AN INVARIANT OF LINKS ON SURFACES VIA HOPF ALGEBRA BUNDLES

Dissertation

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ABSTRACT

One semi-classical knot invariant involves turning a knot diagram into a curve in \( \mathbb{R}^2 \) which is "decorated" by elements of a ribbon Hopf algebra \( \mathcal{H} \). A decorated curve is turned into an element of \( \mathcal{H} \) using a form of pictoral calculus. The image of this element in a certain quotient space of \( \mathcal{H} \) defines a framed knot invariant. We generalize this process to define an invariant of links in a thickened surface \( \Sigma \times [0, 1] \), where \( \Sigma \) is a connected, oriented surface.

In this process, we develop a theory of decorated curves in an arbitrary smooth manifold \( M \) using a balanced, flat ribbon Hopf algebra bundle \( E \to M \) with typical fiber \( \mathcal{H} \). The link invariant is defined using decorated curves in \( T^1\Sigma \), the unit tangent bundle of \( \Sigma \), and takes values in the quotient space of a semi-direct product \( k[\pi_1(M, b_0)] \rtimes \mathcal{H} \).

We also define local diagrams to picture the decorated curves. The original pictoral calculus for decorated curves in \( \mathbb{R}^2 \) is recaptured by viewing decorated curves in \( T^1\Sigma \) through these local diagrams.
For my parents, my first teachers
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CHAPTER 1
INTRODUCTION

Invariants of oriented links using ribbon Hopf algebras have a long history. The earliest invariants used solutions to the Quantum Yang-Baxter Equation (QYBE), as in Turaev’s papers [33, 34]. The use of an $R$-matrix element to find solutions of the QYBE were first studied for statistical mechanics purposes, as in Jimbo [12]. These first $R$-matrices lived in $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$, where $\mathfrak{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, a finite-dimensional simple Lie algebra. The study of deformations of $\mathfrak{U}(\mathfrak{g})$ led to the more general notion of Hopf algebras, as in Jimbo [11] and Drinfel’d [8, 9].

The formal approach to these link invariants uses the monoidal category of framed tangles, as developed by Turaev [34], Freyd-Yetter [10, 38], and Joyal-Street [13]. A colored tangle is a tangle along with a choice of an element of $\text{Rep}(\mathcal{H})$ assigned to each component of the tangle. The set of representations, $\text{Rep}(\mathcal{H})$, is also a monoidal category, and the modern formulations of these invariants are defined as functors from the category of colored tangles to $\text{Rep}(\mathcal{H})$. These are sometimes called colored quantum invariants. One crowning achievement of these invariants was recovering the Jones polynomial by using the standard representation of the Hopf algebra $U_q(\mathfrak{sl}_2)$, see, for example, [28]. These types of link invariants were strongly shaped by Witten’s success in describing the Jones polynomial using Chern-Simons notion of quantum
theory [37], leading to Atiyah’s pioneering work defining the topological quantum field theories [1].

A standard graphical calculus arose to calculate these colored invariants. Soon, similar invariants were defined taking values in the Hopf algebra $\mathcal{H}$ itself, as done by Lawrence [19] and Reshetikhin [27]. We will call these universal quantum invariants, because colored quantum invariants can be recovered from these $\mathcal{H}$-valued invariants. There are now many versions, including variations for framed or unoriented links. A central aspect is the use of decorated curves, a form of graphical calculus using immersed curves in which points are labeled by elements of $\mathcal{H}$. Further, these methods typically do not define an element of $\mathcal{H}$ itself, but an equivalence class in some linear quotient space of $\mathcal{H}$. For $S$ the antipode map of $\mathcal{H}$, let $C_{S^2}(\mathcal{H}) \subset \mathcal{H}$ be the linear subspace generated by all elements of the form $xy - S^2(y)x$, for all $x,y \in \mathcal{H}$. The version which we use in this paper is defined in Chapter 3, which takes values in the linear quotient space $\mathcal{H} / C_{S^2}(\mathcal{H})$. For a slightly different version, and for explicit details relating to the colored invariants, see [24].

The major goal of this dissertation is to define an extended version of this invariant for links in an arbitrary thickened surface.

A partial extension is given in [4], where an invariant for general tangle diagrams on the annulus is defined. This paper was inspired by Turaev’s idea of knotoids, [35], as knotoids can be seen as a certain quotient of 1-1 tangles on the annulus. When evaluated using the fundamental representation of $U_q(\mathfrak{sl}_2)$, the invariant specializes to Turaev’s extended bracket polynomial for knotoids. The extended bracket polynomial for knotoids is, in turn, a generalization of the Kauffman bracket of links, which can be thought of as a version of the Jones polynomial.
The invariant of [4] depends on the choice of a ribbon automorphism of the ribbon algebra $\mathcal{H}$, which accounts for the additional parameter in [35]. A ribbon automorphism is a Hopf algebra automorphism which respects the ribbon structure; see Chapter 4 for precise definitions. This ribbon automorphism is used to keep track of how a curve wraps around the cylinder. Because $\mathbb{R}^2$ has no non-trivial curves, this structure is not needed in the classical universal quantum invariant.

Defining the correct generalization of these decorated curves constitutes a large part of this paper. We require two generalizations to move to an arbitrary oriented surface $\Sigma$. First is the ribbon automorphism, which defines a group anti-homomorphism $\pi_1(C) \cong \mathbb{Z} \to \text{RAut}(\mathcal{H})$, where $\text{RAut}(\mathcal{H})$ is the group of ribbon automorphisms of $\mathcal{H}$; again, see Chapter 4. This element generalizes to a right group action of $\pi_1(\Sigma)$ on $\mathcal{H}$ via ribbon automorphisms; this action is equivalently defined by a group anti-homomorphism $\pi_1(\Sigma) \to \text{RAut}(\mathcal{H})$.

Second is the height function, which is implicit for both the plane and the cylinder. This function is used to define local maximums and minimums, which are important when defining decorated curves. A natural generalization of this would be a choice of non-singular vector field on $\Sigma$; unfortunately, many surfaces do not admit any such vector field.

Instead of choosing a vector field, we leave the surface $\Sigma$ and instead work with $T^1\Sigma$, the unit tangent bundle lying over $\Sigma$. Instead of a right-action by $\pi_1(\Sigma)$, we use a right-action by $\pi_1(T^1\Sigma)$, defined by a group anti-homomorphism

$$\rho : \pi_1(T^1\Sigma, b_0) \to \text{RAut}(\mathcal{H}).$$

This change allows us to define link invariants for a much broader class of surfaces. In order to retain the property that sliding a decorated point once around a circle
applies $S^\pm 2$ to the decoration, we must restrict which flat connections are allowed. This requires a special element of $\pi_1(T^1\Sigma, v_0)$.

For a basepoint $x_0 \in \Sigma$, let $T^1\Sigma_{x_0} \subset T^1\Sigma$ be the circle lying over $x_0$. For a fixed $v_0 \in T^1\Sigma_{x_0}$, let $f : [0, 1] \to T^1\Sigma_{x_0}$ be a simple closed loop with $f(0) = f(1) = v_0$. The homotopy class $[f]$ and its inverse $[f]^{-1}$ are the two generators of $\pi_1(T^1\Sigma_{x_0}, v_0) \subseteq \pi_1(T^1\Sigma, v_0)$. One of these two generators agrees with the orientation of $T^1\Sigma$, which is inherited from $\Sigma$; without loss of generality, suppose $[f]$ is this generator. This element is called the fiber generator over $x_0$ based at $v_0$. A group anti-homomorphism $\rho : \pi_1(T^1\Sigma, v_0) \to RAut(H)$ is called balanced if $\rho([f]) = S^2$.

The rule for applying $S^\pm 2$ at local extrema, defined by a local non-singular vector field, arises naturally from such a balanced anti-homomorphism. The map $S^2$ is also now on an equal footing as the other ribbon automorphisms arising from $\rho$.

These anti-homomorphisms are related to flat bundles over $T^1\Sigma$. As seen in Theorem 5.3.8, the map $\rho$ uniquely defines, up to isomorphism, a triple $(P, p_0, H)$, where $P$ is a principal $RAut(H)$-bundle over $T^1\Sigma$, $p_0 \in P$ is a basepoint lying over $b_0$, and $H$ is a flat connection on $P$ such that its related holonomy map is $\rho$. This principal bundle then defines an associated Hopf algebra bundle over $T^1\Sigma$ with typical fiber $H$, again with flat connection, as shown in Section 5.4. A flat connection on a principal $RAut(H)$-bundle over $T^1\Sigma$ is called balanced if its related holonomy anti-homomorphism is balanced. In Chapter 6, flat Hopf algebra bundles are used to define a notion of a decorated curve in an arbitrary manifold. The results are applied to the 3-manifold $T^1\Sigma$ in Chapter 7. In particular, we show how local flat charts naturally lead to a local calculus of decorated curves in the style of Lawrence [19] and Reshetikhin [27].
The final piece is finding the proper space for the invariant to take values in. Using a balanced anti-homomorphism \( \rho \), we define the semi-direct Hopf algebra

\[
\mathcal{SD} = \mathcal{SD}(\pi_1(T^1\Sigma, b_0), \rho, \mathcal{H}) = \pi_1(T^1\Sigma, b_0) \rtimes_\rho \mathcal{H}.
\]

We then take two quotients of this Hopf algebra, as defined in Section 4.5. The first requires the fiber generator \([f]\), as well as the balancing element \( \kappa \in \mathcal{H} \) as in Lemma 4.3.8. Notice that

\[
\rho_{[f]}(x) = S^2(x) = \kappa x \kappa^{-1}.
\]

Thus by Lemma 4.5.3, the element \(( [f], \kappa) \in \mathcal{SD} \) is central and group-like. Define \( \mathcal{SD} \) to be the algebraic quotient of \( \mathcal{SD} \) by \(( [f], \kappa) \), as in Lemma 4.5.2. This quotient is again a Hopf algebra. Next, we define \( \mathcal{SD} \) to be the commutator quotient space of \( \mathcal{SD} \), as defined in Definition 4.5.5. This is the linear quotient of \( \mathcal{SD} \) by the linear subspace generated by elements of \( xy - yx \) for \( x, y \in \mathcal{SD} \).

The set of links on \( \Sigma \), denoted \( \mathcal{L}(\Sigma) \), is defined in Section 3.2 as the equivalence classes of link diagrams on \( \Sigma \) modulo ambient isotopy and the Reidemeister Moves. This set is naturally associated to the set of links in the thickened surface \( \Sigma \times [0, 1] \). Similarly, the set of framed links on \( \Sigma \), denoted \( \mathcal{FL}(\Sigma) \), is the set of equivalence classes of link diagrams modulo the framed Reidemeister Moves. The subsets of links and framed links with \( m \) components are denoted \( \mathcal{L}(\Sigma)^m \) and \( \mathcal{FL}(\Sigma)^m \), respectively.

With these generalizations we can state our main theorem.

**Theorem 1.0.1.** Consider a connected, oriented surface \( \Sigma \) which is not diffeomorphic to the sphere, and basepoint \( b_0 \in \Sigma \). Given a balanced anti-homomorphism

\[
\rho : \pi_1(T^1\Sigma, b_0) \to RAut(\mathcal{H}),
\]


there are invariants of framed and unframed links on $\Sigma$, written

$$\mathcal{J}_{\text{fr}}: \mathcal{FL}(\Sigma)^m \to \mathcal{S}\mathcal{D}(\pi_1(T^1\Sigma, b_0), \rho, \mathcal{H})^\otimes m$$

and

$$\mathcal{J}_{\text{ufr}}: \mathcal{L}(\Sigma)^m \to \mathcal{S}\mathcal{D}(\pi_1(T^1\Sigma, b_0), \rho, \mathcal{H})^\otimes m.$$  

If $\rho'$ is another balanced anti-homomorphism defined by

$$\rho'(c) = g\rho(c)g^{-1}$$

for some $g \in \text{RAut}(\mathcal{H})$, and $(\mathcal{J}_{\text{fr}})'$ and $(\mathcal{J}_{\text{ufr}})'$ are its related link invariants, then

$$id \otimes g: \mathcal{S}\mathcal{D}(\pi_1(T^1\Sigma, b_0), \rho, \mathcal{H}) \to \mathcal{S}\mathcal{D}(\pi_1(T^1\Sigma, b_0), \rho', \mathcal{H})$$

is a Hopf algebra isomorphism which induces a linear isomorphism

$$\lambda^\otimes m: \mathcal{S}\mathcal{D}(\pi_1(T^1\Sigma, b_0), \rho, \mathcal{H})^\otimes m \to \mathcal{S}\mathcal{D}(\pi_1(T^1\Sigma, b_0), \rho', \mathcal{H})^\otimes m$$

satisfying

$$(\mathcal{J}_{\text{fr}})' = \lambda^\otimes m \circ \mathcal{J}_{\text{fr}} \quad \text{and} \quad (\mathcal{J}_{\text{ufr}})' = \lambda^\otimes m \circ \mathcal{J}_{\text{ufr}}.$$  

This theorem follows from a more general result which uses a ribbon Hopf algebra over $T^1\Sigma$ with a balanced, flat connection. This is because the anti-homomorphism $\rho$ defines, up to isomorphism, a triple $(P, p_0, H)$, where $P \to T^1\Sigma$ is a principal $\text{RAut}(\mathcal{H})$-bundle, $p_0 \in P_{b_0}$ is a fixed basepoint, and $H$ is a balanced flat connection on $P$. This, in turn, defines the associated Hopf algebra bundle $E \to T^1\Sigma$, along with a balanced flat connection on it. The basepoint $p_0$ defines a trivialization $\phi_0: E_{b_0} \to \mathcal{H}$. With this, we can apply the following proposition.
Proposition 1.0.2. Consider a connected, oriented surface $\Sigma$ which is not diffeomorphic to the sphere along with a ribbon Hopf algebra bundle $\pi : E \to T^1\Sigma$ with a balanced flat connection. For a fixed $b_0 \in T^1\Sigma$, let $E_0 = \pi^{-1}(b_0)$ and define the holonomy anti-homomorphism

$$\gamma : \pi_1(T^1\Sigma, b_0) \to \text{RAut}(E_0).$$

Then there is an invariant of framed links on $\Sigma$ which, for a link with $m$ strands, takes values in the quotient space $SD(\pi_1(T^1\Sigma, b_0), \gamma, E_0)^\otimes m$:

$$\mathcal{J}^{\text{fr}} = \mathcal{J}^{\text{fr}}(\Sigma, \pi, b_0) : \mathcal{F}\mathcal{L}(\Sigma)^m \to SD(\pi_1(T^1\Sigma, b_0), \gamma, E_0)^\otimes m.$$

There is also a normalized invariant for unframed links,

$$\mathcal{J}^{\text{ufr}} : \mathcal{L}(\Sigma)^m \to SD(\pi_1(T^1\Sigma, b_0), \gamma, E_0)^\otimes m.$$

If $E' \to T^1\Sigma$ is another ribbon Hopf algebra bundle with balanced flat connection, holonomy map $\gamma'$, related link invariants $(\mathcal{J}^{\text{fr}})'$ and $(\mathcal{J}^{\text{ufr}})'$, and $F : E \to E'$ is a flat bundle isomorphism which is the identity on the base space $T^1\Sigma$, then

$$\text{id} \otimes F|_{E_0} : SD(\pi_1(T^1\Sigma, b_0), \gamma, E_0) \to SD(\pi_1(T^1\Sigma, b_0), \gamma', E'_0)$$

is a Hopf algebra isomorphism which induces a linear isomorphism

$$\lambda^{\otimes m} : SD(\pi_1(T^1\Sigma, b_0), \gamma, E_0)^\otimes m \to SD(\pi_1(T^1\Sigma, b_0), \gamma', E'_0)^\otimes m$$

satisfying

$$(\mathcal{J}^{\text{fr}})' = \lambda^{\otimes m} \circ \mathcal{J}^{\text{fr}} \quad \text{and} \quad (\mathcal{J}^{\text{ufr}})' = \lambda^{\otimes m} \circ \mathcal{J}^{\text{ufr}}.$$
1.1 Outline of Paper

Chapter 2 defines a version of the universal invariant for links. This is the version which we refer to and generalize throughout the paper. Chapter 3 defines links in a thickened surface, and relates them to link diagrams on the surface itself. Chapter 4 defines the basic Hopf algebra structures which we will need. In particular, the semidirect Hopf algebra construction is defined in Section 4.2.

Chapter 5 defines ribbon Hopf algebra bundles. These are used in Chapter 6, where flat Hopf algebra bundles are used to define decorated curves in an arbitrary surface. Section 6.3 discusses the connection between decorated curves and the semidirect Hopf algebra SD($b_0, \Gamma, E_0$). In Section 6.4, we introduce a method of adding new decorations to a curve, and discuss when this is well-defined.

Chapter 7 applies the previous chapters to the 3-manifold $T^1\Sigma$. Section 7.2 is particularly important, as the local diagrams introduced there will be our main method of working with decorated curves. In Section 7.3, the two quotients of the semidirect product SD are defined and discussed.

Chapter 8 fully defines the invariant. The main results, Theorem 1.0.1 and Proposition 1.0.2, are proven, a general method of calculation is shown, and examples are given.

1.2 Future Research

There are several possible avenues for future research. We outline a few of them here.

First, it should be possible to generalize the technical details. Extending the invariant to infinite-dimensional Hopf algebras is a natural generalization. Generalizing
from links to tangles should also be possible. A natural way of doing this might be to create a category of surfaces with tangle diagrams and define the invariant as a functor, or even a 2-functor.

Another possible application is to create an invariant of virtual links. Virtual links were defined by Kauffman [15]. Carter, Kamado and Saito have shown that virtual links are equivalent to stable equivalence classes of links on surfaces [31]. An invariant of links on surfaces defines an invariant of virtual links if it is constant under the so-called stabilization and de-stabilization moves. The difficulty is that these moves change not just the link, but the surface itself. Showing invariance under these moves requires a method to canonically define new Hopf algebra bundles on the new surface.
CHAPTER 2
CLASSICAL INVARIANT

We will introduce a version of the classical universal invariant described in the introduction. The new invariants $\mathcal{I}_{fr}$ and $\mathcal{I}_{ufr}$ are a generalization of this invariant, and as such an understanding of this chapter is very useful for the rest of the paper. We stay light on technical details in this chapter, but many of the terms used will be explained in detail later. For more details about the classical universal invariant see, for example, [24].

Let $\mathcal{H}$ be a ribbon Hopf algebra as defined in Section 4.3. This comes equipped with an antipode map $S$, an $R$-matrix $R = \sum_i a_i \otimes b_i \in \mathcal{H}^{\otimes 2}$ and a ribbon element $\nu \in \mathcal{H}$. Also recall the special element $u = \sum_i S(b_i)a_i$, and the balancing element $\kappa = uv^{-1}$ as seen in Lemma 4.3.8.

Let $C_{S^2}(\mathcal{H}) \subset \mathcal{H}$ be the linear subspace generated by elements $xy - S^2(y)x$ for all $x, y \in \mathcal{H}$. Then define the linear quotient space $\hat{\mathcal{H}} = \mathcal{H}/C_{S^2}(\mathcal{H})$. The universal invariant takes values in $\hat{\mathcal{H}}$.

Let $\hat{h} \in \hat{\mathcal{H}}$ denote the image of an element $h \in \mathcal{H}$ under the linear projection $\mathcal{H} \to \hat{\mathcal{H}}$. Notice that, because $\nu$ is central in $\mathcal{H}$, for any $\hat{h} \in \hat{\mathcal{H}}$ the element $\nu^m\hat{h} \in \hat{\mathcal{H}}$ is well-defined, independent of the choice of representative $h$ of $\hat{h}$.

The calculation process for a knot $k$ can be summarized in three steps:
1. Flatten crossings and add decorations according to Figure 2.1.

2. Simplify the resulting decorated curve and collect decorations into one element 
   \( I(k) \in \mathcal{H} \), using the relations shown in Figures 2.4 - 2.7.

3. Project onto \( \hat{\mathcal{H}} \).

![Diagram](attachment:image.png)

Figure 2.1: Definition of universal invariant at crossings.

*Remark* 2.0.1. We only gave a rule for adding decorations at a crossing when both 
crossing strands are oriented downward. The rule for adding decorations in other 
cases can be deduced from Figure 2.1 and the equivalence moves shown in Figures 
2.4 - 2.7. An example of this is seen in Figure 2.2.

For the normalized invariant of unframed knots, we need the *writhe* of a knot 
diagram. For an oriented knot we call a crossing as in the left side of Figure 2.1 a 
*positive crossing*, and one as in the right side of Figure 2.1. For a knot diagram \( k \),
let $c_+(k)$ denote the number of positive crossings, and $c_-(k)$ the number of negative crossings. Then the writhe is

$$\text{wr}(k) = c_+(k) - c_-(k).$$

We can now define the invariant.

**Lemma 2.0.2.** The element $\hat{I}(k) \in \hat{\mathcal{H}}$ is an invariant of framed knots. The element $\nu^{\text{wr}(k)} I(k)$ is an invariant of unframed knots, where $\nu$ is the central ribbon element of $\mathcal{H}$.

We will show these steps in an example: Consider the figure-eight knot $k$ shown in Figure 2.3. The first step in calculating the invariant is to flatten all crossings, and
add decorations to the resulting curve. Decorations corresponding to \( \mathcal{R} \) or \( \mathcal{R}^{-1} \) are added at every crossing, as shown in Figure 2.1. This process applied to \( k \) is shown in Figure 2.3. The resulting object is called a *decorated curve* in \( \mathbb{R}^2 \). We will use the following precise definition.

![Figure 2.3: First step of invariant applied to a figure eight knot](image)

**Definition 2.0.3.** A *decorated curve* in \( \mathbb{R}^2 \) is a triplet \( (\mathcal{C}, \overline{x}, h) \), where

1. \( \mathcal{C} = \{c_1, \cdots, c_n\} \) is a set of immersed curves \( c_i : S^1 \to \mathbb{R}^2 \) such that the image \( c_1 \cup \cdots \cup c_n \) is in general position, as defined below.

2. \( \overline{x} = \{x_1, \cdots, x_m\} \) is a set of distinct points lying on the image of \( \mathcal{C} \); that is, \( x_i = c_j(t) \) for some \( j \) and \( t \), and \( x_i \neq x_j \) for all \( i \neq j \).

3. \( h \in \mathcal{H}^\otimes n \).
The element $h$ may be written as a sum

$$h = \sum_i h_i^1 \otimes h_i^2 \otimes \cdots \otimes h_i^m.$$ 

In diagrams, the decorated points $x_j$ are represented by dots, and the decoration $h_i^j$ is printed nearby. This is seen in the example.

![Diagram](image)

(a) RI-type Move  (b) RI-type Move

(c) RII-type Move  (d) RIII-type Move

Figure 2.4: Reidemeister type equivalence moves for decorated curves

Arbitrary curves in $\mathbb{R}^2$ may contain a variety of singularities. This is avoided by requiring decorated curves to be in *general position*, meaning that the following conditions hold:

1. All intersection points are transverse intersection of two strands.
2. The set of local extrema is discrete and non-degenerate, and disjoint from the set of intersection points.

3. The set of decorated points are disjoint from the intersection points and the local extrema.

Notice that defining local extrema implicitly uses a height function on \( \mathbb{R}^2 \). We assume that this function is fixed throughout, and that diagrams are drawn such that this function is given by \((x, y) \mapsto y\). A local extrema is non-degenerate if it is a local maximum or minimum.

![Diagram](image)

(a) Sliding a decoration past an intersection point  
(b) Two decorations being multiplied

Figure 2.5: Equivalence moves involving critical points

Decorated curves have an equivalence relation similar to that of links, defined by ambient isotopy and the set of equivalence moves shown in Figures 2.4 - 2.7. The equivalence rules in Figure 2.4 are analogous to the Reidemeister moves for link diagrams. Figure 2.5 shows how a decoration may be moved past an intersection point. It also shows how two decorations which lie adjacent to each other may be
multiplied. The set of moves shown in Figure 2.6 describe how to move a decoration past local extrema by possibly applying $S^\pm 2$ depending on the case. Finally, the set of moves shown in Figure 2.7 is needed because we require decorated curves to be in general position.

Using these moves, we may transform any decorated curve into a simple embedded loop. Then the decorations can be multiplied in turn, until only one is left. This decoration is decorated by the element $I(k) \in \mathcal{H}$. This process applied to our example $k$ is shown in Figure 2.8. The end result is the element

$$I(k) = \sum_{i,j,k,n} b_n S^2(a_k) b_j S^{-3}(a_i) S^{-2}(b_k) a_n b_i S^{-1}(a_j).$$

The writhe for $k$ is 0, so in this case the unframed knot invariant is the same

$$\nu^{wr(k)} I(k) = \overline{I(k)}.$$
The decoration is not completely well-defined. We can slide the decorated point clockwise around the curve, which transforms $h$ into $S^2(h)$. More generally, if we can write $h = h_1 h_2$, then we can slide the decoration $h_2$ around the curve and leave the decoration $h_1$ still. This transforms the decoration $h_1 h_2$ into $S^2(h_2) h_1$. This is the reason why we must project onto the quotient space $\hat{\mathcal{H}}$. 

Figure 2.7: A few more equivalence moves for decorated curves.
$S^{-1}(a_j) b_i \\ b_j S^{-1}(a_i) \\ b_k S^2(a_k) \\ S^2(a_n) b_n$

Figure 2.8: Simplifying a decorated curve
CHAPTER 3

LINKS

In this chapter, we define the generalized notion link diagrams on a connected, oriented surface $\Sigma$, which represent links in the thickened surface $M_\Sigma = \Sigma \times [-1,1]$.

3.1 Links in Thickened Surfaces

In this section we define links in a thickened surface $M_\Sigma$. Isotopies of $M_\Sigma$ act on embedded curves, and a link is defined as an equivalence class of this action.

**Definition 3.1.1.** For $m$ a positive integer an *embedded curve system in $M_\Sigma$ with $m$ components* is a smooth embedding

$$C = \prod_{i=1}^{m} C_i : \prod_{i=1}^{m} S^1 \to M_\Sigma$$

such that

$$\text{Im} \ C \cap \partial M_\Sigma = \emptyset.$$

The set of curve systems in $M_\Sigma$ is denoted $\text{ECS}(M_\Sigma)$, and the subset of curve systems with $m$ components is denoted $\text{ECS}(M_\Sigma)^m$.

The group of smooth diffeomorphisms of $M_\Sigma$ is denoted $\text{Diff}(M_\Sigma)$. For $f \in \text{Diff}(M_\Sigma)$ and $C \in \text{ECS}(M_\Sigma)$, we let $f \circ C$ denote the curve system where each component is composed by $f$. 

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Lemma 3.1.2. The map \((f, C) \mapsto f \circ C\) defines a left group action of \(\text{Diff}(M_\Sigma)\) on \(\text{ECS}(M_\Sigma)\).

Let \(\text{Diff}_0(M_\Sigma) \subset \text{Diff}(M_\Sigma)\) denote the connected component containing the identity. Notice that \(\text{Diff}_0(M_\Sigma)\) is a subgroup of \(\text{Diff}(M_\Sigma)\). This subgroup can also be defined as diffeomorphisms which are isotopic to the identity, as a path in \(\text{Diff}(M_\Sigma)\) corresponds to an isotopy.

We now define links by restricting the diffeomorphism action to \(\text{Diff}_0(M_\Sigma)\).

Definition 3.1.3. Two curve systems \(C_1, C_2\) are amblytly isotopic if \(C_2 = f \circ C_1\) for some \(f \in \text{Diff}_0(M_\Sigma)\).

Equivalently, we can say that two curve systems are amblytly isotopic if they lie in the same orbit under the \(\text{Diff}_0(M_\Sigma)\)-action.

Ambient isotopy defines an equivalence relation on \(\text{CS}(M_\Sigma)^m\). Any such equivalence class is called a link in \(M_\Sigma\) with \(m\) components. A link with 1 component is called a knot. The set of links and knots is denoted \(\mathcal{L}(M_\Sigma)\) and \(\mathcal{K}(M_\Sigma)\) respectively.

Remark 3.1.4. We note that for any smooth isotopy of a curve extends to an isotopy of \(M_\Sigma\. While we use ambient isotopy, defining isotopy in the usual way would yield the same theory.

3.2 Link Diagrams on Oriented Surfaces

It is well known that links in \(\mathbb{R}^3\) are equivalent to link diagrams in \(\mathbb{R}^2\) modulo the three Reidemeister moves. In this section we will define link diagrams in \(\Sigma\), and
then show that links in $M_\Sigma$ are equivalent to link diagrams in $\Sigma$ modulo the same Reidemeister moves.

**Definition 3.2.1.** For $m$ a positive integer, an immersed curve system in $\Sigma$ with $m$ components is a smooth immersion

$$\mathcal{C} = \prod_{i=1}^m c_i : \prod_{i=1}^m S^1 \to \Sigma$$

such that

$$\text{Im} \mathcal{C} \cap \partial \Sigma = \emptyset.$$  

An immersed curve system is in general position if it has only finitely many self-intersection points, each of which is a transverse double point. At a transverse double point, crossing data is a choice of one of the two intersecting strands, thought of as the over-crossing strand. A link diagram with $y$ components is an immersed curve system with $y$ components which is in general position, along with crossing data for each of its intersection points. The set of link diagrams is denoted $\mathbb{L}\mathcal{D}(\Sigma)$.

Let $\text{Diff}(\Sigma)$ be the group of diffeomorphisms of $\Sigma$, and $\text{Diff}_0(\Sigma)$ the connected component which contains the identity. This component is also a subgroup.

For $f \in \text{Diff}(\Sigma)$ and $L$ a link diagram, the composition $f \circ L$ is also a link diagram.

**Definition 3.2.2.** Two link diagrams $L_1, L_2$ are ambiently isotopic if $L_2 = f \circ L_1$ for some $f \in \text{Diff}_0(\Sigma)$.

An isotopy of $\Sigma$ induces an isotopy of $M_\Sigma$. This defines a subgroup $\text{Diff}_0(\Sigma) \hookrightarrow \text{Diff}_0(M_\Sigma)$. However, this restricted isotopy will not capture all of the possible movement of a link in three dimensions. To fix these gaps, a few equivalence moves must
be added. These local moves correspond to the classical Reidemeister Moves, shown in Figure 3.1. We will define two ways of taking equivalence classes of $LD(\Sigma)$. These moves are local, taking place in contractible open sets in $\Sigma$.

**Definition 3.2.3.** Two link diagrams $L, L'$ are *unframed equivalent* if $L$ can be transformed into $L'$ by a finite sequence of ambient isotopy, and the moves 3.1a, 3.1b, and 3.1d. They are *framed equivalent* if $L$ can be transformed into $L'$ by a finite sequence of ambient isotopy, and the moves 3.1c, 3.1b, and 3.1d.

A *framed link* is framed equivalence class of link diagrams, the set of which is denoted $\mathcal{FL}(\Sigma)$. An *(unframed) link* is an unframed equivalence class of link diagrams, the set of which is denoted $\mathcal{L}(\Sigma)$. A knot is a link with only one component, and the set of knots is denotes $\mathcal{K}(\Sigma)$

*Remark 3.2.4.* We use the notation $\mathcal{L}(\ )$ to denote both links in $M_\Sigma$ and unframed links in $\Sigma$. This is for a natural reason, as the two sets are equivalent to each other. This is shown in the next lemma.

Notice that Figure 3.1c is the union of two copies of Figure 3.1a. Hence every framed equivalence is also an unframed equivalence, and there is a surjection $\mathcal{FL}(\Sigma) \to \mathcal{L}(\Sigma)$.

**Lemma 3.2.5.** *There is a natural bijection between $\mathcal{L}(\Sigma)$ and $\mathcal{L}(M_\Sigma)$.*

*Proof.* We give a sketch of this proof. The full details would require invoking machinery such as Sard’s theorem, the Thom-Transversality theorem, and Morse Theory. To see these details used to define a Reidemeister-like theorem for higher-dimensional knotted manifolds, see [30].
First, we define a map which takes an immersed curve system \( c \) in \( \Sigma \) to an embedded curve system \( \hat{c} \) in \( M_\Sigma \). We define \( \hat{c} \) by first mapping the surface \( \Sigma \), including the curve system \( c \), to \( \Sigma \times \{ \frac{1}{2} \} \). Then remove every intersection point in \( c \) by lifting the over-crossing strand slightly.

Any ambient isotopy in \( \Sigma \) lifts to an ambient isotopy in \( M_\Sigma \). Further, the Reidemeister moves have obvious three-dimensional equivalents. Thus, equivalent tangle diagrams in \( \Sigma \) map to equivalent curve systems in \( M_\Sigma \). Thus the map \( c \mapsto \hat{c} \) is well-defined.

We can define an inverse map by applying the projection \( p : M_\Sigma \to \Sigma \) to \( \mathcal{CS}(M_\Sigma) \).
We claim that every tangle in $M_{\Sigma}$ has a representative $c$ such that $p(c)$ is an immersed curve system in $\Sigma$. It is a standard result that inside the set of smooth maps $C^\infty(S^1, \Sigma)$, the set of maps which are in general position forms an open and dense subset. Thus any given representative of a tangle class in $M_{\Sigma}$ can be perturbed to find a representative $c$ which maps to an immersed curve $p(c)$. The intersection points of $p(c)$ can then be given over-crossing and under-crossing data using the $[0,1]$ component of $M_{\Sigma}$.

Next, we must show that if two curve systems $c_1, c_2$ represent the same tangle in $M_{\Sigma}$, then $p(c_1), p(c_2)$ both represent the same tangle diagram in $\Sigma$. A homotopy connecting $c_1$ to $c_2$ descends via $p$ to a homotopy between $p(c_1)$ and $p(c_2)$. This also defines a path $\rho : [0, 1] \rightarrow C^\infty(S, \Sigma)$. We cannot say that $\rho(t)$ is in general position for each $t$, but we can at least say that $\rho(t)$ has a singularity at only one point, and the singularity has dimension 1. This means that it is either a triple intersection point, with not all three strands co-linear, or a non-transversal double intersection point. These two cases are then covered by the second and third Reidemeister moves.

Thus we have a well-defined map $\overline{p} : \mathcal{L}(M_{\Sigma}) \rightarrow \mathcal{L}(\Sigma)$. Clearly $\overline{p}$ is the inverse map to $c \mapsto \hat{c}$. Thus both maps are bijections. \hfill \qed
CHAPTER 4

HOPF ALGEBRAS

In this chapter we will introduce Hopf algebras, including everything which will be needed later. Our main reference is Kassel [14].

All vector spaces are assumed to be over a fixed field $k$.

4.1 Flavors of Algebra

Definition 4.1.1. An algebra is a triplet $(V, \mu, \eta)$, where $V$ is a finite-dimensional vector space, and $\mu : V \otimes V \to V$ and $\eta : k \to V$ are linear maps satisfying the following commutative diagrams:

\[
\begin{array}{ccc}
V \otimes V \otimes V & \xrightarrow{\mu \otimes id} & V \otimes V \\
\downarrow{id \otimes \mu} & & \downarrow{\mu} \\
V \otimes V & \xrightarrow{\mu} & V \\
\end{array}
\]

\[
\begin{array}{ccc}
k \otimes V & \xrightarrow{\eta \otimes id} & V \otimes V \\
\downarrow{\cong} & & \downarrow{\cong} \\
V & \xleftarrow{id \otimes \eta} & V \otimes k \\
\end{array}
\]

An algebra is commutative if it satisfies the additional axiom

\[
\begin{array}{ccc}
V \otimes V & \xrightarrow{\tau_{V,V}} & V \otimes V \\
\downarrow{\mu} & & \downarrow{\mu} \\
V & & V \\
\end{array}
\]
where $\tau_{V,V}(a \otimes b) = b \otimes a$.

A coalgebra is a triplet $(V, \Delta, \epsilon)$, where $\Delta : V \to V \otimes V$ and $\epsilon : V \to \mathbb{k}$ are linear maps satisfying the following commutative diagrams:

\[
\begin{array}{ccc}
V & \xrightarrow{\Delta} & V \otimes V \\
\downarrow{\Delta} & & \downarrow{id \otimes \Delta} \\
V \otimes V & \xrightarrow{\Delta \otimes id} & V \otimes V \otimes V
\end{array}
\quad \quad
\begin{array}{ccc}
\mathbb{k} \otimes V & \xrightarrow{\epsilon \otimes id} & V \otimes V \\
\downarrow{\cong} & & \downarrow{\cong} \\
V & \xrightarrow{id \otimes \epsilon} & V \otimes \mathbb{k}
\end{array}
\]

The coproduct $\Delta$ is cocommutative if the following diagram also commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{\Delta} & V \\
\downarrow{\Delta} & & \downarrow{\tau_{V,V}} \\
V \otimes V & \xrightarrow{\Delta} & V \otimes V
\end{array}
\]

Remark 4.1.2. The coproduct of an element $v \in V$ is generally a sum

$$\Delta(v) = \sum_i v_i' \otimes v_i''.$$ 

We will use Sweedler’s sunless notation, which drops the sum and the subscript, and simply write

$$\Delta(v) = v' \otimes v''.$$ 

A bialgebra is a vector space which is simultaneously an algebra and a coalgebra, with the two structures being compatible in the sense of the following theorem.

**Theorem 4.1.3** ([14], Theorem III-2.1). The following two statements are equivalent

1. The maps $\mu$ and $\eta$ are morphisms of coalgebras.

2. The maps $\Delta$ and $\epsilon$ are morphisms of algebras.
The fact that $\mu$ and $\eta$ are morphisms of coalgebras mean that the following diagrams commute:

\[
\begin{array}{ccc}
V \otimes V & \xrightarrow{\mu} & V \\
\downarrow{(id \otimes \tau \otimes id)(\Delta \otimes \Delta)} & & \downarrow{\Delta} \\
(V \otimes V) \otimes (V \otimes V) & \xrightarrow{\mu \otimes \mu} & V \otimes V
\end{array}
\quad
\begin{array}{ccc}
V \otimes V & \xrightarrow{\epsilon \otimes \epsilon} & k \\
\downarrow{\mu} & & \downarrow{\epsilon} \\
V & \xrightarrow{id} & k
\end{array}
\]

**Definition 4.1.4.** A bialgebra is $(V, \mu, \eta, \Delta, \epsilon)$ such that $(V, \mu, \eta)$ is an algebra, $(V, \Delta, \epsilon)$ is a coalgebra, and they satisfy the conditions of the previous theorem.

An antipode for a bialgebra is a linear isomorphism $S : V \rightarrow V$ which satisfies the axiom

\[S(v')v'' = v'S(v'') = \epsilon(v)1 \forall v \in V\]

A Hopf algebra is a bialgebra with antipode.

For two Hopf algebras $(\mathcal{H}, \mu, \eta, \Delta, \epsilon, S)$ and $(\mathcal{H}', \mu', \eta', \Delta', \epsilon', S')$, a linear map $f : \mathcal{H} \rightarrow \mathcal{H}'$ is a Hopf algebra morphism if it respects the Hopf algebra structure, meaning it satisfies the following relations.

\[\mu' \circ (f \otimes f) = f \circ \mu, \quad f \circ \eta = \eta', \quad (f \otimes f) \circ \Delta = \Delta' \circ f, \quad \epsilon = \epsilon' \circ f, \quad f \circ S = S' \circ f.\]

The set of Hopf algebra homomorphisms is denoted $\text{Hom}(\mathcal{H}, \mathcal{H}')$, and the set of automorphisms of $\mathcal{H}$ is denoted $\text{Aut}(\mathcal{H})$.  

