New Imaging Approaches for Process Tomography Based on Capacitive Sensors

Dissertation

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Process tomography is the investigation and imaging of a physical process in region of interest (RoI), such as fluid flow for example, on time and spatial scales around those of the process dynamics. The data gathered from the RoI may be utilized for diverse purposes such as characterization of industrial monitoring and control, design and optimization of industrial hardware, combustion flame imaging, and flow imaging, to name just a few. Due to nature of these applications, the associated sensors often need to be operated in harsh environments under very high pressure and/or temperature conditions. This reduces the currently available sensing modalities to a handful of choices as the possible candidates. Among these modalities, electrical capacitance tomography (ECT) holds great potential due to its relatively fast, non-invasive, non-intrusive imaging characteristics in addition to lightweight and inexpensive hardware. These attractive characteristics also carry over to electrical capacitance volume tomography (ECVT) which find applications in petroleum, chemical, and biochemical industries. Despite all these benefits, ECT and ECVT systems also have a few challenges that demand research efforts. First, typical operational frequencies are below 10 MHz, which make these “soft-field” modalities yield relatively low resolution compared with “hard-field” imaging counterparts such as X-ray. Second, current hardware design imply a high degree of correlation between mutual capacitance measurements and therefore an highly ill-conditioned inverse (imaging) problem. In addition, with
the increasing demand for volume tomography, more challenging applications are being sought after by industry such as exploration of larger RoI with better resolutions. Therefore, these scenarios imply increased computational costs for the volumetric imaging problem and make it more difficult real-time ECVT imaging applications.

In this dissertation, we introduce displacement-current phase tomography (DCPT) for process tomography. The operation principle of DCPT is based on the imaging of the imaginary part of the permittivity inside the RoI, which is complementary to real-part permittivity imaging obtained by ECT. While using the same ECT hardware, DCPT provides better resolution for certain classes of applications involving lossy media. This method is also extended to 3D volume tomography based on the use of ECVT hardware. DCPT is also extended to velocimetry applications, where the objective is to image the flow velocity in the RoI, based on ECVT hardware. Finally, a faster reconstruction approach for ECT/ECVT systems based on sparse representation of images in the Fourier domain is proposed and studied to facilitate real-time imaging for applications involving volumetric RoIs.
This work is dedicated to the ones I love.
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Chapter 1: Introduction

In November 1895, Wilhelm Conrad Roentgen’s discovery with X-Rays opened a new window on imaging technologies [1]. Not long after, he foresaw the potential medical applications of this discovery. He brought his wife into his laboratory and they left with an X-ray photography of the bones in her hand also showing the ring on her finger. This early advance inspired many researchers to work on the field of tomography [2, 3, 4]. Tomography spans a plethora of applications including, but not limited to, archaeology [5], biology [6, 7], target detection [8], atmospheric science [9], geophysics [10], oceanography [11], plasma physics [12], material science [13], astrophysics [14], non-destructive evaluation (NDE) [15], and quantum information [16].

The etymological root of the word *tomography* originates from the Greek word “tome” (slice, section) and “graphia” (description of); however, over the years, the concept of tomography widened to include not only 2D cross section imaging, but also a third dimension as in volume tomography and a fourth dimension as in process tomography and real-time imaging applications [17, 18]. Therefore, a general definition of tomography can be stated as the process of investigating the internal characteristics of a particular region of interest (RoI) based on boundary measurements.

Although studies on tomography technologies over the decades gave rise to a wide range of imaging modalities applied on various branches of science and engineering,
there are a few important factors to consider when assessing the performance of these techniques: cost, complexity of implementation and operation, rate of capture and post processing, flexibility to fit various application sizes and shapes, resolution as a percentage of imaged volume, and safety to both the user and the process under investigation. The safety for the process can be further classified into non-invasive tomography, which requires no direct contact with the domain, and non-intrusive tomography, which does not disturb the nature of the process under investigation. Similar to the other areas of science and engineering, there is no panacea or a tomographic modality that can address all these challenges with optimal performance. This ultimately means that particular imaging modalities are best suited for different applications. Typically, higher spatial resolution is achieved on the tomography methods that operate with high-frequency electromagnetic radiation, such as magnetic resonance imaging (MRI) [19, 20], positron-emission tomography (PET) [21], X-ray [22, 23], γ-ray [24], optical coherence tomography (OCT) [25], and microwave tomography [26, 27, 28, 29]. MRI scanners can be relatively fast but they constitute costly and bulky units. X-ray and γ-ray units can be slow and they have safety issues associated with ionizing radiation containment. PET scans do not offer on-line tomography and their operating is relatively costly. OCT is fast but provides limited penetration power [30]. Microwave imaging can be effective but it is prone to interference from other sources and nearby instruments [31, 32, 33]. In addition to these methods based on high-frequency electromagnetic radiation, some other tomography methods utilize acoustic and (low-frequency) electrical sensors [34, 35, 36, 37, 38]. Although different sensor systems exist, the basic underlying principle consisting in
measuring the contrast in physical properties (permittivity, resistivity, etc.) of the domain under investigation.

The focus of this study is on process tomography, whereby imaging techniques are used to provide information about a multiphase (gas/fluid/solid) flow inside a vessel in real-time. Process tomography is used for better design and control of industrial processes. Although most of the aforementioned methods can also be used in the context of process tomography, due to costs and technological feasibility the most common technologies are: (a) electrical impedance tomography (EIT) [39, 40], which is sensitive to electrical conductivity and permittivity, (b) resistivity tomography [41, 42, 43], which is sensitive to conductivity (or, equivalently, resistivity) only, (c) magnetic induction tomography (MIT) [44, 45, 46], which is sensitive to electrical conductivity and permeability, and (d) electrical capacitance tomography (ECT) [47, 48], which is sensitive to electric permittivity. Among these methods, MIT and ECT do not need contact with the domain under investigation whereas EIT and resistivity tomography sensors requires electrical contact. For volume imaging on the process tomography, a useful modality is electrical capacitance volume tomography (ECVT) [49, 50], which is an extension of ECT to 3D (volumetric) imaging. ECVT can be effectively used for flow velocimetry applications as well [51]. A recent modality introduced for process tomography is displacement-current phase tomography (DCPT) as illustrated in Fig. 1.1. DCPT relies on the same hardware as ECT and is sensitive to dielectric losses in the medium [52, 53].

In this dissertation, we develop new approaches for process tomography based on DCPT, ECT and ECVT systems.
1.1 Overview of ECT and ECVT

Electrical capacitance tomography (ECT) is a non-invasive and non-intrusive imaging modality used to image the spatial distribution of electric permittivity in a RoI from mutual capacitance measurements between electrodes at the RoI boundary [54, 55]. Initial works on this soft-field tomography date back to 1980s [56]. The name soft-field originates from the fact that typical excitation frequencies for ECT systems are below 10 MHz causing the electric field that penetrates the RoI to fall in the quasi-static regime. In other words, for typical applications, the characteristic size of the RoI is much smaller than the wavelength [57]. ECT sensors are lightweight, have a sturdy structure, are low cost, and have relatively high-speeds. Coupled with good immunity to electrical noise and interference, these features make ECT attractive for many industrial applications involving real-time monitoring of multiphase
flows. ECT sensor analysis can be decomposed into the forward problem and inverse problem, as shown in Fig 1.2. The forward problem for ECT seeks to obtain the mutual capacitance values between sensing electrodes for a given sensor geometry and RoI. The inverse problem focuses on reconstructing the spatial distribution of the electric permittivity in the RoI from the given mutual capacitance values between sensing electrodes. The number of the electrodes in ECT hardware depends on the application or the nature of the flow to be imaged, but typically they are chosen to be 8, 12, 18, 24, and 36. These electrodes are positioned next to the wall of the vessel that encloses the RoI. In typical ECT applications, the wall corresponds to a plastic material (tube), with relative permittivity around $\varepsilon_{\text{wall}} = 3$. The effect of this insulating wall is minimized during the sensor calibration process, special care
needs be to taken for problems with other types of walls [58]. In practice, ECT operation starts by applying a voltage excitation to a given electrode and measuring the current on another one while the remaining electrodes are grounded. This process is repeated for all electrodes so that for an ECT hardware with \( N \) electrodes there are \( M = N(N - 1)/2 \) distinct mutual capacitance measurements. The electric field distribution inside the RoI is governed by Poisson’s equation:

\[
\nabla \cdot (\varepsilon(x, y) \nabla \phi(x, y)) = -\rho(x, y).
\]

(1.1)

where \( \varepsilon(x, y) \), \( \phi(x, y) \) and \( \rho(x, y) \) represent the electric permittivity, the electric potential and charge density, respectively, in the point \((x, y)\). The relation between the measured capacitance \( C \) and the spatial permittivity distribution \( \varepsilon(x, y) \) in the RoI can be written as

\[
C = \frac{Q}{V} = -\frac{1}{V} \int_S \varepsilon(x, y) \nabla \phi(x, y) \cdot \hat{n} \, dS,
\]

(1.2)

where \( V \) is the applied voltage on the excitation electrode, \( S \) is the surface of the electrode where the current is being measured, and \( \hat{n} \) is the unit vector normal to \( S \). It is clear from eq. (1.2) that the relation between measured capacitance \( C \) and the permittivity distribution inside the RoI is nonlinear because \( \phi(x, y) \) depends on \( \varepsilon(x, y) \). In cases where the permittivity contrast and/or the permittivity fluctuation region are sufficiently small, the (first-order) Born approximation can be used, which linearizes the problem. In this case, the permittivity and electric field distribution become effectively independent from each other. By applying the Born approximation and by discretizing the RoI into \( P \) small pixels (or voxels), eq. (1.2) can be rewritten in matrix form as

\[
[c]_M = [S]_{M \times P} \begin{bmatrix} g_c \end{bmatrix}_P,
\]

(1.3)
where \( c \) and \( g_c \) are the (measured) capacitance and image vectors related through the sensitivity matrix \( S \). Again, the Born approximation for the ECT problem is only adequate when the permittivity fluctuations are small, either due to low contrast or to small spatial extension of the permittivity perturbation. In particular, this linear approximation is not expected to remain accurate in the presence of high permittivity materials such as water when these comprise a significant area (or volume) fraction of the RoI. In ECVT, the final discretized linear equation has the same basic form of eq. (1.3), with the \( g_c \) vector now representing voxel permittivities instead of pixel permittivities. To address the inverse problem in ECT/ECVT, different reconstruction algorithms can be used \[59\].

### 1.1.1 Challenges

Current research efforts on ECT/ECVT can be roughly classified into three categories: sensor design, acquisition hardware design, and reconstruction algorithms. These three research threads also summarize the main inherent challenges of ECT an ECVT systems. The high degree of correlation between some mutual capacitance measurements and the requirement of a minimum area size of the electrodes (as dictated by the minimum signal-to-noise ratio) poses a challenge for ECT/ECVT technology as the typical number of pixels/voxels that discretize RoI with adequate resolution is much larger than the number of electrode pairs. This results in a severely underdetermined problem. In theory, the number of plates could be increased to obtain a larger amount of distinct measurements, but this would result in smaller plates, which in turn decreases the signal to noise ratio (SNR) to inadequate levels. An alternative to increase the number of independent measurements is the use of adaptive
or reconfigurable sensors plates. Such adaptive ECVT (AECVT) systems provide the flexibility to form *synthetic* capacitance plates from a combination of many small plate segments [60]. Furthermore, if the amplitude and phase of each segment can in principle be controlled independently. This means that not only synthetic plate can be formed but also different voltage excitation patterns can be attained on each synthetic plate. AECVT has the potential for alleviating SNR constraints and add more degrees of freedom in the sensor design. As mentioned before, the relationship between the changing capacitance value due to changing permittivity value inside RoI is highly nonlinear. This is a consequence of the “soft-field” nature of the problem [61]. Although the Born approximation can often times be made to linearize the problem, in some cases this assumption does not hold true. To address this problem, nonlinear reconstruction methods can be used; however, nonlinear reconstruction methods are time consuming and are thus difficult to use for real-time applications [62, 63]. Finally, due to the Laplacian (self-averaging) nature of the interrogating field [64], the achievable resolution of ECT/ECVT systems is not uniform inside the RoI [65]: in a circular domain for example, the achievable resolution is higher near the periphery and lower towards the center of the RoI due to the spatial spreading of the field lines. Optimal reconstructions methods for ECT/ECVT would need to account for this fact. It should also be noted that although ECVT inherits all attractive features of ECT, it also inherits its challenges and some of the latter are exacerbated. Due to the larger number of plates in ECVT compared to ECT and the larger number of voxels that discretize the RoI in typical ECVT problems than pixels in typical ECT problems, ECVT systems require more computational resources and time to operate. Unless these issues are mitigated, real-time ECVT becomes difficult [66].
1.2 Contributions

In this dissertation, the formulation for DCPT is derived and an extensive comparison with conventional ECT is provided. In addition to providing extra information about the RoI, DCPT has an extended linear range for the measured admittance phase and material loss distribution when compared to ECT measured capacitance and electric permittivity distribution. This feature mitigates issues associated with the nonlinear nature of ECT for problems involving media with higher permittivity values. Exploration of this feature is also implemented for volumetric DCPT applications. In addition, we extend DCPT for velocimetry application, where the objective is not to measure a (snapshot) of the permittivity distribution inside the RoI but rather the velocity of the fluid flow inside a vessel. Finally, a faster reconstruction approach for ECT/ECVT systems based on sparse representation of images in the Fourier domain is proposed and studied to facilitate real-time imaging for applications involving volumetric RoIs.

1.3 Organization of the Thesis

The remainder of this dissertation is organized as following. Chapter 2 presents the basics of the DCPT formulation and a comparison with ECT using numerical simulation and experimental cases. Chapter 3 is devoted to the extension of DCPT to volumetric applications and the application thereof to velocity profiling. This chapter also evaluates the velocimetry performance with numerical simulation and experimental results. Chapter 4 describes the ECT/ECVT image reconstruction approach using a sparse representation based on a Fourier basis. Specifically, a simple filtering is used as a sparsity-promoting procedure in the Fourier domain, as informed by the physics
of the imaging problem. This formulation is compared with traditional pixel- and
voxel-based reconstruction techniques through both numerical simulations and ex-
perimental results. Chapter 5 provides a few final remarks and recommendations for
future work.
Chapter 2: Displacement-Current Phase Tomography

Electrical capacitance tomography (ECT) is a noninvasive and nonintrusive soft-field imaging technology used to image the spatial distribution of the electrical permittivity inside of a region of interest (RoI) [54]. ECT is based on mutual capacitance values obtained from a multi-electrode sensor surrounding the RoI. ECT systems have good immunity to electrical noise and interference. In addition, they are lightweight, low-cost, and relatively high-speed, which makes them a popular choice for real-time monitoring of multi-phase flows [56, 67, 68, 47, 69, 58, 70]. Another popular soft-field tomography technique is electrical impedance tomography (EIT) [71], which finds widespread use in medical and industrial imaging systems [72, 73, 74]. EIT can be used for simultaneous reconstruction permittivity and conductivity imaging in a region of interest; however, unlike ECT, EIT necessitates electrical contact and is based on the injection of electrical currents into the RoI. In many applications, this may be neither feasible nor desirable. For example, in many industrial systems, the flow occurs inside insulated vessels that do not allow electrical contact for current injection. Moreover, direct current injection might disturb the characteristics of the flow itself, thus potentially affecting the quality of the measurement results.

ECT measurements are often performed using time-harmonic currents. As such, not only the quadrature component of the measured current can be measured (as
done to compute the mutual capacitances in ECT) but also the in-phase component. The magnitude of the in-phase component is relatively small and, as detailed below, depends on loss mechanisms present in the RoI. Nevertheless, this information is valuable since it can be used to estimate the conductivity (or dielectric loss) distribution present in the RoI. In the past, this measurement component was utilized directly [75, 76] or in combination with ECT acquisitions [77, 78]. In the former case, the terminology “electric field tomography” was adopted to describe the system. Here, we instead adopt the term displacement current phase tomography (DCPT) because the current measured by any electrode pair excited by time-harmonic voltages surrounding the RoI is associated to the displacement current [79, 80] inside the RoI. Moreover, the term electric field tomography is ambiguous because ECT is also a form of electric field tomography as well. Displacement current sensing was also studied in [81] to measure the movement of charged particles. The system was intended for charge measurement based on capacitance input and displacement current measurement due to charge movement based on resistance input. In addition, a displacement current-based ECT system was proposed in [82]. This system was still based on capacitance measurements and used to provide permittivity reconstruction. In a recent study, the real part of the measured admittance has been used separately to image the conductive properties in the RoI [83].

