INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

Bell & Howell Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600

UMI®
Applications of the Novel Spectral Acceleration (NSA) Algorithm for the Computation of Scattering from Rough Surfaces

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Danai Torrungrueng, M.S., B.S.

* * * * *

The Ohio State University

2000

Dissertation Committee:

J. T. Johnson, Adviser
E. H. Newman
P. H. Pathak

Approved by

Adviser

Department of Electrical Engineering
ABSTRACT

Rough surface scattering plays an important role in many electromagnetic (EM) applications. Analytical theories exist and are limited in their regimes of validity. At present, scattering from surfaces whose properties render the analytical theories invalid can be accurately calculated only through the use of numerical methods. However, numerical scattering models for rough surfaces are usually computationally expensive for general use in most practical applications. In this dissertation, an extremely efficient and accurate iterative method of moments based on a novel spectral acceleration (NSA) algorithm is developed for the computation of scattering from both one-dimensional (1-D) and two-dimensional (2-D) large-scale rough surfaces corresponding to 2-D and 3-D vector wave problems, respectively. For fixed frequency and surface roughness statistics, it is shown that the computational cost and memory storage requirement of the NSA algorithm is $O(N_{tot})$ as the surface size increases, where $N_{tot}$ is the total number of unknowns to be solved. The contribution of this dissertation can be summarized as follows:

- The original 1-D NSA algorithm has been generalized for the fast computation of radiation/scattering from 1-D extremely large-scale quasi-planar structures (QPS). New analytical formulas associated with the 1-D NSA parameters are also derived, resulting in more flexibility in optimizing the 1-D NSA algorithm.
In addition, the “multilevel” concept is introduced to improve the accuracy of the original NSA algorithm in the case of 1-D extremely large-scale QPS.

- The 1-D NSA algorithm is extended to treat 2-D random rough surfaces, and the new 2-D NSA algorithm for these surfaces is derived. Appropriate spectral integral representations of the 3-D free space scalar Green's function and the “multilevel” algorithm are developed to efficiently compute scattered fields from extremely large-scale surfaces with relatively large surface cross-range size.

- Numerical studies of the backscattering enhancement phenomenon for both 1-D and 2-D random rough surfaces are performed by varying the following parameters: surface statistics, surface material, polarization, and incident angle. It is found that the backscattering enhancement strongly depends on the above parameters.

Physical insight gained from numerical results can potentially aid in the development and assessment of future extended analytical theories.
This dissertation is dedicated to

Idappaccayatā

(the law of nature)

and

my family
ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor, Professor Joel T. Johnson, for his extreme patience, invaluable discussions, academic training, encouragement, and excellent guidance throughout my Ph.D.'s course years and also this dissertation. Thanks go to him also for his willingness to share knowledge and for introducing me to the "EIII" concept (Enduring, Ingenious, Iterative, and Initiative) for doing research and problem solving.

I also would like to express my gratitude to my great electromagnetic gurus, Professor Prabhakar H. Pathak, Professor Edward H. Newman, Professor Joel T. Johnson, and Professor Benedikt A. Munk, for their excellent teaching, invaluable discussions, and encouragement throughout the length of my stay at the Ohio State University (OSU). Thanks go to my dissertation reading committee, Professor Joel T. Johnson, Professor Edward H. Newman, and Professor Prabhakar H. Pathak, for their patience in reading my dissertation, as well as their valuable comments and suggestions. In addition, I would like to thank other professors, researchers, and friends at OSU for their instruction and sharing of knowledge, especially Professor Ulrich H. Gerlach, Professor Urbashi Mitra, Dr. Nan N. Wang, Dr. Hsi-Tseng Chou and Professor Robert J. Burkholder.

In the remote sensing group at the ElectroScience Laboratory (ESL), I would like to acknowledge all of the former and current members for their friendship and
sharing of their knowledge: Jatupum Jenwatanavet, Hyunjun Kim, Baran Ungan, Min Zhang, Yongyao Cai, and Judith Salvati. Thanks also to other ESL members for their friendship and support, especially Kyu-Pyung Hwang, Vakur B. Erturk, Ozlem Ozbay, Dilek Colak, Kishore Rama Rao, and Kwan Ho Lee. In addition, special thanks should also be given to the ESL staff for their assistance and support: Ann Dominek, Molly Wambold, Mary Ann Hrabiey, Frank Paynter and his cadre of computer experts.

I also would like to acknowledge all Thai Students at OSU for their support and the camaraderie we have shared, especially Tasanee Thanaprachoom, Suwanakiet Sawangkoon, Jatupum Jenwatanavet, Panuwat Janpugdee, and Nuttawit Surittikul.

Finally, last but not least, I would like to express my deepest gratitude to all my family for their love, support, and encouragement throughout my life.

The support of ONR contracts N00014-97-1-0541 and N00014-00-1-0399 and NSF project ECS-9701678 are acknowledged. Use of the IBM SP system at the Maui High Performance Computing Center is acknowledged, sponsored by the Air Force Research Laboratory, Air Force Materiel Command under cooperative agreement F29601-93-2-0001. Opinions, interpretations, conclusions, and recommendations contained in this dissertation are those of the author and are not necessarily endorsed by the United States Air Force, Air Force Research Laboratory, or the U.S. Government. Use of the CRAY T3E at the Ohio Supercomputing Center and the SGI Origin 2000 at the National Computational Science Alliance (grant ECS990008N) is also acknowledged.
VITA

November 26, 1973 .............................................. Born - Bangkok, Thailand

May 1993 ......................................................... B.S. Electrical Engineering, Chulalongkorn University, Bangkok, Thailand

May 1993 - October 1993 ...................... Loxley Public Company Limited, Bangkok, Thailand

September 1996 ............................................. M.S. Electrical Engineering, The Ohio State University, Columbus, Ohio

April 1995-present ................................. Graduate Research Associate, The ElectroScience Laboratory, The Ohio State University

PUBLICATIONS

Research Publications

Journal Articles


Conference Papers


Academic Reports

- D. Torrungrueng, “Applications of the Novel Spectral Acceleration (NSA) Algorithm for the Computation of Scattering from Large-Scale Rough Surfaces with an Iterative Method.” Ph.D Research Proposal, November 1999. The Ohio State University, Department of Electrical Engineering, Columbus, Ohio.


FIELDS OF STUDY

Major Field: Electrical Engineering

Studies in:

Electromagnetics

Prof. Edward H. Newman
Prof. Joel T. Johnson
Prof. Prabhakar H. Pathak
Prof. Benedikt A. Munk
Prof. Robert Lee

Microwave Circuits: Analysis and Design

Prof. Patrick Roblin

Communications and Signal Processing

Prof. Urbashi Mitra
Prof. Randolph L. Moses
Prof. Kim L. Boyer
Prof. Lee C. Potter

Mathematics

Prof. Ulrich H. Gerlach
## TABLE OF CONTENTS

Abstract ...................................................................................................................... ii

Dedication .................................................................................................................... iv

Acknowledgments ....................................................................................................... v

Vita .................................................................................................................................... vii

List of Figures .............................................................................................................. xv

**LIST OF ABBREVIATIONS** ...................................................................................... xxviii

Page

Chapters:

1. Introduction ............................................................................................................. 1

   1.1 Background and Motivation ............................................................................. 1
   1.1.1 Approximate Analytical Theories ............................................................ 3
   1.1.2 Monte-Carlo Simulations ........................................................................ 5
   1.1.3 The Backscattering Enhancement Phenomenon .................................... 6

1.2 Literature Review .................................................................................................. 7
   1.2.1 Numerical Computation of Scattering from Rough Surfaces ............... 7
   1.2.2 Previous Studies of Backscattering Enhancement ............................... 14

1.3 Definitions of Scattering Quantities ................................................................. 18

1.4 Organization of This Dissertation ................................................................... 23

2. Development of the Novel Spectral Acceleration Algorithm for One-Dimensional Rough Surface Scattering Problems .......................................................... 30

   2.1 Introduction ..................................................................................................... 30
7.3 A Concluding Remark ....................................................... 272

Appendices:

A. Random Rough Surface Generation ................................................ 274
   A.1 1-D Random Rough Surface Generation ....................................... 274
   A.2 2-D Random Rough Surface Generation ....................................... 279

B. Topology of the Complex Planes Associated with the Integral Represen-
tation of the Free Space 2-D Scalar Green’s Function \( g(\rho, \rho') \) ......... 282
   B.1 Topology Associated with the Spectral Integral Representation of
       \( g(\rho, \rho') \) for a Lossless Medium ............................................. 285
   B.2 Topology Associated with the Angular Spectral Integral Representa-
tion of \( g(\rho, \rho') \) for a Lossless Medium ........................................ 292
   B.3 Derivation of the SDP and SAP Contours Associated with the Inte-
gral Representation of \( g(\rho, \rho') \) in the Complex \( k_z \) and \( \phi \) Planes for
       the Real Propagation Constant .................................................. 293
   B.4 Topology Associated with the Spectral Integral Representation of
       \( g(\rho, \rho') \) for a Lossy Medium ............................................... 301
   B.5 Topology Associated with the Angular Spectral Integral Representa-
tion of \( g(\rho, \rho') \) for a Lossy Medium ........................................... 306
   B.6 Derivation of the SDP and SAP Contours Associated with the Inte-
gral Representation of \( g(\rho, \rho') \) in the Complex \( k_z \) and \( \phi \) Planes for
       the Complex Propagation Constant ........................................... 313
   B.7 A Summary and Conclusions ..................................................... 319

C. Topology of Two Coupled Complex \( k_y \) and \( k_z \) Planes Associated with the
   2-D NSA Algorithm ........................................................................ 321
   C.1 Topology Associated with the Double Spectral Integral Representa-
tion of \( g(\mathbf{r}, \mathbf{r}') \) in the Complex \( k_z \) Plane .......................... 323
   C.2 Topology Associated with the Double Spectral Integral Representa-
tion of \( g(\mathbf{r}, \mathbf{r}') \) in the Complex \( k_y \) Plane .......................... 325
   C.3 A Summary and Conclusions ..................................................... 332

D. Derivation of Formulas Associated with Integration Parameters of the
   Standard 1-D NSA Algorithm .......................................................... 333
   D.1 Derivation of Analytical Formula for \( \delta \) .................................. 333
   D.2 Derivation of Analytical Formula for \( \phi_{\text{max}} \) .......................... 336
   D.3 Derivation of Analytical Formula for \( \Delta \phi \) ............................. 337
E. Derivation of Formulas Associated with Integration Parameters of the 2-D NSA Algorithm .................................................. 339
   E.1 Derivation of Analytical Formula for $\delta_k$ ........................................ 339
F. Muller's Method .......................................................... 344
Bibliography .............................................................. 346
**LIST OF FIGURES**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 A 3-D coordinate systems used to described incident and scattered field directions</td>
<td>17</td>
</tr>
<tr>
<td>1.2 A 2-D coordinate systems used to described incident and scattered field directions</td>
<td>20</td>
</tr>
<tr>
<td>2.1 A 1-D finite dielectric surface profile, ( z = f(x) ), is illuminated by a TM incident magnetic field ( \mathbf{H}^i(\rho) = j\mathbf{H}_y^i(\rho) ). The region above the surface profile ( L ) is the free space with ( (\mu, \varepsilon) ), and the region below the surface profile is the nonmagnetic region with ( (\mu, \varepsilon_1) )</td>
<td>33</td>
</tr>
<tr>
<td>2.2 (a) Forward region ( L_f^f ) in the forward direction (b) Backward region ( L_b^b ) in the backward direction</td>
<td>37</td>
</tr>
<tr>
<td>2.3 Strong region ( L_s^f ) and weak region ( L_w^f ) in the forward direction. ( L_s ) is a neighborhood distance</td>
<td>39</td>
</tr>
<tr>
<td>2.4 The original contour ( C_0 ) and the deformed contour ( C_\delta ) of integration in the periodic complex ( \phi ) plane</td>
<td>41</td>
</tr>
<tr>
<td>2.5 The plot of (</td>
<td>I(\phi)</td>
</tr>
<tr>
<td>2.6 The plots of the real and imaginary parts of ( I(\phi) ) versus ( \phi_R ) and their discrete fourier transforms versus ( f ).</td>
<td>54</td>
</tr>
<tr>
<td>2.7 A flat PEC half plane is illuminated by the TE plane wave.</td>
<td>57</td>
</tr>
<tr>
<td>2.8 The plot of ( e^{-\kappa_i \phi_h} ), where ( i = 1 ) or ( 2 ), and their difference versus ( \phi_R ).</td>
<td>61</td>
</tr>
<tr>
<td>2.9 The decomposition of the very large weak region of size ( L_w ) into ( M ) weak regions of size ( L_{w,j} ), where ( j = 1, \cdots, M ) and ( L_w = \sum_{j=1}^{M} L_{w,j} )</td>
<td>63</td>
</tr>
</tbody>
</table>
2.10 Normalized bistatic RCS in dB comparing between the standard 1-D FB and 1-D FB/NSA methods: (a) HH polarization (b) VV polarization. 70

2.11 Plots of the 1-D NSA parameters in the first weak region versus the size of the strong region $L_s$ for the case of $\Delta z_{max} = 5.0\lambda$. 71

2.12 Plots of the size of the first region $L_{w,1}$ and the relative error (in %) in computing the free space Green's function in the angular spectral domain versus $L_s$ for the case of $\Delta z_{max} = 5.0\lambda$. 73

2.13 Plots of the 1-D NSA parameters in the first weak region versus $L_s$ for the case of $\Delta z_{max} = 15.0\lambda$. 75

2.14 Plots of $L_{w,1}$ and the relative error (in %) in computing the free space Green's function in the angular spectral domain versus $L_s$ for the case of $\Delta z_{max} = 15.0\lambda$. 76

2.15 Plots of the relative error (in %) from the one-level and two-level NSA algorithms versus $\Delta x = x - x_t$ (in $\lambda$): (a) $\Delta z_{max} = 5.0\lambda$ (b) $\Delta z_{max} = 15.0\lambda$. 78

3.1 Comparisons between Monte-Carlo normalized incoherent backscattering radar cross sections, computed via the 1-D FB/NSA and analytical results computed via the first-order SPM at 14 GHz with $G = 5$ for both HH and VV polarizations and for different values of $\epsilon_r$: (a) PEC surfaces (b) $\epsilon_r = 38.0 + i40.0$ (c) $\epsilon_r = 10.0 + i10.0$ (d) $\epsilon_r = 6.5 + i1.0$. Gaussian surface statistics for this case are: $h = 0.014\lambda, l = 0.397\lambda$, and $\sigma_s = 0.05$. 88

3.2 Comparisons between Monte-Carlo normalized incoherent bistatic radar cross sections, computed via the 1-D FB/NSA and Monte-Carlo PO, and analytical results computed via the first-order KA (PO) at 14 GHz with $\theta_t = 50^\circ$ and $G = 5$ for both HH and VV polarizations and for PEC rough surfaces. Gaussian surface statistics for this case are: $h = 0.20\lambda, l = 5.657\lambda$, and $\sigma_s = 0.05$. 90
3.3 Comparisons between Monte-Carlo normalized incoherent bistatic radar cross sections in the specular direction, computed via the 1-D FB/NSA, and analytical results computed via the first-order KA (PO) at 14 GHz with $G = 5$ for both HH and VV polarizations and for PEC rough surfaces. Gaussian surface statistics for this case are: $h = 0.20\lambda, l = 5.657\lambda$, and $\sigma_s = 0.05$.

3.4 Convergence study of the normalized HH incoherent bistatic RCS in Monte-Carlo simulations with the 1-D FB/NSA method at $\theta_i = 20^\circ$ for two different PEC Gaussian surfaces: (a) Surface # 3 (b) Surface # 4.

3.5 Normalized HH incoherent bistatic RCS versus $\theta_s$ for PEC rough surfaces at $\theta_i = 20^\circ$: (a) Surface # 1 (b) Surface # 2 (c) Surface # 3 (d) Surface # 4.

3.6 Normalized VV incoherent bistatic RCS versus $\theta_s$ for PEC rough surfaces at $\theta_i = 20^\circ$: (a) Surface # 1 (b) Surface # 2 (c) Surface # 3 (d) Surface # 4.

3.7 Normalized incoherent bistatic RCS versus $\theta_s$ for Surface # 4 at $\theta_i = 20^\circ$: (a) HH polarization (b) VV polarization.

3.8 Normalized incoherent bistatic RCS versus $\theta_s$ for Surface # 4 at $\theta_i = 20^\circ$ average over 256 surface realizations. The normalized incoherent bistatic RCS for the IBC surfaces is scaled such that its maximum is equal to the maximum of normalized incoherent bistatic RCS for the PEC surfaces, and the scaling factor is given as a number in a square bracket appended the value of $\varepsilon_{r1}$ in each plot: (a) HH polarization (b) VV polarization.

3.9 Normalized incoherent bistatic RCS versus $\theta_s$ for Surface # 4 at $\theta_i = 50^\circ$: (a) HH polarization (b) VV polarization.

3.10 Normalized incoherent bistatic RCS versus $\theta_s$ for Surface # 4 at $\theta_i = 50^\circ$ average over 256 surface realizations. The normalized incoherent bistatic RCS for the IBC surfaces is scaled such that its maximum is equal to the maximum of normalized incoherent bistatic RCS for the PEC surfaces, and the scaling factor is given as a number in a square bracket appended the value of $\varepsilon_{r1}$ in each plot: (a) HH polarization (b) VV polarization.
3.11 Normalized incoherent bistatic RCS versus $\theta_s$ for PEC rough surfaces at $\theta_i = 20^\circ$: (a) Surface # 1 (b) Surface # 2 (c) Surface # 3 (d) Surface # 4. .......................................................... 109

3.12 Normalized incoherent bistatic RCS versus $\theta_s$ for Surface # 4 at $\theta_i = 50^\circ$: (a) PEC (b) IBC: $\varepsilon_{r1} = 38.0 + i40.0$. .......................................................... 110

3.13 Normalized incoherent bistatic RCS versus $\theta_s$ for PEC rough surfaces and using Surface # 4: (a) HH polarization (b) VV polarization. .................................................. 114

3.14 Normalized incoherent bistatic RCS versus $\theta_s$ averaged over 256 surface realizations for PEC rough surfaces and using Surface # 4. The normalized incoherent bistatic RCS is scaled such that its maximum is equal to the maximum of normalized incoherent bistatic RCS for $\theta_i = 0.1^\circ$. and the scaling factor is given as a number in a square bracket appended the value of $\theta_i$ in each plot: (a) HH polarization (b) VV polarization. .................................................. 115

3.15 Normalized incoherent bistatic RCS versus $\theta_s$ for IBC rough surfaces ($\varepsilon_{r1} = 38.0 + i40.0$) and using Surface # 4: (a) HH polarization (b) VV polarization. .................................................. 116

3.16 Normalized incoherent bistatic RCS versus $\theta_s$ averaged over 256 surface realizations for IBC rough surfaces ($\varepsilon_{r1} = 38.0 + i40.0$) and using Surface # 4. The normalized incoherent bistatic RCS is scaled such that its maximum is equal to the maximum of normalized incoherent bistatic RCS for $\theta_i = 0.1^\circ$. and the scaling factor is given as a number in a square bracket appended the value of $\theta_i$ in each plot: (a) HH polarization (b) VV polarization. .................................................. 117

3.17 Normalized incoherent bistatic RCS versus $\theta_s$ for PEC rough surfaces and using Surface # 4: (a) HH polarization (b) VV polarization. .................................................. 119

3.18 Normalized incoherent bistatic RCS versus $\theta_s$ averaged over 256 surface realizations for PEC rough surfaces and using Surface # 4. The normalized incoherent bistatic RCS is scaled such that its maximum is equal to the maximum of normalized incoherent bistatic RCS for $\theta_i = 50^\circ$, and the scaling factor is given as a number in a square bracket appended the value of $\theta_i$ in each plot: (a) HH polarization (b) VV polarization. .................................................. 120

xviii
3.19 Normalized HH incoherent bistatic RCS versus $\theta_s$ for PEC rough surfaces: (a) $\theta_i = 60^\circ$ (b) $\theta_i = 70^\circ$. ............................................... 121

3.20 Comparisons between Monte-Carlo normalized incoherent bistatic radar cross sections, computed via the single Neumann iteration, the 1-D FB/NSA and Monte-Carlo PO, and analytical results computed via the PO at 14 GHz with $\theta_i = 50^\circ$ and $G = 5$ for VV polarization and for PEC rough surfaces. Gaussian surface statistics for this case are: $h = 0.20\lambda$, $l = 5.65\lambda$, and $\sigma_s = 0.05$. ...................................................... 126

3.21 Comparisons of Monte-Carlo VV normalized incoherent bistatic radar cross sections computed via the 1-D FB/NSA, the single Neumann iteration, the second-order term of the KA, and Monte-Carlo PO at 14 GHz with $G = 5$ for PEC Surface # 4: (a) $\theta_i = 0.1^\circ$ (b) $\theta_i = 20^\circ$ (c) $\theta_i = 50^\circ$ (d) $\theta_i = 85^\circ$. ...................................................... 127

3.22 Comparisons of Monte-Carlo VV normalized incoherent bistatic radar cross sections computed via the 1-D FB/NSA, the single Neumann iteration, the second-order term of the KA, and Monte-Carlo PO at 14 GHz with $\theta_i = 20^\circ$ and $G = 5$ for PEC surfaces: (a) Surface # 2 (b) Surface # 4. .......................................................... 132

4.1 A 2-D PEC rough surface profile $S$ illuminated by a tapered incident field $\mathbf{E}'(x, y, z)$ centered at the origin and propagating in direction $\hat{k}_i = \hat{x} \sin \theta_i \cos \phi_i + \hat{y} \sin \theta_i \sin \phi_i - \hat{z} \cos \theta_i$. ........................................... 140

4.2 Discretization of a rectangular surface area $D_x \times D_y$ of $S_{xy}$ into an $N \times M$ rectangular grid. Each grid cell has the dimension $\Delta x \times \Delta y$. .................................................. 141

4.3 Forward-stepping (FS) and backward-stepping (BS) processes: (a) FS process (b) BS process (c) Alternative FS process (d) FS and BS projected surfaces. .......................................................... 145

4.4 Strong and weak regions in the FS direction. ................................. 151

4.5 Integration contour of $g(r,r')$ on the complex $k_z$ plane. $C_{k_z}$ is the original contour and $C_{k_z+}$ is the deformed contour. The SDP and SAP contours for a flat surface ($\Delta z = 0$) are also shown. .................................................. 155
4.6 Integration contour of $g(\mathbf{r}, \mathbf{r}')$ on the complex $k_z$ plane. $C_{SDP}(\Delta z = \Delta z_{max})$ and $C_{SAP}(\Delta z = \Delta z_{max})$ are the SDP and SAP contours when $\Delta z = \Delta z_{max}$ respectively. ........................................ 155

4.7 Integration contour of $g(\mathbf{r}, \mathbf{r}')$ on the complex $k_y$ plane for a fixed value of $k_z$. $C_{k_y}$ is the original contour and $C_{4k_y}$ is the deformed contour. . 156

4.8 The worst-case configuration of a pair of source and field points for computing $g(\mathbf{r}, \mathbf{r}')$ in the complex $k_y$ plane. .......................... 163

4.9 Decomposition of regions in the FS direction for extremely large-scale 2-D RSS problems. .................................................... 165

4.10 The FS process in the 2-D FB/NSA algorithm using both $x$- and $y$- expansions with the following parameters: $N = 5, M = 5, N_x = 3$ and $N_y = 2$. ....................................................... 169

4.11 Integration contour of $g(\mathbf{r}, \mathbf{r}')$ on the complex $k_x$ plane for a fixed value of $k_z$. $C_{k_x}$ is the original contour and $C_{4k_x}$ is the deformed contour. . 173

4.12 Comparison of computational efficiency between the conventional FB method and the FB/NSA method using the $x$- expansion for 2-D PEC RSS problems: (a) CPU time per iteration versus number of unknowns (b) CPU memory versus number of unknowns. ................. 181

4.13 Normalized bistatic RCS in dB: (a) HH polarization (b) VH polarization. 186

4.14 Radiation function $F^{(k)}(\mathbf{r}, k_z, k_y)$ in the complex $k_z$ and $k_y$ planes for the last backward sweep ($k = 3$) and the receiving element located at the $(x_1, y_1)$ element of the rough surface. ..................... 188

4.15 Comparison between Monte-Carlo simulation results for moderately rough surfaces computed via the 2-D FB/NSA method using the $x$- expansion and the experimental data obtained from the University of Washington, Seattle. ................................. 189

4.16 A comparison of the normalized bistatic RCS in dB when $\phi_i = \phi_s = 0^\circ$ computed by the standard 2-D FB/NSA using the $x$- expansion and the 2-D FB/NSA algorithm using both $x$- and $y$- expansions: (a) HH polarization (b) VH polarization. ........................... 192
4.17 A comparison of the normalized bistatic RCS in dB when $\phi_i = \phi_s = 0^\circ$ computed by the 2-D *two-level* FB/NSA method and the SMCG method: (a) HH polarization (b) VH polarization. .......................... 194

4.18 A comparison of the normalized bistatic RCS in dB when $\phi_i = \phi_s = 0^\circ$ computed by the 2-D *one-level* FB/NSA method and the SMCG method: (a) HH polarization (b) VH polarization. .......................... 195

5.1 Integration contour of $g(r, r')$ and $\nabla g(r, r')$ on the complex $k_z$ plane. The original contour $C_{k_z}$ is deformed to the new contour $C'_{k_z}$. .......................... 211

5.2 Integration contour of $g(r, r')$ and $\nabla g(r, r')$ on the complex $k_y$ plane for a fixed value of $k_z$. $C_{k_y}$ is the original contour and $C'_{k_y}$ is the deformed contour. .......................... 211

5.3 A comparison of the normalized bistatic RCS in dB computed by the standard 2-D FB method and the 2-D FB/NSA method for a *deterministic* IBC rough surface with $\epsilon_r = 10.0 + i10.0$ of size $128\lambda \times 16\lambda$ illuminated by a vertical polarized (TM) tapered plane wave with the taper parameter $g'/a = 6$ at an incident angle of 40°. The surface of interest is a realization of a Gaussian random process described by a Gaussian spectrum with $l_x = l_y = 1.414\lambda$ and $h = 1.0\lambda$: (a) HV polarization (b) VH polarization. .......................... 217

5.4 Monte-Carlo simulation results computed via the 2-D FB/NSA method and averaged over 150 realizations for the same scenario as considered in Figure 5.3: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization. .......................... 218

6.1 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($0^\circ \leq \theta_i \leq 40^\circ$) for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization. .......................... 229

6.2 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($60^\circ \leq \theta_i \leq 70^\circ$) for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization. .......................... 230
6.3 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($0^\circ \leq \theta_i \leq 40^\circ$) for the case of IBC Gaussian surfaces with relative permittivity $\varepsilon_{r1} = 38.0 + i40.0$ and an isotropic Gaussian spectrum $h/\lambda = \sigma_z = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$:
(a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization .................................................... 231

6.4 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($60^\circ \leq \theta_i \leq 70^\circ$) for the case of IBC Gaussian surfaces with relative permittivity $\varepsilon_{r1} = 38.0 + i40.0$ and an isotropic Gaussian spectrum $h/\lambda = \sigma_z = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$:
(a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization .................................................... 232

6.5 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($0^\circ \leq \theta_i \leq 40^\circ$) for the case of IBC Gaussian surfaces with relative permittivity $\varepsilon_{r1} = 10.0 + i10.0$ and an isotropic Gaussian spectrum $h/\lambda = \sigma_z = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$:
(a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization .................................................... 233

6.6 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($60^\circ \leq \theta_i \leq 70^\circ$) for the case of IBC Gaussian surfaces with relative permittivity $\varepsilon_{r1} = 10.0 + i10.0$ and an isotropic Gaussian spectrum $h/\lambda = \sigma_z = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$:
(a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization .................................................... 234

6.7 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($0^\circ \leq \theta_i \leq 40^\circ$) for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_z = 0.5$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization .................................................... 235

6.8 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($60^\circ \leq \theta_i \leq 70^\circ$) for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_z = 0.5$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization .................................................... 236
6.9 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying surface materials at $\theta_i = 0^\circ$ for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_x = 1.0$ and $l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization. ................................................................. 240

6.10 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying surface materials at $\theta_i = 20^\circ$ for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_x = 1.0$ and $l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization. ................................................................. 241

6.11 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying surface materials at $\theta_i = 40^\circ$ for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_x = 1.0$ and $l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization. ................................................................. 242

6.12 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying surface materials at $\theta_i = 60^\circ$ for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_x = 1.0$ and $l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization. ................................................................. 243

6.13 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying surface materials at $\theta_i = 70^\circ$ for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_x = 1.0$ and $l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization. ................................................................. 244

6.14 Comparison of the normalized incoherent backscattering RCS (dB), computed by the Monte-Carlo 2-D FB/NSA method (150 realizations), averaged (magnitudes of backscattered fields) over three consecutive angles in one degree steps (including the backscattering angle) with varying surface materials for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_x = 1.0$ and $l_y = \sqrt{2}\lambda$: (a) Co-polarization (b) Cross-polarization. ....................................................... 247
6.15 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying rms surface slopes at $\theta_i = 20^\circ$ for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $l_x = l_y = \sqrt{2\lambda}$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization. 249

6.16 Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying rms surface slopes at $\theta_i = 70^\circ$ for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $l_x = l_y = \sqrt{2\lambda}$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization. 250

6.17 Comparison of the normalized incoherent backscattering RCS (dB), computed by the Monte-Carlo 2-D FB/NSA method (150 realizations), averaged over three consecutive angles in one degree steps (including the backscattering angle) and the Monte-Carlo SSA (up to 1st order and using 100 realizations) with varying rms surface slopes for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $l_x = l_y = \sqrt{2\lambda}$: (a) Co-polarization (b) Cross-polarization. 251

6.18 Comparison of the HH/VV polarization ratios (dB), computed by the Monte-Carlo 2-D FB/NSA method (150 realizations), averaged over three consecutive angles in one degree steps (including the backscattering angle) and the Monte-Carlo SSA (up to 1st order and using 100 realizations) with varying rms surface slopes for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $l_x = l_y = \sqrt{2\lambda}$. 252

6.19 Comparison of the HH normalized incoherent backscattering RCS between the 1-D and 2-D surface models, computed by the Monte-Carlo 2-D FB/NSA method with 256 and 150 surface realizations respectively, with varying incident angles for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$: (a) $\theta_i = 0^\circ$ (b) $\theta_i = 20^\circ$ (c) $\theta_i = 60^\circ$ (d) $\theta_i = 70^\circ$. 255

6.20 Comparison of the VV normalized incoherent backscattering RCS between the 1-D and 2-D surface models, computed by the Monte-Carlo 2-D FB/NSA method with 256 and 150 surface realizations respectively, with varying incident angles for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$: (a) $\theta_i = 0^\circ$ (b) $\theta_i = 20^\circ$ (c) $\theta_i = 60^\circ$ (d) $\theta_i = 70^\circ$. 256
6.21 Comparison of the HH normalized incoherent backscattering RCS between the 1-D and 2-D surface models, computed by the Monte-Carlo 2-D FB/NSA method with 256 and 150 surface realizations respectively, with varying incident angles for the case of IBC Gaussian surfaces ($\epsilon_{r1} = 38.0 + i40.0$) with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$: (a) $\theta_i = 0^\circ$ (b) $\theta_i = 20^\circ$ (c) $\theta_i = 60^\circ$ (d) $\theta_i = 70^\circ$.

6.22 Comparison of the VV normalized incoherent backscattering RCS between the 1-D and 2-D surface models, computed by the Monte-Carlo 2-D FB/NSA method with 256 and 150 surface realizations respectively, with varying incident angles for the case of IBC Gaussian surfaces ($\epsilon_{r1} = 38.0 + i40.0$) with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$: (a) $\theta_i = 0^\circ$ (b) $\theta_i = 20^\circ$ (c) $\theta_i = 60^\circ$ (d) $\theta_i = 70^\circ$.

B.1 The original contour of integration $C_\phi$ of the angular spectral integration of $g(p, p')$ in the periodic complex $\phi$ plane.

B.2 The plots of the hyperbolic curves $\text{Re}[k_z^2] = 0$ and $\text{Im}[k_z^2] = 0$ as well as the regions for which $\text{Re}[k_z^2] > 0$, for the case of the slightly lossy medium.

B.3 The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_z$ plane, for the case of the slightly lossy medium.

B.4 The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_z$ plane, for the case of the lossless medium.

B.5 The plot of the topology associated with the angular spectral integral representation of $g(p, p')$ and the original contour of integration $C_\phi$ in the periodic complex $\phi$ plane for the case of the lossless medium.

B.6 Topology associated with the angular spectral integral representation of $g(p, p')$ for the case of $\phi = \gamma = \frac{\pi}{4}$ with $x = x' = z = z' = 5\lambda$: (a) The plot of $\ln |I_1(\phi, p, p')| = \kappa \text{Re}[f_1(\phi)]$ as a function of $\phi_R$ and $\phi_I$ in the vicinity of the first-order saddle point (b) The associated contour plot of $\ln |I_1(\phi, p, p')|$, the SDP and SAP contours in the complex $\phi$ plane.
B.7 The contour plot of $\ln|f_2(k_z, \rho, \rho_1)|$ and the corresponding SDP and SAP contours (for $\gamma = \frac{\pi}{4}$ rad. with $x - x' = z - z' = 5\lambda$) in both proper and improper Riemann sheets ...................................................... 302

B.8 The plots of the straight lines $Re[k_z^2] = 0$ and the hyperbolic curves $Im[k_z^2] = 0$ as well as the regions for which $Re[k_z^2] > 0$, for the case of the lossy medium with $k_I = k_R = k_1$. .............................. 304

B.9 The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_z$ plane, for the case of the lossy medium with $k_I = k_R = k_1$. 305

B.10 The plots of the hyperbolic curves $Re[k_z^2] = 0$ and $Im[k_z^2] = 0$ as well as the regions for which $Re[k_z^2] > 0$, for the case of the lossy medium with $k_I > k_R$. .............................. 306

B.11 The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_z$ plane, for the case of the lossy medium with $k_I > k_R$. 307

B.12 The topology in the complex $\phi$ plane obtained from the mapping $k_z = k\sin \phi$ when $k$ is complex with $k_I < k_R$ ($\Delta = \tan^{-1}\left(\frac{k_I}{k_R}\right) < \frac{\pi}{4}$). ...... 309

B.13 The topology in the complex $\phi$ plane obtained from the mapping $k_z = k\sin \phi$ when $k$ is complex with $k_I = k_R$ ($\Delta = \tan^{-1}\left(\frac{k_I}{k_R}\right) = \frac{\pi}{4}$). ...... 310

B.14 The topology in the complex $\phi$ plane obtained from the mapping $k_z = k\sin \phi$ when $k$ is complex with $k_I > k_R$ ($\Delta = \tan^{-1}\left(\frac{k_I}{k_R}\right) > \frac{\pi}{4}$). ...... 311

B.15 Topology associated with the angular spectral integral representation of $g(\rho, \rho_1)$ for the lossy case with $k_I = k_R = 2\pi$ rad./\(\lambda\) ($\Delta = \frac{\pi}{4}$ rad.) and with $\phi = \gamma = \frac{\pi}{4}$ with $x - x' = z - z' = 5\lambda$: (a) The plot of $\ln|I_1(\phi, \rho, \rho_1)| = \kappa Re[f_1(\phi)]$ as a function of $\phi_R$ and $\phi_I$ in the vicinity of the first-order saddle point (b) The associated contour plot of $\ln|I_1(\phi, \rho, \rho_1)|$, the SDP and SAP contours in the complex $\phi$ plane. 318

B.16 The contour plot of $\ln|f_2(k_z, \rho, \rho_1)|$ and the corresponding SDP and SAP contours for the lossy medium with $k_I = k_R = 2\pi$ rad./\(\lambda\) ($\gamma = \frac{\pi}{4}$ rad. and $x - x' = z - z' = 5\lambda$) in both proper and improper Riemann sheets .............................. 319

xxvi
C.1 The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_y$ plane, for Case 1. .......................................................... 327

C.2 The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_y$ plane, for Case 2. .......................................................... 328

C.3 The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_y$ plane, for Case 3. .......................................................... 329

C.4 The contour plot of $\ln |I_b(k_y, \kappa, \rho, \rho')|$ and the corresponding SDP and SAP contours for the lossy medium with $k_I = k_R = 2\pi$ rad. /$\lambda$ ($\gamma = \pi$ rad. and $x - \xi = z - \zeta = 5\lambda$) in both proper and improper Riemann sheets .......................................................... 331

D.1 The original contour $C_\phi$, the deformed contour $C_\delta$, and the SDP and SAP contours when $\Delta z = \Delta z_{\text{max}}$ ($C_{SDP}(\Delta z = \Delta z_{\text{max}})$ and $C_{SAP}(\Delta z = \Delta z_{\text{max}})$, respectively) in the periodic complex $\phi$ plane. $I_\phi$ is the intersection point between $C_\delta$ and $C_{S\delta}(\Delta z = \Delta z_{\text{max}})$. ........................................ 334
LIST OF ABBREVIATIONS

1-D ................................. One-Dimensional
2-D ................................. Two-Dimensional
3-D ................................. Three-Dimensional
AIM ................................ Adaptive Integral Method
BMIA ............................. Banded Matrix Iterative Approach
BS ................................. Backward-Stepping
CAG .............................. Canonical Grid
CG ................................. Conjugate-Gradient
CMT ............................... Current Marching Technique
DFT ............................... Discrete Fourier Transform
EM ............................... Electromagnetic
FB ............................... Forward-Backward
FDTD ............................ Finite-Difference Time-Domain
FEM ............................. Finite-Element Method
FFT ............................... Fast Fourier Transform
FMM ............................. Fast Multipole Method
FS ............................... Forward-Stepping
FT ............................... Fourier Transform
FVTD ............................ Finite-Volume Time-Domain
H .................................Horizontal
HH ...............................Horizontal-Horizontal
HV ...............................Horizontal-Vertical
IBC ..............................Impedance Boundary Condition
IEM ..............................Integral Equation Model
iEMMS ...........................IEM with Multiple Scattering
KA ...............................Kirchhoff Approximation
LGA ..............................Low-Grazing-Angle
MFIE .............................Magnetic Field Integral Equation
MLFMA ..........................Multilevel Fast Multipole Algorithm
MM ...............................Moment Methods
MMW .............................Millimeter-Wave
MOMI ............................Method of Ordered Multiple Interaction
MPI ...............................Message Passing Interface
NDT ..............................Non-Destructive Testing
NSA ..............................Novel Spectral Acceleration
P2SC .............................Power2 Super Chip
pdf ...............................probability density function
PE ...............................Processing Element
PEC ..............................Perfect Electric Conducting
PO ...............................Physical Optics
PR ...............................Pseudo-Residual
psd ...............................power spectral density
QPS ..............................Quasi-Planar Structure

xxix
RCS ................................................. Radar Cross Section
RHS ................................................. Right-Hand Side
rms .................................................. root-mean-square
RSS .................................................. Rough Surface Scattering
SAP ................................................... Steepest Ascent Path
SDFMM ............................................... Steepest Descent-Fast Multipole Method
SDP ................................................... Steepest Descent Path
SMCG ............................................... Sparse-Matrix Canonical Grid
SMP ................................................... Symmetric Multiprocessing
SPM ................................................... Small Perturbation Method
SSA ................................................... Small Slope Approximation
SW ................................................... Scattering Width
TE .................................................... Transverse Electric
TM .................................................... Transverse Magnetic
TMSR ................................................. Total Memory Storage Requirement
TOC .................................................. Total Operational Count
V ...................................................... Vertical
VH .................................................... Vertical-Horizontal
VV .................................................... Vertical-Vertical
CHAPTER 1

Introduction

1.1 Background and Motivation

The problem of scattering from a rough surface has aroused the interest of physicists and engineers for the last 90 years. This problem was examined for the first time as a problem of mathematical physics in 1907 by Rayleigh [1]. Rough surface scattering (RSS) plays an important role in many electromagnetic (EM) applications, including both active and passive remote sensing, wave propagation, radio astronomy, biomedicine, surface physics, ultrasonic non-destructive testing (NDT), and optical and radar system design. For examples, radio communication over the ocean is affected by the surface roughness due to rapidly fluctuating reflections from facets of the ocean surface known as sea returns or sea clutter. Effective measures against sea clutter depend on the knowledge of the scattering characteristics. In radar astronomy, waves scattered from planetary surfaces are analyzed to deduce the surface characteristics: e.g., the surface roughness of the moon is estimated by studying the distortion of the radar pulse backscattered from the moon to the earth. In biological media, rough interfaces between different organs and tissues affect propagation and scattering characteristics of a wave.
In general, rough surface scattering has the following characteristics. If the surface is smooth, there is only a specular (coherent) component of EM waves reflected from the surface. If the surface is slightly rough, the specular component is attenuated slightly due to scattering and the power corresponding to this decrease of the reflected power, called the diffuse (incoherent) component, is scattered in all directions, although certain privileged directions may receive more power than others. If the surface becomes very rough, the specular component almost disappears and the diffuse component dominates. In addition, the same surface may be rough for some EM wavelengths and smooth for others, or for the same wavelength it may be either rough or smooth for different angles of incidence. These are the well-known results from the Rayleigh hypothesis [2]. The important point of this hypothesis is that the roughness of any scattering surface is not an intrinsic property of that surface but depends on the properties of the wave being scattered. Both the wavelength and angle of the incident wave determine how rough any surface appears to be: i.e. its "effective roughness"; a surface appears rougher the smaller the incident wavelength or the closer the incident angle is to the surface normal. The scale of the roughness, relative to the incident wavelength, is crucial in determining how energy is scattered. The role of the incident angle is clearly illustrated with reference to the phenomenon of road glare, or a sunset over the sea [3]. In both cases, bright reflections occur whenever the sun is low on the horizon. The scattering surface (road or sea) is behaving as though it was smooth, and is reflecting strongly in the specular direction. The same surface does not lead to bright reflections when the sun is high above the horizon, as it is then rough to this incident wave.
The rest of this section is organized as follows. A brief overview of approximate analytical theories employed in rough surface scattering is given in Section 1.1.1. Section 1.1.2 provides an introduction to Monte-Carlo simulations, an alternative approach for solving RSS problems. In this dissertation, the backscattering enhancement phenomenon, one of the most interesting phenomena in rough surface scattering [4]–[22], is studied intensively, and Section 1.1.3 gives an overview of this phenomenon.

1.1.1 Approximate Analytical Theories

EM wave scattering from random rough surfaces has been studied extensively [2, 3], [23]–[27]. Two classical methods that are most commonly employed in rough surface scattering are the small perturbation method (SPM) and the Kirchhoff approximation (KA) (or the physical optics (PO) approximation). These methods are limited in their regimes of validity. The SPM originally developed in [28] requires that surface heights be much smaller than an incident EM wavelength and small surface slopes in order for the perturbation series to converge. The zeroth-order solution yields the reflected and transmitted fields from a flat surface. The first-order solution gives the lowest-order incoherent reflected and transmitted intensities; however, it does not provide the depolarization effect in the backscattering direction. The second-order solution yields the lowest-order correction to the coherent reflection and transmission coefficients and the depolarization of the backscattering intensity. In the KA, tangential fields on the surface profile are assumed to be the same as those that would exist on a plane tangent to each point on the surface [2]. Shadowing and feature diffraction effects are neglected in this approximation, although attempts have been made to incorporate the shadowing effect through the use of shadowing functions in
an analytical averaging procedure [29]. The KA requires surfaces with fairly small slopes and large radii of curvature, which limit multiple scattering and diffraction effects, and near-forward scattering observations, which limit shadowing effects. It is noted that the KA solution predicts no differences between horizontal-horizontal (HH) and vertical-vertical (VV) results for perfect electric conducting (PEC) rough surfaces, and also predicts no cross-polarized fields in the backscattered direction due to the single scattering nature of the KA solution [30].

Another approximate method, the composite surface (two scale) model, has also been widely used for the RSS problems. This method is applied for surfaces containing a large range of spatial scales such as an ocean surface, so that small surface variations can be considered to overly larger scale variations, and involves a hybridization of the KA solution modelling the large spatial variations and the SPM solution modelling the small spatial scale contributions [31]-[38]. The composite surface model has shown to match experimental backscattering data from the ocean at various frequencies [37]. However, the theoretical basis of this model still remains unclear about a scale-dividing parameter (between small and large scales) which can be in fact arbitrary chosen within rather wide limits. Other approximate analytical theories have also been proposed [26],[39]-[55]: however, these approaches have not been as widely employed as the three theories discussed above. At present, scattering from surfaces whose properties render the analytical theories invalid can be accurately calculated only through the use of numerical methods based on Monte-Carlo simulations [56]. Physical insight gained from numerical results can potentially aid in the development and assessment of future extended analytical theories. In the next section, a brief overview of Monte-Carlo simulations is given.
1.1.2 Monte-Carlo Simulations

It is well known that solution of many scientific and applied problems can be reduced to the calculation of integrals or to solution of differential, integral or other mathematical equations. For RSS problems, for example, their solution can also be obtained by solving associated surface integral equations as will be discussed in detail in Chapters 2, 4 and 5 of this dissertation. To solve such a problem, the Monte-Carlo method can be applied to provide approximate solutions of a mathematical problem by performing statistical sampling experiments on a computer. Historically, the method was called after the city in the Monaco principality due to a roulette, a simple random number generator. The name and the systematic development of Monte-Carlo methods date from about 1944, however there are a number of isolated and undeveloped instances on much earlier occasions as well. Mathematically, the Monte-Carlo method consists of the generation of a random quantity with probability or probability density function (pdf) such that the corresponding expected value is equal to the integral needed to calculate or obeys the equation needed to solve. It should be pointed out that the method can be applied to problems with no probabilistic content as well as to those with inherent probabilistic structure (RSS problems, for example). For rough surface scattering problems, first a sufficient number of random rough surfaces with a specified statistics are generated numerically. To obtain the scattered fields, associated surface scattering problems for each surface realization are then solved via appropriate techniques. Finally, performing statistical average on the obtained scattered fields results in the desired statistical scattering quantities such as the coherent and incoherent radar cross sections (RCS) as will be described

5
in Section 1.3. Monte-Carlo simulations have been successfully applied in solving various rough surface scattering problems [16]–[22].

1.1.3 The Backscattering Enhancement Phenomenon

One of the most interesting phenomena associated with RSS is the backscattering enhancement ("retroreflectance" or "opposition") effect [4]–[22]. This phenomenon is associated with the appearance of a well-defined peak in the backscattering direction of the intensity of the incoherently scattered component of the EM field, and it is a direct consequence of the coherent interference of multiple scattered waves which interfere constructively in the backscattering direction. Enhanced backscattering associated with rough surfaces is usually divided into the cases of large and small height variations due to different mechanisms causing the backscattering peak in each case [4]. In addition, this phenomenon is also important in applications to lidar, surface optics, ocean acoustics, geophysics, and electron localization in disordered media. It allows obtaining important information on the scattering multiplicity, scatterer structure, scattering medium characteristics, etc. [6]. Physicists working in solid state physics realized that this phenomenon is essentially the same as the electron localization effect predicted by Anderson in 1958. Under certain conditions, the electromagnetic wave can be strongly localized due to the multiple-scattering effects, and the diffusion constant that is related to the propagation of the incoherent wave becomes very small. This phenomenon is known as strong localization.

In the next section, a literature review of rough surface scattering is discussed in detail.
1.2 Literature Review

EM wave scattering from random rough surfaces has been studied extensively, and several approximate analytical methods have been developed as discussed earlier in Section 1.1.1. However, most of analytical techniques have restricted regions of validity in terms of surface slope and roughness, and are computationally inefficient, often requiring the evaluation of many multidimensional integrals. At present, outside the regime of validity of analytical theories, numerical methods based on Monte-Carlo simulations can be applied to accurately solving RSS problems. However, numerical models are usually computationally expensive for general use in most practical applications. In this section, a literature review of the numerical computation of scattering from rough surfaces and previous studies of backscattering enhancement is provided.

1.2.1 Numerical Computation of Scattering from Rough Surfaces

With the advent of more powerful computers, numerical approaches based on Monte Carlo simulations of wave scattering by random rough surfaces have become more practical and are receiving greater attention. Several rigorous numerical techniques have been proposed to solve RSS problems with its own distinct advantages and disadvantages such as integral equation method, finite-element method (FEM), finite-difference time-domain (FDTD) method and finite-volume time-domain (FVTD) method [57]–[80]. Among these, the surface integral equation approach and its solution by the moment methods (MM) [81]–[86] are the most commonly used. This is due to the fact that the MM requires discretization in a maximum of two dimensions compared to the three dimensional discretization needed in FEM and
FDTD techniques, resulting in a much smaller number of unknowns to be determined by the numerical method. However, the primary factor limiting the use of the MM in calculation of EM scattering from large-scale rough surfaces is that one must solve a very large dense linear system of equations obtained from discretization of the surface integral equation. Standard factorization techniques of the system matrix, such as LU decomposition, become intractable because of their computational complexity of $O(N_{tot}^3)$, where $N_{tot}$ is the number of unknowns in the system. In addition, the standard MM also requires memory storage of $O(N_{tot}^2)$ to store all elements of a dense impedance matrix. Due to CPU time and memory constraints, direct solution of the MM matrix equation is restricted to moderate-length RSS only. To improve computational efficiency of the standard MM, one must resort to efficient iterative MM techniques.

To increase computational efficiency in solving the surface integral equation, several efficient methods based on iterative versions of the MM have been developed. The first method, the Kirchhoff iteration method (Neumann expansion), has been employed [59]-[62] and shown to be mathematically equivalent to one of the stationary iterative algorithms, named the Jacobi iteration [87]. The Neumann series is attractive in that each term corresponds physically to the contribution of an individual multiple scattering term. However, it has been shown that the Neumann series has convergence problems when used at large incident angles and/or for surfaces with large slopes [62].

Another efficient method, termed the sparse-matrix canonical grid (SMCG) method has been first developed by Tsang et al. for 1-D and 2-D RSS problems [68]-[79]. In the SMCG method, interactions between two points on the surface are distinguished
as either neighborhood strong interaction or far-field (weak) interaction. This classification allows the original matrix to be decomposed into the superposition of a neighborhood strong matrix, a block Toeplitz flat-surface part, and a weak remainder. The iteration stops when a convergence is achieved for the error norm of the matrix equation. By performing the Taylor-series expansion about a flat surface and the use of the fast Fourier transform (FFT) algorithm [88] to calculate the weak matrix-vector multiplication, the method has shown to be an extremely efficient $O(N_{\text{tot}} \log_2 N_{\text{tot}})$ algorithm for small rms height surfaces. However, as pointed out in [75,80], the method becomes inefficient as surface heights increase. It should be pointed out that other approaches utilizing the FFT algorithm have also been proposed such as the fast multipole method (FMM) - FFT algorithm [22] and the adaptive integral method (AIM) [89]-[92].

The third approach involves use of FMM-based techniques [22],[93]-[101]. Briefly, the FMM is a technique for fast computation of a matrix-vector multiply and can be readily incorporated into an iterative matrix solver such as the conjugate-gradient (CG)-based algorithms. In the FMM, the scatterers are decomposed into several groups and the interaction between neighborhood groups is computed using direct matrix-vector multiplication while the interaction between distant groups is obtained using the multipole expansion involving plane-wave expansions and the translation operator [94]. The key to the efficiency of the FMM is that the translation operator is diagonal; i.e. there is no interaction between field components propagating in different directions. For quasi-planar scatterers such as rough surfaces, modified FMM techniques have been proposed to take advantage of this special geometry such as the
FMM-FFT algorithm and the steepest descent-fast multipole method (SDFMM) [98]-[101]. In the FMM-FFT algorithm, the FMM group centers are placed on a regular 3-D grid and the discretized translation operator becomes convolutional, which can be efficiently computed using the FFT algorithm resulting in the memory storage of $O(N_{\text{tot}})$ and $O(N_{\text{tot}} \log_2 N_{\text{tot}})$ CPU time per matrix-vector multiply [22]. As in the SMCG method, the FMM-FFT algorithm has been illustrated to be very efficient for small rms height surfaces solely. Unlike the FMM-FFT algorithm, the SDFMM [98] employs the free-space dyadic Green's function in terms of a rapidly converging Sommerfeld steepest descent integral and a multilevel fast multipole algorithm (MLFMA) [96, 97] is used in efficiently evaluating the Hankel function arising in the integral for multiple source and observation points. The MLFMA is an accelerated version of the FMM, in which the scatterers are subdivided recursively into smaller and smaller groups, and the FMM is applied in a nested manner. The computational efficiency of the SDFMM has been shown to be $O(N_{\text{tot}})$ for both CPU time and memory storage requirements.

Another approach, the forward-backward (FB) method developed by Holliday et al., has been shown in many cases to provide a more rapid convergence than standard CG-like iterative algorithms [102]-[109]. This method is functionally identical to the method of ordered multiple interaction (MOMI) [110]-[116] or the current marching technique (CMT) [117]. In the FB method, fields at each current element on the surface are decomposed into a superposition of the forward and backward contributions. The forward propagating field is contributed by the incident field and the radiation of surface currents preceding the receiving element. The surface currents following the receiving element contribute to the backward propagating field. The forward current
due to the forward propagating field contribution is first computed for every receiving element on the surface and then used to calculate the backward current in an iterative manner. It should be pointed out that the FB (MOMI) method is not guaranteed to converge. In fact, it diverges for some rough surface profiles and its convergence strongly depends on to the order in which the current elements are updated [111]. The FB method has been shown to be an $\mathcal{O}(N_{tot}^3)$ rapidly-convergent algorithm for many rough surface scattering problems. Due to its rapidly-convergent property, it is worthwhile to determine ways to improve its computational efficiency.

Previous use of the FB method has emphasized the 1-D surface (2-D scattering problem). In the past few years, several fast techniques has been proposed to improve the computational efficiency of the 1-D FB (MOMI) method. One fast technique, the novel spectral acceleration (NSA) algorithm with the FB method (FB/NSA), originally developed by Chou et al., has been shown to be an extremely efficient technique for the computation of EM wave scattering from 1-D rough surfaces [104,105]. As in the SMCG method, a neighborhood distance around each receiving element on the surface is defined to separate the strong interaction region from the weak interaction region. Direct matrix-vector multiplication is performed when the source points are in the strong interaction region. The NSA algorithm is employed to rapidly compute weak interactions between widely separated points in the conventional FB method, and is based on a spectral domain representation of source currents and the free space scalar Green’s function. It is emphasized that, unlike the FMM-based algorithms, the method creates only one large source group for the weak interaction computations. This large source group keeps modifying as the forward or backward sweep proceeds and continuously builds up the coefficients of
the source element plane wave expansion. For fixed surface roughness statistics, the computational cost and memory storage requirements of the 1-D FB/NSA method have been shown to be $O(V_{tot})$ as the surface size increases. The NSA algorithm, as other fast techniques [71,94], can be easily incorporated into other iterative techniques in the case that the FB method diverges by performing only matrix-vector multiplies without updating currents. Furthermore, the NSA algorithm can be applied to solve EM problems associated with quasi-planar structures such as planar arrays [118,119]. Another fast technique, the FMM-based algorithm, was applied to the MOMI, termed the FMM/MOMI method [115]. As shown in [115], it is an $O(V_{tot} \log_2 V_{tot})$ method when the multilevel fast-multipole algorithm (MLFMA) is used. Tran et al. also applied the banded matrix iterative approach with canonical grid expansion (BMIA/CAG) to the MOMI, termed a banded MOMI [116]. Like the SMCG method, a banded MOMI is an $O(V_{tot} \log_2 V_{tot})$ algorithm and it becomes inefficient as surface heights increase. It should also be pointed out that the derivations and limitations of some formulas associated with the 1-D NSA algorithm in [104] are unclear and need to be clarified (see Appendix D).

In this dissertation, the original 1-D FB/NSA method is generalized for the fast computation of radiation/scattering from 1-D extremely large-scale rough surfaces. New analytical formulas associated with the 1-D NSA parameters are also derived, resulting in more flexibility in optimizing the 1-D FB/NSA algorithm. In addition, the 1-D FB/NSA method is extended from 1-D to 2-D RSS problems as will be discussed in detail in Chapters 4 and 5. It is noted that computation of EM scattering from 1-D rough surfaces may be adequate for co-polarized scattering; however, to account for cross-polarization scattering, a 2-D RSS computation (corresponding to
a full 3-D scattering computation) is required. Cross-polarized scattering can provide additional information regarding the scattering properties of the surface as well as being a good tool for studying backscattering enhancement. Note that backscattering enhancement, like cross-polarized scattering, is a multiple-scattering effect and therefore should be more easily studied with the use of cross-polarized scattering.

The FB method for 2-D rough surfaces (3-D scattering) has previously been studied by Tran [111], again using an $O(N_{tot}^2)$ algorithm. To accelerate the FB (MOMI) method, the 2-D NSA algorithm is developed and employed to perform matrix-vector multiplication for weak group interactions. The NSA algorithm for this case involves a double spectral integral representation of source currents and the 3-D free space scalar Green’s function. The coupling between two spectral variables makes the problem more challenging. It is pointed out that the FB/NSA method is specifically designed for 2-D large-scale finite rectangular surfaces. Preliminary results show that the 2-D FB/NSA method yields very accurate results, and its computational efficiency is $O(N_{tot})$ for both CPU time and memory storage requirements as one of the surface dimensions increases [107]–[109]. In addition, the method still remains very efficient for moderately-rough large-scale surfaces (surface height variations up to 10 \( \lambda \) where \( \lambda \) is the electromagnetic wavelength in free space). For such surfaces, the 2-D FB/NSA method seems to be more efficient than the SMCG method, and it is comparable to the SDFMM method. It is also observed that the 2-D FB/NSA method requires much less memory storage requirement than the SDFMM method by at least a factor of 10 for this surface type. Thus, the method is well suited for studying RSS problems at low grazing angles (LGA) [120], which require a large-scale rectangular rough surface,
and also for studies of backscattering enhancement [4,5], which require considerably rough surfaces.

1.2.2 Previous Studies of Backscattering Enhancement

Enhanced backscattering from rough surfaces has been first predicted theoretically by McGurn et al. since 1985 [121], and it has been extensively studied theoretically [48]–[55], experimentally [8]–[15], and numerically [16]–[22] from several rough surface types. One such type involves surfaces with relatively large heights and slopes for which multiple surface scattering dominates. The standard KA or SPM are inaccurate for this surface type due to their small slope limitations. In [7], Maradudin et al. concluded that the existence of enhanced backscattering from moderately rough reflecting random surfaces appears not to depend strongly on the nature of the surface profile function. Provided that these surfaces can scatter electromagnetics waves multiply, enhanced backscattering can occur for various surface profiles: i.e. the multiple-scattering property of surfaces determines the existence of backscattering enhancement.

Recently, two promising extended analytical theories has been developed to explain the backscattering enhancement phenomenon for surfaces with large heights and slopes: i.e. the "higher-order Kirchhoff approximation" theory [55] and the integral equation model with multiple scattering (IEMMS) [52, 53]. The former developed by Ishimaru et al. is based on the first- and second-order Kirchhoff approximations with angular and propagation shadowing corrections to take care of multiple scattering beyond second-order scattering. This theory is developed for rough surfaces with high rms slopes and roughness. It employs an appropriate integral representation of
the associated Green's function to incorporate both large height variations and large slopes in the statistical moment calculations. However, this theory is based on several approximations, and the exact form of the shadowing functions for second-order scattering still needs further investigation. In addition, this theory yield poor results for lossless rough surfaces. Another analytical theory, the IEMMS method, is based on the integral equation model (IEM) [26]. It is extended to include multiple surface scattering for surfaces with high rms slopes and roughness by considering the propagation effects due to the phase term in the associated Green's function. By doing this, the upward and downward scattered fields can be identified, and appropriate shadowing effects can be included to produce the correct amount of multiple scattering. It should be pointed out that both analytical theories involve many multidimensional integrals, which must be approximated or evaluated with time consuming. Although both theories can predict the backscattering enhancement for several rough surface scattering problems, they are still limited in their domain of validity.

Due to the lack of general analytical theories, experimental study is an alternative way to investigate the backscattering enhancement phenomenon. To compare the experimental results with a theory or numerical simulations, it is required to know the detailed characteristics of the rough surface of interest, and thus controlled experiments are indispensable for yielding reliable results. Backscattering enhancement of EM waves from random rough surfaces was first observed experimentally by O’Donnell et al. [9,10]. This pioneering experiments were conducted with carefully fabricated very rough surfaces of known characteristics at optical wavelengths, and the findings spurred a surge of research on theoretical, experimental and numerical studies on
backscattering enhancement. In addition, controlled experiments at millimeter wave-lengths has been performed extensively at the University of Washington [13]–[15]. It has been shown that the millimeter-wave (MMW) system is very useful for comparisons between the experimental data and the numerical and theoretical calculations. Thus, controlled experiments are very useful for studying backscattering enhancement.

The last approach for studying backscattering enhancement is to employ numerical models based on the Monte-Carlo simulations [16]–[22]. As pointed out earlier, numerical models are usually computational expensive, and thus efficient and accurate numerical methods are indispensable to obtain effective and reliable results. Several extremely efficient iterative MM techniques of \( O\left(N_{tot} \log_2 N_{tot}\right) \) or \( O(N_{tot}) \) have been developed as discussed earlier in this section, and they have been successful in predicting backscattering enhancement [18]–[22]. Among these, an \( O(N_{tot}) \) method based on the NSA algorithm seems to be a good candidate for studies backscattering enhancement, especially at LGA for which large-scale surface sizes are required. In summary, theoretical, experimental and numerical studies are useful for understanding the backscattering enhancement phenomenon. Numerical and experimental results can provide more physical insight into the phenomenon. This assist in the the future development of entended analytical theories.

In the next section, several scattering quantities are defined and used throughout this dissertation.
Figure 1.1: A 3-D coordinate systems used to described incident and scattered field directions
1.3 Definitions of Scattering Quantities

In this dissertation, the computation of scattering from both 1-D and 2-D rough surfaces, corresponding to 2-D and 3-D scattering problems respectively, will be investigated in detail. An $e^{-i\omega t}$ harmonic time convention is assumed and suppressed throughout this dissertation, and the propagation constant is defined as $k = \omega \sqrt{\mu \varepsilon}$. Where $\omega$ is the radian frequency, $i = \sqrt{-1}$ is a complex constant, and $\mu$ and $\varepsilon$ are the permeability and permittivity of free space respectively. First, define incident and scattered field directions using a spherical coordinate system. For the 3-D scattering problem, consider an incident EM plane wave, $\hat{e}_i \, e^{ik_i \cdot r}$, where $k_i$ is a propagating vector of the incident field, $\hat{e}_i$ is an incident field polarization, and $r$ is a position vector. Each quantity is defined mathematically as follows:

\[
k_i = ik_{ix} + yk_{iy} - zk_{iz} \tag{1.1}\]

\[
\hat{e}_i = \begin{cases} \hat{h}_i & \text{for the horizontal polarization} \\ \hat{v}_i & \text{for the vertical polarization} \end{cases} \tag{1.2}\]

\[
r = \hat{x}x + \hat{y}y + \hat{z}z. \tag{1.3}\]

Where

\[
k_{ix} = k \sin \theta_i \cos \phi_i \tag{1.4}\]

\[
k_{iy} = k \sin \theta_i \sin \phi_i \tag{1.5}\]

\[
k_{iz} = k \cos \theta_i \tag{1.6}\]

\[
\hat{h}_i = \frac{\hat{z} \times \hat{k}_i}{|\hat{z} \times \hat{k}_i|} = -\hat{x} \sin \phi_i + \hat{y} \cos \phi_i \tag{1.7}\]

\[
\hat{v}_i = \hat{h}_i \times \hat{k}_i = -\hat{x} \cos \theta_i \cos \phi_i - \hat{y} \cos \theta_i \sin \phi_i - \hat{z} \sin \theta_i, \tag{1.8}\]

Where $\theta_i$ and $\phi_i$ refer to the incident polar and azimuthal angles respectively, as shown in Figure 1.1. Similarly, for a scattered field, $\hat{e}_s \, e^{ik_s \cdot r}$, observed in the far field, its
propagation and polarization vectors are defined as follows:

\[
\mathbf{k}_s = \hat{x}k_{sx} + \hat{y}k_{sy} + \hat{z}k_{sz} \\
\hat{e}_s = \begin{cases} 
\hat{h}_s & \text{for the horizontal polarization} \\
\hat{v}_s & \text{for the vertical polarization}
\end{cases}
\]

where

\[
k_{sx} = k \sin \theta_s \cos \phi_s \\
k_{sy} = k \sin \theta_s \sin \phi_s \\
k_{sz} = k \cos \phi_s \\
\hat{h}_s = \frac{\hat{x} \times \hat{k}_s}{|\hat{x} \times \hat{k}_s|} = -\hat{x} \sin \phi_s + \hat{y} \cos \phi_s \\
\hat{v}_s = \hat{k}_s \times \hat{h}_s = \hat{x} \cos \phi_s \cos \theta_s + \hat{y} \cos \phi_s \sin \theta_s - \hat{z} \sin \phi_s,
\]

where \( \theta_s \) and \( \phi_s \) refer to the scattered polar and azimuthal angles respectively. as shown in Figure 1.1. In this notation, the forward scattering direction corresponds to \((\theta_s, \phi_s) = (\theta_i, \phi_i)\), while the backscattering direction is represented by either \((\theta_s, \phi_s) = (-\theta_i, \phi_i)\) or \((\theta_s, \phi_s) = (\theta_i, \phi_i + \pi)\).

For the 2-D scattering problem as shown in Figure 1.2, the above definitions for propagating and polarization vectors can still be employed by setting \( \phi_i = \phi_s = 0 \) rad. (in plane scattering) as shown below:

\[
k_i = \hat{x}k_{ix} - \hat{z}k_{iz} \\
\hat{e}_i = \begin{cases} 
\hat{y} & \text{for the horizontal polarization} \\
-\hat{x} \cos \theta_i - \hat{z} \sin \theta_i & \text{for the vertical polarization}
\end{cases} \\
k_s = \hat{x}k_{sz} + \hat{z}k_{sz} \\
\hat{e}_s = \begin{cases} 
\hat{y} & \text{for the horizontal polarization} \\
\hat{x} \cos \theta_s - \hat{z} \sin \theta_s & \text{for the vertical polarization}
\end{cases}.
\]
Figure 1.2: A 2-D coordinate systems used to described incident and scattered field directions

where

\[ k_{iz} = k \sin \theta_i \] (1.21)
\[ k_{iz} = k \cos \theta_i \] (1.22)
\[ k_{sz} = k \sin \theta_s \] (1.23)
\[ k_{sz} = k \cos \theta_s \] (1.24)

In this dissertation, most of numerical results are presented in terms of the normalized radar cross section \( \sigma^0 \) defined as

\[ \sigma^0 = \frac{\sigma}{A} \] (1.25)

where \( \sigma \) is the RCS and \( A \) is the area illuminated by the transmitted antenna pattern.

The RCS \( \sigma \) is defined as the area intercepting the amount of power that, when
scattered isotropically, produces at the receiver a density that is equal to the density scattered by the actual target [122]. Note that $\sigma$ is a function of frequency, incident and scattered angles, polarization, and target physical properties. In addition, $\sigma$ contains only scattered field amplitudes with no phase information. For a 2-D target, the scattering parameter is referred to the scattering width (SW) or the RCS per unit length.

For 2-D deterministic surfaces, the normalized bistatic RCS $\sigma_{aJ}(\theta_s, \phi_s, \theta_i, \phi_i)$ is defined for a scattered wave in $\alpha$-polarization and an incident wave in $J$-polarization as

$$
\sigma_{aJ}(\theta_s, \phi_s, \theta_i, \phi_i) = \lim_{r \to \infty} \frac{4\pi r^2|\mathbf{E}_{aJ}|^2 \cos \theta_i}{2\eta \int_{\Sigma^a} \mathbf{S}^J \cdot \hat{n}_{in} \, d\Sigma}. \quad (1.26)
$$

where $\mathbf{E}_{aJ}$ is the $\alpha$-polarized scattered field of the $J$-polarized incident wave. $\eta$ is the free space intrinsic impedance. $\hat{n}_{in}$ is a unit normal vector pointing into the rough surface from the free space region. $\mathbf{S}^J$ is the time average Poynting vector of the $J$-polarized incident wave, and $\Sigma$ is the 2-D rough surface profile of interest.