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Remark 4.1.5. This definition implies that the antipode is an anti-homomorphism and anti-cohomomorphism. Not every bialgebra has an antipode. If it does have an antipode, it is unique. [14]

Example 4.1.6. Let $G$ be a group with identity element $1_G$. A Hopf algebra structure exists for the group algebra

$$\mathbb{k}[G] = \left\{ \sum_{g \in G} k_g g \mid k_g = 0 \text{ for all but finitely many } g \in G \right\}.$$  

This vector space is generated by elements of $G \subset \mathbb{k}[G]$. The Hopf algebra operations are defined as follows for elements $g, h \in G$, and then linearly extended to $\mathbb{k}[G]$.

$$\eta(1) = 1_G, \quad \mu(g, h) = gh,$$
$$\eta(g) = 1_k, \quad \Delta(g) = g \otimes g, \quad S(g) = g^{-1}.$$  

Example 4.1.7. When $G$ is a finite group we can define the dual Hopf algebra to the previous example. This is $\text{Fun}(G)$, the set of $\mathbb{k}$-valued functions on $G$. The Hopf algebra structure on $\text{Fun}(G)$ is defined as follows for $f, g \in \text{Fun}(G)$ and $x, y \in G$.

$$\epsilon(f) = f(1_G), \quad \mu(f, g)(x) = f(x)g(x),$$
$$\Delta(f)(x \otimes y) = f(xy), \quad \epsilon(f) = f(1_G), \quad S(f)(x) = f(x^{-1})$$  

4.2 Semidirect Product

This section gives another example of a Hopf algebra using the semidirect product construction. A Hopf algebra of this type will be used to construct the codomain of
our invariant. This type of construction is explained in detail in [23], where the term smash product is used.

Let \( \mathcal{H} \) be a Hopf algebra over \( \mathbb{k} \), and \( G \) a group acting on it by a group anti-homomorphism \( \rho : G \to \text{Aut}(\mathcal{H}) \); that is, \( \rho(gh) = \rho(h) \circ \rho(g) \) for all \( g, h \in G \). We use the notation \( \rho_g = \rho(g) \). This action extends linearly to a right \( \mathbb{k}[G] \)-action on \( \mathcal{H} \).

**Definition 4.2.1.** The *semidirect product*

\[
\text{SD}(G, \rho, \mathcal{H}) := \mathbb{k}[G] \rtimes_{\rho} \mathcal{H}
\]

is isomorphic, as a vector space, to \( \mathbb{k}[G] \otimes_{\mathbb{k}} \mathcal{H} \). This tensor space is generated linearly by pairs \( (g, x) = g \otimes x \) for \( g \in G \) and \( x \in \mathcal{H} \). We define the following linear maps on the generators:

**Mult:** \( \mu((g, x) \otimes (h, y)) = (gh, \rho_h(x)y) \)

**CoMult:** \( \Delta(g, x) = \sum (g, x') \otimes (g, x'') \)

**Unit:** \( 1_{\text{SD}} = (1_G, 1_{\mathcal{H}}) \)

**CoUnit:** \( \epsilon(g, x) = \epsilon(x) \)

**Antipode:** \( S(g, x) = (g^{-1}, \rho_{g^{-1}}(S(x))) \)

**Lemma 4.2.2.** The vector space \( \text{SD} \) with operations defined above is a Hopf algebra.

**Proof.** \( \text{SD} \) is clearly an algebra and a coalgebra. To prove it is a bialgebra, we show the following diagrams commute:
For the first diagram, we first calculate

$$(\Delta \circ \mu)((g, x) \otimes (h, y)) = \Delta(gh, \rho_h(x)y)$$

$$= (gh, (\rho_h(x)y') \otimes (gh, (\rho_h(x)y''))).$$

Running the other way around the diagram, we see

$$[(\mu \otimes \mu) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta)][(g, x) \otimes (h, y)] =$$

$$= [\mu \otimes \mu][(g, x') \otimes (h, y') \otimes (g, x'') \otimes (h, y'')]$$

$$= (gh, \rho_h(x')y') \otimes (gh, \rho_h(x'')y'').$$

These two values agree because

$$(\rho_h(x)y') \otimes (\rho_h(x)y'') = \Delta(\rho_h(x)y)$$

$$= ((\rho_h \otimes \rho_h) \circ \Delta(x))\Delta(y)$$

$$= \rho_h(x')y' \otimes \rho_h(x'')y''.$$ 

For the second diagram, we calculate

$$\epsilon((g, x)(h, y)) = \epsilon(gh, \rho_h(x)y).$$
\[= \epsilon(\rho_h(x)y)\]
\[= \epsilon(\rho_h(x))\epsilon(y)\]
\[= \epsilon(x)\epsilon(y)\]
\[= \epsilon(g, x)\epsilon(h, y).\]

For the third diagram,
\[
\Delta(1_G, 1_H) = (1_G, 1'_H) \otimes (1_G, 1''_H) = (\Delta \otimes \Delta)(1 \otimes 1).
\]

And for the final equation,
\[
\epsilon(1_G, 1_H) = \epsilon(1_H) = 1.
\]

Thus \(\mathbb{k}[G] \ltimes \mathcal{H}\) is a bialgebra. To make this a Hopf algebra, we must show that \(S\) is an antipode by checking that the following diagram commutes:

First notice that
\[
(\eta \circ \epsilon)(g, x) = \epsilon(x)(1_G, 1_H).
\]

For the left side of the diagram, we have
\[
S(g, x')(g, x'') = (g^{-1}, \rho_{g^{-1}}(S(x')))(g, x'')
\]
\[
= (1_G, (\rho_g \circ \rho_{g^{-1}})(S(x'))x'')
\]
\[
= (1_G, \epsilon(x)1_H))
\]
\[
= \epsilon(x)(1_G, 1_H).
\]
Running around the right side, we calculate

\[ (g, x')S(g, x'') = (g, x')(g^{-1}, \rho_{g^{-1}}(S(x''))) \]
\[ = (1_G, \rho_{g^{-1}}(x') \cdot \rho_{g^{-1}}(S(x'')))) \]
\[ = (1_G, \rho_{g^{-1}}(x'S(x'')))) \]
\[ = (1_G, \epsilon(x)1_H) \]
\[ = \epsilon(x)(1_G, 1_H). \]

\[\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}\]

\[\mathcal{R} \Delta(x)\mathcal{R}^{-1} = (\tau \circ \Delta)(x) \text{ for all } x \in \mathcal{H}.\]

\[\Delta \otimes 1)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}.\]

\[1 \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}.\]

where if \(\mathcal{R} = \sum a_i \otimes b_i\), then \(\mathcal{R}_{13} = \sum a_i \otimes 1 \otimes b_i\), \(\mathcal{R}_{23} = 1 \otimes \mathcal{R}\), and \(\mathcal{R}_{12} = \mathcal{R} \otimes 1\).

The element \(\mathcal{R}\) is called the \(R\)-matrix.

\[\text{Remark 4.3.2.} \text{ Braided Hopf algebras are also commonly called quasi-triangular Hopf algebras.}\]
This $R$-matrix satisfies many nice properties. We cite the following:

**Theorem 4.3.3** (*[14]* Theorem VIII.2.4). Let $(\mathcal{H}, \mu, \eta, \Delta, \epsilon, R)$ be a braided bialgebra.

(a) Then the universal $R$-matrix $R$ satisfies

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \quad (4.3.1)$$

and we have

$$(\epsilon \otimes id)(R) = 1 = (id \otimes \epsilon)(R).$$

(b) If, moreover, $\mathcal{H}$ has an invertible antipode $S$, then

$$(S \otimes id)(R) = R^{-1} = (id \otimes S)(R) \quad (4.3.2)$$

and

$$(S \otimes S)(R) = R.$$ 

The $R$-matrix is particularly famous for solving the Yang-Baxter Equation 4.3.1.

We continue the notation for $R$ and its inverse:

$$R = \sum_i a_i \otimes b_i, \quad R^{-1} = \sum_i \overline{a}_i \otimes \overline{b}_i.$$ 

A particularly important element in a braided Hopf algebra is

$$u = \sum_i S(b_i)a_i.$$ 

We cite its most useful properties.

**Proposition 4.3.4** (*[14]* Proposition VIII.4.1). Let $\mathcal{H}$ be a braided Hopf algebra with an invertible antipode. Then the element $u$ is invertible in $\mathcal{H}$ with inverse given by

$$u^{-1} = \sum_i S^{-1}(\overline{b}_i)\overline{a}_i$$ 

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and for all \( x \in \mathcal{H} \), we have
\[
S^2(x) = uxu^{-1}.
\]

**Proposition 4.3.5** ( [14] Proposition VIII.4.5 ). The element \( uS(u) \in \mathcal{H} \) is central and satisfies:
\[
S(uS(u)) = uS(u), \quad \epsilon(uS(u)) = 1,
\]
\[
\Delta(uS(u)) = (R_{21}R_{12})^{-2}(uS(u) \otimes uS(u)).
\]

Creating a link invariant requires a fixed square root of the element \( uS(u) \). This motivates the following definition.

**Definition 4.3.6.** For \( \mathcal{H} \) a braided Hopf algebra, a **ribbon element** is a central element \( \nu \in \mathcal{H} \) satisfying:
\[
\epsilon(\nu) = 1, \quad S(\nu) = \nu, \quad \text{and} \quad \Delta(\nu) = (R_{21}R_{12})^{-1}(\nu \otimes \nu)
\]

A **ribbon Hopf algebra** is a braided Hopf algebra with fixed ribbon element.

**Remark 4.3.7.** If \( \mathcal{H} \) is finite-dimensional, this definition implies that \( \nu^2 = uS(u) \); see Corollary XIV.6.3 of [14].

Given a ribbon algebra \( \nu \), we can also define the **balancing element** \( \kappa = \nu^{-1} \). We will note some of its properties, which are easily checked, for use later.

**Lemma 4.3.8.** The element \( \kappa = \nu^{-1} \) is group-like, \( \kappa^2 = uS(u)^{-1} \), and for all \( x \in \mathcal{H} \),
\[
S^2(x) = \kappa x \kappa^{-1}.
\]

Given a braided Hopf algebra \( \mathcal{H} \) with \( R \)-matrix \( R \) and a group-like element \( \kappa \) satisfying these properties, the element \( \nu = \kappa \kappa^{-1} \) is a ribbon element for \( \mathcal{H} \). Thus the choice of ribbon element is equivalent to the choice of balancing element.
Example 4.3.9. A good first example of a braided Hopf algebra is the universal enveloping algebra \( U(\mathfrak{g}) \) of a Lie algebra \( \mathfrak{g} \). To define \( U(\mathfrak{g}) \), we let \( T(\mathfrak{g}) \) denote the tensor algebra of \( \mathfrak{g} \), and let \( I(\mathfrak{g}) \subset T(\mathfrak{g}) \) be the two-sided ideal generated by all elements of the form \( xy - yx - [x,y] \) for \( x, y \in \mathfrak{g} \). Then define

\[
U(\mathfrak{g}) = T(\mathfrak{g})/I(\mathfrak{g}).
\]

This quotient is again an algebra. It may be given a coalgebra structure by linearly extending the formulas

\[
\Delta(x) = 1 \otimes x + x \otimes 1 \quad \text{and} \quad \epsilon(x) = 0 \quad \text{for} \ x \in \mathfrak{g}.
\]

The antipode map is defined by

\[
S(x_1x_2\cdots x_n) = (-1)^n x_n x_{n-1} \cdots x_1,
\]

where each \( x_i \in \mathfrak{g} \).

Finally, we may make \( U(\mathfrak{g}) \) into a ribbon Hopf algebra with the trivial \( R \)-matrix \( R = 1 \otimes 1 \), and the trivial ribbon element \( \nu = 1 \). For more details and proofs, see [14].

It is an interesting fact that, while the set of Lie algebras is discrete, there may exist smooth deformations of \( U(\mathfrak{g}) \) in the space of Hopf algebras. These deformations have been an important topic in the theory of Hopf algebras since they were first studied by Drinfel’d [8, 9] and Jimbo [11]. The next example is a one-variable deformation of the universal enveloping algebra of \( \mathfrak{sl}_2 \). We follow the treatment given in [14] XVII.4.

Example 4.3.10. Fix a complex number \( q \) which is not a root of unity.

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We define $U_q = U_q(\mathfrak{sl}_2)$ as the algebra generated over $k$ by the four variables $E, F, K, K^{-1}$ with the relations

$$KK^{-1} = K^{-1}K = 1,$$

$$KE = q^2EK,$$

$$KF = q^{-2}FK,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

$U_q$ becomes a Hopf algebra when given the comultiplication $\Delta$, counit $\epsilon$, and antipode $S$ defined by

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$

$$\Delta(K^\pm) = K^\pm \otimes K^\pm,$$

$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = \epsilon(K^{-1}) = 1,$$

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K.$$

Proposition VI.1.4 from [14] says that the set $\{E^i F^j K^l\}_{i,j \in \mathbb{N}, l \in \mathbb{Z}}$ is a linear basis of $U_q$.

While $U_q$ has no $R$-matrix, it does inject into a larger Hopf algebra which does have one. We will not go into these details. Instead, we will only consider the $R$-matrix for the finite-dimensional quotient $\overline{U}_q$, as defined in the next example.

Example 4.3.11. The algebra $U_q(\mathfrak{sl}_2)$ can also be defined when $q$ is a root of unity. In these cases, a certain quotient of $U_q$ inherits a well-defined Hopf algebra structure from $U_q$. This finite-dimensional Hopf algebra has a well-defined $R$-matrix. This case was studied by Lusztig [21, 22]. We quickly summarize this space.

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Fix a complex number $q$ which is a $d$-th root of unity for an odd integer $d$ larger than 1. Let $\overline{U}_q$ be the quotient of $U_q$ by the two-sided ideal generated by the set 
\{ $E^d$, $F^d$, $K^d - 1$ $\}$.

Proposition VI.5.8 from [14] says that the set \{ $E^i F^j K^l$ $\}_{0 \leq i, j, l \leq d-1}$ is a linear basis of $U_q$. Proposition IX.6.1 shows that the natural quotient $U_q \rightarrow \overline{U}_q$ induces a well-defined Hopf algebra structure on $\overline{U}_q$.

The $R$-matrix, first given by Reshetikhin and Turaev [29], uses the quantum integer

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$$

and the quantum factorial

$$[k]_q! = [1]_q [2]_q \cdots [k]_q.$$ 

The formula is

$$\overline{R}_q = \frac{1}{d} \sum_{i,j,k=0}^{d-1} \frac{(q - q^{-1})^k}{[k]_q!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E^k K^i \otimes F^k K^j.$$

The ribbon element and balancing element, respectively, are

$$\nu = u K^{-1}, \text{ and } \kappa = K.$$ 

Example 4.3.12. The following example will be used throughout the paper. It is another quotient of $U_q$, this time defined for $q$ a 2nd root of unity. It is originally from Radford [26] and was introduced to the author by Kerler [17], who defined an $R$-matrix and ribbon element for it.

Let $N$ be the algebra over $k$ generated by elements $K, \eta, \theta$, and subject to the relations

$$K^2 = 1, \quad \eta^2 = \theta^2 = 0,$$
\[ \eta \theta = -\theta \eta, \quad \eta K = -K \eta, \quad \theta K = -K \theta. \]

This algebra is eight-dimensional as a vector space. Define coproduct, counit, and antipode maps by

\[ \Delta (K) = K \otimes K, \quad \Delta (x) = x \otimes 1 + K \otimes x \quad \text{for all } x \in \{\eta, \theta\}, \]
\[ \epsilon (K) = 1, \quad \epsilon (x) = 0 \quad \text{for all } x \in \{\eta, \theta\}, \]
\[ S (K) = K, \quad S (x) = x K \quad \text{for all } x \in \{\eta, \theta\}. \]

These operations make \( \mathcal{N} \) into a Hopf algebra. An \( R \)-matrix for \( \mathcal{N} \) can be defined by

\[ R = \frac{1}{2} (1 \otimes 1 - \eta \otimes \theta K) \sum_{i,j=0}^{1} (-1)^{ij} K^i \otimes K^j. \]

We can calculate the elements \( u \) and \( u^{-1} \) as

\[ u = (1 - \eta \theta) K, \quad u^{-1} = (1 + \eta \theta) K. \]

Next we choose the ribbon element \( \nu = 1 - \eta \theta \). With this, \( \mathcal{N} \) is a ribbon Hopf algebra. The balancing element is

\[ \kappa = \nu^{-1} u = K. \]

**Example 4.3.13.** Let \( \mathcal{H} \) be a finite-dimensional Hopf algebra. An important example of a braided Hopf algebra is the quantum double of \( \mathcal{H} \), denoted \( D(\mathcal{H}) \). For details, see [14], Chapter IX.4. This braided Hopf algebra was introduced by Drinfel’d [9]. A necessary and sufficient condition for \( D(\mathcal{H}) \) to have a ribbon element was found by Kauffman and Radford [16].

Let \( \mu \) denote the multiplication map in \( \mathcal{H} \), and define the **opposite multiplication map** \( \mu' = \mu \circ \tau \), where \( \tau : \mathcal{H}^\otimes 2 \to \mathcal{H}^\otimes 2 \) is defined by \( \tau (a \otimes b) = b \otimes a \). The **opposite Hopf**
algebra $\mathcal{H}^{op}$ is identical to $\mathcal{H}$ except that $\mu$ is replaced with the opposite multiplication map $\mu'$.

As a vector space $D(\mathcal{H}) = (\mathcal{H}^{op})^* \otimes \mathcal{H}$, where $(\mathcal{H}^{op})^*$ is the vector space dual to $\mathcal{H}^{op}$. We define the bialgebra structure by the following equations:

$$\eta_{D(\mathcal{H})}(1) = \epsilon \otimes 1, \quad \epsilon_{D(\mathcal{H})}(f \otimes a) = \epsilon(a)f(1),$$

$$\Delta_{D(\mathcal{H})}(f \otimes a) = (f' \otimes a') \otimes (f'' \otimes a''),$$

$$\mu_{D(\mathcal{H})}((f \otimes a) \otimes (g \otimes b)) = fg(S^{-1}(a'')?a') \otimes a''b,$$

where $g(S^{-1}(a'')?a')$ stands for the map $x \mapsto g(S^{-1}(a'')xa')$. The antipode of $D(\mathcal{H})$ is given by

$$S_{D(\mathcal{H})}(f \otimes h) = (\epsilon \otimes S(h))(f \circ S, 1) = (f(S^2(h'') S(?) S(h'')) \otimes S(h'')).$$

An important property of the quantum double is that $1 \otimes \mathcal{H}$ and $(\mathcal{H}^{op})^* \otimes 1$ are both sub-Hopf algebras, isomorphic to $\mathcal{H}$ and $\mathcal{H}^*$ respectively. Let $i_{\mathcal{H}} : \mathcal{H} \to D(\mathcal{H})$ and $i_{\mathcal{H}^*} : \mathcal{H}^* \to D(\mathcal{H})$ denote the injection maps.

Now we can describe the $R$-matrix. First, we use the canonical vector space isomorphism

$$\lambda : \mathcal{H}^* \otimes \mathcal{H} \to \text{End}(\mathcal{H}).$$

Then we let $\rho = \lambda^{-1}(id_{\mathcal{H}})$, and define

$$R = (i_{\mathcal{H}} \otimes i_{\mathcal{H}^*})(\rho) \in D(\mathcal{H}) \otimes D(\mathcal{H}).$$

For a more explicit formulation, fix a linear basis $\{b_i\}$ of $\mathcal{H}$, with dual basis $\{b^i\}$ of $\mathcal{H}^*$. Then $\rho = \sum_i b_i \otimes b^i$, and the $R$-matrix is given by

$$R = \sum_i (1 \otimes b_i) \otimes (b^i \otimes 1).$$
Notice that this formula is independent of the choice of linear basis. This is proven to be an $R$-matrix in [14] Theorem IX.4.4.

A well-known theorem from Radford [26] says that for $\mathcal{H}$ a finite-dimensional Hopf algebra over a field $k$, there exist distinguished group-like elements $g \in \mathcal{H}$ and $\alpha \in \mathcal{H}^*$ such that for any $x \in \mathcal{H}$,

$$S^4(x) = g(\alpha \rightarrow x \leftarrow \alpha^{-1})g^{-1} = \alpha \rightarrow (gxg^{-1}) \leftarrow \alpha^{-1},$$

where $\alpha \rightarrow x$ and $x \leftarrow \alpha^{-1}$ denotes the left- and right- $\mathcal{H}^*$ actions on $\mathcal{H}$, respectively, defined by

$$\alpha \rightarrow x = \alpha(x'')x' \text{ and } x \leftarrow \alpha^{-1} = \alpha^{-1}(x')x''.$$ 

Kauffman and Radford [16] proved that $D(\mathcal{H})$ has a ribbon element if and only if there are group-like elements $l \in \mathcal{H}$ and $\beta \in \mathcal{H}^*$ such that $l^2 = g$, $\beta^2 = \alpha$, and for all $x \in \mathcal{H}$,

$$S^2(x) = l(\beta \rightarrow x \leftarrow \beta^{-1})l^{-1}.$$ 

In this case, the element $(\beta, l) \in D(\mathcal{H})$ is a ribbon element.

This result was generalized to Hopf algebras over commutative rings (as opposed to fields) by Kerler and Qi [7].

### 4.4 Ribbon Automorphisms

**Definition 4.4.1.** For a ribbon Hopf algebra $\mathcal{H}$ with R-matrix $\mathcal{R}$ and ribbon element $\nu$, a ribbon automorphism is a Hopf algebra automorphism $\phi : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$(\phi \otimes \phi)(\mathcal{R}) = \mathcal{R} \quad \text{and} \quad \phi(\nu) = \nu.$$ (4.4.1)
The set of ribbon automorphisms of $\mathcal{H}$ is denoted $\text{RAut}(\mathcal{H})$. It is a subgroup of $\text{Aut}(\mathcal{H})$, the set of Hopf algebra automorphisms of $\mathcal{H}$.

The property of fixing the $R$-matrix is a strong one. The rest of this section is dedicated to finding the ribbon automorphisms of the example ribbon Hopf algebras given above. For convenience, we list them all first as separate lemmas. We assume from here on that $\mathbb{k}$ has characteristic $0$, and denote the set of non-zero elements of $\mathbb{k}$ by $\mathbb{k}^\neq 0$.

**Lemma 4.4.2.** Let $\mathcal{N}$ be the ribbon Hopf algebra defined in Example 4.3.12. For any $t \in \mathbb{k}^\neq 0$, the function $\phi_t : \mathcal{N} \to \mathcal{N}$ defined by

$$\phi_t(\eta) = t\eta, \quad \phi_t(\theta) = t^{-1}\theta, \quad \phi_t(K) = K,$$

is a ribbon automorphism of $\mathcal{N}$. The mapping $t \mapsto \phi_t$ defines a group isomorphism between $\mathbb{k}^\neq 0$ and $\text{RAut}(\mathcal{N})$.

**Proof.** It is straightforward to check that $\phi_t$ as defined in a ribbon automorphism. We will show that for any $\phi \in \text{RAut}(\mathcal{N})$, we must have $\phi = \phi_t$ for some $t \in \mathbb{R}^*$.

First, every ribbon automorphism fixes the ribbon element $\nu = 1 + \eta\theta$ and balancing element $\kappa = K$. From this we also conclude that $\phi(\eta\theta) = \eta\theta$.

Recall the $R$-matrix

$$\mathcal{R} = \frac{1}{2}(1 \otimes 1 - \eta \otimes \theta K) \sum_{i,j=0}^{1} (-1)^{ij} K^i \otimes K^j.$$

Notice that

$$\sum_{i,j=0}^{1} (-1)^{ij} K^i \otimes K^j$$

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is invertible. Analyzing the equation \((\phi \otimes \phi)(R) = R\), and using the fact that \(\phi(K) = K\), we come to the fundamental equality

\[(\phi \otimes \phi)(\eta \otimes \theta) = \eta \otimes \theta.\]

This is only possible if \(\phi(\eta) = t\eta\) and \(\phi(\theta) = t^{-1}\theta\) for some \(t \in \mathbb{k}^{\neq 0}\).

\[\square\]

**Lemma 4.4.3.** For \(q \in \mathbb{k}\) a root of unity, consider the ribbon Hopf algebra \(\overline{U}_q(\mathfrak{sl}_2)\) defined in Example 4.3.11. For any \(t \in \mathbb{k}^{\neq 0}\), there is a ribbon automorphism \(\phi_t\) of \(\overline{U}_q(\mathfrak{sl}_2)\) defined by

\[
\phi_t(E) = tE, \quad \phi_t(F) = t^{-1}F, \quad \phi_t(K) = K.
\]

Further, the mapping \(t \mapsto \phi_t\) defines a group isomorphism between \(\mathbb{k}^{\neq 0}\) and \(\text{RAut}(\overline{U}_q(\mathfrak{sl}_2))\).

**Proof.** This proof runs very similarly to the proof of Lemma 4.4.2. It is straightforward to check that \(\phi_t\) is a ribbon automorphism of \(\overline{U}_q(\mathfrak{sl}_2)\).

Now consider any \(\phi \in \text{RAut}(\overline{U}_q(\mathfrak{sl}_2))\). It must fix the ribbon element \(\kappa = K\), as well as the \(R\) matrix, which we recall

\[
\overline{R}_q = \frac{1}{d} \sum_{i,j,k=0}^{d-1} \frac{(q - q^{-1})^k q^{k(k-1)/2+2k(i-j)-2ij} E^k K^i \otimes F^k K^j}{[k]!}.
\]

For now denote the coefficients by

\[
c_{i,j,k} = \frac{(q - q^{-1})^k q^{k(k-1)/2+2k(i-j)-2ij}}{[k]!}.
\]

Applying \(\phi \otimes \phi\) to \(\overline{R}_q\) and using that \(\phi(K) = K\), we have

\[
(\phi \otimes \phi)(\overline{R}_q) = \frac{1}{d} \sum_{i,j,k=0}^{d-1} c_{i,j,k} \phi(E)^k K^i \otimes \phi(F)^k K^j.
\]

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Using the fact that the set \( \{E^nK^i \otimes F^mK^j\}_{0 \leq i,j,n \leq d-1} \) is linearly independent, we conclude that \( \phi(E) \otimes \phi(F) = E \otimes F \). This is only possible if \( \phi(E) = tE \) and \( \phi(F) = t^{-1}F \) for some \( t \in k \neq 0 \).

Recall from Example 4.3.13 that for any finite-dimensional Hopf algebra \( \mathcal{H} \) the quantum double \( D(\mathcal{H}) \) is a braided Hopf algebra, and \( D(\mathcal{H}) \) is a ribbon Hopf algebra if certain elements \( l \in \mathcal{H} \) and \( \beta \in \mathcal{H}^* \) exist. The next lemma connects ribbon automorphisms of \( D(\mathcal{H}) \) with Hopf algebra automorphisms of \( \mathcal{H} \).

**Lemma 4.4.4.** Let \( \mathcal{H} \) be a finite-dimensional Hopf algebra, and \( D(\mathcal{H}) \) the double as defined in Example 4.3.13. Then there is a group isomorphism

\[
\text{Aut}(\mathcal{H}) \to \{ \Phi \in \text{Aut}(D(\mathcal{H})) | (\Phi \otimes \Phi)(R) = R \}
\]

defined by the mapping

\[
\phi \mapsto \hat{\phi} = (\phi^{-1})^* \otimes \phi
\]

where \( \phi^* \in \text{Aut}((\mathcal{H}^{op})^*) \) is defined by

\[
\phi^*(f) = f \circ \phi.
\]

If \( D(\mathcal{H}) \) is a ribbon Hopf algebra, with ribbon element \((l, \beta)\) as in Example 4.3.13, then \( \hat{\phi} \) is a ribbon automorphism of \( D(\mathcal{H}) \) if and only if

\[
\phi(l) = l \text{ and } (\phi^{-1})^*(\beta) = \beta \circ \phi^{-1} = \beta,
\]

or

\[
\phi(l) = -l \text{ and } (\phi^{-1})^*(\beta) = \beta \circ \phi^{-1} = -\beta.
\]
Remark 4.4.5. For ease of notation, in this proof we simply write $\mathcal{H}^*$ to stand for $(\mathcal{H}^{\text{op}})^*$.

Proof. This proof has three major parts. In Part One it is shown that for any $\phi \in \text{Aut}(\mathcal{H})$, $\hat{\phi}$ is an automorphism of $D(\mathcal{H})$ and fixes $\mathcal{R}$. It is clear that if

$$\phi(l) = l \quad \text{and} \quad \phi^*(\beta) = \beta$$

or

$$\phi(l) = -l \quad \text{and} \quad \phi^*(\beta) = \beta \circ \phi = -\beta,$$

then

$$(\phi^* \otimes \phi)(\beta, l) = (\beta, l).$$

On the other hand, if

$$(\phi^* \otimes \phi)(\beta, l) = (\beta, l)$$

then we must have $\phi^*(\beta) = t\beta$ and $\phi(l) = t^{-1}l$ for some $t \in \mathbb{k}$. Because $l^2 = g$ is fixed, we must have $t = \pm 1$. In this case, $\hat{\phi}$ is a ribbon automorphism of $D(\mathcal{H})$.

In Part Two, it is shown the subalgebras

$$\epsilon \otimes \mathcal{H} \cong \mathcal{H}, \quad \mathcal{H}^* \otimes 1 \cong \mathcal{H}^*$$

are fixed by any ribbon automorphism $\Phi$ of $D(\mathcal{H})$. Thus, restricting $\Phi$ to these subalgebras defines Hopf algebra isomorphisms of $\mathcal{H}$ and $\mathcal{H}^*$, denoted $\Phi|_\mathcal{H}$ and $\Phi|_{\mathcal{H}^*}$ respectively. Formulas for both of these restrictions are given.

In Part Three it is shown that $\Phi \mapsto \Phi|_\mathcal{H}$ and $\phi \mapsto \hat{\phi}$ are inverse maps. Thus, they define the needed group isomorphisms.

Part One
Let $\phi \in \text{Aut}(\mathcal{H})$. We show that $\hat{\phi}$ is a Hopf algebra automorphism of $D(\mathcal{H})$ and fixes $\mathcal{R}$.

It has an inverse, namely $\hat{\phi}^{-1}$, and is thus a bijection. It is straightforward to verify that $\hat{\phi}$ respects comultiplication, the unit, and the counit in $D(\mathcal{H})$, and these calculations are omitted. The longest step is showing that $\hat{\phi}$ respects multiplication in $D(\mathcal{H})$.

Recall the notation, used when defining $D(\mathcal{H})$, of using a question mark to represent the input variable of a function. For example, $g(h'?h'')$ represents the map $x \mapsto g(h'xh'')$. Then,

$$\hat{\phi}((f \otimes h)(g \otimes k)) = \hat{\phi}(f(?g(S^{-1}(h'')?h') \otimes h''k))$$

$$= [(f \circ \phi^{-1})(?) g(S^{-1}(h''?) \phi^{-1}(?) h')] \otimes \phi(h''k)$$

$$= (f \circ \phi^{-1})(?) (g \circ \phi^{-1})(S^{-1}(\phi(h)'?) \phi(h)') \otimes \phi(h)'' \phi(k)$$

$$= (f \circ \phi^{-1}) \otimes \phi(h))((g \circ \phi^{-1}) \otimes \phi(k))$$

$$= \hat{\phi}(f \otimes h) \hat{\phi}(g \otimes k).$$

The antipode of $D(\mathcal{H})$ commutes with $\hat{\phi}$:

$$\hat{\phi}(S(f \otimes h)) = \hat{\phi}(f(S^2(h')S(?)(S(h'')) \otimes S(h))$$

$$= (f(S^2(h')S(\phi^{-1}(?))S(h'')) \otimes S(h''))$$

$$= (f \circ \phi^{-1})(S^2(\phi(h'))S(\phi(h''))) \otimes S(\phi(h''))$$

$$= S(f \circ \phi^{-1} \otimes \phi(h))$$

$$= S(\hat{\phi}(f \otimes h)).$$

This proves that $\hat{\phi}$ is a Hopf algebra automorphism.
Finally, we show that \( \hat{\phi} \) fixes \( \mathcal{R} \). Fix a basis \( \{b_i\} \) of \( \mathcal{H} \), with dual basis \( \{b^i\} \) of \( \mathcal{H}^* \). Recall the formula for the \( R \)-matrix

\[
\mathcal{R} = \sum_i (1 \otimes b_i) \otimes (b^i \otimes 1).
\]

The set \( \{\phi(b_i)\} \) is also a linear basis of \( \mathcal{H} \). Let \( \{\phi(b_i)^*\} \) be its dual basis of \( \mathcal{H}^* \). These two bases are related by the equation

\[
\phi^*(b^i) = b^i \circ \phi^{-1} = (\phi(b_i))^*.
\]

Then we calculate

\[
(\hat{\phi} \otimes \hat{\phi})(\mathcal{R}) = \sum_i \hat{\phi}(1 \otimes b_i) \otimes \hat{\phi}(b^i \otimes 1) = \sum_i (1 \otimes \phi(b_i)) \otimes ((\phi(b_i))^* \otimes 1) = \mathcal{R}.
\]

**Part Two**

We show that for any \( \Phi \in RAut(D(H)) \),

\[
\Phi(1 \otimes \mathcal{H}) \subseteq 1 \otimes \mathcal{H} \text{ and } \Phi(\mathcal{H}^* \otimes 1) \subseteq \mathcal{H}^* \otimes 1.
\]

We will use the maps

\[
\rho_i : D(\mathcal{H})^* \to D(\mathcal{H}), \ i = 1, 2,
\]

defined by

\[
\rho_1(\chi) = (\text{id} \otimes \chi)(\mathcal{R}) \text{ and } \rho_2(\chi) = (\chi \otimes \text{id})(\mathcal{R}).
\]

Note that the images of \( \rho_1 \) and \( \rho_2 \) are \( \mathcal{H}^* \) and \( \mathcal{H} \), respectively.

Now for any \( \chi \in D(\mathcal{H})^* \) and \( \Phi \in RAut(D(H)) \),

\[
\rho_1(\chi \circ \Phi^{-1}) = (\text{id} \otimes (\chi \circ \Phi^{-1}))(\mathcal{R}) = (\Phi \otimes \chi)(\mathcal{R}) = \Phi(\rho_1(\chi)) = \Phi(\rho_1(\chi)).
\]
An analogous equality holds for $\rho^2$,

$$\rho_2(\chi \circ \Phi^{-1}) = \Phi(\rho_2(\chi)).$$

Now for $h \in \mathcal{H}$, there is a $\chi \in D(\mathcal{H})^*$ such that

$$\rho_2(\chi) = \epsilon \otimes h,$$

and then

$$\Phi(\epsilon \otimes h) = \rho_2(\chi \circ \Phi^{-1}) \in \mathcal{H}.$$

Thus we see that $\Phi(\mathcal{H}) = \mathcal{H}$. Similarly, $\Phi(\mathcal{H}^*) = \mathcal{H}^*$.

**Part Three**

In this part it is shown that $\Phi \mapsto \Phi|_{\mathcal{H}}$ is the inverse map to $\phi \mapsto \hat{\phi}$. It is easy to verify that

$$\left(\hat{\phi}\right)|_{\mathcal{H}} = \phi.$$

We must check the other direction, that $\hat{\Phi}|_{\mathcal{H}} = \Phi$. It suffices to show that, for any $\lambda \in \mathcal{H}^*$,

$$\Phi|_{\mathcal{H}^*}(\lambda) = \lambda \circ \Phi|_{\mathcal{H}}^{-1}.$$

(4.4.2)

We will use a map similar to $\rho_i$ from Part Two. Define $\bar{\rho} : \mathcal{H}^* \to \mathcal{H}^*$ by

$$\bar{\rho}(\lambda) = (\lambda \otimes \text{id})(\mathcal{R}).$$

Then we have, for any $\Phi \in \text{RAut}(D(\mathcal{H}))$,

$$\bar{\rho}(\lambda \circ (\Phi|_{\mathcal{H}})^{-1}) = \Phi|_{\mathcal{H}^*}(\bar{\rho}(\lambda)).$$
We claim that \( \hat{\rho} \) is the identity map, which finishes this part of the proof. This can be seen by setting a basis \( \{ b_i \} \) of \( \mathcal{H} \), with dual basis \( \{ b^i \} \) of \( \mathcal{H}^* \). Then we can calculate
\[
\hat{\rho}(\lambda) = \sum_i \lambda(b_i) b^i = \lambda.
\]

Then the previous equation may be rewritten as
\[
\lambda \circ \Phi|_\mathcal{H} = \Phi|_{\mathcal{H}^*}(\lambda).
\]

As remarked, this proves that \( \Phi|_{\mathcal{H}^*} = (\Phi|_{\mathcal{H}})^* \), and so \( \Phi|_{\mathcal{H}} = \Phi \). Thus, \( \Phi \mapsto \Phi|_\mathcal{H} \) is the inverse map to \( \phi \mapsto \hat{\phi} \).

\[\square\]

### 4.5 Quotients of the Semidirect Product

In this section we revisit the semidirect product as defined in Section 4.2, specializing to the case when the group \( G \) acts on \( \mathcal{H} \) on the right by ribbon automorphisms. This action is defined by a group anti-homomorphism \( \gamma : G \to \text{RAut}(\mathcal{H}) \). We define two quotients, to be taken successively, of the semidirect product
\[
\text{SD} = \text{SD}(G, \gamma, \mathcal{H}) = \mathbb{k}[G] \rtimes_\gamma \mathcal{H}.
\]

The invariant takes values in this double quotient. Specifically, a knot diagram yields an element \( (g,x) \in \text{SD} \), which we project onto the quotient spaces. Because of this we will take note of the image of the subset \( G \times \mathcal{H} \subset G \rtimes \mathcal{H} \) under the quotient maps.

The first quotient may be defined on an arbitrary Hopf algebra \( \mathcal{K} \), and results in another Hopf algebra. It requires the choice of central, group-like element of \( \mathcal{K} \).
First we define this quotient for an arbitrary Hopf algebra, and then specialize to the semidirect product.

**Definition 4.5.1.** For a Hopf algebra $\mathcal{K}$, with central, group-like element $g \in \mathcal{K}$, let $((g)) = (g - 1) \cdot \mathcal{K} \subset \mathcal{K}$ be the linear subspace spanned by elements $gx - x$ for all $x \in \mathcal{K}$. The **algebraic quotient of $\mathcal{K}$ by $g$**, denoted $\mathcal{K}/((g))$, is the linear quotient of $\mathcal{K}$ by $((g))$. The natural quotient map is denoted

$$Q_g : \mathcal{K} \to \mathcal{K}/((g)).$$

**Lemma 4.5.2.** For $g \in \mathcal{K}$ central and group-like, the algebraic quotient $\mathcal{K} = \mathcal{K}/((g))$ is a Hopf algebra, with operations induced by the Hopf algebra structure on $\mathcal{K}$.

**Proof.** We will briefly check that all of the Hopf algebra operations on $\mathcal{K}$ descend to well-defined operations on $\mathcal{K}$. These new operations follow the same relations as the original operations in $\mathcal{K}$. Thus, the quotient is a Hopf algebra.

Multiplication is well-defined because $x((g)) \subseteq ((g)) \supseteq ((g))x$ for all $x \in \mathcal{K}$. This is easily checked, for any $x, y \in \mathcal{K},$

$$x(gy - y) = gxy - xy \in ((g))$$

and

$$(gy - y)x = gyx - yx \in ((g)).$$

Thus,

$$(x + ((g)))(y + ((g))) = xy + ((g))$$

is well-defined.

Co-multiplication is also well-defined. The kernel of

$$Q_g \otimes Q_g : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$$

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is the linear subspace

\[ ((g)) \otimes \mathcal{K} + \mathcal{K} \otimes ((g)). \]

We must show that \( \Delta((g)) \) is contained in this kernel. We check, for \( x \in \mathcal{K} \),

\[
\Delta(gx - x) = (g \otimes g)\Delta(x) - \Delta(x)
= (g \otimes g - 1 \otimes 1)\Delta(x)
= ((g - 1) \otimes g + 1 \otimes (g - 1))\Delta(x)
= (g - 1)x' \otimes gx'' + x' \otimes (g - 1)x''
\in ((g)) \otimes \mathcal{K} + \mathcal{K} \otimes ((g)).
\]

The counit is well-defined because \( \epsilon(g) = 1 \). The unit is the equivalence class \( 1_{\mathcal{K}} + ((g)) \), where \( 1_{\mathcal{K}} \) is the unit in \( \mathcal{K} \).

