{2z\bar{z}} + T_{z(1+\ln|z|^2)} + T_{\bar{z}^3}.$$ 

**Proof.** According to Lemma 2.2.4 and Proposition 2.2.2, we have

$$T_z T_{\bar{z}} = T_{1+\ln|z|^2},$$
and
\[ T_{z\bar{z}}T_{\bar{z}} = T_{2z - \frac{1}{z}}. \]

Moreover, it is well known that \( T_uT_v = T_{uv} \) whenever \( \bar{u} \) is analytic or \( v \) is analytic. Therefore

\[ T^3_{z+\bar{z}} = (T_z + T_{\bar{z}})^3 = T_zT_zT_z + T_zT_zT_{\bar{z}} + T_zT_{\bar{z}}T_z + T_{\bar{z}}T_zT_z \]
\[ + T_zT_{\bar{z}}T_{\bar{z}} + T_{\bar{z}}T_zT_{\bar{z}} + T_{\bar{z}}T_{\bar{z}}T_z. \]
\[ = T_{z^3} + T_{2z - \frac{1}{z}} + T_{z(1+\ln|z|^2)} + T_{2z^2 - \frac{1}{z}} + T_{z\bar{z}} \]
\[ + T_{z^2\bar{z}} + T_{\bar{z}(1+\ln|z|^2)} + T_{\bar{z}^3}. \]

Similarly, we obtain

\[ T^2_{z+\bar{z}} = T_{z^2} + T_{1+\ln|z|^2} + T_{|z|^2} + T_{\bar{z}^2}. \]

\[ \square \]

**Remark 3.4.2.** Here are some comments about Lemma 3.4.4.

i) The symbols \( 1+\ln|z|^2, z(1+\ln|z|^2) \) and \( 2z - \frac{1}{z} \) and their conjugates, that appear in \( T^3_{z+\bar{z}} \) and \( T^2_{z+\bar{z}} \), obviously are not bounded but they are nearly bounded functions. Hence the Toeplitz operators associated to those symbols are all bounded.

ii) It is easy to see, using Corollary 2.2.6 (or [20, Corollary 6.5, p. 533]), that \( T^n_{z+\bar{z}} \) is not a Toeplitz operator whenever \( n \geq 4 \).

The following proposition is the first step in proving the next main result Theorem 3.4.6.

**Proposition 3.4.5.** Let \( f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r) \), where \( N \) is a positive integer; be
a function in $L^1(\mathbb{D}, dA)$ such that the Toeplitz operator $T_f$ is bounded. If $T_f$ commutes with $T_{z+\bar{z}}$, then $N$ must be less than or equal to 3.

**Proof.** We proceed by contradiction, and we assume $N \geq 4$. If $T_f$ commutes with $T_{z+\bar{z}}$, then for all $n \geq 0$

$$T_f T_{z+\bar{z}}(z^n) = T_{z+\bar{z}} T_f(z^n).$$

Thus for all $n \geq 0$,

$$\sum_{k=-\infty}^{N} T_{e^{ik\theta} f_k} T_{z+\bar{z}}(z^n) = \sum_{k=-\infty}^{N} T_{z+\bar{z}} T_{e^{ik\theta} f_k}(z^n).$$

In the above equation, the term of the highest degree in $z$ is $z^{n+N+1}$. On the left hand side, this term comes from the product $T_{e^{iN\theta} f_N} T_z(z^n)$ only, and on the right hand side only $T_z T_{e^{iN\theta} f_N}(z^n)$ provides the monomial $z^{n+N+1}$. Thus by equality we should have

$$T_{e^{iN\theta} f_N} T_z = T_z T_{e^{iN\theta} f_N}.$$

Since the symbol $z$ is analytic then, by Theorem 3.2.2, $e^{iN\theta} f_N$ must be analytic too.

This is possible if and only if $f_N = c_N z^N$ where $c_N$ is a constant, i.e, $e^{iN\theta} f_N = c_N z^N$.

Redoing the same argument for the term $z^{n+N-1}$ on both sides, gives us

$$c_N T_{z z^N} T_z(z^n) + T_{e^{i(N-2)\theta} f_{N-2}} T_z(z^n) = c_N T_z T_{z^N}(z^n) + T_z T_{e^{i(N-2)\theta} f_{N-2}}(z^n), \text{ for all } n \geq 1.$$

Hence for all $n \geq 1$, we have

$$c_N z^N P(\bar{z} z^n) + T_{e^{i(N-2)\theta} f_{N-2}}(z^{n+1}) = c_N P(\bar{z} z^{n+N}) + z T_{e^{i(N-2)\theta} f_{N-2}}(z^n).$$
which, by Lemma 2.3.1 and (3.3), is equivalent to

\[
\left[c_N \frac{n}{n+1} + 2(n+N) \overline{f_{N-2}}(2n+N+2)\right] z^{n+N-1} = \left[c_N \frac{n}{n+1} + 2(n+N-1) \overline{f_{N-2}}(2n+N)\right] z^{n+N-1}.
\]

