INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

Bell & Howell Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0800

UMI®
Material Characterization of Agricultural and Industrial Solutions and Melts in Elongational Processes

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Fengtai Mark Hwang, B.S., M.S.

* * * * *

The Ohio State University

2000

Dissertation Committee:

Dr. Stephen E. Bechtel, Adviser
Dr. Brian D. Harper
Dr. Kurt W. Koelling

Approved by

Dr. Stephen E. Bechtel, Adviser

The Ohio State University
ABSTRACT

Many manufacturing and agriculture processes involve elongational flow of viscoelastic fluids. To model these processes, one must know the properties of the fluid (e.g. viscosity, relaxation time, retardation time etc.) in elongational flows. The research of this dissertation exploits a free surface rheometer that measures the free surface profiles of a filament with resolution sufficient to compute the first and second numerical derivatives of the profile, and that measures the normal stress difference at the nozzle. This rheometer allows for variation of flow rate, windup speed, and filament length. The measurements from the rheometer are combined with analysis to characterize the elongational response of the test fluid. In this dissertation a single test fluid is investigated under a range of flow conditions.

Two characterization procedures are studied in this dissertation: (i) In the first characterization procedure, seven viscoelastic constitutive models are selected, namely, single mode Oldroyd fluid-B model, single mode Giesekus model, single mode FENE-P model, modified single mode FENE-P model, two-mode Oldroyd fluid-B model, and modified two-mode FENE-P model, and three-mode Oldroyd fluid-B model. The experiments for fourteen different take-up flow conditions are performed; to examine reproducibility, one of these flows is duplicated in separate experiments. For each constitutive form, optimal material constants are determined through comparison
with the experimental measurements. The dependence of optimal coefficients in the seven viscoelastic forms on the flow conditions is investigated. (ii) In the second characterization procedure, the uniaxial response of the test fluid is characterized by computing the paths of many different experiments in stress/strain/strain-rate space, and then fitting these paths to a surface in this space; thereby deducing the constitutive functional form of the elongational viscosity of the test fluid and the coefficients in this form.
This is dedicated to my beloved parents, brother, and Hsiao-chin.
ACKNOWLEDGMENTS

I would like to express my gratitude to my adviser, Professor Bechtel, for providing encouragement and guidance throughout the course of this work. I would like to gratefully acknowledge the help of Professor Koelling for his valuable advice. I am also grateful to Professor Harper for serving as a member of my committee. This work was sponsored in part by the National Science Foundation under Grants CTS-9624293, CTS-9711109, and DMS-9704549.
VITA

December 7, 1968 ............................................... Born, Edmonton, Canada

1987 - 1991 .......................................................... B.S. Mathematics,
National Tsing-hua University,
Hsinchu, Taiwan

1993 - 1996 .......................................................... M.S. Applied Mathematics,
University of Minnesota

1997 - present .................................................... GTA/GRA, Engineering Mechanics,
The Ohio State University

FIELDS OF STUDY

Major Field: Engineering Mechanics
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>v</td>
</tr>
<tr>
<td>Vita</td>
<td>vi</td>
</tr>
<tr>
<td>List of Tables</td>
<td>xi</td>
</tr>
<tr>
<td>List of Figures</td>
<td>xiii</td>
</tr>
<tr>
<td>Chapters:</td>
<td></td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Motivations</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Forward problems vs. inverse problems</td>
<td>2</td>
</tr>
<tr>
<td>1.3 Inverse problem of elongational characterization based on fiber spinning</td>
<td>3</td>
</tr>
<tr>
<td>1.4 Two characterization procedures</td>
<td>4</td>
</tr>
<tr>
<td>1.4.1 Characterization procedure I</td>
<td>4</td>
</tr>
<tr>
<td>1.4.2 Characterization procedure II</td>
<td>6</td>
</tr>
<tr>
<td>2. The Fiber Spinning Experiments</td>
<td>7</td>
</tr>
<tr>
<td>2.1 Experimental apparatus and procedures</td>
<td>7</td>
</tr>
<tr>
<td>2.2 Test fluid</td>
<td>8</td>
</tr>
<tr>
<td>2.3 Experimental data</td>
<td>9</td>
</tr>
</tbody>
</table>
### 7. Numerical Implementations of Characterization Procedure I

#### 7.1 The single mode Oldroyd fluid-B model

- **7.1.1 Numerical implementation, results and comparisons of categories** (CF, a, MES, M1f, Old1), (CF, a, MES, M1s, Old1), (CF, a, MES, M2, Old1), (F, a, MES, M1f, Old1), (F, a, MES, M1s, Old1), and (F, a, MES, M2, Old1). 
- **7.1.2 Numerical implementation and results of category (F, a, GA, M2, Old1).**
- **7.1.3 Numerical implementation and results of category (F, b, MES, M2, Old1).**
- **7.1.4 Representation decomposing stress in solvent and polymer stresses.**

#### 7.2 The single mode Giesekus model

#### 7.3 The single mode FENE-P model

- **7.3.1 Numerical implementation and results of category (F, b, MES, M2, FENE1).**

#### 7.4 The modified single mode FENE-P model

#### 7.5 The two-mode Oldroyd fluid-B model

- **7.5.1 Numerical implementation and results of category (F, b, MES, M2, Old2).**

#### 7.6 The three-mode Oldroyd fluid-B model

#### 7.7 The modified two-mode FENE-P model

- **7.7.1 Numerical implementation and results of category (F, b, GA, M2, MFENE2).**

#### 7.8 Comparison of stress ratios at \( \tilde{z} = 0 \)

#### 7.9 Summary and conclusions

### 8. Inverse Formulation and Implementation of Material Characterization Procedure II

#### 8.1 Elongational viscosity as a function of strain \( \varepsilon \) and strain rate \( \dot{\varepsilon} \)

#### 8.2 Numerical implementation and results

#### 8.3 Conclusion
Appendices:

A. Measured and deduced kinematics and spinline force for the experiments 140

Bibliography ................................................................. 231
LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td></td>
</tr>
<tr>
<td>7.1</td>
<td></td>
</tr>
<tr>
<td>7.2</td>
<td></td>
</tr>
<tr>
<td>7.3</td>
<td></td>
</tr>
</tbody>
</table>

2.1 Controlled and measured conditions for each experiment: \( Q = \) flow rate, \( \phi(0) = \) upstream radius, \( \delta = \) LDVT force difference, \( C = \) calibration factor, \( P = \) translation in pixels between successive images, \( n = \) number of overlayed images. .......................... 12

3.1 Dimensionless numbers for each experiment, deduced from the measured test fluid properties and scales of Table 2.1. \( \epsilon \) is the slenderness parameter of the filament, and \( B^{-1}, B^{-1}, B^{-1}, \) and \( Z + S \) are measures of the effects of inertia, gravity, surface tension, and viscosity relative to the characteristic axial force, respectively. .......................................... 22

7.1 Optimal values of material constants \( \lambda, \eta_p, \) and \( \eta_s \) of the single mode Oldroyd fluid-B constitutive model, as returned by all of the different methods (Methods 1-f, 1-s, and 2), using either polynomial curve fitting or filtering; corresponding errors (error \( f \) for Method 1-f, error \( s \) for Method 1-s, and error \( h \) for Method 2 are minimized to identify the optimal material constants). Continued .......................... 96

7.2 Optimal values of \( \lambda, \eta_p, \eta_s \) for single mode Oldroyd fluid-B model, and the corresponding errors, using a Genetic Algorithm with 200 initial populations and mutation rate \( \mu = 0.2 \). .......................... 100

7.3 Optimal constitutive parameters of single mode Oldroyd fluid-B, Giesekus, FENE-P, and modified FENE-P model for each experiment and their corresponding error, subscript \( a \): formulation \( a \), \( b \): formulation \( b \), subscript \( m \): modified. Continued .......................... 101
7.4 Optimal constitutive parameters of two-mode Oldroyd fluid-B and modified FENE-P model for each experiment and their corresponding error. Note that the optimal values of FENE-P model obtained from Genetic Algorithm with 10000 initial populations and mutation rate 0.2. ........................................ 104

7.5 Optimal constitutive parameters of three-mode Oldroyd fluid-B model for each experiment and their corresponding error. Note that the optimal values obtained from Genetic Algorithm with 10000 initial populations and mutation rate 0.2. ........................................ 105

7.6 Stress ratios at $\tilde{z} = 0$ of single mode models: Oldroyd fluid-B, Giesekus, FENE-P, and modified FENE-P for each experiment. .......................... 106

7.7 Stress ratios at $\tilde{z} = 0$ of two-mode Oldroyd fluid-B, three-mode Oldroyd fluid-B, and modified two-mode FENE-P for each experiment. Continued .......................................................... 107
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>The experimental apparatus.</td>
<td>11</td>
</tr>
<tr>
<td>4.1</td>
<td>Typical no-windup experiment, exp. 4-0-3: measured free surface profile $\phi^{nw}$.</td>
<td>31</td>
</tr>
<tr>
<td>4.2</td>
<td>Typical windup experiment, exp. 4-1-3: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the complete profile the discrete functions are indistinguishable; in the blow-up the raw data $\phi^{exp}(z_k)$ is given as (○) and the filtered profile $\phi^{fil}(z_k)$ by (×).</td>
<td>32</td>
</tr>
<tr>
<td>4.3</td>
<td>Typical windup experiment, exp. 4-1-3: piecewise continuous free surface slope $\phi_{poly}^z(z)$ (- - -) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{fil}^z(z_k)$(——) generated by numerical differentiation of the filtered profile $\phi^{fil}(z_k)$.</td>
<td>33</td>
</tr>
<tr>
<td>4.4</td>
<td>Typical windup experiment, exp. 4-1-3: axial velocity $v_z^{exp}(z_k)$ generated by $Q/(\pi \phi^{exp}(z_k))$ and $v_z^{fil}(z_k)$ generated by $Q/(\pi \phi^{fil}(z_k))$. In the blow-up $v_z^{exp}(z_k)$ is given as (○) and the filtered profile $v_z^{fil}(z_k)$ by (×).</td>
<td>34</td>
</tr>
<tr>
<td>4.5</td>
<td>Typical windup experiment, exp. 4-1-3: piecewise continuous velocity gradient $v_{z,fil}^{poly}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z,fil}^{global}(z)$ (⋯), discrete velocity gradient $v_{z,fil}^{fil}(z_k)$ (——) generated by the numerical differentiation of the filtered velocity $v_z^{fil}(z_k)$, and refiltered velocity gradient $v_{z,fil}^{fil}(z_k)$ (- - -) generated by $v_z^{fil}(z_k)$.</td>
<td>35</td>
</tr>
</tbody>
</table>
4.6 Typical windup experiment, exp. 4-1-3: continuous second gradient $v_{*,zz}^{global}(z)$ of velocity (- - -) generated from analytical differentiation of the globally smoothed polynomial fit $v_{*,zz}^{global}(z)$, and discrete second gradient $v_{*,zz}^{d/dt}(z_k)$ of velocity (--), generated from the numerical differentiation of refiltered velocity gradient $v_{*,zz}^{d/dt}(z_k)$. ........................................ 36

5.1 Typical windup experiment, exp. 4-1-3: spinline force calculated from the differential form (3.33) of the momentum equation using polynomial curve fitting of the experimental data (- - -), calculated from the integral form (3.34) of the momentum equation using polynomial fitting (- - -), calculated from eq. (3.33) using filtering (--), and calculated from eq. (3.34) using filtering (- - -). ........................................ 40

5.2 Typical windup experiment, exp. 4-1-3: normal stress difference calculated by directly integrating eq. (3.32) for the stress difference, using polynomial curve fitting (- - -) and using filtering (- - -); normal stress difference calculated by integrating eq. (3.34) for spinline force and then algebraically deducing the stress difference from eq. (3.29), using polynomial fitting (---) and using filtering (- - -). The two solutions using filtering are indistinguishable to the resolution of the enlarged window. 41

5.3 Typical windup experiment, exp. 4-1-3: difference between two stress calculations of fig. 11 when using polynomial curve fitting (---) and when using filtering (- - -). ........................................ 42

7.1 Exps. 4-1-3, 4-2-3, and 4-3-3: Best fits to Oldroyd fluid-B using Method 1-1 (best fit between measured and computed free surface profile) with polynomial curve fitting (- - -) and filtering (- - -), measured profile (---) and computed profiles (top set is exp. 4-1-3, middle set is exp. 4-2-3, and bottom set is exp. 4-3-3). ........................................ 109

7.2 Exps. 4-1-3, 4-2-3, and 4-3-3: Best fits to Oldroyd fluid-B using Method 1-1 (best fit between measured and computed free surface profile) with polynomial curve fitting (- - -) and filtering (- - -), normal stress difference computed from the momentum equation by the measured profile (---) and from coupled momentum and constitutive equations (top set is exp. 4-3-3, middle set is exp. 4-2-3, and bottom set is exp. 4-1-3). ........................................ 110
7.3 Exps. 4-1-3, 4-2-3, and 4-3-3: Best fits to Oldroyd fluid-B using Method 1-s (best fit between stress difference computed from the momentum equation, and coupled momentum and constitutive equations) with polynomial curve fitting (---) and filtering (•••), measured profile (—) and computed profiles (top set is exp. 4-1-3, middle set is exp. 4-2-3, and bottom set is exp. 4-3-3).

7.4 Exps. 4-1-3, 4-2-3, and 4-3-3: Best fits to Oldroyd fluid-B using Method 1-s (best fit between stress difference computed from the momentum equation, and coupled momentum and constitutive equations) with polynomial curve fitting (-----) and filtering (•••), normal stress difference computed from the momentum equation by the measured profile (—) and from coupled momentum and constitutive equations (top set is exp. 4-3-3, middle set is exp. 4-2-3, and bottom set is exp. 4-1-3).

7.5 Exps. 4-1-3, 4-2-3, and 4-3-3: Best fits to Oldroyd fluid-B using Method 2 of the normal stress difference computed from the momentum equation (——) and the constitutive equations with polynomial curve fitting (- - -) and filtering (•••) (top set is exp. 4-3-3, middle set is exp. 4-2-3, and bottom set is exp. 4-1-3).

7.6 Optimal relaxation time $\lambda$ versus windup rate using filtering (top plot) and polynomial curve fitting (bottom plot): Method 1-f (o), Method 1-s (x), and Method 2 (+). Note the lesser spread with filtering.

7.7 Optimal relaxation time $\lambda$ versus nominal filament length: Method 1-f (o), Method 1-s (x), and Method 2 (+) using filtering.

7.8 Optimal relaxation time $\lambda$ versus flow rate: Method 1-f (o), Method 1-s (x), and Method 2 (+) using filtering.

7.9 $\Delta \lambda = \lambda^{opt} - \lambda^{avg}$ for the 14 experiments, where $\lambda^{opt}$ is the optimal relaxation time for the experiment and a given method and $\lambda^{avg}$ is the average value of the 14 optimal relaxation times for that method: Method 1-f (o): $\lambda^{avg} = 2.1684$ s, Method 1-s (x): $\lambda^{avg} = 2.1488$ s, Method 2 (+): $\lambda^{avg} = 2.1590$ s.

7.10 Errors $f$ (o), $s$ (x), and $h$ (+) for the 14 experiments when $\lambda = \lambda^{avg}$ and $\eta_p = 0$ Pa s.
7.11 Exp 4-1-3: Measured free surface profile (---), profile computed from the coupled momentum and single mode Oldroyd fluid-B constitutive equations with $\lambda = \lambda^{\text{avg}} = 2.1684 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$ (- - -), and profile computed from the coupled momentum and single mode Oldroyd fluid-B constitutive equations with $\lambda = \lambda^{\text{opt}} = 3.6907 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$ using Method 1- $f$ (- - -).  

7.12 Exp. 4-1-3: Normal stress differences computed from the momentum equation (---), the coupled momentum and single mode Oldroyd fluid-B constitutive equations with $\lambda = \lambda^{\text{avg}} = 2.1488 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$ (- - -), and the coupled momentum and single mode Oldroyd fluid-B constitutive equations with $\lambda = \lambda^{\text{opt}} = 3.7328 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$ using Method 1- $s$ (- - -).  

7.13 Exp. 4-1-3: Normal stress differences computed from the momentum equation (---), the single mode Oldroyd fluid-B constitutive equation with $\lambda = \lambda^{\text{avg}} = 2.1590 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$ (- - -), and the single mode Oldroyd fluid-B constitutive equation with $\lambda = \lambda^{\text{opt}} = 3.6849 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$ using Method 2 (- - -).  

7.14 The optimal relaxation time $\lambda$ (+) of single mode Oldroyd fluid-B model and the optimal relaxation times $\lambda_1$ (*) and $\lambda_2$ (o) of two-mode Oldroyd fluid-B model versus wind-up rate. Note that $\lambda_1 < \lambda_2$.  

7.15 The optimal relaxation time $\lambda$ (+) of single mode Oldroyd fluid-B model and the optimal relaxation times $\lambda_1$ (*) and $\lambda_2$ (o) of two-mode Oldroyd fluid-B model versus filament length. Note that $\lambda_1 < \lambda_2$.  

7.16 The optimal relaxation time $\lambda$ (+) of single mode Oldroyd fluid-B model and the optimal relaxation times $\lambda_1$ (*) and $\lambda_2$ (o) of two-mode Oldroyd fluid-B model versus flow rate $Q$. Note that $\lambda_1 < \lambda_2$.  

7.17 The optimal relaxation times $\lambda_1$ (o), $\lambda_2$ (*) and $\lambda_3$ (o) of three-mode Oldroyd fluid-B model versus wind-up rate. Note that $\lambda_1 < \lambda_2 < \lambda_3$.  

7.18 The optimal relaxation times $\lambda_1$ (o), $\lambda_2$ (*) and $\lambda_3$ (o) of three-mode Oldroyd fluid-B model versus filament length. Note that $\lambda_1 < \lambda_2 < \lambda_3$.  

7.19 The optimal relaxation times $\lambda_1$ (o), $\lambda_2$ (*), and $\lambda_3$ (o) of three-mode Oldroyd fluid-B model versus flow rate $Q$. Note that $\lambda_1 < \lambda_2 < \lambda_3$.  

xvi
7.20 The optimal relaxation time $\lambda$ (+) of modified single mode FENE-P model and the optimal relaxation times $\lambda_1$ (*) and $\lambda_2$ (o) of modified two mode FENE-P model versus wind-up rate. Note that $\lambda_1 < \lambda_2$.

7.21 The optimal relaxation time $\lambda$ (+) of modified single mode FENE-P model and the optimal relaxation times $\lambda_1$ (*) and $\lambda_2$ (o) of modified two mode FENE-P model versus filament length. Note that assumed that $\lambda_1 < \lambda_2$.

7.22 The optimal relaxation time $\lambda$ (+) of modified single mode FENE-P model and the optimal relaxation times $\lambda_1$ (*) and $\lambda_2$ (o) of modified two mode FENE-P model versus flow rate $Q$. Note that $\lambda_1 < \lambda_2$.

8.1 Paths of 8 experiments in strain/strain-rate plane

8.2 Paths of 8 experiments in elongational viscosity/strain-rate plane

8.3 Paths of 8 experiments in elongational viscosity/strain plane

8.4 Paths of 8 experiments in elongational viscosity/strain/strain-rate space

8.5 Surface fitting of 8 experiments in elongational viscosity/strain/strain-rate space

A.1 Exp. 4-0-3x: measured free surface profile $\phi^{nw}$

A.2 Exp. 2-0-3: measured free surface profile $\phi^{nw}$

A.3 Exp. 1-0-3: measured free surface profile $\phi^{nw}$

A.4 Exp. 8-0-3: measured free surface profile $\phi^{nw}$

A.5 Exp. 4-0-1: measured free surface profile $\phi^{nw}$

A.6 Exp. 4-0-2: measured free surface profile $\phi^{nw}$

A.7 Exp. 4-0-4: measured free surface profile $\phi^{nw}$

A.8 Exp. 4-2-3: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the profile the discrete functions are indistinguishable.
A.9 Exp. 4-2-3: piecewise continuous free surface slope $\phi_{xz}^{\text{poly}}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{xz}^{\text{fil}}(z_k)$ (-----) generated by numerical differentiation of the filtered profile $\phi_{xz}^{\text{fil}}(z_k)$. .......................................................... 148

A.10 Exp. 4-2-3: axial velocity $v_x^{\text{esp}}(z_k)$ generated by $Q/(\pi \phi_{xz}^{\text{esp}}(z_k))$ and $v_x^{\text{fil}}(z_k)$ generated by $Q/(\pi \phi_{xz}^{\text{fil}}(z_k))$. .................................................. 149

A.11 Exp. 4-2-3: piecewise continuous velocity gradient $v_{xz}^{\text{poly}}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{xz}^{\text{global}}(z)$ (-----), discrete velocity gradient $v_{xz}^{\text{fil}}(z_k)$ (-----) generated by the numerical differentiation of the filtered velocity $v_{xz}^{\text{fil}}(z_k)$, and refiltered velocity gradient $v_{xz}^{\text{fil/fil}}(z_k)$ (-----) generated by $v_{xz}^{\text{fil}}(z_k)$. .......................................................................................... 150

A.12 Exp. 4-2-3: continuous second gradient $v_{xz,zz}^{\text{global}}(z)$ of velocity (- - -) generated from analytical differentiation of the globally smoothed polynomial fit $v_{xz,zz}^{\text{global}}(z)$, and discrete second gradient $v_{xz,zz}^{\text{fil/fil}}(z_k)$ of velocity (-----) generated from the numerical differentiation of refiltered velocity gradient $v_{xz,zz}^{\text{fil}}(z_k)$. .......................................................... 151

A.13 Exp. 4-2-3: spinline force calculated from eq. (3.33) using filtering (-----), and calculated from eq. (3.34) using filtering (- - - -). In the profile the two forces are indistinguishable. .......................................................... 152

A.14 Exp. 4-3-3: measured free surface profile $\phi_{xz}^{\text{esp}}(z_k)$ and filtered profile $\phi_{xz}^{\text{fil}}(z_k)$. In the profile the discrete functions are indistinguishable. 153

A.15 Exp. 4-3-3: piecewise continuous free surface slope $\phi_{xz}^{\text{poly}}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{xz}^{\text{fil}}(z_k)$ (-----) generated by numerical differentiation of the filtered profile $\phi_{xz}^{\text{fil}}(z_k)$. .......................................................... 154

A.16 Exp. 4-3-3: axial velocity $v_x^{\text{esp}}(z_k)$ generated by $Q/(\pi \phi_{xz}^{\text{esp}}(z_k))$ and $v_x^{\text{fil}}(z_k)$ generated by $Q/(\pi \phi_{xz}^{\text{fil}}(z_k))$. .................................................. 155

xviii
A.17 Exp. 4-3-3: piecewise continuous velocity gradient $v_{z,z}^{\text{poly}}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z,z}^{\text{global}}(z)$ (--; discrete velocity gradient $v_{z,z}^{f/f}(z_k)$ (-----) generated by the numerical differentiation of the filtered velocity $v_{z}^{f/f}(z_k)$, and refiltered velocity gradient $v_{z,z}^{f/f/f}(z_k)$ (--; generated by $v_{z,z}^{f/f}(z_k)$. ............................... 156

A.18 Exp. 4-3-3: continuous second gradient $v_{z,z}^{\text{global}}(z)$ of velocity (--) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,z}^{\text{global}}(z)$, and discrete second gradient $v_{z,z}^{f/f}(z_k)$ of velocity (-----) generated from the numerical differentiation of refiltered velocity gradient $v_{z,z}^{f/f}(z_k)$. ............................... 157

A.19 Exp. 4-3-3: spinline force calculated from eq. (3.33) using filtering (---), and calculated from eq. (3.34) using filtering (--; - --). In the profile the two forces are indistinguishable. ............................... 158

A.20 Exp. 4-4-3: measured free surface profile $\phi^{\text{exp}}(z_k)$ and filtered profile $\phi^{f/f}(z_k)$. In the profile the discrete functions are indistinguishable. ............................... 159

A.21 Exp. 4-4-3: piecewise continuous free surface slope $\phi_{z,z}^{\text{poly}}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{z,z}^{f/f}(z_k)$ (-----) generated by numerical differentiation of the filtered profile $\phi^{f/f}(z_k)$. ............................... 160

A.22 Exp. 4-4-3: axial velocity $v_{z,z}^{\text{exp}}(z_k)$ generated by $Q/(\pi \phi^{\text{exp}}(z_k))$ and $v_{z}^{f/f}(z_k)$ generated by $Q/(\pi \phi^{f/f}(z_k))$. ............................... 161