The above RCS is defined so that integration of $\sigma_{hJ} + \sigma_{vJ}$ over all scattering angles should yield $4\pi \cos \theta_i$ for power conservation. Note that for a monostatic radar system only the RCS in the backscattering direction is measured. In this dissertation, both monostatic and bistatic radar cross sections are considered. For 2-D random surfaces, the normalized bistatic RCS $\sigma_{a3}(\theta_s, \phi_s, \theta_i, \phi_i)$ is defined in terms of the ensemble average scattered intensity as follows:

$$
\sigma_{a3}(\theta_s, \phi_s, \theta_i, \phi_i) = \sigma_{a3}^c(\theta_s, \phi_s, \theta_i, \phi_i) + \sigma_{a3}^l(\theta_s, \phi_s, \theta_i, \phi_i) \quad (1.27)
$$

$$
\sigma_{a3}^c(\theta_s, \phi_s, \theta_i, \phi_i) = \lim_{r \to \infty} 4\pi r^2 \left\langle \left| \mathbf{E}_{a3}^c \right|^2 \cos \theta_i \right\rangle \quad (1.28)
$$

$$
\sigma_{a3}^l(\theta_s, \phi_s, \theta_i, \phi_i) = \lim_{r \to \infty} 4\pi r^2 \left\langle \left| \mathbf{E}_{a3}^l - \left\langle \mathbf{E}_{a3}^l \right\rangle \right|^2 \cos \theta_i \right\rangle \quad (1.29)
$$
where \( \widetilde{E}_{\alpha,j} \) for 2-D surfaces is defined as

\[
\widetilde{E}_{\alpha,j}^s = \frac{E_{\alpha,j}^s}{[2\eta \int_S S_j \cdot \hat{n} \, ds]^{1/2}},
\]  

(1.30)

\( \sigma_{\alpha,j}^s \) and \( \sigma_{\alpha,j}^i \) are the normalized coherent and incoherent bistatic radar cross sections respectively, and the \( \langle \cdot \rangle \) notation above indicates an ensemble average over realizations of the surface stochastic process. From Eq. (1.30), it is noted that \( \widetilde{E}_{\alpha,j}^s \) (in \( m^{-1} \)) is a random process due to the fact that the scattered field \( E_{\alpha,j}^s \) and the surface profile \( S \) are random processes. For studies of backscattering enhancement illustrated in Chapters 3 and 6, numerical results are presented in terms of \( \sigma_{\alpha,j}^s \) rather than \( \sigma_{\alpha,j}^i \) since backscattering enhancement is associated with the appearance of a well-defined backscattering peak of \( \sigma_{\alpha,j}^i \). Not \( \sigma_{\alpha,j}^i \), as discussed earlier in Section 1.1, and \( \sigma_{\alpha,j}^s \) is very small compared to \( \sigma_{\alpha,j}^i \) even in the specular direction for very rough surfaces with relatively high rms slopes associated with backscattering enhancement.

For 1-D surfaces (2-D scattering problem), no cross-polarized scattering exists (\( \sigma_{\alpha,j} = 0 \) for \( \alpha \neq j \)) and the normalized bistatic RCS \( \sigma_{\alpha,j}(\theta, \theta_i) \) for deterministic and random surfaces is slightly changed as given below. For deterministic surfaces.

\[
\sigma_{\alpha,j}(\theta, \theta_i) = \lim_{\rho \to \infty} \frac{\rho |E_{\alpha,j}^s|^2 \cos \theta_i}{2\eta \int_L S_j \cdot \hat{n} \, ds},
\]  

(1.31)

where \( L \) is a 1-D rough surface profile of interest. For power conservation, 1-D radar cross sections integrated over all scattering angles in the plane of incidence \( (\phi_i = \phi_s = 0 \text{ rad.}) \) should yield \( \cos \theta_i \). For random surfaces.

\[
\sigma_{\alpha,j}^c(\theta, \theta_i) = \lim_{\rho \to \infty} \rho \left\langle |\widetilde{E}_{\alpha,j}^s|^2 \cos \theta_i \right\rangle,
\]

(1.32)

\[
\sigma_{\alpha,j}^i(\theta, \theta_i) = \lim_{\rho \to \infty} \rho \left\langle \left| E_{\alpha,j}^s - \langle \widetilde{E}_{\alpha,j}^s \rangle \right|^2 \cos \theta_i \right\rangle,
\]

(1.33)

22
where $\overline{E}_{\alpha\beta}$ for 1-D surfaces is defined as

$$
\overline{E}_{\alpha\beta} = \frac{E_{\alpha\beta}}{\sqrt{\eta \int_{L_F} S_3 \cdot \hat{n}_{in} ds}}.
$$

(1.34)

However, in the literature the normalized bistatic RCS $\sigma_{\alpha\beta}(\theta_s, \theta_i)$ for 1-D deterministic surfaces due to a tapered incident plane wave is often defined as

$$
\sigma_{\alpha\beta}(\theta_s, \theta_i) = \lim_{\rho \to \infty} \frac{\rho |E_{\alpha\beta}|^2 \cos \theta_i}{2 \eta \int_{L_F} S_3 \cdot \hat{n}_{in} ds}.
$$

(1.35)

where $L_F$ is an infinite flat surface profile and $\hat{n}_{in} = -\hat{z}$ is a unit vector pointing into the flat surface from the free space region. In addition, for 1-D random surfaces $\sigma_{\alpha\beta}^r$ and $\sigma_{\alpha\beta}^i$ are often defined as

$$
\sigma_{\alpha\beta}^r(\theta_s, \theta_i) = \lim_{\rho \to \infty} \frac{\rho \langle |E_{\alpha\beta}|^2 \rangle |^2 \cos \theta_i}{2 \eta \int_{L_F} S_3 \cdot \hat{n}_{in} ds}.
$$

(1.36)

$$
\sigma_{\alpha\beta}^i(\theta_s, \theta_i) = \lim_{\rho \to \infty} \frac{\rho \langle |E_{\alpha\beta}^s - \langle E_{\alpha\beta}^s \rangle |^2 \rangle \cos \theta_i}{2 \eta \int_{L_F} S_3 \cdot \hat{n}_{in} ds}.
$$

(1.37)

Note that the denominators of Eqs. (1.36) and (1.37) are deterministic quantities. Thus, to be consistent with numerical results in the literature, the definitions of the normalized bistatic RCS for 1-D rough surfaces given in Eqs. (1.35) to (1.37) are employed throughout this dissertation. In the next section, the organization of this dissertation is provided.

1.4 Organization of This Dissertation

The remaining portion of this dissertation consists of five chapters, involving theories and applications of the NSA algorithm employed in the computation of scattering from both 1-D and 2-D rough surfaces, followed by a concluding chapter and six appendices.
In Chapter 2, the formulation of the FB/NSA method for 1-D impedance rough surfaces based on the original paper [104,105] is reviewed. As pointed out earlier in Section 1.2, the NSA algorithm can be applied to solve problems related to quasi-planar structures as well. Thus, in this chapter the original NSA algorithm has also been generalized for the fast computation of radiation/scattering from 1-D extremely large-scale quasi-planar structures to avoid the assumption, used in the development of the standard FB/NSA method, that the outermost possible saddle point, $\alpha_{s,\text{max}}$, along the real axis in the complex angular ($\alpha$) plane is small. New analytical formulas associated with the 1-D NSA parameters for an arbitrary value of $\alpha_{s,\text{max}}$ are also provided, resulting in more flexibility in selecting the size of the strong region to compromise between the computation of the contributions of strong and weak regions. Numerical results illustrate that the derived 1-D NSA parameters work well for both small and very rough surfaces. Finally, the “multilevel” concept is proposed to improve the accuracy of the original NSA algorithm in the case of 1-D extremely large-scale quasi-planar structures. It is found that only a few weak regions or even one are required for most practical problems to obtain the desired accuracy, but the accuracy of the original 1-D NSA algorithm can be improved when incorporating the “multilevel” algorithm.

Backscattering enhancement from 1-D random rough surfaces is studied intensively in Chapter 3. Variations in the level and angular width of backscattering enhancement with surface statistics, surface material, polarization, and incident angle are considered. Due to the fact that backscattering enhancement is associated with surfaces with large slopes in this study, it is very difficult to predict surface scattering using existing approximate analytical theories, and the 1-D FB/NSA method.
developed in Chapter 2, combined with parallel computing techniques is employed to study the backscattering enhancement \textit{effectively}. The efficiency of the 1-D FB/NSA method makes investigation of backscattering enhancement phenomenon at LGA for which no results have been reported previously in the literature possible. It is found that the backscattering enhancement \textit{strongly} depends on surface statistics, surface material, polarization, and incident angle. In this study, the backscattering enhancement phenomenon is investigated from normal incidence up to low grazing incidence ($0^\circ - 85^\circ$). The results in this chapter provide some physical insight into the backscattering enhancement phenomenon, and it serves as a background for studying the backscattering enhancement from 2-D random rough surfaces in Chapter 6, which requires extremely intensive computations.

Chapter 4 presents an extension of the NSA algorithm from 1-D to 2-D RSS problems for the PEC case. The formulations of the 2-D FB method and the 2-D FB/NSA method for 2-D PEC rough surfaces, based on the spectral domain expansion in the $x$-direction, are provided. Due to the coupling between two spectral variables, associated with a double spectral integral representation of source currents and the 3-D free space scalar Green's function, the problem is more difficult and challenging. Because of the complexities of the topology of two coupled complex planes as discussed above, only \textit{empirical procedure} for finding values of the 2-D NSA parameters is provided. It should be pointed out that this procedure still provides accurate results even though it may not be convenient for users. It is also found that the \textit{standard} 2-D FB/NSA formulation based on the spectral domain expansion in the $x$-direction may not be efficient if the size of the strong region is large due to increasing cross-range surface sizes. To reduce the size of the strong region, an
additional 2-D FB/NSA formulation based on the spectral domain expansion in the $y$-direction is also incorporated into the original formulation. Note that the spectral domain expansion in the $y$-direction is performed in the same fashion as in the $x$-direction expansion, and the complexity and the memory storage of the algorithm are increased. However, the new expansion can be advantageous when considering 2-D rough surfaces with relatively large surface cross-range sizes. In the case of 2-D extremely large-scale rough surfaces, a “multilevel” algorithm for 2-D rough surfaces, which is analogous to that for the 1-D case, is proposed to improve the accuracy of the 2-D FB/NSA algorithm based on the spectral domain expansion in the $x$-direction. Comparing numerical results between the 2-D FB/NSA method based on the spectral domain expansion in the $x$-direction and the standard 2-D FB method, it is found that the 2-D FB/NSA method using the $x$-expansion yields accurate results with a reduction of CPU time and only a slight increase in memory storage for large-scale surfaces. In addition, results obtained from experimental data from the University of Washington confirm the accuracy of the method. Furthermore, it is found that incorporating the spectral domain expansion in the $y$-direction can indeed improve computational efficiency of the 2-D FB/NSA algorithm for surfaces with relatively large surface cross-range sizes, and the “multilevel” algorithm can indeed improve the accuracy of the standard (one-level) 2-D FB/NSA algorithm for the computation of scattering from 2-D extremely large-scale rough surfaces. Due to the computational efficiency of the 2-D FB/NSA algorithm, studies of backscattering enhancement at large incident angles, which requires moderately-rough large-scale surfaces, is feasible as illustrated in Chapter 6.
In Chapter 5, the 2-D FB/NSA formulation for the PEC case is generalized to include the finite conductivity of rough surfaces. For practical rough surfaces having finite but high conductivity such as ocean and metallic surfaces, the impedance boundary condition (IBC) can be applied to capture the effects of the finite conductivity effectively, and to reduce the number of unknown surface currents significantly. Two additional terms are required in the surface integral formulation for the PEC case in order to include the finite conductivity effects, and a numerical differentiation associated with the surface divergence term (one of the additional terms) is needed as well. However, the standard 2-D FB/NSA algorithm using the $x$-expansion for the IBC case can be formulated successfully, and it is employed to study the effects of material on backscattering enhancement in Chapter 6. The validity of numerical results is confirmed by comparing with the results obtained from the standard 2-D FB method.

Like the PEC case illustrated in Chapter 4, to improve the computational efficiency and accuracy of the standard 2-D FB/NSA algorithm for 2-D extremely large-scale IBC rough surfaces with relatively large surface cross-range sizes, the spectral domain expansion in the $y$-direction can be incorporated into the standard 2-D FB/NSA algorithm combined with the "multilevel" algorithm.

Chapter 6 presents an application of the standard 2-D FB/NSA method derived in Chapters 4 and 5: i.e. numerical studies of the backscattering enhancement from 2-D random rough surfaces. Variations in the characteristics of backscattering enhancement with incident angle, surface material (PEC or IBC), polarization, and surface statistics are investigated. Surface sizes of $128\lambda$ by $16\lambda$ sampled with 262,144 unknowns are employed to ensure the validity of numerical results up to 70°. Previous Monte-Carlo simulations of backscattering enhancement for 2-D rough surfaces
considered only small angles of incidence (i.e. near normal incidence). It should be pointed out that the computation of EM scattering from 1-D rough surfaces as in Chapter 3 may be adequate for co-polarized scattering; however, to account for cross-polarization scattering, a 2-D RSS computation is required. Cross-polarized scattering can provide additional information regarding the scattering properties of the surface as well as being a good tool for studying backscattering enhancement. Note that backscattering enhancement, like cross-polarized scattering, is related to multiple-scattering effects and therefore should be more easily studied with the use of cross-polarized scattering. For co-polarization, the multiple-scattering effects can be obscured by the presence of the first-order scattering. Unlike the co-polarization, the first-order scattering associated with cross-polarization is zero, and the multiple scattering, especially the second-order backscattering enhancement, can be more clearly exhibited for cross-polarization. However, 2-D surfaces greatly increase computational requirements since the surface profile and surface fields must be discretized in both dimensions. The 2-D FB/NSA method combined with parallel computing techniques is employed to study the backscattering enhancement effectively. Like the case of 1-D surfaces illustrated in Chapter 3, it is found that the backscattering enhancement \textit{strongly} depends on incident angle, surface material, polarization, and surface statistics. In this chapter, comparison of the backscattering enhancement results between 1-D and 2-D rough surfaces are also provided. Results obtained from these studies will provide more physical insight to the backscattering enhancement phenomenon and will assist in the future development of analytical theories.
Chapter 7 provides a summary and conclusions of Chapters 2 to 6 and Appendices A to F, including the associated future works. Appendix A describes the procedure of random rough surface generation for both 1-D and 2-D surface models. In Appendix B, the topology of the complex planes associated with the integral representation of the free space 2-D scalar Green's function $g(p, p')$ is described for both lossless and lossy media. In addition, Appendix C illustrates the topology of two coupled complex planes associated with the double spectral integral representation of the free space 3-D scalar Green's function. Furthermore, Appendix D presents the derivation of analytical formulas associated with integration parameters of the standard 1-D NSA algorithm. In Appendix E, the analytical formula for the tilt angle $\delta_k$ in the complex $k_z$ plane associated with the 2-D NSA algorithm is derived. Finally, Appendix F illustrates a simple procedure to obtain a root of a nonlinear equation via Muller's method [88].
CHAPTER 2

Development of the Novel Spectral Acceleration Algorithm for One-Dimensional Rough Surface Scattering Problems

2.1 Introduction

Quasi-planar structures (QPS) play an important role in many electromagnetic (EM) applications including EM radiation/scattering from rough surfaces, microstrip structures, microwave integrated circuits, and optical gratings. Due to their geometrical complexities and to obtain an accurate description of EM scattered fields for large-scale structures, several efficient numerical methods have been proposed recently [77, 91, 98, 104, 109, 119, 123]. One of these methods, the FB/XSA method, originally developed by Chou et al., has been shown to be an extremely efficient technique for the computation of EM wave scattering from both 1-D PEC and impedance rough surfaces [104, 105]. In the FB/NSA method, a neighborhood distance around each receiving element on the surface is defined to separate the strong interaction region from the weak interaction region. Direct matrix-vector multiplication is performed when the source points are in the strong interaction region, and the NSA algorithm is employed to rapidly compute weak interactions between widely separated points in the conventional FB method. Like the steepest descent-fast multipole
method (SDFMM) [98]-[101]. the FB/NSA method is based on a spectral integral representation of source currents and the free space scalar Green's function. Unlike the SDFMM algorithm, the method creates only one large source group for the weak interaction computations. This large source group keeps modifying as the forward or backward sweep proceeds and continuously builds up the coefficients of the source element plane wave expansion. Due to the recursive property of the plane wave spectrum, it has been shown that for fixed surface roughness statistics, the computational cost and memory storage requirements of the 1-D FB/NSA method is an $O(N_{tot})$ algorithm as the surface size increases, where $N_{tot}$ is the total number of unknowns to be solved. The NSA algorithm, as other fast techniques [71, 94], can be easily incorporated into other iterative techniques in the case that the FB method diverges by performing only matrix-vector multiplies without updating currents. In this chapter, the 1-D NSA algorithm is developed and applied to 1-D rough surfaces, an example of quasi-planar structures. However, it can be applied to other quasi-planar structures as well [118, 119].

In the NSA method, the most important issue is to determine the appropriate NSA parameters which include the tilt angle ($\delta$) of the deformed contour in the periodic complex angular $\phi$ domain, the domain of integration ($[-\phi_{max}, \phi_{max}]$) and the integration step size ($\Delta \phi$). In the original paper [104], these parameters are derived based on the assumption that the outermost possible saddle point, $\phi_{s, max}$, along the real axis in the complex angular $\phi$ plane is small. For a given surface height variation, this assumption can be satisfied by adjusting the size of the strong region. However, for rough surfaces with large height variations, the adjusted size of the strong region is typically large resulting in significant increases in computational
time for the strong-region contribution, and degrading overall efficiency of the NSA algorithm [124]. The derivation of analytical formulas associated with the 1-D NSA parameters for an arbitrary value of $\phi_{s,\text{max}}$ will be presented in Section 2.3.

For the case of 1-D extremely large-scale structures, studies based on the physical optics (PO) approximation and the flat surface assumption show that the given 1-D NSA parameters in [124] may yield inaccurate results. Inaccuracy comes from the fact that the complex radiation function (plane wave spectrum) of a source group far separated from the receiving element is rapidly decayed along the deformed contour away from the origin in the complex $\phi$ plane, which requires higher sampling rates. Analytical results obtained from asymptotic evaluations of the radiation integral associated with the PO and flat surface assumptions suggest that the very large weak region associated with 1-D extremely large-scale QPS needs to be decomposed into more than one separate weak region, and the appropriate NSA parameters must be determined separately for each weak region to gain better accuracy. Thus, the new proposed scheme for the NSA algorithm can be classified as a "multilevel" algorithm [124, 125]. Note that the "multilevel" NSA algorithm is distinct from standard multilevel algorithms applied in computational physics [96, 97].

This chapter is organized as follows. The formulation of the original FB/NSA method for 1-D impedance rough surfaces is reviewed in Section 2.2. Section 2.3 presents the derivation of analytical formulas associated with the 1-D NSA parameters for an arbitrary value of $\phi_{s,\text{max}}$. The "multilevel" NSA algorithm for 1-D extremely large-scale QPS is illustrated in Section 2.4. Some numerical results are presented in Section 2.5, and a summary and conclusions can be found in Section 2.6.
Figure 2.1: A 1-D finite dielectric surface profile, $z = f(x)$, is illuminated by a TM incident magnetic field $\mathbf{H}^i(\rho) = \hat{y}H_y^i(\rho)$. The region above the surface profile $L$ is the free space with $(\mu, \epsilon)$, and the region below the surface profile is the nonmagnetic region with $(\mu, \epsilon_1)$.

2.2 Formulation of the Standard FB/NSA Method for 1-D Impedance Rough Surfaces

Consider a scalar Kirchhoff diffraction integral equation on the finite dielectric surface profile $L$ as shown in Figure 2.1 with the 2-D Hankel function form of the Green’s function[126]:

$$\frac{\Psi(\rho)}{2} = \Psi^i(\rho) + \int_{PV:L} dl \left\{ \Psi(\rho') \frac{\partial g(\rho, \rho')}{\partial n'} - g(\rho, \rho') \frac{\partial \Psi(\rho')}{\partial n'} \right\}.$$  \hspace{1cm} (2.1)
where

\[
\Psi(\rho) = \begin{cases} 
E_y(\rho) & \text{for the TE case} \\
H_y(\rho) & \text{for the TM case}
\end{cases}
\]

(2.2)

\[
\Psi^i(\rho) = \begin{cases} 
E_y^i(\rho) & \text{for the TE case} \\
H_y^i(\rho) & \text{for the TM case}
\end{cases}
\]

(2.3)

\[
g(\rho, \rho^i) = i \frac{1}{4} H_0^{(1)}(k|\rho - \rho^i|).
\]

(2.4)

\(H_0^{(1)}(\cdot)\) denotes the zeroth-order Hankel function of the first kind. \(\rho = \hat{x}x + \hat{z}z\) and \(\rho^i = \hat{x}x^i + \hat{z}z^i\) are two-dimensional position vectors, and a principal value integration is implied. Note that “TE” and “TM” in the above equations stand for “Transverse Electric” and “Transverse Magnetic”, corresponding to the horizontal and vertical polarizations, respectively. In addition, \(\rho\) and \(\rho^i\) denote a field point and a source point on the rough surface profile \(L\), respectively. To reduce the edge effect of the finite surface, a tapered incident field \(\Psi^i(\rho)\) based on the Thorsos’s paper [58] is employed:

\[
\Psi^i(\rho) = \Psi_0^i e^{i\mathbf{k}_i \cdot \rho} e^{iW(\rho)} \exp \left\{- \frac{(x + z \tan \theta_i)^2}{g} \right\}.
\]

(2.5)

where \(\Psi_0^i\) is the amplitude of the incident field, and

\[
\Psi_0^i = \begin{cases} 
E_0 & \text{for the TE case} \\
H_0 & \text{for the TM case}
\end{cases}
\]

(2.6)

and \(E_0\) and \(H_0\) are the amplitudes of the incident electric and magnetic fields respectively, where \(E_0 = \eta H_0\) and \(\eta\) is the intrinsic impedance of free space. In addition, \(\mathbf{k}_i\) is a propagating vector of the incident field as previously defined in Eq. (1.16) in Chapter 1. \(g\) is a taper parameter which determines the spot size of the Gaussian beam on the surface, and \(W(\rho)\) is a phase correction term expressed as

\[
W(\rho) = \frac{2(x + z \tan \theta_i)^2}{(k g \cos \theta_i)^2} - 1.
\]

(2.7)
The taper parameter \( g \) should be chosen appropriately for a given incident angle. The appropriate value of \( g \) can be determined using Kapp's criterion for the taper applicability [127]

\[
g = \frac{L_z}{G} > \frac{\sqrt{2}A}{k \left( \frac{\pi}{2} - \theta_i \right) \cos \theta_i},
\]

where \( A \) is a constant and \( G \) is a unitless parameter associated with \( g \). Next, for illustration of the 1-D FB/NSA method, consider only the TM polarization. The 1-D FB/NSA method for the TE polarization can be formulated in the same manner.

Figure 2.1 illustrates a 1-D finite dielectric surface profile \( L. z = f(x) \), illuminated by a TM incident magnetic field \( \mathbf{H}^i(\rho) = \mathbf{y}H_y^i(\rho) \) centered at the origin and propagating in direction \( \hat{k}_i = \hat{x} \sin \theta_i - \hat{z} \cos \theta_i \). Finite 1-D surface profiles with specified statistics are generated using the spectral method [58] as shown in Section A.1 in Appendix A. Note that \( L_z \) is the surface length obtained from the projection of the surface profile \( L \) onto the \( x \) axis. The surface height function \( z = f(x) \) has zero mean and its maximum and minimum height variations are denoted by \( z_{\max} \) and \( z_{\min} \), respectively. The incident magnetic field \( H_y^i(\rho) \) is chosen to be a tapered plane wave, as given in Eq. (2.5), so that the illuminated region on the rough surface profile can be confined to the surface length \( L_z \) as shown in Figure 2.1. Note that Eq. (2.1) has two unknowns \( \Psi \) and \( \frac{\partial \Psi}{\partial n} \), where \( \Psi = H_y \) for the TM case. Use of the impedance boundary condition (IBC) [128] on the surface reduces this to one unknown through the relationship

\[
H_y = \frac{i}{k} \sqrt{\frac{\epsilon_1}{\epsilon}} \frac{\partial H_y}{\partial n} = \frac{i \sqrt{\epsilon r_1}}{k} \frac{\partial H_y}{\partial n}.
\]

where \( \epsilon_1 \) and \( \epsilon r_1 \) are the permittivity and the relative permittivity of the nonmagnetic region below the surface profile. Note that the analysis of surface scattering problems
is simplified considerably by using the IBC. It is also noted that, for a 1-D PEC surface
\( |\epsilon_r| \to \infty \), Eq. (2.9) is reduced to \( \frac{\partial H_y}{\partial n} = 0 \) (i.e. Neumann boundary condition).

Using pulse basis functions and point matching [81] results in the following MM matrix equation:

\[
\overline{Z} \overline{I} = \overline{F}.
\]  

(2.10)

where \( \overline{Z} \) is the \( N_{tot} \times N_{tot} \) MM impedance matrix, \( \overline{F} \) is the \( N_{tot} \times 1 \) excitation vector, \( \overline{I} \) is the \( N_{tot} \times 1 \) unknown vector to be found, and \( N_{tot} \) is the total number of unknowns to be solved. It can be shown that the elements of the impedance matrix \( \overline{Z} \) are given as follows:

\[
Z_{mn} = \begin{cases} 
- \frac{k \Delta x}{4} \left[ i H_1^{(1)}(k \rho_{mn}) \left\{ \frac{\Delta z_{mn} - f'(x_n) \Delta x_{mn}}{\rho_{mn}} \right\} 
- \frac{T_n}{\sqrt{\epsilon_r}} H_0^{(1)}(k \rho_{mn}) \right] & \text{if } m \neq n \\
\frac{1}{2} + \frac{k \Delta x}{4} \frac{T_m}{\sqrt{\epsilon_r}} \left\{ 1 + \frac{2i}{\pi} \left[ \ln \left( \frac{k \Delta x}{4e} T_m \right) + \gamma \right] \right\} \frac{\Delta x}{4\pi} \frac{f''(x_m)}{T^2_m} & \text{if } m = n
\end{cases}
\]  

(2.11)

(2.12)

where \( \Delta x_{mn} = x_m - x_n, \Delta z_{mn} = f'(x_m) - f'(x_n), \rho_{mn} = ((\Delta x_{mn})^2 + (\Delta z_{mn})^2)^{\frac{1}{2}} \). \( \Delta x \) is the step size of discretization. \( T_n = (1 + (f'(x_n))^2)^{\frac{1}{2}} \). \( \gamma \) represents the Euler number 0.577216. \( m \) refers the surface point index of the testing point \( n \) the integration point where \( x_m = m \Delta x \) and \( x_n = n \Delta x \). and \( f'(x_n) = \frac{\partial f(x_n)}{\partial x} \) and \( f''(x_n) = \frac{\partial^2 f(x_n)}{\partial x^2} \) refer to the surface profile first and second derivatives evaluated at the point \( x = x_n \), respectively. Note that the effects of surface curvature involving \( f''(x_n) \) are also included in the self term \( Z_{mn} \) in Eq. (2.12) (i.e. the last term on the right-hand side (RHS) of Eq. (2.12)).

Briefly, in the FB method, fields at each current element on the surface are divided in terms of two contributions: forward propagating field contribution, due to the
incident field and the radiation of surface currents preceding the receiving point \( x_m \), and backward propagating field contribution, due to the surface currents following the receiving point \( x_m \). Let \( L^f \) and \( L^b \) denote the forward and backward regions, respectively, as illustrated in Figure 2.2 (a) and (b), respectively. First, the forward current is computed for every receiving element, and then employed to calculate the backward current in an iterative fashion. The total fields at each current element are the superposition of the forward and backward contributions. Finally, the iterative process is continued in the FB manner until surface currents exhibit convergence to within a specified accuracy criterion.

To be more explicit, consider the mathematical formulation of the FB method. The method starts with the following decompositions of the unknown vector \( \mathbf{T} \) and the impedance matrix \( \mathbf{Z} \):

\[
\mathbf{T} = \mathbf{T}^f + \mathbf{T}^b \quad (2.13)
\]

\[
\overline{\mathbf{Z}} = \overline{\mathbf{Z}}^f + \overline{\mathbf{Z}}^r + \overline{\mathbf{Z}}^b \quad (2.14)
\]
where $\mathbf{I}^f$ is the forward propagating current, and $\mathbf{I}^b$ is the backward propagating current. In Eq. (2.14), $\mathbf{Z}^f$ and $\mathbf{Z}^b$ are impedance matrices associated with the forward and backward propagating directions respectively, and $\mathbf{Z}^s$ is the self impedance matrix. Note that $\mathbf{Z}^f$, $\mathbf{Z}^s$, and $\mathbf{Z}^b$ consist of elements in the lower triangular part, diagonal part, and upper triangular part of $\mathbf{Z}$, respectively. Substituting Eqs. (2.13) and (2.14) into Eq. (2.10), Eq. (2.10) can be separated into two matrix equations:

$$
\mathbf{Z}^s \mathbf{I}^f = \mathbf{V} - \mathbf{Z}^f (\mathbf{I}^f + \mathbf{I}^b) \tag{2.15}
$$

$$
\mathbf{Z}^s \mathbf{I}^b = -\mathbf{Z}^b (\mathbf{I}^f + \mathbf{I}^b). \tag{2.16}
$$

Note that the second term on the RHS of Eq. (2.15) represents the forward propagating field contribution, and the term on the RHS of Eq. (2.16) represents the backward propagating field contribution. Eqs. (2.15) and (2.16) can be solved via an iterative FB sweep procedure by initializing $\mathbf{I}^b(0) = 0$, and at $k$ th sweep ($k \geq 1$),

$$
\mathbf{I}^f(k) = \left[\mathbf{Z}^s\right]^{-1} \left[\mathbf{V} - \mathbf{Z}^f (\mathbf{I}^f(k) + \mathbf{I}^b(k-1))\right] \tag{2.17}
$$

$$
\mathbf{I}^b(k) = -\left[\mathbf{Z}^s\right]^{-1} \left[\mathbf{Z}^b (\mathbf{I}^f(k) + \mathbf{I}^b(k))\right]. \tag{2.18}
$$

Iterations are continued in the FB fashion until the solution vector at the $k$ th iteration $\mathbf{I}^{(k)}$, where $\mathbf{I}^{(k)} = \mathbf{I}^f(k) + \mathbf{I}^b(k)$, converges to within a specified accuracy criterion. Typically, the FB method provides very rapid convergence in many RSS problems of interest. It should be pointed out that the inverse of $\mathbf{Z}^s$ in Eqs. (2.17) and (2.18) results simply in a scalar multiplication due to the fact that the impedance matrix $\mathbf{Z}^s$ is diagonal; i.e. no direct matrix inversion is required in the iterative FB method. However, Eqs. (2.17) and (2.18) require two matrix-vector multiplies for each iteration. Note that a direct computation of the above matrix-vector multiplies produces an
Forward Direction

Figure 2.3: Strong region $L^f_s$ and weak region $L^f_w$ in the forward direction. $L_s$ is a neighborhood distance.

$O(N_{tot})$ algorithm due to the fact that the mutual coupling between all pairs of points on the surface must be evaluated. In addition, the impedance matrix $\overline{Z}$ must be stored at a cost of $O(N_{tot}^2)$ memory storage or all elements of the matrix must be recomputed at each iteration with a time-consuming computation. One way to improve the computational efficiency of the FB method is to incorporate the NSA algorithm into the FB method. Next, consider the 1-D standard NSA algorithm for IBC rough surfaces.

For convenience in illustration of the algorithm, consider only the computation of the field in the forward direction. The computation of the field in the backward direction can also be treated in a similar fashion. Let $L_s$ be a neighborhood distance, as shown in Figure 2.3, within which interactions between points are classified as strong and outside of which interactions are classified as weak. Note that, in Figure 2.3, $L^f_s$ and $L^f_w$ denote the strong and weak regions of the forward region $L^f$, respectively.
where \( L_f = L_f^s + L_f^w \). Using Eq. (2.9), Eq. (2.1) for the TM case can be rewritten as:

\[
H_y^f(\rho) = 2H_y^s(\rho) + H_y^w(\rho) + H_y^f(\rho) \\

H_y^f(\rho) = \int_{L_f^s} d\rho \left\{ \frac{\partial g(\rho, \rho')}{\partial n\rho} + \frac{ik}{\sqrt{\varepsilon_R}} g(\rho, \rho') \right\} H_y^f(\rho') \\

H_y^f(\rho) = \int_{L_f^w} d\rho \left\{ \frac{\partial g(\rho, \rho')}{\partial n\rho} + \frac{ik}{\sqrt{\varepsilon_R}} g(\rho, \rho') \right\} H_y^f(\rho')
\]

where the superscript \( f \) denotes the forward direction, and \( H_y^s(\rho) \) and \( H_y^w(\rho) \) represent the strong and weak region contributions, respectively. Note that \( H_y^s(\rho) \) is computed in the conventional manner and the NSA algorithm is employed to compute \( H_y^f(\rho) \).

The NSA algorithm starts with the angular spectral representation of \( g(\rho, \rho') \) and \( \frac{\partial g(\rho, \rho')}{\partial n\rho} \) for \( x - x' > 0 \):

\[
g(\rho, \rho') = \frac{i}{4\pi} \int_{C_\phi} d\phi e^{ik(\rho - \rho')}
\]

\[
\frac{\partial g(\rho, \rho')}{\partial n\rho} = \frac{k}{4\pi|n|} \int_{C_\phi} d\phi (-f(x)\cos \phi + \sin \phi) e^{ik(\rho - \rho')}
\]

where \( k = ik \cos \phi + ik \sin \phi, \rho - \rho' = \hat{x}(x - x') + \hat{z}(z - z'), n = \hat{z} - \hat{x} f(x), |n| = (1 + (f(x))^2)^{\frac{1}{2}}, \) the normal unit vector \( \hat{n} = \frac{n}{|n|}, \) and the contour of integration \( C_\phi \) is shown in Figure 2.4. Note that \( \phi_R \) and \( \phi_I \) denote the real and imaginary parts of the periodic complex angle \( \phi \), respectively. Note that \( \frac{\partial g(\rho, \rho')}{\partial n\rho} \) in Eq. (2.23) can be found via

\[
\frac{\partial g(\rho, \rho')}{\partial n\rho} = \hat{n} \cdot \nabla g(\rho, \rho').
\]

where \( \nabla g \) denotes the gradient operator in the two-dimensional “prime” coordinate system. The topology in the periodic complex \( \phi \) plane is discussed in detail in Appendix B and [104].
Substituting Eqs. (2.22) and (2.23) into Eq. (2.21) and interchanging the spatial and contour integrations. Eq. (2.21) can be rewritten as:

\[
H_{y,w}^f(\rho) = \frac{k}{2\pi} \int_{C_o} d\phi F(\rho, k, \phi).
\]  

(2.25)

where \( F(\rho, k, \phi) \) is called the complex scalar \textit{radiation} function (or plane wave spectrum), defined as

\[
F(\rho, k, \phi) = \int_{L_x} d\mu \nabla^f(\rho, k, \phi) e^{ik(\rho - \rho')} \quad (2.26)
\]

\[
V^f(\rho, k, \phi) = \left\{ \frac{1}{|\nabla|} (-f(t) \cos \phi + \sin \phi) - \frac{1}{\sqrt{\varepsilon_r}} \right\} H_y^f(\rho').
\]  

(2.27)
The most important property of \( F(\rho, k, \phi) \) is that it can be computed from the currents in the weak region in a recursive manner as follows:

\[
F(\rho_m, k, \phi) = F(\rho_{m-1}, k, \phi) e^{ik(\rho_m - \rho_{m-1})} + \int_{x} F(\rho_{m-1-N_\delta}, k, \phi) e^{ik(\rho_{m-1-N_\delta})} \Delta x \left(1 + \left(\frac{f(x_{m-1-N_\delta})}{x_{m-1-N_\delta}}\right)^2\right)^{\frac{1}{2}}. \tag{2.28}
\]

with \( F(\rho_m, k, \phi) = 0 \) for \( m \leq N_s + 1 \), where \( N_s = \frac{L_\delta}{\Delta x} \), in the forward sweep. and \( \rho_m = \delta x_m + \delta f(x_m) \). From Eq. (2.28), the radiation function of the new test point (located at \( \rho_m \)) can be computed from the one for the old test point (located at \( \rho_{m-1} \)) multiplying by a phase correction term to adjust for a shift of the test point from \( \rho_{m-1} \) to \( \rho_m \), and then adding by the contribution from the new source point (located at \( \rho_{m-1-N_\delta} \)), which just enters into the weak region \( L_\delta \). Note that this recursive property of \( F(\rho_m, k, \phi) \) makes the NSA method an efficient algorithm for both computational and memory requirements. To further gain the computational efficiency, the original contour \( C_\phi \) is deformed to the new contour \( C_\delta \) (see Figure 2.4). This contour deformation is valid due to the Cauchy's residue theorem [129] with the fact that there are no singularities encountered between \( C_\phi \) and \( C_\delta \) for \( x - x_0 \geq L_\delta \). Along the deformed contour \( C_\delta \), smaller integration intervals and smaller sampling rates for evaluating the angular spectral integral involving \( F(\rho_m, k, \phi) \) are possible. It should be pointed out that the contour deformation is indispensable to make the NSA method an \( O(N_{tot}) \) algorithm as well as the recursive property of \( F(\rho_m, k, \phi) \). Without the contour deformation, it is required to increase the sampling rate to evaluate the integral involving \( F(\rho_m, k, \phi) \) accurately as the size of the weak region increases due to more oscillation of \( F(\rho_m, k, \phi) \) along the original contour \( C_\phi \). Increasing the
sampling rate as the surface size increases makes the efficiency of the NSA method degrade considerably. Thus, both the recursive property of \( F(\rho_m, k, \phi) \) and contour deformation are necessary in the NSA algorithm. After discretizing the integral of Eq. (2.25) over \( \phi \) into \( 2Q + 1 \) plane wave directions, Eq. (2.25) can be expressed as:

\[
H_{y,u}(\rho_m) = \frac{k \Delta \phi}{2\pi} e^{-i\phi} \sum_{p=-Q}^{Q} W'(\phi_p) F(\rho_m, k, \phi_p).
\]  
(2.30)

where \( \phi_p = p \Delta \phi e^{-i\phi} \) for \( p = -Q, \ldots, Q \). and \( W'(\phi_p) \) is a weighting function for numerical integration (Simpson's rule is used in this development). Note that the integration parameters \( \delta, \Delta \phi \) and \( Q \) are given as follows:

\[
\delta = \tan^{-1} \left[ \frac{1}{\max\{b, 1\}} \right],
\]  
(2.31)

\[
\Delta \phi = \frac{1}{22} \sqrt{\frac{5}{kR_s}},
\]  
(2.32)

\[
Q = \left\lceil \frac{\phi_{\text{max}}}{\Delta \phi} \right\rceil.
\]  
(2.33)

where

\[
b = \sqrt{\frac{kR_s}{a_{\text{max}}}} \phi_{s,\text{max}} - 1.
\]  
(2.34)

\[
R_s = (L_s^2 + (\Delta z_{\text{max}})^2)^{1/2}, \phi_{s,\text{max}} = \tan^{-1} \left( \frac{\Delta z_{\text{max}}}{L_s} \right), \phi_{\text{max}} = \max \{ \phi_s, \phi_{s,\text{max}} \}. \phi_s = \sqrt{\frac{\zeta}{kL_s}}, \Delta z_{\text{max}} = z_{\text{max}} - z_{\text{min}}, a_{\text{max}} \text{ and } \zeta \text{ are given positive constants (typically } a_{\text{max}} = 5.0 \text{ and } \zeta = 10.0 \text{ yield reasonable values for } \delta \text{ and } \phi_s, \text{ respectively}), \text{ and } \lceil \cdot \rceil \text{ denotes the ceiling operator: i.e. rounding its argument to the nearest integer towards plus infinity. The detail of the derivation of the above 1-D integration parameters can be found in Appendix D. It should be pointed out that the above formulas for the 1-D NSA parameter are derived based on the assumption that } \phi_{s,\text{max}} \text{ is small. For a given surface height variation, this assumption can be satisfied by adjusting the size}
of the strong region. However, for rough surfaces with large height variations, the adjusted size of the strong region is typically large resulting in significant increases in computational time for the strong-region contribution, and degrading overall efficiency of the NSA algorithm. The derivation of analytical formulas associated with the 1-D NSA parameters for an arbitrary value of $\sigma_{s,\text{max}}$ will be presented in Section 2.3. It is emphasized that the neighborhood distance $L_s$ depends on surface statistics, and it must be chosen appropriately to yield accurate results. Generally, $L_s$ is chosen such that the relative errors obtained from computing $g(\rho, \rho')$ using the exact expression (Eq. (2.4)) and the angular spectral domain representation (Eq. (2.22)), for the nearest pair of source-and-field points ($x - x' = L_s$) and the farthest pair of source-and-field points ($x - x' = L_z$) in the weak region, are less than specified tolerances. Assuming that $z - z' = \Delta z_{\text{max}}$, specified tolerances for $x - x' = L_s$ and $x - x' = L_z$ are typically set to be 0.5 % and 2.0 %, respectively.

In the backward direction, the contributions of the weak region $H^b_{y,w}(\rho_m)$ can also be derived in the same manner as $H^f_{y,w}(\rho_m)$. The FB/NSA method is iterated until the relative error of the solutions of adjacent iterations is less than a specified tolerance. Typically, this tolerance is set to be 0.01 to achieve accurate results. Using the 1-D NSA algorithm to compute the contributions of the weak region, it can be shown that the computational cost and memory storage requirement are $O(V_{\text{tot}})$ for fixed surface roughness statistics. Finally, numerical results in Section 2.5 illustrate that the 1-D NSA algorithm can significantly improve the computational efficiency of the standard 1-D FB method without degrading its accuracy.
2.3 Derivation of General Formulas Associated with the 1-D NSA Parameters

Without loss of generality, consider only the 1-D NSA algorithm in the forward sweep for which \( x - x' > 0 \) as illustrated in Figure 2.3. In the weak region, the 1-D NSA algorithm employs the angular spectral representation of the two-dimensional (2-D) scalar free space Green's function. \( H_0^{(1)}(k|\rho - \rho'|) \), along the original contour \( C_\phi \):

\[
H_0^{(1)}(k|\rho - \rho'|) = \frac{1}{2\pi} \int_{C_\phi} d\phi I(\phi), \quad x - x' \geq L_x \tag{2.35}
\]

\[
I(\phi) = e^{ikR\cos(\phi - \phi_s)} \tag{2.36}
\]

where \( R = |\rho - \rho'| = \sqrt{(x - x')^2 + (z - z')^2} \) and the saddle point for a given pair of source and observation points \( \phi_s = \tan^{-1}\left(\frac{z - z'}{x - x'}\right) \). For convenience, first consider the topology in the periodic complex \( \phi \) plane for a single pair of points \( \rho \) and \( \rho' \) on a flat surface. For a flat surface, the steepest descent path (SDP) \( C_{SDP}(\Delta z = 0) \) passes through its saddle point at the origin as shown in Figure 2.4. From an asymptotic analysis, most of the contribution occurs on portions of the SDP path near a saddle point on the real axis. As the distance from the saddle point increases along the SDP path, the integrand \( I(\phi) \) is exponentially attenuated so that the contributions become negligible. Thus, it is numerically advantageous to deform the original contour \( C_\phi \) to the SDP contour for a flat surface \( C_{SDP}(\Delta z = 0) \) as illustrated in Figure 2.4.

However when coupling between many pairs of points is considered as in the weak region contribution to the receiving point as shown in Figure 2.3, there is no longer a unique SDP path for a rough surface along which only attenuation of the integrand is obtained away from a single saddle point. Figure 2.4 also shows the distribution
of all possible saddle points that can exist for a given surface profile. Thus, the deformed contour $C_{\delta}$ as shown in Figure 2.4 must be chosen as a compromise between extreme exponential growth and rapid oscillation of the integrand $I(\phi)$. Note that in the 1-D NSA algorithm, the efficiency of the angular spectral expansion can be improved by deforming the original contour $C_{0}$ to the deformed contour $C_{\delta}$. Employing the Cauchy’s residue theorem [129] with the fact that there are no singularities encountered between $C_{0}$ and $C_{\delta}$ for $x - x_{l} \geq L_{s}$, $H_{0}^{(1)}(k|\rho - \rho|)$ can be rewritten as

$$H_{0}^{(1)}(k|\rho - \rho|) = \frac{1}{\pi} \int_{C_{\delta}} d\phi I(\phi), \quad x - x_{l} \geq L_{s}. \quad (2.37)$$

Let $\phi_{R}$ and $\phi_{I}$ denote the real and imaginary parts of the periodic complex angle $\phi$, respectively. Let $\phi_{s,\text{max}}$ and $\phi_{\text{max}}$ be the outermost possible saddle point and the upper limit of integration along the real axis of $C_{\delta}$, where $\phi_{s,\text{max}} = \tan^{-1}\left(\frac{\Delta z_{\text{max}}}{L_{s}}\right)$ and $\Delta z_{\text{max}} = z_{\text{max}} - z_{\text{min}}$. Note that the linear part of $C_{\delta}$ extends from $-\phi_{\text{max}}$ to $\phi_{\text{max}}$ and the rest, where the integrand $I(\phi)$ behaves like an exponential decaying function, is deformed to connect with $C_{0}$. Along the linear portion, the contour $C_{\delta}$ can be expressed as:

$$\phi = \phi_{R}(1 - i \tan \delta). \quad (2.38)$$

where $\delta \in (0, \frac{\pi}{4}]$. Note that $\delta_{\text{max}} = \frac{\pi}{4}$ rad. is the tilt angle of the steepest descent path $C_{SDP}$ for the flat surface ($\Delta z = 0$) passing through its saddle point at the origin.

Consider the magnitude $|I(\phi)|$ and phase $\psi(\phi)$ of the integrand $I(\phi)$ of $H_{0}^{(1)}(k|\rho - \rho|)$ for a single saddle point $\phi_{s}$:

$$I(\phi) = |I(\phi)| e^{i\psi(\phi)} \quad (2.39)$$

$$|I(\phi)| = e^{-kR\sin(\phi_{R} - \phi_{s})\sinh(\phi_{R}\tan \delta)} \quad (2.40)$$

$$\psi(\phi) = kR\cos(\phi_{R} - \phi_{s})\cosh(\phi_{R}\tan \delta). \quad (2.41)$$
Without loss of generality, consider only \( \phi_s > 0 \); i.e. assuming that \( \Delta z = z - z' > 0 \), to the rest of this chapter. As discussed in the previous section, the 1-D NSA method involves three parameters: the tilt angle \( \delta \) of the deformed contour \( C_\delta \), the domain of integration \( [-\phi_{\text{max}}, \phi_{\text{max}}] \) and the integration step size \( \Delta \phi \). It should be pointed out that these parameters can be derived by considering the integrand \( I(\phi) \) for a pair of source and field points in the worst-case scenario in the \( x - z \) plane; i.e. when \( x - x' = L_s \) and \( \Delta z = \Delta z_{\text{max}} \) (corresponding to \( \phi_s = \phi_{s,\text{max}} \) in the complex \( \phi \) plane). Due to the fact that the magnitude \( |I(\phi)| \) and phase \( \psi(\phi) \) of the integrand \( I(\phi) \) usually vary most rapidly for this configuration. In this section, first an analytical formula associated with \( \delta \) for an arbitrary value of \( \phi_{s,\text{max}} \) is derived, and then the derivation of analytical formulas associated with \( \phi_{\text{max}} \) and \( \Delta \phi \) follows.

### 2.3.1 Analytical Formula for \( \delta \)

One criterion for selecting a possible value of \( \delta \), denoted as \( \tilde{\delta} \), can be obtained by limiting the maximum value of \( |I(\phi)| \) along the contour \( C_\delta \) to \( e^{a_{\text{max}}} \), where \( a_{\text{max}} \) is a given positive constant. Typically, it is found out that \( a_{\text{max}} = 5.0 \) yields a reasonable value for \( \delta \). Figure 2.5 illustrates a typical plot of \( |I(\phi)| \) versus \( \phi_R \) along the deformed contour \( C_\delta \) with a single saddle point \( \phi_s = \phi_{s,\text{max}} \). Note that the maximum value of \( |I(\phi)| \), denoted as \( |I(\phi)|_{\text{max}} \), occurs at \( \phi_R = \phi_{R,0} \), and \( \phi_{R,0} \) is always located between the origin and \( \phi_{s,\text{max}} \). Employing standard calculus, \( \phi_{R,0} \) can be determined by maximizing Eq. (2.40) using \( \phi_s = \phi_{s,\text{max}} \) to obtain the following nonlinear equation for \( \phi_{R,0} \):

\[
\tan \tilde{\delta} \tan(\phi_s - \phi_{R,0}) = \tanh(\phi_{R,0} \tan \tilde{\delta}), \quad \phi_{R,0} \in [0, \phi_s]. \tag{2.42}
\]
Figure 2.5: The plot of \(|I(\phi)|\) versus \(\phi_R\) along the deformed contour \(C_\delta\).
For a given pair of observation and source points, the saddle point $\phi_s$ is fixed. Note that Eq. (2.42) has two unknowns $\phi_{R,o}$ and $\delta$, so that another equation is required to uniquely specify $\delta$. Setting $|I(\phi)|_{max} = e^{a_{max}}$ yields another nonlinear equation:

$$kR\sin(\phi_s - \phi_{R,o}) \sinh(\phi_{R,o} \tan \delta) = a_{max}, \quad \phi_{R,o} \in [0, \phi_s]. \quad (2.43)$$

Typically, it is found that for a given surface, a possible pair of points possessing the largest saddle point $\phi_{s,max}$; i.e. $x - x_I = L_s$ and $z - z_I = \Delta z_{max}$, approximately yields $|I(\phi)|_{max}$ at $\phi_R = \phi_{R,o}$. Thus, for a given $a_{max}$, $\delta$ can be determined analytically via solving Eqs. (2.42) and (2.43) simultaneously with $\phi_s = \phi_{s,max}$ and $R = R_w$, where $R_w$ is the corresponding distance between a pair of points possessing $\phi_{s,max}$; and it is defined as $R_w = \sqrt{L_s^2 + (\Delta z_{max})^2}$. Note that Eqs. (2.42) and (2.43) can be solved simultaneously via a root-finding technique such as Muller's method (see Appendix F for detail) by first initially guessing the value of $\delta$ denoted as $\delta_0$, where $\delta_0 \in (0, \frac{\pi}{4}]$ and then solving for $\phi_{R,o}$ numerically using Eq. (2.42). Substituting this $\phi_{R,o}$ into Eq. (2.43), and then solving for $\delta$ numerically, allows this new $\delta$ to be used as an initial guess for the next iteration. This procedure repeats iteratively until $\delta$ converges to the desired value within a specified tolerance. Finally, once $\delta$ is known, the tilt angle $\delta$ is obtained as follows

$$\delta = \min \{ \hat{\delta}, \frac{\pi}{4} \}, \quad (2.44)$$

since $\delta_{max} = \frac{\pi}{4}$ rad. as discussed in the earlier section. Note that Eqs. (2.42) and (2.43) can be simplified for two limiting cases as shown below:

**Case 1: Small $\phi_{s,max}$**

For small $\phi_{s,max}$ ($0 \leq \phi_{s,max} \sim 0.3 \text{ rad.}$), it is implied that $\phi_{R,o}$ is also small since $\phi_{R,o} \in [0, \phi_{s,max}]$. Performing a Taylor's series expansion and keeping only the first
term, Eqs. (2.42) and (2.43) with \( \phi_s = \phi_{s,\text{max}} \) and \( R = R_w \) can be simplified to the following equations:

\[
\phi_{R,0} \approx 0.5 \phi_{s,\text{max}} \quad (2.45)
\]
\[
\delta = \tan^{-1}\left(\frac{2a_{\text{max}} \sqrt{4 + \phi_{s,\text{max}}^2}}{kR_w \phi_{s,\text{max}}^2}\right). \quad (2.46)
\]

respectively. It should be pointed out that \( \delta \) in [101] is derived based on the assumption that \( \phi_{s,\text{max}} \) is small and the approximation that \( \phi_{R,0} \) occurs at the intersection between the steepest ascent path (SAP) passing through \( \phi_{s,\text{max}} \) (employing the linear approximation of the SAP) and the linear part of the contour \( C_\delta \) as given in Eq. (2.38). It is found that \( \delta \) from both approaches match well when \( \delta \) approaches \( \frac{\pi}{4} \) rad.

**Case 2: Large \( \phi_{s,\text{max}} \)**

In this case, \( \delta \) is expected to be small to avoid excessive exponential increase of \( |I(\phi)| \) along the contour \( C_\delta \). Performing the Taylor’s series expansion and keeping only the first term. Eqs. (2.42) and (2.43) with \( \phi_s = \phi_{s,\text{max}} \) and \( R = R_w \) can be approximated by the following equations to solve for \( \phi_{R,0} \) and \( \delta \):

\[
\tan(\phi_{s,\text{max}} - \phi_{R,0}) = \phi_{R,0}, \quad \phi_{R,0} \in [0, \phi_{s,\text{max}}] \quad (2.47)
\]
\[
\delta = \tan^{-1}\left(\frac{a_{\text{max}}}{kR_w \phi_{R,0} \sin(\phi_{s,\text{max}} - \phi_{R,0})}\right). \quad (2.48)
\]

respectively. Note that once \( \phi_{R,0} \) is known by solving Eq. (2.47), \( \delta \) can be obtained via Eq. (2.48). To further simplify Eq. (2.47), performing the Taylor’s series expansion and keeping up to the second term (since the difference \( \phi_{s,\text{max}} - \phi_{R,0} \) may not be small enough to neglect the second term), the following cubic equation for \( \phi_{R,0} \) is obtained:

\[
(\phi_{s,\text{max}} - \phi_{R,0})^3 + 6(\phi_{s,\text{max}} - \phi_{R,0}) + 3\phi_{s,\text{max}} = 0. \quad (2.49)
\]
Solving Eq. (2.49) analytically [130] and keeping only the real solution and discarding a pair of complex conjugate roots, one obtains

\[ \varphi_{R,0} = \phi_{s,\text{max}} + (u - v)^{\frac{3}{2}} - (u + v)^{\frac{3}{2}}, \quad \phi_{R,0} \in [0, \phi_{s,\text{max}}] \]  

(2.50)

where \( u = 0.5 \sqrt{32 + 9 \phi_{s,\text{max}}^2} \) and \( v = 1.5 \phi_{s,\text{max}} \). To obtain a more accurate value of \( \delta \), one may use \( \delta \) and \( \phi_{R,0} \) given in Eqs. (2.48) and (2.50) respectively, as an initial guess in solving Eqs. (2.42) and (2.43) simultaneously. It is interesting to point out that Eqs. (2.47) and (2.48) can be simplified to Eqs. (2.45) and (2.46) for the case of small \( \phi_{s,\text{max}} \), respectively. Thus, \( \delta \) obtained from Eqs. (2.47) and (2.48) is expected to work well for both small and large \( \phi_{s,\text{max}} \). Numerical results also show that \( \delta \) from Eqs. (2.47) and (2.48) usually provides a good estimate for the exact \( \delta \).

### 2.3.2 Analytical Formula for \( \phi_{\text{max}} \)

The maximum of the domain of integration \( \phi_{\text{max}} \) can be determined analytically by considering the distribution of \( |I(\phi)| \) along the deformed contour \( C_\delta \). It is found that \( \phi_{\text{max}} \) corresponding to \( \phi_s = \phi_{s,\text{max}} \) and \( R = R_v \) is usually the largest, and \( |I(\phi)| \) for this case is exponentially decaying and less than one outside the interval \([0, \phi_{s,\text{max}}]\) as illustrated in Figure 2.5. Thus, \( \phi_{\text{max}} \) can be determined by limiting the value of \( |I(\phi)| \) along \( C_\delta \) to \( e^{-\zeta} \), where \( \zeta \) is typically equal to 6, and a possible value of \( \phi_{\text{max}} \), denoted as \( \tilde{\phi}_{\text{max}} \), can be obtained by solving the following nonlinear equation:

\[ k R_v \sin(\tilde{\phi}_{\text{max}} - \phi_{s,\text{max}}) \sinh(\tilde{\phi}_{\text{max}} \tan \delta) - \zeta = 0, \quad \tilde{\phi}_{\text{max}} > \phi_{s,\text{max}}. \]  

(2.51)

Note that Eq. (2.51) can be solved for \( \tilde{\phi}_{\text{max}} \) via a root-finding technique such as Muller's method (see Appendix F). The condition \( \tilde{\phi}_{\text{max}} > \phi_{s,\text{max}} \) is imposed to avoid another possible invalid solution of \( \tilde{\phi}_{\text{max}} \), denoted as \( \phi_l \) (\( \phi_l < 0 \)), as shown in
Figure 2.5, and the analytical formula for $\phi_{\text{max}}$ is given as follows:

$$\phi_{\text{max}} = \min \{\phi_{\text{max}}, \frac{\pi}{2}\}. \quad (2.52)$$

where the maximum value of $\phi_{\text{max}}$ is equal to $\frac{\pi}{2}$ rad. as shown in Figure 2.4. For the case of very large $\phi_{s,\text{max}}$ ($\phi_{s,\text{max}}$ close to $\frac{\pi}{2}$ rad.), it should be pointed out that $\phi_{\text{max}}$ is usually equal to $\frac{\pi}{2}$ rad.. and the domain of integration should include the portion of the vertical lines at $\pm \frac{\pi}{2}$ rad. as shown in Figure 2.4 to allow the integrand $I(\phi)$ to decay sufficiently. This increase in the domain of integration can be avoided by properly adjusting the size of the strong region $L_s$ to trade off the computation of the contributions from the strong region and the weak region. No results including the vertical portion of the contour at $\pm \frac{\pi}{2}$ rad. are included in this chapter.

### 2.3.3 Analytical Formula for $\Delta \phi$

The integration step size $\Delta \phi$ can be determined by considering the variation of the integrand $I(\phi)$ (see Eq. (2.36)) along the deformed contour $C_\delta$ with $\phi_s = \phi_{s,\text{max}}$ and $R = R_w$, and sampling $I(\phi)$ according to its highest frequency component. Let $f_{\text{max}}$ is the maximum frequency of $I(\phi)$ (cycles per radian) obtained by taking the discrete Fourier transform (DFT) of $I(\phi)$ (via the fast Fourier transform (FFT) algorithm) and searching for its maximum frequency component. Assuming that $I(\phi)$ is a bandlimited function. then the integration step size $\Delta \phi_{\text{FFT}}$ can be expressed as

$$\Delta \phi_{\text{FFT}} = \frac{1}{f_{\text{max}} \cdot N_s}. \quad (2.53)$$

where $N_s$ is the number of samples per cycle and $N_s$ is typically equal to 8. It is emphasized that the number of sampling points used in the FFT algorithm must be sufficient to avoid aliasing effects. For illustration, consider the case of $\Delta z =$
$\Delta z_{\text{max}} = 5.0\lambda$ and $x - x' = L_s = 1.0\lambda$. Using Eqs. (2.42) to (2.44), it is found that the appropriate tilt angle $\delta$ for this case is equal to 19.406°. Figure 2.6 (a) and (b) plots the real and imaginary parts of $I(\phi)$. denoted as $I_R(\phi)$ and $I_I(\phi)$ respectively, along $C_s$ versus the real angle $\phi_R$. From the plots, it is observed that both $I_R(\phi)$ and $I_I(\phi)$ are quite oscillatory. In addition, the plots of the corresponding magnitudes of the DFT of $I_R(\phi)$ and $I_I(\phi)$ versus the frequency variable $\tilde{f}$ are also shown in Figure 2.6 (c) and (d), respectively. Note that both $I_R(\phi)$ and $I_I(\phi)$ are bandlimited functions, and thus their maximum frequency can be determined by finding the frequency such that its DFT magnitude normalized by the maximum DFT magnitude is less than a given tolerance (typically 0.001). Using this procedure, it is found that the maximum frequencies for $I_R(\phi)$ and $I_I(\phi)$, denoted as $f_{\text{max,R}}$ and $f_{\text{max,I}}$ respectively, are equal to 7.0 cycles/ rad. Thus, the maximum frequency $f_{\text{max}}$ of $I(\phi)$, which is the maximum of $f_{\text{max,R}}$ and $f_{\text{max,I}}$, is equal to 7.0 cycles/ rad., and the integration step size $\Delta \phi$ via the “FFT” approach is equal to 1.023°.

For the case of large $\phi_{s,\text{max}}$, which frequently occurs in practice when $L_s$ is optimized for a given $\Delta z_{\text{max}}$, the phase component of $I(\phi)$, $e^{i\theta(\phi)}$, typically varies more rapidly than the magnitude term $|I(\phi)|$. Thus, it is reasonable to consider only the variation of $e^{i\theta(\phi)}$ instead of $I(\phi)$ to determine $\Delta \phi$. It should be pointed out that $|I(\phi)|$ acts as a bandpass filter: i.e. suppressing the components outside the interval $[\phi_l, \phi_{\text{max}}]$, as illustrated in Figure 2.5. Note that $\phi_l$ can be determined by following the same procedure as in Section 2.3.2 resulting in the following equations to solve for $\phi_l$:

$$kR_w \sin(\hat{\phi}_l - \phi_{s,\text{max}}) \sinh(\hat{\phi}_l \tan \delta) - \zeta = 0, \quad \hat{\phi}_l < 0 \quad (2.54)$$

$$\phi_l = \max \{\hat{\phi}_l, -\frac{\pi}{2}\}, \quad (2.55)$$
Figure 2.6: The plots of the real and imaginary parts of $f(\phi)$ versus $\phi_R$ and their discrete fourier transforms versus $f$. 
where $\phi_l$ is a possible value of $\phi_l$ and the parameter $\zeta$ is defined in Section 2.3.2.

Performing the first-order Taylor's series expansion on $\psi(\phi_R)$ at a local point $\tilde{\phi}_R$, where $\tilde{\phi}_R \in [\phi_l, \phi_{max}]$, $\psi(\phi_R)$ and $e^{i\psi(\phi_R)}$ can be approximated as

$$
\psi'(\phi_R) = \psi'(\tilde{\phi}_R) + (\phi_R - \tilde{\phi}_R) \frac{\partial \psi'(\phi_R)}{\partial \phi_R} \bigg|_{\phi_R = \tilde{\phi}_R} \tag{2.56}
$$

$$
e^{i\psi(\phi_R)} = e^{i\psi(\tilde{\phi}_R)} e^{i\psi'(\tilde{\phi}_R) \phi_R} \tag{2.57}
$$

where

$$
\psi'(\phi_R) = k_{R_w} \cos(\phi_R - \phi_{s,max}) \cosh(\phi_R \tan \delta) \tag{2.59}
$$

$$
p(\tilde{\phi}_R) = \psi'(\tilde{\phi}_R) - \tilde{\phi}_R \psi''(\tilde{\phi}_R) \tag{2.60}
$$

$$
q(\tilde{\phi}_R) = \frac{\partial \psi'(\phi_R)}{\partial \phi_R} \bigg|_{\phi_R = \tilde{\phi}_R} \tag{2.61}
$$

$$
= k_{R_w} \left[ \tan \delta \cos(\tilde{\phi}_R - \phi_{s,max}) \sinh(\tilde{\phi}_R \tan \delta) \right. \tag{2.62}
$$

$$
- \cosh(\tilde{\phi}_R \tan \delta) \sin(\tilde{\phi}_R - \phi_{s,max}) \left.] \right. \tag{2.63}
$$

Note that the first term $e^{i\psi(\tilde{\phi}_R)}$ in Eq. (2.58) is a constant, and the second term $e^{i\psi'(\tilde{\phi}_R) \phi_R}$ is a linear phase term. Thus, $\Delta \phi$ can be determined analytically by considering only the highest frequency component contained in the second term which occurs when $|q(\tilde{\phi}_R)|$ is maximum. Let $q_{max}$ denote $\max \{|q(\tilde{\phi}_R)|\}$, where $\tilde{\phi}_R \in [\phi_l, \phi_{max}]$, and the approximate analytical formula of $\Delta \phi$ is given as

$$
\Delta \phi \approx \frac{2\pi}{q_{max} \sqrt{N}} \tag{2.64}
$$

$q_{max}$ can be determined by simply using a numerical searching procedure or using a standard maximization/minimization technique as described in [88]. It should be pointed out that the "FFT" approach takes both magnitude and phase variations of
\( I(\phi) \) into account at the cost of the computation of the Fourier transform of \( I(\phi) \),
and \( \Delta \phi_{FFT} \) is therefore employed as a reference solution. For the case of large \( \phi_{s,\text{max}} \),
umerical tests shows that \( \Delta \phi \) obtained from Eq. (2.64) yields reasonable results as
compared to \( \Delta \phi_{FFT} \) in Eq. (2.53) even though the variation of the magnitude term
\(|I(\phi)|\) is neglected. For the case considered in the "FFT" approach, \( \phi_{s,\text{max}} \) is equal to
78.69°, which is quite large, and \( \Delta \phi \) in Eq. (2.64) can be shown to be 1.398°, which is
close to \( \Delta \phi_{FFT} = 1.023° \). Note that once \( \phi_{max} \) and \( \Delta \phi \) are known, the total number
of plane waves \( Q_{TOT} \) employed in the 1-D NSA algorithm is given as
\[
Q_{TOT} = 2 \left[ \frac{\phi_{max}}{\Delta \phi} \right] + 1. \tag{2.65}
\]

2.4 The 1-D "Multilevel" NSA Algorithm

For the case of 1-D extremely large-scale quasi-planar structures, the given 1-
D NSA parameters in [104] may yield inaccurate results due to the fact that the
complex radiation function \( I(\phi) \) of a source group far separated from the receiving
element is rapidly decayed along the deformed contour \( C_s \) away from the origin in
the periodic complex angular \( \phi \) plane, which requires a significantly higher sampling
rate to resolve. It is emphasized that only one very large weak region employed in
the original 1-D NSA algorithm causes this higher sampling rate for 1-D extremely
large-scale QPS, and overall accuracy of the 1-D NSA algorithm is degraded. To
maintain the accuracy of the 1-D NSA algorithm without significantly degrading its
efficiency, the very large weak region needs to be decomposed into more than one
separate weak region, and appropriate 1-D NSA parameters must be determined
separately for each weak region. The PO approximation of surface currents along
with a flat surface assumption is employed to determine the appropriate sizes of
each weak region. It should be pointed out that the PO approximation is used to obtain analytical formulas for the sizes of each weak region, and the flat surface is a good assumption in studying the sizes of each weak region since the saddle points of all possible pairs of an observation point and source points in the weak regions, except the first weak region, are in the neighborhood of the origin. Note that the new proposed scheme for the 1-D NSA algorithm can be classified as a “multilevel” algorithm although it is different from the standard multilevel algorithms [96, 97].