It follows that for all \( n \geq 1 \)

\[
c_N \frac{n}{n+1} + 2(n+N) \overline{f_{N-2}}(2n+N+2) = c_N \frac{n}{n+1} + 2(n+N-1) \overline{f_{N-2}}(2n+N),
\]

and so for all \( n \geq 1 \)

\[
2(n+N) \overline{f_{N-2}}(2n+N+2) - 2(n+N-1) \overline{f_{N-2}}(2n+N) = c_N \left[ \frac{n}{n+1} - \frac{n}{n+1} \right].
\]

Now, using Corollary 2.3.3, we obtain for \( \Re z \geq 1 \)

\[
2(z+N) \overline{f_{N-2}}(2z+N+2) - 2(z+N-1) \overline{f_{N-2}}(2z+N) = c_N \left[ \frac{z}{z+N+1} - \frac{z}{z+1} \right], \quad (3.11)
\]

For simplicity, we might assume without loss of generality that \( c_N = 1 \). Let \( F \) and \( G \) be the two bounded analytic functions in \( \{ z : \Re z > 1 \} \) defined by

\[
F(z) = 2(z+N-1) \overline{f_{N-2}}(2z+N)
\]

and

\[
G(z) = \frac{z+N-1}{z+N} + \frac{z+N-2}{z+N-1} + \ldots + \frac{z}{z+1}.
\]

Then equation (3.11) implies

\[
F(z+1) - F(z) = G(z+1) - G(z), \text{ for } \Re z \geq 1.
\]

Thus \( F-G \) is 1-periodic function, and using Lemma 3.4.1, we obtain \( F(z) - G(z) = c \)
where \( c \) is a constant. Therefore \( F(z) = c + G(z) \), which implies

\[
(2z + 2N - 2) \hat{f}_{N-2}(2z + N) = c + \sum_{j=1}^{N} \frac{z + N - j}{z + N + 1 - j}.
\]

Thus

\[
\hat{f}_{N-2}(2z + N) = \frac{c}{2z + 2N - 2} + \frac{1}{2z + 2N - 2} \sum_{j=1}^{N} \frac{z + N - j}{z + N + 1 - j} + \frac{z + N - 2}{z + N - 1} \sum_{j=3}^{N} \frac{z + N - j}{z + N + 1 - j} + \frac{c + 1}{2z + 2N} - \frac{1}{2z + 2N - 2} \left( \frac{1}{2z + 2N + 2j} - \frac{j - 1}{2z + 2N - 2j} \right).
\]

Since \( \hat{r}^m(z) = \frac{1}{z + m} \) and \( \hat{r}^m \ln r(z) = -\frac{1}{(z + m)^2} \) for any integer \( m \), we obtain the following

\[
\hat{f}_{N-2}(2z + N) = \hat{r}^{N}(2z + N) + (c + 1)\hat{r}^{N-2}(2z + N) + 2\hat{r}^{N-2} \ln r(2z + N) - \sum_{j=3}^{N} \frac{1}{j - 2} \left( \hat{r}^{N+2-2j}(2z + N) - (j - 1)\hat{r}^{N-2}(2z + N) \right).
\]

Again Corollary 2.3.3 implies

\[
f_{N-2}(r) = r^N + (c + 1)r^{N-2} + 2r^{N-2} \ln r - \sum_{j=3}^{N} \frac{1}{j - 2} \left( r^{N+2-2j} - (j - 1)r^{N-2} \right).
\]

It is easy to see that the radial functions \( r^{N+2-2j} \) are in \( L^1([0, 1], rdr) \) if and only if

\[
N + 2 - 2j + 1 > -1, \text{ for all } 3 \leq j \leq N.
\]

(3.12)
In particular by taking $j = N$ in Equation (3.12), we obtain $N < 4$. This contradicts our assumption. Therefore $N$ has to be less than or equal to 3.

Now we are able to describe the Toeplitz operator $T_f$ that commutes with $T_{z+\bar{z}}$, when the symbol $f$ has a certain polar decomposition form.

**Theorem 3.4.6.** Let $f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r)$ be a function in $L^1(\mathbb{D}, dA)$ such that the Toeplitz operator $T_f$ is bounded. If $T_f$ commutes with $T_{z+\bar{z}}$, then $T_f = Q(T_{z+\bar{z}})$ where $Q$ is a polynomial of degree at most 3.