A.23 Exp. 4-4-3: piecewise continuous velocity gradient $v_{z,z}^{\text{poly}}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z,z}^{\text{global}}(z)$ (--; discrete velocity gradient $v_{z,z}^{f/f}(z_k)$ (-----) generated by the numerical differentiation of the filtered velocity $v_{z}^{f/f}(z_k)$, and refiltered velocity gradient $v_{z,z}^{f/f/f}(z_k)$ (--; generated by $v_{z,z}^{f/f}(z_k)$. ............................... 162

A.24 Exp. 4-4-3: continuous second gradient $v_{z,z}^{\text{global}}(z)$ of velocity (--; --) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,z}^{\text{global}}(z)$, and discrete second gradient $v_{z,z}^{f/f}(z_k)$ of velocity (-----) generated from the numerical differentiation of refiltered velocity gradient $v_{z,z}^{f/f}(z_k)$. ............................... 163
A.25 Exp. 4-4-3: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable. ................................................ 164

A.26 Exp. 4-5-3: measured free surface profile $\phi^{esp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the profile the discrete functions are indistinguishable. ............................................. 165

A.27 Exp. 4-5-3: piecewise continuous free surface slope $\phi^{poly}(z) (- - -)$ generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi^{fil}(z_k)$ (—) generated by numerical differentiation of the filtered profile $\phi^{fil}(z_k)$. ............................................................ 166

A.28 Exp. 4-5-3: axial velocity $v^{esp}(z_k)$ generated by $Q/(\pi\phi^{esp^2}(z_k))$ and $v^{fil}(z_k)$ generated by $Q/(\pi\phi^{fil^2}(z_k))$. ............................................................. 167

A.29 Exp. 4-5-3: piecewise continuous velocity gradient $v^{poly}(z) (- - -)$ generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v^{global}(z)$ (· · ·), discrete velocity gradient $v^{fil}(z_k)$ (——) generated by the numerical differentiation of the filtered velocity $v^{fil}(z_k)$, and refiltered velocity gradient $v^{fil/fil}(z_k)$ (· · · · · ·) generated by $v^{fil}(z_k)$. ............................................................. 168

A.30 Exp. 4-5-3: continuous second gradient $v^{global}(z)$ of velocity (· · · · · · · ) generated from analytical differentiation of the globally smoothed polynomial fit $v^{global}(z)$, and discrete second gradient $v^{fil/fil}(z_k)$ of velocity (——) generated from the numerical differentiation of refiltered velocity gradient $v^{fil}(z_k)$. ............................................................. 169

A.31 Exp. 4-5-3: spinline force calculated from eq. (3.33) using filtering (——), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable. ................................................ 170

A.32 Exp. 4-6-3: measured free surface profile $\phi^{esp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the profile the discrete functions are indistinguishable. ............................................. 171

A.33 Exp. 4-6-3: piecewise continuous free surface slope $\phi^{poly}(z) (- - -)$ generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi^{fil}(z_k)$ (——) generated by numerical differentiation of the filtered profile $\phi^{fil}(z_k)$. ............................................................. 172
A.34 Exp. 4-6-3: axial velocity \( u_{exp}^z(z_k) \) generated by \( Q/(\pi \phi_{exp}^z(z_k)) \) and
\( u^z_{fil}(z_k) \) generated by \( Q/(\pi \phi_{fil}^z(z_k)) \). ................................................ 173

A.35 Exp. 4-6-3: piecewise continuous velocity gradient \( u_{pol}^{\text{poly}}(z) \) (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient \( u^{\text{global}}_{z,z}(z) \) (•••), discrete velocity gradient \( u^z_{fil}(z_k) \) (— — ) generated by the numerical differentiation of the filtered velocity \( u^z_{fil}(z_k) \), and refiltered velocity gradient \( u^{fil}_{z,z}(z_k) \) (- - -) generated by \( u^{fil}_{z,z}(z_k) \). ................................................ 174

A.36 Exp. 4-6-3: continuous second gradient \( u^{\text{global}}_{z,z}(z) \) of velocity (- - -) generated from analytical differentiation of the globally smoothed polynomial fit \( u^{\text{global}}_{z,z}(z) \), and discrete second gradient \( u^{fil}_{z,z}(z_k) \) of velocity (— — ) generated from the numerical differentiation of refiltered velocity gradient \( u^{fil}_{z,z}(z_k) \). ................................................ 175

A.37 Exp. 4-6-3: spinline force calculated from eq. (3.33) using filtering (— — ), and calculated from eq. (3.34) using filtering (- - -). In the profile the two forces are indistinguishable. ...................... 176

A.38 Exp. 4-7-3: measured free surface profile \( \phi_{exp}^{z}(z_k) \) and filtered profile \( \phi_{fil}^{z}(z_k) \). In the profile the discrete functions are indistinguishable. ...................... 177

A.39 Exp. 4-7-3: piecewise continuous free surface slope \( \phi_{pol}^{z,+}(z) \) (- - -) generated by analytical differentiation of the polynomial fit, and discrete free surface slope \( \phi_{z,z}^{fil}(z_k) \) (— — ) generated by numerical differentiation of the filtered profile \( \phi^{fil}_{z,z}(z_k) \). ................................................ 178

A.40 Exp. 4-7-3: axial velocity \( u_{exp}^z(z_k) \) generated by \( Q/(\pi \phi_{exp}^z(z_k)) \) and
\( u^z_{fil}(z_k) \) generated by \( Q/(\pi \phi_{fil}^z(z_k)) \). ................................................ 179

A.41 Exp. 4-7-3: piecewise continuous velocity gradient \( u_{pol}^{\text{poly}}(z) \) (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient \( u^{\text{global}}_{z,z}(z) \) (•••), discrete velocity gradient \( u^z_{fil}(z_k) \) (— — ) generated by the numerical differentiation of the filtered velocity \( u^z_{fil}(z_k) \), and refiltered velocity gradient \( u^{fil}_{z,z}(z_k) \) (- - -) generated by \( u^{fil}_{z,z}(z_k) \). ................................................ 180
A.42 Exp. 4-7-3: continuous second gradient $v^{\text{global}}_{z,zz}(z)$ of velocity (—) generated from analytical differentiation of the globally smoothed polynomial fit $v^{\text{global}}_{z,z}(z)$, and discrete second gradient $v^{\text{fil,fil}}_{z,zz}(z_k)$ of velocity (—) generated from the numerical differentiation of refiltered velocity gradient $v^{\text{fil}}_{z,z}(z_k)$. .......................................................... 181

A.43 Exp. 4-7-3: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable. .......................... 182

A.44 Exp. 4-8-3: measured free surface profile $\phi^{\text{exp}}(z_k)$ and filtered profile $\phi^{\text{fil}}(z_k)$. In the profile the discrete functions are indistinguishable. ... 183

A.45 Exp. 4-8-3: piecewise continuous free surface slope $\phi^{\text{poly}}(z)$ (— —) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi^{\text{fil}}(z_k)$ (——) generated by numerical differentiation of the filtered profile $\phi^{\text{fil}}(z_k)$. ................................. 184

A.46 Exp. 4-8-3: axial velocity $v^{\text{exp}}_{z}(z_k)$ generated by $Q/(\pi \phi^{\text{exp}}(z_k))$ and $v^{\text{fil}}_{z}(z_k)$ generated by $Q/(\pi \phi^{\text{fil}}(z_k))$. .................................................. 185

A.47 Exp. 4-8-3: piecewise continuous velocity gradient $u^{\text{poly}}_{z,z}(z)$ (— —) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $u^{\text{global}}_{z,z}(z)$ (···), discrete velocity gradient $u^{\text{fil}}_{z,z}(z_k)$ (——) generated by the numerical differentiation of the filtered velocity $u^{\text{fil}}_{z,z}(z_k)$, and refiltered velocity gradient $u^{\text{fil,fil}}_{z,z}(z_k)$ (····) generated by $u^{\text{fil}}_{z,z}(z_k)$. .................................................. 186

A.48 Exp. 4-8-3: continuous second gradient $v^{\text{global}}_{z,zz}(z)$ of velocity (— —) generated from analytical differentiation of the globally smoothed polynomial fit $v^{\text{global}}_{z,z}(z)$, and discrete second gradient $v^{\text{fil,fil}}_{z,zz}(z_k)$ of velocity (——) generated from the numerical differentiation of refiltered velocity gradient $v^{\text{fil}}_{z,z}(z_k)$. .................................................. 187

A.49 Exp. 4-8-3: spinline force calculated from eq. (3.33) using filtering (— —), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable. .......................... 188

A.50 Exp. 4-3-3x: measured free surface profile $\phi^{\text{exp}}(z_k)$ and filtered profile $\phi^{\text{fil}}(z_k)$. In the profile the discrete functions are indistinguishable. ... 189

xxii
A.51 Exp. 4-3-3x: piecewise continuous free surface slope $\phi_{\text{poly}}'$ generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{\text{fit}}'(z_k)$ generated by numerical differentiation of the filtered profile $\phi_{\text{fit}}'(z_k)$. ........................................ 190

A.52 Exp. 4-3-3x: axial velocity $v_{\text{exp}}'(z_k)$ generated by $Q/(\pi \phi_{\text{exp}}^2(z_k))$ and $v_{\text{fit}}'(z_k)$ generated by $Q/(\pi \phi_{\text{fit}}^2(z_k))$. ........................................ 191

A.53 Exp. 4-3-3x: piecewise continuous velocity gradient $v_{\text{poly}}''(z)$ generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{\text{fit},z}''(z)$, discrete velocity gradient $v_{\text{fit},z}''(z_k)$ generated by the numerical differentiation of the filtered velocity $v_{\text{fit}}''(z_k)$, and refiltered velocity gradient $v_{\text{fit},z}''(z_k)$ generated by $v_{\text{fit},z}''(z_k)$. ........................................ 192

A.54 Exp. 4-3-3x: continuous second gradient of velocity $v_{\text{global}}''(z)$ generated from analytical differentiation of the globally smoothed polynomial fit $v_{\text{global}}''(z)$, and discrete second gradient $v_{\text{fit},z}''(z_k)$ of velocity generated from the numerical differentiation of refiltered velocity gradient $v_{\text{fit},z}''(z_k)$. ........................................ 193

A.55 Exp. 4-3-3x: spinline force calculated from eq. (3.33) using filtering, and calculated from eq. (3.34) using filtering. In the profile the two forces are indistinguishable. .......................... 194

A.56 Exp. 2-3-3: measured free surface profile $\phi_{\text{exp}}(z_k)$ and filtered profile $\phi_{\text{fit}}'(z_k)$. In the profile the discrete functions are indistinguishable. ........................................ 195

A.57 Exp. 2-3-3: piecewise continuous free surface slope $\phi_{\text{poly}}'(z)$ generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{\text{fit}}'(z_k)$ generated by numerical differentiation of the filtered profile $\phi_{\text{fit}}'(z_k)$. ........................................ 196

A.58 Exp. 2-3-3: axial velocity $v_{\text{exp}}'(z_k)$ generated by $Q/(\pi \phi_{\text{exp}}^2(z_k))$ and $v_{\text{fit}}'(z_k)$ generated by $Q/(\pi \phi_{\text{fit}}^2(z_k))$. ........................................ 197
A.59 Exp. 2-3-3: piecewise continuous velocity gradient $v_{z}^{poly}(z)$ (— - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z}^{global}(z)$ (— ), discrete velocity gradient $v_{z}^{fil}(z_k)$ (— — ) generated by the numerical differentiation of the filtered velocity $u_{z}^{fil}(z_k)$, and refiltered velocity gradient $v_{z}^{fil/fil}(z_k)$ (— — -) generated by $v_{z}^{fil}(z_k)$.

A.60 Exp. 2-3-3: continuous second gradient $v_{z,z}^{global}(z)$ of velocity (— — ) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,z}^{global}(z)$, and discrete second gradient $v_{z,z}^{fil/fil}(z_k)$ of velocity (— — ) generated from the numerical differentiation of refiltered velocity gradient $v_{z}^{fil}(z_k)$.

A.61 Exp. 2-3-3: spinline force calculated from eq. (3.33) using filtering (— — ), and calculated from eq. (3.34) using filtering (— — -). In the profile the two forces are indistinguishable.

A.62 Exp. 1-3-3: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the profile the discrete functions are indistinguishable.

A.63 Exp. 1-3-3: piecewise continuous free surface slope $\phi_{z}^{poly}(z)$ (— - -) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{z}^{fil}(z_k)(— )$ generated by numerical differentiation of the filtered profile $\phi_{z}^{fil}(z_k)$.

A.64 Exp. 1-3-3: axial velocity $u_{z}^{exp}(z_k)$ generated by $Q/(\pi \phi^{exp^2}(z_k))$ and $u_{z}^{fil}(z_k)$ generated by $Q/(\pi \phi^{fil^2}(z_k))$.

A.65 Exp. 1-3-3: piecewise continuous velocity gradient $v_{z,z}^{poly}(z)$ (— - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z,z}^{global}(z)$ (— — ), discrete velocity gradient $v_{z,z}^{fil}(z_k)$ (— — ) generated by the numerical differentiation of the filtered velocity $u_{z}^{fil}(z_k)$, and refiltered velocity gradient $v_{z,z}^{fil/fil}(z_k)$ (— — -) generated by $v_{z,z}^{fil}(z_k)$.

A.66 Exp. 1-3-3: continuous second gradient $v_{z,z}^{global}(z)$ of velocity (— — ) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,z}^{global}(z)$, and discrete second gradient $v_{z,z}^{fil/fil}(z_k)$ of velocity (— — ) generated from the numerical differentiation of refiltered velocity gradient $v_{z,z}^{fil}(z_k)$.
A.67 Exp. 1-3-3: spinline force calculated from eq. (3.33) using filtering (---), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable. .......................... 206

A.68 Exp. 8-3-3: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the profile the discrete functions are indistinguishable. . . 207

A.69 Exp. 8-3-3: piecewise continuous free surface slope $\phi^{poly}_x(z)$ (- - -) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi^{fil}_x(z_k)$ (---) generated by numerical differentiation of the filtered profile $\phi^{fil}(z_k)$. .................................................. 208

A.70 Exp. 8-3-3: axial velocity $u_z^{exp}(z_k)$ generated by $Q/(\pi\phi^{exp}_z(z_k))$ and $u_z^{fil}(z_k)$ generated by $Q/(\pi\phi^{fil}_z(z_k))$. .................................................. 209

A.71 Exp. 8-3-3: piecewise continuous velocity gradient $u_z^{poly}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $u_z^{global}(z)$ (---), discrete velocity gradient $u_z^{fil}(z_k)$ (---) generated by the numerical differentiation of the filtered velocity $u_z^{fil}(z_k)$, and refriltered velocity gradient $u_z^{fil/fil}(z_k)$ (- - -) generated by $u_z^{fil}(z_k)$. .................................................. 210

A.72 Exp. 8-3-3: continuous second gradient $u_z^{global}_z(z)$ of velocity (- - -) generated from analytical differentiation of the globally smoothed polynomial fit $u_z^{global}(z)$, and discrete second gradient $u_z^{fil/fil}(z_k)$ of velocity (---) generated from the numerical differentiation of refriltered velocity gradient $u_z^{fil}(z_k)$. .................................................. 211

A.73 Exp. 8-3-3: spinline force calculated from eq. (3.33) using filtering (---), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable. .......................... 212

A.74 Exp. 4-3-1: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the profile the discrete functions are indistinguishable. . . 213

A.75 Exp. 4-3-1: piecewise continuous free surface slope $\phi^{poly}_x(z)$ (- - -) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi^{fil}_x(z_k)$ (---) generated by numerical differentiation of the filtered profile $\phi^{fil}(z_k)$. .................................................. 214
A.76 Exp. 4-3-1: axial velocity $v^{\text{exp}}(z_k)$ generated by $Q/((\pi \phi^{\text{exp}}(z_k))$ and $v^{\text{fil}}(z_k)$ generated by $Q/((\pi \phi^{\text{fil}}(z_k)))$.

A.77 Exp. 4-3-1: piecewise continuous velocity gradient $v^{\text{poly}}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v^{\text{global}}(z) (\cdots)$, discrete velocity gradient $v^{\text{fil}}(z_k)$ (---) generated by the numerical differentiation of the filtered velocity $v^{\text{fil}}(z_k)$, and refiltered velocity gradient $v^{\text{fil/fil}}(z_k) (\cdots \cdots)$ generated by $v^{\text{fil}}(z_k)$.

A.78 Exp. 4-3-1: continuous second gradient $v^{\text{global}}(z)$ of velocity (- - -) generated from analytical differentiation of the globally smoothed polynomial fit $v^{\text{global}}(z)$, and discrete second gradient $v^{\text{fil/fil}}(z_k)$ of velocity (---) generated from the numerical differentiation of refiltered velocity gradient $v^{\text{fil}}(z_k)$.

A.79 Exp. 4-3-1: spinline force calculated from eq. (3.33) using filtering (---), and calculated from eq. (3.34) using filtering (---). In the profile the two forces are indistinguishable.

A.80 Exp. 4-3-2: measured free surface profile $\phi^{\text{exp}}(z_k)$ and filtered profile $\phi^{\text{fil}}(z_k)$. In the profile the discrete functions are indistinguishable.

A.81 Exp. 4-3-2: piecewise continuous free surface slope $\phi^{\text{poly}}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi^{\text{fil}}(z_k)$ (---) generated by numerical differentiation of the filtered profile $\phi^{\text{fil}}(z_k)$.

A.82 Exp. 4-3-2: axial velocity $v^{\text{exp}}(z_k)$ generated by $Q/((\pi \phi^{\text{exp}}(z_k))$ and $v^{\text{fil}}(z_k)$ generated by $Q/((\pi \phi^{\text{fil}}(z_k)))$.

A.83 Exp. 4-3-2: piecewise continuous velocity gradient $v^{\text{poly}}(z) (- - -)$ generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v^{\text{global}}(z) (\cdots)$, discrete velocity gradient $v^{\text{fil}}(z_k)$ (---) generated by the numerical differentiation of the filtered velocity $v^{\text{fil}}(z_k)$, and refiltered velocity gradient $v^{\text{fil/fil}}(z_k) (\cdots \cdots)$ generated by $v^{\text{fil}}(z_k)$.
A.84 Exp. 4-3-2: continuous second gradient $v_{g,zz}^{\text{global}}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v_{g,zz}^{\text{global}}(z)$, and discrete second gradient $v_{g,zz}^{f\text{il}f\text{il}}(z_k)$ of velocity (—) generated from the numerical differentiation of refiltered velocity gradient $v_{g,zz}^{f\text{il}f\text{il}}(z_k)$. .......................... 223

A.85 Exp. 4-3-2: spinline force calculated from eq. (3.33) using filtering (---), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable. ................................................. 224

A.86 Exp. 4-3-4: measured free surface profile $\phi^{\text{exp}}(z_k)$ and filtered profile $\phi^{f\text{il}}(z_k)$. In the profile the discrete functions are indistinguishable. . . 225

A.87 Exp. 4-3-4: piecewise continuous free surface slope $\phi^{\text{poly}}(z)$ (---) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi^{f\text{il}}(z_k)$ (—) generated by numerical differentiation of the filtered profile $\phi^{f\text{il}}(z_k)$. ................................................. 226

A.88 Exp. 4-3-4: axial velocity $v^{\text{exp}}_z(z_k)$ generated by $Q/\pi \phi^{\text{exp}}(z_k)$ and $v^{f\text{il}}_z(z_k)$ generated by $Q/\pi \phi^{f\text{il}2}(z_k)$. .................................................. 227

A.89 Exp. 4-3-4: piecewise continuous velocity gradient $v^{\text{poly}}_z(z)$ (---) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v^{\text{global}}_z(z)$ (⋯), discrete velocity gradient $v^{f\text{il}}_z(z_k)$ (—), generated by the numerical differentiation of the filtered velocity $v^{f\text{il}}_z(z_k)$, and refiltered velocity gradient $v^{f\text{il}f\text{il}}_z(z_k)$ (— — —) generated by $v^{f\text{il}}_z(z_k)$. .................................................. 228

A.90 Exp. 4-3-4: continuous second gradient $v_{g,zz}^{\text{global}}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v_{g,zz}^{\text{global}}(z)$, and discrete second gradient $v_{g,zz}^{f\text{il}f\text{il}}(z_k)$ of velocity (—) generated from the numerical differentiation of refiltered velocity gradient $v_{g,zz}^{f\text{il}f\text{il}}(z_k)$. .................................................. 229

A.91 Exp. 4-3-4: spinline force calculated from eq. (3.33) using filtering (---), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable. ......................... 230

xxvii
CHAPTER 1

INTRODUCTION

1.1 Motivations

This dissertation is guided by the need for elongational characterization of polymer melts and solutions in manufacturing and agriculture. Elongational characterization means the measurement of material response, for instance viscosity and relaxation and retardation times, in stretching-type flows (as opposed to shearing flows), and the incorporation of these measurements in mathematical equations that can then predict fluid behavior under given flow conditions. To be precise, in an elongational flow the off-diagonal components of the symmetric part of the velocity gradient are zero to leading order; in a shearing flow the diagonal components are zero to leading order.

Agricultural pesticide spray mixtures are polymer solutions. To address overspray of agricultural pesticides (due to spray drift and drop rebound from leaf surfaces) leading to groundwater contamination, the USDA needs to know the elasticity and strain-rate-dependent viscosity of these polymer solutions in the elongational flows of spray formation and drop impact and rebound. In manufacturing there is the need to understand and quantify the response of molten and dissolved polymers in the elongational flow of extrusion. In both agriculture and manufacturing, the elongational
characterization necessary to understand and simulate polymer processing cannot be inferred from standard shear rheometry, but rather must be measured and deduced directly in an elongational flow.

1.2 Forward problems vs. inverse problems

In general, science or engineering problems can be divided into forward (or direct) problems, and inverse problems. Briefly, in the forward problems one computes the response of a system with the known characteristics and known conditions. Forward problems are usually well-posed, i.e. the solution of a forward problem is unique, insensitive to slight perturbations of system characteristics and/or process conditions. In inverse problems one either determines the input from the response of a known system or characterizes the system from given conditions and response from the system. The inverse problems are usually non-linear and ill-posed, and often require new experimental procedures, problem-solving schemes, and computational algorithms.

The forward problems in fluid dynamics are to obtain the velocity fields, and perhaps free surface profiles, in particular flows or processes by solving the equations of mass, momentum, and perhaps energy, coupled with appropriate constitutive forms; in the forward problem the boundary and initial conditions on velocity are known for the process, as well as all coefficients in the constitutive form. Once computed, the velocity fields are used to deduce other information of engineering interest, such as the stress field.

In contrast, the inverse problems are to determine the material properties of the fluid, (namely the form of the constitutive equation and the values of the coefficients in that form), from the measured velocity fields and free surface profile and controlled
process conditions. As in the forward problem, the equations of mass, momentum, and perhaps energy are solved, but with different input and output.

1.3 Inverse problem of elongational characterization based on fiber spinning

In this dissertation devoted to the material characterization of viscoelastic fluids, fiber spinning experiments are conducted and analyzed. Fiber spinning, a process which involves injecting a liquid filament continuously from a nozzle and stretching it with a take-up drum or vacuum suction, has been widely used to investigate the elongational properties of polymer fluid flows [12, 13, 14, 28, 37, 38]. The fiber spinning process is an elongational flow, as defined in section 1.1. In the free surface rheometer based on fiber spinning process, flow rate, filament length, and wind-up rate are controlled, and the profile of the filament free surface and upstream boundary force are measured; from these the elongational material characterization is determined using the equations of conservation of mass and momentum.

The free surface rheometer and the viscoelastic test fluid characterized in this dissertation are described in Chapter 2. Chapter 3 develops the mathematical coupled differential equations that govern the fiber spinning experiment. Chapter 4 describes the manipulation of the experimental measurements provided by the rheometer, in preparation for the solution of the inverse problem.

Other techniques for elongational characterization are bubble collapse, stagnation flows, and entrance flows [26].
1.4 Two characterization procedures

In this study, two different characterization procedures are proposed and employed.

