Automorphic $L$-Functions and Their Derivatives

Dissertation

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By

Shenhui Liu, B.S., M.S.

Graduate Program in Mathematics

The Ohio State University

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Dissertation Committee:

Wenzhi Luo, Advisor

James Cogdell

Roman Holowinsky
Abstract

In this dissertation we investigate automorphic $L$-functions and their derivatives at the central point of the critical strip, by the method of moments and/or the mollification method à la Selberg. In Chapter 1, we introduce basic concepts and facts of the families of automorphic forms considered in this work and state the main results.

In Chapter 2, we study the average and nonvanishing of the central $L$-derivative values of $L(s, f)$ and $L(s, f_{K_D})$ for $f$ in an orthogonal Hecke eigenbasis $\mathcal{H}_{2k}$ of weight $2k$ cusp forms for $SL(2, \mathbb{Z})$ for large odd $k$. Here $f_{K_D}$ is the base change of $f$ to an imaginary quadratic field $K_D = \mathbb{Q}(\sqrt{D})$ with fundamental discriminant $D$. We prove asymptotic formulas for the first and second moments of $L'(\frac{1}{2}, f)$, as well as the first moment of $L'(\frac{1}{2}, f_{K_D})$, over $\mathcal{H}_{2k}$ as odd $k \to \infty$. Further, we employ mollifiers to establish that for sufficiently large $k$ there are positive proportion of Hecke eigenforms $f$ in $\mathcal{H}_{2k}$ with $L'(\frac{1}{2}, f) \neq 0$. We also give applications of our results to Heegner cycles of high weights of the modular curve $X_0(1)$.

In Chapter 3, we establish an asymptotic formula with arbitrary power saving for the first moment of $L(\frac{1}{2}, \text{sym}^2 f)$ for $f \in \mathcal{H}_{2k}$ as $k \to \infty$, where $L(s, \text{sym}^2 f)$ denotes the symmetric square $L$-function of $f$. We extract two secondary main terms from the best known error term $O(k^{-\frac{1}{2}})$ in the asymptotic formula for the first moment of $L(\frac{1}{2}, \text{sym}^2 f)$. Specifically, the secondary main terms involve central values of Dirichlet
$L$-functions of characters $\chi_{-4}$ and $\chi_{-3}$ and depend on the values of $k \pmod{2}$ and $k \pmod{3}$, respectively.

In Chapter 4 we study the central $L$-values of Maass forms of weight 0 for $SL(2, \mathbb{Z})$ and establish a positive-proportional nonvanishing result of such values in the aspect of large spectral parameter in short intervals, which is qualitatively optimal in view of Weyl’s law. As an application of this result and a formula of Katok–Sarnak, we give a nonvanishing result on the first Fourier coefficients of Maass forms of weight $\frac{1}{2}$ for $\Gamma_0(4)$ in the Kohnen plus space.
Dedicated to my parents.
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Vita

2008 .............................................. B.S. in Mathematics,  
Shandong University, Jinan, China

2011 .............................................. M.S. in Mathematics,  
Shandong University, Jinan, China

2011-present ................................. Graduate Associate,  
The Ohio State University.

Publications

“Mollification and non-vanishing of automorphic $L$-functions on GL(3)” (joint with  


“On central $L$-derivative values of automorphic forms”. To appear in Mathematische  
Zeitschrift.

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Fields of Study

Major Field: Mathematics

Specialization: Analytic Number Theory
## Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>v</td>
</tr>
<tr>
<td>Vita</td>
<td>vii</td>
</tr>
<tr>
<td>Chapter 1: Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Preliminaries</td>
<td>2</td>
</tr>
<tr>
<td>1.1.1 The upper half-plane ( \mathbb{H} ) and group action of ( SL(2, \mathbb{R}) )</td>
<td>2</td>
</tr>
<tr>
<td>1.1.2 Holomorphic modular forms for ( \Gamma_0(1) )</td>
<td>4</td>
</tr>
<tr>
<td>1.1.3 Nonholomorphic automorphic forms for ( \Gamma_0(1) )</td>
<td>8</td>
</tr>
<tr>
<td>1.2 Main results</td>
<td>13</td>
</tr>
<tr>
<td>Chapter 2: Central ( L )-derivative values of holomorphic modular forms</td>
<td>16</td>
</tr>
<tr>
<td>2.1 Introduction</td>
<td>16</td>
</tr>
<tr>
<td>2.1.1 Asymptotic formulas and nonvanishing</td>
<td>17</td>
</tr>
<tr>
<td>2.1.2 Applications to Heegner cycles of high weights</td>
<td>21</td>
</tr>
<tr>
<td>2.1.3 Structure of this chapter</td>
<td>23</td>
</tr>
<tr>
<td>2.2 Preparation</td>
<td>23</td>
</tr>
<tr>
<td>2.2.1 Functional equations</td>
<td>24</td>
</tr>
<tr>
<td>2.2.2 Approximate functional equations</td>
<td>24</td>
</tr>
<tr>
<td>2.2.3 The mollifiers ( M_f )</td>
<td>26</td>
</tr>
<tr>
<td>2.2.4 Petersson’s formula</td>
<td>27</td>
</tr>
<tr>
<td>2.2.5 Some properties of ( J_{\nu}(x) )</td>
<td>27</td>
</tr>
<tr>
<td>2.2.6 The Estermann zeta-functions</td>
<td>28</td>
</tr>
<tr>
<td>2.2.7 Results on two complex integrals</td>
<td>29</td>
</tr>
<tr>
<td>2.3 The first moment of ( L'(\frac{1}{2}, f_{KD}) )</td>
<td>30</td>
</tr>
<tr>
<td>2.3.1 Diagonal contribution ( D )</td>
<td>30</td>
</tr>
</tbody>
</table>
2.3.2 Off-diagonal contribution $J$ ........................................... 31
2.4 The mollified first moment of $L'(\frac{1}{2}, f)$ ........................................... 34
  2.4.1 Diagonal contribution $D_1$ ........................................... 35
  2.4.2 Off-diagonal contribution $J_1$ ........................................... 37
2.5 The mollified second moment of $L'(\frac{1}{2}, f)$ ........................................... 38
  2.5.1 Diagonal contribution $D_2$ ........................................... 38
  2.5.2 Off-diagonal contribution $J_2$ ........................................... 41
2.6 The first and second moments of $L'(\frac{1}{2}, f)$ ........................................... 46
  2.6.1 Proof of Theorem 2.1 (a) ........................................... 46
  2.6.2 Proof of Theorem 2.1 (b) ........................................... 47
2.7 Proof of bounds (A) and (B) ........................................... 48

Chapter 3: The first moment of symmetric square $L$-functions ................. 58
  3.1 Introduction ........................................... 58
  3.2 Preparation ........................................... 62
    3.2.1 Functional equations ........................................... 62
    3.2.2 Approximate functional equation ........................................... 63
    3.2.3 Petersson’s formula ........................................... 63
    3.2.4 An integral representation of $J_\nu (x)$ ........................................... 64
    3.2.5 A complex integral ........................................... 64
  3.3 The diagonal contribution ........................................... 65
  3.4 The off-diagonal contribution ........................................... 67
    3.4.1 Partition of $O$ ........................................... 67
    3.4.2 Treatment of $O_R$ ........................................... 68
    3.4.3 Treatment of $O_I$ ........................................... 70

Chapter 4: Central $L$-values of Maass forms ........................................... 78
  4.1 Introduction ........................................... 78
    4.1.1 Nonvanishing ........................................... 78
    4.1.2 An application ........................................... 79
    4.1.3 Proof of Theorem 4.1 and outline of the chapter ........................................... 82
  4.2 Preparation ........................................... 86
    4.2.1 Approximate functional formula ........................................... 87
    4.2.2 Kuznetsov trace formulas ........................................... 88
    4.2.3 Motohashi’s formula ........................................... 89
    4.2.4 Mollifiers ........................................... 90
  4.3 The mollified first moment ........................................... 90
    4.3.1 Diagonal contribution $D$ ........................................... 91
    4.3.2 Continuous spectrum part $C$ ........................................... 93
    4.3.3 Off-diagonal sum $O^+$ ........................................... 93
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.3.4</td>
<td>Off-diagonal sum $O^-$</td>
<td>95</td>
</tr>
<tr>
<td>4.4</td>
<td>The mollified second moment</td>
<td>100</td>
</tr>
<tr>
<td>4.4.1</td>
<td>Case $\sum_1$</td>
<td>101</td>
</tr>
<tr>
<td>4.4.2</td>
<td>Case $\sum_2$</td>
<td>102</td>
</tr>
<tr>
<td>4.4.3</td>
<td>Cases $\sum_\nu (\nu = 3, \ldots, 7)$</td>
<td>105</td>
</tr>
<tr>
<td>4.5</td>
<td>A negative moment of $L(1, \text{sym}^2 u_i)$</td>
<td>106</td>
</tr>
<tr>
<td>4.6</td>
<td>A discussion on Barnes’ formula</td>
<td>109</td>
</tr>
<tr>
<td>4.7</td>
<td>A uniform estimate for $K_{it}(x)$ ($t &gt; 0, \ x &gt; 1$)</td>
<td>112</td>
</tr>
</tbody>
</table>

Bibliography                                                                                       114
Chapter 1: Introduction

Two fascinating areas, number theory, and theory of automorphic forms and their \( L \)-functions, are connected by the Langlands Program, which roughly depicts, among other things, that any \( L \)-function arising from number theory would be the same as certain automorphic \( L \)-function on some \( \text{GL}(n) \). One important aspect of the theory of \( L \)-functions is the study of an \( L \)-function or its derivatives at the central point of symmetry of its functional equation. The investigation of the nonvanishing property of such values is very attractive and useful for two reasons. One is that the nonvanishing is deeply (and conjecturally) connected with the distribution of zeros of \( L \)-functions. The other is that such values are sometimes (expected to be) special values, as in the case of the \( L \)-function associated to an elliptic curve over a number field, for which the Birch–Swinnerton-Dyer conjecture relates such values with invariants of the curve, such as the order of the torsion group, the order of the Tate–Shafarevich group, and the canonical heights of a basis of rational points.

A major approach to study the nonvanising problem for families of automorphic \( L \)-functions is via the method of moments, since the pioneering works of Duke [8] and Iwaniec–Sarnak [23, 24], the latter two of which further employed the mollification method. The method of moments and the mollification method have been very fruitful
in yielding positive proportional nonvanishing results in families (see, for example, [23, 30, 58, 31, 32, 33, 47, 55, 34, 5, 27, 45]).

In this dissertation we apply the method of moments and/or the mollification method to study central $L$- or $L$-derivative values for two families of automorphic forms, the holomorphic modular forms and the Maass forms, both for the modular group $SL(2, \mathbb{Z})$. In the rest of this chapter, we review basic facts on these automorphic forms and their $L$-functions and introduce the main results.

1.1 Preliminaries

In this section we introduce basic concepts and results, based on several references [7, 48, 20, 50, 21].

1.1.1 The upper half-plane $\mathbb{H}$ and group action of $SL(2, \mathbb{R})$

Let $\mathbb{H} = \{x + iy \mid x \in \mathbb{R}, y > 0\}$ be the upper-half plane. The group $SL(2, \mathbb{R})$ acts on $\mathbb{H}$ via Möbius transformations

$$\gamma z = \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R});$$

this action can be extended to the boundary $\mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$ by setting

$$\gamma \infty = \begin{cases} \frac{a}{c}, & \text{if } c \neq 0; \\ \infty, & \text{if } c = 0. \end{cases}$$

On $\mathbb{H}$ we have the metric $ds^2$ and the measure $d\mu(z)$ given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \quad \text{and} \quad d\mu(z) = \frac{dx \, dy}{y^2},$$

both of which are invariant under the action of $SL(2, \mathbb{R})$.  

2
The subgroups of $SL(2, \mathbb{R})$ of interest to us are the **Hecke congruence subgroups** of $SL(2, \mathbb{R})$, which are of the form

$$\Gamma_0(N) := \{(a \ b \newline\ c \ d) \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod{N}\}$$

for a positive integer $N$, which is usually referred to as the **level**. In particular, we will use the modular group $\Gamma_0(1) = SL(2, \mathbb{Z})$ throughout this work and $\Gamma_0(4)$ in Chapter 4. Let $\Gamma$ denote such a subgroup of $SL(2, \mathbb{R})$.

Two points $z_1, z_2 \in \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ are $\Gamma$-equivalent if there exists $\gamma \in \Gamma$ such that $\gamma z_1 = z_2$. A **cusp** of $\Gamma$ is a fixed point of a parabolic element $\gamma$ of $\Gamma$ which satisfies $|\text{tr}(\gamma)| = 2$; a cusp necessarily lies in $\mathbb{Q} \cup \{\infty\}$. For example, $\Gamma_0(1)$ has only one equivalence class of cusps with $\infty$ as a representative, and $\Gamma_0(4)$ has three equivalence classes of cusps with representatives $0, \frac{1}{2}$ and $\infty$. A **fundamental domain** for a subgroup $\Gamma$ of $SL(2, \mathbb{R})$ is an open subset $D \subseteq \mathbb{H}$ such that no two distinct points of $F$ are equivalent under the action of $\Gamma$ and every point $z \in \mathbb{H}$ is $\Gamma$-equivalent to some point in the closure $\overline{D}$ of $D$. A fundamental domain for $\Gamma_0(1)$ is

$$D_1 = \{z \in \mathbb{H} \mid |z| > 1, |\text{Re}(z)| < \frac{1}{2}\}$$

We illustrate $D_1$ and one fundamental domain $D_4$ of $\Gamma_0(4)$ in Figure 1.1 where the solid curves in black are not included in the domains. These fundamental domains are drawn by H. Verrill’s program, which is available on [http://wstein.org/Tables/fundomain/index2.html](http://wstein.org/Tables/fundomain/index2.html)
1.1.2 Holomorphic modular forms for $\Gamma_0(1)$

Let $k \geq 1$ be an integer. A modular form of weight $2k$ for $\Gamma_0(1)$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which obeys

$$f(\gamma z) = (cz + d)^{2k} f(z) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(1),$$

and is holomorphic at the cusp $\infty$. Now we explain the meaning of holomorphy at $\infty$. Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(1)$ implies $f(z + 1) = f(z)$, the function $f$ has a Fourier expansion at the cusp $\infty$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e(nz),$$

where $e(z)$ denotes $e^{2\pi iz}$. Then the holomorphy at $\infty$ means that $a_n = 0$ for all $n < 0$. If $f$ is holomorphic at $\infty$ and $a_0 = 0$, we call $f$ cuspidal at $\infty$, or call $f$ a cusp.
form. Let $M_{2k}$ denote the space of modular forms of weight $2k$ for $\Gamma_0(1)$ and $S_{2k}$ the subspace of cusp forms. These spaces are finite dimensional. In fact, by [48, Corollary 4.1.4] we have for $k \geq 1$:

$$
\dim S_{2k} = \begin{cases} 
0 & \text{if } k = 1, \\
\lfloor 2k/12 \rfloor - 1 & \text{if } k \equiv 1 \pmod{6}, \\
\lfloor 2k/12 \rfloor & \text{if } k \not\equiv 1 \pmod{6}.
\end{cases}
$$

$$
\dim M_{2k} = \begin{cases} 
\lfloor 2k/12 \rfloor & \text{if } k \equiv 1 \pmod{6}, \\
\lfloor 2k/12 \rfloor + 1 & \text{if } k \not\equiv 1 \pmod{6}.
\end{cases}
$$

Here $[\alpha]$ denotes the integer part of a real number $\alpha$. We are interested in cusp forms of large weight, in which case

$$
\dim S_{2k} = \frac{2k - 1}{12} + O(1).
$$

We equip the space $S_{2k}$ with the **Petersson inner product** $(\cdot, \cdot)$:

$$
(f, g) = \int_{\mathcal{D}_1} f(z)\overline{g(z)}y^{2k}d\mu(z), \quad f, g \in S_{2k};
$$

here the omitted dependence of the inner product on $k$ will be clear in the context and will not cause confusion.

Now we define the **Hecke operators** $T_n$ ($n \geq 1$) by

$$
T_n f(z) = \frac{1}{n} \sum_{ad=n, b \geq 0} a^{2k} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right)
$$

for any function $f : \mathbb{H} \to \mathbb{C}$. Then the operators $T_n$ map a modular form to a modular form and a cusp form to a cusp form, and satisfy the multiplicativity property

$$
T_m T_n = \sum_{d|\langle m, n \rangle} d^{2k-1} T_{mnd^{-2}} \quad (m, n \geq 1)
$$

which has an immediate consequence that $T_m T_n = T_n T_m$ for $m, n \geq 1$. Also Hecke operators $T_n$ are self-adjoint on $S_{2k}$ with respect to the Petersson inner product:

$$
(T_n f, g) = (f, T_n g) \quad \text{for all } f, g \in S_{2k}.
$$
See, e.g., [20, Chapter 6] for proofs of these properties. Then there exists an orthogonal basis $H_{2k}$ of $S_{2k}$ which consists of common eigenfunctions of all the Hecke operators $T_n$. Let $f \in S_{2k}$ be such a common eigenfunction with normalized Hecke eigenvalues $\lambda_f(n)$:

$$T_nf = \lambda_f(n)n^{\frac{2k-1}{2}}f, \quad n \geq 1.$$ 

Then the self-adjointness of $T_n$'s implies that $\lambda_f(n)$'s are real. Suppose $f$ has Fourier expansion

$$f(z) = \sum_{m=1}^{\infty} a_f(m)e(mz)$$

Then one can compute that

$$\lambda_f(n)n^{\frac{k-1}{2}}a_f(m) = \sum_{d|\langle m,n \rangle} d^{2k-1}a_f(mnd^{-2})$$

and the special case $m = 1$ implies that

$$\lambda_f(n)n^{\frac{2k-1}{2}}a_f(1) = a_f(n).$$

Thus $a_f(1) \neq 0$ and can be set to equal 1, in which case we call $f$ a **normalized Hecke eigenform**. From now on we require the eigenforms in $H_{2k}$ to be normalized, i.e., each $f \in H_{2k}$ has Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{\frac{2k-1}{2}}e(nz).$$

The values $\lambda_f(n)$ are real and satisfy the Ramanujan Conjecture or the Deligne bound

$$\lambda_f(n) \ll \tau(n)$$

as well as the **Hecke relation**

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f(mnd^{-2}).$$
Here $\tau(n) = \sum_{d\mid n} 1$ is the divisor function.

For a Hecke eigenform $f$, its associated $L$-function $L(s, f)$ is given by

$$L(s, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} \quad (\text{Re}(s) > 1)$$

and has Euler product

$$L(s, f) = \prod_p (1 - \lambda_f(p)p^{-s} + p^{-2s})^{-1} \quad (\text{Re}(s) > 1).$$

The $L$-function $L(s, f)$ has analytic continuation to the whole complex plane and satisfy the functional equation

$$\Lambda(s, f) := (2\pi)^{-s}\Gamma\left(s + \frac{2k - 1}{2}\right)L(s, f) = i^{2k}\Lambda(1 - s, f)$$

(see [20, Theorem 7.6]).

A related $L$-function is the symmetric square $L$-function $L(s, \text{sym}^2 f)$, studied by Shimura [53], Zagier [65] and in a more general context by Gelbart–Jacquet [13]. The $L(s, \text{sym}^2 f)$ is given by

$$L(s, \text{sym}^2 f) = \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n^2)}{n^s} \quad (\text{Re}(s) > 1),$$

has Euler product

$$L(s, \text{sym}^2 f) = \prod_p (1 - \lambda_f(p^2)p^{-s} + \lambda_f(p^2)p^{-2s} - p^{-3s})^{-1} \quad (\text{Re}(s) > 1),$$

has analytic continuation to the whole complex plane, and satisfies the functional equation

$$\Lambda(s, \text{sym}^2 f) := L_\infty(s, \text{sym}^2 f)L(s, \text{sym}^2 f) = \Lambda(1 - s, \text{sym}^2 f)$$

where

$$L_\infty(s, \text{sym}^2 f) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s + 1}{2}\right)\Gamma\left(\frac{s + 2k - 1}{2}\right)\Gamma\left(\frac{s + 2k}{2}\right).$$
Note that we have

\[ \frac{1}{L(1, \text{sym}^2 f)} = \frac{2k - 1}{2\pi^2} \omega_f \quad \text{with} \quad \omega_f := \frac{\Gamma(2k - 1)}{(4\pi)^{2k-1}(f, f)}. \]

We use the harmonic weights \( \frac{1}{L(1, \text{sym}^2 f)} \) in Chapter 2 and \( \omega_f \) in Chapter 3.

1.1.3 Nonholomorphic automorphic forms for \( \Gamma_0(1) \)

An automorphic function of weight 0 for \( \Gamma_0(1) \) is a function \( f : \mathbb{H} \to \mathbb{C} \) such that \( f(\gamma z) = f(z) \) for all \( \gamma \in \Gamma_0(1) \). Let \( L^2(\Gamma_0(1) \setminus \mathbb{H}) \) denote the space of all automorphic functions \( f \) with \( \|f\|^2 = (f, f) < \infty \), where \((\cdot, \cdot)\) is the Petersson inner product given by

\[ (f, g) = \int_{D_1} f(z) \overline{g(z)} \, d\mu(z). \]

Let \( B(\Gamma_0(1) \setminus \mathbb{H}) \) denote the dense subspace of \( L^2(\Gamma_0(1) \setminus \mathbb{H}) \) which consists of functions of \( C^\infty \) class with partial derivatives of rapid decay at the cusp \( \infty \). Here a function \( g(z) \) is of rapid decay means that \( g(z) = O(y^{-M}) \) for any \( M > 0 \) as \( y \to \infty \). Define the Laplace operator \( \Delta_0 \) (of weight 0) on \( \mathbb{H} \) by

\[ \Delta_0 = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \]

One can check that \( \Delta_0 \) is invariant under \( SL(2, \mathbb{R}) \), i.e.,

\[ \Delta_0(f(\gamma z)) = (\Delta_0 f)(\gamma z) \quad \text{for all} \quad \gamma \in SL(2, \mathbb{R}). \]

One can show by Green’s theorem that

\[ (\Delta_0 f, g) = \int_{D_1} \nabla f(z) \cdot \overline{\nabla g(z)} \, y^2 \, d\mu(z), \]

where \( \nabla \) is the usual gradient. Consequently we have

\[ (\Delta_0 f, g) = (f, \Delta_0 g) \quad \text{for} \quad f, g \in B(\Gamma_0(1) \setminus \mathbb{H}). \]
and in particular \((\Delta_0 f, f) > 0\) for nonconstant \(f\). We note that a \(\Delta_0\)-eigenvalue has the form \(s(1 - s)\) with either \(0 \leq s \leq 1\) or \(s = \frac{1}{2} + it\) for some \(t \in \mathbb{R}\). By functional analysis, \(\Delta_0\) has a self-adjoint extension to the whole space \(L^2(\Gamma_0(1)\backslash \mathbb{H})\).

A Maass form of weight 0, or a nonholomorphic automorphic form of weight 0 for \(\Gamma_0(1)\) is a real analytic automorphic function \(f\) that is a \(\Delta_0\)-eigenfunction and of moderate growth, i.e., there exists a positive integer \(N\) such that \(f(z) = O(y^{N})\) as \(y \to \infty\). A Maass form \(f\) is called a Maass cusp form if it also satisfies

\[
\int_{0}^{1} f\left(\left(\begin{array}{c} 1 \\ b \\ 1 \end{array}\right) z\right) db = 0 \quad \text{for all} \quad z \in \mathbb{H},
\]

Let \(C_0(1) := C_0(\Gamma_0(1)\backslash \mathbb{H})\) denote the closure in \(L^2(\Gamma_0(1)\backslash \mathbb{H})\) of the subspace of Maass cusp forms. Due to the invariance under \(\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)\) and the moderate growth condition, a Maass form \(f\) of \(\Delta_0\)-eigenvalue \(s(1 - s)\) has Fourier expansion

\[
f(z) = a_f(0; y) + \sum_{n \neq 0} a_f(n)W_{0,s,-\frac{1}{2}}(4\pi|n|y)e(nx).
\]

where \(a_f(0; y)\) is a linear combination of \((y^s + y^{1-s})/2\) and \(1/(2s - 1)(y^s - y^{1-s})\), and \(W_{\mu,\nu}\) denotes the Whittaker function (see [46, Chapter 7]) that satisfies the differential equation

\[
\frac{d^2u}{dz^2} + \left(-\frac{1}{4} + \frac{\mu}{z} + \frac{1}{4} - \nu^2 \right) u = 0
\]

which has a specialization

\[
W_{0,\nu}(4\pi y) = (4y)^{\frac{1}{2}}K_{\nu}(2\pi y)
\]

where \(K_{\nu}(z)\) is the usual \(K\)-Bessel function. A Maass cusp form \(f\) has the zeroth Fourier coefficient \(a_f(0; y) = 0\); we call \(f\) even (or odd) if \(a_f(n) = -a_f(-n)\) (or \(a_f(n) = a_f(-n)\)) for all \(n\) and set accordingly \(\varepsilon_f = 1\) (or \(-1\)).
An example of Maass forms for \( \Gamma_0(1) \) is the **Eisenstein series**

\[
E(z, s) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma_0(1)} (\text{Im}(\gamma z))^s \quad (\text{Re}(s) > 1),
\]

with Fourier expansion

\[
E(z, s) = y^s + \phi(s)y^{1-s} + \sum_{n \neq 0} \phi(n, s)W_{0,s-\frac{1}{2}}(4\pi|n|y)e(nx),
\]

where

\[
\phi(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2}) \zeta(2s - 1)}{\Gamma(s) \zeta(2s)}
\]

and

\[
\phi(n, s) = \frac{\pi^s}{\Gamma(s) \zeta(s)} \frac{\tau_{s-\frac{1}{2}}(|n|)}{\sqrt{|n|}}.
\]

Here \( \Gamma_{\infty} = \{ \pm (1 \ 0) \ n \in \mathbb{Z} \} \) is the subgroup of \( \Gamma_0(1) \) that fixes \( \infty \) and \( \tau_\alpha(n) = \sum_{n=ab}(a/b)^\alpha \). Then \( E(z, s) \) has meromorphic continuation to \( \mathbb{C} \) (in particular it is holomorphic for \( \text{Re}(s) \geq \frac{1}{2} \) with the exception of a simple pole at \( s = 1 \) with residue \( 3/\pi \)) and satisfies the functional equation

\[
E(z, s) = \phi(s)E(z, 1 - s).
\]

But \( E(z, s) \) is not square-integrable. (See [50, Lemma 1.2] and [21, Theorem 3.4])

Related to \( E(z, s) \) are the **incomplete Eisenstein series** \( E(z \mid \psi) \), given by

\[
E(z \mid \psi) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma_0(1)} \psi(\text{Im}(\gamma z)),
\]

with \( \psi \) being smooth and compactly supported on \( (0, \infty) \). \( E(z \mid \psi) \) are a bounded automorphic functions, belong to \( L^2(\Gamma_0(1)\setminus \mathbb{H}) \), but fail to be an eigenfunction of \( \Delta_0 \).

Note that we have the relation

\[
E(z \mid \psi) = \frac{1}{2\pi i} \int_{(c)} E(z, s)\hat{\psi}(s) \, ds \quad (c > 1)
\]
where \( \hat{\psi}(s) = \int_0^\infty \psi(y)y^{-s-1} \, ds \). Let \( \mathcal{E}_0(1) := \mathcal{E}(\Gamma_0(1) \backslash \mathbb{H}) \) denote the closure in \( L^2(\Gamma_0(1) \backslash \mathbb{H}) \) of the subspace spanned by all such incomplete Einsenstein series.

Then we have the spectral decomposition of \( L^2(\Gamma_0(1) \backslash \mathbb{H}) \) with respect to \( \Delta_0 \)

\[
L^2(\Gamma_0(1) \backslash \mathbb{H}) = \mathbb{C} \oplus \mathcal{C}_0(1) \oplus \mathcal{E}_0(1).
\]

The Laplace operator \( \Delta_0 \) has discrete spectrum \( \lambda_j = \frac{1}{4} + it_j^2 \) for \( t_j > 0 \) in \( \mathcal{C}_0(1) \), which has an orthonormal basis of Maass cusp forms \( u_j \) with eigenvalue \( \lambda_j \); \( \Delta_0 \) has continuous spectrum in \( \mathcal{E}_0(1) \), which covers \([\frac{1}{4}, \infty)\) uniformly with multiplicity one. More precisely, with \( u_0 \equiv \sqrt{\frac{3}{\pi}} \), we have for any \( f \in L^2(\Gamma_0(1) \backslash \mathbb{H}) \),

\[
f(z) = \sum_{j \geq 0} (f, u_j)u_j + \frac{1}{4\pi} \int_{-\infty}^{\infty} (f, E(\cdot, \frac{1}{2} + it))E(z, \frac{1}{2} + it) \, dt,
\]

which converges in the norm topology. For proof of these statements, see [21, Theorems 4.7 & 7.3] and [50, Theorem 1.1, Lemmas 1.9 & 1.11].) Furthermore, the discrete spectrum has infinite cardinality and obeys Weyl’s law (see [59] and [16]), which states that

\[
N(T) := \# \{ j \mid 0 \leq t_j \leq T \} = \frac{1}{12} T^2 - \frac{1}{2\pi} T \log T + c_0 T + O(T(\log T)^{-1}), \quad (1.1)
\]
as \( T \to \infty \), where \( c_0 \) is an absolute constant.

As in the case of holomorphic modular forms, there is a Hecke theory for Maass forms. For \( n \geq 1 \) the Hecke operator \( T_n \) is defined by

\[
T_n f(z) = \frac{1}{\sqrt{n}} \sum_{d|n} \sum_{b=0}^{d-1} f \left( \frac{az + b}{d} \right).
\]

The operators \( T_n \) commute with \( \Delta_0 \), map a Maass form to a Maass form and a Maass cusp form to a Maass cusp form, satisfy the the multiplicative property

\[
T_m T_n = \sum_{d|(m,n)} T_{mn/d^2}, \quad m, n \geq 1,
\]

11
and are self-adjoint with respect to the Petersson inner product
\[(T_n f, g) = (f, T_n g).\]

Thus \(C_0(1)\) has an orthonormal basis of Maass cusp forms which are common eigenfunctions of all \(T_n\) \((n \geq 1)\). Let \(f = \sum_{n \neq 0} a_f(n)W_{0, s - \frac{1}{2}}(4\pi|n|y)e(nx)\) be such an eigenform, called a Hecke–Maass form, with Hecke eigenvalue \(\lambda_j(n)\) \((n \geq 1)\). Then one can compute that
\[\lambda_f(n)a_f(1) = a_f(n)\sqrt{n} \quad (n \geq 1),\]

which implies that \(a_f(1) \neq 0\). The Hecke eigenvalues \(\lambda_f(n)\) satisfies the Hecke relation
\[\lambda_f(m)\lambda_f(n) = \sum_{d|\gcd(m, n)} \lambda(mnd^{-2}) \quad (m, n \geq 1).\]

We have the best known upper bound of \(\lambda_f(n) \ll n^{\frac{7}{64}}\tau(n)\) due to Kim–Sarnak [28], while the still open Ramanujan Conjecture predicts that \(\lambda_f(n) \ll \tau(n)\). We also remark that
\[T_n E(z, s) = \tau_s(n) E(z, s).\]

From now on we fix an orthonormal basis \(\{u_j\}\) of \(C_0(1)\) of Hecke–Maass forms with \(\Delta_0\)-eigenvalues \(\frac{1}{4} + t_j^2\) \((t_j > 0)\) and Hecke eigenvalues \(\lambda_j(n)\) \((n \geq 1)\).

To any Hecke–Maass form \(f\) with \(\Delta_0\)-eigenvalues \(\frac{1}{4} + t_f^2\) \((t_f > 0)\) and Hecke eigenvalues \(\lambda_f(n)\) \((n \geq 1)\), we associate its \(L\)-function
\[L(s, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} \quad \text{for } \text{Re}(s) > 1;\]

the convergence of the Dirichlet series in this region can be obtained by the Rankin-Selberg method (see [7, Exercise 1.6.5]), although the Ramanujan Conjecture is not
available. $L(s, f)$ admits analytic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(s, f) := L_\infty(s, t_f) L(s, f) = \varepsilon_f \Lambda(1-s, f)$$

where

$$L_\infty(s, t) = \pi^{-s} \Gamma\left(\frac{s+it}{2}\right) \Gamma\left(\frac{s-it}{2}\right).$$

Again we have the symmetric square $L$-function

$$L(s, \text{sym}^2 f) = \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n^2)}{n^s}$$

$$= \prod_p (1 - \lambda_f(p^2)p^{-s} + \lambda_f(p^2)p^{-2s} - p^{-3s})^{-1} \quad (\text{Re}(s) > 1),$$

which has analytic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(s, \text{sym}^2 f) := L_\infty(s, \text{sym}^2 f) L(s, \text{sym}^2 f) = \Lambda(1-s, \text{sym}^2 f)$$

where

$$L_\infty(s, \text{sym}^2 f) = \pi^{-\frac{3}{2} s} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+t_f}{2}\right) \Gamma\left(\frac{s-t_f}{2}\right).$$

We note that

$$\frac{1}{L(1, \text{sym}^2 f)} = \frac{1}{2 \cosh(\pi t_f)(f, f)}.$$  

1.2 Main results

Now we are ready to state the main results obtained in three individual projects \[41, 42, 43\], leaving detailed discussions (and applications) in the introductory section of each corresponding chapter.

In Chapter 2 we study the central $L$-derivative values for normalized Hecke eigenforms in $\mathcal{H}_{2k}$ and obtain the following main result.

13
Theorem 1.1. For sufficiently large odd \( k \),
\[
\# \left\{ f \in \mathcal{H}_{2k} \mid L'(\frac{1}{2}, f) \neq 0 \right\} \gg k,
\]
i.e., as \( k \to \infty \) there are positive proportion of Hecke eigenforms in \( \mathcal{H}_{2k} \) with nonvanishing central \( L \)-derivative value.

Remark 1.1. This nonvanishing result is qualitatively optimal by virtue of the size of \( \mathcal{H}_{2k} \), and is a natural extension of the recent result of Luo [45]: for sufficiently large even \( k \)
\[
\# \left\{ f \in \mathcal{H}_{2k} \mid L(\frac{1}{2}, f) \neq 0 \right\} \gg k.
\]

In Chapter 3 we study the values \( L(s, \text{sym}^2 f) \) for \( f \in \mathcal{H}_{2k} \) and obtained the following asymptotic formula.

