TOPOLOGY AND THE PLATONIC SOLIDS

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Topology and the Platonic Solids

Introduction

The nature of mathematics insures a steady supply of new questions. Established areas of math develop until they can no longer address new concepts, and new mathematical systems must develop as a result. For example, the Greeks thought that all numbers could be expressed as a ratio of two whole numbers, or a fraction. These numbers are called rational. Then, while exploring different patterns associated with the Pythagorean Theorem an interesting question arose from considering the hypotenuse of a right triangle with legs each 1 unit long. While they were sure that a number existed representing this length, they found that they could not represent it as a fraction. The actual length, $\sqrt{2}$, did exist, but it was a new sort of number. Modern mathematics uses these sorts of numbers, called irrational, without hesitation, but they lay beyond the ancient Greeks’ conceptual framework. As a geometric discovery, the Pythagorean Theorem stimulated thought that necessitated the development of a new set of numbers to include irrational numbers. Continually in math, generalization, reconceptualization, and growth occur among its branches and spark new questions.

Mathematical knowledge has continued to expand and challenge itself to answer new, emerging questions. What seem to be at first simple, and possibly trivial, questions plant the seed for new ideas to emerge. As the number of these instances increases, these new ideas begin to form a cohesive approach that establishes new areas of math. One such famous “seed-planting” question occurred in the 18th century.
The city of Königsberg, Germany, contained seven bridges, oriented in such a way that observers wondered whether a person could cross all seven bridges just once in a continuous path. Many had tried to devise various paths along the bridges, none of which seemed to work. It was not enough to simply say it could not be done based upon these various attempts, and in 1735 Leonard Euler (pronounced “oiler”) was the first to give a definitive solution to the problem, stating that it was not possible to do. He described the problem as “concerned with the geometry of position,” for it “seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculations help at all.”¹ This implies that the solution did not rely on the specific locations of the bridges in terms of the distances between them, but rather on their positions in relation to one another.

Figure 1 is of Königsberg in the 18th century. The Pregel River is traversed by the seven bridges marked in red. When one tries to trace a path crossing each bridge only once, you can soon see that this does not seem possible (try it yourself).

To fully see what is going on here it is helpful to simplify our picture of the problem without considering the full map of the city. We can model the bridges and

land masses in more basic diagrams. In Figure 2, we can see that the situation involves four land masses, labeled A, B, C, and D, and the seven bridges, indicated by dashes.

Now, consider Figure 3. Would you expect this to yield the same result as the above diagrams? The points A, B, C, and D are the above-labeled land masses and the lines connecting them are the seven bridges. You can again try tracing a path—passing through a point (land mass) more than once, but not tracing along each line (bridge) more than once.

Why would we expect the two simpler (yet very different) diagrams to yield the same result? What at first seemed to be a geographical and, therefore, geometrical problem transformed into one focused on basic structures and configurations and the relationships between them. Euler used these simplified diagrams, abstracting the problem from its geometrical context to more easily see that a proof of why this was not possible depended on the relationship among the number of paths meeting at each bridge, and not the distances between them. The 7 bridges of Königsberg problem is widely known as the earliest example of a problem in topology—a branch of math that did not grow into its own as a serious field of
mathematical study until the last century. Its central idea—that problems of a seemingly geometrical nature could be thought of by simplifying them down to an essential structure—kept returning in various ways in the following centuries, prompting investigation into the common ideas and notions that would eventually comprise topology as we know it today.

For example, Euler’s continuing work in geometry built on this method of discovery to play another foundational role in the development of topology. His focus was on the Platonic solids—a special class of polyhedra that, since their discovery in ancient Greek times, have held an important place in the imaginations of mathematicians, astronomers, and philosophers. Polyhedra, in general, are 3-dimensional shapes made of polygonal faces. A soccer ball, for example, is a polyhedron that has 12 pentagonal faces and 20 hexagonal faces, or sides. The Platonic solids, or regular polyhedra, are special in that all of their faces are made of the same congruent regular polygons (polygons whose sides are all the same length and angles are all the same measure). These faces fit together in the same way across the entire polyhedron, creating a sort of perfectly fitting, symmetrical structure. Each is named for the number of faces it has; there are the tetrahedron, cube, octahedron, dodecahedron, and icosahedron, pictured in Figure 4, respectively. While one can construct an infinite amount of regular polygons, these five polyhedra are the only regular polyhedra—a unique characteristic that has ensured their popularity through the ages and captured the attention of the Greeks some 2,500 years ago.
For Plato, these polyhedra were so significant that in his dialogue, *Timaeus*, these objects represented the four elements and the universe itself. The tetrahedron, cube, octahedron, and icosahedron represented fire, earth, air, and water, respectively, and the dodecahedron symbolized the “essence of the universe.”

That the Platonic solids are the only five regular polyhedra was proven in Euclid’s *Elements*, the groundbreaking, first-known geometry text written in 300 B.C., around 60 years before Plato’s dialogue gave the solids their name. Euclid’s proof involved a geometric interpretation of the polyhedra, deducing their existence following a strict line of reasoning that eliminated all other possibilities based upon the angle and measurement requirements necessary to meet the criterion of regularity.

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In the 16th century we see the Platonic solids related to the universe again in Kepler’s *Mysterium cosmographicum*.⁴ Kepler modeled the solar system, then thought of as containing the six planets—Saturn, Jupiter, Mars, Earth, Venus, and Mercury, by inscribing the Platonic solids in the spheres representing each planet. In his model, “if one inscribed a cube into the sphere of Saturn, the faces of the cube would be tangent to the sphere of Jupiter. Similarly, inscribing a tetrahedron in the sphere of Jupiter made the faces tangent to the sphere of Mars.”⁵ This inscription continued in order of the planets towards the Sun with the dodecahedron, icosahedron, and then the octahedron. For Kepler, there had to exist some connection between the Platonic solids and the universe that “the Creator” had determined.⁶

While these philosophical understandings of the Platonic solids “have not stood the test of time,” the unique existence and characteristics of the solids have ensured their prominence in mathematics.⁷ Later work addressing these regular polyhedra concerned not just their existence but also ways to understand and classify them. For Euler in the 18th century, for example, studying and classifying these regular polyhedra led to a very interesting discovery. Euler noted in 1758 that “polygons…could be classified very easily according to the number of their sides, which of course is always equal to the number of their angles, [while] the classification of polyhedra…represent a much more difficult problem, since the

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⁵ Berlinghoff, Gouvêa, 134.
⁶ Newman, 720.
⁷ Berlinghoff, Gouvêa, 134.
number of faces alone is insufficient for this purpose." Euler’s attempts to classify these objects resulted in his published work, *Elementa Doctrinae Solidorum*—the first work to discuss a specific relationship between the number of faces (sides), $F$, edges, $E$, and vertices (points where the edges meet), $V$, of a regular polyhedron. He noticed that $V - E + F = 2$ for all of these regular polyhedra. Take a moment to count the number of vertices, edges, and faces of a cube to see this relationship.