In contrast to these works, DCPT as proposed here utilizes directly the phase information of the displacement current in the electroquasistatic (EQS) regime [84]. to provide information about the distribution of the loss factor (or loss tangent) within the RoI. An attractive feature of DCPT is that the relationship between the measured phase and the loss factor inside the RoI has a more extended linear range than
the relationship between the measured capacitances in ECT and the permittivity distribution. As illustrated below, this is especially advantageous in reconstructing multiphase flows where the continuous phase is water. Since DCPT can be implemented using the same hardware as used for ECT, DCPT acquisition can be employed alongside ECT acquisition to provide additional data for reconstruction purposes. We also note that DCPT does not require any electrical contact with the RoI as opposed to EIT, for example. In this dissertation, we compare DCPT and ECT reconstruction results using the Landweber algorithm and identical sensor configurations. We include both numerical examples and experimental results.

2.1 ECT and DCPT

2.1.1 Conventional ECT sensitivity matrix

Despite their advantages, ECT systems have a number of design challenges, which can be classified into three main aspects: sensor design, acquisition hardware design, and image reconstruction algorithm. Extensive studies on sensor and hardware designs can be found in literature [85]. The challenges on reconstruction algorithm will be briefly restated here for understanding the potential added value that DCPT offers. A typical ECT system consists of a number of capacitive electrodes surrounding the region to be imaged. The basic principle of ECT system relies on the measurement of the mutual capacitances between every electrode pair to reconstruct the permittivity distribution inside the region. In practice, this is done by applying a voltage excitation on a given electrode and measuring the current on the remaining (grounded) electrodes. This process is repeated for all electrodes. Due to reciprocal nature of the problem, for an ECT hardware with \( N_e \) electrodes, there are \( N_m = N_e(N_e - 1)/2 \).
independent measurements. The ECT reconstruction problem is severely underdetermined as the typical number of pixels/voxels to be reconstructed in the RoI is much larger than the number of electrode pairs. A number of strategies can be used to ameliorate this problem by increasing the number of independent measurements, such as the use of adaptive (AECT) and reconfigurable sensors [60]-[86]. Although AECT combats this problem to some extent, the issue of ill-conditioning is still present. Another major challenge in ECT systems is the nonlinear nature of the reconstruction problem [87]. This is easily understood from the relation between the measured capacitance $C$ and the permittivity distribution $\epsilon(x, y)$ which writes as

$$C = \frac{Q}{V} = -\frac{1}{V} \int_S \epsilon(x, y) \nabla \phi(x, y) \cdot \hat{n} dS$$  \hspace{1cm} (2.1)$$

where $V$ is the applied voltage, $\phi(x, y)$ is the electric potential, $\hat{n}$ is a normal unit vector, and the surface integral is performed over the electrode area $S$. The nonlinearity comes from the fact that $\phi(x, y)$ depends on $\epsilon(x, y)$. A common linearization approach for this problem is to assume a Born approximation, i.e., to neglect variations on $\phi(x, y)$ due to perturbations on $\epsilon(x, y)$ [88, 89], i.e. $\phi(x, y)$ is assumed to remain equal to that of an empty RoI even in the presence of non-uniform material distribution. By discretizing the RoI into pixels (or voxels), perturbations on the pixel permittivities $\Delta \epsilon_j$, for $j = 1, \ldots, N_p$, where $N_p$ is the number of pixels, can then be linearly mapped to perturbations on the measured capacitances $\Delta C_i$, for $j = 1, \ldots, N_m$ via the sensitivity (Jacobian) matrix elements $S_{ij}$ [90, 91]:

$$\Delta C_i = \sum_{j=1}^{N_p} S_{ij} \Delta \epsilon_j$$  \hspace{1cm} (2.2)$$

Strictly speaking, the Born approximation for the ECT problem is only adequate when the permittivity fluctuations are small, either due to low contrast or to small
spatial extension of the permittivity perturbation. In particular, this approximation is not expected to remain accurate in the presence of high permittivity materials such as water, when these comprise a significant area (or volume) fraction of the RoI. In the next section, we discuss DCPT properties in more detail and how they can be used to help alleviate this limitation.

2.1.2 Admittance between electrode terminals

Most fluids, including water, have dielectric responses with both real and imaginary components. From the general form of time-harmonic Ampere’s law, we can write [79, 80]

\[
\nabla \times \mathbf{H} = j \omega \mathbf{D} + \mathbf{J} = j \omega \epsilon \mathbf{E} + \sigma \mathbf{E} \\
= j \omega (\epsilon' - j \epsilon'') \mathbf{E} + \sigma \mathbf{E} \\
= j \omega \left[ \epsilon' - j (\epsilon'' + \frac{\sigma}{\omega}) \right] \mathbf{E}
\]

where \( \mathbf{H}, \mathbf{D}, \mathbf{J}, \mathbf{E} \) represent the magnetic field, the electric flux density, the current density, and the electric field, respectively, and it is seen the imaginary component of the permittivity (loss) arises from the presence of a conductivity \( \sigma \) in (2.3) and/or a dielectric damping factor loss \( \epsilon'' \) given in (2.4) [79]. Note that magnetic effects will be omitted within the scope of this dissertation. In fact, ECT acquisitions typically rely on induced voltages on sensor plates with excitation frequencies below a few MHz, which implies the electroquasistatic field (EQS) regime [84, 57]. So, in what follows we consider a time-harmonic excitation in the EQS regime.
The electrical properties of the medium in the RoI can in general be characterized by means of a effective permittivity as

\[ \epsilon = \epsilon' - j(\epsilon'' + \frac{\sigma}{\omega}) = \epsilon' - j\epsilon' \tan \delta, \]  

(2.6)

where \( \tan \delta \equiv \epsilon'' / \epsilon' + \sigma / (\omega \epsilon') \) is the so-called loss tangent [79, 80]. Note that the equation above refers to the absolute permittivity, not the relative one. During each mutual capacitance measurement, only two electrode plates are effectively being used: one (sender plate) that is excited with a reference input voltage and the other (receiver plate) where the output current is being measured. Therefore, it is convenient to consider the RoI between these two electrodes as a 2-terminal device for each separate measurement. Under a linear approximation and assuming a uniform permittivity distribution in the RoI, the relation between the admittance \( Y \) between the terminals and a uniform permittivity distribution can be expressed in the generic form \( Y = j\omega k_g \epsilon \), where \( k_g \) is a parameter depending on the frequency of operation and the particular plate arrangement and geometry. For example, in the case of a uniformly filled parallel-plate capacitor, \( k_g = A / d \), where \( A \) is the area of the plates and \( d \) is their separation. However, this formula is only valid if the electric field is assumed uniform within the plates and fringing field effects are ignored (which is a good approximation only if \( d \) is very small). In practice, these assumptions may break down and the general relation between \( Y \) and the permittivity distribution becomes nonlinear. In particular, nonlinear effects in ECT ensue because, among other reasons, fringing field effects cannot be ignored given the distance between the electrode plates. In the following, we provide an analysis of the admittance based on the use of complex power to meets two objectives: (1) to help identify at which stages of the derivation of the
ECT sensitivity matrix a linear approximation is invoked and (2) to show that, under a similar approximation, DCPT can employ the same sensitivity matrix as ECT.

In a generic 2-terminal device, we have

$$P_c = \frac{1}{2} VI^*$$  \hspace{1cm} (2.7)

where $P_c$ is the complex power (the imaginary part of which represents the reactive power), and the admittance of the system can be written as

$$Y = \frac{P_c^*}{|V|^2/2} = \frac{2}{|V|^2} \left[ \langle P_d \rangle + j2\omega\langle W_e \rangle \right],$$  \hspace{1cm} (2.8)

where

$$\langle P_d \rangle = \int_V \frac{1}{2} \bar{E} \cdot \bar{J}^* \, dv$$  \hspace{1cm} (2.9)

and

$$\langle W_e \rangle = \int_V \frac{1}{4} \epsilon |\bar{E}|^2 \, dv,$$  \hspace{1cm} (2.10)

are integrals over the RoI volume $V$ representing the average dissipated power and the average stored energy over one cycle, with $|\bar{E}|^2 = \bar{E} \cdot \bar{E}^*$. In the two-dimensional scenario, these integrals become area integrals over the cross-section of the sensor.

Assuming EQS, we ignore the energy stored in the (negligible) magnetic field. The admittance of the system can be rewritten in terms of the loss factor $\varrho \equiv \sigma + \omega \epsilon''$ as

$$Y = \frac{1}{|V|^2} \left[ \int_V \varrho |\bar{E}|^2 \, dv + j\omega \int_V \epsilon' |\bar{E}|^2 \, dv \right]$$  \hspace{1cm} (2.11)

As a result, the phase of the admittance between plates can be expressed as $\pi/2 - \varphi$ with $\varphi$ being a small angle perturbation written as

$$\varphi = \tan^{-1} \left( \frac{\int_V \varrho |\bar{E}|^2 \, dv}{\int_V \omega \epsilon' |\bar{E}|^2 \, dv} \right)$$  \hspace{1cm} (2.12)
Next, we will describe the evaluation of the sensitivity matrix based on $\varphi$. Because the current being measured in (2.7) is primarily a *displacement current* (i.e. produced by the term $j\omega D$ in Ampere’s law and not as a conduction current), it is appropriate, as noted before, to denote this proposed imaging modality as a displacement-current phase tomography.

### 2.1.3 Sensitivity matrix based on displacement current phase

The Born approximation can be applied to establish a relationship between the loss factor $\varrho$ and the measured phases for each electrode pair. Since $\tan^{-1} \alpha \cong \alpha$ for small $\alpha$, the equation (2.12) can be simplified to

$$
\varphi = \frac{\int_{V} \varrho |E|^2 \, dv}{\int_{V} \omega \epsilon |E|^2 \, dv}
$$

(2.13)

if low losses are assumed. Accordingly, the variation of on the phase due to a perturbation over a small volume $\delta V$ (voxel in three-dimensions or pixel in two-dimensions) in the RoI can be written as

$$
\delta \varphi = \frac{\int_{\delta V} \delta \varrho |E|^2 \, dv}{\int_{\delta V} \omega \epsilon |E|^2 \, dv + \int_{(V-\delta V)} \omega \epsilon |E|^2 \, dv}
$$

(2.14)

Since $\delta V \ll V$ in this small angle regime, the second term in the denominator can be neglected and the above equation simplified to

$$
\delta \varphi = \frac{\delta \varrho \int_{\delta V} |E|^2 \, dv}{\int_{V} \omega \epsilon |E|^2 \, dv}
$$

(2.15)

where the perturbation $\delta \varrho$ is assumed uniform over the voxel/pixel $\delta V$ and hence can be moved outside the integral. Equation (2.15) defines the sensitivity matrix for DCPT, by expressing the change in the phase factor caused by a perturbation on the loss factor in a voxel/ pixel $\delta V$ inside $V$. Traditional ECT hardware utilizes phase-sensitive demodulation technique, therefore both magnitude and phase information of
the equivalent lumped admittance of each measurement can be obtained through this circuitry [92, 93]. In other words, the phase in l.h.s. of this equation can be obtained directly from the admittance measurement using ECT hardware, so no extra hardware is necessary. The denominator of this equation does not provide extra information after row normalization of sensitivity matrix and the volume integral appearing in the numerator of (2.15) recovers the expression of the conventional ECT sensitivity matrix [91]. The final formulation for DCPT can be expressed in a form similar to (2.2) as

$$\Delta \varphi_i = \sum_{j=1}^{N_p} S_{ij} \Delta \varrho_j$$

Because of this similarity, ECT reconstruction approaches can be used for DCPT by replacing the capacitance variation with the admittance phase variation and the dielectric perturbation on a voxel/pixel by a loss factor perturbation. Before proceeding, we should again stress that the above relation is predicated on the Born approximation, which assumes small perturbations on the loss factor $\varrho$. As the perturbation on the loss factor in the RoI increases, this assumption will cease to be valid. In particular, an increase on frequency of operation will increase $\varrho$ and decrease the skin depth [79]. If the latter becomes smaller that the RoI dimensions, the imaging is not expected to be accurate.

### 2.2 Numerical Examples

The problem considered here consists of two main parts, the forward problem and the inverse (reconstruction) problem. The forward problem focuses on obtaining mutual capacitance (or admittance) values between electrode pairs for a given RoI, with or without material inclusions. This part of the simulation is conducted
using the commercial-grade finite element (FEM) solver COMSOL™. To simulate a time-harmonic AC signal in the EQS regime, the COMSOL™ AC/DC Module is used. Once the unknown nodal voltage values are calculated, the dependent current density parameter can be evaluated everywhere inside the domain. After the current values are calculated through the integration of current density values on the electrode terminals, this module of COMSOL™ provides admittance system matrix for all the terminals as global lumped parameter. The admittance values from each sensor pair are extracted through an automated Java-language script controlling the FEM solver for the simulation of three cases: the first case corresponds to the object in the given background comprising the RoI, the second case corresponds to an uniform background without object (background-filled RoI), and the third case corresponds to the a uniform background set to a permittivity value equal to that of the object (“full” RoI). The latter two cases are used for capacitance normalization purposes.

The mutual capacitance are associated to the imaginary part of the admittance values between two electrodes. The admittance phases are related, on the other hand, to material loss factors inside the RoI as discussed in the previous Section. A standard normalization procedure is adopted to obtain the ECT reconstructed image from the measured set of mutual capacitances [94, 95]. The second part (inverse problem) of the problem deals with reconstruction of images using the capacitance or phase values obtained in the first part. This part of the simulation is conducted based on a commercial-grade Matlab™ software implementing the Landweber iterative reconstruction algorithm with optimal step length [96]. This reconstruction method is coupled with a projection filter as regularization approach [94]. The choice for the iteration number for each image is done empirically as the Landweber method shows
semi-convergence characteristics [59]. In the results shown below, the actual air/water boundary is represented by a solid circle in each figure to assess the quality of the reconstruction. ECT images are mapped back to permittivity values such that the colorbar values correspond to the relative permittivity (unitless quantity). This mapping can be done given a priori knowledge of the permittivity of the material phases inside the RoI. Since DCPT does not use an equivalent mapping at the present, the unit of the colorbars are expressed directly in terms of radians, which are nevertheless associated to the material loss distribution inside the RoI.

2.2.1 Single circular object with varying radii

This numerical example aims at providing a comparative assessment of ECT and DCPT for the imaging of circular-shaped objects (corresponding to columnar and annular flows) of different radii placed inside the RoI. The examples covers the cases of water-in-air (columnar water flow) and vice-versa (annular water flow) shown in Fig.2.1(a) and Fig.2.2(a). We assume distilled water with $\epsilon_r = 80$ and $\sigma = 5.5 \times 10^{-6}$ S/m. As illustrated in Fig.2.1-(a), the ECT setup used in this simulation consists of 12 electrode plates separated by angular gaps comprising 25% of the plate angular extension. The sensors are placed within a dielectric wall with $\epsilon_r = 3$. The (circular) region of interest comprises 2,365 pixels out of the $64 \times 64$ pixels used in the RoI discretization. The sensors are placed around a circular domain of 10 cm radius with the wall thickness of 0.5 cm making the RoI a circular domain of 9 cm radius. The frequency of operation is 300 kHz. ECT and DCPT reconstruction results are shown in Fig.2.1(b),(d),(f) and Fig.2.1(c),(e),(g) respectively where first column represents ECT results and the second column DCPT results. Since the permittivity contrast
between air and water is high, the ECT results seen in Fig. 2.1(b), (d), (f) are affected by the nonlinearity of the problem, which distorts the shape of the reconstructed object; in particular, the center of each reconstructed image is shifted towards the nearest sensor location (where the field is stronger). The permittivity values predicted by the ECT images decrease with the radius of the object. The DCPT results, on the other hand, show a more linear behavior. This point will be further discussed below.

In the second case, we consider air-in-water, as shown in Fig. 2.2(a). The corresponding ECT results in Fig. 2.2(b), (d), (f) shows that the imaging of air region surrounded by water by ECT is very challenging [97, 98, 99, 100]. The contrast of each ECT image decreases; therefore, it cannot be deduced whether the inner region corresponds to a mixture of different materials or is truly a homogeneous one. The location of the object on the other hand is successfully identified for all radii considered. DCPT results on the other hand show better contrast and location prediction. Compared to ECT, DCPT shows more stable behavior with respect to changes in the inner region area. DCPT reconstructed values are larger in the air region rather than in the lossy water periphery, just as in the ECT case. The reason for capturing the air region rather than lossy material is due to the different normalization processes: while ECT results are normalized with fill and empty conditions as noted above (which can be viewed as a calibration process prior the actual measurement), in DCPT the only reference are the background values. Although DCPT is quite accurate in predicting the spatial distribution of the image, quantitative information should be complemented from ECT imaging in this case and in the DCPT plots the colorbar convention is inverted to match the ECT profile convention.
Figure 2.1: Comparison of ECT and DCPT for air-background columnar flow: (a) Simulation setup. (b) (d) (f) ECT results for objects of radius 3 cm, 4 cm, and 5 cm, respectively. (c) (e) (g) DCPT results for objects of radius 3 cm, 4 cm, and 5 cm, respectively.
Figure 2.2: Comparison of ECT and DCPT for water-background columnar flow: (a) Simulation setup. (b) (d) (f) ECT results for objects of radius 3 cm, 4 cm, and 5 cm, respectively. (c) (e) (g) DCPT results for objects of radius 3 cm, 4 cm, and 5 cm, respectively.
2.2.2 Two circular objects

The next numerical example aims to compare ECT and DCPT results for imaging two circular object placed inside different backgrounds (air-in-water and vice-versa). All other simulation parameters are kept the same as before. The reconstructed images for water-in-air case are shown in Fig.2.3(b),(c). Similar to before, ECT shows more nonlinear behavior, with the center of the inner region shifted towards the periphery of the RoI. DCPT shows better performance on predicting the location and shape of the inner region. Fig.2.4 shows the air-in-water case, which is very challenging for ECT because the high permittivity of water masks the inner air region.
As a result, the ECT image is very blurred. On the other hand, the DCPT image shows a sharper reconstruction.