Theorem 1.2. For any \( B > 0 \) and sufficiently large \( k \) we have
\[
\sum_{f \in \mathcal{H}_{2k}} \omega_f L(\frac{1}{2}, \text{sym}^2 f) = \\
\psi(2k - \frac{1}{2}) + 2\gamma + \frac{1}{2}\psi\left(\frac{3}{4}\right) - \log(2\pi^\frac{3}{2}) \\
+ i^{-2k} \sqrt{\frac{\pi}{2}} L\left(\frac{1}{2}, \chi_{-4}\right) \Gamma(k - \frac{1}{4}) \Gamma(k + \frac{1}{4}) \\
+ \sqrt{2\pi i^{-2k}} L\left(\frac{1}{2}, \chi_{-3}\right) \left(\frac{2}{\sqrt{3}}\right)^{2k - \frac{1}{2}} F\left(k - \frac{1}{4}, k - \frac{1}{4}, \frac{1}{4}; -\frac{1}{3}\right) \Gamma(k - \frac{1}{4}) \Gamma(k + \frac{1}{4}) \\
+ O_B(k^{-B}).
\]

Here \( \gamma \) is the Euler constant, \( \psi(z) = \Gamma'(z)/\Gamma(z) \), \( \chi_D(\cdot) = (\frac{D}{\cdot}) \) denotes the Kronecker symbol, and \( F(a, b; c; z) \) is the Gauss hypergeometric function, initially defined for \( |z| < 1 \) by
\[
F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},
\]
where \( (\alpha)_0 = 1 \) and \( (\alpha)_n = \alpha(\alpha+1) \cdots (\alpha+n-1) \) for \( n \geq 1 \).
Remark 1.2. This result refines the best known result due to Fomenko \[12\] and Sun \[57\]:

$$\sum_{f \in \mathcal{H}_{2k}} \omega_f L(\frac{1}{2}, \text{sym}^2 f) = \psi(2k - \frac{1}{2}) + 2\gamma + \frac{1}{2} \psi\left(\frac{3}{4}\right) - \log(2\pi \frac{3\sqrt{3}}{2}) + O(k^{-\frac{1}{2}}).$$

In Chapter 4, we consider the natural analogue of holomorphic cusp forms, Maass forms and their \(L\)-functions, and establish the following nonvanishing result.

**Theorem 1.3.** Let \( T > 0 \) be a large parameter and \( M = T^{\eta} \) with \( 0 < \eta < 1 \). We have

$$\# \left\{ j \mid |t_j - T| \leq M, L(\frac{1}{2}, u_j) > 0 \right\} \gg TM.$$

**Remark 1.3.** This result is qualitatively optimal in view the Weyl law, and is a natural analogue of Luo’s work \[45\] for the holomorphic modular forms of large weight.

Finally, we comment that except using the concepts, facts and notations in Chapter 1, each of Chapters 2, 3, 4 has its own system of notations.
Chapter 2: Central $L$-derivative values of holomorphic modular forms

2.1 Introduction

Let $k$ be a (large) positive integer and consider the orthogonal Hecke eigenbasis $H_{2k}$ of holomorphic cusp forms of weight $2k$ for $\Gamma_0(1)$. Let $K_D = \mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field with fundamental discriminant $D$. Let $\chi_D$ denote the Kronecker symbol $(\frac{D}{\cdot})$. Let $f \in H_{2k}$ and $f_{K_D}$ be the base change of $f$ to $K_D$. We have the associated $L$-function $L(s, f)$ and its twist $L(s, f \otimes \chi_D)$ by

$$L(s, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} \text{ and } L(s, f \otimes \chi_D) = \sum_{n \geq 1} \frac{\lambda_f(n)\chi_D(n)}{n^s} \quad (\text{Re}(s) > 1),$$

as well as the $L$-function $L(s, f_{K_D})$ for the base change $f_{K_D}$, which has factorization (see [7, p. 99])

$$L(s, f_{K_D}) = L(s, f)L(s, f \otimes \chi_D).$$

In this chapter we study average behaviors and nonvanishing of $L'(\frac{1}{2}, f)$ and $L'(\frac{1}{2}, f_{K_D})$ for $f \in H_{2k}$ and give applications in arithmetic geometry, as such values, according to Zhang [66] and Xue [64], are closely related to height pairings of certain Heegner cycles of high weights. To study these values, we use the method of moments and the mollification method. The moments of interest to us in this chapter
are weighted sums of the following form

\[ \sum_{f \in H_{2k}} \frac{\alpha_f}{L(1, \text{sym}^2 f)}, \]

where \(\alpha_f\) is a number attached to \(f\).

### 2.1.1 Asymptotic formulas and nonvanishing

For average behaviors of the central \(L\)-derivative values, we prove the following.

**Theorem 2.1.** Let \(A > 0\) and \(B > 2\) be arbitrary. For sufficiently large odd \(k\),

\[
\sum_{f \in H_{2k}} \frac{L'(\frac{1}{2}, f)}{L(1, \text{sym}^2 f)} = \frac{2k - 1}{2\pi^2} \left\{ 2\psi(k) - 2 \log 2\pi + O_A(k^{-A}) \right\}, \tag{a}
\]

\[
\sum_{f \in H_{2k}} \frac{L'(\frac{1}{2}, f)^2}{L(1, \text{sym}^2 f)} = \frac{2k - 1}{2\pi^2} \left\{ c_3 \psi^3(k) + c_2 \psi^2(k) + c_1 \psi(k) \right. \tag{b}
\]

\[+ c_0 + c_{-2} \psi''(k) + O_B(k^{-B}) \left\}. \right.

and for a fundamental discriminant \(D\) with \(-k/(2 \log k) < D < 0\)

\[
\sum_{f \in H_{2k}} \frac{L'(\frac{1}{2}, f_{K_D})}{L(1, \text{sym}^2 f)} = \frac{2k - 1}{2\pi^2} \left\{ 4L(1, \chi_D) \left[ \psi(k) - \log 2\pi + \frac{L'(1, \chi_D)}{L(1, \chi_D)} \right] \tag{c}
\]

\[+ O(|D|^{\frac{3}{2}} k^{-\frac{3}{2}} (\log k)^4) \right\}, \right.

where \(\psi(z) = \Gamma'(z)/\Gamma(z)\) is the digamma function,

\[
c_3 = \frac{4}{3}, \quad c_2 = 4(\gamma_0 - \log 2\pi), \quad c_1 = -8\gamma_1 - 8\gamma_0 \log 2\pi + 4(\log 2\pi)^2,
\]

\[
c_0 = 4\gamma_2 + 8\gamma_1 \log 2\pi + 4\gamma_0 (\log 2\pi)^2 - \frac{4}{3}(\log 2\pi)^3, \quad c_{-2} = -\frac{2}{3},
\]

and \(\gamma_n\)'s are the Stieltjes constants in the expansion of the Riemann zeta-function \(\zeta(s)\) at \(s = 1\).

**Remark 2.1.** Note that \(L'(\frac{1}{2}, f)\) and \(L'(\frac{1}{2}, f_{K_D})\) are real-valued for \(f \in H_{2k}\) and that \(\psi(k) = \log k + O(k^{-1})\) for large \(k\). Fix a fundamental discriminant \(D < 0\).
So Theorem 2.1 implies that for sufficiently large odd $k$ there exist $f, g \in \mathcal{H}_{2k}$ with $L'(\frac{1}{2}, f) > 0$ and $L'(\frac{1}{2}, g_{K_D}) > 0$. Also Theorem 2.1 can be viewed as evidence to the conjectural positivity of $L'(\frac{1}{2}, f)$ and $L'(\frac{1}{2}, f_{K_D})$, which is a consequence of the Riemann Hypothesis.

In Theorem 2.1(c), the requirement $-k/(2\log k) < D < 0$ is needed to put contributions from different parts together in the $O$-term (see §2.3.2). The appearance of $L'(1, \chi_D)$ makes the expression “non-pure”, according to [24]. In order to have $\psi(k)$ dominate $\frac{L'(1,\chi_D)}{L(1,\chi_D)}$, in view of that $|L'(1,\chi_D)| \leq M(\log |D|)^2$ for an absolute $M > 0$, we need a lower bound of $L(1,\chi_D)$, which concerns the existence of Landau–Siegel zeros. But even with Siegel’s lower bound

$$L(1,\chi_D) \geq \frac{c(\varepsilon)}{|D|^\varepsilon}$$

with the ineffective $c(\varepsilon)$ for $\varepsilon \in (0, \frac{1}{2})$, we still need $k \geq e^{M|D|^{c(\log |D|)^2/c(\varepsilon)}}$ to achieve the desired dominance. Hence unfortunately we are not able to make Theorem 2.1(c) an asymptotic formula which holds uniformly for $D$ in a range like $0 < |D| < k^\alpha$ for some power $\alpha > 0$.

**Remark 2.2.** Theorem 2.1 should be compared with the results by Lau–Tsang [36] in the weight aspect and with the results by Duke [8] in the level aspect.

In view of the power saving in Theorem 2.1(a) and (b), we further estimate the mollified moments.

**Proposition 2.1.** Let $k$ be odd and $M_f$ the mollifiers in (2.3). For any fixed $0 < \delta < \frac{1}{2}$ and sufficiently large $k$

$$\sum_{f \in \mathcal{H}_{2k}} \frac{L'(\frac{1}{2}, f)M_f}{L(1, \text{sym}^2 f)} = \frac{(3 + 2\delta^{-1})}{\pi^2} k + O\left(\frac{k}{\log k}\right),$$
and with further restriction $0 < \delta < \frac{1}{32}$

$$\sum_{f \in \mathcal{H}_{2k}} \frac{L'(\frac{1}{2}, f)^2 M_f^2}{L(1, \text{sym}^2 f)} \ll k.$$ (b)

With Proposition 2.1 for a chosen $\delta$ in $(0, \frac{1}{32})$ and the bound [45, (3)] (or see [37, Proposition 6.1]), that is,

$$\sum_{f \in \mathcal{H}_{2k}} \frac{1}{L(1, \text{sym}^2 f)^2} \ll k,$$

we deduce by Hölder’s inequality that

$$k \ll \left| \sum_{f \in \mathcal{H}_{2k}} \frac{L'(\frac{1}{2}, f) M_f}{L(1, \text{sym}^2 f)} \right|$$

$$\ll \left| \sum_{f \in \mathcal{H}_{2k}, L'(\frac{1}{2}, f) \neq 0} 1 \cdot \left| \sum_{f \in \mathcal{H}_{2k}} \frac{1}{L(1, \text{sym}^2 f)^2} \right| \cdot \left| \sum_{f \in \mathcal{H}_{2k}} \frac{L'(\frac{1}{2}, f)^2 M_f^2}{L(1, \text{sym}^2 f)^2} \right|^{\frac{1}{2}} \right|$$

$$\ll k \cdot \left| \sum_{f \in \mathcal{H}_{2k}, L'(\frac{1}{2}, f) \neq 0} 1 \right|^{\frac{1}{2}}$$

and arrive at the main result of this chapter, a nonvanishing result for $L'(\frac{1}{2}, f)$.

**Theorem 2.2** (Restatement of Theorem 1.1). For sufficiently large odd $k$,

$$\# \{ f \in \mathcal{H}_{2k} \mid L'(\frac{1}{2}, f) \neq 0 \} \gg k,$$

i.e., as $k \to \infty$ there are positive proportion of Hecke eigenforms in $\mathcal{H}_{2k}$ with nonvanishing central $L$-derivative value.

**Remark 2.3.** In the weight aspect, Theorem 2.2 should be compared with the nonvanishing result of Luo [45]: for sufficiently large even $k$

$$\# \{ f \in \mathcal{H}_{2k} \mid L(\frac{1}{2}, f) \neq 0 \}$$

We note that Theorem 2.2 is also true for large even $k$, which follows from Luo’s result and the observation that if $k$ is even and $L(\frac{1}{2}, f) \neq 0$, logarithmic differentiation of
the functional equation at the central point gives

\[
\frac{L'(\frac{1}{2}, f)}{L(\frac{1}{2}, f)} = \log 2\pi - \psi(k),
\]

where \(\psi(k) = \log k + O(k^{-1})\).

It is also interesting to compare Theorem 2.2 with results in the level aspect: it is known that \(L'(\frac{1}{2}, f) \geq 0\) for any newform \(f \in S_2(\Gamma_0(N))\) (see Gross–Kohnen–Zagier [15]), and it is shown (e.g. in VanderKam [58] and Kowalski–Michel [31]) that for sufficiently large prime level \(N\) there are positive proportion of primitive newforms in \(S_2(\Gamma_0(N))\) with nonvanishing central \(L\)-derivative values.

**Remark 2.4.** The mollifiers \(M_f\) defined in (2.3) are slight modification of those used by Luo [45], which are of analytic nature and bring the bounds of the mollified moments comparable to the size of \(\mathcal{H}_{2k}\), that is, to kill the \(\log k\) factors in Theorem 2.1. In similarity with [45], the use of mollifiers leads to certain sums which are difficult to bound in the analysis of \(\sum_{f \in \mathcal{H}_{2k}} L'(\frac{1}{2}, f)^2 M_f^2 / L(1, \text{sym}^2 f)\); see the bounds (A) in §2.5.1 and (B) in §2.5.2. However, due to the presence of extra poles in the test function \(V\) of the approximate functional equation of \(L'(\frac{1}{2}, f)^2\) (see Lemma 2.1), the bounds (A) and (B) are more complex and difficult to deal with than Luo’s cases (see [45], (5), (6) and p. 228).

A problem common to both [45] and our work is that it seems very difficult to derive an asymptotic formula for the mollified second moment using the “analytic” mollifiers. It would be interesting to use other mollifiers, say, the ones determined by optimizing certain quadratic forms, to compute a percentage of the nonvanishing proportions in the family \(\mathcal{H}_{2k}\), for both the central \(L\)-values (when \(k\) is even) and central \(L\)-derivative values (when \(k\) is odd). We mention that Balkanova–Frolenkov
recently use a different choice of mollifiers to obtain a weighted count that, in our notation, for any \( \varepsilon > 0 \) and sufficiently large even \( k \)

\[
\sum_{f \in \mathcal{H}_{2k}, L(\frac{1}{2}, f) \geq (\log k)^{-2}} \frac{1}{L(1, \text{sym}^2 f)} \geq \left( \frac{1}{5} - \varepsilon \right) \frac{2k - 1}{2\pi^2}.
\]

Here we point our that one can use this result and [37, Proposition 6.1], an asymptotic formula of \( \sum_{f \in \mathcal{H}_{2k}} \frac{1}{L(1, \text{sym}^2 f)^2} \), to obtain a true percentage of forms \( f \in \mathcal{H}_{2k} \) with \( L(\frac{1}{2}, f) \geq (\log k)^{-2} \).

### 2.1.2 Applications to Heegner cycles of high weights

In his work [66] generalizing Gross–Zagier [14] to high weight forms, Zhang defines the global height pairing \( \langle \cdot, \cdot \rangle \) between CM-cycles in certain Kuga–Sato varieties and obtains an identity between the height pairings of Heegner cycles and Fourier coefficients of certain cusp forms of higher weights. Recently, Xue [64] proves a high weight version of the Gross–Kohnen–Zagier theorem [15], assuming the unproven positive-definiteness of the height pairing \( \langle \cdot, \cdot \rangle \). The results [66, Corollary 0.3.2] and [64, Theorem 2] relate the values \( L'(\frac{1}{2}, f_{K_D}) \) and \( L'(\frac{1}{2}, f) \) with height pairings of certain Heegner cycles. We will apply our results to these height pairings after introducing necessary notions and notations.

For \( K_D \) as before let \( H_D \) be its Hilbert class field and \( h_D \) the class number of \( K_D \). A Heegner point \( x \in X_0(1)(H_D) \) is determined by the residue class of \( r \mod 2 \), where \( r^2 \equiv D \mod 4 \). Let \( P_{D,r} \) be the formal sum of the \( h_D \) Heegner points which are permuted by \( \text{Gal}(H_D/K_D) \) (see [15]). Let \( R \) be an integral domain flat over \( \mathbb{Z} \). For a Heegner point \( x \in X_0(1)(H_D) \) of discriminant \( D \) rational over \( R \), one can define the Heegner cycle \( s_{2k}(x) \) of weight \( 2k \) over \( X_0(1)_R \) and the space \( \text{Heeg}_{2k}(X_0(1)_R) \otimes \mathbb{R} \)
generated by all such Heegner cycles \( s_{2k}(x) \) (see [66] for detailed construction and [64] for a brief account). Now define a cycle

\[
\mathcal{S}^*_{D,r} = \sum_{x \in \mathcal{P}_{D,r}} s_{2k}(x) + \sum_{x \in \mathcal{P}_{D,r}} \overline{s_{2k}(x)} \in \text{Heeg}_{2k}(X_0(1)_Q) \otimes \mathbb{R}
\]

where the bar means complex conjugation, and let \((\mathcal{S}^*_{D,r})_f\) be the \(f\)-isotypical component for \(f \in \mathcal{H}_{2k}\).

For odd \(D < 0\) [66, Corollary 0.3.2] implies

\[
\langle (\mathcal{S}^*_{D,r})_f, (\mathcal{S}^*_{D,r})_f \rangle = \sqrt{|D|} \frac{2\pi^2}{\pi} \frac{L'(1/2, f_{\mathcal{K}_D})}{2k - 1 L(1, \text{sym}^2 f)}
\]

so that Theorem 2.1(c) yields the following quantitative result on height pairings.

**Corollary 2.1.** For any odd fundamental discriminant \(D < 0\) and sufficiently large odd \(k\), we have

\[
\sum_{f \in \mathcal{H}_{2k}} \langle (\mathcal{S}^*_{D,r})_f, (\mathcal{S}^*_{D,r})_f \rangle = \frac{4\sqrt{|D|} L(1, \chi_D)}{\pi} \left\{ \psi(k) - \log 2 \pi + \frac{L'(1, \chi_D)}{L(1, \chi_D)} \right\}
\]

\[
+ O\left(|D|^{2k-\frac{1}{2}} (\log k)^4\right).
\]

Thus there exists \(f \in \mathcal{H}_{2k}\) such that \(\langle (\mathcal{S}^*_{D,r})_f, (\mathcal{S}^*_{D,r})_f \rangle > 0\).

Corollary 2.1 provides some evidence (though not strong) for the following unproven conjecture, which we assume in the rest of this subsection.

**Assumption.** The height pairing \(\langle \cdot, \cdot \rangle\) is positive-definite.

To \(f \in \mathcal{H}_{2k}\) one can associate a Jacobi form

\[
\phi_f(\tau, z) = \sum_{n,r \in \mathbb{Z}, r^2 \leq 4n} c(n,r)e(n\tau)e(rz) \quad (\tau \in \mathbb{H}, z \in \mathbb{C})
\]

they share the same Hecke eigenvalues (see Skoruppa–Zagier [54]). Let \(\text{Heeg}_{2k,f}\) denote the subspace of \(\text{Heeg}_{2k}(X_0(1)_Q) \otimes \mathbb{R}\) generated by \((\mathcal{S}^*_{D,r})_f \mid (D, 2) = 1\). Then
Theorem 2\ref{2} says that there exists \(s_f^* \in (\text{Heeg}_{2k}(X_0(1)_{\mathbb{Q}}) \otimes \mathbb{R})_f\) independent of odd \(D < 0\) and \(r\) such that
\[
|D|^{\frac{2k-1}{2}}(s_{D,r}^*)_f = c\left(\frac{r^2 - D}{4}, r\right)s_f^* \quad \text{and} \quad \langle s_f^*, s_f^* \rangle = \frac{\Gamma(2k - 1)L'(\frac{1}{2}, f)}{2^{2k-1} \pi^k \Gamma(k)(\phi_f, \phi_f)},
\]
and \(\dim \text{Heeg}_{2k,f} = 1\) if \(L'(\frac{1}{2}, f) \neq 0\), and \(= 0\) if \(L'(\frac{1}{2}, f) = 0\). Hence Theorem \ref{2.2} implies that under the Assumption we expect to have many nontrivial subspaces \(\text{Heeg}_{2k,f}\) of \(\text{Heeg}_{2k}(X_0(1)_{\mathbb{Q}}) \otimes \mathbb{R}\). Or more precisely, we have the following.

**Corollary 2.2.** For sufficiently large odd \(k\), there are positive proportion of \(f \in \mathcal{H}_{2k}\) such that \(\langle s_f^*, s_f^* \rangle \neq 0\) and \(\dim \text{Heeg}_{2k,f} = 1\).

### 2.1.3 Structure of this chapter

In §\ref{2.2} we introduce necessary ingredients and tools. In §\ref{2.3} we prove Theorem 2.1\ref{c}. Then we prove Proposition 2.1\ref{a} and (b) in §\ref{2.4} and §\ref{2.5}, respectively. For completeness we give in §\ref{2.6} only a sketchy proof of Theorem 2.1\ref{a} and (b), since they can be viewed as a “baby” version of Proposition 2.1. In §\ref{2.6} we establish some technical bounds needed for the proof of Proposition 2.1.

### 2.2 Preparation

In this section, we include necessary ingredients and gather some preparatory results to be used in later sections of this chapter. Unless specified otherwise, we let \(k\) be a positive odd integer and \(D < 0\) a fundamental discriminant throughout the rest of this work. We reserve \(p\) for prime numbers. We use \((a, b)\) and \([a, b]\) for the g.c.d. and the l.c.m of two integers \(a\) and \(b\), respectively. We use \(\varepsilon\) and \(\delta\) for small positive reals while \(B\) for big ones. We adopt the conventions that these constants
may take different values in different paragraphs, and that implied constants may
depend on $\varepsilon$ or $\delta$.

### 2.2.1 Functional equations

For $f \in \mathcal{H}_{2k}$ the $L$-functions $L(s, f)$ and $L(s, f \otimes \chi_D)$ have analytic continuation
to the whole $s$-plane and their complete $L$-functions

$$
\Lambda(s, f) = \left( \frac{1}{2\pi} \right)^s \Gamma(s + k - \frac{1}{2}) L(s, f)
$$

and

$$
\Lambda(s, f \otimes \chi_D) = \left( \frac{|D|}{2\pi} \right)^s \Gamma(s + k - \frac{1}{2}) L(s, f \otimes \chi_D)
$$
satisfy the functional equations (see, e.g., [22, Theorem 14.17])

$$
\Lambda(s, f) = i^{2k} \Lambda(1 - s, f) \quad \text{and} \quad \Lambda(s, f \otimes \chi_D) = \varepsilon_{k,D} \Lambda(1 - s, f \otimes \chi_D),
$$

(2.1)

where $\varepsilon_{k,D} = i^{2k} \tau^2(\chi_D)|D|^{-1} = 1$ as the Gauss sum $\tau(\chi_D)$ equals $\sqrt{|D|} \text{sign}(\chi_D)$.

Since $k$ is odd, we have

$$
L\left(\frac{1}{2}, f\right) = 0 \quad \text{and} \quad L'\left(\frac{1}{2}, f_{K_D}\right) = L'\left(\frac{1}{2}, f\right)L\left(\frac{1}{2}, f \otimes \chi_D\right).
$$

### 2.2.2 Approximate functional equations

To avoid the convergence issue of relevant Dirichlet series, we use approximate
functional equations in Lemma 2.1 to represent $L'\left(\frac{1}{2}, f\right)$, $L'\left(\frac{1}{2}, f\right)^2$, and $L'\left(\frac{1}{2}, f_{K_D}\right)$.

Let $G(u)$ be an even holomorphic function with $G(0) = 1$ and of rapid decay in any
fixed vertical strips. Here we take $G(u) = e^{-u^4}$ and put

$$
H(u) = \frac{G(u)}{u^2}, \quad K(u) = \frac{G(u)}{u^3}, \quad \gamma(u) = \frac{\Gamma(k + u)}{\Gamma(k)}.
$$
Lemma 2.1. The following hold

\[ L'(\frac{1}{2}, f) = 2 \sum_{n \geq 1} \frac{\lambda_f(n) U(n)}{\sqrt{n}}, \]  
\[ L'(\frac{1}{2}, f)^2 = 2 \sum_{n \geq 1} \frac{\lambda_f(n) \tau(n) V(n)}{\sqrt{n}}, \]  
\[ L'(\frac{1}{2}, f_KD) = 2 \sum_{n \geq 1} \frac{\lambda_f(n) \tau_D(n) W(n)}{\sqrt{n}}, \]

where

\[ U(y) = \frac{1}{2\pi i} \int_{(3)} y^{-u(2\pi)^{-u} \gamma(u)} H(u) du, \]
\[ V(y) = \frac{1}{2\pi i} \int_{(3)} y^{-u(2\pi)^{-2u} \zeta(1+2u) \gamma^2(u)} K(u) du, \]
\[ W(y) = \frac{1}{2\pi i} \int_{(3)} y^{-u|D|^u(2\pi)^{-2u} L(1+2u, \chi_D) \gamma^2(u) H(u)} du, \]

and \( \tau(n) \) denotes the divisor function and \( \tau_D(n) = \sum_{n=ab \chi_D(a)} \).

Proof. We mimic the proof of [22, Theorem 5.3]. Shifting the integrals

\[ I_1 = \frac{1}{2\pi i} \int_{(3)} \Lambda(\frac{1}{2} + u, f) H(u) du, \quad I_2 = \frac{1}{2\pi i} \int_{(3)} \Lambda^2(\frac{1}{2} + u, f) K(u) du, \]

and

\[ I_3 = \frac{1}{2\pi i} \int_{(3)} \Lambda(\frac{1}{2} + u, f) \Lambda(\frac{1}{2} + u, f \otimes \chi_D) H(u) du \]

to \( \text{Re}(u) = -3 \), picking up at most a simple pole at \( u = 0 \), and applying (2.1), we have

\[ (2\pi)^{-\frac{1}{2}} \Gamma(k) L'(\frac{1}{2}, f) = 2I_1, \quad (2\pi)^{-1} \Gamma^2(k) L'(\frac{1}{2}, f)^2 = 2I_2, \]

and

\[ (2\pi)^{-1} \Gamma^2(k) L'(\frac{1}{2}, f_KD) = 2I_3. \]
Then the lemma follows from expanding the involved Dirichlet series in $I_1$, $I_2$ and $I_3$, where for the latter two we use for $\text{Re}(s) > 1$

$$L^2(s, f) = \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n) \tau(n)}{n^s} \quad \text{and} \quad L(s, f_K) = L(2s, \chi_D) \sum_{n \geq 1} \frac{\lambda_f(n) \tau_D(n)}{n^s}$$

in view of the Hecke relation $\lambda_f(m) \lambda_f(n) = \sum_{d|(m,n)} \lambda_f(mn d^{-2})$.

Here we also record the following consequence of Stirling’s formula

$$\gamma(u) \ll_{\text{Re}(u)} k^\text{Re}(u), \quad (2.2)$$

which shall be frequently used.

2.2.3 The mollifiers $M_f$

Let $0 < \delta < \frac{1}{2}$ and $\xi = k^\delta$. For $f \in \mathcal{H}_{2k}$, we define its mollifier

$$M_f = \sum_{n \geq 1} \frac{\lambda_f(n) \mu(n)a_n}{\sqrt{n}}, \quad (2.3)$$

where

$$a_n = \frac{1}{2\pi i} \int_{(2)} \frac{(\frac{\xi}{n})^z - (\frac{\xi}{n})^{-z}}{z^3 (\log \xi)^2} \, dz.$$  

By the discontinuous contour integral

$$\frac{1}{2\pi i} \int_{(2)} \frac{y^z}{z^3} \, dz = \begin{cases} \frac{1}{2} (\log y)^2, & \text{if } y \geq 1, \\ 0, & \text{if } 0 < y \leq 1, \end{cases}$$

we have

$$a_n = \begin{cases} \frac{1}{2} \left( \frac{\log \frac{\xi^2}{n}}{\log \xi} \right)^2 - \left( \frac{\log \frac{\xi}{n}}{\log \xi} \right)^2, & 1 \leq n \leq \xi, \\ \frac{1}{2} \left( \frac{\log \frac{\xi^2}{n}}{\log \xi} \right)^2, & \xi \leq n \leq \xi^2, \\ 0, & n \geq \xi^2, \end{cases}$$

and thus $0 \leq a_n \ll 1$. 

26
2.2.4 Petersson’s formula

For a nonempty Hecke eigenbasis \( \mathcal{H}_{2k} \), Petersson’s formula (see, e.g., [22, Corollary 14.23]) states that

\[
\frac{\Gamma(2k-1)}{(4\pi)^{2k-1}} \sum_{f \in \mathcal{H}_{2k}} \frac{\lambda_f(m)\lambda_f(n)}{\langle f, f \rangle} = \delta_{m,n} + 2\pi i^{-2k} \sum_{c \geq 1} \frac{S(m, n; c)}{c} J_{2k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right).
\]

Here \( \delta_{m,n} \) is the Kronecker delta symbol; \( S(m, n; c) \) denotes the classical Kloosterman sum

\[
S(m, n; c) = \sum_{\substack{x \pmod{c} \atop x^2 \equiv 1 \pmod{c}}} e \left( \frac{mx + nx}{c} \right),
\]

which satisfies Weil’s bound \( |S(m, n; c)| \leq \tau(c)(m, n, c) \frac{c}{2\sqrt{\nu} + 1} \) ([22, (1.60)]); \( J_\nu(z) \) is the Bessel function of the first kind of order \( \nu \). For convenience, we will denote the summation symbol in \( S(m, n; c) \) by \( \sum'_{x(c)} \).

2.2.5 Some properties of \( J_\nu(x) \)

For \( \nu \geq 0 \), we have

\[
|J_\nu(x)| \leq 1 \quad \text{for } x \in \mathbb{R}
\]

and by [61, 3.31(1)] and [51, (5.6.1)] that

\[
|J_\nu(x)| \leq \frac{e}{\sqrt{2\pi}} \frac{1}{\sqrt{\nu + 1}} \left( \frac{ex}{2\nu + 2} \right)^\nu.
\]

Combining these, we get the following bound which is sufficient for our purpose: for any positive integer \( k \) it holds that

\[
J_{2k-1}(x) \ll \begin{cases} 2^{-k}k^{-\frac{3}{2}}x, & \text{if } 0 < x < k, \\ 1, & \text{if } x \geq k. \end{cases}
\]

(2.4)

We also need the following Mellin-Barnes integral

\[
J_\nu(x) = \frac{1}{2\pi i} \int_{(\sigma)} x^{-s-1} 2^s \Gamma \left( \frac{\nu+1+s}{2} \right) \Gamma \left( \frac{\nu+1-s}{2} \right) ds
\]

for \( x > 0 \) and \( -1 - \text{Re}(\nu) < \sigma < 0 \) (see [45, p. 226]).
2.2.6 The Estermann zeta-functions

Such zeta-functions are first introduced and studied by Estermann \cite{11}:

$$E\left(s, \frac{x}{c}\right) = \sum_{n \geq 1} \tau(n)e\left(\frac{nx}{c}\right)n^{-s},$$

where $c > 0$ and $x$ are coprime integers. Lau–Tsang \cite[Theorem A]{36} carefully studies the analytic properties of generalized Estermann zeta-functions, replacing $\tau(n)$ by $\tau_{\chi_1,\chi_2}(n) = \sum_{n=ab} \chi_1(a)\chi_2(b)$ for two characters $\chi_1$ and $\chi_2$. In this paper, we will need $E(s, \frac{x}{c})$ and another special case $E(s, D, \frac{x}{c})$ with $\chi_1 = \chi_D$ and $\chi_2 \equiv 1$:

$$E\left(s, D, \frac{x}{c}\right) = \sum_{n \geq 1} \tau_D(n)e\left(\frac{nx}{c}\right)n^{-s}.$$

These zeta-functions play an important role in the analysis of the so-called “off-diagonal terms” resulting from Petersson’s formula. We state the following properties of them and refer the reader to \cite{36} for details.