The fact that the numbers of vertices, edges, and faces of a regular polyhedron will always demonstrate the exact same relationship, regardless of which polyhedron is used, implies a relationship between the Platonic solids inherent in their structure that goes beyond their geometric properties. If we imagine a cube made out of a balloon-like rubber and blow air into it, the cube will stretch outward and expand into a sphere. This same thing occurs for the four other regular polyhedra. In fact, the Platonic solids are all specific representations of the sphere related to the uniqueness of their structure as regular polyhedra. Euler’s discovery contributed to topology much in the same way that the bridges of Königsberg problem helped to motivate topological thought—the regular polyhedra are all spheres in the same way that each model of the bridges of Königsberg problem represents the same relationship. A

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9 Lakatos, 6.
topological approach helps us to demonstrate how the Platonic solids are spheres and also provides a new perspective as to why they are the only regular polyhedra.
**Thesis Focus**

The goal of this thesis is to develop a full understanding of the abstract point of view Euler developed of the Platonic solids. To reach this understanding, I present here a comprehensive, introductory look at topology, exploring the basic ideas and concepts upon which it is based. This introductory consideration involves looking at topological surfaces such as the sphere, as well as the torus, which is shaped much like a donut. For example, one can look at a ping pong ball and a donut and consider what they have in common. Both are objects on which a smooth path can be drawn between any two points on the surface. They are also objects with no distinct perimeter, or boundary edge, but yet they have a definite surface area. And in much the same way that an ant’s perspective of the Earth yields a flat area all around it, an ant in proportional size on a ping pong ball or donut would not see the curve of the object but would instead understand view the surface as flat.

While these may seem to be trivial similarities, when we abstract these understandings on standard representations, called surfaces, to explore these relationships, we come to see a general pattern and structure. This abstraction to topological surfaces allows us to generalize and understand whether objects that may seem similar have the same properties.

Also important to understanding topological surfaces is developing methods of classifying these surfaces that help illuminate their properties. Euler’s formula does just that. His approach to the Platonic solids can be generalized to allow one to notice similarities and relationships between objects outside of strict notions of geometry,
increasing the possibilities for other understandings of objects in a way that Euclid’s understanding of surfaces could not. I will explore these methods of classification to provide a solid basis for understanding topological surfaces. From this basis I will explore the Platonic solids, defining the ways in which they model and represent the sphere, allowing me to provide a topological explanation as to why they are the only regular polyhedra.
History and Motivations for Topological Ideas

Euler’s work in the 18th century expanded upon ideas in geometry by taking new approaches to classical modes of thinking. Specifically, his work on the regular polyhedra helped to challenge typical understandings of geometry because the concepts involved were “concerned only with the numbers of the vertices, edges, and faces, and not with lengths, areas, straightness, cross-ratios, or any of the usual concepts of elementary or projective geometry.”¹¹ Because of this, Euler’s formula could be generalized to many other forms of polyhedra besides those made of straight lines and allows us to apply these new relationships to objects in various areas of geometry. The ability to transform one object into another and still find the same relationship inherent to the structure is an idea central to topology.

Projective geometry is one area of geometry that was influenced by the development of topological notions. It emerged in the early 1800s as a different way of considering objects: looking at properties that developed through the projection of the lines and images of objects. During the Renaissance, painters developed ways of creating more realistic works by projecting points of an image on a canvas based upon a specific eye’s perspective. This is how railroad tracks in a painting have a vanishing point at which they meet. While in reality the tracks are parallel, our eye would see them seeming to come together in the distance. This perspective changed the way in which artists drew and painted images and offered mathematicians a new way to perceive space. Two lines in projective geometry eventually come together and meet at a point, eliminating the possibility of parallel lines that are found in traditional

¹¹ Newman, 585.
geometry. Reflecting these differences in perspective, this branch of mathematics took on the name “geometria situs,” meaning “geometry of position.”\textsuperscript{12} Remember that Euler described the 7 bridges of Königsberg problem as concerned with the “geometry of position” in the 18\textsuperscript{th} century.\textsuperscript{13} These changing conceptions of geometry had been leading up to a new understanding of position, relationships, and how to study them for over half a century.

Another term emerging in mathematics relating to position was “analysis situs.” This term was used by the mathematician Riemann in 1857 concerning his work in analysis. He considered certain kinds of functions that contained “variables… studied completely independently of numerical relations, from the point of view of just the spatial relations of mutual location and connection arising between them.”\textsuperscript{14} These functions did not strictly manipulate numbers but were applied to abstract objects, transforming points from one object to another. The properties of these objects preserved by the transformation of these points were what was of interest in creating an “analysis of position.”

Many more mathematicians of the 19\textsuperscript{th} century developed work that contributed to the growth of topological ideas, leading to the coining of the term “topology” by Johann Benedikt Listing in 1847. Listing published his paper “Vorstudien zur Topologie,” introducing the term instead of using “Analysis situs (the

\textsuperscript{13} Alexanderson, 568.
\textsuperscript{14} Kolmogorov and Yushkevich, 102.
term used by Leibniz in different sense and later by Riemann in the sense of topology) in order to avoid confusion with *Geometria situs.*”¹⁵

In the 1900s, renewed interest in topological ideas led to a boom in topological research and its formation as a serious branch of mathematics as the creation of university courses in topology occurred and the text “General Topology” by John Kelley became, in 1955, a major work for topological study.¹⁶ While the majority of foundational topological ideas emerged in the 1800s, topology as we know it today has only been a serious branch of mathematical study for a little over eighty years.

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¹⁵ Kreyszig, 360.
What Is Topology?

Formally, topology is “the study of those properties preserved by homeomorphisms;” that is, the study of an object’s configuration and characteristics and the ways in which you can transform the object without changing the essential structure. Those types of smooth transformations that allow this preservation are called homeomorphisms. A common joke in mathematics that helps to illuminate this process is that a topologist cannot tell the difference between a coffee cup and a donut. The transformation is pictured here:


Imagine the donut as being made out of a very stretchable and moldable material allows us to follow the stretching and re-forming of one object into another in a smooth fashion. Nowhere in the process did we break or tear the object to get to the next step. The object we begin and end with, regardless of whether we start with the coffee cup or donut, is an object with one hole and no other openings (what you might consider to be an opening in the coffee cup is just a new molding of the material to have a dip in it; this dip is just another curve in the surface and does not change the basic structure of the object). In topology we can look at seemingly different objects

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and identify if they are, in fact, the same basic thing, abstracting them from our everyday reality and discovering new relationships.

We can also understand this idea by looking at another example involving objects very familiar to us: the letters of the English alphabet. For example, when one looks at the letters typed in uppercase, one can divide them into groups based on their structure.

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A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
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- \{A R\} \{P Q\} \{D O\} \{B\}
- \{C G I J L M S U V W Z\} \{Y T\} \{E F H\} \{K X\}
  - \{one hole, two tails\} \{one hole, one tail\} \{one hole, no tails\} \{two holes, no tails\}
  - \{no holes, one tail\} \{no holes, two tails\} \{no holes, three tails\} \{no holes, four tails\}

Fig. 7. Homeomorphism groupings of the English alphabet.

The groups are defined by how you classify your structure: how many “holes” and “tails” form the letter. One can imagine the letters in the “one tail” group being transformed into each other by a smooth process of stretching, pulling, smashing, and pinching—but no breaking. These letters are homeomorphic to each other, and therefore essentially the same objects. In each group, we could take one letter as representative, knowing that it is essentially the same letter as the others in its group.