Figure 2.4: Comparison of ECT and DCPT results for two 3 cm radius circular air-in-water regions: (a) Setup of the model. (b) ECT result. (c) DCPT result.

2.2.3 Stratified flow

The next example considers a stratified model of air-water flow, as shown in Fig.2.5(a). The left-hand side of the RoI is filled with water while the right-hand side of the RoI with air. The sensor has the same geometry and the Roi has the same dimensions as before. Again, the iterative Landweber algorithm is applied. The reconstructed results in Fig.2.5(b),(c) shows that ECT obtains a better image quality
in this case. Since DCPT is still in preliminary stage, more optimum normalization approaches could improve its performance in this scenario.

Figure 2.5: Comparison of ECT and DCPT results for stratified air-water flow: (a) Model setup. (b) ECT result. (c) DCPT result.

2.2.4 Nonlinearity effects in ECT and DCPT

Image reconstruction techniques used for ECT are based on a linearization of the dependency between the permittivity and the capacitance perturbations, as expressed by (2.2). In reality this relation is nonlinear and deviations from the linearity assumption affect the image quality. These deviations become larger when the Born approximation assumptions cease to valid, i.e. when the permittivity fluctuations
are large in magnitude and/or comprise a large spatial extent of the RoI. This section aims at comparing the deviations from the linearity in ECT measurements and DCPT measurements. This is done by considering water-in-air scenarios in which the radius of the circular zone where the water \((\epsilon_r = 80 \text{ and } \sigma = 5.5 \times 10^{-6} \text{ S/m})\) exists is gradually increased from 0.1 m to 0.7 m, as depicted in Fig. 2.6. The remaining parameters are the same as in the previous section. When the radius is small, the Born approximation is valid and the linear approximation is accurate. As the radius is increased, we should expect increasingly larger deviations from the linear behavior. The normalized capacitance \(\frac{|C_{\text{obj}} - C_e|}{|C_f - C_e|}\) and current phase \(\frac{\varphi_{\text{obj}} - \varphi_e}{|C_f - C_e|}\) are plotted with respect to object area as shown in Fig. 2.7-(b),(c). Here, \(C_{\text{obj}}\) is the measured capacitances between plates, while \(C_e\) and \(C_f\) are the capacitances assuming the RoI completely filled with air or water, which is a typical normalization used in ECT. In addition, \(\varphi_{\text{obj}}\) is the measured current phase with the water present in the RoI and \(\varphi_e\) is the measured phase in an empty RoI. Since the channel capacitance values have different orders of magnitude, each channel results is subsequently normalized by its peak value. This is also done for the inner circular area so that the traces are mapped to the unit square. Channels 3, 4, 5, and 6 are chosen in this study since the sensitivity maps of these channels are (spatially) nearly uniform near the center of the domain. On the other hand, channels 1 and 2 have sensitivity maps with more abrupt spatial variation across the circular zones of different radii, which would mask the dependency on the spatial extension of the water zone. As a measure of nonlinearity, the areas between the measurement curves and the linear trace as illustrated in Fig. 2.7-(a), which define a “linearity deviation parameter” denoted as \(\zeta\), are calculated and listed in Table 2.1. It can be seen that \(\zeta\) is larger in the ECT
case than in the DCPT case, which supports the observations made in the previous Section.

![Simulation setup for analysis of the linearity deviation in capacitance and phase measurements.](image)

Figure 2.6: Simulation setup for analysis of the linearity deviation in capacitance and phase measurements. (a) A circular region of water exists at the center of the RoI, otherwise filled with air. The radius of the center region is gradually increased. (b) Channel numbering adopted for the mutual capacitance measurements.

It is also illustrative to look at the difference between the actual capacitance and phase values and those provided by the linear estimates (2.2) and (2.16). For that, the values of $|C_{est} - C_{meas}|/|C_e|$ and $|\varphi_{est} - \varphi_{meas}|$ are plotted versus object area. Here, $C_{est}$ is the estimated capacitance value using (2.2), whereas $C_{meas}$ and $C_e$ are the actual capacitances computed with a finite element algorithm considering the RoI with and without a water-filled region (object), respectively. In addition, $\varphi_{est}$ is the estimated current phase using (2.16) and $\varphi_e$ is the actual phase computed using finite elements. The plots shown in Fig.2.8-(a), (b) corresponds to ECT and DCPT results for a inner region with material properties $\epsilon_r = 3$ and $\sigma = 5.5 \times 10^{-6}$ S/m. The plots shown in Fig.2.8-(c), (d) corresponds to ECT and DCPT results with same conductivity but $\epsilon_r = 80$. In this case, the deviation from nonlinearity becomes larger in ECT due to a higher $\epsilon_r$. In contrast, the deviation becomes smaller in DCPT since
Figure 2.7: Mutual capacitance and current phase measurements for channels 3 to 6. (a) Mapping rule from which the area under the curve (denoted as ζ) serves to quantify the amount of deviation from the linear behavior. (b) Mapped capacitance values. (c) Mapped current phase values.
the loss tangent decreases. Finally, Fig.2.9-(a), (b) shows the nonlinearity deviation of the DCPT results for tap water ($\epsilon_r = 80, \sigma = 5 \times 10^{-3}$ S/m) and sea water ($\epsilon_r = 80, \sigma = 5$ S/m), respectively. These results show higher deviation compared to pure water case due to higher losses, as expected. However, they are still in acceptable range and not much larger than the previous results for $\sigma = 5.5 \times 10^{-6}$ S/m. This shows that the linearity of the phase data remains valid over conductivity (or, equivalently, imaginary part of the permittivity) values spanning several order of magnitude.

<table>
<thead>
<tr>
<th>Channel #</th>
<th>ECT</th>
<th>DCPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.168</td>
<td>0.114</td>
</tr>
<tr>
<td>4</td>
<td>0.104</td>
<td>0.043</td>
</tr>
<tr>
<td>5</td>
<td>0.079</td>
<td>0.035</td>
</tr>
<tr>
<td>6</td>
<td>0.073</td>
<td>0.036</td>
</tr>
</tbody>
</table>

2.3 Experimental Results

We have conducted a series of experiments to verify the feasibility of DCPT for imaging within lossy media. Our ECT hardware utilizes a phase-sensitive demodulation technique [92] based on an AC system [93] that can extract both magnitude and phase information of the admittance. The ECT hardware includes 12 electrode plates of 1.5 cm width and 5.08cm length placed around a 11.43 cm outer diameter vessel, as shown in Fig.2.12. The vessel dielectric wall has 0.32 cm thickness, so that the diameter of the circular RoI is 10.79 cm. The experiments are conducted
Figure 2.8: ECT versus DCPT in terms of deviation from the linear regime: (a) ECT deviation for object with $\epsilon_r = 3$ and $\sigma = 5.5 \times 10^{-6}$ S/m. (b) DCPT deviation for object with $\epsilon_r = 3$ and $\sigma = 5.5 \times 10^{-6}$ S/m. (c) ECT deviation for object with $\epsilon_r = 80$ and $\sigma = 5.5 \times 10^{-6}$ S/m. (d) DCPT deviation for object with $\epsilon_r = 80$ and $\sigma = 5.5 \times 10^{-6}$ S/m.
Figure 2.9: DCPT deviation from the linear regime: (a) DCPT deviation for object with $\epsilon_r = 80$ and $\sigma = 50 \times 10^{-3}$ S/m. (b) DCPT deviation for object with $\epsilon_r = 80$ and $\sigma = 5$ S/m.

at 2 MHz with a 500 fps data acquisition system. A plastic tube made of a thermoplastic polymer, acrylonitrile butadiene styrene (ABS), with $\epsilon_r \sim 2.75 \pm 0.75$ and $\tan \delta \sim (12 \pm 7) \times 10^{-3}$ is placed in the presence of tap water in the RoI as depicted schematically in Fig.2.11.

This type of scenario is especially challenging for ECT because the high permittivity of the water effectively screens the low permittivity of the ABS tubes inside it. We will show that DCPT is able to provide more satisfactory images in this case. The data acquisition system was first calibrated using an empty RoI. The RoI was later filled with tap water for a second set of measurements. After the calibration process, the actual experiment was conducted for two different ABS tubes with diameters 3.81 cm and 4.45 cm. These tubes were placed inside the domain with the help of a 3D printed holder as shown in Fig.2.12(b) (in this figure, the 3D printed holder is shown for 4.45 cm diameter tube only, another 3D printed holder was prepared for
the 3.81 cm diameter ABS tube. Each ABS tube was placed at the center and at four different positions along the periphery of the RoI, and the corresponding capacitance and phase data was collected. The ECT hardware was operated at 500 fps and each capacitance and phase data was averaged over 1 sec (500 frames). The maximum coefficient of variation, $c_v = \sigma/|\mu|$ where $\sigma$ is the standard deviation and $\mu$ is the mean.
value of the data, for the obtained phase values (for all experiments) is $c_v \approx 4\%$.

This amount of deviation from the mean values among all of these frames indicate that ECT hardware provided stable data. Following the normalization procedure [95] explained in Section II, the images were reconstructed using the iterative Landweber algorithm along with a projection filter as before.

The results are shown in Fig. 2.10, where first two rows show DCPT results for the tubes with diameters 3.81 cm and 4.45 cm, respectively, and last two rows show the respective ECT results. The colorbar values are larger compared to simulation cases due since the measured phases are expressed in degrees and because of the different domain sizes. Although ECT has difficulty in imaging this problem because of the high permittivity of the background, which, as noted above, screens the air region, DCPT can still provide good images.

![Figure 2.11: Experimental reconstruction of an ABS object placed in water.](image)

Figure 2.11: Experimental reconstruction of an ABS object placed in water.
2.4 Summary

We provided a comparison between ECT and DCPT for imaging of lossy media based on the iterative Landweber algorithm. Losses in the media can be the combination of conductive losses or dielectric absorption losses. DCPT provides an useful alternative technique for reconstruction of flows that provide information in the phase on the admittance measured between electrode pairs. One attractive feature of DCPT is that it can be based on the same hardware as ECT. As a result, DCPT can be easily employed either as a standalone or as a supplementary modality to ECT for lossy media imaging. DCPT provides a more extended linear regime for the relationship between the dielectric losses and the measured admittance phase than the equivalent relationship in ECT between permittivity and the measured capacitance values, for
typical values of these quantities encountered in practice. This feature makes it especially useful for applications involving high permittivity media, such as water, which is far off from the linear regime of ECT. Simulations of air bubbles in water and experiments involving plastic objects in water have shown the advantages of DCPT versus ECT in scenarios where the water in the continuous background phase. On the other hand, ECT have yielded a better performance than DCPT for stratified air-water scenarios. Presently, DCPT has a lack of absolute reference information and hence cannot distinguish the absolute level of the tangent loss in the reconstructed image; however, this information can be inferred from ECT data based on the same hardware and inserted into the DCPT reconstruction algorithm or estimated from the measurement of the total power loss in the RoI. For highly conductive media and sufficiently high frequencies, the skin depth may become too small to allow for sufficient field penetration into lossy media. In this case, DCPT imaging will deteriorate. It should be noted that ECT also has limitations at higher frequencies since the EQS regime may cease to remain valid. As a possible future work, DCPT can be evaluated for joint inversion algorithms with ECT. In addition, calibration approaches can be devised to provide absolute reference data in DCPT and improved sensitivity maps and/or reconstruction algorithms can be tailored to correct for nonlinear effects in DCPT problems with higher losses.
Chapter 3: DCPT Velocity Profiling

Displacement-current phase tomography is a developing soft-field imaging modality used for 2D spatial loss distribution imaging for a given RoI [52]. Using electrical capacitance tomography (ECT) hardware, DCPT supports all the popular features of ECT such as non-invasive, non-intrusive operation with relatively fast reconstruction time along with inexpensive, lightweight, sturdy hardware appropriate for industrial applications. Being a relatively immature tomography that exploits the admittance phase information [75, 76], DCPT does not have as widespread application span as ECT which can be used in cryogenic two phase flow imaging [101], presymptomatic disease detection for plant roots [102], monitoring moisture in cement-based materials [103], bubbling and slugging detection in fluidized beds [104] and other fields [105, 106, 107]. However, DCPT holds a great potential for future development since it extracts complementary material property information when compared with ECT from the same hardware. Fundamental operation for ECT relies on the perturbation on the measured capacitance due to perturbation on the spatial distribution of electric permittivity for a given RoI. DCPT, on the other hand relies on the perturbation on the measured displacement-current phase due to perturbation on the spatial distribution of loss. This lossy nature under the electroquasistatic (EQS) excitation could come from the material conductivity or electric permittivity loss of the RoI
Although, both for ECT and DCPT, these relationships between boundary measurements and interior material property content of RoI are nonlinear by nature, DCPT supports a more extended linear range as compared to ECT one for the aforementioned perturbations. Therefore, for flow distributions with lossy material inclusions, DCPT supports larger set of cases in comparison with ECT that still falls within linear Born approximation. This feature is especially useful for challenging problems for ECT such as imaging low permittivity materials in the presence of a high permittivity background.

Electrical capacitance volume tomography (ECVT) is the natural extension of ECT into 3D volume tomography case with the additional dimension information is obtained with the help of multi-layer sensor formations surrounding RoI \[49\]. ECVT is gaining more attention from the scientific community due to popular benefits of ECT translated into the vast applications of volume tomography. These applications cover a variety of use in petroleum, chemical and biochemical industries \[108, 109, 110\]. In addition to volume tomography use, velocity profiling of multi-phase flows can also be conducted with these systems, where each discretization within the domain is assigned with a velocity vector. This transient information of the domain under investigation is as valuable as the single step tomographic image for the understanding of the system under investigation \[111\]. The early approaches on velocity profiling with ECT and ECVT used cross correlation of the output images between time frames \[112, 113\]. However, the drawback of this method is its computational requirements, which limits the real time flow use of this technique. For this front, a recent novel method that uses sensitivity matrix gradient is proposed for ECVT systems \[114\]. Sensitivity matrix is the pre-computed Jacobian matrix for ECT/ECVT systems that maps
variation in the spatial distribution of electric permittivity values to variation in the mutual capacitance [115, 90]. By utilization of the different time frame measurement data packs, displacement of the spatial electric permittivity can be related with the gradient of this linear sensitivity matrix. And with the time information obtained from the frame rate, velocity vectors at each point in the discretized RoI can be determined.

In this chapter, we extend DCPT into 3D for volume tomography and use velocity profiling of multiphase flows as case study. Based on the same approach as ECVT velocity profiling, sensitivity gradient mapping can be used to relate alteration in the spatial distribution of material loss to the alteration of the measured displacement-current phase. And along with time-frame information, velocity profiling of flows can be obtained as it was conducted for ECVT case. Since these types of velocity reconstruction is based on the performance of the soft field tomography, the advantages of DCPT in comparison with ECT can be directly transferred to the velocity profiling applications.

In subsequent sections, derivation for the forward problem of DCPT velocity profiling is presented. The solution of the corresponding inverse problem is demonstrated and implemented with Landweber iteration. Simulation results from COMSOL are presented along with experimental results as further verification for the proposed method.
3.1 Formulation

3.1.1 Displacement-Current Phase Tomography

DCPT imaging is a two segment operation consisting of data collection and data manipulation. In the former segment, traditional computer controlled ECT hardware is utilized to collect boundary measurements from a RoI, where this step is referred as forward problem. In the latter segment, the collected boundary data is fed to a chosen reconstruction algorithm to obtain the interior spatial material loss distribution images for the RoI, where this step is referred as the inverse problem. Data collection part of the procedure is achieved by sensing electrodes surrounding the RoI for non-invasive, non-intrusive measurements. These non-contact sensing electrodes typically wrap the RoI to form a single layer arrangement with equidistant gaps in between \[89, 47\]. In addition, the hardware supports the flexibility of planar or curved sensor designs, resulting in ease of implementation to fit complex RoI structures partially or fully to collect boundary measurements \[54\]. The implementation of the forward problem is achieved by activation of a particular sensing plate with AC voltage and measuring the resulting displacement current by another sensing plate while all the other sensing plates are grounded. Typical operation frequency is less than 10\(MHz\), resulting in EQS regime and hence the name soft-field tomography for this imaging modality \[64\]. This process is repeated until all the possible sensor pairs are scanned and due to reciprocity of the problem this results in \(N_p(N_p - 1)/2\) number of independent measurements for a ECT hardware composed of \(N_p\) different sensing plates.