Finally, the antipode \( S \) is well-defined because \( S((g)) \subset ((g)) \). We check, for \( x \in \mathcal{K} \),

\[
S(gx - x) = S(x)g^{-1} - S(x) = -[g(g^{-1}S(x)) - g^{-1}S(x)] \in ((g)).
\]

To apply this to \( \text{SD} \), we will use a group-like element which exists under certain circumstances.

**Lemma 4.5.3.** Let \( G \) be a finite group with a right action on a ribbon Hopf algebra \( \mathcal{H} \) by a group anti-homomorphism

\[ \gamma : G \to \text{RAut}(\mathcal{H}). \]

Suppose \( z \in \mathcal{H} \) is group-like and fixed by \( \text{RAut}(\mathcal{H}) \), and \( f \in G \) is central such that for all \( x \in \mathcal{H} \),

\[ \gamma_f(y) = zxz^{-1}. \]
Then the element

\((f, z) \in k[G] \ltimes_\gamma \mathcal{H}\)

is central and group-like.

Let \(Q_{(f,z)} : SD \to SD\) be the algebraic quotient of \(SD\) by \((f,z)\). Consider the quotient group \(G/ \langle f \rangle\), where \(\langle f \rangle\) is the central subgroup generated by \(f\). Suppose there is a subset \(\mathcal{G} \subset G\) which maps bijectively onto \(G/ \langle f \rangle\) under the quotient map. Then \(Q_{(f,z)}\) maps \(k[G] \otimes \mathcal{H} \subset SD\) bijectively onto \(SD\).

Remark 4.5.4. We note that if \(\mathcal{G}\) is a subgroup, then \(k[G] \ltimes \mathcal{H}\) is a sub-Hopf algebra of \(SD\), and \(Q_{(f,z)}\) is a Hopf algebra isomorphism when restricted to \(k[G] \ltimes \mathcal{H}\).

The existence of a subgroup \(\mathcal{G}\) is a homological condition. A choice of such a subgroup is equivalent to the choice of a splitting of the short exact sequence

\[1 \to \langle f \rangle \to G \to G/ \langle f \rangle \to 1.\]

**Proof.** It is easily checked that \((f,z)\) is a central element. For any other element \((g, x) \in G \times \mathcal{H}\), we have

\[(f, z)(g, x) = (fg, \gamma_g(z)x) = (gf, zx) = (gf, \gamma_f(x)z) = (g, x)(f, z).\]

This equality uses the fact that \(f \in G\) is central, and \(z \in \mathcal{H}\) is fixed by any ribbon automorphism, in particular \(\gamma_g\).

Now we look at the restriction of \(Q_{(f,z)}\) to \(\mathcal{G} \times \mathcal{H}\). Let \((g, x) \in G \times \mathcal{H}\). We claim that there is a unique

\[(h, y) \in \mathcal{G} \times \mathcal{H} \subset G \times \mathcal{H}\]
such that $Q_{(f,z)}(g, x) = Q_{(f,z)}(h, y)$. This is because $g$ lies in a unique equivalence class of the quotient $G/\langle f \rangle$. By assumption, there is a unique $h \in G$ which also lies in this class. Then $g = hf^k$ for some $k$, and so

$$Q_{(f,z)}(g, x) = Q_{(f,z)}(hf^k, x) = Q_{(f,z)}(h, z^{-k}(x)).$$

Because the elements $(g, x)$ linearly generate $SD$, this shows that $Q_{(f,z)}$ is surjective when restricted to $k[G] \otimes \mathcal{H}$. Next we show that this restricted map is injective.

It is clear that for any $h \in G$, the map

$$Q_{(f,z)} : h \times \mathcal{H} \to SD$$

is an injection. In other words, for $x \neq y$,

$$Q_{(f,z)}(h, x) \neq Q_{(f,z)}(h, y).$$

It is also clear that for $h_1 \neq h_2 \in G$,

$$Q_{(f,z)}(h_1, x) \neq Q_{(f,z)}(h_2, y)$$

for any $x, y \in \mathcal{H}$.

These two facts imply that $Q_{(f,z)}$ is injective when restricted to $k[G] \otimes \mathcal{H}$. Thus $Q_{(f,z)}$ restricted to $k[G] \otimes \mathcal{H}$ defines a bijection.

The second quotient is taken as a vector space, and the resulting space is no longer a Hopf algebra. We define it for a general algebra $A$, then discuss the specifics of $SD$.

**Definition 4.5.5.** For an algebra $A$, the *commutator subspace* $\text{Com}(A) \subset A$ is the linear subspace of $A$ spanned by elements of the form $xy - yx$ for $x, y \in A$. The
commutator quotient of $A$ is the linear quotient $\hat{A} = A/\text{Com}(A)$. Elements of $\hat{A}$ are commonly called characters on $A$.

For $\phi$ an algebra automorphism of $A$, the twisted commutator subspace defined by $\phi$, $\text{Com}_\phi(A) \subset A$, is the linear subspace spanned by elements of the form $xy - \phi(y)x$ for $x, y \in A$. The twisted commutator quotient of $A$ is the linear quotient $\hat{A}_\phi = A/\text{Com}_\phi(A)$.

The quotient map is denoted

$$Q^\phi : A \to \hat{A}_\phi,$$

and in particular $Q^\text{id}$ is the commutator quotient map.

Recall the quotient map $Q_{(f,z)} : \mathcal{SD} \to \mathcal{SD}$. Define $\mathcal{S}\mathcal{D}$ to be the commutator quotient space of $\mathcal{SD}$, with quotient map

$$Q^\text{id} : \mathcal{SD} \to \mathcal{S}\mathcal{D}.$$

We will give an explicit description of the subspace $\text{Com}(\mathcal{S}\mathcal{D})$. Let $\Lambda$ denote the set of conjugacy classes of $G$. The subgroup $\langle f \rangle$ acts on $\Lambda$. This action is well-defined because $f \in G$ is central, so $x$ is conjugate to $y$ if and only if $fx$ is conjugate to $fy$.

Let $\mu_1, \ldots, \mu_n \in \Lambda$ be a choice of representative for each $\langle f \rangle$-orbit of $\Lambda$. For each $i$, define

$$F_i = \bigoplus_{g \in \mu_i} g \otimes \mathcal{H}$$

and

$$\text{Com}_i = \text{Com} \cap F_i = \langle (a, x)(b, y) - (b, y)(a, x) : ab, ba \in \mu_i \rangle.$$
We note that $SD = \bigoplus_{i=1}^{n} F_i$, and and so

$$\text{Com} = \bigoplus_{i=1}^{n} F_i \cap \text{Com} = \bigoplus_{i=1}^{n} \text{Com}_i.$$

The next lemma will show that $SD$ is isomorphic (as a vector space) to the direct sum of the quotient spaces $F_i / \text{Com}_i$. To study the subspace $\text{Com}_i$, pick a representative $m_i \in \mu_i$. Then for every $c \in \mu_i$, fix a $\tau_c \in G$ such that $\tau_c m_i \tau_c^{-1} = c$. In particular, fix $\tau_{m_i} = 1 \in G$.

Then there is a (non-direct) sum

$$\text{Com}_i = T_i + X_i + Y_i,$$

where

$$T_i = \langle (c, x) - (m_i, \gamma_{\tau_c}(x)) : c \in \mu_i, x \in \mathcal{H} \rangle,$$

$$X_i = \langle (m_i, \gamma_i(y)x - xy) : x, y \in \mathcal{H} \rangle, \text{ and }$$

$$Y_i = \langle (m_i, \gamma_s(x) - x) : s \in C_G(m_i), x \in \mathcal{H} \rangle.$$

With this notation, we can analyze the structure of $SD$.

**Lemma 4.5.6.** Suppose $G, \mathcal{H}, \gamma, (f, z)$, and $Q_{(f, z)} : SD \to SD$ are defined as in Lemma 4.5.3. Let $\Lambda$ be the set of conjugacy classes of $G$, $\mu_1, \cdots, \mu_n \in \Lambda$ a choice of representative for each $\langle f \rangle$-orbit of $\Lambda$, and $m_i \in \mu_i$ a choice of representative for each $\mu_i$. Let $F_i, \text{Com}_i, T_i, X_i, Y_i$ be defined as in the previous discussion. Then there is a linear isomorphism

$$\alpha : \bigoplus_{i=1}^{n} (F_i / \text{Com}_i) \to SD.$$

Further, there is a linear isomorphism for each $i$,

$$F_i / \text{Com}_i \sim \{m_i\} \times \mathcal{H} / (X_i + Y_i).$$
Proof. Let \( \mathcal{G} = \bigcup_{i=1}^{n} \mu_i \). By definition, \( \mathcal{G} \) has exactly one representative for each \( \langle f \rangle \)-orbit of \( G \) and so maps bijectively onto the quotient \( G/\langle f \rangle \), satisfying the conditions of Lemma 4.5.2. This shows that the map

\[
\mathbb{k}[\mathcal{G}] \otimes \mathcal{H} = \bigoplus_{i=1}^{n} F_i \rightarrow SD
\]

is a bijection. The restriction

\[
\bigoplus_{i=1}^{n} \text{Com}_i \rightarrow \text{Com}(SD)
\]

is also a bijection. These two linear isomorphisms then define an isomorphism

\[
\left( \bigoplus_{i=1}^{n} F_i \right) \big/ \left( \bigoplus_{i=1}^{n} \text{Com}_i \right) \rightarrow SD / \text{Com}(SD).
\]

The linear isomorphism \( \alpha \) is defined as the composition of linear isomorphisms

\[
\bigoplus_{i=1}^{n} (F_i / \text{Com}_i) \cong \left( \bigoplus_{i=1}^{n} F_i \right) \big/ \left( \bigoplus_{i=1}^{n} \text{Com}_i \right) \rightarrow SD / \text{Com}(SD) = SD,
\]

where the first isomorphism comes from the direct sum decomposition.

To prove the final claim, pick an \( m_i \in \mu_i \). We use the notation

\[
x = y \pmod{T_i}
\]

to mean that \( x - y \in T_i \). The same notation holds for \( X_i \) and \( Y_i \).

We will show shortly that \( \text{Com}_i = T_i + X_i + Y_i \). Suppose for the moment that this equality holds.

Recall the fixed elements \( \tau_c \in G \) such that \( \tau_c m_i \tau_c^{-1} = c \). Then for every \( (c, x) \in G \times \mathcal{H} \) we have

\[
(c, x) = (m_i, \gamma_c(x)) \pmod{T_i}.
\]
While the choice of $\tau_c$ is not unique, any other $\tau'_c$ satisfying $\tau'_c m_i \tau'_c^{-1} = c$ are related by $\tau_c^{-1} \tau'_c \in C_G(m_i)$. Thus we have the relation

$$(m_i, \tau_c(x)) = (m_i, \tau'_c(x)) \pmod{Y_i}.$$ 

So every element of $F_i / \text{Com}_i$ has a representative of the form $(m_i, x)$, and two such representatives are equivalent after factoring out $Y_i$. Finally,

$$(T_i \cup X_i \cup Y_i) \cap \{m_i\} \times \mathcal{H} = X_i + Y_i,$$

and so $X_i + Y_i$ is the kernel of the quotient map when restricted to $\{m_i\} \times \mathcal{H}$. This proves the claim. The only thing left is to show

$$\text{Com}_i = T_i + X_i + Y_i.$$ 

It is straightforward to check that $T_i + X_i + Y_i \subseteq \text{Com}_i$; we will show the opposite inclusion.

The proof will come in a sequence of steps, beginning with the commutator of two arbitrary elements $(a, x), (b, y) \in G \times \mathcal{H}$:

$$(a, x)(b, y) - (b, y)(a, x) = (ab, \gamma_b(x)y) - (ba, \gamma_a(y)x). \quad (4.5.1)$$

Substituting $c = ba$ and $z = \gamma_a(y)$, this element is written as

$$(aca^{-1}, \gamma_{ca^{-1}}(x) \gamma_{a^{-1}}(z)) - (c, zx). \quad (4.5.2)$$

We use the facts that

$$(aca^{-1}, \gamma_{ca^{-1}}(x) \gamma_{a^{-1}}(z)) = (m_i, \gamma_{aca^{-1}}(\gamma_{ca^{-1}}(x) \gamma_{a^{-1}}(z))) \pmod{T_i}$$

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and
\[(c, zx) = (m_i, \gamma_{\tau_c}(zx)) \pmod{T_i}.\]

Using these we see that the element in Equation 4.5.2 is equal to
\[(m_i, \gamma_{\tau_{aca}^{-1}}(\gamma_{\tau_{ca}^{-1}}(x)\gamma_{a^{-1}}(z))) - (m_i, \gamma_{\tau_c}(zx)) \pmod{T_i}. \tag{4.5.3}\]

We make a second substitution, using \(\bar{x} = \gamma_{\tau_c}(x), \bar{z} = \gamma_{\tau_c}(z),\) and \(d = \tau_c^{-1}a^{-1}\tau_{aca}^{-1}.\)

It is worth noting here that, in general, \(\tau_c^{-1} \neq \tau_{aca}^{-1}.\) Then we calculate
\[
\gamma_{\tau_{aca}^{-1}}(\gamma_{\tau_{ca}^{-1}}(x)) = \gamma_{\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{x})
= \gamma_{m_i\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{x})
= \gamma_{m_i\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{x})
= \gamma_{m_i\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{x}).
\]

And,
\[
\gamma_{\tau_{aca}^{-1}}(\gamma_{a^{-1}}(z)) = \gamma_{\tau_{aca}^{-1}\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{z})
= \gamma_{\tau_{aca}^{-1}\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{z})
= \gamma_{\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{z})
= \gamma_{m_i\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{z}).
\]

Using these two equalities, the element in Equation 4.5.3 is written as
\[(m_i, \gamma_{m_i\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{x})) - (m_i, \gamma_{m_i\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{z})) \pmod{T_i}.\]

Now, \(d\) commutes with \(m_i\), so \(d \in C_G(m_i).\) Thus,
\[(m_i, \gamma_{m_i\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{x})) = (m_i, \gamma_{m_i\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{x})) = (m_i, \gamma_{m_i\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{x})) \pmod{T_i}. \tag{Y_i}\]

Moving onto the second element of Equation 4.5.3, we have
\[(m_i, \gamma_{m_i\tau_c^{-1}a^{-1}\tau_{aca}^{-1}}(\bar{z})) \pmod{T_i}. \tag{X_i}\]
Using this and the previous equation, the element in Equation 4.5.3 is equal to

\[(m_i, \gamma_{m_i}(\bar{x})\bar{z}) - (m_i, \gamma_{m_i}(\bar{x})\bar{z}) \pmod{T_i + X_i + Y_i}\]

This element, of course, lies inside \(T_i + X_i + Y_i\). This chain of logic then implies that the commutator element in Equation 4.5.1 lies in \(T_i + X_i + Y_i\), and so \(\text{Com}_i \subseteq T_i + X_i + Y_i\). We already have the other inclusion, and so

\[\text{Com}_i = T_i + X_i + Y_i.\]

\[\square\]

**Example 4.5.7.** We revisit the eight-dimensional ribbon algebra \(\mathcal{N}\) from Example 4.3.12. It is generated as an algebra over \(\mathbb{k}\) by the set \(\{K, \eta, \theta\}\). Recall the relations

\[\eta K = -K \eta, \theta K = -K \theta, \text{ and } \eta \theta = -\theta \eta.\]

Every ribbon automorphism of \(\mathcal{N}\) is \(\phi_t\) for some \(t \in \mathbb{k} \neq 0\), defined by

\[\phi_t(\eta) = t \eta, \phi_t(\theta) = t^{-1} \theta \text{ and } \phi_t(K) = K.\]

We will investigate the double quotient of \(\text{SD}(G, \rho, \mathcal{N})\) in the case where \(G = \mathbb{Z}^3\). This will be used in Section 8.5 to calculate the invariant for knots on the torus \(\mathbb{T}\). This is discussed in detail later. Put short, the unit tangent bundle over \(\mathbb{T}\) is trivial because the torus has a non-singular vector field. Hence it is diffeomorphic to \((S^1)^3\), the 3-dimensional torus. The fundamental group of this space is a free abelian group generated by three elements, that is, \(\mathbb{Z}^3\).

Say that \(G = \mathbb{Z}^3\) is generated by the three elements \(a, b, f\). Let \(\rho\) be the balanced anti-homomorphism defined by

\[\rho(a) = \phi_t, \quad \rho(b) = \phi_s, \quad \rho(f) = S^2,\]

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where \( t, s \in \mathbb{K}^\neq 0 \).

The central element which defines the algebraic quotient is \((f, K) \in \textbf{SD}\). We note here that \( S^2 = \phi_{-1} \). For example,

\[
S^2(\theta) = S(\theta K) = K\theta K = -\theta.
\]

For now we study an arbitrary element \( m_i \in \mu_i \) and find the subspaces \( T_i, X_i, Y_i \). \( T_i \) is immediately trivial. Because \( G \) is abelian, every conjugacy class contains only one element.

Write \( m_i = a^pb^qf^r \) for some integers \( p, q, r \). Then,

\[
\rho(m_i) = S^{2r} \circ \phi_i^p \circ \phi_s^q = \phi_{-1}^r \circ \phi_{\theta} \circ \phi_s^q = \phi_{(-1)^r \theta^q s^q}.
\]

Setting \( z = (-1)^r t^p s^q, \) this summarizes as \( \rho(m_i) = \phi_z \). Then,

\[
X_i = \langle (m_i, \phi_z(y)x - xy) : x, y \in \mathcal{N} \rangle = \{m_i\} \otimes \text{Com}_{\phi_z}(\mathcal{N}).
\]

Now we find \( Y_i \). Note that \( C_G(m_i) = G \) because \( G \) is abelian. It is easy to directly calculate the elements of \( Y_i \). If \( z = 1 \), then clearly \( Y_i = \emptyset \). For \( z \neq 1 \), \( \phi_z \) fixes only the linear span \( \langle 1, K, \eta \theta, \eta \theta K \rangle \). Then,

\[
Y_i = \langle (m_i, \phi_z(x) - x \mid x \in \mathcal{N} \rangle
= \langle (m_i, x) : x \in \{\eta, \theta, \eta K, \theta K\} \rangle.
\]

We cite the following lemma for the twisted commutator quotients of \( \mathcal{N} \); it is proved at the end of this section. Recall the definitions of the twisted commutator subspace and the twisted commutator quotient:

\[
\text{Com}_{\phi_t}(\mathcal{N}) = \langle xy - \phi_t(y)x : x, y \in \mathcal{N} \rangle \quad \text{and} \quad \hat{\mathcal{N}}_{\phi_t} = \mathcal{N} / \text{Com}_{\phi_t}(\mathcal{N}).
\]
Lemma 4.5.8. For \( t \neq \pm 1 \),

\[
\text{Com}_\phi(N) = \langle \eta, \theta, \eta K, \theta K, \eta \theta, \eta \theta K \rangle.
\]

For \( t = \pm 1 \), we have

\[
\text{Com}_{id}(N) = \langle \eta, \theta, \eta K, \theta K, \eta \theta \rangle.
\]

\[
\text{Com}_{\phi^{-1}}(N) = \langle \eta, \theta, \eta K, \theta K, \eta \theta K \rangle.
\]

Using this lemma case, it is clear that

\[
Y_i \subset \{ m_i \} \otimes \text{Com}_z(N) = X_i.
\]

So, \( X_i + Y_i = X_i \). In particular, we have isomorphisms

\[
F_i / \text{Com}_i \cong \{ m_i \} \times N / (X_i + Y_i) \cong N / \text{Com}_z(N) = \mathcal{N}_{\phi_2}.
\]

And so \( \mathcal{S}D(\mathbb{Z}^3, \rho, N) \) is isomorphic to a direct sum of twisted commutator quotients of \( N \). Using the choice of generators \( \{ a, b, f \} \), we can fix nice representative elements \( m_{p,q} = a^p b^q \). The linear span of these elements maps bijectively onto \( \mathcal{S}D \).

Notice that \( \rho(m_{p,q}) = \phi_{p,q} \). Then, for our choice of \( \rho \), we have

\[
\mathcal{S}D(\mathbb{Z}^3, \rho, N) \cong \bigoplus_{p,q \in \mathbb{Z}} \mathcal{N}_{\phi_{p,q}}.
\]

This isomorphism is not canonical, as it depends on the choice of representative elements \( m_{p,q} \). We finish this section by proving the previous lemma.

Proof of Lemma 4.5.8. In this proof we use the shorthand \( \text{Com}_t = \text{Com}_{\phi_t}(N) \).

The subspace \( \text{Com}_t \) is given by \( \text{Im}(g) \), where

\[
g : \mathcal{N} \otimes \mathcal{N} \to \mathcal{N}
\]

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is defined by
\[ g(x \otimes y) = xy - \phi_t(y)x. \]

\( \mathcal{N} \) breaks up linearly into the direct sum of three subspaces
\[ \mathcal{N}_0 = \langle 1, K \rangle, \mathcal{N}_1 = \langle \eta, \theta, \eta K, \theta K \rangle \quad \text{and} \quad \mathcal{N}_2 = \langle \eta \theta, \eta \theta K \rangle. \]

These subspaces define a grading on \( \mathcal{N} \). It respects multiplication in the sense that
\[ \mu : \mathcal{N}_i \otimes \mathcal{N}_j \to \mathcal{N}_{i+j}. \]

Note \( \mathcal{N}_j = 0 \) for \( j \geq 3 \), so it suffices to consider the images under \( g \) of the subspaces
\[ \mathcal{N} \otimes \mathcal{N}_0, \mathcal{N}_0 \otimes \mathcal{N}, \text{and} \mathcal{N}_1 \otimes \mathcal{N}_1. \]

Further, since \( g \) is linear, it suffices to find the images of bases of these subspaces.

We will consider separately the cases \( t = \pm 1 \). For now, suppose that \( t \neq \pm 1 \).

Consider an element, \((x, y) \in \mathcal{N} \otimes \mathcal{N}_0 \).

If \( y = 1 \), then \( g(x \otimes 1) = 0 \). So \( \mathcal{N} \otimes 1 \) lies in the kernel of the mapping.

Suppose \( y = K \). Let \( w = \eta^i \theta^j \) for \( i, j \in \{0, 1\} \). The image of \( w \otimes K \) is calculated to be
\[ g(w \otimes K) = wK - \phi_t(K)w = wK - Kw = (1 - (-1)^{i+j}) wK. \tag{4.5.4} \]

For \( w = \eta \) or \( w = \theta \), this image is a multiple of \( \eta K \) and \( \theta K \) respectively. Thus \( \eta K, \theta K \in \text{Com}_\eta \). However, \( 1 \otimes K \) and \( \eta \theta \otimes K \) both map to 0, and thus lie in the kernel of \( g \).

To finish analyzing this subspace, consider the element \( wK \otimes K \in \mathcal{N} \otimes \mathcal{N}_0 \). We calculate the image
\[ g(wK \otimes K) = w - KwK = (1 - (-1)^{i+j}) w. \tag{4.5.5} \]
This equation is very similar to the Equation 4.5.4. If \( w = \eta \) or \( w = \theta \), then the image is a multiple of \( w \). Thus \( \eta, \theta \in \text{Com}_v \). On the other hand, \( K \otimes K \) and \( \eta \theta K \otimes K \) lie in the kernel of the mapping.

This exhausts \( \mathcal{N} \otimes \mathcal{N}_0 \). Next we consider an element \( (x, y) \in \mathcal{N}_0 \otimes \mathcal{N} \).

If \( x = 1 \), then \( g(1 \otimes y) = y - \phi_t(y) \). If \( y \in \{1, K, \eta \theta, \eta \theta K\} \), then this image is 0. However, if \( y \in \{\eta, \theta, \eta K, \theta K\} \) and \( t \neq 1 \), then the image is \((1 - t^\pm 1)y\). We see again that, since \( t \neq 1 \),

\[
\eta, \theta, \eta K, \theta K \in \text{Com}_v.
\]

Suppose \( x = K \), and let \( w = \eta^i \theta^j \) as before. The image of \( K \otimes w \) is calculated as

\[
g(K \otimes w) = K w - \phi_t(w)K = ((-1)^{i+j} - t^{i+j})wK.
\]

This is either a multiple of \( wK \), if \( w = \eta \) or \( w = \theta \), or it is 0, if \( w = 1 \) or \( w = \eta \theta \).

The image of \( K \otimes wK \) is calculated to be

\[
g(K \otimes wK) = KwK - \phi_t(wK)K = ((-1)^{i+j} - t^{i+j})wK.
\]

This is either a multiple of \( w \), if \( w = \eta \) or \( w = \theta \), or it is 0, if \( w = 1 \) or \( w = \eta \theta \).

Next we consider \( \mathcal{N}_1 \otimes \mathcal{N}_1 \). Let \( x, y \in \{\eta, \theta\} \) and \( i, j \in \{0, 1\} \). Notice that \( xK^i \otimes xK^j \) lies in the kernel of \( g \).

So suppose that \( x \neq y \). Let \( e = 1 \) if \( y = \eta \), and \( e = -1 \) if \( y = \theta \). Then we calculate

\[
g(xK^i \otimes yK^j) = xK^i yK^j - \phi_t(yK^j)xK^i
\]

\[
= (-1)^i xyK^{i+j} - (-1)^j t^e yxK^{i+j}
\]

\[
= (((-1)^i + (-1)^j t^e)xyK^{i+j}. \tag{4.5.6}
\]

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In the case that \( i = j \), this element is a non-zero multiple of \( xy \). In the case that \( i \neq j \), this element is a nonzero multiple of \( xyK \). Putting these together, we have that as long as \( t \neq \pm 1 \),

\[
\eta \theta, \eta \theta K \in \text{Com}_t.
\]

Altogether we have shown that for \( t \neq \pm 1 \), \( \text{Com}_t \) is spanned by \( \{\eta, \theta, \eta K, \theta K, \eta \theta, \eta \theta K\} \).

For \( t = \pm 1 \), the equations above are still valid. In particular, Equation 4.5.4 and 4.5.5 show that

\[
\eta, \theta, \eta K, \theta K \in \text{Com}_{\pm 1}.
\]

Differences occur when considering Equation 4.5.6. For \( t = 1 \) the coefficient \((-1)^i + (-1)^j\) is 0 if \( i \neq j \), and is 2 if \( i = j \). This implies that

\[
\eta \theta \in \text{Com}_1, \text{ but } \eta \theta K \notin \text{Com}_1.
\]

However, if \( t = -1 \) then the coefficient is \((-1)^i - (-1)^j\), which is 0 if and only if \( i = j \). This implies that

\[
\eta \theta K \in \text{Com}_{-1}, \text{ but } \eta \theta \notin \text{Com}_{-1}.
\]

\[\square\]
CHAPTER 5

BUNDLES

This chapter reviews topics in fiber bundles, leading to flat connections of principal bundles over a surface. Our main reference for this chapter is [32].

5.1 Principal Group Bundles

Before continuing, we fix some notation. For $G$ a group, we let $r_g : G \to G$ be the right-multiplication map $r_g(h) = hg$. This is a right group-action of $G$ on itself. For $A, B$ any two sets, let $pr_1 : A \times B \to A$ be the projection onto the first component.

**Definition 5.1.1.** Let $M$ be a smooth manifold, and $G$ a Lie group. A *principal $G$-bundle over $M$* is a smooth manifold $P$ along with a smooth right $G$-action on $P$, denoted $R_g(p) = pg$, and a smooth projection $\pi : P \to M$ such that:

1. For every $x \in M$, $G$ acts freely and transitively on the fiber, and

2. For every $x \in M$, there is a neighborhood $U \ni m$ and a diffeomorphism $\phi :$
\[ \pi^{-1}(U) \to U \times G \] satisfying the following two commutative diagrams, :

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\phi} & U \times G \\
\downarrow{\pi} & & \downarrow{pr_1} \\
U & & U
\end{array}
\quad
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\phi} & U \times G \\
\downarrow{R_g} & & \downarrow{id \times r_g} \\
\pi^{-1}(U) & \xrightarrow{\phi} & U \times G
\end{array}
\]

A local chart is a pair \((U, \phi)\), where \(U \subset M\) is open, and \(\phi : \pi^{-1}(U) \to U \times G\) satisfies the commutative diagrams above. A trivialization is a diffeomorphism \(\phi_x : P_x \to G\) satisfying \(\phi_x(pg) = \phi_x(p)g\). By definition, a restriction of a local chart to \(P_x\) is a trivialization.

For two principal \(G\)-bundles \(\pi : P \to M\) and \(\pi' : P' \to M'\), with right \(G\)-action \(R_g, R'_g\) respectively, a \(G\)-bundle morphism is a pair of smooth maps

\[ F : P \to P' \text{ and } f : M \to M' \]

such that the following diagrams commute:

\[
\begin{array}{ccc}
P & \xrightarrow{F} & P' \\
\downarrow{R_g} & & \downarrow{R'_g} \\
P & \xrightarrow{F} & P'
\end{array}
\quad
\begin{array}{ccc}
P & \xrightarrow{F} & P' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
P & \xrightarrow{E} & E'
\end{array}
\]

A \(G\)-bundle morphism \(F\) is an isomorphism if it has an inverse.

For two local charts \((U_\alpha, \phi_\alpha)\) and \((U_\beta, \phi_\beta)\) such that

\[ U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset, \]

consider the map

\[ \phi_\beta \circ \phi_\alpha^{-1} : U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G. \]
This map is the identity on the first component, and so we define a transition map \( t_{\alpha\beta} : U_{\alpha\beta} \to G \) by
\[
(\phi_\alpha \circ \phi_\beta^{-1})(m, g) = (m, t_{\alpha\beta}(m)g)
\]

Transition maps will be used often. The following standard lemma will be used implicitly from here on.

**Lemma 5.1.2.** For three local charts \((U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta), (U_\gamma, \phi_\gamma)\) of a principal \(G\)-bundle, the transition maps satisfy the following properties:

1. \( t_{\alpha\alpha}(x) = 1_G \) for all \( x \in U_\alpha \).
2. \( t_{\beta\alpha}(x) = t_{\alpha\beta}^{-1}(x) \) for all \( x \in U_{\alpha\beta} \).
3. \( t_{\alpha\gamma}(x)t_{\gamma\beta}(x)t_{\beta\alpha}(x) = 1_G \) for all \( x \in U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma \).

It turns out that a principal \(G\)-bundle is uniquely defined, up to isomorphism, by its transition maps. We give an explicit construction of a \(G\)-bundle using coherent local transition maps.

**Definition 5.1.3.** A \(G\)-bundle structure on \(M\), \( \mathcal{A} = (\{U_\alpha\}, \{t_{\alpha\beta}\}) \), is an open cover \( \{U_\alpha\} \) of \(M\) along with structure maps \( t_{\alpha\beta} : U_{\alpha\beta} \to G \) which satisfy the properties in Lemma 5.1.2.

**Lemma 5.1.4.** A \(G\)-bundle structure \( \mathcal{A} = (\{U_\alpha\}, \{t_{\alpha\beta}\}) \) on \(M\) uniquely defines a principal \(G\)-bundle over \(M\) with canonical charts \((U_\alpha, \phi_\alpha)\) such that the transition map between \(\phi_\alpha\) and \(\phi_\beta\) is \(t_{\alpha\beta}\).

**Proof.** Consider the disjoint union of trivial bundles,
\[
N = \coprod_{\alpha} U_\alpha \times G \times \{\alpha\}.
\]
There is an equivalence relation on \( N \), given by

\[
(x, g, \alpha) \sim (y, h, \beta)
\]

if \( x = y \), and \( g = t_{\alpha \beta} h \). Let \([x, g, \alpha]\) denote its equivalence class.

There is a right \( G \)-action on \( N \) inherited from the trivial bundles. For \( h \in G \), define

\[
[x, g, \alpha].h = [x, gh, \alpha].
\]

This right-action is well-defined on \( N \) because the transition maps act on the left. Specifically, for \((x, g, \alpha) \sim (y, h, \beta)\) and \( f \in G \), we have

\[
[x, g, \alpha].f = [x, gf, \alpha] = [y, t_{\alpha \beta}gf, \beta] = [y, h, \beta].f.
\]

For each \( \alpha \) there is a canonical chart \((U_\alpha, \phi_\alpha)\) which is the inverse of the inclusion

\[
\phi_\alpha^{-1}(x, g) = [x, g, \alpha].
\]

It is clear that \( \phi_\alpha \) satisfies the axioms of Definition 5.1.3. For \( x \in U_{\alpha \beta} \) the transition map \( \phi_\alpha \circ \phi_\beta^{-1} \) is found by

\[
\phi_\alpha^{-1}(x, g) = [x, g, \alpha] = [x, t_{\alpha \beta}g, \beta] = \phi_\beta^{-1}(x, t_{\alpha \beta}g).
\]

\[\square\]

### 5.2 Vector Bundles

**Definition 5.2.1.** Let \( M \) be a smooth manifold and \( V \) a vector space. A **vector bundle over \( M \) with typical fiber \( V \)** is a smooth manifold \( E \) along with smooth surjection \( \pi : E \to M \) such that
1. For all \( x \in M \), the fiber \( E_x = \pi^{-1}(x) \) has a vector space structure.

2. Every \( x \in M \) has a neighborhood \( U \) and diffeomorphism \( \phi : \pi^{-1}(U) \to U \times V \) such that \( \phi|_x : E_x \to \{x\} \times V \) is a linear isomorphism for all \( x \), and the following diagram commutes:

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\phi} & U \times V \\
\downarrow{\pi} & & \downarrow{pr_1} \\
U & & U
\end{array}
\]

Such a pair \((U, \phi)\) is called a local chart for \( E \).

For two local charts \((U_\alpha, \phi_\alpha)\) and \((U_\beta, \phi_\beta)\), the change of basis map \( \phi_\alpha \circ \phi_\beta^{-1} \) preserves fibers. The transition map

\[
t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)
\]

is defined by

\[(\phi_\alpha \circ \phi_\beta^{-1})(x, v) = (x, t_{\alpha\beta}(v)).\]

For two vector bundles

\[
\pi : E \to M \text{ and } \pi' : E' \to M',
\]

a vector bundle morphism is a pair of smooth maps

\[
F : E \to E' \text{ and } f : M \to M'
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{F} & E' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
M & \xrightarrow{f} & M'
\end{array}
\]
and \( F|_x : E_x \to E'_{f(x)} \) is a linear map for all \( x \in M \).

A vector bundle morphism \( F \) is an isomorphism if it has an inverse.

Transition maps for vector bundles play the same role as those for principal group bundles. When we define associated bundles in Section 5.4, this connection is made explicit. For now, we notice that they satisfy the same properties as those in Lemma 5.1.2

**Lemma 5.2.2.** For three local charts \((U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta), (U_\gamma, \phi_\gamma)\) of a vector bundle, the transition maps satisfy the following properties:

1. \( t_{\alpha\alpha}(x) = id \) for all \( x \in U_\alpha \).
2. \( t_{\beta\alpha}(x) = t_{\alpha\beta}^{-1}(x) \) for all \( x \in U_{\alpha\beta} \).
3. \( t_{\alpha\gamma}(x)t_{\gamma\beta}(x)t_{\beta\alpha}(x) = id \) for all \( x \in U_{\alpha\beta\gamma} \).

**Remark 5.2.3.** Just as with principal bundles, vector bundle may be pieced together from local charts and coherent gluing data. Suppose \( \{U_\alpha\}_{\alpha \in \mathcal{A}} \) is an open cover of \( M \) and for every intersection \( U_{\alpha\beta} \) there is a map \( t_{\alpha\beta} : U_{\alpha\beta} \to GL(V) \) satisfying the conditions in Lemma 5.2.2.

We take the disjoint union \( \coprod_\alpha (U_\alpha \times V \times \{\alpha\}) \), modulo the equivalence relation \((x, v, \alpha) \sim (x', v', \beta)\) if \( x = x' \in U_{\alpha\beta} \), and \( v' = t_{\beta\alpha}(x)(v) \). The resulting space is a vector bundle.

### 5.3 Flat Connections and Parallel Transport

The notion of connections on bundles has a rich mathematical history. Because of this, there are several ways to define them. We will use the parallel transport definition,
as pioneered by Knebler [18]. It is well known that this approach is equivalent to the more usual definitions, see Barrett [3], Caetano and Picken [6], and Vassiliou [36].

We will only need flat connections, which will allow us to simplify our treatment considerably. At the end of this section, we explain how to classify flat connections using group anti-homomorphisms.

First, some notation. Unless otherwise specified, a path in \( M \), where \( M \) is still a smooth manifold, will refer to a piece-wise smooth map \( c : [0,1] \to M \). The set of such paths is organized into a category \( \mathcal{P}^1(M) \). The objects of \( \mathcal{P}^1(M) \) are elements of \( M \). For \( x, y \in M \), the morphism set \( \mathcal{P}^1(M, x,y) \) is the set of paths with \( c(0) = x \) and \( c(1) = y \). Let \( \text{id}_x \in \mathcal{P}^1(M, x,x) \) be the constant path \( \text{id}_x(t) = x \). For \( c \in \mathcal{P}^1(x, y) \), the reverse path \( c^{-1} \in \mathcal{P}^1(y, x) \) is defined by \( c^{-1}(t) = c(1 - t) \). For two paths \( c, c' \) with \( c(1) = c'(0) \), their composition is the usual concatenation of paths

\[
c \circ c'(t) = \begin{cases} 
c(2t) & \text{if } t \leq \frac{1}{2} \\
c'(2t - 1) & \text{if } t > \frac{1}{2}. \end{cases}
\]

Two paths \( c, c' \in \mathcal{P}^1(M, x,y) \) are homotopic if there exists a homotopy

\[
H : [0,1] \times [0,1] \to M
\]
such that \( H(t, \cdot) \in \mathcal{P}^1(M, x,y) \), \( H(0, \cdot) = c \), and \( H(1, \cdot) = c' \). Homotopy defines an equivalence relation on \( \mathcal{P}^1(M) \). The homotopy class of a path \( c \) is denoted \([c]\), and the set of homotopy classes of \( \mathcal{P}^1(M, x,y) \) is denoted \( \Pi_1(M, x,y) \).

We briefly note that there are other types of path homotopy, namely continuous and smooth homotopies. It is well known that any continuous path \([0,1] \to M\) is continuously homotopic to a smooth path. Further, two smooth paths which are
continuously homotopic are smoothly homotopic as well. For details, see Theorem 6.19 and Proposition 6.20 in [20] or Corollary 17.8.1 in [5].

This implies that the homotopy classes are the same for any of these types of homotopy. As such, we will typically not specify these details. In particular, \( \Pi_1(M, x, x) \) is the usual fundamental group of \( M \) based at \( x \). We refer to this group with the usual notation, \( \pi_1(M, x) \).

The homotopy classes of paths in \( M \) are also organized into the \textit{fundamental groupoid of} \( M \), \( \Pi_1(M) \). Objects of \( \Pi_1(M) \) are elements of \( M \). For \( x, y \in M \), the set of morphisms from \( x \) to \( y \) is the set of homotopy classes \( \Pi_1(M, x, y) \).

The identity element in \( \Pi_1(M, x, x) \) is \([\text{id}_x]\), the homotopy class of the constant path. For \( c \in \Pi_1(M, x, y) \), the compositions \( cc^{-1} \) and \( c^{-1}c \) are homotopic to \( \text{id}_x \) and \( \text{id}_y \) respectively. Thus inverse morphisms are defined by \([c]^{-1} = [c^{-1}]\).

First we give a standard definition of a connection. Let \( \pi : P \to M \) be a principal \( G \)-bundle.

\textbf{Definition 5.3.1.} For \( u \in P \), the \textit{vertical subspace} \( V_u P \subset T_u P \) is the kernel of \( \pi_* : TP \to TM \).