1.4.1 Characterization procedure I

In characterization procedure I, a constitutive form is proposed, and the optimal coefficients with in that form are determined through comparison with the experimental measurements. Characterization procedure I is presented in Chapter 6 and implemented in Chapter 7. Within this procedure three different methods are pursued:

In Methods 1—f and 1-s the coupled momentum/constitutive equations are integrated using the experimentally measured upstream force and profile boundary conditions and trial material coefficients. One seeks values of material coefficients that produce the best agreement between either (i) (section 6.2.1) the free surface profile computed from the coupled momentum/constitutive problem and the measured profile (Method 1—f), or (ii) (section 6.2.2) the normal stress difference computed from this coupled problem and the stress difference computed from the measured free surface profile and its slope inserted into the momentum equation (Method 1-s).

In Method 2 (section 6.2.3) the measured free surface profile is differentiated to produce all kinematical gradients that appear in the momentum and constitutive equations, thereby decoupling these equations. One seeks the material coefficients in the constitutive equation that produce the best agreement between the stress difference computed from the constitutive equation and the stress difference computed from the momentum equation.
A major challenge in the viscoelastic modeling of fiber spinning processes in general, and characterization procedure I in particular, is to produce the necessary in-flow boundary values of the stress components. In both Methods 1 and 2 the differential constitutive equations need to be integrated, either coupled to the momentum equation (Method 1) or uncoupled (Method 2); this integration demands upstream boundary values of axial and radial normal stress. When employing a single-relaxation-time constitutive model there are two necessary in-flow stress boundary values. When employing a multi-relaxation-time viscoelastic constitutive model, the challenge of producing the necessary in-flow boundary values of stress is much greater, since their number is greatly increased from two (note that in shear experiment there is no in-flow boundary, the extension of the shear characterization, from single to multi-mode is straightforward).

Two formulations have been developed to deduce the required stress boundary values from the measurements provided by the rheometer. Formulation a (section 6.3.1) employs only measurements at the upstream boundary of the filament. Formulation b (section 6.3.2) employs measurements along entire filament. Formulation a works only for single mode viscoelastic models; formulation b works for both single and multi-mode models.

Characterization procedure I (using either Method 1 or Method 2, and either formulation a or formulation b) involves searching for the material coefficients in an assumed constitutive form that produce optimal fit to experimental measurements. When a single mode model constitutive model and formulation a are employed, the dimension of the search space is small, e.g. two dimensions for the single mode Oldroyd fluid-B model, three dimensions for the single mode Giesekus model, the
single mode FENE-P model, and the modified single mode FENE-P model. When a multi-mode constitutive model is employed, the dimension of searching space is large, e.g. seven dimensions for the two-mode Oldroyd fluid-B model, nine dimensions for the modified two-mode FENE-P model, and eleven dimensions for the three-mode Oldroyd fluid-B model. Three optimal search algorithms are therefore presented. For relatively small dimensional searching spaces an exhaustive or modified exhaustive search algorithm is employed (sections 6.4.1 and 6.4.2, respectively). Employing these exhaustive search algorithms on a large dimensional searching space is impractical; for these a genetic search algorithm based on probabilistic natural selections is employed (section 6.4.3).

1.4.2 Characterization procedure II

Characterization procedure II is a new, characterizing material properties without any proposed constitutive form: The stress in the filament is computed from the momentum equation and the measurements of the free surface profile, the strain is deduced from the measured profile, and the strain rate is obtained by the differentiation of the profile measurement. A constitutive functional form as well as the values of coefficients in this form are then deduced by computing the paths of many experiments of a fluid in stress/strain/strain-rate space and then fitting these paths to a surface in this space. Characterization procedure II is presented and implemented in Chapter 8.
CHAPTER 2

THE FIBER SPINNING EXPERIMENTS

2.1 Experimental apparatus and procedures

Figure 2.1 is a schematic diagram of the experimental apparatus used for the fiber spinline experiments [15, 29, 33]. The test fluid is pumped through a thin walled stainless steel delivery tube, inside diameter of 0.246 cm, using an infusion/withdrawal syringe pump (Harvard Apparatus 55-2083). The fluid is ejected from the delivery tube in the form of a vertical filament which is taken up tangentially by a rotating windup drum 5 cm in diameter. The drum is mounted on a movable assembly which enables the length of fiber to be varied from 0 to 0.198 m, and its rotation speed is controlled and variable; a flexible torque transmission cable places the driving motor away from the apparatus and minimizes transmitted vibrations. A scraper is attached to the bottom of the drum to remove excess fluid.

The delivery tube structurally is a cantilever beam, deflecting downward due to the tensile force of the filament. This deflection, measured by a frictionless, no contact linear variable displacement transducer (LVDT) is used to measure the spinline force and normal stress difference at the nozzle. The diameter of filament as a function of axial position is recorded using a charged coupled device video camera (COHU)
mounted on a constant speed vertically traveling stage. A major challenge in the data acquisition is to capture the free surface, rather than an internal or external shadow. The apparatus incorporates a diffuse background illumination system (Lumitex Inc.), which provided the best and most authentic resolution of the free surface. The images are recorded by using a Super VHS video cassette recorder (Panasonic AG-1970).

2.2 Test fluid

The test fluid is a Boger fluid [9] consisting of 0.15 wt% of a high molecular weight polyisobutylene (PIB Exxon Vistanex L-120, \( M_w = 1.2 \times 10^6 \text{ gm/mol} \)), 4.85 wt% of hydrocarbon solvent, tetradecane (Fisher), and 95 wt% of viscous low molecular weight polybutene (Amoco Polybutene H-100, \( M_w = 10^3 \text{ gm/mol} \)). This fluid exhibits a strong elastic behavior with a Newtonian shear dependence. The density and surface tension of this test fluid were measured to be \( \rho = 890 \text{ kg/m}^3 \) and \( \sigma = 0.024 \text{ N m}^{-1} \), respectively. This fluid has been investigated in several different flow geometries [27, 32, 11].

In a steady shear flow, the symmetric part of velocity gradient is

\[
\mathbf{D} = \begin{bmatrix}
0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}. \tag{2.1}
\]

The zero shear rate viscosity is defined as

\[
\eta_0 = \lim_{\dot{\gamma} \to 0} \frac{T_{12}}{\dot{\gamma}}, \tag{2.2}
\]

where \( \dot{\gamma} \) is the shear rate. The zero shear rate viscosity of the test fluid from steady shear viscosity measurements is \( \eta_0 = 11.3 \text{ Pa s} \) [15].
2.3 Experimental data

A series of fifteen experiments were run and analyzed in this study [15]. In these experiments three conditions were varied: flow rate, windup rate, and nominal filament length. Flow rate $Q$ is controlled by the syringe pump, windup rate by the powerstat motor, and nominal length by the position of the take-up drum. For each family of experiments with common flow rate and nominal length, an auxiliary experiment was conducted with the same flow rate and nominal length but no wind-up, so that fluid flows only under the effect of gravity. Table 2.1 summarizes the experimental conditions for the fifteen take-up experiments, together with eight corresponding no-windup experiments, and details of the force and profile measurements.

The LVDT output signal was monitored in each experiment to assure that the experiment was steady. The force difference $\delta$ in Table 2.1 is the difference between the output signal of the LVDT deflection measurements for the experiment and the output signal of the LVDT deflection measurement for the corresponding no-windup experiment (so that, for a no-windup experiment, $\delta = 0$). A number of video images captured by the translating camera are overlayed to produce the complete filament profile; $n$ is the number of overlapped video images that comprise the complete filament and $P$ is the measured translation in pixels between successive images. Calibration factor $C$ is measured from the diameter in pixels of the nozzle in the digitized image profile, divided by the diameter of the nozzle, 0.003048 m, measured with a caliper. Filament length $L$ and upstream radius $\phi(0)$ are taken from the digitized concatenated profile image.

The experiments are labeled by a series of three numbers, according to the following scheme:
• The first number relates to the flow rate: 1, 2, 4, and 8 correspond to $Q = 10$, 20, 40, and 80 mm$^3$ s$^{-1}$, respectively.

• The second number relates to the windup rate: 0 corresponds to no-windup, and 1, 2, 3, 4, 5, 6, 7, and 8 correspond to 11, 20, 26, 36, 44, 60, 87, and 126 rpm of the take-up drum, respectively.

• The third number relates to nominal filament length: 1, 2, 3, and 4 correspond to 0.025, 0.06, 0.10, and 0.15 m, respectively.

Note from Table 2.1 that the measured length $L$ is different from the nominal length. For the experiments with windup, the location $z = 0$ is selected to be an axial position on the measured profile just downstream of the location of maximum swell (ensuring that the thin-filament approximation is valid), and $z = L$ is in general the axial location beyond which the take-up drum interferes with the resolution if the free surface profile, due to the blocking of illumination or the creation of shadows. There is a degree of user latitude in these choices, and in this study the sensitivity of the material characterization is investigated as the $z = L$ selections vary. A lower case letter "a" after the three numeral label indicates choices of $z = L$ that use less than the entire experimental profile.

Experiments 4-0-3x and 4-3-3x are separate experiments that duplicate experiments 4-0-3 and 4-3-3, respectively, for the purpose of examining reproducibility.
Figure 2.1: The experimental apparatus.
<table>
<thead>
<tr>
<th>exp.</th>
<th>$Q$ (mm$^3$/s)</th>
<th>windup rate (rpm)</th>
<th>length $L$ (m)</th>
<th>$\phi(0)$ (mm)</th>
<th>$\delta$ ($10^{-3}$N)</th>
<th>$C$ (mm/pix)</th>
<th>$P$ (pix)</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-0-3</td>
<td>40</td>
<td>No windup</td>
<td>0.1133</td>
<td>1.782</td>
<td>0</td>
<td>0.010732</td>
<td>127</td>
<td>81</td>
</tr>
<tr>
<td>4-1-3</td>
<td>40</td>
<td>11</td>
<td>0.1028</td>
<td>1.857</td>
<td>2.966</td>
<td>0.010732</td>
<td>124</td>
<td>75</td>
</tr>
<tr>
<td>4-1-3a</td>
<td>40</td>
<td>11</td>
<td>0.0974</td>
<td>1.857</td>
<td>2.966</td>
<td>0.010732</td>
<td>124</td>
<td>71</td>
</tr>
<tr>
<td>4-2-3</td>
<td>40</td>
<td>20</td>
<td>0.1072</td>
<td>1.583</td>
<td>10.10</td>
<td>0.010809</td>
<td>167</td>
<td>58</td>
</tr>
<tr>
<td>4-3-3</td>
<td>40</td>
<td>26</td>
<td>0.0918</td>
<td>1.481</td>
<td>17.72</td>
<td>0.010657</td>
<td>158</td>
<td>53</td>
</tr>
<tr>
<td>4-4-3</td>
<td>40</td>
<td>36</td>
<td>0.1070</td>
<td>1.345</td>
<td>27.52</td>
<td>0.010847</td>
<td>169</td>
<td>57</td>
</tr>
<tr>
<td>4-5-3</td>
<td>40</td>
<td>44</td>
<td>0.0619</td>
<td>1.047</td>
<td>40.40</td>
<td>0.010847</td>
<td>156</td>
<td>35</td>
</tr>
<tr>
<td>4-6-3</td>
<td>40</td>
<td>60</td>
<td>0.1092</td>
<td>1.099</td>
<td>57.88</td>
<td>0.010886</td>
<td>166</td>
<td>59</td>
</tr>
<tr>
<td>4-6-3a</td>
<td>40</td>
<td>60</td>
<td>0.0983</td>
<td>1.099</td>
<td>57.88</td>
<td>0.010866</td>
<td>166</td>
<td>53</td>
</tr>
<tr>
<td>4-7-3</td>
<td>40</td>
<td>87</td>
<td>0.1056</td>
<td>1.096</td>
<td>70.51</td>
<td>0.010847</td>
<td>161</td>
<td>59</td>
</tr>
<tr>
<td>4-8-3</td>
<td>40</td>
<td>126</td>
<td>0.1068</td>
<td>1.045</td>
<td>82.48</td>
<td>0.010886</td>
<td>168</td>
<td>57</td>
</tr>
<tr>
<td>4-8-3a</td>
<td>40</td>
<td>126</td>
<td>0.0885</td>
<td>1.045</td>
<td>82.48</td>
<td>0.010866</td>
<td>168</td>
<td>47</td>
</tr>
<tr>
<td>4-0-3x</td>
<td>40</td>
<td>no windup</td>
<td>0.1077</td>
<td>1.979</td>
<td>0</td>
<td>0.010695</td>
<td>164</td>
<td>60</td>
</tr>
<tr>
<td>4-3-3x</td>
<td>40</td>
<td>26</td>
<td>0.1074</td>
<td>1.454</td>
<td>17.83</td>
<td>0.010770</td>
<td>168</td>
<td>58</td>
</tr>
<tr>
<td>2-0-3</td>
<td>20</td>
<td>no windup</td>
<td>0.1142</td>
<td>1.664</td>
<td>0</td>
<td>0.010770</td>
<td>173</td>
<td>60</td>
</tr>
<tr>
<td>2-3-3</td>
<td>20</td>
<td>26</td>
<td>0.1111</td>
<td>1.239</td>
<td>10.73</td>
<td>0.010770</td>
<td>168</td>
<td>60</td>
</tr>
<tr>
<td>1-0-3</td>
<td>10</td>
<td>no windup</td>
<td>0.1066</td>
<td>1.356</td>
<td>0</td>
<td>0.010809</td>
<td>166</td>
<td>58</td>
</tr>
<tr>
<td>1-3-3</td>
<td>10</td>
<td>26</td>
<td>0.1066</td>
<td>0.989</td>
<td>5.661</td>
<td>0.010809</td>
<td>166</td>
<td>58</td>
</tr>
<tr>
<td>8-0-3</td>
<td>80</td>
<td>no windup</td>
<td>0.1074</td>
<td>2.391</td>
<td>0</td>
<td>0.010770</td>
<td>171</td>
<td>57</td>
</tr>
<tr>
<td>8-3-3</td>
<td>80</td>
<td>26</td>
<td>0.1036</td>
<td>1.767</td>
<td>20.75</td>
<td>0.010809</td>
<td>164</td>
<td>57</td>
</tr>
<tr>
<td>4-0-1</td>
<td>40</td>
<td>no windup</td>
<td>0.0271</td>
<td>2.101</td>
<td>0</td>
<td>0.010583</td>
<td>166</td>
<td>14</td>
</tr>
<tr>
<td>4-3-1</td>
<td>40</td>
<td>26</td>
<td>0.0253</td>
<td>0.942</td>
<td>60.20</td>
<td>0.010583</td>
<td>166</td>
<td>13</td>
</tr>
<tr>
<td>4-0-2</td>
<td>40</td>
<td>no windup</td>
<td>0.0613</td>
<td>2.039</td>
<td>0</td>
<td>0.010732</td>
<td>166</td>
<td>33</td>
</tr>
<tr>
<td>4-3-2</td>
<td>40</td>
<td>26</td>
<td>0.0617</td>
<td>1.308</td>
<td>31.89</td>
<td>0.010809</td>
<td>166</td>
<td>33</td>
</tr>
<tr>
<td>4-0-4</td>
<td>40</td>
<td>no windup</td>
<td>0.1522</td>
<td>1.932</td>
<td>0</td>
<td>0.010732</td>
<td>166</td>
<td>84</td>
</tr>
<tr>
<td>4-3-4</td>
<td>40</td>
<td>26</td>
<td>0.1468</td>
<td>1.545</td>
<td>10.19</td>
<td>0.010732</td>
<td>166</td>
<td>81</td>
</tr>
</tbody>
</table>

Table 2.1: Controlled and measured conditions for each experiment: $Q =$ flow rate, $\phi(0) =$ upstream radius, $\delta =$ LDVT force difference, $C =$ calibration factor, $P =$ translation in pixels between successive images, $n =$ number of overlayed images.
CHAPTER 3

MATHEMATICAL MODEL OF THE FIBER SPINNING EXPERIMENT: MASS AND MOMENTUM EQUATIONS

In this chapter a mathematical model for fiber spinning experiment is constructed. In section 3.1 the equations for mass and momentum conservations are derived without any assumed constitutive structures. In section 3.2 the spinline force, upstream force, normal stress difference, and normal stress difference gradient are computed using the momentum equation.

3.1 The fiber spinning model: mass/momentum equations

Assuming the fluid flow is incompressible, the 3-D governing equations of mass and momentum are:

- 1. The conservation of mass:

  \[ \nabla \cdot \mathbf{v} = 0, \tag{3.1} \]

  where \( \mathbf{v} \) is the velocity and \( \nabla \cdot \) is Eulerian divergence operator.

- 2. The conservation of linear momentum:

  \[
  \rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = \nabla \cdot \mathbf{T} - \nabla p + \rho \mathbf{g}, \tag{3.2}
  \]
where $\rho$ is the mass density, $p$ is the constraint pressure, $g$ is the acceleration of gravity, $\frac{\partial}{\partial t}$ is the Eulerian differentiation with respect to time, $\nabla$ is the Eulerian gradient operator, and $\hat{T}$ is the determinate part of Cauchy stress tensor $T$ ($= -pI + \hat{T}$).

The boundary conditions at a free surface are

- **3. The kinematic boundary condition:**

  The free surface of the fluid jet

  $$\mathcal{F}(x, t) = 0 \quad (3.3)$$

  satisfies the kinematic boundary condition

  $$\left( \frac{\partial \mathcal{F}}{\partial t} + \mathbf{v} \cdot \nabla \mathcal{F} \right)|_{\partial} = 0, \quad (3.4)$$

  where $|_{\partial}$ denotes that all variables are evaluated on the free surface (3.3) of fluid jet.

- **4. The kinetic boundary condition:**

  $$[T_a - T]|_{\partial} \mathbf{n} = \sigma \kappa \mathbf{n}, \quad (3.5)$$

  where $T_a$ is the stress tensor of the ambient atmosphere, $\mathbf{n}$ is the outward normal unit vector to the free surface, $\sigma$ is the surface tension, and $\kappa$ is the mean curvature of the free surface. The ambient atmosphere is assumed passive, so that

  $$T_a = -p_a I, \quad (3.6)$$

  where $p_a$ is the constant ambient pressure.
In the eighteen experiments analyzed in this dissertation, the fluid filament is axisymmetric and torsionless, and flows in the direction of gravity. Adopting an Eulerian cylindrical coordinate system \((r,\theta,z)\), with \(z\)-axis coincident with the symmetry axis of the jet and the flow in the direction of increasing \(z\), the governing equations (3.1)-(3.6) reduce to:

The mass equation:

\[
\frac{1}{r} u_r + u_{r,r} + u_{z,z} = 0. \tag{3.7}
\]

The momentum equations:

\[
\rho(u_{r,t} + u_r u_{r,r} + u_z u_{r,z}) = \hat{T}_{zr,z} + \frac{1}{r}(\hat{T}_{rr} - \hat{T}_{\theta\theta}) + \hat{T}_{rr,r} - p_{,r}, \tag{3.8}
\]

\[
\rho(u_{z,t} + u_r u_{z,r} + u_z u_{z,z}) = \hat{T}_{zz,z} + \frac{1}{r}(\hat{T}_{rz} + \hat{T}_{rz,r}) - p_{,z} + \rho g, \tag{3.9}
\]

where \(g = ge_z\).

The free surface:

\[
\mathcal{F}(r,\theta,z,t) = r - \phi(z,t) = 0, \tag{3.10}
\]

where \(\phi(z,t)\) is the radius of the filament.

The kinematic boundary condition:

\[
(-\phi_{,t} + u_r - u_z \phi_{,z}) |_{r=\phi(z,t)} = 0, \tag{3.11}
\]

and the kinetic boundary conditions:

\[
(-p_a + p - \hat{T}_{rr} + \phi_{,z} \hat{T}_{rz}) |_{r=\phi(z,t)} = 0 \tag{3.12}
\]

\[
[-\hat{T}_{rz} - \phi_{,z}(-p_a + p - \hat{T}_{zz} - \sigma \kappa)] |_{r=\phi(z,t)} = 0, \tag{3.13}
\]

where

\[
\kappa = \frac{1}{\phi(1 + \phi_{,z}^2)^2} - \frac{\phi_{,zz}}{(1 + \phi_{,z}^2)^{3/2}} \tag{3.14}
\]
is the curvature of the free surface (3.10), and ",, z", ,", r", and " t" denote the differ-entiation with respect to z, r, and t respectively.

The axial force along the filament is

\[ F_z = 2\pi \int_0^{\phi(z)} (T_{zz} - p + p_a) r dr. \]  \(3.15\)

To obtain 1-D slender-jet models from 3-D fluid flow field equations, a high-order perturbation technique is employed [2, 3]. First, the characteristic scales of time \(t_0\), velocity \(v_0\), force \(f_0\), radial length \(r_0\), and axial length \(z_0\) are selected; the slenderness parameter \(\epsilon\) is defined by

\[ \epsilon = \frac{r_0}{z_0}. \]  \(3.16\)

The dimensionless forms of coordinates, free surface radius, velocities, stress components, and axial force are:

\[ \tilde{r} = \frac{r}{r_0}, \quad \tilde{z} = \frac{z}{z_0}, \quad \tilde{t} = \frac{t}{t_0}, \quad \tilde{\phi} = \frac{\phi}{\phi_0}, \quad \tilde{v}_r = \frac{v_r}{v_0}, \quad \tilde{v}_z = \frac{v_z}{v_0}, \]

\[ \tilde{T}_{rr} = \frac{r_0^2}{f_0} \tilde{T}_{rr}, \quad \tilde{T}_{rz} = \frac{r_0^2}{f_0} \tilde{T}_{rz}, \quad \tilde{T}_{\theta\theta} = \frac{r_0^2}{f_0} \tilde{T}_{\theta\theta}, \quad \tilde{T}_{rz} = \frac{r_0^2}{f_0} \tilde{T}_{rz}, \quad \tilde{F}_z = \frac{F_z}{f_0}. \]  \(3.17\)

The dimensionless free surface radius, constraint pressure, velocities, and stress component are expanded as power series in \(\tilde{r}\), and slenderness parameter \(\epsilon\):

\[ \tilde{\phi}(\tilde{z}, \tilde{t}) = \sum_{m \geq 0} \epsilon^m \phi^{(m)}(\tilde{z}, \tilde{t}) = \phi^{(0)}(\tilde{z}, \tilde{t}) + O(\epsilon^2) \]

\[ \tilde{p}(\tilde{r}, \tilde{z}, \tilde{t}) = \sum_{n \geq 0} \sum_{m \geq 0} \epsilon^{2n+m+1} r^{2n} p^{n,m}(\tilde{z}, \tilde{t}) = p^{0,0}(\tilde{z}, \tilde{t}) + O(\epsilon^2) \]

\[ \tilde{v}_r(\tilde{r}, \tilde{z}, \tilde{t}) = \sum_{n \geq 0} \sum_{m \geq 0} \epsilon^{2n+m+1} r^{2n+1} v_r^{n,m}(\tilde{z}, \tilde{t}) = \epsilon \tilde{v}_r^{0,0}(\tilde{z}, \tilde{t}) + O(\epsilon^3) \]

\[ \tilde{v}_z(\tilde{r}, \tilde{z}, \tilde{t}) = \sum_{n \geq 0} \sum_{m \geq 0} \epsilon^{2n+m+1} r^{2n} v_z^{n,m}(\tilde{z}, \tilde{t}) = \epsilon \tilde{v}_z^{0,0}(\tilde{z}, \tilde{t}) + O(\epsilon^3) \]

\[ \tilde{T}_{rr}(\tilde{r}, \tilde{z}, \tilde{t}) = \sum_{n \geq 0} \sum_{m \geq 0} \epsilon^{2n+m+1} r^{2n} T_{rr}^{m,n}(\tilde{z}, \tilde{t}) = \epsilon T_{rr}^{0,0}(\tilde{z}, \tilde{t}) + O(\epsilon^2) \]
Equations (3.17) and (3.18) are inserted into eqs. (3.7), (3.8), (3.9), (3.11), (3.12), (3.13), and (3.14) and the coefficients of the leading order terms in $\epsilon$ and $\tilde{r}$ are extracted. The leading order equations of 1-D slender-jet model are:

The mass equation:

$$2v_r^{0,0} + v_{z,z}^{0,0} = 0. \quad (3.19)$$

The momentum equations:

$$T_{rr}^{0,0} - T_{\theta\theta}^{0,0} = 0, \quad (3.20)$$

$$Tv_{z,z}^{0,0} + v_z^{0,0}v_{z,z}^{0,0} = B(T_{zz,z}^{0,0} + 2T_{zz}^{0,0} - p_{z,z}^{0,0}) + \frac{1}{r}. \quad (3.21)$$

The kinematic boundary condition:

$$-T\phi_z^{(0)} + \phi_z^{(0)}v_r^{0,0} - v_z^{0,0}\phi_z^{(0)} = 0. \quad (3.22)$$

The kinetic boundary conditions:

$$T_{rr}^{0,0} - p^{0,0} + P_a = -\frac{1}{BW} \left( \frac{1}{\phi^{(0)}} \right), \quad (3.23)$$

$$\phi_z^{(0)}T_{zz}^{0,0} + \phi_z^{(0)}(p^{0,0} - P_a - T_{zz}^{0,0}) = \frac{1}{BW} \left( \frac{\phi_z^{(0)}}{\phi^{(0)}} \right). \quad (3.24)$$

These dimensionless equations are called 1-D since the unknowns $v_r^{0,0}$, $v_z^{0,0}$, $T_{rr}^{0,0}$, $T_{\theta\theta}^{0,0}$, $T_{zz}^{0,0}$, $T_{rz}^{0,0}$, $p^{0,0}$, and $\phi^{(0)}$ are functions of the single spatial coordinate $\tilde{z}$ (not $\tilde{r}$ and $\tilde{\theta}$).