First, $E(s, \frac{x}{c})$ admits analytic continuation to the whole $s$-plane except a double pole at $s = 1$, has the Laurent expansion at $s = 1$

$$E\left(s, \frac{x}{c}\right) = \frac{1}{c(s - 1)^2} + \frac{2(\gamma_0 - \log c)}{c(s - 1)} + \cdots,$$

and satisfies the functional equation

$$E\left(s, \frac{x}{c}\right) = 2(2\pi)^{2s-2}\Gamma^2(1-s)c^{1-2s}\left[E\left(1 - s, \frac{x}{c}\right) - \cos(\pi s)E\left(1 - s, -\frac{x}{c}\right)\right].$$

On the other hand, $E(s, D, \frac{x}{c})$ admits analytic continuation to the whole $s$-plane except a simple pole at $s = 1$ with residue

$$\mathop{\text{res}}_{s=1} E\left(s, D, \frac{x}{c}\right) = \delta_1(c)c^{-1}\chi_D(c)L(1, \chi_D) + \delta_2(c)c^{-1}\chi_D(x)\tau(\chi_D)L(1, \chi_D)$$

where

$$\delta_1(c) = \begin{cases} 1, & \text{if } |D|, c = 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\delta_2(c) = \begin{cases} 1, & \text{if } |D| \text{ divides } c \\ 0, & \text{otherwise} \end{cases}.$$
It satisfies the functional equation

\[ E\left(s, D, \frac{x}{c}\right) = -2i \frac{\Gamma^2(1-s)(2\pi)^{2s-2}}{|D|, c} \sin(\pi s) \sum_{a=1}^{\phi} \chi_D(a) \varphi\left(1 - s, a, D, \frac{x}{c}\right). \]

To define \( \varphi(s, a, D, \frac{x}{c}) \), we write \( \Delta = (|D|, c), c = \Delta \gamma, D = \Delta \delta, \) and \( \pi(\beta) \) for the inverse of \( \alpha \) modulo \( \beta \), and set \( \eta(\Delta) = 1 \) if \( \Delta > 1 \) and \( \eta(\Delta) = 0 \) if \( \Delta = 1 \). Define

\[
a_{m,n}(D, \frac{x}{c}) = \begin{cases} 
e^{-\eta(\Delta)(\Delta \delta)^{\gamma} + a \eta(\Delta)(\Delta \delta)^{\gamma}} \left( \frac{|D|}{c} \right)_{\gamma} & \text{if } \Delta \mid \eta(\Delta)ax + n \\
0, & \text{otherwise} \end{cases}
\]

and

\[
\tau(\ell, a, D, \frac{x}{c}) = \ne^{-\frac{\pi(\Delta) \delta(\gamma)^{\gamma}}{c} \ell} \sum_{\ell=mn} a_{m,n}(D, \frac{x}{c}).
\]

Then for \( \text{Re}(s) > 1 \), the function \( \varphi(s, a, D, \frac{x}{c}) \) is given by

\[
\varphi\left(s, a, D, \frac{x}{c}\right) = \sum_{\ell=1}^{\infty} \tau(\ell, a, D, \frac{x}{c}) \ell^{-s}.
\]

### 2.2.7 Results on two complex integrals

Let \( T_1(z) = \sin z \) and \( T_2(z) = \cos z \). We will need to deal with the following complex integrals

\[
\mathcal{I}_j(u, y) = \frac{1}{2\pi i} \int (y) \Gamma^2\left(\frac{2k-2u-z}{2}\right) \Gamma^2\left(\frac{\pi z}{2}\right) T_j\left(\frac{\pi z}{2}\right) y^{-\frac{\pi}{2}} \text{d}z.
\]

**Lemma 2.2.** Let \( k > 2 \) be any integer and \( y > 0 \). For \( 0 < \text{Re}(u) < k - 2 \) we have

\[
\mathcal{I}_2(u, 1) = 2 (-1)^k \cos(\pi u) \frac{\Gamma^2(k-u)}{\Gamma^2(k+u)}.
\]

**Proof.** This follows directly from [36, Lemma 2.3] and [10, §6.8, (36)].
Lemma 2.3. Fix $B > \frac{1}{4}$. For all large $k$ and $\frac{1}{4} \leq \text{Re}(u) \leq B$, we have

$$\mathcal{I}_j(u, y) \ll_B \begin{cases} (|\text{Im}(u)| + 1) y^{-\frac{1}{2}} k^{-2\text{Re}(u) - \frac{1}{2}} (\log k)^2, & \text{if } (2k)^{-1} \leq y \leq 1; \\ e^{\pi |\text{Im}(u)|} y \text{Re}(u) - k^{-2\text{Re}(u) - 1}, & \text{if } y > 1. \end{cases}$$

Proof. The first inequality follows from [36, Lemma 2.3] and [36, Lemma 2.4 (b)]. The second follows from the argument of [45, (8)]. \qed

2.3 The first moment of $L'(\frac{1}{2}, f_{K_D})$

By Lemma 2.1 (c) and Petersson’s formula, we have

$$\sum_{f \in \mathcal{H}_{2k}} \frac{L'(\frac{1}{2}, f_{K_D})}{L(1, \text{sym}^2 f)} = \frac{2k - 1}{\pi^2} \sum_{n \geq 1} \frac{\tau_D(n)W(n)}{\sqrt{n}} \frac{\Gamma(2k - 1)}{(4\pi)^{2k-1}} \sum_{f \in \mathcal{H}_{2k}} \frac{\lambda_f(n)\lambda_f(1)}{(f, f)}$$

$$= \frac{2k - 1}{\pi^2} \sum_{n \geq 1} \frac{\tau_D(n)W(n)}{\sqrt{n}} \left[ \delta_{n, 1} - 2\pi \sum_{c \geq 1} S(n, 1; c) \frac{S(n, 1; c)}{c} J_{2k-1} \left( \frac{4\pi \sqrt{n}}{c} \right) \right]$$

$$= \frac{2k - 1}{2\pi^2} (2D - 2J)$$

where

$$D = W(1) \quad \text{and} \quad J = 2\pi \sum_{n \geq 1} \frac{\tau_D(n)W(n)}{\sqrt{n}} \sum_{c \geq 1} S(n, 1; c) \frac{S(n, 1; c)}{c} J_{2k-1} \left( \frac{4\pi \sqrt{n}}{c} \right).$$

2.3.1 Diagonal contribution $D$

Inserting the definition of $W$, we have

$$D = \frac{1}{2\pi i} \int_{(3)} \left| D \right|^u (4\pi^2)^{-u} \gamma^2(u) L(1 + 2u, \chi_D) H(u) \, du$$

$$= \text{res}_{u=0} \left| D \right|^u (4\pi^2)^{-u} \gamma^2(u) L(1 + 2u, \chi_D) H(u) + \frac{1}{2\pi i} \int_{(-\frac{1}{2})} \cdots$$

$$= (2\psi(k) + \log |D| - 2\log 2\pi) L(1, \chi_D) + 2L'(1, \chi_D) + O_B(|D|^\frac{B}{2} k^{-B}),$$

30
by the bound \( (2.2) \) and that for \( u = -B/2 + it \) one has

\[
L(1 + 2u, \chi_D) = L(1 - B + it, \chi_D) \ll (|D|(|t| + 2))^{B - \frac{1}{2}}.
\]

### 2.3.2 Off-diagonal contribution \( J \)

By \( (2.5) \) we have

\[
J_{2k-1} \left( \frac{4\pi \sqrt{n}}{c} \right) = \frac{c}{4\pi \sqrt{n}} \frac{1}{2\pi i} \int_{(-1)} \left[ \frac{c}{2\pi \sqrt{n}} \right]^s \frac{\Gamma(\frac{2k+s}{2})}{\Gamma(\frac{2k-s}{2})} ds.
\]

Then we open the Kloosterman sums to get

\[
J = \frac{1}{2\pi i} \int_{(3)} |D|^u (4\pi^2)^{-u} \gamma^2(u) H(u) L(1 + 2u, \chi_D) \frac{1}{2} \sum_{c \geq 1} \sum' x(c) e \left( \frac{\overline{x}}{c} \right) \times \frac{1}{2\pi i} \int_{(-1)} \left( \frac{c}{2\pi} \right)^s \frac{\Gamma(\frac{2k+s}{2})}{\Gamma(\frac{2k-s}{2})} E \left( 1 + \frac{s}{2} + u, D, \frac{x}{c} \right) ds du.
\]

Shifting the \( s \)-integral to \( \text{Re}(s) = -B < 0 \) and picking up a pole at \( s = -2u \), we can write \( J \) as the sum of the residue part and the integral part

\[
J = J_R + J_I.
\]

By the residue of \( E(s, D, \frac{x}{c}) \) at \( s = 1 \) (see §[2.2.6]), we have

\[
J_R = \frac{1}{2\pi i} \int_{(3)} |D|^u \gamma(u) \gamma(-u) H(u) L(1 + 2u, \chi_D) \times \sum_{c \geq 1} \frac{1}{c^{1+2u}} \sum' x(c) e \left( \frac{\overline{x}}{c} \right) \left[ \delta_1(c) \chi_D(c) + \delta_2(c) \chi_D(x) \tau(\chi_D) \right] L(1, \chi_D) du.
\]

By the definition of \( \delta_1(c) \)

\[
\sum_{c \geq 1} \frac{1}{c^{1+2u}} \sum' x(c) e \left( \frac{\overline{x}}{c} \right) \delta_1(c) \chi_D(c) = \sum_{c \geq 1} \chi_D(c) \mu(c) = \sum_{c \geq 1} \frac{\chi_D(c) \mu(c)}{c^{1+2u}} = L(1 + 2u, \chi_D)^{-1}.
\]

31
By the definition of \( \delta_2(c) \)

\[
\sum_{c \geq 1} \frac{1}{c^{1+2u}} \sum_{x(c)} e\left(\frac{\overline{x}}{c}\right) \delta_2(c) \chi_D(x)
\]

\[
= \frac{1}{|D|^{1+2u}} \sum_{n \geq 1} \frac{1}{n^{1+2u}} \sum_{d|n|D} \mu(d) \sum_{x(n|D), d|x} \chi_D(x) e\left(\frac{x}{n|D|}\right)
\]

\[
= \frac{1}{|D|^{1+2u}} \sum_{n \geq 1} \frac{1}{n^{1+2u}} \sum_{d|n,(d,D)=1} \chi_D(d) \mu(d) \sum_{a(n|D|d^{-1})} \chi_D(a) e\left(\frac{a}{n|D|d^{-1}}\right)
\]

\[
= \frac{1}{|D|^{1+2u}} \sum_{\ell \geq 1} \frac{1}{\ell^{1+2u}} \sum_{d \geq 1} \chi_D(d) \mu(d) \sum_{a(\ell|D|)} \chi_D(a) e\left(\frac{a}{\ell|D|}\right)
\]

\[
= \frac{1}{|D|^{1+2u}} L(1 + 2u, \chi_D)^{-1} \sum_{h(\ell|D|)} \chi_D(h) e\left(\frac{h}{|D|}\right)
\]

\[
= \frac{\tau(\chi_D)}{|D|^{1+2u}} L(1 + 2u, \chi_D)^{-1},
\]

since by writing \( a = j|D| + h \) with \( j \) (mod \( \ell \)) and \( h \) (mod \( |D| \)) we easily see that

\[
\sum_{a(\ell|D|)} \chi_D(a) e\left(\frac{a}{\ell|D|}\right) \neq 0 \quad \text{only if} \quad \ell = 1.
\]

Hence

\[
J_R = L(1, \chi_D) \cdot \frac{1}{2\pi i} \int_{(3)} \left(|D|^u - |D|^{-u}\right) \gamma(u) \gamma(-u) H(u) \, du \tag{2.7}
\]

\[
= L(1, \chi_D) \cdot \frac{1}{2} \operatorname{Res}_{u=0} \left(|D|^u - |D|^{-u}\right) \gamma(u) \gamma(-u) H(u)
\]

\[
= L(1, \chi_D) \log |D|,
\]

where we used the fact that the integrand is an odd function of \( u \).
Now we are in the position of bounding $J_I$. By applying the functional equation of $E(s, D, \frac{z}{c})$, the $s$-integral in $J_I$ becomes

$$
2i \frac{(4\pi^2)^u}{c^{1+2u}} \frac{D}{|D|} \frac{1}{2\pi i} \int_{(-B)} \left[ \frac{\Gamma((|D|, c)}{|D|} \right]^{\frac{z}{2} + u} \frac{\Gamma(\frac{2k+u}{2})}{\Gamma(\frac{2k+2u}{2})} \times \Gamma^2 \left( -\frac{s}{2} - u \right) \sin \left( \pi(s + 2u) \right) \sum_{a=1}^{|D|} \chi_D(a) \varphi \left( -\frac{s}{2} - u, a, \frac{x}{c} \right) ds
$$

where we made a substitution $z = -s - 2u$ and expanded the Dirichlet series $\varphi(\frac{z}{2}, a, \frac{x}{c})$. So by the definition of $\mathcal{I}_1(u, y)$ in §2.2.7 we can write

$$
J_I = i \sum_{c \geq 1} \frac{1}{c} \int \frac{\chi(c)}{(x,c)} \frac{|D|}{|D|} \frac{1}{2\pi i} \int_{(B,c)} \frac{|D|^u}{c^{2u}} \gamma^2(u) H(u) L(1 + 2u, \chi_D) \times \sum_{a=1}^{|D|} \chi_D(a) \sum_{\ell \geq 1} \tau(\ell, a, D, \frac{x}{c}) \mathcal{I}_1 \left( u, \frac{(|D|, c)\ell}{|D|} \right) du,
$$

where $B_c$ depends on $c$. In view of Lemma 2.3 we further divide $J_I$ as a sum of $J_{I,1}$ and $J_{I,2}$, where the former denotes the contribution from $\ell \leq \frac{|D|}{(|D|, c)}$ and the latter from $\ell > \frac{|D|}{(|D|, c)}$. By Lemma 2.3 and taking $B_c = \frac{1}{2}$ for $c \leq k$ and $B_c = 1$ for $c > k$, we have

$$
J_{I,1} \ll \sum_{c \geq 1} \frac{\phi(c)}{c} \frac{|D|^2}{|D|} \frac{|D|^{B_c+1}}{c^{2B_c}} k^{2B_c} \sum_{\ell \leq \frac{|D|}{(|D|, c)}} \tau(\ell) \left[ \frac{(|D|, c)\ell}{|D|} \right]^{-\frac{1}{4}} k^{-2B_c - \frac{1}{2}} \times \int_{(B,c)} |H(u) L(1 + 2u, \chi_D)(|\text{Im}(u)| + 1)| du \ll k^{-\frac{1}{2}} (\log k)^2 \log |D| \left\{ \left| D \right|^2 \sum_{c \leq k} \frac{1}{c^2} + \left| D \right|^2 \sum_{c > k} \frac{1}{c^2} \right\} \ll |D|^2 k^{-\frac{1}{2}} (\log k)^2 \log |D|;
$$

33
on the other hand, by Lemma 2.3 and taking $B_c = 1$, we have

$$J_{1,2} \ll \sum_{c \geq 1} \frac{\phi(c)}{c} \frac{(|D|, c) |D|^2}{|D| c^2} \sum_{\ell > (|D|, c)} \tau(\ell) \left[ \frac{(|D|, c) \ell}{|D|} \right]^{1-k} k^{2-1}$$

$$\times \int_{(1)} |H(u) L(1 + 2u, \chi_D) (|\text{Im}(u)| + 1)| \, \text{d}u$$

$$\ll |D|^{2k-1} \sum_{c \geq 1} \frac{1}{c^2} \frac{(|D|, c)}{|D|} \sum_{\ell > (|D|, c)} \frac{\tau(\ell)}{\ell^2} \left( \frac{|D|^2}{(|D|, c)^2} \right)^{3-k}$$

$$\ll |D|^{3k-1} \sum_{c \geq 1} \frac{1}{c^2} \sum_{\ell > (|D|, c)} \tau(\ell) \left( 1 + \frac{1}{|D|} \right)^{3-k}$$

$$\ll |D|^{3k-1} e^{-\frac{k}{\log k}}.$$  

Here $\phi(n)$ denotes the Euler totient function. Hence, upon taking $|D| < k/(2 \log k)$ we have

$$J_I \ll |D|^3 k^{-\frac{3}{2}} (\log k)^4. \quad (2.8)$$

So Theorem 2.1(c) follows from (2.6), (2.7), and (2.8).

### 2.4 The mollified first moment of $L'(\frac{1}{2}, f)$

In this section we prove Proposition 2.1(a). Fix a $\delta$ in $(0, \frac{1}{2})$. By Lemma 2.1(a) and Petersson's formula

$$\sum_{f \in \mathcal{H}_{2k}} \frac{L'(\frac{1}{2}, f) M_f}{L(1, \text{sym}^2 f)}$$

$$= \frac{2k-1}{\pi^2} \sum_{n \geq 1} \mu(n) a_n \sum_{m \geq 1} \frac{U(m)}{\sqrt{m}} \left( \frac{\Gamma(2k-1)}{(4\pi)^{2k-1}} \right) \sum_{f \in \mathcal{H}_{2k}} \lambda_f(m) \lambda_f(n)$$

$$= \frac{2k-1}{\pi^2} \sum_{n \geq 1} \mu(n) a_n \sum_{m \geq 1} \frac{U(m)}{\sqrt{m}} \left[ \delta_{m,n} - 2\pi \sum_{c \geq 1} \frac{S(m, n; c)}{c} J_{2k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right) \right]$$

$$= \frac{2k-1}{2\pi^2} (2D_1 - 2J_1),$$

34
where
\[ D_1 = \sum_{n \geq 1} \frac{\mu(n) a_n U(n)}{n} \]

and
\[ J_1 = 2\pi \sum_{n \geq 1} \frac{\mu(n) a_n}{\sqrt{n}} \sum_{m \geq 1} \frac{U(m)}{\sqrt{m}} \sum_{c \geq 1} \frac{S(m, n; c)}{c} J_{2k-1}(\frac{4\pi \sqrt{mn}}{c}). \]

### 2.4.1 Diagonal contribution \( D_1 \)

Let \( B > 1 \). By the definitions of \( a_n \) and \( U \), we have

\[
D_1 = \frac{1}{2\pi i} \int_{(B)} \frac{\xi^{2z} - \xi^z}{z^3(\log \xi)^2} \frac{1}{2\pi i} \int_{(B)} \frac{H(u)}{(2\pi)^u} \frac{\gamma(u)}{\zeta(1 + u + z)} du dz
\]

\[
= \frac{1}{2\pi i} \int_{(B)} \frac{\xi^{2z} - \xi^z}{z^3(\log \xi)^2} \left[ \text{res}_{u=0} \frac{H(u)}{(2\pi)^u} \frac{\gamma(u)}{\zeta(1 + u + z)} \right] du dz
\]

\[
= D_{1,R} + D_{1,I},
\]

where

\[
D_{1,R} = \frac{1}{2\pi i} \int_{(B)} \frac{\xi^{2z} - \xi^z}{z^3(\log \xi)^2} \text{res}_{u=0} \frac{H(u)}{(2\pi)^u} \frac{\gamma(u)}{\zeta(1 + u + z)} dz
\]

\[
= \frac{1}{2\pi i} \int_{(B)} \frac{\xi^{2z} - \xi^z}{z^3(\log \xi)^2} \left[ \psi(k) - \log 2\pi - \frac{\zeta'(1 + z)}{\zeta(1 + z)} - \frac{\zeta'(1 + z)}{\zeta^2(1 + z)} \right] dz
\]

and

\[
D_{1,I} = \frac{1}{(2\pi i)^2} \int_{(-B+1)} \int_{(-B+1)} \frac{H(u)}{(2\pi)^u} \frac{\gamma(u)}{\zeta(1 + u + z)} du dz.
\]

First we treat \( D_{1,R} \). For \( \varepsilon > 0 \), define the contour \( C_\varepsilon = C_{\varepsilon,1} \cup C_{\varepsilon,2} \) which starts from \(-i\infty\), where

\[
C_{\varepsilon,1} = \left\{ z = \varepsilon e^{i\theta} \mid \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \right\} \quad \text{and} \quad C_{\varepsilon,2} = \{ z = it \mid |t| \geq \varepsilon \}.
\]
Fix $\varepsilon$ so small that $C_\varepsilon$ is inside the classical zero-free region of the Riemann zeta-function. Then we shift the integral in $D_{1,R}$ to $C_\varepsilon$, picking up a pole at $z = 0$ with residue

$$
\text{res}_{z=0} \frac{\xi^{2z} - \xi^z}{z^3(\log \xi)^2} \left[ \frac{\psi(k) - \log 2\pi}{\zeta(1+z)} - \frac{\zeta'(1+z)}{\zeta^2(1+z)} \right] = \frac{\psi(k) - \log 2\pi}{\log \xi} + \frac{3}{2} - \frac{2\gamma_0}{\log \xi},
$$

which is computed from the expansions $\zeta(1+z)^{-1} = z + \cdots$,

$$
\frac{\xi^{2z} - \xi^z}{z^3(\log \xi)^2} = \frac{1}{z^2 \log \xi} + \frac{3}{2z} + \cdots \quad \text{and} \quad -\frac{\zeta'(1+z)}{\zeta^2(1+z)} = 1 - 2\gamma_0 z + \cdots.
$$

Thus we have

$$
D_{1,R} = \frac{\psi(k) - \log 2\pi}{\log \xi} + \frac{3}{2} - \frac{2\gamma_0}{\log \xi} + \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{\xi^{2z} - \xi^z}{z^3(\log \xi)^2} \left[ \frac{\psi(k) - \log 2\pi}{\zeta(1+z)} - \frac{\zeta'(1+z)}{\zeta^2(1+z)} \right] dz = \left( \frac{3}{2} + \frac{1}{\delta} \right) + O\left( \frac{1}{\log k} \right).
$$

For the contour shifting business and bounding the integral over $C_\varepsilon$, we used the classical fact that there exists an absolute constant $C > 0$ such that $\zeta(s) \neq 0$ in the region

$$
\{ s = \sigma + it \mid \sigma \geq 1 - C(\log |t|)^{-1}, \ |t| \geq 2 \}
$$

in which the following bounds hold:

$$
\frac{\zeta'(s)}{\zeta(s)} \ll \log |t| \quad \text{and} \quad \frac{1}{\zeta(s)} \ll \log |t|.
$$

For $D_{1,I}$, we have $\text{Re}(1 + u + z) = 2$ and thus $\zeta(1 + u + z)^{-1} \ll 1$. So with (2.2) and $|\xi^{2z} - \xi^z| \ll e^{2\text{Re}(z)\log \xi} = k^{2\delta B}$, we get by taking suitably large $B$ that

$$
D_{1,I} \ll_b \int_{(B)} \frac{k^{2\delta B} \log \xi^2 |dz|}{|z|^3} \int_{(-B+1)} k^{1-B} \frac{e^{-u^2}}{|u|^2} |du| \ll_b k^{-(1-2\delta)B+1} \ll k^{-10}.
$$
Therefore
\[ \mathcal{D}_1 = \left( \frac{3}{2} + \frac{1}{\delta} \right) + O\left( \frac{1}{\log k} \right). \] (2.9)

### 2.4.2 Off-diagonal contribution $\mathcal{J}_1$

We write $\mathcal{J}_1 = \mathcal{J}_{1,1} + \mathcal{J}_{1,2}$, by splitting the $c$-sum according to $c \leq \frac{4\pi \sqrt{mn}}{k}$ and $c > \frac{4\pi \sqrt{mn}}{k}$. Let $B > 1$. Note that the $n$-sum is in fact up to $\xi^2 = k^{2\delta}$ by the definition of $a_n$.

If $c \leq \frac{4\pi \sqrt{mn}}{k}$, then $m \geq \frac{c^{2k^{2-2\delta}}}{16\pi^2} \geq k^{2-2\delta}$. By (2.2), (2.4) and Weil’s bound we have
\[
\mathcal{J}_{1,1} \ll \sum_{n \leq \xi^2} \frac{|a_n|}{\sqrt{n}} \sum_{m \geq \frac{c^{k^{2-2\delta}}}{16\pi^2}} \frac{|U(m)|}{\sqrt{m}} \sum_{c \leq \frac{4\pi \sqrt{mn}}{k}} \frac{|S(m, n; c)|}{c} \\
\ll_B \sum_{n \leq \xi^2} \frac{1}{\sqrt{n}} \sum_{m \geq \frac{c^{k^{2-2\delta}}}{16\pi^2}} \frac{k^B m^{\frac{1}{2} + B}}{m^{\frac{1}{2}} c^{4\pi \sqrt{mn}}} \sum_{c \leq 4\pi \sqrt{mn}} \frac{n^{\frac{1}{2}} c^{\frac{1}{2}} \tau(c)}{c} \\
\ll_B k^{-(1-2\delta)B+3} \ll k^{-10},
\]
upon taking a sufficiently large $B$.

If $c > \frac{4\pi \sqrt{mn}}{k}$, i.e., $\frac{4\pi \sqrt{mn}}{c} < k$, we have by (2.4)
\[
\mathcal{J}_{2k-1}\left( \frac{4\pi \sqrt{mn}}{c} \right) \ll 2^{-k} k^{-\frac{3}{2}} \frac{\sqrt{mn}}{c}.
\]
Then by (2.2) and Weil’s bound, we have
\[
\mathcal{J}_{1,2} \ll 2^{-k} k^{-\frac{3}{2}} \sum_{n \leq \xi^2} \frac{|a_n|}{\sqrt{n}} \sum_{m \geq 1} \frac{|U(m)|}{\sqrt{m}} \sum_{c > \frac{4\pi \sqrt{mn}}{k}} \frac{|S(m, n; c)| \sqrt{mn}}{c} \\
\ll 2^{-k} k^{-\frac{3}{2}} \sum_{n \leq \xi^2} \frac{1}{\sqrt{n}} \sum_{m \geq 1} \frac{k^{\frac{3}{2}}}{m^{\frac{1}{2}} c^{\frac{1}{2}} + \frac{3}{2}} \sum_{c \geq 1} \frac{n^{\frac{1}{2}} c^{\frac{1}{2}} \tau(c) \sqrt{mn}}{c} \\
\ll k^{-10}.
\]

As a result,
\[
\mathcal{J}_1 \ll k^{-10} \tag{2.10}
\]
and Proposition 2.1(a) follows from (2.9) and (2.10).
2.5 The mollified second moment of $L'(\frac{1}{2}, f)$

In this section we prove Proposition 2.1(b). First fix a $\delta$ in $(0, \frac{1}{32})$. By the Hecke relation we can write

$$M_f^2 = \sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} A_{r,n}$$

where

$$A_{r,n} = \sum_{n=n_1n_2} \mu(r_n)a_{r_1n_1}\mu(r_2n_2).$$

Then by Lemma 2.1(b) and Petersson’s formula, we have

$$\sum_{f \in \mathcal{H}_{2k}} L'(\frac{1}{2}, f)^2M_f^2$$

$$= \frac{2k-1}{\pi^2} \sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} \frac{A_{r,n}}{\sqrt{n}} \sum_{m \geq 1} \frac{\tau(m)V(m)\Gamma(2k-1)}{\sqrt{m}} \sum_{f \in \mathcal{H}_{2k}} \frac{\lambda_f(m)\lambda_f(n)}{(f, f)}$$

$$= \frac{2k-1}{\pi^2} \sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} \frac{A_{r,n}}{\sqrt{n}} \sum_{m \geq 1} \frac{\tau(m)V(m)}{\sqrt{m}}$$

$$\times \left[ \delta_{m,n} - 2\pi \sum_{c \geq 1} \frac{S(m,n;c)}{c} J_{2k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right) \right]$$

$$= \frac{2k-1}{2\pi^2} (2D_2 - 2J_2),$$

where

$$D_2 = \sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} A_{r,n} \frac{\tau(n)V(n)}{n}$$

and

$$J_2 = 2\pi \sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} \frac{A_{r,n}}{\sqrt{n}} \sum_{m \geq 1} \frac{\tau(m)V(m)}{\sqrt{m}} \sum_{c \geq 1} \frac{S(m,n;c)}{c} J_{2k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right).$$

2.5.1 Diagonal contribution $D_2$

By the definition of $A_{r,n}$ and

$$\tau(m_1m_2) = \sum_{s \mid (m_1, m_2)} \mu(s)\tau\left( \frac{m_1}{s} \right)\tau\left( \frac{m_2}{s} \right)$$
we rewrite $D_2$ as

$$D_2 = \sum_{r \leq \xi^2} \frac{1}{r} \sum_{s \leq \xi^2} \frac{\mu(s)}{s^2} \sum_{n_1, n_2} \frac{\tau(n_1) \mu(rsn_1) a_{rsn_1}}{n_1} \frac{\tau(n_2) \mu(rsn_2) a_{rsn_2}}{n_2} V(s^2 n_1 n_2).$$

By shifting the integral for $V(s^2 n_1 n_2)$ to Re$(u) = -\frac{B}{2}$ for $B > 0$, we get

$$V(s^2 n_1 n_2) = \frac{1}{2\pi i} \int_{(3)} \zeta(1 + 2u) \frac{\gamma^2(u) K(u)}{(4\pi^2 s^2 n_1 n_2)^u} du$$

$$= \text{res}_{u=0} \zeta(1 + 2u) \frac{\gamma^2(u) K(u)}{(4\pi^2 s^2 n_1 n_2)^u} + \frac{1}{2\pi i} \int_{(-\frac{B}{2})} \zeta(1 + 2u) \frac{\gamma^2(u) K(u)}{(4\pi^2 s^2 n_1 n_2)^u} du$$

$$= V_R(s^2 n_1 n_2) + V_I(s^2 n_1 n_2),$$

according to which we put $D_2 = D_{2,R} + D_{2,I}$. We claim that

$$D_{2,R} \ll 1. \quad (2.11)$$

First we compute the residue at the pole $u = 0$ of order 4

$$V_R(s^2 n_1 n_2) = \text{res}_{u=0} \zeta(1 + 2u) \frac{\gamma^2(u) K(u)}{(4\pi^2 s^2 n_1 n_2)^u}$$

$$= \frac{2}{3} \psi^3(k) + (2\gamma_0 - 2 \log 2\pi s \sqrt{n_1 n_2}) \psi^2(k)$$

$$- \left[ 4\gamma_1 + 4\gamma_0 \log 2\pi s \sqrt{n_1 n_2} - 2(\log 2\pi s \sqrt{n_1 n_2})^2 \right] \psi(k)$$

$$+ \left[ 2\gamma_2 + 4\gamma_1 \log 2\pi s \sqrt{n_1 n_2} + 2\gamma_0 (\log 2\pi s \sqrt{n_1 n_2})^2 \right.$$  

$$- \frac{2}{3}(\log 2\pi s \sqrt{n_1 n_2})^3 \right] + \psi(k) \psi'(k)$$

$$+ (\gamma_0 - \log 2\pi s \sqrt{n_1 n_2}) \psi'(k) + \frac{1}{6} \psi''(k).$$
by using the expansions $K(u) = u^3 - u + \cdots$,
\[
\zeta(1 + 2u) = \frac{1}{2u} + \gamma_0 - 2\gamma_1 u + 2\gamma_2 u^2 + \cdots,
\]
\[
\gamma^2(u) = 1 + 2\psi(k)u + [2\psi^2(k) + \psi'(k)]u^2
+ \frac{1}{3} [4\psi^3(k) + 6\psi(k)\psi'(k) + \psi''(k)]u^3 + \cdots,
\]
\[
(4\pi^2 s^2 n_1 n_2)^{-u} = 1 - 2(\log 2\pi s\sqrt{n_1 n_2})u + 2(\log 2\pi s\sqrt{n_1 n_2})^2 u^2
- \frac{4}{3}(\log 2\pi s\sqrt{n_1 n_2})^3 u^3 + \cdots.
\]

Thus by inspecting the terms in (2.12), we need only the following bounds in order to show the claimed bound (2.11):
\[
(\log k)^a \sum_{r \leq \xi^2} \frac{1}{r} \sum_{s \leq \xi^2} \frac{\mu(s)}{s^2} \sum_{n_1} \tau(n_1) \mu(rsn_1) (\log n_1)^b \sum_{n_2} \tau(n_2) \mu(rsn_2) (\log n_2)^c \ll 1,
\]
for integers $a, b, c \geq 0$ with $b \geq c$ and $a + b + c = 3$.

**Remark 2.5.** Since we will need to bound similar sums arising in §2.5.2, we leave the involving establishment of such bounds to the last section. See Remark 2.6 in the last section for why the above bounds are sufficient.

Next we shall see that $D_{2,I}$ is negligible in size. By (2.2), $s^2 n_1 n_2 \ll k^{4\delta}$ and that $\zeta(1 + 2u) = \zeta(1 - B + 2it) \ll |2t|^{B - \frac{1}{2}} \log |2t|$ for $u = -B^2 + it$ and $|t| \geq 1$, we have
\[
V_1(s^2 n_1 n_2) \ll_B \left( \frac{s^2 n_1 n_2}{kB} \right)^{B/2} \left( 1 + \int_{|t| \geq 1} \frac{|2t|^{B-\frac{1}{2}} \log |2t| e^{-\left(\frac{B^2}{2} + it\right)}}{|-B^2 + it|^3} \, dt \right)
\ll_B k^{-(1-2\delta)B}.
\]

Hence, we have
\[
D_{2,I} \ll_B k^{-(1-2\delta)B} \sum_{r \leq \xi^2} \frac{1}{r} \sum_{s \leq \xi^2} \frac{1}{s^2} \sum_{n_1, n_2 \leq \xi^2} \frac{\tau(n_1) \tau(n_2)}{n_1 n_2} \tau(n_1) \tau(n_2) \ll_B k^{-(1-2\delta)B} (\log k)^5 \ll k^{-10},
\]
by choosing a suitably large $B$.

### 2.5.2 Off-diagonal contribution $\mathcal{J}_2$

By \((2.5)\) we have

$$J_{2k-1}(\frac{4\pi\sqrt{mn}}{c}) = \frac{c}{4\pi\sqrt{n} \ 2\pi i} \int (-1) \left[ \frac{c}{2\pi\sqrt{n}} \right]^{s} \frac{\Gamma(\frac{2k+s}{2})}{\Gamma(\frac{2k-s}{2})} \frac{1}{m^{\frac{s}{2}} + \frac{s}{2}} ds.$$  

Then by inserting the definitions of $A_{r,n}$ and $V$ and opening the Kloosterman sums, we get

$$\mathcal{J}_2 = \sum_{r \geq 1} \frac{1}{r} \sum_{n_1, n_2} \frac{\mu(rn_1)a_{rn_1}\mu(rn_2)a_{rn_2}}{n_1n_2} \mathcal{J}_2(n_1n_2)$$  

where

$$\mathcal{J}_2(n_1n_2) = \frac{1}{2} \sum_{c \geq 1} \sum_{x(c)} \frac{e(\frac{n_1n_2}{c})}{2\pi i} \int (4\pi^2)^{-u} \zeta(1 + 2u)\gamma^2(u)K(u)$$

$$\times \frac{1}{2\pi i} \int (-1) \left[ \frac{c}{2\pi\sqrt{n_1n_2}} \right]^{s} \frac{\Gamma(\frac{2k+s}{2})}{\Gamma(\frac{2k-s}{2})} E\left(1 + s + u, \frac{x}{c}\right) ds du$$

and $E(s, \frac{x}{c})$ denotes the Estermann zeta-function.

Next we shift the $s$-integral in $\mathcal{J}_2(n_1n_2)$ to $\text{Re}(s) = -B$ for $B > 6$ to get

$$\mathcal{J}_2(n_1n_2) = \mathcal{J}_{2,R}(n_1n_2) + \mathcal{J}_{2,I}(n_1n_2),$$

where

$$\mathcal{J}_{2,R}(n_1n_2) = \frac{1}{2} \sum_{c \geq 1} \sum_{x(c)} \frac{e(\frac{n_1n_2}{c})}{2\pi i} \int (4\pi^2)^{-u} \zeta(1 + 2u)\gamma^2(u)K(u)$$

$$\times \left\{ \text{res}_{s=-2u} \left[ \frac{c}{2\pi\sqrt{n_1n_2}} \right]^{s} \frac{\Gamma(\frac{2k+s}{2})}{\Gamma(\frac{2k-s}{2})} E\left(1 + s + u, \frac{x}{c}\right) \right\} du$$
and

\[ J_{2,1}(n_1n_2) = \frac{1}{2} \sum_{c \geq 1} \sum_{x(c)}' e\left(\frac{n_1n_2x}{c}\right) \frac{1}{2\pi i} \int_{(3)} (4\pi^2)^{-u} \zeta(1+2u) \gamma^2(u) K(u) \]

\[ \times \frac{1}{2\pi i} \int_{(-B)} \left[ \frac{c}{2\pi \sqrt{n_1n_2}} \right] * \frac{\Gamma\left(\frac{2k+s}{2}\right)}{\Gamma\left(\frac{2k-s}{2}\right)} E\left(1 + \frac{s}{2} + u, \frac{x}{c}\right) ds du. \]

Correspondingly, we write \( J_2 = J_{2,R} + J_{2,I} \).

We claim that

\[ J_{2,R} \ll 1 \]

(2.15)

and argue as follows. In \( J_{2,R}(n_1n_2) \) we have

\[ \text{res}_{s=-2u} \left[ \frac{c}{2\pi \sqrt{n_1n_2}} \right] * \frac{\Gamma\left(\frac{2k+s}{2}\right)}{\Gamma\left(\frac{2k-s}{2}\right)} E\left(1 + \frac{s}{2} + u, \frac{x}{c}\right) = 4 \left(\frac{4\pi^2 n_1n_2}{c^{1+2u}} \right) \frac{\Gamma(k-u)}{\Gamma(k+u)} \Psi_{n_1n_2}(u), \]

where \( \Psi_{n_1n_2}(u) = \gamma_0 - \log 2\pi \sqrt{n_1n_2} + \frac{1}{2}[\psi(k-u) + \psi(k+u)] \). With \( \tau_u(n) := \sum_{n=ab(a/b)^u} \), we have

\[ \tau_u(n_1n_2) = \sum_{s|n_1n_2} \mu(s) \tau_u\left(\frac{n_1}{s}\right) \tau_u\left(\frac{n_2}{s}\right) \]

and

\[ (n_1n_2)^u \sum_{c \geq 1} \frac{S(0, n_1n_2; c)}{c^{1+2u}} = \frac{\tau_u(n_1n_2)}{\zeta(1+2u)}, \]

and have

\[ J_{2,R}(n_1n_2) = \frac{2}{2\pi i} \int_{(3)} \gamma(-u) \gamma(u) \Psi_{n_1n_2}(u) \tau_u(n_1n_2) K(u) du \]

\[ = \text{res}_{u=0} \gamma(-u) \gamma(u) \Psi_{n_1n_2}(u) \tau_u(n_1n_2) K(u) \]

\[ = \sum_{s|n_1n_2} \mu(s) \cdot \text{res}_{u=0} \gamma(-u) \gamma(u) \Psi_{n_1n_2}(u) \tau_u\left(\frac{n_1}{s}\right) \tau_u\left(\frac{n_2}{s}\right) K(u), \]

since the integrand is an odd function in \( u \).
Setting \( \tau(m, n) := \frac{\partial^2}{\partial u^2} \bigg|_{u=0} \tau_u(m) \tau_u(n) \), we compute

\[
\text{res}_{u=0} \gamma(-u) \gamma(u) \Psi_{n_1 n_2}(u) \tau_u \left( \frac{n_1}{s} \right) \tau_u \left( \frac{n_2}{s} \right) K(u)
\]

\[
= \frac{1}{2} \tau \left( \frac{n_1}{s} \right) \tau \left( \frac{n_2}{s} \right) \psi(k) + \frac{1}{2} \left( \gamma_0 - \log 2\pi \sqrt{n_1 n_2} \right) \tau \left( \frac{n_1}{s} \right) \tau \left( \frac{n_2}{s} \right) \psi'(k)
\]

\[
+ \frac{1}{2} \tau \left( \frac{n_1}{s} \right) \tau \left( \frac{n_2}{s} \right) \psi''(k),
\]

by the expansions \( K(u) = u^{-3} - u + \cdots \),

\[
\tau_u(m) \tau_u(n) = \tau(m) \tau(n) + \frac{1}{2} \tau(m, n) u^2 + \cdots,
\]

\[
\gamma(-u) \gamma(u) = 1 + \psi'(k) u^2 + \cdots,
\]

\[
\frac{1}{2} [\psi(k - u) + \psi(k + u)] = \psi(k) + \frac{1}{2} \psi''(k) u^2 + \cdots.
\]

Hence we infer that

\[
J_{2,R} = \sum_{r \leq \xi^2} \sum_{s \leq \xi^2} \frac{\mu(s)}{s^2} \sum_{n_1, n_2} \frac{\mu(r n_1) a_{rs n_1} \mu(r n_2) a_{rs n_2}}{n_1 n_2} \left[ \tau(n_1) \tau(n_2) \psi(k) + \frac{1}{2} \left( \gamma_0 - \log 2\pi \sqrt{n_1 n_2} \right) \tau(n_1) \tau(n_2) \psi'(k) \right.
\]

\[
+ \frac{1}{2} \tau(n_1) \tau(n_2) \psi''(k) \right] .
\]

To achieve the claimed (2.15), on the one hand we apply the bounds (A) to the summands in (2.16) with factor \( \tau(n_1) \tau(n_2) \); on the other hand we apply the bounds (B) below, to be proved in the last section, to those summands in (2.16) with factor \( \tau(n_1, n_2) \):

\[
(\log k)^6 \sum_{r \leq \xi^2} \sum_{s \leq \xi^2} \frac{\mu(s)}{s^2} \left( \log n_1 \right)^b \ll 1,
\]

\[
\times \sum_{n_1, n_2} \tau(n_1, n_2) \mu(r n_1) a_{rs n_1} \mu(r n_2) a_{rs n_2} \left( \log n_1 \right)^b \ll 1.
\]
where \( a, b \geq 0 \) are integers with \( a + b = 1 \).