A \textit{connection} \( H \) on \( P \) is a choice of subspace \( H_u P \subset T_u P \) for each \( u \in P \), satisfying the following:

\begin{enumerate}
  \item \( T_u P = H_u P \oplus V_u P \) for all \( u \in P \).
  \item \( dR_g(H_u P) = H_{ug} P \) for all \( u \in P \) and \( g \in G \).
  \item For every smooth vector field \( X \) on \( P \), there are smooth vector fields \( X^H \) and \( X^V \) such that \( X^H(u) \in H_u P, \ X^V(u) \in V_u P \), and \( X = X^H + X^V \).
\end{enumerate}
Two bundles $P, P'$ with connections $H, H'$ are isomorphic if there is a bundle isomorphism $F : P \to P'$ such that the derivative map $dF : TP \to TP'$ maps $H_u P$ isomorphically onto $H'_{f(u)} P$ for every $u \in P$.

In this paper, the main purpose of a connection is to define their related parallel transport and holonomy maps. We summarize the following standard definitions and lemmas.

**Definition 5.3.2.** Let $P \to M$ be a principal $G$-bundle with connection $H$. A smooth path $c$ in $P$ is horizontal if $\dot{c}(t) \in H_{c(t)} P$ for all $t \in [0, 1]$, where $\dot{c}(t)$ is the tangent vector of $c$ at $t$.

**Lemma 5.3.3.** For $c$ a smooth path in $M$ and $u \in P_{c(0)}$, there is a unique horizontal path $c^u$ in $P$ such that $c^u(0) = u$ and $\pi \circ c^u(t) = c(t)$ for all $t \in [0, 1]$. The association $u \mapsto c^u(1)$ defines a bijection

$$\Gamma_c : P_x \to P_y.$$ 

This map extends to piecewise smooth paths. For $c \in \mathcal{P}^1(M, x, y), c' \in \mathcal{P}^1(M, y, z)$, $p \in P_x$ and $g \in G$, $\Gamma$ satisfies the following properties:

1. $\Gamma_{id_x} = id_{P_x}$,
2. $\Gamma_c(pg) = \Gamma_c(p)g$,
3. $\Gamma_{c^{-1}} = \Gamma^{-1}_c$,
4. $\Gamma_{cc'} = \Gamma_{c'} \circ \Gamma_c$.

**Definition 5.3.4.** $\Gamma$ is called the parallel transport map associated to $H$. The connection $H$ is flat if $\Gamma_c = \Gamma_{c'}$ for any two homotopic paths $c, c' \in \mathcal{P}^1(M, x, y)$. In this case, the map $\Gamma$ is also called flat.
Lemma 5.3.5. Let $x_0 \in M$ and $b_0 \in \pi^{-1}(x_0)$ be fixed. For $c \in \pi_1(M, x_0)$, there is a unique element $\gamma(c) \in G$ such that

$$b_0 \gamma(c) = \Gamma_c(b_0).$$

This association defines an anti-homomorphism

$$\gamma : \pi_1(M, x_0) \rightarrow G.$$

Definition 5.3.6. The map $\gamma$ is called the holonomy map associated to the triplet $(P, H, b_0)$.

Proof. The element $\gamma(c)$ exists and is unique because $G$ acts freely and transitively on fibers of $P$. The identity element of $\pi_1(M, x_0)$ is represented by the constant path $\text{id}_{x_0}(t) = x_0$, and it is clear that $\gamma(\text{id}_{x_0}) = 1_G$. We can also check that $\gamma$ is an anti-homomorphism; for any two $c_1, c_2 \in \pi_1(M, x_0)$, we have

$$b_0 \gamma(c_1 c_2) = \Gamma_{c_1 c_2}(b_0)$$

$$= \Gamma_{c_2} (\Gamma_{c_1}(b_0))$$

$$= \Gamma_{c_2} (b_0 \gamma(c_1))$$

$$= (\Gamma_{c_2}(b_0)) \gamma(c_1)$$

$$= (b_0 \gamma(c_2)) \gamma(c_1)$$

$$= b_0 (\gamma(c_2) \gamma(c_1)).$$

Every triple $(P, H, b_0)$ thus defines a group anti-homomorphism. Bundle isomorphism allows us to define a natural equivalence relation for such triplets. These equivalence classes are then characterized by their holonomy map.
**Definition 5.3.7.** Two triples \((P, H, b_0)\) and \((P', H', b'_0)\) are **isomorphic** if there is an isomorphism of flat bundles \(F : P \to P'\) such that \(F(b_0) = b'_0\).

**Theorem 5.3.8.** Given \(x_0 \in M\), there is a bijection between anti-homomorphisms \(\pi_1(M, x_0) \to G\) and isomorphism classes of triples \((P, H, b_0)\), where \(P \to M\) is a principal \(G\)-bundle, \(H\) a flat connection on \(P\), and \(b_0 \in \pi^{-1}(x_0)\).

This theorem is proved by Caetano and Picken [6] in a slightly different guise. The theorem presented there is slightly broader as it characterizes all connections on \(P\), while we only look at flat connections. Further, the holonomy maps defined there are group homomorphisms, while ours are anti-homomorphisms. This is because they define the holonomy of an element \(c \in \pi_1(M, x_0)\) to be the element \(g_c \in G\) such that

\[
b_0 = \Gamma_c(b_0)g_c,
\]

while our holonomy map \(\gamma\) satisfies

\[
b_0\gamma(c) = \Gamma_c(b_0).
\]

These are related by \(\gamma(c) = g^{-1}_c\), giving the usual correspondence between group homomorphisms and anti-homomorphisms.

We note that it is possible to similarly characterize pairs \((P, H)\), where \(P\) is a principal \(G\)-bundle over \(M\), and \(H\) a flat connection on \(P\). The dependence on \(b_0 \in \pi^{-1}(x_0)\) may be removed by using conjugacy classes of homomorphisms. The following definition and theorem are from a paper of Vassiliou [36]. It will not be used later, so we state it without comment.

**Definition 5.3.9.** Two homomorphisms \(f, f' \in \text{Hom}(\pi_1(M, x_0), G)\) are **conjugate** if there is a \(g \in G\) such that \(f = I(g) \circ f'\), where \(I(g)\) denoted conjugation by \(g\):
\( I(g)(h) = ghg^{-1} \). Conjugacy defines an equivalence relation on \( \text{Hom}(\pi_1(M, x_0), G) \).

Let \( S(M, G) \) denote the set of such conjugacy classes, and \( B(M, G) \) the set of isomorphism classes of flat principal \( G \)-bundles over \( M \).

**Theorem 5.3.10 ( [36] ).** For \( M \) a connected smooth manifold and \( G \) a Lie group, there is a bijection between \( B(M, G) \) and \( S(M, G) \).

### 5.3.1 Flat Charts

In this section, we define local charts which respect the flat connection. We will use these flat charts extensively when working with decorations.

**Definition 5.3.11.** Let \( (P, H) \) be a flat principal \( G \)-bundle over a smooth manifold \( M \) with parallel transport map \( \Gamma \). A chart \( \phi : \pi^{-1}(U) \to U \times G \) is flat with respect to \( \Gamma \) if for every path \( c : [0, 1] \to U \), we have

\[
\phi \circ \Gamma_c \circ \phi^{-1}(c(0), g) = (c(1), g).
\]

An atlas is flat with respect to \( \Gamma \) if all of its charts are.

First, some basic results about flat charts. For a chart \( \phi \) we will use the notation \( \phi_x : P_x \to G \) to denote the map defined by

\[
\phi(p) = (x, \phi_x(p)),
\]

for all \( p \in P_x \).

**Lemma 5.3.12.** Let \( (U, \phi) \) and \( (U', \phi') \) be two charts of \( P \) which are flat with respect to \( \Gamma \). Then the transition map between them is locally constant.
Proof. Let \( x, y \) lie in a connected component of \( U \cap U' \), and \( c \in \mathcal{P}^1(U \cap U', x, y) \). Consider the following commutative diagram. The vertical arrows are the transition maps over \( x \) and \( y \), and the horizontal arrows represent parallel transport in local coordinates. Because \( \phi \) and \( \phi' \) are flat with respect to \( \Gamma \), the two horizontal maps are the identity. Thus the two vertical maps must be equal.

\[
\begin{array}{c}
G \\
\downarrow \phi_y \circ \Gamma_c \circ \phi_x^{-1} \\
G \\
\downarrow \phi_y' \circ \phi_x^{-1} \\
G \\
\end{array}
\]

\( \square \)

If a path is covered by flat charts, then parallel translation along the path can be calculated by composing the transition maps.

Lemma 5.3.13. Suppose \( p \) is a path in \( M \), and there are points

\[ 0 = t_1 < t_2 < t_3 < \cdots < t_{k+1} = 1 \]

and flat charts \( (U_1, \phi_1), \cdots, (U_k, \phi_k) \) such that for \( 1 \leq i \leq k \),

\[ p([t_i, t_{i+1}]) \subset U_i. \]

In particular, \( p(t_1) \in U_1 \), \( p(t_{k+1}) \in U_k \), and

\[ p(t_i) \in U_{i-1} \cap U_i \text{ for all } 2 \leq i \leq k. \]

Let \( \Phi_i \in G \) be the transition map from \( \phi_i \) to \( \phi_{i+1} \), and

\[ \Phi = \Phi_k \circ \Phi_{k-1} \circ \cdots \circ \Phi_1. \]
Then
\[ \phi_k \circ \Gamma_p \circ \phi_0^{-1}(p(0), h) = (p(1), \Phi(h)). \]

Proof. It suffices to prove this for two charts \((U_1, \phi_1)\) and \((U_2, \phi_2)\). Let \(p\) be a path in \(U_1 \cup U_2\) such that for some \(t_0 \in (0, 1)\),
\[ p([0, t_0]) \subset U_1, \text{ and } p([t_0, 1]) \subset U_2. \]

Let \(p_1\) be the restriction of \(p\) to \([0, t_0]\), and \(p_2\) the restriction to \([t_0, 1]\). Then because \(\phi_1, \phi_2\) are flat charts, we have
\[ \phi_1 \circ \Gamma_{p_1} \circ \phi_1^{-1}(p(0), h) = (p(t_0), h) \]
and
\[ \phi_2 \circ \Gamma_{p_2} \circ \phi_2^{-1}(p(t_0), h) = (p(1), h). \]

Then,
\[ \phi_2 \circ \Gamma_p \circ \phi_1 = (\phi_2 \circ \Gamma_{p_2} \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1}) \circ (\phi_1 \circ \Gamma_{p_1} \circ \phi_1^{-1}) \]
and so,
\[ (\phi_2 \circ \Gamma_p \circ \phi_1)(p(0), h) = (p(1), t(h)) \]
where \(t\) is the locally constant transition map from \(\phi_1\) to \(\phi_2\), defined by
\[ (\phi_2 \circ \phi_1^{-1})(x, h) = (x, t(h)). \]

\[ \square \]

Lemma 5.3.14. For \(M\) a smooth manifold, \(U \subset M\) connected and simply-connected, \(x \in U\), and any trivialization \(\phi_0 : P_x \rightarrow G\), there exists a unique flat chart \(\phi\) over \(U\) which extends \(\phi_0\).
Proof. First we prove uniqueness. Suppose \( \phi, \phi' \) are two flat charts extending \( \phi_0 \). The transition map between them at the point \( x \) must be the identity map. This transition map is locally constant, and so \( \phi = \phi' \).

To prove existence, we explicitly construct the chart. For \( p \in \pi^{-1}(U) \), with \( \pi(p) = y \), pick a path \( c \) in \( U \) such that \( c(0) = y \) and \( c(1) = x \). Notice that because \( \Gamma \) is flat and \( U \) is simply connected, \( \Gamma_c \) does not depend on the choice of \( c \). Then we define

\[
\phi(p) = (y, (\phi_0 \circ \Gamma_c)(p)).
\]

Notice that, because \( \phi_0 \) and \( \Gamma_c \) are \( G \)-equivariant, we have, for any \( g \in G \),

\[
\phi(pg) = (y, (\phi_0 \circ \Gamma_c)(pg)) = (y, (\phi_0 \circ \Gamma_c)(p))g.
\]

This \( \phi \) defines our flat bundle chart.

We can use a single flat chart \( \phi \), along with the \( G \)-action, to find all of the flat charts over \( U \). For \( g \in G \), define the flat chart \( \phi^g \) by

\[
\phi^g(p) = g \cdot \phi(p).
\]

**Lemma 5.3.15.** Any two flat charts \( \phi, \phi' \) over \( U \) with constant transition map \( t \in G \) are related by \( \phi' = \phi^t \). Consequently, every flat chart over \( U \) lies in the orbit

\[
\{ \phi^g | g \in G \}.
\]

Finally, we revisit \( G \)-bundle structures as in Lemma 5.1.4. We have seen that if two charts are flat with respect to \( \Gamma \), then their transition map is locally constant. This relation goes both ways; if a \( G \)-bundle structure has locally constant transition maps, then it has a canonical flat connection.
**Definition 5.3.16.** A $G$-bundle structure is *flat* if all transition maps are locally constant. These will simply be called flat structures when the group $G$ can be assumed.

**Lemma 5.3.17.** A flat $G$-bundle structure on $M$ defines a flat $G$-bundle on $M$.

*Proof.* For a trivial bundle $U \times G$, the tangent space splits as

$$T(U \times G)_{(x,g)} = TU_x \oplus TU_g.$$  

There is a canonical flat connection on $U \times G$ defined by

$$H_{(x,g)} = TU_x \oplus \{0\}.$$  

It is easily checked that this choice satisfies the axioms in Definition 5.3.1. In particular, for $h \in G$ and $R_h$ the right action $R_h(x,g) = (x, gh)$ we have

$$(dR_h)(H_{(x,g)}) = H_{(x,gh)},$$  

where $dR_h$ denotes the derivative map.

Now let $\{(U_\alpha, \{t_{\alpha\beta}\})\}$ be a flat $G$-bundle structure on $M$. The induced bundle is glued together starting with the disjoint union

$$N = \bigsqcup_\alpha U_\alpha \times G \times \{\alpha\}.$$  

The bundle $E$ is then defined as the equivalence classes of $N$, where

$$(x, g, \alpha) \sim (y, h, \beta)$$

if $x = y$ and $g = t_{\alpha\beta} h$.  

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Consider $U_\alpha, U_\beta$ two open neighborhoods in the flat $G$-bundle structure. Their locally constant transition map $t$ defines an isomorphism

$$L_t : U_{\alpha\beta} \times G \times \{\alpha\} \to U_{\alpha\beta} \times G \times \{\beta\}.$$

The trivial connections on each connected component of $N$ together define a flat connection on $N$. This gluing isomorphism $L_t$ respects the canonical connection on each component, as

$$(dL_t)(H_{(x,g,\alpha)}) = H_{(x,tg,\beta)}.$$

Thus the quotient space $E$ inherits a well-defined connection. By construction, we have trivial flat charts

$$\phi_\alpha : U_\alpha \times G \times \{\alpha\} \to U_\alpha \times G,$$

which are just the projections onto the first two coordinates. These flat charts cover $M$ and, by assumption, have locally constant transition maps. These conditions are enough to ensure that $\Gamma$ is flat.

We show this by calculating the associated parallel transport map $\Gamma$ for a contractible loop $c$ in $M$. Any such loop can be written as a composition of paths $c = p_1p_2\cdots p_n$, where each $p_i$ is a path in one neighborhood $U_{\alpha_i}$, $p_n(1) = p_1(0)$, and for each $i < n$,

$$p_i(1) = p_{i+1}(0) \in U_{\alpha_i} \cap U_{\alpha_{i+1}}.$$

Let $t_i$ be the transition map between $\phi_{\alpha_i}$ and $\phi_{\alpha_{i+1}}$. Then, using Lemma 5.3.13, $\Gamma_c$ in local $\phi_{\alpha_1}$-coordinates can be calculated as the composition of these maps. Namely,

$$(\phi_{\alpha_1} \circ \Gamma_c \circ \phi_{\alpha_1}^{-1})(c(0), g) = (c(0), t_n t_{n-1} \cdots t_1 g).$$
We claim that $t_n \cdots t_1 = 1 \in G$. This is a result of the cocycle condition which transition maps satisfy. Because $c$ is contractible, there is a simply connected open neighborhood $U$ which contains the image of $c$. For each $x_i = p_i(0)$, let $q_i$ be a path from $x_i$ to $x_1$ which lies in $U$. Then $q_i^{-1} p_i q_{i+1}$ is a loop in $U$, hence contractible. By further subdividing the paths $q_i$ if necessary, we can assume that the image of $q_i$ lies in $U_{\alpha_1} \cup U_{\alpha_i}$, and that

$$U_{\alpha_1} \cap U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset.$$

Then the transition maps satisfy the cocycle condition

$$t_{\alpha_1 \alpha_{i+1}} t_{\alpha_{i+1} \alpha_i} t_{\alpha_i \alpha_1} = 1_G,$$

and thus

$$
\left( \phi_{\alpha_1}^{-1} \circ \Gamma_{q_i}^{-1} p_i q_{i+1} \circ \phi_{\alpha_1} \right) (c(0), g) = (c(0), g)
$$

Now we finish the proof by writing $c$ as the composition

$$c = p_1 p_2 \cdots p_n = (p_1 p_2 q_2)(q_2^{-1} p_3 q_3) \cdots (q_n^{-1} p_{n-1} p_n),$$

so that

$$1_G = (t_{\alpha_1 \alpha_{n-1}} t_{\alpha_{n-1} \alpha_{n-2}} t_{\alpha_{n-2} \alpha_1})(t_{\alpha_1 \alpha_{n-2}} t_{\alpha_{n-2} \alpha_{n-3}} t_{\alpha_{n-3} \alpha_1}) \cdots (t_{\alpha_1 \alpha_3} t_{\alpha_3 \alpha_2} t_{\alpha_2 \alpha_1})$$

$$= t_n t_{n-1} \cdots t_1.$$

$\square$
5.4 Associated Bundles

Suppose we have a Lie group $G \subset \text{GL}(V)$, and a principal $G$-bundle $\pi : P \to M$. Then we can define the associated vector bundle. We start with the bundle $P \times V \to M$, which has a $G$-action defined by

$$g.(p,v) = (pg^{-1}, g(v)).$$

This action fixes the fibers of $P \times V \to M$. The associated vector bundle is the orbit space $\hat{\pi} : (P \times V)/G \to M$.

We construct local charts for $\hat{\pi}$. Let $(U, \phi)$ be a local chart of $P$, where $U \subset M$ is open and $\phi : \pi^{-1}(U) \to U \times G$. In the coordinates $\phi(p) = (x, r)$, the $G$-action is

$$g.(x,r,v) = (x, rg^{-1}, \rho(g)(v)).$$

Thus in the neighborhood $(U \times G \times V)/G$, every element has a unique representative of the form $(x, 1, v) \in U \times \{1\} \times V$. Then the local chart

$$\hat{\phi} : (\pi^{-1}(U) \times V)/G \to U \times V$$

is defined, for $\phi(p) = (x, r)$, by

$$\hat{\phi}(p,v) = (x, \rho(r)v).$$

We can find the transition maps between local maps of this form. Suppose $(U_i, \phi_i)$, $i = 1, 2$, are two charts of $P$ with non-zero intersection, with transition function $t : U_1 \cap U_2 \to G$ defined by

$$(\phi_2 \circ \phi_1^{-1})(x,g) = (x,t_xg).$$

Then the transition map of the associated vector bundle over $U_1 \cap U_2$ is $\rho(t)$; that is,

$$(\hat{\phi}_2 \circ \hat{\phi}_1^{-1})(x,1,v) = (x,t_xv) = (x,1,\rho(t_x)(v)).$$
Remark 5.4.1. Notice that isomorphic principal $G$-bundles yield isomorphic associated bundles. Suppose $F : P \to P'$ is an isomorphism of principal $G$-bundles over $M$. Then there is an isomorphism $F \times id : P \times V \to P' \times V$. This isomorphism commute with the $G$-action, thus also defining an isomorphism

$$\hat{F} : (P \times V) / G \to (P' \times V) / G.$$  

There is also a method of going from a vector space to a principal bundle, which is an inverse operation to the associated vector bundle. This is explained with the next lemma.

Definition 5.4.2. Let $E \to M$ be a vector bundle with typical fiber $V$. Suppose there exists an atlas $\{U_a, \phi_a\}$ such that all transition maps $t_{a\beta}$ have images lying in a subgroup $G \subset GL(V)$. In the case, the atlas is said to define a $G$-structure for the bundle. For a basepoint $b \in M$, two linear trivialization $\phi_0, \phi_1 : E_b \to V$ are $G$-equivalent if the transition map $\phi_1 \circ \phi_0^{-1} \in GL(V)$ lies in the subgroup $G$.

Notice that being $G$-equivalent over a point $b$ is an equivalence relation. Indeed, if the transition map $\phi_2 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_0^{-1}$ lie in $G$, then their product $\phi_2 \circ \phi_0^{-1}$ does as well.

Lemma 5.4.3. For a vector space $V$ and subgroup $G \subset GL(V)$, there is a bijection between isomorphism classes of vector bundles over $M$ with fiber $V$ and structure group $G$, and isomorphism classes of principal $G$-bundles over $M$.

If $P$ and $E$ are a $G$-bundle and its associated vector bundle, respectively, then for a fixed point $b \in M$, there is a canonical association $\phi_0 \mapsto \hat{\phi}_0$ which maps the set of trivializations $\phi_0$ of $P_b$ bijectively onto a $G$-equivalence class of trivializations $\hat{\phi}_0$ of $E_b$. 

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Proof. Given a principal $G$-bundle, we take the vector bundle associated to the inclusion $G \to GL(V)$.

Now suppose we have a vector bundle and a bundle atlas $\{U_i, \phi_i\}$ with structure group $G$. Then a principal $G$-bundle can be constructed using the same atlas. The transition maps $t_{ij} : U_i \cap U_j \to G$ may be read as transition maps for a $G$-bundle. Then we may use the gluing procedure described before to create a principal $G$-bundle over $M$. Specifically, it is the equivalence classes of the disjoint union $\Pi_i U_i \times G \times \{i\}$, with the equivalence $(u,g,i) \sim (u',g',j)$ is $u = u' \in U_i \cap U_j$, and $t_{ji}(g) = g'$.

Next suppose $\phi_0 : P_b \to G$ is a trivialization, and let $p_0 = \phi_0^{-1}(1) \in P_b$. Recall that the fiber $E_b$ is defined as the quotient space $(P_b \times V)/G$. Then we can define

$$\hat{\phi}_0^{-1}(v) = [p_0, v].$$

This map is clearly linear. It is a bijection because every element of $E_b$ has a unique representative of the form $(p_0, v)$. Thus $\hat{\phi}_0$ is a linear isomorphism from $E_b$ to $V$.

We now prove the second claim. First we show that for any two trivializations $\phi_0, \phi_1$ of $P_b$, the related maps $\hat{\phi}_0$ and $\hat{\phi}_1$ are $G$-equivalent. Recall the elements $p_i = \phi_i^{-1}(1) \in P_b$. Let $t \in G$ be the unique element such that $p_0 = p_1 t$. We claim that $t$ is exactly the transition map $\hat{\phi}_1 \circ \hat{\phi}_0^{-1} \in GL(V)$. This is because for any $v \in V$, we have

$$\hat{\phi}_0^{-1}(v) = [p_0, v] = [p_1 t, v] = [p_1, t(v)] = \hat{\phi}_1^{-1}(t(v)).$$

Thus the trivializations $\hat{\phi}_0$ and $\hat{\phi}_1$ are $G$-equivalent.

Next we show that the association $\phi_0 \mapsto \hat{\phi}_0$ is injective. Consider two trivialization
\( \phi_0, \phi_1 \) of \( P_b \) with \( \hat{\phi}_0 = \hat{\phi}_1 \). Then \( \hat{\phi}_0^{-1}(1) = \hat{\phi}_1^{-1}(1) \). G-equivariant maps are completely defined by the image of one element, so this is enough to conclude that \( \phi_0 = \phi_1 \).

Finally we show surjectivity onto the G-equivalence class. Suppose we have a linear trivialization \( \psi_0 : E_b \to V \) which is G-equivalent to \( \hat{\phi}_0 \), for some trivialization \( \phi_0 : P_b \to G \) with \( p_0 = \phi_0^{-1}(1) \). Then we have the transition map

\[
t = \hat{\phi}_0 \circ \psi_0^{-1} \in G.
\]

This map is well defined by the relation

\[
\psi_0^{-1}(v) = \hat{\phi}_0^{-1}(t(v)) = [p_0, t(v)].
\]

Let \( p_1 = \phi_0^{-1}(t) \in P_b \), and \( \phi_1 \) the trivialization of \( P_b \) uniquely defined by \( \phi_1^{-1}(1) = p_1 \). Notice that \( p_1 = p_0 t \). Then,

\[
\hat{\phi}_1^{-1}(v) = [p_1, v] = [p_0 t, v] = [p_0, t(v)] = \psi_0^{-1}(v).
\]

We see that \( \psi_0 = \hat{\phi}_1 \), and so every trivialization of \( E_b \) which is G-equivalent to \( \hat{\phi}_0 \) is itself induced some trivialization of \( P_b \).

\[ \square \]

### 5.4.1 Parallel Transport in Associated Bundles

Let \( P \to M \) be a principal \( G \)-bundle, \( G \subseteq GL(V) \) a Lie group, and \( E = P \times V / G \) the associated vector bundle. Suppose that \( P \) has a connection with parallel transport map \( \Gamma^P \). Then we may define parallel transport in \( E \), called \( \Gamma^E \), as follows. Pick a curve \( c \in \mathcal{P}^1(M) \), and a point \( e \in E_{c(0)} \) with representation \( (p, v) \in P \times V \). The curve defines the parallel transport map \( \Gamma^P_c : P_{c(0)} \to P_{c(1)} \). Then the result of parallel transport of \( e \) along \( c \) is the equivalence class

\[
\Gamma^E_c[p, v] = [\Gamma^P_c(p), v] \in E_{c(1)}.
\]
This definition is independent of choice of representation of $e$, as

$$[\Gamma_c^P(pg^{-1}), gv] = [\Gamma_c^P(p)g^{-1}, gv] = [\Gamma_c^P(p), v].$$

For a subgroup $G \subseteq GL(V)$, a vector bundle $E \to M$ with typical fiber $V$ is $G$-flat if it has an atlas $\{(U_\alpha, \phi_\alpha)\}$ whose transition maps lie in $G$ and are locally constant. By Lemma 5.3.17, this atlas defines a flat principal $G$-bundle over $M$, and $E$ is the associated bundle to this principal bundle. With such a structure, we can define a flat parallel transport map for $E$. The following is a corollary of Lemma 5.4.3 and Theorem 5.3.8.

**Corollary 5.4.4.** For a vector space $V$ and subgroup $G \subseteq GL(V)$, there is a bijection between isomorphism classes of flat vector bundles over $M$ with typical fiber $V$ and structure group $G$, and isomorphism classes of flat principal $G$-bundles over $M$.

In particular, a holonomy anti-homomorphism $\gamma : \pi_1(M, x_0) \to G$ defines an isomorphism class of triples $(E, \Gamma^E, \phi_0)$ where $E$ is a vector bundle over $M$ with typical fiber $V$ and structure group $G$, $\Gamma^E$ is a flat parallel transport map on $E$, and $\phi_0$ is a linear trivialization of the fiber lying over $b_0$ such that for all $v \in V$ and $c \in P^1(M, b_0)$, we have

$$\phi_0 \circ \Gamma^E_c \circ \phi_0^{-1} = \gamma([c]) \in G \subseteq GL(V).$$

**Proof.** If the connection on $P$ is flat, then parallel transport in its associated vector bundle is invariant of homotopy. Thus, a flat principal $G$-bundle defines a flat vector bundle.

On the other hand, if $E \to M$ is a flat vector bundle with structure group $G$, we can find a flat structure on $M$ with structure group $G$. This flat $G$-structure is used to define the associated principal $G$-bundle, which will then also be flat.
Now we show that the anti-homomorphism $\gamma$ defines a triple $(E, \Gamma^E, \phi_0)$. By Theorem 5.3.8, this anti-homomorphism defines, up to isomorphism, a triple $(P, H, b_0)$, where $P$ is a principal $G$-bundle over $M$, $H$ is a flat connection on $P$, and $b_0 \in P_{x_0}$ is a fixed basepoint. We now define $E$ to be the vector bundle associated to $P$. The point $b_0$ defines a trivialization $\phi_0^{-1} : V \to E_0$ by

$$\phi_0^{-1}(v) = [b_0, v].$$

By construction the holonomy map $\gamma$ satisfies, for all $[c] \in \pi_1(M, x_0)$,

$$b_0 \gamma([c]) = \Gamma^P_c(b_0).$$

Then the associated parallel transport map $\Gamma^E$ on $E$ is defined by

$$\Gamma^E_c[b_0, v] = [b_0 \gamma([c]), v] = [b_0, \gamma([c])(v)].$$

From this equality, it is clear that $\phi_0$ satisfies the relation

$$\phi_0 \circ \Gamma^E_c \circ \phi_0^{-1} = \gamma([c]).$$

\[\square\]

5.5 Some Constructions

Later in this thesis, we will need three particular bundle constructions: the external tensor product bundle, configuration bundle, and the pullback bundle. They are defined in this section.
5.5.1 External Tensor Product

Let $\pi_1 : E \to M$ and $\pi_2 : F \to N$ be two vector bundles with typical fibers $V, W$ respectively. We define the external tensor product

$$\pi_1 \boxtimes \pi_2 : E \boxtimes F \to M \times N.$$ 

The fiber over a point $(x, y)$ is

$$(E \boxtimes F)_{(x, y)} = E_x \otimes F_y.$$ 

In particular, it has typical fiber $V \otimes W$.

Suppose $(U_1, \phi_1)$ is a chart in $E$, and $(U_2, \phi_2)$ a chart in $F$. Then we create a new chart for $E \boxtimes F$ whose inverse map

$$(\phi_1 \otimes \phi_2)^{-1} : (U_1 \times U_2) \times (V \otimes W) \to \pi_1^{-1}(U_1) \otimes \pi_2^{-1}(U_2),$$

is defined by

$$(\phi_1 \otimes \phi_2)^{-1}((x, y), \sum_{\nu} v_{\nu} \otimes w_{\nu}) = \sum_{\nu} \phi_1^{-1}(x, v_{\nu}) \otimes \phi_2^{-1}(y, w_{\nu}).$$

This sum lies in $E_x \otimes F_y$.

For a vector bundle $\pi : E \to M$, we will use the notation $\pi^{[n]} : E^{[n]} \to M^n$ to denote the $n$-fold external tensor product of $E$ as a bundle over the Cartesian product of $n$-copies of $M$.

A flat connection on $E$ induces a flat connection on $E^{[n]}$. In particular, it induces a parallel transport map. Suppose $\Gamma$ is the parallel transport functor for a flat connection on $E$. Let

$$\xi = (c_1, \ldots, c_n) : [0, 1] \to M^n$$

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be a piecewise smooth path. Then we define $\Gamma_\mathcal{L}$ by

$$\Gamma_\mathcal{L} = \bigotimes_{i=1}^n \Gamma_{c_i} : \bigotimes_{i=1}^n E_{c_i(0)} \to \bigotimes_{i=1}^n E_{c_i(1)}.$$  

The fiber over $c(t)$, for $t = 0, 1$, is $\bigotimes_{i=1}^n E_{c_i(t)}$.

### 5.5.2 Pullback Bundles

Let $f : M \to N$ be a smooth map, and $\pi : E \to N$ a vector bundle. We will define the pullback bundle $f^*\pi : f^*E \to M$. The total space $f^*E$ is

$$f^*E = \{(x, e) \in M \times E : f(x) = \pi(e)\}$$

The map $f^*\pi$ is given by the projection from $M \times E$ onto $M$. The projection map onto the second factor is a bundle map $f_* : f^*E \to E$ such that the following diagram commutes:

$$\begin{array}{ccc}
  f^*E & \xrightarrow{f_*} & E \\
  \downarrow{f^*\pi} & & \downarrow{\pi} \\
  M & \xrightarrow{f} & N
\end{array}$$

Suppose $(U, \phi)$ is a local chart of $E$. Then $(f^{-1}(U), \phi')$ is a local chart of $f^*E$, where

$$\phi' : (f^*\pi)^{-1}(f^{-1}(U)) \to f^{-1}(U) \times V : (x, e) \mapsto (x, pr_2(\phi(e))).$$

If $E$ has a flat connection, then we can define the pullback flat connection over $f^*E$. We will not need this entire construction, and only define the parallel transport map $f^*\Gamma$.

Let $p$ be a path in $M$. Then $f \circ p$ is a path in $N$. Further, we have for each $x \in M$ and $t \in [0, 1]$ that

$$\left(f^*E\right)_{p(t)} := E_{f(p(t))}.$$
Then the parallel transport is defined by

$$(f^*\Gamma)_p = \Gamma_{f(p)} : E_{f(p(0))} \to E_{f(p(1))}.$$  

The pullback will be applied later to embedded curves. The following lemma is a useful observation.

**Lemma 5.5.1.** Suppose $f : M \to N$ is a smooth embedding, and $\pi : E \to N$ is a vector bundle. Then $f^*E \to E|_{\text{Im} f}$ is an isomorphism of vector bundles.

### 5.5.3 Configuration Bundles

Decorations will lie over "decorated points" which need to be distinct. In this section we discuss configuration spaces, to be used in the following sections to precisely define decorations.

For any smooth manifold $M$, define the fat diagonal

$$\Delta_n(M) = \{(x_1, \ldots, x_n) \in M^\times^n : x_i = x_j \text{ for some } i \neq j\}.$$  

Its complement is the $n$-th configuration space on $M,$

$$\text{Conf}_n(M) = M^n - \Delta_n(M).$$

Now suppose $\pi : E \to M$ is a vector bundle with typical fiber $V$. The tensor product from Section 5.5.1 is given by

$$\pi^{[n]} : E^{[n]} \to M^n.$$  

Restricting this bundle to $\text{Conf}_n(M)$ defines the *configuration bundle*

$$\text{Conf}_n(E) = E^{[n]}|_{\text{Conf}_n(M)}.$$
This sub-bundle can also be defined as the pullback of $E^{[n]}$ along the smooth inclusion $\text{Conf}_n(M) \hookrightarrow M^n$. Lemma 5.5.1 proves that it is a vector bundle. Further, if $E$ has a flat connection, then it induces a flat connection on $E^{[n]}$, and hence also on $\text{Conf}_n(E)$.

**Lemma 5.5.2.** For a vector bundle $\pi : E \rightarrow M$ with structure group $G$, the configuration bundle

$$\text{Conf}_n(\pi) : \text{Conf}_n(E) \rightarrow \text{Conf}_n(M)$$

is a vector bundle with typical fiber $V^\otimes n$, and structure group $G^\times n$. A flat connection on $E$ induces a flat connection on $\text{Conf}_n(E)$.

### 5.6 Ribbon Hopf algebra bundles

In this section we give the basic definitions for ribbon Hopf algebra bundles. This framework will be used for the invariant, particularly defining decorated curves in arbitrary manifolds in Chapter 6. Recall that for a ribbon Hopf algebra $\mathcal{H}$, the set of ribbon automorphisms of $\mathcal{H}$ is denoted $\text{RAut}(\mathcal{H})$.

**Definition 5.6.1.** Let $\mathcal{H}$ be a ribbon Hopf algebra. A vector bundle $E \rightarrow M$ is called a **ribbon Hopf algebra bundle** if it has typical fiber $\mathcal{H}$ and structure group $\text{RAut}(\mathcal{H})$. A **flat ribbon Hopf algebra bundle** is a ribbon Hopf algebra bundle along with an $\text{RAut}(\mathcal{H})$-flat structure. When the context is clear, we refer to these as Hopf bundles for short.

**Remark 5.6.2.** Recall that the $\text{RAut}(\mathcal{H})$-flat structure defines a parallel translation map for $E$. 

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Definition 5.6.3. Let $\pi_i : E_i \to M$ be two Hopf bundles and $(F, f)$ a vector bundle morphism. Then $(F, f)$ is a ribbon Hopf algebra bundle morphism if $F|_{\pi_1^{-1}(x)} : \pi_1^{-1}(x) \to \pi_2^{-1}(f(x))$ is a Hopf algebra homomorphism for every $x \in M$.

The following result is an immediate corollary of Theorem 5.3.8.

Corollary 5.6.4. The associated bundle construction defines a bijection between isomorphism classes of principal $\text{RAut}(\mathcal{H})$-bundles and isomorphism classes of ribbon Hopf algebra bundles with typical fiber $\mathcal{H}$.

An associated ribbon Hopf algebra bundle has a flat connection if and only if the related principal $\text{RAut}(\mathcal{H})$-bundle has a flat connection.

The following is almost a restatement of Corollary 5.4.4, but will be specifically useful later.

Corollary 5.6.5. A holonomy anti-homomorphism $\gamma : \pi_1(M, x_0) \to \text{RAut}(\mathcal{H})$ defines an isomorphism class of triples $(E, \Gamma^E, \phi_0)$, where $E$ is a ribbon Hopf algebra bundle with typical fiber $\mathcal{H}$, $\Gamma^E$ a flat parallel transport map on $E$, and $\phi_0$ a trivialization of the fiber lying over $b_0$ such that for all $h \in \mathcal{H}$ and $c \in \mathcal{P}^1(M, b_0)$, we have

$$\phi_0 \circ \Gamma^E_c \circ \phi_0^{-1} = \gamma([c]) \in \text{RAut}(\mathcal{H}).$$

5.6.1 Bundles over the Circle

For a flat bundle $E \to T^1\Sigma$ and an embedded curve $c : S^1 \to T^1\Sigma$, we can pull the bundle back to a flat bundle over the circle. This allows us to work over a simpler space, which is used to define a decoration on $c$. This section introduces the basics for working with these bundles.
We will use a fixed parametrization of the circle

\[ S^1 = [0, 1]/(0 \sim 1). \]

Define the basepoint \(* \in S^1\) to be the point represented by 0 (or 1). Let \( \ell : [0, 1] \to S^1 \) be the loop defined by \( \ell(t) = t \). The homotopy class \([\ell]\) generates \( \pi_1(S^1, *) \sim \mathbb{Z} \).

Any homomorphism \( \pi_1(S^1, *) \to \text{RAut}(\mathcal{H}) \) is determined by the image of the generator \([\ell]\). The following corollary is immediate from Theorem 5.3.10 and Theorem 5.3.8.

**Corollary 5.6.6.** Given a ribbon Hopf algebra \( \mathcal{H} \), the set of isomorphism classes of principal \( \text{RAut}(\mathcal{H}) \) bundles over \( S^1 \) is bijective with the set of conjugacy classes of \( \text{RAut}(\mathcal{H}) \).

There is a bijection between \( \text{RAut}(\mathcal{H}) \) and the equivalence classes of triples \((P, \Gamma, p)\), where \( P \to S^1 \) is a principal \( \text{RAut}(\mathcal{H}) \) bundle, \( \Gamma \) a flat connection on \( P \), and \( p \in E_\ast \) an element of the fiber lying over the basepoint \(*\).

Given \( \Psi \in \text{RAut}(\mathcal{H}) \), the flat principal \( \text{RAut}(\mathcal{H}) \)-bundle with parallel transport map \( \Gamma \) defined by \( \Gamma_\ell = \Psi \) has the following standard form:

\[ S_{\Psi}^{\text{RAut}(\mathcal{H})} := [0, 1] \times \text{RAut}(\mathcal{H}) / (1, f) \sim (0, \Psi \circ f). \]

Its associated ribbon Hopf algebra bundle has the standard form

\[ S_{\Psi}^{\mathcal{H}} := [0, 1] \times \mathcal{H} / (1, h) \sim (0, \Psi(h)). \]

**Definition 5.6.7.** The space \( S_{\Psi}^{\mathcal{H}} \) is called the standard \((\text{Hopf algebra})\) bundle over \( S^1 \) associated to \( \Psi \). The trivial chart on \( S_{\Psi}^{\mathcal{H}} \) over \((0, 1) \subset S^1\) is the identity map.
We prove that these are, indeed, bundles.