The dimensionless combinations appearing in the equations are

$$B = \frac{f_0}{\rho r_0^2 v_0^2}, \quad W = \frac{\rho r_0 v_0^2}{\sigma}, \quad F = \frac{v_0}{g z_0}, \quad T = \frac{z_0}{v_0 t_0}, \quad P_a = \frac{p_a r_0^2}{f_0}. \quad (3.25)$$
The eighteen experiments are conducted so that they steady in time. The steady forms of eqs (3.19) - (3.24) are

The mass equation:

$$-\frac{\phi(0)}{2} v_{z,0}^0 = v_{z,0}^0 \phi(z,0).$$  \hspace{1cm} (3.26)

The momentum equation:

$$\left(T_{zz}^0 - T_{rr}^0\right)_{,z} + \frac{1}{BW} \frac{\phi(z,0)}{\phi(0)^2} \phi(z,0) + 2(T_{zz}^0 - T_{rr}^0) \frac{\phi(z,0)}{\phi(0)} + \frac{1}{BF} + 2 \frac{\phi(z,0)^2}{B} \frac{\phi(z,0)}{\phi(0)} = 0,$$  \hspace{1cm} (3.27)

where $v_r^0, T_{\theta\theta}^0, T_{zz}^0$ and $p_r^0$ have been algebraically eliminated. The dimensionless 1-D equations (3.26) and (3.27) include the slenderness parameter $\epsilon = r_0/z_0$ and the dimensionless measures $(BW)^{-1}$, $(BF)^{-1}$, and $B^{-1}$ of the importance of surface tension, gravity, and inertia relative to the characteristic force $f_0$.

The characteristic scales are selected as follows: $r_0$ is selected to be the upstream radius $\phi(0)$ of the filament, $z_0$ to be its length $L$, $v_0 = Q/\pi \phi^2(0)$ to be the axial velocity at $z = 0$, and $f_0$ for a windup experiment to be the LVDT force difference $\delta$.

Table 3.1 gives the formulas (3.25) for dimensionless numbers in terms of the material properties $\rho, \sigma, \eta_0, g$ and measured conditions $Q, \phi(0), L, \delta$, and their values for each windup experiment, as well as the ratio $Z + S$ of viscous force, defined as the zero strain rate viscosity times $Q/\pi L$, to characteristic force $\delta$.

With choices $r_0 = \phi(0), v_0 = Q/\pi \phi^2(0)$ of scales, eq. (3.26) is integrated to obtain

$$v_z^0 \phi(z,0)^2 = 1.$$  \hspace{1cm} (3.28)
3.2 Derivations of spinline force, and upstream force, normal stress difference, and normal stress gradient

Equations (3.17) and (3.18) are inserted into definition (3.15); the leading order component of axial force $F_{z}^{(0)}$ is seen to be

$$F_{z}^{(0)} = \pi(T_{zz}^{0,0} - p^{0,0} + P_{a})\phi^{(0)2} = \pi(T_{zz}^{0,0} - T_{rr}^{0,0} - \frac{1}{BW}\phi^{(0)})\phi^{(0)2},$$

(3.29)

where eq. (3.23) has been used to eliminate $p^{0,0}$.

Using eq. (3.29), the momentum equation (3.27) can be rewritten in terms of the axial force gradient $F_{z,\tilde{z}}^{(0)}$:

$$F_{z,\tilde{z}}^{(0)} + \frac{2\pi}{BW}\phi^{(0)} + \frac{\pi}{BF}\phi^{(0)2} + 2\frac{\pi}{B}v_{z}^{0,0}\phi^{(0)}\phi^{(0)} = 0.$$  (3.30)

The integrated form of eq. (3.30) is

$$F_{z}^{(0)}(\tilde{z}) = F_{z}^{(0)}(0) - \frac{2\pi}{BW}(\phi^{(0)}(\tilde{z}) - \phi^{(0)}(0)) - \frac{\pi}{BF}\int_{0}^{\tilde{z}}\phi^{(0)2}d\tilde{z} + \frac{\pi}{B}\left[\phi^{(0)2}(\tilde{z}) - \phi^{(0)2}(0)\right].$$

(3.31)

It can be seen from Table 3.1 that all eighteen experiments the value of $B^{-1}$ is much smaller than the values of $(BF)^{-1}$ and $(BW)^{-1}$; therefore in each of the experiments reported here surface tension $(BW)^{-1}$ and gravity $(BF)^{-1}$ are dominant effects in our experiment whereas inertia $B^{-1}$ is weak, so that the inertia term can safely be neglected, and eqs. (3.27), (3.30), and (3.31) reduce to

$$\left(T_{zz}^{0,0} - T_{rr}^{0,0}\right)_{,\tilde{z}} + \frac{1}{BW}\frac{\phi^{(0)}}{\phi^{(0)2}} + 2\left(T_{zz}^{0,0} - T_{rr}^{0,0}\right)\phi^{(0)}\phi^{(0)} + \frac{1}{BF} = 0,$$

(3.32)

$$F_{z,\tilde{z}}^{(0)} + \frac{2\pi}{BW}\phi^{(0)} + \frac{\pi}{BF}\phi^{(0)2} = 0,$$

(3.33)

$$F_{z}^{(0)}(\tilde{z}) = F_{z}^{(0)}(0) - \frac{2\pi}{BW}(\phi^{(0)}(\tilde{z}) - \phi^{(0)}(0)) - \frac{\pi}{BF}\int_{0}^{\tilde{z}}\phi^{(0)2}d\tilde{z}.$$  (3.34)

As discussed in Chapter 2, for each pair of flow rate and nominal filament lengths, force and profile measurement were recorded both with and without windup. This is
so that the upstream boundary condition $F_z(0)$ at $z = 0$ can be deduced from the LVDT measurement of force difference $\delta$ as follows:

The notations $L$, $\phi$, and $F_z$ are used for a filament length, profile, and tensile force with windup, and $L^{nw}$, $\phi^{nw}$, $L^{nw}$, and $F_z^{nw}$ for a filament length, profile, and force without windup. In a no-windup case the jet is disrupted by a scraper at the take-up location, so that the dimensionless axial force $F_z^{nw}(0)(\frac{L^{nw}}{L})$ at the take-up location is zero. Hence, evaluating eq. (3.34) for a no-windup case at $\bar{z} = \frac{L^{nw}}{L}$ gives:

$$F_z^{nw}(0)(\frac{L^{nw}}{L}) = 0 = F_z^{nw}(0)(0) - 2 \frac{\pi}{BW} \left[ \bar{\phi}^{nw}(0)(\frac{L^{nw}}{L}) - \phi^{nw}(0)(0) \right] - \frac{\pi}{BF} \int_0^{L^{nw}} \phi^{nw}(0)^2 d\bar{z},$$

(3.35)

so that in a no-windup experiment the upstream axial force $F_z^{nw}(0)(0)$ is balanced by surface tension and the weight of the filament. Then the upstream axial force $F_z^{(0)}(0)$ in a windup case is

$$F_z^{(0)}(0) = 1 + F_z^{nw}(0)(0) = 1 + 2 \frac{\pi}{BW} \left[ \phi^{nw}(0)(\frac{L^{nw}}{L}) - \phi^{nw}(0)(0) \right] + \frac{\pi}{BF} \int_0^{L^{nw}} \phi^{nw}(0)^2 d\bar{z}.$$ 

(3.36)

Knowing the upstream force $F_z^{(0)}(0)$, the upstream normal stress difference is derived from eq. (3.29):

$$(T_{zz}^{(0)} - T_{rr}^{(0)})(0) = \frac{F_z^{(0)}(0)}{\pi} + \frac{1}{BW},$$

(3.37)

so that

$$(T_{zz}^{(0)} - T_{rr}^{(0)})(0) = \frac{1}{\pi} + \frac{2}{BW} \left[ \phi^{nw}(0)(\frac{L^{nw}}{L}) - \phi^{nw}(0)(0) \right] + \frac{1}{BF} \int_0^{L^{nw}} \phi^{nw}(0)^2 d\bar{z} + \frac{1}{BW}.$$ 

(3.38)
The upstream gradient of the normal stress difference is derived by evaluating eqs. (3.32) at \( \tilde{z} = 0 \):

\[
(T^{0,0}_{zz} - T^{0,0}_{rr}) - \tilde{t}ildex(0) = -2(0)(T^{0,0}_{zz} - T^{0,0}_{rr})(0) - \frac{\phi^{(0)}(0)}{BW} - \frac{1}{BF}. \tag{3.39}
\]

Note that axial force \( F^{(0)}(\tilde{z}) \), normal stress difference \( (T^{0,0}_{zz} - T^{0,0}_{rr})(\tilde{z}) \), and stress gradient \( (T^{0,0}_{zz,\tilde{z}} - T^{0,0}_{rr,\tilde{z}})(\tilde{z}) \) are computed for all \( 0 \leq \tilde{z} \leq 1 \), including \( \tilde{z} = 0 \), from experimental profile measurements and momentum considerations alone, independent of any assumption of constitutive material response.

Thereafter, the symbols \( \tilde{T}_{rr}, \tilde{T}_{zz}, \tilde{\phi}, \) and \( \tilde{v}_z \) are used to represent the leading order contributors \( T^{0,0}_{rr}, T^{0,0}_{zz}, \phi^{(0)} \), and \( v^{2,0}_z \) of radial stress, axial stress, free surface profile, and axial velocity, respectively.
<table>
<thead>
<tr>
<th>exp.</th>
<th>$\epsilon$</th>
<th>$\frac{BL^2}{\epsilon^2}$</th>
<th>$\frac{\rho Q^2}{\Delta f(\epsilon)}$</th>
<th>$\frac{\rho L\Delta f(\epsilon)}{\delta}$</th>
<th>$\frac{\rho L^2}{\delta}$</th>
<th>$\frac{\rho Q}{\pi L^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-1-3</td>
<td>$1.807 \times 10^{-2}$</td>
<td>$1.411 \times 10^{-5}$</td>
<td>$1.043 \times 10^{0}$</td>
<td>$1.503 \times 10^{-2}$</td>
<td>$4.721 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>4-1-3a</td>
<td>$1.905 \times 10^{-2}$</td>
<td>$1.411 \times 10^{-5}$</td>
<td>$0.989 \times 10^{0}$</td>
<td>$1.503 \times 10^{-2}$</td>
<td>$4.979 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>4-2-3</td>
<td>$1.477 \times 10^{-2}$</td>
<td>$5.695 \times 10^{-5}$</td>
<td>$2.322 \times 10^{-1}$</td>
<td>$3.761 \times 10^{-3}$</td>
<td>$1.328 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>4-3-3</td>
<td>$1.613 \times 10^{-2}$</td>
<td>$3.710 \times 10^{-3}$</td>
<td>$9.927 \times 10^{-2}$</td>
<td>$2.006 \times 10^{-3}$</td>
<td>$8.844 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>4-4-3</td>
<td>$1.257 \times 10^{-2}$</td>
<td>$2.898 \times 10^{-5}$</td>
<td>$6.139 \times 10^{-2}$</td>
<td>$1.173 \times 10^{-3}$</td>
<td>$4.886 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>4-5-3</td>
<td>$1.692 \times 10^{-2}$</td>
<td>$3.260 \times 10^{-5}$</td>
<td>$1.465 \times 10^{-2}$</td>
<td>$6.218 \times 10^{-4}$</td>
<td>$5.757 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>4-6-3</td>
<td>$1.007 \times 10^{-2}$</td>
<td>$2.062 \times 10^{-5}$</td>
<td>$1.990 \times 10^{-2}$</td>
<td>$4.559 \times 10^{-4}$</td>
<td>$2.278 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>4-6-3a</td>
<td>$1.118 \times 10^{-2}$</td>
<td>$2.062 \times 10^{-5}$</td>
<td>$1.793 \times 10^{-2}$</td>
<td>$4.559 \times 10^{-4}$</td>
<td>$2.529 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>4-7-3</td>
<td>$1.037 \times 10^{-2}$</td>
<td>$1.705 \times 10^{-8}$</td>
<td>$1.570 \times 10^{-2}$</td>
<td>$3.729 \times 10^{-4}$</td>
<td>$1.932 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>4-8-3</td>
<td>$9.789 \times 10^{-3}$</td>
<td>$1.602 \times 10^{-5}$</td>
<td>$1.234 \times 10^{-2}$</td>
<td>$3.041 \times 10^{-4}$</td>
<td>$1.634 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>4-8-3a</td>
<td>$1.181 \times 10^{-2}$</td>
<td>$1.602 \times 10^{-5}$</td>
<td>$1.023 \times 10^{-2}$</td>
<td>$3.041 \times 10^{-4}$</td>
<td>$1.972 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>4-9-3x</td>
<td>$1.353 \times 10^{-2}$</td>
<td>$3.829 \times 10^{-5}$</td>
<td>$1.112 \times 10^{-1}$</td>
<td>$1.958 \times 10^{-3}$</td>
<td>$7.513 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>2-3-3</td>
<td>$1.115 \times 10^{-2}$</td>
<td>$2.191 \times 10^{-5}$</td>
<td>$1.386 \times 10^{-1}$</td>
<td>$2.770 \times 10^{-3}$</td>
<td>$6.036 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>1-3-3</td>
<td>$9.279 \times 10^{-3}$</td>
<td>$1.629 \times 10^{-8}$</td>
<td>$1.608 \times 10^{-1}$</td>
<td>$4.192 \times 10^{-3}$</td>
<td>$5.961 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>8-3-3</td>
<td>$1.706 \times 10^{-2}$</td>
<td>$8.907 \times 10^{-5}$</td>
<td>$1.361 \times 10^{-1}$</td>
<td>$2.044 \times 10^{-3}$</td>
<td>$1.339 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>4-3-1</td>
<td>$3.722 \times 10^{-2}$</td>
<td>$2.702 \times 10^{-7}$</td>
<td>$3.256 \times 10^{-3}$</td>
<td>$3.755 \times 10^{-4}$</td>
<td>$9.445 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>4-3-2</td>
<td>$2.119 \times 10^{-2}$</td>
<td>$2.645 \times 10^{-8}$</td>
<td>$2.891 \times 10^{-2}$</td>
<td>$9.843 \times 10^{-2}$</td>
<td>$7.310 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>4-3-4</td>
<td>$1.053 \times 10^{-2}$</td>
<td>$5.927 \times 10^{-6}$</td>
<td>$3.003 \times 10^{-1}$</td>
<td>$3.639 \times 10^{-3}$</td>
<td>$9.616 \times 10^{-5}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: Dimensionless numbers for each experiment, deduced from the measured test fluid properties and scales of Table 2.1. $\epsilon$ is the slenderness parameter of the filament, and $B^{-1}, B_{eff}^{-1}, B_{eff}^{-1}$, and $Z + S$ are measures of the effects of inertia, gravity, surface tension, and viscosity relative to the characteristic axial force, respectively.
CHAPTER 4

MANIPULATION OF EXPERIMENT DATA

In this chapter the experiment data from the free surface measurements are manipulated to produce the kinematical quantities, for instance free surface radius, slope, axial velocity, velocity gradient, and second velocity gradient. Section 4.1 describes a treatment of a typical no windup experiment. Section 4.2 demonstrates two different techniques, curve fitting and filtering, employed to a typical windup experiment to compute the first and second spatial derivatives of the experimentally measured profile, polynomial curve fitting of the experimental data followed by analytical differentiation, and filtering of the data followed by numerical differentiation.

4.1 A typical no-windup case: Experiment 4-0-3

The no-windup experiments are used only to compute the upstream spinline force \( F_z^{(0)}(0) \) in the windup experiments, via eq. (3.36). In eq. (3.36) only the free surface profile \( \phi^{nw(0)}(z) \) of the no-windup experiment is employed, not any spatial derivatives. This section describes a treatment of experiment 4-0-3, which is typical of how all no-windup experiments are treated.

Figure 4.1, the free surface profile from Exp. 4-0-3, is an overlay of 81 digitized images. Note that all figures in this chapter have reverted to dimensional quantities.
Each original image is 448 pixels wide in axial direction; before overlaying 24 pixels are removed from both sides of each image to avoid possible optical aberration. Hence fig. 4.1 is an overlay of 81 images, each 400 pixels wide. In overlaying the 81 images, there are axial sub domains in which the measurement from two or three images coincide at the same pixel, since the translation \( P = 127 \) pixels between each image is less than the image width 400 pixels and the shift is an integer number of pixels. In these sub domains the radius is defined to be the average. Pixels are converted to length through the calibration factor \( C = 0.010732 \) m/pixel, so that in the overlay each digitized profile translates a distance \( \Delta \bar{z} \) downstream from the previous one, with \( \Delta \bar{z} = CP/L^w = 1.2029 \times 10^{-2} \) m/m. In the integration in eq. (3.36) \( \bar{\phi}^w \) is consider as a piecewise constant function.

### 4.2 A typical windup case: Experiment 4-1-3

In this research two different methods for material characterization are pursued (see chapter 6 to follow). These methods employ different kinematical quantities deduced from the experimental measurement of the free surface profile \( \bar{\phi}(z) \) of the take-up experiment: For both Method 1 and 2 need to know (i) \( \bar{\phi}(0), \bar{\phi}_z(0), \bar{v}_z(0), \bar{v}_{zz}(0), \), and \( \bar{v}_{z,zz}(0) \) at the upstream boundary \( \bar{z} = 0 \), and (ii) \( \bar{\phi}(\bar{z}), \bar{\phi}_z(\bar{z}) \) as function of \( \bar{z} \) along the length of filament, \( 0 \leq \bar{z} \leq 1 \). For Method 2 also need to know (iii) \( \bar{v}_z(\bar{z}), \bar{v}_{zz}(\bar{z}), \) and \( \bar{v}_{z,zz}(\bar{z}) \) as functions along the length of the filament. To obtain these kinematical quantities (i), (ii), and (iii) from the free surface measurements of the experiments, two different techniques are employed: polynomial curve fitting [10] and filtering [30] [35].
In the following two subsections the polynomial curve fitting and filtering, respectively, of the measured data of experiment 4-1-3, which is typical of all windup experiments are described. For the windup experiments the experimentally measured profile \( \tilde{\phi}^{\exp}(\tilde{z}) \) is assembled the same way discussed above for the no-windup case; in particular, the experiment data in \( \tilde{\phi}^{\exp}(\tilde{z}) \) fig. 4.2 consists of 75 images overlayed in intervals \( \Delta \tilde{z} = CP/L = 1.2945 \times 10^{-2} \text{ m/m} \), (note: table 2.1 shows that the values of \( C, P \) and \( L \) vary from experiment to experiment). In the experimental measurement of the free surface profile the total number of pixels in the axial domain of the filament is \( N = P(n-1) + 400 \), where \( P \) is the translation in pixels between successive images and \( n \) is the number of frames. The axial location of the \( k \)-th pixel is denoted by \( \tilde{z}_k \), and the experimental measurement of free surface radius at that location by \( \tilde{\phi}^{\exp}(\tilde{z}_k) \).

Recall that there are some axial sub domains in which the measurement from two or three images coincide at the same pixel; in these sub domains the radius is defined to be the average.

Each digitized frame for dimensionless free surface radius is used to construct the axial velocity \( \tilde{v}_f^{\exp} = 1/(\tilde{\phi}^{\exp})^2 \) (see eq. (3.28)); fig. 4.4 displays an overlay the 75 images. Each original image is 448 pixels wide but again 24 pixels are removed from each side.

4.2.1 Polynomial curve fitting

When employing polynomial curve fitting, if a formula calls for free surface radius \( \tilde{\phi} \) or axial velocity \( \tilde{v}_z \), the raw experimental measurements \( \tilde{\phi}^{\exp} \) and \( \tilde{v}_z^{\exp} \) are employed directly. The free surface slope \( \tilde{\phi}_{z,i}^{\text{ploy}}(\tilde{z}) \) is obtained from the measured free surface profile \( \tilde{\phi}^{\exp} \) as follows: Each profile image is fitted by a 5th degree polynomial; in this
experiment, 

\[ \tilde{\phi}_i(x) = \tilde{a}_{i0} + \tilde{a}_{i1}\tilde{x} + \tilde{a}_{i2}\tilde{x}^2 + \tilde{a}_{i3}\tilde{x}^3 + \tilde{a}_{i4}\tilde{x}^4 + \tilde{a}_{i5}\tilde{x}^5, \quad 0 \leq \tilde{x} \leq 400 \text{ pixel } C/L = 4.1759 \times 10^{-2}, \]  

(4.1)

where \( i \) is the frame number (\( i = 1, 2, \ldots, 75 \) in this experiment), and \( \tilde{x} \) is the local coordinate. (Note that if the polynomial is of degree less than 5 then it cannot fit the variation of an individual image well; if the degree is greater than 5, the derivative has too much oscillation.) The coefficients \( \tilde{a}_{ij} \) are computed from all 448 data points \( \tilde{z}_k \) in the image using a least square fit, but after fitting 24 pixels are removed from each side as above, to eliminate possible optical aberration at the edges of the image. This results in the axial domain \( 0 \leq \tilde{x} \leq 400 \text{ pixel } C/L = 4.1759 \times 10^{-2} \) m/m given in eq. (4.1). The slope is obtained by first analytically computing the slope in each frame from the 5th degree polynomial fit (4.1) of the radius,

\[ \frac{d\tilde{\phi}_i}{d\tilde{x}}(\tilde{x}) = \tilde{a}_{i1} + 2\tilde{a}_{i2}\tilde{x} + 3\tilde{a}_{i3}\tilde{x}^2 + 4\tilde{a}_{i4}\tilde{x}^3 + 5\tilde{a}_{i5}\tilde{x}^4, \quad 0 \leq \tilde{x} \leq 400 \text{ pixel } C/L = 4.1759 \times 10^{-2}, \]  

(4.2)

then converting from the local coordinate \( \tilde{x} \) to the global coordinate \( \tilde{z} \) in each frame,

\[ \frac{d\tilde{\phi}_i}{d\tilde{z}}(\tilde{z}) = \begin{cases} \frac{d\tilde{\phi}_i}{d\tilde{x}}(\tilde{x} + (i - 1)\Delta\tilde{z}) & \text{for } (i - 1)\Delta\tilde{z} \leq \tilde{z} \leq i\Delta\tilde{z} \\ 0 & \text{otherwise} \end{cases}, \]  

(4.3)

where \( \tilde{z} \) spans the entire length of the filament from 0 to 1 . The 75 functions \( \frac{d\tilde{\phi}_i}{d\tilde{z}}(\tilde{z}) \) are overlayed; except for the first and last \( \Delta\tilde{z} \), along the length of the filament either two or three frames overlap at each location \( \tilde{z} \) since the translation distance \( \Delta\tilde{z} = 1.2945 \times 10^{-2} \) m/m between frames is less than width \( (4.1759 \times 10^{-2} \) m/m) of each frame. The new polynomial is defined

\[ \tilde{\phi}_i^{\text{poly}}(\tilde{z}) = 0.5\frac{d\tilde{\phi}_i}{d\tilde{z}}(\tilde{z}) + 0.5\frac{d\tilde{\phi}_{i+1}}{d\tilde{z}}(\tilde{z}) \]  

(4.4)
for the axial domains where two frame overlap, and the polynomial

$$\tilde{\phi}_{i,\tilde{z}}^{\text{poly}}(\tilde{z}) = 0.25 \frac{d\phi_{i-1}}{d\tilde{z}}(\tilde{z}) + 0.5 \frac{d\phi_{i}}{d\tilde{z}}(\tilde{z}) + 0.25 \frac{d\phi_{i+1}}{d\tilde{z}}(\tilde{z})$$  \hspace{1cm} (4.5)

for the axial domains where three frames overlap. Note that the assembled slope is a piecewise continuous function, with discontinuities at the beginning and end of each frame in the overlay (at these locations the profile changes from the average of two frames to the average of three frames, or vice versa). Figure 4.3 displays the slope $\tilde{\phi}_{i,\tilde{z}}^{\text{poly}}$ generated by the polynomial fit of the filament free surface. When employing polynomial fitting, if a formula calls for $\tilde{\phi}_{i,\tilde{z}}$, the data $\tilde{\phi}_{i,\tilde{z}}^{\text{poly}}(\tilde{z})$ of fig. 4.3 is used.