Next we treat \( J_{2,1} \) by analyzing each \( J_{2,1}(n_1n_2) \). By the functional equation of \( E(s, \frac{z}{c}) \), we have

\[
J_{2,1}(n_1n_2) = J_{2,1}^+(n_1n_2) + J_{2,1}^-(n_1n_2)
\]

and deal with each term separately.

By \((2.2)\) and \( n_1n_2 \ll k^{4\delta} \), we get for sufficiently large \( B \)

\[
J_{2,1}^+(n_1n_2) \ll k^{2\delta B} \int_{(3)} \int_{(-B)} k^6 |K(u)| (k + |\text{Im}(u + z/2)|)^{-B} |\Gamma \left( \frac{z}{2} \right) |^2 |dz||du| \\
\ll_B k^{-(1-2\delta)B+6} \ll k^{-10},
\]

where we also used the bound obtained by Stirling’s formula

\[
\frac{\Gamma(k-w)}{\Gamma(k+w)} \ll_{\text{Re}(w)} (k + |\text{Im}(w)|)^{-2\text{Re}(w)}.
\]

For \( J_{2,1}^-(n_1n_2) \), we first expand \( E(s, \frac{z}{c}) \) into Dirichlet series to get

\[
J_{2,1}^-(n_1n_2) = \sum_{c \geq 1} \sum'_{x(c)} e \left( \frac{n_1n_2x}{c} \right) \frac{1}{(2\pi i)^2} \int_{(3)} \frac{\zeta(1+2u)}{c^{1+2u}} \gamma^2(u)K(u) \int_{(-B)} \frac{1}{(n_1n_2)^2} \frac{\Gamma \left( \frac{2k+s}{2} \right)}{\Gamma \left( \frac{2k-s}{2} \right)} \\
\times \Gamma^2 \left( -\frac{s}{2} - u \right) \left[ E \left( -\frac{s}{2} - u, \frac{\pi}{c} \right) + \cos \frac{\pi(s+2u)}{2} E \left( -\frac{s}{2} - u, -\frac{\pi}{c} \right) \right] ds du \\
\times \frac{\Gamma \left( \frac{2k-2u-z}{2} \right)}{\Gamma \left( \frac{2k+2u+z}{2} \right)} \Gamma \left( \frac{z}{2} \right) \left[ E \left( \frac{z}{2}, \frac{\pi}{c} \right) + \cos \frac{\pi z}{2} E \left( \frac{z}{2}, -\frac{\pi}{c} \right) \right] dz du,
\]

where we made a substitution \( z = -s - 2u \). According to \( E(\frac{s}{2}, \frac{\pi}{c}) \) and \( E(\frac{s}{2}, -\frac{\pi}{c}) \) in the above, we write

\[
J_{2,1}^-(n_1n_2) = J_{2,1}^+ (n_1n_2) + J_{2,1}^- (n_1n_2)
\]
For $n_1n_2 = 1$, (2.2) and Lemma 2.2 imply that the contribution to $J_{21}^-(1)$ from $n = 1$ is

$$\begin{align*}
&\frac{-2}{2\pi i} \int (\sum_{c \geq 1} c^{1+2u} \zeta(1+2u) \gamma^2(u) K(u) \cos(\pi u) \frac{\Gamma^2(k-u)}{\Gamma^2(k+u)} du \\
&= \frac{-2}{2\pi i} \int \zeta(2u) \cos(\pi u) \gamma^2(-u) K(u) du \\
&= \frac{-2}{2\pi i} \int \zeta(2u) \cos(\pi u) \gamma^2(-u) K(u) du \\
&\ll_B k^{-B},
\end{align*}$$

for any $B > 0$, where we used the fact that $\zeta(s - 1)/\zeta(s) = \sum_{n \geq 1} \phi(n)n^{-s}$ for $\text{Re}(s) > 2$; note that our definition of $K(u)$ is more than enough to suppress the growth of $\cos(\pi u)$. By (2.2) and Lemma 2.3, the contribution to $J_{21}^-(1)$ from $n \geq 2$ is

$$\ll \int (k^6 |K(u)| e^{\pi |\text{Im}(u)|} \sum_{n \geq 2} \frac{\tau(n)}{n^{k-3}} k^{-7} |du| \ll k^{-1} 2^{-k}.$$ 

Thus for any $B > 0$

$$J_{21}^-(1) \ll_B k^{-B}.$$ 

For $n_1n_2 \geq 2$ we estimate $J_{21}^-(n_1n_2)$ based on the $n$-sum according to $n \leq n_1n_2$ or $n > n_1n_2$. By (2.2) and Lemma 2.3 we have that the contribution to $J_{21}^-(n_1n_2)$ from $n \leq n_1n_2$ is

$$\ll \int (k^6 (n_1n_2)^3 |K(u)| \sum_{n \leq n_1n_2} \tau(n)(|\text{Im}(u)| + 1) \left(\frac{n_1n_2}{n}\right)^{\frac{1}{2}} k^{-6-\frac{1}{2}} (\log k)^2 |du| \ll (n_1n_2)^{4} k^{-\frac{1}{2}} (\log k)^3$$

and that the contribution to $J_{21}^-(n_1n_2)$ from $n > n_1n_2$ is

$$\ll \int (k^6 (n_1n_2)^3 |K(u)| e^{\pi |\text{Im}(u)|} \sum_{n > n_1n_2} \frac{\tau(n)}{n^{k-3}} (n_1n_2)^{k-3} k^{-7} |du| \ll (n_1n_2)^{4} k^{-1} \log k.$$
To summarize, we have that for all $n_1, n_2$ with $n_1n_2 \geq 1$

$$J_{2,1}(n_1n_2) \ll (n_1n_2)^4k^{-\frac{1}{2}}(\log k)^3$$

and thus that

$$J_{2,1} \ll k^{-\frac{1}{2}}(\log k)^3 \sum_{r \leq \xi^2} \sum_{n_1, n_2 \leq \xi^2} (n_1n_2)^3 \ll k^{-\frac{1}{2}+16\delta}(\log k)^4. \quad (2.17)$$

Finally, Proposition 2.1 (b) follows from (2.11), (2.14), (2.15), and (2.17).

2.6 The first and second moments of $L'(\frac{1}{2}, f)$

2.6.1 Proof of Theorem 2.1 (a)

By Lemma 2.1 (a) and Petersson’s formula, we have

$$\sum_{f \in \mathcal{H}_{2k}} \frac{L'(\frac{1}{2}, f)}{L(1, \text{sym}^2 f)} = \frac{2k-1}{2\pi^2} (2D_1 - 2J_1),$$

where

$$D_1 = U(1) \quad \text{and} \quad J_1 = 2\pi \sum_{n \geq 1} \frac{U(n)}{\sqrt{n}} \sum_{c \geq 1} \frac{S(n, 1; c)}{c} J_{2k-1} \left( \frac{4\pi \sqrt{n}}{c} \right).$$

First we have

$$D_1 = \frac{1}{2\pi i} \int_{(3)} (2\pi)^{-u} \gamma(u) H(u) \, du$$

$$= \text{res}_{u=0} (2\pi)^{-u} \gamma(u) H(u) + \frac{1}{2\pi i} \int_{(-B)} (2\pi)^{-u} \gamma(u) H(u) \, du$$

$$= \psi(k) - \log 2\pi + \frac{1}{2\pi i} \int_{(-B)} (2\pi)^{-u} \gamma(u) H(u) \, du$$

$$= \psi(k) - \log 2\pi + O_B(k^{-B})$$

for any $B > 0$, in view of (2.2). While for $J_1$ we again write $J_1 = J_{1,1} + J_{1,2}$ by splitting the $c$-sum according to $c \leq \frac{4\pi \sqrt{n}}{k}$ and $c > \frac{4\pi \sqrt{n}}{k}$. Then arguments similar to that for $J_1$ in §2.4.2 show that $J_1 = O_B(k^{-B})$ for $B > 0$. Thus Theorem 2.1 (a) follows.
2.6.2 Proof of Theorem 2.1(b)

By Lemma 2.1(b) and Petersson’s formula, we have

\[
\sum_{f \in \mathcal{H}_{2k}} \frac{L'(1/2, f)^2}{L(1, \text{sym}^2 f)} = \frac{2k - 1}{2\pi^2} (2D_2 - 2J_2),
\]

where

\[
D_2 = V(1) \quad \text{and} \quad J_2 = 2\pi \sum_{n \geq 1} \frac{\tau(n)V(n)}{\sqrt{n}} \sum_{c \geq 1} \frac{S(n, 1; c)}{c} J_{2k-1} \left( \frac{4\pi \sqrt{n}}{c} \right).
\]

From §2.5.1 we see that \( D_2 = V_R(1) + V_I(1) =: D_{2,R} + D_{2,I} \), for which (2.12) and (2.13) imply that

\[
D_{2,R} = \frac{2}{3} \psi^3(k) + 2(\gamma_0 - \log 2\pi)\psi^2(k)
\]

\[
\quad + \left[ -4\gamma_1 - 4\gamma_0 \log 2\pi + 2(\log 2\pi)^2 \right] \psi(k)
\]

\[
\quad + \left[ 2\gamma_2 + 4\gamma_1 \log 2\pi + 2\gamma_0 (\log 2\pi)^2 - \frac{2}{3} (\log 2\pi)^3 \right]
\]

\[
\quad + \psi(k)\psi'(k) + (\gamma_0 - \log 2\pi)\psi'(k) + \frac{1}{6} \psi''(k)
\]

and that for any \( B > 0 \)

\[
D_{2,I} \ll_b k^{-B}.
\]

It is not difficult to see that \( J_2 \) equals \( J_2(1) \) in §2.5.2. Then proceeding as in §2.5.2 and noticing that \( \tau(1, 1) = 0 \), we have

\[
J_{2,R} := J_{2,R}(1) = \psi(k)\psi'(k) + (\gamma_0 - \log 2\pi)\psi'(k) + \frac{1}{2} \psi''(k)
\]

and for any \( B > 0 \)

\[
J_{2,I} := J_{2,I}(1) \ll_b k^{-B}.
\]
Then (2.18) and (2.20) give the main term in Theorem 2.1 (b). From

\[ \psi(z) = \log z - \frac{1}{2z} - 2 \int_0^\infty \frac{t \, dt}{(t^2 + z^2)(e^{2 \pi t} - 1)} \] (see [62, p. 251])

it follows that \( \psi''(k) = -k^{-2} + O(k^{-3}) \) for large \( k \). So requiring \( B > 2 \) in (2.21) gives the error term in Theorem 2.1 (b).

2.7 Proof of bounds (A) and (B)

We introduce some auxiliary notations, which look strange at first glance but enable us to shorten expressions which are originally very lengthy. Define a (finite) Dirichlet series

\[ D_{u,r,s}(z) = \sum_n \frac{\tau_u(n) \mu(rs n) a_{rs n}}{n^{k+z}} \]

for which we use the usual notation \( D_{u,r,s}^{(j)}(z) \) for its \( j \)-th derivative in \( z \). Also define an operator

\[ P = \left. \frac{\partial^2}{\partial u^2} \right|_{u=0} \]

for which we use \( P[h] \) to denote its action on a function \( h \). Then the bounds (A) are equivalent to the following set of bounds

\[ (\log k)^a \sum_{r \leq \xi^2} \frac{1}{r} \sum_{s \leq \xi^2} \frac{\mu(s)}{s^2} D_{0,r,s}^{(b)}(0) D_{0,r,s}^{(c)}(0) \ll 1, \quad (A') \]

where \( a, b, c \geq 0 \) are integers with \( b \geq c \) and \( a + b + c = 3 \). Since \( \tau(n_1, n_2) = P[\tau_u(n_1) \tau_u(n_2)] \) by definition and the involved sums are finite, the bounds (B) are equivalent to the following bounds

\[ (\log k)^a \sum_{r \leq \xi^2} \frac{1}{r} \sum_{s \leq \xi^2} \frac{\mu(s)}{s^2} P[D_{u,r,s}^{(b)}(0) D_{u,r,s}(0)] \ll 1, \]
where \(a, b \geq 0\) are integers with \(a + b = 1\). Taking the derivatives in \(\mathcal{P}[D^{(b)}_{u,r,s}(0)D_{u,r,s}(0)]\) by the product rule and noticing that \(D_{u,r,s}(z)\) and \(D'_{u,r,s}(z)\) are even functions in \(u\), we see that it suffices to show

\[
(\log k)^a \sum_{r \leq \xi^2} \frac{1}{r} \sum_{s \leq \xi^2} \frac{\mu(s)}{s^2} \mathcal{P}[D^{(b)}_{u,r,s}(0)]D^{(c)}_{0,r,s}(0) \ll 1, \tag{B'}
\]

where \(a, b, c \geq 0\) are integers with \(a + b + c = 1\).

In order to show \((A')\), it suffices to establish the bounds

\[
D^{(j)}_{0,r,s}(0) \ll (\log k)^{j-2} |\mu(rs)| \prod_{p|r} \left(1 + \frac{1}{\sqrt{p}}\right), \quad j = 0, 1, 2, 3, \tag{D_j}
\]

since these imply the following inequalities: for integers \(a, b, c \geq 0\) with \(b \geq c\) and \(a + b + c = 3\)

\[
(\log k)^a D^{(b)}_{0,r,s}(0)D^{(c)}_{0,r,s}(0) \ll \frac{1}{\log k} |\mu(rs)| \prod_{p|r} \left(1 + \frac{1}{\sqrt{p}}\right)^2,
\]

and then \((A')\) follows in view of Lemma 2.4 below. Similarly, to get \((B')\), it is enough to show

\[
\mathcal{P}[D^{(j)}_{u,r,s}(0)] \ll (\log k)^j |\mu(rs)| \prod_{p|r} \left(1 + \frac{1}{\sqrt{p}}\right), \quad j = 0, 1, \tag{PD_j}
\]

**Lemma 2.4.** For large \(X > 0\), we have

\[
\sum_{r \leq X} \frac{|\mu(r)|}{r} \prod_{p|r} \left(1 + \frac{1}{\sqrt{p}}\right)^2 \ll \log X
\]

and for any \(\sigma > 0\)

\[
\sum_{r \leq X} \frac{|\mu(r)|}{r^{1+\sigma}} \prod_{p|r} \left(1 + \frac{1}{\sqrt{p}}\right)^2 \ll_{\sigma} 1.
\]

**Remark 2.6.** A careful reader notices that when one tries to bound \(D_{2,R} (\S 2.5.1)\) and \(J_{2,R} (\S 2.5.2)\), there are certain cases with a factor \((\log s)^j\) appearing in the \(s\)-sum for
some $1 \leq j \leq 3$. Such concern can be remedied by the second bound in Lemma 2.4.

The proofs for Lemma 2.4 and the upcoming Lemma 2.5 will be provided at the end of this section.

We consider the nontrivial case $1 \leq r s \leq \xi^2$ with $(r, s) = 1$. Restrict $u$ to the interval $(-\frac{\xi}{2}, \frac{\xi}{2})$. Then by the definition of $a_n$ one can show that

$$D_{u, r, s}(0) = \frac{\mu(rs)}{2\pi i} \int_{(2)} T_{u, r, s}(z) R_{r, s}(z) \, dz,$$

where

$$T_{u, r, s}(z) = \sum_{n, r, s = 1} \frac{\tau_u(n)\mu(n)}{n^{1+z}} \quad \text{and} \quad R_{r, s}(z) = \begin{cases} \left(\frac{\xi^2}{rs}\right) - \left(\frac{\xi}{rs}\right)^z, & 1 \leq rs < \xi, \\ z^3(\log \xi)^2, & \xi \leq rs < \xi^2. \end{cases}$$

Also one can easily check that

$$D_{0, r, s}^{(j)}(0) = \frac{\mu(rs)}{2\pi i} \int_{(2)} T_{0, r, s}^{(j)}(z) R_{r, s}(z) \, dz, \quad j = 0, 1, 2, 3,$$

$$\mathcal{P}[D_{u, r, s}^{(j)}(0)] = \frac{\mu(rs)}{2\pi i} \int_{(2)} \left[ \frac{d^j}{dz^j} \mathcal{P}[T_{u, r, s}(z)] \right] R_{r, s}(z) \, dz, \quad j = 0, 1.$$

The procedure for proving bounds $(D_j)$ and $(\mathcal{P}D_j)$ is to shift the contour of the above integrals to the contour $C_\varepsilon$ defined in §2.4.1 and estimate the possible residues and the resulting integrals over $C_\varepsilon$. Along the arguments we will point out analytic properties of $T_{u, r, s}(z)$ and related functions when we need them. First, with $u \in (-\frac{\xi}{2}, \frac{\xi}{2})$, we impose further $\varepsilon < \frac{1}{10}$ and write for large $\text{Re}(z)$

$$T_{u, r, s}(z) = \frac{L_{u, r, s}(z)}{\zeta(1-u+z)\zeta(1+u+z)} \prod_{p|r, s} \left(1 - \frac{\tau_u(p)}{p^{1+z}}\right)^{-1},$$

where by Euler products

$$L_{u, r, s}(z) = e^{i(u)} \prod_{p|r, s, p<10} \left(1 - \frac{\tau_u(p)}{p^{1+z}}\right) \cdot \prod_{p<10} \left(1 - \frac{1}{p^{1-u+z}}\right)^{-1} \left(1 - \frac{1}{p^{1+u+z}}\right)^{-1}.$$
and
\[ l_u(z) = \sum_{p>10} \sum_{m \geq 2} \frac{p^{mu} + p^{-mu} - \tau_u(p)^m}{mp^m(1+z)}. \]
Since the series representing \( l_u(z) \) converge absolutely in \( \text{Re}(z) \geq -\frac{1}{2} + \varepsilon \), \( L_{u,r,s}(z) \) is analytic and bounded in \( \text{Re}(z) \geq -\frac{1}{2} + \varepsilon \).

Remark 2.7. Note that \( T_{0,r,s} \) has a double zero at \( z = 0 \) if \( 2 \mid rs \), or a triple zero if \( 2 \nmid rs \), and that \( R_{r,s} \) has a double pole at \( z = 0 \) if \( 1 \leq rs \leq \xi \), or a triple pole if \( \xi < rs \leq \xi^2 \). As for establishing \( [D] \) and \( [PD] \), we will show details only for the case with \( 2 \mid rs \) and \( 1 \leq rs \leq \xi \) since other cases are similar. To further simplify notations in the rest of this section, we mute the appearance of \( 0, r, s \) in the subscripts of the involved functions; for example, we write \( T_u(z) \) for \( T_{u,r,s}(z) \) and \( D_{0,r,s}(0) \) for \( D(0) \).

First we focus on establishing \( [D] \). Although \( (D_0) \) and \( (D_1) \) were already obtained in [45], we need parts of their proof for other cases and thus derive them here with details.

Case \( (D_0) \). Since \( T \cdot R \) is analytic at \( z = 0 \), we have
\[ D(0) = \frac{\mu(rs)}{2\pi i} \int_{C_\varepsilon} (T \cdot R)(z) \, dz. \]
By the boundedness of \( L(z) \) in \( \text{Re}(z) \geq -\frac{1}{2} + \varepsilon \), that \( \zeta(1+z)^{-1} \ll 1 \) on \( C_{\varepsilon,1} \) and \( \ll \max\{1, \log |\text{Im}(z)|\} \) on \( C_{\varepsilon,2} \), and the elementary bound
\[ \left|1 - \frac{2}{p^{1+z}}\right|^{-1} < 1 + \frac{1}{\sqrt{p}} \quad \text{for all } p > 10 \text{ and } \text{Re}(z) \geq -\frac{1}{2} + \varepsilon, \] (2.22)
we have
\[ (T \cdot R)(z) \ll \begin{cases} \frac{1}{(\log \xi)^2} \prod_{p \mid rs} \left(1 + \frac{1}{\sqrt{p}}\right), & z \in C_{\varepsilon,1}, \\ \max\{1, (\log |\text{Im}(z)|)^2\} \prod_{p \mid rs} \left(1 + \frac{1}{\sqrt{p}}\right), & z \in C_{\varepsilon,2}, \end{cases} \] (2.23)
which holds for all \(1 \leq rs \leq \xi^2\) with \((r, s) = 1\). So \((D_0)\) holds.

**Remark 2.8.** To treat the more complicated cases, we will make free use of \((2.22)\), \((2.23)\), and the following classical results for the Riemann zeta-function: there exists a constant \(C' > 0\) such that in the region \(\{s = \sigma + it \mid \sigma \geq 1 - C'(\log |t|)^{-1}, |t| \geq 2\}\) it holds that

\[
\zeta^{(j)}(s) \ll (\log |t|)^{j+1}, \quad \frac{\zeta'(s)}{\zeta(s)} \ll \log |t|, \quad \frac{1}{\zeta(s)} \ll \log |t|.
\]

The following lemma is also useful.

**Lemma 2.5.** Let \(j > 0\) be an integer. For any \(0 \leq \alpha \leq \frac{1}{2}\) and large \(r\) we have

\[
\sum_{p|rs} \frac{(\log p)^j}{p^{1-\alpha}} \ll (\log r)^\alpha (\log \log r)^j.
\]

**Case (D\(_1\)).** We compute

\[
T'(z) = T(z) \left[ -2 \frac{\zeta'(1+z)}{\zeta(1+z)} + \sum_{p>10} \sum_{m \geq 2} \frac{(2^m - 2) \log p}{p^{m(1+z)}} \right. \\
\left. + \sum_{p<10} \frac{-2 \log p}{p^{1+z} - 1} + \sum_{p|rs, p<10} \frac{2 \log p}{p^{1+z} - 2} + \sum_{p|rs, p>10} \frac{-2 \log p}{p^{1+z} - 2} \right]
\]

\[= T(z) \sum_{\ell=1}^{5} g_\ell(z).\]

Here \(\ell\) denotes the order of appearance of a summand inside the above brackets. Since \(T \cdot R \cdot g_1\) has a simple pole at \(z = 0\) and \(T \cdot R \cdot g_\ell\ (2 \leq \ell \leq 5)\) are analytic at \(z = 0\), we have

\[
D'(0) = \mu(rs) \mathop{\text{res}}_{z=0} T \cdot R \cdot g_1 + \frac{\mu(rs)}{2\pi i} \sum_{\ell=1}^{5} \int_{C_\epsilon} T \cdot R \cdot g_\ell(z) \, dz.
\]

For the residue, we compute, in view of \(z g_1(z) = 2 - 2\gamma_0 z + \cdots\), to get

\[
\mathop{\text{res}}_{z=0} T \cdot R \cdot g_1 = \frac{2L(0)}{\log \xi} \prod_{p|rs, p>10} \left(1 - \frac{2}{p}\right)^{-1} \ll \frac{1}{\log \xi} \prod_{p|rs} \left(1 + \frac{1}{\sqrt{p}}\right).
\]
We observe that \( g_\ell(z) \) \((2 \leq \ell \leq 4)\) and its derivatives up to order 3 are bounded in 
\[ \text{Re}(z) \geq -\frac{1}{2} + \varepsilon. \]
So
\[
\sum_{\ell=1}^{4} \int_{C_\varepsilon} (T \cdot R \cdot g_\ell)(z) \, dz \ll \frac{1}{(\log \xi)^2} \prod_{p|rs} \left( 1 + \frac{1}{\sqrt{p}} \right).
\]
By Lemma 2.5
\[
\int_{C_\varepsilon} (T \cdot R \cdot g_5)(z) \, dz \ll \frac{(\log \xi)^{\varepsilon}}{(\log \xi)^2} \prod_{p|rs} \left( 1 + \frac{1}{\sqrt{p}} \right).
\]
Thus \((D_1)\) holds.

Case \((D_2)\). By (2.24) we have
\[
T''(z) = T(z) \left[ \left( \sum_{\ell=1}^{5} g_\ell(z) \right)^2 + \sum_{\ell=1}^{5} g'_\ell(z) \right],
\] (2.25)
where
\[
g'_1(z) = 2 \left[ \frac{\zeta'(1+z)^2}{\zeta(1+z)^2} - \frac{\zeta''(1+z)}{\zeta(1+z)} \right],
\]
\[
g'_2(z) = \sum_{p>10} \sum_{m \geq 2} \frac{m(2 - 2^m)(\log p)^2}{p^m(1+z)}, \quad g'_3(z) = \sum_{p<10} \frac{2p^{1+z}(\log p)^2}{(p^{1+z} - 1)^2},
\]
\[
g'_4(z) = \sum_{p|rs \atop p<10} \frac{-2p^{1+z}(\log p)^2}{(p^{1+z} - 2)^2}, \quad g'_5(z) = \sum_{p|rs \atop p>10} \frac{2p^{1+z}(\log p)^2}{(p^{1+z} - 2)^2}.
\]
Inside the brackets of \( T''(z) \), only \( g_1^2 + g'_1 \) and \( g_1 g_\ell \) \((2 \leq \ell \leq 5)\) contribute poles at \( z = 0 \). First, \( T \cdot R \cdot (g_1^2 + g'_1) \) has a double pole at \( z = 0 \) with residue
\[
\text{res}_{z=0} T \cdot R \cdot (g_1^2 + g'_1) = \frac{2L(0)}{\log \xi} \left[ \frac{L'(0)}{L(0)} - \sum_{p|rs \atop p>10} \frac{2 \log p}{p - 2} - 6\gamma_0 + \frac{3}{2} \log \xi - \log rs \right] \prod_{p|rs \atop p>10} \left( 1 - \frac{2}{p} \right)^{-1}
\]
\[
\ll \prod_{p|rs} \left( 1 + \frac{1}{\sqrt{p}} \right),
\]
53
which can be seen from the expansions

\[
L(z) \prod_p \left(1 - \frac{2}{p^{1+z}}\right)^{-1} = L(0) \prod_p \left(1 - \frac{2}{p}\right)^{-1} \left[1 + \left(L'(0) \frac{L(0)}{L(0)} - \sum_{p|rs} \frac{2 \log p}{p-2}\right) z + \cdots\right],
\]

(2.26)

\[
\zeta(1+z)^{-2} R(z) = \frac{1}{\log \xi} + \left(\frac{3}{2} - \frac{\log rs}{\log \xi} - \frac{2\gamma_0}{\log \xi}\right) z + \cdots,
\]

(2.27)

\[
z^2(g_1^2 + g_1')(z) = 2 - 8\gamma_0 z + \cdots.
\]

(2.28)

Next, \(T \cdot R \cdot g_1 g_\ell (2 \leq \ell \leq 5)\) has a simple pole at \(z = 0\) with residue

\[
\text{res}_{z=0} T \cdot R \cdot g_1 g_\ell = \frac{2L(0)g_\ell(0)}{\log \xi} \prod_p \left(1 - \frac{2}{p}\right)^{-1} \ll \frac{\log \log \xi}{\log \xi} \prod_p \left(1 + \frac{1}{\sqrt{p}}\right).
\]

We easily see, especially with Lemma 2.5, that all resulting integrals over \(C_\varepsilon\) yield smaller contribution. Hence (D2) holds.

Case (D3). By (2.24) and (2.25) we have

\[
T'''(z) = T(z) \left[\left(\sum_{\ell=1}^5 g_\ell(z)\right)^3 + 3\left(\sum_{\ell=1}^5 g_\ell(z)\right)\left(\sum_{\ell=1}^5 g_\ell'(z)\right) + \sum_{\ell=1}^5 g_\ell''(z)\right],
\]

where

\[
g_1''(z) = -2\frac{\zeta'''(1+z)}{\zeta(1+z)} - 4\frac{\zeta'(1+z)^3}{\zeta(1+z)^3} + 6\frac{\zeta'(1+z)\zeta''(1+z)}{\zeta(1+z)^2},
\]

\[
g_2''(z) = \sum_{p>10} \sum_{m \geq 2} \frac{m^2(2m^2 - 2)(\log p)^3}{p^{m(1+z)}}, \quad g_3''(z) = \sum_{p<10} \frac{-2p^{1+z}(p^{1+z} + 1)(\log p)^3}{(p^{1+z} - 1)^3},
\]

\[
g_4''(z) = \sum_{p|rs} \frac{2p^{1+z}(p^{1+z} + 2)(\log p)^3}{(p^{1+z} - 2)^3}, \quad g_5''(z) = \sum_{p|rs} \frac{-2p^{1+z}(p^{1+z} + 2)(\log p)^3}{(p^{1+z} - 2)^3}.
\]

Inside the brackets of \(T'''(z)\), we find four types of terms, neglecting those nonessential coefficients:

a) \(g_1^3 + 3g_1 g_1' + g_1''\),  

b) \((g_2^2 + g_1')\left(\sum_{j=2}^5 g_j\right)\),  
c) \(g_1\left[\left(\sum_{j=2}^5 g_j\right)^2 + \sum_{j=2}^5 g_j^2\right]\).
and d) others; here only type (d) is analytic at $z = 0$. For type (a), $T \cdot R \cdot (g_1^3 + 3g_1 g'_1 + g''_1)$ has a double pole at $z = 0$ with residue

$$\text{res}_{z=0} T \cdot R \cdot (g_1^3 + 3g_1 g'_1 + g''_1) \cdot R = \frac{-12\gamma_0 L(0)}{\log \xi} \left[ \frac{L'(0)}{L(0)} - \sum_{p|rs \atop p > 10} \frac{2 \log p}{p - 2} - \frac{4\gamma_1}{\gamma_0} - 6\gamma_0 - \frac{3}{2} \log \xi - \log rs \right]$$

$$\ll \prod_{p|rs} \left(1 + \frac{1}{\sqrt{p}}\right),$$

which can be seen from expansions (2.26), (2.27) and

$$z^2(g_1^3 + 3g_1 g'_1 + g''_1)(z) = -12\gamma_0 + 48(\gamma_1 + \gamma_0^2)z + \cdots.$$

For type (b), $T \cdot R \cdot (g_1^2 + g'_1)(\sum_{\ell=2}^5 g_\ell)$ has a double pole at $z = 0$ with residue

$$\text{res}_{z=0} T \cdot R \cdot (g_1^2 + g'_1)(\sum_{\ell=2}^5 g_\ell) = \frac{2L(0)}{\log \xi} \left[ \frac{L'(0)}{L(0)} - \sum_{p|rs \atop p > 10} \frac{2 \log p}{p - 2} - 6\gamma_0 \right.\left. + \frac{\sum_{\ell=2}^5 g'_\ell(0)}{\sum_{\ell=2}^5 g_\ell(0)} - \frac{3}{2} \log \xi - \log rs \prod_{p|rs \atop p > 10} \left(1 - \frac{2}{p}\right)^{-1}\right]$$

$$\ll \log \xi \cdot \prod_{p|rs} \left(1 + \frac{1}{\sqrt{p}}\right),$$

in view of (2.26) $\sim$ (2.28) and Lemma 2.5. For type (c), $T \cdot R \cdot g_1 \cdot \left[ (\sum_{\ell=2}^5 g_\ell)^2 + \sum_{\ell=2}^5 g'_\ell \right]$ has a simple pole at $z = 0$ with residue

$$\text{res}_{z=0} T \cdot R \cdot g_1 \cdot \left[ (\sum_{\ell=2}^5 g_\ell)^2 + \sum_{\ell=2}^5 g'_\ell \right] = \frac{2L(0)}{\log \xi} \left[ (\sum_{\ell=2}^5 g_\ell(0))^2 + \sum_{\ell=2}^5 g'_\ell(0) \right] \prod_{p|rs \atop p > 10} \left(1 - \frac{2}{p}\right)^{-1}$$

$$\ll \prod_{p|rs} \left(1 + \frac{1}{\sqrt{p}}\right).$$

55
Similar to previous cases, the resulting integrals over $C_\varepsilon$ yield smaller contribution. So (D_3) holds.

Now we turn to (PD_j). It is not hard to obtain

\[
\mathcal{P}[T_u(z)] = T(z) \left[ 2\left( \frac{\zeta'(1+z)^2}{\zeta(1+z)^2} - \frac{\zeta''(1+z)}{\zeta(1+z)} \right) + \sum_{p>10} \sum_{m \geq 2} \frac{(2m - 2^m)(\log p)^2}{p^m(1+z)} \right.
\]
\[
+ \sum_{p<10} \frac{2p^{1+z}(\log p)^2}{(p^{1+z} - 1)^2} + \sum_{p|rs, p<10} \frac{-2(\log p)^2}{p^{1+z} - 2} + \sum_{p|rs, p>10} \frac{2(\log p)^2}{p^{1+z} - 2} \left. \right]
\]
\[
= : T(z) \sum_{\ell=1}^5 h_\ell(z).
\]

Again, $\ell$ denotes the order of the appearance of a summand in the brackets. We notice that $\mathcal{P}[T_u(z)]$ is very similar to $T''(z)$, especially in that $h_1(z) = g'_1(z)$. So a similar (but easier) argument as that of (D_2) shows that (PD_0) holds. Differentiating $\mathcal{P}[T_u(z)]$ with respect to $z$ gives

\[
\frac{d}{dz} \mathcal{P}[T_u(z)] = T(z) \left[ \left( \sum_{\ell=1}^5 g_\ell(z) \right) \left( \sum_{\ell=1}^5 h_\ell(z) \right) + \sum_{\ell=1}^5 h'_\ell(z) \right],
\]

where

\[
h'_1(z) = f''_1(z), \quad h'_2(z) = \sum_{p>10} \sum_{m \geq 2} \frac{m(2^m - 2^m)(\log p)^3}{p^m(1+z)}, \quad h'_3(z) = g''_3(z),
\]
\[
h'_4(z) = \sum_{p|rs, p<10} \frac{2p^{1+z}(\log p)^3}{(p^{1+z} - 2)^2}, \quad h'_5(z) = \sum_{p|rs, p>10} \frac{-2p^{1+z}(\log p)^3}{(p^{1+z} - 2)^2}.
\]

Inside the brackets of $\frac{d}{dz} \mathcal{P}[T_u(z)]$ we find four type of terms:

a) $g_1 h_1 + h'_1$,  
b) $h_1 \left( \sum_{\ell=2}^5 g_\ell \right)$,  
c) $g_1 \left( \sum_{\ell=2}^5 h_\ell \right)$,  
and d) others.