In the previous works, DCPT is implemented as cross section modality, meaning that all the sensors arranged on a single layered design and displacement-current phase measurements were related with the 2D spatial loss distribution.
For the volume tomography version of DCPT, additional layers of sensing electrodes are necessary for the third dimension info on spatial distribution of material loss, which can be accomplished using ECVT hardware. Since the collected data is structured by the repetition of the same process both for 2D and 3D imaging with DCPT, it is sufficient to observe one of those operations where only two different sensors are relevant. In fact, these two sensors responsible from mutual admittance measurements can be modeled as a two port circuit. Therefore the mutual admittance $Y$ of each measurement under the voltage excitation of magnitude $V$ and angular frequency of $\omega$ for a RoI with volume of $V$ can be expressed as [52]

$$Y = \frac{1}{|V|^2} \left[ \int_V \theta |E|^2 \, dv + j \omega \int_V \epsilon' |E|^2 \, dv \right], \quad (3.1)$$

where the integrals represent average dissipated power due to loss factor $\theta$ and average storage energy due to electric permittivity $\epsilon'$ within resultant the electric field distribution of $E$. Notice that $\theta \equiv \sigma + \omega \epsilon''$ includes both material conductivity $\sigma$ and dielectric loss $\epsilon''$ given that general electric permittivity can be represented by $\epsilon = \epsilon' - j \epsilon''$. This equation holds true within EQS regime and without the presence of magnetic material properties. With the eq. (3.1), the phase of the admittance can be expressed as $\pi/2 - \varphi$ where $\varphi$ is a small angle perturbation written as

$$\varphi = \tan^{-1} \left( \frac{\int_V \theta |E|^2 \, dv}{\int_V \omega \epsilon' |E|^2 \, dv} \right). \quad (3.2)$$

The operational principle of DCPT relies on the eq. (3.2), which allows the observation of two immediate limitations. These constraints are nonzero operational frequency and presence of loss, otherwise the measured admittance phase will be constant at 0 and $\pi/2$ respectively. However, real life flows generally have loss associated with
them and various excitation frequency based operation for both ECVT and DCPT is getting more important [116, 83].

When it comes to the data manipulation segment of the DCPT imaging, it is desired to simplify the eq. (3.2) into a matrix equation. Common approach is to follow Born approximation, which allows the perturbed term to be taken out of the integral under the assumption that perturbation on this term is small. The significance of this assumption is to decouple the nonlinear relationship between the loss factor present and resultant electric field distribution in the RoI due to AC excitation. This form of linearization assumes that within each small voxel, the loss factor is constant and any perturbation on the loss factor of this voxel does not disturb the electric field distribution in and around it. From a practical perspective, as the perturbation on the loss factor in the RoI increases, this assumption will cease to be valid. In particular, an increase in the loss factor \( \rho \) due to increase on the frequency of operation will decrease the skin depth. And the imaging is not expected to be accurate if the skin depth becomes smaller than the RoI dimensions. After this linearization step, proper discretization process allows eq. (3.3) to be reduced to the matrix equation written as

\[
\varphi = S \delta,
\]

where the perturbation on the measured phase vector \( \varphi \) due to the perturbation on the loss factor vector \( \delta \) present in the RoI is related through the so called sensitivity matrix \( S \). The reverse problem can then be defined as the searching for the unknown spatial loss factor distribution vector \( \delta \) given mutual admittance phase measurement vector \( \varphi \) and pre-calculated sensitivity matrix \( S \). The reverse problem also has its own challenges as the system matrix is ill-posed and ill-conditioned [117]. To solve this
matrix equation, there are different reconstruction algorithms to consider including single step, iteration based and nonlinear optimization methods [59]. For the scope of this chapter, iterative Landweber method (ILM) will be used since it is widely accepted and exploring a new reconstruction algorithm is not the scope of this method. ILM method can be written as [59]

\[
\delta^{k+1} = \delta^k + \vartheta_k S^t (\varphi - S \delta^k) \quad k \geq 0 ,
\]

\[
\delta^{(0)} = S^t \delta ,
\]

where the superscript \( k \) indicates iteration number. The initial guess \( \delta^{(0)} \) for the ILM method is the linear back projection (LBP) solution. For this solution \( S^t \) transpose of the matrix \( S \) is simply multiplied with the normalized measurement vector. The optimal step size \( \vartheta_k \) for the update procedure can be written as [96]

\[
\vartheta_k = \frac{\| S^t \delta^{(k)} \|^2}{\| S S^t \delta^{(k)} \|^2} .
\]

It is also a common practice to incorporate projection operation to increase stability of solution.

### 3.1.2 Velocity Reconstruction with DCPT

In this section DCPT will be formulated for velocity profiling applications of multiphase flows. The objective is to find the velocity vector at each voxel within the RoI by incorporating the sensitivity gradient with successive time frame data packages. This operation will follow the same process outlined in the previous section yet instead of finding the spatial loss factor distribution (contrast magnitude value), the velocity profile (vector value) of this distribution will be studied. The implementation starts by focusing on a perturbation on the voxel-\( \alpha \) and relocating this perturbation.
component to voxel-β between two successive time frame data sets. Following the standard normalization process for collected data, sensitivity matrix and the loss factor contrast, the relationship between the variation in the loss factor and resultant variation in the measured phase can be written as

\[ \varphi_{t_1} = S_\alpha \delta_s, \]  

(3.7)

where \( \varphi_{t_1} \) shows the change in the phase value at the time frame \( t_1 \) due to small perturbation \( \delta_s \) on the loss factor at voxel-α. These values are connected by the sensitivity matrix value \( S_\alpha \) at the same location. If this perturbation is relocated to voxel-β at the time frame of \( t_2 \) and the corresponding equation can be written similar to above for that voxel and the difference between them can be expressed as

\[ \varphi_{t_2} - \varphi_{t_1} = (S_\beta - S_\alpha) \delta_s. \]  

(3.8)

Assuming a small displacement from voxel-α to voxel-β, the sensitivity value difference at those locations can be approximated by the gradient of the sensitivity matrix calculated at the initial position and therefore the above equation can be rewritten as

\[ \varphi_{t_2} - \varphi_{t_1} \approx (\nabla S_\alpha \cdot \mathbf{v}_{\beta\alpha}) \delta_s, \]  

(3.9)

where \( \nabla S_\alpha \) is the sensitivity gradient calculated at the initial location voxel-α and \( \mathbf{v}_{\beta\alpha} \) is the displacement vector for the perturbation of the loss factor from voxel-α to voxel-β. This equation above shows the idea of using the successive time data sets to deduce the displacement amount between each measurements. In fact, with the information of displacement amount for a given time difference, the unknown velocity \( \mathbf{v}_{\beta\alpha} \) can be calculated. Notice that this velocity component will have 3 dimensions for the case of volume tomography and within the scope of this dissertation cartesian
coordinate system will be used. To further clarify the above equation, loss factor contrast image $\delta_s$ can be combined with the sensitivity gradient calculation at the initial voxel-\(\alpha\) given as

$$\varphi_{t_2} - \varphi_{t_1} \approx (\delta_s \nabla S_{\alpha}) \cdot \mathbf{v}_{\beta \alpha}.$$  \hfill (3.10)

The equation above summarizes the velocity profiling of a loss factor distribution confined to voxel-\(\alpha\) and voxel-\(\beta\) at the time frames of \(t_1\) and \(t_2\). To adapt this equation for a general loss factor distribution inside the RoI, perturbation magnitude $\delta_s$ needs to be replaced with the loss factor distribution image vector $\delta_{t_1}$ at the time frame $t_1$. Therefore the relevant voxels for the sensitivity gradient will be decided according to this contrast image. Let gradient of the sensitivity matrix can be decomposed into cartesian coordinate dimensions as $\nabla S = \hat{x}F_x + \hat{y}F_y + \hat{z}F_z$ where $\hat{x}, \hat{y}, \hat{z}$ represents the unit vectors of cartesian coordinate components. Also define a matrix $G = [\delta_{t_1} \delta_{t_1} \ldots \delta_{t_1}]^t$ with the same size of the sensitivity matrix whose columns are stacking of contrast image $\delta_{t_1}$. Then the velocity profile equation for DCPT can be written as

$$\Delta \varphi = (G \odot F_x)\mathbf{v}_x + (G \odot F_y)\mathbf{v}_y + (G \odot F_z)\mathbf{v}_z,$$  \hfill (3.11)

where $\Delta \varphi = \varphi_{t_2} - \varphi_{t_1}$ is the change in the collected phase data between time frames and $\odot$ represents the element-wise multiplication of the two matrices. The above equation defines the forward problem of DCPT for the velocity profiling of multi phase flows where the unknowns vector components $\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z$ are related with the measured phase difference vector $\Delta \varphi$ between two time frames, pre-calculated sensitivity gradient $F$ and also the contrast image $G$ as a window function for the sensitivity gradient. For the inverse problem, Iterative Landweber Method can be
modified to be used with vector valued function where the step length will be dimension dependent for each vector however, residual for iteration will be dependent on the other dimensions as well. The use of ILM method for the velocity reconstruction is outlined in [114]. Notice that eq.(3.11) outlines the reconstruction of velocity based on two separate frames, however, the continuous operation will follow the same procedure where the schematics of this shown in Fig. 3.1.

![ECT and ECVT operational chart.](image)

**Figure 3.1:** ECT and ECVT operational chart.

### 3.2 Numerical Examples

We conducted a series of numerical simulations to verify DCPT implementation for 3D velocity profiling of multiphase flows described in the previous section. The solution of the forward problem for obtaining the mutual admittance values between sensing plates was found using the commercial-grade finite element (FEM) solver COMSOL™. For the inverse problem, proper normalizations of the small phase
angle vector and sensitivity matrix in eq. (3.3) were performed following the standard
approaches described in [52, 114]. A cylindrical RoI with 3 cm radius and 9 cm height
is considered for the simulation cases. An insulating wall made of a dielectric ($\varepsilon_r = 3$

Figure 3.2: Simulation setups for Frame-1 and Frame-2 are shown in the first and
second row, third row shows the reconstruction results. ILM for velocity reconstruc-
tion uses $[\vartheta_x, \vartheta_y, \vartheta_z] = [0.15, 0.15, 0.22]$. (a) Results produced with 30 ILM frame
iteration, 12 ILM velocity iteration. (b) Results produced with 30 ILM frame itera-
tion, 3 ILM velocity iteration. (c) Results produced with 30 ILM frame iteration, 7
ILM velocity iteration. (d) Results produced with 30 ILM frame iteration, 60 ILM
velocity iteration.
and $\sigma = 0$ S/m) with 0.3 cm thickness separates the RoI from the sensing plates. The RoI is surrounded by 4 rows with 6 square electrodes each, as shown in Fig. 3.2. Within each row, plate size to gap ratio is kept at 75% and between rows staggered sensor design is inherited, meaning that each row is off-centered by 30° as compared to adjacent rows. The lateral size of each sensor plate is 1.6 cm and there exist 0.867 cm of axial gap between adjacent plates. The plates are excited with 2 MHz voltage signal. The domain is discretized by a regular grid with $20 \times 20 \times 30$ voxels, resulting in 8281 voxels inside the cylindrical RoI. The dispersed phase (red sphere) is composed of air ($\epsilon_r = 1$ and $\sigma = 0$ S/m) and the continuous phase is composed of tap water ($\epsilon_r = 80$ and $\sigma = 5$ mS/m).

Four simulation examples consisting of three different cases for a single air bubble and one case for double air bubbles dispersed in the water domain are investigated. For the single air bubble, displacement in horizontal, vertical and combination of both are studied whereas for the double air bubbles case only vertical motion is considered. When it comes to the implementation, air bubbles in the tap water domain are modeled as spheres and for each of these four cases, two separate frames are created for the reconstruction of the velocity. These two frames, named as Frame-1 and Frame-2 are shown in the first and second row of Fig. 3.2 respectively. For these frame images, white arrows are added to emphasize the direction of simulated air bubble for easy reference. For the first experiment, an air ball of radius 2 cm is placed at $(0, 0, 0)$ [cm] and then relocated to $(0, 0, 1)$ [cm] as shown in Fig. 3.2(a). For the second and third simulations, the same ball is moved from $(0, 0, -0.5)$ [cm] and $(0, 0, 0)$ [cm] locations to $(0, 0, 0.5)$ [cm] and $(0.6, 0, 0.8)$ [cm] respectively as shown in Fig. 3.2(b)-(c). The fourth simulation considers two spheres of radius 1.4 cm that
are displaced from (1, 0, 2) \text{[cm]} and (−1, 0, −2) \text{[cm]} to (1, 0, 3) \text{[cm]} and (−1, 0, −3) \text{[cm]} respectively as shown in Fig. 3.2(d). In all of the simulation examples, this displacement of each sphere is adjusted to be 1 cm between frames and frame rate is assumed to be 1 \text{fps}, meaning that these two frames are captured 1 s apart.

As shown in Fig. 3.1, the inverse problem consists of two parts, first one being the reconstruction of \textit{Frame-1} and \textit{Frame-2} and the second part being the velocity reconstruction based on these results. For all the simulation cases, initial stage was conducted with ILM as shown in eq. (3.4) with optimal step length \( \vartheta \) that is updated at every iteration using eq. (3.6). The second stage is completed with ILM method for velocity reconstruction by using constant step length \( \vartheta_x, \vartheta_y, \vartheta_z \) for each dimension as outlined in [114]. DCPT use for the 3D velocity reconstruction of multi-phase flows shows good performance as can be observed from the last of row of Fig. 3.2. Each columns in this figure are put in the same orientation axis view where the orientation coordinates are only included for first row images. In addition, the reconstruction results are shown where they include both velocity information and DCPT reconstruction of the individual \textit{Frame-1} as a separate color-bar. In these plots, certain sections of 3D cylindrical RoI is removed to provide best view for each simulation case. To highlight these removed sections, a white edge is featured on resultant surfaces of RoI. 3D DCPT reconstruction of \textit{Frame 1} is included as the surface plot where color-bar scale is normalized to 0 – 1 as DCPT currently only provides quantitative data [52]. On the same surface, velocity vector field is plotted as circular right cones. Each discretized domain contains 8281 voxel inside cylindrical RoI that supports a vector in 3D space. The magnitude of information of these vectors are represented by both the color and size of the cones. The direction information of these 3D vectors
are shown as vectors that originates from the center and directs towards the vertex of these cones. All of the results are put under the same color-bar which is only included for the first row of reconstructed images. The importance of the inclusion of only Frame 1 comes from the fact that, this vector acts as an envelope for the sensitivity matrix as given in eq (3.11). In addition Frame 2 result can be deduced from the combination of Frame 1 and velocity data. This envelope feature can be clearly seen from the reconstructed results, where vector field magnitude and the DCPT Frame 1 magnitude distributions are correlated. Vector field magnitudes are also inline with the 1cm displacement distance under the assumption of 1s time slot in between.

Figure 3.3: Experimental setup images. (a) Side view of the PVC tube filled with tap water and sensors installed. (b) Top view of the PVC tube filled with tap water and table tennis ball is placed with 3D printed holder (c) Data acquisition hardware, courtesy of Tech4Imaging LLC.
3.3 Experimental Results

To verify the performance of DCPT for the 3D velocity reconstruction of multiphase flows, we conducted a series of controlled experiments. To this end, 3 layers of 12 electrode plates each, having 1.09 cm of radial width and 3 cm of axial height were wrapped around a cylindrical PVC tube with an outer diameter of 6.3 cm. The plates are separated by 0.55 cm gaps in the azimuth direction and by 1 cm gaps in the axial direction. The PVC dielectric vessel is 0.3 cm thick and has $\epsilon_r = 3$ and $\sigma = 0$ S/m. The RoI has a circular diameter of 6 cm and a height of 11 cm. The complete experimental setup is shown in Fig. 3.3(a). The RoI is discretized using a regular grid with $20 \times 20 \times 20$ voxels, from which 6320 voxels cover the RoI. The experiments are conducted at 2 MHz (electroquasistatic regime) with a 100 fps data acquisition system.
Three separate experiments are conducted exploring the velocity reconstruction of an air bubble dispersed in tap water for horizontal, vertical and both horizontal and vertical displacements. For the implementation, a table tennis ball of 4 cm diameter is used as the air bubble ($\epsilon_r = 1.5$ and $\sigma = 0 \text{ S/m}$) and positioned inside of an tap water-filled ($\epsilon_r = 80$ and $\sigma = 5 \text{ mS/m}$) RoI, as shown in Fig. 3.3(a). Similar to simulation cases, between two frames, objects are displaced by an amount of 1 cm, frame rate is assumed to be 1 fps. The table tennis ball is placed inside of water-filled PVC tube with the help of 3D printed holder as shown in Fig. 3.3(b). For Frame-1 measurements, the ball was placed in center the of RoI and then with help of 3D printed holder, it was moved by aforementioned displacement amounts as for the Frame-2 measurements. Measurements were conducted by computer controlled ECVT hardware as shown in Fig. 3.3(c).