This is one way to understand how the Platonic solids are all spheres. They are all homeomorphic to the sphere because with a smooth transformation of stretching and smoothing, one can make the octahedron, for example, into a sphere. Again, you can imagine the polyhedron as being made out a balloon-like rubber and blowing air into it. The sides will smooth out and the essential shape will be a sphere.
These “properties preserved by homeomorphisms” not only tell us when two objects are essentially the same object but also when they are not. The comparison made previously between the ping pong ball and the donut demonstrates that two objects can have similar properties, but that does not necessarily ensure that they are the same object. The donut, or torus, as it is called in topology, is not homeomorphic to the sphere because there is no way in which you can transform the torus to get a sphere—no amount of expanding, stretching, or smashing will eliminate the hole in the middle of the surface to create a sphere.

Certain surfaces are of major importance in building an understanding of the sorts of topological properties we are interested in and in helping us build a knowledge from which we can develop topological concepts even further. In order to understand these, a more rigorous definition of topological surfaces is needed.
Surfaces

The term “surface” can make one think of a writing surface or a table’s surface—a smooth area upon which things can be placed. Or one might envision the outer layer of a baseball, which we call the surface of the ball. In topological contexts, surfaces can resemble objects in our 3-dimensional world, but they have certain properties that distinguish them from real-life objects. Surfaces are contained, connected abstract objects. They are “connected,” meaning that if someone wanted to draw a path from one point to another on the surface, they would be able to; there are no cut-out or separate, disconnected parts of the surface. We specifically call this property “path-connected.” Surfaces are “contained” in that they are bounded, in a sense. They sit in a finite area of space and do not stretch to or have isolated areas at infinity.

Also, if you imagine shrinking down onto a surface so that you are living on it, you would see an area around you that looks like a flat circle, much like the perspective of an ant on Earth, which sees the world as completely flat and extending in every direction. This perspective is defined on a small enough scale so that “locally” the surface is a flat plane even though “globally” it is a curved object.\(^\text{18}\) This property holds for any part of a surface one looks at locally.

Such an object is called a “surface even though it isn’t the surface of anything,” i.e., one needs to keep in mind that these surfaces don’t hold quite the same characteristics as objects existing in our world.\(^\text{19}\) We live in a 3-dimensional world—


\(^{19}\) Ibid., 24.
our objects have a depth measured in length, width, and height. We model things in two dimensions, for example, by painting their image on a canvas or drawing them on a piece of paper. This representation now has a length and a width, but no height. Even when imagining things represented in 2-dimensional space on a piece of paper, it is important to note that this is not completely accurate, as the paper has a thickness to it, however small. There is no thickness to objects existing abstractly in 2-dimensional space.

This understanding of our world and its dimensions represents what is called Euclidean space. Building upon this, we can conceptualize dimensions of any number. For example, 4-dimensional space includes the three measures of 3-dimensional space, length, width, and height, plus another measure existing in a way generalized from the previous dimensions. In 4 dimensions, we have a new way of moving in space that wasn’t afforded to us in 3 dimensions. While we cannot easily picture these higher dimensions, we have ways of conceptualizing them, which provide interesting areas of study. Objects can also exist in other abstract ways but for our purposes we will imagine them in Euclidean space.

Specifically, topological surfaces are objects that may sit in 3- or 4-dimensional space, but are made out of 2-dimensional material and so do not have any thickness. This distinction will be important in helping us understand certain properties of surfaces discussed later. We can imagine surfaces as being made out of an infinitesimally thin sheet of rubber or clear plastic to help us understand this.
Modeling surfaces in 3-dimensional space allows for an easier picture and conceptualization, but we must remember that they are still inherently 2-dimensional.


The sphere is the simplest topological surface. Denoted by \( S^2 \), the sphere can be imagined like the surface of our Earth, so that it is easy to see that it is path-connected, contained, and, by imagining zooming down onto the surface, the area would look like a flat plane.

Remember, however, that there is no thickness to this surface, unlike our Earth. The sphere can be imagined to be like the outer layer of a basketball. The “material” it is made out of is 2-dimensional so that there is no depth.

![Fig. 9. The Torus. Retrieved from Paul Bourke, “The Torus and Supertoroid,” last modified May, 1990, http://paulbourke.net/geometry/torus/](image)

The torus is another simple and important topological surface. Denoted by \( T \), the torus is shaped like a donut. There is no thickness to a torus like there is a donut, but it can be thought of as the basic outer shell, or “surface” of a donut. Take a moment to examine the properties of the torus that make it a surface.

Looking at \( T \) and \( S^2 \), we note that these are two distinct topological surfaces. The obvious difference is that there is a hole in one object, the torus, and not the other. If we were to try to stretch the torus out into a sphere, we would find that we cannot get rid of the hole.
Likewise, the sphere cannot be transformed to look like the torus, no matter how it is squished and stretched. Cutting a hole through the sphere and then regluing to get a torus is not an acceptable transformation. Remember, only smooth transformations like stretching and compressing are permitted.

Another important note is that surfaces in our 3-dimensional world can have a distinct boundary, or edge. If we think of a piece of paper, the surface of the paper has a definitive edge. Now, think of a donut. There is a definite measurable space in which the donut exists, yet there is no one distinct boundary edge. On a piece of paper one can trace his or her finger across the surface until it leaves the sheet of paper, or, “falls of the edge.” On a donut, one can trace his or her finger across the surface without ever “falling off.” A person would have to pick up his or her finger to remove it from the donut’s surface. This distinction in topology helps us classify “surfaces” and “surfaces with boundary.”\textsuperscript{20} Note, however, that a surface with boundary “is not a surface that has an additional property, but rather a different though closely related topological” object.\textsuperscript{21} The existence of an edge necessarily changes the essential nature of the surface under consideration.

One surface with boundary of particular topological interest is the Möbius strip. You can imagine making this out of a long piece of paper or wide ribbon—take the two ends of the paper, twist one end once

\textsuperscript{20} Goodman, 37.
\textsuperscript{21} Ibid.
over, and then glue the ends together. The properties necessary to characterize this as a surface exist at every point of the Möbius strip except at the points of the boundary edge. This is the distinction between a surface and a surface with boundary. Locally, the Möbius strip looks like a flat plane everywhere except at the boundary edge. Points along the boundary do not give the perspective of a flat plane in all directions, but that of a cliff, where the surface ends. Compare this to Christopher Columbus, who believed he was sailing on a “surface” that would lead him back to the other side of the Earth, while others believed he was sailing on a “surface with boundary” and would fall off of the Earth.