Type (a) and type (b) have a double pole at $z = 0$, type (c) has a simple pole at $z = 0$, and type (d) is analytic at $z = 0$. Then analysis similar to that of (D_3) leads to (PD_1).
Proof of Lemma 2.4. Since the argument has a similar spirit as our treatment of \(|D_j|\), we will omit certain details. The trick is to extend the range of summation and work instead with

\[
\sum_{r \geq 1} \frac{|\mu(r)|}{r^{1+\sigma}} \prod_{p \mid r} \left(1 + \frac{1}{\sqrt{p}}\right)^2 e^{-\frac{r}{X}} \quad (\sigma \geq 0).
\]

We can write

\[
\sum_{r \geq 1} \frac{|\mu(r)|}{r^{1+s}} \prod_{p \mid r} \left(1 + \frac{1}{\sqrt{p}}\right)^2 = \zeta(1+s)L(s)
\]

and show as before by using Euler products that \(L(0) \neq 0\) and that \(L(s)\) and \(L'(s)\) are analytic and bounded in \(\text{Re}(s) \geq -\frac{1}{2} + \varepsilon\) for sufficiently small \(\varepsilon > 0\). Let \(\sigma \geq 0\).

Using the Mellin inversion for \(e^{-x}\) we have for some \(c > 0\)

\[
\sum_{r \geq 1} \frac{|\mu(r)|}{r^{1+\sigma}} \prod_{p \mid r} \left(1 + \frac{1}{\sqrt{p}}\right)^2 e^{-\frac{r}{X}} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s)\zeta(1 + s)L(\sigma+s)X^s ds.
\]

Then the lemma follows from shifting the integral to \(C_\varepsilon\) and estimating the residue at \(s = 0\) and the resulting integrals over \(C_\varepsilon\).

Proof of Lemma 2.5. By \(\sum_{p \leq X} \frac{\log p}{p} = \log X + O(1)\) and partial summation, we have

\[
\sum_{p \leq X} \frac{(\log p)^j}{p} = \frac{1}{j} (\log X)^j + O((\log X)^{j-1})
\]

and thus

\[
\sum_{p \leq X} \frac{(\log p)^j}{p^{1-\alpha}} \ll X^\alpha (\log X)^j.
\]

Since \(x^{\alpha-1}(\log x)^j\) is decreasing for \(x \geq \log r\) (if \(r \geq e^{2j}\)), we can split \(\sum_{p \mid r}\) as the sum of \(\sum_{p \mid r, p < \log r}\) and \(\sum_{p \mid r, p \geq \log r}\) and obtain the lemma by applying the last displayed and the bound \(\sum_{p \mid r} 1 \ll \log r\) to the two sums, respectively.

57
Chapter 3: The first moment of symmetric square $L$-functions

3.1 Introduction

In this chapter we study the average behavior of central values of the symmetric square $L$-functions $L(s, \text{sym}^2 f)$ for $f \in \mathcal{H}_k$ for large even integers $k$\footnote{One reason why we use in this chapter weight $k$ instead of weight $2k$ is that the sign of functional equation (3.5) of $L(s, \text{sym}^2 f)$ for $f \in \mathcal{H}_k$ is always positive, whereas that of $L(s, f)$ depends on $k$.}. Several authors (Lau \cite{35}, Fomenko \cite{12}, Khan \cite{26}, and Sun \cite{57}) obtained asymptotic formulas for the first moment of central values of symmetric square $L$-functions by using Petersson’s formula and various analytic techniques, among which the strongest results is

$$\sum_{f \in \mathcal{H}_k} \omega_f L(\frac{1}{2}, \text{sym}^2 f) = \psi(k - \frac{1}{2}) + 2\gamma + \frac{1}{2} \psi\left(\frac{3}{4}\right) - \log(2\pi^{\frac{3}{4}}) + O(k^{-\frac{1}{2}}),$$

due to Fomenko and Sun independently. Here we recall that for $f \in \mathcal{H}_k$

$$\omega_f = \frac{\Gamma(k - 1)}{(4\pi)^{k-1}(f, f)}.$$ 

Also $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function and $\gamma$ stands for the Euler constant.

We shall prove an improved asymptotic formula with arbitrary power saving when $k$ is large in the following theorem, extracting two secondary main terms from the error term $O(k^{-\frac{1}{2}})$ in previous studies.
**Theorem 3.1** (Restatement of Theorem 1.2). For any $B > 0$ and sufficiently large even integer $k > 0$ we have

$$
\sum_{f \in H_k} \omega_f L\left(\frac{1}{2}, \text{sym}^2 f\right) = \\
\psi\left(k - \frac{1}{2}\right) + 2\gamma + \frac{1}{2} \psi\left(\frac{3}{4}\right) - \log\left(2\pi^{\frac{3}{2}}\right)
$$

(M1)

$$
+ i^{-k} \sqrt{\pi} L\left(\frac{1}{2}, \chi_4\right) \frac{\Gamma\left(k - \frac{1}{2}\right)}{\Gamma\left(k + \frac{1}{2}\right)}
$$

(M4)

$$
+ \sqrt{2\pi} i^{-k} L\left(\frac{1}{2}, \chi_3\right) \left(\frac{2}{\sqrt{3}}\right)^{k - \frac{1}{2}} F\left(\frac{k - \frac{1}{2}}{2}, \frac{k - \frac{1}{2}}{2}, \frac{1}{3}; -1\right) \frac{\Gamma\left(k - \frac{1}{2}\right)}{\Gamma\left(k + \frac{1}{2}\right)}
$$

(M3)

$$
+ O_B\left(k^{-B}\right).
$$

Here $\chi_D(\cdot) = (\frac{D}{\cdot})$ denotes the Kronecker symbol and $F(a, b; c; z)$ is the Gauss hypergeometric function, initially defined for $|z| < 1$ by

$$
F\left(a, b; c; z\right) = \sum_{n \geq 0} \frac{(a)_n (b)_n z^n}{(c)_n n!},
$$

where $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$ for $n \geq 1$.

**Remark 3.1.** In the process of preparing this chapter of the work for publication, the author learned that Balkanova–Frolenkov [2] independently obtained a slightly stronger result by using a different method, replacing the arbitrary power saving by an exponential decay error term.

Now we give the plan of this chapter with some comments. Our approach is based on Petersson’s formula, as in the works by Lau, Khan and Sun, but is different in that we use a different approximate functional equation (Lemma 3.1) at the very beginning of all analysis. After including necessary ingredients and tools in §3.2 we obtain the primary main term $M_1$ in §3.3. The new approximate functional equation makes a
real difference in §3.4 where we analyze the off-diagonal contribution: it gives us a handle to reduce the analysis at various stages to the problem of counting solutions of certain quadratic congruence equations, and thus allows us to extract the secondary main terms $M_{-4}$ and $M_{-3}$. The labeling of the main terms $M_D$ with $D$ a discriminant is used to indicate that the corresponding analysis involves the zeta function $\zeta_{K_D}(s)$ of the quadratic field $K_D$ with discriminant $D$, where we treat $\mathbb{Q}$ as the degenerate “quadratic” field with discriminant 1 (see (3.7), Lemma 3.3 and Lemma 3.4). We also note that the central values of the Dirichlet $L$-functions in $M_{-4}$ and $M_{-3}$ are nonzero: by Mathematica one sees

$$L(\frac{1}{2}, \chi_{-4}) \approx 0.667691 \quad \text{and} \quad L(\frac{1}{2}, \chi_{-3}) \approx 0.480868.$$ 

We remark that the treatment of the off-diagonal contribution is inspired by the work of Lau–Tsang [36].

Now we need to say more about $M_{-4}$ and $M_{-3}$. The main term $M_{-3}$ looks bizarre at first glance but can be shown by further analysis (§4.3.1) to be

$$3^{\frac{3}{4}} \sqrt{2\pi} L(\frac{1}{2}, \chi_{-3}) \frac{\Gamma(k - \frac{1}{2})}{\Gamma(k)} [S(k) + O(k^{-1})], \quad (M'_{-3})$$

where

$$S(k) = \begin{cases} -1, & \text{if } k \equiv 2 \pmod{6}, \\ 0, & \text{if } k \equiv 4 \pmod{6}, \\ 1, & \text{if } k \equiv 0 \pmod{6}. \end{cases}$$

By Barnes’ formula [9, (1.18.12)] we have for large $k$

$$\frac{\Gamma\left(\frac{k-\frac{1}{2}}{2}\right)}{\Gamma\left(\frac{k+\frac{1}{2}}{2}\right)} = \sqrt{2} k^{-\frac{1}{2}} e^{O(k^{-1})} \quad \text{and} \quad \frac{\Gamma(k - \frac{1}{2})}{\Gamma(k)} = k^{-\frac{1}{2}} e^{O(k^{-1})}.$$ 

Hence the secondary main term $M_{-4}$ is of order exactly $k^{-\frac{3}{2}}$, and so is $M_{-3}$ when $k \not\equiv 2 \pmod{4}$; also $M_{-3}$ is $O(k^{-\frac{3}{2}})$ if $k \equiv 2 \pmod{3}$. 

60
We remark that our result, especially the presence of \( M_{-4} \) and \( M_{-3} \), exhibits a connection between \( L\left(\frac{1}{2}, \text{sym}^2 f\right) \) and quadratic fields. This connection is foreshadowed by the remarkable result of Zagier (see [65, Theorem 1]) that a modular form \( \Phi_s(z) \) of weight \( k \) is constructed with Fourier coefficients being infinite linear combinations of zeta functions of quadratic fields such that

\[
(\Phi_s, f) = \frac{i^k \pi}{2^{k-3} (k-1)} \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} L\left(s, \text{sym}^2 f\right).
\]

It is worth pointing out that Kohnen–Sengupta [29] and Fomenko [12] used the following consequence (see [29, (2)]2 or [12, (17)]) of the above identity to investigate the first moment in question:

\[
\sum_{f \in H_k} \omega_f L\left(\frac{1}{2}, \text{sym}^2 f\right) = \frac{i^k \pi 2^{k-2} \Gamma(k)}{\sqrt{\pi} \Gamma(k - \frac{1}{2})} \left\{ \sum_{t \geq 0, t \neq \pm 1} \left[ I_k(t^2 - 4, t; \frac{1}{2}) + I_k(t^2 - 4, -t; \frac{1}{2}) \right] L\left(\frac{1}{2}, t^2 - 4\right) \right. \\
\left. + \lim_{s \to \frac{1}{2}} \left[ 2(I_k(0, 2; s) + I_k(0, -2; s))\zeta(2s - 1) + \frac{i^k \Gamma(s + k - 1)\zeta(2s)}{2^{2s+k-3} \pi^{s-1} \Gamma(k)} \right] \right\}. \tag{3.1}
\]

We note that in the above formula \( L\left(\frac{1}{2}, t^2 - 4\right) = L\left(\frac{1}{2}, \chi_{t^2 - 4}\right) \) for \( t = 0, \pm 1 \), and refer the reader to [65, Section 4] for the definitions of \( L(s, \Delta) \) and \( I_k(\Delta, t; s) \) where \( \Delta = t^2 - 4m \) (\( m = 1 \) for our case). Without explicitly stating an asymptotic formula, Kohnen and Sengupta isolated the secondary main term \( M_{-4} \) (the “\( t = 0 \)” term in (3.1)). Fomenko obtained an asymptotic formula with error term of size \( O(k^{-1/2}) \).

By comparing (3.1) and Theorem 3.1 we can see that the total contribution from all \(|t| \geq 3\) in (3.1) is \( O_k(k^{-B}) \) for any fixed \( B > 0 \) and sufficiently large \( k \), which is difficult to obtain by direct computation.

\[2\text{In Eq. (2) of [29], } 2^{3k-3} \text{ should be replaced by } 2^{3k-4}, \text{ while } I_k(-4, 0; \frac{1}{2}) \text{ by } 2I_k(-4, 0; \frac{1}{2}).\]
3.2 Preparation

In this section, we set up notations and gather some preparatory results to be used later. We write $e(z)$ for $e^{2\pi iz}$, use $B$ for large real numbers, and reserve $p$ for prime numbers.

3.2.1 Functional equations

We include here functional equations of zeta-functions and $L$-functions which will be used at various points.

The Riemann zeta-function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \quad (s \neq 0, 1). \quad (3.2)$$

The periodic zeta-function function $F(s,a) = \sum_{n \geq 1} e(na)n^{-s}$ ($\Re(s) > 1, 0 < a < 1$) satisfies the functional equation

$$F(s,a) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ e\left(\frac{1-s}{4}\right)\zeta(1-s,a) + e\left(\frac{s-1}{4}\right)\zeta(1-s,1-a) \right\}, \quad (3.3)$$

where $\zeta(s,a) = \sum_{n \geq 0} (n+a)^{-s}$ is the Hurwitz zeta-function.

For a primitive Dirichlet character $\chi \mod q$, the $L$-function $L(s,\chi)$ is entire. Let $\delta_{\chi} = \frac{1}{2}(1 - \chi(-1))$. The complete $L$-function $\Lambda(s,\chi) = (q/\pi)^{\frac{s+\delta_{\chi}}{2}}\Gamma(\frac{s+\delta_{\chi}}{2})L(s,\chi)$ satisfies the functional equation

$$\Lambda(s,\chi) = \varepsilon(\chi)\Lambda(1-s,\chi) \quad (3.4)$$

where the root number $\varepsilon(\chi) = i^{-\delta_{\chi}}\tau(\chi)q^{-\frac{1}{2}}$ and $\tau(\chi)$ is the Gauss sum. In particular, $\varepsilon(\chi_{-4}) = \varepsilon(\chi_{-3}) = 1$.

For the symmetric square $L$-function $L(s,\text{sym}^2f)$ let

$$L_{\infty}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right).$$
Then the complete $L$-function $\Lambda(s, \text{sym}^2 f) = L_\infty(s) L(s, \text{sym}^2 f)$ satisfies the functional equation

$$\Lambda(s, \text{sym}^2 f) = \Lambda(1 - s, \text{sym}^2 f). \quad (3.5)$$

### 3.2.2 Approximate functional equation

Unlike previous studies we use

$$L(s, \text{sym}^2 f) = \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n)^2}{n^s}$$

to derive the following the approximate functional equation for $L(\frac{1}{2}, \text{sym}^2 f)$.

**Lemma 3.1.** Let $H(u) = e^{-u^2} u^{-1}$. We have

$$L(\frac{1}{2}, \text{sym}^2 f) = 2 \sum_{n \geq 1} \frac{\lambda_f(n)^2}{\sqrt{n}} V_k(n),$$

where

$$V_k(y) = \frac{1}{2\pi i} \int_{(3)} y^{-u} H(u) R(u) \frac{\zeta(1 + 2u)}{\zeta(\frac{1}{2} + u)} du$$

and

$$R(u) = \frac{L_\infty(\frac{1}{2} + u)}{L_\infty(\frac{1}{2})} = \pi^{-\frac{3}{4}u} R_a(u) = (2\pi^2)^{-u} R_b(u)$$

with

$$R_a(u) = \frac{\Gamma(k - \frac{1}{2} + u)}{\Gamma\left(\frac{k - 1}{2}\right)} \frac{\Gamma\left(\frac{k + \frac{1}{2}}{2} + u\right)}{\Gamma\left(\frac{k + 1}{2}\right)} \frac{\Gamma\left(\frac{3}{4} + \frac{u}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \text{ and } R_b(u) = \frac{\Gamma(k - \frac{1}{2} + u)}{\Gamma(k - \frac{1}{2})} \frac{\Gamma\left(\frac{3}{4} + \frac{u}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}.$$

The proof is standard (see [22, Theorem 5.3]). We will use in the sequel whichever version of $R(u)$ that is more convenient for computation.

### 3.2.3 Petersson’s formula

We recall Petersson’s formula (see [22, Corollary 14.23])

$$\frac{\Gamma(k - 1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{H}_k} \frac{\lambda_f(m) \lambda_f(n)}{(f, f)} = \delta_{m,n} + 2\pi i^{-k} \sum_{c \geq 1} S(m, n; c) \frac{S(m, n; c)}{c} J_{k-1}\left(\frac{4\pi \sqrt{mn}}{c}\right).$$
Here $\delta_{m,n}$ is the Kronecker delta symbol; $S(m, n; c)$ denotes the classical Kloosterman sum

$$S(m, n; c) = \sum_{x \equiv 1 (\text{mod} \ c)} e \left( \frac{mx + n\overline{x}}{c} \right);$$

$J_\nu(z)$ is the Bessel function of the first kind of order $\nu$. For convenience, we will denote the summation symbol in $S(m, n; c)$ by $\sum'_{x(c)}$.

### 3.2.4 An integral representation of $J_\nu(x)$

We need the following Mellin-Barnes integral

$$J_\nu(x) = \frac{1}{2\pi i} \int_{(\sigma)} x^{-s-1} 2^{s} \frac{\Gamma\left(\frac{\nu+1+s}{2}\right)}{\Gamma\left(\frac{\nu+1-s}{2}\right)} \cos\left(\frac{\pi z}{2}\right) y^{-z} \, dz,$$

for $x > 0$ and $-1 - \text{Re}(\nu) < \sigma < 0$ (see [46, p. 82] or [45, p. 226]).

### 3.2.5 A complex integral

In the analysis of the contribution from off-diagonal terms resulting from Petersson’s formula, we will encounter the complex integral

$$I(u; y) = \frac{1}{2\pi i} \int_{(3)} \frac{\Gamma\left(\frac{k-\frac{1}{2}-u-z}{2}\right)}{\Gamma\left(\frac{k+\frac{1}{2}+u+z}{2}\right)} \Gamma(z) \cos\left(\frac{\pi z}{2}\right) y^{-z} \, dz,$$

for which we have the following.

**Lemma 3.2.** For $y > 0$ and $\frac{1}{2} < \text{Re}(u) < k - 4$,

$$I(u; y) = \begin{cases} 
\frac{\Gamma\left(\frac{k-\frac{1}{2}-u}{2}\right)}{\Gamma\left(\frac{k+\frac{1}{2}+u}{2}\right)} \frac{\Gamma\left(k-\frac{1}{2}-u\right)}{\Gamma\left(k-\frac{1}{2}\right)} F\left(\frac{k-\frac{1}{2}-u}{2}, \frac{-k+\frac{3}{2}-u}{2}; \frac{1}{2}; \frac{y^2}{4}\right), & y < 2, \\
\frac{i^k 2^{2u} \cos\left(\frac{\pi}{2}\left(\frac{1}{2} + u\right)\right)}{\sqrt{\pi}} \frac{\Gamma(u)}{\Gamma(k-\frac{1}{2}+u)} \Gamma(k-\frac{1}{2}+u), & y = 2, \\
\frac{2i^k \cos\left(\frac{\pi}{2}\left(\frac{1}{2} + u\right)\right)}{y^{k-\frac{1}{2}-u}} \frac{\Gamma(k-\frac{1}{2}-u)}{\Gamma(k)} F\left(\frac{k-\frac{1}{2}-u}{2}, \frac{k+\frac{1}{2}-u}{2}; k; \frac{4}{y^2}\right), & y > 2,
\end{cases}$$

where $F(a, b; c; z)$ denotes the Gauss hypergeometric function.
Proof. Since $\mathcal{I}(u; y)$ is holomorphic in $u$ for $\frac{1}{2} < \text{Re}(u) < k - 4$, we only need to establish the result for $u > 1$, say. First we have the Mellin inversion (see [52, Vol. 1, (2.5.3.10)] or [10, (1.3.1)])

$$
\cos(x) = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \Gamma(z) \cos\left(\frac{\pi z}{2}\right) x^{-z} \, dz.
$$

We also note that

$$
\frac{\Gamma(k+s)}{\Gamma(k-s)} = \int_0^\infty J_{k-1}(x)\left(\frac{x}{2}\right)^s \, dx, \quad -k < \text{Re}(s) < -\frac{1}{2}.
$$

Shifting the contour of $\mathcal{I}(u; y)$ to $\text{Re}(z) = \frac{1}{2}$, we get

$$
\mathcal{I}(u; y) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} \left( \int_0^\infty J_{k-1}(x)\left(\frac{x}{2}\right)^{-\frac{1}{2} - u - z} \, dx \right) \Gamma(z) \cos\left(\frac{\pi z}{2}\right) y^{-z} \, dz
$$

$$
= 2^{\frac{1}{2} + u} \int_0^\infty J_{k-1}(x)x^{-\frac{1}{2} - u} \left( \frac{1}{2\pi i} \int_{(\frac{1}{2})} \Gamma(z) \cos\left(\frac{\pi z}{2}\right)\left(\frac{xy}{2}\right)^{-z} \, dz \right) \, dx
$$

$$
= 2^{\frac{1}{2} + u} \int_0^\infty J_{k-1}(x) \cos\left(\frac{xy}{2}\right) x^{-\frac{1}{2} - u} \, dx.
$$

Then the lemma follows from [52, Vol. 2, (2.12.15.3), (2.12.15.4), and (2.12.15.19)] (or see [10, (1.12.13) and (6.8.11)]) \qed

### 3.3 The diagonal contribution

In this section we set up the proof of Theorem 3.1 and analyze the diagonal contribution.

By Lemma 3.1 and Petersson’s formula, we have

$$
\sum_{f \in \mathcal{H}_{2k}} \omega_f L\left(\frac{1}{2}, \text{sym}^2 f\right) = 2 \sum_{n \geq 1} \frac{V_k(n)}{\sqrt{n}} \sum_{f \in \mathcal{H}_{2k}} \omega_f \lambda_f(n)^2
$$

$$
= 2 \sum_{n \geq 1} \frac{V_k(n)}{\sqrt{n}} \left\{ 1 + 2\pi i^{-k} \sum_{c \geq 1} \frac{S(n, n; c)}{c} J_{k-1}\left(\frac{4\pi n}{c}\right) \right\}
$$

$$
=: \mathcal{D} + \mathcal{O},
$$

65
where

\[ \mathcal{D} = 2 \sum_{n \geq 1} \frac{V_k(n)}{\sqrt{n}} \quad \text{and} \quad \mathcal{O} = 4\pi i^{-k} \sum_{n \geq 1} \frac{V_k(n)}{\sqrt{n}} \sum_{c \geq 1} \frac{S(n, n; c)}{c} J_{k-1} \left( \frac{4\pi n}{c} \right). \]

Using the definition of \( V_k \) (in Lemma 3.1) and shifting the contour to \( \text{Re}(s) = -B < 0 \), we have

\[ \mathcal{D} = 2 \frac{2 \pi i}{2\pi i} \int_{(-B)} H(u) R(u) \zeta(1 + 2u) \frac{1}{\zeta(\frac{1}{2} + u)} \sum_{n \geq 1} \frac{1}{n^{\frac{1}{2} + u}} \, du \]

\[ = 2 \frac{2 \pi i}{2\pi i} \int_{(-B)} H(u) R(u) \zeta(1 + 2u) \, du \]

\[ = 2 \text{res}_{u=0} H(u) R(u) \zeta(1 + 2u) + 2 \cos \left( \frac{\pi u}{2} \right) \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) \zeta(1 + 2u) \, du \]

\[ =: \mathcal{D}_R + \mathcal{D}_I. \]

**Note:** We adopt the convention that after a contour shifting the subscript “\( R \)” indicates the residue part while the subscript “\( I \)” the part from the resulting integral.

For the double pole \( u = 0 \), we can easily compute

\[ \mathcal{D}_R = 2 \lim_{u \to 0} \frac{d}{du} \frac{u^2 H(u)(2\pi^2)^{-u} R_b(u) \zeta(1 + 2u)}{\zeta(0) \zeta(1 + 2u)} \]

\[ = \psi(k - \frac{1}{2}) + 2\gamma + \frac{1}{2} \psi \left( \frac{3}{4} \right) - \log(2\pi^2). \]

On the other hand we have

\[ \mathcal{D}_I = 2 \frac{2 \pi i}{2\pi i} \int_{(-B)} H(u)(2\pi^2)^{-u} R_b(u) \zeta(1 + 2u) \, du \]

\[ = 2 \frac{2 \pi i}{2\pi i} \int_{(-B)} H(u)(2\pi^2)^{-u} \frac{\Gamma(k - \frac{1}{2} + u)}{\Gamma(k - \frac{1}{2})} \zeta(1 + 2u) \, du \]

\[ \ll B \kappa^{-B}, \]

in view of the classical bound of \( \zeta \) and the bound

\[ \frac{\Gamma(x + z)}{\Gamma(x)} \ll \text{Re}(z) x^\text{Re}(z) \quad \text{for } z \text{ with fixed Re}(z) \text{ and large } x > 0. \]
Note: We will make free use of such bounds in the sequel without mention.

Hence the diagonal contribution is

$$D = \psi(k - \frac{1}{2}) + 2\gamma + \frac{1}{2} \psi(\frac{3}{4}) - \log(2\pi^\frac{3}{2}) + O_B(k^{-B}).$$

(3.8)

### 3.4 The off-diagonal contribution

In this section, we deal with the more delicate off-diagonal contribution \(O\).

#### 3.4.1 Partition of \(O\)

For further analysis we first divide \(O\) into several parts (see (3.10)). To begin with, we have by (3.6)

$$J_{k-1}(\frac{4\pi n}{c}) = \frac{c}{4\pi n} \int_{(-1)} \left( \frac{c}{2\pi n} \right)^s \frac{\Gamma(\frac{k+s}{2})}{\Gamma(\frac{k-s}{2})} ds.$$

Then by inserting the definition of \(V_k\) and opening the Kloosterman sums, we have

$$O = \frac{i^{-k}}{(2\pi i)^2} \int_{(3)} H(u) R(u) \zeta(1 + 2u) \sum_{c \geq 1} \sum' \int_{(-1)} \left( \frac{c}{2\pi} \right)^s \frac{\Gamma(\frac{k+s}{2})}{\Gamma(\frac{k-s}{2})} \sum_{n \geq 1} e\left(\frac{n x + \pi r(x,c)}{c}\right) \frac{1}{n^{\frac{3}{2} + s + u}} ds du.$$

Upon letting

$$r(x,c) \equiv x + \pi \Mod{c} \quad \text{with} \quad 1 \leq r(x,c) \leq c \quad \text{and} \quad r_{x,c} = \frac{r(x,c)}{c}$$

(3.9)

we get

$$O = \frac{i^{-k}}{(2\pi i)^2} \int_{(3)} H(u) R(u) \zeta(1 + 2u) \sum_{c \geq 1} \sum' \int_{(-1)} \left( \frac{c}{2\pi} \right)^s \frac{\Gamma(\frac{k+s}{2})}{\Gamma(\frac{k-s}{2})} F\left(\frac{3}{2} + s + u, r_{x,c}\right) ds du,$$

where \(F(s, a)\) denotes the periodic zeta-function as in §3.2.1. It is natural to write

$$O = O_1 + O_2$$
according to $r_{x,c} = 1$ (for which $F(s, r_{x,c}) = \zeta(s)$) and $1/c \leq r_{x,c} < 1$ (in which case $F(s, r_{x,c})$ is entire). For $O_1$, we shift the $s$-integral to $\text{Re}(s) = -B < 0$, pick up a simple pole at $s = -\frac{1}{2} - u$, and get

$$O_1 = O_R + O_{1,I}. \tag{3.10}$$

For $O_2$, we also shift its $s$-integral to $\text{Re}(s) = -B < 0$ but get only

$$O_2 = O_{2,I},$$

since there is no residue part. Hence we arrive at the partition of $O$

$$O = O_R + O_I \tag{3.11}$$

where $O_I = O_{1,I} + O_{2,I}$. In the sequel, we analyze $O_R$ (see (3.11)) and $O_I$ (see (3.13)) separately.

### 3.4.2 Treatment of $O_R$

Define

$$N(c) = \sum_{x(c) \leq c} 1 \quad \text{and} \quad L(s, N) = \sum_{c \geq 1} \frac{N(c)}{c^s} \quad (\text{Re}(s) > 2).$$

Then using

$$\text{res}_{s = -\frac{1}{2} - u} \left( \frac{c}{2\pi} \right)^s \frac{\Gamma(k+s/2)}{\Gamma(k-s/2)} \zeta(\frac{3}{2} + s + u) = \frac{(2\pi)^{1+u}}{c^{1+u}} \frac{\Gamma\left(\frac{k-\frac{1}{2} - u}{2}\right)}{\Gamma\left(\frac{k+\frac{3}{2} + u}{2}\right)}$$

and $R(u) = \pi^{-\frac{3}{2}u} R_a(u)$, we have

$$O_R = \frac{\pi^{\frac{3}{2} - k} i^{-k}}{2 \cdot 2\pi i} \int_{(3)} H(u) \frac{\Gamma\left(\frac{k-\frac{1}{2} + u}{2}\right)}{\Gamma\left(\frac{k-\frac{3}{2}}{2}\right)} \frac{\Gamma\left(\frac{k-\frac{1}{2} - u}{2}\right)}{\Gamma\left(\frac{k+\frac{3}{2}}{2}\right)} \times \left(\frac{4}{\pi}\right)^{\frac{s}{2} + \frac{k}{2}} \frac{\Gamma\left(\frac{s}{2} + \frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \zeta(1 + 2u) \zeta\left(\frac{1}{2} + u\right) L\left(\frac{1}{2} + u, N\right) du. \tag{3.11}$$

In order to evaluate $O_R$ we need the following result.
Lemma 3.3. We have

\[ \zeta(2s)L(s, N) = \zeta(s)L(s, \chi_{-4}) = \zeta_{\mathbb{Q}(i)}(s). \]

For information on Dedekind zeta functions \( \zeta_K(s) \) and especially their factorization when \( K/\mathbb{Q} \) is Abelian, see, e.g., [22, Section 5.10].

Proof. It suffices to consider \( \text{Re}(s) > 2 \). It is clear from (3.9) that

\[ N(c) = \#\{x \pmod{c} \mid (x, c) = 1, x^2 \equiv -1 \pmod{c} \}. \]

By the characterization for an element of \((\mathbb{Z}/c\mathbb{Z})^\times\) to be a quadratic residue mod \( c \) (see [38, Theorem 5.1], for example)

\[ N(c) = \begin{cases} 2^{\omega(n)}R(-1, n), & \text{if } c = 2^a n \text{ with } (2, n) = 1, a = 0 \text{ or } 1, \\ 0, & \text{otherwise}, \end{cases} \]

where \( \omega(n) \) is the number of distinct prime divisors of \( n \) and

\[ R(a, n) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue mod } n, \\ 0, & \text{otherwise}. \end{cases} \]

Thus we have

\[ L(s, N) = \sum_{a=0 \text{ odd}}^{1} \sum_{n \geq 1} \frac{2^{\omega(n)}R(-1, n)}{(2^a n)^s} = \left(1 + \frac{1}{2^s}\right) \sum_{n \geq 1} \frac{2^{\omega(n)}R(-1, n)P(n)}{n^s}, \]

where \( P(n) = 1 \) if \( n \) is odd and \( = 0 \) if \( n \) is even. The arithmetical function \( 2^{\omega(n)}R(-1, n)P(n) \) is multiplicative and for any odd prime power \( p^\alpha \) \((\alpha \geq 1)\)

\[ 2^{\omega(p^\alpha)}R(-1, p^\alpha)P(p^\alpha) = 2R(-1, p). \]
Hence we have the Euler products

\[
\frac{L(s, N)}{\zeta(s)} = \left(1 - \frac{1}{2^s}\right) \prod_{p > 2} \left(1 - \frac{1}{p^s}\right) \times \left(1 + \frac{1}{2^s}\right) \prod_{p > 2} \left(1 + \frac{2R(-1, p)}{p^s} + \frac{2R(-1, p)}{p^{2s}} + \cdots\right) \\
= \left(1 - \frac{1}{2^{2s}}\right) \prod_{p > 2} \left(1 + \frac{2R(-1, p) - 1}{p^s}\right) \\
= \left(1 - \frac{1}{2^{2s}}\right) \prod_{p > 2} \left(1 + \frac{\chi(-4(p))}{p^s}\right)
\]

and

\[
\frac{L(s, N)}{L(s, \chi(-4)\zeta(s))} = \left(1 - \frac{1}{2^{2s}}\right) \prod_{p > 2} \left(1 - \frac{\chi(-4(p))}{p^s}\right) \left(1 + \frac{\chi(-4(p))}{p^s}\right) = \frac{1}{\zeta(2s)}.
\]

Applying Lemma 3.3 to (3.11) we get

\[
O_R = \frac{i^{-k} \pi^{\frac{2}{4}}}{2 \Gamma \left(\frac{3}{4}\right)} \frac{1}{2 \pi i} \int_{(3)} H(u) \frac{\Gamma \left(\frac{k - \frac{1}{2} + u}{2}\right)}{\Gamma \left(\frac{k - \frac{1}{2}}{2}\right)} \frac{\Gamma \left(\frac{k - \frac{1}{2} - u}{2}\right)}{\Gamma \left(\frac{k + \frac{1}{2}}{2}\right)} \Lambda \left(\frac{1}{2} + u, \chi(-4)\right) du
\]

\[
= \frac{i^{-k} \pi^{\frac{2}{4}}}{4 \Gamma \left(\frac{3}{4}\right)} \text{res}_{u=0} H(u) \frac{\Gamma \left(\frac{k - \frac{1}{2} + u}{2}\right)}{\Gamma \left(\frac{k - \frac{1}{2}}{2}\right)} \frac{\Gamma \left(\frac{k - \frac{1}{2} - u}{2}\right)}{\Gamma \left(\frac{k + \frac{1}{2}}{2}\right)} \Lambda \left(\frac{1}{2} + u, \chi(-4)\right)
\]

\[
= i^{-k} \sqrt{\pi} \frac{2}{2} L \left(\frac{1}{2}, \chi(-4)\right) \frac{\Gamma \left(\frac{k - \frac{1}{2}}{2}\right)}{\Gamma \left(\frac{k + \frac{1}{2}}{2}\right)}
\]

where we used that the integrand of the \(u\)-integral is an odd function in \(u\) due to the functional equation (3.4).