Although in principle the velocity gradient $\tilde{v}_{z,\tilde{z}}$ may be obtained from the measured free surface radius $\tilde{\phi}^{\text{exp}}$ and calculated slope $\tilde{\phi}_{i,\tilde{z}}^{\text{poly}}$ of fig. 4.2 and 4.3 through $\tilde{v}_{z,\tilde{z}} = -2\frac{d\phi}{d\tilde{z}}$ (the derivative of eq. (3.28)), in practice this formula amplifies the experimental noise too much downstream near the take-up location. This is because as the radius $\tilde{\phi}^{\text{exp}}$ gets small, the noise becomes greater as a fraction of $\tilde{\phi}^{\text{exp}}$, and this noise amplifies when $\tilde{\phi}^{\text{exp}}$ is put in the denominator and cubed. Instead, the function $\tilde{v}_{z,\tilde{z}}$ (and the function $\tilde{u}_{z,\tilde{z}}$) are obtained as follows:

Each piecewise constant image of $\tilde{v}_{z,\tilde{z}}^{\text{exp}}(\tilde{z})$ is fitted by a 3rd degree polynomial,

$$\tilde{v}_{i}(\tilde{z}) = c_{i0} + c_{i1}\tilde{x} + c_{i2}\tilde{x}^2 + c_{i3}\tilde{x}^3, \quad 0 \leq \tilde{x} \leq 400 \text{ pixel} \quad C/L = 4.1759 \times 10^{-2}, \hspace{1cm} (4.6)$$

where $i$ is the frame number, and $\tilde{x}$ is the local coordinate. (Note that if the polynomial is of degree less than 3 then it cannot fit the variation of an individual image well; if the degree is greater than 3, the derivative has too much oscillation.) The coefficients $c_{ij}$ are computed from all 448 data points using a least square fit. After the fit 24/448 of the image are removed from each side, to avoid possible optical aberration. The velocity gradient $\tilde{v}_{z,\tilde{z}}^{\text{poly}}(\tilde{z})$ shown in fig. 4.5 is obtained by first computing
the slope for each frame from the 3rd degree polynomial fit (4.6) of the velocity,

\[
\frac{d\bar{u}_i}{d\bar{x}} = \bar{c}_{i1} + 2\bar{c}_{i2}\bar{z} + 3\bar{c}_{i3}\bar{z}^2, \quad 0 \leq \bar{z} \leq 4.1759 \times 10^{-2},
\]

(4.7)
then converting from the local coordinate \( \bar{x} \) to the global coordinate \( \bar{z} \) in each frame,

\[
\frac{d\bar{u}_i}{d\bar{z}}(\bar{z}) = \begin{cases} 
\frac{d\bar{u}_i}{d\bar{x}}(\bar{x} + (i - 1)\Delta\bar{z}) & \text{for } (i - 1)\Delta\bar{z} \leq \bar{z} \leq i\Delta\bar{z} \\
0 & \text{otherwise},
\end{cases}
\]

(4.8)
where \( \bar{z} \) ranges from 0 to 1, and finally overlaying the 75 functions to produce \( \bar{v}_{z,i}^{\text{poly}}(\bar{z}) \).

When a formula calls for \( \bar{u}_{z,i} \), this piecewise continuous function is used.

Figure 4.5 also displays a globally smoothed plot \( \bar{v}_{z,i}^{\text{global}}(\bar{z}) \) of the velocity gradient \( \bar{v}_{z,i}^{\text{global}} \). This global continuous function is obtained by using a 10th degree polynomial \( h(\bar{z}) = \sum_{j=0}^{10} \bar{d}_j \bar{z}^j \), where \( \bar{z} \) is the global coordinate, and the coefficients \( \bar{d}_j \) are computed from all data points of \( \bar{v}_{z,i}^{\text{poly}}(\bar{z}) \). (Note that if the degree of the polynomial is less than 10, then the polynomial cannot adequately fit the global variation of the velocity gradient; if its degree is greater than 10 then the fitting experiences numerical instability.) The velocity gradient \( \bar{v}_{z,i}^{\text{global}}(\bar{z}) \) in fig. 4.5 is therefore a continuous function. It is not employed directly, but rather is used to obtain the second derivative \( \bar{v}_{z,i}^{\text{global}}(\bar{z}) \) shown in fig. 4.6. The \( \bar{v}_{z,i}^{\text{global}}(\bar{z}) \) is the second gradient of velocity, obtained by differentiating the globally continuous function \( \bar{v}_{z,i}^{\text{global}}(\bar{z}) \), i.e. \( \frac{d^2\bar{h}}{d\bar{z}^2}(\bar{z}) = \sum_{j=1}^{10} j\bar{d}_j \bar{z}^{j-1} \). Note that the second gradient \( \bar{v}_{z,i}^{\text{global}}(\bar{z}) \) is a continuous function. The second derivative of velocity \( \bar{v}_z \) is obtained from the globally continuous derivative of \( \bar{v}_{z,i}^{\text{global}} \) rather than piecewise continuous derivative of \( \bar{v}_{z,i}^{\text{poly}} \) because the direct differentiation of \( \bar{v}_{z,i}^{\text{poly}} \) is too noisy.

4.2.2 Filtering

The second way to obtain the functions \( \bar{\phi}_i, \bar{\phi}_z, \bar{u}_z, \bar{v}_{z,i}, \) and \( \bar{v}_{z,i} \) from the experimental measured free surface profile \( \bar{\phi}^{\text{expt}}(\bar{z}_k) \) in fig. 4.2 is filtering. Filtering removes
the high frequency noise of the experiment data, so that the fourth order Runge-Kutta method can be employed to integrate the differential equations with these data with more confidence (see section 5). The noisy experimental data \( \tilde{\phi}^{\exp}(z_k) \) is smoothed by filtering it with its neighboring values,

\[
\tilde{\phi}^{fil}(z_k) = \int_{-\infty}^{\infty} K_\epsilon(z - z_k)\tilde{\phi}^{exp,lin}(z)dz,
\]

where \( K_\epsilon \) is a kernel function with parameter \( \epsilon \), and \( \tilde{\phi}^{exp,lin}(z) \) is the piecewise linear function connecting the discrete values \( \tilde{\phi}^{\exp}(z_k) \). Note that \( \tilde{\phi}^{fil}(z_k) \), like \( \tilde{\phi}^{\exp}(z_k) \), is a discrete functions, but less noisy. After investigating several candidates, the truncated Gaussian function is selected as the kernel function:

\[
K_\epsilon(z) = \begin{cases} 
0 & \text{if } \tilde{z} < -3\epsilon, \\
\frac{1}{\sqrt{2\pi\epsilon}}e^{-(\frac{z}{\epsilon})^2} & \text{if } -3\epsilon \leq \tilde{z} \leq 3\epsilon \\
0 & \text{if } 3\epsilon < \tilde{z}.
\end{cases}
\]

If \( \epsilon \) is too small the filtered profile will retain too much of the noise of experimental data, and if it is too large one will lose local information; a good selection is \( \epsilon = 100C/L(= 0.1044 \times 10^{-2} \text{ in this experiment}) \). To fix the boundary value \( \tilde{\phi}^{fil}(z_1 = 0) \) of the filtered function to be the same as the boundary value \( \tilde{\phi}^{\exp}(z_1 = 0) \) of the experimental data, and to calculate the slope at the boundaries, one extends above the boundary \( z_1 = 0 \) through \( \tilde{\phi}^{\exp}(-z_{k-1}) = 2\tilde{\phi}^{\exp}(z_1) - \tilde{\phi}^{\exp}(z_k), k = 1, \ldots, 400 \), and beyond the boundary \( z_N = 1 \) through \( \tilde{\phi}^{\exp}(z_k) = \tilde{\phi}^{\exp}(z_N), k = N + 1, \ldots, N + 400 \). In fig. 4.2 the filtered profile is overlayed on the experiment data. When employing filtering, if a formula calls for \( \tilde{\phi} \) the filtered profile \( \tilde{\phi}^{fil}(z_k) \) is used.

The filtered profile \( \tilde{\phi}^{fil}(z_k) \) is smooth enough so that one can compute the derivative numerically,

\[
\tilde{\phi}^{fil}_{zz}(z_k) = \frac{L}{12C}[-\tilde{\phi}^{fil}(z_{k+2}) + 8\tilde{\phi}^{fil}(z_{k+1}) - 8\tilde{\phi}^{fil}(z_{k-1}) + \tilde{\phi}^{fil}(z_{k-2})],
\]

29
where $C$ is the calibration factor and $L$ is the axial length. In fig. 4.3 the numerical derivative $\dot{\phi}_{s,k}^{fil}(\tilde{z}_k)$ is overlayed on $\dot{\phi}_{s,k}^{poly}(\tilde{z})$ from the polynomial curve fitting approach. Even though $\dot{\phi}_{s,k}^{fil}(\tilde{z}_k)$ is a discrete function, the pixels are so dense as to make it appear continuous.

To obtain the filtered velocity, eq. (3.28) is used, so that $\tilde{\nu}^{fil}(\tilde{z}_k) = 1/(\dot{\phi}_{s,k}^{fil})^2(\tilde{z}_k)$. In fig. 4.4 the filtered velocity $\tilde{\nu}^{fil}$ is overlayed on $\tilde{\nu}^{exp}$. The velocity gradient is computed numerically through

$$\tilde{\nu}^{fil}(\tilde{z}_k) = \frac{L}{12C}[ -\tilde{\nu}^{fil}(\tilde{z}_{k+2}) + 8\tilde{\nu}^{fil}(\tilde{z}_{k+1}) - 8\tilde{\nu}^{fil}(\tilde{z}_{k-1}) + \tilde{\nu}^{fil}(\tilde{z}_{k-2})]. \quad (4.12)$$

In fig. 4.5 the numerical gradient $\tilde{\nu}_{s,k}^{fil}(\tilde{z})$ is overlayed on $\tilde{\nu}_{s,k}^{poly}(\tilde{z})$ from the polynomial curve fitting approach.

To calculate the second derivative $\tilde{\nu}_{s,k}^{fil}$, one first refilter $\tilde{\nu}_{s,k}^{fil}$ over a larger window $6\epsilon$,

$$\tilde{\nu}_{s,k}^{fil}(\tilde{z}_k) = \int_{-\infty}^{\infty} \tilde{K}_{6\epsilon}(\tilde{z} - \tilde{z}_k)\tilde{\nu}_{s,k}^{fil}(\tilde{z}_k - \tilde{z}) d\tilde{z}, \quad (4.13)$$

where $filfil$ denotes that the data has been filtered twice, and $\tilde{\nu}_{s,k}^{fil}(\tilde{z})$ is the piecewise linear function connecting the discrete values $\tilde{\nu}_{s,k}^{fil}(\tilde{z}_k)$; $\tilde{\nu}_{s,k}^{fil}(\tilde{z}_k)$ is shown in fig. 4.5. Then $\tilde{\nu}_{s,k}^{filfil}$ is the numerical derivative,

$$\tilde{\nu}_{s,k}^{filfil}(\tilde{z}_k) = \frac{L}{12C}[ -\tilde{\nu}_{s,k}^{filfil}(\tilde{z}_{k+2}) + 8\tilde{\nu}_{s,k}^{filfil}(\tilde{z}_{k+1}) - 8\tilde{\nu}_{s,k}^{filfil}(\tilde{z}_{k-1}) + \tilde{\nu}_{s,k}^{filfil}(\tilde{z}_{k-2})]. \quad (4.14)$$

In fig. 4.6 $\tilde{\nu}_{s,k}^{filfil}(\tilde{z})$ is overlayed on the global fit $\tilde{\nu}_{s,k}^{global}(\tilde{z})$; note the large difference.
Figure 4.1: Typical no-windup experiment, exp. 4-0-3: measured free surface profile $\phi^{nw}$. 
Figure 4.2: Typical windup experiment, exp. 4-1-3: measured free surface profile \( \phi_{\text{exp}}(z_k) \) and filtered profile \( \phi_{\text{filt}}(z_k) \). In the complete profile the discrete functions are indistinguishable; in the blow-up the raw data \( \phi_{\text{exp}}(z_k) \) is given as (○) and the filtered profile \( \phi_{\text{filt}}(z_k) \) by (×).
Figure 4.3: Typical windup experiment, exp. 4-1-3: piecewise continuous free surface slope $\phi_x^{\text{poly}}(z)$ (-----) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_x^{\text{fil}}(z_k)$ (---) generated by numerical differentiation of the filtered profile $\phi_x^{\text{fil}}(z_k)$.
Figure 4.4: Typical windup experiment, exp. 4-1-3: axial velocity $v_\text{exp}^z(z_k)$ generated by $Q/(\pi \phi \exp^2(z_k))$ and $v_\text{fil}^z(z_k)$ generated by $Q/(\pi \phi \mu^2(z_k))$. In the blow-up $v_\text{exp}^z(z_k)$ is given as (○) and the filtered profile $v_\text{fil}^z(z_k)$ by (×).
Figure 4.5: Typical windup experiment, exp. 4-1-3: piecewise continuous velocity gradient \( u_{z,z}^{\text{poly}}(z) \) (---) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient \( u_{z,z}^{\text{global}}(z) \) (・・・), discrete velocity gradient \( u_{z,z}^{\text{f1}}(z_k) \) (——) generated by the numerical differentiation of the filtered velocity \( u_z^{\text{f1}}(z_k) \), and refiltered velocity gradient \( u_{z,z}^{\text{f1f1}}(z_k) \) (・・・) generated by \( u_{z,z}^{\text{f1}}(z_k) \).
Figure 4.6: Typical windup experiment, exp. 4-1-3: continuous second gradient $v_{z,x}^{global}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,x}^{global}(z)$, and discrete second gradient $v_{z,xx}^{fil}(z_k)$ of velocity (—) generated from the numerical differentiation of refiltered velocity gradient $v_{z,x}^{fil}(z_k)$. 
CHAPTER 5

CALCULATION OF SPINLINE FORCE AND NORMAL STRESS DIFFERENCE: A SUPERIORITY OF FILTERING

5.1 Spinline force and normal stress difference for exp. 4-1-3

In section 3.2 two ways are afforded to deduce spinline force $\tilde{F}_z$ along the filament, namely via differential or integral forms (eqs. (3.33) and (3.34), respectively) of the momentum equation; this is the spinline force that must be present in the filament, from momentum considerations and independent of constitutive assumption, to balance gravity and surface tension. Although the two forms are mathematically equivalent, they will not in general produce the same output results, since eq. (3.33) involves numerical differentiation of experimental data, and eq. (3.34) involves numerical integration of experimental data. It is found that when filtered experimental data is used, the two ways to calculate spinline force are numerically as well as mathematically equivalent. When the data is fit to polynomial functions, the numerically equivalence is not as good. These results are now demonstrated for exp. 4-1-3:

As can be seen from the representative calculations displayed in fig. 5.1: the spinline forces for exp. 4-1-3 calculated from differential form (3.33) and integral form (3.34) using $\tilde{\phi}^f_{\mu}$ and $\tilde{\phi}^f_{\mu}$ from figs. 4.2 and 4.3 differ by at most 0.0027%. When
polynomial curve fitting of the experimental data are employed, the predictions of the two forms are still close but not as close, as can also be seen from fig. 5.2: the spinline forces for exp. 4-1-3 calculated both from the differential form (3.33) and integral form (3.34) with $\tilde{\Phi}^{exp}$ and $\tilde{\Phi}^{poly}$ from figs. 4.2 and 4.3 differ by up to 0.013%. (The upstream boundary value $\tilde{F}_z(0)$ for the four calculations are deduced from eq. (3.36), with $\tilde{\Phi}^{nw}(0)$ known from fig. 4.1.)

Section 3.2 also provides two ways to compute the normal stress difference $\tilde{T}_{zz} - \tilde{T}_{rr}$ along the spinline, either by integrating eq. (3.33) for the spinline force $\tilde{F}_z$ and then algebraically solving eq. (3.29) for $\tilde{T}_{zz} - \tilde{T}_{rr}$, or by integrating eq. (3.32) directly for $\tilde{T}_{zz} - \tilde{T}_{rr}$. Again, the two ways are mathematically equivalent, but they are not numerically equivalent: In the first way the experimental data is introduced twice, once as known functions $\tilde{\Phi}$ and $\tilde{\Phi}_z$ in eq. (3.33), and then again as the known functions $\tilde{\Phi}$ and $\tilde{\Phi}^{-1}$ in eq. (3.29). In the second way the experimental data in introduced once, as the known functions $\tilde{\Phi}_z / \tilde{\Phi}$ and $\tilde{\Phi}_z / \tilde{\Phi}^2$ in eq. (3.32). Figures 5.2 and 5.3 demonstrate (with the data of exp. 4-1-3) that the two mathematically equivalent ways to calculate the normal stress difference numerically produce the effectively identical results when filtering is employed, but somewhat different results when polynomial curve fitting is employed.

These two results reflect that $\tilde{\Phi}^{exp}$ and $\tilde{\Phi}^{poly}$ from figs. 4.2 and 4.3 contain enough high frequency noise to make numerical integration of the differential equations (3.32) and (3.33) using the fourth order Runge-Kutta method unreliable to some degree, whereas the filtering removes this high frequency noise, and one can integrate with confidence. *A result of this dissertation is:* Filtering of the experimentally measured profile produces data that is both more physical (removing the artificial high frequency
noise created by the digitations of the video recording) and more stable in numerical computations.

5.2 Measured and deduced kinematics and spinline force for the remaining experiments

Experiments 4-0-3x, 2-0-3, 1-0-3, 8-0-3, 4-0-1, 4-0-2, and 4-0-4 are the other no-windup cases; the data are manipulated from these experiments as described in section 4.1 for exp. 4-0-3. The resulting digitized free surface profiles are shown in Appendix A. Experiments 4-2-3, 4-3-3, 4-4-3, 4-5-3, 4-6-3, 4-7-3, 4-8-3, 4-3-3x, 2-3-3, 1-3-3, 8-3-3, 4-3-1, 4-3-2, 4-3-4 are the other windup cases; the data are manipulated from these experiments as described in section 4.2 for exp. 4-1-3. The resulting digitized free surface profile, slope, axial velocity, velocity gradient, second derivative of velocity, and spinline force are given for these experiments in Appendix A.
Figure 5.1: Typical windup experiment, exp. 4-1-3: spinline force calculated from the differential form (3.33) of the momentum equation using polynomial curve fitting of the experimental data (---), calculated from the integral form (3.34) of the momentum equation using polynomial fitting (---), calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (•••).
Figure 5.2: Typical windup experiment, exp. 4-1-3: normal stress difference calculated by directly integrating eq. (3.32) for the stress difference, using polynomial curve fitting (-----) and using filtering (•••); normal stress difference calculated by integrating eq. (3.34) for spinline force and then algebraically deducing the stress difference from eq. (3.29), using polynomial fitting (——) and using filtering (.). The two solutions using filtering are indistinguishable to the resolution of the enlarged window.
Figure 5.3: Typical windup experiment, exp. 4-1-3: difference between two stress calculations of fig. 11 when using polynomial curve fitting (—) and when using filtering (-----).
CHAPTER 6

INVERSE FORMULATIONS OF MATERIAL CHARACTERIZATION PROCEDURE I

In this chapter characterization procedure I is presented. In characterization procedure I, a constitutive form is proposed, and the optimal values of coefficients with in that form are determined through comparison with the experimental measurements. In section 6.1, seven constitutive forms are proposed. In section 6.2, three different methods for material characterization are described. In section 6.3, two formulations are presented to determine the values of the in-flow boundary stresses. In section 6.4, three optimal search algorithms are described, and two of them are employed.

To summarize: characterization procedure I can be implemented in each of three methods. In each method there is a choice of two formulations to determine the in-flow stress values, and a choice of two different search algorithms. The data can be either filtered or fit to polynomial forms (see Chapter 4) In this dissertation seven different constitutive forms are proposed. Hence characterization procedure I can be implemented in \( 3 \times 2 \times 2 \times 2 \times 7 = 168 \) different ways. In next chapter the results of 14 of these implementations are presented. In Chapter 7 there is also a discussion of which of the 168 different ways are best.
6.1 Proposed constitutive forms

In this dissertation seven different proposed constitutive equations are investigated; for each proposed form, characterization procedure I produces the optimal coefficients within that form. The constitutive equations are all finite deformational and viscoelastic, reflecting the elasticity of the test fluid and the large strains associated with the filament spinning experiment, and differential (rather than integral).

In general, for a differential viscoelastic model the Cauchy stress is given in the form

\[ T = T^s + T^p, \quad (6.1) \]

with

\[ T^s = 2\eta_s D, \quad T^p = \sum_{k=1}^{K} T^{p_k}, \quad (6.2) \]

where \( D \) is the symmetric part of velocity gradient, and \( \eta_s \) is the solvent viscosity. \( T^s \) is termed the solvent contribution to the stress and \( T^p \) is the polymer contribution; each of \( T^{p_k} \) has a distinct differential constitutive equation.

6.1.1 Single mode models

For \( K = 1 \) in eqs. (6.1) and (6.2) the model is called a single mode model. Four single mode models are investigated.

The single mode Oldroyd fluid-B model

The single mode Oldroyd fluid-B constitutive model \([7, 18, 21, 25]\) is

\[ \dot{T} = \dot{T}^s + \dot{T}^p, \quad \dot{T}^s = 2\eta_s D, \quad \dot{T}^p + \lambda \frac{D}{Dt} \dot{T}^p = 2\eta_p D, \quad (6.3) \]
where
\[
\frac{D}{Dt}(\cdot) = (\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla)(\cdot) + (\cdot)\mathbf{W} - \mathbf{W}(\cdot) - (\cdot)\mathbf{D} - \mathbf{D}(\cdot). \tag{6.4}
\]

For the steady axisymmetric torsionless flow of the filament, the leading order dimensionless form of (6.3) is
\[
\frac{\tilde{T}}{\tau_0} = \tilde{T}_{rr}^s + \tilde{T}_{rr}^p, \quad \frac{\tilde{T}}{\tau_0} = \tilde{T}_{zz}^s + \tilde{T}_{zz}^p
\]
\[
\tilde{T}_{rr}^s = -S\tilde{\nu}_{zz}, \quad \tilde{T}_{zz}^s = 2S\tilde{\nu}_{zz},
\]
\[
\tilde{T}_{rr}^p + \Lambda(\tilde{\nu}_{zz}^p + \tilde{\nu}_{zz}^p) = -Z\tilde{\nu}_{zz},
\]
\[
\tilde{T}_{zz}^p + \Lambda(\tilde{\nu}_{zz}^p + 2\tilde{\nu}_{zz}^p) = 2Z\tilde{\nu}_{zz}. \tag{6.5}
\]

Equation (6.3) can be collapsed to
\[
\dot{T} + \lambda \frac{D}{Dt} \dot{T} = 2\eta_s \lambda \frac{D}{Dt} \mathbf{D} + 2(\eta_s + \eta_p)\mathbf{D}. \tag{6.6}
\]

The coefficients within this form, to be determined by characterization procedure I, are the relaxation time \(\lambda\), viscosity \(\eta_s\) of the solvent, and the zero strain rate viscosity \(\eta_p\) of the polymer; \(\lambda, \eta_s,\) and \(\eta_p\) are all constant.