**Proof.** Since $T_f$ commutes with $T_{z+\bar{z}}$, Proposition 3.4.5 implies that $N \leq 3$. So $f$ is in fact
\[
f(re^{i\theta}) = \sum_{k=-\infty}^{3} e^{ik\theta} f_k(r),\]
and for all $n \geq 0$,
\[
\sum_{k=-\infty}^{3} T_{e^{ik\theta}} f_k T_{z+\bar{z}}(z^n) = \sum_{k=-\infty}^{3} T_{z+\bar{z}} e^{ik\theta} f_k(z^n).
\]

In the equation above, the monomial in $z$ of highest degree on both sides is $z^{n+4}$, which comes on the left hand side from the product $T_{e^{i3\theta}} f_3 T_z(z^n)$, and on the right hand side from $T_z T_{e^{i3\theta}} f_3(z^n)$. Thus by equality we must have
\[
T_{e^{i3\theta}} f_3 T_z = T_z T_{e^{i3\theta}} f_3.
\]

Since the symbol $z$ is analytic, by Theorem 3.2.2, $e^{i3\theta} f_3$ must be analytic too. This is possible if and only if $f_3(r) = c_3 r^3$ where $c_3$ is a constant, i.e., $e^{i3\theta} f_3(r) = c_3 z^3$.

By Lemma 3.4.4, $T^3_{z+\bar{z}}$ is the sum of quasihomogeneous Toeplitz operators each of degree not equal to 2. Thus when we subtract $c_3 T^3_{z+\bar{z}}$ from $T_f$, $T_{e^{i3\theta}} f_3$ will be the quasihomogeneous Toeplitz operator of highest degree in the semi-finite sum $\sum_{k=-\infty}^{3} T_{e^{ik\theta}} f_k(r) - c_3 T^3_{z+\bar{z}}$. Now, the operator $\sum_{k=-\infty}^{3} T_{e^{ik\theta}} f_k(r) - c_3 T^3_{z+\bar{z}}$ also com-
mutes with $T_{z+\bar{z}}$, so for all $n \geq 0$

$$
\left( \sum_{k=-\infty}^{3} T_{e^{i\theta}f_k(r)} - c_3 T_{z+\bar{z}}^3 \right)^{n} = T_{z+\bar{z}}^n \left( \sum_{k=-\infty}^{3} T_{e^{i\theta}f_k(r)} - c_3 T_{z+\bar{z}}^3 \right)^n. \quad (3.13)
$$

In equation (3.13), the terms in $z$ of degree $n + 3$ come, on the left hand side from $T_{e^{2i\theta}f_2} T_z(z^n)$ only, and on the right hand side from $T_z T_{e^{2i\theta}f_2}(z^n)$ only. Thus by equality we have

$$
T_{e^{2i\theta}f_2} T_z(z^n) = T_z T_{e^{2i\theta}f_2}(z^n), \quad \text{for all } n \geq 0.
$$

Again using Theorem 3.2.2, we conclude that $T_{e^{2i\theta}f_2} = c_2 T_z$, where $c_2$ is a constant. By Lemma 3.4.4, $T_{z+\bar{z}}^3$ is the sum of quasihomogeneous Toeplitz operators all of them are of degree different from 1. So when subtracting $c_2 T_{z+\bar{z}}^2$ from $\sum_{k=-\infty}^{3} T_{e^{i\theta}f_k(r)} - c_3 T_{z+\bar{z}}^3$, the operator $T_{e^{i\theta}(f_1 - c_3\phi)}$, where $\phi(r) = (2r - \frac{1}{r}) + r(1 + \ln r^2) + r^3$, will be the only quasihomogeneous Toeplitz operator of degree 1 in $\sum_{k=-\infty}^{3} T_{e^{i\theta}f_k(r)} - c_3 T_{z+\bar{z}}^3 - c_2 T_{z+\bar{z}}^2$.

In fact, $T_{e^{i\theta}\phi}$ is the quasihomogeneous Toeplitz operator of degree 1 that appears in $T_{z+\bar{z}}^3$. More precisely, $T_{e^{i\theta}\phi} = T_{z(1+\ln |z|^2)} + T_{z^2\bar{z} + T_{z\bar{z}} - \frac{1}{2}}$. Since $T_f = c_3 T_{z+\bar{z}}^3 - c_2 T_{z+\bar{z}}^2$ commutes with $T_{z+\bar{z}}$, by the same argument as before $T_{e^{i\theta}(f_1 - c_3\phi)}$ must commute with $T_z$, and so $T_{e^{i\theta}(f_1 - c_3\phi)} = c_1 T_z$ where $c_1$ is a constant. Hence $f_1 = c_1 r + c_3\phi$. Redoing the same technique, we observe that in the sum $\sum_{k=-\infty}^{3} e^{i\theta}f_k(r) - c_3 T_{z+\bar{z}}^3 - c_2 T_{z+\bar{z}}^2 - c_1 T_{z+\bar{z}}^1$, the quasihomogeneous Toeplitz operator of highest degree is $T_{(f_0 - c_2\psi)}$ where $\psi(r) = (1 + \ln r^2) + r^2$. Here $\psi$ is the symbol of the sum $T_{1+\ln |z|^2} + T_{|z|^2}$ that appears in $T_{z+\bar{z}}^3$. Clearly $T_f - c_3 T_{z+\bar{z}}^3 - c_2 T_{z+\bar{z}}^2 - c_1 T_{z+\bar{z}}^1$ commutes with $T_{z+\bar{z}}$, and so the same argument implies that $T_{f_0 - c_2\psi}$ commutes with $T_z$. Thus $T_{f_0 - c_2\psi} = c_0 I$, where $I$ is the identity operator on $L^2_{a}(\mathbb{D}, dA)$. Hence $f_0 = c_0 + c_2\psi$. Finally, in the sum