### 3.4.3 Treatment of \(O_I\)

We still need a few steps to get a reformulation (3.14) of \(O_I = O_{1,I} + O_{2,I}\). Applying the functional equation (3.2) to \(O_{1,I}\) followed by the substitution \(z = -\frac{1}{2} - s - u\) we
Similarly, by the functional equation (3.3) we get

\[
\mathcal{O}_{1,I} = \frac{i^{-k}}{(2\pi i)^2} \int_{(-\infty)} (2\pi)^{1/2 + u} R(u) \frac{\zeta(1 + 2u)}{\zeta(1/2 + u)} \sum_{c \geq 1} \sum_{x(c)}' \left( \frac{c}{2\pi} \right) \frac{\Gamma\left(\frac{k+x}{2}\right)}{\Gamma\left(\frac{k-x}{2}\right)} \zeta\left(\frac{3}{2} + s + u\right) ds du
\]

\[
= \frac{2i^{-k}}{2\pi i} \int_{(3)} (2\pi)^{1/2 + u} R(u) \frac{\zeta(1 + 2u)}{\zeta(1/2 + u)} \sum_{c \geq 1} \frac{1}{c^{1/2 + u}} \sum_{x(c)}' \left( \frac{c}{2\pi} \right) \frac{\Gamma\left(\frac{k-x}{2}\right)}{\Gamma\left(\frac{k+x}{2}\right)} \zeta\left(\frac{3}{2} + s + u\right) ds du
\]

\[
\times \frac{1}{2\pi i} \int_{(B-z)} \frac{1}{c^z} \frac{\Gamma\left(\frac{k-x}{2}\right)}{\Gamma\left(\frac{k+x}{2}\right)} \Gamma(z) \cos\left(\frac{\pi z}{2}\right) \zeta(z) dz du.
\]

Similarly, by the functional equation (3.3) we get

\[
\mathcal{O}_{2,I} = \frac{i^{-k}}{2\pi i} \int_{(3)} (2\pi)^{1/2 + u} R(u) \frac{\zeta(1 + 2u)}{\zeta(1/2 + u)} \sum_{c \geq 1} \frac{1}{c^{1/2 + u}} \sum_{x(c)}' \frac{1}{2\pi i}
\]

\[
\times \frac{1}{2\pi i} \int_{(B-z)} \frac{1}{c^z} \frac{\Gamma\left(\frac{k-x}{2}\right)}{\Gamma\left(\frac{k+x}{2}\right)} \Gamma(z) \left( e\left(\frac{z}{4}\right) \zeta(z, r_{x,c}) + e\left(-\frac{z}{4}\right) \zeta(z, 1 - r_{x,c}) \right) dz du
\]

\[
= \frac{i^{-k}}{2\pi i} \int_{(3)} (2\pi)^{1/2 + u} R(u) \frac{\zeta(1 + 2u)}{\zeta(1/2 + u)} \sum_{c \geq 1} \frac{1}{c^{1/2 + u}} \sum_{x(c)}' \frac{1}{2\pi i}
\]

\[
\times \frac{1}{2\pi i} \int_{(B-z)} \frac{1}{c^z} \frac{\Gamma\left(\frac{k-x}{2}\right)}{\Gamma\left(\frac{k+x}{2}\right)} \Gamma(z) \left( e\left(\frac{z}{4}\right) + e\left(-\frac{z}{4}\right) \right) \zeta(z, r_{x,c}) dz du,
\]

since \(1 - r_{x,c} = r_{-x,c}\) for \(x\) coprime with \(c\) and if \(x\) runs through reduced classes mod \(c\) with \(0 < r_{x,c} < 1\) then \(-x\) also runs through reduced classes mod \(c\) with \(0 < r_{-x,c} < 1\).

Now we can rewrite \(\mathcal{O}_I = \mathcal{O}_{1,I} + \mathcal{O}_{2,I}\) as

\[
\mathcal{O}_I = \frac{2i^{-k}}{2\pi i} \int_{(3)} (2\pi)^{1/2 + u} R(u) \frac{\zeta(1 + 2u)}{\zeta(1/2 + u)} \sum_{c \geq 1} \frac{1}{c^{1/2 + u}} \sum_{x(c)}' \frac{1}{2\pi i}
\]

\[
\times \frac{1}{2\pi i} \int_{(B-z)} \frac{1}{c^z} \frac{\Gamma\left(\frac{k-x}{2}\right)}{\Gamma\left(\frac{k+x}{2}\right)} \Gamma(z) \cos\left(\frac{\pi z}{2}\right) \zeta(z, r_{x,c}) dz du
\]
\[
\frac{2i^{-k}}{2\pi i} \int H(u)(2\pi)^{\frac{1}{2} + u} R(u) \frac{\zeta(1 + 2u)}{\zeta\left(\frac{1}{2} + u\right)} \sum_{c \geq 1} \frac{1}{c^{\frac{1}{2} + u}} \sum' \sum_{x(c)} \sum_{n \geq 0} \frac{1}{c^{\frac{1}{2} + u}} \sum' \sum_{x(c)} \sum_{n \geq 0} I(u; cn + r(x, c)) \, du,
\]

by recalling the definition of \( I(u; y) \) (see §2.5). In view of Lemma 3.2 we write

\[
\mathcal{O}_I = \mathcal{O}_{I,\text{i}} + \mathcal{O}_{I,\text{ii}} + \mathcal{O}_{I,\text{iii}}, \tag{3.14}
\]

where the summands correspond to the cases (i) \( cn + r(x, c) < 2 \), (ii) \( cn + r(x, c) = 2 \), and (iii) \( cn + r(x, c) > 2 \), respectively.

**Treatment of \( \mathcal{O}_{I,\text{i}} \)**

Clearly \( cn + r(x, c) = 1 \) if and only if \( n = 0 \) and \( r(x, c) = 1 \). Define

\[
M(c) = \sum' \frac{1}{x(c)} \quad \text{and} \quad L(s, M) = \sum_{c \geq 1} \frac{M(c)}{c^s} \quad (\text{Re}(s) > 2).
\]

Then

\[
\mathcal{O}_{I,\text{i}} = \frac{2i^{-k}}{2\pi i} \int H(u)(2\pi)^{\frac{1}{2} + u} R(u) \frac{\zeta(1 + 2u)}{\zeta\left(\frac{1}{2} + u\right)} L\left(\frac{1}{2} + u, M\right) I(u; 1) \, du.
\]

Similar to the treatment of \( \mathcal{O}_R \), we need the following in order to evaluate \( \mathcal{O}_{I,\text{i}} \).

**Lemma 3.4.** We have

\[
\zeta(2s)L(s, M) = \zeta(s)L(s, \chi_{-3}) = \zeta_{Q(\sqrt{-3})}(s).
\]

**Proof.** By definition

\[
M(c) = \#\{ x \, (\text{mod} \, c) \mid (x, c) = 1 \text{ and } x^2 - x + 1 \equiv 0 \, (\text{mod} \, c)\}.
\]

72
There are two steps to solve a quadratic equation $Ax^2 + Bx + C \equiv 0 \pmod{D}$ with $(A, D) = 1$: first solve $t^2 \equiv B^2 - 4AC \pmod{4D}$, $-D < t \leq D$, then solve $Ax \equiv (t - B)/2 \pmod{D}$, $0 \leq x < D$. For our case we first need to solve $t^2 \equiv -3 \pmod{4c}$ for $-c < t \leq c$. It is clear that $t = c$ is a solution if and only if $c = 1$ or $3$, for which $M(c) = 1$. So we assume $c \neq 1, 3$ and solve $t^2 \equiv -3 \pmod{4c}$ for $-c < t < c$. Then it is not hard to see that $t = 2c$ is not a solution and that

$$
\# \{ -c < t < c \mid t^2 \equiv -3 \pmod{4c} \} = \frac{1}{2} \# \{ -2c < t < 2c \mid t^2 \equiv -3 \pmod{4c} \},
$$

(3.15)

Also note that for $b \geq 1$ we have $(-3, 3^b) = 3$ and the equation $t^2 \equiv -3 \pmod{3^b}$ has a solution if $b = 1$ and no solution if $b > 1$. Let $c = 2^a3^bn$ with $(n, 6) = 1$. Then by virtue of (3.15) and [38, Theorem 5.1], solving $t^2 \equiv -3 \pmod{2^{a+2}3^bn}$ gives

$$M(c) = \begin{cases} 2^{\omega(n)}R(-3, n), & \text{if } c = 2^a3^bn \text{ with } (6, n) = 1, \ a, b = 0, \ or \ a = 0 \text{ and } b = 1, \\ 0, & \text{otherwise}. \end{cases}
$$

Note that the above formula for $M(c)$ also includes the cases $c = 1$ and $c = 3$. Thus

$$L(s, M) = \sum_{b=0}^{1} \sum_{\substack{n \geq 1 \\ (n, 6) = 1}} \frac{2^{\omega(n)}R(-3, n)}{(3^bn)^s} = \left(1 + \frac{1}{3^s}\right) \sum_{n \geq 1} \frac{2^{\omega(n)}R(-3, n)Q(n)}{n^s},$$

where $Q(n) = \sum_{d|(6, n)} \mu(d)$. Again it is easy to check that $2^{\omega(n)}R(-3, n)Q(n)$ is multiplicative and for any odd prime power $p^\alpha$ ($p > 3$ and $\alpha \geq 1$)

$$2^{\omega(p^\alpha)}R(-3, p^\alpha)Q(p^\alpha) = 2R(-3, p).$$

Hence the desired equality follows from an Euler product argument similar to that of Lemma 3.3.
By Lemma 3.2, Lemma 3.4, that $R(u) = \pi^{-\frac{3}{2}u}R_a(u)$, and the well-known linear transformation ([46, p. 47])

$$F(a, b; c; z) = (1 - z)^{-a}F\left(a, c - b; c; \frac{z}{z - 1}\right),$$

we have

$$O_{l,i} = \frac{2\sqrt{2\pi}i^{-k}}{\Gamma(\frac{3}{4})} \left(\frac{\pi}{3}\right)^{\frac{3}{4}} \frac{1}{2\pi i} \int_{(3)} H(u) \frac{\Gamma\left(\frac{k}{2} + \frac{u}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \frac{\Gamma\left(\frac{k}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{k}{2} + \frac{1}{2}\right)} \Lambda\left(\frac{1}{2} + u, \chi_{-3}\right)$$

$$\times \left(\frac{2}{\sqrt{3}}\right)^u F\left(\frac{k - \frac{1}{2} + u}{2}, -\frac{k + \frac{3}{2} - u}{2}, \frac{1}{2}; \frac{1}{4}\right) du$$

$$= \frac{2\sqrt{2\pi}i^{-k}}{\Gamma(\frac{3}{4})} \left(\frac{\pi}{3}\right)^{\frac{3}{4}} \frac{1}{2\pi i} \int_{(3)} H(u) \frac{\Gamma\left(\frac{k}{2} + \frac{u}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \frac{\Gamma\left(\frac{k}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{k}{2} + \frac{1}{2}\right)} \Lambda\left(\frac{1}{2} + u, \chi_{-3}\right)$$

$$\times \left(\frac{2}{\sqrt{3}}\right)^{k - \frac{1}{2}} F\left(\frac{k - \frac{1}{2} - u}{2}, \frac{k - \frac{1}{2} + u}{2}, \frac{1}{2}; -\frac{1}{3}\right) du$$

which becomes, since the integrand is odd in $u$ by the functional equation [3.4],

$$= \frac{\sqrt{2\pi}i^{-k}}{\Gamma(\frac{3}{4})} \left(\frac{\pi}{3}\right)^{\frac{3}{4}} \left(\frac{2}{\sqrt{3}}\right)^{k - \frac{1}{2}} \text{res}_{u=0} \left\{ H(u) \frac{\Gamma\left(\frac{k}{2} + \frac{u}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \frac{\Gamma\left(\frac{k}{2} - \frac{u}{2}\right)}{\Gamma\left(\frac{k}{2} + \frac{1}{2}\right)} \Lambda\left(\frac{1}{2} + u, \chi_{-3}\right) \right\}$$

$$\times F\left(\frac{k - \frac{1}{2} - u}{2}, \frac{k - \frac{1}{2} + u}{2}, \frac{1}{2}; -\frac{1}{3}\right) \right\}$$

$$= \sqrt{2\pi}i^{-k} L(\frac{1}{2}, \chi_{-3}) \left(\frac{2}{\sqrt{3}}\right)^{k - \frac{1}{2}} F\left(\frac{k - \frac{1}{2}}{2}, \frac{k - \frac{1}{2}}{2}, \frac{1}{2}; -\frac{1}{3}\right) \frac{\Gamma\left(\frac{k - \frac{1}{2}}{2}\right)}{\Gamma\left(\frac{k - \frac{1}{2}}{2}\right)}.$$}

Next we analyze the behavior of $(\frac{2}{\sqrt{3}})^{k - \frac{1}{2}} F\left(\frac{k - \frac{1}{2}}{2}, \frac{k - \frac{1}{2}}{2}, \frac{1}{2}; -\frac{1}{3}\right)$. To this end we need [46, Line 1 on p. 54], that is,

$$F\left(a, b; \frac{1}{2}; -x\right) = \frac{2^{a-b-1}}{\sqrt{\pi}} \Gamma\left(a + \frac{1}{2}\right) \Gamma(1 - b)(1 + x)^{-a+b}$$

$$\times \left\{ P_{a+b-1}^{b-a} \left(\sqrt{\frac{x}{1+x}}\right) + P_{a+b-1}^{b-a} \left(-\sqrt{\frac{x}{1+x}}\right) \right\} \ (0 < x < \infty)$$

74
as well as the asymptotic formula for Legendre function $P^\mu_\nu$ \[ (3.9.1.2) \], that is,

$$P^\mu_\nu(\cos \theta) = \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} \sqrt{\frac{2}{\pi \sin \theta}} \left\{ \cos \left[ (\nu + \frac{1}{2})\theta - \frac{\pi}{4} + \frac{\mu\pi}{2} \right] + O(\nu^{-1}) \right\},$$

where $\mu$ and $\nu$ are real, $0 < \varepsilon < \theta < \pi - \varepsilon$, and $|\nu| \gg \varepsilon^{-1}$. With the above formulas we see that

$$\left( \frac{2}{\sqrt{3}} \right)^{k-\frac{3}{2}} F \left( \frac{k - \frac{1}{2}, k - \frac{1}{2}; \frac{1}{2}; -\frac{1}{3} \right)$$

$$= \Gamma \left( \frac{k + \frac{1}{2}}{2} \right) \Gamma \left( 1 - \frac{k - \frac{1}{2}}{2} \right) \left\{ P^0_{k - \frac{1}{2}} \left( \frac{1}{2} \right) + P^0_{k - \frac{1}{2}} \left( -\frac{1}{2} \right) \right\}$$

$$= -\left( \frac{2}{\sqrt{3}} \right)^{\frac{1}{2}} i^{k} \Gamma \left( k - \frac{1}{2} \right) \left( \frac{k + \frac{1}{2}}{2} \right) \left( \frac{k - \frac{1}{2}}{2} \right) \left\{ C(k) + O(k^{-1}) \right\}$$

where

$$C(k) = \cos \left( \frac{k\pi}{3} - \frac{7\pi}{12} \right) + \cos \left( \frac{2k\pi}{3} - \frac{11\pi}{12} \right) = \begin{cases} \sqrt{\frac{3}{2}}, & \text{if } k \equiv 2 \pmod{6}, \\ 0, & \text{if } k \equiv 4 \pmod{6}, \\ -\sqrt{\frac{3}{2}}, & \text{if } k \equiv 0 \pmod{6}. \end{cases}$$

Hence we have

$$O_{I,i} = 3^{\frac{1}{4}} \sqrt{2\pi L\left( \frac{1}{2}, \chi_{-3} \right)} \Gamma \left( k - \frac{1}{2} \right) \frac{\Gamma( k - \frac{1}{2} )}{\Gamma(k)} \left[ S(k) + O(k^{-1}) \right],$$

where

$$S(k) = \begin{cases} -1, & \text{if } k \equiv 2 \pmod{6}, \\ 0, & \text{if } k \equiv 4 \pmod{6}, \\ 1, & \text{if } k \equiv 0 \pmod{6}. \end{cases}$$

**Treatment of $O_{I,ii}$**

Clearly $cn + r(x,c) = 2$ if and only if $c = n = 1$ and $r(x,c) = 1$ or $n = 0$ and $r(x,c) = 2$ for all $c \geq 2$. By Lemma \[3.2\] and that $R(u) = (2\pi^2) u R_b(u)$, we have
\[
O_{I,\text{ii}} = \frac{2i^{-k}}{2\pi i} \int H(u)(2\pi)^{\frac{1}{2}+u}(2\pi)^{-\frac{3}{4}} \Gamma(k - \frac{1}{2} + u) \frac{\Gamma\left(\frac{3}{4} + \frac{y}{2}\right)}{\Gamma\left(\frac{3}{4}ight)} \frac{\zeta\left(1 + 2u\right)}{\zeta\left(\frac{1}{2} + u\right)}
\]

\[\times \frac{i^{k+2u}}{\sqrt{\pi}} \frac{\cos\left(\frac{\pi}{2} \left(\frac{1}{2} + u\right)\right)}{(\frac{y}{2})^{k - \frac{1}{2} - u}} \Gamma\left(k - \frac{1}{2} + u\right) \Gamma\left(\frac{1}{2} + u\right) \left\{ 1 + \sum_{c \geq 2} \frac{1}{c^{2+u}} \sum'_{r(x,c) = 2} 1 \right\} du \]

\[= \frac{2\sqrt{2}}{2\pi i} \int H(u) \left(4\sqrt{\pi}\right)^{u} \Gamma\left(k - \frac{1}{2} - u\right) \frac{\Gamma\left(\frac{3}{4} + \frac{y}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \Gamma\left(\frac{1}{2} + u\right) \cos\left(\frac{\pi}{2} \left(1 + 2u\right)\right) \frac{\zeta\left(1 + 2u\right)}{\zeta\left(\frac{1}{2} + u\right)} \]

\[\times \left\{ 1 + \sum_{c \geq 2} \frac{1}{c^{2+u}} \sum'_{r(x,c) = 2} 1 \right\} du \]

\[\ll_{B} k^{-B}, \]

by shifting the integral to \(\text{Re}(u) = B\) for large \(B\).

**Treatment of \(O_{I,\text{iii}}\)**

By Lemma 3.2, the symmetry \(F(b,a;c;z) = F(a,b;c;z)\), and the Euler integral representation for \(F(a,b;c;z) (\text{[16], p. 54})\) with \(\text{Re}(c) > \text{Re}(b) > 0\) and \(|\text{arg}(1-z)| < \pi\), we have for \(y > 2\)

\[I(u;y) = \frac{i^{k}}{\sqrt{\pi}} \frac{\cos\left(\frac{\pi}{2} \left(\frac{1}{2} + u\right)\right)}{(\frac{y}{2})^{k - \frac{1}{2} - u}} \frac{\Gamma\left(\frac{k + \frac{1}{2} - u}{2}\right)}{\Gamma\left(\frac{k + \frac{1}{2} + u}{2}\right)} \times \int_{0}^{1} t^{k - \frac{1}{2} - u - 1}(1-t)^{k + \frac{1}{2} + u - 1} \left(1 - \frac{4}{y^{2}} t\right)^{-k + \frac{1}{2} - u} dt \]

where the \(t\)-integral equals

\[\int_{0}^{1} t^{k - \frac{1}{2} - u - 1}(1-t)^{u-1} \left(\frac{1 - t}{1 - \frac{4}{y^{2}} t}\right)^{k + \frac{1}{2} - u} dt \]

\[\ll \int_{0}^{1} t^{k - \frac{1}{2} - \text{Re}(u) - 1}(1-t)^{\text{Re}(u)-1} dt \]

\[= \frac{\Gamma\left(k - \frac{1}{2} - \text{Re}(u)\right)\Gamma(\text{Re}(u))}{\Gamma\left(k - \frac{1}{2} + \text{Re}(u)\right)} \ll_{\text{Re}(u)} k^{-\text{Re}(u)}. \]
Hence by shifting the contour to \( \text{Re}(u) = B \) and using \( R(u) = \pi^{-\frac{3}{2}} R_a(u) \) and (3.19) we get

\[
O_{I,iii} = \frac{2i^{-k}}{2\pi i} \int_{(B)} H(u)(2\pi)^{\frac{1}{2}+u} R(u) \frac{\zeta(1+2u)}{\zeta(\frac{1}{2}+u)}
\]

\[
\times \sum_{c \geq 1} \frac{1}{c^{\frac{1}{2}+u}} \sum_{x(c) \neq 0} \sum_{n \geq 0} \mathcal{I}(u, cn + r(x, c)) du
\]

\[
= \frac{2\sqrt{2}}{2\pi i} \int_{(B)} H(u) \left( \frac{2}{\sqrt{\pi}} \right)^u \frac{\Gamma \left( \frac{k-\frac{1}{2}+u}{2} \right)}{\Gamma \left( \frac{k-\frac{1}{2}}{2} \right)} \frac{\Gamma \left( \frac{k+\frac{1}{2}-u}{2} \right)}{\Gamma \left( \frac{k+\frac{1}{2}}{2} \right)} \frac{\Gamma \left( \frac{3}{4} + \frac{u}{2} \right)}{\Gamma \left( \frac{3}{4} \right)} \cos \left( \pi \left( \frac{1}{2} + u \right) \right)
\]

\[
\times \frac{\zeta(1+2u)}{\zeta(\frac{1}{2}+u)} \left\{ \sum_{c \geq 1} \frac{1}{c^{\frac{1}{2}+u}} \sum_{x(c) \neq 0} \sum_{n \geq 0} \left( \frac{cn + r(x, c)}{2} \right)^{-k+\frac{1}{2}+u} \right\}
\]

\[
\times \int_0^1 t^{\frac{k-\frac{1}{2}-u}{2}} (1-t)^{\frac{k+\frac{1}{2}+u}{2}-1} \left( 1 - \frac{4t}{(cn + r(x, c))^2} \right)^{-\frac{k+\frac{1}{2}-u}{2}} dt \right\} du
\]

\[
\ll_B k^{-B} \sum_{c \geq 1} \frac{1}{c^{\frac{1}{2}+B}} \sum_{x(c) \neq 0} \sum_{n \geq 0} \left( \frac{cn + r(x, c)}{2} \right)^{-k+\frac{1}{2}+B}
\]

\[
\ll_B k^{-B} \left\{ \sum_{c \geq 1} \frac{\phi(c)}{c^{\frac{1}{2}+B}} + \sum_{c \geq 1} \frac{\phi(c)}{c^{\frac{1}{2}+B}} \sum_{n \geq 1} \frac{1}{n^{\frac{k+\frac{1}{2}-B}{2}}} \right\}
\]

where \( \phi \) denotes the Euler totient function.

Finally, Theorem 1 follows from collecting the diagonal contribution (3.8) and the off-diagonal contribution (3.12), (3.16), (3.18), and (3.20).
Chapter 4: Central $L$-values of Maass forms

4.1 Introduction

4.1.1 Nonvanishing

Let $\{u_j\}$ be the fixed orthonormal basis of Hecke–Maass cusp forms of weight 0 for $\Gamma_0(1)$, where each $u_j$ has $\Delta_0$-eigenvalue $\frac{1}{4} + t_j^2$ ($t_j \geq 0$) and Hecke eigenvalues $\lambda_j(n)$. In the rest of this chapter we always let $T > 0$ be a large parameter and assume $T^\eta < M < T(\log T)^{-1}$ with a fixed small $0 < \eta < 1$. The main result of this chapter is the following

**Theorem 4.1** (Restatement of Theorem 1.3). We have

$$\# \left\{ j \mid |t_j - T| \leq M, L(\frac{1}{2}, u_j) > 0 \right\} \gg TM.$$ 

By Weyl’s law (1.1) we have

$$N(T + M) - N(T - M) = \frac{1}{3}TM + O(T),$$

i.e., there are $\approx TM$ many $t_j$’s in the interval $[T - M, T + M]$. Hence Theorem 4.1 implies that for Hecke–Maass forms in the basis $\{u_j\}$ with spectral parameter $t_j \in [T - M, T + M]$, there are positive proportion of them with nonvanishing central $L$-values. This is analogous to Luo’s nonvanishing result [45] for central $L$-values of
holomorphic cusp forms of large weight for $\Gamma_0(1)$, which is our main motivation. It is worth mentioning that Xu [63] obtains a positive-proportion nonvanishing result of the $L(\frac{1}{2} + it_j, u_j)$ for $t_j$ in short intervals, using mollifiers and moments but with different treatment.

In view of Theorem 1.1, our nonvanishing result on central $L$-derivative values of holomorphic cusp forms for $\Gamma_0(1)$ of large weight, one expects a similar nonvanishing result for $L'(\frac{1}{2}, u_j)$ for odd Hecke–Maass eigenforms $u_j$ ($\varepsilon_j = -1$). A possible approach to prove this, say, is to adapt Motohashi’s formula (Lemma 4.6) to treat a twisted moment of $L'(\frac{1}{2}, u_j)^2$ and apply the mollification analysis in Chapter 2.

### 4.1.2 An application

Before outlining the proof of Theorem 4.1, we give an application of this result and a formula of Katok–Sarnak (4.1). We use Katok–Sarnak [25] as a reference for the involved Maass forms of weight $\frac{1}{2}$ for $\Gamma_0(4)$. Similar to Maass forms of weight 0 for $\Gamma_0(1)$, we can define Maass forms of weight $\frac{1}{2}$ for $\Gamma_0(4)$. In particular, we define the space $C_{1/2}(4) = C_{1/2}(\Gamma_0(4) \backslash \mathbb{H})$ of **Maass cusp forms of weight $\frac{1}{2}$ for $\Gamma_0(4)$** to be the set

$$C_{1/2}(4) = \left\{ F \in L^2(\Gamma_0(4) \backslash \mathbb{H}) \mid \begin{array}{l}
F(\gamma z) = J(\gamma, z)f(z), \forall \gamma \in \Gamma_0(1), F \text{ is cuspidal} \\
\Delta_{1/2} F = \left( \frac{1}{4} + t_F^2 \right) F \text{ for some } F \geq 0
\end{array} \right\},$$

where the automorphy factor $J(\gamma, z) = \theta(\gamma z)/\theta(z)$ with $\theta(z) = y^{1/4} \sum_{n \in \mathbb{Z}} e(n^2 z)$, and the Laplace operator

$$\Delta_{1/2} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{i}{2} y \frac{\partial}{\partial x}.$$
Here a form $F$ is cuspidal means that its zeroth Fourier coefficient at each cusp of $\Gamma_0(4)$ is zero. In particular, each $F \in C^1_\frac{1}{2}(4)$ has a Fourier expansion at $\infty$

$$F(z) = \sum_{n \neq 0} b_F(n) W_{\frac{1}{4} \text{sgn}(n), itp}(4\pi |n|y)e(nx).$$

Define as in [25] Hecke operators $T_{p^2}$ for all primes $p \neq 2$ by

$$T_{p^2} F(z) = \sum_{n \neq 0} \left\{ pb_F(n p^2) + \left( \frac{n}{p} \right) \frac{1}{\sqrt{p}} b_F(n) + \frac{1}{p} b_F \left( \frac{n}{p^2} \right) \right\} W_{\frac{1}{4} \text{sgn}(n), itp}(4\pi |n|y)e(nx),$$

where $b_F(r) \neq 0$ if $r \not\in \mathbb{Z}$ and $\left( \frac{n}{p} \right)$ denotes the Legendre symbol. Define an operator $L : C^1_\frac{1}{2}(4) \rightarrow C^1_\frac{1}{2}(4)$ by

$$LF(z) = \frac{1}{4} e \left( \frac{\pi}{8} \right) \left( \frac{z}{|z|} \right)^{-\frac{1}{2}} \sum_{\nu \mod 4} F \left( \frac{4\nu z - 1}{16z} \right).$$

Then $L$ is self-adjoint, commutes with $\Delta_{\frac{1}{2}}$ and all $T_{p^2}$, and satisfies $(L - 1)(L + \frac{1}{2}) = 0$ (see [25] Proposition 1.4]). The Kohnen plus space $C^+_{\frac{1}{2}}(4) \subseteq C^1_\frac{1}{2}(4)$ is the eigenspace of $L$ with eigenvalue 1. We have

$$F \in C^1_\frac{1}{2}(4) \text{ lies in } C^+_{\frac{1}{2}}(4) \text{ if and only if } b_F(n) = 0 \text{ for } n \equiv 2, 3 \pmod{4}.$$  

Then we can find an orthonormal basis $\{F_j\}$ of $C^+_{\frac{1}{2}}(4)$ consisting of common eigenfunctions of all $T_{p^2} (p \neq 2)$. For $F \in C^1_\frac{1}{2}(4)$ we define its Shimura lift $ShF$ by

$$ShF(z) = \sum_{n \neq 0} a_{ShF}(n) W_{0, 2itp}(4\pi |n|y)e(nx),$$

where

$$a_{ShF}(n) = \sum_{m = \ell n} \sqrt{|m|/\ell} b_F(\ell^2).$$

Then $Sh \circ T_{p^2} = T_p \circ Sh$; if $F \in C^+_{\frac{1}{2}}(4)$ then $ShF \in C^1_0(1)$ with $\Delta_0$-eigenvalue $\frac{1}{4} + (2t_F)^2$; if $F \in C^+_{\frac{1}{2}}(4)$ is a common eigenfunction of $T_{p^2} (p \neq 2)$, then $T_{p^2} F = \lambda_{ShF}(p) F$;
$b_F(1) \neq 0$ if and only if $\text{Sh}F \neq 0$. (See [25, Proposition 4.1] and its proof for details.)

Now we record the following

**Katok–Sarnak formula** ([25, (0.19)]). For a normalized Hecke–Maass form $f \in \mathcal{C}_0(1) (a_f(1) = 1)$

$$\frac{\Lambda(\frac{1}{2}, f)}{\langle f, f \rangle} = 12\sqrt{\pi} \sum_{\text{Sh}F_j = b_{F_j}(1)} |b_{F_j}(1)|^2.$$

(4.1)

Here in the sum we have $\text{Sh}F_j = b_{F_j}(1)f$, which is different from [25, (19)] where the nonzero Shimura lifts are arithmetically normalized. An immediate and famous consequence of this formula is the positivity of $L(\frac{1}{2}, f)$. We comment that Baruch–Mao [3] shows that the Kohnen plus space $\mathcal{C}_{\frac{1}{2}}^+(f, 4)$ for an individual Hecke–Maass form $f \in \mathcal{C}_0(1)$, given by

$$\mathcal{C}_{\frac{1}{2}}^+(f, 4) = \left\{ F \in \mathcal{C}_{\frac{1}{2}}(4) \left| \begin{array}{l}
\Delta_{\frac{1}{2}} F = \frac{1}{4}(1 + t_F^2)F \\
T_p^2 F = \lambda_f(p)F \text{ for all } p > 2
\end{array} \right. \right\},$$

is one-dimensional. It may well be true that the spanning set of $\mathcal{C}_{\frac{1}{2}}^+(f, 4)$ contains only one $F_j$, i.e., that the sum on the right-hand side of (4.1) consists of only one summand.

By Theorem 4.1, the Katok–Sanark formula (4.1), and the fact that every weight $\frac{1}{2}$ Maass form lifts to a weight 0 Maass form, we have the following

**Theorem 4.2.** For the Hecke–Maass forms in the basis $\{F_j\}$ of $\mathcal{C}_{\frac{1}{2}}^+(4)$ with $t_{F_j} \in [T - M, T + M]$, there are positive proportion of them whose first Fourier coefficient $b_{F_j}(1) \neq 0$. 

81
4.1.3 Proof of Theorem 4.1 and outline of the chapter

In the following we outline the structure of this chapter and give the proof of Theorem 4.1 and some comments. We approach the nonvanishing problem in Theorem 4.1 via the study of the harmonic moments

$$\sum_j \frac{L(\frac{1}{2}, u_j)^k M_j^k}{L(1, \text{sym}^2 u_j)} h_0(t_j) \quad (k = 1, 2) \quad \text{and} \quad \sum_j \frac{h_0(t_j)}{L(1, \text{sym}^2 u_j)^2}.$$  

Here the test function $h_0(t)$ is given by

$$h_0(t) = T^{-2} h(t),$$  \hspace{1cm} (4.2)

where

$$h(t) = (t^2 + \frac{1}{4}) \omega(t) \quad \text{and} \quad \omega(t) := \omega_{T,M}(t) = e^{-(\frac{t+T}{\pi})^2} + e^{-(\frac{t-T}{\pi})^2};$$

and $M_j \ (j \geq 1)$ are mollifiers defined in (4.9). We remark that $h_0(t)$ gives a more natural counting than $h(t)$ but in the actual computation we use $h(t)$ in place of $h_0(t)$ to avoid writing the factor $T^{-2}$ everywhere. One reason for including the extra factor $t^2 + \frac{1}{4}$ in $h(t)$ is that Motohashi’s formula (Lemma 4.6), which we use to treat the second moment, requires that $h(\pm \frac{T}{2}) = 0$.

For completeness we record the following asymptotic formulas for the unmollified moments with power saving, which seem not to have been stated in the literature.

**Proposition 4.1.** We have

$$\sum_j \frac{L(\frac{1}{2}, u_j)}{L(1, \text{sym}^2 u_j)} h_0(t_j) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\zeta(\frac{1}{2} + it)^2}{|\zeta(1 + 2it)|^2} h_0(t) dt = \frac{2}{\pi^{3/2}} TM + O(T^{-\frac{1}{2} + \varepsilon} M)$$

and

$$\sum_j \frac{L(\frac{1}{2}, u_j)^2}{L(1, \text{sym}^2 u_j)} h_0(t_j) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\zeta(\frac{1}{2} + it)^4}{|\zeta(1 + 2it)|^4} h_0(t) dt$$

$$= \frac{2}{\pi^{3/2}} \left(T M \log T + (\gamma - \log 2\pi) TM\right) + O(T^{-2} M^3 \log T),$$

82
where $\gamma$ is the Euler constant.

The power saving in the above indicates that there is room to insert mollifiers à la Selberg to kill the extra $\log T$ in the second moment, i.e., to bring the mollified moments to comparable size as in Lemma 4.1 whose proof constitutes the major part of this chapter. The proof of Proposition 4.1 can be viewed as a simplified version of that for Lemma 4.1. For example, the second asymptotic formula in Proposition 4.1 follows from Motohashi’s formula (Lemma 4.6) for $n = 1$ and the estimates (4.26) and (4.27). We remark that closely related to the asymptotics in Proposition 4.1 are the following upper bounds for the unmollified and unweighted moments

$$\sum_{T-M \leq t_j \leq T+M} \frac{L(\frac{3}{2}, u_j)}{L(1, \text{sym}^2 u_j)} = \pi^3/2 TM + o(TM)$$

and

$$\sum_{T-M \leq t_j \leq T+M} \frac{L(\frac{3}{2}, u_j)^2}{L(1, \text{sym}^2 u_j)} = TM \log T,$$

due to Ivić–Jutila [19] and Motohashi [49], respectively.

Next we explain the use of the mollified moments and prove Theorem 4.1. After some preparation in §4.2, we establish the following estimates for the mollified moments in §§4.3–4.4.

**Lemma 4.1.** Let $\delta$ be the number which appears in the definition of $M_j$ (see (4.9)). If $0 < \delta < \frac{3}{10}$ we have

$$\sum_j \frac{L(\frac{3}{2}, u_j)M_j}{L(1, \text{sym}^2 u_j)} h_0(t_j) = \frac{1}{\pi^{3/2}} TM + o(TM)$$
and if $0 < \delta < \frac{1}{4}$ we have

$$\sum_j \frac{L(\frac{1}{2}, u_j) M_j^2}{L(1, \text{sym}^2 u_j)} h_0(t_j) \ll TM.$$ 

For the mollified first moment we apply an approximate functional equation (Lemma 4.3) for $L(\frac{1}{2}, u_j)$ and the Kuznetsov trace formula over even forms (Lemma 4.5).