Without loss of generality, consider a TE plane wave impinging on a flat PEC half plane as illustrated in Figure 2.7, where

$$E' = \hat{y} E_0 e^{-i(k_x x + k_z z)}.$$  \hspace{1cm} (2.66)

$$k_x = k \sin \theta_i, \quad k_z = k \cos \theta_i, \quad \theta_i$$ is an incident angle measured from the $z-$ axis. $E_0$ is the amplitude constant of the incident field, $x_n$ is a receiving point on the half
plane, \( L_s \) is the size of the strong region. and the weak region of size \( L_w \) starts from the origin to the source point \( x_w \). It should be pointed out that this procedure also illustrates the NSA method with many pairs of observation and source points rather than considering only the specific pair of interest as employed in determining the NSA parameters in the previous section. Using the PO approximation, the electric field of the weak region at the receiving point \( x_n \), denoted as \( E_w(x_n) \), can be expressed as

\[
E_w(x_n) = \hat{y} E_w(x_n) = -\frac{k\eta}{4} \int_0^{x_w} dx' J_{PO}(x') H_0^{(1)}(k(x_n - x')).
\]  

(2.67)

where \( x_n - x' > 0 \), \( x' \in [0, x_w] \), \( \eta = \sqrt{\epsilon} \), and

\[
J_{PO}(x') = \hat{y} \frac{2E_0 k_z}{k\eta} e^{-ik_z x'}. \]

(2.68)

Employing Eq. (2.35) with \( R = x_n - x' \) and \( \phi_s = 0 \): i.e.

\[
H_0^{(1)}(k(x_n - x')) = \frac{1}{\pi} \int_{C_{\phi}} d\phi I_f(\phi), \quad x_n - x' \geq L_s
\]

(2.69)

\[
I_f(\phi) = e^{ik(x_n - x') \cos \phi}.
\]

(2.70)

\( E_w(x_n) \) defined in Eq. (2.67) can be rewritten as

\[
E_w(x_n) = -\frac{E_0 k_z}{2\pi} \int_0^{x_w} dx' e^{-ik_z x'} \int_{C_\phi} d\phi I_f(\phi).
\]

(2.71)

Interchanging the spatial domain integration and the contour integration, and then performing the spatial domain integration analytically, \( E_w(x_n) \) can be expressed in terms of the plane wave spectrum of the flat surface \( F(\phi) \) as follows:

\[
E_w(x_n) = -\frac{iE_0 k_z}{2\pi} \int_{C_\phi} d\phi F(\phi).
\]

(2.72)
where

\[
F(\phi) = -ie^{ikx_n \cos \phi} \int_0^{x_w} dx e^{-i(kz + k \cos \phi)z} = G_1(\phi) - G_2(\phi)
\]  \hspace{1cm} (2.73)

\[
G_1(\phi) = \frac{e^{i(kL \cos \phi - kx_w)}}{k \cos \phi + k_x}
\]  \hspace{1cm} (2.74)

\[
G_2(\phi) = \frac{e^{ikx_n \cos \phi}}{k \cos \phi + k_x}.
\]  \hspace{1cm} (2.75)

Note that \( F(\phi) \) consists of the contribution \( G_1(\phi) \) from the nearest source point in the weak region \((x = x_w)\) to the observation point \(x_n\) and the contribution \( G_2(\phi) \) from the farthest source point \((x = 0)\) to \(x_n\). For the large strong region \((kL_s > > 1)\), \( E_w(x_n) \) can be evaluated asymptotically using the method of steepest descent [131,132], and the following asymptotic result is obtained:

\[
E_w(x_n) \sim -E_0 \left( \frac{k_z}{k + k_x} \right) \frac{e^{i\kappa_1^2}}{\sqrt{2\pi}} \left[ e^{-ikz_xw} \frac{e^{i\kappa_1}}{\sqrt{\kappa_1}} - e^{i\kappa_2} \frac{e^{i\kappa_2}}{\sqrt{\kappa_2}} \right].
\]  \hspace{1cm} (2.76)

where \( \kappa_1 = k L_s, \kappa_2 = k x_n \), and \( \kappa_2 >> \kappa_1 \). Note that the first and second terms in Eq. (2.76) corresponds to the contribution from \( G_1(\phi) \) and \( G_2(\phi) \) respectively, and the first term is usually much larger than the second term. As the strong region \( L_s \) increases, the contribution from the weak region decreases at the rate of \( 1/\sqrt{\kappa_1} \).

In the 1-D NSA algorithm, \( E_w(x_n) \) in Eq. (2.72) is evaluated numerically along the deformed contour \( C_\delta \) instead of \( C_\phi \), where \( \delta = \frac{\pi}{4} \) rad. for the flat surface case. Using Eq. (2.38) with \( \delta = \frac{\pi}{4} \) rad., the contributions \( G_1(\phi) \) and \( G_2(\phi) \) along \( C_\delta(\delta = \frac{\pi}{4}) \) can be expressed explicitly as

\[
G_1(\phi) = \frac{e^{-\kappa_1 \sin \sigma_R \sinh \sigma_R e^{i(\kappa_1 \cos \sigma_R \cosh \sigma_R - kx_w)}}}{D(\phi)}
\]  \hspace{1cm} (2.77)

\[
G_2(\phi) = \frac{e^{-\kappa_2 \sin \sigma_R \sinh \sigma_R e^{ikx_n \cos \sigma_R \cosh \sigma_R}}}{D(\phi)},
\]  \hspace{1cm} (2.78)

where \( D(\phi) = k(\cos \phi_R \cosh \phi_R - i \sin \phi_R \sinh \phi_R) + k_x \). Note that \( G_1(\phi) \) and \( G_2(\phi) \) decay exponentially due to the fact that \( \sin \phi_R \sinh \phi_R > 0 \) along \( C_\delta(\delta = \frac{\pi}{4}) \). Thus, in
the neighborhood of the saddle point \( \phi_j = 0 \). \( G_1(\phi) \) and \( G_2(\phi) \) can be approximated as

\[
G_1(\phi) = \frac{e^{-\kappa_1 \sigma_R^2} e^{i(\kappa_1 - k_\ell \phi)}}{k + k_\ell - i k \sigma_R^2}
\]

\( G_2(\phi) = \frac{e^{-\kappa_2 \sigma_R^2} e^{i\kappa_2}}{k + k_\ell - i k \sigma_R^2}. \]

(2.79)  

(2.80)

Note that the distribution of \( G_1(\phi) \) and \( G_2(\phi) \) is approximately Gaussian centered at the origin, and the half widths of \( G_i(\phi) \), denoted as \( \phi_{\text{HW},G_i} \), can be determined by setting (analogous to the derivation of analytical formula for \( \phi_{\text{max}} \) of the standard 1-D NSA algorithm in Appendix D)

\[
e^{-\kappa_i \phi_{\text{HW},G_i}} = e^{-\tau}.
\]

(2.81)

where \( i = 1 \) or \( 2 \), and \( \tau \) is a positive constant (typically equal to 10). Solving for \( \phi_{\text{HW},G_i} \) in Eq. (2.81) yields

\[
\phi_{\text{HW},G_i} = \sqrt{\frac{\tau}{\kappa_i}}.
\]

(2.82)

Performing the same analysis as above, it can also be shown that \( \phi_{\text{HW},G_1} \) and \( \phi_{\text{HW},G_2} \) correspond to the halfwidths of the integrand of the Hankel function \( I_f(\phi) \) along the deformed contour \( C_\delta(\delta = \frac{x}{\ell}) \), where \( x_l = x_w \) and 0, respectively. Thus, based on the PO and flat surface assumptions, one can consider only the source points located at the edges of the weak region instead of the whole source group, as far as the bandwidth of the integrand \( F(\phi) \) is concerned. Figure 2.8 shows the plot of \( e^{-\kappa_i \sigma_R^2} \), where \( i = 1 \) or 2, and their difference versus \( \phi_R \), where \( L_s = 10\lambda \) and \( x_n = 5000\lambda \). Using Eq. (2.82) with \( \tau = 10 \), it is found that \( \phi_{\text{HW},G_1} = 22.855^\circ \) and \( \phi_{\text{HW},G_2} = 1.02^\circ \). Note that \( \phi_{\text{HW},G_2} << \phi_{\text{HW},G_1} \), and the difference varies most rapidly in the neighborhood of the origin.
Figure 2.8: The plot of $e^{-\kappa \phi_R^2}$, where $i = 1$ or $2$, and their difference versus $\phi_R$. 
Let $\Delta \phi, \Delta \phi_{G_1}$, and $\Delta \phi_{G_2}$ be appropriate angular step sizes employed in sampling $F(\phi), G_1(\phi)$, and $G_2(\phi)$, respectively. Due to the Gaussian distribution of $G_i(\phi)$, it is found that

$$\Delta \phi_{G_i} = \frac{1}{22} \sqrt{\frac{\tau}{2\kappa_i}}. \quad (2.83)$$

where $i = 1$ or 2, yields accurate results (analogous to the derivation of analytical formula for $\Delta \phi$ of the standard 1-D NSA algorithm in Appendix D). If $F(\phi)$ is sampled using uniform sampling rate, $\Delta \phi$ must be set to be $\Delta \phi_{G_2}$ to obtain accurate results, and as the weak region increases, $\Delta \phi$ decreases inversely proportional to $\sqrt{K_{\text{n}}}$, which degrades the efficiency the 1-D NSA algorithm significantly. However, it should be emphasized that the contribution from $G_2(\phi)$ is much less than the contribution from $G_1(\phi)$ as can be seen from Eq. (2.76). To maintain the efficiency of the 1-D NSA algorithm, it is reasonable to consider only the contribution from $G_1(\phi)$ for $F(\phi)$; i.e. tradeoff the accuracy for the efficiency of the 1-D NSA algorithm, and this procedure is employed in the development of the original NSA algorithm [104].

One way to maintain the accuracy of the 1-D NSA algorithm without significantly degrading its efficiency is to decompose the very large weak region into more than one weak region as illustrated in Figure 2.9. Figure 2.9 shows the decomposition of the very large weak region of size $L_w$ into $M$ weak regions of size $L_{w,j}$, where $j = 1, \cdots, M$ and $L_w = \sum_{j=1}^{M} L_{w,j}$. It will be shown later that, for a given accuracy, typically $L_{w,j+1} \gg L_{w,j}$ and each weak region, except for the first weak region in the case of rough surfaces, requires the same number of plane waves. For the new
Figure 2.9: The decomposition of the very large weak region of size $L_w$ into $M$ weak regions of size $L_{w,j}$, where $j = 1, \ldots, M$ and $L_w = \sum_{j=1}^{M} L_{w,j}$.

proposed scheme shown in Figure 2.9, it is preferable to rewrite $F(\phi)$ as follows:

$$F(\phi) = \sum_{j=1}^{M} F_j(\phi)$$  \hspace{1cm} (2.84)

$$F_j(\phi) = -ie^{ikx_n \cos \phi} \int_{x_w,j}^{x_w,j-1} dx'e^{-i(kx + k \cos \phi)x'}.$$  \hspace{1cm} (2.85)

where $x_{w,0} = x_w$ and $x_{w,M} = 0$. Next consider the procedure in determining the size of each weak region $L_{w,j}$.

For illustration, consider the first weak region of size $L_{w,1}$ located between $x_{w,1}$ and $x_{w,0}$ as shown in Figure 2.9. As pointed out earlier in this section, considering only the source points located at the edges of the weak region of interest is sufficient to obtain information about the bandwidth of the entire source group in that weak region. In this case, the bandwidth of the integrand $F_1(\phi)$ associated with the first weak region
is governed by the edge points \( x_{w,1} \) and \( x_{w,0} \). Let \( BW_{w,1} \) be the bandwidth of the integrand \( I(\phi; R = x_n - x_{w,1}, \phi_s = 0) \) along \( C_\delta \), which is given by (see Eq. (2.82)):

\[
BW_{w,1} = 2 \sqrt{\frac{\tau}{k(x_n - x_{w,1})}} = 2 \sqrt{\frac{\tau}{k(L_s + L_{w,1})}}. \tag{2.86}
\]

If the edge point \( x_{w,1} \) is not too far-separated from the observation point \( x_n \), the angular step size \( \Delta \phi_1 \) for the flat surface case, employed in accurately sampling the integrand \( F_1(\phi) \), is equal to the angular step size associated with the integrand \( I(\phi; R = L_s, \phi_s = 0) \): i.e.

\[
\Delta \phi_1 = \frac{1}{22} \sqrt{\frac{\tau}{2kL_s}}. \tag{2.87}
\]

For a given accuracy, the largest size of the first weak region \( L_{w,1} \) for the flat surface case can be determined by setting \( \Delta \phi_1 \) to be a fraction of \( BW_{w,1} \): i.e.

\[
\Delta \phi_1 = \frac{BW_{w,1}}{\alpha}. \tag{2.88}
\]

where \( \alpha \) is a positive constant (\( \alpha \geq 1 \)), and solving for \( L_{w,1} \) resulting in the following formula for \( L_{w,1} \):

\[
L_{w,1} = \left[ 2 \left( \frac{44}{\alpha} \right)^2 - 1 \right] L_s. \tag{2.89}
\]

Note that, for a given \( L_s \), \( L_{w,1} \) only depends on \( \alpha^{-2} \): i.e. small \( \alpha \) yields large \( L_{w,1} \) with less accuracy and vice versa. Typically, \( \alpha = 8 \) yields quite accurate results, and \( L_{w,1} \) is found to be 59.5 \( L_s \). Using the same procedure as described above for other weak regions, it can be shown that the appropriate sizes of each weak region are given by the following equation:

\[
L_{w,j} = \left[ 2 \left( \frac{44}{\alpha} \right)^2 - 1 \right] \sum_{n=0}^{j-1} L_{w,n}, \quad j = 1, \ldots, M. \tag{2.90}
\]
where \( L_{w,0} \) is defined to be \( L_s \). From Eq. (2.90) with \( \alpha = 8 \), it is clear that \( L_{w,j} \gg L_{w,j-1} \); e.g., \( L_{w,1} = 59.5 \, L_s \), \( L_{w,2} = 3.599.75 \, L_s \), \( L_{w,3} = 217.784.875 \, L_s \), and \( L_{w,4} = 13.175.984.9375 \, L_s \). In addition, the contribution from the \( j^{th} \) weak region of interest decreases at the rate of \( \frac{1}{\sqrt{\kappa_{w,j}}} \) as predicted by Eq. (2.76), where \( \kappa_{w,j} = k \sum_{n=0}^{j-1} L_{w,n} \) and \( j = 1, \cdots, M \). Thus, for 1-D extremely large-scale QPS in most practical problems, only a few weak regions are necessary to obtain accurate results. It should be pointed out that the appropriate sizes of each weak region \( L_{w,j} \) in Eq. (2.90) are based on the PO and flat surface assumptions. As pointed out earlier in this section, the flat surface is a good assumption to determine \( L_{w,j} \) except for the size of the first weak region \( L_{w,1} \) since the saddle points of all possible pairs of an observation point and source points in the first weak region may not be located in the neighborhood of the origin (the saddle point of the flat surface). However, \( L_{w,1} \) can still be determined via Eq. (2.88) except that \( \Delta \theta_1 \) derived for the case of rough surfaces must be employed; i.e. using \( \Delta \theta_1 \) as given in Eq. (2.53) or Eq. (2.64) for the case of large \( \phi_{s,\text{max}} \). Thus, solving for \( L_{w,1} \) in Eq. (2.88) yields

\[
L_{w,1} = \frac{\tau}{k} \left( \frac{2}{\alpha \, \Delta \theta_1} \right)^2 - L_s. \tag{2.91}
\]

and the appropriate sizes of other weak regions for the case of rough surfaces are still given as in Eq. (2.90) except that \( j \) starts from 2 to \( M \). As in the case of the flat surface, only a few weak regions (or even one) are required for most practical problems to obtain accurate results. Up till now, the appropriate sizes of each weak region have been determined completely. Next, consider the formulas associated with the 1-D NSA parameters \( \delta_j \), \( \phi_{\text{max},j} \) and \( \Delta \phi_j \), where \( j = 1, \cdots, M \), employed in the 1-D "multilevel" NSA algorithm.
2.4.1 Analytical formula for $\delta_j$

For the first weak region, the tilt angle $\delta_1$ is given as in Eq. (2.44) (also see Eqs. (2.46) and (2.48) for the two limiting cases). For other weak regions, $\delta_j$ is set to be the tilt angle for the flat surface: i.e.

$$\delta_j = \frac{\pi}{4}, \quad j = 2, \cdots, M.$$  \hspace{1cm} (2.92)

due to the fact that the saddle points associated with these weak regions are in the neighborhood of the saddle point of the flat surface: i.e. the flat surface is a good approximation to obtain $\delta_j$.

2.4.2 Analytical formula for $\phi_{max,j}$

The maximum of the domain of integration of the first weak region $\phi_{max,1}$ is given as in Eq. (2.52). For other weak regions, based on the PO and flat surface assumptions, it can be shown that the associated integrands $F_j(\phi)$ (see Eq. (2.85)) along the deformed contour $C_\delta(\delta = \frac{\pi}{4})$ are distributed approximately as the superposition of wide and narrow Gaussians centered at the origin. In addition, the support of each $F_j(\phi)$ is governed by the source point located at the leading edge (closer to the observation point $\ell_n$ as shown in Figure 2.9), and thus $\phi_{max,j}$ is given by the following equation (see Eq. (2.82)):

$$\phi_{max, j} = \sqrt{\frac{\tau}{\kappa_{w,j}}}, \quad j = 2, \cdots, M.$$ \hspace{1cm} (2.93)

where $\kappa_{w,j} = k \sum_{n=0}^{j-1} L_{w,n}$. Note that $\phi_{max,j}$ decreases significantly for one weak region to the next since $L_{w,j+1} >> L_{w,j}$.

66
2.4.3 Analytical formula for $\Delta \phi_j$

For the first weak region, the appropriate angular step size $\Delta \phi_1$ is given as in Eq. (2.53) or Eq. (2.64) for the case of large $\phi_{s, \text{max}}$, and the number of plane waves $Q_{\text{TOT},1}$ is given as in Eq. (2.65). Due to the Gaussian-like distribution of $F_j(\phi)$ for other weak regions as discussed in Section 2.4.2, the angular step size $\Delta \phi_j$ is given as follows (see Eq. (2.83)):

$$\Delta \phi_j = \frac{1}{22} \sqrt{\frac{\tau}{2K_{u,j}}}.$$  \hfill (2.94)

Note that the number of plane waves $Q_{\text{TOT},j}$ required for each weak region (except the first weak region) is the same, and it is given as follows:

$$Q_{\text{TOT},j} = 2 \left[ \frac{\phi_{\text{max},j}}{\Delta \phi_j} \right] + 1 = 65. \quad j = 2, \ldots, M.$$  \hfill (2.95)

2.5 Numerical Results

In this section, the information of the CPU time is based on a Pentium II 333 MHz computer with 128 Mbytes RAM. To illustrate the computational efficiency and accuracy of the standard 1-D FB/NSA algorithm developed in Section 2.2 for impedance rough surfaces, consider a deterministic 1024\lambda IBC rough surface with $\epsilon_r = 38.0+i40.0$ illuminated by a 14 GHz tapered plane wave with the taper parameter $G = 5$ (see Eq. (2.8)) at an incident angle of 85°. With this surface size, incident fields are approximately 54.3 dB down at the surface edges, and thus the edge effects are negligible. The surface is a realization of a zero-mean Gaussian stochastic process described by a Gaussian spectrum: i.e.

$$W(k_j) = \frac{h^2 l}{2\sqrt{\pi}} e^{-\frac{s^2 k_j^2}{l}}.$$  \hfill (2.96)
as defined in Appendix A, where $W(k_f)$ represents the spectrum amplitude in $m^3$, $k_f$ represents the spatial wavenumber of the surface in rad./m, $h$ refers to the root-mean-square (rms) of surface heights in $m$, and $l$ is the surface correlation length in $m$. In this study, the parameters of the Gaussian spectrum $W(k_f)$ are chosen as follows: $l = 1.414\lambda$ and $h = 1.0\lambda$ having $z_{\text{min}} = -3.609\lambda$, $z_{\text{max}} = 3.393\lambda$, and $\Delta z_{\text{max}} = 7.002\lambda$. The surface is sampled with 8 points per $\lambda$ resulting in 8192 unknowns for the surface currents. The neighborhood distance of the 1-D FB/NSA method is chosen to be $128.0\lambda$ to obtain accurate results as discussed in Section 2.2, and the 1-D NSA parameters are chosen according to the formulas in Eqs. (2.31) to (2.33). The standard 1-D FB method is employed to compare with the standard 1-D FB/NSA method and requires 6 and 3 iterations to converge to within 1% accuracy for HH- and VV- polarizations, respectively. Its total CPU times for this example are equal to 23.67 and 11.65 minutes for HH- and VV- polarizations. The standard 1-D FB/NSA method requires the same number of iterations to converge within the same accuracy as in the 1-D FB method. However, its total CPU times are only 5.23 and 2.64 minutes for HH- and VV- polarizations, respectively. Thus, with the 1-D NSA algorithm, a CPU time reduction of 4.53 and 4.41 is achieved in this case for HH- and VV- polarizations, respectively. In addition, Figure 2.10 illustrates plots of the normalized bistatic RCS in dB versus the scattering angle ($\theta_s$) for both HH- and VV- polarizations, comparing between the standard 1-D FB and 1-D FB/NSA methods. Note that the normalized bistatic RCS for 1-D deterministic surfaces is defined as in Eq. (1.35) in Chapter 1, and for the tapered incident field $\Psi^t(\rho)$ given in Eq. (2.5),
the denominator of Eq. (1.35) can be shown to be

\[ 2\eta \int_{L_f} S_j \cdot \hat{n}_{in} \, ds = g \sqrt{\frac{\pi}{2}} \left[ 1 - \frac{1 + 2 \tan^2 \theta_j}{2(kg \cos \theta_j)^2} \right] |\Psi_0|^2 \cos \theta_i, \tag{2.97} \]

where \( \hat{n}_{in} = -\hat{z} \) is a unit vector pointing into the flat surface from the free space region. From the plots, the normalized bistatic radar cross sections obtained from both methods are in good agreement. Thus, it can be concluded that the 1-D NSA algorithm can indeed improve the computational efficiency of the standard 1-D FB algorithm without degrading its accuracy. It should be pointed out that as the surface size increases the standard 1-D FB/NSA algorithm is much more efficient than the standard 1-D FB algorithm since the former is an \( O(N_{tot}) \) algorithm while the latter is an \( O(N_{tot}^2) \) algorithm.

To investigate the validity of the 1-D NSA parameters derived in Sections 2.3 and 2.4, the 1-D NSA parameters are evaluated numerically for two different surface heights: i.e. Case A: \( \Delta z_{max} = 5.0\lambda \) and Case B: \( \Delta z_{max} = 15.0\lambda \). First, it is interesting to see the variations of the 1-D NSA parameters for the first weak region as a function of the size of the strong region \( L_s \). Figure 2.11 illustrates the 1-D NSA parameters versus \( L_s \) (in \( \lambda \)) for Case A. Figure 2.11 (a) plots the tilt angle \( \delta_1 \) (in deg.) calculated by using the exact and approximated formulas as described in Section 2.3.1 versus \( L_s \). From the plot, the \( \delta_1 \) based on the small \( \phi_{s,\text{max}} \) approximation (see Eqs. (2.44) and (2.46)) has some discrepancy with the exact \( \delta_1 \) (see Eqs. (2.42) to (2.44)) for small \( L_s \); i.e. large \( \phi_{s,\text{max}} \). Note that, for a given roughness, \( \phi_{s,\text{max}} \) decreases as \( L_s \) increases as shown in Figure 2.11 (c). As \( L_s \) increases, the \( \delta_1 \) based on the small \( \phi_{s,\text{max}} \) approximation and the exact \( \delta_1 \) approach the \( \frac{\pi}{4} \) tilt angle for the flat surface, and finally they are equal to \( \frac{\pi}{4} \text{ rad. (45°)} \) at \( L_s \approx 7.0\lambda \) (\( \phi_{s,\text{max}} = 35.535° \)). In addition, the \( \delta_1 \) based on the large \( \phi_{s,\text{max}} \) approximation (see Eqs. (2.44) and (2.48)) and the
Figure 2.10: Normalized bistatic RCS in dB comparing between the standard 1-D FB and 1-D FB/NSA methods: (a) HH polarization (b) VV polarization.
Figure 2.11: Plots of the 1-D NSA parameters in the first weak region versus the size of the strong region $L_s$ for the case of $\Delta z_{\text{max}} = 5.0\lambda$
exact $\delta_1$ are in good agreement for all values of $L_s$, even for moderate and large $L_s$, corresponding to moderate and small $\phi_{s,max}$, respectively. Thus, the $\delta_1$ based on the large $\phi_{s,max}$ approximation seems to provide a good estimate for the exact $\delta_1$. Figure 2.11 (b) exhibits a plot of the integration step size $\Delta \phi_1$ (in deg.) versus $L_s$ for both the "FFT" approach (see Eq. (2.53)) and the approximate formula as given in Eq. (2.64). Note that $\Delta \phi_1$ calculated by these approaches follow the same trend only when $L_s$ is small due to the fact that the approximate $\Delta \phi_1$ is derived based on the assumption that the phase component of the integrand $I(\phi)$ varies more rapidly than its magnitude, which is valid only when $\phi_{s,max}$ is large: i.e. $L_s$ is small. In general, $\Delta \phi_{FFT}$ provides better results, and should be employed in the 1-D NSA algorithm. Furthermore, Figure 2.11 (c) plots the maximum domain of integration $\phi_{max,1}$ and the maximum saddle point $\phi_{s,max}$ (in deg.) versus $L_s$. Note that $\phi_{max,1}$ is always greater than $\phi_{s,max}$ as expected, and both $\phi_{max,1}$ and $\phi_{s,max}$ decrease as $L_s$ increases. Figure 2.11 (d) manifests the plot of the number of plane waves $Q_{TOT,1}$ computed by using $\Delta \phi_{FFT}$ versus $L_s$. Note that $Q_{TOT,1}$ tends to decrease as $L_s$ increases: i.e. trade off between the weak region computation, which is related to the number of plane waves, and the strong region computation, which depends on the size of the strong region. In practice, this plot serves as a tool in selecting the appropriate $L_s$ to optimize the 1-D NSA algorithm.

In addition, Figure 2.12 (a) and (b) plots the size of the first region $L_{w,1}$ (see Eq. (2.91)) and the relative error (in %) in computing the 2-D scalar free space Green’s function in the angular spectral domain for $\Delta x = L_s$ (the shortest horizontal distance between the observation point and the source point in the first weak region) and $\Delta x = L_s + L_{w,1}$ (the farthest horizontal distance between the observation point
Figure 2.12: Plots of the size of the first region $L_{w.1}$ and the relative error (in %) in computing the free space Green's function in the angular spectral domain versus $L_s$ for the case of $\Delta z_{max} = 5.0\lambda$
and the source point in the first weak region) versus $L_s$ for Case A, respectively. Note that both $L_{w,1}$ and the relative error are computed using $\Delta \phi_{FFT}$. From Figure 2.12 (a), $L_{w,1}$ tends to increase as $L_s$ increases; i.e. a larger strong region yields a larger size of the first weak region as expected. From Figure 2.12 (b), the relative error for $\Delta x = L_s$ is almost constant (0.015 % on average), and thus it is implied that the derived 1-D NSA parameters work quite well. In addition, the relative error for $\Delta x = L_s + L_{w,1}$ is about 1.0 %. Note that as the horizontal distance $\Delta x$ increases, the relative error tends to increase, but the contribution from a source point to the observation point of interest tends to decrease as expected. It will be shown later in this section that by incorporating the second weak region into the NSA algorithm, the accuracy can be improved. Next, consider the 1-D NSA parameters for the rougher surface Case B.

Figure 2.13 plots the 1-D NSA parameters in the first weak region versus $L_s$ for Case B. Figure 2.13 (a) demonstrates the plot of $\delta_1$ (in deg.) versus $L_s$. As in Case A, the exact $\delta_1$ and the $\delta_1$ based on the large $\phi_{s,max}$ approximation are in very good agreement for all values of $L_s$. In addition, the $\delta_1$ based on the small $\phi_{s,max}$ approximation is slightly different from the exact $\delta_1$ for small $L_s$ (large $\phi_{s,max}$), and the former tends to approach the latter as $L_s$ increases. Figure 2.13 (b), (c) and (d) plots $\Delta \phi_1$ (in deg.), $\phi_{ax,1}$, and $Q_{TOT,1}$ versus $L_s$, respectively. Similar conclusions can be drawn as in Case A described earlier. Comparing between Figure 2.11 (d) and Figure 2.13 (d), it is observed that for a given $\phi_{s,max}$, $Q_{TOT,1}$ tends to increase as $\Delta z_{max}$ increases; i.e. as the roughness increases, more plane waves are required to obtain the same accuracy. Thus, $Q_{TOT,1}$ depends on both $\phi_{s,max}$ and $\Delta z_{max}$ in general. Furthermore, Figure 2.14 (a) and (b) plots $L_{w,1}$ and the relative error (in
Figure 2.13: Plots of the 1-D NSA parameters in the first weak region versus $L_s$ for the case of $\Delta z_{max} = 15.0\lambda$. 
Figure 2.14: Plots of $L_{w,1}$ and the relative error (in %) in computing the free space Green's function in the angular spectral domain versus $L_s$ for the case of $\Delta z_{\text{max}} = 15.0\lambda$.
for $\Delta x = L_s$ and $\Delta x = L_s + L_{w,1}$ versus $L_s$ for Case B, respectively. As in Case A, $L_{w,1}$ tends to increase as $L_s$ increases. Comparing $L_{w,1}$ between Case A and Case B, one can see that for a given $s_{max}$, $L_{w,1}$ seems to increase as $\Delta z_{max}$ increases. This is due to the fact that $L_{w,1}$ depends on $s_{max}$ (see Eq. (2.91)), and $s_{max}$ tends to decrease as the roughness increases. From Figure 2.14 (b), the average relative errors for $\Delta x = L_s$ and $\Delta x = L_s + L_{w,1}$ are about 0.025% and 1.0%, respectively. Thus, the derived 1-D NSA parameters work reasonably well for both cases. It should be pointed out that for a very flat surface case, the derived 1-D NSA parameters still work very well. Note that the tilt angle $\delta_1$ for the first weak region for this case is usually equal to $\pi/4$ rad. as expected.

Finally, it is of interest to see the accuracy improvement of the 1-D NSA algorithm by incorporating the multilevel approach. Figure 2.15 (a) and (b) plots the relative error (in %) using the one-level and two-level NSA algorithms versus the horizontal distance between source and observation points $\Delta x = x - x'$ (in $\lambda$) for both Case A and Case B, respectively. It should be emphasized that for the one-level NSA algorithm, the NSA parameters are chosen based on the strong distance $L_s$ and the maximum surface height variation $\Delta z_{max}$; previous plots examined variations in these parameters with $L_s$. For the two-level NSA algorithm, however, the first weak region employs the same NSA parameters derived in the one-level NSA algorithm for a fixed $L_s$, and uses these same parameters as the source to observation distance $\Delta x$ increases within the first weak region. In addition, the second weak region employs the NSA parameters chosen based on the distance $L_s + L_{w,1}$ (assuming that $\Delta z_{max} \ll L_s + L_{w,1}$) as described in the previous section, and uses these same parameters as $\Delta x$ increases, where $\Delta x \geq L_s + L_{w,1}$. From the plots in Figure 2.15, the minimum
Figure 2.15: Plots of the relative error (in %) from the one-level and two-level NSA algorithms versus $\Delta x = x - x^*$ (in $\lambda$): (a) $\Delta z_{max} = 5.0\lambda$ (b) $\Delta z_{max} = 15.0\lambda$
\( \Delta x \) is equal to \( L_s \) in the 1-D NSA algorithm. Note that Case A and Case B employ 
\( L_s = 1.5 \lambda \left( \phi_{s,\text{max}} = 73.281^\circ \right) \) and \( L_s = 4.0 \lambda \left( \phi_{s,\text{max}} = 75.057^\circ \right) \), respectively. The 1-D NSA parameters for Case A are given as follows: \( \delta_1 = 21.429^\circ \), \( \phi_{\text{max},1} = 39.553^\circ \), \( \Delta \phi_1 = 0.974^\circ \), \( L_{w,1} = 339.7 \lambda \), \( \delta_2 = \frac{\pi}{4} \) rad., \( \phi_{\text{max},2} = 3.913^\circ \), \( \Delta \phi_2 = 0.126^\circ \), and \( L_{w,2} = 20.302.6 \lambda \). For Case B, the 1-D NSA parameters are given as follows: \( \delta_1 = 7.276^\circ \), \( \phi_{\text{max},1} = \frac{\pi}{2} \) rad., \( \Delta \phi_1 = 0.13^\circ \), \( L_{w,1} = 171\theta.2 \lambda \), \( \delta_2 = \frac{\pi}{4} \) rad., \( \phi_{\text{max},2} = 1.73^\circ \), \( \Delta \phi_2 = 0.055^\circ \), and \( L_{w,2} = 103.777.7 \lambda \). From both plots, it is obvious that incorporating the second weak region yields appreciable reduction of the relative error. Note that as the surface size increases, additional weak regions are required to maintain the accuracy. However, as pointed out earlier, \( L_{w,j} \gg L_{w,j-1} \), and the contribution from the \( j \)th weak region of interest decreases at the rate of \( \frac{1}{\sqrt{\kappa_{w,j}}} \) as predicted by Eq. (2.76) based on the PO and flat surface assumptions, where \( \kappa_{w,j} = k \sum_{n=0}^{j-1} L_{w,n} \) and \( j = 1, \ldots, M \). Thus, only a few weak regions (or even one) are required for most practical problems to obtain accurate results.

2.6 A Summary and Conclusions

In this chapter, the formulation of the FB/NSA method for 1-D impedance rough surfaces based on the original paper [104, 105] is reviewed. It is found out that both recursive property of the complex scalar radiation function \( F(p, k, \phi) \) and appropriate contour deformation are indispensable to make the NSA method an \( O(N_{\text{tot}}) \) algorithm for both computational and memory requirements. In addition, the original NSA algorithm has been generalized for the fast computation of radiation/scattering from 1-D extremely large-scale quasi-planar structures. New analytical formulas associated with the 1-D NSA parameters for an arbitrary value of \( \phi_{s,\text{max}} \) are also presented.
resulting in more flexibility in selecting $L_s$ to compromise between the computation of the contributions of strong and weak regions. The plot of the number of plane waves $Q_{TOT}$ versus the strong region $L_s$ serves as a tool for a user to properly select $L_s$ to optimize the 1-D NSA algorithm. Numerical results show that the 1-D NSA algorithm can improve the computational efficiency of the standard 1-D FB algorithm without degrading its accuracy. In addition, numerical results also illustrate that the derived 1-D NSA parameters work well for both small and very rough surfaces. Furthermore, a theoretical study based on the PO and flat surface approximations leads to the "multilevel" concept to improve the accuracy of the original NSA algorithm in the case of 1-D extremely large-scale QPS. It is found that only a few weak regions or even one are required for most practical problems to obtain the desired accuracy, but the accuracy of the original 1-D NSA algorithm indeed can be improved when incorporating the "multilevel" algorithm.
CHAPTER 3

Numerical Studies of Backscattering Enhancement for One-Dimensional Random Rough Surfaces

3.1 Introduction

Enhanced backscattering of EM waves from random rough surfaces was first observed experimentally by O'Donnell et al. [9, 10]. This phenomenon is associated with the appearance of a well-defined peak in the backscattering direction of the intensity of the incoherently scattered component of the EM field, and it is a direct consequence of the coherent interference of multiple scattered waves which interfere constructively in the backscattering direction. Enhanced backscattering has been observed experimentally for several rough surface types, and is usually divided into the cases of large and small height variations due to different mechanisms causing the backscattering peak in each case [4]. One such type involves surfaces with relatively large heights and slopes for which multiple surface scattering dominates [4]: i.e. $h \geq 0.5\lambda$ and $\sigma_s \geq 0.5$, where $h$ and $\sigma_s$ are the root-mean-square (rms) surface height and the rms surface slope, respectively, and $\lambda$ is the EM wavelength in free space. Two classical analytical methods, the Kirchhoff approximation (KA), or the physical optics (PO), and the small-perturbation method (SPM) [23, 25], are not applicable for this case.
due to the small slope limitations of these theories. Although other approximate theories such as higher-order KA [54, 55], integral equation model (IEM) [26, 52, 53] and full wave [49]-[51] theories, have been developed to explain the backscattering enhancement phenomenon, they remain restricted in their domain of validity. The limitations of these approximate analytical methods and more powerful modern computers have increased interest in efficient numerical techniques for large-height and large-slope surfaces based on Monte-Carlo simulations of EM waves scattered from random rough surfaces [16]-[22].

Numerical models are usually computationally expensive for general use in random rough surface scattering problems due to the fact that Monte-Carlo simulations and relatively large-scale surfaces (especially as the angle of incidence \( \theta_i \) increases toward grazing angle) are required. As illustrated in Chapter 2, the 1-D FB/NSA algorithm is an extremely efficient and accurate \( \mathcal{O}(N_{tot}) \) method for the computation of scattering from 1-D rough surfaces, where \( N_{tot} \) is the total number of surface unknowns. In addition, the method is still very efficient for moderately-rough large-scale surfaces. Thus, the 1-D FB/NSA algorithm is a good candidate to be employed in performing numerical studies of backscattering enhancement at low grazing angles (LGA).

Numerical studies at LGA are typically difficult due to edge scattering and angular resolution problems. Edge scattering is caused by the finite size of surface realizations discretized in a numerical study. These surface edge effects typically can be minimized through use of a “tapered beam” incident field [58] which reduces incident fields on surface edges, however careful tests are required to insure the validity of numerical results: i.e. results under tapered beam illumination model those under the plane wave illumination usually considered in analytical models. Even without
edge scattering effects, use of a finite size surface reduces the angular resolution of scattered fields compared to the infinite size surface modeled in analytical studies due to the fact that the “spatial window” applied to create a finite surface results in an angular averaging of infinite surface scattered fields to produce fields for the finite surface. It is found that the appropriate tapered beam spot size on the surface profile to minimize both problems discussed above must increase dramatically as low grazing incidence is approached, resulting in large surface profiles which increase computational requirements. Thus, appropriate surface sizes and taper parameters of the incident field must be chosen appropriately by comparing numerical results with analytical results as will be discussed in detail in Section 3.3.

In this study, the 1-D standard FB/NSA algorithm for the IBC case developed in Chapter 2 is employed to perform Monte-Carlo simulations of EM scattering from 1-D random rough surfaces exhibiting backscattering enhancement at 14 GHz ($\lambda = 0.02143 \text{m}$). Throughout this chapter, the parameter $a_{\text{max}}$, employed in the 1-D standard FB/NSA algorithm (see Eq. (2.34) in Chapter 2), is set to be equal to 20.0 as in [104,105]. Variations in the level and angular width of backscattering enhancement with surface statistics, surface material (PEC or impedance), polarization, and incident angle $\theta_i$ are considered. In this study, surfaces with relatively large heights and large slopes ($h \geq 0.5 \lambda$ and $\sigma_s \geq 0.5$) are considered. It should be pointed out that previous Monte-Carlo simulations of backscattering enhancement considered only incident angles near normal incidence [17]–[19], [22,98,99]. Thus, with the 1-D FB/NSA method, incident angles ranging from normal incidence to low-grazing incidence ($\theta_i = 0.1^\circ$ to $85^\circ$) are considerable in this study. To accelerate the intensive computation for this numerical study, parallel computing techniques are
incorporated into the 1-D FB/NSA method, primarily to perform Monte-Carlo simulations in parallel. In addition, to provide a physical interpretation concerning to the multiple-scattering effects associated with the backscattering enhancement, comparisons of numerical results and a single Neumann iteration are also made. Note that EM scattering from 1-D rough surfaces does not account for the cross-polarization effect, however, its less computational complexity compared to the computation of two-dimensional (2-D) rough surface scattering allow us to consider the backscattering enhancement phenomenon at LGAs.

This chapter is organized as follows. Section 3.2 briefly describes parallel computing techniques, and Section 3.3 provides comparisons of numerical simulation results with predictions of the SPM and the KA to verify the accuracy of the 1-D FB/NSA method for LGA problems. Numerical results and discussions of this study are illustrated in Section 3.4. A physical interpretation of the backscattering enhancement phenomenon based on a single Neumann iteration and further discussions are shown in Section 3.5. Finally, Section 3.6 provides a summary and conclusions of this study.

3.2 Parallel Processing

Briefly, the parallel computer is a collection of several workstations, networked through a high performance communication system to allow groups of nodes to operate in combination as a parallel processor. Software libraries are employed to implement inter-process communications using simple routine calls. Parallel computing techniques are incorporated into the 1-D FB/NSA method, primarily to perform Monte-Carlo simulations in parallel. Monte-Carlo simulations for 1-D rough surfaces in this chapter were performed on two supercomputers, the CRAY T3E and
the SGI Origin 2000, located in Ohio and Illinois, respectively. The CRAY T3E at the Ohio Supercomputing Center is a stand-alone system and it has a total of 136 processing elements (PE's), where each PE includes a 300 MHz DEC 21164 CPU with 600 MFLOPS and 16 Mwords (128 Mbytes) of memory (see more detail at http://www.osc.edu/). The SGI Origin 2000 at the National Computational Science Alliance (NCSA) is a scalable shared memory multiprocessor (S2MP). (see more detail at http://www.ncsa.uiuc.edu/). The parallel algorithm uses the message passing interface (MPI) for task and processor controls. In the next section, comparisons of numerical simulation results with predictions of classical analytical methods are illustrated.

3.3 Comparisons of Numerical Results with the SPM and the KA

In performing Monte-Carlo simulations using the 1-D FB/NSA method, surfaces of finite size and taper parameters of the incident field must be chosen appropriately to avoid angular averaging and surface edge effects. To achieve these, the SPM and KA are used to validate numerical results obtained from the 1-D FB/NSA method with the IBC formulation as described in Section 2.2. In this study, 1-D surfaces to be used in Monte-Carlo simulations are modeled as realizations of a zero-mean Gaussian stochastic process, and the surface spectrum is chosen to be a Gaussian spectrum \( W(k_f) \) (in \( m^3 \)) as previously defined in Eq. (2.96) in Chapter 2. Note that \( k_f \) represents the spatial wavenumber of the surface in rad./m. Let \( h \) denote the root-mean-square (rms) of surface heights in \( m \) and \( l \) denote the surface correlation length in \( m \). In case of a Gaussian surface with a Gaussian spectrum, the slope \( s \) is a Gaussian variate with rms fluctuation \( \sigma_s = \frac{\sqrt{2}h}{l} \) [133]. In this chapter, results will
be presented in terms of the normalized incoherent bistatic radar cross section $\sigma_{a,d}$ in the plane of incidence, defined for a scattered wave in $\alpha$-polarization and an incident wave in $\beta$-polarization, as given in Eq. 1.37 in Section 1.3. For the tapered incident field $\Psi^i(\rho)$ given in Eq. (2.5) in Chapter 2, the denominator of Eq. (1.37) is given as in Eq. (2.97). Note that no cross-polarized scattering exists for 1-D surfaces: i.e. $\sigma_{a,d} = 0$ for $\alpha \neq \beta$.

The normalized incoherent co-polarized bistatic radar cross sections $\sigma_{aa}^{i}$ computed using the first-order SPM and the first-order (conventional) KA are given below. For the first order SPM. $\sigma_{aa}^{i}$ of a 1-D dielectric rough surface in the plane of incidence is [134]:

$$\sigma_{aa}^{i}(\theta_s, \theta_i) = 4k^3 \cos^2 \theta_s \cos^2 \theta_i f_{aa} W(kzd). \quad (3.1)$$

where

$$f_{HH} = \left| \frac{k_i^2 - k^2}{(k_{ix} + k_{iz})(k_{iiz} + k_{iz})} \right|^2 \quad (3.2)$$

$$f_{VV} = \frac{(k_i^2 - k^2)(k_i^2 k^2 \sin \theta_s \sin \theta_i - k^2 k_{ix} k_{iiz})}{(k_i^2 k_z + k^2 k_{iiz})(k_i^2 k_{iiz} + k^2 k_{iz})} \right|^2 \quad (3.3)$$

$$k_{ix} = k \cos \theta_i, \ k_{iiz} = k \sin \theta_i, \ k_x = k \sin \theta_s, \ k_z = k \cos \theta_s, \ k_{iiz} = (k_i^2 - k_{ix}^2)^{1/2}, \ kzd = k_{iiz} - k_z. \ and$$

$$k_{iiz} = \begin{cases} \sqrt{k_i^2 - k_x^2} & \text{for } k_x \leq k_i \\ i \sqrt{k_i^2 - k_x^2} & \text{for } k_x > k_i \end{cases} \quad (3.4)$$

Note that the subscripts H and V denote horizontal and vertical polarizations respectively, and $\theta_i$ and $\theta_s$ denote the incident and scattering angles, respectively. For the first-order (conventional) KA, $\sigma_{aa}^{i}(\theta_s, \theta_i)$ of a 1-D PEC rough surface with Gaussian
power spectrum $W(k_f)$ in the plane of incidence is expressed as:

$$
\sigma_{HH}^i(\theta_x, \theta_y) = \frac{kl}{2\sqrt{\pi}} \left| 1 + \cos(\theta_x + \theta_y) \right|^2 e^{-(k_{zd})^2} e^{-\frac{(k_{zd})^2}{4m}} \sum_{m=1}^{\infty} \frac{(k_{zd})^{2m}}{m!},
$$

(3.5)

where $k_{zd} = k_{ix} - k_x$ and $k_{zd} = -(k_{iz} + k_z)$.

In this study, the largest angle of incidence measured from the normal to the mean plane of the surface is equal to 85°, and the appropriate surface length $L_x$ chosen for $\theta_i = 85°$ is fixed for all angles of incidence. The incident field $\Psi^i(\rho)$ is the Thorsos tapered Gaussian-beam incident field [58] as previously defined in Eqs. (2.3) and (2.5) in Chapter 2. Note that the taper parameter $g$ should be chosen appropriately for a given incident angle. Using Eq. (2.8) in Chapter 2 with $A = 6.64$ (reasonable value for $\theta_i = 85°$) and $G = 5$, the conditions on $L_x$ and $G$ are: $L_x > 982.5\lambda$ and $g > 196.5\lambda$.

Thus, an overall surface size of $L_x = 1024\lambda = 21.944 m$ (at 14 GHz) is employed in the simulation so that incident fields are approximately 54.3 dB down at the surface edges. According to [74] (see Eq. (6) in [74]), it is noted that this value of $g$ yields an approximate 3-dB beamwidth of 1.212°, which should provide reasonable accuracy for backscattering predictions, as shown in the following tests.

For the first test, consider Gaussian surfaces with small height and small slope fluctuations: $h = 0.014\lambda, l = 0.397\lambda$, and $\sigma_s = 0.05$. In this case, the first-order SPM should yield accurate results. In the 1-D FB/NSA method with the IBC formulation, surfaces of size 1024$\lambda$ are used, and they are sampled with 8 unknowns per $\lambda$ resulting in 8192 unknowns. The neighborhood distance $L_s$ is chosen to be $8\lambda$ to achieve desired accuracy as discussed in Section 2.2; i.e. the specified tolerances for $x - x' = L_s$ and $x - x' = L_x$ are set to be 0.5 % and 2.0 %, respectively. In addition, the 1-D FB/NSA parameters employed throughout this chapter are also given.

87
Figure 3.1: Comparisons between Monte-Carlo normalized incoherent backscattering radar cross sections, computed via the 1-D FB/NSA, and analytical results computed via the first-order SPM at 14 GHz with $G = 5$ for both HH and VV polarizations and for different values of $\varepsilon_{r1}$: (a) PEC surfaces (b) $\varepsilon_{r1} = 38.0 + i40.0$ (c) $\varepsilon_{r1} = 10.0 + i10.0$ (d) $\varepsilon_{r1} = 6.5 + i1.0$. Gaussian surface statistics for this case are: $\mu = 0.014\lambda, l = 0.397\lambda, \text{and } \sigma_z = 0.05$. 
in Section 2.2. On average, the 1-D FB/NSA method converges within 8 iterations for HH (TE) polarization and within 2 iterations for VV (TM) polarization, where TE and TM stand for transverse electric and transverse magnetic, respectively. The number of surface realizations used in Monte-Carlo simulation in this case is equal to 64, indicated as "(64)" in the legend of each plot. Figure 3.1 illustrates the comparisons between Monte-Carlo normalized incoherent backscattering radar cross sections, computed via the 1-D FB/NSA, and analytical results computed via the first-order SPM at 14 GHz with \( G = 5 \) for both HH and VV polarizations and for different values of \( \epsilon_r \): PEC (\(|\epsilon_r| \to \infty\), 38.0 + \( i40.0 \), \( 10.0 + i10.0 \) and \( 6.5 + i1.0 \). From these plots, the VV backscattering radar cross sections are greater than or equal to the HH backscattering radar cross sections as expected from the SPM theory. In addition, the normalized incoherent backscattering radar cross section for both polarizations tends to decrease as \( \epsilon_r \) decreases (i.e., surface impedance increases). It is also noted that the results of the 1-D FB/NSA method and the SPM are in very good agreement, even at \( \theta_i = 85^\circ \), except for Figure 3.1 (d) at high angles of incidence for HH polarization. The reason for this discrepancy in the case of \( \epsilon_r = 6.5 + i1.0 \) may come from the fact that the impedance boundary condition as shown in Eq. (2.9) is not a good approximation for the small value of \( \text{Re}[\epsilon_r] \) with small loss. Thus, in the next section, variations in the level and angular width of backscattering enhancement with PEC surfaces and impedance surfaces with \( \epsilon_r = 38.0 + i40.0 \) and \( 10.0 + i10.0 \) will be considered only. Note that, with the surface of size 1024\( \lambda \) and \( G = 5 \), the 1-D FB/NSA method seems to yield accurate results compared to the SPM prediction.

For another test, consider PEC Gaussian surfaces with small slope fluctuation and moderately rough: \( h = 0.20\lambda, l = 5.657\lambda \), and \( \sigma_s = 0.05 \), illuminated by the tapered
Figure 3.2: Comparisons between Monte-Carlo normalized incoherent bistatic radar cross sections, computed via the 1-D FB/NSA and Monte-Carlo PO, and analytical results computed via the first-order KA (PO) at 14 GHz with $\theta_i = 50^\circ$ and $G = 5$ for both HH and VV polarizations and for PEC rough surfaces. Gaussian surface statistics for this case are: $h = 0.20\lambda$, $l = 5.657\lambda$, and $\sigma_s = 0.05$. 
Figure 3.3: Comparisons between Monte-Carlo normalized incoherent bistatic radar cross sections in the *specular* direction, computed via the 1-D FB/NSA, and analytical results computed via the first-order KA (PO) at 14 GHz with $G = 5$ for both HH and VV polarizations and for PEC rough surfaces. Gaussian surface statistics for this case are: $h = 0.20\lambda, l = 5.657\lambda$, and $\sigma_z = 0.05$. 
incident field with $G = 5$ at $\theta_i = 50^\circ$. In this case, the first-order KA (PO) should yield accurate normalized RCSs, especially near the specular direction. In the 1-D FB/NSA method with the IBC formulation, the same surface size and sampling rate as in the first test are used. The neighborhood distance $L_s$ is chosen to be $8\lambda$ in this case to achieve desired accuracy. The number of surface realizations used in Monte-Carlo simulations is equal to 256. Figure 3.2 illustrates the comparisons between Monte-Carlo normalized incoherent bistatic radar cross sections, computed via the 1-D FB/NSA and Monte-Carlo PO (using tapered incident field), and analytical results computed via the PO at 14 GHz for both HH and VV polarizations. Note that the PO theory predicts the same normalized incoherent bistatic RCSs for both polarizations. From the plot, results for all methods are in very good agreement near the specular direction, as predicted by the PO. Note that the 1-D FB/NSA predicts almost the same normalized RCSs for both HH and VV polarizations near the specular direction. Away from the specular direction, the normalized HH and VV RCSs are distinct, and the normalized VV RCS tends to be greater than the normalized HH RCS as expected. It is observed that results from Monte-Carlo PO and analytical PO are in very good agreement. In addition, for the same surface statistics and incident field as considered above. Figure 3.3 shows the comparisons between Monte-Carlo normalized incoherent bistatic radar cross sections in the specular direction computed via the 1-D FB/NSA and analytical results computed via the PO, as $\theta_i$ changes from normal incidence to grazing incidence. From the plot, results from the PO and the 1-D FB/NSA method are in very good agreement, except for large angles of incidence (near grazing incidence). It is known that the PO does not provide accurate results for large $\theta_i$ due to increases in the multiple-scattering effects at low grazing angles. Note also that
the 1-D FB/NSA provides identical results for both HH and VV polarizations except a few dB difference at \( \theta_i = 85^\circ \).

From these tests, with the surface of size 1024\( \lambda \) sampling with 8 unknowns per \( \lambda \) and the taper parameter \( G = 5 \), the 1-D FB/NSA method yields very accurate results compared to the SPM and the KA. Thus, the following parameters, \( L_x = 1024\lambda \) and \( G = 5 \), will be used throughout this numerical study of the backscattering enhancement phenomenon using 1-D random rough surfaces. The numerical results of this study and discussions are presented in the next section.

### 3.4 Numerical Results and Discussions

In this study, variations in the level and angular width of backscattering enhancement, using 1-D 1024\( \lambda \) random rough surfaces and \( G = 5 \), with surface statistics, surface material (PEC or impedance), polarization (HH or VV), and incident angle \( \theta_i \) are considered. Four different 1-D Gaussian surfaces (with Gaussian spectrum \( W(k_f) \) as defined in Eq. (2.96)) with relatively large heights and large slopes \( (h \geq 0.5\lambda \) and \( \sigma_s \geq 0.5 \) ) are considered (due to the requirement of the existence of the backscattering enhancement for this surface type [4]) as follows:

- Surface # 1: \( h = 0.5\lambda \), \( l = 1.414\lambda \) and \( \sigma_s = 0.5 \)
- Surface # 2: \( h = 1.0\lambda \), \( l = 2.828\lambda \) and \( \sigma_s = 0.5 \)
- Surface # 3: \( h = 0.5\lambda \), \( l = 0.707\lambda \) and \( \sigma_s = 1.0 \)
- Surface # 4: \( h = 1.0\lambda \), \( l = 1.414\lambda \) and \( \sigma_s = 1.0 \)

Note that Surface # 1 is the smoothest surface compared to other surfaces considered above since its \( h \) and \( \sigma_s \) are the smallest, whereas Surface # 4 is the roughest surface.
since its $h$ and $\sigma_z$ are the largest. As discussed in the previous section, PEC surfaces and impedance surfaces with $\epsilon_{r1} = 38.0 + i40.0$ and $10.0 + i10.0$ are employed only in studying the variations in the level and angular width of backscattering enhancement with surface material. To study the effects of the incident angle on the backscattering enhancement, the following values of $\theta_i$ are used in studying the variations with incident angle: $0.1^\circ, 20^\circ, 50^\circ, 60^\circ, 70^\circ,$ and $85^\circ$. It should be pointed out that the impedance boundary condition employed in the surface integral formulation in Chapter 2 (see Eq. (2.9)) still yields quite accurate numerical results even for the case of the smallest magnitude of $\epsilon_{r1}$ considered in this study: i.e. $\epsilon_{r1} = 10.0 + i10.0$. This is confirmed by obtaining very good agreement of the normalized radar cross sections obtained from the impedance surface model and the lossy dielectric surface model for the roughest surface (Surface # 4) and for all incident angles of interest. From numerical experiments on the roughest surface (Surface # 4), numerical results using 8 unknowns per $\lambda$ and 16 unknowns per $\lambda$ are in good agreement. Thus, in this study, surfaces are sampled with 8 unknowns per $\lambda$ resulting in 8192 unknowns. In this section, the vertical solid line in figures corresponds to the backscattering direction of interest. First, consider the variations with surface statistics.

In the 1-D FB/NSA method, the neighborhood distance $L_s$ depending on $h$ must be chosen appropriately as discussed in Section 2.2. From numerical studies, Surface # 1 and Surface # 3 ($h = 0.5\lambda$) require $L_s = 42.0\lambda (N_s = 336)$, whereas Surface # 2 and Surface # 4 ($h = 1.0\lambda$) require $L_s = 128.0\lambda (N_s = 1024)$. In addition, when performing Monte-Carlo simulations, an appropriate number of surface realizations must be used to obtain accurate statistics of scattered fields: e.g., the coherent and incoherent powers, especially in case of the surfaces with relatively large heights and
Figure 3.4: Convergence study of the normalized HH incoherent bistatic RCS in Monte-Carlo simulations with the 1-D FB/NSA method at $\theta_i = 20^\circ$ for two different PEC Gaussian surfaces: (a) Surface # 3 (b) Surface # 4.
large slopes employed in this study. Note that an appropriate number of surface realizations depend on surface statistics; i.e. as surfaces get rougher, the required number of surface realizations tends to increase. Figure 3.4 shows the convergence study of the normalized HH incoherent bistatic RCS in Monte-Carlo simulations with the 1-D FB/NSA method at $\theta_i = 20^\circ$ for two different PEC Gaussian surfaces: i.e. Surface # 3 and Surface # 4. In Figure 3.4 (a), it is noted that 128 surface realizations of Surface # 3 seems to provide reasonable normalized HH incoherent bistatic RCS. For Surface # 4 as shown in Figure 3.4 (b), 128 surface realizations are not enough to provide convergent results especially near the backscattering direction of interest ($\theta_s = -20^\circ$) as shown by the vertical line. Thus, 256 surface realizations are employed in Monte-Carlo simulations for Surface # 4. In addition, from numerical tests for Surface # 1 and Surface # 2, it seems that 128 surface realizations are enough to provide reasonable results. In conclusion, only Surface # 4 employs 256 surface realizations, whereas other surfaces use 128 surface realizations in Monte-Carlo simulations.

Prior to discussing the variations in the characteristics of backscattering enhancement with parameters of interest, it is interesting to consider the statistical distribution of some scattering quantities related to scattered fields. Let $U_{\alpha\beta}(\theta_s, \theta_i)$ be a random process associated with the normalized incoherent bistatic RCS $\sigma_{\alpha\beta}^i(\theta_s, \theta_i)$ (see Eq. (1.37) in Chapter 1; i.e.

$$U_{\alpha\beta}(\theta_s, \theta_i) = \lim_{\rho \to \infty} \frac{\rho |E^s_{\alpha\beta} - \langle E^s_{\alpha\beta} \rangle|^2 \cos \theta_i}{2\eta \int_L \mathbf{S}_j \cdot \hat{n}_{in} \, ds} \tag{3.6}$$

where the above parameters are defined as in Eq. (1.37). Note that the statistical average of $U_{\alpha\beta}(\theta_s, \theta_i)$ is equal to $\sigma_{\alpha\beta}^i(\theta_s, \theta_i)$: i.e.

$$\sigma_{\alpha\beta}^i(\theta_s, \theta_i) = \langle U_{\alpha\beta}(\theta_s, \theta_i) \rangle \tag{3.7}$$
where the \( \langle \cdot \rangle \) notation above indicates an ensemble average over realizations of a random variable. In addition, let define a random process \( V_{a,j}(\theta_s, \theta_i) \) as

\[
V_{a,j} = \sqrt{U_{a,j}}.
\]

(3.8)

The random processes \( U_{a,j}(\theta_s, \theta_i) \) and \( V_{a,j}(\theta_s, \theta_i) \) may be thought as the “normalized power process” and “normalized field magnitude process”, respectively. For given incident and scattering angles, numerical results based on Monte-Carlo simulations in this section show that the mean and standard deviation of \( U_{a,j}(\theta_s, \theta_i) \) are almost identical, and its histogram looks like an exponential distribution. Thus, the probability density function (pdf) of \( U_{a,j}(\theta_s, \theta_i) \), denoted as \( p_{U_{a,j}}(u) \), for fixed \( \theta_i \) and \( \theta_s \) tends to be an exponential pdf: i.e.

\[
p_{U_{a,j}}(u) = \begin{cases} \frac{1}{u_0} e^{-\frac{u}{u_0}} & u \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]

(3.9)

where \( u_0 \) denotes the mean of \( U_{a,j} \); i.e. \( u_0 = \langle U_{a,j} \rangle \), and the standard deviation of \( U_{a,j} \), denoted as \( \sigma_{U} \), can be shown to be equal to \( u_0 \). From Eqs. (3.8) and (3.9) and the well-known fact from the probability theory [135], the pdf of \( V_{a,j}(\theta_s, \theta_i) \) for fixed incident and scattering angles tends to be a Rayleigh pdf: i.e.

\[
p_{V_{a,j}}(v) = \begin{cases} \frac{v}{\nu_0^2} e^{-\frac{v^2}{2\nu_0^2}} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]

(3.10)

where \( \nu_0 \) is related to the mean of \( V_{a,j} \), denoted as \( \langle V_{a,j} \rangle \), as follows:

\[
\nu_0 = \frac{2}{\pi} \langle V_{a,j} \rangle.
\]

(3.11)

The rest of this section will investigate the variations in the level and angular width of backscattering enhancement with surface statistics, surface material, polarization, and incident angle \( \theta_i \). For convenience in discussion, define the angular regions \(-90^\circ \leq \)
\( \theta_s \leq 0^\circ \) and \( 0^\circ \leq \theta_s \leq 90^\circ \) to be the “forward” and “backward” angular regions, respectively.

### 3.4.1 Variations in the Level and Angular Width of Backscattering Enhancement with Surface Statistics

Consider a 1-D PEC rough surface illuminated by a tapered incident field with \( G = 5 \) at \( \theta_i = 20^\circ \). Using the 1-D FB method, Figure 3.5 illustrates the plots of the normalized HH incoherent bistatic RCS \( \sigma_{HH}(\theta_s, \theta_i) \) versus \( \theta_s \) for Surfaces # 1 to # 4. From the plots, results for Surface # 1 and Surface # 2 (both have \( \sigma_s = 0.5 \)), as shown in Figure 3.5 (a) and (b), respectively, do not exhibit a well-defined peak in the backscattering direction \( (\theta_s = -20^\circ) \) as shown by the vertical line, and their backscattering and specular levels are nearly equal (about 0.5). On the other hand, results for Surface # 3 and Surface # 4 (both have \( \sigma_s = 1.0 \)), as shown in Figure 3.5 (c) and (d), respectively, exhibit a well-defined peak in the backscattering direction \( (\theta_s = -20^\circ) \), and the backscattering level is higher than the specular level. Note that the backscattering peak level of Surface # 3 (about 0.8) is slightly higher than the backscattering peak level of Surface # 4 (about 0.7), however, the backscattering angular width of Surface # 4 is wider than the backscattering angular width of Surface # 3. In addition, the specular level of Surface # 4 is higher than the specular level of Surface # 3.

Figure 3.6 illustrates the same plots as in Figure 3.5 except for VV polarization. In this case, results for Surface # 1 and Surface # 2, as shown in Figure 3.6 (a) and (b), respectively, seem to exhibit a small backscattering peak at \( \theta_s = -20^\circ \), and their backscattering and specular levels are nearly equal (about 0.5). On the other hand, results for Surface # 3 and Surface # 4, as shown in Figure 3.6 (c) and (d),
Figure 3.5: Normalized HH incoherent bistatic RCS versus $\theta_s$ for PEC rough surfaces at $\theta_i = 20^\circ$: (a) Surface # 1 (b) Surface # 2 (c) Surface # 3 (d) Surface # 4.
Figure 3.6: Normalized VV incoherent bistatic RCS versus $\theta$, for PEC rough surfaces at $\theta_i = 20^\circ$: (a) Surface # 1 (b) Surface # 2 (c) Surface # 3 (d) Surface # 4.
respectively, exhibit a well-defined backscattering peak at \( \theta_s = -20^\circ \). For Surface # 3, the backscattering level is higher than the specular level. It is interesting that \( \sigma_{1V}(\theta_s, \theta_l) \) for Surface # 4 also exhibits a well-defined peak in the specular direction at \( \theta_s = 20^\circ \), and its backscattering and specular levels are nearly equal (about 0.55). It should be pointed out that Surface # 4 is considered to be too rough to produce a specular reflection due to the first-order scattering. Thus, the specular peak should come from the multiple-scattering effects. As in the HH case, the backscattering peak level of Surface # 3 (about 0.65) is slightly higher than the backscattering peak level of Surface # 4 (about 0.55), and the backscattering angular width of Surface # 4 is wider than the backscattering angular width of Surface # 3. In addition, the specular level of Surface # 4 is also higher than the specular level of Surface # 3.

From the above numerical results, it is clear that a well-defined backscattering peak exists only for certain surface statistics, even for relatively large-height and large-slope surfaces. In addition, the parameter \( \sigma_s \) seems to be more important than the parameter \( h \) (compare results for Surface # 2 and Surface # 3). For surfaces with large \( \sigma_s \) (Surface # 3 and Surface # 4), as \( h \) increases, the backscattering peak level tends to decrease, but the backscattering angular width tends to increase. The reason for the decrease of the backscattering peak level as \( h \) increases may be due to the fact that the multiple-scattering effects increase (as \( h \) increases), and it contributes more energy into other directions than the backscattering direction: e.g., the specular direction as shown in Figure 3.6 (d) for Surface # 4. It is also noted that polarization affects scattering mechanisms associated with the backscattering enhancement phenomenon considerably (compare \( \sigma_{HH}(\theta_s, \theta_l) \) and \( \sigma_{1V}(\theta_s, \theta_l) \) for Surface # 4). Effects of polarization will be discussed in detail in Section 3.4.3. Note that from the above
numerical results, Surface # 4 seems to clearly exhibit the backscattering enhancement phenomenon. Next, consider the effects of surface material to the backscattering enhancement.

3.4.2 Variations in the Level and Angular Width of Backscattering Enhancement with Surface Material

Consider a 1-D rough surface of Surface # 4 illuminated by a tapered incident field with $G = 5$. Using the 1-D FB method. Figure 3.7 shows the plots of the normalized incoherent bistatic RCS $\sigma_{\alpha\alpha}^i(\theta_s, \theta_i)$ versus $\theta_s$ at $\theta_i = 20^\circ$ and for PEC surfaces ($|\varepsilon_r|=\infty$) and impedance surfaces with $\varepsilon_{r1} = 38.0 + i40.0$ and $10.0 + i10.0$. From the plots for both polarizations, the overall level of $\sigma_{\alpha\alpha}^i(\theta_s, \theta_i)$ for PEC surfaces is the highest, and impedance surfaces with $\varepsilon_{r1} = 10.0 + i10.0$ yield the lowest level of $\sigma_{\alpha\alpha}^i(\theta_s, \theta_i)$ as expected. Qualitatively, as the surface impedance decreases (i.e. $\varepsilon_{r1}$ increases), the overall level of $\sigma_{\alpha\alpha}^i(\theta_s, \theta_i)$ increases due to the increase of the effective reflection coefficients. For convenience in comparison, the co-polarized normalized incoherent bistatic RCS for the IBC surfaces is scaled such that its maximum is equal to the maximum of normalized incoherent bistatic RCS for the PEC surfaces, and the scaling factor is given as a number in a square bracket appended the value of $\varepsilon_{r1}$ in each plot as shown in Figure 3.8. For HH polarization as shown in Figure 3.8 (a), the shapes of $\sigma_{HH}^i(\theta_s, \theta_i)$ for different surface materials tend to be the same, however, the backscattering peak level decreases as the surface impedance increases as can be seen from the increasing of the numbers in the brackets as $|\varepsilon_{r1}|$ decreases. It is observed that the backscattering angular width tends to be invariant with the change of surface impedances for HH polarization. In addition, the backscattering level is higher than the specular level for all surface impedances of interest. For
Figure 3.7: Normalized incoherent bistatic RCS versus $\theta_s$ for Surface #4 at $\theta_i = 20^\circ$: (a) HH polarization (b) VV polarization.
Figure 3.8: Normalized incoherent bistatic RCS versus $\theta_i$ for Surface # 4 at $\theta_i = 20^\circ$ average over 256 surface realizations. The normalized incoherent bistatic RCS for the IBC surfaces is scaled such that its maximum is equal to the maximum of normalized incoherent bistatic RCS for the PEC surfaces, and the scaling factor is given as a number in a square bracket appended the value of $\epsilon_r I$ in each plot: (a) HH polarization (b) VV polarization.
Figure 3.9: Normalized incoherent bistatic RCS versus $\theta_s$ for Surface # 4 at $\theta_t = 50^\circ$: (a) HH polarization (b) VV polarization.
Figure 3.10: Normalized incoherent bistatic RCS versus $\theta_i$ for Surface # 4 at $\theta_i = 50^\circ$ average over 256 surface realizations. The normalized incoherent bistatic RCS for the IBC surfaces is scaled such that its maximum is equal to the maximum of normalized incoherent bistatic RCS for the PEC surfaces. and the scaling factor is given as a number in a square bracket appended the value of $\varepsilon_r$ in each plot: (a) HH polarization (b) VV polarization.
Vv polarization as shown in Figures 3.7 (b) and 3.8 (b), the backscattering and specular peak levels decrease considerably as the surface impedance increases, and the shapes of $\sigma_{1V}^i(\theta_s, \theta_i)$ are quite different for different surface materials, especially in the backward angular region. The backscattering angular width for VV polarization is dependent on the surface impedances; i.e. it tends to be wider as the surface impedance increases. In addition, the scattered energy tends to distribute more in the backward angular region. Note that the result of the impedance surfaces with $\varepsilon_{r1} = 10.0 + i10.0$ does not exhibit the well-defined peaks in the backscattering direction ($\theta_s = -20^\circ$), but there exists a broad peak centered near $\theta_s = -40^\circ$ instead. The $\sigma_{1V}^i(\theta_s, \theta_i)$ of impedance surfaces with $\varepsilon_{r1} = 38.0 + i40.0$ still exhibits the well-defined peaks in the backscattering and specular directions, even though they are not distinct as in the PEC case. In addition, the backscattering and specular levels are nearly equal for the PEC surface and the IBC surface with $\varepsilon_{r1} = 38.0 + i40.0$.