Reconstruction results for the experimental case is shown in Fig. 3.4. These results include both the 3D velocity for displacements and DCPT results for Frame-1 measurements. For both of the presented data, certain parts of the RoI is removed to provide best view. These cuts are highlighted with white edges and resultant surface plot is allocated for normalized DCPT results. On the top that velocity results are included with cones. Each of these results share the same color-bar scales and individual orientation axis information is added as the coordinate axis. Similar to simulation results, DCPT use for the 3D velocity reconstruction of multi-phase flows shows reasonable performance. However, some artifacts are present especially on the top layer of the domain. These artifacts are results of Fringe effects of the sensing plates, since only three layer inclusions are implemented. Notice that, even if these
artifacts are present on the DCPT results, they do not translate into velocity reconstruction results since these artifacts are also present in the Frame 2, the differential operation removes them.

3.4 Summary

In this study, we extend the use of DCPT from 2D cross section imaging to 3D volume imaging and showed its potential for velocity profiling of multiphase flows. Being able to use the same hardware as ECVT, DCPT can tackle some of the more challenging flow imaging applications as compared to ECVT. This benefit can be used in conjunction with ECVT reconstruction for the volume tomography case and it can also be translated into velocity reconstruction of multiphase flows. The simulation and experiment verifications show that DCPT has potential on the flow medias that includes loss. In addition, this work provides the platform for the dual usage of ECVT and DCPT for 3D volume applications to be conducted in parallel as dual modality. This dual reconstruction can provide the complete picture of the materials under investigation since electrical permittivity information of ECT and material loss information of DCPT are complementary.
Chapter 4: Acceleration of ECVT Imaging by Fourier-Based Sparse Representations

Electrical capacitance tomography (ECT) is a soft-field imaging modality used for estimation of the electric permittivity distribution within a region of interest (RoI) [54]. ECT relies on non-invasive and non-intrusive mutual capacitance measurements obtained by electrode plates surrounding the RoI. These non-contact sensing electrodes are typically placed to form a single layer (plane) arrangement around the RoI, implying a two-dimensional cross-sectional reconstruction. Some of the advantages of ECT are its relatively fast operation and good immunity to electrical noise and interference. In addition, ECT hardware is inexpensive, lightweight, and suitable for operation in harsh (e.g. high temperature, high pressure, etc.) environments. Because of these characteristics, ECT is popular in applications related to multiphase flow imaging, combustion imaging, and industrial process monitoring, velocimetry, among others [118, 119, 70, 120, 121, 122]. Increasing attention has been given in recent years to extending ECT to volumetric imaging, also known as electrical capacitance volume tomography (ECVT). The hardware in ECVT consists of multiple layers (planes) of sensing plates as opposed to the single layer operation in conventional ECT. The majority of ECT advantages carry over to ECVT [109, 123, 124, 125, 86, 126, 114].
Despite these advantages, ECT/ECVT systems have important challenges to be addressed for increasing the image resolution. The first challenge originates from the nonlinear relation between the measured capacitances and the electric permittivity inside the RoI [61]. Although this relation can be extricated by nonlinear reconstruction methods, the computational burden implied by the latter often impedes real-time applications [62, 63, 68]. Secondly, since the measured capacitances are proportional to the electrode areas, the latter cannot be reduced below a value dictated by the minimum signal-to-noise ratio required for operation. This fact in turn implies that the total number of possible electrode pair combinations (independent measurements) is typically much smaller than the number of unknowns that results from the spatial discretization of the RoI into pixels or voxels. This situation is exacerbated in ECVT as compared to ECT due to the larger number of unknowns. Adaptive sensing methods have been proposed to alleviate this problem through, for example, fine-stepped scanning of the RoI by synthetic electrodes [60]. Although this approach increases the number of independent measurements and can play an important role in improving resolution, it also implies more complex hardware and reconstruction process [65]. There have been a number of prior works that have examined the problem of speeding up ECT/ECVT reconstruction from the perspective of data acquisition [127, 66] or by accelerating the forward problem solution [128, 129]. Other studies have explored non-conventional ECT image basis representations for different purposes [130, 131]. These include a few works which focused on applying sparse representations and compressive sensing ideas to the ECT/ECVT reconstruction problem [132, 133]. In particular, the use of Fourier domain has been explored within the context of statistical principles for shape identification and compressive sensing [134, 135]. The use
of Fourier domain bases for promoting sparsity in the inverse reconstruction process has also been employed in microwave imaging applications [136, 137].

In this study we adopt a Fourier basis representation as a means to introduce sparsity in the reconstruction of ECT/ECVT images and to reduce the number of unknowns. This in turn helps to reduce the computational burden of the aforementioned potential strategies aimed at increasing ECT/ECVT resolution. The motivation behind the Fourier representation comes from the physical nature of the interrogating ECT/ECVT field (potential): ECT/ECVT operates in the electroquasistatic regime and the interrogating field is consequently Laplacian in nature. As such, it is self-averaging in space. This is equivalent to a low-pass filter effect that suppresses any sharp field variations in space, especially in regions sufficiently away from the electrodes [64]. In other words, many elements of the representation set spanned by pixels and voxels bases correspond to sharp-varying spatial permittivity distributions in space that cannot be reproduced by the interrogating field and hence captured by any meaningful mutual capacitance measurement based on boundary electrodes. Therefore, mapping these conventional bases into the Fourier domain and removing the sharp-varying components that do not contribute to the measurement prior to the reconstruction process can lead to a great reduction in the number of unknowns. As noted in the previous paragraph, some effort has been devoted in the past towards this end, but compared to those efforts, our work is the first to focus directly on reducing the computational burden in the ECT/ECVT image reconstruction process while exploiting the underlying physical nature of the problem. The more general perspective developed here also opens the door for studying a wider range of possible sparse basis
representations such as Discrete Cosine Transform (DCT) or Hankel Transform (HT) that might be relevant to reconstructions in a RoI of cylindrical shape.

The remainder of this chapter is organized as follows. In Section II we detail the motivation and implementation of Fourier-domain based representation for ECT/ECVT problems. In Sections III and IV we present numerical and experimental results, respectively. In Section V we provide a comparison of the costs associated with the Fourier-based reconstruction versus those of conventional approaches in ECT/ECVT. Concluding remarks are provided in Section VI.

4.1 Fourier Domain Image Representation

4.1.1 Fourier Analysis of Permittivity Distribution Patterns

Fourier analysis is a powerful tool in image processing for noise removal, pattern/texture recognition, filtering (convolution, deconvolution), image analysis, image compression, and image reconstruction [138]. The use of a Fourier basis for representing the spatial permittivity distribution in ECT/ECVT is particularly attractive since it provides a natural decomposition of the distribution in terms of its spatial frequency components. In addition, it enables efficient computational processing through the use of a Fast Fourier Transform (FFT). Consider for example the images shown in Fig. 4.1(a)-(e). These images are 64 × 64 pixel-based binary representations of typical permittivity distributions encountered in ECT applications associated with different types of flows (bubbly, columnar, annular, and stratified). As a first step in the analysis, it is convenient to investigate the spectral energy distribution of these images with the help of ideal low pass (ILP) filters, where each of these images is mapped into the Fourier domain and the high-frequency content is removed based on different cutoff
Figure 4.1: Binary images of various ECT flow types. The magenta lines delineate the boundary of the circular RoI, with unit radius. (a) Flow type A (bubbly or columnar pattern) includes a single circular object located at \((-0.2,0)\) and radius 0.35 with 12% object volume fraction. (b) Flow type B (bubbly or columnar pattern) includes a single circular object located at \((0,-0.6)\) and radius 0.14 with 2% object volume fraction. (c) Flow type C (bubbly or columnar pattern) includes two circular objects located at \((0.35, -0.35)\), \((0.35, 0.35)\) and radius 0.25 with 6% object volume fraction. (d) Flow type D is an annular flow pattern spanning radius larger than 0.7 with 37% object volume fraction. (e) Flow type E is a stratified flow pattern (typically of horizontal flows) spanning below the \(x = -0.34\) line with 21% object volume fraction. (f) Ideal low pass filter with cut-off radius 13 and pass region of 540 pixels. (g) Fourier spectrum energy distribution of the aforementioned flows.
frequencies. An example of such a filter is depicted in Fig. 4.1(f) for cutoff radius of 13 pixels in the $64 \times 64$ spectral domain. This particular ILP filter retains 540 Fourier spectral components out of the original 4096 components. The energy distribution (square norm of the image) of the aforementioned flows as a function of the cutoff radii is shown in Fig. 4.1(g). This quantity represents the percentage of the energy that remains after the high-frequency components have been removed. From Fig. 4.1(g), we observe that most of the energy is indeed contained in the low frequency region. This is a direct consequence of the fact that these images have a structured pattern, i.e. in general a high correlation exists among permittivity values of nearby pixels. Because of this, an ILP filter with an appropriate cutoff frequency can be effectively applied to discard high frequency components and reduce the number of unknowns. The localization of the spectral energy on the lower portion of the spectrum is not limited to the prototypical distributions presented above. Indeed, many industrial flow processes consist of permittivity distributions with gradual variations (due to intermixing of the various phases) which further suppresses high-frequency content.

4.1.2 ECT/ECVT Implementation

The linearized form of relation between the measured capacitances and the electrical permittivity distribution inside the RoI is given by

$$c = S g .$$  \hspace{1cm} (4.1)

where $c \in \mathbb{R}^M$ represents the mutual capacitance vector with $M = N_p(N_p - 1)/2$ being the number of independent mutual capacitance measurements and $N_p$ being the number of electrodes, $S \in \mathbb{R}^{M \times N}$ is the so-called sensitivity (Jacobian) matrix, which is obtained from the discretization of RoI into $N$ pixels (voxels in the case of
ECVT) [49, 139], and $\mathbf{g} \in \mathbb{R}^N$ is the unknown vector representing the spatial electrical permittivity distribution across all pixels (voxels) in the RoI. With the objective of introducing a sparse Fourier basis in the ECT/ECVT model, consider the following transformation to eq. (4.1):

$$\mathbf{c} = \mathbf{S} \mathcal{F}^{-1} \mathcal{F} \mathbf{g},$$

(4.2)

where $\mathcal{F} \in \mathbb{C}^{N \times N}$ represents the (discrete) Fourier transform matrix, and $\mathcal{F}^{-1} \in \mathbb{C}^{N \times N}$ represents its inverse. Note that since $\mathcal{F}^{-1} \mathcal{F} = \mathcal{I}$, where $\mathcal{I} \in \mathbb{R}^{N \times N}$ is identity matrix, our ECT/ECVT model remains unaltered at this stage. Eq. (4.2) can be rewritten as

$$\mathbf{c} = \mathbf{P} \mathbf{y},$$

(4.3)

where $\mathbf{P} \in \mathbb{C}^{M \times N}$ denotes the new system matrix given by $\mathbf{S} \mathcal{F}^{-1}$ and $\mathbf{y} \in \mathbb{C}^{N}$ denotes the (unknown) vector of the spatial frequency coefficients given by $\mathcal{F} \mathbf{g}$.

The analysis provided in the previous section showed that most of the energy of the unknown vector $\mathbf{y}$ lies on the low frequency spectrum. Therefore, an a priori elimination of high frequency components can lead to a sparse representation of the unknown permittivity distribution and a substantial reduction on the problem size.

The reduced problem can be simply obtained by discarding the rows of the vector $\mathbf{y}$ and the columns of the matrix $\mathbf{P}$ corresponding to frequencies above the cutoff value. By letting $\mathbf{P}_f \in \mathbb{C}^{M \times K}$ denote the truncated system matrix and $\mathbf{y}_f \in \mathbb{C}^{K}$ the truncated vector of unknowns, with $K \ll N$, the reduced system becomes

$$\mathbf{c} = \mathbf{P}_f \mathbf{y}_f.$$  

(4.4)
The optimal choice for $K$ as determined by the ILP cutoff is problem dependent, as shown in Fig. 4.1(g). However, as a general rule, the sparse (truncated problem) solution converges to the original problem solution as $K$ is increased.

There are two basic types of algorithms to solve for the image $g$ given the measured data $c$ in ECT/ECVT. The first type consists of non-iterative image reconstruction algorithms and the second of iterative algorithms. Non-iterative algorithms such as linear back projection (LBP) are preferred for faster reconstruction and mainly used for qualitative purposes in real-time applications [59]. Iterative algorithms are preferred for quantitative imaging but in general considerably slower. In this chapter, we adopt LBP and the projected iterative Landweber method (ILM) with optimal step length [96] for experimental and simulation results, respectively. The main reason for the use of LBP and ILM is the fact that they are widely used algorithms in ECT/ECVT due to their relative simplicity and robustness. This choice facilitates comparison against results from other studies in the context of ECT/ECVT applications. Despite our focus on these reconstruction techniques, the advantages of Fourier-based reconstruction should translate to other reconstruction techniques as well.

The update equation of the ILM method used here is given by

$$ g^{(k+1)} = g^{(k)} + \alpha_k S^t (c - S g^{(k)}) \quad k \geq 0 \quad (4.5) $$

$$ g^{(0)} = S^t c \quad (4.6) $$

where the superscript $k$ indicates iteration number. The initial guess $g^{(0)}$ for the ILM method is the LBP solution as indicated above, where $S^t$ denotes the transpose
of the matrix $S$. The parameter $\alpha_k$ represents the step size of the update procedure. It is possible to incorporate a projection operator at each iteration. With this operator represented as $T$, the resulting projected ILM method may provide more stable solutions. The parameter $\alpha_k$ and the operator $T$ are given by the following expressions:

$$\alpha_k = \frac{\|S^t g^{(k)}\|^2}{\|SS^t g^{(k)}\|^2} \quad (4.7)$$

$$T[g] = \begin{cases} 
0, & \text{if } g_i < 0 \\
1, & \text{if } g_i > 1, \quad i = 1, \ldots, N \\
g_i, & \text{if } 0 \leq g_i \leq 1
\end{cases} \quad (4.8)$$

In contrast to the original real-valued system in eq. (4.1), the entries in the system matrix and the unknown vector in eq. (4.4) are complex-valued. As a result, the ILM update formulas need to be properly modified to

$$y_f^{(k+1)} = y_f^{(k)} + \beta_k P_f^* (c - P_f y_f^{(k)}), \quad k \geq 0 \quad (4.9)$$

$$y_f^{(0)} = P_f^* c, \quad (4.10)$$

where $P_f^*$ denotes the conjugate transpose of $P_f$ and the step length $\beta_k$ is given by

$$\beta_k = \frac{\|P_f y_f^{(k)}\|^2}{\|P_f P_f^* y_f^{(k)}\|^2} \quad (4.11)$$

For the Fourier-based reconstruction, once the solution vector $y_s$ is obtained, it is zero padded to match the original size of the array (before truncation) and next transformed back into the pixel domain by applying an inverse FFT. For the modified version of ILM, the projection operator given in eq. (4.8) cannot be directly implemented at each iteration since non-linearity of this function prevents finding a straightforward counterpart in the Fourier domain. However, once the iterations have concluded and the unknown vector has been mapped back to the pixel domain, the
projection can be applied once to reduce any image artifacts. Although the projection can still be employed in every iteration for the conventional ECT reconstructions, it is used only after the last iteration here for providing a fair comparison with FECT. Some alternatives to be used in conjunction with FECT can be investigated in the future, including thresholding methods in the spatial frequency domain and different normalization techniques for sensitivity matrix [140].

4.2 Numerical Examples

We conducted a series of numerical simulations of 2D and 3D models to compare traditional ECT and ECVT reconstruction results with their counterparts based on the Fourier sparsification described in the previous section. In what follows, we denote the latter reconstructions by FECT and FECVT respectively. The solution of the forward problem for obtaining the mutual capacitances was found using the commercial-grade finite element (FEM) solver COMSOL\textsuperscript{TM}. For the inverse problem, proper normalizations of the capacitance vector and sensitivity matrix in eq. (4.1) were performed following the standard approaches described in [94].