**Lemma 5.6.8.** The space $S^\text{RAut(H)}_\Psi$ is a flat principal $\text{RAut(H)}$ bundle over $S^1$. Its associated flat ribbon Hopf algebra bundle is isomorphic to $S^H_\Psi$, and the standard chart is flat.

*Proof.* We define two charts for $S^\text{RAut(H)}_\Psi$ which cover $S^1$ and whose transition maps are locally constant. This defines a flat $\text{RAut(H)}$-bundle structure on $S^1$. This proves that $S^\text{RAut(H)}_\Psi$ is a flat bundle, and allows us to define the associated bundle $S^H_\Psi$. One of the two charts will be the trivial chart, and so it is a flat chart by construction.

Let $U_1 = (0, 1) \subset S^1$, and for some $0 < \epsilon < 1/2$ let

$$U_2 = (1 - \epsilon, 1] \cup [0, \epsilon).$$

The chart over $U_1$ is the trivial one. The chart over $U_2$, say $\phi_2 : \pi^{-1}(U_2) \to U_2 \times H$, is defined by

$$\phi_2(x, h) = \begin{cases} (x, \Psi(h)) & \text{if } x > 1 - \epsilon \\ (x, h) & \text{if } x < \epsilon. \end{cases}$$

This map is well-defined at the point $\ast$, as

$$\phi_2(1, h) = (\ast, \Psi(h)) = \phi_2(0, \Psi(h)).$$

Now, $U_1 \cap U_2 = (1 - \epsilon, 1) \cup (0, \epsilon)$. Let $t_{12} : U_1 \cap U_2 \to \text{RAut(H)}$ be the transition map. Over $(0, \epsilon)$, this map is the identity. Over $(1 - \epsilon, 1)$, it is $\Psi$. These are the two connected components of the intersection. Thus, we see that the transition map is locally constant. Being the only transition map, it automatically satisfies the requirements for a flat structure. Thus, this is a flat structure on $M$. 

\[\square\]
Notice that for any other $\Phi \in \mathbf{RAut}(\mathcal{H})$, the bundles $S_{\Phi}^{\mathbf{RAut}(\mathcal{H})}$ and $S_{\Phi}^{\mathcal{H}}$ are isomorphic to $S_{\Phi \circ \Phi \circ \Phi^{-1}}^{\mathbf{RAut}(\mathcal{H})}$ and $S_{\Phi \circ \Phi \circ \Phi^{-1}}^{\mathcal{H}}$, respectively, via the maps $(s, f) \mapsto (s, \Phi \circ f)$ and $(s, h) \mapsto (s, \Phi(h))$. Thus, conjugate elements define isomorphic flat bundles, as required by Corollary 5.6.6.

While every flat Hopf bundle over $S^1$ is isomorphic to some standard bundle, this isomorphism is not canonical. Fixing the choice of isomorphism with $S_{\Phi}^{\mathbf{RAut}(\mathcal{H})}$ is equivalent to choosing a trivialization over some fixed $s \in S^1$. The following lemma makes this observation precise.

**Lemma 5.6.9.** For a flat ribbon Hopf algebra bundle $E \to M$ with typical fiber $\mathcal{H}$ and parallel transport map $\Gamma$, let $c : [0, 1] \to M$ be a smooth embedded loop with $c(0) = c(1) = b_0$. Then the pullback bundle $c^*E \to S^1$ is isomorphic (as a flat Hopf bundle) to the standard bundle over $S^1$ associated to $\Gamma_c$.

If $x = c(s) \neq b_0$ lies on the curve and $\phi_0 : E_x \to \mathcal{H}$ is a trivialization, then there is a unique isomorphism

$$\Phi_c : c^*E \to S_{\Gamma_c}^{\mathcal{H}}$$

which is the identity on the base space $S^1$ and such that for any $v \in E_x = (c^*E)_s$, we have

$$\Phi_c(v) = (s, \phi_0(v)) \in \{s\} \times \mathcal{H}.$$
CHAPTER 6
DECORATIONS

Equivalence classes of decorated curves on the plane lie at the heart of the classical invariant. In this chapter, the theory of decorated curves is generalized to arbitrary manifolds. A decorated curve in a smooth manifold $M$ requires a flat ribbon Hopf algebra bundle $E \to M$, and the decorations are defined as elements of a configuration bundle of a pullback of $E$.

6.1 Decorations in Bundles over $S^1$

In this section, we define decorations in a fixed flat ribbon Hopf algebra bundle $\pi : E \to S^1$. For $s \in S^1$, let $E_s = \pi^{-1}(s)$; specifically, let $E_0 = \pi^{-1}(*)$, where $*$ is the basepoint of $S^1$. We assume that $S^1$ has a fixed orientation.

Recall the notation from Subsection 5.5.3, specifically the configuration bundle

$$\text{Conf}_n(\pi) : \text{Conf}_n(E) \to \text{Conf}_n(S^1).$$

This is a flat ribbon Hopf algebra bundle with standard fiber isomorphic to $H^\otimes n$.

**Definition 6.1.1.** A decoration in $E$ is an element $D \in \text{Conf}_n(E)$ lying over some $s = (s_1, s_2, \cdots, s_n) \in \text{Conf}_n(S^1)$. The set of decorations in $E$ is denoted $\text{Dec}(E)$.
We define an equivalence relation of decorations in $E$ with respect to the basepoint $\star$. This relation is generated by the two equivalence moves shown in Figure 6.1, where $\star$ represents the basepoint on $S^1$, the decorated points are marked by dots, and the decoration is

$$h = \sum_{\mu} h_1^\mu \otimes h_2^\mu \otimes h_3^\mu \otimes h_4^\mu \in \bigotimes_{i=1}^4 E_{s_i}.$$ 

We also note one more move, the zero move, which adds or removes a decorated point labeled by $1 \in \mathcal{H}$. This move will typically be taken for granted without explicit mention.

$$p = (p_1, p_2, \cdots, p_n) : [0, 1] \rightarrow \text{Conf}_n(S^1 \setminus \star).$$

Let

$$\Gamma_p = \bigotimes_{i=1}^n \Gamma_{p_i} : \bigotimes_{i=1}^m E_{p_i(0)} \rightarrow \bigotimes_{i=1}^m E_{p_i(1)}.$$
Then for any decoration $D$ lying over $p(0)$, we define the equivalence

$$D \sim \Gamma_p(D).$$

This move is seen in the left equality of Figure 6.1, where we use the notation $\Gamma_i = \Gamma_{p_i}$.

The second equivalence move is multiplication of two decorated points $s_i \neq s_j \in S^1$ which do not have any other decorated points lying between them. Specifically, suppose $p : [0, 1] \to S^1$ is a smooth path with $p(0) = s_i$ and $p(1) = s_j$, and whose image is disjoint from all other decorated points and the basepoint $\ast$. Without loss of generality, suppose $j = i + 1$. We further assume that $p$ is orientation-preserving, where $[0, 1]$ has the standard orientation from $0$ to $1$.

Let $d_i : \text{Conf}_n(S^1) \to \text{Conf}_{n-1}(S^1)$ be the map which forgets the $i$-th coordinate. The decorations are multiplied using the map

$$m = \mu \circ (\Gamma_p \otimes 1) : E_{s_i} \otimes E_{s_j} \to E_{s_j},$$

where $\mu$ is the multiplication map in the Hopf algebra $(E_{s_j})$. To multiply the correct components in the tensor product, let

$$m_i = \text{id}^{\otimes i-1} \otimes m \otimes \text{id}^{\otimes n-i-1}.$$

Then for any decoration $D$ lying over $s_i$, the element $m_i(D)$ is a decoration lying over $d_i(s_i)$, and we say

$$D \sim m_i(D).$$

This move is seen in the right equality of Figure 6.1.

**Definition 6.1.2.** Two decorations in $E$ are congruent if they both can be transformed into the same decoration by a series of the above two moves or their reverses.
Congruence defines an equivalence relation on $\text{Dec}(E)$. The set of congruence classes is denoted $\text{Dec}(E, \ast)$.

For any decoration, it is possible to continue multiplying decorated points until only one remains. A decoration lying over a single point is called a simple decoration. The single decorated point can then slide around $S^1$, following its orientation, until it reaches the basepoint $\ast$. This simple operation translates any decoration into an element of $E_0$.

We define this operation more precisely. Number the decorated points $s_1, \ldots, s_n$ following the orientation of $S^1$ so that $\ast$ lies between $s_n$ and $s_1$. Figure 6.1 is labeled this way, for an example. For each $s_i$, let $p_i : [0, 1] \to S^1$ be a smooth, injective, orientation-preserving path with $p_i(0) = s_i$ and $p_i(1) = \ast$. Then let

$$\Gamma_i = \Gamma_{p_i} : E_{s_i} \to E_0,$$

and let $\mu_0$ be the multiplication map in $E_0$. Then we define the element $\zeta_E(D) \in E_0$ by

$$\zeta_{E, \ast}(D) = \mu_0^{\otimes n - 1} \circ \left( \bigotimes_{i=1}^n \Gamma_i \right) (D). \quad (6.1.1)$$

We will show that this operation respects congruence, and defines an isomorphism between $\text{Dec}(E, \ast)$ and $E_0$.

**Lemma 6.1.3.** The operation from Equation 6.1.1 defines a bijection

$$\zeta_E : \text{Dec}(E, \ast) \to E_0.$$

**Proof.** We make a brief note $\zeta_E$ is clearly invariant under the zero move, adding or removing a decorated point labeled by $1 \in \mathcal{H}$.
We will show that $\zeta_E(D) = \zeta_E(D')$ when $D'$ can be obtained from $D$ by one of the other two equivalence moves.

Let $D$ be a decoration over $\mathcal{C} = (s_1, \cdots, s_n) \in \text{Conf}_n(S^1)$. First we suppose $D'$ is obtained from $D$ by sliding the decorated points around. Say there is a path $\mathcal{C}$ in $\text{Conf}_n(S^1)$ with $c_i(0) = s_i$ and $c_i(t) \neq \ast$ for each $i$ and $t \in [0,1]$. The decoration $D$ transforms into

$$D' = \left( \bigotimes_{i=1}^{n} \Gamma_{c_i} \right) (D).$$

The decorated points do not cross during this move, so the order of multiplication is well-defined. Let $p'_i$ be the orientation-preserving path from $c_i(1)$ to $\ast$, as defined for Equation 6.1.1. Because $S^1 \setminus \ast$ is simply-connected, it is easy to see that $p_i^{-1} c_i p'_i$ is null-homotopic. Thus, $p_i$ and $c_i p'_i$ are homotopic paths, so that $\Gamma_{p_i} = \Gamma_{c_i p'_i}$ and in particular

$$\left( \bigotimes_{i=1}^{n} \Gamma_{p_i} \right) (D) = \left( \bigotimes_{i=1}^{n} \Gamma_{p'_i} \right) (D').$$

Next suppose $D'$ is obtained from $D$ by multiplication of the decorated points $s_i$ and $s_{i+1}$. Let $c$ be an orientation-preserving path from $s_i$ to $s_{i+1}$. Then

$$D' = \left( \text{id}^\otimes i^{-1} (\mu_{i+1} \circ (\Gamma_c \otimes \text{id})) \otimes \text{id}^{n-i-1} \right) (D),$$

where $\mu_{i+1}$ denotes multiplication in $E_{s_{i+1}}$.

Multiplication is associative, so it suffices to focus on the $i$ and $i+1$ components; for now, suppose $D$ is a decoration over just the two points $s_i, s_{i+1}$. Let $p_i$ and $p_{i+1}$ be the orientation-preserving paths from $s_i$ and $s_{i+1}$, respectively, to $\ast$. It is clear that $p_i$ and $cp_{i+1}$ are homotopic. Then,

$$\zeta_E(D') = \Gamma_{p_{i+1}}(D)$$

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\[
= (\Gamma_{p_{i+1}} \circ \mu_{i+1} \circ (\Gamma_c \circ \text{id})) (D) \\
= (\mu_0 \circ (\Gamma_{p_{i+1}} \otimes \Gamma_{p_{i+1}})) (D) \\
= (\mu_0 \circ (\Gamma_{p_i} \otimes \Gamma_{p_{i+1}})) (D) \\
= \zeta_E(D).
\]

The map \(\zeta\) behaves well under bundle isomorphism, as the next lemma shows.

**Lemma 6.1.4.** Suppose \(\pi : E \to S^1\) and \(\pi' : E' \to S^1\) are two flat ribbon Hopf algebra bundles, and \((\xi, f)\) is an isomorphism of flat bundles between them, where \(\xi : E \to E'\) and \(f : S^1 \to S^1\) are oriented diffeomorphisms. Let \(* \in S^1\) be a fixed basepoint.

Then the induced isomorphism of flat bundles

\[
Conf_n(\xi) : Conf_n(E) \to Conf_n(E')
\]

takes decorations in \(E\) with basepoint \(*\) to decorations in \(E'\) with basepoint \(f(*)\).

This map respects the congruence relation for decorations, and hence \(\xi\) also defines a bijection

\[
\text{Dec}(\xi) = \text{Dec}(\xi, *) : \text{Dec}(E, *) \to \text{Dec}(E', f(*)).
\]

Let \(E_0 = E_*\), \(E'_0 = E'_{f(*)}\), and \(\xi_0 = \xi|_{E_0}\). Then the following diagram commutes:

\[
\begin{array}{ccc}
\text{Dec}(E, *) & \xrightarrow{\text{Dec}(\xi)} & \text{Dec}(E', f(*)) \\
\downarrow{\xi_{E, *}} & & \downarrow{\xi_{E', f(*)}} \\
E_0 & \xrightarrow{\xi_0} & E'_0
\end{array}
\]
Proof. Because $f$ is a diffeomorphism, it clearly extends to a map

$$\text{Conf}_n(f) : \text{Conf}_n(S^1 \setminus \ast) \to \text{Conf}_n(S^1 \setminus f(\ast)).$$

The isomorphism $\text{Conf}_n(\xi)$ is simply the restriction of $\xi^{[n]} : E^{[n]} \to (E')^{[n]}$ to the bundle over $\text{Conf}_n(S^1 \setminus \ast)$. For this proof we will abuse notation and write $\xi^{[n]}$ instead of $\text{Conf}_n(\xi)$.

We must show that $\text{Conf}_n(\xi)$ maps congruent decorations with respect to $\ast$ to congruent decorations with respect to $f(\ast)$. Let $D$ be a decoration over $\xi \in \text{Conf}_n(S^1 \setminus \ast)$.

First, suppose $\underline{p} = (p_1, \cdots, p_n)$ is a path in $\text{Conf}_n(S^1 \setminus \ast)$ with $\underline{p}(0) = \xi$. For each $i$ define the path in $S^1$, $p_i' = f \circ p_i$. Clearly, $p_i'$ is a path in $S^1 \setminus f(\ast)$, and $\underline{p}' = (p_1', \cdots, p_n')$ is a path in $\text{Conf}_n(S^1 \setminus f(\ast))$. And for each $i$,

$$\xi \circ \Gamma_{p_i} = \Gamma_{p_i'} \circ \xi,$$

and in particular

$$\left( \xi^{[n]} \circ \bigotimes_{i=1}^{n} \Gamma_{p_i} \right) (D) = \left( \left( \bigotimes_{i=1}^{n} \Gamma_{p_i'} \right) \circ \xi^{[n]} \right) (D).$$

Thus $\xi^{[n]}(D)$ is congruent to

$$\left( \xi^{[n]} \circ \bigotimes_{i=1}^{n} \Gamma_{p_i} \right) (D).$$

Suppose $D'$ is obtained from $D$ by multiplying two points. Without loss of generality, suppose $D$ lies over two decorated points $s_1$, $s_2$, with $p$ an orientation-preserving path in $S^1$ with $p(0) = s_1$ and $p(1) = s_2$. Then $f \circ p$ is a path in $S^1$ from $f(s_1)$ to $f(s_2)$, and

$$\xi(D') = \xi \circ \mu \circ (\Gamma_p \otimes \text{id})(D)$$
\[
\mu \circ (\Gamma_{f_{\text{op}}} \otimes \text{id}) \circ \xi^{[2]}(D).
\]

So \(\xi(D')\) is congruent to \(\xi^{[n]}(D)\) by multiplying the decorated points \(f(s_1)\) and \(f(s_2)\).

Finally, we check the commutative diagram. Let \(D\) be a decoration lying over \(\xi \in \text{Conf}_n(S^1 \setminus \ast)\). Then \(D' = \text{Conf}_n(\xi)(D)\) is a decoration lying over \((f(s_1), \cdots, f(s_n))\).

Let \(p_i\) be the orientation-preserving path from \(s_i\) to \(\ast\). Then \(p_i' = f \circ p_i\) is an orientation-preserving path from \(f(s_i)\) to \(f(\ast)\). Then

\[
\zeta_{E, \ast}(D) = \mu_0^{\otimes n-1} \circ \left( \bigotimes_{i=1}^n \Gamma_{p_i} \right)(D),
\]

while

\[
\zeta_{E', f(\ast)}(D') = \mu_0^{\otimes n-1} \circ \left( \bigotimes_{i=1}^n \Gamma_{p_i'} \right)(D').
\]

And so we have

\[
(\xi_0 \circ \zeta_{E, \ast})(D) = \mu_0^{\otimes n-1} \circ \left( \bigotimes_{i=1}^n (\xi \circ \Gamma_{p_i}) \right)(D)
\]

\[
= \mu_0^{\otimes n-1} \circ \left( \bigotimes_{i=1}^n \Gamma_{p_i'} \circ \xi \right)(D)
\]

\[
= \mu_0^{\otimes n-1} \circ \left( \bigotimes_{i=1}^n \Gamma_{p_i'} \right) \circ \zeta^{[n]}(D)
\]

\[
= \zeta_{E', f(\ast)} \circ \zeta^{[n]}(D).
\]

\[\square\]

### 6.2 Tethered Decorated Curves

In this section we use decorations on the standard bundle to define decorated curves in a flat ribbon Hopf algebra bundle over an arbitrary manifold \(M\). This will be applied in Chapter 8 to the circle tangent bundle \(T^1\Sigma\) of an oriented surface \(\Sigma\).
In this section, a \textit{loop in} \( M \) will be a smooth map \( S^1 \to M \). We assume a parametrization on \( S^1 \), so that a loop is given by a smooth map \( c : [0, 1] \to M \), with \( c(0) = c(1) \). We continue the notation of letting \( * = [0] \) be the basepoint of \( S^1 \), and say \( c(*) \) is the basepoint of \( c \).

A flat ribbon Hopf algebra bundle \( \pi : E \to M \) can be pulled back along the loop \( c \), defining a flat ribbon Hopf algebra bundle

\[
c^* \pi : c^* E \to S^1 = [0, 1] / (0 \sim 1).
\]

\textbf{Definition 6.2.1.} A \textit{decorated curve} in a flat ribbon Hopf algebra bundle \( \pi : E \to M \) is a pair \((c, D)\), where \( c \) is a loop in \( M \), and \( D \in \text{Conf}_n(c^* E) \). The element \( D \) is called a \textit{decoration} on \( c \). The set of decorated curves in \( E \) is denoted \( \mathcal{DC}(E) \).

\textbf{Remark 6.2.2.} The pullback bundle \( c^* E \) is isomorphic to \( S^1_{\Gamma_c} \), though not canonically.

Recall the map \( c_* : c^* E \to E \) which fits into the following commutative diagram:

\[
\begin{array}{ccc}
c^* E & \xrightarrow{c_*} & E \\
\downarrow{c^* \pi} & & \downarrow{\pi} \\
S^1 & \xrightarrow{c} & M
\end{array}
\]

A decoration \( D \) lies over an element \( s = (s_1, s_2, \cdots, s_n) \in \text{Conf}_n(S^1) \). Let \( x_i = c(s_i) \) for \( 1 \leq i \leq n \). We will often associate \( D \) with its image \( (f_*)^{\otimes n}(D) \in E[n] \), which lies over \( \bar{x} \in M^n \). The points \( \{x_i\} \) are called the \textit{decorated points in} \( M \); notice that they may not all be distinct.

In general we deal with several curves which are organized into a \textit{curve system}. We use the notation \( S = \coprod_{i=1}^n S^1 \), where \( \coprod \) represents a disjoint union.
**Definition 6.2.3.** A *curve system in* \(M\) *with* \(m\) *components* is a smooth map*

\[
\mathcal{C} : \prod_{i=1}^{m} S^1 \to M.
\]

For each \(i\), there is an injection \(\text{inj}_i : S^1 \hookrightarrow \mathcal{S}\) which maps onto the \(i\)-th copy of \(S^1\). Define the smooth loop \(c_i\) as the composition

\[
S^1 \xhookrightarrow{\text{inj}_i} \prod_{i=1}^{m} S^1 \xrightarrow{\pi_i} M.
\]

The curves \(c_i\) are called the *components* of \(\mathcal{C}\). The set of curve systems in \(M\) is denoted \(\mathbb{C}(M)\), and the subset of systems with exactly \(m\) components is \(\mathbb{C}(M)^m\).

**Remark 6.2.4.** Notice that until we add tethers and/or decorations, the basepoint has no role to fill. We will need decorations to be distinct from \(*\); for this we use the notation

\[
\mathcal{S} \setminus \{*\} = \prod_{i=1}^{m} (S^1 \setminus \{*\}).
\]

**Definition 6.2.5.** A *decorated curve system in* \(E\) *is a pair* \((\mathcal{C}, D)\), *where* \(\mathcal{C}\) *is a curve system in* \(M\) *with* \(m\) *components*, and \(D\) is a decoration living in the configuration bundle

\[
\text{Conf}_n(\mathcal{C}^*E) \to \text{Conf}_n(\mathcal{S} \setminus \{*\}).
\]

In this case, we say that \(D\) is a decoration *on the curve system* \(\mathcal{C}\). The set of decorated curve systems in \(M\) is denoted \(\mathbb{DC}(E)\), and the subset of systems with \(m\) components is \(\mathbb{DC}(E)^m\).

A decorated curve system is *simple* if each component contains exactly one decorated point.
Next we define tethers for decorated curve systems, building up from a tether for a single loop. Tethers again use the basepoint \( * \in S^1 \). A good way to visualize a tethered curve is as a map of the oriented "tadpole", seen in Figure 6.2, into \( M \). For our purposes the tether and the loop act in different roles, so we keep them separate.

\[ \begin{array}{c}
\text{Figure 6.2: A tadpole.}
\end{array} \]

**Definition 6.2.6.** For a fixed \( b_0 \in M \) and a smooth loop \( c \) with basepoint \( c(*) \), a *tether for \( c \) based at \( b_0 \) is a smooth path \( \ell \) in \( M \) such that \( \ell(0) = c(*) \), and \( \ell(1) = b_0 \). The point \( \ell(0) \) is called the *tethered point of \( c \).*

A *tethered curve system based at \( b_0 \) is a pair \((c, \ell)\), such that \( \ell_i \) is a tether for \( c_i \) based at \( b_0 \). The set of tethered curve systems based at \( b_0 \) is denoted \( \text{TCS}(M, b_0) \), and the subset of systems with \( m \) components is \( \text{TCS}(M, b_0)^m \).

A *tethered decorated curve based at \( b_0 \) is a triple \((c, \ell, D)\), where \((c, D)\) is a decorated curve, and \( \ell \) is a tether for \( c \).

A *tethered decorated curve system based at \( b_0 \) is a triple \((c, \ell, D)\) such that \((c, D)\) is a decorated curve system and \((c, \ell)\) is a tethered curve system based at \( b_0 \). The set of tethered decorated curves systems is denoted \( \text{TDCS}(E, b_0) \), and the subset of
tethered decorated curve systems based at $b_0$ with exactly $m$ components is denoted $\text{TDCS}(E, b_0)^m$.

Figure 6.3 shows an example of a tethered curve system on the torus, with two curves $c_1, c_2$ and respective tethers $\ell_1, \ell_2$.

![Diagram of a tethered curve system on a torus]

Figure 6.3: A tethered curve system on the torus.

There is a natural equivalence relation for (tethered) decorated curves arising from congruence of the decorations and homotopy of the underlying curve. First we define homotopy of (tethered) decorated curve systems.

**Definition 6.2.7.** For two curves $c, c'$ in $M$, a *homotopy of curves* from $c$ to $c'$ is a smooth map $H : [0, 1]^2 \to M$ such that for all $t \in [0, 1]$, $H(0, t) = c(t)$ and $H(1, t) = c'(t)$.

For two curve systems $\mathbf{c} = (c_1, \cdots, c_m)$ and $\mathbf{c}' = (c'_1, \cdots, c'_m)$, a *homotopy of curve systems in $M$* from $\mathbf{c}$ to $\mathbf{c}'$ is a sequence $\mathbf{H} = (H_1, \cdots, H_m)$ such that for each $i$, $H_i$ is a homotopy from $c_i$ to $c'_i$. 
For two tethered curves based at \(b_0\), \((c, \ell)\) and \((c', \ell')\), a homotopy of tethered curves between them is a pair \((H, l)\), where \(H, l : [0, 1]^2 \to M\) are both smooth maps satisfying:

1. For each \(t \in [0, 1]\), the pair \((H(t, \cdot), l(t, \cdot))\) is a tethered curve system based at \(b_0\),

2. \(H\) is a homotopy of curves from \(c\) to \(c'\),

3. \(l(0, t) = \ell(t)\) and \(l(1, t) = \ell'(t)\) for each \(t \in [0, 1]\).

For two tethered curve systems based at \(b_0\) with \(m\) components, \((c, \ell)\) and \((c', \ell')\), with a homotopy of tethered curve systems between them is a pair \((\underline{H}, \underline{l})\), where \(\underline{H} = (H_1, \cdots, H_m)\) and \(\underline{l} = (l_1, \cdots, l_m)\), such that for each \(i\), \((H_i, l_i)\) is a homotopy from \((c_i, \ell_i)\) to \((c'_i, \ell'_i)\).

Remark 6.2.8. The different components of a tethered curve system do not interact during a homotopy. Hence, a homotopy of tethered curve systems is nothing more than a homotopy for each component of the system.

Homotopy lets us define an equivalence relation of (tethered) decorated curve systems. The image of a curve changes throughout a homotopy, and the decorated points must move along with it. The decoration itself then transforms following the parallel transport.

We will first discuss this for decorated curves with one component. Suppose \((c, D)\) and \((c', D')\) are two decorated curves in \(M\). The decoration \(D\) lies over some \(\underline{s} = (s_1, \cdots, s_n) \in \text{Conf}_n(S^1)\).
Consider a smooth homotopy $H$ from $c$ to $c'$. The homotopy allows us to define the pullback bundle

$$H^*\pi : H^*E \to S^1 \times [0, 1].$$

This pullback bundle inherits a flat connection from $E$. The restriction of $H^*E$ to $S^1 \times \{0\}$ is exactly $c^*E$, and the restriction to $S^1 \times \{1\}$ is $(c')^*E$. This is a standard construction commonly used to prove that homotopic maps yields isomorphic pullback bundles. See, for example, Lemma 1.4.3 of [2].

Define the paths $p_i : [0, 1] \to S^1 \times [0, 1]$ by

$$p_i(t) = (s_i, t).$$

Each $p_i$ is smooth and defines the parallel transport map $\Gamma_{p_i}$. We can now define congruent decorated curves, and the natural extension to congruent (tethered) decorated curve systems.

**Definition 6.2.9.** Using notation from the preceding discussion, two decorated curves $(c, D)$ and $(c', D)$ are *congruent in $E$* if there is a homotopy $H$ from $c$ to $c'$ such that $D'$ and $\left( \bigotimes_{i=1}^n \Gamma_{p_i} \right)(D)$ are congruent decorations in $\text{Dec}((c')^*E)$.

Two tethered decorated curves $(c, \ell, D)$ and $(c', \ell', D')$ are *congruent in $E$* if there is a homotopy $(H, l)$ from $(c, \ell)$ to $(c', \ell')$ such that $D'$ and $\left( \bigotimes_{i=1}^n \Gamma_{p_i} \right)(D)$ are congruent decorations in $\text{Dec}(c'_*E)$. The set of congruence classes of tethered curve systems is denoted $\mathcal{TCS}(M, b_0)$.

The congruence relation for (tethered) decorated curve systems takes a bit more care to define. First suppose $H$ is a homotopy of curve systems with $m$ components,
from \( \mathcal{C} \) to \( \mathcal{C}' \). Denote the pullback bundles of these curve systems by \( \mathcal{\mathcal{E}} \) and \( \mathcal{\mathcal{E}}' \), respectively. Consider the disjoint union of maps

\[
\prod_{i=1}^{m} H_i : \prod_{i=1}^{m} [0, 1]^2 \to M.
\]

This smooth map pulls back to a bundle \( \mathcal{H} \) over the base space \( \prod_{i=1}^{m} (S^1 \times [0, 1]) \).

Let \( D \) be a decoration on \( \mathcal{C} \) so

\[
D \in \text{Conf}_n (\mathcal{E}),
\]

and lies over some points \( (s_1, \ldots, s_n) \in \text{Conf}_n (\mathcal{S} \setminus \{*\}) \). There are again natural inclusions

\[
\mathcal{E} \rightarrow \mathcal{H} \leftarrow \mathcal{E}',
\]

and again we can define the path for each \( i \),

\[
p_i(t) = (s_i, t) \in \prod_{i=1}^{m} S^1 \times [0, 1].
\]

The parallel transport maps \( \Gamma_{p_i} \) then translates the decoration \( D \) into a decoration of \( \mathcal{E}' \), namely

\[
\left( \bigotimes \Gamma_{p_i} \right) (D) \in \text{Conf}_n (\mathcal{E}').
\]

**Definition 6.2.10.** Two decorated curve systems \( (\mathcal{C}, D), (\mathcal{C}', D') \in DCS(E)^m \) are congruent if there is a homotopy of curve systems, \( \mathcal{H} \) such that \( D' \) and

\[
\left( \bigotimes_{i=1}^{n} \Gamma_{p_i} \right) (D)
\]

are congruent decorations in \( \mathcal{E}' \).

The set of congruence classes of decorated curve systems in \( E \) with \( m \) components is denoted \( DCS(E, b_0)^m \). The set of all congruence classes of decorated curve systems is denoted \( DCS(E, b_0) \).
Two tethered decorated curve systems \((\mathcal{C}, \ell, D), (\mathcal{C}', \ell', D') \in \text{TDCS}(E, b_0)^m\) are congruent if there is a homotopy of tethered curve systems, \((\mathcal{H}, \mathcal{L})\) such that \(D'\) and
\[
\left( \bigotimes_{i=1}^{n} \Gamma_{p_i} \right) (D)
\]
are congruent decorations in \(\mathcal{E}'\).

The set of congruence classes of tethered decorated curve systems in \(E\) is denoted \(\text{TDCS}(E, b_0)\), and the subset of such systems with \(m\) components is \(\text{TDCS}(E, b_0)^m\).

**Remark 6.2.11.** The definition implies that if two tethered decorated curve systems are congruent, then the underlying tethered curve systems are homotopic. Specifically, they have the same number of components. Without this, the subsets \(\text{TDCS}(E, b_0)^m\) would not be well-defined.

Recall that two decorations in a pullback bundle \(c^* E\) are congruent if one can be transformed into the other by a series of sliding and multiplying decorated points. Multiplying adjacent decorated points does not change the congruence class of a tethered decorated curve.

**Lemma 6.2.12.** Let \((c, \ell, D) \in \text{TDCS}(E, b_0)\) with \(D\) lying over \(\mathfrak{s} \in \text{Conf}_n(S^1 \setminus \{*\})\). Suppose that \(s_i, s_{i+1}\) are decorated points and \(p : [0, 1] \to S^1\) is a smooth, orientation-preserving path such \(p(0) = s_i, p(1) = s_{i+1}\), and the image \(p((0, 1))\) is disjoint from all decorated points and the basepoint \(*\). Recall the map \(m_i\), which is defined in Section 6.1 by
\[
m_i = id^\otimes i-1 \otimes [\mu \circ (\Gamma_p \otimes 1)] \otimes id^\otimes n-i-1.
\]
Then \(m_i(D)\) is a decoration in \(c^* E\), and \((c, \ell, D)\) is congruent to \((c, \ell, m_i(D))\).
Proof. The decorations $D$ and $m_i(D)$ are, by definition, congruent decorations in $c^*E$. Thus the decorated curves $(c, \ell, D)$ and $(c, \ell, m_i(D))$ are congruent as well. □

Of course, there is nothing stopping us from multiplying decorated points until only one remains. This observation yields the following corollary.

**Corollary 6.2.13.** Every tethered decorated curve system is congruent to a simple tethered decorated curve system.

### 6.3 Data in a Decorated Curve

In this section, we will describe a standard method of extracting information from elements of $\mathcal{TDCS}(E, b_0)^1$. Specifically, we classify congruence classes of tethered decorated curves using an element of $\pi_1(M, b_0)$ and an element of the fiber $E_0 = \pi^{-1}(b_0)$.

Recall the map $\zeta_{c^* E, *}: \text{Dec}(c^* E, *) \to (c^* E)_*$ from Section 6.1, where $(c^* E)_*$ is the fiber over the basepoint $* \in S^1$. By definition, $(c^* E)_* = E_{c(*)} = E_{\ell(0)}$.

**Lemma 6.3.1.** There is a bijection

$$\xi_{E,b_0}: \mathcal{TDCS}(E, b_0)^1 \to \pi_1(M, b_0) \times E_0$$

defined on a decorated curve $(c, \ell, D) \in \mathcal{TDCS}(E, b_0)^1$ by

$$\xi_{E,b_0}(c, \ell, D) = ([\ell^{-1} c \ell], (\Gamma_{\ell} \circ \zeta_{c^* E, *})(D)).$$

**Remark 6.3.2.** The composition $\ell^{-1} c \ell$ is a loop based at $b_0$. As such, its homotopy class lies in $\pi_1(M, b_0)$. 

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Proof. We must show that if \((c, \ell, D)\) and \((c', \ell', D')\) are congruent tethered decorated curves, then \(\xi_{E,b_0}(c, \ell, D) = \xi_{E,b_0}(c', \ell', D')\).

First consider two congruent decorations \(D, D'\) on a single tethered curve \((c, \ell) \in \text{TCS}(E, b_0)^1\) with basepoint \(c(*)\). Then according to Lemma 6.1.3, \(\zeta_{c^*E,*}(D) = \zeta_{c^*E,*}(D')\). The paths \(\ell\) and \(c\) also do not change, so \(\xi_{E,b_0}(c, D) = \xi_{E,b_0}(c, D')\).

Now consider a smooth based homotopy \((H, l)\) from \((c, \ell)\) to another tethered curve \((c', \ell')\), also based at \(b_0\). The existence of such a homotopy proves that

\[
[\ell^{-1}c\ell] = [\ell'^{-1}c'\ell'] \in \pi_1(M, b_0).
\]

Define the pullback bundle \(H^*E\) with flat connection induced from \(E\); denote the parallel transport map by \(\Gamma^H\). The bundles \(c^*E\) and \((c')^*E\) inject into \(H^*E\) as subbundles.

Let \(D\) be a decoration over \(c\) lying over \(s \in \text{Conf}_n(S^1)\). For each decorated point \(s_i\), define the path

\[p_i(t) = (s_i, t) \in S^1 \times [0, 1],\]

and its parallel transport map \(\Gamma^H_{p_i}\). Define the decoration \(D'\) over \(c'\) by

\[D' = \left( \bigotimes_{i=1}^n \Gamma^H_{p_i} \right) (D).
\]

We want to show that \(\xi_{E,b_0}(c, \ell, D) = \xi_{E,b_0}(c', \ell', D')\). We already showed that the fundamental group element is unchanged. All that is left is to show

\[(\Gamma_\ell \circ \Gamma_{\alpha_{c,\ell}} \circ \zeta_{c^*E,*})(D) = (\Gamma_\ell' \circ \Gamma_{\alpha_{c',\ell'}} \circ \zeta_{(c')^*E,*})(D').\]

It suffices to show this for a simple decoration \(D\) over a single point \(s_1 \in S^1\).  

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Let $\alpha$ be a smooth, orientation-preserving, injective path in $S^1$ with $\alpha(0) = s_1$ and $\alpha(1) = *$; then define $\beta = c \circ \alpha$. Then

$$(\Gamma_\ell \circ \zeta_{E,s})(D) = \Gamma_{\beta_\ell}(D).$$

Let $p_1$ be, as before, the path $p_1(t) = (s_1, t)$. Then the decoration $D' = \Gamma_{p_1}^H(D)$ lies over $p_1(1)$.

Let $\beta'$ be a similar parametrization of the segment of $c'$ lying between $p_1(1)$ and $(*, 1)$. Then,

$$\Gamma_{\ell'} \circ \zeta_{(c')}^*E,s(D') = \Gamma_{p_1,\beta'}E(D).$$

The homotopy $H$ tells us that the piecewise smooth curve $\ell^{-1}\beta^{-1}p_1\beta'\ell'$ is null-homotopic. A simple example of this is shown in Figure 6.4. This implies that

$$\Gamma_{\beta_\ell}(D) = \Gamma_{p_1,\beta'\alpha'}(D) = \Gamma_{\beta'\alpha'}(D'),$$

which finishes the proof $\xi(c, \ell, D) = \xi(c', \ell', D')$.

Next we show that the map is injective. First, define the trivial tether based at $b_0$ to be the constant tether, $b_0(t) = b_0$. It is easy to see that every tethered decorated curve is congruent to a tethered decorated curve with the trivial tether. For $(c, \ell, D)$ a tethered decorated curve, let $\ell^{-1}\widehat{\ell}\ell$ be a smooth curve which is homotopic to the piecewise-smooth curve $\ell^{-1}\ell$, and which is equal to $c$ on neighborhoods of the decorated points. Then $D$ is still a decoration on $\ell^{-1}\ell$, and $(c, \ell, D)$ is congruent to $(\ell^{-1}\ell, b_0, D)$.

Suppose $(c, b_0, D)$ and $(c', b_0, D')$ are two decorated curves with trivial tether based at $b_0$ such that

$$\xi(c, b_0, D) = \xi(c', b_0, D').$$

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Then in particular, $[c] = [c']$ and there must be a smooth based homotopy $H$ from $c$ to $c'$. Because both tethered curves have the trivial tether, we can assume that $H(0, t) = H(1, t) = b_0$ for all $t \in [0, 1]$. As we have done before, the decorated points $s_i$ under $D$ can then slide through the path $p_i(t) = (s_i, t)$, defining a new decoration on $c'$,

$$\left( \bigotimes_i \Gamma_{p_i} \right)(D).$$

By definition, $(c, b_0, D)$ is congruent to $\left( c', b_0, \left( \bigotimes_i \Gamma_{p_i} \right)(D) \right)$. Thus it suffices to consider two decoration which lie over the same curve $c$.

So, now suppose $D$ and $D'$ are two decorations over the same tethered curve $(c, b_0)$, with

$$(\Gamma_\ell \circ \Gamma_\alpha \circ \zeta_{E,*})(D) = (\Gamma_\ell \circ \Gamma_\alpha \circ \zeta_{E,*})(D').$$
The parallel transport maps $\Gamma_{\alpha}$ and $\Gamma_{\ell}$ are also isomorphisms. The map $\zeta_{E,*}$ is an injection from $\mathcal{D}ec(c^*E, *)$ to $(c^*E)_0 = E_{c(\alpha)}$, and so $\zeta_{E,*}(D) = \zeta_{E,*}(D')$ if and only if $D$ and $D'$ are congruent decorations in $c^*E$. This implies that $(c, b_0, D)$ and $(c, b_0, D')$ are congruent decorated curves in $E$.

Finally, it is clear that this map is surjective, as we can construct a curve with the appropriate fundamental class and decoration. \hfill \Box

The parallel transport map $\Gamma$ defines an action of $\pi_1(M, b_0)$ on the Hopf algebra $E_0$. This action satisfies $\Gamma_{cc'} = \Gamma_{c'} \circ \Gamma_c$, thus we can define the semidirect product, as defined in Section 4.2,

$$SD = SD(\pi_1(M, b_0), \Gamma, E_0) = \mathbb{k}[\pi_1(M, b_0)] \rtimes_{\Gamma} E_0.$$  

The cross product $\pi_1(M, b_0) \times E_0$ injects into $SD$, leading to the following corollary. The algebraic structure on $SD$ lets us define quotient spaces, as in 4.5. For a geometric view of this particular semidirect product, see Subsection 6.3.1.