An interesting characteristic of the Möbius strip is that it is “one-sided.” This fact often confuses people: How does one start with a piece of paper that has two sides, play around with it, and end up with something that has only one side? This can be seen by imagining the two ends glued together without twisting the paper. You can trace your finger around an inside and an outside of this paper loop, distinguishing two sides. However, take a moment to imagine tracing along this Möbius strip (or create one yourself and try). Pick a starting point and trace along the strip until you arrive back where you started. When you trace along the Möbius strip, you first arrive at your starting point, but on the opposite side of the paper. If you continue along, you end back at your starting point in one smooth path. Not once do you have to pick up your finger to get to the “other side” of the ribbon, yet you traverse the whole strip (it may help to even draw on your Möbius strip to see that your pen never has to flip over the paper to get to the other side).
Understanding the one-sidedness of the Möbius strip helps reiterate that surfaces are 2-dimensional so that shrinking yourself down onto a surface is actually shrinking yourself into the dimensions of length and width. This is confusing at first, but if you take a minute to let it sink in, this gives a new perspective to the Möbius strip. When you imagined tracing your finger along the Möbius strip, you saw that one loop around led your finger back to its original point but on the other side of the paper. In 2-dimensional Euclidean space, however, we do not arrive back to “the other side of the paper, but actually at our original point. There is still a process of ‘flipping over’ to what would be the other side of the paper, yet it is a flipped orientation—you become ‘mirror-reversed.’”

When you look in a mirror, the image you see is you, but with your characteristics switched from either side of your face. If you have a dimple in your right cheek, it is now in your mirror image’s left cheek. This same sort of switching

22 Weeks, 48.
from right to left happens on the Möbius strip. The twist in the strip makes it so that if you imagined yourself traversing the Möbius strip you would come back with your perspective of left and right flipped.

Such a path on a surface is called an “orientation-reversing path.”23 If we took that path twice along the Möbius strip, we would arrive back to the original point with our original orientation. Of course we could also find a path on the Möbius strip which is not orientation-reversing at all. The Möbius strip is a “surface with boundary” as discussed before, so that a path that does not follow along the strip, but rather, across the strip, takes one off the edge but does not reverse his or her orientation. Also, a small circular path, or loop, on a section of the Möbius strip that neither goes off the edge or around the whole strip would preserve one’s orientation.

Compare this with the torus, T. We can find many paths along the torus, but we essentially have three types—a path which traces around the torus, through the hole in the middle; a path that traces around the hole in the middle; and a path that does not go around the hole at all. The torus is inherently two-dimensional, so, again, imagine zooming down into the surface. If we start at one point and follow each type of path, we come back to our starting point with the same perspective (much like traveling a path around the Earth does not change our perspective). None of the paths that can be made on the torus reverses our orientation so that we become mirror-reversed.

23 Ibid., 58.
Types of paths on $T$:

Fig. 12. Paths on a surface.

The previous discussion highlights an important distinction among surfaces. A surface is called “non-orientable” if it contains an orientation-reversing path. If a surface does not have an orientation-reversing path, it is called “orientable.” Of course, there are any number of paths one can identify on a surface, accounting for variations in position and curves of the path, but all of these paths are classified as orientation-reversing, or not. Looking at the Möbius strip, there are various types of paths one could draw, but because one of those types of paths—the one along the strip—is an orientation-reversing path, the Möbius strip is classified as non-orientable. The torus, however, is orientable, because no matter what type of path is drawn on the torus, nothing about its structure changes our orientation. On a local scale, if one were to imagine traveling these paths on the surfaces within their two dimensions, they would all seem to be the same. Just like an ant travels the Earth without realizing it is

\[24\text{ Ibid.}\]
on a sphere, one traveling a path on a surface would not be able to tell whether the surface was a sphere, torus, or Möbius strip. Non-orientability is a property that requires consideration of the entire surface on a global scale, not just locally.

Orientability is topological property that is very important in helping to distinguish among various surfaces. If two surfaces are homeomorphic, an orientation-reversing path would exist either in both or neither of the surfaces. No homeomorphism can create or eliminate such a path, as we would have to insert or remove a sort of twist in the surface to alter the orientation, requiring a cutting and re-gluing of the surface.

If such an orientation-reversing path exists, it can also be said, “a Möbius [strip] is contained in the surface.”

We can see one example of this by examining another famous topological surface, the Klein bottle. One can visualize this surface by “[holding] one end of [a] Slinky in each hand, letting it hang between your hands. Slide the two ends together” so that the Slinky intersects itself and the two ends are identified as one.

Note that in 3-dimensional space the Klein bottle has one circle of intersection. This surface, $K$, is modeled in 3-dimensional space so that its characteristics can be more easily understood, but in its natural habitat of 4-dimensional space it has no self-intersection. One way to begin to

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25 Goodman, 55.
26 Ibid., 51-52.
understand the Klein bottle is to again examine several paths on the surface. If you trace a circle around the cylindrical part of the Klein bottle, as pictured in red in Figure 14, you get a path much like the ones traveled around a torus. Your orientation does not change along this path, nor would it change if you were to trace a small loop on any part of the surface as well.

Another path could begin at the rim and go “inside” the bottle where you end up coming back to the “outside” of the bottle by passing through the self-intersection and continuing along, passing through the intersection again, and returning to your original point. This path is modeled in orange in Figure 15. They are labeled “inside” and “outside” in quotations to again make it clear that the Klein bottle is a 2-dimensional surface; there is no “inside” or “outside” to the Klein bottle, only the 3-dimensional space model which makes it seem so. The path you follow is not along “sides” of the bottle but within the surface. With this in mind, a closer look at this path yields a familiar shape—a Möbius strip.

Fig. 14. Non-orientation-reversing path on the Klein bottle.

Fig. 15. Orientation-reversing path on the Klein bottle.
Since the Klein bottle contains a Möbius strip, it is non-orientable. In fact, the path just defined cuts the Klein bottle in half so that we see that "a Klein bottle is simply two Möbius [strips] glued together along their boundary circles." Imagining gluing two Möbius strips together along their respective boundary edges can create a complicated picture, but the essence is that we now have a non-orientable surface distinct from the above-mentioned sphere and torus.

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27 Ibid., 52.
Plane Models

To understand these surfaces, it was helpful to imagine constructing them from basic shapes and objects, like a piece of paper, or stretchy rubber. For example, to create the Möbius strip, we imagined taking a strip of paper, half-twisting one end, and then gluing both ends together. You can also imagine taking a piece of paper, gluing the long edges together to create a cylinder, and then stretching the ends around and gluing them together along the edges of the circles, to create a torus. These constructions start from very basic shapes that are then molded and glued to create a model of something new. For some, it is easier to think about these other ways to model surfaces. Just like Euler created a model of the bridge/path system in Königsberg that simplified the picture and helped him to see the critical characteristics of the situation, topologists model these surfaces in a simpler way in 2-dimensional space, enabling them to understand the surfaces’ properties in different ways.

These new models are called “plane models.” The edges are labeled by letters to indicate which are to be glued, or identified, together. The arrows indicate the direction in which the points along the edges will be identified. For example, one can imagine creating the sphere by taking a round piece of rubber, folding it in half, and then gluing the two edges together from one end of the fold around to the other. If you could blow up this rubber like a balloon, you would get a round surface, just like a sphere. This sort of gluing is represented in the plane model for the sphere. The two edges are labeled \(a\) to indicate that they are identified together.

\[ \text{Fig. 16. Plane model for the sphere, } S^2. \]

28 Ibid., 42.
into one, and the arrows pointing in the same direction indicate that the points located on the corresponding parts of the edges are identified together.