4.2.1 Comparison of ECT and FECT results

A circular RoI with 10 cm radius is considered in the 2D case. An insulating wall separating the RoI from the sensing plates is modeled as a dielectric layer of thickness 0.5 cm and with $\epsilon_r = 3$ and $\sigma = 0$ S/m. The RoI is surrounded by 12 electrode plates. The gap between two consecutive plates is set to 25% of the plate size. The mutual capacitance values for different electrode pairs are obtained with an excitation voltage signal of 1 MHz, which results in a electroquasistatic regime given the RoI size [52]. The dielectric distribution patterns used follow the ones
described in Fig. 4.1 with the black regions filled with oil ($\epsilon_r = 3$ and $\sigma = 0$ S/m) and the white regions (background) filled with air ($\epsilon_r = 1$ and $\sigma = 0$ S/m). A $64 \times 64$ rectangular grid of pixels is used to discretize the RoI, which results in 2828 pixels inside the circular RoI. The reconstruction results obtained with ILM are shown in Fig. 4.2. As noted, the projection operator was applied once in the last iteration. Each row in these plots has the same color-bar scale and axis values as the image in the first column of the corresponding row. For reference, the actual air/oil boundaries are indicated by black lines in each plot. Fig. 4.2(a) column shows conventional ILM reconstruction results based on 2828 pixel-based unknowns, and Fig. 4.2(b)-(e) columns show ILM/FECT results based on 32, 52, 80, 112, and 1246 unknowns (Fourier components), respectively.

<table>
<thead>
<tr>
<th>Flow Type</th>
<th>Iteration</th>
<th>$R_E$ (%)</th>
<th>$R_I$</th>
<th>$R_E$ (%)</th>
<th>$R_I$</th>
<th>$R_E$ (%)</th>
<th>$R_I$</th>
<th>$R_E$ (%)</th>
<th>$R_I$</th>
<th>$R_E$ (%)</th>
<th>$R_I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>50</td>
<td>62.86</td>
<td>0.8926</td>
<td>62.09</td>
<td>0.9001</td>
<td>60.81</td>
<td>0.9056</td>
<td>61.12</td>
<td>0.9062</td>
<td>61.84</td>
<td>0.8989</td>
</tr>
<tr>
<td>B</td>
<td>300</td>
<td>80.79</td>
<td>0.7116</td>
<td>86.59</td>
<td>0.5928</td>
<td>81.29</td>
<td>0.6982</td>
<td>78.18</td>
<td>0.7386</td>
<td>79.06</td>
<td>0.7439</td>
</tr>
<tr>
<td>C</td>
<td>150</td>
<td>66.01</td>
<td>0.8396</td>
<td>71.89</td>
<td>0.7873</td>
<td>63.31</td>
<td>0.8681</td>
<td>64.61</td>
<td>0.8465</td>
<td>65.28</td>
<td>0.8412</td>
</tr>
<tr>
<td>D</td>
<td>50</td>
<td>29.59</td>
<td>0.9395</td>
<td>28.28</td>
<td>0.9394</td>
<td>26.08</td>
<td>0.9464</td>
<td>23.28</td>
<td>0.9592</td>
<td>27.68</td>
<td>0.9567</td>
</tr>
<tr>
<td>E</td>
<td>4</td>
<td>39.49</td>
<td>0.9187</td>
<td>38.36</td>
<td>0.9073</td>
<td>41.04</td>
<td>0.8924</td>
<td>41.09</td>
<td>0.8949</td>
<td>40.68</td>
<td>0.9033</td>
</tr>
</tbody>
</table>

From the results shown in Figure 4.2 it can be seen that FECT gives satisfactory performance as compared to ECT, even after a significant reduction in the number of unknowns. As the number of unknowns increases (going from column (b) to column (e)), the FECT estimation of permittivity distribution converges to the original ECT result (column (a)). One can also see the presence of Gibbs artifacts, especially in the reconstructions of flow types D and E. When it comes to the assessment of FECT
Figure 4.2: Comparison of FECT with ECT for the flow types A,B,C,D and E. (a) ECT reconstruction with 2828 unknowns. (b) FECT reconstruction with 32 unknowns. (c) FECT reconstruction with 52 unknowns. (d) FECT reconstruction with 80 unknowns. (e) FECT reconstruction with 112 unknowns. (f) FECT reconstruction with 1246 unknowns.
results for different number of unknowns, the performance varies according to the flow type considered. For example, FECT results with 32 unknowns shown in column (b) of Fig. 4.2 do not exhibit good performance for flow types B and C as compared to FECT results with more unknowns. This is due to the fact that flow types B and C are the two flows that have the least amount of energy on the low-frequency components of the spectrum, as shown in Fig. 4.1(g). On the other hand, FECT reconstruction has best performance (which we analyze quantitatively later in this section) for the flow type E, which has the highest amount of energy concentrated on the low-frequency components of the spectrum. This might suggest that ECT should provide the best reconstruction for flow types B and C, since the ECT reconstruction covers the entire spectrum range. However, it is seen that FECT reconstructions with 80 and 52 unknowns provide the best solutions for flow types B and C, respectively. Therefore, even though a higher number of unknowns is required in those cases, the optimal performance occurs for a number of unknowns that is still significantly less than in the original problem. This seems to indicate that indeed a conventional pixel-based representation for the unknowns is indeed not the best choice. Clearly, the selection of the optimal cutoff frequency of the ILP filter is based on obtaining the largest possible reduction in the number of unknowns while maintaining acceptable performance. As shown in Fig. 4.1(g), all flow type images maintain at least 90% of the energy with a number of Fourier components ranging from 200 to 600 on a regular grid of 64×64 pixels. As noted before, the FECT reconstruction examples with 52 and 80 unknowns were seen to outperform the ECT results for flow types B and C. This is a consequence of the limited number of measurements as well as to the nonlinear
and soft-field nature of the ECT problem. Taken together, these features cause pixel-based ECT reconstructions to be highly ill-conditioned. For example, standard ECT reconstructions may not even reach the 90% of the energy of the true distribution (see the color bar scales of the images in column (a) of Fig. 4.2 for flow types A, B and C). This again points to the fact that, in an optimal and hence sparse representation, the number of unknowns for optimal reconstruction should not be drastically different from the number of measurements (the system considered has 12 electrode plates, resulting in 66 distinct mutual capacitance measurements). The energy distribution for the Fourier components of the reconstructed image may serve as a rough guideline for determination of the ILP cutoff frequency. The guideline is not precise due to the nonlinear nature of the problem and the fact that the iterative reconstruction and projection procedure prevents direct correspondence of the color-bar scale with the signal energy percentage parameters.

Figure 4.3: Projected ILM reconstruction of flow type D and E. (a) Iteration #: 30, $R_E = 22.86\%$, $R_I = 0.9603$. (b) Iteration #: 4, $R_E = 36.29\%$, $R_I = 0.9239$.

It should be pointed out that ILM with projection operation applied at each and every iteration can lead to better results for some flow distributions. Examples of such
reconstructions for flow types D and E are shown in Fig. 4.3. Incorporation of this feature will be part of the future work for on sparse Fourier-based reconstructions.

In what follows, we adopt two complementary error metrics for quantitative performance comparison. The first is the relative image error denoted by $R_E$ and the second is the image correlation coefficient denoted by $R_I$. They are expressed as:

$$R_E = \frac{\|g - q\|}{\|q\|}$$  \hspace{1cm} (4.12)

$$R_I = \frac{\sum_{i=1}^{N} [(g_i - \bar{g}) \cdot (q_i - \bar{q})]}{\left[\sum_{i=1}^{N}(g_i - \bar{g})^2 \cdot \sum_{i=1}^{N}(q_i - \bar{q})^2\right]^{1/2}}$$  \hspace{1cm} (4.13)

where $\|\cdot\|$ denotes Euclidean norm, $g$ denotes the vector of the reconstructed spatial permittivity distribution and $q$ represents the true permittivity distribution vector. Their $i$-th elements are denoted by $g_i$ and $q_i$, respectively. In addition, $\bar{g}$, $\bar{q}$ denote their mean values [138]. It is clear that small values of $R_E$ and large values of $R_I$ are desirable. The values of these parameters along with the number of iterations used for the reconstruction results shown in Fig. 4.2 are presented in Table. 4.1.

### 4.2.2 Comparison of ECVT and FECVT results

A cylindrical RoI with 10 cm radius and 30 cm height is considered in the 3D case. An insulating wall made of a dielectric ($\epsilon_r = 3$ and $\sigma = 0$ S/m) with 0.53 cm thickness separates the RoI from the sensing plates. The RoI is surrounded by 4 rows with 6 square electrodes each, as shown in Fig. 4.4(a). The lateral size of each sensor plate is 6.52 cm and there exist 1.3 cm of axial gap between adjacent plates. The plates are excited with 1 MHz voltage signal. The RoI is discretized by a regular grid with $20 \times 20 \times 30$ voxels. Flow types F and G are considered as shown in Fig. 4.4(a)-(b) where the dispersed phase (black region) is composed of oil ($\epsilon_r = 3$ and $\sigma = 0$ S/m)
and the continuous phase is composed of air ($\epsilon_r = 1$ and $\sigma = 0$ S/m). ECVT and

<table>
<thead>
<tr>
<th>Flow Type</th>
<th>Iteration #</th>
<th>$R_E$ (%)</th>
<th>$R_I$</th>
<th>$R_E$ (%)</th>
<th>$R_I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>200</td>
<td>70.92</td>
<td>0.8227</td>
<td>71.91</td>
<td>0.8091</td>
</tr>
<tr>
<td>G</td>
<td>200</td>
<td>65.16</td>
<td>0.8717</td>
<td>65.87</td>
<td>0.8529</td>
</tr>
</tbody>
</table>

FECVT reconstruction results for flow types F and G are shown in Fig. 4.4(c)-(f). The corresponding error analysis is tabulated in Table 4.2. Projected ILM ECVT reconstructions with 8281 unknowns are shown in the second row, whereas FECVT with 279 unknowns are shown in the last row. Note that the color-bar scale is kept constant for the reconstructions of the same flow type. From these plots it can be observed that FECVT provides very similar results to conventional ECVT, with the advantage of a significant reduction in the number of unknowns. This can also be concluded from examining the $R_E$ and $R_I$ values in Table 4.2.

### 4.3 Experimental Results

We conducted a series of controlled experiments to verify the performance of FECVT as compared to ECVT. For this purpose, 4 layers of 6 electrode plates each, having 2.77 cm of radial width and 2 cm of axial height were placed around a cylindrical PVC tube with an outer diameter of 6 cm. The plates are separated by 0.56 cm gaps in the azimuth direction and by 0.5 cm gaps in the axial direction. The PVC dielectric vessel is 0.32 cm thick and has $\epsilon_r = 3$ and $\sigma = 0$ S/m. The RoI has a
Figure 4.4: Comparison of the ECVT and FECVT reconstruction results for the flow types F and G (some plates are removed in this image to make the objects in the RoI fully visible). (a) Flow type F, which considers a single spherical object located at $(-2, 0, 0)[cm]$ with 3.5 cm radius, resulting in a 1.9% dispersed phase volume fraction. (b) Flow type G, which considers two spherical objects located at $(-4, 0, -4)[cm]$ and $(4, 0, 4)[cm]$ with radii 4 cm, resulting in a 5.7% dispersed phase volume fraction. (c) ECVT reconstruction with 8281 unknowns for flow type F. (d) ECVT reconstruction with 8281 unknowns for flow type G. (e) FECVT reconstruction with 279 unknowns for flow type F. (f) FECVT reconstruction with 279 unknowns for flow type G.
Figure 4.5: Experimental setups. (a) Cylindrical RoI with a single table tennis ball filled with glass beads. (b) Cylindrical RoI with two table tennis balls filled with glass beads. (c) Experiment setup with data acquisition hardware shown along with RoI.
Table 4.3: $R_E$ and $R_I$ for experimental results shown in Fig. 4.6

<table>
<thead>
<tr>
<th>Experiment #</th>
<th>$R_E$ (%)</th>
<th>$R_I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.35</td>
<td>0.9977</td>
</tr>
<tr>
<td>2</td>
<td>9.44</td>
<td>0.9948</td>
</tr>
<tr>
<td>3</td>
<td>14.97</td>
<td>0.9846</td>
</tr>
<tr>
<td>4</td>
<td>21.75</td>
<td>0.9676</td>
</tr>
</tbody>
</table>

circular diameter of 5.68 cm and a height of 9.5 cm. The complete experimental setup is shown in Fig. 4.5(c). The RoI is discretized using a regular grid with $20 \times 20 \times 20$ voxels, from which 6320 voxels cover the RoI. The experiments are conducted at 250 kHz (electroquasistatic regime) with a 100 fps data acquisition system.

For the first experiment, a table tennis ball of 4 cm diameter is filled with glass beads ($\epsilon_r = 1.5$ and $\sigma = 0$ S/m) and positioned at (0, 0, 1.5)[cm] on an air-filled RoI, as shown in Fig. 4.5(a). For the second experiment, the filled ball was shifted 1 cm upwards to the location (0, 0, 2.5)[cm]. In the third experiment, two filled balls were placed at (1, 0, 2)[cm] and (−1, 0, −2)[cm]. And finally for the fourth experiment, the two balls from the third experiment were shifted 1 cm upwards and 1 cm downwards, respectively.

The LBP method was employed for the reconstructions based on experimental data. The LBP method was adopted in this case to provide more variety of setups and to understand the potential of the proposed Fourier-based representations for direct reconstruction methods. Despite typically lower resolutions, direct reconstruction methods such as LBP are suitable when real-time operation with stringent high
Figure 4.6: LBP reconstruction results from ECVT and FECVT. (a) ECVT result with 6320 unknowns for experiment #1. (b) FECVT result with 279 unknowns for experiment #1. (c) ECVT result with 6320 unknowns for experiment #2. (d) FECVT result with 279 unknowns for experiment #2. (e) ECVT result with 6320 unknowns for experiment #3. (f) FECVT result with 279 unknowns for experiment #3. (g) ECVT result with 6320 unknowns for experiment #4. (h) FECVT result with 279 unknowns for experiment #4.
frame-speed requirements preclude the use of iterative methods in problems involving volumetric RoIs. Since the LBP method provides a qualitative image, we normalize our obtained images to a $[0,1]$ colorbar scale. The results are shown in Fig. 4.6, where the first column corresponds to ECVT with 6320 unknowns and the second column corresponds to FECVT with 279 unknowns for all four experiments described above. In the plots of Fig. 4.6 we can see that FECVT provides very similar results to ECVT with the advantage of a significant reduction in the number of unknowns, which decreases from 6320 to 279 (as in the simulation results). The results of the first experiment (Fig. 4.6(a) and (b)) show a nearly identical performance of ECVT and FECVT. The results of the second experiment (Fig. 4.6(c), (d)) show a minor discrepancy in the top layer of the discretization, where ECVT seem to yield a slight better contrast than FECVT. This behavior is due to the fringe effect present for the capacitance measurements associated with the top-most and bottom-most layers. For the results of the third and fourth experiments shown in Fig. 4.6(e)-(h), in addition to the fringe effects, large field gradients near the gap region between adjacent plates can also be observed [47]. Thanks to the low-pass filtering involved in FECVT, the influence of these effects is reduced in the FECVT results. To assess the quantitative performance of FECVT, we take the ECVT result as the reference result and calculate $R_E$ and $R_I$ accordingly. The results are tabulated in Table 4.3. The general trends from the experiments are similar to the ones observed from the simulations, but $R_E$ and $R_I$ indicate a progressively larger difference between ECVT and FECVT results from experiment #1 to #4.
4.4 Performance Analysis

We compared the computational performance of FECT/FECVT reconstructions versus those of conventional ECT/ECVT in the 2D and 3D setups considered before. The tests were performed using on a Lenovo W530 laptop with an Intel Core i7 3630QM processor and 16GB of DDR3 RAM. These results are presented in Table 4.4, where each test was conducted 1000 times and the obtained values were averaged. This was done to minimize the effects of any residual variations on the computer loading due to the presence of background running processes and other factors. The results were obtained considering LBP reconstructions based on eqs. (4.6) and (4.10) and implemented on Matlab. The cost for obtaining $S^t$, $P^*_t$ is not included since these are precomputed and stored beforehand. The LBP method was used for the comparison because of its suitability to obtain high-frame speeds in real-time applications. However, it should be stressed that the LBP results yield a lower bound on the speed gain for the Fourier-based approach: when iterative methods are used, we would expect to see an increase in the speed gain in favor of Fourier-based reconstruction as the system matrix size is reduced. For example, conventional ECT/ECVT reconstructions based on ILM with constant step size require $O(N^2)$ operations in the asymptotic limit, where $N$ is the number of unknowns. In FECT/FECVT, this reduces to $O(K^2)$ where $K \ll N$. On the other hand, in the LBP case these asymptotic limits reduce to $O(MN)$ operations for ECT/ECVT and $O(KM + N \log N)$ operations for FECT/FECVT, where $M$ is the number of number of measurements.