Figure 3.9 shows the plots of the normalized incoherent bistatic RCS $\sigma_{aa}^i(\theta_s, \theta_i)$ versus $\theta_s$ at $\theta_i = 50^\circ$ and for PEC surfaces and impedance surfaces with $\varepsilon_{r1} = 38.0 + i40.0$. From the plots for both polarizations, the overall level of $\sigma_{aa}^i(\theta_s, \theta_i)$ for PEC surfaces is higher than impedance surfaces with $\varepsilon_{r1} = 38.0 + i40.0$. For convenience in comparison, Figure 3.10 shows the same results as in Figure 3.9, but the results for the IBC surfaces are scaled such that their maximums are equal to the maximum of the PEC results. and the scaling factor is given as a number in a square bracket appended the value of $\varepsilon_{r1}$ in each plot. For HH polarization as shown in Figures 3.9 (a) and 3.10 (a), same conclusions can be drawn as in the case of $\theta_i = 20^\circ$ for HH polarization. For VV polarization as shown in Figure 3.10 (b), the shapes of $\sigma_{1V}^i(\theta_s, \theta_i)$ are different for different surface materials in the forward
angular region between 0° and 20°: i.e. $\sigma_{\text{V}}(\theta_{s}, \theta_{i})$ for the PEC surfaces seem to exhibit another broad peak in that region, whereas the one for the IBC surfaces with $\varepsilon_{r1} = 38.0 + i40.0$ does not exhibit the well-defined peak.

From the above numerical results, the backscattering enhancement strongly depends on surface material: i.e. the backscattering peak level decreases as the surface impedance increases. In addition, the backscattering angular width for HH polarization tends to be invariant with the change of surface impedances. However, for VV polarization the backscattering angular width is dependent on the surface impedances at least for small incident angles: i.e. it tends to be wider as the surface impedance increases. Note that polarization and incident angle affect scattering mechanisms associated with the backscattering enhancement considerably. Effects of incident angle will be discussed in details in Section 3.4.4. Next, consider the effects of polarization to the backscattering enhancement.

### 3.4.3 Variations in the Level and Angular Width of Backscattering Enhancement with Polarization

Consider a 1-D rough surface illuminated by a tapered incident field with $G = 5$. Using the 1-D FB/NSA method. Figure 3.11 shows the plots of the normalized incoherent bistatic RCS versus $\theta_{s}$ at $\theta_{i} = 20^\circ$ for PEC Surfaces # 1 to # 4 and for both HH and VV polarizations. From the plots, results for Surface # 1 and Surface # 2, as shown in Figure 3.11 (a) and (b), respectively, do not strongly depend on polarization. Thus, for these surfaces, polarization does not strongly affect the backscattering enhancement. On the other hand, results for Surface # 3 and Surface # 4, as shown in Figure 3.11 (c) and (d), respectively, exhibit polarization dependency. For Surface # 3, the HH backscattering peak level at $\theta_{s} = -20^\circ$ (about 0.8) is
Figure 3.11: Normalized incoherent bistatic RCS versus $\theta_s$ for PEC rough surfaces at $\theta_i = 20^\circ$: (a) Surface # 1 (b) Surface # 2 (c) Surface # 3 (d) Surface # 4.
Figure 3.12: Normalized incoherent bistatic RCS versus $\theta_s$ for Surface $\# 4$ at $\theta_i = 50^\circ$: (a) PEC (b) IBC: $\epsilon_r = 38.0 + i40.0$. 
higher than the VV backscattering peak level (about 0.65), and the VV backscattering angular width seems to be wider than the HH backscattering angular width. For Surface #4, $\sigma_{HH}^i(\theta_s, \theta_i)$ and $\sigma_{VV}^i(\theta_s, \theta_i)$ are distinct, especially in the backscattering ($\theta_s = -20^\circ$) and specular ($\theta_s = 20^\circ$) directions. Note that the HH backscattering peak level (about 0.7) is higher than the VV backscattering peak level (about 0.55). However, $\sigma_{VV}^i(\theta_s, \theta_i)$ is greater than $\sigma_{HH}^i(\theta_s, \theta_i)$ in the specular region. In addition, the VV backscattering angular width seems to be wider than the HH backscattering angular width. It is interesting that $\sigma_{VV}^i(\theta_s, \theta_i)$ also exhibits a well-defined peak in the specular direction, whereas $\sigma_{HH}^i(\theta_s, \theta_i)$ does not clearly exhibit this specular peak.

It is observed that the levels of both peaks of $\sigma_{VV}^i(\theta_s, \theta_i)$ are almost the same.

In addition, to study more about the effects of polarization, Figure 3.12 illustrates the plots of the normalized incoherent bistatic RCS versus $\theta_s$ at $\theta_i = 50^\circ$ for Surfaces #4 and for PEC surfaces and impedance surfaces with $\epsilon_{r1} = 38.0 + i40.0$. For PEC surfaces as shown in Figure 3.12 (a), same conclusions can be drawn as in the case of $\theta_i = 20^\circ$ for PEC Surface #4 (see Figure 3.11 (d)). Note that as $\theta_i$ increases from $20^\circ$ to $50^\circ$, the backscattering peak level decreases. Figure 3.12 (b) illustrates the same plot as in Figure 3.12 (a) except for the impedance surfaces with $\epsilon_{r1} = 38.0 + i40.0$. It is interesting to point out that $\sigma_{HH}^i(\theta_s, \theta_i)$ exceeds $\sigma_{VV}^i(\theta_s, \theta_i)$ almost all of observation angles. Note that the HH backscattering peak level at $\theta_s = -50^\circ$ (about 0.25) is higher than the VV backscattering peak level (about 0.18), and the VV backscattering angular width seems to be wider than the HH backscattering angular width.

From the above numerical results for very rough surfaces exhibiting the backscattering enhancement, the backscattering enhancement strongly depends on polarization; i.e. the HH backscattering peak level exceeds the VV backscattering peak level.
In addition, the VV backscattering angular width seems to be wider than the HH backscattering angular width. Furthermore, for some rough surfaces exhibiting the backscattering enhancement with small incident angles (i.e. near normal incidence), $\sigma_{VV}(\theta_s, \theta_i)$ exhibits a well-defined peak in the specular direction, whereas $\sigma_{HH}(\theta_s, \theta_i)$ does not clearly exhibit this specular peak. Next, consider the effects of incident angle to the backscattering enhancement.

### 3.4.4 Variations in the Level and Angular Width of Backscattering Enhancement with Incident Angle

Consider a 1-D rough surface illuminated by a tapered incident field with $G = 5$. Using the 1-D FB method. Figure 3.13 shows the plots of the normalized incoherent bistatic RCS versus $\theta_s$ for PEC Surface # 4 and for both HH and VV polarizations. From the plots, the overall level of $\sigma_{\alpha\alpha}(\theta_s, \theta_i)$ tends to decrease as $\theta_i$ increases. For convenience in comparison, Figure 3.14 shows the same results as in Figure 3.13, but the results are scaled such that their maximums are equal to the maximum of the results for $\theta_i = 0.1^\circ$. and the scaling factor is given as a number in a square bracket appended the value of $\theta_i$ in each plot. For HH polarization as shown in Figures 3.13 (a) and 3.14 (a), as $\theta_i$ increases the backscattering peak level decreases, but the backscattering angular width tends to increase. It is interesting that $\sigma_{HH}(\theta_s, \theta_i)$ for $\theta_i = 85^\circ$ does not exhibit a backscattering peak at $\theta_s = -85^\circ$, but a broad peak exists in the backward angular region instead. In addition, another well-defined peak near the specular region exists, and it exceeds the former peak. For VV polarization as shown in Figures 3.13 (b) and 3.14 (b), same conclusions can be drawn as in HH polarization. In addition, $\sigma_{VV}(\theta_s, \theta_i)$ seems to exhibit the secondary peak except for $\theta_i = 0.1^\circ$. Note that $\sigma_{VV}(\theta_s, \theta_i)$ for $\theta_i = 85^\circ$ exhibits a small peak near the
specular region as well, and its magnitude is much less than another broad peak in the backward angular region. To see the effect of surface material, Figures 3.15 and 3.16 illustrate the same plots as in Figures 3.13 and 3.14 (for the PEC case) respectively, except using impedance surfaces with $\epsilon_r = 38.0 + i40.0$. From the plots, same conclusions can be drawn as in the PEC case, except that for VV polarization as shown in Figure 3.15 (b), the backscattering peak level for $\theta_i = 50^\circ$ seems to be slightly higher than the backscattering peak level for $\theta_i = 20^\circ$. This is an unusual phenomenon, i.e. the VV backscattering peak at larger incident angles can exceed the one for smaller angles of incidence, observed in the scattering from IBC rough surfaces. It should be pointed out that this phenomenon is more pronounced for impedance surfaces with lower dielectric constant: e.g. $\epsilon_r = 10.0 + i10.0$.

To further investigate the backscattering enhancement at large incident angles, Figure 3.17 shows the plots of the normalized incoherent bistatic RCS versus $\theta_s$ for PEC Surface # 4 and for $\theta_i = 50^\circ, 60^\circ, 70^\circ$ and $85^\circ$. It can be seen from both plots that the overall level of $\sigma^{i}_{aa}(\theta_s, \theta_i)$ tends to decrease as $\theta_i$ increases. In addition, $\sigma^{i}_{aa}(\theta_s, \theta_i)$ exhibits the backscattering enhancement (in the backscattering direction) only for $\theta_i = 50^\circ$ and $60^\circ$, even though $\sigma^{i}_{aa}(\theta_s, \theta_i)$ for $\theta_i = 70^\circ$ and $85^\circ$ seem to exhibit more energy in the backward angular region. For convenience in comparison, Figure 3.18 shows the same results as in Figure 3.17, but the results are scaled such that their maximums are equal to the maximum of the results for $\theta_i = 50^\circ$, and the scaling factor is given as a number in a square bracket appended the value of $\theta_i$ in each plot. For large incident angles, it can be seen from the plots that scattered energy tends to migrate from the backward angular region to the forward angular region as the incident angle increases toward grazing incidence. Thus, it is implied
Figure 3.13: Normalized incoherent bistatic RCS versus $\theta_s$ for PEC rough surfaces and using Surface $\# 4$: (a) HH polarization (b) VV polarization.
Figure 3.14: Normalized incoherent bistatic RCS versus $\theta_s$ averaged over 256 surface realizations for PEC rough surfaces and using Surface # 4. The normalized incoherent bistatic RCS is scaled such that its maximum is equal to the maximum of normalized incoherent bistatic RCS for $\theta_i = 0.1^\circ$, and the scaling factor is given as a number in a square bracket appended the value of $\theta_i$ in each plot: (a) HH polarization (b) VV polarization.
Figure 3.15: Normalized incoherent bistatic RCS versus $\theta_s$ for IBC rough surfaces ($\epsilon_r = 38.0 + i40.0$) and using Surface # 4: (a) HH polarization (b) VV polarization.
Figure 3.16: Normalized incoherent bistatic RCS versus $\theta_s$ averaged over 256 surface realizations for IBC rough surfaces ($\epsilon_{r1} = 38.0 + i40.0$) and using Surface # 4. The normalized incoherent bistatic RCS is scaled such that its maximum is equal to the maximum of normalized incoherent bistatic RCS for $\theta_i = 0.1^\circ$, and the scaling factor is given as a number in a square bracket appended the value of $\theta_i$ in each plot: (a) HH polarization (b) VV polarization.
that for fixed surface statistics, rough surfaces look “smoother” as the incident angle increases. This observation agrees with the Rayleigh hypothesis [2] as discussed earlier in Section 1.1 in Chapter 1. It is emphasized that PEC Surface # 4 is employed in this figure. Rougher surfaces may be required to better see the backscattering enhancement phenomenon at large incident angles.

To see the effects of surface statistics to the backscattering enhancement at large angles of incidence, consider Surface # 5, which is rougher than Surface # 4:

- Surface # 5: \( h = 1.414 \lambda, l = 1.414 \lambda \) and \( \sigma_s = 1.414 \)

From numerical studies, the neighborhood distance \( L_s \) for Surface # 5 is chosen to be 256\( \lambda \). Figure 3.19 illustrates the plots of the normalized HH incoherent bistatic RCS versus \( \theta_s \) at \( \theta_i = 60^\circ \) and \( 70^\circ \) for PEC Surfaces # 4 and # 5 and for both HH and VV polarizations. From both plots, the results for Surface # 5 seem to exhibit more energy in the backward region than the results for Surface # 4. For \( \theta_i = 60^\circ \) as shown in Figure 3.19 (a), the backscattering peak level for Surface # 5 is higher than the backscattering peak level for Surface # 4. For \( \theta_i = 70^\circ \) as shown in Figure 3.19 (b), \( \sigma_{HH}^i(\theta_s, \theta_i) \) for Surface # 5 also exhibits a small peak in the backscattering direction. Rougher surfaces may be required for \( \theta_i = 70^\circ \) to exhibit a well-defined peak in the backscattering direction.

From the above numerical results, the backscattering enhancement strongly depends on incident angle. As \( \theta_i \) increases, the backscattering peak level decreases, but the backscattering angular width tends to increase. However, there exists an unusual phenomenon associated with some IBC rough surfaces with relatively low dielectric constant, i.e., the VV backscattering peak at larger incident angles is possible to
Figure 3.17: Normalized incoherent bistatic RCS versus $\theta_s$ for PEC rough surfaces and using Surface # 4: (a) HH polarization (b) VV polarization.
Figure 3.18: Normalized incoherent bistatic RCS versus $\theta_i$ averaged over 256 surface realizations for PEC rough surfaces and using Surface #4. The normalized incoherent bistatic RCS is scaled such that its maximum is equal to the maximum of normalized incoherent bistatic RCS for $\theta_i = 50^\circ$, and the scaling factor is given as a number in a square bracket appended the value of $\theta_i$ in each plot: (a) HH polarization (b) VV polarization.
Figure 3.19: Normalized HH incoherent bistatic RCS versus $\theta_s$ for PEC rough surfaces: (a) $\theta_i = 60^\circ$ (b) $\theta_i = 70^\circ$.  

121
exceed the one for smaller angles of incidence. It is also interesting that the backscattering enhancement at large angles of incidence depends strongly on surface statistics; i.e. as surface roughness increases (increasing both $h$ and $\sigma_z$), the backscattering peak level tends to increase.

It should be pointed out that the backscattering peak in some cases presented in this section is not well defined, even though surfaces of interest are very rough. This may be due to presenting numerical results in terms of the total incoherent power $\sigma^i_{\text{tot}}(\theta_s, \theta_l)$. As will be shown in the next section, the backscattering enhancement is mostly contributed from the second-order scattering, not from the first-order (single) scattering as predicted by the PO. Thus, the backscattering peak may be seen more clearly if numerical results are presented in terms of the second-order contribution to the total incoherent power rather than the total incoherent power. Note that the first-order scattering may dominate the second-order scattering in some cases. For these cases, it is better to present numerical results in terms of the second-order contribution. However, the first-order and second-order contributions must be computed appropriately: i.e. the shadowing effects must be included properly, as will be discussed more in the next section.

3.5 Physical Interpretation and Further Discussions

In this section, a simple physical interpretation based on a single Neumann iteration (series or expansion) is provided to explain the causes of the backscattering enhancement phenomenon. The Neumann iteration (Kirchhoff or Jacobi iteration) has been used to compute the solutions of the magnetic field integral equation (MFIE) for PEC Gaussian rough surfaces. The Neumann series is attractive in that
each term corresponds physically to the contribution of individual multiple-scattering
term. In [9], it was conjectured that the backscattering enhancement is associated
with the multiple-scattering mechanisms. Thus, the Neumann iteration is suitable
for this case. However, it has been shown that the Neumann series has convergence
problems when used at large incident angles and/or for surfaces with large slopes [62].
Since the rms slopes of surfaces in this backscattering enhancement study are rela-
tively large ($\sigma_s = 0.5$ and $1.0$), only the single Neumann iteration is considered in
this study.

To illustrate the Neumann iteration, consider a 1-D finite PEC rough surface
illuminated by a TM incident magnetic field at $\theta$, as shown in Figure 2.1. For a PEC
surface ($\partial H_y / \partial n = 0$), Eq. (2.1) is reduced to

$$\frac{H_y(\rho)}{2} = H_y^0(\rho) + \int_{\Gamma_{PV:C}} dl H_y(\rho) \frac{\partial g(\rho, \rho')}{\partial n}.$$  \hspace{1cm} (3.12)

Using pulse basis functions and point matching [81] results in a matrix equation, and
elements of the impedance matrix $\bar{Z}$ are (see Eqs. (2.12) and (2.12) with $|\varepsilon_{\tau 1}| \rightarrow \infty$):

$$Z_{mn} = -\frac{k \Delta x}{4} \left[ i H_1^{(1)}(k \rho_{mn}) \left\{ \frac{\Delta z_{mn} - f n(x_m) \Delta x_{mn}}{\rho_{mn}} \right\} \right]. \quad m \neq n \quad (3.13)$$

$$Z_{mm} = \frac{1}{2} - \frac{\Delta x}{4\pi} \frac{f n(x_m)}{T_m^2}. \quad (3.14)$$

Note that the effects of surface curvature are included in the self term $Z_{mm}$ in Eq.
(3.14). Let $\bar{Z}$, $\bar{I}$ and $\bar{1}$ denote the impedance matrix, the solution vector, and the
RHS vector (excitation vector), respectively: i.e.

$$\bar{Z} \bar{I} = \bar{1}.$$  \hspace{1cm} (3.15)

where

$$\bar{Z} = \frac{1}{2} \bar{1} + \hat{Z}. \quad (3.16)$$
Note that elements of the modified impedance matrix \( \tilde{Z} \) are

\[
\tilde{Z}_{mn} = Z_{mn}, \quad m \neq n \tag{3.17}
\]

\[
\tilde{Z}_{mm} = -\frac{\Delta x f_m(x_m)}{4\pi T_m^2} \tag{3.18}
\]

Substituting Eq. (3.16) into Eq. (3.15) and rearranging, Eq. (3.15) can be rewritten in the iterative form as

\[
\vec{I}^{(k)} = 2 \vec{I} - 2 \tilde{Z} \vec{I}^{(k-1)} \tag{3.19}
\]

with

\[
\vec{I}^{(0)} = 2 \vec{I}. \tag{3.20}
\]

where \( \vec{I}^{(k)} \) is the solution vector for the \( k \)th iteration. Using Eq. (3.19), the single Neumann-iteration solution of Eq. (3.15) is:

\[
\vec{I}^{(1)} = 2 \vec{I} - 4 \tilde{Z} \vec{I} \tag{3.21}
\]

The first term in Eq. (3.21) is the PO solution (without shadowing corrections), named the first-order KA \( (K^{(1)}) \) solution, and the second term in that equation is called the second-order KA \( (K^{(2)}) \) solution: i.e.

\[
K^{(2)} = -4 \tilde{Z} \vec{I}. \tag{3.22}
\]

Note that Eqs. (3.21) and (3.22) involves a matrix-vector multiply. Thus, the 1-D NSA algorithm developed for the 1-D FB method can be applied to efficiently compute a matrix-vector multiply, especially for large-scale surface-scattering problems, by performing only a matrix-vector multiply without updating the solution vector associated the FB method.
Let $\sigma_{aa}^{N^{(1)}}(\theta_s, \theta_i)$, $\sigma_{aa}^{K^{(1)}}(\theta_s, \theta_i)$, and $\sigma_{aa}^{K^{(2)}}(\theta_s, \theta_i)$ denote the normalized incoherent bistatic radar cross sections obtained from the single Neumann-iteration solution, the first-order KA solution and the second-order KA solution, respectively. It is emphasized that

$$
\sigma_{aa}^{N^{(1)}}(\theta_s, \theta_i) \neq \sigma_{aa}^{K^{(1)}}(\theta_s, \theta_i) + \sigma_{aa}^{K^{(2)}}(\theta_s, \theta_i).
$$

(3.23)

due to the interference between $K^{(1)}$ and $K^{(2)}$ solutions. If this interference is small, it can be shown that

$$
\sigma_{aa}^{N^{(1)}}(\theta_s, \theta_i) \approx \sigma_{aa}^{K^{(1)}}(\theta_s, \theta_i) + \sigma_{aa}^{K^{(2)}}(\theta_s, \theta_i).
$$

(3.24)

To validate numerical results obtained from the single Neumann iteration, consider PEC Gaussian surfaces with small slope fluctuation and moderately rough: $h = 0.20\lambda$, $l = 5.657\lambda$, and $\sigma_s = 0.05$, illuminated by the tapered incident field with $G = 5$ at $\theta_i = 50^\circ$ (same scenario as in Figure 3.2). Figure 3.20 illustrates the comparisons between Monte-Carlo normalized incoherent bistatic radar cross sections, computed via the single Neumann iteration, the 1-D FB/NSA and Monte-Carlo PO, and analytical results computed via the PO at 14 GHz for VV polarization. From the plot, results for all methods are in good agreement near the specular region. Note that the single Neumann iteration and the 1-D FB/NSA yield almost identical results for all observation angles, whereas the single Neumann iteration and Monte-Carlo PO yield different results away from the specular region. Thus, it can be concluded that the single Neumann iteration provide a higher-order correction to the first-order KA (PO), and it yields almost the exact solution (using the 1-D FB/NSA method) for this particular problem.
Figure 3.20: Comparisons between Monte-Carlo normalized incoherent bistatic radar cross sections, computed via the single Neumann iteration, the 1-D FB/NSA and Monte-Carlo PO, and analytical results computed via the PO at 14 GHz with $\theta_i = 50^\circ$ and $G = 5$ for VV polarization and for PEC rough surfaces. Gaussian surface statistics for this case are: $h = 0.20\lambda$, $l = 5.657\lambda$, and $\sigma_s = 0.05$. 
Figure 3.21: Comparisons of Monte-Carlo VV normalized incoherent bistatic radar cross sections computed via the 1-D FB/NSA, the single Neumann iteration, the second-order term of the KA, and Monte-Carlo PO at 14 GHz with $G = 5$ for PEC Surface # 4: (a) $\theta_i = 0.1^\circ$ (b) $\theta_i = 20^\circ$ (c) $\theta_i = 50^\circ$ (d) $\theta_i = 85^\circ$. 
For better understanding about the backscattering enhancement phenomena, consider a 1-D PEC Surface # 4 illuminated by the TM tapered incident field with $G = 5$ at 14 GHz. Let $\sigma_{\alpha\beta}^{i,nsa}(\theta_s, \theta_i)$ denote the normalized incoherent bistatic radar cross sections obtained from the 1-D FB/NSA method. Figure 3.21 illustrates the comparisons of Monte-Carlo VV normalized incoherent bistatic radar cross sections computed via the 1-D FB/NSA, the single Neumann iteration, the second-order term of the KA $(K^{(2)})$, and Monte-Carlo PO for $\theta_i = 0.1^\circ, 20^\circ, 50^\circ, \text{ and } 85^\circ$. For $\theta_i = 0.1^\circ$ as shown in Figure 3.21 (a), it is very interesting that the single Neumann iteration can predict a backscattering peak at $\theta_s = -0.1^\circ$ denoted by the vertical line, even though the overall level of $\sigma_{\alpha\alpha}^{i,NN1}(\theta_s, \theta_i)$ exceeds the level of $\sigma_{\alpha\beta}^{i,nsa}(\theta_s, \theta_i)$. Note that the overall shape of $\sigma_{\alpha\alpha}^{i,NN1}(\theta_s, \theta_i)$ is similar to $\sigma_{\alpha\beta}^{i,nsa}(\theta_s, \theta_i)$ except in the angular regions near $\theta_s = \pm 90^\circ$. In these regions, $\sigma_{\alpha\alpha}^{i,NN1}(\theta_s, \theta_i)$ exhibits unphysical peaks since $\sigma_{\alpha\beta}^{i,nsa}(\theta_s, \theta_i)$ approaches zero as $\theta_s$ approaches $\pm 90^\circ$. To understand these results, Figure 3.21 (a) also illustrates the plot of $\sigma_{\alpha\alpha}^{i,K^{(1)}}(\theta_s, \theta_i)$ and $\sigma_{\alpha\beta}^{i,K^{(2)}}(\theta_s, \theta_i)$ versus $\theta_s$. It is observed that the first-order KA (PO) yields symmetric $\sigma_{\alpha\alpha}^{i,K^{(1)}}(\theta_s, \theta_i)$, however, it cannot predict the backscattering peak. In addition, $\sigma_{\alpha\alpha}^{i,K^{(1)}}(\theta_s, \theta_i)$ does not approach zero as $\theta_s$ approaches $\pm 90^\circ$. It is emphasized that the PO is not applicable for large heights and large slopes surfaces of Surface # 4 due to significant multiple-scattering effects. To account for these multiple-scattering effects, higher-order solutions of the KA must be employed appropriately. It is interesting that the second-order term $K^{(2)}$ can predict a backscattering peak, even though the level of the peak exceeds the backscattering peak level predicted by the 1-D FB/NSA method. In addition, $\sigma_{\alpha\alpha}^{i,K^{(2)}}(\theta_s, \theta_i)$ exhibits unphysical peaks in the angular regions near $\theta_s = \pm 90^\circ$, and these peaks contribute to the unphysical peaks associated with $\sigma_{\alpha\beta}^{i,NN1}(\theta_s, \theta_i)$. From $\sigma_{\alpha\alpha}^{i,K^{(1)}}(\theta_s, \theta_i)$.
and $\sigma_{VV}^{i,K^{(2)}}(\theta_s, \theta_i)$, it seems that, for this case, the shape of $\sigma_{VV}^{i,N^{(1)}}(\theta_s, \theta_i)$ is governed by the second-order term $K^{(2)}$, and it is scaled by the PO solution $\sigma_{VV}^{i,K^{(1)}}(\theta_s, \theta_i)$.

Figure 3.21 (b) illustrates the same scenario as in Figure 3.21 (a) except for $\theta_i = 20^\circ$. From the plot, the single Neumann iteration can still predict a sharp backscattering peak at $\theta_s = -20^\circ$, even though the overall level of $\sigma_{VV}^{i,N^{(1)}}(\theta_s, \theta_i)$ exceeds the level of $\sigma_{VV}^{i,nsa}(\theta_s, \theta_i)$ as in the case of $\theta_i = 0.1$. In addition, the single Neumann iteration can also predict a broad specular peak, even though it is not sharp as a backscattering peak. Note that the overall shape of $\sigma_{VV}^{i,N^{(1)}}(\theta_s, \theta_i)$ is similar to $\sigma_{VV}^{i,nsa}(\theta_s, \theta_i)$ except in the angular regions near $\theta_s = \pm 90^\circ$. The unphysical peaks exhibit near $\theta_s = \pm 90^\circ$, and the peak near $\theta_s = -90^\circ$ is stronger than the peak near $\theta_s = 90^\circ$. This is due to the fact that $\sigma_{VV}^{i,N^{(1)}}(\theta_s, \theta_i)$ and $\sigma_{VV}^{i,K^{(2)}}(\theta_s, \theta_i)$ exhibits more energy in the angular region near $\theta_s = -90^\circ$ than in the angular region near $\theta_s = 90^\circ$. It is observed that the first-order KA (PO) does not predict the backscattering peak, and $\sigma_{VV}^{i,K^{(1)}}(\theta_s, \theta_i)$ does not approach zero as $\theta_s$ approaches $\pm 90^\circ$. In addition, $\sigma_{VV}^{i,K^{(1)}}(\theta_s, \theta_i)$ is not symmetric in this case, and the second-order term $K^{(2)}$ contributes to both peaks: i.e. excessive backscattering and specular peaks. From Figure 3.21 (a) and (b), it is found that the second-order Kirchhoff scattering contributes to the backscattering enhancement; however, it contains excessive power. The first-order KA (PO) cannot predict the backscattering enhancement, which is associated with multiple-scattering mechanisms, due to its single-scattering nature of the method.

To investigate more about the single Neumann iteration at large incident angles, Figure 3.21 (c) and (d) shows $\sigma_{VV}^{i}(\theta_s, \theta_i)$ at $\theta_i = 50^\circ$ and $85^\circ$, respectively. As pointed out earlier in this section, the Neumann series has convergence problems
when used at large incident angles. Thus, only the single Neumann iteration is considered. However, it appears that the single Neumann iteration provides some physical insight of the scattering mechanisms associated with very rough surfaces at large angles of incidence as described below. For $\theta_i = 50^\circ$ as illustrated in Figure 3.21 (c), the single Neumann iteration can predict two broad peaks in the forward and backward angular regions, and again the overall level of $\sigma_{VV}^{i,N(1)}(\theta_s, \theta_i)$ exceeds the level of $\sigma_{VV}^{i,nsa}(\theta_s, \theta_i)$. It is observed that the single Neumann iteration does not predict a well-defined peak in the backscattering direction at $\theta_s = -50^\circ$ as in the cases of $\theta_i = 0.1^\circ$ and $20^\circ$. Note also that $\sigma_{VV}^{i,N(1)}(\theta_s, \theta_i)$ still exhibits unphysical peaks near $\theta_s = \pm 90^\circ$, which come from the second-order Kirchhoff scattering as shown in the plot. It is interesting to point out that the first-order KA (PO) predicts a slightly-excessive broad peak in the backward angular region, whereas the second-order Kirchhoff term predicts a slightly-excessive broad peak in the forward angular region. In addition, it is observed that the second-order Kirchhoff term also predicts a small sharp peak in the backscattering direction. As in the cases of $\theta_i = 0.1^\circ$ and $20^\circ$, $\sigma_{VV}^{i,K(1)}(\theta_s, \theta_i)$ still does not approach zero as $\theta_s$ approaches $\pm 90^\circ$, and $\sigma_{VV}^{i,K(2)}(\theta_s, \theta_i)$ still exhibits unphysical peaks in the angular regions near $\theta_s = \pm 90^\circ$. For $\theta_i = 85^\circ$ as illustrated in Figure 3.21 (d), the single Neumann iteration predicts three peaks: i.e. two broad peaks in the backward angular region and one peak in the forward region. Again, the overall level of $\sigma_{VV}^{i,N(1)}(\theta_s, \theta_i)$ exceeds the level of $\sigma_{VV}^{i,nsa}(\theta_s, \theta_i)$. In this case, the first-order KA (PO) predicts an excessive broad peak in the backward angular region, and $\sigma_{VV}^{i,K(1)}(\theta_s, \theta_i)$ does approach zero as $\theta_s$ approaches $\pm 90^\circ$. The second-order Kirchhoff term also predicts three peaks as in the single Neumann iteration. Note that $\sigma_{aa}^{i,nsa}(\theta_s, \theta_i)$ also exhibits three broad peaks, even though the peak near
\( \theta_s = -85^\circ \) is questionable. It is emphasized that the second-order Kirchhoff term still predicts the broad backscattering peak at \( \theta_s = -85^\circ \). From Figure 3.21 (c) and (d), it is found that the second-order Kirchhoff scattering contributes to the backscattering enhancement, however, it contains excessive power. The first-order KA (PO) still \textit{cannot} predict the backscattering enhancement, even though \( \sigma_{VV}^{i,K^{(1)}}(\theta_s, \theta_i) \) exhibits more energy in the backward angular region.

For the last numerical example, consider the same scenario as in Figure 3.21 except using only \( \theta_i = 20^\circ \) and using smaller slope surfaces: i.e. Surface \# 2 instead of Surface \# 4. It is expected that in this case the single Neumann iteration should predict \( \sigma_{VV}^{i,\nu}(\theta_s, \theta_i) \) better than the previous case using Surface \# 4 due to the smaller slope surfaces of Surface \# 2. Figure 3.22 illustrates the comparisons of Monte-Carlo VV normalized incoherent bistatic radar cross sections computed via the 1-D FB/NSA, the single Neumann iteration, the second-order term of the KA \((K^{(2)})\), and Monte-Carlo PO for \( \theta_i = 20^\circ \) and for PEC Surfaces \# 2 and \# 4. Note that Figure 3.22 (b) for Surface \# 4 is replotted for better comparisons with Figure 3.22 (a). For Surface \# 2 as shown in Figure 3.22 (a), \( \sigma_{VV}^{i,\nu}^{(1)}(\theta_s, \theta_i) \) and \( \sigma_{VV}^{i,\text{nsa}}(\theta_s, \theta_i) \) are in better agreement than the results for the case of Surface \# 4 except in the angular regions near \( \theta_s = \pm 90^\circ \). It is noted that the first-order scattering dominates the second-order scattering in this case since the overall level of \( \sigma_{VV}^{i,K^{(1)}}(\theta_s, \theta_i) \) is greater than the overall level of \( \sigma_{VV}^{i,K^{(2)}}(\theta_s, \theta_i) \). In this case, \( \sigma_{VV}^{i,K^{(1)}}(\theta_s, \theta_i) \) still does not approach zero as \( \theta_s \) approaches \( \pm 90^\circ \), and the second-order Kirchhoff term still predicts the backscattering peak at \( \theta_s = -20^\circ \). In addition, the unphysical peaks still exist in the angular regions near \( \theta_s = \pm 90^\circ \) for both \( \sigma_{VV}^{i,\nu}^{(1)}(\theta_s, \theta_i) \) and \( \sigma_{VV}^{i,K^{(2)}}(\theta_s, \theta_i) \).
Figure 3.22: Comparisons of Monte-Carlo VV normalized incoherent bistatic radar cross sections computed via the 1-D FB/NSA, the single Neumann iteration, the second-order term of the KA, and Monte-Carlo PO at 14 GHz with $\theta_i = 20^\circ$ and $G = 5$ for PEC surfaces: (a) Surface # 2 (b) Surface # 4.
It is emphasized that the single Neumann iteration does not take the shadowing effects into consideration resulting in excessive $\sigma_{\text{\textscript{T}1}}(\theta_s, \theta_i)$ prediction. In [136], it is pointed out that the conventional KA (PO) shows no backscattering enhancement and does not satisfy the law of energy conservation, and the second-order Kirchhoff scattering contains the essential physics of the backscattering enhancement process and the effects of the higher-order Kirchhoff scattering are included by a shadowing correction. In addition, the backscattering enhancement is mostly contributed from the second-order scattering, and when the surface slope increases, the contributions from the second-order Kirchhoff scattering increase. Furthermore, for very rough surfaces as considered in this study, the second-order Kirchhoff scattering contains excessive power, however, it exhibits the backscattering peak [54]. This implies that the higher-order Kirchhoff scattering, which is included by a shadowing correction, cancels some of the second-order scattering field without altering the essence of the physical phenomena. It is also pointed out in [54] that the shadowing functions are approximate, and their improvement requires more refined analysis. In the next section, a summary and conclusions of this 1-D backscattering enhancement study are provided.

3.6 A Summary and Conclusions

In this chapter, the backscattering enhancement phenomenon for 1-D random rough surfaces is studied effectively using the 1-D FB/NSA method combined with parallel computing techniques. Only surfaces with relatively large heights and slopes for which multiple surface scattering dominates are considered in this study. Numerical results of the 1-D FB/NSA method are validated through comparison with the
SPM and KA, especially at $\theta_i = 85^\circ$, to determine the appropriate surface size $L_x$ and the appropriate taper parameter $G$. Variations in the level and angular width of backscattering enhancement, using 1-D 1024$\lambda$ random rough surfaces and $G = 5$, with surface statistics, surface material (PEC or impedance), polarization (HH or VV), and incident angle $\theta_i$ are also considered. From this study, it is found that

- For surfaces with large $\sigma_s$ (e.g., Surfaces # 3 and # 4), as $h$ increases, the backscattering peak level tends to decrease, but the backscattering angular width tends to increase. In addition, the rms surface slope $\sigma_s$ seems to be more important than $h$ for the backscattering enhancement.

- The backscattering peak level decreases as the surface impedance increases. In addition, the backscattering angular width for the HH polarization tends to be invariant with the change of surface impedances. However, for VV polarization the backscattering angular width is dependent on the surface impedances at least for small incident angles; i.e. it tends to be wider as the surface impedance increases.

- For very rough surfaces exhibiting the backscattering enhancement, the HH backscattering peak level exceeds the VV backscattering peak level, and the VV backscattering angular width seems to be wider than the HH backscattering angular width. In addition, for some rough surfaces exhibiting the backscattering enhancement with small incident angles (i.e. near normal incidence), the VV normalized incoherent bistatic RCS exhibits a well-defined peak in the specular direction, whereas the HH normalized incoherent bistatic RCS does not clearly exhibit this specular peak.
• As $\theta_i$ increases, the backscattering peak level decreases, but the backscattering angular width tends to increase. However, there exists an unusual phenomenon associated with some IBC rough surfaces with relatively low dielectric constant, i.e. the VV backscattering peak at larger incident angles is possible to exceed the one for smaller angles of incidence.

In addition, a physical interpretation based on a single Neumann iteration is also provided to explain the causes of the backscattering enhancement phenomenon. It is found that the second-order Kirchhoff scattering contributes to the backscattering enhancement, however, it contains excessive power. The higher-order Kirchhoff scattering, which is included by a shadowing correction, is required to cancel some of the second-order scattering field without altering the essence of the physical phenomena. Thus, the backscattering enhancement is associated with multiple-scattering and shadowing effects.

Numerical results presented in this chapter provide some physical insight into the backscattering enhancement phenomenon, and it serves as a background for studying the backscattering enhancement from 2-D random rough surfaces considered in Chapter 6, which requires more intensive computation than the 1-D random rough surfaces. In addition, due to its less computational requirements for the 1-D case, it is permissible to consider higher angles of incidence compared to the 2-D case. Finally, general analytical theory for the prediction of the backscattering enhancement for an arbitrary angle of incidence and arbitrary rough surfaces still remains to be discovered. Numerical results obtained in this chapter will provide more physical insight to the backscattering enhancement phenomenon, and will assist in the future development of analytical theories.
CHAPTER 4

Development of the Novel Spectral Acceleration Algorithm for Two-Dimensional Perfect Electric Conducting Rough Surface Scattering Problems

4.1 Introduction

The NSA algorithm has been shown to be an extremely efficient and accurate iterative MM technique for the computation of scattering from 1-D PEC and impedance rough surfaces as illustrated earlier in Chapters 2 and 3. The NSA algorithm is employed to rapidly compute interactions between widely separated points in the conventional FB method, and is based on a single spectral integral representation of source currents and the associated Green’s function. In this chapter, an extension of the NSA algorithm from 1-D to 2-D rough surface scattering (RSS) problems is discussed in detail. The NSA algorithm for this case involves a double spectral integral representation of source currents and the 3-D free space scalar Green’s function. The coupling between two spectral variables makes the problem more challenging, and efficiency improvements obtained for 2-D surfaces are appreciable but not as dramatic as those for 1-D surfaces. However the computational efficiency of the FB/NSA method for 2-D rough surfaces remains $O(N_{tot})$ as one of the surface dimensions increases. As pointed out in Section 2.1, the NSA algorithm can be readily incorporated into
any iterative technique. However, for 2-D rough surfaces of interest, the FB method seems to provide better convergence than other iterative algorithms. It should be pointed out that the FB method is \textit{not} guaranteed to converge for general scattering problems, however the method converges for surfaces of interest. Thus, it is advantageous to incorporate the 2-D NSA algorithm into the FB method. Note that the FB/NSA method is specifically designed for 2-D large-scale finite rectangular surfaces and remains very efficient for moderately-rough large-scale surfaces. Use of a large rectangular surface makes the method well suited for studying RSS at low grazing angles (LGA), and also for studies of backscattering enhancement as will be presented in Chapter 6.

To understand the basic idea of the NSA algorithm applied for 2-D rough surfaces, first the NSA formulation based on the spectral domain representation of the free space 3-D scalar Green's function in the along-range ($x$) direction is considered. The method improves the efficiency of the standard FB method, but it is found that for relatively large surface cross-range sizes $D_y$ the method's efficiency decreases due to the increase of the direct computation of the mutual coupling in the strong region. To improve the computational efficiency of the strong-region computation, an additional NSA formulation based on the spectral domain representation of the Green's function in the cross-range ($y$) direction is incorporated as discussed in detail in Section 4.5. In addition, as in the case of 1-D surfaces, the 2-D FB/NSA algorithm with only one large weak region may yield inaccurate results for extremely large-scale rough surfaces. Inaccuracy comes from the fact that the complex vector radiation function of a source group far separated from the receiving element is \textit{rapidly decayed} along the deformed contours away from the origin in the complex planes. A "multilevel"
algorithm for 2-D extremely large-scale rough surfaces analogous to that for the 1-D case (presented in Section 2.4 in Chapter 2) is proposed to solve the above problem as discussed in detail in Section 4.6.

For large-scale PEC rough surfaces, comparisons of numerical results between the conventional FB method and the 2-D FB/NSA method based on the spectral domain representation of the free space 3-D scalar Green's function in the $x$- direction show that the latter yields identical results to the former with a reduction of CPU time and only a slight increase in memory storage. In addition, numerical results of the FB/NSA method are in good agreement with experimental data obtained from the University of Washington. In addition, it is found that for rough surfaces with relatively large surface cross-range size $D_y$ the computational efficiency is indeed improved via the the NSA formulation including the spectral domain representation of the Green's function in both $x$- and $y$- directions at the cost of increasing algorithmic complexity and memory requirements.

This chapter is organized as follows. The standard FB method for 2-D PEC RSS problems [111] is illustrated in Section 4.2. The tapered incident field used to reduce the edge effects for finite surface under consideration is delineated in Section 4.3. Section 4.4 presents the formulation of the 2-D FB/NSA algorithm using the spectral domain representation of the Green's function in the $x$- direction. In Section 4.5, an additional NSA formulation based on the spectral domain representation of the Green's function in the $y$- direction is discussed in detail. The "multilevel" FB/NSA algorithm for 2-D extremely large-scale rough surfaces is investigated in Section 4.6. Section 4.7 discusses the computational cost and memory storage requirements of the
2-D FB/NSA method, and numerical results are illustrated in Section 4.8. Finally, Section 4.9 presents conclusions of the 2-D FB/NSA method for the PEC case.

### 4.2 Forward-Backward Method for 2-D RSS Problems

Consider a 2-D PEC rough surface profile $S$ illuminated by an incident field $E^i(x, y, z)$ centered at the origin and propagating in direction $\hat{\mathbf{k}}_i = \hat{x}\sin\theta_i\cos\sigma_i + \hat{y}\sin\theta_i\sin\sigma_i - \hat{z}\cos\theta_i$, as shown in Figure 4.1. Finite 2-D surface profiles with specified statistics are generated using a Fourier transform (FT) technique [15] as shown in Section A.2 in Appendix A. The surface height function $z = f(x, y)$ has zero mean and its maximum and minimum height variations are denoted by $z_{\text{max}}$ and $z_{\text{min}}$, respectively. The incident field $E^i(x, y, z)$ is tapered with a Gaussian beam amplitude pattern confining the illuminated rough surface to the rectangular surface area $D_x \times D_y$ so that surface edges do not contribute strongly to the scattered fields. The tapered incident field will be described in more detail in the next section.

Let $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$ and $\mathbf{r}' = \hat{x}x' + \hat{y}y' + \hat{z}z'$ denote a field point and a source point on the rough surface, respectively. Then, the magnetic field integral equation (MFIE) on the PEC rough surface is given by

\[
J(\mathbf{r}) = J_{\mathcal{P}O}(\mathbf{r}) + 2\pi \int_{\mathcal{P} \setminus S_{xy}} dx' dy' \nabla g(\mathbf{r}, \mathbf{r}') \times J(\mathbf{r}').
\]  
(4.1)

where the above integral is a principal-value integral.

\[
g(\mathbf{r}, \mathbf{r}') = \frac{e^{ikR}}{4\pi R}
\]  
(4.2)

\[
\nabla g(\mathbf{r}, \mathbf{r}') = G(R)\mathbf{R}
\]  
(4.3)

\[
G(R) = \frac{e^{ikR}}{4\pi R^2} \left( ik - \frac{1}{R} \right).
\]  
(4.4)
Figure 4.1: A 2-D PEC rough surface profile $S$ illuminated by a *tapered* incident field $E'(x, y, z)$ centered at the origin and propagating in direction $k_i = \hat{x} \sin \theta_i \cos \phi_i + \hat{y} \sin \theta_i \sin \phi_i - \hat{z} \cos \theta_i$. 
\[ V \] denotes the gradient operator in the three-dimensional coordinate system. \( J(r) = n \times H(r) \). \( J_{P\theta}(r) = 2n \times H'(r) \). \( n = \hat{z} - \hat{x} \frac{\partial}{\partial x} - \hat{y} \frac{\partial}{\partial y} \). \( R = r - r' \) and \( R = |R| \). where \( H' \) is the incident magnetic field associated with \( E' \). The normal vector \( n \) (not a unit vector) points upward from the PEC surface and \( S_{xy} \) is the surface obtained from the projection of the rough surface \( S \) onto the \( x - y \) plane. As usual, this surface integral equation can be solved using a standard point-matching MM technique \([81]-[86]\).

For the purpose of the MM formulation, the rectangular surface area \( D_x \times D_y \) of \( S_{xy} \) is discretized into the \( N \times M \) rectangular grid as shown in Figure 4.2. Each \((n,m)\) th element has the dimension \( \Delta x \times \Delta y \) and its center is located at \((x_n, y_m)\) in the coordinate system, where \( n = 1, \ldots , N \) and \( m = 1, \ldots , M \). Let \( N_{tot} = 2NM \) be the total number of unknowns on the surface \( S \) (due to 2 independent vector components for \( J(r) \)). Using pulse basis functions and delta testing functions (i.e. point matching), the above MFIE can be discretized into the following MM matrix
\begin{equation}
\vec{Z} \vec{I} = \vec{V},
\end{equation}

where \( \vec{Z} \) is the \( N_{tot} \times N_{tot} \) MM impedance matrix, \( \vec{V} \) is the \( N_{tot} \times 1 \) excitation vector and \( \vec{I} \) is the \( N_{tot} \times 1 \) solution vector. It is noted that \( x \)- and \( y \)-components of \( \mathbf{J}(\vec{r}) \) for all current elements are stored in \( \vec{I} \) as shown below:

\begin{equation}
\vec{I} = [J_{x}^{(1,1)}, J_{y}^{(1,1)}, \ldots, J_{x}^{(1,M)}, J_{y}^{(1,M)}, J_{x}^{(2,1)}, J_{y}^{(2,1)}, \ldots, J_{x}^{(N,1)}, J_{y}^{(N,1)}, \ldots, J_{x}^{(N,M)}, J_{y}^{(N,M)}]^{T},
\end{equation}

where \( J_{p}^{(n,m)} \) denotes a \( p \)-polarized \( (p = x \) or \( y \) \) current at the \( (n,m) \) th element on the rough surface \( S \) and the superscript \( T \) denotes the transpose operation. It is not necessary to store the \( z \)- component of \( \mathbf{J}(\vec{r}) \) since it totally depends on \( x \)- and \( y \)-components. Using pulse basis functions, point matching and a linear surface model over each grid cell (no surface curvature), it can be shown that the principal-value integration over the self \( (n,m) \) th element \( S_{self} \) vanishes: i.e.

\begin{equation}
2\mathbf{n} \times \int_{pV:S_{self}} dx dy \, G(\vec{r}) \mathbf{R} \times \mathbf{J}(\vec{r}) = 0.
\end{equation}

Thus, for each current element, the \( x \)- and \( y \)-components of the current are uncoupled and Eq. (4.5) can be written more explicitly as follows:

\begin{equation}
\begin{bmatrix}
1 & 0 & X & X & \cdots & X & X & \cdots & X & X \\
0 & 1 & X & X & \cdots & X & X & \cdots & X & X \\
X & X & 1 & 0 & \cdots & X & X & \cdots & X & X \\
X & X & 0 & 1 & \cdots & X & X & \cdots & X & X \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
X & X & X & X & \cdots & 1 & 0 & \cdots & X & X \\
X & X & X & X & \cdots & 0 & 1 & \cdots & X & X \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
X & X & X & X & \cdots & X & X & \cdots & 1 & 0 \\
X & X & X & X & \cdots & X & X & \cdots & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
J_{x}^{(1,1)} \\
J_{y}^{(1,1)} \\
J_{x}^{(1,2)} \\
J_{y}^{(1,2)} \\
\vdots \\
J_{x}^{(n,m)} \\
J_{y}^{(n,m)} \\
J_{x}^{(N,M)} \\
J_{y}^{(N,M)}
\end{bmatrix}
= \begin{bmatrix}
J_{PO,x}^{(1,1)} \\
J_{PO,y}^{(1,1)} \\
J_{PO,x}^{(1,2)} \\
J_{PO,y}^{(1,2)} \\
\vdots \\
J_{PO,x}^{(n,m)} \\
J_{PO,y}^{(n,m)} \\
J_{PO,x}^{(N,M)} \\
J_{PO,y}^{(N,M)}
\end{bmatrix}.
\end{equation}
where $X$ denotes a mutual impedance between a pair of currents and $J_{PO,p}^{(n,m)}$ denotes a $p$-polarized ($p = x$ or $y$) PO current at the $(n,m)$th element on the rough surface $S$.

To develop the FB method for the 2-D RSS, it is required to make the following decomposition for the current vector $J(r)$:

$$J(r) = J^f(r) + J^b(r).$$

where $J^f(r)$ and $J^b(r)$ are the forward-stepping (FS) and backward-stepping (BS) currents respectively. The FS and BS processes as described in Tran [111] are illustrated in Figure 4.3 (a) and (b) respectively. For the FS process as shown in Figure 4.3 (a), the procedure starts from the $(x_1, y_1)$ current element and moves from bottom to top and then from left to right until reaching the $(x_n, y_m)$ current element, called the receiving element hereinafter. For the BS process, the sweep starts from the $(x_N, y_M)$ current element and moves from top to bottom and then from right to left until reaching the receiving element as shown in Figure 4.3 (b). It should be mentioned that these stepping processes can be defined differently. For example, for the FS process, one may start from the $(x_1, y_1)$ current element and move from left to right first and then from bottom to top as illustrated in Figure 4.3 (c). However, for the finite rectangular surface where $D_x > D_y$, it can be shown numerically that the FS and BS schemes given in Figure 4.3 (a) and (b) require a smaller number of plane waves when the NSA algorithm is incorporated into the FB method. In addition, Tran observed that these schemes shown in Figure 4.3 (a) and (b) usually provide better convergence except for some types of surfaces that exhibit surface shape resonances. Finally, it should be pointed out that the FS and BS processes for 2-D rough surfaces are not unique as discussed above, while those for 1-D rough surfaces are
unique as illustrated in Section 2.2 in Chapter 2. For the 1-D case, the FS and BS processes correspond to the “forward” and “backward” propagations of EM waves on 1-D rough surfaces respectively, where the “forward” direction is defined to be the direction of the incident field propagating along rough surfaces, and the “backward” direction is defined to be the opposite of the “forward” direction. For the 2-D case, it is not possible to define FS and BS processes such that they are exactly corresponding to the “forward” and “backward” propagations since the formers always involve with two propagating directions, while the latters involve only one propagating direction.

Substituting Eq. (4.9) into Eq. (4.1). Eq. (4.1) can be separated into two coupled integral equations:

\[
J^f(r) = J_{PO}(r) + 2n \int \int_{P \setminus S^b_{xy}} dx \, dy \nabla g(r, r') \times [J^f(r') + J^b(r')]
\] (4.10)

\[
J^b(r) = 2n \int \int_{P \setminus S^f_{xy}} dx \, dy \nabla g(r, r') \times [J^f(r') + J^b(r')]
\] (4.11)

where \(S^f_{xy}\) and \(S^b_{xy}\) are the surfaces corresponding to the FS and BS processes respectively, as shown in Figure 4.3 (d). It is noted that adding Eq. (4.10) and Eq. (4.11) results in Eq. (4.1) due to the fact that \(S_{xy} = S^f_{xy} + S^b_{xy}\). The integral term on the right-handed side (RHS) of Eq. (4.10) represents the FS contribution due to the radiation of current elements belonging to the surface \(S^f_{xy}\). Similarly, the integral term on the RHS of Eq. (4.11) represents the BS contribution due to the radiation of current elements belonging to the surface \(S^b_{xy}\). Eq. (4.10) and (4.11) can be solved using an iterative method, the FB method, by first initializing \(J^{b(0)}(r) = 0\), and at the \(k\)th
Figure 4.3: Forward-stepping (FS) and backward-stepping (BS) processes: (a) FS process (b) BS process (c) Alternative FS process (d) FS and BS projected surfaces.
(k ≥ 1) iteration,

\[ J^{f,(k)}(r) = J_{PO}(r) + 2n \times \int \int_{\mathcal{P}_V \setminus \mathcal{S}_y^r} dx \, dy \nabla g(r, r) \times [J^{f,(k)}(r) + J^{b,(k-1)}(r)] \] (4.12)

\[ J^{b,(k)}(r) = 2n \times \int \int_{\mathcal{P}_V \setminus \mathcal{S}_y^r} dx \, dy \nabla g(r, r) \times [J^{f,(k)}(r) + J^{b,(k)}(r)] \] (4.13)

The currents \( J^{f,(k)}(r) \) in Eq. (4.12) are first solved for all receiving elements, and then employed in Eq. (4.13) to solve for the currents \( J^{b,(k)}(r) \) for all receiving elements. The iterative process is continued in the FB fashion until the surface currents exhibit convergence to within a specified accuracy criterion. The normalized pseudo-residual (\( PR_N^{(k)} \)) and normalized residual (\( R_N^{(k)} \)) are used to monitor the convergence of the FB method, and are defined as follows for the kth iteration:

\[ PR_N^{(k)} = \frac{|| \overline{T}(k) - \overline{T}(k-1) ||}{|| \overline{T}(k) ||} \] (4.14)

\[ R_N^{(k)} = \frac{|| \overline{r} - \overline{ZT}(k) ||}{|| \overline{r} ||} \] (4.15)

where || ⋅ || is the vector norm. The convergence test based on \( PR_N^{(k)} \) is employed first. Once the \( \overline{T}(k) \) vector satisfies this test, another convergence test based on \( R_N^{(k)} \) is employed to ascertain accuracy of the final solution. If \( \overline{T}(k) \) does not satisfy a specified accuracy criterion for \( R_N^{(k)} \), the accuracy criterion for \( PR_N^{(k)} \) is reduced further and the two-step testing is started again until both residuals of \( \overline{T}(k) \) are satisfied. The reason for performing this testing procedure is to reduce the number of matrix-vector multiplies used in the \( R_N^{(k)} \) convergence test while keeping the desired accuracy. It is noted that the \( PR_N^{(k)} \) convergence test does not require a matrix-vector multiply; however, it may not be an acceptable stopping test for general scattering problems. In contrast, the \( R_N^{(k)} \) convergence test is a desirable stopping test since the solution
errors are tied directly to the accuracy of the elements of the impedance matrix $\mathbf{Z}$ and the excitation vector $\mathbf{V}$ [87]. However, it requires one more matrix-vector multiply per testing. From numerical experience in 2-D RSS problems of interest, the convergence test based on $PR_{\\chi}^{(k)}$ seems to yield quite accurate results for tolerance of 0.01. Thus, from now on, the $PR_{\\chi}^{(k)}$ convergence test is employed only.

Although the FB method provides very rapid convergence in many RSS problems, it requires a direct computation of the matrix-vector multiplies to compute the mutual coupling between all pairs of points on the rough surface. In addition, the impedance matrix $\mathbf{Z}$ must be stored at a cost of $\mathcal{O}(N^2_{tot})$ memory storage or all elements of the matrix must be recomputed at each iteration with a time-consuming computation. One way to dramatically improve the computational efficiency of the FB method is to incorporate the NSA algorithm into the FB method. This algorithm will be discussed in Section 4.4. In the next section, the tapered incident field is described in detail.

### 4.3 Tapered Incident Field

In order to reduce the edge effect of the finite surface used in numerical models of RSS problems, a tapered incident field is employed in Monte Carlo simulations. The incident field is tapered with a Gaussian beam amplitude pattern so that the illuminated rough surface can be confined to the surface area $D_x \times D_y$. To satisfy Maxwell's equations exactly, a spectrum of vector plane waves is utilized. Consider an incident field centered in direction $\hat{k}_i = \hat{x} \sin \theta_i \cos \phi_i + \hat{y} \sin \theta_i \sin \phi_i + \hat{z} \cos \theta_i$ with wave number $k$. where $\theta_i$ and $\phi_i$ refer to the incident polar and azimuthal angles, respectively. The general plane wave spectral form of an arbitrary incident field is
given by

\[
E^t(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{ik \cdot r} \Psi(k_x, k_y) \left[ -E_{TM} \hat{h}(-k_z) + E_{TE} \hat{e}(-k_z) \right] (4.16)
\]

\[
H^t(r) = -\frac{1}{\eta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{ik \cdot r} \Psi(k_x, k_y) \left[ E_{TM} \hat{e}(-k_z) \\
+ E_{TE} \hat{h}(-k_z) \right]. (4.17)
\]

where

\[
\hat{e}(-k_z) = \frac{x k_y}{k_{\rho}} - \frac{y k_x}{k_{\rho}} (4.18)
\]

\[
\hat{h}(-k_z) = \frac{x k_z k_y}{kk_{\rho}} + \frac{y k_y k_z}{kk_{\rho}} + \frac{z k_{\rho}}{k} (4.19)
\]

\[ k_{\rho} = \sqrt{k_x^2 + k_y^2}, \quad k_z = (k^2 - k_{\rho}^2)^{1/2}, \quad k = \omega \sqrt{\mu \varepsilon}, \quad \text{and} \quad \eta \text{ is the}
\]

impedance of free space [25]. In Eqs. (4.16) and (4.17), \( \Psi(k_x, k_y) \) is the spectrum of

the incident field defined as

\[
\Psi(k_x, k_y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dy \, e^{-i(k_x x + k_y y)} e^{i(k_{x} x + k_{y} y)} e^{-t} (4.20)
\]

where

\[
t = t_x + t_y (4.21)
\]

\[
t_x = \frac{(x \cos \theta_i \cos \phi_i + y \cos \theta_i \sin \phi_i)^2}{g_x^2 \cos^2 \theta_i} (4.22)
\]

\[
t_y = \frac{(-x \sin \phi_i + y \cos \phi_i)^2}{g_y^2} (4.23)
\]

where \( g_x \) and \( g_y \) are taper parameters defined as \( g_x = \frac{D_x}{g'} \) and \( g_y = \frac{D_y}{g'} \) and \( g' \) is a constant. Note that Eqs. (4.16) and (4.17) can be evaluated efficiently using the 2-D fast Fourier transform (FFT) algorithm [88].

The parameters \( g_x \) and \( g_y \) control the tapering of the incident field. The term \( t \) is introduced to approximate the tapered wave solution that was previously used for
the scalar wave case [58]. For TE incidence, $E_{TE} = 1$ and $E_{TM} = 0$, while for TM incidence, $E_{TE} = 0$ and $E_{TM} = 1$. However, this tapered beam choice lacks of clear polarization at normal incidence as pointed out in [137], and another tapered beam choice is proposed as follows:

$$E'(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{ik_r} \Psi_n(k_x, k_y) \left[ E_{TE} \hat{h}_n(-k_z) \right.$$

$$+ E_{TM} \hat{e}_n(-k_z) \right]$$

(4.24)

$$H'(r) = \frac{1}{\eta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{ik_r} \Psi_n(k_x, k_y) \left[ E_{TE} \hat{e}_n(-k_z) \right.$$  

$$- E_{TM} \hat{h}_n(-k_z) \right].$$

(4.25)

where

$$\hat{e}_n(-k_z) = \hat{x} + \frac{k_x}{k_z}$$

(4.26)

$$\hat{h}_n(-k_z) = \frac{k_x k_y}{kk_z} - \frac{k_x^2 + k_y^2}{kk_z} - \frac{k_y}{k}$$

(4.27)

$$\Psi_n(k_x, k_y) = \frac{g_x g_y}{4\pi} e^{-\frac{1}{2}(k_x^2 g_x^2 + k_y^2 g_y^2)}.$$
4.4 Formulation of the 2-D FB/NSA Algorithm Using Spectral Domain Representation of the Green’s Function in the x-Direction

As shown in Section 4.2, the FB method requires computation of the mutual coupling between all pairs of points on the rough surface resulting in an $O(N_{tot}^2)$ algorithm. In order to accelerate this mutual coupling computation, the NSA algorithm is employed to obtain $O(N_{tot})$ for both CPU time and memory storage requirements. For convenience in understanding the FB/NSA method, the computation of the FS process is considered only: the BS computation can also be treated in a similar fashion. The NSA algorithm starts with selection of a neighborhood distance in the x-direction, $L_x$, within which interactions between points are classified as strong and outside of which interactions are classified as weak, as illustrated in Figure 4.4. It can be seen that the FS surface $S_{xy}$ is decomposed into the FS strong ($S_{xy,s}$) and weak ($S_{xy,w}$) regions. Using this decomposition, Eq. (4.10) can be rewritten as

$$J_{f,(k)}(r) = J_{PO}(r) + J_{s,(k)}(r) + J_{w,(k)}(r).$$  (4.29)

where

$$J_{s,(k)}(r) = 2n \times \int_{PV,S_{xy,s}} dx dy \frac{G(R) R}{J_{f,(k)}(r)} dR$$  (4.30)

$$J_{w,(k)}(r) = 2n \times \int_{S_{xy,w}} dV dy \frac{\nabla g(r)}{J_{f,(k)}(r)} dR$$  \times [J_{f,(k)}(r) + J_{b,(k-1)}(r)].$$  (4.31)

and $S_{xy} = S_{xy,s} + S_{xy,w}$. The terms $J_{f,(k)}(r)$ and $J_{w,(k)}(r)$ in Eq. (4.29) represent the strong and weak region contributions, respectively. The current $J_{s,(k)}(r)$ is computed in the conventional manner and the NSA algorithm is employed to compute the current $J_{w,(k)}(r)$. 

150
For convenience in illustration, only the NSA formulation based on the spectral domain representation of the free space 3-D scalar Green's function in the $x$-direction is considered in this section, and in the next section an additional NSA formulation based on the spectral domain representation of the Green's function in the $y$-direction is incorporated into the former formation to improve the computational efficiency of the strong-region computation associated with rough surfaces with relatively large surface cross-range size $D_y$. The former NSA algorithm starts with the spectral integral representation of the Green's function $g(r,r')$ for $x - x' > 0$:

$$g(r,r') = \frac{i}{8\pi^2} \int_{C_{k_x}} \int_{C_{k_y}} dk_z dk_y \frac{e^{ik\cdot R}}{k_x}.$$  \hspace{1cm} (4.32)
where

\[ k = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z. \quad (4.33) \]

\[ k_x = (k^2 - k_y^2 - k_z^2)^{1/2}, \] and \( x \) and \( x' \) denote the \( x \)-coordinates of a field (receiving) point and a source point, respectively [131,132]. The contours \( C_{k_x} \) and \( C_{k_y} \) are the original contours of integration on the real axes in the complex \( k_x \) and \( k_y \) planes, respectively. Note that the order of integration in Eq. (4.32) is arbitrary at this point. and one spectral variable (\( k_y \) or \( k_z \)) always depends on another spectral variable (\( k_z \) or \( k_y \), respectively) via \( k_z \) as defined above. In addition, it should be pointed out that spectral integral representation of \( g(\mathbf{r}, \mathbf{r'}) \) is employed instead of the angular spectral integral representation since it can be shown numerically that the former is more efficient than the latter when performing the numerical integration: i.e. the former requires less number of integration points than the latter although the former topology in the complex planes is more sophisticated than the latter topology. The topology associated with the spectral integral representation of \( g(\mathbf{r}, \mathbf{r'}) \) is studied in detail in Appendix C.

Applying the gradient operator (\( \nabla \)) to \( g(\mathbf{r}, \mathbf{r'}) \) in Eq. (4.32), the spectral integral representation of \( \nabla g(\mathbf{r}, \mathbf{r'}) \) is obtained as

\[ \nabla g(\mathbf{r}, \mathbf{r'}) = -\frac{1}{8\pi^2} \int_{C_{k_x}} \int_{C_{k_y}} dk_x dk_y \frac{ke^{i\mathbf{k}\cdot\mathbf{R}}}{k_z}. \quad (4.34) \]

Substituting Eq. (4.34) into Eq. (4.31) and interchanging the spatial and contour integrations, Eq. (4.31) can be rewritten as

\[ J_w^{(k)}(\mathbf{r}) = -\frac{1}{4\pi^2} \mathbf{n} \times \int_{C_{k_x}} \int_{C_{k_y}} dk_x dk_y \frac{F^{(k)}(\mathbf{r}, k_z, k_y)}{k_z}. \quad (4.35) \]
where $F^{(k)}(r, k_z, k_y)$ is called the complex vector radiation function (or plane wave spectrum) on the $k$th iteration, defined as

$$F^{(k)}(r, k_z, k_y) = \int_0^L \int_0^{2\pi} dx dy \, \mathbf{V}^{(k)}(r) e^{i\mathbf{k} \cdot \mathbf{R}}.$$  \hspace{1cm} (4.36)

where

$$\mathbf{V}^{(k)}(r) = \mathbf{k} \times \left[ J_{f^{(k)}}(r) + J_{b^{(k-1)}}(r) \right].$$  \hspace{1cm} (4.37)

Thus, weak element contributions $J_{w^{(k)}}(r)$ can be obtained through a spectral domain integral of the complex vector radiation function $F^{(k)}(r, k_z, k_y)$. As in the 1-D surface case discussed earlier in Chapter 2, the most important property of $F^{(k)}(r, k_z, k_y)$ in the NSA approach is that it can be computed from currents in the weak region in a recursive manner. To see this, let $r_{(n,m)}$ denote a position vector starting from the origin of the coordinate system to the center of the $(n,m)$th element on the surface $S$, where $n = 1, \ldots, N$ and $m = 1, \ldots, M$. Discretizing the double spatial integral on the RHS of Eq. (4.36), the radiation function $F^{(k)}(r, k_z, k_y)$ in the FS direction can be recursively computed through a "phase shifting" process as follows. For $n > N_x$; i.e. the weak region $S_{fyy,w}$ exists,

$$F^{(k)}(r_{(n,m)}, k_z, k_y) = \begin{cases} F^{(k)}(r_{(n-1,M)}, k_z, k_y) e^{i\Psi_1} + F_{add}^{(k)} & m = 1 \\ F^{(k)}(r_{(n,m-1)}, k_z, k_y) e^{i\Psi_3} & m \neq 1 \end{cases} \hspace{1cm} (4.38)$$

$$F_{add}^{(k)} = \Delta S_{xy} \sum_{i=1}^M \mathbf{V}^{(k)}(r_{(n-N_x,i)}) e^{i\Psi_2}. \hspace{1cm} (4.39)$$

where

$$F^{(k)}(r_{(n,m)}, k_z, k_y) = 0, \hspace{1cm} 1 \leq n \leq N_x \hspace{1cm} (4.40)$$
\[ \Psi_1 = k \cdot [r_{(n,1)} - r_{(n-1,M)}] \]  
(4.41)

\[ \Psi_2 = k \cdot [r_{(n,1)} - r_{(n-Nx,i)}] \]  
(4.42)

\[ \Psi_3 = k \cdot [r_{(n,m)} - r_{(n,m-1)}] \]  
(4.43)

\[ N_x = \frac{L_x}{\Delta x}, \quad \text{and} \quad \Delta S_{xy} = \Delta x \Delta y. \]

Eq. (4.38) illustrates that the function \( F^{(k)}(r_{(n,m)}, k_z, k_y) \) continuously updates when the receiving element is at the bottom of the 2-D grid \( S_{xy} \) \((m = 1)\) as a new set of source element enters the weak interaction group. For \( m \neq 1 \), the function \( F^{(k)}(r_{(n,m)}, k_z, k_y) \) can be computed by simply multiplying the previous value \( F^{(k)}(r_{(n,m-1)}, k_z, k_y) \) by a "phase" function. Note that this recursive property of \( F^{(k)}(r, k_z, k_y) \) makes the NSA method an efficient algorithm for both computational and memory requirements. Next, consider the topology of two coupled complex \( k_y \) and \( k_z \) planes associated with the double spectral integral representation of \( g(r, rt) \). 

The topology of two coupled complex \( k_y \) and \( k_z \) planes totally depends on the order of integration in Eq. (4.32). For the 2-D NSA formulation in this chapter, the integration on \( k_y \) is performed first, and then follow by the integration on \( k_z \): i.e. the spectral integral representation of \( g(r, rt) \) in Eq. (4.32) is rearranged as follows:

\[ g(r, rt) = \frac{i}{8\pi^2} \int_{C_{k_z}} dk_z e^{ik_z(z-z')} \int_{C_{k_y}} dk_y \frac{e^{i[k_z(x-x') + k_y(y-y')]} k_z}{k_z}. \]  
(4.44)

It is obvious from the above equation that the spectral variable \( k_y \) is dependent on \( k_z \) through \( k_z \) resulting in the dependence of the topology in the complex \( k_y \) plane on the topology in the complex \( k_z \) plane as will be discussed in detail later. It should be pointed out that another order of integration (i.e. integrating on \( k_z \) first and then integrating on \( k_y \)) is also possible. and it might yield some advantages in the 2-D NSA algorithm. This issue will be investigated later for future research. Figures 4.5, 4.6
Figure 4.5: Integration contour of \( g(\tau, r/l) \) on the complex \( k_z \) plane. \( C_{k_z} \) is the original contour and \( C_{\delta k_z} \) is the deformed contour. The SDP and SAP contours for a flat surface (\( \Delta z = 0 \)) are also shown.