The plane model for the Möbius strip in Figure 17 shows that we have two edges which are identified together. The arrows are going in opposite directions, indicating the twist that occurs before identifying the edges together. The points at the bottom of the right edge become identified with the points at the top of the left edge. If we imagine ourselves as zoomed down into this model of the Möbius strip, tracing a path across from left to right demonstrates, again, the notion of an orientation-reversing path. Walking off of the bottom right edge of the plane model brings you back to the top left edge, mirror-reversed, as shown above. The colors in the image demonstrate the way the twist in the Möbius strip reverses, or flips, one’s orientation.

The plane model for the torus, shown in Figure 18, has two sets of edges to be identified. One can imagine identifying the top and bottom edges, labeled $a$, together to create a cylinder. Next, you stretch and glue the edges labeled $b$ to get a torus.
Taking note of the arrows on edges $a$ and $b$, we see that each edge is identified together straight along as if we had zipped up the sides. Changing this identification for one of our edges produces a new plane model, shown in Figure 19. The direction of the identification of edge $b$ has changed so that one arrow is going down and the other is going up. Note that this edge identification is the same way the Möbius strip was constructed, indicating that “a Möbius [strip] is contained in the surface,” and, hence, this is one of the non-orientable surfaces.\(^\text{29}\)

Take a moment to pause and consider what this surface might be. Edge $a$ is identified to create a cylinder. Next, edge $b$ needs to be identified, but the arrows are now going in different directions. Instead of gluing the ends together head on, there is a twist which requires them to be identified in the same direction. This is much like the way in which the ends of the slinky were identified by sliding one into the other, so we can see that it is a plane model of the Klein bottle.

If you were uncertain of the fact that the Klein bottle is constructed from two Möbius strips, the plane model provides another way to demonstrate this point. Refer back to page 30 to see the path on the Klein bottle which was the Möbius strip.

\(^{29}\) Ibid., 55.
path cut the Klein bottle in half while preserving the “twist” in the surface. This means that the path, when represented on the plane model should not interrupt, or break, the identification of edge $b$ which creates the twist, but instead go along with it. It is labeled path $a$ on the diagram in Figure 20. Notice that this path appears twice on the plane model even though it is one connected path because of the twist in identification of edge $b$.\textsuperscript{30} We use this path to cut the plane model and by rearranging the cuts while preserving the identification of edge $a$, we get two pieces of the Klein bottle which are each Möbius strips. While this concept was more difficult to model in 3-dimensional space, the plane model provides a new and useful perspective.

We can make another change to this previous plane model to produce a new one. By reversing the orientation of the bottom edge $a$, we now have two sets of edges that have a twist in their identification. If we imagine identifying edge $a$ together first, we have a Möbius-type object with another set of edges requiring a twist as well. In trying to identify edge $b$ together, we can imagine a very convoluted process of twisting and gluing along the edge that would require many points of intersection along the object. A simpler way to instead imagine this surface is by stretching our plane model out into a circle and forming this piece into a bowl. The arrows along the edge indicate that each point on the “rim” of the bowl is identified with the point diametrically

\textsuperscript{30} Ibid., 52.
opposite to it. This means that by trying to glue the identified points together, like we have for each of the previous constructions, the surface must cross through itself multiple times.

This surface is called the projective plane, denoted by $P$. Similar to the Klein bottle, the projective plane exists “more naturally” in 4-dimensional space where, because of the higher dimension, self-intersections do not occur.\(^3\) Many other ways to model the surface have been developed, each trying to capture the various mental images the indicated identifications can create for a person. The model in Figure 22, however, proves to be a simple way to conceptualize the projective plane.

In the interest of simplicity, there is another plane model used to represent the projective plane. While the circular model above was useful in demonstrating the surface’s construction because its edges are not clearly defined it is not used. The plane model in Figure 23 is the standard plane model for $P$. It is even simpler than the square model, as it has two edges and is similar to the plane model of the sphere. Notice that instead of gluing straight along the edge as with the sphere, the arrows going in opposite directions model the twist in the identification that creates the projective plane.

\(^3\) Ibid., 48.
These standard plane models help one to identify the properties of a surface as well as the similarities and differences among them. Yet up to this point, they have been relatively simple structures, made of only two or four edges. We can imagine a more complex plane model, say with 12 sides, but how can we be sure what surface, if any, that plane model represents? We begin to wonder how many surfaces actually exist and how we can begin to understand and classify them. Plane models are a key ingredient in resolving these questions.

Fig. 24. Is this a surface?
Classification Theorem

We can create new surfaces from old ones by creating a connected sum. Connected sums are formed by taking two small disks out of each of our surfaces and gluing the surfaces together at those disk edges. For example, imagine taking two tori, shown in Figure 25. The resulting surface is the connected sum of two tori, denoted $T \# T$, or $2T$. Repeating this operation yields $2T \# T$, or $3T$, and so on, creating a chain of tori.

![Diagram of connected sums of tori](image)

Fig. 25. The connected sum of two tori. Retrieved from Weeks, 71.

We can take this idea of a connected sum from an abstract perspective to a geometric representation by using plane models. Using the standard plane model for each torus, one draws a disk emerging from one corner of each torus. This disk becomes its own edge on the surface, call it $c$. Imagine creating the torus from this plane model by identifying edges $a$ and $b$ to convince yourself that this is just a disk cut out of the surface. Stretching the plane model out now to account for the new edge, the shape is a pentagon. To represent gluing the two tori together at the disk edges to make the connected sum, we identify the edge $c$ together, creating an octagon. The
identification of edges $a$ and $b$ on each torus is still preserved in this model so that we see two tori connected along the disk edge $c$.

![Two Tori:](image)

![Tori with disk edges:](image)

![Connected sum 2T:](image)

This plane model in Figure 27 is the plane model of the connected sum of two tori. It does not provide a very intuitive picture of what the connected sum of two tori looks like, but it does provide a systematic way of representing the construction. We can go one step further in attempting to simplify this representation by reading the edge labels around the plane model. Starting at the top, the edge labels produce the “word” $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$, where “$-1$” is read “inverse,” indicating...
that “you travel that [edge] against the arrow.” In addition to the space model and the plane model, this word also represents the connected sum of two tori. It contains all of the information about the construction of the surface that we would need to know, indicating the number of edges of the plane model and the direction of their identification. For the connected sum of any number of tori, \( nT \), you would repeat this construction \( n \) times to get a \( 4n \)-gon. That is, a polygonal plane model with 4 sides for each torus in the connected sum. The word around this plane model would read

\[ a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1} \]

This provides us with an even simpler way to capture the surface, taking the concept of representation from a geometric context to an algebraic one.

The usefulness of plane models and word representation is demonstrated especially when considering the connected sum of projective planes. Because projective planes are so difficult to model in 3-dimensional space, it is even more difficult to picture what the connected sum of two projective planes would look like. However, this connected sum can be modeled simply using plane models.

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32 Ibid., 45.
Two projective planes:

Connected sum, $2\mathbf{P}$:

Fig. 28. Plane model of two projective planes.