If \(\mathbf{D}\) is set equal to \(\frac{\tau}{2} \mathbf{e}_1 \otimes \mathbf{e}_2 + \frac{\tau}{2} \mathbf{e}_2 \otimes \mathbf{e}_1\), and the fluid is assumed to be a steady, homogeneous flow, eq. (6.3) reduced
\[
\begin{bmatrix}
T_{11}^s & T_{12}^s & 0 \\
T_{12}^s & T_{22}^s & 0 \\
0 & 0 & T_{33}^s
\end{bmatrix}
= \eta_s
\begin{bmatrix}
0 & \dot{\gamma} & 0 \\
\dot{\gamma} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
T_{11}^p & T_{12}^p & 0 \\
T_{12}^p & T_{22}^p & 0 \\
0 & 0 & T_{33}^p
\end{bmatrix}
- \lambda
\begin{bmatrix}
\dot{\gamma}T_{12}^p & \dot{\gamma}T_{22}^p & 0 \\
\dot{\gamma}T_{22}^p & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
= \eta_p
\begin{bmatrix}
0 & \dot{\gamma} & 0 \\
\dot{\gamma} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}. \tag{6.7}
\]

The solutions \(T_{12}^s\) and \(T_{12}^p\) of eq. (6.7) are
\[
T_{12}^s = \eta_s \dot{\gamma}, \quad T_{12}^p = \eta_p \dot{\gamma}. \tag{6.8}
\]
so that

\[ \eta_0 = \lim_{\gamma \to 0} \frac{T_{12}}{\dot{\gamma}} = \lim_{\gamma \to 0} \frac{T_{12}^p + T_{12}^p}{\dot{\gamma}} = \lim_{\gamma \to 0} \frac{\eta_s \dot{\gamma} + \eta_p \dot{\gamma}}{\dot{\gamma}} = \eta_s + \eta_p. \]  

(6.9)

Recall that \( \eta_0 \) of the test fluid is known from shear measurements to be 11.3 Pa s (see section 2.2). Hence the material coefficients in form (6.5) are constrained by \( \eta_0 = \eta_s + \eta_p = 11.3 \) Pa s.

For the steady axisymmetric torsionless flow of the filament, the leading order dimensionless form of the Oldroyd fluid-B constitutive model (6.6) is

\[ \tilde{T}_{rr} + \Lambda (\tilde{v}_z \tilde{T}_{rr,\tilde{z}} + \tilde{v}_{z,\tilde{z}} \tilde{T}_{rr}) = -\Lambda S (\tilde{v}_z \tilde{v}_{z,\tilde{z}} + \tilde{v}_z^2) - (S + Z) \tilde{v}_{z,\tilde{z}}, \]

\[ \tilde{T}_{zz} + \Lambda (\tilde{v}_z \tilde{T}_{zz,\tilde{z}} - 2\tilde{v}_{z,\tilde{z}} \tilde{T}_{zz}) = 2\Lambda S (\tilde{v}_z \tilde{v}_{z,\tilde{z}} - 2\tilde{v}_z^2) + 2(S + Z) \tilde{v}_{z,\tilde{z}}, \]  

(6.10)

where \( \Lambda = \frac{\lambda \eta_p}{\eta_s}, S = \frac{\eta_p \eta_s^2}{\eta_s \eta_p}, \) and \( Z = \frac{\eta_p \eta_s^2}{\eta_s \eta_p} \). Note that eqs. (6.10) are two uncoupled first order linear differential equations of \( \tilde{T}_{rr} \) and \( \tilde{T}_{zz} \), respectively.

**The single mode Giesekus model**

The single mode Giesekus constitutive model [7, 18, 21, 25] is

\[ \dot{T} = \dot{T}^s + \dot{T}^p, \quad \dot{T}^s = 2\eta_s \mathbf{D}, \quad \dot{T}^p + \lambda \frac{D}{Dt} \dot{T}^p + \alpha \frac{\lambda}{\eta_p} \dot{T}^p = 2\eta_p \mathbf{D}, \]  

(6.11)

or in the collapsed form,

\[ \dot{T} + \lambda \frac{D}{Dt} \dot{T} + \frac{\alpha \lambda}{\eta_p} \dot{T}^2 - 2 \frac{\alpha \lambda \eta_s}{\eta_p} (\dot{T} \mathbf{D} + \mathbf{D} \dot{T}) \]

\[ = 2\lambda \eta_s \frac{D}{Dt} \mathbf{D} + 2(\eta_s + \eta_p) \mathbf{D} - 4 \frac{\alpha \lambda \eta_s^2}{\eta_p} \mathbf{D}^2, \]  

(6.12)

where \( \dot{T} = \dot{T}^s + \dot{T}^p \). The coefficients in this form are the relaxation time \( \lambda \), the viscosity \( \eta_s \) of the solvent, he zero strain rate viscosity \( \eta_p \) of the polymer, and the mobility parameter \( \alpha \); \( \lambda, \eta_s, \eta_p, \) and \( \alpha \) are all constant.
If \( D \) is set equal to \( \frac{1}{2} e_1 \otimes e_2 + \frac{1}{2} e_2 \otimes e_1 \), and the fluid is assumed to be a steady, homogeneous flow, eq. (6.11) reduced

\[
\begin{bmatrix}
T_{11} \hspace{1cm} T_{12} \hspace{1cm} 0 \\
T_{12} \hspace{1cm} T_{22} \hspace{1cm} 0 \\
0 \hspace{1cm} 0 \hspace{1cm} T_{33}
\end{bmatrix} = \eta_s \begin{bmatrix}
0 \hspace{1cm} \dot{\gamma} \hspace{1cm} 0 \\
\dot{\gamma} \hspace{1cm} 0 \hspace{1cm} 0 \\
0 \hspace{1cm} 0 \hspace{1cm} 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
T_{11}^p \hspace{1cm} T_{12}^p \hspace{1cm} 0 \\
T_{12}^p \hspace{1cm} T_{22}^p \hspace{1cm} 0 \\
0 \hspace{1cm} 0 \hspace{1cm} T_{33}^p
\end{bmatrix} - \lambda \begin{bmatrix}
\dot{\gamma} T_{12}^p \hspace{1cm} \dot{\gamma} T_{22}^p \hspace{1cm} 0 \\
\dot{\gamma} T_{22}^p \hspace{1cm} 0 \hspace{1cm} 0 \\
0 \hspace{1cm} 0 \hspace{1cm} 0
\end{bmatrix}
\]

\[
+ \alpha \frac{\lambda}{\eta_p} \begin{bmatrix}
T_{11}^{p2} + T_{12}^{p2} \hspace{1cm} T_{11}^{p2} T_{12}^p + T_{22}^{p2} T_{12}^p \hspace{1cm} 0 \\
T_{12}^{p2} T_{11}^p + T_{22}^{p2} T_{12}^p \hspace{1cm} T_{22}^{p2} + T_{12}^{p2} \hspace{1cm} 0 \\
0 \hspace{1cm} 0 \hspace{1cm} T_{33}^{p2}
\end{bmatrix} = \eta_p \begin{bmatrix}
0 \hspace{1cm} \dot{\gamma} \hspace{1cm} 0 \\
\dot{\gamma} \hspace{1cm} 0 \hspace{1cm} 0 \\
0 \hspace{1cm} 0 \hspace{1cm} 0
\end{bmatrix}. \quad (6.13)
\]

The solutions \( T_{12}^s \) and \( T_{12}^p \) [7, 34] of (6.13) are

\[
T_{12}^s = \eta_s \dot{\gamma}, \quad T_{12}^p = \eta_p \dot{\gamma} \left[ \frac{(1 - \psi)^2}{1 + (1 - 2\alpha)\psi} \right], \quad (6.14)
\]

where

\[
\psi = \frac{1 - \chi}{1 + (1 - 2\alpha)\chi}, \quad (6.15)
\]

and

\[
\chi = \sqrt{\frac{(1 + 6\alpha(1 - \alpha)(\dot{\gamma}\lambda)^2)^{1/2} - 1}{8\alpha(1 - \alpha)(\dot{\gamma}\lambda)^2}}. \quad (6.16)
\]

Note that

\[
\lim_{\dot{\gamma} \to 0} \chi = 1, \quad \lim_{\chi \to 1} \psi = 0, \quad (6.17)
\]

so that

\[
\eta_0 = \lim_{\dot{\gamma} \to 0} \frac{T_{12}}{\dot{\gamma}} = \lim_{\dot{\gamma} \to 0} \frac{T_{12}^s + T_{12}^p}{\dot{\gamma}} = \lim_{\dot{\gamma} \to 0} \left[ \eta_s + \eta_p \frac{(1 - \psi)^2}{1 + (1 - 2\alpha)\psi} \right] = \eta_s + \eta_p. \quad (6.18)
\]

Hence the material coefficients in form (6.11) are constrained by \( \eta_0 = \eta_s + \eta_p = 11.3 \) Pa s.
For the steady axisymmetric torsionless flow of the filament, the leading order dimensionless form of the Giesekus constitutive model (6.12) is

$$
\begin{align*}
\tilde{T}_{rr} & = \Lambda (\tilde{u}_z \tilde{T}_{rr,z} + \tilde{v}_{z,z} \tilde{T}_{rr}) + \frac{\alpha \Lambda}{Z} \tilde{T}_{rr}^2 + \frac{2 \alpha \Lambda S}{Z} \tilde{v}_{z,z} \tilde{T}_{rr} \\
& = -\Lambda S (\tilde{u}_z \tilde{v}_{z,z} + \tilde{v}_{z,z}^2) - (S + Z) \tilde{v}_{z,z} - \frac{\alpha \Lambda S^2}{Z} \tilde{v}_{z,z}^2,
\end{align*}
$$

with

$$
\Lambda = \frac{\lambda u_0}{z_0}, \quad S = \frac{\eta_p r_0^2 u_0}{z_0 f_0}, \quad Z = \frac{\eta_p r_0^2 u_0}{z_0 f_0}.
$$

To avoid the singularity at $Z = 0$ in the search algorithm to follow in section 6.4, eqs. (6.19) are reformulated to:

$$
\begin{align*}
\tilde{T}_{rr} & = \Lambda (\tilde{u}_z \tilde{T}_{rr,z} + \tilde{v}_{z,z} \tilde{T}_{rr}) + \frac{\gamma \lambda f_0 \tilde{T}_{rr}^2}{r_0^2} + 2 \frac{\gamma \lambda f_0 S}{r_0^2} \tilde{v}_{z,z} \tilde{T}_{rr} \\
& = -\Lambda S (\tilde{u}_z \tilde{v}_{z,z} + \tilde{v}_{z,z}^2) - (S + Z) \tilde{v}_{z,z} - \frac{\gamma \lambda f_0 S^2}{r_0^2} \tilde{v}_{z,z}^2,
\end{align*}
$$

with $\gamma = \frac{\alpha}{\eta_p}$. Note that eqs. (6.20) are two uncoupled first order non-linear differential equations of $\tilde{T}_{rr}$ and $\tilde{T}_{zz}$, respectively.

**The single mode FENE-P model**

The single mode FENE-P (Finite Extendable Nonlinear Elastic) constitutive model [6, 7, 8, 18] is

$$
\begin{align*}
\tilde{T} & = \tilde{T}^* + \tilde{T}^p, \quad \tilde{T}^* = 2 \eta_p \tilde{D}, \\
\Psi \tilde{T}^p & + \lambda \frac{D}{Dt} \tilde{T}^p - \lambda \left[ \tilde{T}^p + (1 - \xi b) \frac{\eta_p \Psi}{\lambda} \right] \frac{d}{dt} \ln \Psi = 2(1 - \xi b) \eta_p \tilde{D}, \quad (6.21)
\end{align*}
$$
with

\[ \Psi = 1 + \frac{3}{b}(1 + \lambda \frac{\text{tr} \hat{T}^p}{3\eta_p}), \quad \xi = \frac{2}{b(b + 2)}, \]

where \( I \) is the unit tensor, \( \text{tr} \) is the trace of a second order tensor, and the material time derivative \( \frac{d}{dt} \) is defined by

\[ \frac{d}{dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + \mathbf{v} \cdot \nabla(\cdot). \tag{6.22} \]

The coefficients in this form are the relaxation time \( \lambda \), the viscosity \( \eta_s \) of the solvent, the zero strain rate viscosity \( \eta_p \) of the polymer, and the dumbbell value \( b \); \( \lambda, \eta_s, \eta_p, \) and \( b \) are all constant.

If \( D \) is set equal to \( \hat{\mathbf{e}}_1 \otimes \mathbf{e}_2 + \frac{1}{2} \mathbf{e}_2 \otimes \mathbf{e}_1 \), and the fluid is assumed to be a steady, homogeneous flow, eq. (6.29) reduced

\[
\begin{bmatrix}
T^p_{11} & T^p_{12} & 0 \\
T^p_{12} & T^p_{22} & 0 \\
0 & 0 & T^p_{33}
\end{bmatrix}
= \eta_p \begin{bmatrix} 0 & \dot{\gamma} & 0 \\
\dot{\gamma} & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}, \tag{6.23}
\]

where

\[ \Psi = 1 + \frac{3}{b} + \frac{\lambda}{b\eta_p}(T^p_{11} + T^p_{22} + T^p_{33}), \quad \xi = \frac{2}{b(b + 2)}. \]

The solutions \( T^p_{12} \) and \( T^p_{12} \) of (6.23) are

\[ T^p_{12} = \eta_p \dot{\gamma}, \quad T^p_{12} = \frac{(1 - \xi b)\dot{\gamma} \eta_p}{1 + \frac{3}{b} + \frac{\lambda}{b\eta_p} T^p_{11}}, \tag{6.24} \]

where \( T^p_{11} \) is a solution of the cubic equation,

\[ T^p_{11} + 2\frac{b\eta_p}{\lambda} (1 + \frac{3}{b}) T^p_{11}^2 + (\frac{b\eta_p}{\lambda})^2 (1 + \frac{3}{b})^2 T^p_{11} - 2(\frac{b\eta_p}{\lambda})^2 \eta_p \lambda (1 - \xi b) \dot{\gamma}^2 = 0. \tag{6.25} \]
The coefficients in this equation are defined such that this cubic equation always has one real root and two conjugate complex roots. This equation can be solved analytically using the formulas presented in [4]. The zero shear rate viscosity is

\[ \eta_0 = \lim_{\gamma \to 0} \frac{T_{12}}{\gamma} = \lim_{\gamma \to 0} \frac{T_{12}' + T_{12}''}{\gamma} \]

\[ = \lim_{\gamma \to 0} \left[ \eta_s + \eta_p \frac{1}{1 + \frac{b}{3} + \frac{\lambda}{b\eta_p} T_{11}''} \right] = \eta_s + \eta_p \frac{1}{1 + \frac{b}{3} + \frac{\lambda}{b\eta_p}}. \quad (6.26) \]

Hence the material coefficients in form (6.29) are constrained by \[ \eta_0 = \eta_s + \eta_p \frac{1}{1 + \frac{b}{3} + \frac{\lambda}{b\eta_p}} = 11.3 \text{ Pa s}. \]

For the steady axisymmetric torsionless flow of the filament, the leading order dimensionless form of the FENE-P constitutive model (6.21) is

\[ \ddot{T}_{rr} = \ddot{T}_{rr}' + \ddot{T}_{rr}'', \quad \ddot{T}_{zz} = \ddot{T}_{zz}' + \ddot{T}_{zz}'', \quad \ddot{T}_{rr}' = -S\ddot{v}_{s,3}, \quad \ddot{T}_{zz}' = 2S\ddot{v}_{s,3}, \]

\[ \Phi \ddot{T}_{rr}' + \Lambda(\ddot{v}_s \ddot{T}_{rr}', + \ddot{v}_{s,3} \ddot{T}_{rr}') - \frac{\Lambda^2}{bZ\Phi} \ddot{v}_s (\ddot{T}_{zz}' + 2\ddot{T}_{rr}') \left[ \ddot{T}_{rr}' + (1 - \xi b)\frac{Z}{\Lambda} \right] \]

\[ = -(1 - \xi b)Z\ddot{v}_{s,3}, \]

\[ \Phi \ddot{T}_{zz}' + \Lambda(\ddot{v}_s \ddot{T}_{zz}', - 2\ddot{v}_{s,3} \ddot{T}_{zz}') - \frac{\Lambda^2}{bZ\Phi} \ddot{v}_s (\ddot{T}_{zz}' + 2\ddot{T}_{rr}') \left[ \ddot{T}_{zz}' + (1 - \xi b)\frac{Z}{\Lambda} \right] \]

\[ = 2(1 - \xi b)Z\ddot{v}_{s,3}, \quad (6.27) \]

where

\[ \Phi = 1 + \frac{3}{b} \left[ 1 + \frac{\Lambda}{3Z}(\ddot{T}_{pp}' + 2\ddot{T}_{rr}') \right], \quad \xi = \frac{2}{b(b + 2)}, \]

\[ \Lambda = \frac{\lambda v_0}{z_0}, \quad S = \frac{\eta s f_0^2 v_0}{z_0 f_0}, \quad Z = \frac{\eta p f_0^2 v_0}{z_0 f_0}. \]

To avoid the singularity at \( Z = 0 \) when the optimal search algorithm is employed, eqs. (6.27) are reformulated to

\[ \ddot{T}_{rr} = \ddot{T}_{rr}' + \ddot{T}_{rr}'', \quad \ddot{T}_{zz} = \ddot{T}_{zz}' + \ddot{T}_{zz}'', \quad \ddot{T}_{rr}' = -S\ddot{v}_{s,3}, \quad \ddot{T}_{zz}' = 2S\ddot{v}_{s,3}, \]

50
\[ \Phi \tilde{T}_{rr} = \lambda (\tilde{v}_{z,\tilde{z},\tilde{z}} + \tilde{v}_{z,\tilde{z}} \tilde{T}_{rr}) - \Pi \frac{\Lambda f_0}{b \Phi_0} \tilde{v}_z (\tilde{T}_{zz,\tilde{z}} + 2 \tilde{T}_{rr,\tilde{z}}) \left[ \tilde{T}_{rr} + (1 - \xi b) \frac{Z}{\Lambda} \right] \]
\[ = -(1 - \xi b) Z \tilde{v}_z, \]
\[ \Phi \tilde{T}_{zz} + \lambda (\tilde{v}_z \tilde{T}_{zz,\tilde{z}} - 2 \tilde{v}_{z,\tilde{z}} \tilde{T}_{zz}) - \Pi \frac{\Lambda f_0}{b \Phi_0} \tilde{v}_z (\tilde{T}_{zz,\tilde{z}} + 2 \tilde{T}_{rr,\tilde{z}}) \left[ \tilde{T}_{zz} + (1 - \xi b) \frac{Z}{\Lambda} \right] \]
\[ = 2(1 - \xi b) Z \tilde{v}_z, \tag{6.28} \]

where
\[ \Phi = 1 + \frac{3}{b} \left[ 1 + \Pi \frac{f_0}{3 \eta_0} (\tilde{T}_{zz} + 2 \tilde{T}_{rr}) \right], \quad \xi = \frac{2}{b(b + 2)}, \quad \Pi = \frac{\lambda}{\eta_p}. \]

Note that eqs. (6.28) are two coupled first order non-linear differential equations of \( \tilde{T}_{rr} \) and \( \tilde{T}_{zz} \), respectively.

**The modified single mode FENE-P model**

The modified single mode FENE-P constitutive model is

\[ \dot{T} = \dot{T}^s + \dot{T}^p, \quad \dot{T}^s = 2 \eta_2 \dot{D}, \]
\[ \Psi \dot{T}^p + \lambda \frac{D}{Dt} \dot{T}^p - \lambda \left[ \dot{T}^p - (1 - \xi b) \frac{\eta_p \dot{T}^p}{\lambda} \right] \frac{d}{dt} \ln \Psi = 2(1 - \xi b) \eta_p \dot{D}, \tag{6.29} \]

with
\[ \Psi = 1 + \frac{3}{b} \left( 1 - \lambda \frac{tr \dot{T}^p}{3 \eta_p} \right), \quad \xi = \frac{2}{b(b + 2)}. \]

The coefficients in this form are the relaxation time \( \lambda \), the viscosity \( \eta_s \) of the solvent, the zero strain rate viscosity \( \eta_p \) of the polymer, and the dumbbell value \( b \); \( \lambda \), \( \eta_s \), \( \eta_p \), and \( b \) are all constant.

If \( \dot{D} \) is set equal to \( \frac{1}{2} \mathbf{e}_1 \otimes \mathbf{e}_2 + \frac{1}{2} \mathbf{e}_2 \otimes \mathbf{e}_1 \), and the fluid is assumed to be a steady, homogeneous flow, eq. (6.29) reduced

\[
\begin{bmatrix}
T_{11}^s & T_{12}^s & 0 \\
T_{12}^s & T_{22}^s & 0 \\
0 & 0 & T_{33}^s
\end{bmatrix} = \eta_s \begin{bmatrix}
0 & \dot{\gamma} & 0 \\
\dot{\gamma} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

51
\[
\begin{align*}
\Psi \begin{bmatrix}
T_{11}^p & T_{12}^p & 0 \\
T_{12}^p & T_{22}^p & 0 \\
0 & 0 & T_{33}^p
\end{bmatrix} - \lambda \begin{bmatrix}
\dot{\gamma} T_{12}^p & \dot{\gamma} T_{22}^p & 0 \\
\dot{\gamma} T_{12}^p & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} &= \eta_p (1 - \xi b) \begin{bmatrix}
0 & \dot{\gamma} & 0 \\
\dot{\gamma} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad (6.30)
\end{align*}
\]

where
\[
\Psi = 1 + \frac{3}{b} - \frac{\lambda}{b \eta_p} (T_{11}^p + T_{22}^p + T_{33}^p), \quad \xi = \frac{2}{b(b + 2)}.
\]

The solutions \(T_{12}^p\) and \(T_{12}^s\) of (6.30) are
\[
T_{12}^s = \eta_s \dot{\gamma}, \quad T_{12}^p = \frac{(1 - \xi b) \eta_p \dot{\gamma}}{1 + \frac{3}{b} - \frac{\lambda}{b \eta_p} T_{11}^p}, \quad (6.31)
\]

where \(T_{11}^p\) is a solution of the cubic equation,
\[
T_{11}^{p3} - 2 \frac{b \eta_p}{\lambda} (1 + \frac{3}{b}) T_{11}^{p2} + (\frac{b \eta_p}{\lambda})^2 (1 + \frac{3}{b}) T_{11}^p - 2 (\frac{b \eta_p}{\lambda})^2 \eta_p \lambda (1 - \xi b) \dot{\gamma}^2 = 0. \quad (6.32)
\]

The coefficients in this equation are defined such that this cubic equation has one real root and two conjugate complex roots, if
\[
\dot{\gamma}^2 < \frac{2b(1 + \frac{3}{b})^3}{27b^2(1 - \xi b)}. \quad (6.33)
\]

Again this equation can be solved analytically. The zero shear rate viscosity is
\[
\eta_0 = \lim_{\dot{\gamma} \to 0} \frac{T_{12}}{\dot{\gamma}} = \lim_{\dot{\gamma} \to 0} \frac{T_{12}^s + T_{12}^p}{\dot{\gamma}} = \lim_{\dot{\gamma} \to 0} \left[ \eta_s + \eta_p \frac{b}{1 + \frac{3}{b} + \frac{\lambda}{b \eta_p} T_{11}^p} \right] = \eta_s + \eta_p \frac{1}{1 + \frac{3}{b} + \frac{1}{b \eta_p}}. \quad (6.34)
\]

Hence the material coefficients in form (6.29) are constrained by \(\eta_0 = \eta_s + \eta_p \frac{1}{1 + \frac{3}{b} + \frac{1}{b \eta_p}} = 11.3 \text{ Pa s.}\)

For the steady axisymmetric torsionless flow of the filament, the leading order dimensionless form of this modified FENE-P constitutive model (6.29) is
\[
\tilde{T}_{rr} = \tilde{T}_{rr}^s + \tilde{T}_{rr}^p, \quad \tilde{T}_{zz} = \tilde{T}_{zz}^s + \tilde{T}_{zz}^p, \quad \tilde{T}_{rr}^s = -S \tilde{v}_{s,\tilde{z}}, \quad \tilde{T}_{zz}^s = 2S \tilde{v}_{s,\tilde{z}},
\]

\[52\]
\[
\Phi \ddot{T}_{rr} + \Lambda (\ddot{u}_z \ddot{T}_{rr,z} + \ddot{u}_z \ddot{T}_{rr}) + \frac{\Lambda^2}{b Z \Phi} \ddot{u}_z (\ddot{T}_{rz,z} + 2 \ddot{T}_{rr,z}) \left[ \ddot{T}_{rr} - (1 - \xi b) \frac{Z}{\Lambda} \right] \\
= -(1 - \xi b) Z \ddot{u}_z, \]
\[
\Phi \ddot{T}_{zz} + \Lambda (\ddot{u}_z \ddot{T}_{rz,z} - 2 \ddot{u}_z \ddot{T}_{zz}) + \frac{\Lambda^2}{b Z \Phi} \ddot{u}_z (\ddot{T}_{rz,z} + 2 \ddot{T}_{rr,z}) \left[ \ddot{T}_{zz} - (1 - \xi b) \frac{Z}{\Lambda} \right] \\
= 2(1 - \xi b) Z \ddot{u}_z, \tag{6.35}
\]

with
\[
\Phi = 1 + \frac{3}{b} \left[ 1 - \frac{\Lambda}{3Z}(\ddot{T}_{zz} + 2 \ddot{T}_{rr}) \right], \quad \xi = \frac{2}{b(b + 2)}, \\
\Lambda = \frac{\lambda v_0}{z_0}, \quad S = \frac{\eta_\alpha r_0 v_0}{z_0 f_0}, \quad Z = \frac{\eta_p r_0 v_0}{z_0 f_0}.
\]

To avoid the singularity at \(Z = 0\) when the optimal search algorithm is employed, eqs. (6.35) are reformulated to
\[
\ddot{T}_{rr} = \ddot{T}_{rr}^s + \ddot{T}_{rr}^p, \quad \ddot{T}_{zz} = \ddot{T}_{zz}^s + \ddot{T}_{zz}^p, \quad \ddot{T}_{rr}^p = -S \ddot{u}_z, \quad \ddot{T}_{zz}^p = 2S \ddot{u}_z
\]
\[
\Phi \ddot{T}_{rr}^p + \Lambda (\ddot{u}_z \ddot{T}_{rr,z}^p + \ddot{u}_z \ddot{T}_{rr}) + \Pi \frac{f_0}{b \Phi r_0^2} \ddot{u}_z (\ddot{T}_{zz,z}^p + 2 \ddot{T}_{rr,z}^p) \left[ \ddot{T}_{rr}^p - (1 - \xi b) \frac{Z}{\Lambda} \right] \\
= -(1 - \xi b) Z \ddot{u}_z
\]
\[
\Phi \ddot{T}_{zz}^p + \Lambda (\ddot{u}_z \ddot{T}_{rz,z}^p - 2 \ddot{u}_z \ddot{T}_{zz}^p) + \Pi \frac{f_0}{b \Phi r_0^2} \ddot{u}_z (\ddot{T}_{nz,z}^p + 2 \ddot{T}_{rr,z}^p) \left[ \ddot{T}_{zz}^p - (1 - \xi b) \frac{Z}{\Lambda} \right] \\
= 2(1 - \xi b) Z \ddot{u}_z, \tag{6.36}
\]

where
\[
\Phi = 1 + \frac{3}{b} \left[ 1 - \Pi \frac{f_0}{3r_0^2}(\ddot{T}_{zz} + 2 \ddot{T}_{rr}) \right], \quad \xi = \frac{2}{b(b + 2)}, \quad \Pi = \frac{\lambda}{\eta_p}.
\]

Note that eqs. (6.36) are two coupled first order non-linear differential equations of \(\ddot{T}_{rr}^p\) and \(\ddot{T}_{zz}^p\), respectively.