**Corollary 6.3.3.** There is a injective map

$$\iota_{b_0} = \iota(E, M, b_0, \Gamma) : T\mathcal{D}CS(E, b_0)^1 \hookrightarrow SD(\pi_1(M, b_0), \Gamma, E_0)$$

whose image is the set of elements $g \otimes x$ for $(g, x) \in \pi_1(M, b_0) \times E_0$.

**Definition 6.3.4.** For two points $b_0, b_1 \in M$ with a path $b$ from $b_0$ to $b_1$, the adjoint map for the fundamental group,

$$ad_b : \pi_1(M, b_0) \to \pi_1(M, b_1),$$
is defined by
\[ ad_b([c]) = [b^{-1}cb]. \]

The adjoint map for tethered curves
\[ Ad_b : TDCS(E, b_0)^1 \to TDCS(E, b_1)^1 \]
is defined by
\[ Ad_b(c, \ell, D) = (c, \ell b, D). \]

**Remark 6.3.5.** It is easy to see that these maps are well-defined and depend only on the homotopy class of the path \( b \). They are both bijections as well, with inverse \( ad_{b^{-1}} \) and \( Ad_{b^{-1}} \). These adjoint maps fit into a commutative diagram with \( \iota_{b_0} \) and \( \iota_{b_1} \), as seen in Lemma 6.3.6.

Before then, consider that the fiber \( E_0 \) is isomorphic to the Hopf algebra \( \mathcal{H} \), but not canonically. Fix a trivialization \( \phi_0 : E_0 \to \mathcal{H} \). Then we have a bijection
\[ \text{id} \times \phi_0 : \pi_1(M, b_0) \times E_0 \to \pi_1(M, b_0) \times \mathcal{H}. \]

It will be useful to consider the image of \( \iota_{b_0} \) using these local coordinates.

On the algebraic side, \( \phi_0 \) along with the right action of \( \pi_1(M, b_0) \) on \( E_0 \) lets us define a right \( \pi_1(M, b_0) \) action on \( \mathcal{H} \). For \( g \in \pi_1(M, b_0) \), define \( \Psi_g \in \text{R}Aut(\mathcal{H}) \) by
\[ \Psi_g(x) = (\phi_0 \circ \Gamma_g \circ \phi_0^{-1})(x). \]

This action satisfies
\[ \Psi_{g'} = \Psi_{g'} \circ \Psi_g, \]
and thus defines the semidirect product
\[ SD(\pi_1(M, b_0), \Psi, \mathcal{H}) = \mathbb{K}[\pi_1(M, b_0)] \rtimes_\Psi \mathcal{H}. \]
The bijection $\text{id} \times \phi_0$ extends to an isomorphism of Hopf algebras

$$\text{id} \otimes \phi_0 : \text{SD}(\pi_1(M, b_0), \Gamma, E_0) \rightarrow \text{SD}(\pi_1(M, b_0), \Psi, \mathcal{H}).$$

Now suppose $b$ is a path in $M$ from $b_0$ to another basepoint $b_1$, and suppose $\phi_1$ is a trivialization over $b_1$. Then we can extend the change-of-basepoint isomorphism using $\phi_1 \circ \Gamma_b \circ \phi_0^{-1} \in \text{RAut}(\mathcal{H})$. All of these maps are related by the commutative diagram in the following lemma.

**Lemma 6.3.6.** If $b_0, b_1$ are two basepoints with trivializations $\phi_i : E_i \rightarrow \mathcal{H}$, $b$ a path from $b_0$ to $b_1$, and $t = \phi_1 \circ \Gamma_b \circ \phi_0^{-1}$, then the following diagram commutes:

$$
\begin{array}{ccc}
\text{TDCS}(E, b_0)_1 & \xrightarrow{\iota_{b_0}} & \text{SD}(\pi_1(M, b_0), \Gamma, E_0) \xrightarrow{id \otimes \phi_0} \text{SD}(\pi_1(M, b_0), \Psi, \mathcal{H}) \\
\downarrow \text{Ad}_b & & \downarrow \text{ad}_b \otimes \Gamma_b & & \downarrow \text{ad}_b \otimes t \\
\text{TDCS}(E, b_1)_1 & \xrightarrow{\iota_{b_1}} & \text{SD}(\pi_1(M, b_1), \Gamma, E_1) \xrightarrow{id \otimes \phi_1} \text{SD}(\pi_1(M, b_1), \Psi, \mathcal{H})
\end{array}
$$

**Proof.** The square on the right commutes by definition.

For the square on the left, consider a decorated curve $(c, \ell, D)$ with basepoint $c(\ast)$. Suppose

$$\zeta(c^*E, \ast)(D) = h \in (c^*E)_\ast = E_{c(\ast)},$$

where $(c^*E)_\ast$ again denotes the fiber of $c^*E$ over the basepoint $\ast$. Then the image of $(c, \ell, D)$ in $\text{SD}$ is

$$\iota_{b_0}(c, \ell, D) = ([\ell^{-1}c\ell], \Gamma_\ell(h)).$$

Applying $\text{ad}_b \otimes \Gamma_b$ gives

$$((\text{ad}_b \otimes \Gamma_b) \circ \iota_{b_0})(c, \ell, D) = ([b^{-1}\ell^{-1}c\ell b], (\Gamma_b \circ \Gamma_\ell)(h)).$$

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The adjoint map \( Ad_b \) changes only the tether, so the same decoration \( h \) is used to calculate

\[
\iota_{\ell b}(c, \ell b, D = (b^{-1}\ell^{-1}c\ell b, \Gamma_{\ell b}(h)).
\]

And \( \Gamma_{\ell b} = \Gamma_b \circ \Gamma_\ell \), so the left square commutes.

\[\square\]

### 6.3.1 The Geometric View

The semidirect product

\[
SD = \mathbb{k}[\pi_1(M, b_0)] \rtimes \Gamma E_0
\]

is central to the invariants \( \mathcal{I}^{fr} \) and \( \mathcal{I}^{ufx} \). We introduced it without context, but this new Hopf algebra arises naturally from the study of decorated loops based at \( b_0 \). In this subsection we discuss this connection. The purpose is just to give the picture, so details are kept light. This subsection will not be needed anywhere else in this dissertation.

Define the set of smooth loops based at \( b_0 \),

\[
CS(M, b_0)^1 = \{ c : [0, 1] \to M : c(0) = c(1) = b_0 \}.
\]

Homotopy defines an equivalence relation on \( CS(M, b_0)^1 \), and the set of homotopy classes is the fundamental group \( \pi_1(M, b_0) \); the homotopy class of an element \( c \in CS(M, b_0)^1 \) is given by \([c]\). Concatenation of paths defines the usual group operation \([c][c'] = [cc']\). We note that the path \( cc' \) is typically not smooth; we gloss over the details of identifying homotopy classes of smooth paths with homotopy classes of piece-wise smooth paths.

A decorated curve based at \( b_0 \) is a pair \((c, D)\) where \( c \in CS(M, b_0)^1 \) and \( D \) is a decoration on \( c \), the set of which is defined by \( DCS(E, b_0)^1 \). Homotopy defines an
equivalence relation on \( \mathcal{DCS}(E, b_0) \); this relation is very similar to the congruence relation on \( \mathcal{TDCS}(E, b_0) \), so we will not repeat it.

The map \( c \mapsto (c, 1) \) defines an injection \( \mathcal{CS}(M, b_0)^1 \to \mathcal{DCS}(E, b_0) \), where \( 1 \) refers to the trivial decoration. This injection respects homotopy, and thus defines an injective group homomorphism \( \pi_1(M, b_0) \to \mathcal{DCS}(E, b_0) \).

The group operation on the fundamental group extends to a group operation on \( \mathcal{DCS}(E, b_0) \). For \( (c, D), (c', D') \in \mathcal{DCS}(E, b_0)^1 \), this operation is defined by

\[
(c, D)(c', D') = (cc', D \otimes D').
\]

**Remark 6.3.7.** Again, we are glossing over some details. The element \( D \otimes D' \) is supposed to lie in the some configuration space of the pullback bundle \( (cc')^* E \). However, the product \( cc' \) is generally not smooth, and the pullback generally is not smooth. There is always a smooth loop \( r \in \mathcal{CS}(M, b_0)^1 \) such that \( [r] = [cc'] \), which does pull back to a smooth bundle. Parallel transport along the homotopy from \( cc' \) to \( r \) translates the decorations \( D \) and \( D' \) into a decoration on \( r \).

There is a bijection between \( \mathcal{DCS}(E, b_0) \) and \( \pi_1(M, b_0) \times E_0 \), which is again very similar to the same bijection for \( \mathcal{TDCS}(E, b_0) \). This bijection is demonstrated as follows, for an element \( (c, D) \in \mathcal{DCS}(E, b_0) \). Multiply the decorated points until only one remains. We slide this decorated point around \( S^1 \), following its orientation, until it arrives at \( b_0 = c(1) \). Parallel transport over this sliding translates \( D \) into an element \( e \in E_0 \). Then \( ([c], e) \) is a well-defined element which depends only on the congruence class of \( (c, D) \).

Consider the product of two decorated curves using these coordinates. For \( ([c_i], e_i) \in \pi_1(M, b_0) \times E_0, i = 0, 1 \), their product can be viewed geometrically as follows. The
product curve $c_1 c_2$ is decorated by $e_1 \otimes e_2$. The point lying under $e_2$ lies at the tail end of $c_2$, which is already the tail end of the product $c_1 c_2$. However, the point under $e_1$ lies at the tail end of $c_1$, which is only halfway through $c_1 c_2$. Thus we slide $e_1$ around the curve $c_2$, transforming the decoration into $\Gamma_{c_2}(e_1) \otimes e_2$. We finish by multiplying the two decorations in $E_0$. This yields the definition

$$(c_1, e_1)(c_2, e_2) = (c_1 c_2, \Gamma_{c_2}(e_1)e_2). \quad (6.3.1)$$

Multiplication in the semidirect product $\mathbf{SD}(\pi_1(M, b_0), \Gamma, E_0)$ corresponds exactly to the multiplication defined in Equation 6.3.1. Thus the injective map $\mathcal{D}CS(E, b_0)^1 \to \mathbf{SD}$ respects multiplication.

### 6.4 Adding Decorations

In the definition of $\mathfrak{F}$, we will create decorated curves by adding decorations to an ordinary curve. We fix notation to be used throughout this section. Let $\pi : E \to M$ be a smooth ribbon Hopf algebra bundle. Let $c$ be a smooth curve in $M$ with basepoint $c(*)$, and $\mathfrak{a} = (s_1, \cdots, s_n) \in \text{Conf}_n(S^1)$ such that each $s_i \neq *$. Assume that for each $s_i$, $c$ is a local embedding on some neighborhood $U_i$ of $s_i$. This implies that $c_* : (c^*E) \to E$ is a local embedding on $(c^*\pi)^{-1}(U_i)$. Let $x_i = c(s_i)$; then $\mathfrak{x} = (x_1, \cdots, x_n) \in \text{Conf}_n(M)$ and all points are distinct from the basepoint $c(*)$. In this section particularly, we will identify the fibers $(c^*E)_{s_i}$ and $E_{x_i}$; notice that the restriction

$$(c_*)|(c^*\pi)^{-1}(s_i) : (c^*E)_{s_i} \to E_{x_i}$$

is the identity map.
This method uses a chart $\phi$ of $E$ to map $(c(s), h) \in M \times \mathcal{H}$ into the decoration
$\phi^{-1}(c(s), h) \in E_{c(s)} = (c^*E)_s$. Generally, this decoration depends on the choice of
chart. However, we can remove this dependence if $h$ is fixed by the structure group $\text{RAut}(\mathcal{H})$.

**Definition 6.4.1.** Let $\text{RFix}_n^H \subset \mathcal{H}^\otimes n$ denote the subset of elements which are fixed
by $f^\otimes n$ for every $f \in \text{RAut}(\mathcal{H})$.

First we use a single point $x = c(s) \in M$ and show how an element of $\text{RFix}_n^H$ gives
a well-defined element of the tensor product $E_x^\otimes n = (c^*E)^\otimes n$.

We make a note of the notation for the restriction of $\phi$, $\phi_x : E_x \to \mathcal{H}$, and

$$\phi_x^{-\otimes n} : \mathcal{H}^\otimes n \to E_x^\otimes n$$

to represent the $n$-fold tensor product of $\phi_x^{-1}$.

**Lemma 6.4.2.** Let $\pi : E \to M$ be a ribbon Hopf algebra bundle with typical fiber $\mathcal{H}$,
$x \in M$, and $\phi_1, \phi_2 : E_x \to \mathcal{H}$ two trivializations. For all $n \geq 1$ and $h \in \text{RFix}_n^H$, we have

$$\phi_1^{-\otimes n}(h) = \phi_2^{-\otimes n}(h).$$

**Proof.** Let $t \in \text{RAut}(\mathcal{H})$ be the transition map between $\phi_1$ and $\phi_2$. Then for $h \in \text{RFix}_n^H$, we have

$$\phi_1^{-\otimes n}(h) = \phi_2^{-\otimes n}(t^\otimes n(h)) = \phi_2^{-\otimes n}(h).$$

For $n > 1$, the transition map between $\phi_1^\otimes n$ and $\phi_2^\otimes n$ is $t^\otimes n$. Thus for $h \in \text{RFix}_n^H$,

$$\phi_1^{-\otimes n}(h) = \phi_2^{-\otimes n} \circ t^\otimes n(h) = \phi_2^{-\otimes n}(h).$$

$\square$
This lemma shows that for any \( x \), an element \( h \in \text{RFix}_{\mathcal{H}}^n \) can be assigned to a unique decoration in \( E_x^\otimes_n \). Assuming that \( x = c(s) \) and \( c \) is a local embedding on a neighborhood of \( s \), this translates into an element of \( (c^*E)^\otimes_n \). We will denote this assignment by

\[
\text{RFix}_{\mathcal{H}}^n \ni h \mapsto \hat{h}_x \in \text{RFix}_{\mathcal{H}_x}^n.
\]

The more general process of adding decorations over distinct points requires a flat connection on \( E \), which has not been needed yet. There are two equivalent methods to decorate the points \( (x_1, x_2, \cdots, x_n) \in \text{Conf}_n(M) \) with a decoration \( h \in \mathcal{H}^\otimes_n \). We will describe each of these methods in turn, and they can both be seen in Figure 6.5. Recall that \( x_i = c(s_i) \) are distinct and \( c \) is a local embedding on a neighborhood of \( s_i \).

![Diagram](image)

Figure 6.5: Two methods for adding decorations to a curve.

The first method uses smooth paths \( p_i : [0, 1] \to M \), for \( i \geq 2 \), with \( p(0) = x_1 \) and
$p(1) = x_i$. These paths are not required to lie on the image of $c$. Let $\Gamma_i = \Gamma_{p_i}$ be the parallel transport map defined by $p_i$. Then we can define the element

$$(1 \otimes \Gamma_2 \otimes \Gamma_3 \otimes \cdots \otimes \Gamma_n)(\hat{h}) \in \bigotimes_{i=1}^{n} E_{x_i}.$$ 

This element depends on the point $x_1$ and the homotopy class of the paths $p_i$. Because of our local embedding assumption, it yields a well-defined decoration

$$(c_s^{-\otimes n} \circ (1 \otimes \Gamma_2 \otimes \Gamma_3 \otimes \cdots \otimes \Gamma_n))(\hat{h}) \in \bigotimes_{i=1}^{n} (c^* E)_{s_i} \subset \text{Conf}_n(c^* E).$$

The second method uses a simply connected neighborhood $U$ containing all of the points $x_i$. Let $\phi$ be a flat chart over $U$, and $\phi_i = \phi|_{x_i} : E_{x_i} \to \mathcal{H}$. Consider the tensor product

$$
\phi_{\underline{\underline{x}}}^{-\otimes n} = \left( \bigotimes_{i=1}^{n} \phi_i^{-1} \right) : \mathcal{H}^{\otimes n} \to \bigotimes_{i=1}^{n} E_{x_i}.
$$

This map turns $h \in \mathcal{H}^{\otimes n}$ into a bundle element lying over $\underline{\underline{x}}$. This, in turn, defines a decoration over $c$, namely

$$(c^{-\otimes n} \circ \phi_{\underline{\underline{x}}}^{-\otimes n})(h).$$

This decoration is actually independent of the chart $\phi$, as proved by the following lemma. Interestingly, it does depend on the choice of simply connected neighborhood $U$.

**Lemma 6.4.3.** For $(U, \phi)$ and $(V, \psi)$ two flat charts of $E$ with $U, V$ simply connected, an element $\underline{x} = (x_1, \cdots, x_n) \in \text{Conf}_n(M)$ such that all $x_i$ lie in the same connected component of $U \cap V$, and $h \in \text{RFix}_K^n$, we have

$$
\phi_{\underline{\underline{x}}}^{-\otimes n}(h) = \psi_{\underline{\underline{x}}}^{-\otimes n}(h).
$$
Proof. Let \( t \in \text{RAut}(\mathcal{H}) \) be the transition map for the component of \( U \cap V \) containing the points \( x_i \). Because \( \phi \) and \( \psi \) are both flat charts, the transition map is constant on the connected component containing the \( x_i \). Thus, on that connected component the transition map between \( \phi^{-\otimes n}_U \) and \( \psi^{-\otimes n}_U \) is \( t^{\otimes n} \). Thus,

\[
\phi^{-\otimes n}_U(h) = \psi^{-\otimes n}_U(t^{\otimes n}(h)) = \psi^{-\otimes n}_U(h).
\]

This lemma applies, in particular, to all flat charts lying over a simply connected \( U \subset M \). Thus for any \( \underline{x} \in U^n \), an element \( h \in \text{RFix}^n_\mathcal{H} \) defines a unique element \( \hat{h} \in E^{\otimes n}_\underline{x} \).

While the decoration is independent of the choice of flat chart over \( U \), it does depend on the choice of \( U \) itself. Suppose \( U, V \) are two simply connected neighborhoods containing the points \( x_1, x_2 \). If the decorated points lie in different connected components of \( U \cap V \), then the transition map is no longer of the form \( t^{\otimes n} \) for some \( t \in \text{RAut}\mathcal{H} \). In this case, the different simply connected neighborhoods may lead to different decorations.

We now show that the two methods for adding decorations are equivalent in a specific way.

**Lemma 6.4.4.** Let \( (\phi, U) \) be a flat chart with \( U \) simply connected, and \( \underline{x} \in U^n \). For \( i \geq 2 \), let \( c_i \) be a path in \( U \) connecting \( x_1 \) to \( x_i \), with parallel transport map \( \Gamma_i \). Then for any \( h \in \text{RFix}^n_\mathcal{H} \) with corresponding \( \hat{h} \in \text{RFix}^n_{E_1} \), we have

\[
(1 \otimes \Gamma_2 \otimes \Gamma_3 \otimes \cdots \otimes \Gamma_n)(\hat{h}) = \phi^{-\otimes n}_U(h).
\]

Proof. By definition of a flat chart, for any path \( c \) in \( U \) and \( \hat{h} \in E_{c(0)} \), we have

\[
\phi_{E_{c(1)}} \circ \Gamma_c(\hat{h}) = \phi_{E_{c(0)}}(\hat{h}).
\]

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Taking the tensor product of this relation, we see for $h \in \text{RFix}_H^n$ as we assumed,

$$\phi_{\frac{1}{H}}^n \circ (1 \otimes \Gamma_2 \otimes \Gamma_3 \otimes \cdots \otimes \Gamma_n)(\hat{h}) = \phi_{x_1, x_1, \ldots, x_1}^n(\hat{h}) = h.$$ 

Applying $\phi_{\frac{1}{H}}^{-n}$ yields the equation we wanted. □

This lemma shows us how to relate the two methods of adding decorations. Notice that both methods require a choice: the first method depends on the homotopy class of the paths, while the second method depends on the choice of neighborhood. The equivalence between these methods reflects this choice.


CHAPTER 7
THE CASE OF $\mathcal{T}^1\Sigma$

Let $c \subset \mathbb{R}^2$ be an embedded loop, $x \in c$ on the right side, and $h \in \mathcal{H}$ some decoration over $x$. One central aspect of the classical link invariant is that pushing the point $x$ once clockwise around the loop $c$ applies the map $S^2$ to the decoration.

We want to capture this behavior for an arbitrary oriented surface $\Sigma$. Let $T\Sigma$ be the tangent bundle, with zero section $0$. We view the circle bundle $\mathcal{T}^1\Sigma$ as a quotient of $T\Sigma \setminus 0$ by the $\mathbb{R}^+$ action. Let $\Phi : \mathcal{T}^1\Sigma \to \Sigma$ be the natural projection map. This circle bundle, instead of $\Sigma$ itself, will be used as the base space of a ribbon Hopf algebra bundle.

We will refer to a section $U \to \mathcal{T}^1U$ as a (non-singular) vector field, to avoid confusion with possible sections of other bundles floating around. This is only a slight abuse of notation, as an actual non-singular vector field defines such a section.

7.1 Balanced Connections

This section will detail some basics about $\mathcal{T}^1\Sigma$, starting with some results on the fundamental group. We assume from here on that $\Sigma$ is an orientable surface such that $\pi_2(\Sigma)$ is trivial. This includes all cases except for the sphere.
Lemma 7.1.1. For a smooth, connected, oriented surface $\Sigma$ which is not diffeomorphic to the sphere, $v \in T^1\Sigma$ and $x = \Phi(v)$, there is a short exact sequence:

$$0 \to \pi_1(S^1, s) \xrightarrow{i_*} \pi_1(T^1\Sigma, v) \xrightarrow{\Phi_*} \pi_1(\Sigma, x) \to 0,$$

where $\Phi_*$ is induced by $\Phi$, and $\iota_* : \pi_1(S^1, s) \to \pi_1(T^1\Sigma, v)$ is induced by the injection $\iota : S^1 \to T^1\Sigma_x$ with $\iota(s) = v$.

Proof. Any fibration has an associated long exact sequence of homotopy groups. In this case, because $\pi_2(\Sigma, x) = 0$, we obtain this short exact sequence. \hfill \square

One consequence of this lemma is that the inclusion

$$i : T^1\Sigma_x \to T^1\Sigma$$

defines an injective group homomorphism

$$i_* : \pi_1(S^1, s) \to \pi_1(T^1\Sigma, v).$$

There are two elements which generate this subgroup; the one which agrees with the orientation on $T^1\Sigma$ will be called the fiber generator in $T^1\Sigma$ based at $v$. Let $c : [0, 1] \to T^1\Sigma_x$ be a smooth, orientation-preserving map with $c(0) = c(1) = v$ and which is injective on $(0, 1)$. The curve $f_v := i \circ c$ is called the standard representation of the fiber generator.

Lemma 7.1.2. The fiber generator based at $v$ is a central element in $\pi_1(T^1\Sigma, v)$.

Proof. We continue the notation of letting $S^1 = [0, 1] / (0 \sim 1)$, and letting the basepoint $*$ to be the image of 0 in this quotient. Let $c : [0, 1] \to T^1\Sigma$ be a smooth
curve with $c(*) = v$. We will show that $[c]$ and $[f_v]$ commute. Define the pullback bundle

$$M = (\Phi \circ c) \ast T^1 \Sigma.$$  

$M$ is an oriented circle bundle over the circle, and thus is diffeomorphic to a torus. In particular, its fundamental group is abelian.

Recall the definition

$$(\Phi \circ c) \ast T^1 \Sigma = \{ (s, v) \in S^1 \times T^1 \Sigma : \Phi(c(s)) = \Phi(v) \}.$$  

The map $(\Phi \circ c)_*$ is the projection onto the second coordinate. This projection induces a group homomorphism

$$d : \pi_1(M, (*, v)) \to \pi_1(T^1 \Sigma, v).$$

Define two new curves in $M$ by

$$f'(s) = (*, f_v(s)) \quad \text{and} \quad c'(s) = (s, c(s)).$$

It is straightforward to check that $(\Phi \circ c)_* \circ f' = f_v$ and $(\Phi \circ c)_* \circ c' = c$. Thus $d([f']) = [f_v]$ and $d([c']) = [c]$. However, $[f'], [c']$ live in an abelian fundamental group and thus commute. Thus $[c]$ and $[f_v]$ must commute as well. \hfill \Box

Now let $\mathcal{H}$ be a ribbon Hopf algebra with antipode $S$, and $E$ a flat ribbon Hopf algebra bundle over $T^1 \Sigma$ with typical fiber $\mathcal{H}$. For $v \in T^1 \Sigma$, let $S_v$ denote the antipode of the fiber $E_v$.

**Definition 7.1.3.** A flat ribbon ribbon Hopf algebra bundle $\pi : E \to T^1 \Sigma$ is balanced if for all $v \in T^1 \Sigma$ and $c$ a loop in $T^1 \Sigma$ based at $v$ and representing the fiber generator, we have

$$\Gamma_c = S_v^2 \in RAut(E_v).$$
As long as $\Sigma$ is connected, it suffices for this equality to hold for just one point $v \in T^1\Sigma$.

**Lemma 7.1.4.** Suppose $\Sigma$ is a smooth, orientable, connected surface, $v_0, v_1 \in T^1\Sigma$, and $f_i$ a loop representing the fiber generator at $v_i$ for $i = 0, 1$. Then $\Gamma_{f_0} = S^2_{v_0}$ if and only if $\Gamma_{f_1} = S^2_{v_1}$.

**Proof.** Suppose $\Gamma_{f_0} = S^2_{v_0}$. Let $p : [0, 1] \to T^1\Sigma$ be a smooth path with $p(0) = v_0$ and $p(1) = v_1$. Then $\Gamma_p$ is a ribbon automorphism, and so

$$\Gamma_p \circ S^2_{v_0} = S^2_{v_1} \circ \Gamma_p.$$ 

And, using that $S^2_{v_0} = \Gamma_{f_0}$, we have

$$S^2_{v_1} = \Gamma_p \circ S^2_{v_0} \circ \Gamma_p^{-1} = \Gamma_{p^{-1}f_0p}.$$ 

We claim that $[p^{-1}f_0p] = [f_1]$, which would show that $\Gamma_{f_1} = S^2_{v_1}$, finishing the proof. This can be seen by considering the pullback bundle $M = (\Phi \circ p)^*T^1\Sigma$, which is a circle bundle over $[0, 1]$. Define the curves in $M$

$$p'(t) = (t, p(t)) \text{ for } t \in [0, 1], \text{ and } f'_i(s) = (i, f_i(s)) \text{ for } i = 1, 2, s \in [0, 1].$$

It is clear that

$$(\Phi \circ p)_* \circ f'_0 = f_0, \quad (\Phi \circ p)_* \circ f'_1 = f_1, \quad \text{and} \quad (\Phi \circ p)_* \circ p' = p.$$ 

It is not difficult to see that $(p')^{-1}f'_0p'$ is homotopic to $f'_1$ in the pullback bundle $M$. Because $M$ is a circle bundle over $[0, 1]$, it is diffeomorphic to a cylinder. Both $f'_1$ and $(p')^{-1}f'_0p'$ wrap once around this cylinder, both in the same direction, and so are homotopic.

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Let \( d: \pi_1(M, (s, v)) \to \pi_1(T^1\Sigma, v) \) be the group homomorphism defined by \((\Phi \circ p)_*\). Then
\[
[p^{-1}f_0p] = d((p')^{-1}f_0'p') = d([f_1]) = [f_1].
\]
This finishes the proof.

If it exists, we can use a section of \( T^1\Sigma \) to measure the winding of a curve. This is done by splitting the short exact sequence from Lemma 7.1.1. If \( V: \Sigma \to T^1\Sigma \) is a smooth section, then a curve \( c: [0, 1] \to \Sigma \) lifts to \( \ell_c = V \circ c \).

**Lemma 7.1.5.** For a smooth \( V: \Sigma \to T^1\Sigma \), the induced map \( V_*: \pi_1(\Sigma, x) \to \pi_1(T^1\Sigma, v) \) splits the short exact sequence in Lemma 7.1.1. Thus, there is a non-canonical isomorphism
\[
\pi_1(T^1\Sigma, v) \cong \mathbb{Z} \oplus \pi_1(\Sigma, x).
\]

**Proof.** Because the fundamental groups might not be abelian, the existence of a right-splitting map does not immediately imply the direct sum isomorphism. However, it is easy to see that the image of \( V_* \) is normal, because the quotient
\[
\pi_1(T^1\Sigma, v) / \text{Im}(V_*)
\]
is abelian. This ensures that
\[
\pi_1(T^1\Sigma, v_0) = \text{Im}(i_*) \oplus \text{Im}(V_*).
\]
The isomorphism is realized, because \( i_* \) and \( V_* \) are isomorphisms onto their images. \(\square\)
Remark 7.1.6. Generally, a global section of \( \Phi \) does not exist, limiting the usefulness of the previous lemma. Usually, we apply the above lemma to subsets \( T^1U \subset T^1\Sigma \), where \( U \subset \Sigma \) is simply connected. There always exists a local section over these neighborhoods.

7.2 Local Diagrams

The process of translating a link diagram on \( \Sigma \) into a tethered decorated curve system in \( T^1\Sigma \) does not depend on any local charts. However, it is difficult to understand the process, and almost impossible to realistically calculate, without a good method of picturing it. In this section, we show how to use a local diagram to picture a (segment of a) decorated curve in \( T^1\Sigma \). First we discuss how to represent the segment of a curve living in a neighborhood \( U \subset \Sigma \) and \( T^1U \subset T^1\Sigma \). Then we discuss how to represent decorated curves in \( T^1\Sigma \). A flat local chart is used for this, which requires a simply connected neighborhood of \( T^1\Sigma \). To finish the section, we show how local diagrams recapture the equivalence moves defined in Chapter 2.

Start with a simply connected neighborhood \( U \subset \Sigma \). As with every simply connected 2-manifold, \( U \) is diffeomorphic to the interior of the rectangle shown in Figure 7.1, considered as a neighborhood in \( \mathbb{R}^2 \). We will always choose such a smooth embedding \( \xi : U \rightarrow \mathbb{R}^2 \) so that the orientation of \( \Sigma \) corresponds to clockwise rotation.

Now for a link diagram on \( \Sigma \), the piece of the diagram which intersects with \( U \) may then be pictured in our rectangle. These are called local diagrams of the link diagram; see Figure 7.2 for an example. Recall that by definition, the curve under the link diagram is always in general position in \( \Sigma \).
A simply connected neighborhood $U$ of $x \in \Sigma$ defines a neighborhood $T^1U \subset T^1\Sigma$ for any $v \in T^1\Sigma_x$. The smooth embedding $\xi$ of $U$ induces a smooth embedding

$$d\xi : T^1U \hookrightarrow T^1\mathbb{R}^2.$$ 

For $c : [0, 1] \to T^1\Sigma$, let $c_\Sigma = \Phi \circ c$ be its projection curve in $\Sigma$. Then the smooth embedding $d\xi$ of $U$ lets us represent the segments of the curve $\text{Im}(c) \cap T^1U$ using the same rectangular diagram that we used for $U$. An example of this is seen in Figure 7.1: A simply connected neighborhood in $\Sigma$

Figure 7.2: A local diagram showing a piece of a link diagram in $\Sigma$

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7.3, where the curve $c_\Sigma$ is shown and the actual vectors $c(t)$ are represented by small arrows drawn along the curve.

![Figure 7.3: A general curve in $T^1 U$](image)

For this dissertation, we will often not need these most general curves in $T^1 U$. The curves we deal with are mostly tangent lifts of curves.

**Definition 7.2.1.** For $x \in \Sigma$ and a non-zero vector $v \in T\Sigma_x$, denote its equivalence class in the unit tangent bundle by $|v| \in T^1 \Sigma_x$.

Suppose $c : [0, 1] \to \Sigma$ is a smooth immersion, with derivative

$$Dc : T[0, 1] \to T\Sigma.$$ 

We use the standard representation $T[0, 1] = [0, 1] \times \mathbb{R}$. Then the tangent lift
\( \hat{c} : [0, 1] \to \mathcal{T}^1 \Sigma \) is defined by

\[
\hat{c}(t) = |Dc(t, 1)|.
\]

Figure 7.4: A curve in \( U \) (left) and its tangent lift in \( \mathcal{T}^1 U \) (right)

Figure 7.4 shows the process of taking the tangent lift of a curve \( c \) in \( U \). Vector arrows are drawn tangent to the curve in the direction of its orientation. Often we will not explicitly draw the vector arrows, and simply use the curve \( c \) to represent the tangent lift \( \hat{c} \). It will be clear from context if a diagram represents the curve \( c \) in \( U \) or the curve \( \hat{c} \) in \( \mathcal{T}^1 U \). In particular, all decorated curves live in \( \mathcal{T}^1 U \).

Now suppose \( \pi : E \to \mathcal{T}^1 \Sigma \) is a flat ribbon algebra bundle. We want to use a flat chart to define local coordinates over \( \mathcal{T}^1 U \). Unfortunately, \( \mathcal{T}^1 U \cong U \times S^1 \) is not simply connected, and thus no flat charts exist over it. The remedy is a local
smooth section $V: U \to \mathcal{T}^1 U$. Removing the image of this vector field yields a simply connected neighborhood

$$\mathcal{T}^1 U_V = \mathcal{T}^1 U \setminus (\text{Im} V). \quad (7.2.1)$$

Figure 7.5 shows an example of how we represent the simply connected neighborhood $\mathcal{T}^1 U_V$. The simply connected $U \subset \Sigma$ is again represented by a smooth embedding $\xi$ of $U$ into a dashed, rounded rectangle. The derivative $d\xi$ then defines a correspondence between vector fields on the rectangle and vector fields on $U$. We always choose $V$ to be a vector field which corresponds to a parallel vector field on the rectangle. This vector field is represented by the dotted arrow in the top-left corner of the diagram. Our standard will be to have the vector pointing right, though any choice is valid and sometimes useful.

![Diagram](image)

**Figure 7.5:** Two representations of simply connected neighborhoods in $\mathcal{T}^1 \Sigma$

For ease of notation, from here on we will implicitly identify $\mathcal{T}^1 U$ with our local diagrams, and will not explicitly mention the diffeomorphism $\xi$.  

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These local diagrams of $T^1U_V$ can be used to picture a piece of a decorated curve system in $T^1\Sigma$. However, there can be some ambiguities. These can be resolved by placing some light conditions on the neighborhood $T^1U_V$.

**Definition 7.2.2.** Consider a series of paths

$$p = (p_1, \cdots, c_n) : \prod_{i=1}^{n} [0,1] \to T^1U.$$

For a section $V : U \to T^1U$, a point $p_i(s)$ with $\Phi(p(s)) = x$ is a critical point (with respect to $V$) if $p(s) = V_x$. The point $x$ is a crossing point if $T^1U_x \subset T^1U$ contains at least two points of the image of the path.

A local diagram of decorated curves is admissible if the set of critical points is discrete, and the sets of crossing points, critical points, and decorated points are mutually disjoint. The example shown in Figure 7.6 is admissible.

**Remark 7.2.3.** Any diagram of decorated curves can be turned into an admissible diagram by applying a local homotopy to the curve. Recall that two decorated curves are congruent if there is such a homotopy between them.

Critical points in a local diagram are represented by an open circle on the curve. Decorated points are represented by a filled in circle.

If $\tilde{p}$ is the tangent lift of a path $p$ in $\Sigma$, then the crossing points of $\tilde{p}$ are just regular intersection points of the projected curve in $\Sigma$.

If the decorated points on a curve lie in $T^1U$, then the decoration can also be represented in the local diagram. Let $D$ be a decoration on $\widehat{c}$ over some points $s \in \text{Conf}_n(S^1)$, and let $v_i = \widehat{c}(s_i)$. Then

$$D = \sum_{\mu} \left( \bigotimes_{i=1}^{n} d_{\mu}^{i} \right) \in \bigotimes_{i=1}^{n} (c^*E)_{s_i} = \bigotimes_{i=1}^{n} E_{v_i}.$$
Generally, only some of the points \( v_i \) lie inside \( T^1U \); without loss of generality, suppose that

\[
v_1, \ldots, v_j \in T^1U \quad \text{and} \quad v_{j+1}, \ldots, v_n \in \left( T^1\Sigma \setminus T^1U \right).
\]

We can always pick a vector field \( V \) on \( U \) such that for all \( x \in U \) and \( 1 \leq i \leq j \), \( V(x) \neq v_i \). Suppose \( \phi \) is a flat chart over \( T^1U_V \). Recall the notation \( \phi_v : E_v \to H \) to denote the map, for \( v \in T^1\Sigma \), defined by

\[
\phi(v) = (x, \phi_v(h)).
\]

Then

\[
\left( \bigotimes_{i=1}^{j} \phi|_{v_i} \otimes \text{id}^{\otimes n-j} \right)(D) = \sum_{\mu} h_{\mu}^{i} \otimes \cdots \otimes h_{\mu}^{j} \otimes d_{j+1}^{\mu} \otimes \cdots \otimes d_{n}^{\mu} \in H^{\otimes j} \otimes \left( \bigotimes_{i=j+1}^{n} E_{v_i} \right),
\]

where \( h_{\mu}^{k} \in H \) and \( d_{k}^{\mu} \in E_{v_k} \).

**Remark 7.2.4.** In our discussion, we often apply some transformation \( \phi \in \text{Aut}(H^{\otimes j}) \) to a decoration. When this happens, we are implicitly using the tensor product \( \phi \otimes \text{id}^{\otimes n-j} \), and do not explicitly mention the decorated points lying outside of \( T^1U \).

In local diagrams we will represent this decoration by placing the label \( h_{\mu}^{k} \) near the filled dot representing the decorated point \( v_k \). The flat chart and summation index \( \mu \) are left out when they can be assumed. An example of this is shown in Figure 7.6.

We finish this section by showing how local diagrams can recover the local equivalence moves of decorated curves in the plane, which were defined in Chapter 2. The moves shown in Figure 2.4a and 2.4b, replacing a small kink with the decoration \( \kappa \), is discussed in the following section. Here we discuss every other equivalence move shown in Figures 2.4 - 2.7.
Many of the equivalence moves do not involve decorated points, and are a simple consequence of homotopy invariance of the underlying curve.

Several of the moves do require sliding decorated point in $\mathcal{T}^1\Sigma$. We investigate this case for a curve $c$ with decoration $D$ which lies over a single decorated point $x = c(s) \in \mathcal{T}^1U_V$. This decoration lives in

$$D \in (c^*E)_s = E_x,$$

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and we will often identify these two fibers. This identification simplifies our notation considerably.

If the decorated point slides along a path $p$, the decoration transforms following the parallel transport map $\Gamma_p$. We can use local $\phi$-coordinates to study this parallel transport. For a smooth path $p : [0, 1] \to T^1U$ with $p(0), p(1) \in T^1U_V$, define $\Psi_p \in RAut(\mathcal{H})$ by

$$(\phi \circ \Gamma_p \circ \phi^{-1}) (p(0), h) = (p(1), \Psi_p(h)).$$

The map $\Psi_p$ is simply the parallel transport map over $\rho$ in local $\phi$-coordinates. It depends only on the homotopy class of the path $p$, as well as the flat chart $\phi$. If $p, p'$ are two smooth paths with $p(1) = p'(0)$, then $\Psi_{pp'} = \Psi_p \circ \Psi_{p'}$.

Recall that, by definition of a flat chart, $\Psi_p = id$ whenever the image of $p$ lies entirely in $T^1U_V$. Thus sliding a decorated point generally does not change the local coordinates for its decoration. It is only when a decorated point leaves the neighborhood $T^1U_V$ that a non-trivial change can happen.

This is seen in Figure 7.7, which shows an equivalence relation analogous to that shown in Figure 2.5a. In this example, the decorated point slides through a path lying entirely in $T^1U_V$, and so the local coordinate for the decoration is constant. Notice that these curves do not intersect in $T^1U_V$.