From Table 4.4, it can be observed that, under LBP, no speed gain was achieved with FECT in 2D. This is because the 2D problem has a relatively smaller number
Table 4.4: Run times for the ECT/ECVT and FECT/FECVT for the models considered

<table>
<thead>
<tr>
<th></th>
<th>Number of Measurements</th>
<th>Number of Unknowns</th>
<th>Average Run Time (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ECT</td>
<td>66</td>
<td>2828</td>
<td>0.0829</td>
</tr>
<tr>
<td>FECT</td>
<td>66</td>
<td>80</td>
<td>0.1326</td>
</tr>
<tr>
<td>ECVT</td>
<td>276</td>
<td>6320</td>
<td>1.0976</td>
</tr>
<tr>
<td>FECVT</td>
<td>276</td>
<td>276</td>
<td>0.4581</td>
</tr>
<tr>
<td>ECVT</td>
<td>496</td>
<td>8281</td>
<td>2.1639</td>
</tr>
<tr>
<td>FECVT</td>
<td>496</td>
<td>276</td>
<td>0.7705</td>
</tr>
</tbody>
</table>

of unknowns and the FFT operations add an extra computational load that the utilization of even fewer unknowns cannot compensate for. However, when the number of unknowns and/or measurements increases, as in the FECVT experimental and simulation cases in 3D, a significant speed gain is obtained. Note that, from a practical point of view, it is precisely for scenarios with large number of measurements and/or unknowns that a gain in speed is critical for realizing real-time operation. This also suggests that, in 2D cases, the reconstruction strategy proposed here can help in speeding up the reconstruction only if a larger number of plates is utilized such as in adaptive ECT [60] or possibly for reconstruction algorithms other than LBP. Although this was not pursued here, it is possible to further increase the speed of the operation of FECT/FECVT systems by using parallelization techniques, along with GPUs, for the FFT computation.
4.5 Summary

In this work we explored the use of a sparse Fourier basis to represent the unknown permittivity distribution in the RoI and accelerate image reconstruction in ECT/ECVT imaging. By using an ideal low pass filter, it is possible to reduce the number of unknowns with only minimal deterioration on the image quality. The underlying rationale of this approach comes from the sparsity-promoting nature of the Fourier domain along with the inherent low-pass-filtering nature of the interrogating field in ECT/ECVT. With the reduced basis, a gain in the speed of reconstruction was achieved for ECVT. Because of the relatively low number of electrode plates in conventional ECT, the reduction in the number of unknowns did not compensate the added cost of FFT computations. From a practical point of view, irrespective of whether the application is ECT or ECVT, for the cases with a low number of measurement pairs and unknowns (pixels), the increase in the reconstruction speed is less relevant because such systems already have low computational costs and high speeds. However, for systems with more demanding hardware that entail large number of measurements and/or unknowns, such as ECVT or adaptive systems (AECT/AECVT), Fourier-based reconstruction holds great potential for speeding up the reconstruction process. From the image quality point of view, Fourier-based results provide satisfactory image quality when compared to traditional ECT/ECVT methods for the most part. However, when the ECT/ECVT scenario involves a flow type whose adequate reconstruction is predicated on the ILM projection procedure, the performance of Fourier-based reconstruction may degrade in a significant way. With the purpose of remedying this issue, implementation of different normalization methods for the sensitivity matrix and capacitance vectors may be considered as a possible
future work. In addition, different filters in the Fourier domain can be also exploited to obtain improvements on this front.
Chapter 5: Conclusions and Look Ahead

Process tomography deals with observing internal dynamics of processes using imaging technologies. With the expanding needs of industry, these technologies are more needed than ever to help design, control, and optimize industrial processes. Industrial processes may require tomography hardware to be easily deployable, portable, and able to work on harsh environmental conditions such as under very high temperature and/or pressure. ECT/ECVT systems provide non-invasive, non-intrusive process tomography solution with relatively low cost and high speed. In addition, ECT/ECVT systems support flexible sensor designs with good immunity to electrical noise and interference coupled to lightweight and sturdy hardware.

In this dissertation, a new “soft-field” tomography method based on ECT hardware and denoted as displacement-current phase tomography (DCPT) is formulated and compared with ECT for cross-section process tomography. For imaging scenarios that include media with dielectric losses, DCPT provide extra information about the process under imaging and can yield higher resolution than conventional ECT while relying on the same basic hardware. DCPT is also extended to volume tomography, where it is verified that these benefits carry over for ECVT-related imaging scenarios as well. DCPT is next adapted to provided velocity profiling of multiphase flows and implemented with various examples. It is shown that DCPT provides an
effective low-cost velocimetry technique complementary to that based on ECT. Finally, a Fourier-based representation is introduced for ECT and ECVT where sparse representation of the permittivity image allows great reduction in the number of unknowns. These approach did not provide significantly faster operation compared to conventional pixel-based ECT imaging, but did so compared to conventional voxel-based ECVT imaging.

Many possible future avenues of research exist in this area. A particular aspect ripe for future development is the determination of a proper normalization/calibration process for DCPT-based imaging. This is because presently DCPT provides image reconstruction without quantitative estimates on the permittivity contrast. This is sufficient for the imaging of, for example, two-phase flows where the constitutive properties of the two phases are known a priori but not for arbitrary multiphase flow where the constitutive phase properties are themselves unknown. In order to provide the necessary information, the magnitude of the permittivity contrast should be related to loss factor in the media through an adequate normalization/calibration process. Another potential future development would be to combine ECT and DCPT during the image reconstruction process to enable a multimodal image reconstruction. Using the same hardware, these two sensing modalities can provide complementary information regarding the material electric permittivity and loss in the RoI. To improve image resolution, the development of adaptive or electronically reconfigurable sensor hardware (AECT/AECVT) is highly desirable. Such hardware can provide for example a fine-stepped electronic scanning of the RoI based on the sequential activation of synthetic plates. These synthetic plates are composed of many small segment plates that have their individual signals combined electronically. The much larger
number of (synthetic) plates and hence mutual capacitance measurements enabled by AECVT sensors will demand significantly higher computational resources when compared with traditional ECVT. Consequently, acceleration and sparsity-promoting approaches such as FFT-based reconstructions designed to speed up the image reconstruction process will be necessary in order to allow real time operation of these systems. Implementation of FFT-based reconstruction in multimodal DCPT/AECT systems will also help volumetric reconstruction problems to be more computationally efficient for more challenging multiphase flow imaging conditions. Finally, other sparsity-promoting basis alternatives can be investigated and compared with the results of this work in terms of image quality and computational demands. To this end, a particular sparsity promoting basis that exploits the cylindrical geometry of typical process tomography RoI, such as Hankel transforms, can be one of the candidates. Another candidate could be edge element basis of finite element method (FEM) where the unknowns can be defined on the edges of the discretization for example triangles for ECT or tetrahedrons for ECVT. With the proper triangulation of the domain such as smaller elements near sensors and larger elements towards RoI center, this basis could represent the unknowns in more compact form as well as compensating for the non-uniform spatial resolution due to Laplacian nature of the excitation.
Appendix A: Analytical Derivation of 2D Sensitivity Matrix

The Laplace equation on a disk in polar coordinates can be written as [141]:

$$\nabla^2 V = r \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial \varphi^2} = 0$$

(A.1)

The unknown potential $V(r, \varphi)$ can be written as the product of two functions $R(r), F(\varphi)$ of only one variable each as $V(r, \varphi) = R(r)F(\varphi)$ by the usage of separation of variables technique. The Laplace equation can then be written as,

$$r F \frac{\partial}{\partial r} \left( r \frac{\partial R}{\partial r} \right) + R \frac{\partial^2 F}{\partial \varphi^2} = 0$$

(A.2)

$$r \frac{\partial}{R \partial r} \left( r \frac{\partial R}{\partial r} \right) = -\frac{1}{F} \frac{\partial^2 F}{\partial \varphi^2} = m^2$$

(A.3)

where $m^2$ is a constant. Now that the variables are separated, the above equation can be solved separately for each variable as,

$$r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} - m^2 R = 0$$

(A.4)

and

$$\frac{\partial^2 F}{\partial \varphi^2} + m^2 F = 0$$

(A.5)

Notice that for $F$ function to be periodical, $m$ must be integer and for the potential inside the disk. The general solution for $R$ and $F$ when $m = 0$ can be found as

$$R = A_1 + A_2 \ln(r)$$

(A.6)
\begin{align*}
F &= B_1 + B_2 \phi \\
R &= C_1 r^m + C_2 r^{-m} \quad (A.8) \\
F &= D_1 \sin(m \varphi) + D_2 \cos(m \varphi) \quad (A.9)
\end{align*}

For the interior of the disk, for \( r = 0 \), \( B_2 \) coefficient will be undefined so it needs to be zero and for the same case \( A_2 \) coefficient needs to be zero as well to avoid singularity. Therefore \( m = 0 \) case only results in a constant for the interior, so it can be combined with \( R \) function. Then the general solution for the interior of the disk can be written as

\begin{align*}
R &= C_1 r^m \quad m = 0, 1, 2, 3... \quad (A.10) \\
F &= D_1 \sin(m \varphi) + D_2 \cos(m \varphi) \quad m = 1, 2, 3... \quad (A.11)
\end{align*}

The general form of the potential inside this the disk can then be expressed as in equation (A.12) where \( C_1 D_1, C_1 D_2 \) terms are combined into coefficients \( a_m, b_m \).

\begin{equation}
V(r, \varphi) = \sum_{m=1}^{\infty} a_m r^m \sin(m \varphi) + \sum_{m=0}^{\infty} b_m r^m \cos(m \varphi) \quad (A.12)
\end{equation}

\( a_m \) and \( b_m \) coefficients can be expressed by a known potential \( V(a, \varphi) = V(\varphi) \) at the boundary. The calculation of these coefficients are based on the orthogonality principle in the Fourier series.

\begin{align*}
b_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} V(\varphi) d\varphi \quad (A.13) \\
a_m a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} V(\varphi) \sin(m \varphi) d\varphi \quad m = 1, 2, 3... \quad (A.14) \\
b_m a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} V(\varphi) \cos(m \varphi) d\varphi \quad m = 1, 2, 3... \quad (A.15)
\end{align*}
For a circle of radius $a$, the general form of the Laplace equation in the interior of the domain can then be written as,

$$V(r, \varphi) = b_0 + \sum_{m=1}^{\infty} r^m (a_m \sin(m\varphi) + b_m \cos(m\varphi))$$ (A.16)

with,

$$b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} V(\phi) d\phi$$ (A.17)

$$a_m = \frac{1}{a^m \pi} \int_{-\pi}^{\pi} V(\phi) \sin(m\phi) d\phi \quad m = 1, 2, 3...$$ (A.18)

$$b_m = \frac{1}{a^m \pi} \int_{-\pi}^{\pi} V(\phi) \cos(m\phi) d\phi \quad m = 1, 2, 3...$$ (A.19)

Assuming the following potential function at the boundary: $V(a, \varphi) = 1$ for $\varphi \in [-15, 15]$ degrees and zero elsewhere, the equation (A.16) derived can be used to find the interior potential for the given boundary condition. First the coefficients needs to be calculated as given in (A.17), (A.18) and (A.19) for $V(a, \varphi) = V(\phi) = 1$ for $\phi \in [-15, 15]$ degrees and zero elsewhere.

$$b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} V(\phi) d\phi = \frac{1}{2\pi} \int_{-\pi \over 12}^{\pi \over 12} 1 d\phi = \frac{1}{12}$$ (A.20)

$$a_m = \frac{1}{a^m \pi} \int_{-\pi}^{\pi} V(\phi) \sin(m\phi) d\phi = \frac{1}{a^m \pi} \int_{-\pi \over 12}^{\pi \over 12} \sin(m\phi) d\phi = 0$$ (A.21)

$$b_m = \frac{1}{a^m \pi} \int_{-\pi}^{\pi} V(\phi) \cos(m\phi) d\phi = \frac{1}{a^m \pi} \int_{-\pi \over 12}^{\pi \over 12} \cos(m\phi) d\phi = \frac{2 \sin(m\pi \over 12)}{a^n \pi n}$$ (A.22)

Therefore, the interior potential for a disk of radius $a$ with the given boundary condition can be expressed as,

$$V(r, \varphi) = \frac{1}{12} + \sum_{m=1}^{\infty} \frac{2 \sin(m\pi \over 12)}{a^n \pi m} r^m \cos(m\varphi)$$ (A.23)

An example model of radius $r = 10cm$ is shown in Fig.(A.1). The model is plotted on $200 \times 200$ pixels with summation is truncated after 300th element.
The electric field inside the disk can be calculated with the gradient operator in polar coordinates.

\[
E(r, \varphi) = -\nabla V(r, \varphi) = -\frac{\partial V(r, \varphi)}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial V(r, \varphi)}{\partial \varphi} \hat{\varphi} = E_r \hat{r} + E_\varphi \hat{\varphi} \quad (A.24)
\]

\[
E_r = -\sum_{m=1}^{\infty} \frac{r^{m-1}}{a^m \pi} \left[ (\cos(m\phi_1) - \cos(m\phi_2)) \sin(m\varphi) + (\sin(m\phi_2) - \sin(m\phi_1)) \cos(m\varphi) \right] \quad (A.25)
\]

\[
E_\varphi = -\sum_{m=1}^{\infty} \frac{r^{m-1}}{a^m \pi} \left[ (\cos(m\phi_1) - \cos(m\phi_2)) \cos(m\varphi) - (\sin(m\phi_2) - \sin(m\phi_1)) \sin(m\varphi) \right] \quad (A.26)
\]

For the verification, the magnitude of the electric field for the problem case of section 2.2 is plotted in Fig.(A.3). Using these equation, the sensitivity matrix can be calculated for the pixelated domain as,

\[
S_{ij} = \nu_j \frac{E_{ti}(r, \varphi) E_{ri}(r, \varphi)}{V_{ti} V_{ri}} \quad (A.27)
\]

where \(E_{ti}, E_{ri}\) are the transmitter and receiver responses when the other plates are grounded, \(V_{ti}, V_{ri}\) are the applied voltages to the transmitter and receiver cases, and
Figure A.2: Sensitivity Matrices for a domain surrounded by 12 plates, \( r = 10cm \), (a) S11. (b) S12. (c) S13. (d) S14. (e) S15. (f) S16.