6.1.2 Multimode models

When \(K > 1\) in eqs. (6.1) and (6.2) the viscoelastic model is called a multimode model. Three multimode models are investigated.
Two-mode Oldroyd fluid-B model

The two-mode Oldroyd fluid-B model is

\[
\mathbf{T} = \mathbf{T}^s + \mathbf{T}^{p_1} + \mathbf{T}^{p_2},
\]
\[
\mathbf{T}^s = 2\eta_s \mathbf{D}, \quad \mathbf{T}^{p_k} + \lambda_k \frac{D\mathbf{T}^{p_k}}{Dt} = 2\eta_{p_k} \mathbf{D}, \quad k = 1, 2. \tag{6.37}
\]

The coefficients within this form are the relaxation times \(\lambda_1, \lambda_2\), viscosity \(\eta_s\) of the solvent, and the zero strain rate viscosities \(\eta_{p_1}, \eta_{p_2}\) of the polymer; \(\lambda_1, \lambda_2, \eta_s, \eta_{p_1}, \) and \(\eta_{p_2}\) are all constant.

If \(\mathbf{D}\) is set equal to \(\frac{1}{2} \mathbf{e}_1 \otimes \mathbf{e}_2 + \frac{1}{2} \mathbf{e}_2 \otimes \mathbf{e}_1\), and the fluid is assumed to be a steady, homogeneous flow, eq. (6.37) reduced

\[
\begin{bmatrix}
\mathbf{T}_{11}^s & \mathbf{T}_{12}^s \\
\mathbf{T}_{12}^s & \mathbf{T}_{22}^s
\end{bmatrix}
= \eta_s
\begin{bmatrix}
0 & \dot{\gamma} & 0 \\
\dot{\gamma} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{T}_{11}^{p_k} & \mathbf{T}_{12}^{p_k} \\
\mathbf{T}_{12}^{p_k} & \mathbf{T}_{22}^{p_k}
\end{bmatrix}
- \lambda_k
\begin{bmatrix}
\dot{\gamma}\mathbf{T}_{11}^{p_k} & \dot{\gamma}\mathbf{T}_{12}^{p_k} \\
\dot{\gamma}\mathbf{T}_{12}^{p_k} & \dot{\gamma}\mathbf{T}_{22}^{p_k}
\end{bmatrix}
= \eta_{p_k}
\begin{bmatrix}
0 & \dot{\gamma} & 0 \\
\dot{\gamma} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad k = 1, 2. \tag{6.38}
\]

The solutions \(T_{12}^s\) and \(T_{12}^{p_k}\) of eq. (6.7) are

\[
T_{12}^s = \eta_s \dot{\gamma}, \quad T_{12}^{p_k} = \eta_{p_k} \dot{\gamma}, \quad k = 1, 2, \tag{6.39}
\]

so that

\[
\eta_0 = \lim_{\dot{\gamma} \to 0} \frac{T_{12}}{\dot{\gamma}} = \lim_{\dot{\gamma} \to 0} \frac{T_{12}^s + T_{12}^{p_1} + T_{12}^{p_2}}{\dot{\gamma}} = \lim_{\dot{\gamma} \to 0} \frac{\eta_s \dot{\gamma} + \eta_{p_1} \dot{\gamma} + \eta_{p_2} \dot{\gamma}}{\dot{\gamma}} = \eta_s + \eta_{p_1} + \eta_{p_2}. \tag{6.40}
\]

Hence the material coefficients in form (6.37) are constrained by \(\eta_0 = \eta_s + \eta_{p_1} + \eta_{p_2} = 11.3\) Pa s.

For the steady axisymmetric torsionless flow of the filament, the leading order dimensionless form of the two-mode Oldroyd fluid-B constitutive model (6.37) is

\[
\mathbf{T}_{rr} = \mathbf{T}_{rr}^s + \mathbf{T}_{rr}^{p_1} + \mathbf{T}_{rr}^{p_2}, \quad \mathbf{T}_{zz} = \mathbf{T}_{zz}^s + \mathbf{T}_{zz}^{p_1} + \mathbf{T}_{zz}^{p_2},
\]

\[
54
\]
The three-mode Oldroyd fluid-B model

The three-mode Oldroyd fluid-B model is

\[ \begin{align*}
T_{rr} & = -S\vec{u}_{r,\hat{r}}, \\
T_{zz} & = 2S\vec{u}_{z,\hat{r}}, \\
\dot{T}_{rr}^{p_k} + \Lambda_k(\vec{u}_z \dot{T}_{rr}^{p_k} + \vec{v}_{r,\hat{z}} \dot{T}_{rr}^{p_k}) & = -Z_k\vec{v}_{r,\hat{z}}, \\
\dot{T}_{zz}^{p_k} + \Lambda_k(\vec{v}_z \dot{T}_{zz}^{p_k} - 2\vec{v}_{z,\hat{r}} \dot{T}_{zz}^{p_k}) & = 2Z_k\vec{v}_{r,\hat{z}},
\end{align*} \]

(6.41)

The coefficients within this form are the relaxation times \(\lambda_1, \lambda_2, \lambda_3\), viscosity \(\eta_s\) of the solvent, and the zero strain rate viscosities \(\eta_{p1}, \eta_{p2}, \eta_{p3}\) of the polymer; \(\lambda_1, \lambda_2, \lambda_3, \eta_s, \eta_{p1}, \eta_{p2}, \eta_{p3}\) are all constant.

The material coefficients in form (6.42) are constrained by \(\eta_0 = \eta_s + \eta_{p1} + \eta_{p2} + \eta_{p3} = 11.3 \text{ Pa s}\).

For the steady axisymmetric torsionless flow of the filament, the leading order dimensionless form of the three-mode Oldroyd fluid-B constitutive model (6.42) is

\[ \begin{align*}
\dot{T}_{rr} & = \dot{T}_{zz} - \vec{v}_{r,\hat{z}}, \\
\dot{T}_{zz} & = 2\vec{v}_{z,\hat{r}}, \\
\dot{T}_{rr}^{p_k} + \Lambda_k(\vec{u}_z \dot{T}_{rr}^{p_k} + \vec{v}_{r,\hat{z}} \dot{T}_{rr}^{p_k}) & = -Z_k\vec{v}_{r,\hat{z}}, \\
\dot{T}_{zz}^{p_k} + \Lambda_k(\vec{v}_z \dot{T}_{zz}^{p_k} - 2\vec{v}_{z,\hat{r}} \dot{T}_{zz}^{p_k}) & = 2Z_k\vec{v}_{r,\hat{z}}.
\end{align*} \]

(6.43)

The modified two-mode FENE-P model

The modified two-mode FENE-P model is

\[ \begin{align*}
\hat{T} & = \hat{T}^s + \hat{T}^{p1} + \hat{T}^{p2}, \\
\hat{T}^s & = 2\eta_s \hat{D},
\end{align*} \]
The coefficients in this form are the relaxation times $\lambda_1, \lambda_2$, the viscosity $\eta_s$ of the solvent, the zero strain rate viscosities $\eta_p1, \eta_p2$ of the polymer, and the dumbbell values $b_1, b_2; \lambda_1, \lambda_2, \eta_s, \eta_p1, \eta_p2, b_1, \text{and} b_2$ are all constant.

If $D$ is set equal to $\frac{1}{2} e_1 \otimes e_2 + \frac{1}{2} e_2 \otimes e_1$, and the fluid is assumed to be a steady, homogeneous flow, eq. (6.29) reduced

$$
\begin{bmatrix}
T_{11}^p & T_{12}^p & 0 \\
T_{12}^p & T_{22}^p & 0 \\
0 & 0 & T_{33}^p
\end{bmatrix}
= \eta_s
\begin{bmatrix}
0 & \dot{\gamma} & 0 \\
\dot{\gamma} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

$$
\psi_k
\begin{bmatrix}
T_{11}^p & T_{12}^p & 0 \\
T_{12}^p & T_{22}^p & 0 \\
0 & 0 & T_{33}^p
\end{bmatrix}
- \lambda_k
\begin{bmatrix}
\dot{\gamma}T_{12}^p & \dot{\gamma}T_{22}^p & 0 \\
\dot{\gamma}T_{22}^p & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
= \eta_p(k) (1 - \xi_k b_k)
\begin{bmatrix}
0 & \dot{\gamma} & 0 \\
\dot{\gamma} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad k = 1, 2,
$$

where

$$
\psi_k = 1 + \frac{3}{b_k} (1 - \frac{\eta_p}{3\eta_p}) + \frac{\eta_p}{b_k b_k} (T_{11}^p + T_{22}^p + T_{33}^p).
$$

The solutions $T_{12}^p$ and $T_{12}^p$ of (6.46) are

$$
T_{12}^p = \eta_s \dot{\gamma}, \quad T_{12}^p = \frac{(1 - \xi_k b_k) \dot{\gamma} \eta_p}{1 + \frac{3}{b_k} - \frac{\lambda_k}{\eta_p} T_{11}^p},
$$

where $T_{11}^p, k = 1, 2$ are solutions of the cubic equations,

$$
T_{11}^p - 2 \frac{\eta_p}{\lambda_k} \frac{b_k \eta_p}{\lambda_k} (1 + \frac{3}{b_k}) T_{11}^p2 + \frac{(b_k \eta_p)^2}{\lambda_k} (1 + \frac{3}{b_k})^2 T_{11}^p
- 2 (\frac{b_k \eta_p}{\lambda_k})^2 \eta_p \lambda_k (1 - \xi_k b_k) \dot{\gamma}^2 = 0, \quad k = 1, 2.
$$

56
The coefficients in these two equations are defined such that these cubic equations have one real root and two conjugate complex roots, if

\[ \gamma^2 < \frac{2b_k(1 + \frac{3}{b_k})^3}{27\lambda_k^2(1 - \xi_k b_k)}, \quad k = 1, 2. \]  

(6.49)

These two equations can be solved analytically.

The zero shear rate viscosity is

\[
\eta_0 = \lim_{\gamma \to 0} \frac{T_{12}}{\gamma} = \lim_{\gamma \to 0} \frac{T_{12}^s + T_{12}^{p_1} + T_{12}^{p_2}}{\gamma} \\
= \lim_{\gamma \to 0} \left[ \eta_s + \eta_{p_1} \frac{b_1}{1 + \frac{3}{b_1} + \frac{\lambda_1}{b_1 \eta_{p_1} T_{11}^p}} + \eta_{p_2} \frac{b_2}{1 + \frac{3}{b_2} + \frac{\lambda_2}{b_2 \eta_{p_2} T_{11}^p}} \right] \\
= \eta_s + \eta_{p_1} \frac{1}{1 + \frac{5}{b_1} + \frac{6}{b_1^2}} + \eta_{p_2} \frac{1}{1 + \frac{5}{b_2} + \frac{6}{b_2^2}}.
\]

(6.50)

Hence the material coefficients in form (6.44) are constrained by \( \eta_0 = \eta_s + \eta_{p_1} \frac{1}{1 + \frac{5}{b_1} + \frac{6}{b_1^2}} + \eta_{p_2} \frac{1}{1 + \frac{5}{b_2} + \frac{6}{b_2^2}} = 11.3 \text{ Pa s} \).

For the steady axisymmetric torsionless flow of the filament, the leading order dimensionless form of this modified two-mode FENE-P constitutive model (6.44) is

\[
\tilde{T}_{rr} = \tilde{T}_{rr}^s + \tilde{T}_{rr}^{p_1} + \tilde{T}_{rr}^{p_2}, \quad \tilde{T}_{zz} = \tilde{T}_{zz}^s + \tilde{T}_{zz}^{p_1} + \tilde{T}_{zz}^{p_2}, \\
\tilde{T}_{zz}^s = -S\tilde{u}_z\tilde{z}, \quad \tilde{T}_{zz}^{p_1} = 2S\tilde{u}_z\tilde{z} \\
\phi_k \tilde{T}_{rr}^{p_k} + \lambda_k (\tilde{u}_z\tilde{T}_{zz,\tilde{z}}^{p_k} + \tilde{u}_{\tilde{z}}\tilde{T}_{rr,\tilde{z}}^{p_k}) + \frac{\lambda_k^2}{b_k Z_k \phi_k} \tilde{u}_z (\tilde{T}_{zz}^{p_k} + 2\tilde{T}_{rr}^{p_k}) \left[ \tilde{T}_{rr}^{p_k} - (1 - \xi_k b_k) Z_k \right] \Lambda_k \\
= -(1 - \xi_k b_k) Z_k \tilde{u}_z \tilde{z,} \\
\phi_k \tilde{T}_{zz}^{p_k} + \lambda_k (\tilde{u}_z\tilde{T}_{zz,\tilde{z}}^{p_k} - 2\tilde{u}_{\tilde{z}}\tilde{T}_{zz}^{p_k}) + \frac{\lambda_k^2}{b_k Z_k \phi_k} \tilde{u}_z (\tilde{T}_{zz}^{p_k} + 2\tilde{T}_{rr}^{p_k}) \left[ \tilde{T}_{zz}^{p_k} - (1 - \xi_k b_k) Z_k \right] \Lambda_k \\
= 2(1 - \xi_k b_k) Z_k \tilde{u}_z \tilde{z,}
\]

(6.51)

where

\[
\phi_k = 1 + \frac{3}{b_k} \left[ 1 - \frac{\lambda_k}{3Z_k} (\tilde{T}_{zz}^{p_k} + 2\tilde{T}_{rr}^{p_k}) \right], \quad \xi_k = \frac{2}{b_k(b_k + 2)}, \\
\lambda_k = \frac{\lambda_k v_0}{z_0}, \quad S = \frac{\eta_s r_0^2 v_0}{z_0 f_0}, \quad Z_k = \frac{\eta_{p_k} r_0^2 v_0}{z_0 f_0}, \quad k = 1, 2.
\]

57
To avoid singularity at \( Z = 0 \) when the optimal search algorithm is employed, eqs. (6.51) are reformulated to

\[
\tilde{T}_{rr} = \tilde{T}_{rr}^p + \tilde{T}_{rr}^p + \tilde{T}_{rr}^{p1}, \quad \tilde{T}_{zz} = \tilde{T}_{zz}^p + \tilde{T}_{zz}^{p1} + \tilde{T}_{zz}^{p2},
\]

\[
\tilde{T}_{ss}^s = -S\tilde{u}_{z,\tilde{z}}, \quad \tilde{T}_{zz}^s = 2S\tilde{u}_{z,\tilde{z}},
\]

\[
\Phi_k\tilde{T}_{rr}^p + \Lambda_k(\tilde{u}_{z,\tilde{z}}\tilde{T}_{rr}^{p1} + \tilde{u}_{z,\tilde{z}}\tilde{T}_{rr}^p) + \Pi_k\frac{\Lambda_k f_0}{b_k \Phi_k \tilde{T}_{0}^2} \tilde{u}_z(\tilde{T}_{zz}^{p1} + 2\tilde{T}_{rr}^{p1}) \left[ \tilde{T}_{rr}^{p1} - (1 - \xi_k b_k) \frac{Z_k}{\Lambda_k} \right]
\]

\[
= -(1 - \xi_k b_k) Z_k \tilde{u}_{z,\tilde{z}},
\]

\[
\Phi_k\tilde{T}_{zz}^p + \Lambda_k(\tilde{u}_{z,\tilde{z}}\tilde{T}_{zz}^{p1} - 2\tilde{u}_{z,\tilde{z}}\tilde{T}_{zz}^p) + \Pi_k\frac{\Lambda_k f_0}{b_k \Phi_k \tilde{T}_{0}^2} \tilde{u}_z(\tilde{T}_{zz}^{p1} + 2\tilde{T}_{zz}^{p1}) \left[ \tilde{T}_{zz}^{p1} - (1 - \xi_k b_k) \frac{Z_k}{\Lambda_k} \right]
\]

\[
= 2(1 - \xi_k b_k) Z_k \tilde{u}_{z,\tilde{z}}, \quad (6.52)
\]

where

\[
\Phi_k = 1 + 3 \frac{1}{b_k} \left[ 1 - \Pi_k f_0 \frac{3 r_0^2}{2} (\tilde{T}_{zz}^{p1} + 2\tilde{T}_{zz}^{p1}) \right], \quad \xi_k = \frac{2}{b_k (b_k + 2)}, \quad \Pi_k = \frac{\lambda_k}{\eta_{p_k}}, k = 1, 2.
\]

### 6.2 Three methods to approach the inverse problem

Three different methods of employing the experimental data in the inverse problem for material characterization have been developed and implemented. All three methods demand a proposed constitutive form, with coefficients to be determined through comparison with the experimental measurements.

#### 6.2.1 Method 1-f

In Method 1–f the coupled momentum/constitutive equations (eq. (3.32) and one of eqs. (6.10), (6.19), (6.36), (6.41), (6.43), and (6.52)) are integrated with measured in-flow upstream stress boundary conditions and trial material coefficients. The computed radius and normal stress components that solve this coupled problem are denoted \( \tilde{\phi}_{\text{comp}}^{\text{rr}}, \tilde{\phi}_{\text{zz}}^{\text{comp}}, \) and \( \tilde{T}_{rr}^{\text{comp}} \). Values of coefficients are sought to minimize
the cost function

\[ f = \frac{1}{[N/2]} \sum_{k=1}^{[N/2]-1} [\phi^{\text{comp}}(\bar{z}_{2k-1}) - \phi^{\text{meas}}(\bar{z}_{2k-1})]^2, \quad (6.53) \]

where \( N = (P(n-1) + 400) \) is the total number of pixels in the axial domain \( 0 \leq \bar{z} \leq 1 \), 
\([N/2]\) is the greatest integer which is less or equal to \( N/2 \), \( \bar{z}_{2k-1} \) is the location of \( 2k-1 \)-th pixel, and \( \phi^{\text{meas}} \) is the measured free surface profile.

### 6.2.2 Method 1-s

In Method 1-s values of material coefficients are sought to minimize the cost function

\[ s = \frac{1}{[N/2]} \sum_{k=1}^{[N/2]-1} [(\hat{T}_{zz}^{\text{comp}} - \hat{T}_{rr}^{\text{comp}})(\bar{z}_{2k-1}) - (\hat{T}_{zz}^{\text{mom}} - \hat{T}_{rr}^{\text{mom}})(\bar{z}_{2k-1})]^2, \quad (6.54) \]

where \( \hat{T}_{zz}^{\text{mom}} - \hat{T}_{rr}^{\text{mom}} \) is the stress difference computed via the momentum equation and measured profile, as obtained in section 4.2.1 and 4.2.2.

### 6.2.3 Method 2

In Method 2 all kinematical quantities are regarded as known (either by direct measurement or numerical differentiation of the measurements). With these known kinematical quantities, the momentum equation (3.32) and proposed constitutive equations (one of eqs. (6.10), (6.19), (6.28), (6.36), (6.41), (6.43) and (6.52)) decouple. One seeks the coefficients of the proposed constitutive form to produce the best agreement between the stress difference by minimizing the computed

\[ h = \frac{1}{[N/2]} \sum_{k=1}^{[N/2]-1} [(\hat{T}_{zz}^{\text{cons}} - \hat{T}_{rr}^{\text{cons}})(\bar{z}) - (\hat{T}_{zz}^{\text{mom}} - \hat{T}_{rr}^{\text{mom}})(\bar{z}_{2k-1})]^2, \quad (6.55) \]

where \( (\hat{T}_{zz}^{\text{cons}} - \hat{T}_{rr}^{\text{cons}})(\bar{z}) \) denotes the normal stress difference computed from the constitutive equations, and \( (\hat{T}_{zz}^{\text{mom}} - \hat{T}_{rr}^{\text{mom}})(\bar{z}) \), as above, denotes the stress difference computed from the momentum equation (3.32) and measured profile only.
6.3 Determination of the in-flow stress boundary conditions

Recall the momentum equation (3.32):

\[ (\ddot{T}_{zz} - \ddot{T}_{rr})_{z} + \frac{1}{BW} \ddot{\phi}_{zz} + 2(\ddot{T}_{zz} - \ddot{T}_{rr}) \frac{\ddot{\phi}_{z}}{\dot{\phi}} + \frac{1}{BF} = 0. \]

To integrate eq. (3.32) one needs the upstream stress difference \((\ddot{T}_{zz} - \ddot{T}_{rr})(0)\). However, to integrate the differential equations from the single mode constitutive forms (6.10) (Oldroyd fluid-B model), (6.20) (Giesekus model), (6.28) (FENE-P model), or (6.36) (modified FENE-P model) the upstream normal stress components \(\ddot{T}_{zz}(0)\) and \(\ddot{T}_{rr}(0)\) are required separately. (as well as trial values of the material constants).

Recall from section 2.3 that for each experiment with a take-up force an auxiliary experiment with the same flow conditions but no wind-up is run, and the profile \(\ddot{\phi}_{nw}\) of the no windup filament and the force difference \(\delta\) between the LVDT output signal for the windup experiment and the LVDT output signal for the corresponding no-windup experiment are recorded. The upstream boundary stress difference \((\ddot{T}_{zz} - \ddot{T}_{rr})(0)\) necessary to integrate eq. (3.32) is derived from these measurements eq. (3.38):

\[ (\ddot{T}_{zz} - \ddot{T}_{rr})(0) = \frac{1}{\pi} + \frac{2}{BW} \left[ \ddot{\phi}_{nw} \left( \frac{L_{nw}}{L} - \ddot{\phi}_{nw}(0) \right) \right] + \frac{1}{BF} \int_{0}^{L_{nw}} \ddot{\phi}_{nw} d\ddot{z} + \frac{1}{BW}, \]

where \(L_{nw}\) and \(L\) are the filament lengths of no-windup and the corresponding windup experiments respectively.