Multiplication shown in Figure 2.5b is a direct result of multiplication of decorations in the pullback bundle.

Now consider the moves shown in Figure 2.6, sliding a decorated point past a local extrema. Figure 7.8 shows analogous moves for a local diagram over $T^1U_V$.

Recall that critical points are marked by an open circle. In the two diagrams on
the right side of Figure 7.8, the decorated point never leaves the neighborhood $T^1U_V$.
Thus the local coordinate for the decoration is kept constant.

The two curves shown in the left side of the figure, by contrast, run through a
critical point, and so do not lie entirely in $T^1U_V$. We investigate how the decoration
transforms when the decorated point slides through such a critical point.

**Lemma 7.2.5.** The equivalence moves shown in Figure 7.8 do not change the con-
gruence class of a decorated curve.

**Proof.** This can be seen using a second simply-connected neighborhood, namely
$T^1U_{-V}$, where $(-V)_x = -V_x$. It is easy to see that their intersection has two con-
ected components, say

$$T^1U_V \cap T^1U_{-V} = A \coprod B$$
Then define

\[ E^+ = \pi^{-1}(\mathcal{T}^1U_V) \text{ and } E^- = \pi^1(\mathcal{T}^1U_{-V}). \]

Suppose \( \phi^+, \phi^- \) are flat charts on \( E^+ \) and \( E^- \) respectively. The transition map between them is locally constant, so there are unique \( t_A, t_B \in \text{RAut}(\mathcal{H}) \) such that

\[ \Phi(x, h) := (\phi^+ \circ (\phi^-)^{-1})(x, h) = \begin{cases} (x, t_A(h)) & \text{if } x \in A \\ (x, t_B(h)) & \text{if } x \in B \end{cases} \]

By substituting the chart \( \phi^- \) with \((\text{id} \times t_B) \circ \phi^-\), we may assume that \( t_B = \text{id} \).

Now we claim \( t_A = S^{\pm 2} \). We show this by relating \( t_A \) with the monodromy around the fiber generator. Restricting \( A \) and \( B \) to the fiber over \( x \) defines connected components

\[ A_x, B_x \subset (\mathcal{T}^1U_x) \setminus \{V_x, -V_x\}. \]

![Diagram](image_url)

Figure 7.9: \( A_x \) and \( B_x \) in \( \mathcal{T}^1U \).

This is shown in Figure 7.9, where we use the standard that \( V_x \) points right, \( A_x \) is the bottom half of the circle, and \( B_x \) the top half. Pick basepoints \( b_1 \in A_x \) and
$b_2 \in B_x$, and let $f$ be the standard representative of the fiber generator at $b_1$. The loop $f$ first runs from $b_1$ to $b_2$, staying inside $E^+$ while doing so. It then runs from $b_2$ back to $b_1$, staying inside $E^-$. Then using Lemma 5.3.13, we must have that

$$\left( \phi^+ \circ \Gamma_f \circ (\phi^+)^{-1} \right) (b_1, h) = (b_1, (t_A \circ t_B^{-1})(h)) = (b_1, t_A(h)).$$

And because $\Gamma$ comes from a balanced connection, we must have $t_A = S^2$.

Now we can use the transition map $\Phi$ to translate between local diagrams of $T^1 U_V$ and $T^1 U_{-V}$, so long as all decorated points are not a critical points with respect to either $V$ or $-V$. Again, we assume that $V$ is parallel on the rectangle and points to the right, and $A$ consists of vectors pointing upwards while $B$ consists of vectors which points downwards.

![Diagram](image)

**Figure 7.10: Translating between local diagrams**

This translation is shown in Figure 7.10. For clarity, we add symbols in a convenient corner to represent the chart used for each diagram. Recall that $\Phi$ is the identity on $B$, and so $\Phi$ acts as the identity decorated points lying in $B$. However, if a decorated point lies in $A$ then its decoration, in local coordinates, is transformed by an
application of $S^\pm 2$. Specifically, $\Phi$ translates from $\phi^-$-coordinates to $\phi^+$-coordinates, and applies $S^2$, while $\Phi^{-1}$ translates from $\phi^+$-coordinates to $\phi^-$-coordinates, and applies $S^{-2}$.

With this setup, the moves can be proved by translating from $\phi^+$ to $\phi^-$, sliding the decorated point in $E^-$, and then translating back into $\phi^+$. This is shown in Figure 7.11. Again, we add symbols in a convenient corner to represent the chart used in each local diagram.

Figure 7.11: Using $\Phi$ to slide decorated points past critical points
We give another method, slightly informal, for proving Lemma 7.2.5 using only one neighborhood $\mathcal{T}^1U_V$ with one chart.

Consider the loop in $\mathcal{T}^1U$ shown in Figure 7.12, which is the tangent lift $\hat{c}$ of a curve $c$ in $U$. Let $v_0$ be the basepoint of $\hat{c}$. In $\Sigma$, $c$ is null-homotopic. However, $\hat{c}$ winds once clockwise around the circle in $\mathcal{T}^1U$, and so it represents the fiber generator at $v_0$. Thus we know that $\Gamma_{\hat{c}}(D) = S^2_{v_0}(D)$ for any $D \in E_{v_0}$. In local coordinates $h = \phi_{v_0}(D) \in \mathcal{H}$, this becomes the identity

$$\Psi_{\hat{c}}(h) = S^2(h).$$

Break $\hat{c}$ into a top half and a bottom half. Let $\hat{c}_b$ be the bottom half, running from $v_0$ to $v_1$, and let $\hat{c}_t$ be the top-half, running from $v_1$ to $V_0$. Then $\hat{c}_b \hat{c}_t = \hat{c}$. The bottom half $\hat{c}_b$ lies in $\mathcal{T}^1U_V$, and so $\Psi_{\hat{c}_b} = \text{id}$. Then,

$$\Psi_{\hat{c}_t} = \Psi_{\hat{c}} = S^2.$$

This identity is demonstrated in the bottom-left diagram of Figure 7.8. The
top-left identity can be demonstrated similarly, using the same circle with opposite orientation.

7.3 Quotients and Equivalence Moves

In this section, we discuss the equivalence moves in Figures 2.4a and 2.4b. A generalized version of these equivalence moves for local diagrams is shown in Figure 7.13.

![Equivalence Moves](image)

(a) \( \sim \) \( \kappa \) (b) \( \sim \) \( \kappa^{-1} \)

Figure 7.13: RI-type Equivalence Moves

This equality does not hold for decorated curves themselves; the two decorated curves are not congruent. The equality arises only after applying the injection

\[ \iota_{b_0} : TDCS(E, b_0) \rightarrow SD(\pi_1(\mathcal{T}^1 \Sigma, b_0), \Gamma, E_0) = k[\pi_1(\mathcal{T}^1 \Sigma, b_0)] \bowtie \Gamma E_0, \]

and then taking an algebraic quotient of \( SD \), as defined in Lemma 4.5.2. The moves in Figure 7.13 only hold after taking this quotient. We discuss this in more details.
Figure 7.14: Homotopic curves in $T^1U$

First we note that the small kink can be viewed as a loop representing the fiber generator. Let $c$ be the curve in $U$ shown in the left side of Figure 7.14. It has tangent lift $\tilde{c}$ in $T^1U$. The curve $c$ is trivial in $U$, and we can use a homotopy to straighten it. However, $\tilde{c}$ in $T^1U$ is not trivial, as the vectors rotate once clockwise. This is shown in the right side of Figure 7.14, where we use arrows along the curve to represent $\tilde{c}$.

Let $b \in T^1\Sigma$ be a basepoint and $x = \Phi(b)$. Let $T^1\Sigma_x \cong S^1$ be the fiber of $T^1\Sigma$ lying over $x$, and let $f_b$ be the standard representation of the fiber generator at $b$. Also recall the balancing element $\kappa = \nu^{-1} \in \text{RFix}^1_{\nu}$. The following result is a direct application of Lemmas 7.1.2 and 4.5.3.

**Lemma 7.3.1.** For $E \to T^1\Sigma$ a flat, balanced ribbon Hopf algebra bundle, let $\kappa_b \in E_0$ be the balancing element. Then the element $([f_b],\kappa_b)$ is group-like and central in $SD(\pi_1(T^1\Sigma,b),\Gamma,E_b)$. Thus there is a quotient map

$$Q_{([f_b],\kappa_b)}: SD \to SD,$$

where $SD$ is again a Hopf algebra.

**Proof.** We showed in Lemma 7.1.2 that $[f_b]$ is central.
Because the connection is balanced, we know that $\Gamma_{f_b} = S^2$. Similarly, $\kappa_b h \kappa_b^{-1} = S_b h$ for any $h \in E_b$, where $S_b$ is the antipode in $E_b$. Further, $\kappa_b$ is group-like and fixed by $\text{RAut}(E_b)$. Then an application of Lemma 4.5.3 shows that $([f_b], \kappa_b) \in SD$ is central and group-like. The same lemma also defines the quotient map $Q_{([f_b], \kappa_b)}$. $\square$

Let $(([f_b], \kappa_b))$ be the linear subspace of SD spanned by elements

$$([f_b], \kappa_b)(c, h) - (c, h) \in SD.$$  

Then $\mathcal{SD}$ is defined as the linear quotient of $SD$ by $(([f_b], \kappa_b))$. Lemma 4.5.2 shows that $\mathcal{SD}$ is a Hopf algebra. For an element $(c, h) \in SD$, denote its quotient equivalence class by $[c, h] \in \mathcal{SD}$.

The equivalence moves in Figure 7.13 are a result of the equality $[[f_b], \kappa_b] = 1$ in $\mathcal{SD}$.

In Figure 7.13a, notice that the oriented curve on the left rotates once counterclockwise, and thus represents the inverse of the fiber generator in $\pi_1(T^1 \Sigma, b)$. This equivalence move is a result of the equality

$$[[f_b^{-1}], 1] = [1, \kappa_b].$$

The oriented curve on the left of Figure 7.13b rotates once clockwise, and so represents the fiber generator. This equivalence move arises from the equality

$$[[f_b], 1] = [1, \kappa_b^{-1}].$$
CHAPTER 8
INARIANT DEFINITION

8.1 From Link Diagrams to Decorated Curves

To prepare for the invariant definition, we will use this section to define the flattening map

\[ \mathcal{F} : \text{LD}(\Sigma)^m \to \text{DCS}(T^1\Sigma)^m. \]

Suppose that \( \pi : E \to T^1\Sigma \) is a flat ribbon Hopf algebra bundle with typical fiber \( \mathcal{H} \) and parallel transport map \( \Gamma \). Let \( \mathcal{L} \in \text{LD}(\Sigma)^m \) be a smooth framed link diagram. Calculating \( \mathcal{F}(\mathcal{L}) \in \text{DCS}(E)^m \) can be summarized by the steps:

1. Flatten the crossings of \( \mathcal{L} \), giving a smooth immersed curve system in \( \Sigma \), which we call

\[ \mathcal{C} = \prod_{i=1}^{m} c_i : \prod_{i=1}^{m} S^1 \to \Sigma. \]

2. Take the tangent lift of \( \mathcal{C} \). Because \( \mathcal{L} \) was assumed to be in general position, this lift is an embedded curve system in \( T^1 \Sigma \), which we call

\[ \hat{\mathcal{C}} = \prod_{i=1}^{m} \hat{c}_i : \prod_{i=1}^{m} S^1 \to T^1\Sigma. \]

3. Define a decoration on \( \hat{\mathcal{C}} \) by adding decorated points at every crossing point.
We discuss adding the decoration in detail, using notation from Sections 6.4 and 7.2. Recall that a decoration on $\hat{\xi}$ is defined to be an element of the configuration bundle of the pullback bundle

$$\hat{\xi}^* \pi : \hat{\xi}^* E \to \prod_{i=1}^{m} S^1.$$ 

We will use the notation

$$\mathcal{E} = \hat{\xi}^* E \text{ and } \mathcal{S} = \prod_{i=1}^{m} S^1.$$ 

We also continue the notation, for $s \in \mathcal{S}$,

$$\mathcal{E}_s = (\hat{\xi}^* \pi)^{-1}(s).$$

First consider an arbitrary intersection point $x \in \Sigma$. The two crossing strands defines two tangent vectors $v_1, v_2 \in T^1 \Sigma_x$; let $v_1$ correspond to the over-crossing strand. Because $\hat{\xi}$ is an embedding, there are unique $s_1, s_2 \in \mathcal{S}$ which map to $v_1$ and $v_2$, respectively. We will define an element in the fiber $\mathcal{E}_{s_1} \otimes \mathcal{E}_{s_2} = E_{v_1} \otimes E_{v_2}$.

Figure 8.1: The fiber $T^1 \Sigma_x$, with clockwise orientation

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Let \( \mathcal{R} \) be the \( R \)-matrix in \( \mathcal{H} \), with inverse \( \mathcal{R}^{-1} \). Recall that because \( \mathcal{R}, \mathcal{R}^{-1} \in \text{RFix}_2^2 \), there are well-defined elements \( \hat{\mathcal{R}}, \hat{\mathcal{R}}^{-1} \in E_{v_1}^2 \) defined by

\[
\hat{\mathcal{R}} = (\phi^{-1} \otimes \phi^{-1}) (\mathcal{R}) \quad \text{and} \quad \hat{\mathcal{R}}^{-1} = (\phi^{-1} \otimes \phi^{-1}) (\mathcal{R}^{-1})
\]

for any trivialization \( \phi : E_{v_1} \to \mathcal{H} \). This element does not depend on the choice of \( \phi \), as seen in Lemma 6.4.3. For general details about adding decorations, see Section 6.4.

To turn these into elements of \( E_{v_1} \otimes E_{v_2} \), we use a path connecting \( v_1 \) to \( v_2 \). Because \( \underline{L} \) was in general position, we know that \( v_1 \neq \pm v_2 \), so the four distinct vectors \( \pm v_1, \pm v_2 \) cut the circle \( T^1 \Sigma_x \) into four pieces. Let \( p : [0, 1] \to T^1 \Sigma_x \) be a path such that \( p(0) = v_1, p(1) = v_2, \) and \( p(t) \neq -v_1, -v_2 \) for all \( t \); see Figure 8.1. These condition uniquely describe the path up to homotopy, giving a well-defined map \( \Gamma_p : E_{v_1} \to E_{v_2} \). Sliding the second decorated vector along \( p \) yields the decorations

\[
(id \otimes \Gamma_p)(\hat{\mathcal{R}}) \quad \text{and} \quad (id \otimes \Gamma_p)(\hat{\mathcal{R}}^{-1}) \in E_{v_1} \otimes E_{v_2}.
\]

Give the interval \([0, 1]\) the standard orientation from 0 to 1, and recall \( T^1 \Sigma_x \) has an orientation inherited from \( \Sigma \). Then the path \( p \), being injective, is either orientation-preserving or orientation-reversing.

**Definition 8.1.1.** For a crossing at \( x \in \Sigma \), with \( v_1, v_2 \in T^1 \Sigma_x \) the tangent vector of the over-crossing strand and under-crossing strand, respectively, let \( p \) be the path from \( v_1 \) to \( v_2 \) as described in the preceding discussion.

The crossing at \( x \) is a *negative crossing* if the path \( p \) is orientation-preserving. It is a *positive crossing* if the path \( p \) is orientation-reversing.
Remark 8.1.2. We note that the orientation of a crossing is the opposite of the orientation of the path $p$; an orientation-preserving path corresponds to a negative crossing, and an orientation-reversing path corresponds to a positive crossing.

Both of these cases are shown in Figure 8.1. The positive crossing is shown in 8.1a, and the negative crossing in 8.1b.

Now suppose that the link diagram $L$ has $n$ distinct crossing points $x_1, \ldots, x_n$. Associated to each crossing point $x_i$ is: two vectors $v_{i,1}, v_{i,2} \in T^i \Sigma_x$, two unique points $s_{i,1}, s_{i,2} \in \mathcal{S}$, and a path $p_i$ with $p_i(0) = v_{i,1}$ and $p_i(1) = v_{i,2}$. Finally, for each $i$ let $k_i = 1$ if the crossing at $x_i$ is positive, and $k_i = -1$ if it is negative. Then we define

$$D_i = (\text{id} \otimes \Gamma_{p_i}) (\hat{R}^{k_i}) \in E_{v_{i,1}} \otimes E_{v_{i,2}},$$

Finally we define

$$D = \bigotimes_{i=1}^n D_i \in \bigotimes_{i=1}^n \left( E_{v_{i,1}} \otimes E_{v_{i,2}} \right).$$

We check that $D$ lies in the correct fiber to define a decoration on $\hat{\gamma}$. Taken together, the points $s_{i,j}$ define

$$\underline{s} = (s_{1,1}, s_{1,2}, s_{2,1}, \ldots, s_{n,2}) \in \text{Conf}_{2n}(\mathcal{S}).$$

The fiber of $\text{Conf}_{2n}(\mathcal{E})$ lying over $\underline{s}$ is exactly

$$\text{Conf}_{2n}(\mathcal{E})_{\underline{s}} := \bigotimes_{i=1}^n \left( \mathcal{E}_{s_{i,1}} \otimes \mathcal{E}_{s_{i,2}} \right) = \bigotimes_{i=1}^n \left( E_{v_{i,1}} \otimes E_{v_{i,2}} \right).$$

Definition 8.1.3. Using the notation of the previous discussion, define

$$\mathcal{F}(L) = (\hat{\gamma}, D).$$
This map can be difficult to handle. The process of adding decorations can be clarified by using a local diagram. Let $U \subset \Sigma$ be a simply connected neighborhood of a crossing point $x$. As detailed in Section 7.2, a simply connected neighborhood of $\mathcal{T}^1\Sigma$ lying over $U$ can be defined by specifying a non-zero vector field on $U$. Let $V : U \rightarrow \mathcal{T}^1U$ be such a vector field, and $\mathcal{T}^1U_V = \mathcal{T}^1U \setminus V$ the neighborhood over $U$. The neighborhood is \textit{admissible} as long as for each crossing point $x$ with associated vectors $v_1, v_2 \in \mathcal{T}^1\Sigma_x$, 

$$V_x \neq v_1, v_2.$$ 

The process of adding decorations is seen in Figures 8.2 and 8.3, where for clarity we slide the decorated points slightly off of the crossing point. Notice that the left diagram in each figure shows a positive crossing, while the right diagram shows a negative crossing. Recall that our rectangle always has a clockwise orientation; with this standard, positive and negative crossings in $U$ correspond to the usual notion of positive and negative crossings in the plane.

Let $p$ be the path in $\mathcal{T}^1U_x$ with $p(0) = v_1$, $p(1) = v_2$, and $-v_1, -v_2 \notin \text{Im}(p)$. There are two cases for the vector field $V$. If $V_x \notin \text{Im}(p)$, then $\mathcal{T}^1U_V$ is called a \textit{preferred neighborhood with respect to the crossing at $x$}. If $V_x \in \text{Im}(p)$, then $\mathcal{T}^1U_V$ is called a \textit{non-preferred neighborhood}.

First we suppose that $\mathcal{T}^1U_V$ is a preferred neighborhood, as seen in Figure 8.2, where for clarity we slide the decorated points slightly off of the crossing point. Fix a flat chart over $\mathcal{T}^1U_V$,

$$\phi : \pi^{-1}(\mathcal{T}^1U_V) \rightarrow \mathcal{T}^1U_V \times \mathcal{H},$$

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and recall the notation for the restriction map, for $v \in \mathcal{T}^1U_V$, 

$$\phi_v : E_v \to \mathcal{H}$$

defined for $e \in E_v$ by

$$\phi(e) = (v, \phi_v(e)).$$

The elements $\hat{R}^\pm \in E_v^{\otimes 2}$ have the local coordinates

$$(\phi_{v_1} \otimes \phi_{v_1})(\hat{R}) = \sum_i a_i \otimes b_i, \quad \text{and} \quad (\phi_{v_1} \otimes \phi_{v_1})(\hat{R}^{-1}) = \sum_i S(a_i) \otimes b_i.$$

The isomorphism $\text{id} \otimes \Gamma_p$ is then applied to $\hat{R}^\pm$, turning it into an element of $E_{v_1} \otimes E_{v_2}$. Because $\phi$ is a flat chart and $p$ is a path lying in $\mathcal{T}^1U_V$, we have

$$\left(\phi^{-1} \circ \Gamma_p \circ \phi\right)(v_0, x) = (v_1, x).$$

Thus parallel translation along $p$ does not change the local $\phi$-coordinates of the decoration. That is,

$$((\phi_{v_1} \otimes \phi_{v_2}) \circ (\text{id} \otimes \Gamma_p))(\hat{R}) = \sum_i a_i \otimes b_i,$$

and

$$((\phi_{v_1} \otimes \phi_{v_2}) \circ (\text{id} \otimes \Gamma_p))(\hat{R}^{-1}) = \sum_i S(a_i) \otimes b_i.$$

We also consider the case when $\mathcal{T}^1U_V$ is not preferred, meaning that $V_x \in \text{Im}(p)$. We claim that in local $\phi$-coordinates, sliding the decorated point past $V_x$ applies the map $S^{\pm 2}$ to the decoration, depending on the direction of rotation of the vector in $\mathcal{T}^1U$. This is the same situation as in Lemma 7.2.5, and can be proved using the same argument.
Figure 8.2: Definition of $\tilde{\mathcal{F}}$ at crossings using a preferred neighborhood

Specifically, if $p$ rotates clockwise past $V_x \in \mathcal{T}^1 U_x$, then $\Gamma_p$ in local coordinates is given by $S^2$. If $p$ rotates counter-clockwise past $V_x$, then $\Gamma_p$ is given by $S^{-2}$.

Figure 8.3 shows this case, where again for clarity the decorated point is slid slightly off of the crossing point. For both positive and negative crossings, the decorated vector starts at the over-crossing strands and rotates to the under-crossing strand.

The left diagram shows a positively-oriented link crossing, so we start with the local coordinates

$$(\phi_{v_1} \otimes \phi_{v_1})(\hat{R}) = \sum_i a_i \otimes b_i.$$  

The second decorated point, the one decorated by $b_i$, then slides along the path $p$ to $v_2$. During this slide, it rotates counter-clockwise past $V_x$, and so $S^{-2}$ is applied to the decoration. It should be remembered that this is only applied to the local coordinates, and the underlying decoration transforms as normal by $\Gamma_p$.

The right diagram shows a negatively-oriented link crossing, so we start with the local coordinates

$$(\phi_{v_1} \otimes \phi_{v_1})(\hat{R}^{-1}) = \sum_i S(a_i) \otimes b_i.$$  

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Recall that \( v_1 \) is the vector tangent to the over-crossing strand. When the point decorated by \( b_i \) slides along \( p_i \) it rotates clockwise past \( V_x \). In local coordinates, the decoration \( b_i \) transforms into \( S^2(b_i) \). Then we use the identity

\[
\sum_i S(a_i) \otimes S^2(b_i) = \sum_i S^{-1}(a_i) \otimes b_i
\]

to finish with the local coordinates as shown.

![Diagram](image)

**Figure 8.3:** Definition of \( \mathfrak{S} \) using a non-preferred neighborhood

The congruence class of \( \mathfrak{S}(L) \) does not define a link invariant. However, we show that it is invariant under RII and RIII moves, as well as regular isotopy. Invariance under the first Reidemeister move requires more work, and will be discussed when defining the full invariant.

**Lemma 8.1.4.** Let \( L, L' \) be two link diagrams which can be connected by a sequence of regular isotopy, RII, and RIII moves. Then \( \mathfrak{S}(L) \) and \( \mathfrak{S}(L') \) are congruent decorated curve systems, relative to any basepoint \( * \) whose image is disjoint from.
This lemma is proved by using local diagrams to check the oriented Reidemeister moves. There are, up to rotation, 4 distinct oriented RII moves and 8 distinct oriented RIII moves. To cut down on the work, we use the following lemma, which is Lemma 2.6 in [25].

**Lemma 8.1.5** ([25]). All oriented RIII moves can be generated by a sequence of isotopy, the four oriented RII moves in Figure 8.4, and the two oriented RIII moves shown in Figure 8.5.

![Diagram](image)

Figure 8.4: Oriented RII Moves

*Proof of Lemma 8.1.4.* The fact that the congruence class of $\mathcal{F}(L)$ is invariant under regular homotopy is clear. Any homotopy of the link diagram defines a corresponding homotopy of the resulting decorated curve system.
We use local diagrams to show that the congruence class of $\mathfrak{F}(L)$ is invariant under the necessary oriented Reidemeister moves. For each move, we pick a simply-connected neighborhood $U$ in which the Reidemeister move is applied. Let $L'$ be the link diagram resulting from applying the Reidemeister move inside of $U$, and which is identical to $L$ outside of $U$. Then the resulting decorated curves $\mathfrak{F}(L)$ and $\mathfrak{F}(L')$ are identical outside of $\mathcal{T}^1U$. Thus, we only need to show that the part of the decorated curves lying inside $\mathcal{T}^1U$ are congruent.

The only oriented RII move that we explicitly check is the RIIId move. Checking the other RII variants is a very similar process. To check the RIIId move we use the calculation shown in Figure 8.6. The first diagram represents the result from applying the RIIId move to a trivial link diagram. The second diagram shows the resulting decorated curve after applying $\mathfrak{F}$. Notice that we also chose a suitable vector field $V$. The third diagram shows a congruent decorated curve resulting from multiplying the decorated points, as well as a basic local homotopy in $\mathcal{T}^1U$ to pull
the strands apart. We calculate the resulting decoration in local coordinates:

\[
\sum_{i,j} a_i S(a_j) \otimes b_j S^{-2}(b_i) = \sum_{i,j} S^2(a_i) S(a_j) \otimes b_j b_i \\
= \sum_{i,j} S(a_j S(a_i)) \otimes b_j b_i \\
= (S \otimes id)(\mathcal{R} \mathcal{R}^{-1}) \\
= 1 \otimes 1.
\]

This is the trivial decoration. The decorated curve is then congruent to the decorated curve resulting from applying \( \mathfrak{F} \) to the trivial curve. For the equality in the first line, we use the fact that

\[(id \otimes S^{-2})(\mathcal{R}) = (S^2 \otimes id)(\mathcal{R}).\]

![Diagram](image_url)

Figure 8.6: Checking the R1Id move

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Next we check the Reidemeister III moves. Checking these relations requires the Yang-Baxter Equation
\[ \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \]

The top-left and top-right diagrams of Figure 8.7 show the segment of two link diagram \( L, L' \) which are related by the RIIIa move. We assume that the link diagrams are identical outside of \( U \). The second row of the figure shows a local diagram representing the decorated curves \( \mathfrak{F}(L) \) and \( \mathfrak{F}(L') \). Remember that now these curves represent the tangent lift in \( \mathcal{T}^1U \).

The third row shows a congruent decorated curve resulting from multiplying the decorated points lying on the same component of the curve. It is left unshown that there is a homotopy between these two decorated curves which is the identity outside of \( \mathcal{T}^1U \). We can assume that during this homotopy the decorated points stay inside \( \mathcal{T}^1U_V \), and so the local coordinates remain constant. Now we calculate this decoration using the local coordinates as given in the figure. For the left side, this decoration is:
\[
\sum_{i,j,k} S(a_i) a_k \otimes S^{-2}(b_k) a_j \otimes b_j b_i = \sum_{i,j,k} S(a_i) S(a_k) \otimes S^{-1}(b_k) a_j \otimes b_j b_i \\
= \sum_{i,j,k} S(a_k a_i) \otimes S^{-1}(S(a_j) b_k) \otimes b_j b_i \\
= (S \otimes S^{-1} \otimes \text{id})(\mathcal{R}^{-1}_{23}\mathcal{R}_{12}\mathcal{R}_{13}).
\]

We make crucial use of the identity \( \sum_i S(a_i) \otimes S(b_i) = \sum_i a_i \otimes b_i \).

Now we calculate the decoration for the right diagram, making sure to use the correct ordering of tensor products:
\[
\sum_{i,j,k} a_k S(a_i) \otimes a_j S^{-2}(b_k) \otimes b_i b_j = \sum_{i,j,k} S(a_k a_i) \otimes S^{-1}(b_i S(a_j)) \otimes b_k b_j
\]
\[
(S \otimes S^{-1} \otimes \text{id})(R_{13}R_{12}R_{23}^{-1}).
\]

Finally, we have the form of the Yang-Baxter equation,

\[
R_{23}^{-1}R_{12}R_{13} = R_{13}R_{12}R_{23}^{-1},
\]

which shows that these decorations are identical elements of the relevant tensor product. This finishes our checking of the RIIIa move.

Checking invariance under the RIIIb move is very similarly done. Figure 8.8 shows a component of a link diagram, along with the resulting curve from an application of the RIIIb move. Applying $\delta$ to these link diagrams gives the decorated curves shown in the second row. The third row shows congruent decorated curves resulting from multiplying the decorated points. As before, there is a local homotopy which takes the decorated curves on the left to the curve on the rights. This proof again comes down to calculating the decorations using local coordinates. The decoration in the bottom-left local diagram is

\[
\sum_{i,j,k} a_j a_k \otimes b_j a_i \otimes S^{-2}(b_k)S^{-2}(b_i) = \sum_{i,j,k} a_j S^2(a_k) \otimes b_j S^2(a_i) \otimes b_k b_i
\]

\[
= \sum_{i,j,k} S(S(a_k)a_j) \otimes S(S(a_i)b_j) \otimes b_k b_i
\]

\[
= (S \otimes S \otimes \text{id})(R_{13}^{-1}R_{23}^{-1}R_{12}).
\]

The decoration in the bottom-right diagram, using the same ordering, is

\[
\sum_{i,j,k} a_k a_j \otimes a_i b_j \otimes S^{-2}(b_i)S^{-2}(b_k) = \sum_{i,j,k} S^2(a_k)S(a_j) \otimes S^2(a_i)S(b_j) \otimes b_i b_k
\]

\[
= \sum_{i,j,k} S(a_j S(a_k)) \otimes S(b_j S(a_i)) \otimes b_i b_k
\]

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Figure 8.7: Checking the RIIIa Move
Figure 8.8: Checking the RIIIb Move
\[(S \otimes S \otimes \text{id})(R_{12}R_{23}^{-1}R_{13}^{-1}).\]

To see that these decorations are identical, finishing our check of the RIIb move, we have the form of the Yang-Baxter equation

\[R_{13}^{-1}R_{23}^{-1}R_{12} = R_{12}R_{23}^{-1}R_{13}^{-1}.\]

\[\square\]

### 8.2 The Framed Invariant

In this section, we define the map of framed knots

\[\Gamma_f = \Gamma_f(\Sigma, \pi, b_0) : LD(\Sigma)^m \to SD(\pi_1(\mathcal{T}^1\Sigma, b_0), \Gamma, E_0)^m\]

which depends on the following data:

1. An oriented surface \(\Sigma\) with tangent circle bundle \(\mathcal{T}^1\Sigma\),

2. A ribbon Hopf algebra bundle \(\pi : E \to \mathcal{T}^1\Sigma\) with balanced flat connection,

3. A fixed basepoint \(b_0 \in \mathcal{T}^1\Sigma\).

It is the composition of six maps:

\[\Gamma_f = Q_{\text{Com}} \circ Q_{(f_b)} \circ Q_{\text{DCS}} \circ S \circ \mathcal{F}.

The first map applied is \(\mathcal{F} : LD(\Sigma)^m \to DCS(\mathcal{T}^1\Sigma)^m\), as defined in the previous section.

Second is an arbitrary but fixed choice of basepoints and tethers for each decorated curve system, written \(S : DCS(\mathcal{T}^1\Sigma) \to TDCS(\mathcal{T}^1\Sigma, b_0)\). We will show that \(\Gamma_f\) does not depend on this choice.
Third is the projection which takes a tethered decorated curve system to its congruence class,

\[ Q_{\text{TDCS}} : \text{TDCS}(T^1\Sigma, b_0) \to \mathcal{TDCS}(T^1\Sigma, b_0). \]

Fourth, we use the basepoint \( b_0 \) and Lemma 6.3.6 to define the map

\[ t_{b_0} : \text{TDCS}(T^1\Sigma, b_0)^m \to \text{SD}(\pi_1(T^1\Sigma, b_0), \Gamma, E_0)^{\otimes m}. \]

Fifth is the algebraic quotient map of \( \text{SD} \) by the element \((f_{b_0}, \kappa)\),

\[ Q_{(f_{b_0}, \kappa)}^{\otimes m} : \text{SD}^{\otimes m} \to \mathcal{SD}^{\otimes m}. \]

Finally, we compose with the commutator quotient map

\[ Q_{\text{Com}}^{\otimes m} : \mathcal{SD}^{\otimes m} \to \mathcal{SD}^{\otimes m}. \]

These quotients are defined in Section 4.5, and the algebraic quotient by the element \(([f_{b_0}], \kappa)\) is discussed specifically in Section 7.3. The commutator quotient map was called \( Q^{\text{id}} \) in Section 4.5, to differentiate it from the twisted commutator quotient maps \( Q^{\phi} \). From here on we use the new notation.

We will now prove the main theorem

**Theorem 8.2.1.** The element \( \mathbf{I}^{fr}(L) \) depends only on the framed link class of \( L \) in \( \Sigma \), and so \( \mathbf{I}^{fr} \) defines a map

\[ \mathcal{I}^{fr} : \mathcal{FL}(\Sigma)^m \to \mathcal{SD}^{\otimes m}. \]

Proving this theorem, in steps, will take the rest of this section. Lemma 8.1.4 shows that the congruence class of \( \mathfrak{S} \), and hence \( \mathbf{I}^{fr} \) as well, is invariant under regular
isotopy and the Reidemeister moves RII and RIII. Next, we show that applying $Q((f_{b_0}, \kappa_{b_0}))$ creates invariance under the framed RI move, and finish the section by showing that $Q^{id}$ creates independence of the choice of tethers $s$. This finishes the proof of the theorem.

**Lemma 8.2.2.** The element $Q((f_{b_0}, \kappa_{b_0})) \circ \iota_{b_0} \circ Q_{TDCS} \circ s \circ \mathcal{F}(L) \in SD^m \otimes$ is invariant under the framed RI move.

**Proof.** There are several framed, oriented RI moves. We only check one of these moves; the rest are very similar.

Only one component of the link diagram is used, so we can assume that $L$ is a knot diagram. Let $U$ be a neighborhood such that $U \cap \text{Im}(L)$ is a single embedded curve in $U$. We will apply the framed RI move to $L$ in the neighborhood $U$, yielding a new link diagram $L'$, and show that $L$ and $L'$ map to equivalent elements under $Q((f_{b_0}, \kappa_{b_0})) \circ \iota_{b_0} \circ Q_{TDCS} \circ s \circ \mathcal{F}$.

Figure 8.9 shows $U$ and the piece of $L'$ where we applied the RI move. The second diagram shows the resulting decorated curve after applying $\mathcal{F}$.

The second and third diagrams are congruent decorated curves. To go from the second to the third, we use the equivalence moves for local diagrams as described in Sections 7.2 and 7.3.

The upper pair of decorated points are collected by sliding the $b_i$ point counterclockwise around the loop, and then multiplying it by the $a_i$ decorated point. Sliding $b_i$ around the loop requires an application of the moves shown in Figure 7.8. This transforms the element $b_i$ into $S^{-2}(b_i)$. After multiplying, this yields the single point
decorated by

\[ \sum_i S^{-2}(b_i)a_i = u^{-1}. \]

The bottom pair of decorations are collected similarly. This time, the point decorated by \( S(a_j) \) slides counter-clockwise, turning the local coordinate into \( S^{-1}(a_j) \). Multiplying yields the single point decorated by

\[ \sum_j S^{-1}(a_j)b_j = S(u). \]

Finally, slide the point decorated by \( u^{-1} \) down next to the point decorated by
$S(u)$. This requires applying $S^2$, but $S^2(u^{-1}) = u^{-1}$ and so the local coordinate is unchanged. Multiplying, we get $u^{-1}S(u) = \kappa^{-2}$.

So far, all operations have been done on the level of decorated curves. To finish, we apply $Q_{([f_{b_0}], \kappa_{b_0})} \circ t_{b_0}$ and work in the quotient space $\mathcal{SD}$. Recall from Section 7.3 that this map is invariant under the moves shown in Figure 7.13. Using this move, we can remove the small kinks and replace each with a point decorated by the element $\kappa$. We point out that these are no longer congruent decorated curves - they are decorated curves mapping to the same element under $Q_{([f_{b_0}], \kappa_{b_0})} \circ t_{b_0} \circ Q_{\text{TDCS}}$.

Thus the two kinks become a point decorated by $\kappa^2$. This new decorated point is finally multiplied by the previous decorated point. These decorations cancel, resulting in a trivial, embedded strand with no decorated points.

This shows that $Q_{([f_{b_0}], \kappa_{b_0})} \circ t_{b_0} \circ Q_{\text{TDCS}} \circ \mathfrak{s} \circ \mathfrak{F}$ is invariant under the framed RI move.

\[ \square \]

All that is left is to show that $\mathbf{I}^\text{fr}$ does not depend on the choice of tethers $\mathfrak{s}$. The map $Q_{([f_{b_0}], \kappa_{b_0})} \circ t_{b_0} \circ Q_{\text{TDCS}} \circ \mathfrak{s}$ actually does depend on this choice; however, this dependence disappears after the projection to $\mathcal{SD}$.

**Lemma 8.2.3.** The map $\mathbf{I}^\text{fr}$ does not depend on the choice of tethers $\mathfrak{s}$.

**Proof.** It suffices to prove this for a decorated curve $(c, D)$ with a single component. Let $\ell_1, \ell_2$ be two tethers for $(c, D)$ based at $b_0$, with tethered points

\[ b_i := c(s_i) = \ell_i(0) \text{ for } i = 1, 2. \]
Notice that when choosing these tethers, we also choose the basepoints \( s_1, s_2 \in S^1 \).

We will show that

\[
(Q_{\text{Com}} \circ Q_{([f_{b_0}], \kappa_{b_0})} \circ \iota_{b_0} \circ Q_{\text{TDCS}})(c, \ell_1, D) = (Q_{\text{Com}} \circ Q_{([f_{b_0}], \kappa_{b_0})} \circ \iota_{b_0} \circ Q_{\text{TDCS}})(c, \ell_2, D).
\]

This picture is shown in Figure 8.10. The points \( b_1, b_2 \) cut the curve \( c \) into two segments. Without loss of generality, we may multiply any decorated points which lie on the same segment and thus assume that \( D \) consists of two decorated points lying on \( c \), say \( r \) and \( s \), with decoration

\[
\sum_{\mu} h^\mu_r \otimes h^\mu_s \in E_r \otimes E_s.
\]

This decoration actually lives in the pullback bundle \( c^*E \), but for this proof we use the identity \( (c^*E)_x = E_{c(x)} \).

The points \( b_1, b_2, r, s \) cut the curve \( c \) into four pieces. We call these pieces \( \alpha, \beta, \gamma, \delta \).

We now calculate the image of the two tethered decorated curves under \( \iota_{b_0} \). This image consists of a homotopy class in \( \pi_1(T^1\Sigma, b_0) \) and a decoration in \( E_0 \).