$$
\sum_{k=-\infty}^{3} T_{e^{i\theta}f_k(r)} - c_3 T_{z+\bar{z}}^3 - c_2 T_{z+\bar{z}}^2 - c_1 T_{z+\bar{z}} - c_0 I,
$$

the quasihomogeneous Toeplitz operator of highest degree is $T_{e^{-\theta}(f_1 - c_1 r - c_3\phi)}$. Be-
cause $T_f - c_3 T_{3}^{3} + c_2 T_{2}^{2} - c_1 T_{1} - c_0 I$ commutes with $T_{z+\bar{z}}$, $T_{e^{-i\theta(f_1 - c_1 r - c_3 \phi)}}$ must commute with $T_z$. However, since $T_{e^{-i\theta(f_1 - c_1 r - c_3 \phi)}}$ is of quasihomogeneous degree -1, it cannot be analytic Toeplitz operator unless its symbol $e^{-i\theta(f_1 - c_1 r - c_3 \phi)}$ is zero, i.e. $f_1 = c_1 r + c_3 \phi$. In other words,

$$\sum_{k=-\infty}^{3} T_{e^{ik\theta f_k(r)}} - c_3 T_{3}^{3} + c_2 T_{2}^{2} - c_1 T_{1} - c_0 I$$

does not contain any quasihomogeneous Toeplitz operator of degree -1. Thus, the operator $T_{e^{-2i\theta(f_2 - c_2 r^2)}}$ is the quasihomogeneous Toeplitz operator of highest degree in $\sum_{k=-\infty}^{3} T_{e^{ik\theta f_k(r)}} - c_3 T_{3}^{3} + c_2 T_{2}^{2} - c_1 T_{1} - c_0 I$. This again implies that $T_{e^{-2i\theta(f_2 - c_2 r^2)}}$ commutes with $T_z$, and therefore $e^{-2i\theta(f_2 - c_2 r^2)}$ must be analytic which is impossible unless $f_2 - c_2 r^2 = 0$. Now the quasihomogeneous Toeplitz operator of highest degree in the sum

$$\sum_{k=-\infty}^{3} T_{e^{ik\theta f_k(r)}} - c_3 T_{3}^{3} + c_2 T_{2}^{2} - c_1 T_{1} - c_0 I$$

is $T_{e^{-3i\theta(f_3 - c_3 r^3)}}$, and it commutes with $T_z$, so that $f_3 - c_3 r^3 = 0$. As a consequence, for each $k \leq -4$, $T_{e^{ik\theta f_k}}$ will be the quasihomogeneous Toeplitz operator of highest degree in $\sum_{k=-\infty}^{3} T_{e^{ik\theta f_k(r)}} - c_3 T_{3}^{3} + c_2 T_{2}^{2} - c_1 T_{1} - c_0 I$. Therefore for each $k \leq -4$, $T_{e^{ik\theta f_k}}$ must commute with $T_z$. Since $T_{e^{ik\theta f_k}}$ cannot be analytic for all $k \leq -4$ unless it is the zero operator, $f_k = 0$ for all $k \leq -4$. Hence, when we reconstitute the Toeplitz operator $T_f$, it appears that

$$T_f = \sum_{k=-3}^{3} T_{e^{ik\theta f_k}}$$

$$= c_3 T_{3}^{3} + c_2 T_{2}^{2} + c_1 T_{1} + c_3 T_{e^{i\theta \phi}} + c_0 I + c_2 T_{\psi}$$

$$+ c_1 T_{z} + c_3 T_{e^{-i\theta \phi}} + c_2 T_{z+\bar{z}} + c_3 T_{z}^{3}$$

$$= c_0 I + c_1 T_{z+\bar{z}} + c_2 T_{z+\bar{z}} + c_3 T_{z+\bar{z}}^{3}$$

where the last equality is obtained using Lemma 3.4.4.
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