Figure 4.6: Integration contour of \( g(\tau, r/l) \) on the complex \( k_z \) plane. \( C_{SDP}(\Delta z = \Delta z_{max}) \) and \( C_{SAP}(\Delta z = \Delta z_{max}) \) are the SDP and SAP contours when \( \Delta z = \Delta z_{max} \) respectively.
Figure 4.7: Integration contour of $g(r, r')$ on the complex $k_y$ plane for a fixed value of $k_z$. $C_{k_y}$ is the original contour and $C_{\delta k_y}$ is the deformed contour.

and 4.7 illustrate various contours in the complex $k_z$ and $k_y$ planes (see Appendix C for detail). A solid line indicates that a contour is in the proper Riemann sheet which satisfies the radiation condition for the $e^{-i\omega t}$ harmonic time convention, while a dashed line indicates that a contour is in the improper Riemann sheet. Note also that wavy lines denote branch cuts. To gain numerical efficiency when performing the numerical contour integration, the original contours $C_{k_z}$ and $C_{k_y}$ are deformed to the new contours $C_{\delta k_z}$ and $C_{\delta k_y}$ respectively as shown in the above figures. This is possible since there are no singularities in between the original and deformed contours. These contour deformations yield smaller integration intervals and smaller sampling rates for evaluating the double spectral integral involving $F^{(k)}(r, k_z, k_y)$. This is due to the fact that the function $F^{(k)}(r, k_z, k_y)$ is relatively smooth and localized in the complex
$k_z$ and $k_y$ planes along the deformed complex contours, as illustrated via numerical examples later in Section 4.8, whereas $F^{(k)}(r, k_z, k_y)$ is highly oscillatory along the real axes, especially as the size of the weak region increases. Like the 1-D NSA algorithm discussed earlier in Section 2.2 in Chapter 2, these contour deformations are necessary to make the 2-D NSA method an $O(N_{tot})$ algorithm as well as the recursive property of $F^{(k)}(r, k_z, k_y)$. Without these contour deformations, it is required to increase the sampling rate to evaluate the double spectral integral involving $F^{(k)}(r, k_z, k_y)$ accurately as the size of the weak region increases due to more oscillation of $F^{(k)}(r, k_z, k_y)$ along the original contours $C_{k_z}$ and $C_{k_y}$. Increasing the sampling rate as the surface size increases makes the efficiency of the 2-D NSA method degrade considerably. Thus, it is emphasized that both the recursive property of $F^{(k)}(r, k_z, k_y)$ and contour deformations are necessary in the 2-D NSA algorithm.

Advantageous deformed contours $C_{\delta_{k_z}}$ and $C_{\delta_{k_y}}$ can be determined by considering the spectral integral representation of $g(r,r')$. Eq. (4.44) can be rewritten as

$$g(r,r') = \frac{i}{8\pi^2} \int_{C_{\delta_{k_z}}} dk_z e^{ik_z(z-z')} \int_{C_{\delta_{k_y}}} dk_y \frac{e^{i(k_z(z-z') + k_y(y-y')}}}{k_z}.$$  (4.45)

It can be seen from Eq. (4.45) that the two spectral variables $k_z$ and $k_y$ are coupled through $k_x = (k^2 - k_y^2 - k_z^2)^{\frac{1}{2}}$. As $k_z$ changes along the contour $C_{\delta_{k_z}}$, the topology in the complex $k_y$ plane is modified. For convenience, first consider the topology in the complex $k_z$ plane for a single pair of points $r$ and $r'$ on a flat surface. In Figure 4.5, there are a pair of branch cuts originating at the branch points $\pm k$. For a flat surface, the steepest descent path (SDP) $C_{SDP}(\Delta z = 0)$ and steepest ascent path (SAP) $C_{SAP}(\Delta z = 0)$ intersect at the saddle point at the origin in the complex $k_z$ plane. From an asymptotic analysis, most of the contribution occurs on portions of the SDP path near a saddle point on the real axis. As the distance from the saddle
point increases along the SDP path, the integrand of Eq. (4.45) with respect to \( k_z \) is exponentially attenuated so that the contributions become negligible. Thus, it is numerically advantageous to deform the original contour \( C_k \) to the SDP contour for a flat surface \( C_{SDP}(\Delta z = 0) \).

An SDP path can also be determined for a pair of points \( r \) and \( r' \) with \( \Delta z = |z - z'| \neq 0 \), for which the saddle point is located at

\[
k_{z,s} = \frac{k(z - z')}{\sqrt{|\rho - \rho'|^2 + (z - z')^2}},
\]

where \( \rho = \hat{x}x + \hat{y}y \) and \( \rho' = \hat{x}x' + \hat{y}y' \) as illustrated in Figure 4.6, but the path equation is more complicated. However when coupling between many pairs of points is considered as in the weak region contribution to the \((n, m)\) th receiving element, there is no longer a unique SDP path for a rough surface along which only attenuation of the integrand is obtained away from a single saddle point. Thus, the deformed contour \( C_{\delta k_z} \) must be chosen as a compromise between extreme exponential growth and rapid oscillation of the integrand. Let \( \Delta z_{\text{max}} \) denote the largest surface variation, which is equal to \( z_{\text{max}} - z_{\text{min}} \). The saddle points in this case are distributed along the real axis between \( \pm k_{z,s_{\text{max}}} \), where \( k_{z,s_{\text{max}}} \) is the outermost possible saddle point on the real axis when \( \Delta z = \Delta z_{\text{max}} \) and \( |\rho - \rho'| = L_z \) (i.e. \( x - x' = L_x \) and \( y - y' = 0 \)). Consider contours \( C_{SDP}(\Delta z = \Delta z_{\text{max}}) \) and \( C_{SAP}(\Delta z = \Delta z_{\text{max}}) \) which intersect at the saddle point \( k_{z,s_{\text{max}}} \), computed numerically in Figure 4.6. It is noted that the portion of the deformed contour \( C_{\delta k_z} \) between \( \pm k_{z,s_{\text{max}}} \) mixes both descent and ascent paths where the integrand of Eq. (4.45) with respect to \( k_z \) may exponentially increase. Note that the magnitude of the exponential integrand in the complex \( k_z \) plane is usually found to be maximum near the intersection point \( I_z \), as shown in Figure 4.6, where the SAP contour \( C_{SAP}(\Delta z = \Delta z_{\text{max}}) \) intersects the deformed contour \( C_{\delta k_z} \), and thus the
pair of points possessing $\pm k_{z,\text{max}}$ (i.e. $\Delta z = \pm \Delta z_{\text{max}}$ and $|\rho - \rho| = L_z$) corresponds to the worst-case scenario in the $\rho - z$ plane. For convenience when performing the numerical integration, the contour $C_{\delta k_z}$ is defined to be a straight line making an angle $\delta k_z$ with respect to the negative real axis in the interval from $-k_{z,\text{max}}$ to $k_{z,\text{max}}$, where $k_{z,\text{max}}$ is the upper limit of integration which will be specified later. Outside this interval, the contour $C_{\delta k_z}$ is deformed and joined to the original contour $C_k$ (the real axis). It is noted that the contribution of the integral along $C_{\delta k_z}$ outside the interval is negligible due to either exponential attenuation or fast oscillation of the integrand.

For a flat surface, $\delta k_z$ is chosen to be $\frac{\pi}{4}$ which is the angle of $C_{\text{SAP}}(\Delta z = 0)$ measured with respect to the negative real axis. For a rough surface, $\delta k_z$ must be chosen to be smaller than or equal to $\frac{\pi}{2}$ to avoid exponential growth rates in the interval from $-k_{z,\text{max}}$ to $k_{z,\text{max}}$, where the contour $C_{\delta k_z}$ mixes both descent and ascent paths.

Several tests illustrate that $\delta k_z = \frac{\pi}{4}$ still provides accurate numerical results when $\tan^{-1}\left(\frac{\Delta z_{\text{max}}}{L_z}\right) \leq 0.1$. However, when $\tan^{-1}\left(\frac{\Delta z_{\text{max}}}{L_z}\right) > 0.1$, $\delta k_z$ can be obtained by limiting the maximum of the integrand on the contour $C_{\delta k_z}$ to $e^{a_{\text{max}}}$. where $a_{\text{max}}$ is some constant to be determined later. Based on several numerical experiments, $a_{\text{max}}$ is typically found to be less than 3. Employing this criterion and performing a first-order Taylor's series expansion of the SAP contour $C_{\text{SAP}}(\Delta z = \Delta z_{\text{max}})$ in the neighborhood of the saddle point $k_{z,\text{max}}$ results in a nonlinear equation which must be solved to determine $\delta k_z$. A two-step procedure is involved: first solving the following nonlinear equation for $\xi$, where $\xi$ is a possible value of $\frac{1}{\tan\delta k_z}$.

$$L_z\sqrt{\tau_1 - \tau_2 + \tau_3} = 0. \quad (4.47)$$
where \( \tau_1 = 0.5 \sqrt{4r_2^2 + r_1^2} \), \( \tau_2 = 0.5 \left[ k^2(1 + x)^2 + k_z^2 \Delta z_{max}(1 - \xi^2) \right] \), \( \tau_3 = (1 + x)a_{max} - k_z \Delta z_{max} \), and \( \tau_4 = 2\xi k_z^2 \). Once \( \xi \) is solved, \( \delta_{k_z} \) can be determined as follows:

\[
\delta_{k_z} = \tan^{-1} \left[ \frac{1}{\max\{x, 1\}} \right] \cdot \tan^{-1} \left( \frac{\Delta z_{max}}{L_x} \right) > 0.1. \quad (4.48)
\]

Note that Eq. (4.47) can be solved numerically for \( \xi \) via a standard root-finding technique such as Muller’s method (see Appendix F). It should be pointed out that the above procedure for determining the tilt angle \( \delta_{k_z} \) for the 2-D NSA algorithm is analogous to the procedure for determining the tilt angle \( \delta \) for the standard 1-D NSA algorithm as illustrated in Section D.1 in Appendix D. The detail of the derivation of Eq. (4.47) can be found in Section E.1 in Appendix E. It should be pointed out that Eq. (4.47) may not work for all cases since its derivation is based on the assumption that the magnitude of the exponential integrand in the complex \( k_z \) plane is maximum near the intersection point \( I_z \). Thus, it is still challenging to rederive the analytical formula for \( \delta_{k_z} \) for an arbitrary of \( k_z \) in the future work.

Next, consider the topology in the complex \( k_y \) plane, which varies with \( k_z \) as illustrated in Eq. (4.45). Note also that surface roughness is not involved in Eq. (4.45) but the \( y \) dimension of the surface \( (D_y) \) appears as a “roughness” parameter. For a fixed value of \( k_z \), the topology in the complex \( k_y \) plane is illustrated in Figure 4.7. A pair of branch cuts originating from the branch points \( \pm \kappa \), where \( \kappa = (k^2 - k_z^2)^{1/2} = \kappa_l + i\kappa_l \), are observed in Figure 4.7 with \( \kappa_l \) and \( \kappa_l \) defined to be greater than or equal to 0. It is noted that \( \kappa \) acts as a complex propagation constant in the \( k_y \) plane. In general, an SDP path can be determined from a pair of points \( r \) and \( r' \) with the saddle point located at

\[
k_{y,s} = \frac{\kappa(y - y')}{|\rho - \rho'|}. \quad (4.49)
\]
Like the analysis in the complex $k_z$ plane, there is no unique SDP path for the rough surface case. The saddle points in this case are distributed along the straight line joining the branch points $\pm \kappa$. The saddle point real coordinate ranges from $-Re[k_y,smaz]$ to $Re[k_y,smaz]$, where $k_y,smaz$ is the outermost saddle point on the above line when $y - y_t = D_y$ and $|\rho - \rho'| = \sqrt{L_x^2 + D_y^2}$ and $Re[\cdot]$ denotes the real part of its argument. The contours $C_{SDP}(y - y_t = D_y)$ and $C_{SAP}(y - y_t = D_y)$ intersect at the saddle point $k_y,smaz$. Let $Re[k_y,ma_z]$ denote the upper limit of integration on the real axis which is specified later. In the interval between $-Re[k_y,ma_z]$ and $Re[k_y,ma_z]$, the deformed contour $C_{\delta_k_y}$ is a straight line with the slope $-\tan(\delta_{k_y})$. Outside this interval, the contour $C_{\delta_k_y}$ is deformed and joined to the original contour $C_{k_y}$. As in the case of the angle $\delta_k$, it can be shown numerically that $\delta_{k_y} = \frac{\pi}{2}$ provides accurate results when $\tan^{-1}\left(\frac{D_y}{L_z}\right) \leq 0.1$. However, when $\tan^{-1}\left(\frac{D_y}{L_z}\right) > 0.1$, it is quite difficult to obtain an analytical formula for $\delta_{k_y}$ due to the complexity of the topology in the complex $k_y$ plane. For convenience in discussion later, let $\gamma (0 < \gamma \leq \frac{\pi}{2})$ be the value of $\delta_{k_y}$ for this case. One solution to this problem is to determine $\delta_{k_y}$ empirically by comparing the analytical solution of $g(r, r')$ to the solution obtained from its spectral domain representation. This empirical procedure of determining $\gamma$ for given a finite surface size, surface statistics, and $L_z$ will be explained in more detail later. Note that the appearance of $D_y$ as a roughness parameter makes the method most suited for rectangular surfaces with large $\frac{D_y}{L_z}$ ratios.

Once the deformed contours $C_{\delta_k_z}$ and $C_{\delta_k_y}$ are known, Eq. (4.35) can be rewritten in terms of $C_{\delta_k_z}$ and $C_{\delta_k_y}$ as

$$J_w^{(k)}(r) = -\frac{1}{4\pi^2} n \times \int_{C_{\delta_k_z}} \int_{C_{\delta_k_y}} dk_z dk_y \frac{F^{(k)}(r, k_z, k_y)}{k_x}.$$  \hspace{1cm} (4.50)
For convenience when performing the numerical double contour integration, the double contour integral of Eq. (4.50) is discretized and mapped to the real axis according to the following mappings: $dk_z \rightarrow \Delta k_z e^{-i\delta k_z}, k_z \rightarrow k_{zp} = p\Delta k_z e^{-i\delta k_z}$ for $p = -P, \ldots, P$, $dk_y \rightarrow \Delta k_y e^{-i\delta k_y}$, and $k_y \rightarrow k_{yp} = q\Delta k_y e^{-i\delta k_y}$ for $q = -Q_p, \ldots, Q_p$, where $\Delta k_z$ and $\Delta k_y$ are the integration step sizes in the complex $k_z$ and $k_y$ planes respectively, and $2P + 1$ is the number of plane waves in the $k_z$ plane. It is noted that $Q_p$ depends on $p$ and it can be shown that $Q_{-p} = Q_p$. Thus, for a fixed $k_z$ ($p$ is fixed), the number of plane waves in the $k_y$ plane is equal to $2Q_p + 1$ and the total number of plane waves in both planes $Q_{TOT,x}$ is given by

$$Q_{TOT,x} = \sum_{p=-P}^{P} (2Q_p + 1),$$

(4.51)

where $P = \frac{k_{z,max}}{\Delta k_z} + 1$ and $Q_p = \frac{Re[k_{y,max}]}{\Delta k_y} + 1$. Using the above mapping, the discretized version of Eq. (4.50) can be written as:

$$J_{w}^{L(k)}(r) = -\frac{1}{4\pi^2} \Delta \Omega \sum_{p=-P}^{P} \sum_{q=-Q_p}^{Q_p} \frac{W(k_{zp}, k_{yp})[\mathbf{F}(k)(r, k_{zp}, k_{yp})]}{k_{zp,yp}} e^{-i\delta k_z} e^{-i\delta k_y}.$$  

(4.52)

where $k_{zp,yp} = (k^2 - k_{y}^2 - k_{z}^2)^{\frac{1}{2}}$, $\Delta \Omega = \Delta k_y \Delta k_z$ and $W(k_{zp}, k_{yp})$ is a weighting function for numerical integration. Various integration parameters in the $k_z$ and $k_y$ planes are given as follows:

$$k_{z,max} = \begin{cases} \sqrt{\frac{20k}{L_z}}, & \tan^{-1}\left(\frac{\Delta z_{max}}{L_z}\right) \leq 0.1 \\ k_{z,smax} + k_{z,tail}, & \tan^{-1}\left(\frac{\Delta z_{max}}{L_z}\right) > 0.1 \end{cases},$$

(4.53)

$$Re[k_{y,max}] = \begin{cases} \sqrt{\frac{20k}{L_z}}, & \tan^{-1}\left(\frac{D_{y}}{L_z}\right) \leq 0.1 \\ Re[k_{y,smax}] + k_{y,tail}, & \tan^{-1}\left(\frac{D_{y}}{L_z}\right) > 0.1 \end{cases},$$

(4.54)

and $k_{z,smax} = \frac{k_{z,smax}}{R_{xz}}$, $\Delta k_z = \frac{1}{2k} \sqrt{C_{z}k}$, $k_{y,smax} = \frac{k_{y}}{R_{xy}}$, and $\Delta k_y = \frac{1}{2k} \sqrt{C_{y}k}$, where $R_{xz} = \sqrt{L_z^2 + (\Delta z_{max})^2}$, $R_{xy} = \sqrt{L_y^2 + D_y^2}$. The constants $k_{z,tail}$, $k_{y,tail}$, $C_z$ and $C_y$
Figure 4.8: The worst-case configuration of a pair of source and field points for computing \( g(\mathbf{r}, \mathbf{r}') \) in the complex \( k_y \) plane.

are to be determined. The first lines of Eqs. (4.53) and (4.54) and the forms for \( \Delta k_z \) and \( \Delta k_y \) are obtained by studying the behaviour of the integrand of Eq. (4.32) for the flat surface case and employing the physical optics approximation for the current distribution on the rough surface. and numerical tests confirm the accuracy of these expressions. It is noted that the constants \( k_{z,\text{tail}} \) and \( k_{y,\text{tail}} \) are some tolerances added to \( k_{z,\text{max}} \) and \( k_{y,\text{max}} \) respectively, to ensure that the integrand of Eq. (4.32) is exponentially decayed. Note that increasing \( C_z \) and \( C_y \) results in a smaller number of plane waves required, while increasing \( k_{z,\text{tail}} \) and \( k_{y,\text{tail}} \) results in a larger number of plane waves.

One way to determine the unknown constants \( a_{\text{max}}, \gamma, k_{z,\text{tail}}, k_{y,\text{tail}}, C_z \) and \( C_y \) for large-scale surfaces is to study the scalar Green's function \( g(\mathbf{r}, \mathbf{r}') \) for a pair of
source and field points in the two worst-case scenarios (i.e. one in the $\rho - z$ plane and another in the $x - y$ plane) for computing $g(r, r')$ in the spectral domain. Comparing the exact solution of $g(r, r')$ to its spectral domain solution within a specified accuracy, these unknown constants can be determined \textit{empirically}. For convenience in illustration, consider only the FS direction. The unknown constants $a_{\text{max}}$ and $k_{z,\text{tail}}$ can be determined \textit{first} by considering only the worst-case scenario in the $\rho - z$ plane as discussed earlier, and then employed to determine the rest of unknown constants by considering the worst-case scenario in the complex $x - y$ plane as shown in Figure 4.8. Figure 4.8 illustrates this worst-case configuration of a pair of source and field points for computing $g(r, r')$ in the complex $k_y$ plane. In this figure, $S_i$ and $F_i$ denote source and field points: respectively, where $i = 1, 2$. In the spectral domain point of view, this configuration of source and field points ($S_1$ and $F_1$ or $S_2$ and $F_2$) yields the outermost saddle point, $k_{y,\text{max}}$ in the $k_y$ plane. The SAP path $C_{S,A,P}(y - y' = D_y)$ passes through $k_{y,\text{max}}$ and intersects the deformed contour $C_{\delta_k}$ at $I_y$, where the integrand is maximum, as shown in Figure 4.7. In addition, $k_{z,s}$ in this configuration is close to $k_{z,\text{max}}$ since $x - x'$ is minimum ($x - x' = L_x$ and assuming that $\Delta z \rightarrow \Delta z_{\text{max}}$). Other source points in the region $S_{\text{zy},w}^I$ yield $-k_{z,\text{max}} < k_{z,s} < k_{z,\text{max}}$ and $-\text{Re}[k_{y,\text{max}}] < \text{Re}[k_{y,s}] < \text{Re}[k_{y,\text{max}}]$. Numerical values of these constants for specific examples are provided in Sections 4.7 and 4.8. In the next section, the computational cost and memory storage requirement of the FB/NSA method for 2-D RSS problems are discussed.

It is noted that the integration parameters given in this section (for large-scale surfaces) may not yield accurate results for a pair of points $r$ and $r'$ where $x - x' \gg L_x$, corresponding to \textit{extremely} large-scale surfaces ($D_z \gg L_x$ and $D_x > D_y$), since
they are determined by considering the worst-case configurations in the $\rho - z$ and $x - y$ planes only (i.e. separated by $x - x_l = L_x$). As discussed earlier in Section 2.4 for the case of 1-D rough surfaces, one way to solve this problem is to decompose the old weak region $S^f_{xy, w}$ into more than one weak regions as illustrated in Figure 4.9 for the FS direction: appropriate integration parameters are then determined for each separate region. This "multilevel" FB/NSA algorithm for 2-D rough surfaces will be discussed in detail later in Section 4.6. In addition, extremely large-scale surfaces with relatively large $D_y$ require a large strong region of approximate size $L_x \times D_y$ or a significant increase in the total number of plane waves $Q_{TOT}$.

To reduce the size of the strong region $S^f_{xy, s}$, the old $L_x \times D_y$ approximate strong region can be decomposed into three separate regions as also shown in Figure 4.9. The surface
$S^f_{xy,rs}$ is the reduced strong region. The surfaces $S^f_{xy,yt}$ and $S^f_{xy,yb}$ are the weak regions which employ the spectral domain expansion in the $y-$ direction for $y < yt$ and $y > yt$, respectively. Numerical results in Section 4.8 illustrate that these decompositions for the old strong region computation can be advantageous when considering 2-D rough surfaces with relatively large $D_y$, and are discussed in detail in the next section.

4.5 Formulation of the 2-D FB/NSA Algorithm Using Spectral Domain Representation of the Green's Function in the $x$- and $y$- Directions

Like the formulation of the 2-D FB/NSA algorithm using the $x-$ expansion in Section 4.4, only the computation of the FS process is considered to illustrate the concept of applying the spectral domain representation of the Green’s function in the $y-$ direction to reduce the computational complexity of the old strong region $S^f_{xy,s}$ for the case of extremely large-scale rough surfaces with relatively large surface cross-range size $D_y$. As shown in Figure 4.9, the strong region contribution $J^f_{s,k}(r)$ defined in Eq. (4.30) can be rewritten as follows:

$$J^f_{s,k}(r) = J^f_{rs,k}(r) + J^f_{w,yt}(r) + J^f_{w,yb}(r).$$ (4.55)

where

$$J^f_{rs,k}(r) = 2\pi \times \int_{P \setminus S^f_{xy,rs}} dxdy G(R) \times [J^f_{s,k}(r) + J^{b,(k-1)}_s(r)]$$ (4.56)

$$J^f_{w,yt}(r) = 2\pi \times \int_{S^f_{xy,yt}} dxdy \nabla g(r, r') \times [J^f_{s,k}(r) + J^{b,(k-1)}_s(r)]$$ (4.57)

$$J^f_{w,yb}(r) = 2\pi \times \int_{S^f_{xy,yb}} dxdy \nabla g(r, r') \times [J^f_{s,k}(r) + J^{b,(k-1)}_s(r)].$$ (4.58)
\[ S_{r,rs}^f = S_{r,rs}^f + S_{r,gt}^f + S_{r,gb}^f. \]

The terms \( J_{r,g}^{f,(k)}(r), J_{w,gt}^{f,(k)}(r) \) and \( J_{w,gb}^{f,(k)}(r) \) in Eq. (4.55) represent the new strong region contribution, and the weak region contributions using the spectral domain representation in the \( y \)-direction for \( y < y_1 \) and \( y > y_1 \), respectively. Note that the term \( J_{r,g}^{f,(k)}(r) \) is computed in the conventional manner, and the NSA algorithm is employed to compute the terms \( J_{w,gt}^{f,(k)}(r) \) and \( J_{w,gb}^{f,(k)}(r) \). Let \( L_y \) be a neighborhood distance in the \( y \)-direction as illustrated in Figure 4.9. Note that the size of the strong region \( S_{r,rs}^f \) is reduced to \( S_{r,rs}^f \) of approximate size \( L_x \times 2L_y \); i.e. the reduction of the direct computation, but the complexity of the NSA algorithm increases. Thus, a neighborhood distance \( L_y \) must be chosen appropriately to compromise between the direct computation of \( J_{r,g}^{f,(k)}(r) \) and the NSA computation of \( J_{w,gt}^{f,(k)}(r) \) and \( J_{w,gb}^{f,(k)}(r) \).

To illustrate the 2-D FB/NSA algorithm using both \( x \)- and \( y \)-expansions, consider the FS process as shown in Figure 4.10 for a simple configuration with the following parameters: \( N = 5, M = 5, N_x = 3 \) and \( N_y = 2 \), where \( N_x = \frac{L_x}{\Delta x} \) and \( N_y = \frac{L_y}{\Delta y} \). The FS process starts from Figure 4.10 (a) and ends with Figure 4.10 (y). Note that the receiving element moves from bottom to top, and then from left to right as illustrated earlier in Figure 4.3 (a). In addition, the weak region employing the NSA algorithm with the \( x \)- expansion is the same as shown in Figure 4.4, and only the old strong region is modified. Thus, it is sufficient to consider only the FS process when the receiving element moves in the vertical direction; i.e. consider along a column instead of a row. Without loss of generality, it is convenient to consider only the "middle" column of Figure 4.10: i.e. Figures 4.10 (k) to 4.10 (o). When the receiving element is located at the bottom as shown in Figure 4.10 (k), only the regions \( S_{r,rs}^f \) (which employs direct computation) and \( S_{r,gt}^f \) (which applies the NSA algorithm with the
y-direction for $y < y_f$) contribute. Note that there is no region $S^f_{xy,yb}$ employing the NSA algorithm with the y-direction for $y > y_f$ in Figure 4.10 (k). As the receiving element moves upward, the region $S^f_{xy,yt}$ decreases, but the region $S^f_{xy,yb}$ increases. Note that the region $S^f_{xy,rs}$ tends to maintain the same size except when the receiving element is near the bottom or the top. Finally, as the receiving element approaches the top (see Figures 4.10 (u) and 4.10 (v)), there are only the regions $S^f_{xy,rs}$ and $S^f_{xy,yb}$, i.e. the region $S^f_{xy,yt}$ vanishes. Next, consider the formulation of the 2-D FB/NSA algorithm using the y-expansion in detail.

Without loss of generality, only the term $J^f_{w,yb}(r)$ associated with the spectral domain representation in the y-direction for $y > y_f$ is considered. The term $J^f_{w,yt}(r)$ for $y < y_f$ can be formulated in a similar fashion, except for minor changes as will be discussed later in this section. The NSA algorithm associated with $J^f_{w,yb}(r)$ starts with the spectral integral representation of the Green's function $g(r, r')$ and $\nabla g(r, r')$ propagating in the y-direction for $y - y_f > 0$ as follows:

$$g(r, r') = \frac{i}{8\pi^2} \int_{C_{k_z}} \int_{C_{k_y}} dk_z dk_y \frac{e^{ik_R}}{k_y} \quad (4.59)$$

$$\nabla g(r, r') = -\frac{1}{8\pi^2} \int_{C_{k_z}} \int_{C_{k_x}} dk_z dk_x \frac{ke^{ik_R}}{k_y} \quad (4.60)$$

where $k$ is defined as in Eq. (4.33) and $k_y = (k^2 - k_z^2 - k_x^2)^{1/2}$. The contours $C_{k_z}$ and $C_{k_x}$ are the original contours of integration on the entire real axis starting from $-\infty$ to $\infty$ in the complex $k_z$ and $k_x$ planes, respectively. Substituting Eq. (4.60) into Eq. (4.58) and interchanging the spatial and contour integrations, Eq. (4.58) can be
Figure 4.10: The FS process in the 2-D FB/NSA algorithm using both x- and y-
 expansions with the following parameters: $N = 5, M = 5, N_x = 3$ and $N_y = 2$. 

169
rewritten as follows:

\[
J_{w,yb}^{(k)}(\mathbf{r}) = -\frac{1}{4\pi^2} \mathbf{n} \times \int_{C_{k_z}} \int_{C_{k_x}} dk_z dk_x \frac{F_{yb}^{(k)}(\mathbf{r}, k_z, k_x)}{k_y}
\]

\[
F_{yb}^{(k)}(\mathbf{r}, k_z, k_x) = \int \int_{S_{xy,yb}^f} d\mathbf{r} d\mathbf{y} \mathbf{V}_{yb}^{(k)}(\mathbf{r}) e^{i\mathbf{k} \cdot \mathbf{r}},
\]

where \( \mathbf{V}_{yb}^{(k)}(\mathbf{r}) \) is defined as

\[
\mathbf{V}_{yb}^{(k)}(\mathbf{r}) = \mathbf{k} \times [J_{f, (k)}^{(k)}(\mathbf{r}) + J_{b, (k-1)}^{(k)}(\mathbf{r})].
\]

As in the case of the \( x \)-expansion, the discretized version of the complex vector radiation function \( F_{yb}^{(k)}(\mathbf{r}_{(n,m)}, k_z, k_x) \) can be computed from currents in the region \( S_{xy,yb}^f \) in a recursive manner through a “phase shifting” process, where \( \mathbf{r}_{(n,m)} \) denotes a position vector starting from the origin of the coordinate system to the center of \( (n,m) \) th element on the surface \( S \). To see this, first consider the region \( S_{xy,yb}^f \) when the receiving element moves in the vertical direction. The radiation function \( F_{yb}^{(k)}(\mathbf{r}_{(n,m)}, k_z, k_x) \) can be computed numerically as follows. For \( 1 \leq m \leq N_y \), where \( N_y = \frac{L_y}{\Delta y} \):

\[
F_{yb}^{(k)}(\mathbf{r}_{(n,m)}, k_z, k_x) = 0
\]

since the weak region \( S_{xy,yb}^f \) does not exist. For \( m > N_y \); i.e. the weak region \( S_{xy,yb}^f \) exists,

\[
F_{yb}^{(k)}(\mathbf{r}_{(n,m)}, k_z, k_x) = \begin{cases} 
F_{1}^{(k)}(\mathbf{r}_{(n,m)}, k_z, k_x) & n = 1 \\
F_{2}^{(k)}(\mathbf{r}_{(n,m)}, k_z, k_x) & 1 < n \leq N_x \\
F_{3}^{(k)}(\mathbf{r}_{(n,m)}, k_z, k_x) & n > N_x
\end{cases}
\]
where

\[
F^{(k)}_1(r_{(n,m)}, k_z, k_x) = \begin{cases} 0 & m \leq N_y \\ F^{(k)}_4(r_{(n-1,m)}, k_z, k_x) e^{i\Psi_{yb,1}} + F^{(k)}_{1,add} & m > N_y \end{cases}
\]

\[
F^{(k)}_2(r_{(n,m)}, k_z, k_x) = F^{(k)}_{yb}(r_{(n-1,m)}, k_z, k_x) e^{i\Psi_{yb,2}} + F^{(k)}_4(r_{(n,m)}, k_z, k_x)
\]

\[
F^{(k)}_3(r_{(n,m)}, k_z, k_x) = \left[ F^{(k)}_{yb}(r_{(n-1,m)}, k_z, k_x) - F^{(k)}_4(r_{(n-1,m)}, k_z, k_x) \right] e^{i\Psi_{yb,2}} + F^{(k)}_4(r_{(n,m)}, k_z, k_x).
\]

and

\[
F^{(k)}_4(r_{(n-1,m)}, k_z, k_x) = \begin{cases} 0 & m \leq N_y \\ F^{(k)}_4(r_{(n-1,m-1)}, k_z, k_x) e^{i\Psi_{yb,1}} + F^{(k)}_{1,add} & m > N_y \end{cases}
\]

\[
F_{1,add}^{(k)} = \Delta S_{xy} V_{yb}^{(k)}(r_{(n,m)}, k_z, k_x) e^{i\Psi_{yb,1}}
\]

\[
F_{4,add}^{(k)} = \Delta S_{xy} V_{yb}^{(k)}(r_{(n-1,m)}, k_z, k_x) e^{i\Psi_{yb,5}}.
\]

The "phase" functions \( \Psi_{yb,j} \), where \( j = 1, \ldots, 5 \), are defined as follows:

\[
\Psi_{yb,1} = k \cdot [r_{(n,m)} - r_{(n,m-1)}]
\]

\[
\Psi_{yb,2} = k \cdot [r_{(n,m)} - r_{(n-1,m)}]
\]

\[
\Psi_{yb,3} = k \cdot [r_{(n-1,m)} - r_{(n-1,m-1)}]
\]

\[
\Psi_{yb,4} = k \cdot [r_{(n,m)} - r_{(n,m-N_y)}]
\]

\[
\Psi_{yb,5} = k \cdot [r_{(n-1,m)} - r_{(n-N_y,m-N_y)}].
\]

To implement the 2-D FB/NSA algorithm associated with the \( y \)-expansion efficiently, it is required to store \( F^{(k)}_{yb}(r_{(n,m)}, k_z, k_x) \) for each row associated with the weak region \( S'_{xy,yb} \); i.e. requiring an additional memory requirement of \( O(MQ_{TOT,y}) \), where \( Q_{TOT,y} \) is the total number of plane waves associated with the \( y \)-expansion. Thus, for the case of \textit{extremely} large-scale 2-D rough surfaces with relatively large \( D_y \) the efficiency
of the standard 2-D FB/NSA algorithm can be improved by incorporating the $y$-expansion, however the total memory requirement is increased as well.

After deforming the contours $C_{k_z}$ and $C_{k_x}$ to $C_{\delta_{k_z}}$ and $C_{\delta_{k_x}}$ respectively. Eq. (4.61) can be discretized as follows:

$$J_{w,yb}^{(k)}(\mathbf{r}) = -\frac{1}{4\pi^2} \Delta \Omega \sum_{u=-U}^{U} \sum_{v=-V}^{V} \frac{W(k_{zu}, k_{xz})[\mathbf{n} \times \mathbf{F}_{ybu}^{(k)}(\mathbf{r}, k_{zu}, k_{xz})]}{K_{ybu}} e^{-i\delta_{k_z}} e^{-i\delta_{k_x}}.$$  

(4.77)

where $k_{wu,v} = (k^3 - k^2_{zu} - k^2_{xz})^{\frac{1}{2}}, W(k_{zu}, k_{xz})$ is a weighting function for numerical integration. $2U + 1$ and $2V + 1$ are the number of plane waves in the complex $k_z$ and $k_x$ planes respectively, and $\delta_{k_z}$ and $\delta_{k_x}$ are the tilt angles of the deformed contours $C_{\delta_{k_z}}$ and $C_{\delta_{k_x}}$ with respect to the negative real axis as shown in Figures 4.6 and 4.11, respectively. The total number of plane waves for $y$-expansion is given by

$$Q_{TOT,y} = \sum_{u=-U}^{U} (2V_u + 1).$$  

(4.78)

Note that the topology in the complex $k_z$ and $k_x$ planes for the $y$-expansion is similar to the topology in the complex $k_z$ and $k_y$ planes for the $x$-expansion, respectively. Thus, the 2-D NSA parameters associated with the $y$-expansion can be obtained using the same approach as employed in the $x$-expansion. Comparing the weak region $S_{xy,w}^f$ for the $x$-expansion and the weak region $S_{xy,yb}^f$ for the $y$-expansion with $y > y_f$, it is observed that the 2-D NSA parameters for the $y$-expansion can be obtained from the corresponding 2-D NSA parameters for the $x$-expansion by simply interchanging the following quantities: $L_z \rightarrow L_y, D_y \rightarrow L_x, k_y \rightarrow k_x, k_z \rightarrow k_y$ and $R_{xz} \rightarrow R_{yz}$.

The last issue associated with the $y$-expansion is that the radiation function $\mathbf{F}_{ybu}^{(k)}(\mathbf{r}_{(n,m)}, k_z, k_x)$ can grow geometrically as the receiving element moves from left
Figure 4.11: Integration contour of $g(r, r')$ on the complex $k_z$ plane for a fixed value of $k_z$. $C_{k_z}$ is the original contour and $C_{\delta k_z}$ is the deformed contour.

to right in the FS process. To see this, consider Eqs. (4.65), (4.67) and (4.68) with an observation that computing $F_y^{(k)}(r_{(n,m)}, k_z, k_x)$ for $n > 1$ involves the previous calculated radiation function $F_y^{(k)}(r_{(n-1,m)}, k_z, k_x)$ and the term $e^{i\Psi_{yb,2}}$. Note that $\Psi_{yb,2}$ defined in Eq. (4.73) can be simplified as follows:

$$\Psi_{yb,2} = k_z \Delta x + k_z \Delta y \cdot [r_{(n,m)} - r_{(n-1,m)}]. \tag{4.79}$$

In general, the magnitude of the term $e^{i\Psi_{yb,2}}$ along the deformed contours $C_{\delta k_z}$ and $C_{\delta z}$ may be greater than, less than or equal to 1.0; i.e.

$$|e^{i\Psi_{yb,2}}| \leq 1.0 \tag{4.80}$$

From Eq. (4.79), it is noted that $\Psi_{yb,2}$ does not involve the term $k_y \Delta y$, which typically assists in controlling $|e^{i\Psi_{yb,2}}|$ to be less than one. If $|e^{i\Psi_{yb,2}}| > 1.0$ along $C_{\delta k_z}$ and

173
as the receiving element moves from left to right, \( F_{y}(k)(r_{(n,m)}, k_{z}, k_{x}) \) will grow geometrically since it involves a repeatedly multiplication process with \( e^{i\Psi_{y,2}} \). It should be pointed out that the above problem arises due to the fact that the NSA parameters for the \( y \)-expansion are chosen based on the two worst-case scenarios (i.e. one in the \( \rho - z \) plane and another in the \( x - y \) plane) as in the case of \( x \)-expansion. The maximum of \( x - x' \) employed in determining the NSA parameters in the complex \( k_{z} \) plane is equal to \( L_{x} \). However, the weak region \( S_{x,y,yb}^{f} \) moves to the right with the approximate distance \( D_{x} \), which is greater than the distance \( L_{x} \) employed in determining the NSA parameters in the FS process. Note that the above problem can be avoided if the maximum of \( x - x' \) is set to be \( D_{x} \) when determining the NSA parameters. However, the total number of plane waves \( Q_{TOT,y} \) increases dramatically if using \( \max\{x - x'\} = D_{x} \) since typically \( D_{x} \gg L_{x} \), and the efficiency of the NSA algorithm for the \( y \)-expansion degrades significantly.

One way to solve this problem is to regenerate the complex vector radiation function \( F_{y}(k)(r_{(n,m)}, k_{z}, k_{x}) \) for a pre-specified horizontal distance \( L_{r} \) to stop the geometrically growing process before \( |F_{y}(k)(r_{(n,m)}, k_{z}, k_{x})| \) grows too large. The distance \( L_{r} \) can be determined by solving the following equation for \( N_{r} \), where \( L_{r} = N_{r} \Delta x \).

\[
\text{Along } C_{\delta_{x}} : \quad \chi^{N_{r}} = 10^{b_{\text{max}}} \tag{4.81}
\]

or

\[
\text{Along } C_{\delta_{x}} : \quad N_{r} = \frac{b_{\text{max}}}{\log \chi} \tag{4.82}
\]

where \( \chi \) is the maximum of \( |e^{ik_{z}\Delta x}| \) and \( b_{\text{max}} \) is a given positive constant. Typically, \( b_{\text{max}} \in [4.0, 6.0] \) yields accurate results with an appropriate value for \( L_{r} \). After \( F_{y}(k)(r_{(n,m)}, k_{z}, k_{x}) \) is generated, it can be computed recursively again using Eqs. (4.65)
to (4.76) until it is regenerated for the next $L_r$. This process continues until the receiving element reaches the final $(N, M)$ th element on the surface $S$.

For the NSA algorithm using the $x$- expansion in Section 4.4. it should be pointed out that the radiation function $F^{(k)}(r_{(n,m)}, k_z, k_y)$ defined in Eq. (4.38) does not grow up too large as the receiving element moves in the $y$- direction from bottom to top in the FS process. The reason for this is that the maximum of $|y - y'|$ employed in determining the NSA parameters in the complex $k_y$ plane is equal to $D_y$, which is equal to the maximum vertical distance $D_y$ that the receiving element can move in the FS process: i.e. the NSA parameters for the $x$- expansion are chosen appropriately. Thus, it is not necessary to regenerate $F^{(k)}(r_{(n,m)}, k_z, k_y)$ for the $x$- expansion.

As discussed earlier, the NSA algorithm associated with $J^f_{w,yt}(r)$ for $y < y_t$ can be formulated in a similar fashion as the NSA algorithm associated with $J^f_{w,yt}(r)$ for $y > y_t$. However, it should be pointed out that the radiation function $F^{(k)}_{yt}(r_{(n,m)}, k_z, k_x)$ associated with $J^f_{w,yt}(r)$ must be computed recursively in an increasing manner of the weak region $S_{x,y,yt}$ otherwise $F^{(k)}_{yt}(r_{(n,m)}, k_z, k_x)$ will grow geometrically. For example, for the middle column of Figure 4.10, $F^{(k)}_{yt}(r_{(n,m)}, k_z, k_x)$ is computed recursively by starting at Figure 4.10 (m) and ending with Figure 4.10 (k). Like $F^{(k)}_{yt}(r_{(n,m)}, k_z, k_x)$ for $y > y_t$, $F^{(k)}_{yt}(r_{(n,m)}, k_z, k_x)$ is also regenerated for every horizontal distance $L_r$. Finally, numerical results in Section 4.8.2 will illustrate that the NSA algorithm employing both $x$- and $y$- expansions yields less CPU requirements than the one using the $x$- expansion for the case of relatively large $D_y$. In the next section, the 2-D “multilevel” FB/NSA algorithm for extremely large-scale surfaces is illustrated.
4.6 The 2-D “Multilevel” FB/NSA Algorithm for Extremely Large-Scale Surfaces

For 2-D extremely large-scale surfaces, the NSA parameters for the \( x \)- expansion given in Section 4.4 may yield inaccurate results for a pair of source and observation points where \( x - x' \gg L_x \). Like the 1-D NSA algorithm, this problem can be solved by decomposing the old weak region \( S^j_{xy,w} \) into more than one weak region as shown in Figure 4.9 for the FS direction. For 1-D extremely large-scale surfaces discussed earlier in Chapter 2, it is found that only a few weak region (or even one) are usually required for most practical problems to obtain the desired accuracy due to the fact that contributions from other far-away weak regions are negligible compared to those for the first few regions. Thus, it is expected that only a few weak regions are required for most practical 2-D extremely large-scale surface problems as well.

Due to mathematical complexity, it is difficult to determine the appropriate sizes of the weak regions \( S^j_{xy,w} \) \textit{analytically}. where \( j = 1, \ldots, T \) and \( T \) is the total number of the weak regions for a given surface size, however the appropriate size of each weak region can be determined \textit{empirically}. Note that \( T \) is an unknown, which will be determined later in this section. To illustrate the above concept, consider the first weak region \( S^{1}_{xy,w} \). The horizontal distance \( L_{w,1} \) of \( S^{1}_{xy,w} \) is chosen such that the relative error obtained from computing the scalar Green's function \( g(r, r') \) using the exact expression (see Eq. (4.2)) and the spectral integral representation (see Eq. (4.45)), for the case of \( x - x' = L_x + L_{w,1}, y - y' = D_y \) and \( \Delta z = \Delta z_{max}, \) is less than a specified tolerance, which is typically set to be 5.0%. In other words, the horizontal distance \( L_{w,1} \) is adjusted, when the variations of the source and observation points in the \( y \)- and \( z \)- directions are maximum, such that the relative error satisfies
a specified tolerance. In the first weak region $S_{2y,w}^1$, it should be pointed out that
the 2-D NSA parameters are fixed before the horizontal distance $L_{w,1}$ is determined
by studying the scalar Green’s function $g(r, r')$ for a pair of source and observation
points in the two worst-case scenarios as discussed earlier in Section 4.4. Once $L_{w,1}$
is known, the 2-D NSA parameters employed in the second weak region $S_{2y,w}^2$ can be
determined empirically by following the same approach as discussed in Section 4.4.
Note that the effective strong distance employed in the calculation of the 2-D NSA
parameters in $S_{2y,w}^2$ is equal to $L_z + L_{w,1}$. This process continues until the following
condition is satisfied:

$$L_x + \sum_{j=1}^{T} L_{w,j} \geq D_z,$$

where $L_{w,j}$ is the horizontal distance of the $j$th weak region $S_{2y,w}^j$, and $T$ is the
minimum integer that satisfies Eq. (4.83). Note that in Eq. (4.83) $L_{w,j}$ is known.
Numerical results in Section 4.8.3 confirm that the accuracy of the standard 2-D
FB/NSA algorithm (using only $x$-expansion) is improved when incorporating the
“multilevel” algorithm for the case of extremely large-scale surfaces. In the next
section, the computational cost and memory storage requirements of the FB/NSA
method for 2-D RSS problems are discussed.

4.7 Computational Cost and Memory Storage Requirement
of the 2-D FB/NSA Method

For convenience, the computational cost and memory storage requirement of the
2-D FB/NSA algorithm using only the $x$-expansion are estimated first for large-scale
surfaces. As discussed in Sections 4.5 and 4.6, for extremely large-scale surfaces with
relatively large $D_y$ it is recommended to employed the 2-D “multilevel” FB/NSA
method using both $x$- and $y$- expansions, and its computational requirements are estimated later in this section.

### 4.7.1 Computational Requirements of the 2-D FB/NSA Method Using the $x$- Expansion

The total operation count ($TOC$) of the 2-D FB/NSA method using the $x$- expansion to compute the strong-region contribution via direct matrix-vector multiply, and the weak-region contribution with the NSA algorithm (see Eqs. (4.38) and (4.52)) is estimated as follows:

$$TOC \sim C_s N_x N_{elmt} + C_w Q_{TOT,x} (N_{elmt} - N_s).$$  \hspace{1cm} (4.84)

where $N_{elmt} = N M = 0.5 N_{tot}$, $N_s = N_x M$ and $C_s$ are constants. It is noted that $N_{elmt}$ is the total number of elements on the surface $S_{xy}$ and $N_s$ is approximately the total number of elements in the strong region. The first term on the RHS of Eq. (4.84) is the number of operations involved in the computation of the strong-region contribution for $N_{elmt}$ receiving elements and the second term involves the number of operation count to compute $Q_{TOT,x}$ plane waves in Eq. (4.52) for $N_{elmt} - N_s$ source elements in the weak region. In addition, the total memory storage requirement ($TMSR$) of the 2-D FB/NSA method using the $x$- expansion is estimated as follows:

$$TMSR \sim C_v N_{elmt} + C_Q Q_{TOT,x}.$$  \hspace{1cm} (4.85)

where $C_s$ are some constants. The first term on the RHS of Eq. (4.85) accounts for the storage of necessary matrices and vectors employed in the 2-D FB/NSA method using the $x$- expansion and the second term involves the storage of total number of plane waves. It is noted from Eq. (4.85) that there is no storage for matrix elements.
associated with the strong region since they are recalculated on each FB iteration to reduce overall memory storage.

Numerical tests show that the neighborhood distance $L_x$ is dependent on $D_y$: as $D_y$ increases, $L_x$ should be increased to compromise between the total number of plane waves employed in the 2-D FB/NSA algorithm using the $x$-expansion and the approximate size $L_x \times D_y$ of the strong region. However, for fixed $D_y$, frequency and surface roughness, the parameters $L_x$, $N_x$ and $Q_{TOT,x}$ are fixed, and as $D_x$ increases it can be seen from Eqs. (4.84) and (4.85) that $TOC$ and $TMSR$ are $O(N_{tot})$. Thus, the 2-D FB/NSA method is an extremely efficient method for studying low-grazing-angle RSS problems in which $D_x >> D_y$.

Figure 4.12 (a) and (b) illustrate plots of CPU time per iteration and CPU memory versus number of unknowns respectively, based on a Pentium II 200 MHz computer with 128 Mbytes RAM. Due to the $D_y$-dependency of the 2-D FB/NSA method using the $x$-expansion, two surface sizes of different $D_y$ are considered. Surfaces sizes of interest are scaled in terms of the electromagnetic wavelength $\lambda$ and sampled with 8 points per $\lambda$. Case 1 has $D_x(\lambda) \times 8\lambda$ Gaussian surfaces with a Gaussian spectrum given by

$$W(k_x, k_y) = \frac{l_x l_y h^2}{4\pi} e^{-\frac{1}{4}(k_x^2 l_x^2 + k_y^2 l_y^2)}.$$  (4.86)

where $W(k_x, k_y)$ represents the spectrum amplitude in $m^4$. $l_x$ and $l_y$ are the correlation lengths in the $x-$ and $y-$ directions respectively, $h$ is the surface root-mean-square (rms) height, and $k_x$ and $k_y$ are the spatial frequencies in the $x-$ and $y-$ directions, respectively. The maximum value of $D_x$ considered is equal to 1024$\lambda$ resulting in 1,048,576 unknowns. In this case, the surface spectrum parameters are $h = 0.5\lambda$.
and $l_z = l_y = 1.414\lambda$. In addition, the 2-D FB/NSA method using the $x$-expansion employs the following parameters to obtain 1% accuracy in the scalar Green’s function for $x - x_f = L_x$; $L_x = 3.5\lambda$. $a_{max} = 1.0$, $\delta_{k_y} = 0.1$, $k_{z,tail} = 0.35k$, $k_{y,tail} = 0.25Re[\kappa]$, $C_z = 30.0$ and $C_y = 11.0$. Another case, Case 2, involves $D_x(\lambda) \times 32\lambda$ Gaussian surfaces with a power law spectrum given by

$$W(k, \phi) = a_0 k^{-1}, \quad k_{dl} < k < k_{du}. \quad (4.87)$$

where $W(k, \phi)$ represents the spectral amplitude, $k$ denotes the spatial wave number of the surface, $\phi$ represents the azimuthal angle of the two dimensional spectrum, $a_0$ is a specified constant, and $k_{dl}$ and $k_{du}$ are the lower and upper cutoff spatial wave numbers, respectively. The maximum value of $D_x$ considered is equal to 256$\lambda$ resulting in 1,048,576 unknowns. In this case, the surface spectrum parameters are $a_0 = 0.6365 \times 10^{-1}$, $k_{dl} = 210.0$ rad./$\lambda$ and $k_{du} = 27,393.3$ rad./$\lambda$. Furthermore, the 2-D FB/NSA method using the $x$-expansion employs the following parameters to obtain 1% accuracy in the scalar Green’s function for $x - x_f = L_x$; $L_x = 3.0\lambda$. $a_{max} = 1.0$, $\delta_{k_y} = 0.03$, $k_{z,tail} = 0.45k$, $k_{y,tail} = 0.20Re[\kappa]$, $C_z = 25.0$ and $C_y = 11.0$.

Figure 4.12 (a) illustrates a comparison of CPU time per iteration (in seconds) between the conventional 2-D FB method and the 2-D FB/NSA method using the $x$-expansion versus number of unknowns for two cases on a log-log scale. It is noted that the CPU time per iteration of the standard 2-D FB method is independent of $D_y$; thus, both Case 1 and Case 2 provide the same CPU time. In the log-log scale, the plot of CPU time versus number of unknowns is found to be a straight line for a large number of unknowns. The plot of the conventional 2-D FB method has a slope approximately equal to 2, while the 2-D FB/NSA method using the $x$-expansion has a slope approximately equal to 1 for both cases. However, the CPU time of the 2-D
Figure 4.12: Comparison of computational efficiency between the conventional FB method and the FB/NSA method using the $\epsilon$- expansion for 2-D PEC RSS problems: (a) CPU time per iteration versus number of unknowns (b) CPU memory versus number of unknowns.
FB/NSA using the \( x \)-expansion method for Case 2 is greater than for Case 1. This is due to the fact that the size of the strong region in Case 2 (3\( \lambda \times 32\lambda \)) is greater than in Case 1 (3.5\( \lambda \times 8\lambda \)), and Case 2 requires a larger number of plane waves \( Q_{TOT,x} \) in the 2-D NSA algorithm using the \( x \)-expansion than Case 1. Figure (4.12) (b) shows a comparison of the CPU memory (Mbytes) versus number of unknowns between the conventional 2-D FB and 2-D FB/NSA methods using the \( x \)-expansion.

For convenience, it is assumed that the maximum number of plane waves \( Q_{TOT,x,\max} \) is equal to \( 10^5 \). This is sufficient for both cases to provide accurate results. As the number of unknowns increases, the memory requirements of the 2-D FB/NSA method using the \( x \)-expansion approach those of the 2-D FB method. From the plot, large-scale surfaces with 1.048,576 unknowns require the CPU memory approximately 100 Mbytes, a remarkably low memory storage requirement compared to other fast algorithms. Thus, it can be concluded from these plots that the 2-D NSA algorithm using the \( x \)-expansion can yield a reduction in CPU time for very large problems while only slightly increasing the memory storage requirement. In the next section, the computational requirements of the 2-D “multilevel” FB/NSA algorithm using both \( x \)- and \( y \)-expansions are derived.

### 4.7.2 Computational Requirements of the 2-D “Multilevel” FB/NSA Method Using Both \( x \)- and \( y \)-Expansions

The total operation count \( TOC \) of the 2-D “multilevel” FB/NSA method using both \( x \)- and \( y \)-expansions is estimated as follows:

\[
TOC \sim C_r N_{s,r} N_{elm,x} + C_y Q_{TOT,y} N_{elm} + \sum_{j=1}^{T} C_j Q_{TOT,x,j} (N_{elm} - N_{s,j}),
\]

(4.88)
where \( N_{s,r} = 2N_xN_y \), \( Q_{TOT,y} \) denotes the total number of plane waves for the \( y \)-expansion as defined in Eq. (4.78). \( Q_{TOT,x,j} \) denotes the total number of plane waves for the \( x \)-expansion employed in the \( j \) th weak region \( S_{zy,w}^f \), where \( j = 1, \ldots, T \). \( C_j \) are constants, and

\[
N_{s,j} = \left( N_x + \sum_{m=1}^{J-1} N_{w,m} \right) M, \quad j = 2, \ldots, T. \tag{4.89}
\]

where \( N_{s,1} = N_xM \) and \( N_{w,m} = \frac{L_{x,m}}{\Delta x} \), where \( m = 1, \ldots, T - 1 \). It is noted that \( N_{s,r} \) is approximately the total number of elements in the reduced strong region \( S_{xy,rs}^f \), and \( Q_{TOT,x,1} = Q_{TOT,x} \) by definition (see Eq. (4.51)). The first term on the RHS of Eq. (4.88) is the number of operations involved in the computation of the reduced strong-region contribution for \( N_{elm} \) receiving elements and the second term involves the number of operation count to compute \( Q_{TOT,y} \) plane waves associated with \( J_{w,yb}(r) \) and \( J_{w,yt}(r) \) for \( N_{elm} \) receiving elements. Note that the operation count of the regeneration process of complex vector radiation functions \( F_{y}(k)(r_{n,m}, k_z, k_x) \) and \( F_{y}(k)(r_{n,m}, k_z, k_x) \) for every horizontal distance \( L_r \) (see Eq. (4.82) is proportional to \( N_{elm} \). Finally, the last term is the number of operations involved in the computation of \( Q_{TOT,x,j} \) plane waves associated with the \( j \) th weak region \( S_{zy,w}^f \).

In addition, the total memory storage requirement \( TMSR \) of the 2-D “multilevel” FB/NSA algorithm using both \( x \)- and \( y \)-expansions can be estimated as follows:

\[
TMSR \sim C_wN_{elm} + C_mMQ_{TOT,y} + \sum_{j=1}^{T} D_jQ_{TOT,x,j}, \tag{4.90}
\]

where \( C_j \) and \( D_j \) are constants. The first term on the RHS of Eq. (4.90) accounts for the storage of necessary matrices and vectors employed in the 2-D “multilevel” FB/NSA algorithm using both \( x \)- and \( y \)-expansions. In addition, the second and third terms involve the storage of the total number of plane waves associated with
the $y$-expansion and the $x$-expansions employed in the “multilevel” algorithm, respectively. As pointed out earlier in Section 4.5. to implement the 2-D FB/NSA algorithm associated with the $y$-expansion efficiently, it is required to store the number of plane waves for each row associated with the weak regions $S_{xy,yb}^f$ and $S_{xy,yt}^f$, resulting in an additional memory requirement of $O(MQ_{TOT,y})$.

In practice, it should be pointed out that only few weak regions $S_{zy,w}^f$ are sufficient to yield accurate results for 2-D extremely large-scale surfaces as in the case of 1-D surfaces illustrated earlier in Chapter 2. Note that the neighborhood distances $L_x$ and $L_y$ must be chosen appropriately to compromise between the direct computation in the strong region and the NSA computation in the weak regions. However, for fixed $D_y$, frequency and surface roughness, the parameters $L_x, L_y, Q_{TOT,x,j}$ and $Q_{TOT,y}$ are fixed, and as $D_x$ increases it is implied from Eqs. (4.88) and (4.90) that the computational cost and memory storage requirement of the 2-D “multilevel” FB/NSA algorithm using both $x$- and $y$-expansions are still $O(N_{tot})$, where $N_{tot} = 2N_{eimt}$. Finally, it should be emphasized that the 2-D “multilevel” FB/NSA algorithm using both $x$- and $y$-expansions improves the efficiency and accuracy of the standard 2-D FB/NSA algorithm using only the $x$-expansion at the cost of increasing memory storage requirements and algorithmic complexity. In the next section, several numerical results are illustrated.

4.8 Numerical Results

In this section, several numerical results of rough surface scattering computed by the 2-D FB/NSA algorithm are illustrated. For large-scale rough surfaces with moderate $D_y$, the 2-D FB/NSA algorithm using only the $x$-expansion is sufficient
to obtain accurate results with relatively high computational efficiency compared to the standard FB method, and these results are illustrated in Section 4.8.1. For rough surfaces with *relatively large* $D_y$, it is recommended to employ the 2-D FB/NSA algorithm using both $x$- and $y$- expansions as described in Section 4.5, and its results are presented in Section 4.8.2. The 2-D "multilevel" FB/NSA algorithm is suitable for *extremely* large-scale surfaces, and its numerical results are illustrated in Section 4.8.3.

The numerical and experimental results obtained from the standard FB method and the measurement from the University of Washington, Seattle, are employed in the comparisons with the numerical results obtained from the 2-D FB/NSA algorithm. Note that the information of the CPU time in this section is based on a Pentium II 200 MHz computer, with 128 Mbytes RAM (employed in Sections 4.8.1 and 4.8.3) or with 256 Mbytes RAM (employed in Section 4.8.2).

### 4.8.1 The 2-D FB/NSA Algorithm Using the $x$- Expansion

To illustrate the computational efficiency and accuracy of the 2-D FB/NSA algorithm using the $x$- expansion for large-scale surfaces, first consider a *deterministic* $128\lambda \times 8\lambda$ PEC rough surface illuminated by a tapered plane wave with the taper parameter $g' = 6$ (see Section 4.3) at an incident angle of $20^\circ$. The surface is a realization of a Gaussian random process described by a Gaussian spectrum with $l_x = l_y = 1.414\lambda$ and $h = 0.5\lambda$ having $z_{\text{min}} = -1.888\lambda$ and $z_{\text{max}} = 2.098\lambda$. The surface is sampled with 8 points per $\lambda$ resulting in 131,072 unknowns for $x$- and $y$-polarization surface currents. The standard 2-D FB method is employed to compare with the 2-D FB/NSA method using the $x$- expansion and requires 3 iterations to converge to within 1 \% accuracy based on the $PR_N^{(k)}$ test. Its total CPU time for
this example is 1976.2 minutes. The 2-D FB/NSA method using the $x$-expansion employs the parameters of Case 1 and requires the same number of iterations to converge within the same accuracy as in the 2-D FB method. However, its total CPU time is 359.8 minutes. Thus, with the 2-D NSA algorithm using the $x$-expansion, a CPU time reduction of 5.5 is achieved in this case.

Numerical results are presented in terms of the normalized bistatic radar cross section (RCS) $\sigma_{\alpha\beta}(\theta, \phi)$ in the plane of incidence, defined for a scattered wave in $\alpha$-polarization and an incident wave in $\beta$-polarization, as given in Eq. 1.26 in Section 1.3. Figure 4.13 (a) and (b) illustrates plots of the normalized bistatic RCS in dB: (a) HH polarization (b) VH polarization.
dB when $\phi_i = \phi_s = 0^\circ$ (in plane scattering) versus the scattering angle ($\theta_s$) for HH- and VH- polarizations respectively, comparing between the standard 2-D FB and 2-D FB/NSA methods using the $x$- expansion. From the plots, the normalized bistatic radar cross sections obtained from both methods are in good agreement. In addition, it is also found that the average relative error of the magnitude of surface currents in the main beam is about 0.1%. Thus, the 2-D FB/NSA method using the $x$-expansion provides very accurate results with a reduction of CPU time, which stems from the properties of the complex vector radiation function $F^{(k)}(r, k_z, k_y)$ which can be computed recursively via Eq. (4.38). $F^{(k)}(r, k_z, k_y)$ is also relatively smooth (compared to the function obtained from integrating along the original contours) and localized in the complex $k_z$ and $k_y$ planes as illustrated in Figure 4.14. In this figure, the magnitude of $F^{(k)}(r, k_z, k_y)$ of the above example is plotted versus $\text{Re}[k_z]$ and $\text{Re}[k_y]$ for the last backward sweep ($k = 3$) with the receiving element located at the $(x_1, y_1)$ element of the rough surface. From the plot, $F^{(k=3)}(r, k_z, k_y)$ is indeed localized and relatively smooth in the complex planes as a result of contour deformation from the real axis.

Another example illustrates the application of the 2-D FB/NSA method using the $x$- expansion for moderately rough surfaces for studies of the backscattering enhancement phenomenon. In this example, the rough surface statistics and the taper parameters of the incident beam are the same as in the first example, except $h$ is increased from $0.5\lambda$ to $1.0\lambda$ in order to see the backscattering enhancement phenomenon more clearly. The 2-D FB/NSA method using the $x$- expansion is employed to perform a Monte-Carlo simulation numerically for 100 realizations. The following 2-D FB/NSA parameters are used: $L_x = 4.5\lambda$, $a_{max} = 1.0$, $\delta_{k_z} = 0.1$, $k_{z,\text{tail}} =$
Figure 4.14: Radiation function $F^{(k)}(r, k_z, k_y)$ in the complex $k_z$ and $k_y$ planes for the last backward sweep ($k = 3$) and the receiving element located at the $(x_1, y_1)$ element of the rough surface.

$0.15k, k_{y, tail} = 0.20Re[\kappa], C_z = 10.0$ and $C_y = 20.0$. Figure 4.15 shows the plots of the normalized bistatic RCS versus the scattering angle $\theta_s$, for both co-polarizations (HH and VV) and cross-polarizations (VH and HV). Results are compared with millimeter wave experimental data obtained from the University of Washington, Seattle [17]. Note that the use of millimeter wave frequencies for the experiment allows a calibrated comparison of scattered power to incident power, so that the absolute value of bistatic scattering cross sections as normalized by the incident power is measured. Thus, absolute values of normalized bistatic scattering cross sections are compared between Monte-Carlo simulations and experiments without the use of arbitrary units. The backscattering enhancement phenomenon can be observed clearly at $\theta_s = -20^\circ$ corresponding to the backscattering direction of an incident angle of $20^\circ$ used in
Figure 4.15: Comparison between Monte-Carlo simulation results for moderately rough surfaces computed via the 2-D FB/NSA method using the $x$-expansion and the experimental data obtained from the University of Washington, Seattle.
this simulation. In addition, Monte-Carlo simulations based on the 2-D FB/NSA method using the $x$- expansion are shown to reproduce the overall trends and level of the experimental data. Minor differences between the numerical and experimental Monte-Carlo simulations may come from the differences between the numerical and experimental antenna patterns and the rough surface profiles that were fabricated for the experiment, as described in [17]. In the next section, numerical results obtained from the 2-D FB/NSA algorithm using both $x$- and $y$- expansions are illustrated.

4.8.2 The 2-D FB/NSA Algorithm Using Both $x$- and $y$- Expansions

To illustrate the computational efficiency and accuracy of the 2-D FB/NSA algorithm using both $x$- and $y$- expansions for large-scale surfaces with relatively large $D_y$, consider a deterministic $128\lambda \times 32\lambda$ PEC rough surface sampled with 16 points per $\lambda$ resulting in $2,097,152$ unknowns. The surface is illuminated by a tapered plane wave with $g' = 6$ at an incident angle of $80^\circ$. The surface is a realization of a Gaussian random process described by a power law spectrum (see Eq. (4.87)) with $a_0 = 0.6365 \times 10^{-3}$, $k_d = 210.0$ rad./$\lambda$ and $k_{du} = 27.393.3$ rad./$\lambda$. and its maximum surface variation $\Delta z_{\text{max}}$ is equal to $1.6435\lambda$ ($z_{\text{min}} = -0.8311\lambda$ and $z_{\text{max}} = 0.8124\lambda$). In this case, numerical tests show that only the first weak region $S_{ly,w}^{1,1}$ employed for the $x$- expansion is sufficient to yield accurate results. The 2-D FB/NSA parameters for this case are listed below:

- For the $x$- expansion: $L_x = 3.0\lambda$, $\delta_{k_x} = 0.03$ rad., $a_{\text{max}} = 1.0$, $k_{z,\text{tail}} = 0.60k$.
  
  $k_{y,\text{tail}} = 0.25 \, \text{Re}[\kappa], C_z = 30.0, C_y = 11.0$

- For the $y$- expansion: $L_y = 3.0\lambda$, $\delta_{k_y} = 0.3$ rad., $a_{\text{max}} = 1.0$, $k_{z,\text{tail}} = 0.60k$.
\[ k_{z,\text{tail}} = 0.30 \ Re[k], \ C_z = 40.0, \ C_x = 40.0, \ b_{\text{max}} = 4.0 \]

Note that the \( y \)-expansion employs \( b_{\text{max}} = 4.0 \), and from Eq. (4.82) it is required to regenerate the radiation function \( F_{y}^{(k)}(r_{(n,m)}, k_z, k_x) \) for every 75 points (i.e. \( L_r = 4.6875 \lambda \)). The standard 2-D FB/NSA algorithm using the \( x \)-expansion is employed to compare with the 2-D FB/NSA algorithm using both \( x \)- and \( y \)-expansions, and the former employs the same 2-D NSA parameters as the latter for the \( x \)-expansion. It is found that both methods require 2 iterations to converge to within 1\% accuracy based on \( PR_{N}^{(k)} \) convergence test. Figure 4.16 (a) and (b) illustrates plots of the normalized bistatic RCS in dB when \( \phi_{t} = \phi_{s} = 0^\circ \) versus the scattering angle \( \theta_{s} \) for HH- and VH-polarizations respectively, comparing between the former and the latter. From the plots, the results obtained from both methods are in excellent agreement. The former requires the total CPU time of 21.622.5 minutes, and the latter’s CPU time is equal to 13.755.3 minutes. Thus, it can be concluded that the latter provides very accurate results, and the former’s efficiency can be indeed improved by incorporating the \( y \)-expansion. It should be pointed out that the \( y \)-expansion shows better improvement for larger problems with relatively large \( D_y \). In the next section, numerical results of the 2-D “multilevel” FB/NSA algorithm are illustrated.