After removing the two disks from each projective plane and gluing the two surfaces together along the boundary edges, the resulting plane model is a simple square. While repeating this process an infinite number of times does not allow for an as easily imagined space model as the connected sums of tori, the plane model’s utility is demonstrated here, allowing us to model these surfaces in a clear way. Repeating the construction above, the connected sum of $m$ projective planes, $m\mathbf{P}$, produces a $2m$-gon. That is, a polygon with $2m$ sides—two sides for each projective plane in the connected sum. The word around this surface would read $a_1a_1a_2a_2...a_ma_m$. Notice that there are no inverses in this word—each edge has the same orientation along the plane model, contrasted with the word for the connected sum of $n\mathbf{T}$, in which each edge has an inverse.

Following the above discussion, we can see that we can define infinite lists of connected sums of tori and projective planes. In fact, these infinite lists denote all possible surfaces. The Classification Theorem in topology states that all surfaces are
“homeomorphic to a sphere, a connected sum of tori or a connected sum of projective planes.” Any surface we may imagine must be among one of those categories. The list of connected sums of tori contains all orientable surfaces, and the list of connected sums of projective planes contain all non-orientable surfaces. The sphere is included in the theorem because it is distinct from both the torus and the projective plane and cannot be constructed out of either of those surfaces.

This theorem is critical to topology because we now have a way of classifying all surfaces in existence. Pausing to reflect on the surfaces discussed previously, one may notice that the Klein bottle and the Mӧbius strip are not included in the theorem. Remember that the Mӧbius strip is a surface with boundary, and thus different from the surfaces considered here. The Klein bottle is a non-orientable surface and so must be within the list of connected sums of projective planes. To clearly see this, consider the plane model of the Klein bottle. By cutting the Klein bottle along the path labeled $c$ in Figure 29, and regluing the pieces together along edge $b$, we can see that we now have a plane model of the connected sum of two projective planes, $2P$. So, $P\#P$ is homeomorphic to $K$.

![Fig. 29. The Klein bottle as two projective planes. Retrieved from Goodman, 58.](image)

Now that we have a list of all possible surfaces, we must consider how to easily distinguish between these surfaces. For example, refer back to the previous

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33 Ibid., 57.
example of the 12-sided plane model, on page 37.

If we suppose that the edges were identified as such in Figure 30, we can read the word around this surface as

\[a_1a_1a_2a_2^{-1}b_2^{-1}a_3a_3^{-1}b_3^{-1}a_4b_4,\]

and by checking the edges for their inverses, see that this surface must be among the connected sums of projective planes, as not every edge has an inverse. We could try to build this surface from its word, but where would we start in beginning to visualize this surface? For surfaces that we cannot easily visualize, we need another method of characterization that helps us to distinguish among them.
**Euler Characteristic**

Euler’s formula, \( V - E + F = 2 \), characterized the sphere by the relationship among the vertices, edges, and faces of its representations. A standard sphere does not have any edges or vertices and is made of one smooth face. Yet by modeling the sphere through polyhedra that do have these characteristics, Euler was able to explore the various representations of the sphere and discover that no matter which polyhedron was chosen to represent the sphere, one would always yield \( V - E + F = 2 \). The Platonic solids are a specific and unique type of polyhedra that can represent the sphere, but any polyhedral representation will demonstrate this relationship, as seen in Figure 31, where \( V = 12, E = 24 \), and \( F = 14 \) on a cuboctahedron, and therefore \( V - E + F = 2 \).

We can apply Euler’s idea to the rest of the topological surfaces, defining systematic ways of representing them that allow us to explore relationships similar to those Euler discovered on the sphere.

We can naturally model all surfaces with vertices, edges, and faces by using their plane models. The edges of the plane model come together at points we call vertices, and enclose the area in the center, the face. These vertices, edges, and faces define a cell complex—a sort of grid on a surface. For example, the standard plane model shows that there is one face, one edge—as the two edges labeled \( a \) are identified together and therefore count as one—


![Figure 32. Cell complex on the sphere.](image)
and two vertices—as the vertex at the bottom of the plane model is not identified with vertex at the top, and, therefore, each is counted separately. So, \( V = 2, \ E = 1, \text{ and } F = 1, \) which give us \( V - E + F = 2. \)

This number is called the Euler characteristic of the sphere and is unique to the sphere. Each surface, in fact, has a unique Euler characteristic. The torus, for example, has an Euler characteristic of 0. Looking at the plane model of \( T \) in Figure 33, we see one face—the square area in the middle. There are two edges, edge \( a \) and edge \( b. \) The trickiest part is in identifying the number of vertices. It appears that there are four, denoted by the four dots at which the edges connect. However, we must look at the identification of the vertices, just like the identification of the edges. If we imagine building the torus from the plane model, identifying edge \( a \) also identifies the top two vertices with the bottom two vertices so that we now have just two vertices on our model. Next, in stretching the cylinder just created so that we identify edge \( b, \) we identify the vertex on the left with the vertex on the right. From this construction we can see that the four vertices denoted on the plane model are all identified as one vertex. So, \( V = 1, \ E = 2, \text{ and } F = 1 \) give us \( V - E + F = 1 - 2 + 1 = 0. \)

The Euler characteristic for \( T, \) denoted \( \chi(T) \), is 0, regardless of the cell complex used. Different cell decompositions of the surface produce a different sort of
grid, but do not change the surface in any way, and therefore, do not change its Euler characteristic. The standard plane model for $T$ demonstrated one cell complex, but we could also define another cell complex shown in Figure 34. The addition of four edges has now created four faces instead of one. To determine the number of vertices, we can again imagine constructing the torus from this plane model. By identifying the top and bottom edges and creating a cylinder, we identify 3 sets of vertices. Then, by identifying the left and right ends of that cylinder, we identify two sets of vertices and see that we are left with 4 unidentified vertices altogether. This same process also demonstrates that there are 8 unidentified edges. So $V=4$, $E=8$, and $F=4$ give us $V - E + F = 4 - 8 + 4 = 0$. Again we see that $\chi(T) = 0$, despite having used a different cell decomposition on $T$.