(Note that $L(\frac{1}{2}, u_j) = 0$ for odd forms $u_j$.) The treatment of the off-diagonal sum $O^+$ involving the $J$-Bessel function $J_{2it}(x)$ is inspired by Li’s work [39, 40]. While for the off-diagonal sum $O^-$ involving the $K$-Bessel function $K_{2it}(x)$, we split the $c$-sum of Kloosterman sums $S(m, n; c)$ into two ranges, treat small $c$ by Li’s idea, and for large $c$ perform repeated integration by parts using an asymptotic formula of $K_{2it}(x)$.

For the mollified second moment, we employ Motohashi’s formula (Lemma 4.6) at the outset, instead of using an approximate functional equation for $L(\frac{1}{2}, u_j)^2$. The benefit is that the right-side of Motohashi’s formula does not involve any Kloosterman sums or Bessel functions, but only shifted sums of the divisor function and certain functions $\Psi^\pm$ for which Motohashi’s work [49, 50] and Ivić’s work [18] provide convenient resources. On the other hand, Luo’s work [45] also reduces the expected high load of analysis for the mollified second moment, since Luo’s successful mollification analysis can be applied directly right after we apply Motohashi’s formula.

Now we give the deeper reason for using Motohashi’s formula. If one would proceed with an approximate functional equation for $L(\frac{1}{2}, u_j)^2$ and Kuznetsov over even forms, one then wishes to perform analysis analogous to holomorphic modular form cases as in [36, 45, 41], namely, to extract information from the off-diagonal terms resulting from Kuznetsov over even forms by using properties of Estermann zeta-functions. But this would not be easy since the Mellin–Barnes representation of
$J_x(x)$ gives very narrow room for contour shifting. And in fact, this is not necessary, for in the derivation of Motohashi’s formula ([50, Section 3.3]) one already uses analysis involving Estermann zeta-functions, and more importantly the outset of the derivation gives the advantage of getting rid of the “cumbersome” $J$-Bessel term, which is inevitable if one uses Kuznetsov over even forms (see also the penultimate paragraph on p. 113 of [50]).

In §4.5, we prove the following upper bound, which is a short-interval version of [44, Lemma 5].

**Lemma 4.2.**

$$\sum_j \frac{h_0(t_j)}{L(1, \text{sym}^2 u_j)^2} \ll TM.$$ 

Finally we are ready to prove Theorem 4.1. By Lemma 4.1, Lemma 4.2 and Hölder’s inequality, we have

$$TM \ll \sum_j \frac{L(\frac{1}{2}, u_j) M_j}{L(1, \text{sym}^2 u_j)} h_0(t_j)$$

$$\ll \left( \sum_{L(\frac{1}{2}, u_j) \neq 0} h_0(t_j) \right)^{\frac{1}{2}} \left( \sum_j \frac{h_0(t_j)}{L(1, \text{sym}^2 u_j)^2} \right)^{\frac{1}{2}} \left( \sum_j \frac{L(\frac{1}{2}, u_j)^2 M_j^2}{L(1, \text{sym}^2 u_j) h_0(t_j)} \right)^{\frac{1}{2}}$$

$$\ll \left( \sum_{L(\frac{1}{2}, u_j) \neq 0} h_0(t_j) \right)^{\frac{1}{2}} (TM)^{\frac{3}{4}},$$

Hence we have

$$TM \ll \sum_{L(\frac{1}{2}, u_j) \neq 0} h_0(t_j)$$

$$\ll \sum_{|t_j - T| \leq M \log T} \frac{t_j^2}{T^2} e^{-\left(\frac{t_j - T}{M}\right)^2} \ll \sum_{|t_j - T| \leq M \log T} e^{-\left(\frac{t_j - T}{T}\right)^2}$$

85
and thus

$$TM \ll \sum_{L(\frac{1}{2}, u_j) \neq 0} e^{-\left(\frac{t_j-T}{M}\right)^2}. \quad (4.3)$$

Next we follow Luo [44] to remove the weight. By partial summation we see that for any fixed $A > 0$

$$\sum_{AM \leq t_j-T \leq 2AM} e^{-\left(\frac{t_j-T}{M}\right)^2} \ll TM \int_A^{2A} e^{-t^2} dt. \quad (4.3)$$

Then applying this inequality to $2^k AM \leq t_j-T \leq 2^{k+1} AM$ and summing over $k$, we get

$$\sum_{t_j \geq T+AM} e^{-\left(\frac{t_j-T}{M}\right)^2} \ll TM \int_A^\infty e^{-t^2} dt \ll TMe^{-A^2}. \quad (4.4)$$

A similar argument shows that

$$\sum_{t_j \leq T-AM} e^{-\left(\frac{t_j-T}{M}\right)^2} \ll TMe^{-A^2} \quad (4.5)$$

With a sufficiently large $A$, $(4.3) \sim (4.5)$ imply that

$$TM \ll \sum_{|t_j-T| \leq AM} e^{-\left(\frac{t_j-T}{M}\right)^2}. \quad (4.3)$$

Replacing $M$ by $M/A$ in the above yields

$$TM \ll \sum_{|t_j-T| \leq M} e^{-\left(\frac{t_j-T}{M}\right)^2} \ll \sum_{|t_j-T| \leq M} 1 \quad (4.3)$$

and Theorem 4.1 follows.

### 4.2 Preparation

In the following we introduce the tools required for the study of the relevant harmonic moments.
4.2.1 Approximate functional equation

For even \(u_j\) in the eigenbasis of \(C_0(1)\), we need an approximate functional equation to represent \(L(\frac{1}{2}, u_j)\), whose proof is standard as in [22, Theorem 5.3].

Lemma 4.3.

\[
L(\frac{1}{2}, u_j) = 2 \sum_{m \geq 1} \frac{\lambda_j(m)}{\sqrt{m}} U(m, t_j),
\]

where

\[
U(y, t) = \frac{1}{2\pi i} \int_{(A)} y^{-u} G(u) \gamma(u, t) du,
\]

with any fixed \(A > 0\), \(G(u) = u^{-1} e^{-u^4}\), and

\[
\gamma(u, t) = \frac{L_{\infty}(\frac{1}{2} + u, t)}{L_{\infty}(\frac{1}{2}, t)} = \pi^{-u} \frac{\Gamma\left(\frac{1}{2} + u - it\right) \Gamma\left(\frac{1}{2} + u + it\right)}{\Gamma\left(\frac{1}{2} - it\right) \Gamma\left(\frac{1}{2} + it\right)}.
\]

An easy consequence of the functional equation of the Riemann zeta-function is that

\[
|\zeta(\frac{1}{2} + it)|^2 = 2 \sum_{m \geq 1} \frac{\tau_{it}(m)}{\sqrt{m}} U(m, t).
\]

(4.6)

In view of Barnes’s formula (see Proposition 4.2 in §4.6), we have for fixed \(u\), fixed \(\sigma \geq 0\), and large \(t\)

\[
\gamma(u, t - i\sigma) = \left(\frac{|t|}{2\pi}\right)^{u} e^{O(P(|u| + |\sigma|)|t|^{-1})},
\]

(4.7)

where \(P(x)\) is a cubic polynomial with positive coefficients. Hence for any \(A > 0\) and \(\sigma < A + \frac{1}{2}\), the function \(U(y, t)\) is holomorphic in the strip \(-\sigma \leq \text{Im}(t) \leq 0\), and

\[
U(y, t - i\sigma) \ll_{A, \sigma} \begin{cases} 
  y^{-A}, & \text{if } |t| \leq 1 \\
  |t|^A y^{-A}, & \text{if } |t| \geq 1.
\end{cases}
\]

(4.8)
4.2.2 Kuznetsov trace formulas

In our notation the Kuznetsov traces formulas are as follows.

**Lemma 4.4.** Let \( h(t) \) be an even function which is holomorphic in \(|\text{Im}(t)| \leq \frac{1}{2} \) with \( h(t) \ll (1 + |t|)^{-2-\varepsilon} \) with some \( \varepsilon > 0 \). Then for integers \( m, n \geq 1 \)

\[
\begin{align*}
2 \sum_{j} \frac{\lambda_j(m) \lambda_j(\pm n)}{L(1, \text{sym}^2 u_j)} h(t_j) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\tau_u(m) \tau_{\bar{u}}(n)}{|\zeta(1 + 2it)|^2} h(t) \, dt \\
= \delta_{m,n} H + \sum_{c>0} S(m, \pm n; c) \frac{1}{c} H^\pm \left( \frac{4\pi \sqrt{mn}}{c} \right).
\end{align*}
\]

Here \( \tau_s(n) = \sum_{n=ab} (a/b)^s = \tau_{-s}(n) \),

\[
H = \frac{1}{\pi^2} \int_{-\infty}^{\infty} t \tanh(\pi t) h(t) \, dt,
\]

\[
H^+(x) = \frac{2i}{\pi} \int_{-\infty}^{\infty} \frac{J_{2it}(x)}{\cosh(\pi t)} h(t) \, dt,
\]

\[
H^-(x) = \frac{4}{\pi^2} \int_{-\infty}^{\infty} t \sinh(\pi t) K_{2it}(x) h(t) \, dt,
\]

and \( J_\nu(z) \) and \( K_\nu(z) \) are the usual Bessel functions.

This is a restatement of [21, Theorem 9.3] or [50, Theorem 2.2 and 2.4], in view of the relation

\[
\frac{|2a_{f}(1)|^2}{\cosh(\pi t_f)(f,f)} = \frac{2}{L(1, \text{sym}^2 f)}
\]

for any Hecke–Maass form \( f \in C_0(1) \). As a consequence of Lemma 4.4, we have the Kuznetsov trace formulas over even forms:

**Lemma 4.5.** Let \( h \) be as in the previous lemma. Then for integers \( m, n \geq 1 \)

\[
\begin{align*}
2 \sum_{\varepsilon_j=1} \frac{\lambda_j(m) \lambda_j(n)}{L(1, \text{sym}^2 u_j)} h(t_j) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\tau_u(m) \tau_{\bar{u}}(n)}{|\zeta(1 + 2it)|^2} h(t) \, dt \\
= \frac{1}{2} \delta_{m,n} H + \sum_{c>0} \frac{1}{c} \left\{ S(m, n; c) H^+ \left( \frac{4\pi \sqrt{mn}}{c} \right) + S(m,-n; c) H^- \left( \frac{4\pi \sqrt{mn}}{c} \right) \right\}.
\end{align*}
\]

88
4.2.3 Motohashi’s formula

To treat the second moment, we employ a formula of Motohashi. For any even entire function \( h(t) \) such that \( h(\pm \frac{1}{2} i) = 0 \) and

\[
h(t) \ll e^{-c|t|^2}
\]

for some fixed \( c > 0 \) in any fixed horizontal strip, define

\[
\hat{h}(s) = \int_{-\infty}^{\infty} \frac{\Gamma(s + it)}{\Gamma(1 - s + it)} th(t) dt,
\]

\[
\Psi^+(x; h) = \int_{(\beta)} \Gamma(\frac{1}{2} - s)^2 \tan(\pi s) \hat{h}(s) x^s ds,
\]

\[
\Psi^-(x; h) = \int_{(\beta)} \Gamma(\frac{1}{2} - s)^2 \frac{\hat{h}(s)}{\cos(\pi s)} x^s ds,
\]

where \(-\frac{3}{2} < \beta < \frac{1}{2}\). A restatement of Motohashi’s formula [50, Lemma 3.8] in our context is as follows.

**Lemma 4.6.** For the test function \( h(t) \) as in the last paragraph, we have

\[
\mathcal{H}(n; h) := \sum_j L(\frac{1}{2}, u_j)^2 \lambda_j(n) \frac{L(1, \text{sym}^2 u_j)}{L(1, \text{sym}^2 u_j)} h(t_j) = \sum_{\nu=1}^{7} \mathcal{H}_\nu(n; h),
\]

where

\[
\mathcal{H}_1(n; h) = -\frac{i}{\pi^3} \left\{ (\gamma - \log(2\pi \sqrt{n}) (\hat{h})'(\frac{1}{2}) + \frac{1}{4} (\hat{h})''(\frac{1}{2}) \right\} \frac{\tau(n)}{\sqrt{n}},
\]

\[
\mathcal{H}_2(n; h) = \frac{1}{2\pi^3} \sum_{m \geq 1} \frac{\tau(m)\tau(m+n)}{\sqrt{m}} \Psi^+\left(\frac{m}{n}; h\right),
\]

\[
\mathcal{H}_3(n; h) = \frac{1}{2\pi^3} \sum_{m \geq 1} \frac{\tau(m)\tau(m+n)}{\sqrt{m+n}} \Psi^-\left(1 + \frac{m}{n}; h\right),
\]

\[
\mathcal{H}_4(n; h) = \frac{1}{2\pi^3} \sum_{m = 1}^{n-1} \frac{\tau(m)\tau(n-m)}{\sqrt{m}} \Psi^-\left(\frac{m}{n}; h\right).
\]
\[ H_5(n; h) = -\frac{1}{4\pi^3} \frac{\tau(n)}{\sqrt{n}} \Psi^{-}(1; h), \]
\[ H_6(n; h) = -\frac{6i}{\pi^2} d_{\frac{1}{2}}(n) h'(-\frac{1}{2}i), \]
\[ H_7(n; h) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} |\xi(\frac{1}{2} + it)|^4 \tau(t) h(t) dt. \]

### 4.2.4 Mollifiers

For convenience, we define as in Luo’s work [45] the mollifier \( M_j \) for \( u_j \) by

\[ M_j = \sum_{n \geq 1} a_n \mu(n) \lambda_j(n) \sqrt{n}, \quad (4.9) \]

where

\[ a_n = \begin{cases} 
\frac{1}{2} \log^2(T^{2\delta}/n) - \log^2(T^\delta/n), & 1 \leq n \leq T^\delta, \\
\frac{1}{2} \log(T^\delta), & T^\delta \leq n \leq T^{2\delta}, \\
0, & n \geq T^{2\delta}
\end{cases} \quad (4.10) \]

for some \( \delta > 0 \). It is easy to see that \( 0 \leq a_n \leq 2 \log T \). Also the discontinuous integral

\[ \frac{1}{2\pi i} \int_{(2)} \frac{y^s}{s^3} ds = \begin{cases} 
\frac{1}{2} \log^2(y) & \text{if } y \geq 1, \\
0 & \text{if } 0 < y \leq 1,
\end{cases} \]

gives the analytic form of \( a_n \)

\[ a_n = \frac{1}{2\pi i} \int_{(2)} \frac{(\xi/n)^s - (\xi/n)^s}{s^3} \log \xi \quad (4.11) \]

with \( \xi = T^\delta \).

### 4.3 The mollified first moment

In this section we prove the asymptotic formula for the mollified first moment in Lemma 4.1. Note that we use the test function \( h(t) \) instead of \( h_0(t) \) in the derivation (see 4.2). By Lemma 4.3 and the definition of \( M_j \), we have

\[ \sum_{\epsilon_j = 1} \frac{L(\frac{1}{2}, u_j) M_j}{L(1, \text{sym}^2 u_j)} h(t_j) = \sum_{n \geq 1} a_n \mu(n) \sum_{m \geq 1} \frac{1}{\sqrt{m}} \left( 2 \sum_{\epsilon_j = 1} \lambda_j(m) \lambda_j(n) \frac{L(1, \text{sym}^2 u_j) h_m(t_j)}{L(1, \text{sym}^2 u_j)} \right), \]
where
\[ h_m(t) = h(t)U(m, t). \]

Since \( h_m(t) \) satisfies the conditions in Lemma 4.5, the above becomes
\[
\sum_{\varepsilon_j=1}^{\varepsilon} \frac{L(\frac{1}{2}, u_j)M_j}{L(1, \text{sym}^2 u_j)} h(t_j) = D - C + \mathcal{O}^+ + \mathcal{O}^-,
\] (4.12)

where
\[
D = \frac{1}{2} \sum_{n \geq 1} \frac{a_n \mu(n)}{n} H_n,
\]
\[
C = \frac{1}{\pi} \sum_{n,m} \frac{a_n \mu(n)}{\sqrt{nm}} \int_{-\infty}^{\infty} \frac{\tau_{it}(m)\tau_{it}(n)}{|\zeta(1 + 2it)|^2} h_m(t) \, dt,
\]
\[
\mathcal{O}^+ = \frac{1}{2} \sum_{n,m} \frac{a_n \mu(n)}{\sqrt{nm}} \sum_{c>0} S(m, n;c) \frac{H_m^+(\frac{4\pi \sqrt{mn}}{c})}{c},
\]
\[
\mathcal{O}^- = \frac{1}{2} \sum_{n,m} \frac{a_n \mu(n)}{\sqrt{nm}} \sum_{c>0} S(m, -n;c) \frac{H_m^-(\frac{4\pi \sqrt{mn}}{c})}{c}.
\]

In the following we analyze the above terms on the right-hand side of (4.12).

### 4.3.1 Diagonal contribution \( D \)

We claim that for \( 0 < \delta < \frac{1}{2} \)
\[
D = \frac{1}{\pi^{3/2}} T^3 M + o(T^3 M). \tag{4.13}
\]

By (4.11) and the definition of \( U(y, t) \),
\[
D = \frac{1}{\pi^2} \int_0^\infty t \tanh(\pi t) h(t) \frac{1}{2\pi i} \int_0^\infty \frac{\xi^{2w}}{w^3 \log \xi} \frac{1}{2\pi i} \int_0^\infty \frac{G(u)\gamma(u, t)}{\zeta(1 + w + u)} \, du \, dw \, dt.
\]

Moving the \( w \)-integral to \( \text{Re}(w) = -\frac{1}{2} + \varepsilon \) for a small \( \varepsilon > 0 \), we pick up a simple pole at \( u = 0 \) with residue \( \zeta(1 + w)^{-1} \) and have
\[ D = \frac{1}{\pi^2} \int_0^\infty t \tanh(\pi t) h(t) \left\{ \frac{1}{2\pi i} \int_{(2)} \frac{\xi^{2w} - \xi^w}{w^3 \log \xi} \frac{dw}{\zeta(1+w)} + \frac{1}{(2\pi i)^2} \int_{(-\frac{1}{2} + \epsilon)} (2) \int_{(2)} \frac{\xi^{2w} - \xi^w}{w^3 \log \xi} \frac{G(u)\gamma(u,t)}{\zeta(1+w+u)} \frac{du}{dw} \right\} dt \]

\[ = \frac{1}{\pi^2} \int_0^\infty t \tanh(\pi t) h(t) \left\{ 1 + \frac{1}{2\pi i} \int_{C_\epsilon} \frac{\xi^{2w} - \xi^w}{w^3 \log \xi} \frac{G(u)\gamma(u,t)}{\zeta(1+w+u)} \frac{du}{dw} \right\} dt \]

\[ = \frac{1}{\pi^2} \int_0^\infty t \tanh(\pi t) h(t) \left\{ 1 + \Theta \left( \frac{1}{\log \xi} \right) + \frac{1}{(2\pi i)^2} \int_{(-\frac{1}{2} + \epsilon)} (2) \int_{(2)} \frac{\xi^{2w} - \xi^w}{w^3 \log \xi} \frac{G(u)\gamma(u,t)}{\zeta(1+w+u)} \frac{du}{dw} \right\} dt, \]

where \( C_\epsilon \) denotes the contour

\[ C_\epsilon = \{ it \mid |t| \geq \epsilon \} \cup \{ \epsilon e^{i\theta} \mid \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \} \]

(4.14)

which starts from \(-i\infty\).

It is easy to compute that

\[ \int_0^\infty t \tanh(\pi t) h(t) dt = \sqrt{\pi} T^3 M + O(TM^3). \]

Thus we are left with

\[ \int_0^\infty t \tanh(\pi t) h(t) \int_{(-\frac{1}{2} + \epsilon)} (2) \int_{(2)} \frac{\xi^{2w} - \xi^w}{w^3 \log \xi} \frac{G(u)\gamma(u,t)}{\zeta(1+w+u)} \frac{du}{dw} \frac{dt}{dw} \]

\[ = \int_{(-\frac{1}{2} + \epsilon)} (\frac{1}{2}) \frac{\xi^{2w} - \xi^w}{w^3 \zeta(1+w+u) \log \xi} \int_0^\infty t \tanh(\pi t) h(t) \gamma(u,t) dt \frac{du}{dw}. \]

By (4.7) and considering \( t \) in and outside of \([T - M \log T, T + M \log T]\), we see that

\[ \int_0^\infty t \tanh(\pi t) h(t) \gamma(u,t) dt \ll T^{\frac{5}{2} + \epsilon} M e^{O(P(|u|))} \]

So the triple integral in the above is

\[ \ll \frac{\xi T^{\frac{5}{2} + \epsilon} M}{\log \xi} \int_{(-\frac{1}{2} + \epsilon)} (\frac{1}{2}) \frac{|G(u)| e^{O(P(|u|))}}{|w|^3 |\zeta(1+w+u)|} |du| \]

\[ \ll T^{\frac{5}{2} + \delta + \epsilon} M (\log T)^{-1}. \]
Then the claimed asymptotic formula (4.13) holds for \( \delta < \frac{1}{2} \).

### 4.3.2 Continuous spectrum part \( \mathcal{C} \)

By (4.6), as well as that \( a_n \ll \log T \) and \( a_n = 0 \) for \( n \geq \xi^2 \), we have

\[
\mathcal{C} = \frac{1}{\pi} \sum_{n \geq 1} \frac{a_n \mu(n)}{\sqrt{n}} \int_0^\infty \frac{h(t)\tau_u(n)}{|\zeta(1+2it)|^2} \left( 2 \sum_{m \geq 1} \frac{\tau_u(m)}{\sqrt{m}} U(m, t) \right) dt
\]

\[
= \frac{1}{\pi} \sum_{n \geq 1} \frac{a_n \mu(n)}{\sqrt{n}} \int_0^\infty \frac{|\zeta(\frac{1}{2}+it)|^2 h(t)\tau_u(n) dt}{|\zeta(1+2it)|^2} \]

\[
\ll \log T \sum_{n \leq \xi^2} \frac{\tau(n)}{\sqrt{n}} \int_0^\infty \frac{|\zeta(\frac{1}{2}+it)|^2 h(t) dt}{|\zeta(1+2it)|^2}
\]

\[
\ll T^\delta (\log T)^4 \int_{T-M \log T}^{T+M \log T} t^{2+\frac{1}{3}} \omega(t) dt
\]

\[
\ll T^{2+\frac{1}{3}+\delta} M (\log T)^4,
\]

which is \( o(T^2 M) \) upon letting \( \delta < \frac{2}{3} \). Here we used the classical bounds

\[
\zeta(1+it)^{-1} \ll \log |t| \quad (|t| \geq 1) \quad \text{and} \quad \zeta(\frac{1}{2}+it) \ll |t|^\frac{1}{2}.
\]

### 4.3.3 Off-diagonal sum \( \mathcal{O}^+ \)

In the following we show that \( \mathcal{O}^+ \) is negligible. Here and in the sequel a quantity being negligible means that its size is \( O_A(T^{-A}) \) for any \( A > 0 \). We start with

\[
H_n^{+}(\frac{4\pi \sqrt{mn}}{c})
\]

and abuse the notation \( X = \frac{4\pi \sqrt{mn}}{c} \) for convenience. Note that \( J_\nu(z) \) is entire in \( \nu \) for fixed \( z \neq 0 \) and by the integral representation (see [61, 3.3(5)]

\[
J_\nu(z) = \frac{2(z/2)^\nu}{\sqrt{\pi} \Gamma(\frac{1}{2}+\nu)} \int_0^\frac{\pi}{2} \sin^{2\nu}(\theta) \cos(z \cos(\theta)) d\theta, \quad \text{Re}(\nu) > -\frac{1}{2}
\]

and Stirling’s formula we have for \( x > 0 \)

\[
J_{\sigma+it}(x) \ll \sigma \begin{cases} 
  x^\sigma, & |t| \leq 1, \\
  x^\sigma |t|^{-\sigma} e^{\pi |t|/2}, & |t| \geq 1.
\end{cases}
\]  

(4.15)
Note that \(\cosh(\pi t)\) has simple zeros at \(t = i(\frac{1}{2} - k)\) for integers \(k\). Let \(K > 0\) be an integer and \(A > K + \frac{1}{2}\), both to be chosen later. By shifting the \(t\)-integral to \(\text{Im}(t) = -K\) for a positive integer \(K\), we have

\[
H_m^+(X) = \frac{2i}{\pi} \int_{-\infty}^{\infty} \frac{J_{2it}(X)}{\cosh(\pi t)} \text{th}_m(t) \, dt
\]

\[
= 4 \sum_{k=2}^{K'} \text{res}_{t=i(\frac{1}{2}-k)} J_{2it}(X) \text{th}_m(t)
\]

\[
+ \frac{2i}{\pi} \int_{-\infty}^{\infty} \frac{J_{2K+2it}(X)}{\cosh(\pi (t-iK))}(t - iK)h(t - iK)U(m, t - iK) \, dt
\]

\[
=: R_m^+(X) + I_m^+(X).
\]

Here the notation \(\sum'\) means that at most one of the summand is replaced by zero, since there is at most one \(k \geq 2\) such that \(J_{2k-1}(X) = 0\) due to the fact that no two of the functions \(J_n(z)\) \((n = 0, 1, 2, \ldots)\) have any common strictly positive zeros (see [61 15.28]). First, the residue part becomes

\[
R_m^+(X) = 4 \sum_{k=2}^{K'} \lim_{t \to i(\frac{1}{2}-k)} \frac{t - i(\frac{1}{2} - k)}{\cosh(\pi t)} J_{2it}(X) \text{th}_m(t)
\]

\[
= \sum_{k=1}^{K'} c_k J_{2k-1}(X) \omega(i(\frac{1}{2} - k))U(m, i(\frac{1}{2} - k)),
\]

where

\[
c_k = 4\pi^{-1}(-1)^k(k - \frac{1}{2})(k^2 - k).
\]

Here we recall that \(\omega(t) = e^{-\left(\frac{t+X}{2}\right)^2} + e^{-\left(\frac{t-X}{2}\right)^2}\). Then with \(|\omega(i(\frac{1}{2} - k))| \leq e^{-\frac{T^2}{2T^2}} e^{\frac{K^2}{2T^2}}\), (4.15), and (4.8), we see that

\[
R_m^+(X) \ll e^{-\frac{T^2}{m^2}} \frac{n^{K-\frac{1}{2}}}{m^{A-K+\frac{1}{2}}}.
\]

In the following we omit the dependence on \(A\) and \(K\) of the implied constants.

By

\[
\omega(t - iK) \ll \omega(t) \quad \text{and} \quad |\cosh(\pi(t - iK))| = \cosh(\pi t) \gg e^{\pi|t|}
\]

94
and (4.15), we have
\[
I_m^+(X) \ll \frac{n^K}{m^{A-K}c^{2K}} \int_{|t| \leq 1} \omega(t) \, dt + \frac{n^K}{m^{A-K}c^{2K}} \int_{|t| \geq 1} |t|^{A-2K+3} \omega(t) \, dt
\]
\[
\ll T^{A-2K+3} M \frac{n^K}{m^{A-K}c^{2K}}.
\]

Then (4.17) and (4.18) imply the bound
\[
H_m^+(X) \ll e^{-\frac{\pi^2}{32}} \frac{n^{K-\frac{1}{2}}}{m^{A-K+\frac{1}{2}} c} + T^{A-2K+3} M \frac{n^K}{m^{A-K}c^{2K}},
\]
This, together with Weil’s bound on Kloosterman sums, yields
\[
O^+ = \sum_{n \geq 1} \frac{\alpha_n \mu(n)}{\sqrt{n}} \sum_{m \geq 1} \frac{1}{\sqrt{m}} \sum_{c > 0} S(m, n; c) \frac{H_m^+ \left(4\pi \sqrt{mn} \frac{c}{c} \right)}{c} \nu
\]
\[
\ll e^{-\frac{\pi^2}{32}} \log T \sum_{n \leq \xi^2} n^{K-\frac{1}{2}} \sum_{m \geq 1} \frac{1}{m^{A-K+\frac{1}{2}}} \sum_{c \geq 1} \frac{\tau(c)}{c^{3/2}}
\]
\[
+ T^{A-2K+3} M \log T \sum_{n \leq \xi^2} n^{K-\frac{1}{2}} \sum_{m \geq 1} \frac{1}{m^{A-K+1}} \sum_{c \geq 1} \frac{\tau(c)}{c^{2K}}
\]
\[
\ll T^{(A-K)+3+\delta+(2\delta-1)K} M \log T,
\]
which is negligible upon taking \( \delta < \frac{1}{2} \), sufficiently large \( K \) and suitable \( A \) with \( A - K > \frac{1}{2} \).

### 4.3.4 Off-diagonal sum \( O^- \)

We write
\[
O^- = O_1^- + O_2^-,
\]
by splitting the \( c \)-sum into two ranges: \( c \geq \sqrt{mn} \) (\( X \leq 4\pi \)) and \( c < \sqrt{mn} \) (\( X > 4\pi \)).

Here recall the notation \( X = \frac{4\pi \sqrt{mn}}{c} \).

Case \( O_1^- \) (\( c \geq \sqrt{mn} \)). We start with the identity (\[61\ 3.7(6)]\)
\[
K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi \nu)},
\]

95
where the $I$-Bessel function $I_\nu(z)$ is entire for fixed $z \neq 0$ and has integral representation (61 3.71(9))

$$I_\nu(z) = \frac{(z/2)\nu}{\sqrt{\pi} \Gamma(\frac{1}{2} + \nu)} \int_0^\pi e^{z \cos(\theta)} (\sin \theta)^{2\nu} \, d\theta, \quad \text{Re}(\nu) > -\frac{1}{2}. $$

Thus for $x > 0$

$$I_{\sigma + it}(x) \ll_{\sigma} \begin{cases} x^\sigma e^x, & |t| \leq 1, \\ x^\sigma |t|^{-\sigma} e^{\pi |t| / 2}, & |t| \geq 1. \end{cases}$$

By $\sin(i2z) = i \sinh(2z) = 2i \sinh(z) \cosh(z)$, we have

$$H_{m}^-(X) = \frac{4}{\pi^2} \int \frac{t \sinh(\pi t) h_m(t) I_{-2it}(X) - I_{2it}(X)}{2i \sinh(\pi t) \cosh(\pi t)} \, dt$$

$$= \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{I_{2it}(X)}{\cosh(\pi t)} h_m(t) \, dt - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{I_{-2it}(X)}{\cosh(\pi t)} h_m(t) \, dt$$

$$= \frac{2i}{\pi} \int_{-\infty}^{\infty} \frac{I_{2it}(X)}{\cosh(\pi t)} h_m(t) \, dt.$$

Then by a very similar argument as the treatment for $O^+$, we see that $O_1^-$ is also negligible in size.

**Case $O_2^-$ ($c < \sqrt{mn}$).** We write

$$H_{m}^-(X) = \frac{8}{\pi^2} \int_0^\infty t \sinh(\pi t) K_{2it}(X) h_m(t) \, dt = H_{m,1}^-(X) + H_{m,2}^-(X) + H_{m,3}^-(X),$$

by splitting the $t$-integral into $\int_0^{T-M \log T}$, $\int_{T-M \log T}^{T+M \log T}$, and $\int_{T+M \log T}^\infty$. By $K_{2it}(X) \ll e^{-\pi t - \frac{1}{2}}$ (Proposition 4.3 in §4.7), we have

$$H_{m,1}^-(X) + H_{m,3}^-(X) \ll_A T^{A + 4 - \log T} \frac{1}{m^A}.$$ 

It follows that for fixed $A > 1$, the contribution of

$$\sum_{n,m} \frac{a_n \mu(n)}{\sqrt{nm}} \sum_{c < \sqrt{mn}} c^{-1} S(-m, n; c)(H_{m,1}^-(X) + H_{m,3}^-(X))$$

96
is negligible.

To achieve
\[
\sum_{n,m} \frac{a_n \mu(n)}{\sqrt{nm}} \sum_{\substack{c<\sqrt{mn} \atop c}} c^{-1} S(-m, n; c) H_{m,2}^{-}(X) = o(T^{3}M),
\]
we split the sum in \( m \) into two ranges: \( m \geq T^{1+\varepsilon} \) and \( m < T^{1+\varepsilon} \).

In view of the range of \( t \) in \( H_{m,2}^{-}(X) \), if \( m \geq T^{1+\varepsilon} \), then for sufficiently large \( A \)
\[
U(m, t) \ll_A \frac{t^A}{m^A} \ll_{A,\varepsilon} \frac{1}{m^{2}}.
\]
This and the estimate \( K_{2it}(X) \ll e^{-\pi t} t^{-\frac{1}{3}} \) imply
\[
H_{m,2}^{-}(X) \ll T^{3-\frac{1}{3}} M \frac{1}{m}.
\]
Hence we have
\[
\sum_{n \leq \xi^2} \frac{a_n \mu(n)}{\sqrt{n}} \sum_{m \geq T^{1+\varepsilon}} \frac{1}{\sqrt{m}} \sum_{\substack{c<\sqrt{mn} \atop c}} c^{-1} S(-m, n; c) H_{m,2}^{-}(X)
\ll T^{3-\frac{1}{3}} M \log T \sum_{n \leq \xi^2} \sum_{m \geq T^{1+\varepsilon}} \frac{1}{m^{2+1/2}} \sum_{\substack{c<\sqrt{mn} \atop c}} \frac{\tau(c)}{\sqrt{c}}
\ll T^{3-\frac{1}{3}} M (\log T)^2 \sum_{n \leq \xi^2} \sum_{m \geq T^{1+\varepsilon}} \frac{1}{m^{2+1/4}}
\ll T^{3+\frac{1}{2}\delta-\frac{1}{3}-\frac{5}{4}} M
\]
which is \( o(T^{3}M) \) provided \( \delta < \frac{19}{30} \).