### 4.8.3 The 2-D “Multilevel” FB/NSA Algorithm

To see that the “multilevel” algorithm can improve the accuracy of the standard (one-level) 2-D FB/NSA algorithm, consider a deterministic 256\( \lambda \times 32\lambda \) PEC rough surface sampled with 8 points per \( \lambda \) resulting in 1.048.576 unknowns. The surface is a realization of a Gaussian random process described by a power law spectrum with \( a_0 = 0.6365 \times 10^{-3} \), \( k_{ds} = 210.0 \text{ rad.}/\lambda \) and \( k_{du} = 27,393.3 \text{ rad.}/\lambda \), and its maximum
Figure 4.16: A comparison of the normalized bistatic RCS in dB when $\phi_i = \phi_s = 0^\circ$ computed by the standard 2-D FB/NSA using the $x$-expansion and the 2-D FB/NSA algorithm using both $x$- and $y$-expansions: (a) HH polarization (b) VH polarization.
surface variation $\Delta z_{\text{max}}$ is equal to 2.361$\lambda$ ($z_{\text{min}} = -1.359\lambda$ and $z_{\text{max}} = 1.002\lambda$). The surface is illuminated by a tapered plane wave with $gt = 6$ at an incident angle of $80^\circ$. The 2-D two-level FB/NSA parameters using only the $x$- expansion for this case are given as follows:

- The first level: $L_x = 3.0\lambda$, $L_w,1 = 50.0\lambda$, $\delta_{k_y} = 0.03$ rad., $a_{\text{max}} = 1.0$.

  $$k_{z,\text{tail}} = 0.45k, \quad k_{y,\text{tail}} = 0.20 \Re[k], \quad C_z = 25.0, \quad C_y = 11.0$$

- The second level: $L_x + L_w,1 = 53.0\lambda$, $\delta_{k_y} = 0.20$ rad., $a_{\text{max}} = 1.0$, $k_{z,\text{tail}} = 0.20k$.

  $$k_{y,\text{tail}} = 0.10 \Re[k], \quad C_z = 30.0, \quad C_y = 15.0$$

It is found that the 2-D two-level FB/NSA algorithm requires 2 iterations to converge to within 1% accuracy based on the $P R^{(k)}$ convergence test. The sparse-matrix canonical grid (SMCG) method [68]-[79] is employed to compare with the 2-D FB/NSA algorithm. Figure 4.17 (a) and (b) shows plots of the normalized bistatic RCS in dB when $\phi_i = \phi_s = 0^\circ$ versus $\theta_s$ for HH- and VH- polarizations respectively, comparing between the 2-D two-level FB/NSA algorithm and the SMCG method. Note that the results obtained from both methods are in good agreement. In addition, Figure 4.18 illustrates the same plots as in Figure 4.17, but comparing between the 2-D one-level FB/NSA method and the SMCG method. It is found that the 2-D one-level FB/NSA method requires 2 iterations as well as the 2-D two-level FB/NSA method. From the plots, the discrepancy of the results obtained from the 2-D one-level FB/NSA method and the SMCG method is noticeable. Note that the 2-D one-level and two-level FB/NSA methods require the total CPU time of 4994.1 and 8327.2 minutes, respectively. Thus, the “multilevel” algorithm can indeed improve the accuracy of the 2-D one-level FB/NSA algorithm at the cost of increasing the total CPU time.
Figure 4.17: A comparison of the normalized bistatic RCS in dB when $\phi_i = \phi_s = 0^\circ$ computed by the 2-D two-level FB/NSA method and the SMCG method: (a) HH polarization (b) VH polarization.
Figure 4.18: A comparison of the normalized bistatic RCS in dB when $\phi_t = \phi_s = 0^\circ$ computed by the 2-D one-level FB/NSA method and the SMCG method: (a) HH polarization (b) VH polarization.
4.9 Conclusions

The FB/NSA method has been shown to be an extremely efficient method for 1-D moderately rough surfaces as discussed earlier in Chapter 2. In this chapter, it is extended to treat 2-D PEC random rough surfaces. The new NSA algorithm for the 2-D rough surfaces is derived. First, the 2-D FB/NSA algorithm using only the x-expansion is illustrated. It involves a double spectral integral representation of source currents and the 3-D free space scalar Green’s function \( g(\mathbf{r}, \mathbf{r}') \). The coupling between two spectral variables is complex, and the method is most suited for rectangular surfaces due to the treatment of surface cross-range size as a “roughness” parameter. The 2-D NSA parameters for this case can be obtained by comparing the exact and spectral domain representation solutions of the 3-D free space scalar Green’s function \( g(\mathbf{r}, \mathbf{r}') \) for the two worst-case scenarios (one in the \( \rho - z \) plane and another in the \( x - y \) plane). It is found that the 2-D FB/NSA algorithm using the x-expansion is suitable for large-scale rough surfaces with moderate surface cross-range size \( D_y \).

For rough surfaces with relatively large \( D_y \), it is more efficient to employ the 2-D FB/NSA algorithm using both x- and y-expansions. The y-expansion is employed to reduce the computational complexity of the large strong region \( S_{xy,a} \), and it can be formulated in a similar fashion as the x-expansion. In addition, the NSA parameters associated with the y-expansion can be obtained empirically as in the case of the x-expansion. Note that the 2-D FB/NSA algorithm using both x- and y-expansions requires an additional memory storage of \( O(MQ_{TOT,y}) \) compared to the standard 2-D FB/NSA algorithm using only the x-expansion. Numerical results illustrate that incorporating the y-expansion to the standard 2-D FB/NSA algorithm can indeed improve its computational efficiency without degrading its accuracy.
For extremely large-scale surfaces, the 2-D "multilevel" FB/NSA algorithm is employed to improve the accuracy of the one-level (standard) 2-D FB/NSA algorithm, however the "multilevel" algorithm increases both total CPU time and memory storage requirement. Like 1-D surfaces, only a few weak regions are sufficient to obtain accurate results for most practical 2-D extremely large-scale surface problems. For given surface sizes and surface statistics, it is difficult to determine the appropriate sizes of the weak regions \( S^j \) analytically, but they can be found empirically instead. It is emphasized that the 2-D NSA parameters for the \( x \)-expansion must be determined appropriately for each weak region as well. Numerical results show that the "multilevel" algorithm can indeed improve the accuracy of the one-level 2-D FB/NSA algorithm at the cost of increasing the total CPU time and memory storage requirement.

The computational efficiency of the FB/NSA method for 2-D RSS problems is shown to be \( O(N_{tot}) \) for fixed \( D_y \), frequency and surface roughness. The memory storage requirement of the method is remarkably low for surfaces with moderate surface cross-range size \( D_y \) resulting in larger surface sizes that can be run on present computers. In addition, the FB/NSA method still remains very efficient for moderately rough surfaces with large rms slopes. Comparisons of numerical results for large-scale surfaces between the standard 2-D FB method and the 2-D FB/NSA method using the \( x \)-expansion have shown that the 2-D FB/NSA method using the \( x \)-expansion yields very accurate results with a reduction of CPU time and only slightly larger memory storage requirements. Thus, it is a good candidate for studying the physics of moderately large-scale rough surface scattering at low grazing angles. Monte-Carlo simulations for this case are considered in Chapter 6.
For the future work, it is challenging to determine the 2-D NSA parameters associated with the 2-D FB/NSA algorithm, including the appropriate size of each weak region, analytically for given surface sizes and surface statistics. This may involve the choice of order of integration associated with the double spectral integral representation of $g(r, r')$. The same approach illustrated in Sections 2.3 and 2.4 in Chapter 2 for 1-D surfaces will be extended to determine these parameters of interest.
5.1 Introduction

In general, rough surfaces are considered to be electrically large and penetrable scatterers, and the rigorous formulation of the scattering from such surfaces usually involves surface integral equations for both upper (free space) and lower regions resulting in very large number of unknowns, especially if the lower medium is dense. However, practical rough surfaces such as ocean and metallic surfaces have finite but high conductivity which can be appropriately modeled using the impedance boundary condition (IBC), also known as the Leontovich boundary condition [138]–[142]. This condition was first introduced by Leontovich [138] in attempt to solve problems dealing with the propagation of radio waves over the earth. The IBC relates the tangential components of the electric to the magnetic surface fields via a surface impedance \( \eta_s \), defined by the EM properties of the scatterer, and is mathematically stated as

\[
\hat{n} \times \mathbf{E} = \eta_s \hat{n} \times \hat{n} \times \mathbf{H}.
\]  

(5.1)
where \( \mathbf{n} \) is a unit normal vector pointing out of the rough surface into the upper region. Typically, the surface impedance \( \eta_s \) is set to be the characteristic impedance of the lower medium; i.e., \( \eta_s = \frac{R}{\sqrt{\varepsilon_1}} \). Since this approximate boundary condition relates only the fields outside the scatterer, the scattered fields can be evaluated without involving the fields below the interface: thus, the analysis of RSS problems is simplified considerably. Note that the IBC also has its limits and range of validity, which are discussed in detail in [139]-[142].

In this Chapter, the 2-D FB/NSA algorithm for 2-D impedance rough surfaces is formulated. The goal of this development is to employ this algorithm to study the effects of surface material to the backscattering enhancement phenomenon as will be considered in Chapter 6. Basically, this algorithm is an extension of the 2-D FB/NSA algorithm developed for 2-D PEC rough surfaces as given in Chapter 4 to take the finite conductivity of the medium below the surface profile into account. It is found that two additional terms are required in the MFIE for the PEC case to account for nonzero surface impedance, and one of them involves the surface divergence of the associated currents which require a numerical differentiation. Unlike the PEC case, each current element on the surface is always coupled to another even for a linear surface model. In addition, for the self-term calculation in the IBC case, the associated Green's function has singularity contributions when testing and integration points overlap which require careful consideration. The 2-D NSA algorithm for the IBC case is similar to the PEC case, thus only necessary steps are included, and the 2-D NSA parameters associated with the IBC case are summarized for convenience. It is emphasized that the computational efficiency of the FB/NSA method for 2-D IBC rough surfaces still remains \( \mathcal{O}(N_{tot}) \) as one of the surface dimensions increases.
For large-scale IBC rough surfaces, comparisons of numerical results between the conventional FB method and the 2-D FB/NSA method based on the spectral domain representation of the free space 3-D scalar Green's function in the $x$-direction show that the latter yields identical results to the former with a reduction of CPU time.

This chapter is organized as follows. The MM formulation with the FB method for 2-D IBC RSS problems is described in Section 5.2. Section 5.3 presents the formulation of the 2-D FB/NSA algorithm for 2-D impedance rough surfaces using the spectral domain representation of the Green's function in the $x$-direction. To illustrate the accuracy and efficiency of the 2-D FB/NSA algorithm for IBC rough surfaces, numerical results are illustrated in Section 5.4. Finally, Section 5.5 presents conclusions of the 2-D FB/NSA method developed for the IBC case.

### 5.2 MM Formulation with the FB Method for Impedance Rough Surfaces

Consider a 2-D rough surface profile $S$ illuminated by an incident field $E'(x, y, z)$ centered in direction $\hat{k}_i = \hat{x} \sin \theta_i \cos \phi_i + \hat{y} \sin \theta_i \sin \phi_i - \hat{z} \cos \theta_i$, as shown in Figure 4.1 in Chapter 4. The region above the surface profile is assumed to be free space, and the region below to be a homogeneous, nonmagnetic and isotropic medium described by electric permittivity $\epsilon_i$ and the magnetic permeability $\mu$. The surface height function $z = f(x, y)$ has zero mean and its maximum and minimum height variations are denoted by $z_{\text{max}}$ and $z_{\text{min}}$, respectively. Let $r = \hat{x}x + \hat{y}y + \hat{z}z$ and $r' = \hat{x}x' + \hat{y}y' + \hat{z}z'$ denote a field point and a source point on the rough surface, respectively. The incident field $E'(x, y, z)$ is tapered with a Gaussian beam amplitude pattern confining the illuminated rough surface to the rectangular surface area $D_x \times D_y$ so that surface
edges do not contribute strongly to obtained scattered fields as discussed earlier in Chapter 4.

Consider the Stratton-Chu integral equation for the magnetic field in the upper region above the surface profile [126]:

\[
\frac{n \times H(r)}{2} = n \times H' + n \times \int_{\partial \Omega_{\Sigma y}} d\tau d\sigma \{ -i \omega \varepsilon g(r, \tau) [n' \times E(\tau)] + \nabla g(r, \tau) \times [n' \times H(\tau)] - \frac{|n'|}{\omega \mu} \nabla g(r, \tau) \nabla l_s \cdot [n' \times E(\tau)] \}.
\]

where the above integral is a principal-value integral. \(\nabla l_s\) is the surface divergence operator.

\[
g(r, \tau) = \frac{e^{ikR}}{4\pi R}
\]

\[
\nabla g(r, \tau) = G(R) R
\]

\[
G(R) = \frac{e^{ikR}}{4\pi R^2} \left( ik - \frac{1}{R} \right)
\]

\[
R = \hat{x}(x - x') + \hat{y}(y - y') + \hat{z}(f(x, y) - f(x', y'))
\]

\[
n = \hat{z} - \hat{x} \frac{\partial f}{\partial z} - \hat{y} \frac{\partial f}{\partial y}, \quad n' = \hat{z} - \hat{x} \frac{\partial f}{\partial z} - \hat{y} \frac{\partial f}{\partial y}, \quad R = |R|. \quad \text{and} \quad H' \text{ is the incident magnetic field associated with } E'. \quad \text{The normal vectors } n \text{ and } n' (\text{not unit vectors}) \text{ point upward from the rough surface } S, \text{ and } S_{xy} \text{ is the surface obtained from the projection of the rough surface } S \text{ onto the } xy \text{ plane. For convenience in analysis, define the following quantities: } J(r) = n \times H(r), \quad J_{PO}(r) = 2 n \times H'(r), \quad M(r) = E(r) \times n, \quad \text{and } T(r) = \frac{M(r)}{|n|}. \quad \text{From the impedance boundary condition as defined in Eq. (5.1), } J(r) \text{ and } M(r) \text{ are related via the following equation:}
\]

\[
M(r) = -\eta_s \hat{n} \times J(r).
\]
Using the above definitions, Eq. (5.2) can be expressed as the magnetic field integral equation (MFIE) for an impedance surface as follows:

\[
\mathbf{J}(\mathbf{r}) = \mathbf{J}_{PO}(\mathbf{r}) + \mathbf{J}_{PEC}(\mathbf{r}) \\
+ 2\mathbf{n} \times \int_{\partial \Omega_S} \int_{\partial \Omega_S} d\mathbf{x} d\mathbf{y} \left\{ i\omega \mathbf{e}(\mathbf{r}, \mathbf{r}) \mathbf{M}(\mathbf{r}) - \frac{\nabla g(\mathbf{r}, \mathbf{r}) |\mathbf{n}|}{i\omega \mu} \nabla \mathbf{J}_{\varepsilon}(\mathbf{r}) \cdot \mathbf{T}(\mathbf{r}) \right\}
\]

\[
\mathbf{J}_{PEC}(\mathbf{r}) = 2\mathbf{n} \times \int_{\partial \Omega_S} \int_{\partial \Omega_S} d\mathbf{x} d\mathbf{y} \nabla g(\mathbf{r}, \mathbf{r}) \times \mathbf{J}(\mathbf{r}).
\]

where the above integrals are also principal-value integrals. Note that Eq. (5.8) is reduced to the MFIE for the PEC case when \( \eta_s = 0 \) and two additional terms are required to account for nonzero surface impedance. Unlike the MFIE for the PEC case as discussed in Chapter 4, Eq. (5.8) involves both \( g(\mathbf{r}, \mathbf{r}) \) and \( \nabla g(\mathbf{r}, \mathbf{r}) \), and the \( x- \) and \( y- \) components of the current, denoted as \( J_\varepsilon(\mathbf{r}) \) and \( J_\gamma(\mathbf{r}) \) respectively, for each current element on the surface are coupled together even for a linear surface model as shown later this section. Applying a standard point-matching MM technique [81] to Eq. (5.8) requires expansion of the two unknown scalar functions on the surface into pulse basis functions, \( P_{nm}(\mathbf{r}) \), as

\[
J_\varepsilon(\mathbf{r}) = \hat{x} \cdot \mathbf{J}(\mathbf{r}) = \sum_{n=1}^{N} \sum_{m=1}^{M} J_\varepsilon^{(n,m)} P_{nm}(\mathbf{r}),
\]

\[
J_\gamma(\mathbf{r}) = \hat{y} \cdot \mathbf{J}(\mathbf{r}) = \sum_{n=1}^{N} \sum_{m=1}^{M} J_\gamma^{(n,m)} P_{nm}(\mathbf{r}).
\]

where \( n \) and \( m \) refer to the \( x \) and \( y \) indices of the pulse basis functions used on the surface rectangular grid as shown in Figure 4.2 in Chapter 4, and \( J_p^{(n,m)} \) denotes a \( p- \) polarized \( (p = x \ or \ y) \) current at the \((n, m)\) th element on the rough surface \( S \). Note that \( \Delta x \) and \( \Delta y \) in Figure 4.2 refer to the spacing between these basis functions in the \( x \)- and \( y \)-directions respectively, and each \((n, m)\) th element has the
dimension $\Delta x \times \Delta y$ and its center is located at $(x_n, y_m)$ in the coordinate system.
where $n = 1, \ldots, N$ and $m = 1, \ldots, M$. Substituting these expansions, Eqs. (5.10) and (5.11), into dot products of Eq. (5.8) with $\hat{x}$ and $\hat{y}$ respectively results in the following two scalar equations:

\[
J_x(r) = J_{x,PO}(r) + J_{x,PEC}(r) + 2 \int_{P_x,S_{xy}} dx dy \left\{ -i\omega g(r, r') \left[ M_x(r') \frac{\partial f}{\partial y} \frac{\partial f}{\partial x} + M_y(r') \left(1 + \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \right) \right] \right. \\
+ \left. \frac{G(R)|n|}{i\omega \mu} \left( dy + \frac{\partial f}{\partial y} dz \right) \nabla \nu_s \cdot \mathbf{T}(r') \right\}.
\]

\[ (5.12) \]

\[
J_y(r) = J_{y,PO}(r) + J_{y,PEC}(r) + 2 \int_{P_y,S_{xy}} dx dy \left\{ i\omega g(r, r') \left[ M_x(r') \left(1 + \frac{\partial f}{\partial x} \frac{\partial f}{\partial x} \right) + M_y(r') \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \right] \right. \\
- \left. \frac{G(R)|n|}{i\omega \mu} \left( dx + \frac{\partial f}{\partial x} dz \right) \nabla \nu_s \cdot \mathbf{T}(r') \right\}.
\]

\[ (5.13) \]

where $J_{x,PO}(r) = \hat{x} \cdot J_{PO}(r)$. $J_{y,PO}(r) = \hat{y} \cdot J_{PO}(r)$. $J_{x,PEC}(r) = \hat{x} \cdot J_{PEC}(r)$. $J_{y,PEC}(r) = \hat{y} \cdot J_{PEC}(r)$. $M_x(r) = \hat{x} \cdot M(r)$, and $M_y(r) = \hat{y} \cdot M(r)$. Using Eq. (5.7), it can be shown readily that the $x-$ and $y-$ components of $\mathbf{J}(r)$ and $\mathbf{M}(r)$ are coupled together through the following matrix equation:

\[
\begin{bmatrix}
M_x(r) \\
M_y(r)
\end{bmatrix} = \begin{bmatrix}
\alpha & \beta \\
\gamma & -\alpha
\end{bmatrix}
\begin{bmatrix}
J_x(r) \\
J_y(r)
\end{bmatrix}.
\]

\[ (5.14) \]

where

\[
\alpha = \frac{\eta_s}{|n|} \frac{\partial f}{\partial x} \frac{\partial f}{\partial y}
\]

\[ (5.15) \]

\[
\beta = \frac{\eta_s}{|n|} \left[ 1 + \left(\frac{\partial f}{\partial y}\right)^2 \right]
\]

\[ (5.16) \]

\[
\gamma = -\frac{\eta_s}{|n|} \left[ 1 + \left(\frac{\partial f}{\partial x}\right)^2 \right].
\]

\[ (5.17) \]
These scalar equations, Eqs. (5.12) and (5.13), are then tested at the center points of the pulse basis functions to form the following MM matrix equation:

\[ \overline{Z} \overline{T} = \overline{V}. \]  

(5.18)

where \( \overline{Z} \) is the \( N_{tot} \times N_{tot} \) MM impedance matrix, \( \overline{V} \) is the \( N_{tot} \times 1 \) excitation vector, \( \overline{T} \) is the \( N_{tot} \times 1 \) solution vector, and \( N_{tot} = 2 NV \) be the total number of unknowns on the surface \( S \). It is noted that \( x \) - and \( y \) - components of \( J(\mathbf{r}) \) for all current elements are stored in \( \overline{T} \) as shown below

\[ \overline{T} = [J_x^{(1,1)}, J_y^{(1,1)}, \ldots, J_x^{(1,M)}, J_y^{(1,M)}, J_x^{(2,1)}, J_y^{(2,1)}, \ldots, J_x^{(N,1)}, J_y^{(N,1)}, \ldots, J_x^{(N,M)}, J_y^{(N,M)}]^T. \]  

(5.19)

where the superscript \( T \) denotes the transpose operation. Using pulse basis functions, point matching and a linear surface model (no surface curvature), it can be shown that the principal-value integrals involving \( \nabla g(\mathbf{r}, \mathbf{r}) \), as can be seen in Eqs. (5.12) and (5.13), are zero when testing and integration points overlap (self-term calculation) for surfaces approximated by collections of planes. However, the 3-D scalar free space Green's function \( g(\mathbf{r}, \mathbf{r}) \) has singularity contributions when the testing and integration points overlap which require careful consideration. A small argument expansion of \( g(\mathbf{r}, \mathbf{r}) \) is integrated analytically, and its remaining portion is numerically integrated to insure accurate calculation of the self terms in the impedance matrix \( \overline{Z} \).

Note also that the surface divergence term in Eqs. (5.8), (5.12) and (5.13) requires a numerical differentiation, a centered difference derivative, of the unknown function \( T(\mathbf{r}) \) as follows:

\[ \nabla t_s \cdot T(\mathbf{r}) \approx \frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y}, \]  

(5.20)

\[ \approx \frac{T_x^{(n+1,m)} - T_x^{(n-1,m)}}{2\Delta x} + \frac{T_y^{(n,m+1)} - T_y^{(n,m-1)}}{2\Delta y}. \]  

(5.21)
Derivatives at points on surface edges are computed by assuming currents at the adjacent point are zero; note this is not a large source of error since a tapered incident field will be used to eliminate fields at surface edges.

Due to the fact that the principal-value integration over the self \((n, m)\) th element \(S_{self}\) does not vanish, \(J_x(r)\) and \(J_y(r)\) are coupled together, and Eq. (5.18) can be written more explicitly as follows:

\[
\begin{bmatrix}
A_{11} & B_{11} & X & X & \ldots & X & X & \ldots & X & X \\
C_{11} & D_{11} & X & X & \ldots & X & X & \ldots & X & X \\
X & X & A_{12} & B_{12} & \ldots & X & X & \ldots & X & X \\
X & X & C_{12} & D_{12} & \ldots & X & X & \ldots & X & X \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
X & X & X & X & \ldots & A_{nm} & B_{nm} & \ldots & X & X \\
X & X & X & X & \ldots & C_{nm} & D_{nm} & \ldots & X & X \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
X & X & X & X & \ldots & X & X & \ldots & A_{NM} & B_{NM} \\
X & X & X & X & \ldots & X & X & \ldots & C_{NM} & D_{NM}
\end{bmatrix}
\begin{bmatrix}
J_{z}^{(1,1)} \\
J_{y}^{(1,1)} \\
J_{z}^{(1,2)} \\
J_{y}^{(1,2)} \\
J_{z}^{(n,m)} \\
J_{y}^{(n,m)} \\
J_{z}^{(N,M)} \\
J_{y}^{(N,M)}
\end{bmatrix}
= 
\begin{bmatrix}
J_{PO,z}^{(1,1)} \\
J_{PO,y}^{(1,1)} \\
J_{PO,z}^{(1,2)} \\
J_{PO,y}^{(1,2)} \\
J_{PO,z}^{(n,m)} \\
J_{PO,y}^{(n,m)} \\
J_{PO,z}^{(N,M)} \\
J_{PO,y}^{(N,M)}
\end{bmatrix}
\tag{5.22}
\]

where \(X\) denotes a mutual impedance between a pair of currents, \(J_{PO,p}^{(n.m)}\) denotes a \(p\)-polarized (\(p = x\) or \(y\)) PO current at the \((n, m)\) th element on the rough surface \(S\), and the \(2 \times 2\) sub-matrix \(\overline{Z}_{nm}^{s}\) associated with each self \((n, m)\) th element is defined as

\[
\overline{Z}_{nm}^{s} = \begin{bmatrix}
A_{nm} & B_{nm} \\
C_{nm} & D_{nm}
\end{bmatrix}.
\tag{5.23}
\]

where \(A_{nm}, B_{nm}, C_{nm},\) and \(D_{nm}\) are defined as follows:
\[ A_{nm} = 1 + 2 \int_{PV:S_{xy}^{nm}} dx_1 dy_1 \left\{ i \omega g(r, \mathbf{r}) \left[ \frac{\partial f}{\partial y} \frac{\partial f}{\partial x} \right] \alpha \right. \\
+ \left\{ 1 + \left( \frac{\partial f}{\partial y} \frac{\partial f}{\partial y} \right) \right\} \gamma \right\} \]  
(5.24)

\[ B_{nm} = 2 \int_{PV:S_{xy}^{nm}} dx_1 dy_1 \left\{ i \omega g(r, \mathbf{r}) \left[ \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \right] \beta \right. \\
- \left\{ 1 + \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial x} \right) \right\} \alpha \right\} \]  
(5.25)

\[ C_{nm} = -2 \int_{PV:S_{xy}^{nm}} dx_1 dy_1 \left\{ i \omega g(r, \mathbf{r}) \left[ \frac{\partial f}{\partial x} \frac{\partial f}{\partial x} \right] \gamma \right. \\
+ \left\{ 1 + \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial x} \right) \right\} \beta \right\} \]  
(5.26)

\[ D_{nm} = 1 - 2 \int_{PV:S_{xy}^{nm}} dx_1 dy_1 \left\{ i \omega g(r, \mathbf{r}) \left[ -\frac{\partial f}{\partial x} \frac{\partial f}{\partial x} \right] \alpha \right. \\
+ \left\{ 1 + \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial x} \right) \right\} \beta \right\} \]  
(5.27)

and \( S_{xy}^{nm} \) is the projected surface on the \( xy \) plane for each self \((n, m)\) th element. For a linear surface model, each entry of \( \overline{\mathbf{Z}}_{nm} \) can be simplified as follows:

\[ A_{nm} = 1 + i2\omega \left[ \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \alpha + \left( 1 + \left( \frac{\partial f}{\partial y} \right)^2 \right) \gamma \right] I_g \]  
(5.28)

\[ B_{nm} = i2\omega \left[ \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \beta - \left( 1 + \left( \frac{\partial f}{\partial y} \right)^2 \right) \alpha \right] I_g \]  
(5.29)

\[ C_{nm} = -i2\omega \left[ \frac{\partial f}{\partial x} \frac{\partial f}{\partial x} \gamma + \left( 1 + \left( \frac{\partial f}{\partial x} \right)^2 \right) \alpha \right] I_g \]  
(5.30)

\[ D_{nm} = 1 - i2\omega \left[ -\frac{\partial f}{\partial x} \frac{\partial f}{\partial x} \alpha + \left( 1 + \left( \frac{\partial f}{\partial x} \right)^2 \right) \beta \right] I_g \]  
(5.31)

where \( \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y} \) are constants for each \( S_{xy}^{nm} \), and the integral \( I_g \) is defined as

\[ I_g = \int_{PV:S_{xy}^{nm}} dx_1 dy_1 g(r, \mathbf{r}). \]  
(5.32)
which requires careful consideration as discussed earlier. Finally, Eq. (5.18) can be solved iteratively via a rapid-convergent iterative technique, named the FB method.

The first step in the FB method is to make the following decomposition for the current vector \( \mathbf{J}(\mathbf{r}) \):

\[
\mathbf{J}(\mathbf{r}) = \mathbf{J}^f(\mathbf{r}) + \mathbf{J}^b(\mathbf{r}).
\]

where \( \mathbf{J}^f(\mathbf{r}) \) and \( \mathbf{J}^b(\mathbf{r}) \) are the forward-stepping (FS) and backward-stepping (BS) currents, respectively. The FS and BS processes are shown in Figure 4.3 (a) and (b) respectively, and these processes are discussed in detail in Chapter 4. Note that the surfaces \( S^f_{xy} \) and \( S^b_{xy} \) corresponding to the FS and BS processes respectively, where \( S_{xy} = S^f_{xy} + S^b_{xy} \) as shown in Figure 4.3 (d). Substituting Eq. (5.33) into Eq. (5.8) and using Eq. (5.7). Eq. (5.8) can be separated into two coupled integral equations as follows:

\[
\mathbf{J}^f(\mathbf{r}) = \mathbf{J}_{p0}(\mathbf{r}) + 2 \mathbf{n} \times \int \int_{P.V.;S^f_{xy}} \, dx \, dy \, \nabla g(\mathbf{r}, \mathbf{r}') \times [\mathbf{J}^f(\mathbf{r}') + \mathbf{J}^b(\mathbf{r}')] \\
+ 2 \mathbf{n} \times \int \int_{P.V.;S^f_{xy}} \, dx \, dy \, \{-i \omega \epsilon \eta_s g(\mathbf{r}, \mathbf{r}') \hat{n} \times [\mathbf{J}^f(\mathbf{r}') + \mathbf{J}^b(\mathbf{r}')] \\
+ \frac{\nabla g(\mathbf{r}, \mathbf{r}') |\mathbf{n}| \eta_s}{i \omega} \nabla \mathbf{i}_s \cdot \left[ \frac{\hat{n} \times (\mathbf{J}^f(\mathbf{r}') + \mathbf{J}^b(\mathbf{r}'))}{|\mathbf{n}|} \right] \} \tag{5.34}
\]

\[
\mathbf{J}^b(\mathbf{r}) = 2 \mathbf{n} \times \int \int_{P.V.;S^b_{xy}} \, dx \, dy \, \nabla g(\mathbf{r}, \mathbf{r}') \times [\mathbf{J}^f(\mathbf{r}') + \mathbf{J}^b(\mathbf{r}')] \\
+ 2 \mathbf{n} \times \int \int_{P.V.;S^b_{xy}} \, dx \, dy \, \{-i \omega \epsilon \eta_s g(\mathbf{r}, \mathbf{r}') \hat{n} \times [\mathbf{J}^f(\mathbf{r}') + \mathbf{J}^b(\mathbf{r}')] \\
+ \frac{\nabla g(\mathbf{r}, \mathbf{r}') |\mathbf{n}| \eta_s}{i \omega} \nabla \mathbf{i}_s \cdot \left[ \frac{\hat{n} \times (\mathbf{J}^f(\mathbf{r}') + \mathbf{J}^b(\mathbf{r}'))}{|\mathbf{n}|} \right] \} \tag{5.35}
\]

Note that adding Eq. (5.34) and Eq. (5.35) results in Eq. (5.8) as expected, and Eqs. (5.34) and (5.35) can be solved using the FB method by first initializing \( \mathbf{J}^{b,(0)}(\mathbf{r}) = \)
0, and at the kth \((k \geq 1)\) iteration,

\[
J^{f,(k)}(r) = J_{pO}(r) + 2n \times \int_{P^\ast \cap S^k_y} dx dy \nabla g(r, r) \times [J^{f,(k)}(r) + J^{b,(k-1)}(r)]
\]

\[
+ 2n \times \int_{P^\ast \cap S^k_y} dx dy \left\{ -i \omega \eta_s g(r, r) \hat{n} \times [J^{f,(k)}(r) + J^{b,(k-1)}(r)] \right\} + \nabla g(r, r) \cdot \left[ \hat{n} \times \frac{[J^{f,(k-\frac{1}{2})}(r) + J^{b,(k-1)}(r)]}{|n|} \right] \}
\]

(5.36)

\[
J^{b,(k)}(r) = 2n \times \int_{P^\ast \cap S^k_y} dx dy \nabla g(r, r) \times [J^{f,(k)}(r) + J^{b,(k)}(r)]
\]

\[
+ 2n \times \int_{P^\ast \cap S^k_y} dx dy \left\{ -i \omega \eta_s g(r, r) \hat{n} \times [J^{f,(k)}(r) + J^{b,(k)}(r)] \right\} + \nabla g(r, r) \cdot \left[ \hat{n} \times \frac{(J^{f,(k-\frac{1}{2})}(r) + J^{b,(k)}(r))}{|n|} \right] \}
\]

(5.37)

In Eqs. (5.36) and (5.37), the surface divergence term associated \(J^{f,(k-\frac{1}{2})}(r)\) can be approximately computed via a centered difference derivative (see Eq. (5.21)) as

\[
\nabla l_s \cdot U^{(k-\frac{1}{2})}(r) \approx \frac{U^{(k-1),(n+1,m)}_x - U^{(k),(n,m)}_x}{2\Delta x} + \frac{U^{(k-1),(n,m+1)}_y - U^{(k),(n,m)}_y}{2\Delta y}.
\]

(5.38)

where the vector \(U^{(k-\frac{1}{2})}(r)\) is defined as

\[
U^{(k-\frac{1}{2})}(r) = \frac{\hat{n} \times J^{f,(k-\frac{1}{2})}(r)}{|n|}.
\]

(5.39)

Note that \(U^{(k),(n,m)}_j\) denotes the \(j\)th component of \(U(r)\) at the \((n, m)\)th element on the rough surface \(S\) for the \(k\)th iteration, and the computation of \(\nabla l_s \cdot U^{(k-\frac{1}{2})}(r)\) in Eq. (5.38) involves the FS currents \(J^{f}(r)\) at \(k\)th and \((k-1)\)th iterations. The reason for using \(J^{f,(k-1)}(r)\) is that the FS currents \(J^{f,(k)}(r)\) at the \(k\)th iteration have not been computed yet in the FS process for the elements above and on the right of the test point located in the \((n, m)\)th element. In Eq. (5.36), the currents \(J^{f,(k)}(r)\) are first solved for all receiving elements, and then employed in Eq. (5.37) to solve
for the currents $J^{b,(k)}(r)$ for all receiving elements. It should be pointed out that for each $(n, m)$th receiving element, it is required to invert the $2 \times 2$ sub-matrix $\overline{Z}_{nm}$ analytically in order to solve for $J^{f,(k)}(r)$ or $J^{b,(k)}(r)$ due to the coupling between $J_x(r)$ and $J_y(r)$ in matrix self-terms. The iterative process is continued in the FB fashion until the surface currents exhibit convergence to within a specified accuracy criterion, and the normalized pseudo-residual ($PR_{N}^{(k)}$), defined as

$$PR_{N}^{(k)} = \frac{||\overline{I}^{(k)} - \overline{I}^{(k-1)}||}{||\overline{I}^{(k)}||}.$$  \hspace{1cm} (5.40)

is employed to monitor the convergence of the FB method, where $\overline{I}^{(k)}$ is the solution vector at the $k$th iteration. Typically, the tolerance of 0.01 seems to yield quite accurate results for RSS problems of interest.

The FB method usually provides very rapid convergence in many RSS problems of interest; however it is an $O(N_{\text{tot}}^2)$ iterative method due to its direct computation of the matrix-vector multiplies to compute the mutual coupling between all pairs of points on the rough surface. In addition, the impedance matrix $\overline{Z}$ must be stored at a cost of $O(N_{\text{tot}}^2)$ memory storage or all elements of the matrix must be recomputed at each iteration with a time-consuming computation. To accelerate the FB method, the NSA algorithm is employed to achieve $O(N_{\text{tot}})$ for both CPU time and memory storage requirements. In the next section, the novel spectral acceleration algorithm with the FB method for the case of impedance rough surfaces is discussed in detail.
Figure 5.1: Integration contour of $g(r, r')$ and $\nabla g(r, r')$ on the complex $k_z$ plane. The original contour $C_{k_z}$ is deformed to the new contour $C_{k_z}$.

Figure 5.2: Integration contour of $g(r, r')$ and $\nabla g(r, r')$ on the complex $k_y$ plane for a fixed value of $k_z$. $C_{k_y}$ is the original contour and $C_{\delta k_y}$ is the deformed contour.
5.3 Novel Spectral Acceleration Algorithm for 2-D Impedance Rough Surfaces Using the Spectral Domain Representation of the Green's Function in the x-Direction

In this section, the NSA formulation based on the spectral domain representation of the free space 3-D scalar Green's function in the x-direction is considered. It should be pointed out that the 2-D NSA formulation for the IBC case is similar to the one for the PEC case illustrated earlier in Section 4.4 in Chapter 4, and thus only the key steps are emphasized here. For convenience in discussion, the computation of the FS process associated with the computation of the weak region contribution $J_{w}^{f,(k)}(r)$ (see Figure 4.4) is considered only. Following the same procedure as in the PEC case, $J_{w}^{f,(k)}(r)$ can be expressed as

$$J_{w}^{f,(k)}(r) = 2n \times \int \int_{S_{xy,w}} dx \, dy \, \nabla g(r, r_t) \times [J_{f}^{f,(k)}(r_t) + J_{b}^{b,(k-1)}(r_t)]$$

$$+ 2n \times \int \int_{S_{xy,w}} dx \, dy \left\{ -i \omega \eta \eta_s g(r, r_t) \hat{n} \times [J_{f}^{f,(k)}(r_t) + J_{b}^{b,(k-1)}(r_t)] \right\}$$

$$+ \frac{\nabla g(r, r_t) |n| \eta_s}{i \omega \mu} \nabla t_s \cdot \left[ \hat{n} \times \left( J_{f}^{f,(k-\frac{1}{2})}(r_t) + J_{b}^{b,(k-1)}(r_t) \right) \right] \right\}.$$  (5.41)

where $S_{xy,w}$ is the weak region in the FS process as shown in Figure 4.4. The current $J_{w}^{f,(k)}(r)$ are computed using the 2-D NSA algorithm for IBC surfaces.

Using the spectral integral representation of the free space 3-D scalar Green's function in the x-direction with appropriate contour deformation (see Eq. (4.45)), $J_{w}^{f,(k)}(r)$ can be reexpressed in terms of the complex vector radiation function $F^{(k)}(r, k_z, k_y)$ on the $k$th iteration as follows:

$$J_{w}^{f,(k)}(r) = -\frac{1}{4\pi^2} n \times \int_{C_{k_z}} \int_{C_{k_y}} dk_z \, dk_y \frac{F^{(k)}(r, k_z, k_y)}{k_x}.$$  (5.42)
where $F^{(k)}(r, k_z, k_y)$ is defined as in Eq. (4.36) with

$$
V^{(k)}(r) = (k - i\omega \eta_s \hat{n}) \times \left[ J^{(k)}(r) + J^{b,(k-1)}(r) \right] + \frac{|n|\eta_s k}{i\omega \mu} \nabla I_s \cdot \left[ \hat{n} \times \left( J^{(k-\frac{1}{2})}(r) + J^{b,(k-1)}(r) \right) \right],
$$

(5.43)

and $C_{\delta z}$ and $C_{\delta y}$ are the deformed contours in the complex $k_z$ and $k_y$ planes as shown in Figures 5.1 and 5.2, respectively.

Like the PEC case, the vector function $F^{(k)}(r, k_z, k_y)$ can be computed from weak element currents in a recursive fashion as shown in Eq. (4.38). The topology in both complex planes is discussed in detail in Section 4.4. Eq. (5.42) can be discretized and mapped to the real axis according to the following mappings: $dk_z \rightarrow \Delta k_z e^{-i\delta k_z}$, $k_z \rightarrow k_{zp} = p\Delta k_z e^{-i\delta k_z}$ for $p = -P, \ldots, P$. $dk_y \rightarrow \Delta k_y e^{-i\delta k_y}$, and $k_y \rightarrow k_{yp} = q\Delta k_y e^{-i\delta k_y}$ for $q = -Q_p, \ldots, Q_p$, where $\Delta k_z$ and $\Delta k_y$ are the integration step sizes in the complex $k_z$ and $k_y$ planes respectively. $P = \frac{k_z{\max}}{\Delta k_z} + 1$, $Q_p = \frac{Re[k_y{\max}]}{\Delta k_y} + 1$, $k_z{\max}$ and $Re[k_y{\max}]$ are the maximum domains of integration in the complex $k_z$ and $k_y$ planes as shown in Figures 5.1 and 5.2 respectively, and the total number of plane waves in both planes $Q_{TOT,x}$ is given by

$$Q_{TOT,x} = \sum_{p=\pm P} P (2Q_p + 1).
$$

(5.44)

Finally, the discretized version of Eq. (5.42) can be written as

$$J_w^{(k)}(r) = -\frac{1}{4\pi^2} \Delta \Omega \sum_{p=\pm P} \sum_{q=\pm Q_p} \frac{W(k_{zp}, k_{yp})[n \times F^{(k)}(r, k_{zp}, k_{yp})]}{k_{zp,q}} e^{-i\delta k_z} e^{-i\delta k_y},
$$

(5.45)

where $k_{zp,q} = (k^2 - k_{yz}^2 - k_{zp}^2)^{\frac{1}{2}}$, $\Delta \Omega = \Delta k_y \Delta k_z$ and $W(k_{zp}, k_{yp})$ is a weighting function for numerical integration. For convenience, the 2-D NSA parameters in both complex
$k_z$ and $k_y$ planes given in Section 4.4 are summarized here:

\begin{align*}
\delta k_z &= \begin{cases} 
\frac{\pi}{4}, & \tan^{-1}\left(\frac{\Delta z_{\text{max}}}{L_z}\right) \leq 0.1 \\
\tan^{-1} J, & \tan^{-1}\left(\frac{\Delta z_{\text{max}}}{L_z}\right) > 0.1 
\end{cases} \\
k_{z, \text{max}} &= \begin{cases} 
\sqrt{\frac{2\alpha k}{L_z}}, & \tan^{-1}\left(\frac{\Delta z_{\text{max}}}{L_z}\right) \leq 0.1 \\
k_{z, \text{sm}} + k_{z, \text{tail}}, & \tan^{-1}\left(\frac{\Delta z_{\text{max}}}{L_z}\right) > 0.1 
\end{cases} \\
\Delta k_z &= \frac{1}{22} \sqrt{\frac{C_z k}{L_z}} \\
\delta k_y &= \gamma \\
\text{Re}[k_{y, \text{max}}] &= \begin{cases} 
\sqrt{\frac{2\alpha k}{L_y}}, & \tan^{-1}\left(\frac{\Delta y_{\text{max}}}{L_y}\right) \leq 0.1 \\
\text{Re}[k_{y, \text{sm}}] + k_{y, \text{tail}}, & \tan^{-1}\left(\frac{\Delta y_{\text{max}}}{L_y}\right) > 0.1 
\end{cases} \\
\Delta k_y &= \frac{1}{22} \sqrt{\frac{C_y k}{R_{xy}}},
\end{align*}

where $k_{z, \text{sm}} = \frac{k_{z, \text{max}}}{R_{xz}}$, $k_{y, \text{sm}} = \frac{k_{y, \text{max}}}{R_{xy}}$, $R_{xz} = \sqrt{L_z^2 + (\Delta z_{\text{max}})^2}$, $R_{xy} = \sqrt{L_y^2 + D_y^2}$, and $J = \frac{1}{\max\{\xi, 1\}}$, where $\xi$ is the solution of the nonlinear equation as shown in Eq. (4.47) in Chapter 4. and $a_{\text{max}}$ is some constant (typically found to be less than 3). Note that $\xi$ can be solved numerically via a standard root-finding technique such as Muller’s method (see Appendix F). In addition, there are six unknown constants in the above formulas: $\gamma, a_{\text{max}}, k_{z, \text{tail}}, k_{y, \text{tail}}, C_z$ and $C_y$, and these unknowns can be determined empirically by comparing the analytical solution of $g(r, r')$ to the solution obtained from its spectral domain representation as discussed in detail in Section 4.4.

So far, the development of the 2-D NSA algorithm for the IBC rough surfaces using the spectral domain representation of the free space 3-D scalar Green’s function in the $x$-direction is completed. As in the PEC case, it is straightforward to incorporate an additional NSA formulation based on the spectral domain representation of the Green’s function in the $y$-direction and the “multilevel” algorithm for improving the...
computational efficiency of the strong-region computation and accuracy of the standard (one-level) 2-D NSA algorithm for 2-D extremely large-scale IBC rough surfaces with relatively large surface cross-range sizes. Thus, it is not necessary to illustrate the detail of this additional NSA formulation in this chapter. In the next section, some numerical results are illustrated to show the accuracy and efficiency of the 2-D FB/NSA algorithm using the \( x \)-expansion for IBC large-scale rough surfaces.

5.4 Numerical Results

In this section, the 2-D FB/NSA algorithm employs only the \( x \)-expansion for large-scale surfaces of interest. To illustrate the computational efficiency and accuracy of the 2-D FB/NSA algorithm using the \( x \)-expansion for the IBC large-scale rough surfaces, consider a deterministic IBC rough surface with \( \epsilon_r = 10.0 + i10.0 \) of size \( 128\lambda \times 16\lambda \) illuminated by a vertical polarized (TM) tapered plane wave with the taper parameter \( g' = 6 \) (see Section 4.3) at an incident angle of \( 40^\circ \). The surface of interest is a realization of a Gaussian random process described by a Gaussian spectrum with \( l_x = l_y = 1.414\lambda \) and \( h = 1.0\lambda \) having \( z_{min} = -3.651\lambda \), \( z_{max} = 4.103\lambda \) and \( \Delta z_{max} = 7.754\lambda \). The surface is sampled with 8 points per \( \lambda \) resulting in 262,144 unknowns for \( x \)- and \( y \)-polarization surface currents. The following results on CPU time for the above deterministic case are based on a Pentium II 333 MHz computer with 128 Mbytes RAM. The standard 2-D FB method is employed to compare with the 2-D FB/NSA method and requires 6 iterations to converge to within 1 % accuracy based on the normalized pseudo-residual \( PR_{x}^{(k)} \) test (see Eq. (4.15) in Chapter 4). Its total CPU time for this example is 20990.88 minutes. The 2-D FB/NSA method employs the following 2-D NSA parameters: \( L_x = 4.5\lambda \), \( \gamma = 0.08 \)
rad., \( a_{\text{max}} = 2.0, k_{z,\text{tail}} = 0.20k, k_{y,\text{tail}} = 0.20\ Re[\kappa], C_z = 10.0, \) and \( C_y = 15.0, \) and requires the same number of iterations to converge within the same accuracy as in the 2-D FB method. However, its total CPU time is 2368.23 minutes. Thus, with the 2-D FB/NSA algorithm, a CPU time reduction of 8.86 is achieved in this case.

Numerical results are presented in terms of the normalized bistatic radar cross section (RCS) \( \sigma_{\alpha,\beta}(\theta_s, \varphi_s) \) in the plane of incidence, defined for a scattered wave in \( \alpha \)-polarization and an incident wave in \( \beta \)-polarization, as given in Eq. 1.26 in Section 1.3. Figure 5.3 (a) and (b) plots the normalized bistatic RCS in dB versus the scattering angle \( \theta_s \) when \( \varphi_s = \varphi_i = 0^\circ \) (in plane scattering) for \( H \)- and \( V \)-polarizations respectively, comparing between the standard 2-D FB and 2-D FB/NSA methods. From the plots, the normalized bistatic RCS obtained from both methods are in very good agreement. Thus, the 2-D FB/NSA method provides very accurate results with a great reduction of CPU time. In addition, Figure 5.4 shows the Monte-Carlo simulation results computed via the 2-D FB/NSA method and averaged over 150 realizations for the same scenario as in the previous example. The Monte-Carlo simulations for this case were performed on the CRAY T3E located in Ohio. The CRAY T3E at the Ohio Supercomputing Center is a stand-alone system and it has a total of 136 processing elements (PE’s), where each PE includes a 300 MHz DEC 21164 CPU with 600 MFLOPS and 16 Mwords (128 Mbytes) of memory (see more details at http://www.osc.edu/). From the plots, the backscattering enhancement phenomenon can be observed clearly for all polarizations at \( \theta_s = -40^\circ \) corresponding to the backscattering direction of an incident angle of 40° used in this simulation. Thus, the backscattering enhancement phenomenon can be studied effectively using

216
Figure 5.3: A comparison of the normalized bistatic RCS in dB computed by the standard 2-D FB method and the 2-D FB/NSA method for a deterministic IBC rough surface with $\varepsilon_r = 10.0 + i10.0$ of size $128\lambda \times 16\lambda$ illuminated by a vertical polarized (TM) tapered plane wave with the taper parameter $g/t = 6$ at an incident angle of $40^\circ$. The surface of interest is a realization of a Gaussian random process described by a Gaussian spectrum with $l_x = l_y = 1.414\lambda$ and $h = 1.0\lambda$: (a) HV polarization (b) VV polarization.
Figure 5.4: Monte-Carlo simulation results computed via the 2-D FB/NSA method and averaged over 150 realizations for the same scenario as considered in Figure 5.3: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
the 2-D FB/NSA method, and several numerical results will be illustrated in Chapter 6.

### 5.5 Conclusions

In this chapter, the 2-D FB/NSA algorithm for 2-D impedance rough surfaces is developed. The algorithm is basically an extension of the 2-D FB/NSA algorithm developed for 2-D PEC rough surfaces to take the finite conductivity of the medium below the surface profile into account. It is found that two additional terms are required in the MFIE formulation for the PEC case to account for nonzero surface impedance, and one of them involves the surface divergence of the associated currents which require a numerical differentiation. In addition, the 2-D NSA algorithm for the IBC case is similar to the one for the PEC case. Thus, the method remains an $O(N_{tot})$ algorithm for a fixed surface dimension in the $y$-direction $D_y$, and is most suited for rectangular surfaces due to the treatment of surface cross-range size as a “roughness” parameter. Comparisons of numerical results between the standard 2-D FB and 2-D FB/NSA methods for a deterministic surface have shown that the 2-D FB/NSA method yields very accurate results with a great reduction of CPU time. Thus, the 2-D FB/NSA method, developed for the PEC and IBC rough surfaces in Chapters 4 and 5, is a good candidate for studying the effects of the surface material to the backscattering enhancement phenomenon as will be discussed in detail in Chapter 6.
CHAPTER 6

Numerical Studies of Backscattering Enhancement for Two-Dimensional Random Rough Surfaces

6.1 Introduction

One of the most interesting phenomena associated with rough surface scattering (RSS) is the backscattering enhancement [4]-[22], and it has been observed experimentally from several rough surface types [8]-[15]. One such type involves surfaces with relatively large slopes. Most of approximate analytical techniques have restricted region of validity in terms of slope and roughness of the surface and are computationally inefficient, often requiring the evaluation of many multidimensional integrals. The limitations of the approximate analytical methods for large-slope surfaces and more powerful modern computers have increased interest in numerical techniques. However, numerical methods are computationally intensive, particularly when a Monte-Carlo simulation to obtain field statistics is needed. One way to reduce computational requirements is to model surfaces as being rough in one direction only: i.e. a 2-D scattering problem, as employed in Chapter 3. Note that computation of electromagnetic (EM) scattering from 1-D surface models may be adequate for co-polarized
scattering as discussed in Chapter 3; however, to account for cross-polarization scattering, a 2-D surface model corresponding to a full 3-D computation is required. Cross-polarized scattering can provide additional information regarding the scattering properties of the surface as well as being a good tool for studying backscattering enhancement. Note that backscattering enhancement, like cross-polarized scattering, is related to multiple-scattering effects and therefore should be more easily studied with the use of cross-polarized scattering. For co-polarization, the multiple-scattering effects can be obscured by the presence of the first-order scattering. Unlike the co-polarization, the first-order scattering associated with cross-polarization is zero, and the multiple scattering, especially the second-order backscattering enhancement, can be more clearly exhibited for cross-polarization. However, 2-D surfaces greatly increase computational requirements since the surface profile and surface fields must be discretized in both dimensions. Thus, efficient numerical methods are indispensable tools for studying backscattering enhancement numerically.

Several previous studies of backscattering enhancement have been performed with both 1-D and 2-D surfaces, but results have not been reported for large incident angles. From an analytical point of view, studies of backscattering enhancement phenomenon at large incident angles are very challenging due to the non-local interactions and increased shadowing effects involved. From a numerical viewpoint, the absence of Monte-Carlo results for this case is due to the computing limitation of previous studies, caused in part by the unavailability of efficient numerical techniques. The NSA algorithm with the FB method has been shown to be a very efficient and accurate iterative MM technique for both 1-D and 2-D surfaces as discussed in earlier
Due to the fact that the NSA algorithm is still very efficient for moderately-rough large-scale surfaces and its memory storage requirement is very low compared to other fast techniques, numerical studies of backscattering enhancement from 2-D moderately-rough large-scale surfaces, as are required in scattering problems at large incident angles, are feasible.

In this study, variations in the characteristics of backscattering enhancement with incident angle, surface material (impedance or perfect electric conductor (PEC)), polarization, and surface statistics are investigated. Incident angles ranging from normal incidence up to 70° are considered. Due to the fact that larger surface lengths are required primarily in the along range direction and not the cross range direction as the incident angle increases toward grazing incidence, a rectangular surface size is used to reduce the number of unknowns. However, the surface size must be chosen appropriately such that numerical results obtained from Monte-Carlo simulations are accurate. Surface sizes of 128λ by 16λ sampled with 8 unknowns per λ resulting in 262,144 unknowns is employed in this study, where λ is the EM wavelength in free space. This surface size is chosen such that numerical results are reasonably accurate for incident angles up to 70° as discussed later in the next section. For consistency, the same surface size is employed for all incident angles of interest in order to have the same surface realizations, even though smaller surface sizes could be sufficient for smaller incident angles. 150 surface realizations are averaged for Gaussian random processes with an isotropic Gaussian spectrum (0.5λ ≤ h ≤ λ and 0.5 ≤ σ_s ≤ 1.0, where h and σ_s are the rms surface height and the rms surface slope, respectively). It should be pointed out that for this rough surface type both large rms surface heights and slopes are required to exhibit backscattering enhancement. Surfaces with large

222
rms heights or large rms slopes alone are not sufficient [52]. In addition, comparisons of the co-polarized results between the 1-D and 2-D surface models with varying incident angle, surface material and polarization are also discussed in this chapter. It should be emphasized that in this chapter the 2-D FB/NSA algorithm employs only the $x$-expansion since the surface size of interest is considered as a large-scale surface and its $D_y$ is not too large. Although the 2-D FB/NSA method is a very efficient iterative technique, the numerical problem of interest is still extremely computationally intensive. Parallel computing techniques are incorporated into the 2-D FB/NSA method, primarily to perform Monte-Carlo simulations in parallel. Results obtained from these studies will provide more physical insight to the backscattering enhancement phenomenon, and will assist in the future development of analytical theories.

This chapter is organized as follows. Section 6.2 briefly describes parallel computing techniques. Various numerical results and discussion are presented in Section 6.3. Section 6.4 provides comparisons between backscattering enhancement results obtained from 1-D surface and 2-D surface models, and a summary and conclusions can be found in Section 6.5.

6.2 Parallel Processing

Although the 2-D FB/NSA method is a very efficient iterative technique, the numerical problem of interest is still extremely computationally intensive. Parallel computing techniques are incorporated into the 2-D FB/NSA method, primarily to perform Monte-Carlo simulations in parallel. Monte-Carlo simulations for 2-D rough surfaces in this chapter were performed on two supercomputers, the CRAY T3E
and the IBM Power2 Super Chip (P2SC) and Symmetric Multiprocessing (SMP) (P2SC/SMP) system, located in Ohio and Hawaii, respectively. The CRAY T3E at the Ohio Supercomputing Center is a stand-alone system, and it has a total of 136 processing elements (PE's), where each PE includes a 300 MHz DEC 21164 CPU with 600 MFLOPS and 16 Mwords (128 Mbytes) of memory (see more detail at http://www.osc.edu/). The IBM P2SC/SMP offers 224 P2SC batch nodes and two interactive SMP eight-processor nodes (see more detail at http://www.mhpcc.edu/). The parallel algorithm uses the message passing interface (MPI) for task and processor controls. In the next section, numerical results for backscattering enhancement studies are illustrated and discussed in detail.

6.3 Numerical Results and Discussion

In this section, variations in the characteristics of backscattering enhancement with incident angle, surface material, polarization, and surface statistics are investigated. Surface sizes of $128\lambda$ by $16\lambda$ sampled with 8 unknowns per $\lambda$ resulting in 262,144 unknowns are employed, and 150 surface realizations averaged for Gaussian random processes with an isotropic Gaussian spectrum given by

$$W(k_x, k_y) = \frac{l^2 h^2}{4\pi} e^{-\frac{1}{2}(k_x^2 + k_y^2)l^2}.$$  

(6.1)

where $W(k_x, k_y)$ represents the spectrum amplitude in $m^4$ as described in Appendix A, $l$ is the surface correlation length in $m$, $h$ is the rms surface height in $m$, and $k_x$ and $k_y$ are the spatial frequencies in the $x-$ and $y-$ directions in rad./$m$ respectively. Note that the rms surface slopes in the $x-$ and $y-$ directions, denoted as $\sigma_{x,x}$ and
respectively, are identical for the isotropic Gaussian spectrum; i.e.

\[ \sigma_{s,x} = \sigma_{s,y} = \sigma_s, \]  

(6.2)

where \( \sigma_s \) is defined as follows:

\[ \sigma_s = \frac{\sqrt{2} h}{l}. \]  

(6.3)

The total rms surface slopes \( \sigma_{s,tot} \) is defined as

\[ \sigma_{s,tot} = \sqrt{\sigma_{s,x}^2 + \sigma_{s,y}^2}, \]  

(6.4)

and for the isotropic Gaussian spectrum it is reduced to

\[ \sigma_{s,tot} = \sqrt{2}\sigma_s. \]  

(6.5)

The ranges of surface statistical parameters employed in this study are: \( 0.5\lambda \leq h \leq \lambda \) and \( 0.5 \leq \sigma_s \leq 1.0 \) (i.e. \( 0.707 \leq \sigma_{s,tot} \leq 1.414 \)), which are relatively large heights and large slopes. As pointed out in [4], rough surfaces corresponding to these parameters should exhibit backscattering enhancement. Specifically, three isotropic Gaussian spectra with a fixed surface correlation length \( l = \sqrt{2}\lambda \) and \( h/\lambda = \sigma_s = 0.5, 0.707 \) and 1.0 are considered. Surface materials considered in this study are PEC surfaces and impedance surfaces with \( \epsilon_{r1} = 38.0 + i40.0 \) and 10.0 + i10.0, where \( \epsilon_{r1} \) is the relative permittivity of the nonmagnetic region below the surface profile. It should be pointed out that impedance surfaces with the smallest magnitude of \( \epsilon_{r1} \) (|\( \epsilon_{r1} | \)) (i.e. \( \epsilon_{r1} = 10.0 + i10.0 \)) is still a good approximation for lossy dielectric surfaces with the same \( \epsilon_{r1} \). This is confirmed by obtaining a very good agreement between Monte-Carlo simulation results for impedance surfaces and those obtained from the small perturbation method (SPM) for small height 1-D dielectric surfaces up to incident
angle 85° as discussed previously in Section 3.3 in Chapter 3. In addition, Monte-Carlo results obtained from 1-D impedance surfaces and 1-D lossy dielectric surfaces (with \( \epsilon_{r1} = 10.0 + i10.0 \) for both models) for the roughest case (\( h/\lambda = \sigma_s = 1.0 \)) and for all incident angles of interest (\( 0^\circ \leq \theta_i \leq 70^\circ \)) are in very good agreement.

To study polarization effects on backscattering enhancement, both co-polarization (HH and VV) and cross-polarization (VH and HV) are considered, where the symbols H and V stand for the horizontal and vertical polarizations, respectively. It is emphasized that backscattering enhancement should be more easily studied with the use of cross-polarized scattering since both backscattering enhancement and cross-polarized scattering are associated with multiple-scattering effects. In addition, effects of incident angle on backscattering enhancement, where incident angles ranging from normal incidence up to 70°, are considered. The taper parameter \( g_l \) employed in generating the incident field is chosen to be equal to 6.0 (see Section 4.3 in Chapter 4). With this value of \( g_l \) and surface size \( 128\lambda \) by \( 16\lambda \) averaged with 150 surface realizations, numerical results are reasonably accurate for the incident angle up to 70°. This is confirmed by comparing these results with the results obtained from larger surface sizes with a fixed \( g_l \) and obtaining a good agreement.

The first step in applying the 2-D FB/NSA method is to appropriately choose the strong distance \( L_z \) and the 2-D NSA parameters, which depend on the surface statistics and the surface cross-range size \( D_y \) as discussed in Chapter 4. From numerical studies of the spectral domain representation of \( g(\mathbf{r}, \mathbf{r'}) \), it is found that appropriate \( L_z \) and 2-D NSA parameters for the three surface statistics of interest are listed as follows:

- For \( h/\lambda = \sigma_s = 0.5 \): \( L_z = 3.5\lambda \), \( \gamma = 0.08 \) rad., \( a_{\text{max}} = 1.0 \), \( k_{z,\text{tail}} = 0.24k \),
\[ k_{y,\text{tail}} = 0.24 \Re[\kappa], \ C_z = 8.0, \ C_y = 14.0 \]

- For \( h/\lambda = \sigma_s = 0.707 \): \( L_z = 4.0\lambda, \ \gamma = 0.08 \text{ rad.} \ a_{\text{max}} = 2.0, \ k_{z,\text{tail}} = 0.2k, \)
\[ k_{y,\text{tail}} = 0.24 \Re[\kappa], \ C_z = 11.0, \ C_y = 14.0 \]

- For \( h/\lambda = \sigma_s = 1.0 \): \( L_z = 4.5\lambda, \ \gamma = 0.08 \text{ rad.} \ a_{\text{max}} = 2.0, \ k_{z,\text{tail}} = 0.20k, \)
\[ k_{y,\text{tail}} = 0.20 \Re[\kappa], \ C_z = 10.0, \ C_y = 15.0 \]

These parameters are chosen such that the relative error obtained in the evaluation of \( g(r, r') \) in the spectral domain for the two worst-case scenarios (i.e. one in the \( \rho-z \) plane and another in the \( x-y \) plane), as discussed in Section 4.4 in Chapter 4, is less than 0.5%. It is found that the 2-D FB/NSA method is faster than the standard FB method about a factor of 3 to 10 in this study. Numerical results are presented in terms of the normalized incoherent radar cross section (RCS) \( \sigma_{a,3}(\theta_s, \theta_i) \) in the plane of incidence \( (\phi_i = \phi_s = 0 \text{ rad.}) \), defined for a scattered wave in \( \alpha \)-polarization and an incident wave in \( \beta \)-polarization, as given in Eq. (1.29) in Section 1.3. The rest of this section will present and discuss the variations in the characteristics of backscattering enhancement with incident angle, surface material and surface statistics, respectively. Note that polarization effects to the backscattering enhancement are discussed for each scenario as well.

### 6.3.1 Effects of Incident Angle

Variations in the characteristics of backscattering enhancement with incident angle \( \theta_i \) ranging from normal incidence up to 70° for different surface materials and surface statistics are considered in this section. For convenience in discussion, define the angular regions \(-90^\circ \leq \theta_s \leq 0^\circ \) and \(0^\circ \leq \theta_s \leq 90^\circ \) to be the “forward” and “backward” angular regions, respectively. Figures 6.1 and 6.2 shows a comparison of
the co-polarized and cross-polarized normalized incoherent bistatic RCS with varying incident angles for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum \( h/\lambda = \sigma_s = 1.0 \) for \( 0^\circ \leq \theta_i \leq 40^\circ \) and \( 60^\circ \leq \theta_i \leq 70^\circ \). respectively. It can be seen from these plots that the overall level of the normalized incoherent bistatic RCS tends to decrease as the incident angle \( \theta_i \) increases from \( 0^\circ \) to \( 70^\circ \) for both co-polarization and cross-polarization as should be expected. Note that backscattering enhancement exists only for small \( \theta_i \) from \( 0^\circ \) to \( 40^\circ \): i.e. no well-defined backscattering peak exists for the \( 60^\circ \) and \( 70^\circ \) cases. However, the co-polarized scattered energy tends to largely distribute in the backward angular region for all \( \theta_i \) of interest, even with relatively large \( \theta_i \). It is also observed that there are some differences between HH and VV scattering patterns. For example, VV polarization exhibits a secondary peak at the specular direction for \( \theta_i = 20^\circ \), but no such peak clearly exists for HH polarization. On the other hand, for a higher angle of incidence \( \theta_i = 70^\circ \), HH polarization exhibits a secondary peak in the neighborhood of the specular direction \( \theta_s = 70^\circ \) while VV polarization does not. Note also that HH polarization tends to exhibit more scattered energy near the specular direction compared to VV polarization as \( \theta_i \) increases toward grazing incidence. For cross-polarization, VH scattering patterns are similar to those in HV, and as \( \theta_i \) increases the scattering patterns tend to broaden and become more uniform at least for these surfaces. Next, consider variations of scattering patterns with incident angle for IBC Gaussian rough surfaces with the same surface statistics as in the PEC case.

Figures 6.3 and 6.4 show a comparison of the co-polarized and cross-polarized normalized incoherent bistatic RCS with varying \( \theta_i \) for the same surface statistics as the PEC case with IBC Gaussian rough surfaces with \( \epsilon_r = 38.0 + i40.0 \) for \( 0^\circ \leq \)
Figure 6.1: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($0^\circ \leq \theta_i \leq 40^\circ$) for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$ and $l_z = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.2: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($60^\circ \leq \theta_i \leq 70^\circ$) for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_z = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.3: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($0^\circ \leq \theta_i \leq 40^\circ$) for the case of IBC Gaussian surfaces with relative permittivity $\varepsilon_r = 38.0 + i 40.0$ and an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.4: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($60^\circ \leq \theta_i \leq 70^\circ$) for the case of IBC Gaussian surfaces with relative permittivity $\varepsilon_{r1} = 38.0 + i40.0$ and an isotropic Gaussian spectrum $h/\lambda = \sigma_x = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.5: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($0^\circ \leq \theta_i \leq 40^\circ$) for the case of IBC Gaussian surfaces with relative permittivity $\varepsilon_r = 10.0 + i0.0$ and an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.6: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($60^\circ \leq \theta_i \leq 70^\circ$) for the case of IBC Gaussian surfaces with relative permittivity $\varepsilon_r = 10.0 + i0.0$ and an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.7: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($0^\circ \leq \theta_i \leq 40^\circ$) for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 0.5$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.8: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying incident angles ($60^\circ \leq \theta_i \leq 70^\circ$) for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_z = 0.5$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
\( \theta_i \leq 40^\circ \) and \( 60^\circ \leq \theta_i \leq 70^\circ \), respectively. In addition, Figures 6.5 and 6.6 shows a comparison of the co-polarized and cross-polarized normalized incoherent bistatic RCS with varying \( \theta_i \) for the same surface statistics as above with IBC Gaussian rough surfaces with \( \epsilon_{r1} = 10.0 + i10.0 \) for \( 0^\circ \leq \theta_i \leq 40^\circ \) and \( 60^\circ \leq \theta_i \leq 70^\circ \), respectively.

From Figures 6.1 to 6.6, it is observed that as \( |\epsilon_{r1}| \) decreases the overall level of the bistatic RCS decreases noticeably. Note that cross-polarized scattering patterns for both values of \( \epsilon_{r1} \) in Figures 6.3 to 6.6 reduce in amplitude but are otherwise similar to those in the PEC case. As in the PEC case, co-polarized scattered energy with IBC surfaces primarily distributes in the backward angular region for all incident angles of interest. However, some co-polarized scattering characteristics are different between the PEC and IBC cases as discussed below. In Figures 6.3 and 6.4, the decreasing rate of the bistatic RCS level with respect to \( \theta_i \) for \( \epsilon_{r1} = 38.0 + i40.0 \) is slower than the decreasing rate for the PEC case. In Figure 6.5, it is interesting to note that the co-polarized backscattering peak at larger incident angles can exceed the one for smaller angles of incidence. For example, the VV backscattering peak for \( \theta_i = 40^\circ \) exceeds the VV backscattering peak for \( \theta_i = 0^\circ \) and \( 20^\circ \). This is an unusual phenomenon observed in the scattering from IBC rough surfaces, which was observed in 1-D surface results presented in Chapter 3 as well. It is also observed that for \( \theta_i = 70^\circ \) HH polarization exhibits a dominant peak near the specular direction. For both PEC and IBC cases, it is also found that as \( \theta_i \) increases HH polarization exhibits more scattered energy near the specular direction than VV polarization. As \( |\epsilon_{r1}| \) decreases and \( \theta_i \) increases, both HH and VV polarizations tend to exhibit more scattered energy near the specular direction. Note that for all surface materials considered in this study, the backscattering enhancement exists only for relatively
small incident angles $0^\circ \leq \theta_i \leq 40^\circ$. Next, consider variations in the backscattering enhancement with varying $\theta_i$ for smoother surface statistics.