The ability to model the surface in two different ways and yet still find the same Euler characteristic hints at the uniqueness of this characteristic. It is not a property of the surface that is dependent on the way in which we construct the surface through cell complexes, but rather an essential property of the surface itself. This is an extremely important note. The Euler characteristic not only provides another way of classifying a surface but in a way captures the essence of the surface itself, reducing our issue of classification to a simple algebraic equation.
For example, one may consider the Euler characteristic of the connected sum of two tori. You could easily assume that you add the Euler characteristics of each surface, since you just have a connected sum of the two surfaces. In this case, that would mean that $\chi(2T) = 0 + 0 = 0$. However, we know that $\chi(T) = 0$ and since the torus and $2T$ are not homeomorphic to each other; that is, one cannot be smoothly transformed into the other, they cannot have the same Euler characteristic. This problem arises because the Euler characteristic of a connected sum is not simply determined by summing the Euler characteristic of each surface involved in the construction. Consider the plane model for $2T$ in Figure 35. When the two tori are connected together, the number of vertices or faces of the cell complex do not change. We still have $F = 1$ and $V = 1$. However, there are now 4 edges of the plane model instead of 2. The Euler characteristic of $2T$ is then $\chi(2T) = V - E + F = 1 - 4 + 1 = -2$. Notice that this is 2 less than the sum of the Euler characteristic of each torus. So for each torus $T_1$ and $T_2$ in the connected sum, $\chi(2T) = \chi(T_1) + \chi(T_2) - 2$. In fact, that relationship holds for the Euler characteristic of the connected sum of any two surfaces: $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$. Continuing this pattern, the Euler characteristic of a connected sum of tori decreases by 2 for each additional torus in the construction. This necessarily implies that all orientable surfaces have an even Euler characteristic.
It is also important to know that the Euler characteristic for the projective plane is 1. We can understand this by looking at the cell complex defined on the standard plane model for $\mathbb{P}$. We have one face, one edge $a$, and one vertex. Because edge $a$ is identified with the twist in orientation, the vertex at the top is identified with the vertex at the bottom, giving us one vertex (compared to the sphere that had two unidentified vertices). So $V = 1$, $E = 1$, and $F = 1$ give us $V - E + F = 1 - 1 + 1 = 1$. Thus, $\chi(\mathbb{P}) = 1$. Following the relationship concerning the connected sums of surfaces, the Euler characteristic for $2\mathbb{P}$ is the sum of the Euler characteristics of each projective plane, decreased by 2. That is, $\chi(2\mathbb{P}) = 1 + 1 - 2 = 0$. For each additional projective plane included in the connected sum, the Euler characteristic decreases by 1. From these patterns we can discern the Euler characteristic for any surface.

You may have noticed that $\chi(T) = 0$ and also $\chi(2\mathbb{P}) = 0$. Clarification is needed here in reference to my earlier statement that each surface has a unique Euler characteristic. While this is true for each surface, it is only true with respect to orientability. These two surfaces share the same Euler characteristic, but while $T$ is orientable, $2\mathbb{P}$ is not, and so we know these are two different surfaces. The point here is that the Euler characteristic, along with orientability, completely classifies the surface.
**Specific Cell Complexes**

Because the Euler characteristic of a surface is independent of the cell complex used to represent it, any number of cell decompositions exist that one could use to model a surface. There are, however, some specific types of cell complexes that model surfaces in more systematic ways. For example, triangulations are a common and useful type of cell complex. A triangulation divides the surface into a symmetrical web of interlocking triangles, thereby establishing a neat pattern across the surface. The way in which the triangles intersect must be along a complete edge or at a single vertex only, making it so that each surface requires its own minimum number of triangles necessary to satisfy a triangulation. This does not mean that triangulations are more difficult for some surfaces or impossible for others—any surface can be modeled by a triangulation, making this a widely used and useful type of cell complex in areas such as modeling in biology and chemistry.\(^{34}\)

Within computer graphics, these complexes are used to create a sort of wire frame model of a surface. The triangulation of a sphere, shown in Figure 37, gives a very structured model that ensures symmetry along the surface. We can use fewer triangles in our representation, but the more triangles used, the smoother the surface becomes to more closely resemble the sphere, as it exists naturally. Other triangulations of the sphere are found among the Platonic solids. The icosahedron, octahedron, and tetrahedron are triangulations of the sphere that use a

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\(^{34}\) Ibid., 75.
small number of triangles—20, 8, and 4, respectively. The tetrahedron is in fact the simplest triangulation of the sphere, with only four faces.

These models of the sphere are also important in that they are another type of cell complex as well. Remember that the icosahedron, octahedron, and tetrahedron are regular polyhedra, constructed of regular polygonal faces fitting together in a symmetric way. This regularity provides another systematic way to define a cell complex on a surface, called regular complexes.

We characterize regular complexes by their edges and vertices. On a regular complex, “each face has the same number of edges $a$,” and “each vertex has the same valency—say $b$.\textsuperscript{35}” (The valency of a vertex is the number of edges that meet at that vertex). The symmetry of the regular polygons means that the faces fit together in the same way across the entire surface so that at each vertex the same number of edges come together.

Knowing these properties of regular complexes, we can work to understand the ways in which we can construct them. Just like the triangulation of a surface has a minimal number of triangles that will suffice, a regular complex must meet certain criteria. First, we know that the number of edges, $a$, of each face must be the same number, and that this number must also must be greater than 3. This is because no face can be made with less than three sides—two straight edges cannot enclose a space in

\textsuperscript{35} Ibid., 80.
the plane. Attempting to connect both ends at two vertices would just create two overlapping lines. The valency of the vertices must be at least three as well. In the same way that two connected edges cannot create a face, two connected faces cannot meet at a vertex and create a corner at which the surface will be enclosed. At least three faces, and therefore at least three edges, must meet at a vertex.

We denote such a regular complex on a surface, say $M$, by $(a, b)M$. We can work with these values $a$ and $b$ to find interesting properties about our surfaces. Specifically, the following equation tells us a lot about our surface and the possible ways in which we can construct it via a regular complex: $\chi(M) = 2e(\frac{1}{a} + \frac{1}{b} - \frac{1}{2})$. We can derive this equation by noticing a couple of patterns on a surface made of a regular complex. First, all of the faces have the same number of edges $a$. So if we count the number of edges on our surface for each face, we find that we have $aF$ edges for $F$ faces. Also, we know that each face intersects another along one edge, or rather, each edge connects two faces. So by counting the number of edges for each face on the surface, we count each edge twice. That is, we count $2e$. So $2e = aF$.

A similar relationship occurs between the vertices and edges on a regular complex. By counting the number of edges that meet at each vertex, we count $bV$ edges for $V$ vertices. Also, since each edge meets two vertices, by counting the edges at each vertex, we count each edge twice, or $2e$. So we see that $2e = bV$.

By solving $2e = aF$ and $2e = bV$ for $F$ and $V$, respectively, we can substitute these values in Euler’s formula $\chi(M) = V - E + F$ to get $\chi(M) = 2e(\frac{1}{a} + \frac{1}{b} - \frac{1}{2})$.\(^{36}\) This

\(^{36}\)Ibid., 83.
formula is useful in that given any surface, since we know its Euler characteristic based upon the patterns discussed previously, we can find the possible solutions for $a$ and $b$ that satisfy the equation and determine the number and types of regular complexes that can be made on a surface. Establishing this relationship between regular complexes and the Euler characteristic enables us to consider surfaces algebraically, providing a new perspective on the classification and representation of surfaces.

Let us return to the Platonic solids. We can check that each of these solids represents the sphere by checking the Euler characteristic of each. By counting the number of vertices, edges, and faces of each polyhedra, we see that the Euler characteristic of each surface is 2—the Euler characteristic of the sphere. Because we can now look at these polyhedra topologically as a sphere, we can begin to understand why they are the only regular polyhedra.

These polyhedra are constructed versions of regular complexes modeling the sphere. For each solid, each face has the same number of edges, and each vertex has the same valency. For the icosahedron, octahedron, and tetrahedron, the number of edges on each face is three. But the valency of the vertices is different for each surface—the vertices of the icosahedron have a valency of 5, the vertices of the octahedron have a valency of 4, and the vertices of the tetrahedron have a valency of 3. We denote these different complexes by $(3, 5)S^2$, $(3, 4)S^2$, and $(3, 3)S^2$, respectively. The cube is $(4, 3)S^2$ and the dodecahedron is $(5, 3)S^2$. 
How do we know that these are the only regular complexes on the sphere?