\( \nu_j \) is the area of the plate. The first approximation for this calculation is that, there is no gap between plates, since each of the 12 plates span 30° width. Secondly, in this calculation the plates are in direct contact with the domain, whereas in ECT applications, there is a buffer domain, commonly referred as wall for contactless operation of the sensing plates. The magnitude of the electric field for this particular problem is shown in Fig.(A.3) where the image is calculated for 200 \( \times \) 200 pixels with summation truncation at 300th element. Note that for a domain surrounded by 12 plates, there are 6 different sensitivity matrices, the rest can be found from the symmetry, these unique S matrices are shown in the Fig.(A.2).
Figure A.3: Magnitude of electric field distribution on XY plane of the example model with $r = 10\,\text{cm}$, $V(a, \varphi, z) = 1$ for $\varphi \in [-15, 15]$
Appendix B: Analytical Derivation of Electric Field due to ECVT Plate Excitation

This problem deals with finding the values of potential function inside of an infinitely long cylinder due to radial excitation on the boundary. The Laplace equation in cylindrical coordinates can be written as,

\[ \nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad (B.1) \]

The potential function due to a charge distribution can be expressed in terms of Green function as \[ V(\hat{x}) = \frac{1}{4\pi\epsilon_0} \int_\nu \rho(\hat{x}')G(\hat{x}, \hat{x}')d\nu' + \frac{1}{4\pi} \iint_S \left[ G(\hat{x}, \hat{x}') \frac{\partial V(\hat{x}')}{\partial n'} - V(\hat{x}') \frac{\partial G(\hat{x}, \hat{x}')}{\partial n'} \right] dS' \quad (B.2) \]

where the prime terms are used to indicate the source and non prime terms are observation points. In this representation, the first part of the equation is a volume integral over source points inside the domain whereas the second term is a surface integral due to potential originates from these source points. The surface term consists of Neumann and Dirichlet boundary conditions respectively. For the problem of interest, there is no source distribution inside the cylinder but only radial boundary condition of potential. Therefore, the volume integral and surface term
related to Neumann boundary condition can be removed from the equation.

\[ V(\hat{x}) = -\frac{1}{4\pi} \iint_S V(\hat{x}') \frac{\partial G(\hat{x}, \hat{x}')}{\partial n'} dS' \]  
(B.3)

Separation of variables technique can be used to find the form of the Green function for this problem,

\[ G(\hat{x}, \hat{x}') = R(r, r')F(\varphi, \varphi')Q(z, z') \]  
(B.4)

\[ \nabla^2 G = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial G}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 G}{\partial \varphi^2} + \frac{\partial^2 G}{\partial z^2} = 0 \]  
(B.5)

Using eq.(B.4) in eq.(B.5) and dividing by the eq.(B.4) results as,

\[ \frac{1}{r} \frac{R'}{R} + \frac{R''}{R} + \frac{1}{r^2} \frac{Q''}{Q} + \frac{Z''}{Z} = -k^2 \]  
(B.6)

where prime sign is used for differentials according to the relative variable. This equation can be put into three separate differential equation according to the relative variable. This equation can be put into three separate differential equation as,

\[ r^2 R'' + r R' + (k^2 r^2 - m^2) \]  
(B.7)

\[ Q'' + m^2 Q = 0 \]  
(B.8)

\[ Z'' - k^2 Z = 0 \]  
(B.9)

From these set of differential equations, depending on the sign selection of k, the solution set be either \( \{e^{im\varphi}, J_m(N_m), e^{\pm kz}\} \) or \( \{e^{im\varphi}, I_m(K_m), \{\sin(kz), \cos(kz)\}\} \) where \( J_m, N_m \) are Bessel functions of the first and second kind respectively and \( I_m, K_m \) are modified Bessel functions of the first and second kind respectively. For the interior solution only the first kind Bessel is used due to singularity of second kind Bessel function for 0 argument. For the outer part of the cylinder, a linear combination of these functions will be used for general solution. Through Laplace equation, the general
form of the solution for the set of functions is shown, now assuming an infinitesimal
source at the location \((r', \varphi', z')\),

\[
\nabla^2 G = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial G}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 G}{\partial \varphi^2} + \frac{\partial^2 G}{\partial z^2} = -4\pi \frac{\delta(r - r') \delta(\varphi - \varphi') \delta(z - z')}{r} \tag{B.10}
\]

Well known mathematical identities can be shown as,

\[
\delta(\varphi - \varphi') = \frac{1}{2\pi} \sum_{m=\infty}^{\infty} e^{im(\varphi - \varphi')} \tag{B.11}
\]

\[
\delta(z - z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(z - z')} dk \tag{B.12}
\]

Now that the medium is unbounded in \(z\) direction, Fourier transform with respect to \(z\)
direction is used, therefore the form of the solutions can be written as [143, 144, 145],

\[
G_{in}(\hat{x}, \hat{x}') = \sum_{m=\infty}^{\infty} \int_{-\infty}^{\infty} R_m(k) I_m(kr) e^{ikz} e^{im\varphi} \tag{B.13}
\]

\[
G_{ex}(\hat{x}, \hat{x}') = \sum_{m=\infty}^{\infty} \int_{-\infty}^{\infty} F_m(k) [I_m(kr) + Q_m K_m(kr)] e^{ikz} e^{im\varphi} \tag{B.14}
\]

for interior and exterior regions respectively. Since the radial boundary condition is
given at the radius \(a\), Green functions should be zero at the surface where the source
is. This identity is useful for exterior Green function such that,

\[
0 = \sum_{m=\infty}^{\infty} \int_{-\infty}^{\infty} F_m(k) [I_m(ka) + Q_m K_m(ka)] e^{ikz} e^{im\varphi} \tag{B.15}
\]

resulting in \(Q_m = -I_m(ka)/K_m(ka)\) due to orthogonality. Another useful Green
function identity is the continuity of the potential across the boundary, in other
words interior and exterior Green functions should be equal at \(r = r'\),

\[
\sum_{m=\infty}^{\infty} \int_{-\infty}^{\infty} R_m(k) I_m(kr') e^{ikz} e^{im\varphi} = \sum_{m=\infty}^{\infty} \int_{-\infty}^{\infty} F_m(k) [I_m(kr') - \frac{I_m(ka)}{K_m(ka)} K_m(kr')] e^{ikz} e^{im\varphi} \tag{B.16}
\]
due to orthogonality,

$$R_m(k) = F_m(k)[1 - \frac{I_m(ka)K_m(kr')}{I_m(kr')K_m(ka)}]$$  \hspace{1cm} (B.17)

The interior Green function can be expressed as,

$$G_{in}(\hat{x}, \hat{x}') = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} F_m(k)[1 - \frac{I_m(ka)K_m(kr')}{I_m(kr')K_m(ka)}]I_m(kr)e^{ik\zeta}e^{im\varphi}$$  \hspace{1cm} (B.18)

$F_m(k)$ constant can be found by inserting eq.(B.18) back in eq.(B.10) and making use of Wronksian of the modified Bessel differential equation. The final form of the Green function can be found as,

$$G_{in}(\hat{x}, \hat{x}') = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{I_m(kr)}{I_m(ka)}[I_m(ka)K_m(kr')] - I_m(kr')K_m(ka)]e^{ik(z-z')}e^{im(\varphi-\varphi')}dk$$  \hspace{1cm} (B.19)

Then the potential inside the infinitely long cylinder can be calculated as,

$$V(\hat{x}) = -\frac{1}{4\pi} \iint_S V(\hat{x}') \frac{\partial G_{in}(\hat{x}, \hat{x}')}{\partial r'} dS'$$  \hspace{1cm} (B.20)

where the derivative is taken with respect to radial distance due to radial boundary condition.

$$V(\hat{x}) = -\frac{1}{4\pi^2 a} \iint_S V(\hat{x}') \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{I_m(kr)}{I_m(ka)}e^{ik(z-z')}e^{im(\varphi-\varphi')}\Big|_{r'=a}$$  \hspace{1cm} \hspace{1cm} dk dS'$$  \hspace{1cm} (B.21)

Using Wronksian identities, the final form of the potential for the interior due to radial excitation can be given as,

$$V(\hat{x}) = -\frac{1}{4\pi^2 a} \iint_S V(\hat{x}') \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{I_m(kr)}{I_m(ka)}e^{ik(z-z')}e^{im(\varphi-\varphi')}dk dS'$$  \hspace{1cm} (B.22)
Assuming \( V(a, \varphi, z) = 1 \) for \((\varphi, z) \in([-15, 15], [-d, d]) \) [degrees, meters], and zero elsewhere,

\[
V(r, \varphi, z) = \frac{1}{4\pi^2 a} \int_{-\pi}^{\pi} \int_{-d}^{d} V(r', \varphi', z') \sum_{m=\infty}^{\infty} \int_{-\infty}^{\infty} \frac{I_m(kr')}{I_m(ka)} e^{ik(z-z')} e^{im(\varphi-\varphi')} adkdz'd\varphi' \tag{B.23}
\]

\[
V(r, \varphi, z) = \frac{1}{4\pi^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{I_m(kr')}{I_m(ka)} \int_{-d}^{d} e^{ik(z-z')} dz' \int_{-\pi}^{\pi} e^{im(\varphi-\varphi')} d\varphi' dk \tag{B.24}
\]

\[
V(r, \varphi, z) = \frac{1}{\pi^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{m} \frac{I_m(kr')}{I_m(ka)} e^{ikz} e^{im\varphi} \sin\left(\frac{m\pi}{12}\right) \sin(kd) dk \tag{B.25}
\]

Further manipulating this equation gives,

\[
V(r, \varphi, z) = \frac{4}{\pi^2} \sum_{m=0}^{\infty} \frac{\cos(m\varphi) \sin\left(\frac{m\pi}{12}\right)}{m} \int_{0}^{\infty} \frac{I_m(kr) \cos(kz) \sin(kd)}{kI_m(ka)} dk \tag{B.26}
\]

Note that for \( m = 0 \), this equation needs special treatment, with the usage of \( \text{sinc} \) function, the final interior potential for the given boundary condition can be written as

\[
V(r, \varphi, z) = \frac{1}{3\pi} \int_{0}^{\infty} \frac{I_0(kr) \cos(kz) \sin(kd)}{kI_0(ka)} dk \tag{m=0} \tag{B.27}
\]

and

\[
V(r, \varphi, z) = \frac{4}{\pi^2} \sum_{m=1}^{\infty} \frac{\cos(m\varphi) \sin\left(\frac{m\pi}{12}\right)}{m} \int_{0}^{\infty} \frac{I_m(kr) \cos(kz) \sin(kd)}{kI_m(ka)} dk \tag{m \neq 0} \tag{B.28}
\]

Examining eq.(B.27) and eq.(B.28) shows that the unbounded domain in ±z directions creates an integral term with infinity as the upper limit. The numerical methods may be applied to approximate this integral but this type of equations might be challenging. From the physics point of view, the electric field created by the source will decay in \( z \) and \(-z\) directions, therefore it is reasonable to truncate the domain in
$z = \pm L$ planes where $L$ will be a sufficiently large number for the problem. Once the domain is truncated, this continuous spectrum is expected to become a discrete sum. The problem can then be stated as finding the Dirichlet type of Green function for inside region of a cylinder with radius $a$ subject to boundary conditions of $V(a, \varphi, 0) = 0, V(a, \varphi, L) = 0$ and $V(a, \varphi, z) = 1$ for $(\varphi, z) \in ([−15, 15], [−d + \frac{L}{2}, d + \frac{L}{2}])$ with $d < \frac{L}{2}$. Following a similar procedure as section (2.3), general solution for $Z$ in eq.(B.9) can be written as,

$$Z(z) = \sum_{n=1}^{\infty} Z_n \sin(kz) \quad k = n \frac{\pi}{L}$$  \hspace{1cm} (B.29)

since $Z = 0$ for $z = 0$ and $z = L$, after finding the $Z_n$ coefficient with orthogonality, this expression can be written as,

$$\delta(z - z') = \frac{2}{L} \sum_{n=1}^{\infty} \sin(kz) \sin(kz')$$  \hspace{1cm} (B.30)

Then the general form of the Green function can be written as,

$$G_{in}(\hat{x}, \hat{x}') = \frac{1}{\pi L} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \sin(kz) \sin(kz') e^{i m (\varphi - \varphi')} A(r') I_m(kr)$$  \hspace{1cm} (B.31)

$$G_{ex}(\hat{x}, \hat{x}') = \frac{1}{\pi L} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \sin(kz) \sin(kz') e^{i m (\varphi - \varphi')} B(r') [I_m(kr) K_m(ka) - I_m(ka) K_m(kr)]$$  \hspace{1cm} (B.32)

Following a similar approach in section (2.3) with proper properties of the Green function, Dirichlet Green function can be found as,

$$G_{in}(\hat{x}, \hat{x}') = \frac{1}{\pi L} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \sin(kz) \sin(kz') e^{i m (\varphi - \varphi')} \frac{I_m(kr)}{I_m(ka)} [I_m(ka) K_m(kr') - I_m(kr') K_m(ka)]$$  \hspace{1cm} (B.33)
Then the potential inside of the truncated cylinder can be found from eq. (B.20),

\[
V(x) = -\frac{1}{\pi L} \oint_S \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} V(x') \sin(kz') e^{im(\varphi - \varphi')} \frac{I_m(kr)}{I_m(ka)}
\]

\[
\left. \frac{\partial}{\partial r'} [I_m(ka) K_m(kr') - I_m(kr') K_m(ka)] \right|_{r'=a}
\]

(B.34)

Notice that \(1/4\pi\) factor is dropped since, impulse function is not scaled by \(4\pi\) as in B.10. Using Wronskian identities,

\[
V(x) = \frac{1}{\pi L} \oint_S \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} V(x') \frac{I_m(kr)}{I_m(ka)} \sin(kz') e^{im(\varphi - \varphi')} d\varphi' dz'
\]

(B.35)

Now applying \(V(a, \varphi, z) = 1\) for \((\varphi, z) \in [-15, 15], [-d + \frac{L}{2}, d + \frac{L}{2}]\))

\[
V(x) = \frac{1}{\pi L} \int_{-d-L}^{+d} \int_{-\frac{L}{2}}^{\frac{L}{2}} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{I_m(kr)}{m k I_m(ka)} \sin(kz) e^{im(\varphi - \varphi')} d\varphi' dz'
\]

(B.36)

\[
V(x) = \frac{4}{\pi L} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{I_m(kr)}{mk I_m(ka)} \sin(kz) \left[ \cos(k(L/2 - d)) - \cos(k(L/2 + d)) \right] \cos(m\varphi) \sin(m\pi/12)
\]

(B.37)

Since \(k = \frac{n\pi}{L}\), the potential for the interior of this truncated cylinder can be expressed as,

\[
V(r, \varphi, z) = \frac{4}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{I_m(n\pi r/L)}{mn I_m(n\pi a/L)} \cos(m\varphi) \sin(m\pi/12) \sin(n\pi z/L) \left[ \cos(n\pi (L/2 - d)) - \cos(n\pi (L/2 + d)) \right]
\]

(B.38)

As expected, the initial assumption converted the continuous spectrum to discrete one. To compute the sensitivity matrix, gradient of the potential field is necessary along with 3D sensitivity matrix calculation. A quick observation on this expression shows that these summations are convergent as the denominator will always be bigger than the numerator, and the values of the elements of the summation will decrease
due to $mn$ factor as a damping factor on the denominator. The voltage distribution is plotted for an example model of radius 10cm, length $L = 1m$ as shown in Fig.(B.2) and Fig.(B.1). A single plate excitation is employed with $V(a, \varphi, z) = 1$ for $(\varphi, z) \in ([−15, 15], [L/2 − d, L/2 + d])$ with $d = 0.1$. For the calculation of the potential, both of the summation is truncated after 30th elements, XY plane is plotted by $200 \times 200$ pixels and XZ plane is plotted by $65 \times 390$ pixels.

Figure B.1: Potential distribution on XY plane of the example model with $r = 10cm$, $L = 1m$, $d = 10cm$, $V(a, \varphi, z) = 1$ for $(\varphi, z) \in ([−15, 15], [L/2 − d, L/2 + d])$

\[ E(r, \varphi, z) = -\nabla V(r, \varphi) = -\frac{\partial V(r, \varphi, z)}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial V(r, \varphi, z)}{\partial z} \hat{\varphi} - \frac{\partial V(r, \varphi, z)}{\partial \varphi} \hat{z} \quad (B.39) \]

\[ S_{ij} = \nu_j \frac{E_{ti}(r, \varphi, z)E_{ri}(r, \varphi, z)}{V_t V_r} \quad (B.40) \]

To take the derivative of this equation, differentiation of the modified Bessel function of the first kind can be given with the recursive relation as follows,

\[ \frac{\partial I_m(z)}{\partial z} = \frac{m}{z} I_m(z) + I_{m+1}(z) \quad (B.41) \]
Figure B.2: Potential distribution on XZ plane of the example model with \( r = 10 \text{cm} \), \( L = 1 \text{m} \), \( d = 10 \text{cm} \), \( V(a, \varphi, z) = 1 \) for \((\varphi, z) \in ([-15, 15], [L/2 - d, L/2 + d])\)

\[
E_r = -\frac{4}{\pi L} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(m I_m(n \pi r/L) + I_{m+1}(n \pi r/L))}{m I_m(n \pi a/L)} \cos(m \varphi) \sin\left(\frac{m \pi}{12}\right) \sin\left(\frac{n \pi z}{L}\right) \left[ \cos\left(\frac{n \pi}{L}(\frac{L}{2} - d)\right) - \cos\left(\frac{n \pi}{L}(\frac{L}{2} + d)\right) \right]
\] (B.42)

\[
E_\varphi = \frac{4}{\pi^2 r} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{I_m(n \pi r/L)}{n I_m(n \pi a/L)} \sin(m \varphi) \sin\left(\frac{m \pi}{12}\right) \sin\left(\frac{n \pi z}{L}\right) \left[ \cos\left(\frac{n \pi}{L}(\frac{L}{2} - d)\right) - \cos\left(\frac{n \pi}{L}(\frac{L}{2} + d)\right) \right]
\] (B.43)
\[ E_z = -\frac{4}{\pi L} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{I_m(n\pi r)}{m I_m(n\pi a/L)} \cos(m\varphi) \sin\left(\frac{m\pi}{12}\right) \cos\left(\frac{n\pi z}{L}\right) \]

\[ \left[ \cos\left(\frac{n\pi}{L}\left(\frac{L}{2} - d\right)\right) - \cos\left(\frac{n\pi}{L}\left(\frac{L}{2} + d\right)\right) \right] \]

\[ = -\frac{1}{3L} \sum_{n=1}^{\infty} \frac{I_0(n\pi r)}{I_0(n\pi a/L)} \cos\left(\frac{n\pi z}{L}\right) \left[ \cos\left(\frac{n\pi}{L}\left(\frac{L}{2} - d\right)\right) - \cos\left(\frac{n\pi}{L}\left(\frac{L}{2} + d\right)\right) \right] \]

(B.44)

From these vectors, the sensitivity matrix can be calculated with truncation on the summation terms with eq.(B.40). One of the main assumptions in this calculation is the truncation in z-domain.
Bibliography