Now we deal with the case when \( m < T^{1+\varepsilon} \). In view of the range of \( t \) in \( H_{m,2}^{-}(X) \), we have \( X = o(t) \) and the asymptotic formula (see \[46\, p. 142])
\[
K_{2it}(X) = \frac{\sqrt{2\pi} e^{-\pi t}}{(4t^2 - X^2)^{3/4}} \left\{ \sin(\phi_X(t)) + O(t^{-1}) \right\},
\]
where
\[
\phi_X(t) = \frac{\pi}{4} + 2t \cosh^{-1}\left(\frac{2t}{X}\right) - \sqrt{4t^2 - X^2}.
\]
The error term in [46, p. 142] is only \( O(X^{-1}) \) but the error term \( O(t^{-1}) \) follows from the power series expansion for \( K_{2it}(x) \) through that of \( I_{\pm 2it}(x) \). For \( T - M \log T \leq t \leq T + M \log T \), we have

\[
\sinh(\pi t)e^{-\pi t} = \frac{1}{2} + O(e^{-\pi T}),
\]

\[
\frac{1}{(4t^2 - X^2)^{\frac{1}{4}}} = \frac{1}{\sqrt{2t}} + O(X^2 t^{-5/2}),
\]

and

\[
U(m, t) = \frac{1}{2\pi i} \int_{(\frac{1}{2}), |\operatorname{Im}(u)| \leq T^e} \frac{t^u}{(2\pi m)^u} G(u) \, du + O_A(T^{-A}).
\]

We only need to estimate

\[
\int_{T - M \log T}^{T + M \log T} \left( \int_{(\frac{1}{2}), |\operatorname{Im}(u)| \leq T^e} \frac{t^u G(u)}{m^u} \, du \right) e^{-\left(\frac{t-T}{M}\right)^2} \sin(\phi_X(t)) \, dt \tag{4.20}
\]

\[
= \int_{(\frac{1}{2}), |\operatorname{Im}(u)| \leq T^e} \frac{G(u)}{m^u} \left( \int_{T - M \log T}^{T + M \log T} t^{u+\frac{5}{2}} e^{-\left(\frac{t-T}{M}\right)^2} \sin(\phi_X(t)) \, dt \right) \, du.
\]

since it is easy to see that the contribution from other parts to \( O^{-} \) through \( H_{m,2}(X) \) is \( o(T^3 M) \). To handle the \( t \)-integral, we work with

\[
I(X, T) = \int_{T - M \log T}^{T + M \log T} f(t) e^{ig(t)} \, dt,
\]

where

\[
f(t) = t^3 e^{-\left(\frac{t-T}{M}\right)^2} \quad \text{and} \quad g(t) = \operatorname{Im}(u) \log t \pm \phi_X(t).
\]

Notice that

\[
g'(t) = \operatorname{Im}(u)t^{-1} \pm 2\log \left( 2t + \sqrt{4t^2 - X^2} \right) \asymp \pm \log T
\]

and

\[
g''(t) = -\operatorname{Im}(u)t^{-2} \pm 4(4t^2 - X^2)^{-\frac{1}{2}} \asymp T^{-1}.
\]
By Faà di Bruno’s formula for high derivatives of composite functions (see e.g. [60])

\[
\frac{d^n}{dt^n} \left( e^{-\left(\frac{t}{M}\right)^2} \right) = e^{-\left(\frac{t}{M}\right)^2} \sum_{m_1+2m_2=n} \frac{(-1)^{m_1+m_2}}{m_1!m_2!} \left( \frac{2(t - T)}{M} \right)^{m_1} \frac{n!}{M^n} \ll n \, M^{-n} (\log T)^n,
\]

so that

\[
f^{(n)}(t) = \sum_{k=n-3}^{n} \binom{n}{k} (t^3)^{(n-k)} \frac{d^k}{dt^k} \left( e^{-\left(\frac{t}{M}\right)^2} \right) \ll n \, T^3 M^{-n} (\log T)^n
\]

Repeated integration by parts gives

\[
I(X, T) = -i \int_{T-M}^{T+M} \log T \left( \frac{f'(t)}{g'(t)} - \frac{f(t)g''(t)}{(g'(t))^2} \right) e^{ig(t)} \frac{dt}{t} + O(T^{-3} \log T)
\]

\[
= -i \int_{T-M}^{T+M} \frac{f'(t)}{g'(t)} e^{ig(t)} \frac{dt}{t} + O(T^2 M (\log T)^{-2})
\]

\[
= \int_{T-M}^{T+M} \frac{f''(t)}{(g'(t))^2} e^{ig(t)} \frac{dt}{t} + O(T^2 M (\log T)^{-2})
\]

\[
= \int_{T-M}^{T+M} \frac{f''(t)}{(g'(t))^2} e^{ig(t)} \frac{dt}{t} + O(T^2 M (\log T)^{-2})
\]

\[
\ldots
\]

\[
= c_n \int_{T-M}^{T+M} \frac{f^{(n)}(t)}{(g'(t))^n} e^{ig(t)} \frac{dt}{t} + O_n(T^2 M (\log T)^{-2})
\]

where \(|c_n| = 1\). By the estimates for \(f^{(n)}(t)\) and \(g'(t)\), and by \(M > T^\eta\), we take sufficiently large \(n\) to obtain

\[
I(X, T) \ll T^2 M (\log T)^{-2}.
\]

Hence the integral in (18) is

\[
\ll T^2 M (\log T)^{-2} \frac{1}{\sqrt{m}}
\]
and its contribution to $O^-$ through $H^{-2}_{m,2}(X)$ is

$$
\ll T^2 M (\log T)^{-1} \sum_{n \leq \xi^2} \frac{1}{\sqrt{n}} \sum_{m < T^{1+\varepsilon}} \frac{1}{m} \sum_{c < \sqrt{mn}} |S(-m,n;c)| \frac{c}{c}
$$

$$
\ll T^2 M (\log T)^{-1} \sum_{n \leq \xi^2} 1 \sum_{m < T^{1+\varepsilon}} \frac{1}{m} \sum_{c < \sqrt{mn}} \frac{\tau(c)}{\sqrt{c}}
$$

$$
\ll T^2 M \sum_{n \leq \xi^2} \frac{n^{\frac{1}{2}}}{m^{\frac{1}{2}} - \varepsilon} \sum_{c < \sqrt{mn}} \frac{\tau(c)}{\sqrt{c}}
$$

$$
\ll T^{2+\frac{1}{4} + 2\varepsilon + \frac{3}{2} \delta} M
$$

which is $o(T^3 M)$ provided $\delta < \frac{3}{10}$. That is, we have shown the bound (4.19). Now the claimed asymptotic formula for the first mollified moment in Lemma 4.1 follows.

### 4.4 The mollified second moment

In this section we establish the upper bound of the mollified second moment in Lemma 4.1 or the equivalent upper bound (see (4.2))

$$
\sum_j \frac{L(\frac{1}{2}, u_j)^2 M_j^2}{L(1, sym^2 u_j)} h(t) \ll T^3 M.
$$

By the Hecke relation we have

$$
M_j^2 = \sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} \lambda_j(n) \frac{A_{r,n}}{\sqrt{n}},
$$

(4.21)

where

$$
A_{r,n} = \sum_{n=r_1, r_2} a_{rn_1} \mu(rn_1) a_{rn_2} \mu(rn_2).
$$

Then we apply Lemma 4.6 to get

$$
\sum_j \frac{L(\frac{1}{2}, u_j)^2 M_j^2}{L(1, sym^2 u_j)} h(t) = \sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} \frac{A_{r,n}}{\sqrt{n}} \mathcal{H}(n; h)
$$

$$
= \sum_{\nu=1}^7 \sum_{r \geq 1} \sum_{n \geq 1} \frac{A_{r,n}}{\sqrt{n}} \mathcal{H}_\nu(n; h) =: \sum_{\nu=1} \cdots + \sum_7
$$

and treat $\sum_\nu$’s in separate cases.
4.4.1 Case $\sum_1$

We claim that

$$\sum_1 \ll T^3 M.$$  

From [50, (3.3.37) & (3.3.38)]

$$\hat{h}'(\frac{1}{2}) = 2 \int_{-\infty}^{\infty} \frac{\Gamma'(\frac{1}{2} + it)}{\Gamma(\frac{1}{2} + it)} th(t) dt \quad \text{and} \quad \hat{h}''(\frac{1}{2}) = 4 \int_{-\infty}^{\infty} \left\{ \frac{\Gamma'(\frac{1}{2} + it)}{\Gamma(\frac{1}{2} + it)} \right\}^2 th(t) dt,$$

we see

$$\hat{h}'(\frac{1}{2}) = 2 i \pi^3 T^3 M + O(TM^3)$$

and

$$\hat{h}''(\frac{1}{2}) = 8 i \pi^3 T^3 M \log T + O(TM^3 \log T),$$

and thus

$$\sum_1 = \frac{2}{\pi^{3/2}} (T^3 M \log T + O(T^3 M)) \sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} A_{r,n} \frac{\tau(n)}{n}$$

$$+ \frac{2}{\pi^{3/2}} (T^3 M + O(TM^3)) \sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} -A_{r,n} \frac{\tau(n) \log n}{n}.$$  

Thus we need to establish

$$\sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} A_{r,n} \frac{\tau(n)}{n} \ll \frac{1}{\log T} \quad (4.22)$$

and

$$\sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} -A_{r,n} \frac{\tau(n) \log n}{n} \ll 1 \quad (4.23)$$

In view of the identity

$$\tau(n_1 n_2) = \sum_{s | (n_1, n_2)} \mu(s) \tau\left(\frac{n_1}{s}\right) \tau\left(\frac{n_1}{s}\right),$$
and that $a_n = 0$ for $n \geq \xi^2$, we need for (4.22) the bound

$$
\sum_{r \leq \xi^2} \frac{1}{r} \sum_{s \leq \xi^2} \frac{\mu(s)}{s^2} \sum_{n_1, n_2} \frac{\tau(n_1)a_{rsn_1}\mu(rsn_1)\tau(n_2)a_{rsn_2}\mu(rsn_2)}{n_1} \ll \frac{1}{\log T}, \tag{4.24}
$$

and for (4.23) the bound

$$
\sum_{r \leq \xi^2} \frac{1}{r} \sum_{s \leq \xi^2} \frac{\mu(s)}{s^2} \sum_{n_1, n_2} \frac{\tau(n_1)a_{rsn_1}\mu(rsn_1)\log n_1 \tau(n_2)a_{rsn_2}\mu(rsn_2)}{n_1} \ll 1. \tag{4.25}
$$

But these last two bounds do hold, according to the same argument as in [45, Section 2].

### 4.4.2 Case $\sum_2$

We write

$$
\sum_2 = \frac{1}{2\pi^3} \sum_{r \leq \xi^2} \frac{1}{r} \sum_{n \leq \xi^2} \frac{A_{r,n}}{\sqrt{n}} \sum_{m=1}^{n-1} \frac{\tau(m)\tau(m+n)}{\sqrt{m}\sqrt{m}} \Psi^+\left(\frac{m}{n}; h\right)
+ \frac{1}{2\pi^3} \sum_{r \leq \xi^2} \frac{1}{r} \sum_{n \leq \xi^2} \frac{A_{r,n}}{\sqrt{n}} \sum_{m \geq n} \frac{\tau(m)\tau(m+n)}{\sqrt{m}} \Psi^+\left(\frac{m}{n}; h\right)
=: \sum_{2,1} + \sum_{2,2}.
$$

We have the bound which hold uniformly for $x \geq 1$ (see [50, p. 123]):

$$
\Psi^+(x; h) \ll x^{-1}T^{-B} \text{ for any fixed } 0 < B < \sqrt{\log T}. \tag{4.26}
$$

This bound implies that $\sum_{2,2}$ is negligible.

By [50, (3.4.20)], there is an absolute constant $c > 0$ such that for $m < n$

$$
\Psi^+\left(\frac{m}{n}; h\right) \ll T^3M \exp\left(-cM^2\frac{m}{n}\right) + \frac{n}{m}T^{-\log T}.
$$

For the range $T^\delta \log T \leq M < T/\log T$, we have $M^2\frac{m}{n} \geq (\log T)^2$ and thus

$$
\sum_{2,1} \ll T^{4+2\delta-c\log T} + T^{4\delta+\frac{1}{2}\log T},
$$

102
which is negligible.

Next we consider the range $\log T < M < T^\delta \log T$ and follow Ivić [18]. The nontrivial contribution to $\sum_{2,1}^*$ comes from the sum

$$\sum_{2,1}^* = \sum_{r \leq 2} \frac{1}{r} \sum_{M^2 \leq n \leq \log T} \frac{A_r, n}{\sqrt{n}} \sum_{m \leq n \frac{(\log T)^2}{2x}} \frac{\tau(m) \tau(m + n)}{\sqrt{m}} \Psi^+(\frac{m}{n}; h).$$

We abuse the notation $x = \frac{m}{n}$, which is $o(1)$ as $T \to \infty$. By [18, (2.13)], we have

$$\Psi^+(x; h) = \frac{4\pi \sqrt{x}}{\sqrt{x + \sqrt{1 + x}} \int_{-\infty}^{\infty} th(t) \tanh(\pi t) \times \Re \left \{ \left( \frac{\sqrt{x} + \sqrt{1 + x}}{2} \right)^{-2it} \Gamma^2(\frac{1}{2} + it) \Gamma(1 + 2it) \right \} \left( \frac{\sqrt{x} - \sqrt{1 + x}}{\sqrt{x + \sqrt{1 + x}}} \right)^2 dt,$$

where $F(a, b; c; z)$ denotes the Gauss Hypergeometric function, initially defined for $|z| < 1$ by

$$F(a, b; c; z) = \sum_{k \geq 0} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1)$ for $k \geq 1$. Since

$$\left( \frac{\sqrt{x} - \sqrt{1 + x}}{\sqrt{x} + \sqrt{1 + x}} \right)^2 = (\sqrt{x} + \sqrt{1 + x})^{-4} < 1 - 5\sqrt{x},$$

we can use the absolute convergence of the hypergeometric series and write

$$\Psi^+(x; h) = \frac{4\pi \sqrt{x}}{\sqrt{x + \sqrt{1 + x}} \sum_{k \geq 0} \left( \frac{1}{2} \right)_k \frac{k!}{k!} \left( \sqrt{x} + \sqrt{1 + x} \right)^{-4k} \Re(J_k),$$

where

$$J_k = \int_{-\infty}^{\infty} th(t) \tanh(\pi t) \frac{(\frac{1}{2} + it)_k}{(1 + it)_k} \left( \frac{\sqrt{x} + \sqrt{1 + x}}{2} \right)^{-2it} \Gamma^2(\frac{1}{2} + it) \Gamma(1 + 2it) dt.$$

First we do some reduction. According to the concentration effect of $e^{-(\frac{t-T}{T})^2}$ to $+T$ and $e^{-(\frac{t+T}{T})^2}$ to $-T$, we write

$$J_k = J_k^+ + J_k^-$$

103
and treat only $J_k^+$ since the two terms are very similar. For $J_k^+$, we further write

$$J_k^+ = \int_{|t-T| \leq M \log T} \cdots + \int_{|t-T| > M \log T} \cdots$$

By Stirling’s formula and that $|\Gamma(x + iy)| \geq \Gamma(x) \sqrt{\text{sech} \pi y}$, we have

$$\frac{\Gamma^2(\frac{1}{2} + it)}{\Gamma(1 + 2it)} \ll (|t| + 1)^{-\frac{1}{2}}$$

for all real $t$.

Also $\left| \frac{(\frac{1}{2} + it)_k}{(1 + it)_k} \right| \leq 1$. Thus it is easy to see that the second summand of $J_k^+$ contributes $\ll T^{\frac{5}{2} - \log T}$, which can be neglected. For the first summand of $J_k^+$, we bound trivially to obtain

$$\int_{|t-T| \leq M \log T} \cdots \ll \int_{|t-T| \leq M \log T} t^\frac{5}{2} \omega(t) \, dt \ll T^{\frac{5}{2}} M,$$

so that

$$J_k^+ \ll T^{\frac{5}{2}} M.$$

Hence, by the bound (see [18, (2.17)])

$$\frac{4\pi \sqrt{x}}{\sqrt{x + \sqrt{1 + x}}} \sum_{k \geq 0} \frac{(\frac{1}{2})_k}{k!} (\sqrt{x} + \sqrt{1 + x})^{-4k} \ll x^{\frac{3}{4}},$$

we see that the contribution of $J_k^+$ to $\sum_{2,1}^a$ is

$$\ll T^{\frac{5}{2}} M (\log T)^2 \sum_{\frac{M^2}{(\log T)^2} \leq n \leq \zeta^2} \frac{\tau(n)}{n^{\frac{1}{4}}} \sum_{m \leq n \frac{(\log T)^2}{M^2}} \frac{\tau(m) \tau(m + n)}{m^{\frac{1}{4}}}$$

$$\ll T^{\frac{5}{2} + 2\delta} M,$$

which is $o(T^3 M)$ upon letting $\delta < \frac{1}{4}$.

We remark that finer analysis using the machinery in Ivić’s work [18] leads to a larger range of $\delta$. This is only helpful if one could obtain an asymptotic formula

$$\sum_1 = \text{some constant} \cdot T^3 M + o(T^3 M),$$

which seems very difficult to achieve with our choice of the mollifiers $M_j$. 

104
4.4.3 Cases $\sum_\nu$ ($\nu = 3, \cdots, 7$)

We shall see that the contribution of these $\sum_\nu$'s can be neglected.

Case $\sum_3$. The contribution of $\sum_3$ is negligible due to the bound \[4.26\].

Case $\sum_4$. According to [18, Section 4], we have for $m < n$

$$
\Psi^{-}\left(\frac{m}{n}; h\right) \ll TM^{\varepsilon-1}\left(\frac{n}{m}\right)^{\frac{3}{2}-\varepsilon} + T^{\varepsilon-1}M\left(\frac{n}{m}\right)^{\frac{3}{2}}.
$$

From this we see that

$$
\sum_4 \ll T^{1+4\delta} \ll T^2
$$

if we impose $\delta < \frac{1}{4}$ as in the case of $\sum_2$.

Case $\sum_5$. The contribution of $\sum_5$ is negligible because of the following bound (see [50, p. 123])

$$
\Psi^{-}(x; h) \ll x^{-1}T^{-\log T}, \quad (4.27)
$$

which is uniform in $x \geq 1$.

Case $\sum_6$. The contribution of $\sum_6$ is negligible since

$$
\sum_6 = \frac{12}{\pi^2} \left( e^{-\left(\frac{T+i/2}{M}\right)^2} + e^{-\left(\frac{T+i/2}{M}\right)^2} \right) \sum_{r \leq \xi^2} \sum_{n \leq \xi^2} \frac{A_{r,n}}{\sqrt{n}} d_1(n).
$$

Case $\sum_7$. We simply discard $\sum_7$ for its negativity, which is shown below. Define

$$
M_t = \sum_{n \geq 1} \frac{a_n \mu(n) \tau_d(n)}{\sqrt{n}}.
$$
Then
\[ |M_t|^2 = \sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} \frac{\tau_{it}(n) A_{r,n}}{\sqrt{n}}, \]
due to \( \tau_{it}(m) \tau_{it}(n) = \sum_{d \mid (m,n)} \tau_{it}(md^{-2}) \). Thus it follows from the definition of \( \mathcal{H}_t(n; h) \) that
\[ \sum_1 = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{h(t)}{|\zeta(1/2 + it)|^4} \sum_{r \geq 1} \frac{1}{r} \sum_{n \geq 1} \frac{\tau_{it}(n) A_{r,n}}{\sqrt{n}} dt \]
\[ = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\zeta(1/2 + it)|^4 |M_t|^2}{|\zeta(1 + 2it)|^2} dt \]
\[ \leq 0. \]

4.5 A negative moment of \( L(1, \text{sym}^2 u_j) \)

We closely follow Luo [44] to prove Lemma [4.2] or the equivalent bound (see (4.2))
\[ \sum_j \frac{h(t_j)}{L(1, \text{sym}^2 u_j)^2} \ll T^3 M. \]

For \( \text{Re}(s) > 1 \), we write
\[ \frac{1}{L(s, \text{sym}^2 u_j)} = A_j(s) B_j(s), \]
where
\[ A_j(s) = \prod_p (1 - \lambda_j(p^2)p^{-s}) \quad \text{and} \quad B_j(s) = \prod_p \left(1 + \frac{\lambda_j(p^2)p^{-2s}}{1 - \lambda_j(p^2)p^{-s}}\right). \]

Since \( B_j(s) \) is analytic and zero-free in \( \text{Re}(s) \geq \frac{9}{10} \), and \( 1/B_j(s) \) and \( B_j(s) \) are uniformly bounded in this region, it suffices to show
\[ \sum_j \frac{A_j(1)}{L(1, \text{sym}^2 u_j)} h(t_j) \ll T^3 M. \]

We have
\[ A_j(s) = \sum_{n \geq 1} \frac{\lambda_j(n^2)\mu(n)}{n^s} \quad (\text{Re}(s) > 1). \]
and by \[44\] (36)]

\[
A_j(1) = \sum_{n \geq 1} \frac{\lambda_j(n^2)\mu(n)e^{-n/T}}{n} - I_j
\]

where

\[
I_j = \frac{1}{2\pi i} \int_{C_\varepsilon} A_j(s + 1)\Gamma(s)T^s ds
\]

and \(C_\varepsilon\) is the contour given by (4.14). Then we obtain

\[
\sum_j \frac{A_j(1)}{L(1, \text{sym}^2 u_j)} h(t_j) = \sum_{n \geq 1} \frac{\mu(n)}{n} \sum_j \frac{\lambda_j(n^2)}{L(1, \text{sym}^2 u_j)} h(t_j) - \sum_j \frac{I_j}{L(1, \text{sym}^2 u_j)} h(t_j)
\]

\[=: S_1 - S_2\]

For \(S_1\), we apply Lemma 4.4 to get

\[
S_1 = D - C + O^+,
\]

where

\[
D = \frac{1}{2} \sum_{n \geq 1} \frac{\mu(n)}{n} \frac{e^{-n/T}}{\delta_{n,1} H},
\]

\[
C = \frac{1}{2} \sum_{n \geq 1} \frac{\mu(n)}{n} \frac{e^{-n/T}}{\pi} \int_{-\infty}^{\infty} \frac{\tau_t(n^2)}{\zeta(1 + 2it)} h(t) dt,
\]

\[
O^+ = \frac{1}{2} \sum_{n \geq 1} \frac{\mu(n)}{n} \sum_{c > 0} \frac{S(n^2, 1; c)}{c} H^+\left(\frac{4\pi n}{c}\right).
\]

First it is easy to see that

\[
D = \frac{1}{2} e^{-1/T} H \ll \int_{-\infty}^{\infty} t \tanh(\pi t) h(t) dt \ll T^3 M.
\]

Next we deduce that

\[
C \ll \sum_{n \geq 1} \frac{\tau(n^2)e^{-n/T}}{n} \int_{-\infty}^{\infty} \frac{h(t)}{\zeta(1 + 2it)^2} dt
\]

\[
\ll T^2 M(\log T)^2 \sum_{n \geq 1} \frac{\tau(n^2)e^{-n/T}}{n}
\]

\[
\ll T^2 M(\log T)^5
\]

107
since
\[
\sum_{n \geq 1} \frac{\tau(n^2)e^{-n/T}}{n} = \frac{1}{2\pi i} \int_{(2)} \sum_{n \geq 1} \frac{\tau(n^2)}{n^{1+\sigma}} \Gamma(s) T^s \, ds \\
= \frac{1}{2\pi i} \int_{(2)} \frac{\zeta(1+\sigma)^3}{\zeta(2+2\sigma)} \Gamma(s) T^s \, ds \\
= \text{res}_{s=0} \frac{\zeta(1+\sigma)^3}{\zeta(2+2\sigma)} \Gamma(s) T^s + \frac{1}{2\pi i} \int_{C_\epsilon} T^s \Gamma(s) \frac{\zeta(1+\sigma)^3}{\zeta(2+2\sigma)} \, ds \\
\ll (\log T)^3,
\]
where \( C_\epsilon \) is the contour in (4.14). For
\[
O^+ \ll \sum_{n \geq 1} \frac{e^{-n/T}}{n} \sum_{c>0} \frac{\tau(c)}{\sqrt{c}} \left| H^+ \left( \frac{4\pi n}{c} \right) \right|
\]
we only need some control on \( H^+(X) \) where we abuse the notation \( X = \frac{4\pi n}{c} \) for convenience. Shifting the integral of \( H^+(X) \) to \( \text{Im}(t) = -\sigma \) with \( 0 < \sigma < \frac{1}{2} \), we get
\[
H^+(X) = \frac{2i}{\pi} \int_{-\infty}^{\infty} \frac{J_{2it}(X)}{\cosh(\pi t)} t h(t) \, dt \\
= \frac{2i}{\pi} \int_{-\infty}^{\infty} \frac{J_{2\sigma+2it}(X)}{\cosh(\pi(t-i\sigma))} (t-i\sigma) h(t-i\sigma) \, dt \\
\ll T^{3-2\sigma} MX^{2\sigma},
\]
due to (4.15). Taking \( \sigma = \frac{1}{4} + \frac{\epsilon}{2} \), we have
\[
O^+ \ll T^{3-\frac{1}{2}-\epsilon} M \sum_{n \geq 1} \frac{e^{-n/T}}{n^{\frac{1}{2}-\epsilon}} \sum_{c>0} \frac{\tau(c)}{c^{1+\epsilon}} \\
\ll T^{3-\frac{1}{2}-\epsilon} M \cdot \frac{1}{2\pi i} \int_{(2)} \zeta(s+\frac{1}{2}-\epsilon) \Gamma(s) T^s \, ds \\
= T^{3-\frac{1}{2}-\epsilon} M \left( \text{res}_{s=\frac{1}{2}+\epsilon} + \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \cdots \right) \\
\ll T^3 M,
\]
where the contour \( \Gamma_\epsilon = \left\{ \frac{1}{2} + \epsilon + it \mid |t| \geq \epsilon \right\} \cup \left\{ \frac{1}{2} + \epsilon + \epsilon e^{i\theta} \mid \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \right\} \). Summa-

rizing the estimates of \( D, C \) and \( O^+ \), we obtain that \( S_1 \ll T^3 M \).

108
Now it remains to bound $S_2$. Let $J = \{ j \mid |t_j - T| \leq M \log T \}$. We observe that

$$S_2 \ll \sum_{j \in J} \frac{I_j}{L(1, \text{sym}^2 u_j)} h(t_j),$$

since the contribution from $t_j < T - M \log T$ and $t_j > T + M \log T$ is negligible due to the bound

$$I_j = A_j(1) + \frac{1}{2\pi i} \int_{(\frac{1}{2})} A_j(s + 1) \Gamma(s) T^s \, ds \ll (t_j T)^\varepsilon$$

(see [44, p. 501]). For sufficiently small $\eta > 0$, we partition $J$ into $J_1$ and $J_2$, according to whether $L(s, \text{sym}^2 u_j)$ is zero-free in $1 - 10\eta \leq \text{Re}(s) \leq 1$ and $|\text{Im}(s)| \leq (\log T)^3$. By Luo’s argument,

$$I_j \ll T^{-\frac{9}{20}\eta} \quad \text{for } j \in J_1.$$

Hence by $L(1, \text{sym}^2 u_j)^{-1} \ll t_j^\varepsilon$ (see [17]) and Weyl’s law

$$\sum_{j \in J} \frac{I_j}{L(1, \text{sym}^2 u_j)} h(t_j) \ll T^{-\frac{7}{\varepsilon}\eta} \sum_{j \in J_1} h(t_j) \ll T^{2-\frac{7}{\varepsilon}\eta} \sum_{j \in J} 1 \ll T^{3-\frac{2}{\varepsilon}\eta} M \log T \ll o(T^3 M).$$

On the other hand Luo’s argument gives $|J_2| \ll T^{\frac{1}{\varepsilon}}$ and $\sum_{j \in J_2} |I_j| \ll T^{\frac{1}{\varepsilon}}$. Thus

$$\sum_{j \in J_2} \frac{I_j}{L(1, \text{sym}^2 u_j)} h(t_j) \ll T^{2+\varepsilon} \sum_{j \in J_2} |I_j| \ll T^\frac{5}{\varepsilon}.$$

Hence we have shown that $S_2 = o(T^3 M)$ and completed the proof of Lemma 4.2.

### 4.6 A discussion on Barnes’ formula

In [3] Barnes developed the theory of the simple Gamma function $\Gamma_1(z \mid \omega) = \omega^{z-1} \Gamma(z)$ with parameter $\omega$; $\Gamma_1(z \mid \omega)$ becomes $\Gamma(z)$ when $\omega = 1$. Barnes’s formula
for $\Gamma(z)$ [3, §41] states that for fixed $a \notin \mathbb{Z} \cap (-\infty, 0]$ and for all large $z$ which are not in the vicinity of the negative real axis, we have

$$\log \Gamma(z + a) = (z + a - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + \sum_{j=1}^{n} \frac{(-1)^{j+1}B_{j+1}(a)}{j(j+1)z^{j}} + J_{n}(z; a),$$

(4.28)

where $n \geq 0$ and $J_{n}(z; a) = O(|z|^{-n-1})$. Here $B_{j}(x)$ is the $j$-th Bernoulli polynomial, $\log z$ has the negative real axis as a cut and is real when $z$ is real and positive, and $J_{n}(z; a)$ is the contour integral (4.29). It is often useful to know the explicit dependence of the error term $J_{n}(z; a)$ on $a$, when dealing with ratios of Gamma functions. For this purpose, we prove the following

**Proposition 4.2.** Under the conditions for $a$ and $z$ in the above, we have

$$J_{n}(z; a) \ll P_{n+3}(|a|)|z|^{-n-1},$$

where the implied constant is absolute and $P_{n+3}(x)$ is a degree polynomial of degree $n + 3$ whose coefficients are positive and may depend on $n$.

**Proof.** By the argument on [3, p. 121],

$$J_{n}(z; a) = -\int_{(\sigma)} \frac{\pi}{s \sin \pi s} \zeta(s, a)z^{s}ds,$$

(4.29)

where $-n-1 < \sigma < -n$ and $n \geq 0$, and $\zeta(s, a)$ is the Hurwitz zeta function. Here we take $\sigma = -n - \frac{1}{2}$. By the argument in [3, §40], we can estimate $\zeta(s, a)$ with parameter $a$ as follows. For $-n-1 < \text{Re}(s) = \sigma < -n$, we have

$$|\zeta(s, a)| \leq \sum_{m \leq |a|} \left| \frac{1}{(m+a)^s} \right| + C|a|^{n+2} \sum_{m \geq |a|+1} \frac{1}{m^{s+n+2}} + \sum_{m=0}^{n+1} \frac{|a|^m |\zeta^{(m)}(s)|}{m!},$$

where $C > 0$ is an absolute constant and $[\alpha]$ denotes the integer part of a real number $\alpha$. For $\sigma = -n - \frac{1}{2}$, we have

$$\sum_{m \leq |a|} \left| \frac{1}{(m+a)^s} \right| \leq \sum_{m \leq |a|} (m + |a|)^{n+1}$$
and
\[ \frac{C|a|^{n+2}}{(n+1)!} \sum_{m \geq |a|+1} \left| \frac{1}{m^{s+n+2}} \right| \leq \frac{C|a|^{n+2}}{(n+1)!} \sum_{m=1}^{\infty} \frac{1}{m^{3/2}}. \]

We also claim that the integrals
\[
\int_{(-n-\frac{1}{2})} \left| \frac{\pi}{s \sin \pi s} \zeta^{(m)}(s) \right| |ds|
\]
are convergent for \(0 \leq m \leq n+1\) and will give the proof later. Collecting these estimates, we have
\[ J_n(z; a) \ll Q_{n+2}(|a|)|z|^{-n-\frac{1}{2}}, \]
where \(Q_{n+2}(x)\) is a polynomial of degree \(n+2\) with coefficients possibly dependent on \(n\). With this bound to \(J_{n+1}(z; a)\) and that
\[ J_n(z; a) = \frac{(-1)^{n+2}B_{n+2}(a)}{(n+1)(n+2)} z^{n+1} + J_{n+1}(z, a) \quad \text{(see [3, p. 120])}, \]
the proposition follows.

Now we prove the claim. We have the functional equations
\[ \zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \]
and
\[ \zeta^{(m)}(s) = (-1)^m 2(2\pi)^{s-1} \sum_{j=0}^{m} \sum_{k=0}^{m} \left( a_{jmk} \sin\left(\frac{\pi s}{2}\right) + b_{jmk} \cos\left(\frac{\pi s}{2}\right) \right) \Gamma^{(j)}(1-s)\zeta^{(k)}(1-s), \]
\(m = 1, \ldots, n+1\), as considered in [56], where \(a_{jmk}\) and \(b_{jmk}\) are constants independent of \(s\). As shown in [56], for \(s\) with \(|s| \geq 1\) and \(|\arg(s)| < \pi\) one has
\[ \Gamma^{(j)}(s) = \Gamma(s) \left[ (\log s)^j + \sum_{\ell=0}^{j-1} E_{\ell j}(s)(\log s)^\ell \right], \]
where \( E_{ij}(s) = O(|s|^{-1}) \). Then, for \( s = -n - \frac{1}{2} + it \),
\[
\pi \frac{1}{s \sin \pi s} \zeta^{(m)}(s) = (-1)^m \pi (2\pi)^{s-1} \sum_{j=0}^{m} \sum_{k=0}^{m} \left( \frac{a_{jmk}}{\cos(\pi s/2)} + \frac{b_{jmk}}{\sin(\pi s/2)} \right) \zeta^{(k)}(1 - s)
\times \Gamma(1 - s) \left[ (\log(1 - s))^2 + \sum_{\ell=0}^{j-1} E_{ij}(1 - s)(\log(1 - s))^{\ell} \right]
\]
is of exponential decay in \( t \) as \( t \to \infty \). Thus we have shown the claimed convergence of the integrals
\[
\int_{(-n - \frac{1}{2})} \left| \frac{\pi}{s \sin \pi s} \zeta^{(m)}(s) \right| \, |ds|, \quad m = 1, \ldots, n + 1.
\]

\[\square\]

### 4.7 A uniform estimate for \( K_{it}(x) \) \((t > 0, \ x > 1)\)

Here we give a simple consequence of the work of Booker–Strömbergsson–Then \[6\] on the \( K \)-Bessel function.

**Proposition 4.3.** For all \( t > 0 \) and \( x > 1 \),
\[
K_{it}(x) \ll e^{-\pi^2 t^2} t^{-\frac{1}{3}}. \tag{4.30}
\]

**Proof.** First \[6\] Proposition 2 implies that for \( x \geq t > 0 \)
\[
0 < K_{it}(x) \leq e^{-\pi^2 t} e^{-\sqrt{x^2 - t^2} + t \cos^{-1}(t/x)} \min \left( \frac{\sqrt{\pi}}{2}, \frac{\Gamma(\frac{1}{3})}{2^{\frac{2}{3}}3^{\frac{1}{3}}t^{-\frac{1}{3}}} \right)
\ll e^{-\frac{\pi^2}{2} t^{-\frac{1}{3}}} e^{-\sqrt{x^2 - t^2} + t \cos^{-1}(t/x)}.
\]

Then (4.30) holds for \( x \geq t > 0 \), since
\[
-\sqrt{x^2 - t^2} + t \cos^{-1}(t/x) = t \left( \cos^{-1}(t/x) - \sqrt{(t/x)^{-2} - 1} \right) \leq 0
\]
in view of
\[
\frac{d}{du} \left( \cos^{-1}(u) - \sqrt{u^{-2} - 1} \right) = \frac{\sqrt{1 - u^2}}{u^2} > 0 \quad \text{for} \ 0 < u < 1.
\]

112
On the other hand [6, Proposition 2] says that for $1 \leq x < t$

$$|K_{it}(x)| < e^{-\frac{x}{2}t} \begin{cases} \frac{5}{(t^2-x^2)^{\frac{1}{4}}} & \text{if } x \leq t - \frac{1}{2}t^{\frac{1}{3}}, \\ 4t^{-\frac{1}{3}} & \text{if } x \geq t - \frac{1}{2}t^{\frac{1}{3}}. \end{cases}$$

For $x \leq t - \frac{1}{2}t^{\frac{1}{3}}$, we have $t^2 - x^2 \geq t^{\frac{3}{4}} - \frac{1}{4}t^{\frac{2}{3}}$. In addition, $t > 1$ implies $t^{\frac{3}{4}} - \frac{1}{4}t^{\frac{2}{3}} \geq \frac{1}{2}t^{\frac{3}{4}}$. So (4.3) holds when $1 \leq x < t$. \qed
Bibliography