Figures 6.7 and 6.8 illustrate a comparison of the normalized co-polarized and cross-polarized incoherent bistatic radar cross sections with varying incident angles for PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 0.5$. Comparing Figures 6.7 and 6.8 with Figures 6.1 and 6.2, it is found that the overall bistatic RCS level decreases noticeably as $\sigma_s$ decreases from 1.0 to 0.5. In Figure 6.7 and 6.8, the dominant backscattering peak exists for $\theta_i = 0^\circ$ and $20^\circ$ only. Thus, it can be concluded that higher surface slopes and rougher surfaces can contribute more to the backscattering enhancement. The effects of surface statistics to the backscattering enhancement will be discussed in detail later in Section 6.3.3. From the plots in Figures 6.7 and 6.8, it is also observed that cross-polarized results seem to exhibit backscattering enhancement more clearly than co-polarization for this case due to the fact that the multiple-scattering effects are not obscured by the first-order scattering for cross-polarization, as discussed earlier. In addition, both VH and HV polarizations for $\theta_i = 20^\circ$ exhibit the backscattering peak at $\theta_s = -20^\circ$ as well as the specular peak at $\theta_s = 20^\circ$. For co-polarization, scattering characteristics are again different between HH and VV polarizations. Like the case of rougher surface with $\sigma_s = 1.0$, HH polarization exhibits more scattered energy near the specular direction than VV polarization as $\theta_i$ increases toward grazing incidence. Unlike the case of $\sigma_s = 1.0$, the co-polarized scattered energy largely distributes in the backward angular region only for small $\theta_i$. As $\theta_i$ increases toward grazing incidence, both co-polarized and cross-polarized scattered energies tend to migrate from the backward angular region to the forward angular region.
In summary, the backscattering enhancement strongly depends on incident angle as well as surface material, polarization, and surface statistics. As the incident angle increases, the overall bistatic RCS level tends to decrease, and backscattering enhancement does not exist for larger incident angles as in the case of 1-D rough surfaces discussed earlier in Section 3.4.4 in Chapter 3. According to the Rayleigh criterion [2], rough surfaces appear to be smoother as the incident angle increases, which possibly explains the disappearance of the backscattering enhancement at higher angles of incidence for both 1-D and 2-D rough surfaces. However, increasing surface slopes and surface roughness can contribute to backscattering enhancement as in the 1-D case (see Figure 3.19). For the polarization dependence, cross-polarization exhibits backscattering enhancement more clearly than co-polarization. In addition, scattering characteristics are different between HH and VV polarizations, and HH polarization exhibits more scattered energy near the specular direction than VV polarization as \( \theta_i \) increases toward grazing incidence. Finally, an unusual phenomenon is observed in the scattering from IBC rough surfaces: i.e. the co-polarized backscattering peak at larger incident angles can exceed the one for smaller angles of incidence. In the next section, effects of surface material to the backscattering enhancement are considered in detail.

### 6.3.2 Effects of Surface Material

Variations in backscattering enhancement characteristics with PEC and IBC surfaces for different incident angles are investigated in this section. Figures 6.9 to 6.13 illustrate a comparison of the normalized co-polarized and cross-polarized incoherent bistatic radar cross sections with varying surface materials for Gaussian surfaces with
Figure 6.9: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying surface materials at $\theta_i = 0^\circ$ for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_r = 1.0$ and $t_x = t_y = \sqrt{2} \lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.10: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying surface materials at $\theta_s = 20^\circ$ for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_z = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.11: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying surface materials at $\theta_i = 40^\circ$ for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.12: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying surface materials at $\theta_t = 60^\circ$ for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$ and $l_z = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.13: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying surface materials at $\theta_i = 70^\circ$ for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_z = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
an isotropic Gaussian spectrum \( h/\lambda = \sigma_s = 1.0 \) for \( \theta_i = 0^\circ, 20^\circ, 40^\circ, 60^\circ \) and \( 70^\circ \), respectively. Note that the vertical line in each plot of these figures corresponds to the backscattering direction. For convenience in comparison, the normalized incoherent bistatic RCS for the IBC surfaces is scaled such that its maximum is equal to the maximum for the PEC surfaces, and the scaling factor is given as a number in a parenthesis appended the value of \( \varepsilon_{r1} \) in each plot. From these figures, it can be seen that the overall bistatic RCS level increases as the surface material becomes denser (i.e. increasing \(|\varepsilon_{r1}|\)) due to the stronger reflections of scattered electromagnetic fields for the denser surface material. In addition, for a fixed incident angle the scattering patterns for each surface material follow similar trends in both co-polarization and cross-polarization. From Figures 6.9 to 6.11 the dominant backscattering peak clearly exists for all four polarizations, and \( \text{VH} \) and \( \text{HV} \) scattering patterns look very similar. In Figures 6.9 and 6.10, the co-polarized scattering pattern for the denser surfaces tends to be more concentrated in the backscattering direction, and the cross-polarized scattering patterns for each material are almost the same. In Figure 6.10, the secondary peak in the specular direction at \( \theta_s = 20^\circ \) for \( \text{VV} \) polarization also exists for the PEC surfaces and the IBC surfaces with \( \varepsilon_{r1} = 38.0 + i40.0 \). In Figures 6.12 and 6.13, the dominant peak does not exist in the backscattering direction. For co-polarization, the shift in the backscattering peak is also observed, and most of the scattered energy still distributes largely in the backward angular region. \( \text{HH} \) polarization again exhibits more scattered energy near the specular direction than \( \text{VV} \) polarization for all surface materials of interest. In Figure 6.13, the peak in the forward angular region for \( \text{HH} \) polarization can exceed the peak in the backward angular region as \(|\varepsilon_{r1}|\) decreases. For cross-polarization, the dominant peak tends to exist in
the forward angular region instead of the backward angular region as shown in Figures 6.12 and 6.13. In Figure 6.12, VH and HV polarizations still look quite similar, but they are slightly different in Figure 6.13. As pointed out earlier in Section 6.3.1, the IBC surfaces with $\epsilon_{r1} = 10.0 + i10.0$ exhibit an unusual phenomenon; i.e. the VV backscattering peak at larger incident angles can exceed the one for smaller angles of incidence as shown in Figure 6.5 (d) for $0^\circ < \theta_i \leq 40^\circ$, while the PEC and IBC surfaces with $\epsilon_{r1} = 38.0 + i40.0$ do not. Next, it is interesting to investigate the variations of the backscattering RCS with surface material.

Figure 6.14 shows a comparison of the normalized incoherent backscattering RCS (in dB) with varying surface materials for Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$ for both co-polarization and cross-polarization. The backscattering RCS results computed by the 2-D FB/NSA method in this study are obtained using Monte-Carlo simulations with 150 surface realizations and then averaging the magnitudes of backscattered fields over three consecutive angles in one degree steps (including the backscattering angle). From the plots, it is observed that the normalized co-polarized and cross-polarized incoherent backscattering radar cross sections decrease as $|\epsilon_{r1}|$ decreases. It is also noted that VH and HV backscattering patterns for each surface material are almost identical due to reciprocity as expected. However, HH and VV backscattering patterns are slightly different for each surface material.

In summary, backscattering enhancement strongly depends on surface material. It is observed that the normalized co-polarized and cross-polarized incoherent backscattering RCS decreases as $|\epsilon_{r1}|$ decreases. Next, consider the effects of surface statistics to the backscattering enhancement in detail.
Figure 6.14: Comparison of the normalized incoherent backscattering RCS (dB), computed by the Monte-Carlo 2-D FB/NSA method (150 realizations), averaged (magnitudes of backscattered fields) over three consecutive angles in one degree steps (including the backscattering angle) with varying surface materials for the case of Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$ and $l_x = l_y = \sqrt{2}\lambda$: (a) Co-polarization (b) Cross-polarization.
6.3.3 Effects of Surface Statistics

Variations in backscattering enhancement with surface statistics at \( \theta_i = 20^\circ \) and \( 70^\circ \) for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum are considered in this section. Figures 6.15 and 6.16 manifest a comparison of the normalized co-polarized and cross-polarized incoherent bistatic radar cross sections with varying rms surface slopes \( \sigma_s = 0.5, 0.707 \) and \( 1.0 \) for \( \theta_i = 20^\circ \) and \( 70^\circ \) respectively. In Figure 6.15, it is observed that all surface statistics of interest exhibit backscattering enhancement at \( \theta_s = -20^\circ \) as marked by the vertical line, and cross-polarization exhibits the backscattering enhancement more clearly than co-polarization as expected. It is also noted that as \( \sigma_s \) increases the backscattering peak tends to increase. In addition, \( \text{VH} \) and \( \text{HV} \) scattering patterns are very similar for all surface statistics, but \( \text{HH} \) and \( \text{VV} \) scattering patterns are different. As \( \sigma_s \) decreases (i.e. surfaces become smoother), the scattered energy in the forward angular region tends to increase for both co-polarization and cross-polarization. In Figure 6.16, no surface statistics exhibit a well-defined backscattering peak at \( \theta_s = -70^\circ \) as marked by the vertical line, even for the cross-polarization with the highest surface slope \( \sigma_s = 1.0 \). As \( \sigma_s \) decreases, the scattering patterns tend to migrate from the backward angular region to the forward angular region for both co-polarization and cross-polarization. In addition, \( \text{HH} \) and \( \text{VV} \) polarizations exhibit distinct scattering patterns for all surface statistics of interest, and as \( \sigma_s \) decreases the dominant peak moves from the backward angular region to the forward angular region. For cross-polarization, \( \text{VH} \) and \( \text{HV} \) scattering patterns are also different for all surface statistics, and the overall \( \text{VH} \) bistatic RCS level exceeds the overall \( \text{HV} \) bistatic RCS level for all \( \sigma_s \). Next, consider the variations of the backscattering RCS with surface statistics.
Figure 6.15: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying rms surface slopes at $\theta_i = 20^\circ$ for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.16: Comparison of Monte-Carlo 2-D FB/NSA results (150 realizations) with varying rms surface slopes at $\theta_i = 70^\circ$ for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $l_x = l_y = \sqrt{2}\lambda$: (a) HH polarization (b) VH polarization (c) HV polarization (d) VV polarization.
Figure 6.17: Comparison of the normalized incoherent backscattering RCS (dB), computed by the Monte-Carlo 2-D FB/NSA method (150 realizations), averaged over three consecutive angles in one degree steps (including the backscattering angle) and the Monte-Carlo SSA (up to 1st order and using 100 realizations) with varying rms surface slopes for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $l_x = l_y = \sqrt{2}\lambda$: (a) Co-polarization (b) Cross-polarization.
Figure 6.18: Comparison of the HH/VV polarization ratios (dB), computed by the Monte-Carlo 2-D FB/NSA method (150 realizations), averaged over three consecutive angles in one degree steps (including the backscattering angle) and the Monte-Carlo SSA (up to 1st order and using 100 realizations) with varying rms surface slopes for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $l_z = l_y = \sqrt{2}\lambda$. 

\[ \text{Polarization Ratio HH/VV (dB)} \]

\[ \theta_i (\text{deg.}) \]

- FB/NSA: $\sigma_s = 1.0$
- FB/NSA: $\sigma_s = 0.5$
- SSA: $\sigma_s = 0.1$
Figure 6.17 manifests a comparison of the normalized co-polarized and cross-polarized incoherent backscattering radar cross sections (in dB) with varying rms surface slopes $\sigma_s = 0.1, 0.5$ and 1.0 for PEC Gaussian surfaces. For the relatively smooth surfaces with $\sigma_s = 0.1$, the first-order small slope approximation (SSA) is employed to compute the backscattering RCS due to its efficiency for the case of small slope surfaces [27, 143]. Although analytical ensemble average results can be efficiently computed for the zeroth-order SSA, obtaining ensemble averages for higher order terms required to predict surface cross-polarized backscattering, rapidly becomes impractical. Thus, the first-order SSA employed in this study is applied in a Monte-Carlo simulation to obtain ensemble average SSA predictions and requires an $O(N_{\text{tot}} \log_2 N_{\text{tot}})$ computation. For the SSA computation, surface sizes of $128\lambda$ by $32\lambda$ sampled with 8 unknowns per $\lambda$ are employed, and 100 surface realizations are employed to obtain accurate incoherent backscattering RCS results. For the 2-D FB/NSA computation, the backscattering RCS results are obtained using Monte-Carlo simulations with 150 surface realizations and then averaging the magnitudes of backscattered fields over three consecutive angles in one degree steps (including the backscattering angle). From the plots, it is observed that the co-polarized and cross-polarized backscattering radar cross sections tend to increase as $\sigma_s$ increases, and variations of the backscattering RCS with $\theta_i$ for the rougher surfaces exhibiting backscattering enhancement ($\sigma_s = 0.5$ and 1.0) are typically slower than those for smoother surfaces ($\sigma_s = 0.1$) in both co-polarization and cross-polarization. In addition, VH and HV backscattering patterns are almost identical via reciprocity, but HH and VV backscattering patterns are different although they follow similar trends. To further investigate the differences in the co-polarized backscattering patterns for each
surface statistics in more detail, Figure 6.18 illustrates a comparison of the HH/VV polarization ratios (in dB), computed by the Monte-Carlo 2-D FB/NSA method and the Monte-Carlo first-order SSA as discussed in Figure 6.17, with varying rms surface slopes for the same case considered above. It is found that the HH/VV polarization ratio tends to decrease as $\theta_i$ increases and/or $\sigma_s$ decreases, with smoother surface results decreasing more rapidly with $\theta_i$. Note also that variations of the polarization ratio for the rougher surfaces ($\sigma_s = 0.5$ and 1.0) with $\theta_i$ are less than 5 dB.

In summary, backscattering enhancement strongly depends on surface statistics. As the surface roughness and surface slopes increase, it is found that the backscattering peak, the backscattering RCS and the HH/VV polarization ratio tend to increase. These may be due to increasing in multiple-scattering effects as the surface roughness and the surface slope increase. In addition, variations of the backscattering RCS and the HH/VV polarization ratio with incident angle are slower for the rougher surfaces.

### 6.4 Comparisons Between Backscattering Enhancement Results Obtained from 1-D and 2-D Surface Models

In this section, comparisons between backscattering enhancement results obtained from 1-D and 2-D surface models are investigated, and only the normalized co-polarized incoherent bistatic RCS is employed in these comparisons due to the fact that no cross-polarization exists for 1-D rough surface scattering. For 1-D surfaces, the same surface sizes with the same number of unknowns employed in studying backscattering enhancement in Chapter 3 are used in Monte-Carlo simulations with 256 surface realizations, and the surface statistical parameters are $h/\lambda = \sigma_s = 1.0$ and $l = \sqrt{2}\lambda$, where $\sigma_s$ for 1-D surfaces is defined as in Eq. (6.3). Note that the taper parameter $G$ employed in generating the tapered incident field for these 1-D surfaces
Figure 6.19: Comparison of the HH normalized incoherent backscattering RCS between the 1-D and 2-D surface models, computed by the Monte-Carlo 2-D FB/NSA method with 256 and 150 surface realizations respectively, with varying incident angles for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$: (a) $\theta_i = 0^\circ$ (b) $\theta_i = 20^\circ$ (c) $\theta_i = 60^\circ$ (d) $\theta_i = 70^\circ$. 

255
Figure 6.20: Comparison of the VV normalized incoherent backscattering RCS between the 1-D and 2-D surface models, computed by the Monte-Carlo 2-D FB/NSA method with 256 and 150 surface realizations respectively, with varying incident angles for the case of PEC Gaussian surfaces with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$: (a) $\theta_i = 0^\circ$ (b) $\theta_i = 20^\circ$ (c) $\theta_i = 60^\circ$ (d) $\theta_i = 70^\circ$. 
Figure 6.21: Comparison of the HH normalized incoherent backscattering RCS between the 1-D and 2-D surface models, computed by the Monte-Carlo 2-D FB/NSA method with 256 and 150 surface realizations respectively, with varying incident angles for the case of IBC Gaussian surfaces ($\epsilon_{\infty} = 38.0 + i40.0$) with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$: (a) $\theta_i = 0^\circ$ (b) $\theta_i = 20^\circ$ (c) $\theta_i = 60^\circ$ (d) $\theta_i = 70^\circ$. 

257
Figure 6.22: Comparison of the VV normalized incoherent backscattering RCS between the 1-D and 2-D surface models, computed by the Monte-Carlo 2-D FB/NSA method with 256 and 150 surface realizations respectively, with varying incident angles for the case of IBC Gaussian surfaces ($\epsilon_{\varepsilon_1} = 38.0 + i40.0$) with an isotropic Gaussian spectrum $h/\lambda = \sigma_s = 1.0$: (a) $\theta_i = 0^\circ$ (b) $\theta_i = 20^\circ$ (c) $\theta_i = 60^\circ$ (d) $\theta_i = 70^\circ$. 

258
is equal to 5.0 (see Eq. (2.8) in Chapter 2). For 2-D surfaces, the same surface sizes with the same number of unknowns as in Section 6.3 are employed in these comparisons with the same surface statistical parameters as in the case of 1-D surfaces and with 150 surface realizations. It should be emphasized that for this case the rms surface slopes in the $x -$ and $y -$ directions for 2-D surfaces are equal to $\sigma_s = 1.0$ for 1-D surfaces, i.e.,

$$\sigma_{s,x} = \sigma_{s,y} = 1.0.$$

(6.6)

but the total rms surface slope $\sigma_{s,\text{tot}}$ for 2-D surfaces of interest is equal to $\sqrt{2}\sigma_s = 1.414$, which is different from $\sigma_s$ for 1-D surfaces of interest. Both PEC and IBC surfaces with $\epsilon_{r1} = 38.0 + i40.0$ are considered, and the incident angles varying from $0^\circ$ to $70^\circ$ are employed in these comparisons. The results are presented in terms of the normalized incoherent bistatic RCS as defined in Eqs. (1.37) and (1.29) in Section 1.3 for the cases of 1-D and 2-D surface models, respectively. For better comparison, the normalized incoherent bistatic RCS for 1-D surface model is scaled such that its maximum is the same as the maximum of the result obtained from 2-D surface model. Figure 6.19 shows a comparison of the HH normalized incoherent bistatic RCS between the 1-D and 2-D surface models with varying incident angles for the case of PEC Gaussian surfaces. The number in the parenthesis adjacent to the "1-D" in the legend of each plot denotes the scaling factor as discussed above. From the plots, it is observed that the results for both 1-D and 2-D surface models follow the same trend for all incident angles of interest even though there are some discrepancies among the results. For example, in Figure 6.19 (d) (at $\theta_i = 70^\circ$) the result from the 2-D surface model tends to exhibit more energy distribution in the neighborhood of the specular direction than the one from 1-D surface model. It should be pointed out that if one of
the surface correlation lengths of a 2-D surface is much larger than the other, the 2-D surface is approximately a 1-D surface for the purpose of the numerical calculations. Since the 2-D surface statistics employed in these comparisons is an isotropic Gaussian spectrum with a small correlation length \( l_x = l_y = \sqrt{2} \lambda \), the numerical results for 1-D and 2-D surfaces are not expected to be identical. Furthermore, for another polarization Figure 6.20 illustrates a comparison of the VV normalized incoherent bistatic RCS between the 1-D and 2-D surface models as in Figure 6.19. In general, the results from both surface models follow the same trend for all incident angles again. Note that in Figure 6.20 (b) (at \( \theta_i = 20^\circ \)) the result from the 1-D surface model tends to exhibit more energy distribution in the neighborhood of the specular direction than the one from 2-D surface model.

For another surface material, the IBC surface with \( \epsilon_r = 38.0 + i40.0 \). Figures 6.21 and 6.22 show a comparison of the normalized incoherent bistatic RCS between the 1-D and 2-D Gaussian surface models with varying incident angles for the cases of HH and VV polarizations, respectively. From the plots in both figures, the results for both surface models exhibit some discrepancies, however they follow the same trend for all incident angles. Comparing Figures 6.19 (d) and 6.21 (d) for the case of 70° incident angle, it is observed that the discrepancy between the 1-D and 2-D surface models in the neighborhood of the specular direction increases as \( |\epsilon_r| \) decreases.

In summary, the co-polarized results from both 1-D and 2-D surface models generally follow the same trend in the above comparisons. However, it should be emphasized that the 2-D surface model can provide more information about the multiple-scattering effects such as the cross-polarization, which can assist in studying backscattering enhancement as discussed earlier in this chapter.
6.5 Conclusions

In this chapter, intensive numerical studies of backscattering enhancement from 2-D large-scale random rough surfaces using the 2-D FB/NSA method combined with parallel computing techniques are presented. Variations in the characteristics of backscattering enhancement with incident angle, surface material, polarization and surface statistics for very rough Gaussian surfaces with relatively large surface slopes and with an isotropic Gaussian spectrum are investigated in this study. It is found that backscattering enhancement strongly depends on incident angle, surface material, polarization and surface statistics as follows:

- As incident angle increases, smaller backscattering enhancement effects at backscattering direction are observed, even with the very rough and large-slope surfaces considered. As a result, backscattering enhancement should not be a significant scattering mechanism at large incident angles at least for Gaussian random processes having an isotropic Gaussian spectrum with $\sigma_{s,\text{tot}} \leq 1.414$.

- For the dependence on surface material, the normalized incoherent backscattering RCS tends to increase as surfaces become denser.

- For the polarization dependence, cross-polarization exhibits backscattering enhancement more clearly than co-polarization due to the absence of the cross-polarized first-order scattering. In addition, the scattering characteristics are different between HH and VV polarizations, with HH polarization typically exhibiting more scattered energy near the specular direction than VV polarization as $\theta_i$ increases toward grazing incidence.
• For the dependence on surface statistics, the backscattering peak, the backscattering RCS, and the HH/VV polarization ratio tend to increase as the surface roughness and the surface slope increase. In addition, variations of the backscattering RCS and the HH/VV polarization ratio with incident angle are slower for the rougher surfaces.

It is interesting that an unusual phenomenon is observed in the scattering from IBC rough surfaces for both 1-D and 2-D cases: i.e. the co-polarized backscattering peak at larger incident angles can exceed the one for smaller angles of incidence. Finally, comparisons of the co-polarized results between 1-D and 2-D surface models with varying incident angle, surface material and polarization show that the results from both surface models generally follow the same trend. Results in this chapter show that the 2-D surface model can provide more information about multiple-scattering effects such as the cross-polarization, which can indeed assist in studying backscattering enhancement. Results obtained from these studies provide more physical insight to the backscattering enhancement phenomenon, and will assist in the future development of analytical theories.
CHAPTER 7

Conclusions

This dissertation shows the development and some potential applications of an efficient and accurate numerical technique, the novel spectral acceleration (NSA) algorithm with an iterative method, for the computation of scattering from both 1-D and 2-D rough surfaces. The NSA algorithm is employed to rapidly compute mutual couplings between widely separated points on rough surfaces, and is based on a spectral domain representation of source currents and the associated Green's function. The efficiency of the NSA algorithm stems from the fact that the associated radiation function can be calculated recursively, and the contour deformation assists in maintaining the sampling rate in the numerical evaluation of the spectral integral associated with the NSA algorithm, as the size of the weak region increases. For extremely large-scale rough surfaces, the "multilevel" algorithm is incorporated into the standard NSA algorithm to improve its accuracy. For fixed surface roughness statistics and frequency, it is shown that the computational cost and memory storage requirement of the NSA algorithm is $O(N_{tot})$ as the surface size increases for both 1-D and 2-D rough surfaces. Interesting features of the method are the following: the remarkably low memory storage requirement compared to other efficient techniques for surfaces with moderate surface cross-range size $D_y$, and the method's efficiency
even for moderately rough surfaces. Like other fast algorithms, the NSA algorithm can be readily incorporated into any iterative technique. However, only the forward-backward (FB) method is employed throughout this dissertation since it provides better convergence than other iterative algorithms for surfaces of interest. In addition, the NSA algorithm seems to be competitive with other proposed fast methods such as the sparse-matrix canonical grid (SMCG) method and the steepest descent-fast multipole method (SDFMM), and is suitable for the computation of scattering from moderately-rough large-scale surfaces, which are useful for studies of LGA problems and the backscattering enhancement phenomenon.

The rest of this chapter is organized as follows. Section 7.1 provides a summary and conclusions of Chapters 2 to 6 and Appendices A to F. In addition, future works are also proposed in Section 7.2. Finally, Section 7.3 provides a concluding remark.

7.1 A Summary and Conclusions of This Dissertation

In Chapter 2, the standard 1-D FB/NSA algorithm for impedance rough surfaces [104,105] is reviewed. The associated 1-D NSA parameters given in [104] (also see their derivation in Appendix D of this dissertation) are applicable only when the outermost saddle point $\phi_{s,\text{max}}$ in the periodic complex $\phi$ plane is small. In practice, this assumption can be achieved only by adjusting the size of the strong region $L_s$ appropriately resulting in significant increases in computational time in computing the strong-region contribution for the case of rough surfaces with large height variations. To prevent the unnecessary degradation of the NSA algorithm, the new derivation of the 1-D NSA parameters for an arbitrary value of $\phi_{s,\text{max}}$ is derived resulting in more flexibility in selecting $L_s$ to compromise between the computation of the strong-region
and weak-region contributions. With the *new* 1-D NSA parameters, it is possible to optimize the 1-D NSA algorithm by considering the plot of the number of plane waves versus $L_s$. In addition, for the case of *extremely* large-scale surfaces the "multilevel" algorithm is incorporated into the *standard* 1-D FB/NSA algorithm using the *new* 1-D NSA parameters to improve its accuracy. For most practical problems, only a few weak regions or even one are sufficient to obtain the desired accuracy. Numerical results show that the *new* 1-D NSA parameters are valid, and the "multilevel" algorithm can improve the accuracy of the *standard* 1-D FB/NSA algorithm.

In Chapter 3, an application of the 1-D FB/NSA algorithm for studying the backscattering enhancement from 1-D random rough surfaces with relatively large heights and slopes is illustrated. Performing Monte-Carlo simulations using the 1-D FB/NSA method combining with parallel computing techniques, it is found that the backscattering enhancement *strongly* depends on surface statistics, surface material, polarization and incident angle. Due to its efficiency, numerical studies of the backscattering enhancement phenomenon at low grazing angles (LGA), for which no results have been reported previously in the literature, are tractable. For surfaces with large rms surface slopes, as the rms surface height increases, the backscattering peak level tends to decrease, but the backscattering angular width tends to increase. For surface material dependence, as the surface impedance increases the backscattering peak level tends to decrease, and the backscattering angular width for the HH polarization tends to be *invariant* with the change of surface impedances. However, for the VV polarization the backscattering angular width is *dependent* on the surface impedances at least for small incident angles; i.e. it tends to be wider as the surface impedance increases. For very rough surfaces exhibiting the backscattering
enhancement, the HH backscattering peak level exceeds the VV backscattering peak level, and the HH backscattering angular width tends to be narrower than the VV backscattering angular width. In addition, for some rough surfaces exhibiting the backscattering enhancement with small incident angles (i.e. near normal incidence), the VV normalized incoherent bistatic RCS exhibits a well-defined peak in the specular direction, whereas the HH normalized incoherent bistatic RCS does not clearly exhibit this specular peak. For the dependence on the incident angle, the backscattering peak level decreases, but the backscattering angular width tends to increase as the incident angle increases toward grazing angles. It is interesting that an unusual phenomenon is observed in the scattering from IBC rough surfaces: i.e. the co-polarized backscattering peak at larger incident angles can exceed the one for smaller angles of incidence. In addition, a single Neumann iteration is employed to explain the causes of the backscattering enhancement phenomenon. It is found that the second-order Kirchhoff scattering contributes to the backscattering enhancement, however it contains excessive power since the single Neumann iteration does not take the shadowing effects into consideration. Note that the backscattering enhancement is mostly contributed from the second-order scattering, and when the surface slope increases, the contributions from the second-order Kirchhoff scattering increase. Finally, numerical results in this chapter provide some physical insight into the the backscattering enhancement phenomenon, which is helpful in studying the backscattering enhancement from 2-D random rough surfaces in Chapter 6.

In Chapter 4, the 1-D FB/NSA algorithm is generalized to compute the scattering from 2-D PEC rough surfaces. For convenience in understanding, the 2-D FB/NSA
algorithm based on the $x$- expansion is developed first. It is found that this algorithm is suitable for large-scale rough surfaces with moderate surface cross-range size $D_y$. Due to the coupling between two spectral variables associated with a double spectral integral representation of the 3-D free-space scalar Green’s function $g(r, r')$. the problem of interest is more difficult and challenging than the 1-D case, and only empirical-based formulas of the 2-D NSA parameters are obtained by considering the associated worse-case scenarios of a pair of source and field points for computing $g(r, r')$. For extremely large-scale 2-D rough surfaces with relatively large cross-range sizes, the 2-D “multilevel” FB/NSA algorithm using both $x$- and $y$- expansions is employed to obtain more efficient and accurate solutions at the cost of increasing the memory storage requirement and the algorithmic complexity. For fixed surface cross-range size, frequency and surface roughness, the computational efficiency and the memory requirement of the 2-D FB/NSA algorithm remain $O(N_{tot})$, where $N_{tot}$ is the total number of unknowns to be solved, as the surface size increases. Numerical results confirm the efficiency and accuracy of the 2-D FB/NSA algorithm for different surface sizes and surface statistics. The 2-D FB/NSA algorithm based on the $x$- expansion is employed extensively to study the backscattering enhancement for moderately large-scale rough surfaces with large rms slopes as will be presented in Chapter 6.

Chapter 5 presents the generalization of the 2-D FB/NSA algorithm for PEC surfaces to take the finite conductivity of rough surface into account via the impedance boundary condition. It is found that two additional terms and a numerical differentation are required in the IBC surface integral formulation compared to the PEC case, and the 2-D FB/NSA algorithm for the IBC case is similar to the one
for the PEC case. Like the PEC case, it is straightforward to incorporate an additional NSA formulation based on the $y$-expansion and the “multilevel” algorithm into the standard 2-D FB/NSA algorithm to improve its computational efficiency and accuracy for the case of extremely large-scale 2-D rough surfaces with relatively large surface cross-range sizes $D_y$. For fixed $D_y$, frequency and surface roughness, the 2-D FB/NSA algorithm for the IBC case is still $O(N_{tot})$. Numerical results confirm that the 2-D FB/NSA algorithm for IBC surfaces yields very accurate results with a great reduction of CPU time compared to the standard 2-D FB/NSA algorithm. Finally, the 2-D FB/NSA algorithm for PEC and IBC surfaces developed in Chapters 4 and 5 respectively, is employed to study the effects of surface material to the backscattering enhancement phenomenon in Chapter 6.

Chapter 6 illustrates numerical studies of the backscattering enhancement phenomenon from 2-D random rough surfaces to account for both co-polarized and cross-polarized scattering. Due to significantly increasing in computational requirements for 2-D surfaces, the 2-D FB/NSA algorithm is combined with parallel computing techniques to study the backscattering enhancement effectively. Like the case of 1-D surfaces, it is found that the backscattering enhancement from 2-D surfaces still strongly depends on incident angle, surface material, polarization, and surface statistics. As incident angle increases, smaller backscattering enhancement effects at backscattering direction are observed, and thus the backscattering enhancement should not be a significant scattering mechanism at large incident angles at least for Gaussian random process having an isotropic Gaussian spectrum with $\sigma_{s,tot} \leq 1.414$, where $\sigma_{s,tot}$ is the total rms surface slope for 2-D isotropic surfaces as defined in Eq. (6.5) in Chapter 6. In addition, as surfaces become denser the normalized incoherent backscattering
RCS tends to increase as expected. For polarization dependence, cross-polarization exhibits backscattering enhancement more clearly than co-polarization due to the absence of the cross-polarized first-order scattering. In addition, scattering characteristics are different between HH and VV polarizations, with HH polarization typically exhibiting more scattered energy near the specular direction than VV polarization as \( \theta_i \) increases toward grazing incidence. As the surface roughness and the surface slope increase, the backscattering peak, the backscattering RCS, and the HH/VV polarization ratio tend to increase. In addition, variations of the backscattering RCS and the HH/VV polarization ratio with incident angle are slower for the rougher surfaces. Like 1-D surfaces, an unusual phenomenon is also observed in the scattering from IBC rough surfaces: i.e. the co-polarized backscattering peak at larger incident angles can exceed the one for smaller angles of incidence. Finally, comparisons of the co-polarized results between 1-D and 2-D surface models illustrate that they generally follow the same trend, however the 2-D surface model can provide more information about multiple scattering effects, such as the cross-polarization, at the cost of great increase in computational requirements. Numerical results obtained from Chapters 3 and 6 provide more physical insight to the backscattering enhancement phenomenon, and will assist in the future development of analytical theories.

Appendix A describes the procedure of random rough surface generation for both 1-D and 2-D surface models using a spectral method with the fast Fourier transform (FFT) algorithm. It is assumed that the surface height distributions are Gaussian throughout this appendix. In Appendix B, the topology of the complex planes associated with the integral representation of the free space 2-D scalar Green’s function
$g(\rho, \rho')$, employed in the development of the 1-D NSA algorithm in Chapter 2, is described for both lossless and lossy media. In addition, the derivation of the steepest descent and ascent contours associated with the integral representation of $g(\rho, \rho')$ in both complex spectral and angular spectral planes is illustrated for both lossless and lossy media as well. Theoretical studies of the topology associated with the integral representation of $g(\rho, \rho')$ in Appendix B assists in understanding the more complicated topology associated with the 2-D NSA algorithm described in Appendix C. Appendix C illustrates the topology of two coupled complex planes associated with the double spectral integral representation of the free space 3-D scalar Green’s function $g(r, r')$ used in the development of the 2-D NSA algorithm in Chapters 4 and 5. In addition, the steepest descent and ascent contours associated with the double spectral integral representation of $g(r, r')$ in each complex planes are derived as well.

Appendix D presents the derivation of analytical formulas associated with integration parameters of the standard 1-D NSA algorithm. These formulas are derived based on the assumption that the outermost possible saddle point $\phi_{s,\text{max}}$ in the complex $\phi$ plane is small. In practice, for a given surface height variation this assumption can be satisfied by adjusting the size of the strong region $L_s$. With this assumption, simple analytical formulas for the 1-D NSA parameters can be obtained. Note that the more general formulas of these parameters are illustrated in Section 2.3 in Chapter 2. In Appendix E, the analytical formula for the tilt angle $\delta_{k_z}$ in the complex $k_z$ plane associated with the 2-D NSA algorithm is derived. It involves a nonlinear equation, and a standard root-finding technique such as Muller’s method is required to solve for $\delta_{k_z}$. Finally, for completeness Appendix F illustrates a simple procedure to obtain a root of a nonlinear equation via Muller’s method [88]. The method is a
generalization of the secant method [88] by employing quadratic interpolation among
three given points instead of linear interpolation between two. Muller's method can
provide complex roots and it is required complex arithmetic in implementing the
method. In the next section, future works associated with the NSA algorithm and its
applications are proposed.

7.2 Future Works

The NSA algorithm developed in this dissertation has been applied successfully
to the rough surface scattering problems. However, some technical issues associated
with the NSA algorithm still remain to be solved, and it is also interesting to apply the
NSA algorithm to solve other EM problems associated with quasi-planar structures
such as planar arrays [118,119]. In this section, potential future works associated
with the NSA algorithm and its applications are proposed below.

• Due to the fact that the new 1-D NSA parameters can provide a way to optimize
the 1-D NSA algorithm, it is interesting to apply the 1-D "multilevel" NSA
algorithm with these new NSA parameters to scattering problems with very
large surface height variations such as the terrain propagation problems. In
addition, in this dissertation the 1-D NSA algorithm is developed only for PEC
and IBC surfaces. However, numerical models with lossy dielectric surfaces are
required for some applications such as the calculation of emissivity in passive
microwave remote sensing. Thus, the 1-D NSA algorithm will be extended to the
case of lossy dielectric rough surfaces. Careful investigation of the appropriate
deformed contour associated with the angular spectral integral representation
of the free space 2-D scalar Green's function $g(\rho, \rho')$ for the lossy medium,
including the determination of the new 1-D NSA parameters, is required. The physics-based two-grid method (PBTG) developed by Li et al. [77]-[79] may be incorporated into the 1-D NSA algorithm to speed up the matrix-vector product.

- For the 2-D NSA algorithm, the 2-D NSA parameters including the appropriate size of each weak region will be determined analytically for given surface sizes and surface statistics as in the 1-D case presented in Sections 2.3 and 2.4 in Chapter 2. This may involve the choice of order of integration associated with the double spectral integral representation of $g(r, r')$. Like the 1-D NSA algorithm, the 2-D NSA algorithm developed for PEC and IBC surfaces in this dissertation will be extended to treat 2-D lossy dielectric rough surfaces.

- The NSA algorithm will be applied to other quasi-planar structures such as microstrip structures, microwave integrated circuits, and optical gratings. It is also challenging to extend the algorithm to treat other structures such as quasi-cylindrical structures. Some applications related to these structures are the analysis and design of large conformal antenna arrays mounted on a circular cylinder and the measurement of microscopic surface roughness of optical fibers (an example of rough surface cylinders) [144].

In the next section, a concluding remark of this dissertation is provided.

### 7.3 A Concluding Remark

Although the NSA algorithm is an extremely efficient numerical method for computing scattering from rough surfaces, the problem of electromagnetic fields scattered
by an *arbitrary* rough surface profile is still challenging. Numerical results obtained from the NSA algorithm combined with parallel computing techniques provide more physical insight into some previously unanswered questions, however many issues still remain to be addressed. Physical insight gained from these numerical solutions will assist in the future development of improved analytical theories, which remains the principle goal of surface scattering research.
APPENDIX A

Random Rough Surface Generation

In this appendix, the procedure of random rough surface generation is described for both 1-D and 2-D surface models. Surface realizations are generated numerically using a *spectral* method [58, 145] with the fast Fourier transform (FFT) algorithm [88]. The rough surface is typically described in terms of its deviation from a smooth "reference plane". In general, the reference plane is assumed to be located at $z = 0$, where $z$ is a height function. The random fluctuations from this reference plane are denoted by the function $z = f(x)$ and $z = f(x, y)$ for 1-D and 2-D surface models, respectively. For ease of numerical and analytical modeling, the surface height distributions of random rough surfaces are assumed to be Gaussian. This assumption will be employed throughout this appendix. To illustrate the idea of the spectral method, first consider 1-D random rough surface generation. The 2-D random rough surface generation is an extension of the 1-D case, and will be considered afterwards.

A.1 1-D Random Rough Surface Generation

For 1-D rough surfaces, the *one-point* statistics of the rough surface are described via the probability density function (pdf) of the surface height $p(z)$, and the statistical properties can be controlled by the rms surface height $h$. For the Gaussian surface
height distribution, the pdf is given by

\[ p(z) = \frac{1}{h\sqrt{2\pi}} \exp\left(-\frac{z^2}{2h^2}\right) \]  

Note that the rms surface height \( h \) can be determined via the following formula:

\[ h = \left( \langle z^2 \rangle - \langle z \rangle^2 \right)^{\frac{1}{2}}. \]  

where the \( \langle \cdot \rangle \) notation above indicates an ensemble average over realizations of the surface stochastic process. The specification of a height distribution and rms surface height does not discriminate between surfaces with different “length scales”. To distinguish these surfaces, the two-point statistics are needed. For 1-D rough surfaces formed by stationary stochastic processes, the surface autocovariance function \( C(\tau) \) describing the two-point statistics on the surface is given as follows:

\[ C(\tau) = \langle f(x)f(x+\tau) \rangle. \]  

From the above equation, the surface autocovariance function describes the spatial coherence between two different points on the surface separated by a distance \( \tau \) and it has the units of \( m^2 \). The normalized form of the surface autocovariance function \( C(\tau) \) is called the surface correlation (or autocorrelation) function \( C_0(\tau) \): i.e.

\[ C_0(\tau) = \frac{C(\tau)}{h^2}. \]  

Note that the surface correlation function has the property that \( C_0(0) = 1 \). As \( \tau \) increases, \( C_0(\tau) \) will usually decay to zero, with the shape of this decay being dependent on the type of surface and with the rate of decay being dependent on the distance over which points become uncorrelated.

Alternatively, the surfaces can also be described in the spectral domain via the power spectral density (psd) function of the surface (or the surface spectrum) \( W(k) \).
Note that the psd is related to the autocovariance function by a 1-D Fourier transform, and the Fourier transform pair is defined as follows:

\[
W(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\tau) e^{ik\tau} d\tau \quad \text{(A.5)}
\]
\[
C(\tau) = \int_{-\infty}^{\infty} W(k) e^{-ik\tau} dk. \quad \text{(A.6)}
\]

Physically, $k$ represents the spatial frequency in $m^{-1}$, and the psd is the average distribution of each spatial frequency component of the random rough surface profile in $m^3$. Note that surfaces with rapid variations and sharp edges possess a non-zero spectrum value at large spatial frequency components. From Eq. (A.6), it is noted that

\[
\int_{-\infty}^{\infty} W(k) dk = C(0) = h^2. \quad \text{(A.7)}
\]

i.e. the total area under the power spectrum gives the variance, or "power", of the surface. In addition, the psd $W(k)$ is related to the surface profile $f(x)$ by the following equation [146]:

\[
W(k) = \lim_{L \to \infty} \frac{1}{2\pi L} \left\langle \left| \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{i k x} dx \right|^2 \right\rangle. \quad \text{(A.8)}
\]

One commonly used autocovariance function is the Gaussian autocovariance function,

\[
C(\tau) = h^2 \exp \left( -\frac{\tau^2}{l^2} \right). \quad \text{(A.9)}
\]

where $l$ is the surface correlation length, and the corresponding psd is given by

\[
W(k) = \frac{h^2 l}{2\sqrt{\pi}} \exp \left( -\frac{k^2 l^2}{4} \right). \quad \text{(A.10)}
\]

Note that the corresponding rms slope is defined as $\sigma_s = \frac{\sqrt{2h}}{l}$. Qualitatively, for a given $l$ small values of $h$ give rise to slightly rough surfaces, whereas large values of
result in very rough surfaces. Alternatively, for a given $h$ large values of $l$ give rise
to slightly rough surfaces, whereas small values of $l$ result in very rough surfaces (i.e.
large rms slope). Thus, for surfaces characterized by Gaussian roughness statistics,
a given pair of $h$ and $l$ controls the surface roughness, and statistically defines the
surface profile.

To generate a rough surface numerically, a sequence of normally (Gaussian) dis-
tributed random numbers are employed. Let $\mathcal{N}(0,1)$ denote a sample of a normally
distributed random variable with zero mean and unit variance. Surface realizations
are needed at a set of $\mathcal{N}$ points with spacing $\Delta x$ over length $L = \mathcal{N}\Delta x$. Using the
discrete Fourier transform (DFT) method, the rough surface profile $z = f(x_n)$ at
points $x_n = n\Delta x$ ($n = 1, \ldots, \mathcal{N}$) is related to the 1-D DFT of the surface spectrum
$F(k_j)$ as follows:

$$
f(x_n) = \frac{1}{L} \sum_{j=-\mathcal{N}/2}^{\mathcal{N}/2-1} F(k_j) e^{-ik_j x_n}. \tag{A.11}
$$

where for $j \geq 0$.

$$
F(k_j) = \sqrt{2\pi L W(k_j)} \begin{cases} 
\mathcal{N}(0,1) + i\mathcal{N}(0,1) & , j = 1, \ldots, \frac{\mathcal{N}}{2} - 1 \\
\mathcal{N}(0,1) & , j = 0, \frac{\mathcal{N}}{2} 
\end{cases} \tag{A.12}
$$

and for $j \leq 0$.

$$
F(k_j) = F^*(-k_j). \tag{A.13}
$$

where $k_j = \frac{2\pi j}{L}$ and the superscript "*" denotes the complex conjugate. Note that
Eq. (A.11) can be computed efficiently using the 1-D FFT algorithm, and Eq. (A.13)
is imposed in order to generate a desired real sequence of the rough surface profile. In
addition, the one-point surface statistics are governed by the random modulation in
the phase of the Fourier coefficients, and the two-point surface statistics are governed


277
by the magnitude of the Fourier spectrum which follows the surface spectrum $W(k)$.

The random phase modulation in the Fourier coefficients assures that the resulting surface has a Gaussian height distribution. The use of the DFT in the rough surface generation implies that the resulting rough surface is a periodic function in which the surface height is periodic in space. To avoid the spectral aliasing in the resulting surface, it is required that the surface length $L$ must be at least five correlation length [147]. Due to a finite surface length, the surface autocovariance does not completely decay to zero, and some oscillations exist. Thus, to obtain the psd via the inverse Fourier transform of the actual autocovariance function, a windowing function must be applied to the original sequence to avoid aliasing and edge effects [147].

For the scattering computation from 1-D rough surfaces, surface realizations of the first and second derivatives of the surface profile are also needed at a set of $N$ points. Let $f'(x_n)$ and $f''(x_n)$ denote the first and second derivatives of the surface profile at points $x_n = n\Delta x$, where $n = 1, \ldots, N$. These derivatives can also be obtained by using the spectral method as follows:

$$
f'(x_n) = \frac{1}{L} \sum_{j=-N/2}^{N/2-1} (-ik_j) F(k_j) e^{-ik_jx_n}
$$

(A.14)

$$
f''(x_n) = \frac{1}{L} \sum_{j=-N/2}^{N/2-1} (-ik_j)^2 F(k_j) e^{-ik_jx_n}.
$$

(A.15)

where $F(k_j)$ is defined the same as in Eqs. (A.12) and (A.13). Note that surface realizations of the first and second derivatives of the surface profile have similar issues as in surface realizations of the surface height as discussed earlier. Next, the 2-D random rough surface generation is discussed in detail.
A.2 2-D Random Rough Surface Generation

The 2-D rough surface is described by \( z = f(x, y) \), which is a random function of position \((x, y)\). Similar techniques presented earlier for the 1-D case can readily be extended to generate 2-D random rough surfaces. For 2-D rough surfaces, the one-point statistics of the rough surface are described via the probability density function (pdf) of the surface height \( p(z) \) as previously defined in Eq. (A.1), and the statistical properties can be controlled the rms surface height \( h \). A rough surface is isotropic if the surface statistics are independent of direction along the surface. Let \( l_x \) and \( l_y \) denote the surface correlation lengths in the \( x \)- and \( y \)-directions, respectively. Note that the surface is isotropic if \( l_x = l_y \), and anisotropic if \( l_x \neq l_y \). If one of the correlation lengths is much larger than the other, the 2-D surface is approximately a 1-D surface for the purposes of the numerical calculations. For the Gaussian surface height distribution, the pdf is given as in Eq. (A.1). For 2-D rough surfaces formed by stationary stochastic processes, the surface autocovariance function \( C(\tau_x, \tau_y) \) in \( m^2 \) describing the two-point statistics on the surface is given as follows:

\[
C(\tau_x, \tau_y) = \langle f(x, y) f(x + \tau_x, y + \tau_y) \rangle. \tag{A.16}
\]

where \( \tau_x \) and \( \tau_y \) describe the separation between any two points along the \( x \)- and \( y \)-directions, respectively. Like the 1-D case, the normalized form of the surface autocovariance function \( C(\tau_x, \tau_y) \) is called the surface correlation (or autocorrelation) function \( C_0(\tau_x, \tau_y) \): i.e.

\[
C_0(\tau_x, \tau_y) = \frac{C(\tau_x, \tau_y)}{h^2}. \tag{A.17}
\]

From the above equation, the surface autocovariance function describes the spatial coherence between two different points on the surface separated by a distance.
\[ d = \sqrt{\tau_x^2 + \tau_y^2}. \]
The psd function of the 2-D surface \( W(k_x, k_y) \) is related to the surface autocovariance function \( C(\tau_x, \tau_y) \) by a 2-D Fourier transform, and the Fourier transform pair is defined as follows:

\[
W(k_x, k_y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(\tau_x, \tau_y) e^{ik_x\tau_x} e^{ik_y\tau_y} d\tau_x d\tau_y \tag{A.18}
\]

\[
C(\tau_x, \tau_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(k_x, k_y) e^{-ik_x\tau_x} e^{-ik_y\tau_y} dk_x dk_y. \tag{A.19}
\]

A commonly used autocovariance function for the 2-D surface is the Gaussian autocovariance function.

\[
C(\tau_x, \tau_y) = \exp \left( -\frac{\tau_x^2}{l_x^2} - \frac{\tau_y^2}{l_y^2} \right). \tag{A.20}
\]

and the corresponding psd is given by

\[
W(k_x, k_y) = \frac{h^2 l_x l_y}{4\pi} \exp \left[ -\frac{1}{4} \left( \frac{k_x^2 l_x^2}{l_y^2} \right) \right]. \tag{A.21}
\]

Note that the corresponding rms slopes in the \( x \)- and \( y \)-directions are defined as \( \sigma_{s,x} = \frac{\sqrt{2h}}{l_x} \) and \( \sigma_{s,y} = \frac{\sqrt{2h}}{l_y} \), respectively.

Similarly to the 1-D case, 2-D surface realizations of size \( D_x \times D_y \) are needed at a set of \( N \times M \) points with the spacings \( \Delta x \) and \( \Delta y \) in the \( x \)- and \( y \)-directions, respectively. Using the DFT method, the rough surface profile \( z = f(x_n, y_m) \) at the pairs of points \( (x_n, y_m) \), where \( x_n = n \Delta x \) and \( y_m = m \Delta y \), is related to the 2-D DFT of the surface spectrum \( F(k_x, k_y, q) \) as follows:

\[
f(x_n, y_m) = \frac{1}{D_x D_y} \sum_{p=-\frac{N}{2}}^{\frac{N}{2}-1} \sum_{q=-\frac{M}{2}}^{\frac{M}{2}-1} F(k_x, k_y, q) e^{-ik_x x_n} e^{-ik_y y_m}. \tag{A.22}
\]

where

\[
F(k_x, k_y, q) = c_{p,q} \begin{cases} 
\sqrt{N(0,1)} & p = 1, \ldots, \frac{N}{2} - 1; q = 1, \ldots, \frac{M}{2} - 1 \\
\sqrt{2} \sqrt{N(0,1)} & p = 0, \frac{N}{2}; q = 0, \frac{M}{2} 
\end{cases} \tag{A.23}
\]

\[
c_{p,q} = 2\pi \sqrt{D_x D_y} W(k_x, k_y, q). \tag{A.24}
\]

280
\(k_{x,p} = \frac{2\pi p}{D_x}, \quad k_{y,q} = \frac{2\pi q}{D_y}, \quad m = 1, \ldots, M\) and \(n = 1, \ldots, N\). Note that \(k_{x,p}\) and \(k_{y,q}\) are the discrete set of spatial frequencies. To generate a real sequence of the 2-D rough surface profile, \(F(k_{x,p}, k_{y,q})\) is required to satisfy the “conjugate symmetry” about the origin (i.e. the reflection of any point about the origin is its complex conjugate) as follows:

\[
F(-k_{x,p}, -k_{y,q}) = F^*(k_{x,p}, k_{y,q}) \quad (A.25)
\]

\[
F(k_{x,p}, -k_{y,q}) = F^*(-k_{x,p}, k_{y,q}) \quad (A.26)
\]

Note also that Eq. (A.22) can be computed efficiently using the 2-D FFT algorithm.

For the scattering computation from 2-D rough surfaces using a linear surface model, surface realizations of the first partial derivatives \(\frac{\partial f(x,y)}{\partial x}\) and \(\frac{\partial f(x,y)}{\partial y}\) of the surface profile are also needed at a set of \(N \times M\) points. Let \(\frac{\partial f(x_n, y_m)}{\partial x}\) and \(\frac{\partial f(x_n, y_m)}{\partial y}\) denote the first partial derivatives of the surface profile in the \(x\)- and \(y\)-directions respectively at pairs of points \((x_n, y_m)\). These derivatives can also be obtained by using the spectral approach as follows:

\[
\frac{\partial f(x_n, y_m)}{\partial x} = \frac{1}{D_x D_y} \sum_{p=-\frac{M}{2}}^{\frac{M}{2}-1} \sum_{q=-\frac{N}{2}}^{\frac{N}{2}-1} (-ik_{x,p}) F(k_{x,p}, k_{y,q}) e^{-ik_{x,p} x_n} e^{-ik_{y,q} y_m} \quad (A.27)
\]

\[
\frac{\partial f(x_n, y_m)}{\partial y} = \frac{1}{D_x D_y} \sum_{p=-\frac{M}{2}}^{\frac{M}{2}-1} \sum_{q=-\frac{N}{2}}^{\frac{N}{2}-1} (-ik_{y,q}) F(k_{x,p}, k_{y,q}) e^{-ik_{x,p} x_n} e^{-ik_{y,q} y_m} \quad (A.28)
\]

where \(F(k_{x,p}, k_{y,q})\) is defined the same as in Eqs. (A.23), (A.25), and (A.26).
APPENDIX B

Topology of the Complex Planes Associated with the Integral Representation of the Free Space 2-D Scalar Green’s Function \( g(\rho, \rho') \)

As discussed earlier in Chapter 2, the computational efficiency of the 1-D NSA algorithm strongly depends on the contour deformation, which involves the topology of the complex planes associated with the integral representation of the free space 2-D scalar Green’s function \( g(\rho, \rho') \). Thus, it is necessary to study the topology of the complex planes associated with \( g(\rho, \rho') \) in detail. Consider the free space 2-D scalar Green’s function \( g(\rho, \rho') \) for \( e^{-i\omega t} \) harmonic time convention:

\[
g(\rho, \rho') = \frac{i}{4} H_0^{(1)}(k|\rho - \rho'|). \tag{B.1}
\]

where \( H_0^{(1)}(\cdot) \) denotes the zeroth-order Hankel function of the first kind, and \( \rho = \hat{x}x + \hat{z}z \) and \( \rho' = \hat{x}x' + \hat{z}z' \) are two-dimensional position vectors representing a field point and a source point, respectively. For simplicity in analysis, first it is assumed that the medium is lossless; i.e. the propagation constant \( k \) is real. The generalization of results for the lossy medium (\( k \) is complex) will be considered later in this appendix.

Without loss of generality, consider only the spectral integral representation of \( g(\rho, \rho') \) propagating in the \( x \)- direction for \( x > x' \) as follows [131, 132]:

\[
g(\rho, \rho') = \frac{i}{4\pi} \int_{C_{k_z}} dk_z \left[ \frac{\exp\left\{i[k_x(x - x') + k_z(z - z')]\right\}}{k_x} \right]. \tag{B.2}
\]
where

\[ k_x = (k^2 - k_z^2)^{1/2}. \]  \hspace{1cm} (B.3)

and \( C_k \) is the entire real axis in the complex plane starting from \(-\infty\) to \(\infty\). Note that \( k_z \) is a complex spectral variable in the above representation, and the branch of \( k_x \) must be chosen properly as will be discussed later. Alternatively, \( q(\rho, \rho') \) can also be represented in terms of the angular spectral integral, where \( \phi \) is a complex angular spectral variable, propagating in the \( x \)-direction for \( x > x' \) via the following mappings:

\[ k_z = k \sin \phi \]  \hspace{1cm} (B.4)

\[ k_x = (k^2 - k_z^2)^{1/2} = k \cos \phi. \]  \hspace{1cm} (B.5)

Substituting Eqs. (B.4) and (B.5) into Eq. (B.2) yields the following result:

\[ g(\rho, \rho') = \frac{i}{4\pi} \int_{C_{\phi}} d\phi \exp \{ik[(x - x')\cos \phi + (z - z')\sin \phi]\}. \]  \hspace{1cm} (B.6)

where \( C_{\phi} \) is the contour of the integration in the periodic complex \( \phi \) plane, corresponding to the original contour of integration (the real axis) in the complex \( k_z \) plane, as shown in Figure B.1. Note that \( \phi_R \) and \( \phi_I \) denote the real and imaginary parts of \( \phi \) respectively, and the entire complex \( k_z \) plane can be mapped into various adjacent sections of "strip" of width \( 2\pi \) in the complex \( \phi \) plane due to the periodicity of the "sin" function employed in the mapping \( k_z = k \sin \phi \). Eq. (B.6) can be simplified further by employing the following changes of variables from the rectangular coordinate to the polar coordinate as follows:

\[ x - x' = |\rho - \rho'| \cos \gamma \]  \hspace{1cm} (B.7)

\[ z - z' = |\rho - \rho'| \sin \gamma. \]  \hspace{1cm} (B.8)
where

\[ \gamma = \tan^{-1} \left( \frac{z - z'}{x - x'} \right). \] (B.9)

and \(-\frac{\pi}{2} < \gamma < \frac{\pi}{2}\). Employing Eqs. (B.7) and (B.8) in Eq. (B.6) yields

\[ g(\rho, \rho') = \frac{i}{4\pi} \int_{C_2} d\phi \exp \{ik|\rho - \rho'| \cos(\phi - \gamma)\}. \] (B.10)

Unlike the spectral integral representation of \(g(\rho, \rho')\) (see Eq. (B.2)), the angular spectral integral representation of \(g(\rho, \rho')\) (see Eq. (B.10)) does not involve any branch-point singularity, which simplifies the analysis considerably. This is the virtue of employing the mappings as shown in Eqs. (B.4) and (B.5). Note that Eqs. (B.2) and (B.10) involve the topology in the complex \(k_z\) plane and the periodic complex \(\phi\) plane, respectively.

Asymptotic analysis, especially the method of steepest descent (saddle-point method), is usually involved with two constant phase paths: i.e. the steepest descent path (SDP) and the steepest ascent path (SAP). Along the SDP path, the associated integrand decays most rapidly away from the saddle point, where the integrand is maximum. On the other hand, the integrand increases most rapidly away from the saddle point, where the integrand is minimum, along the SAP contour. The SDP and SAP paths for each complex plane (\(k_z\) and \(\phi\) planes) will be derived mathematically later in this appendix.

This appendix is organized as follows. Sections B.1 and B.2 describe the topology associated with the integral representation of \(g(\rho, \rho')\) in the complex \(k_z\) plane and the periodic complex \(\phi\) plane for a lossless medium, respectively. The derivation of the SDP and SAP contours for the real propagation constant in both complex planes is illustrated in Section B.3. For a lossy medium, Sections B.4 and B.5 discuss the
topology in the complex $k_z$ and $\phi$ planes respectively, and then the associated SDP and SAP contours are derived in Section B.6. Finally, Section B.7 provides a summary and conclusions of this appendix.

**B.1 Topology Associated with the Spectral Integral Representation of $g(\rho, \rho')$ for a Lossless Medium**

From the spectral integral representation of $g(\rho, \rho')$ as given in Eq. (B.2), its integrand contains branch-point singularities at $k_z = \pm k$. and $k_x$ is a *double-valued* function of the spectral variable $k_z$. In order to specify $k_x$ *uniquely*, it is mathematically convenient to employ the concept of a Riemann surface [148]. Briefly, a Riemann surface is an extension of the ordinary complex plane to a surface that has more than
one “sheet”. For the case of the double-valued function $k_x$, the Riemann surface consists of two sheets connected along the branch cuts, as determined later in this section. In each sheet, $k_x$ is a single-valued and analytic function of $k_z$ except along the branch cuts. In addition, $k_x$ is discontinuous only if a branch cut is crossed, in which the sign of $k_x$ is changed. In general, the choice of the branch cuts is arbitrary. However, to satisfy a physical constraint of the electromagnetic waves at a large distance away from a finite source region in this problem, the appropriate branch of $k_x$ is chosen such that the integrand of Eq. (B.2) satisfies the radiation condition [149]: i.e. the electromagnetic energy must travel away from the source region and the following condition on $k_x$ must be satisfied:

$$\alpha > 0$$

(B.11)

$$\beta > 0.$$  \hspace{1cm}  \text{(B.12)}

where $\alpha$ and $\beta$ are the real and imaginary parts of $k_x$, respectively. Note that Eqs. (B.11) and (B.12) ensures that electromagnetic waves are outgoing and evanescent at a large distance away from the source region. One may define the two branches of $k_x$ as the “proper” and “improper” ones as follows [150, 151]:

Proper branch : $\beta > 0$  \hspace{1cm}  \text{(B.13)}

Im proper branch : $\beta < 0.$  \hspace{1cm}  \text{(B.14)}

Using the above definitions, the branch cuts must be defined by the following equation:

$$\beta = 0.$$  \hspace{1cm}  \text{(B.15)}
For convenience in analysis, it is useful to determine the alternative conditions on the branch cuts by considering the function \( k_z^2 \) instead of \( k_z \) as follows:

\[
k_z^2 = (\alpha + i\beta)^2 = (\alpha^2 - \beta^2) + 2i\alpha\beta. \tag{B.16}
\]

Using Eq. (B.15) in Eq. (B.16), the alternative conditions on the branch cuts are given by the following equations:

\[
\text{Im} [k_z^2] = 0 \quad \text{(B.17)}
\]
\[
\text{Re} [k_z^2] = \alpha^2 > 0. \quad \text{(B.18)}
\]

It is also interesting to note that \( \alpha = 0 \) corresponds to the following conditions on \( k_z^2 \):

\[
\text{Im} [k_z^2] = 0 \quad \text{(B.19)}
\]
\[
\text{Re} [k_z^2] = -\beta^2 < 0. \quad \text{(B.20)}
\]

For analytical convenience, it is assumed that the medium is *slightly lossy*: i.e.

\[
k = k_R + ik_I. \tag{B.21}
\]

where \( k_I < k_R \) and \( k_R \) and \( k_I \) are the real and imaginary parts of \( k \) respectively, and both are real and positive. Note that the results for the *lossless* case can be obtained from the ones for the lossy case by taking the limit as \( k_I \to 0^+ \). Let \( k_{z,R} \) and \( k_{z,I} \) be the real and imaginary parts of \( k_z \), respectively. From Eq. (B.3), \( k_z^2 \) can be expressed as follows:

\[
k_z^2 = (k_R + ik_I)^2 - (k_{z,R} + ik_{z,I})^2. \tag{B.22}
\]
or

\[
Re [k_x^2] = k_R^2 - k_I^2 - k_{z,R}^2 + k_{z,I}^2 \quad (B.23)
\]

\[
Im [k_x^2] = 2(k_R k_I - k_{z,R} k_{z,I}) \quad (B.24)
\]

Next, it is useful to study the curves in the complex \(k_z\) plane on which \(Re [k_x^2] = 0\) and on which \(Im [k_x^2] = 0\), and to determine regions where \(Re [k_x^2] > 0\) as well as where \(Re [k_x^2] < 0\). From Eq. (B.23), \(Re [k_x^2] = 0\) implies the following equation:

\[
k_{z,I}^2 - k_{z,R}^2 = k_I^2 - k_R^2. \quad (B.25)
\]

and from Eq. (B.24), \(Im [k_x^2] = 0\) implies the following equation:

\[
k_{z,R} k_{z,I} = k_R k_I. \quad (B.26)
\]

Note that the right-hand side (RHS) of Eq. (B.25) is less than zero since \(k_I << k_R\) for the slightly lossy medium. Figure B.2 illustrates the plots of the hyperbolic curves \(Re [k_x^2] = 0\) and \(Im [k_x^2] = 0\) as well as the regions for which \(Re [k_x^2] > 0\). From the conditions in Eqs. (B.17) and (B.18) and Figure B.2, it follows that the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex \(k_z\) plane are illustrated in Figure B.3. Note that the original contour of integration \(C_k\) (see Eq. (B.2)) is always in the proper sheet.

In the above figure, the branch cuts are denoted by the wavy lines corresponding to \(\beta = 0\), and the letters “P” and “T” stand for “proper” and “improper”, respectively. In addition, Figure B.3 also shows the curve where \(\alpha > 0\) as well as where \(\alpha < 0\) in both Riemann sheets. These regions can be determined by using Eq. (B.24) with the fact that \(Im [k_x^2] = 2\alpha \beta\) (see Eq. (B.16)), and it is found that

\[
\alpha = \frac{k_R k_I - k_{z,R} k_{z,I}}{\beta}. \quad (B.27)
\]

288
Using Eq. (B.27) for the proper sheet ($\mathcal{J} > 0$), the following conditions are obtained:

$$\alpha > 0 : \quad k_{z, R}k_{z, I} > k_R k_I.$$  \hspace{1cm} (B.28)

Similarly, for the improper sheet ($\mathcal{J} < 0$), employing Eq. (B.27) yields:

$$\alpha > 0 : \quad k_{z, R}k_{z, I} > k_R k_I.$$  \hspace{1cm} (B.29)

For the lossless medium, Figure B.3 is modified by taking the limit as $k_I \to 0^+$, and the branch cuts in both Riemann sheets are shown in Figure B.4.
Figure B.3: The plots of the branch cuts and the *proper* (top) and *improper* (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_z$ plane, for the case of the *slightly lossy* medium.
Figure B.4: The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_z$ plane, for the case of the lossless medium.
B.2 Topology Associated with the Angular Spectral Integral Representation of \( g(\rho, \rho') \) for a Lossless Medium

As pointed out earlier, \( g(\rho, \rho') \) can be represented in terms of the angular spectral integral via the mapping \( k_z = k \sin \phi \), or

\[
\phi = \sin^{-1} \left( \frac{k_z}{k} \right), \quad (B.30)
\]

where \( \phi \) is a complex angular spectral variable. To ensure that the mapping from the complex \( k_z \) plane to the complex \( \phi \) plane is one-to-one, the inverse transformation can be expressed explicitly as follows [151]:

\[
\phi = -i \ln \left( \frac{k_z + ik_z}{k} \right), \quad (B.31)
\]

where

\[
\ln z = \ln |z| + i \arg(z). \quad (B.32)
\]

and \( z = |z| \exp\{i \arg(z)\} \) is a complex number which has the magnitude \( |z| \) and the phase \( \arg(z) \) must be confined to the strip of \( 2\pi \). and the principle branch of \( \arg(z) \) is \(-\pi < \arg(z) < \pi \). It should be pointed out that Eq. (B.31) can be derived using the Euler’s identity: i.e.

\[
e^{i\phi} = \cos \phi + i \sin \phi. \quad (B.33)
\]

where

\[
\sin \phi = \frac{k_z}{k}, \quad (B.34)
\]

\[
\cos \phi = \frac{k_x}{k}. \quad (B.35)
\]

where \( k_z = \alpha + i\beta \). It is emphasized that the branch of \( k_x \) must be chosen properly for the proper and improper Riemann sheets, and the appropriate values of \( \alpha \) and \( \beta \)
for each Riemann sheet in the complex $k_z$ plane are shown in Figure B.4. By using Eq. (B.31), both proper and improper Riemann sheets in the complex $k_z$ plane for a lossless medium as illustrated in Figure B.4 can be mapped into the angular strip $-\pi < \phi_R < \pi$ in the complex $\phi$ plane, where $\phi = \phi_R + i\phi_I$, as shown in Figure B.5. Note that Figure B.5 does not exhibit other strips of width $2\pi$; i.e. it shows only the "principle" strip $-\pi < \phi_R < \pi$. In addition, the mapping of Eq. (B.31) transforms the quadrants of the complex $k_z$ plane into parallel strips of width $\frac{\pi}{2}$ radians in the complex $\phi$ plane, and each strip corresponding to each quadrant in the complex $k_z$ plane is labeled as shown in Figure B.5. Note also that the original contour $C_{k_z}$ (the real axis) and the imaginary axis ($k_z, R = 0$) in the complex $k_z$ plane are mapped to the contour $C_\phi$ and the imaginary axis ($\phi_R = 0$) in the complex $\phi$ plane, respectively. Figure B.5 also illustrated the image of the branch cuts ($J = 0$) in the complex $\phi$ plane. It is emphasized that the topology in the complex $\phi$ plane does not involve any branch-point singularity due to the fact that the mapping $k_z = k \sin \phi$ makes the branch points $k_z = \pm k$ the regular points $\phi_R = \pm \frac{\pi}{2} + 2n\pi$ in the complex $\phi$ plane, where $n = 0, \pm 1, \pm 2, \ldots$.