Again, we consider the equation $\chi(S^2) = 2e\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{2}\right)$. We know that $\chi(S^2) = 2$, so we can substitute that into the equation and get $2 = 2e\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{2}\right)$, or $1 = e\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{2}\right)$.

From this, we see that we must have a positive value in the parentheses for our equation to be satisfied. This means that $\frac{1}{a} + \frac{1}{b} - \frac{1}{2} > 0$, or $\frac{1}{a} + \frac{1}{b} > \frac{1}{2}$. We can begin by checking for values of $a$ at $a = 3$. Solving for the above equation when $a = 3$, we get $b = 3, 4, \text{ and } 5$ as viable solutions. This gives us the complexes $(3, 3)S^2$, $(3, 4)S^2$, and $(3, 5)S^2$, which are the tetrahedron, octahedron, and icosahedron, respectively.

Checking the equation when $a = 4$, the only possible solution for $b$ is $b = 3$.

This is the complex, $(4, 3)S^2$, which is the cube. When checking for when $a = 5$, the only possible solution is $b = 3$, giving us the $(5, 3)S^2$ complex that is the dodecahedron.
If we consider $a = 6$, the equation $\frac{1}{a} + \frac{1}{b} \geq \frac{1}{2}$ dictates that the only possible solution for $b$ is $b < 3$, which contradicts the requirement that $b \geq 3$. For any greater value of $a$, the value of $b$ must necessarily decrease to be less than 3, and by the nature of regular complexes this is impossible.

Also, we do not need to perform the same operations to check for values of $b$ and then solve for values of $a$ because regular complexes have the unique property that for any complex $(a, b) M$, by interchanging the values for $a$ and $b$, you can find a “dual” complex that satisfies the requirements of the previous equation and is a regular complex for the surface.\(^{37}\) We can see that the collection of regular complexes represented above contains all of the duals of the complexes as well. Thus, we see that these five regular complexes are the only possible regular complexes of the sphere.

This, in turn, demonstrates that the Platonic solids are the only regular polyhedra. All convex polyhedra are homeomorphic to the sphere when considered topologically, and so given that the Platonic solids are the only regular complexes of a sphere, we see that they are the only regular polyhedra in existence.

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\(^{37}\) Ibid., 81.
Conclusion

By analyzing the structure of the Platonic solids as specific representations of the sphere, we were able to demonstrate their unique existence outside of the strict notions of geometry to which Euclid was confined. We succeeded in reducing a surface down to a number that we were able to manipulate with simple algebraic operations when we substituted it into an equation. Topology enabled us to reposition our perspective and transform our question from “what are the possible regular solids?” to “what are the solutions to this equation?” This generalization lends itself to applications with other surfaces as well.

For example, we can explore regular complexes on the torus. Since \( \chi(T) = 0 \), we have \( 0 = 2e(\frac{1}{a} + \frac{1}{b} - \frac{1}{2}) \). Because the number of edges is always a positive number and never 0, we know that \( (\frac{1}{a} + \frac{1}{b} - \frac{1}{2}) \) must equal 0 to satisfy the equation. More specifically, \( \frac{1}{a} + \frac{1}{b} = \frac{1}{2} \) means that \( \frac{1}{a} + \frac{1}{b} = \frac{1}{2} \). From this we need to find the values of \( a \) and \( b \) such that, as part of a fraction, their sum equals \( \frac{1}{2} \).

We can start to consider the possible values by remembering the criteria imposed upon \( a \) and \( b \)—both values must be greater than or equal to 3. We can also see that as the values for \( a \) and \( b \) increase, the value of the fractions in the equation decrease so that after a certain value, their sum can never reach \( \frac{1}{2} \). For example, if \( a = 6 \), then \( b = 3 \), and \( \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \). But, if \( a = 7 \), then \( \frac{1}{7} + \frac{1}{b} = \frac{1}{2} \) means that

\[
\frac{1}{b} = \frac{1}{2} \cdot \frac{1}{7} = \frac{5}{14},
\]

which cannot reduce down to an integer value. Thus, any possible value for \( a \) is between 3 and 6. Continuing this algebraic work, we can determine that
the only possible values for $a$ are 3, 4, and 6. The regular complexes made from these $a$ values are $(3, 6)T$, $(4, 4)T$, and $(6, 3)T$. By the duality of regular complexes, $(3, 6)T$ is the dual of $(6, 3)T$ and $(4, 4)T$ is the dual of itself.

The wooden block in Figure 41 is an example of the way in which one could begin to conceptualize a regular complex on a torus. Considering the three possible regular complexes for a torus, we can eliminate the possibility of a $(4, 4)T$ on this block by noticing that at the corner of the block, we have only 3 edges coming together. There is no way to insert a fourth edge at this vertex. However we can make 6 edges come together at this vertex to create a $(3, 6)T$ complex. To do this, we divide each side into 9 squares, which we then divide into triangles, ensuring that they are constructed in such a way that each vertex has a valency of 6. For these surfaces that we can easily visualize, the classification of surfaces has become, in essence, a simple exercise in algebra.

We can perform similar operations to discover all of the types of regular complexes on a projective plane as well, even though we cannot visualize them. As surfaces become more complicated and even more difficult to conceptualize, discovering their regular complexes is not as simple. As the number of surfaces
involved in a connected sum increase, the Euler characteristic of the resulting surface reaches increasingly negative values, which, when substituted into the equation
\( \chi(M) = 2e\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{2}\right) \), do not allow for easy solutions.

What is important here is that the approach Euler developed succeeds in a way that Euclid’s could not. By reducing a surface down to an essential number, its Euler characteristic, we are able to work with that surface algebraically and consider it outside of its geometric dimensions in a new way. This approach towards generalization can in one sense revive an old problem, like that of the Platonic solids, and provide an almost trivialized solution, but it can also expand upon these old problems, providing a new perspective towards other, more complicated questions.

By continuing the pattern of definition, classification, and representation that has defined the work of this thesis, we can generalize to explore higher dimensional objects beyond 2-dimensional surfaces. For example, instead of imagining creating a torus from a piece of paper, you could imagine constructing a torus from a 3-dimensional object, like a room. To construct this torus, imagine stretching the room to identify the ceiling with the floor to create a sort of thick tube. Then imagine stretching this tube so that two walls opposite each other are identified together as well.\(^{38}\) This 3-dimensional torus expands upon the properties demonstrated in the 2-dimensional torus, and so, by clearly defining these surfaces and their properties in a variety of ways, redefinitions that expand upon previous work are possible, and can provide a new perspective on 3-dimensional space itself.

\(^{38}\) Weeks, 21.
The surfaces studied in introductory topology can all be used as a basis upon which we can conceptualize more complex, higher dimensional problems. This work has contributed to exploring even the shape of our universe. While “no one knows what the shape of the real universe is . . . people do know a fair amount about what the possible shapes are.” Topology provides the tools to begin conceptualizing these sorts of questions.

39 Goodman, 11.
Bibliography