The homotopy classes are simple to find. The tethered decorated curve \( (c, \ell_1, D) \) and \( (c, \ell_2, D) \) yields, respectively, the homotopy classes

\[
[\ell_1^{-1}\delta\alpha\beta\gamma\ell_1] \quad \text{and} \quad [\ell_2^{-1}\beta\gamma\delta\alpha\ell_2].
\]

Now we translate the decoration into an element of \( E_0 \). For \( (c, \ell_1, D) \), we slide the points \( r \) and \( s \) around \( c \), and then down the tether \( \ell_1 \). The decorations transform following the parallel transport map, and then we multiply the decorated points, yielding the single decoration

\[
\sum_{\mu} \Gamma_{\alpha\beta\gamma\ell_1}(h^\mu_r) \cdot \Gamma_{\gamma\ell_1}(h^\mu_s) \in E_0.
\]
Performing this procedure for \((c, \ell_2, D)\) yields the element

\[
\sum_{\mu} \Gamma_{\gamma \delta \alpha \ell_2}(h_{s}^\mu) \cdot \Gamma_{\alpha \ell_2}(h_{r}^\mu) \in E_0.
\]

Putting together the curve and the decoration we have calculated the elements in \(\text{SD}(b_0, \Gamma, E_0)\),

\[
\iota_{b_0}(c, \ell_1, D) = \left( [\ell_1^{-1} \delta \alpha \beta \gamma \ell_1], \sum_{\mu} \Gamma_{\alpha \beta \gamma \ell_1}(h_{r}^\mu) \cdot \Gamma_{\gamma \ell_1}(h_{s}^\mu) \right)
\]

and

\[
\iota_{b_0}(c, \ell_2, D) = \left( [\ell_2^{-1} \beta \gamma \delta \alpha \ell_2], \sum_{\mu} \Gamma_{\gamma \delta \alpha \ell_2}(h_{s}^\mu) \cdot \Gamma_{\alpha \ell_2}(h_{r}^\mu) \right).
\]
Now, we must show that these elements are equivalent under the quotient maps $Q_{\text{Com}} \circ Q_{([f_{b_0}], \kappa_{b_0})}$. We do this explicitly by finding elements

\[
([W], h^\mu_W), ([Z], h^\mu_Z) \in \mathbf{SD}(\pi_1(T^1 \Sigma, b_0), \Gamma, E_0)
\]

such that

\[
\sum_{\mu} ([W], h^\mu_W)([Z], h^\mu_Z) = ([WZ], \sum_{\mu} \Gamma_Z(h^\mu_W) h^\mu_Z) = \iota_{b_0}(c, \ell_1, D)
\]

and

\[
\sum_{\mu} ([Z], h^\mu_Z)([W], h^\mu_W) = ([ZW], \sum_{\mu} \Gamma_W(h^\mu_Z) h^\mu_W) = \iota_{b_0}(c, \ell_2, D).
\]

Notice that, by definition,

\[
([W], h^\mu_W)([Z], h^\mu_Z) - ([Z], h^\mu_Z)([W], h^\mu_W)) \in \text{Com}(\mathbf{SD}).
\]

The quotient $Q_{([f_{b_0}], \kappa_{b_0})}$ takes $\text{Com}(\mathbf{SD})$ into $\text{Com}(\mathbf{SD})$, which is the kernel of $Q_{\text{Com}}$. Thus, if such $W, Z, h^\mu_W$, and $h^\mu_Z$ exists, we can conclude

\[
\left(Q_{\text{Com}} \circ Q_{([f_{b_0}], \kappa_{b_0})} \circ \iota_{b_0}\right)(c, \ell_1, D) = \left(Q_{\text{Com}} \circ Q_{([f_{b_0}], \kappa_{b_0})} \circ \iota_{b_0}\right)(c, \ell_2, D).
\]

In this spirit, we define curves

\[
W = \ell_1^{-1}\delta\alpha\ell_2 , \ Z = \ell_2^{-1}\beta\gamma\ell_1
\]

and elements of $E_0$

\[
h^\mu_W = \Gamma_{\alpha\ell_2}(h^\mu_r) , \ h^\mu_Z = \Gamma_{\gamma\ell_1}(h^\mu_s).
\]

Notice that

\[
[WZ] = [\ell_1^{-1}\delta\alpha\beta\gamma\ell_1] \text{ and } [ZW] = [\ell_2^{-1}\beta\gamma\delta\alpha\ell_2].
\]

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The fact that the element of $E_b$ is the correct one follows from the equations

$$
\Gamma_Z(h^\mu_W)h^\mu_Z = \left(\Gamma_{\ell_2^{-1}\beta\gamma\ell_1} \circ \Gamma_{\alpha\ell_2}\right)(h^\mu_r) \cdot \Gamma_{\gamma\ell_1}(h^\mu_s)
$$

$$
= \Gamma_{\alpha\beta\gamma\ell_1}(h^\mu_r) \cdot \Gamma_{\gamma\ell_1}(h^\mu_s),
$$

and

$$
\Gamma_W(h^\mu_Z) \cdot h^\mu_W = \left(\Gamma_{\ell_1^{-1}\delta\alpha\ell_2} \circ \Gamma_{\gamma\ell_1}\right)(h^\mu_s) \cdot \Gamma_{\alpha\ell_2}(h^\mu_r)
$$

$$
= \Gamma_{\gamma\delta\alpha\ell_2}(h^\mu_s) \cdot \Gamma_{\alpha\ell_2}(h^\mu_r).
$$

\[\square\]

### 8.3 The Unframed Invariant

The map $I^r$ defines an invariant of framed link diagrams. We can modify it to create an invariant of unframed link diagrams as well. Central to the new map is the ribbon element $\nu$. Specifically, we will add new decorated points to curves which are decorated by $\nu^m$. Recall that because $\nu \in \text{RFix}_H^1$, for any point $b \in T^1\Sigma$, there is a well-defined corresponding element $\hat{\nu} \in E_b$.

Let $(c, \ell, D)$ be a tethered decorated curve with one component. Then $D \in \text{Conf}_n(c^*E)$ lives over some tuple of points $\underline{s} \in \text{Conf}_n(S^1)$. Let $s_{n+1}$ be any point disjoint from the decorated points, and $x_{n+1} = c(s_{n+1})$. Then, because $\nu \in \text{RFix}_H^1$, there is a well-defined element

$$
\hat{\nu} \in (c^*E)_{s_{n+1}} = E_{x_{n+1}}.
$$

Then $D \otimes \hat{\nu}^m$ is a well-defined decoration on $c$, lying over $(s_1, \cdots, s_n, s_{n+1})$.  

We claim that \((c, \ell, D \otimes \hat{\nu}^m)\) is independent of the choice of decorated point \(s_{n+1}\). This is because \(\nu\) is central in \(\mathcal{H}\). A point decorated by \(\nu\) then commutes with other decorated points lying on the same curve. Thus a different choice of decorated points \(x'_{n+1}\) yields an equivalent decorated curve.

A second way of showing that the decorated curve is independent of the choice of \(x_{n+1}\) is by looking at the image under the injective map \(\iota_{x_0}\). If \(\iota_{x_0}(c, \ell, D) = ([\ell^{-1}c\ell], h)\), then because \(\nu\) is central,

\[
\iota_{x_0}(c, \ell, D \otimes \hat{\nu}^m) = ([\ell^{-1}c\ell], h\nu^m).
\]

This element clearly does not depend on the decorated point \(x_{n+1}\).

The power of \(\nu\) to be added depends on the self-intersection number. Recall the definition of positive and negative crossings from Definition 8.1.1.

**Definition 8.3.1.** Suppose \(L_i\) is a component of a link diagram \(L\). A crossing point of \(L\) is a self-crossing point of \(L_i\) if both crossing strands are part of \(L_i\). Let \(c^i_+\) denote the number of positive self-crossing points of \(L_i\), and \(c^i_-\) the number of negative self-crossing points. Then the self-intersection number of \(L_i\) is

\[
\text{lk}(L_i) = c^i_+ - c^i_-.
\]

This definition depends on the orientation of \(\Sigma\). Recall that we use a fixed clockwise orientation in our diagrams.

We normalize \(\mathbf{I}_{\Sigma}\) by adding points decorated by \(\nu\). Let \(L\) be a link diagram with \(m\) components, and for \(1 \leq i \leq m\) let \(n_i = \text{lk}(L_i)\). Define

\[
\hat{\nu}^\otimes = \bigotimes_{i=1}^m \hat{\nu}^{n_i} \in E_0^{\otimes m}.
\]
Let $c_i$ denote the curve underlying the link component $L_i$, $\ell_i$ a choice of tether for $c_i$, and
\[ \{\ell_i^{-1} c_i \ell_i \} = (\ell_1^{-1} c_1 \ell_1, \cdots, \ell_n^{-1} c_n \ell_n). \]

Then
\[ (\iota_{b_0} \circ Q_{\text{TDCS}} \circ s \circ \mathfrak{F})(L) = (\{[\ell_i^{-1} c_i \ell_i]\}, D) \in \text{SD}^{\otimes m}, \]
where $D \in E_0^{\otimes m}$, and the $i$-th decorated point comes from the $i$-th curve component.

**Definition 8.3.2.** Using notation from the preceding discussion, define the normalized map
\[ \mathcal{I}^{\text{unf}}(L) = (Q_{\text{Com}} \circ Q_{([j_{b_0}], \kappa b_0)})(\{[\ell_i^{-1} c_i \ell_i]\}, D \hat{\nu} \otimes \Delta). \]

**Remark 8.3.3.** Notice that the only difference between $\mathcal{I}^{\text{unr}}$ and $\mathcal{I}^{\text{fr}}$ is the normalization in $\text{SD}^{\otimes m}$. The decoration of the $i$-th component is multiplied by $\hat{\nu}^{n_i}$. We will show that this normalization makes $\mathcal{I}^{\text{unr}}$ an invariant of unframed links.

**Theorem 8.3.4.** The map $\mathcal{I}^{\text{unf}}$ is invariant under the unframed RI, RII, and RIII moves. It thus defines a map
\[ \mathcal{I}^{\text{unr}} : \mathcal{L}(\Sigma)^m \rightarrow \text{SD}^{\otimes m}. \]

**Proof.** The normalization does not affect regular isotopy, the framed RI move, the RII move, or the RIII move. The only move which changes the self-intersection number of a link component is the unframed RI move. We check the unframed, oriented RI moves using local diagrams.
Consider a component $L_i$ of a link diagram $L$. Let $U$ be a neighborhood such that $U \cap \text{Im}(L_i)$ is an embedded interval. Applying an RI move to this trivial strand creates the new link diagram component $L'_i$; replacing $L_i$ with $L'_i$ defines the new link diagram $L'$. There are four RI moves up to rotation. Only two of these moves are actually necessary, but we show all four.

Figure 8.11 shows four possible images of the RI move applied to a segment of a link diagram. Next to each is its image under $\mathfrak{F}$, calculated using a similar calculus as in the proof of Lemma 8.2.2. For the third diagram, we use the equivalence move from Figure 7.13, replacing a simple loop with the decoration $\kappa$ or $\kappa^{-1}$.

We take a moment to recall the definition of the balancing element,

$$\nu = u\kappa^{-1}.$$  

Applying $S$ to this equation, using $S(\nu) = \nu$ and $S(\kappa) = \kappa^{-1}$, also gives

$$\nu = \kappa S(u).$$

Taking inverses, we have

$$\nu^{-1} = \kappa u^{-1} = S(u^{-1})\kappa^{-1}.$$

Now, the top two RI moves increase the self-intersection number of its component by 1. Because the self-intersection number of $L'_i$ is one more than in $L_i$, the normalization adds one extra point decorated by $\nu$. On the other hand, the decoration shown in the third diagram in both of these cases is

$$\kappa u^{-1} = \nu^{-1} = S(u^{-1})\kappa^{-1}.$$  

In both of these cases, the decoration of $\nu^{-1}$ is exactly canceled by the decoration of $\nu$ which the normalization adds. Thus we see that in this case, $\Gamma^{\text{unf}}(L) = \Gamma^{\text{unf}}(L')$. 

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The two RI moves on the bottom decrease the self-intersection number by 1. Their decorations in the third diagrams are

\[ \kappa S(u) = \nu = u\kappa^{-1}. \]

In this case, the extra loop adds a decoration of \( \nu \), which is canceled by the decorations of \( \nu^{-1} \) which the normalization adds. Again, \( \Gamma^{\text{unf}}(L) = \Gamma^{\text{unf}}(L') \).

## 8.4 Calculation via Group Action

In this section, we introduce a simple method of calculating the maps \( \mathcal{S}^{\text{fr}} \) and \( \mathcal{S}^{\text{unfr}} \). The only input is a basepoint \( b_0 = (x_0, v_0) \in T^1\Sigma \) and a balanced group anti-homomorphism \( \rho : \pi_1(T^1\Sigma, b_0) \to \text{RAut}(\mathcal{H}) \).

By Theorem 5.3.8, this action can be associated to the following data:

1. A ribbon Hopf algebra bundle \( \pi : E \to T^1\Sigma \) with typical fiber \( \mathcal{H} \),

2. A trivialization \( \phi_0 : \pi^{-1}(b_0) \to \mathcal{H} \),

3. And a flat connection on \( E \) with parallel transport map \( \Gamma \) such that for any \([c] \in \pi_1(T^1\Sigma, b_0)\) and \( h \in \mathcal{H} \),

\[
\phi_0^{-1} \circ \Gamma_c \circ \phi_0(h) = \rho_c(h).
\]

The trivialization \( \phi_0 \) allows us to work directly with an element of \( \mathcal{H} \), as opposed to an element of \( E_0 \). Define the semidirect product

\[
\text{SD}(\pi_1(T^1\Sigma, b_0), \rho, \mathcal{H}) = k[\pi_1(T^1\Sigma, b_0)] \ltimes_{\rho} \mathcal{H}.
\]
Figure 8.11: Images of RI moves under $Q((f_{b_0}, \kappa_{b_0}) \circ \tilde{\phi}$
The group action $\rho$ is then enough to define the maps

$$\mathcal{T}^r : \mathcal{F}\mathcal{L}(\Sigma)^m \to SD^{\otimes m}(\pi_1(\mathcal{T}^1\Sigma, b_0), \rho, \mathcal{H}),$$

and

$$\mathcal{T}^{ufr} : \mathcal{L}(\Sigma)^m \to SD^{\otimes m}(\pi_1(\mathcal{T}^1\Sigma, b_0), \rho, \mathcal{H}).$$

The idea behind the calculation is to find a simply connected neighborhood $U$ of $x_0 \in \Sigma$ large enough to contain all the crossings of a link diagram $L$. We then pick a local vector field $V$ over $U$ such that at all crossing points of $L$, $V$ is not parallel to either of the crossing strands.

This yields a local diagram $\mathcal{T}^1U_V$ over $U$. Over this local diagram, there is a unique flat chart

$$\phi : \pi^{-1}(\mathcal{T}^1U_V) \to \mathcal{T}^1U_V \times \mathcal{H}$$

which agrees with $\phi_0$, in the sense that for any $x \in E_0$, $\phi(x) = (b_0, \phi_0(x))$.

This local diagram contains all of the link crossings, and so calculating $\Phi$ can be done directly in this local diagram. Further, the chart $\phi$ can be used to translate all decorations into an element of $\mathcal{H}^{\otimes m}$ for some $m$, as opposed to the Hopf algebra $E_0^{\otimes m}$. Recall that while $E_0$ and $\mathcal{H}$ are isomorphic, this isomorphism is not canonical. The chart $\phi$ arising from $\phi_0$ fixes this isomorphism.

The next step is choosing a set of tethers for the resulting decorated curve. These tethers are used in the calculation, but the element $\mathcal{T}^{fr}(L)$ is independent of this choice. The tethered points must be chosen to be distinct from all decorated points and crossing points. The tethers may be chosen so that their images lie completely in $\mathcal{T}^1U_V$. Because this subset of $\mathcal{T}^1\Sigma$ is simply connected, these tethers are unique.
up to path homotopy. Further, because each tether $\ell_i$ lies in $\mathcal{T}^1U_V$ and $\phi$ is a flat chart, we know that

$$\left(\phi \circ \Gamma_{\ell_i} \circ \phi^{-1}\right)(\ell(0), h) = (b_0, h).$$

Thus we can omit these tethers in local diagrams, and simply assume that they lie in $\mathcal{T}^1U_V$.

After choosing tethers, we apply $\iota_{b_0}$. To calculate this for a component $c_i$ of the curve diagram, slide all decorations around $c_i$, following its orientation, until they reach the chosen basepoint. Then all decorated points are multiplied to create one decoration $d_i \in \mathcal{H}$ on the component $c_i$. This decoration is then slid down the tether to the basepoint $b_0$.

Sliding decorated points inside of $\mathcal{T}^1U_V$ does not change the decoration, because $\phi$ is a flat chart. However, it is sometimes necessary to slide a decoration around a curve which leaves the diagram and then returns. In this case, a non-trivial transformation is applied to the decoration. This is a major change compared to the classical invariant from Chapter 2.

In local $\mathcal{H}$-coordinates, the decoration transforms in a straightforward manner. Suppose we want to slide a decoration around a curve parametrized by $\tilde{c} : [0, 1] \to \mathcal{T}^1U$, with $\tilde{c}(0), \tilde{c}(1) \in \mathcal{T}^1U_V$. The result of sliding a decoration with local coordinates $(\tilde{c}(0), h)$ to the endpoint $\tilde{c}(1)$ is the decoration

$$\left(\phi \circ \Gamma_{\tilde{c}} \circ \phi^{-1}\right)(\tilde{c}(0), h).$$

This can be calculated using the original group action $\rho$. Choose paths $\alpha_0, \alpha_1 : [0, 1] \to \mathcal{T}^1U_V$ such that $\alpha_i(0) = b_0$ and $\alpha_i(1) = \tilde{c}(i), i = 0, 1$. These paths are unique up to homotopy, because $\mathcal{T}^1U_V$ is simply connected, thus the path $\alpha_0^{-1}\tilde{c}\alpha_1$ is unique
up to homotopy. This implies that \( \hat{c} \), along with the neighborhood \( T^1 U_V \), defines a unique homotopy class

\[
\xi = [\alpha_0^{-1} \hat{c} \alpha_1] \in \pi_1(T^1 \Sigma, b_0).
\]

Notice that because \( \alpha_i \) lies inside \( T^1 U_V \), we have

\[
(\phi \circ \Gamma_{\alpha_i} \circ \phi^{-1})(\alpha_i(0), h) = (\alpha_i(1), h).
\]

Then we can calculate the parallel transport along \( \hat{c} \) as follows,

\[
(\phi \circ \Gamma_{\hat{c}} \circ \phi^{-1})(\hat{c}(0), h) = [(\phi \circ \Gamma_{\alpha_0}^{-1} \circ \Gamma_{\alpha_1}^{-1} \circ \Gamma_{\hat{c}} \circ \Gamma_{\alpha_0} \circ \phi^{-1}) \circ (\phi \circ \Gamma_{\alpha_0}^{-1} \circ \phi^{-1})](\hat{c}(0), h)
\]

\[
= [(\phi \circ \Gamma_{\alpha_1} \circ \phi^{-1}) \circ (\phi \circ \Gamma_{\alpha_0}^{-1} \circ \Gamma_{\hat{c}} \circ \Gamma_{\alpha_0} \circ \phi^{-1})](b_0, h)
\]

\[
= (\phi \circ \Gamma_{\alpha_1} \circ \phi^{-1})(b_0, \rho_\xi(h))
\]

\[
= (\hat{c}(1), \rho_\xi(h)).
\]

This process is shown in Figure 8.12. A segment of the curve \( \hat{c} \) is shown exiting the right side of the neighborhood, and re-entering on the left. Where \( \hat{c} \) exits the diagram, a label is written representing the homotopy class \( \xi \in \pi_1(T^1 \Sigma, b_0) \). The decoration in
the first diagram is pushed around the curve, resulting in the transformed decoration shown in the second diagram.

This technique is displayed in the next section, where some examples are calculated.

### 8.5 Examples

Finally, we calculate the invariant for two knot diagrams on the torus $\mathcal{T}$. Figure 8.13 shows two knot diagrams $k_1$ and $k_2$. They are identical except for their crossings, which are switched. We will calculate $ufr^1$ and $ufr^2$ for both of these diagrams.

The value of our new invariant is seen in this example. There are no non-trivial knot diagrams in $\mathbb{R}^2$ with only two crossings. The knots $k_1$ and $k_2$ are only non-trivial because of their topological properties. This "topological non-triviality" is seen by the dependence of $ufr^1(k_1)$ and $ufr^2(k_2)$ on the variable $t$.

A naive invariant might be to somehow project a link diagram on $\Sigma$ onto $\mathbb{R}^2$, and then apply the classical invariant. This would lose the topological knotting information. This is seen in our example by setting $t = 1$, or equivalently choosing $\rho$ to be the trivial group anti-homomorphism. In this case, $ufr^1(k_1)$ and $ufr^2(k_2)$ cannot distinguish $k_1$ or $k_2$ from the trivial knot in $\mathcal{T}$. Our invariant is thus stronger than the classical invariant.

We view $\mathcal{T}$ as a rectangle with opposite sides identified, as in Figure 8.14. The figure also shows the curves $\alpha$, which is dotted, and $\beta$, which is dashed. As usual, we view these as tangents lift curves in $\mathcal{T} \mathcal{T}^1 \mathcal{T}$. The fundamental group $\pi_1(\mathcal{T}^1 \mathcal{T}, b_0)$ is a
free abelian group generated by the elements $[\alpha], [\beta], [f_{b_0}]$. Recall that a right action 

$$\rho : \pi_1(\mathcal{T}^1\mathbb{T}, b_0) \to \text{RAut} (\mathcal{H}),$$

is balanced if $\rho_{[f_{b_0}]} = S^2$. Thus a balanced right action is defined by the elements $\rho_{[\alpha]}$ and $\rho_{[\beta]}$.

Because the fundamental group is abelian in this case, we are free to choose the maps $\rho_{[\alpha]}$ and $\rho_{[\beta]}$. For other surfaces, these maps would be required to satisfy certain relationships involving $\rho_{[f_{b_0}]} = S^2$.

For this torus, pick the simply connected neighborhood $U$ to be the interior of the rectangle. The dotted arrow on the top-left of Figure 8.14 represents the vector field $V$, which is actually defined on the entire surface. This is a special case for the torus, as arbitrary surfaces generally do not have any smooth non-singular vector field.

Fix, for now, a balanced action $\rho$ and a basepoint $b_0 \in \mathcal{T}^1\mathbb{T}$. This defines, up to isomorphism, a flat ribbon Hopf algebra bundle $\pi : E \to \mathcal{T}^1\mathbb{T}$ with parallel transport $\Gamma$, as well as a local flat chart

$$\phi : \pi^{-1}(\mathcal{T}^1U_V) \to \mathcal{T}^1U_V \times \mathcal{H}$$

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such that for any \([c] \in \pi_1(\mathcal{T}^1\mathbb{T}, b_0)\),

\[
(\phi \circ \Gamma_c \circ \phi^{-1})(b_0, h) = (b_0, \rho_{[c]}(h)).
\]

The local \(\phi\)-coordinates will be used to calculate \(\mathcal{F}^{fr}\).

The calculation for \(k_1\) is shown in Figure 8.15. The first diagram shows a local
diagram containing both crossing points of the knot. The second diagram shows the
decorated curve \(\mathfrak{F}(k_1)\), along with the basepoint \(b_0 \in \mathcal{T}^1\mathbb{T}\). For the third diagram,
a homotopy is applied to center the loop in the frame, and decorations are collected
on the right side of the curve. Two of the decorated points slide off the right side of
the rectangle and come back on the left. The curve they follow during this slide is
homotopic to the generator \(\alpha\), and so \(\rho_{[\alpha]}\) is applied to the local coordinate for each
decorated point. These same two decorated points slide through a local maximum,
where the curve is parallel to the vector field \(V\). Because of this, \(S^2\) is also applied
to these decorations. Multiplying these together, we get the Hopf algebra element

\[
\sum_{i,j} (\rho_{[\alpha]} \circ S^2)(b_j)(\rho_{[\alpha]} \circ S)(a_i)b_iS(a_j) \in \mathcal{H}.
\]

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We now project this to the quotient space $S\mathcal{D}$. First we simplify the element by moving the $(\rho_{[a]} \circ S^2)(b_j)$ term to the end of the word. Geometrically, this amounts to sliding the decorated point clockwise around the curve. When the point moves past the local critical point, $S^{-2}$ is applied to the local coordinate. This changes the element in $\mathcal{H}$, but is equivalent in the quotient space $S\mathcal{D}$. The new decoration is

\[ A = \sum_{i,j} (\rho_{[a]} \circ S)(a_i)b_iS(a_j)\rho_{[a]}(b_j) \]

\[ = \mu^3 \circ (\rho_{[a]} \otimes \text{id} \otimes \text{id} \otimes \rho_{[a]})(\mathcal{R}^{-1} \otimes \mathcal{R}^{-1}). \]

The homotopy class of the curve is the fiber generator $[f] \in \pi_1(T^1\Sigma, b_0)$, thus we get

\[ \mathfrak{I}^{fr}(k_1) = [[f], A] \in S\mathcal{D}. \]

The same calculation, with all the same steps, is done for $k_2$ and shown in Figure 8.16. This time, the collected decoration is

\[ \sum_{i,j} (S^2 \circ \rho_{[a]})(a_j)(\rho_{[a]})(b_i)a_i b_j. \]

This time, we will move the $b_j$ term to the front of the product, changing it to $S^2(b_j)$. Using the fact that

\[ \sum_j S^2(a_j) \otimes S^2(b_j) = \sum_j a_j \otimes b_j, \]

we get the decoration

\[ B = \sum_{i,j} b_j \rho_{[a]}(a_j)\rho_{[a]}(b_i)a_i = \mu^3 \circ (\text{id} \otimes \rho_{[a]} \otimes \rho_{[a]} \otimes \text{id})(\mathcal{R}_{21} \otimes \mathcal{R}_{21}), \]

where, again, $R_{21} = \sum_i b_i \otimes a_i$. The underlying curve is again $[f]$, and so

\[ \mathfrak{I}^{fr}(k_2) = [[f], B] \in S\mathcal{D}. \]

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Figure 8.15: Finding $\mathcal{J}^{tr}(k_1)$.

Now we normalize these elements to calculate the values of $\mathcal{J}^{ufr}$. Notice that the self-linking numbers for these diagrams are $\text{lk}(k_1) = 2$ and $\text{lk}(k_2) = -2$. Then we multiply the decoration by correct power of $\nu$, yielding

$$\mathcal{J}^{ufr}(k_1) = [[f], A\nu^2]$$

and

$$\mathcal{J}^{ufr}(k_2) = [[f], B\nu^{-2}]$$.

So far we have only dealt with an arbitrary ribbon Hopf algebra. Now we specialize to the Hopf algebra $\mathcal{N}$. In Example 4.5.7, we found various quotient spaces for the Hopf algebra $\mathcal{N}$. We will use this to finish the example.

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First we set \( \rho \). As we said, this depends on the choice of \( \rho_\alpha \) and \( \rho_\beta \), where \( \alpha, \beta \) are the curves shown in 8.14. We set \( \rho_\beta = \text{id} \) and \( \rho_\alpha = \phi_t \), the ribbon automorphism of \( \mathcal{N} \) defined by

\[
\phi_t(\eta) = t\eta, \quad \phi_t(\theta) = t^{-1}\theta \quad \text{and} \quad \phi_t(K) = K.
\]

At this point we claim that

\[
A = 1 + (t + t^{-1})\eta\theta \quad \text{and} \quad B = 1 - (t + t^{-1})\eta\theta.
\]

This is shown by explicit computation at the end of this section. For now we continue with this assumption.
So far, we have

$$\mathcal{U}_{\text{fr}}(k_1) = [[f], \nu^2 A] \text{ and } \mathcal{U}_{\text{fr}}(k_2) = [[f], \nu^{-2} B].$$

Both elements are of the form \([f, h] \in \mathcal{S}\mathcal{D}\) for some \(h \in \mathcal{H}\). This element and \([(1), \kappa^{-1} h]\) project to the same class in \(\mathcal{S}\mathcal{D}\), where \([1]\) denotes the identity element of the fundamental group. Further, recall that for \(\mathcal{N}\),

$$\kappa = K = \kappa^{-1}, \quad \nu = 1 - \eta \theta, \quad \text{and } \nu^{-1} = 1 + \eta \theta.$$

Thus

$$\mathcal{U}_{\text{fr}}(k_1) = [(1), \kappa \nu^2 A] = [(1), K - (2 - t - t^{-1}) \eta \theta K]$$

while

$$\mathcal{U}_{\text{fr}}(k_2) = [(1), \kappa \nu^{-2} B] = [(1), K + (2 - t - t^{-1}) \eta \theta K].$$

We have to make sure that these elements are non-trivial in \(\mathcal{S}\mathcal{D}\). We use the notation from Example 4.5.7 in Section 4.5. In particular, recall the following linear subspaces of \(\mathcal{S}\mathcal{D}\):

$$F_1 = \{1\} \otimes \mathcal{N}$$

$$\text{Com}_1 = \text{Com}(\mathcal{S}\mathcal{D}) \cap F_1$$

$$X_1 = \langle (1, xy - yx) : x, y \in \mathcal{N} \rangle$$

$$Y_1 = \langle ([1], \rho_g(x) - x) : g \in \pi_1(T^1 \Sigma, b_0), x \in \mathcal{N} \rangle$$

The other linear subspace \(T_1\) is the trivial space, as \(\mu_1\) consists of only one element. We showed in Example 4.5.7 that

$$\text{Com}_1 = X_1 + Y_1 = X_1 = \{1\} \otimes \text{Com}(\mathcal{N}) \subset F_1.$$
Thus, there is an isomorphism

\[ F_1 / \text{Com}_1 \to \mathcal{N} / \text{Com}(\mathcal{N}). \]

We also showed that \( \text{Com}(\mathcal{N}) \) is spanned by the elements

\[ \text{Com}(\mathcal{N}) = \langle \eta, \theta, \eta K, \theta K, \eta \theta \rangle \]

In particular, we see that

\[
(K - (2 - t - t^{-1})\eta\theta K) - (K + (2 - t - t^{-1})\eta\theta K) = (-4 + 2t + 2t^{-1})\eta\theta K \notin \text{Com}(\mathcal{N}).
\]

Thus \( k_1 \) and \( k_2 \) are not equivalent knots. This invariant also distinguishes the knots \( k_1 \) and \( k_2 \) from the trivial knot, as the value of the trivial link \( L \) which has clockwise orientation is

\[
\mathcal{I}^{ufr}(L) = [[f], 1] = [[1], \kappa^{-1}] = [[1], K].
\]

To finish this section, we calculate \( A \) and \( B \). First, we have the \( R \)-matrix and its inverse:

\[
\mathcal{R} = \frac{1}{2} [1 \otimes 1 - \eta \otimes \theta K + K \otimes 1 - \eta K \otimes \theta K \\
+ 1 \otimes K - \eta \otimes \theta - K \otimes K + \eta K \otimes \theta]
\]

\[
\mathcal{R}^{-1} = \frac{1}{2} [1 \otimes 1 - \eta K \otimes \theta K + K \otimes 1 + \eta \otimes \theta K \\
+ 1 \otimes K - \eta K \otimes \theta - K \otimes K - \eta \otimes \theta]
\]

To calculate \( A \), we first calculate the elements

\[
\mu \circ (\phi_i \otimes \text{id})(\mathcal{R}^{-1}) = \frac{1}{2} [1 + t\eta \theta + K + t\eta \theta K + K + t\eta \theta K - 1 - t\eta]
\]
\[ = K + t\theta \eta K, \]

and

\[
\mu \circ (\phi_t \otimes \text{id})(R_2^{-1}) = \frac{1}{2} \left[ 1 + t^{-1}\eta\theta + K + t^{-1}\eta\theta K + K + t^{-1}\eta\theta K - 1 - t^{-1}\eta\theta \right]
\]

\[ = K + t^{-1}\eta\theta K. \]

Finally we find \( A \) by multiplying these, yielding

\[ A = (K + t\eta\theta K)(K + t^{-1}\eta\theta K) = 1 + (t + t^{-1})\eta\theta. \]

We calculate \( B \) similarly. First,

\[
\mu \circ (\phi_t \otimes \text{id})(R_{21}) = \frac{1}{2} \left[ 1 - t^{-1}\theta K\eta + K - t^{-2}\theta K\eta K + K - t^{-2}\theta\eta - 1 + t^{-1}\theta\eta K \right]
\]

\[ = \frac{1}{2} \left[ 2K + 2t^{-1}\theta\eta K \right]
\]

\[ = K - t^{-1}\theta\eta K, \]

and,

\[
\mu \circ (\text{id} \otimes \phi_t)(R_{21}) = \frac{1}{2} \left[ 1 - t\theta K\eta + K - t\theta K\eta K + K - t\theta\eta - 1 + t\theta\eta K \right]
\]

\[ = K - t\eta\theta K. \]

Multiplying these yields

\[ B = (K - t^{-1}\eta\theta K)(K - t\eta\theta K) = 1 - (t + t^{-1})\eta\theta. \]
# GLOSSARY OF NOTATION

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<tbody>
<tr>
<td>$\text{Com}(A)$</td>
<td>The commutator subspace of the algebra $A$, defined as the subspace spanned by all elements of the form $xy - yx$ for $x, y \in A$.</td>
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<tr>
<td>$\text{Com}_\phi(A)$</td>
<td>The twisted commutator subspace of the algebra $A$ by a automorphism $\phi \in \text{Aut}(A)$. Defined as the subspace spanned by all elements of the form $xy - \phi(y)x$ for $x, y \in A$.</td>
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<tr>
<td>$\text{Conf}_n(X)$</td>
<td>For a manifold $X$, $\text{Conf}_n(X) = {(x_1, \cdots, x_n) : x_i \neq x_j \text{ for all } i \neq j}$.</td>
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<tr>
<td>$\text{Conf}_n(E)$</td>
<td>For a smooth vector bundle $\pi : E \to X$, $\text{Conf}_n(E) \to \text{Conf}_n(X)$ is the $n$-th configuration bundle of $E$.</td>
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<tr>
<td>$\text{CS}(X)$</td>
<td>The set of smooth curve systems in the smooth manifold $X$.</td>
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<tr>
<td>$\text{DC}(E)$</td>
<td>The set of decorated smooth curves in the smooth oriented Hopf algebra bundle $E \to X$.</td>
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<tr>
<td>$\text{DCS}(E)$</td>
<td>The set of decorated curve systems in the smooth oriented Hopf algebra bundle $E \to X$.</td>
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<tr>
<td>$\mathcal{DCS}(E)$</td>
<td>The congruence classes of decorated smooth curves in the smooth oriented Hopf algebra bundle $E \to X$.</td>
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<tr>
<td>$\text{Dec}(E)$</td>
<td>The set of decorations in the smooth Hopf algebra bundle $\pi : E \to S^1$.</td>
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<tr>
<td>$\mathcal{Dec}(E, \ast)$</td>
<td>The congruence classes of decorations in the smooth Hopf algebra bundle $\pi : E \to S^1$.</td>
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<td>$\mathcal{Dec}(\xi, \ast)$</td>
<td>For $E, E' \to S^1$ two ribbon Hopf algebra bundles and $\xi : E \to E'$ a bundle isomorphism, $\mathcal{Dec}(\xi, \ast) : \mathcal{Dec}(E, \ast) \to \mathcal{Dec}(E, \xi(\ast))$ is a bijection.</td>
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<tr>
<td>$\text{ECS}(M)$</td>
<td>The set of embedded curve systems in a smooth manifold $M$.</td>
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<tr>
<td>$\mathcal{FL}(\Sigma)$</td>
<td>The set of equivalence classes of framed link diagrams on the smooth oriented surface $\Sigma$.</td>
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<tr>
<td>$\mathcal{F}$</td>
<td>The flattening map $\text{LD}(\Sigma) \to \text{DCS}(E, b_0)$ for a ribbon Hopf algebra bundle $E \to \mathcal{T}^1\Sigma$.</td>
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<tr>
<td>$I^{fr}$</td>
<td>The map $\mathcal{L}(\Sigma)^m \to \mathcal{D} \otimes^m$ which induces $\mathcal{J}^{fr}$.</td>
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<tr>
<td>$\mathcal{J}^{fr}$</td>
<td>The framed link invariant $\mathcal{F}(\Sigma)^m \to \mathcal{D} \otimes^m$.</td>
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<tr>
<td>$I^{unf}$</td>
<td>The map $\mathcal{L}(\Sigma)^m \to \mathcal{D} \otimes^m$ which induces $\mathcal{J}^{unf}$.</td>
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<tr>
<td>$\mathcal{J}^{unf}$</td>
<td>The link invariant $\mathcal{L}(\Sigma)^m \to \mathcal{D} \otimes^m$.</td>
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<tr>
<td>$\iota(E,M,b_0,\Gamma)$</td>
<td>The injection $\mathcal{TDCS}(E,b_0)^1 \hookrightarrow \mathcal{SD}(\pi_1(M,b_0),\Gamma,E_0)$. Often called $i_{b_0}$ for short.</td>
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<tr>
<td>$\mathcal{K}(\Sigma)$</td>
<td>The set of knots, meaning a link with one component, on the smooth oriented surface $\Sigma$.</td>
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<tr>
<td>$\mathcal{K}(M)$</td>
<td>The set of equivalence classes of knots, meaning links with one component, in the smooth oriented 3-manifold $M$.</td>
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<tr>
<td>$\mathcal{L}(M)$</td>
<td>The set of equivalence classes of links in the smooth oriented 3-manifold $M$.</td>
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<tr>
<td>$\mathcal{LD}(\Sigma)$</td>
<td>The set of link diagrams on the smooth oriented surface $\Sigma$.</td>
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</tr>
<tr>
<td>$\mathcal{L}(\Sigma)$</td>
<td>The set of equivalence classes of link diagrams on the smooth oriented surface $\Sigma$.</td>
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</tr>
<tr>
<td>$\text{RFix}^n_\mathcal{H}$</td>
<td>For a ribbon Hopf algebra $\mathcal{H}$, $\text{RFix}^n_\mathcal{H} \subset \mathcal{H} \otimes^n$ is the subspace of elements fixed by $\phi \otimes^n$ for all $\phi \in \text{RAut}(\mathcal{H})$.</td>
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<tr>
<td>$S^*_\Psi$</td>
<td>The standard Hopf algebra bundle over $S^1$ with typical fiber $\mathcal{H}$ defined by $\Psi \in \text{RAut}(\mathcal{H})$.</td>
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</tr>
<tr>
<td>$\text{SD}(G, \rho, \mathcal{H})$</td>
<td>For $G$ a discrete group, $\mathcal{H}$ a Hopf algebra, and $\rho : G \to \text{Aut}(\mathcal{H})$ a group anti-homomorphism, $\text{SD}(G, \rho, \mathcal{H}) = k[G] \rtimes \rho \mathcal{H}$ is the semi-direct product Hopf algebra. Often referred to with the shorthand $\text{SD}$.</td>
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<tr>
<td>$\text{SD}(G, \rho, \mathcal{H})$</td>
<td>Refers to the algebraic quotient space of $\text{SD}(G, \rho, \mathcal{H})$ defined by a central, group-like element. It is a Hopf algebra, and often referred to with the shorthand $\text{SD}$.</td>
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<tr>
<td>$\delta \mathcal{D}(G, \rho, \mathcal{H})$</td>
<td>Refers to the commutator quotient space of $\mathcal{D}$. It is a vector space, but not a Hopf algebra. Often referred to with the shorthand $\delta \mathcal{D}$.</td>
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</tr>
<tr>
<td>$\mathcal{T}^1\Sigma$</td>
<td>The unit tangent subbundle of the tangent bundle $T\Sigma$, where $\Sigma$ is a smooth oriented surface.</td>
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<tr>
<td>$\mathcal{T}^1U_V$</td>
<td>Defined as $\mathcal{T}^1U - (\text{Im}V)$ where $U$ is a smooth oriented surface and $V$ is a section of the unit tangent subbundle $\mathcal{T}^1U \to U$.</td>
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<tr>
<td>TCS(X, b_0)</td>
<td>The set of tethered curve systems in the smooth manifold (X) based at (b_0 \in X).</td>
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<tr>
<td>TCS((X, b_0))</td>
<td>The homotopy classes of tethered curve systems in the smooth manifold (X) based at (b_0 \in X).</td>
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<tr>
<td>TDCS((E, b_0))</td>
<td>The set of decorated tethered curves in the smooth oriented Hopf algebra bundle (E \to X) based at (b_0 \in X).</td>
<td>100</td>
</tr>
<tr>
<td>TDCS((E, b_0))</td>
<td>The congruence classes of decorated smooth curves on the smooth oriented Hopf algebra bundle (E \to X) based at (b_0 \in X).</td>
<td>104</td>
</tr>
</tbody>
</table>
BIBLIOGRAPHY