### B.3 Derivation of the SDP and SAP Contours Associated with the Integral Representation of $g(\rho, \rho I)$ in the Complex $k_z$ and $\phi$ Planes for the Real Propagation Constant

For convenience in analysis, it is desirable to determine the SDP and SAP contours associated with the integral representation of $g(\rho, \rho I)$ in the complex $\phi$ plane first. After obtaining the above contours, the corresponding contours in the complex $k_z$ plane can be determined using the mapping $k_z = k \sin \phi$. As pointed out earlier, both SDP and SAP contours are the constant phase paths, and they can be determined
Figure B.5: The plot of the topology associated with the angular spectral integral representation of $g(\rho, \rho')$ and the original contour of integration $C_\phi$ in the periodic complex $\phi$ plane for the case of the lossless medium.
by considering the exponential integrand $I_1(\phi, \rho, \rho')$ of Eq. (B.10) for $x > x'$; i.e.

$$I_1(\phi, \rho, \rho') = e^{\kappa f_1(\phi)}. \quad (B.36)$$

where $\kappa$ and $f_1(\phi)$ are a large parameter and a phase function respectively, and they are defined as follows:

$$\kappa = k|\rho - \rho'| \quad (B.37)$$

$$f_1(\phi) = i \cos(\phi - \gamma)$$

$$= \sin(\phi_R - \gamma) \sinh \phi_I + i \cos(\phi_R - \gamma) \cosh \phi_I. \quad (B.38)$$

where $\gamma$ is defined in Eq. (B.9) and $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$. Before determining the constant phase paths, it is required to determine the "saddle" or "stationary" points of the phase function $f_1(\phi)$. In general, the saddle points are defined by the vanishing of one or more of the derivatives of $f_1(\phi)$. In our case, the saddle points $\phi_s$ can be obtained by setting the derivative of $f_1(\phi)$ with respect to $\phi$ at $\phi = \phi_s$ to zero; i.e.

$$\left. \frac{\partial f_1(\phi)}{\partial \phi} \right|_{\phi = \phi_s} = 0. \quad (B.39)$$

and solving for $\phi_s$ resulting in

$$\phi_s = \gamma + 2n\pi. \quad (B.40)$$

where $n = 0, \pm 1, \pm 2, \ldots$. Note that the saddle points are real and occur periodically with period $2\pi$. However, only $\phi_s = \gamma$ is significant in the principal strip $-\pi < \phi_R < \pi$. Note that $\phi_s$ given in Eq. (B.40) is a first-order saddle point due to the fact that

$$\left. \frac{\partial^2 f_1(\phi)}{\partial \phi^2} \right|_{\phi = \phi_s} \neq 0. \quad (B.41)$$
Next, the constant phase paths can be determined via the following equation:

\[ Im [f_1(\phi)] = Im [f_1(\phi)]|_{\phi=\gamma}, \]  

where

\[ Im [f_1(\phi)] = \cos(\phi_R - \gamma) \cosh \phi_I. \]  

Substituting Eq. (B.43) into Eq. (B.42), the constant phase paths are governed by the following equation:

\[ \cos(\phi_R - \gamma) \cosh \phi_I = 1. \]  

As pointed out earlier in the introduction of this appendix, the exponential integrand \( I_1(\phi, \rho, \rho') \) along the SDP path decays most rapidly away from \( \phi_s \), and thus the SDP path must satisfy the following constraint:

\[ Re [f_1(\phi)] \leq Re [f_1(\phi)]|_{\phi=\gamma}. \]  

where

\[ Re [f_1(\phi)] = \sin(\phi_R - \gamma) \sinh \phi_I. \]  

Substituting Eq. (B.46) into Eq. (B.45), Eq. (B.45) becomes

\[ \sin(\phi_R - \gamma) \sinh \phi_I \leq 0. \]  

Thus, the SDP path is governed by Eqs. (B.44) and (B.47). Alternatively, the exponential integrand \( I_1(\phi, \rho, \rho') \) along the SAP path increases most rapidly away from \( \phi_s \), and thus the SAP path must satisfy the following constraint:

\[ Re [f_1(\phi)] \geq Re [f_1(\phi)]|_{\phi=\gamma}. \]
or

\[ \sin(\phi_R - \gamma) \sinh \phi_I \geq 0. \quad (B.49) \]

and thus the SAP path is governed by Eqs. (B.44) and (B.49). For the first-order saddle point, the SDP and SAP contours are perpendicular to each other at the intersection point: i.e. the saddle point. To see this in our case, taking the derivative of Eq. (B.44) with respect to \( \phi_R \) at \( \phi = \gamma \) yields the following equation:

\[
\left[ \cos(\phi_R - \gamma) \sinh \phi_I \frac{d\phi_I}{d\phi_R} - \sin(\phi_R - \gamma) \cosh \phi_I \right]_{\phi=\gamma} = 0. \quad (B.50)
\]

or

\[
\left. \frac{d\phi_I}{d\phi_R} \right|_{\phi=\gamma} = \frac{\tan(\phi_R - \gamma)}{\tanh \phi_I \left|_{\phi=\gamma} \right.}. \quad (B.51)
\]

which is in the “intermediate” form. Applying L’Hospital’s rule [130], Eq. (B.51) can be rewritten as follows:

\[
\left. \frac{d\phi_I}{d\phi_R} \right|_{\phi=\gamma} = \sec^2(\phi_R - \gamma) \cosh^2 \phi_I \left|_{\phi=\gamma} \right. \frac{d\phi_I}{d\phi_R} \left|_{\phi=\gamma} \right. . \quad (B.52)
\]

or

\[
\left. \frac{d\phi_I}{d\phi_R} \right|_{\phi=\gamma} = \pm 1. \quad (B.53)
\]

Thus, at the saddle point \( \phi = \gamma \) there exists two possible values of slopes as expected: i.e. one for the SDP contour and another for the SAP contour. To distinguish between these two contours, Eqs. (B.47) and (B.49) must be imposed resulting in the following equations:

\[
\text{SDP} : \quad \left. \frac{d\phi_I}{d\phi_R} \right|_{\phi=\gamma} = -1 \quad (B.54)
\]

\[
\text{SAP} : \quad \left. \frac{d\phi_I}{d\phi_R} \right|_{\phi=\gamma} = 1. \quad (B.55)
\]
Since the product of the slopes of the SDP and SAP contours at the saddle point are equal to -1, both contours are indeed perpendicular at $\phi = \gamma$. Figure B.6 (a) illustrates the plot of $\ln |I_1(\phi, \rho, \rho^l)| = \kappa Re [f_1(\phi)]$ as a function of $\phi_R$ and $\phi_I$ in the vicinity of the first-order saddle point $\phi = \gamma = \frac{\pi}{4}$ rad. with $x - x_l = z - z_l = 5\lambda$, where $\lambda = \frac{2\pi}{k}$. Note that $\ln |I_1(\phi, \rho, \rho^l)|$ is plotted instead of $|I_1(\phi, \rho, \rho^l)|$ to avoid the numerical overflow due to the exponential growth of $|I_1(\phi, \rho, \rho^l)|$. It can be seen from the plot that the terrain associated with $\ln |I_1(\phi, \rho, \rho^l)|$ contains both mountains and valleys. Figure B.6 (b) illustrates the associated contour plot of $\ln |I_1(\phi, \rho, \rho^l)|$ (i.e. the constant level plot), the SDP and SAP contours in the complex $\phi$ plane. From the plot, the SDP contour originates from the valley region, passes through the saddle point $\phi = \gamma = \frac{\pi}{4}$, and then finally terminates in the valley region as expected. Alternatively, the SAP contour originates from the mountain region, passes through the saddle point, and then finally terminates in the mountain region as expected. It should be pointed out that the SDP and SAP contours possess the same support along the $\phi_R$ axis; i.e. equal to $\pi$ rad., and they are shifted simultaneously as the saddle point $\gamma$ changes.

For the complex $k_z$ plane, consider the exponential integrand $I_2(k_z, \rho, \rho^l)$ of Eq. (B.2) for $x > x_l$; i.e.

$$I_2(k_z, \rho, \rho^l) = e^{\kappa f_2(k_z)} \quad \text{(B.56)}$$

$$f_2(k_z) = i \left[ \frac{k_z(x - x_l)}{k|\rho - \rho^l|} + \frac{k_z(z - z_l)}{k|\rho - \rho^l|} \right]. \quad \text{(B.57)}$$

where $\kappa = k|\rho - \rho^l|$, $k_z = (k^2 - k^2_z)^{\frac{1}{2}} = \alpha + i\beta$ and the appropriate values of $\alpha$ and $\beta$ in the complex $k_z$ plane are shown in Figure B.4. The saddle point $k_{z,s}$ associated with $I_2(k_z, \rho, \rho^l)$ can be determined by setting the derivative of $f_2(k_z)$ with respect
Figure B.6: Topology associated with the *angular spectral* integral representation of $g(\rho, \rho')$ for the case of $\phi = \gamma = \frac{\pi}{4}$ with $x - xt = z - zt = 5\lambda$: (a) The plot of $\ln |I_1(\phi, \rho, \rho')| = \kappa Re [f_1(\phi)]$ as a function of $\phi_R$ and $\phi_I$ in the vicinity of the first-order saddle point (b) The associated contour plot of $\ln |I_1(\phi, \rho, \rho')|$. the SDP and SAP contours in the complex $\phi$ plane.
to \(k_z\) at \(k_z = k_{z,s}\) to zero; i.e.

\[
\frac{\partial f_2(k_z)}{\partial k_z}_{k_z = k_{z,s}} = 0. \tag{B.58}
\]

and solving for \(k_{z,s}\) resulting in

\[
k_{z,s} = \frac{(z - z_I)}{|\rho - \rho'|} k. \tag{B.59}
\]

Note that the saddle point \(k_{z,s}\) is always real and consistent with the result obtained from the mapping \(k_{z,s} = k \sin \phi_s\).

As pointed out earlier, once the SDP and SAP contours in the complex \(\phi\) plane are obtained, the corresponding contours in the complex \(k_z\) plane can be determined via the transformation \(k_z = k \sin \phi\). It should be pointed out that the SDP and SAP contours in the complex \(k_z\) plane can also be obtained using the same procedure as illustrated above; however the analysis in the complex \(k_z\) plane is more sophisticated since it is involved the branch-point singularity. With the mapping \(k_z = k \sin \phi\), the contour plot of \(\ln |I_2(k_z, \rho, \rho')|\) and the corresponding SDP and SAP contours (for \(\gamma = \frac{\pi}{2}\) rad. with \(x - x_I = z - z_I = 5\lambda\) in both proper and improper Riemann sheets are shown in Figure B.7. It can be readily shown that (see Eq. (B.56))

\[
\ln |I_2(k_z, \rho, \rho')| = - [J(x - x_I) + k_{z,1}(z - z_I)]. \tag{B.60}
\]

where \(J > 0\) for the proper Riemann sheet and \(J < 0\) for the improper Riemann sheet as discussed earlier in Section B.1. From the plots, the SDP contour originates from the proper sheet P2, crosses the upper branch cut and enters to the improper sheet II, then crosses the same branch cut again at \(k_z = k_{z,s}\) and enters to the proper sheet P4, and finally goes to the proper sheet P1. In contrast, the SAP contour originates from the improper sheet I3, crosses the lower branch cut and enters to the proper
sheet P4, then crosses the upper branch cut at $k_z = k_{z,s}$ and enters to the improper sheet II, and finally goes to the improper sheet I4. Note that the above description of the SDP and SAP contours is consistent with the picture of the contours in the complex $\phi$ plane as shown in Figure B.6. It should be pointed out that the SDP and SAP contours intersect only at the saddle point $k_z = k_{z,s}$ in the complex $k_z$ plane, which is consistent with Figure B.6 where the contours intersect only at $\phi = \phi_s = \gamma$ in the complex $\phi$ plane, although they seems to intersect each other again at $k_z = k_{z,f}$.

The reason for this is that at $k_z = k_{z,f}$ the SDP and SAP contours are in different Riemann sheets: i.e. the SDP contour is in the proper sheet whereas the SAP contour is in the improper sheet. Furthermore, it is noted that the SDP contour always comes back to the proper sheet although it may enter to the improper sheet whenever it crosses a branch cut from the proper sheet. In contrast, the SAP contour may not come back to the proper sheet. Finally, the SDP and SAP contours in the complex $k_z$ plane are still perpendicular to each other at the saddle point $k_z = k_{z,s}$ due to the conformal property of the mapping $k_z = k \sin \phi$.

B.4 Topology Associated with the Spectral Integral Representation of $g(\rho, \rho')$ for a Lossy Medium

In this section, the results for the lossless medium in Section B.1 are generalized for the case of the lossy medium. Note that in this case the propagation constant $k$ is complex as shown in Eq. (B.21), and there are three cases to consider: i.e.

- Case 1: $k_I < k_R$
- Case 2: $k_I = k_R = k_1$
- Case 3: $k_I > k_R$

301
Figure B.7: The contour plot of $\ln |I_2(k_z, \rho, \rho')|$ and the corresponding SDP and SAP contours (for $\gamma = \frac{\pi}{4}$ rad. with $x - x' = z - z' = 5\lambda$) in both proper and improper Riemann sheets.
Note that Case 1 is similar to the case of slightly lossy medium \((k_l << k_R)\) as considered earlier in Section B.1, and the topology for this case is similar to the one shown in Figure B.3. It should be pointed out that the analysis of the topology for all three cases follows the same technique as described in Section B.1. Only the sign of the RHS of Eq. (B.25) is different for each case, and this requires some modification of the topology in the complex \(k_z\) plane.

For Case 2, Eq. (B.25) for \(Re [k_z^2] = 0\) becomes

\[
k_{z,l}^2 - k_{z,R}^2 = 0. \tag{B.61}
\]

and Eq. (B.26) for \(Im [k_z^2] = 0\) becomes

\[
k_{z,R} k_{z,l} = k_1^2. \tag{B.62}
\]

Figure B.8 shows the plots of the straight lines \(Re [k_z^2] = 0\) and the hyperbolic curves \(Im [k_z^2] = 0\) as well as the regions for which \(Re [k_z^2] > 0\) for Case 2. From the alternative conditions on the branch cuts in Eqs. (B.17) and (B.18) and Figure B.8, the branch cuts and the proper and improper Riemann sheets in the complex \(k_z\) plane are illustrated in Figure B.9.

For Case 3, Eq. (B.25) for \(Re [k_z^2] = 0\) becomes

\[
k_{z,l}^2 - k_{z,R}^2 = k_l^2 - k_R^2 > 0. \tag{B.63}
\]

and the equation for \(Im [k_z^2] = 0\) in this case is still the same as in Eq. (B.26). Figure B.10 illustrates the plots of the hyperbolic curves \(Re [k_z^2] = 0\) and \(Im [k_z^2] = 0\) as well as the regions for which \(Re [k_z^2] > 0\) for Case 3. From the conditions in Eqs. (B.17) and (B.18) and Figure B.10, the branch cuts and the proper and improper Riemann sheets in the complex \(k_z\) plane are illustrated in Figure B.11.
Figure B.8: The plots of the *straight* lines $\text{Re}[k_z^2] = 0$ and the *hyperbolic* curves $\text{Im}[k_z^2] = 0$ as well as the regions for which $\text{Re}[k_z^2] < 0$, for the case of the *lossy* medium with $k_I = k_R = k_1$. 
Figure B.9: The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_z$ plane, for the case of the lossy medium with $k_I = k_R = k_1$. 
Figure B.10: The plots of the hyperbolic curves $Re[k_z^2] = 0$ and $Im[k_z^2] = 0$ as well as the regions for which $Re[k_z^2] > 0$, for the case of the lossy medium with $k_I > k_R$.

In summary for the topology associated with the lossy medium, although the regions for which $Re[k_z^2] > 0$ are distinct for each case (see Figures B.2, B.8, and B.10), finally the branch cuts for each case are similar except for the locations of the branch points ($k_z = \pm k$), which depends on the exact value of $k$.

### B.5 Topology Associated with the Angular Spectral Integral Representation of $g(\rho, \rho')$ for a Lossy Medium

In this section, the results for the lossless medium in Section B.2 are generalized for the case of the lossy medium, where $k$ is complex. For this case, the transformation is more complicated, and the mapping in Eq. (B.4) becomes [132]:

$$k_{z,R} + ik_{z,I} = (k_R + ik_I) \sin(\beta_R + i\beta_I),$$  \hspace{1cm} (B.64)
Figure B.11: The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_z$ plane, for the case of the lossy medium with $k_l > k_R$. 
or

\[ k_{z,R} = k_R \sin 3_R \cosh 3_I - k_I \cos 3_R \sinh 3_I \]  \hspace{1cm} (B.65) \\
\[ k_{z,I} = k_I \sin 3_R \cosh 3_I + k_R \cos 3_R \sinh 3_I . \]  \hspace{1cm} (B.66)

Alternatively, the inverse transformation in Eq. (B.31) becomes:

\[ \phi_R + i \phi_I = -i \ln z_1 \]  \hspace{1cm} (B.67)
\[ z_1 = \frac{(\alpha + i \beta) - (k_{z,R} + i k_{z,I})}{k_R + i k_I} \]  \hspace{1cm} (B.68)
\[ \alpha + i \beta = \left[ (k_R + i k_I)^2 - (k_{z,R} + i k_{z,I})^2 \right]^{1/2} . \]  \hspace{1cm} (B.69)

or

\[ \phi_R = \arg(z_1) \]  \hspace{1cm} (B.70)
\[ \phi_I = -\ln |z_1|. \]  \hspace{1cm} (B.71)

Note that the values of \( \alpha \) and \( \beta \) must be chosen appropriately for each Riemann sheet in the complex \( k_z \) plane as shown in Figures B.3, B.9 and B.11 for the cases of \( k_I < k_R, k_I = k_R \) and \( k_I > k_R \), respectively.

Using Eq. (B.68), both proper and improper Riemann sheets in the complex \( k_z \) plane for a lossy medium as shown in Figures B.3 (for \( k_I < k_R \)), B.9 (for \( k_I = k_R \)) and B.11 (for \( k_I > k_R \)) can be mapped into the angular strip \(-\pi < \phi_R < \pi\) in the complex \( \phi \) plane as shown in Figures B.12, B.13 and B.14, respectively.

From Eq. (B.66), it is noted that the real axis of the complex \( k_z \) plane (\( k_{z,I} = 0 \)) transforms as follows:

\[ \tan \phi_R = -\frac{k_R}{k_I} \tanh \phi_I . \]  \hspace{1cm} (B.72)
Figure B.12: The topology in the complex $\phi$ plane obtained from the mapping $k_z = k \sin \phi$ when $k$ is complex with $k_I < k_R \left( \Delta = \tan^{-1} \left( \frac{k_L}{k_R} \right) < \frac{\pi}{4} \right)$. 

---

Image of the branch cuts

---

Image of the real axis

( $k_{z, I} = 0$ )

---

Image of the imaginary axis

( $k_{z, R} = 0$ )
Figure B.13: The topology in the complex $\phi$ plane obtained from the mapping $k_z = k \sin \phi$ when $k$ is complex with $k_I = k_R \left( \Delta = \tan^{-1} \left( \frac{k_L}{k_R} \right) = \frac{\pi}{4} \right)$. 
Figure B.14: The topology in the complex $\phi$ plane obtained from the mapping $k_z = k \sin \phi$ when $k$ is complex with $k_I > k_R$ ($\Delta = \tan^{-1} \left( \frac{k_L}{k_R} \right) > \frac{\pi}{4}$).
and its asymptotes as $\phi_I \to \pm \infty$ are

$$
\phi_R \rightarrow \mp \tan^{-1} \left( \frac{k_R}{k_I} \right),
$$

(B.73)
as shown in the above figures. From Eq. (B.66), the imaginary axis of the complex $k_z$ plane ($k_{z,R} = 0$) transforms as follows:

$$
\tan \phi_R = \frac{k_z}{k_R} \tanh \phi_I.
$$

(B.74)

and its asymptotes as $\phi_I \to \pm \infty$ are

$$
\phi_R \rightarrow \pm \Delta.
$$

(B.75)

where

$$
\Delta = \tan^{-1} \left( \frac{k_I}{k_R} \right) \geq 0.
$$

(B.76)
since $k_R > 0$ and $k_I \geq 0$. Note that the mapping of the real ($k_{z,I} = 0$) and imaginary ($k_{z,R} = 0$) axes is conformal: i.e. they are perpendicular to each other at the intersection point in both complex $k_z$ and $\phi$ planes. In addition, in the case of $k_I = 0$ (lossless case) the mapping of the real and imaginary axes is in good agreement with the previous lossless case as shown in Figure B.5. Figures B.12, B.13 and B.14 also illustrate the image of the branch cuts ($\mathcal{J} = 0$) for the lossy medium in the complex $\phi$ plane. Note that the branch cuts ($\mathcal{J} = 0$) of the complex $k_z$ plane transforms as follows:

$$
\text{Im}[k_z] = \text{Im}[k \cos \phi] = 0.
$$

(B.77)

and thus the mapping of the branch cuts is identical to the mapping of the real axis ($k_{z,I} = 0$), except that they are shifted by $\pm \frac{\pi}{2}$ rad. from the origin due to the fact that $k \cos \phi = k \sin(\phi \pm \frac{\pi}{2})$. 312
B.6 Derivation of the SDP and SAP Contours Associated with the Integral Representation of $g(p, \rho')$ in the Complex $k_2$ and $\phi$ Planes for the Complex Propagation Constant

In this section, the results for the real propagation constant in Section B.3 are generalized for the case of the complex propagation constant. As in the lossless case, it is mathematically convenient to determine the constant phase paths associated with the integral representation of $g(p, \rho')$ in the complex $\phi$ plane first. For convenience in analysis, the complex propagation constant $k$ in Eq. (B.21) can be rewritten in the polar form as follows:

$$k = |k|e^{i\Delta}. \quad \text{(B.78)}$$

where

$$|k| = \sqrt{k_R^2 + k_I^2} \quad \text{(B.79)}$$

$$\Delta = \tan^{-1} \left( \frac{k_I}{k_R} \right) \geq 0. \quad \text{(B.80)}$$

Note that $k_R = |k|\cos\Delta$ and $k_I = |k|\sin\Delta$. The large parameter $\kappa$ and the phase function $f_1(\phi)$ associated with the exponential integrand $I_1(\phi, p, \rho')$ (see Eqs. (B.36) to (B.38) for $x > x_l$) must be modified as follows:

$$\kappa = |k||p - \rho'| \quad \text{(B.81)}$$

$$f_1(\phi) = ie^{i\Delta} \cos(\phi - \gamma)$$

$$= [\cos\Delta (\sin(\phi_R - \gamma)\sinh \phi_I - \sin\Delta \cos(\phi_R - \gamma)\cosh \phi_I]$$

$$+ i[\sin\Delta (\sin(\phi_R - \gamma)\sinh \phi_I + \cos\Delta \cos(\phi_R - \gamma)\cosh \phi_I]. \quad \text{(B.82)}$$

Following the same procedure as in the lossless case given in Section B.3, the saddle point $\phi_s$ for the lossy case is found to be the same as given in Eq. (B.40). Note that
the saddle point \( \phi_s \) for the lossy case is still real. Employing Eqs. (B.42) and (B.82), the constant phase paths are governed by the following equation:

\[
\sin \Delta \sin (\phi_R - \gamma) \sinh \phi_I + \cos \Delta \cos (\phi_R - \gamma) \cosh \phi_I = \cos \Delta. \tag{B.83}
\]

or

\[
\tan \Delta \sin (\phi_R - \gamma) \sinh \phi_I \cos (\phi_R - \gamma) \cosh \phi_I = 1. \tag{B.84}
\]

In the limiting case as \( \Delta \to 0^+ \): i.e., approaching the lossless case. Eq. (B.84) is reduced to Eq. (B.44) as expected. Using Eq. (B.45), another constrained equation that the SDP contour must satisfy is:

\[
\cos (\phi_R - \gamma) \cosh \phi_I - \cot \Delta \sin (\phi_R - \gamma) \sinh \phi_I \geq 1. \tag{B.85}
\]

Employing Eq. (B.84) to simplify Eq. (B.85). Eq. (B.85) becomes

\[
(tan \Delta + \cot \Delta) \sin (\phi_R - \gamma) \sinh \phi_I \leq 0. \tag{B.86}
\]

or simply

\[
\sin (\phi_R - \gamma) \sinh \phi_I \leq 0. \tag{B.87}
\]

due to the fact that \( \tan \Delta \geq 0 \) and \( \cot \Delta \geq 0 \). It is interesting to note that Eqs. (B.47) and (B.87) are identical. Thus, the SDP contour for the case of the complex propagation constant is governed by Eqs. (B.84) and (B.87). For the SAP contour, it is required to satisfy Eq. (B.48) as well. and after some manipulation yields the following equation:

\[
\sin (\phi_R - \gamma) \sinh \phi_I \geq 0, \tag{B.88}
\]
which is identical to Eq. (B.49). Thus, the SAP contour is governed by Eqs. (B.84) and (B.88).

As in the lossless case, the SDP and SAP contours are still perpendicular to each other at the intersection point; i.e. the saddle point $\phi = \phi_s$. Following the same approach as illustrated in the lossless case, it can be shown that

$$\text{SDP : } \left. \frac{d\phi_I}{d\phi_R} \right|_{\phi=\gamma} = -\tan \Delta - \sec \Delta < 0 \quad (B.89)$$

$$\text{SAP : } \left. \frac{d\phi_I}{d\phi_R} \right|_{\phi=\gamma} = -\tan \Delta + \sec \Delta > 0. \quad (B.90)$$

Note that the product of the slopes of the SDP and SAP contours at the saddle point are equal to -1, which is implied that the contours are indeed perpendicular at $\phi = \phi_s$. In the limiting case as $\Delta \rightarrow 0^\circ$, Eqs. (B.89) and (B.90) are reduced to Eqs. (B.54) and (B.55) respectively, as expected. Figure B.15 (a) shows the plot of $\ln |I_1(\phi, \rho, \rho')| = \kappa Re[f_1(\phi)]$ as a function of $\phi_R$ and $\phi_I$ in the vicinity of the first-order saddle point $\phi = \gamma = \frac{\pi}{4}$ rad. ($x - x' = z - z' = 5\lambda$) for the lossy case with $k_l = k_R = 2\pi \text{ rad./}\lambda$ ($\Delta = \frac{\pi}{4}$ rad.). where $\lambda = \frac{2\pi}{k_l}$. Note that the terrain associated with $\ln |I_1(\phi, \rho, \rho')|$ contains both mountains and valleys. Comparing Figures B.6 (a) and B.15 (a), it is noted that there are some changes in the terrain due to the loss in the medium. Figure B.15 (b) illustrates the associated contour plot of $\ln |I_1(\phi, \rho, \rho')|$, the SDP and SAP contours in the complex $\phi$ plane. Comparing Figures B.6 (b) and B.15 (b), the contour plots of the lossless and lossy media are noticeably different. In addition, the SDP and SAP contours are also modified considerably. The support of the SDP contour along the real axis for the lossy case can be shown to be equal to $\pi - 2\Delta$ rad.; i.e. it depends on $\Delta$, whereas the one for the lossless case is always fixed and equal to $\pi$ rad. The SDP contour for the lossy case intersects the real axis at the
saddle point \( \phi = \gamma \) with the angle

\[
\phi_{SDP} = \tan^{-1}(\tan \Delta + \sec \Delta). \tag{B.91}
\]

measured from the negative real axis as shown in Figure B.15 (b), whereas \( \phi_{SDP} \) for the lossless case is always fixed and equal to \( \frac{\pi}{4} \) rad. as shown earlier in Figure B.6 (b).

For the SAP contour, its support along the real axis for the lossy case can be shown to be equal to \( \pi + 2\Delta \) rad.: i.e. it depends on \( \Delta \), whereas the one for the lossless case is always fixed and equal to \( \pi \) rad. Its intersection angle \( \phi_{SAP} \) for the lossy case with the positive real axis can be shown to be

\[
\phi_{SAP} = \tan^{-1}(\sec \Delta - \tan \Delta) \tag{B.92}
\]

as shown in Figure B.15 (b), whereas \( \phi_{SAP} \) for the lossless case is always fixed and equal to \( \frac{\pi}{4} \) rad. as shown in Figure B.6 (b). It is emphasized that the supports of the SDP and SAP contours for the lossy case are unequal and depend on \( \Delta \), but they are equal and fixed for the lossless case. Finally, in the limiting case as \( \Delta \to 0^+ \) the results for the lossy case are reduced to the ones for the lossless case as expected.

For the complex \( k_z \) plane, the phase function \( f_2(k_z) \) associated with the exponential integrand \( I_2(k_z, \rho, \rho') \) (see Eqs. (B.56) and (B.57)) for \( x > x^f \) must be modified as follows:

\[
f_2(k_z) = ie^{i\Delta} \left[ \frac{k_z(x - x^l)}{k |\rho - \rho'|} + \frac{k_z(z - z^l)}{k |\rho - \rho'|} \right]. \tag{B.93}
\]

where \( k_x = (k^2 - k_z^2)^{\frac{1}{2}} = \alpha + i3 \) and the appropriate values of \( \alpha \) and 3 in the complex \( k_z \) plane are illustrated in Figures B.3, B.9 and B.11 for \( k_l < k_R, k_l = k_R \) and \( k_l > k_R \), respectively. Note that the large parameter \( \kappa \) employed in Eq. (B.56) must be modified to the one defined in Eq. (B.81). Following the same procedure
as in the lossless case given in Section B.3, the saddle point \( k_{z,s} \) for the lossy case is found to be the same as \( k_{z,s} \) for the lossless case (see Eq. (B.59)). However, \( k_{z,s} \) for the lossy case is complex since \( k \) is complex. The saddle point \( k_{z,s} \) is still consistent with the result obtained from the mapping \( k_{z,s} = k \sin \phi \) for the lossy case. It is interesting to note that for the lossy case the saddle point \( \phi_s \) in the complex \( \phi \) plane is always real, whereas the saddle point \( k_{z,s} \) in the complex \( k \) plane is complex.

As in the lossless case, the SDP and SAP contours for the lossy case in the complex \( k \) plane can be obtained from the precomputed contours in the complex \( \phi \) plane via the mapping \( k = k \sin \phi \). Figure B.16 illustrates the contour plot of \( \ln |I_2(k, \rho, \rho')| \) and the corresponding SDP and SAP contours (for \( \gamma = \frac{\pi}{4} \) rad. with \( x - x' = z - z' = 5\lambda \) and \( k_f = k_R = 2\pi \) rad./\( \lambda \)) in both proper and improper Riemann sheets. It is noted that \( \ln |I_2(k, \rho, \rho')| \) for the lossy case is defined as in Eq. (B.60). From the plots, the SDP contour originates from the proper sheet P2, then enters to the proper sheet P1 without crossing any branch cuts, passes through the saddle point \( k_{z,s} \), and finally stays in the proper sheet P1. Note that the SDP contour in this case totally stays in the proper sheet. In contrast, the SAP contour originates from the improper sheet I3, crosses the branch cut and enters to the proper sheet P3, then enters to the proper sheets P4 and P1 respectively, passes through the saddle point \( k_{z,s} \), crosses another branch cut and enters to the improper sheet I1, and finally goes to the improper sheet I4. Note that the above description of the SDP and SAP contours is consistent with the picture of the contours in the complex \( \phi \) plane as shown in Figure B.15 as expected. Like the lossless case, the SDP and SAP contours intersect only at the saddle point \( k_{z,s} \) although they seem to intersect again at \( k_z = k_{z,f} \), where they are in different Riemann sheets. In addition, the SDP and
Figure B.15: Topology associated with the angular spectral integral representation of \( g(\rho, \rho') \) for the lossy case with \( k_l = k_R = 2\pi \text{ rad.}/\lambda \) (\( \Delta = \frac{\pi}{4} \text{ rad.} \)) and with \( \phi = \gamma = \frac{\pi}{4} \) with \( x - x' = z - z' = 5\lambda \): (a) The plot of \( \ln |I_1(\phi, \rho, \rho')| = \kappa \Re[f_1(\phi)] \) as a function of \( \phi_R \) and \( \phi_I \) in the vicinity of the first-order saddle point (b) The associated contour plot of \( \ln |I_1(\phi, \rho, \rho')| \), the SDP and SAP contours in the complex \( \phi \) plane.
Figure B.16: The contour plot of $\ln |f_2(k_z, \rho, \rho\ell)|$ and the corresponding SDP and SAP contours for the lossy medium with $k_l = k_R = 2\pi\,\text{rad.}/\lambda$ ($\gamma = \frac{\pi}{4}\,\text{rad.}$ and $x - x' = z - z' = 5\lambda$) in both proper and improper Riemann sheets.

SAP contours are perpendicular to each other at the intersection point $k_z = k_{z,s}$ via the conformal property of the mapping $k_z = k \sin \phi$. Finally, it should be pointed out that the SDP and SAP contours change as the complex propagation constant $k$ and the saddle point $k_{z,s}$ changes.

B.7 A Summary and Conclusions

In this appendix, the topology associated with the integral representation of the scalar Green’s function $g(\rho, \rho\ell)$ in both complex $k_z$ and $\phi$ planes is discussed in detail for both lossless and lossy media, where the propagation constant $k$ is real and
complex, respectively. In addition, the constant phase paths; i.e. the SDP and SAP contours, associated with the integral representation of $g(\rho, \rho')$ are rigorously derived for both complex $k_\parallel$ and $\phi$ planes. It is found that the topology associated with the complex $\phi$ plane is *less complicated* than the topology associated with the complex $k_\parallel$ plane due to the fact that there is no branch-point singularity involved in the complex $\phi$ plane. In addition, the topology for the lossy case is *more sophisticated* than the one for the lossless case. Due to less complexity of the topology associated with the *periodic* complex $\phi$ plane, the constant phase paths in this plane are determined first, and the paths in the complex $k_\parallel$ plane are obtained via the mapping $k_\parallel = k \sin \phi$.

Understanding the topology associated with the integral representation of $g(\rho, \rho')$ assists in the development of the 1-D NSA algorithm as illustrated earlier in Chapter 2. It should be pointed out that the 1-D NSA algorithm employs only the *angular spectral* integral representation of $g(\rho, \rho')$ associated with the complex $\phi$ plane. However, for completeness the topology associated with the spectral integral representation of $g(\rho, \rho')$ in the complex $k_\parallel$ plane is investigated in detail as well. Studying these topologies forms a theoretical foundation for understanding the *more complicated* topology associated with the *double* spectral integral representation of the free space 3-D scalar Green’s function employed in the development of the 2-D NSA algorithm as shown in Chapter 4. In the next appendix, the topology associated with the 2-D NSA algorithm is discussed in detail.
APPENDIX C

Topology of Two Coupled Complex \( k_y \) and \( k_z \) Planes Associated with the 2-D NSA Algorithm

In the 2-D NSA algorithm as described in Section 4.4 in Chapter 4, the double spectral integral representation of the free space 3-D scalar Green's function \( g(\mathbf{r}, \mathbf{r}') \) is involved in computing the mutual coupling between an observation point and a group of source points in the weak region. Throughout this appendix, it is assumed that the propagation constant \( k \) of the medium above rough surfaces is *real*: i.e. lossless medium, which typically occurs in practice. Employing the results associated with the lossy medium in Appendix B, it is quite straightforward to generalize the results in this appendix for the lossy case. Without loss of generality, consider only the *double* spectral integral representation of \( g(\mathbf{r}, \mathbf{r}') \) propagating in the \( x \)-direction for \( x > x' \) as follows [131, 132]:

\[
g(\mathbf{r}, \mathbf{r}') = \frac{i}{8\pi^2} \int_{C_{k_y}} \int_{C_{k_z}} dk_z dk_y \left[ \exp \left\{ i \left[ k_z (x - x') + k_y (y - y') + k_z (z - z') \right] \right\} \right],
\]

(C.1)

where

\[
k_x = (\kappa^2 - k_y^2)^{\frac{1}{2}},
\]

(C.2)

\[
\kappa = (k^2 - k_z^2)^{\frac{1}{2}},
\]

(C.3)
\[ r = \rho + \hat{z} z, \quad r' = \rho' + \hat{z} z', \quad \rho = \hat{x} x + \hat{y} y, \quad \rho' = \hat{x} x' + \hat{y} y', \] and the contours \( C_k \) and \( C_{k'} \) are the \emph{original} contours of integration on the real axes in the complex \( k_z \) and \( k_y \) planes as shown in Figures 4.6 and 4.7, respectively. Note that the two spectral variables \( k_z \) and \( k_y \) are coupled through \( k_x \) defined in Eq. (C.2), and the branch of \( k_x \) and \( \kappa \) must be chosen properly as will be discussed later in detail. For convenience in analysis, Eq. (C.1) can be reexpressed in terms of the \emph{single} spectral integral representation with respect to \( k_z \) as follows:

\[ g(r, r') = \frac{i}{8\pi} \int_{C_{k_z}} dk_z e^{ik_z(z-z')} I_{k_y}(\kappa, \rho, \rho'). \tag{C.4} \]

where

\[ I_{k_y}(\kappa, \rho, \rho') = \frac{1}{\pi} \int_{C_{k_y}} dk \left[ \frac{\exp \left\{ i[k_x(x-x') + k_y(y-y')] \right\}}{k_x} \right]. \tag{C.5} \]

Employing Eqs. (B.1) and (B.2) in Appendix B, the closed-form expression for \( I_{k_y}(\kappa, \rho, \rho') \) is

\[ I_{k_y}(\kappa, \rho, \rho') = H_0^{(1)}(\kappa|\rho - \rho'|). \tag{C.6} \]

where \( H_0^{(1)}(\cdot) \) denotes the zeroth-order Hankel function of the first kind, and its large argument approximation \((\kappa|\rho - \rho'| \gg 1)\) is given by

\[ I_{k_y}(\kappa, \rho, \rho') \sim \sqrt{\frac{2}{i\pi \kappa|\rho - \rho'|}} e^{i\kappa|\rho - \rho'|}. \tag{C.7} \]

Substituting Eq. (C.7) into Eq. (C.4), \( g(r, r') \) can be approximated as follows:

\[ g(r, r') \sim \frac{e^{it}}{4\pi \sqrt{2\pi|\rho - \rho'|}} \int_{C_{k_z}} dk_z \left[ \frac{\exp \left\{ i[\kappa|\rho - \rho'| + k_z(z-z')] \right\}}{\sqrt{\kappa}} \right]. \tag{C.8} \]

Note that Eqs. (C.5) and (C.8) involve the topology in the complex \( k_y \) and \( k_z \) planes, respectively. In addition, the topology in the complex \( k_y \) plane changes as the spectral
variable $k_z$ varies along the original contour $C_{k_z}$ or the deformed contour $C_{\delta k_z}$ (see Figure 4.6 in Chapter 4) in the complex $k_z$ plane due to the fact that Eq. (C.5) contains $k_z$ which depends on $k_z$ via Eqs. (C.2) and (C.3).

Although the integral representation of $g(r, r')$ in Eq. (C.8) is not exact, it is sufficient to provide useful information about the topology in the complex $k_z$ plane. The accuracy of Eq. (C.8) can be improved by increasing the size of the strong region $L_x$ associated with the 2-D NSA algorithm, where $x - x' \geq L_x$. As pointed out earlier in Appendix B, the study of the topology associated with the spectral integral representation of the free space 2-D scalar Green's function forms a theoretical foundation for understanding the more complicated topology associated with the spectral integral representation of the free space 3-D scalar Green's function, and the results obtained in Appendix B will be employed extensively in this appendix.

This appendix is organized as follows. For better understanding, first consider the relatively simple topology in the complex $k_z$ plane in Section C.1. Section C.2 then presents the more sophisticated topology in the complex $k_y$ plane. The constant phase paths; i.e. the SDP and SAP contours, associated with the complex $k_z$ and $k_y$ planes are derived in Sections C.1 and C.2, respectively. Finally, Section C.3 provides a summary and conclusions of this appendix.

C.1 Topology Associated with the Double Spectral Integral Representation of $g(r, r')$ in the Complex $k_z$ Plane

To investigate the topology in the complex $k_z$ plane, consider the approximate single spectral integral representation of $g(r, r')$ (see Eq. (C.8)) and its exponential
integrand $I_a(k_z, r, r')$ defined as follows:

$$I_a(k_z, r, r') = e^{\Omega_a f_a(k_z)}.$$  \hfill (C.9)

where the large parameter $\Omega_a$ and the phase function $f_a(k_z)$ are defined as follows:

$$\Omega_a = k|r - r'|$$  \hfill (C.10)

$$f_a(k_z) = i \left[ \frac{\kappa(r - r')}{k|r - r'|} + \frac{k_z(z - z')}{k|r - r'|} \right].$$  \hfill (C.11)

and $\kappa$ is defined in Eq. (C.3). Note that Eqs. (C.8) and (C.9) involve the branch-point singularities at $k_z = \pm k$. and $\kappa$ is a double valued function of the spectral variable $k_z$.

For convenience in comparison, let define $k_z$ and $\kappa$ as follows:

$$k_z = k_{z,R} + ik_{z,I}$$  \hfill (C.12)

$$\kappa = \kappa_R + i\kappa_I.$$  \hfill (C.13)

Note that both $\kappa_R$ and $\kappa_I$ are real, and in general $\kappa_R > 0$ and $\kappa_I \geq 0$ in order to satisfy the radiation condition for the $e^{-\omega t}$ harmonic time convention (see Eq. (C.7)).

Employing Eqs. (C.3) and (C.13) yields the following equations:

$$\kappa_R^2 - \kappa_I^2 = k^2 - k_{z,R}^2 + k_{z,I}^2$$  \hfill (C.14)

$$\kappa_R \kappa_I = -k_{z,R}k_{z,I}.$$  \hfill (C.15)

Due to the fact that the product of $\kappa_R \kappa_I$ is nonnegative, it is implied from Eq. (C.15) that the permissible contour in the complex $k_z$ plane must be restricted only in either P2 or P4 as shown in Figure B.4 since $k_{z,R}k_{z,I} \leq 0$, $\kappa_R > 0$ and $\kappa_I \geq 0$ in these regions. Note that the original contour $C_{k_z}$ and the deformed contour $C_{\delta k_z}$ as shown in Figure 4.6 in Chapter 4 are permissible since the product of $k_{z,R}k_{z,I} \leq 0$ along these contours.
Comparing Eq. (C.8) and Eq. (B.2) in Appendix B for the lossless case, it is found that the associated exponential integrands are identical, except replacing \( x - x' \) in Eq. (B.2) by \(|\rho - \rho'|\). Following the same analysis as illustrated in Section B.1 in Appendix B, it is found that the topology associated with the double spectral integral representation of \( g(r, r') \) in the complex \( k_z \) plane for the lossless case is identical to the one shown in Figure B.4 in Appendix B after changing \( \alpha \) and \( J \) to \( \kappa_R \) and \( \kappa_I \), respectively.

For the constant phase paths associated with the double spectral integral representation of \( g(r, r') \) in the complex \( k_z \) plane. following the same procedure as described in Section B.3 in Appendix B, it is found that the saddle point \( k_{z,s} \) associated with \( I_s(k_z, r, r') \) is given by

\[
k_{z,s} = \frac{(z - z')}{|r - r'|} k_z.
\]  

(C.16)

where

\[
|r - r'| = \sqrt{|\rho - \rho'|^2 + (z - z')^2}.
\]  

(C.17)

The contour plot of \( \ln |I_s(k_z, r, r')| \) and the corresponding SDP and SAP contours in both proper and improper Riemann sheets for \( |\rho - \rho'| = z - z' = 5\lambda \) are the same as illustrated in Figure B.7 in Appendix B, where \( \lambda = \frac{2\pi}{k} \). The description of Figure B.7 has already been provided in Section B.3 in Appendix B. In the next section, the more complicated topology in the complex \( k_y \) plane is discussed in detail.

### C.2 Topology Associated with the Double Spectral Integral Representation of \( g(r, r') \) in the Complex \( k_y \) Plane

As shown in Eq. (C.5), the topology in the complex \( k_y \) plane is associated with the integral \( I_{k_y}(\kappa, \rho, \rho') \) as defined in Eq. (C.5). Let \( k_{y,R} \) and \( k_{y,I} \) be the real and
imaginary part of the spectral variable $k_y$, respectively. Note that $k_x$ defined in Eq. (C.2) involves the effective propagation constant $\kappa$ (see Eq. (C.3)), which is generally complex due to the contour deformation in the complex $k_z$ plane. Comparing the integrands of Eq. (C.5) and Eq. (B.2) in Appendix B, it is found that they are identical, except changing $z$, $z\text{d}k_z$, $C_k$, and $k$ in Eq. (B.2) to $y$, $y\text{d}k_y$, $C_{k_y}$ and $\kappa$, respectively. Thus, the topology associated with $I_{k_y}(\kappa, \rho, \rho')$ in the complex $k_y$ plane is similar to the topology associated with the spectral integral representation of the free space 2-D scalar Green's function for the lossy case as described earlier in Section B.4 in Appendix B. For mathematical convenience, let define $k_x$ as follows:

$$k_x = \alpha_x + i\beta_x.$$  \hspace{1cm} (C.18)

where $\alpha_x$ and $\beta_x$ are the real and imaginary parts of $k_x$, respectively. Like the topology described in Section B.4 in Appendix B, there are three cases of $\kappa$ to be considered: i.e.

- **Case 1:** $\kappa_l < \kappa_R$
- **Case 2:** $\kappa_l = \kappa_R = \kappa_1$
- **Case 3:** $\kappa_l > \kappa_R$

Employing the same procedure as described in Section B.4 in Appendix B, the topology associated with $I_{k_y}(\kappa, \rho, \rho')$ in the complex $k_y$ plane for Cases 1, 2 and 3 are illustrated in Figures C.1, C.2 and C.3, respectively. In these figures, the branch cuts denoted by the wavy lines in both proper and improper Riemann sheets in the complex $k_y$ plane are illustrated. It is noted that they are similar for each case except for the location of the branch points at $k_y = \pm \kappa$.  

326
Figure C.1: The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_y$ plane, for Case 1.
Figure C.2: The plots of the branch cuts and the *proper* (top) and *improper* (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_y$ plane, for Case 2.
Figure C.3: The plots of the branch cuts and the proper (top) and improper (bottom) Riemann sheets, which are connected by the branch cuts, in the complex $k_y$ plane, for Case 3.
It should be emphasized that the topology in the complex \( k_y \) plane changes as \( k_z \) varies along the *original* contour \( C_{k_z} \) or the *deformed* contour \( C_{\delta k_z} \) in the complex \( k_z \) plane (see Figure 4.6 in Chapter 4) due to the fact that the branch points at \( k_y = \pm \kappa \) depend on \( k_z \). Thus, for a *given* pair of source and observation points, the topology in the complex \( k_z \) plane is *fixed* whereas the topology in the complex \( k_y \) plane is *varied*.

The SDP and SAP contours associated with \( I_{k_y}(\kappa, \rho, \rho') \) in the complex \( k_y \) plane can be determined by considering the exponential integrand \( I_b(k_y, \kappa, \rho, \rho') \) of \( I_{k_y}(\kappa, \rho, \rho') \) defined as follows:

\[
I_b(k_y, \kappa, \rho, \rho') = e^{\Omega_b f_b(k_y)}. \tag{C.19}
\]

where the large parameter \( \Omega_b \) and the phase function \( f_b(k_y) \) are defined as follows:

\[
\Omega_b = |\kappa| |\rho - \rho'| \tag{C.20}
\]

\[
f_b(k_y) = i e^{i \Delta_b} \left[ \frac{k_x (x - x')}{|\kappa| |\rho - \rho'|} + \frac{k_y (y - y')}{|\kappa| |\rho - \rho'|} \right]. \tag{C.21}
\]

where \(|\kappa|\) and \( \Delta_b \) are the magnitude and phase of \( \kappa \), respectively. Following the same procedure as described in Section B.6 in Appendix B for the lossy case, it is found that the saddle point \( k_{y,s} \) associated with \( I_b(k_y, \kappa, \rho, \rho') \) is given as follows:

\[
k_{y,s} = \frac{(y - y')}{|\rho - \rho'|} \kappa. \tag{C.22}
\]

Figure C.4 illustrates the contour plot of \( \ln |I_b(k_y, \kappa, \rho, \rho')| \) and the corresponding SDP and SAP contours in both proper and improper Riemann sheets for \( x - x' = y - y' = 5\lambda \) and \( \kappa_l = \kappa_R = 2\pi \) rad./\( \lambda \). where \( \lambda = \frac{2\pi}{|\kappa|} \). Note that Figure C.4 is similar to Figure B.16 in Appendix B, and the same description is applied in Figure C.4 as well.
Figure C.4: The contour plot of $\ln |I_b(k_y, \kappa, p, \rho')|$ and the corresponding SDP and SAP contours for the lossy medium with $k_I = k_R = 2\pi$ rad. /$\lambda$ ($\gamma = \frac{\pi}{4}$ rad. and $x-x' = z-z' = 5\lambda$) in both proper and improper Riemann sheets.
C.3 A Summary and Conclusions

In this appendix, the topology of the two coupled $k_y$ and $k_z$ planes associated with the 2-D NSA algorithm is discussed in detail for the case of lossless medium. In addition, the SDP and SAP contours associated with the double spectral integral representation of $g(r, r')$ in each complex plane are derived as well. Theoretical studies of the topology of the complex spectral plane in Appendix B assist in understanding the more sophisticated topology in this appendix. It is found that the topology in the complex $k_z$ plane is equivalent to the topology associated with the spectral integral representation of the free space 2-D scalar Green's function for the lossless case. In addition, the topology in the complex $k_y$ plane is equivalent to the latter for the lossy case, and it changes as the spectral variable $k_z$ varies along the original contour $C_{k_z}$ or the deformed contour $C_{k_z}$ in the complex $k_z$ plane.
APPENDIX D

Derivation of Formulas Associated with Integration Parameters of the Standard 1-D NSA Algorithm

As discussed in Chapter 2, the standard 1-D NSA method involves three parameters: the tilt angle $\delta$ of the deformed contour $C_\delta$, the domain of integration $[-\phi_{max}, \phi_{max}]$ and the integration step size $\Delta \phi$. In the standard 1-D NSA method [104], the formulas for the 1-D NSA parameter are derived based on the assumption that the outermost possible saddle point $\phi_{s, max}$ in the complex $\phi$ plane is small. For a given surface height variation, this assumption can be satisfied by adjusting the size of the strong region $L_s$. In this appendix, an analytical formula associated with $\delta$ is derived first, and then the derivation of analytical formulas associated with $\phi_{max}$ and $\Delta \phi$ follows.

D.1 Derivation of Analytical Formula for $\delta$

One criterion for selecting a possible value of $\delta$, denoted as $\tilde{\delta}$, can be obtained by limiting the maximum value of the magnitude of the exponential integrand $I(\phi)$, where $I(\phi)$ is defined as in Eq. (2.36) in Chapter 2, along the deformed contour $C_\delta$ to $e^{a_{max}}$, where $a_{max}$ is a given positive constant (typically $a_{max} = 5.0$). Let $\phi_R$ and $\phi_I$ denote the real and imaginary parts of the periodic complex angle $\phi$. 333
Figure D.1: The original contour $C_\phi$, the deformed contour $C_\delta$, and the SDP and SAP contours when $\Delta z = \Delta z_{\text{max}}$ ($C_{SDP}(\Delta z = \Delta z_{\text{max}})$ and $C_{SAP}(\Delta z = \Delta z_{\text{max}})$, respectively) in the periodic complex $\phi$ plane. $I_\phi$ is the intersection point between $C_\delta$ and $C_{SAP}(\Delta z = \Delta z_{\text{max}})$. 
respectively. For small $\phi_{s,max}$, the magnitude of the integrand $|I(\phi)|$ is usually found to be maximum near the intersection point $I_\phi$, as shown in Figure D.1, where the SAP contour $C_{SAP}(\Delta z = \Delta z_{\text{max}})$ intersects the deformed contour $C_{\delta}$, and thus the pair of points possessing $\pm \phi_{s,max}$ (i.e. $\Delta z = \pm \Delta z_{\text{max}}$ and $x - x_t = L_s$) corresponds to the worst-case scenario in the $x-z$ plane, where $\phi_{s,max} = \tan^{-1}\left(\frac{\Delta z_{\text{max}}}{L_s}\right)$. From Figure D.1, it is noted that the linear part of $C_{\delta}$ extends from $-\phi_{\text{max}}$ to $\phi_{\text{max}}$, and the contour $C_{\delta}$ along this linear portion can be expressed as in Eq. (2.38) in Chapter 2:

$$\phi_{s,max} = \tan^{-1}\left(\frac{\Delta z_{\text{max}}}{L_s}\right).$$

Assuming that the maximum of $|I(\phi)|$ occurs at the intersection point $I_\phi$ and employing the above criterion for the worst-case scenario, the following equation is obtained (see Eq. (D.1) and Eq. (2.40) in Chapter 2):

$$kR_s \sin(\phi_{s,max} - \phi_R) \sinh\left(\frac{\phi_R}{b}\right) = a_{\text{max}}.$$  

(D.2)

where $b = \frac{1}{\tan \delta}$. Performing a first-order Taylor's series expansion of the SAP contour $C_{SAP}(\Delta z = \Delta z_{\text{max}})$ in the neighborhood of the saddle point $\phi_{s,max}$ obtains the following approximate linear equation:

$$\phi_I = \phi_R - \phi_{s,max}.$$  

(D.3)

Note that for small $\phi_{s,max}$ Eq. (D.3) is a good approximation to the $C_{SAP}(\Delta z = \Delta z_{\text{max}})$ near $\phi_{s,max}$. To solve for $b$, first solving Eqs. (D.1) and (D.3) simultaneously yields the values of $\phi_{R,I_\phi}$ and $\phi_{I,I_\phi}$ at the intersection point $I_\phi$ as follows:

$$\phi_{R,I_\phi} = \left(\frac{b}{1+b}\right) \phi_{s,max}$$  

(D.4)

$$\phi_{I,I_\phi} = -\left(\frac{1}{1+b}\right) \phi_{s,max}.$$  

(D.5)
Substituting Eq. (D.4) into Eq. (D.2), after some algebraic manipulation Eq. (D.2) evaluated at the intersection point \( I_\sigma \) becomes

\[
\sin \left( \frac{\phi_{s,\text{max}}}{1 + b} \right) \sinh \left( \frac{\phi_{s,\text{max}}}{1 + b} \right) = \frac{a_{\text{max}}}{k R_s}.
\]  

(D.6)

Note that Eq. (D.6) can be simplified for small \( \phi_{s,\text{max}} \) by using the small argument approximations of \( \sin \) and \( \sinh \) functions, i.e.

\[
\sin(\theta) \approx \theta \]  

(D.7)

\[
\sinh(\theta) \approx \theta.
\]  

(D.8)

for small value of \( \theta \). Employing the above approximation in Eq. (D.6), the unknown \( b \) can be approximated as

\[
b = \sqrt{\frac{k R_s}{a_{\text{max}}} \phi_{s,\text{max}}} - 1.
\]  

(D.9)

for small \( \phi_{s,\text{max}} \). Finally, the tilt angle \( \delta \) can be determined as

\[
\delta = \tan^{-1}\left[ \frac{1}{\max\{b, 1\}} \right].
\]  

(D.10)

since \( \delta \) is always less than or equal to \( \frac{\pi}{4} \) rad.

**D.2 Derivation of Analytical Formula for \( \phi_{\text{max}} \)**

One criterion for selecting a possible value of \( \phi_{\text{max}} \), denoted as \( \beta_s \), can be obtained by considering the distribution of \( |I(\phi)| \) along the deformed contour \( C_s \). In [104], the "flat surface" assumption is employed in the derivation of \( \beta_s \); i.e. it is assumed that \( \phi_{s,\text{max}} = 0 \) rad. and \( \delta = \frac{\pi}{4} \) rad. (the tilt angle for the flat surface case). Employing this assumption and the small argument approximations of \( \sin \) and \( \sinh \) functions
(see Eqs. (D.7) and (D.8)), \(|I(\phi)|\) can be approximated as follows (see Eq. (2.40) in Chapter 2):

\[
|I(\phi)| = e^{-kR\phi_R^2}.
\]  

(D.11)

for small \(\phi_R\). Note that \(|I(\phi)|\) distributes like a Gaussian in the neighborhood of the origin: i.e. the saddle point of the flat surface. Thus, \(\beta_s\) can be determined by limiting the value of \(|I(\phi)|\) (evaluated at \(\phi_R = \beta_s\)) along \(C_s = C_{SDP}(\Delta z = 0)\) (see Figure 2.4) to \(e^{-\zeta}\), where \(\zeta\) is typically equal to 10.0, and \(\beta_s\) can be obtained by solving the following equation:

\[
k R \beta_s^2 = \zeta. \tag{D.12}
\]

From Eq. (D.12), it is found that \(R = L_s\) yields the largest \(\beta_s\) for the flat surface, and thus \(\beta_s\) is equal to

\[
\beta_s = \sqrt{\frac{\zeta}{kL_s}}. \tag{D.13}
\]

Finally, to ensure that the integrand \(I(\phi)\) decays properly, \(\phi_{max}\) can be determined as

\[
\phi_{max} = \max \left\{ \beta_s, \phi_{s,max} \right\}. \tag{D.14}
\]

Note that \(\phi_{max}\) in the above equation may not hold for very rough surfaces since it is derived based on the flat surface assumption.

### D.3 Derivation of Analytical Formula for \(\Delta \phi\)

Due to the Gaussian-like distribution of \(|I(\phi)|\) in the neighborhood of the origin as discussed in the previous section, it is found that

\[
\Delta \phi = \frac{1}{22} \sqrt{\frac{\zeta}{2kR_s}.} \tag{D.15}
\]
yields accurate results. Note that $R_s$ is employed in the above equation instead of $L_s$ to *heuristically* take the surface roughness into account. The total number of plane waves $Q_{TOT}$ employed in the *standard* 1-D NSA algorithm is given as

$$Q_{TOT} = 2Q + 1. \quad (D.16)$$

where

$$Q = \left\lceil \frac{\phi_{max}}{\Delta \phi} \right\rceil. \quad (D.17)$$

where $\lceil \cdot \rceil$ denotes the ceiling operator: i.e. rounding its argument to the nearest integer towards plus infinity.

Finally, it should be emphasized that the above analytical formulas for the tilt angle $\delta$, the maximum of the domain of integration $\phi_{max}$ and the integration step size $\Delta \phi$ are valid only when the outermost possible saddle point $\phi_{s,max}$ is small. For an arbitrary value of $\phi_{s,max}$, *new* analytical formulas associated with these 1-D NSA parameters must be rederived, and they are presented in Section 2.3 in Chapter 2. These formulas provide more flexibility in selecting the size of the strong region $L_s$ to compromise between the computation of the contributions of strong and weak regions.
APPENDIX E

Derivation of Formulas Associated with Integration Parameters of the 2-D NSA Algorithm

The 2-D NSA algorithm involves three integration parameters in each complex plane: i.e. the total of six parameters: \( \delta_{k_z}, k_{z,tail}, C_z, \gamma, k_{y,tail} \) and \( C_y \), where these parameters are defined in Section 4.4 in Chapter 4. Among these, only the tilt angle \( \delta_{k_z} \) in the complex \( k_z \) plane can be derived analytically due to very complicated topology involved in the coupled complex \( k_z \) and \( k_y \) planes as discussed earlier in Appendix C. and the rest can be determined empirically as discussed in Section 4.4. Thus, only the derivation of analytical formula for \( \delta_{k_z} \) is derived in this appendix.

E.1 Derivation of Analytical Formula for \( \delta_{k_z} \)

As pointed out earlier in Section 4.4 in Chapter 4, when \( \tan^{-1}\left( \frac{\Delta z_{\text{max}}}{L_z} \right) \leq 0.1 \), \( \delta_{k_z} = \frac{\pi}{4} \) (the tilt angle for the flat surface case) still provides accurate numerical results, where \( L_z \) and \( \Delta z_{\text{max}} \) are the neighborhood distance in the \( z \)-direction and the largest surface variation, respectively. However, when \( \tan^{-1}\left( \frac{\Delta z_{\text{max}}}{L_z} \right) > 0.1 \), \( \delta_{k_z} \) can be determined analytically by considering the behavior of the integrand of the free space 3-D scalar Green's function \( g(\mathbf{r}, \mathbf{r}') \) for the worst-case scenario in the \( \rho-z \) plane; i.e. the pair of observation and source points possessing \( |\mathbf{r}-\mathbf{r}'| = L_x, z-z' = \pm \Delta z_{\text{max}} \)
and $k_{z,s} = \pm k_{z,s_{\text{max}}}$, where the magnitude of the integrand of $g(r, r')$ expects to be maximum for this configuration. For this case, an appropriate value of $\delta_{k_z}$ is expected to be in the following interval: $\delta_{k_z} \in (0, \frac{\pi}{4}]$. It should be pointed out that the tilt angle $\delta_{k_z}$ for this case can be derived by following the same procedure for determining the tilt angle $\delta$ for the standard 1-D NSA algorithm as illustrated in Section D.1 in Appendix D, and it is discussed in detail below.

Let define the spectral variable $k_z = k r_z + i k n_z$, where $k r_z$ and $k n_z$ are the real and imaginary parts of $k_z$, respectively. Employing Eq. (C.8) in Appendix C for $x > x_l$, after deforming the original contour $C_{k_z}$ to the deformed contour $C_{\delta_{k_z}}$, $g(r, r')$ can be approximated in terms of the single spectral integral with respect to $k_z$ as follows:

$$g(r, r') \sim \frac{e^{i k r_z}}{4\pi \sqrt{2\pi |\rho - \rho'|}} \int_{C_{k_z}} dk_z \frac{I(k_z)}{\sqrt{\lambda}}.$$  

where

$$I(k_z) = \exp \{i [\kappa |\rho - \rho'| + k_z (z - z_t)]\} \quad (E.2)$$

$$\kappa = (k^2 - k_z^2)^{\frac{1}{2}} = \alpha + i \beta.$$  

(E.3)

where $\alpha$ and $\beta$ are the real and imaginary parts of $\kappa$ respectively, the contours $C_{k_z}$ and $C_{\delta_{k_z}}$ are shown in Figure 4.6, and the symbol “$\sim$” reads “asymptotically equal to”.

One criterion for selecting a possible value of $\delta_{k_z}$, denoted as $\delta_{k_z}$, can be obtained by limiting the maximum value of the magnitude $|I(k_z)|$ along the deformed contour $C_{\delta_{k_z}}$ to $e^{a_{\text{max}}}$, where $a_{\text{max}}$ is a given positive constant. Based on numerical experiments, $a_{\text{max}}$ is typically found to be less than three. Employing the above criterion for the worst-case scenario discussed above, the following equation is obtained:

$$\text{Re} \left[ i(\kappa L_z + k_z \Delta z_{\text{max}}) \right] = a_{\text{max}}. \quad (E.4)$$
or

\[ 3L_x + k n_z \Delta z_{\text{max}} + a_{\text{max}} = 0. \] (E.5)

Consider the SAP contour \( C_{\text{SAP}}(\Delta z = \Delta z_{\text{max}}) \) passing through the outermost possible saddle point \( k_{z,\text{max}} \) as shown in Figure 4.6 in Chapter 4. Note that the magnitude \( |I(k_z)| \) is usually found to be maximum near the intersection point \( I_z \), where the SAP contour \( C_{\text{SAP}}(\Delta z = \Delta z_{\text{max}}) \) intersects the deformed contour \( C_{\delta k} \), and the intersection point \( I_z \) is located in the proper Riemann sheet (\( \delta > 0 \)). and the value of \( \alpha \) at \( I_z \) is also positive as shown in Figure B.4 in Appendix B. where \( \alpha \) and \( \delta \) are defined as in Eq. (E.3). As pointed out earlier in Section C.1 in Appendix C, the topology associated with the double spectral integral representation of \( g(r, r') \) in the complex \( k_z \) plane for the lossless case is similar to the one shown in Figure B.4 in Appendix B. Performing a first-order Taylor's series expansion of the SAP contour \( C_{\text{SAP}}(\Delta z = \Delta z_{\text{max}}) \) in the neighborhood of the saddle point \( k_{z,\text{max}} \) obtains the following approximate linear equation:

\[ k n_z = k I_z - k_{z,\text{max}}. \] (E.6)

Note that \( C_{\text{SAP}}(\Delta z = \Delta z_{\text{max}}) \) intersects the real axis making an angle of \( \frac{\pi}{4} \) rad. The contour \( C_{\delta k} \) is defined to be a straight line in the interval from \( -k_{z,\text{max}} \) to \( k_{z,\text{max}} \), making an angle \( \delta k_z \) with respect to the negative real axis, and the equation for \( C_{\delta k} \) in this interval is given as:

\[ k n_z = -\frac{k I_z}{\xi}, \quad k I_z \in [-k_{z,\text{max}}, k_{z,\text{max}}] \] (E.7)

where \( \xi = \frac{1}{\tan \delta k_z} \) and \( k_{z,\text{max}} \) is the upper limit of integration in the complex \( k_z \) plane.

Based on the assumption that the magnitude \( |I(k_z)| \) is maximum near the intersection...
point $I_z$, Eqs. (E.5), (E.6) and (E.7) are sufficient to obtain the appropriate value of $\delta_k$, since there are three unknowns involved: i.e. $kt_{z,1}, kn_{z,1}$ and $\xi$, where $kt_{z,1}$ and $kn_{z,1}$ are the values of $kt_z$ and $kn_z$ at the intersection point $I_z$, respectively. To solve for $\xi$, first solving Eqs. (E.6) and (E.7) simultaneously yields the values of $kt_{z,1}$ and $kn_{z,1}$ at the intersection point $I_z$ as follows:

$$kt_{z,1} = \left(\frac{\xi}{1 + \xi}\right) k_{z,smax} \quad \text{(E.8)}$$

$$kn_{z,1} = - \left(\frac{1}{1 + \xi}\right) k_{z,smax} \quad \text{(E.9)}$$

From Eq. (E.3), using the fact that $\alpha$ and $\beta$ are positive at the intersection point $I_z$, the values of $\alpha$ and $\beta$ at $I_z$, denoted as $\alpha_I$ and $\beta_I$ respectively, can be expressed in terms of $kt_{z,1}$ and $kn_{z,1}$ as follows:

$$\alpha_I = -\frac{kt_{z,1}kn_{z,1}}{\beta_I} \quad \text{(E.10)}$$

and

$$\beta_I = \frac{1}{\sqrt{2}} \left[ \sqrt{(k^2 - kt_{z,1}^2 + kn_{z,1}^2)^2 + (2kt_{z,1}kn_{z,1})^2 - (k^2 - kt_{z,1}^2 + kn_{z,1}^2)} \right]^{\frac{1}{2}} \quad \text{(E.11)}$$

Note that $\alpha_I$ in Eq. (E.10) is positive since $kt_{z,1}kn_{z,1} < 0$ and $\beta_I > 0$. Using Eqs. (E.9) and (E.11), after some algebraic manipulation Eq. (E.5) evaluated at the intersection point $I_z$ becomes

$$L_z \sqrt{\tau_1 - \tau_2 + \tau_3} = 0 \quad \text{(E.12)}$$

where $\tau_1 = 0.5 \sqrt{4\tau_2^2 + \tau_4^2}$, $\tau_2 = 0.5 \left[ k^2(1 + \xi)^2 + k_{z,smax}^2(1 - \xi^2) \right]$, $\tau_3 = (1 + \xi) a_{max} - k_{z,smax} \Delta z_{max}$, and $\tau_4 = 2\xi k_{z,smax}^2$, which is identical to Eq. (4.47) in Chapter 4. Once $\xi$ is solved via a standard root-finding technique such as Muller’s method (see Appendix F), $\delta_k$, can be determined as follows:

$$\delta_k = \tan^{-1}\left[ \frac{1}{\max\{\xi, 1\}} \right], \quad \tan^{-1}\left( \frac{\Delta z_{max}}{L_z} \right) > 0.1. \quad \text{(E.13)}$$

342
Note that $\delta_{k_z}$ in Eq. (E.13) is always less than or equal to $\frac{\pi}{4}$ rad. as expected. Finally, for the future work it is still challenging to rederive the analytical formula for $\delta_{k_z}$ for an arbitrary of the outermost possible saddle point $k_{z,s_{\text{max}}}$.
Consider the following \emph{one-dimensional} equation:

\[ f(x) = 0. \]  

\( (F.1) \)

whose solution or solutions are desired, where \( f \) is an arbitrary scalar-valued function and \( x \) is an independent variable. For functions that are smooth near a root, the \emph{secant method} can be applied to find that root \cite{88}. In this method, it is assumed that the function is approximately linear in the local region of interest, and the next improvement in the root is taken as the point where the approximating line crosses the axis. Note that it is required only \textit{two} previous guesses for the root and the values of the function \( f(x) \) at those two points. The iteration starts with any two values of \( x \) that users select. After each iteration, the secant method retains the most recent of the prior estimates; i.e., discarding the previous boundary points in favor of the latest estimate of the root. Mathematically, the method converges near a root of a sufficiently continuous function, however the algorithm is not guaranteed to converge for functions that are \textit{not} sufficiently continuous. Note that local behavior might send it off towards infinity. In addition, the secant method can be applied for determining \textit{real} roots only.
To be able to find complex roots of analytic functions in the complex plane, the secant method has been generalized, named Muller's method, by employing quadratic interpolation among three given points instead of linear interpolation between two. Note that solving for zeros of the quadratic allows Muller's method to obtain complex roots. Given three previous guesses $x_{n-2}, x_{n-1}$ and $x_n$ for the root, and the values of the function $f(x)$ at those three points, the next guess $x_{n+1}$ is calculated via the following formulas [88]:

$$x_{n+1} = x_n - D (x_n - x_{n-1})$$

where

\[ A = q f(x_n) - q(1 + q) f(x_{n-1}) + q^2 f(x_{n-2}) \]  
\[ B = (2q + 1) f(x_n) - (1 + q)^2 f(x_{n-1}) + q^2 f(x_{n-2}) \]  
\[ C = (1 + q) f(x_n) \]  
\[ D = \frac{2C}{B \pm \sqrt{B^2 - 4AC}} \]  
\[ q = \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}. \]

where the sign in the denominator of Eq. (F.6) is chosen such that its absolute value or modulus is as large as possible. Note that the iteration starts with any three values of $x$ that users select, and it is required complex arithmetic in implementing the method.
References


352


353