From eq. (3.38) one now has the upstream value \((\ddot{T}_{zz} - \ddot{T}_{rr})(0)\) necessary to integrate the momentum equation (3.32), but to integrate the single mode constitutive equations (6.10), (6.20), or (6.36), one needs the components \(\ddot{T}_{zz}(0)\) and \(\ddot{T}_{rr}(0)\) separately. Two different formulations are proposed to obtain this added information.
6.3.1 Formulation a: Calculation of upstream stress values entirely from measurements at \( \bar{z} = 0 \)

Formulation a is employed in [19] and [33] to calculate in-flow boundary stress components. This formulation utilizes experimental measurements of \( \bar{\phi}(0) \) (free surface radius at \( \bar{z} = 0 \)), \( \bar{\phi}_{\bar{z}}(0) \) (free surface slope at \( \bar{z} = 0 \)), \( \bar{v}_z(0) \) (axial velocity at \( \bar{z} = 0 \)), \( \bar{v}_{z,\bar{z}}(0) \) (axial velocity gradient at \( \bar{z} = 0 \)), and \( \bar{v}_{z,\bar{z},\bar{z}}(0) \) (second gradient of axial velocity at \( \bar{z} = 0 \)). Recall from Chapter 4 that \( \bar{\phi}(\bar{z}) \) is measured directly, \( \bar{v}_z(\bar{z}) \) is deduced from \( \bar{\phi}(\bar{z}) \) via the flow rule (3.28), and \( \bar{\phi}_{\bar{z}}, \bar{v}_{z,\bar{z}}, \) and \( \bar{v}_{z,\bar{z},\bar{z}} \) are deduced via differentiation of the measured profile \( \bar{\phi}(\bar{z}) \).

Recall that the upstream gradient of the normal stress difference is given by (3.39):

\[
(\bar{T}_{zz,\bar{z}} - \bar{T}_{rr,\bar{z}})(0) = -2\bar{\phi}_{\bar{z}}(0)(\bar{T}_{zz} - \bar{T}_{rr})(0) - \frac{\bar{\phi}_{\bar{z}}(0)}{BW} - \frac{1}{BF}.
\]

Note that eqs. (3.38) and (3.39) are two equations for the unknown upstream values \( \bar{T}_{zz}(0), \bar{T}_{rr}(0), \bar{T}_{zz,\bar{z}}(0), \) and \( \bar{T}_{rr,\bar{z}}(0) \). If a single mode constitutive model is being employed, to close the system the single mode constitutive equations are evaluated at \( \bar{z} = 0 \). For any trial values of material constants, eqs. (3.38), (3.39), and these two differential equations for \( \bar{T}_{rr} \) and \( \bar{T}_{zz} \) evaluated at \( \bar{z} = 0 \) are four equations for the four unknowns \( \bar{T}_{zz}(0), \bar{T}_{rr}(0), \bar{T}_{zz,\bar{z}}(0), \) and \( \bar{T}_{rr,\bar{z}}(0) \). For example, when employing the single mode Oldroyd fluid-B model, eqs. (6.10) evaluated at \( \bar{z} = 0 \) produce

\[
\bar{T}_{rr}(0) + \Lambda \left[ \bar{v}_z(0)\bar{T}_{rr,\bar{z}}(0) + \bar{v}_{z,\bar{z}}(0)\bar{T}_{rr}(0) \right] \\
= -\Lambda S \left[ \bar{v}_z(0)\bar{v}_{z,\bar{z}}(0) + \bar{v}_{z,\bar{z},\bar{z}}^2(0) - (S + Z)\bar{v}_{z,\bar{z}}(0), \right.
\]

\[
\bar{T}_{zz}(0) + \Lambda \left[ \bar{v}_z(0)\bar{T}_{zz,\bar{z}}(0) - 2\bar{v}_{z,\bar{z}}(0)\bar{T}_{zz}(0) \right] \\
= 2\Lambda S \left[ \bar{v}_z(0)\bar{v}_{z,\bar{z}}(0) - 2\bar{v}_{z,\bar{z},\bar{z}}^2(0) \right] + 2(S + Z)\bar{v}_{z,\bar{z}}(0). \quad (6.56)
\]

61
Note that eqs. (3.38), (3.39), and (6.56) are the four equations for the four unknowns \( \tilde{T}_{rr}(0) \), \( \tilde{T}_{zz}(0) \), \( \tilde{T}_{r,z}(0) \), and \( \tilde{T}_{z,z}(0) \).

It is important to note that for multimode constitutive models formulation a does not work: If the multimode constitutive equations are evaluated at \( \tilde{z} = 0 \), the system does not close.

6.3.2 Formulation b: Calculation of upstream stress values using measurements along the entire spine

In formulation b, as in formulation a, eq. (3.32) is integrated to produce the stress difference \( \tilde{T}_{zz}^{\text{mom}}(\tilde{z}) - \tilde{T}_{rr}^{\text{mom}}(\tilde{z}) \) along the filament necessary to satisfy the conservation of momentum, using the upstream value \( \tilde{T}_{zz}(0) - \tilde{T}_{rr}(0) \) deduced from the force measurement \( \delta \) through eq. (3.38).

Formulation b differs from formulation a in that one deduces the separated values of \( \tilde{T}_{zz}(0) \) and \( \tilde{T}_{rr}(0) \) in this stress difference not from eqs. (3.38), (3.39), and the constitutive equations evaluated \( \tilde{z} = 0 \), but directly by the optimal search process: Trial values of material constants, and \( \tilde{T}_{zz}(0) \) are selected, \( \tilde{T}_{rr}(0) \) is deduced from eq. (3.38), and the two constitutive equations are integrated to produce \( \tilde{T}_{zz}^{\text{cons}}(\tilde{z}) \) and \( \tilde{T}_{rr}^{\text{cons}}(\tilde{z}) \). The optimal material constants and \( T_{zz}(0) \) are sought and obtained by minimizing either \( f \) defined in eq. (6.53), \( s \) defined in eq. (6.54), or \( h \) defined in eq. (6.55), depending on the method employed, Methods 1-\( f \), 1-\( s \), or 2, respectively. Formulation b works for both single mode and multimode constitutive models.
6.3.3 Comparison

The fundamental difference between the above two formulations is that in formulation a the upstream stress components $\tilde{T}_{zz}(0)$ and $\tilde{T}_{rr}(0)$ are deduced for a particular choice of material constants entirely from experiment measurements at $\tilde{z} = 0$, i.e. from $\tilde{\phi}_0$, $\tilde{\phi}_{zz}(0)$, $\tilde{u}_z(0)$, $\tilde{u}_{zz}(0)$, and $\tilde{u}_{zzzz}(0)$, and then one iterates on material constants. In formulation b, the value of $\tilde{T}_{zz}(0)$ is sought, together with material constants, to give the best fit of the stress difference computed from the constitutive equation to that computed from the momentum equation; in so doing, the experiment measurements are utilized along the entire length of the filament, not just at $\tilde{z} = 0$.

Formulation b has the conceptual disadvantage of requiring a search on a non-minimal set, but in practice this is not a problem. Its advantages are that it replaces reliance on the suspect value $\tilde{u}_{zzzz}(0)$ with more reliable measurements $\tilde{\phi}$ and $\tilde{\phi}_{zz}$ along the entire length of the filament, and that it includes the formulation a as a special case.

6.4 Optimal search algorithms

To obtain a set of the material characterizations through Methods 1-f, 1-s, or 2 by minimizing either $f$ defined in eq. (6.53), $s$ defined in eq. (6.54) or $h$ defined in eq. (6.55), respectively, an optimal search algorithm is required.

In general, an optimization problem is: find a $p^* \in \Omega$ such that

$$G(p^*) = \min_{p \in \Omega} G(p),$$  \hspace{1cm} (6.57)

where $p = \{p_1, p_2, \ldots, p_{N_{par}}\}$ is $N_{par}$-dimension parameter, $\Omega$ is $N_{par}$-dimension bounded real space, and $G$ is a function to be minimized. For example, under Method 1-f and
formulation a employing the single mode Oldroyd fluid-B model, and $G$ is the function $f$ defined in eq. (6.53), $p = \{\lambda, \eta_p, \eta_s\}$. If formulation b is used, $p = \{\lambda, \eta_p, \eta_s, \tilde{T}_{xx}(0)\}$.

Three optimal algorithms are presented in this work: (1) exhaustive search, (2) modified exhaustive search, and (3) genetic algorithm.

6.4.1 Exhaustive search

In exhaustive search algorithm each parameter $p_m$ is varied within a specified domain,

$$p_m^{\text{min}} \leq p_m \leq p_m^{\text{max}}, \quad m = 1, 2, \ldots, N_{\text{par}} \quad (6.58)$$

where $p_m^{\text{min}}$ and $p_m^{\text{max}}$ are the lower and upper bound of parameter $p_m$. The search space $\Omega = [p_1^{\text{min}}, p_1^{\text{max}}] \times [p_2^{\text{min}}, p_2^{\text{max}}] \times \cdots \times [p_{N_{\text{par}}}^{\text{min}}, p_{N_{\text{par}}}^{\text{max}}]$ is divided by partitioning $[p_1^{\text{min}}, p_1^{\text{max}}], [p_2^{\text{min}}, p_2^{\text{max}}], \ldots, [p_{N_{\text{par}}}^{\text{min}}, p_{N_{\text{par}}}^{\text{max}}]$ into subintervals with the evenly spaced mesh points,

$$p_m^n = p_m^{\text{min}} + \frac{p_m^{\text{max}} - p_m^{\text{min}}}{N_{i_m} - 1}(n - 1), \quad n = 1, 2, \ldots, N_{i_m}, \quad m = 1, 2, \ldots, N_{\text{par}}. \quad (6.59)$$

The function $G$ is then evaluated at each point $p_{m_1m_2\ldots m_{N_{\text{par}}}}^{n_1n_2\ldots n_{N_{\text{par}}}}$ to find a minimum value of $G$, and hence the optimal $p$.

The exhaustive search algorithm needs a huge number of function computations, and the total number of combinations of different parameters is $N_{i_1} \times N_{i_2} \times \cdots \times N_{i_{N_{\text{par}}}}$.

With fine enough sampling, this exhaustive search algorithm can find a global minimum, but it is only practical for a small number of parameters with a limited search space. The algorithm is not employed in the final studies of this dissertation.
6.4.2 Modified exhaustive search

A modified exhaustive search algorithm first searches a coarse sampling of the whole search space, then progressively narrows the search to a promising sub-region with a finer sampling until a minimum is found.

6.4.3 Genetic algorithm

Genetic algorithms were first developed by Holland in [17], advanced by Goldberg [16], and then widely used in the past years. Genetic algorithms have been successfully coded and implemented to a number and variety of applications. The applications range from music [5, 20] to finance [1, 24] to science and engineering problems [22, 39, 41, 42].

A real-valued genetic algorithm is employed. The complete genetic algorithms consist of the following operations:

- Cost function and genetic parameters

  The genetic algorithms start by defining a chromosome \( p = \{p_1, p_2, \ldots, p_{N_{\text{par}}}\} \)
  (an array of parameters) with constraints

  \[ p_{m}^{\text{min}} \leq p_m \leq p_{m}^{\text{max}}, \quad m = 1, 2, \ldots, N_{\text{par}}, \]

  and selecting a cost function \( G(p) \) and defining a chromosome \( p = [p_1, p_2, \ldots, p_{N_{\text{par}}} ] \)
  (an array of parameters) with constraints

  \[ p_{m}^{\text{min}} \leq p_m \leq p_{m}^{\text{max}}, \quad m = 1, 2, \ldots, N_{\text{par}} \]

  to be optimized.
• Initial population

An initial population of $N_i$ chromosomes satisfying the constraints is generated randomly.

• Natural selection

A cost for each chromosome is calculated through the cost function $\mathcal{C}$. The $N_i$ costs and the corresponding chromosomes are listed from lowest cost to highest cost. Only the first $N_{\text{good}}$ ($N_{\text{good}} < N_i$) chromosomes are survived for mating.

• Crossover

A crossover operation is as follows: the $[\frac{N_{\text{good}}}{2}]$ mothers and $[\frac{N_{\text{good}}}{2}]$ fathers are selected randomly from the $N_{\text{good}}$ chromosomes, and paired in a stochastic way. (Note that $[\frac{N_{\text{good}}}{2}]$ is the greatest integer which is less or equal to $\frac{N_{\text{good}}}{2}$).

A typical pair of mother and father is

$$p_{\text{mother}} = \{p_{1\text{mom}}, p_{2\text{mom}}, \ldots, p_{N_{\text{par}}\text{mom}}\},$$

$$p_{\text{father}} = \{p_{1\text{dad}}, p_{2\text{dad}}, \ldots, p_{N_{\text{par}}\text{dad}}\}.$$

A single crossover point $i$ (the $i$-th parameter) is selected randomly for each pair, the new parameters generated from the parents are

$$p_{i\text{child}1} = p_{i\text{mom}} - \gamma(p_{i\text{mom}} - p_{i\text{dad}}),$$

$$p_{i\text{child}2} = p_{i\text{dad}} + \gamma(p_{i\text{mom}} - p_{i\text{dad}}),$$

$$p_{i\text{child}2} = 0.5p_{i\text{dad}} + 0.5p_{i\text{mom}},$$

(6.60)

or

$$p_{i\text{child}1} = -1.5p_{i\text{mom}} + 0.5p_{i\text{dad}},$$

66
\begin{align*}
    p_{i}^{\text{child2}} &= 0.5p_{i}^{\text{mom}} - 1.5p_{i}^{\text{dad}}, \\
    p_{i}^{\text{child2}} &= 0.5p_{i}^{\text{dad}} + 0.5p_{i}^{\text{mom}},
\end{align*}

(6.61)

where \( \gamma \) is a random number between 0 and 1. Three new children now are

\begin{align*}
    p_{i}^{\text{child1}} &= \{p_{1}^{\text{mom}}, p_{2}^{\text{mom}}, \ldots, p_{i}^{\text{child1}}, \ldots, p_{N_{\text{par}}}^{\text{mom}}\}, \\
    p_{i}^{\text{child2}} &= \{p_{1}^{\text{dad}}, p_{2}^{\text{dad}}, \ldots, p_{i}^{\text{child2}}, \ldots, p_{N_{\text{par}}}^{\text{dad}}\}, \\
    p_{i}^{\text{child3}} &= \{p_{1}^{\text{dad}}, p_{2}^{\text{dad}}, \ldots, p_{i}^{\text{child3}}, \ldots, p_{N_{\text{par}}}^{\text{dad}}\}.
\end{align*}

(6.62)

- **Mutation**

The mutation is an operation with one or more parameters changed randomly. It is intended to prevent the genetic algorithm from converging quickly to a local minimum. The mutation rate is given by \( \mu \), the total number of parameters which are mutated for each generation is \( \mu \times N_{\text{good}} \times N_{\text{par}} \).

- **Iteration**

The process repeats the natural selection, crossover, and mutation again and again. The smallest value of the costs in the new generation are always smaller or equal to the smallest value of the costs in the previous generation for reasonable crossover and mutation rates [36].

- **Stop criterion**

A stop criterion needs to be fixed since the genetic algorithm is a stochastic search technique. A practical way for the stop criterion is to fix the number of iteration.
CHAPTER 7

NUMERICAL IMPLEMENTATIONS OF CHARACTERIZATION PROCEDURE I

In this chapter numerical implementations and results of characterization procedure I are presented. In section 4.2.1 and 4.2.2, two techniques, curve fitting and filtering, to manipulate the experimental data are employed. In section 6.1, seven constitutive forms are proposed. In section 6.2, three methods, Method 1-f, Method 1-s, and Method 2, to characterize elongational properties are formulated. In section 6.3, two formulations, formulation a and formulation b, to determine the in-flow boundary stresses are discussed. In section 6.4, three optimal search algorithms, exhaustive search, modified exhaustive search, and genetic algorithm, to obtain the optimal material constants are presented. Finally, the modified exhaustive search and genetic algorithm are employed in this study. Hence, there are 168 \((2 \times 7 \times 3 \times 2 \times 2)\) different characterization categories to characterize a test fluid.

The characterization categories labeled by a series of five notations, according the following schemes:

- The first notation relates to the technique for data manipulations: CF and F correspond to polynomial curve fitting and filtering, respectively.
• The second notation relates to the formulation of determining the in-flow stresses: 
a and b correspond to formulation a and formulation b, respectively.

• The third notation relates to the optimal search algorithm: MES and GA corre­
  spond to modified exhaustive search and genetic algorithm, respectively.

• The fourth notation relates the inverse formulation: M1f, M1s, and M2 corre­
  spond to Method 1-f, Method 1-s, and Method 2, respectively.

• The fifth notation relates the constitutive form: Old1, Old2, Old3, Gies1,
  FENE1, MFENE1, MFENE2 correspond to single mode Oldroyd fluid-B model,
  two-mode Oldroyd fluid-B model, three-mode Oldroyd fluid-B model, single
  mode Giesekus model, single mode FENE-P model, modified single mode FENE-
  P model and modified two-mode FENE-P model, respectively.

7.1 The single mode Oldroyd fluid-B model

Recall that the leading order components (6.10) of the single mode Oldroyd fluid-B
model are

\[ \hat{T}_{rr} + \Lambda (\hat{v}_z \hat{T}_{zz,\hat{r}} + \hat{v}_{z,\hat{r}} \hat{T}_{rr}) = -\Lambda S (\hat{v}_z \hat{v}_{z,\hat{z}} + \hat{v}_{z,\hat{z}}^2) - (S + Z) \hat{v}_{z,\hat{z}}, \]

\[ \hat{T}_{zz} + \Lambda (\hat{v}_z \hat{T}_{zz,\hat{z}} - 2 \hat{v}_{z,\hat{z}} \hat{T}_{zz}) = 2\Lambda S (\hat{v}_z \hat{v}_{z,\hat{z}} - 2 \hat{v}_{z,\hat{z}}^2) + 2(S + Z) \hat{v}_{z,\hat{z}}, \]

where \( \Lambda = \frac{A_{v0}}{z_0}, S = \frac{\eta_v r_0^2 v_{00}}{z_0 f_0}, \) and \( Z = \frac{\eta_v r_0^2 v_{00}}{z_0 f_0}. \)
7.1.1 Numerical implementation, results and comparisons of categories (CF, α, MES, M1f, Old1), (CF, α, MES, M1s, Old1), (CF, α, MES, M2, Old1), (F, α, MES, M1f, Old1), (F, α, MES, M1s, Old1), and (F, α, MES, M2, Old1)

Method 1-f

In this method trial values for material constants $\lambda(\Lambda = \frac{\lambda_0}{\lambda_0})$ and $\eta_p(\eta = \frac{\eta_0}{\eta_0})$ are selected. The in-flow boundary stresses are computed through formulation α and the coupled momentum/constitutive equations (3.32) and (6.10) are integrated to produce the functions $\phi^{comp}(\tilde{z})$, $\tilde{T}_{zz}^{comp}(\tilde{z})$, and $\tilde{T}_{rr}^{comp}(\tilde{z})$ that follow from these trial values.

The order of search space $\{\lambda, \eta_p\}$, where $\lambda \in [0, 5]$ s and $\eta_p \in [0, 11.3]$ Pa s, is two. A set of optimal material constants $(\lambda, \eta_p, \eta_s)$ is obtained by using a modified exhaustive search procedure with coarse samplings of 0.1 s, and 0.1 Pa s, followed by finest sampling increments 0.0001s and 0.01 Pa s, respectively, and minimizing $f$ defined in eq. (6.53),

$$f = \frac{1}{[N/2]} \sum_{k=1}^{[\frac{N}{2}]-1} (\phi^{comp}(\tilde{z}_{2k-1}) - \phi^{meas}(\tilde{z}_{2k-1}))^2,$$

(7.1)

where $\phi^{meas}$ the measured free surface profile is $\phi^{exp}$ when polynomial curve fitting is employed or $\phi^{fil}$ when filtering is employed; $\eta_s$ is then given by $\eta_s = \eta_0 - \eta_p$.

Method 1-s

In Method 1-s a set of optimal material constants $(\lambda, \eta_p, \eta_s)$ is obtained by minimizing $s$ defined in eq. (6.54),

$$s = \frac{1}{[N/2]} \sum_{k=1}^{[\frac{N}{2}]-1} (\tilde{T}_{zz}^{comp} - \tilde{T}_{zz}^{exp})(\tilde{z}_{2k-1}) - (\tilde{T}_{zz}^{mom} - \tilde{T}_{zz}^{mom})(\tilde{z}_{2k-1}))^2,$$

(7.2)
where \( \tilde{T}_{zz}^{\text{mom}} - \tilde{T}_{rr}^{\text{mom}} \) is the stress difference computed via the momentum equation and measured profile \( \tilde{\phi}^{\text{exp}} \) or \( \tilde{\phi}^{\text{ful}} \); \( \eta_s \) is then given by \( \eta_s = \eta_0 - \eta_p \).

Figure 7.1 overlays the measured profile and computed best fit obtained in Method 1-f in the typical experiments 4-1-3, 4-2-3, and 4-3-3, both when using polynomial curve fitting and when using filtering; the fig. 7.2 shows the corresponding fits of the normal stress difference computed from the coupled momentum/constitutive problem and computed from the measured profile inserted in the momentum equation. Figure 7.3 and 7.4 show the best fit of stress difference obtained in Method 1-s in these experiments, both using polynomial curve fitting and using filtering, as well as the corresponding fits of computed and measured profiles. Note that in Method 1 the only experimentally obtained function used is the profile \( \tilde{\phi}(\tilde{\phi}, \tilde{v}_z, \tilde{u}_{z,\tilde{z}}, \text{and} \tilde{u}_{z,\tilde{z}\tilde{z}} \) are not used).

**Method 2**

With velocity \( \tilde{v}_z \), velocity gradient \( \tilde{u}_{z,\tilde{z}} \), and second gradient \( \tilde{u}_{z,\tilde{z}\tilde{z}} \) known, the constitutive equations (6.10) decouple from the momentum equation, becoming two first order differential equations for the normal stress components \( \tilde{T}_{zz}(z) \) and \( \tilde{T}_{rr}(z) \). In Method 2 trial values of \( \Lambda \) and \( Z \) deduced from \( \lambda \) and \( \eta_p \) are inserted into eqs. (6.10), and the predicted evolution of stress components is computed, given the experimentally measured kinematical input \( \tilde{v}_z^{\text{exp}}(\tilde{z}) \), \( \tilde{v}_{z,\tilde{z}}^{\text{poly}}(\tilde{z}) \), and \( \tilde{v}_{z,\tilde{z}\tilde{z}}^{\text{global}}(\tilde{z}) \) (when employing polynomial curve fitting) or \( \tilde{v}_z^{\text{ful}}(\tilde{z}) \), \( \tilde{v}_{z,\tilde{z}}^{\text{ful}}(\tilde{z}) \), and \( \tilde{v}_{z,\tilde{z}\tilde{z}}^{\text{ful}}(\tilde{z}) \) (when employing filtering).

The order of this search space \( \{\lambda, \eta_p\} \), where \( \lambda \in [0, 5] \text{ s} \) and \( \eta_p \in [0, 11.3] \text{ Pa s} \), is two. A set of optimal material constants \((\lambda, \eta_p, \eta_s)\) are obtained by using a modified
exhaustive search procedure with coarse samplings of 0.1 s and 0.1 Pa s, followed by finest sampling increments 0.0001s and 0.01 Pa s, respectively, and minimizing

\[ h = \frac{1}{[N/2]} \sum_{k=1}^{[N/2]-1} [(\tilde{T}_{\text{cons}}^{zz} - \tilde{T}_{\text{cons}}^{rr})(\tilde{z}_{2k-1}) - (\tilde{T}_{\text{mom}}^{zz} - \tilde{T}_{\text{mom}}^{rr})(\tilde{z}_{2k-1})]^2; \]  

(7.3)

\( \eta_s \) is then given by \( \eta_s = \eta_0 - \eta_p \).

Figure 7.5 overlays the normal stress difference \( T_{zz}^{\text{mom}} - T_{rr}^{\text{mom}} \) that must be in the spinline to satisfy conservation of momentum and the best prediction of the Oldroyd fluid-B to match this difference, in the typical experiments 4-1-3, 4-2-3, and 4-3-3, both using polynomial curve fitting and using filtering.

Results and discussion

The optimal values for all of the experiments of \( \lambda, \eta_p, \) and \( \eta_s \), as returned by all of the different methods, are listed in Table 7.1. For each experiment there are six different sets of \( \lambda, \eta_p, \) and \( \eta_s \), that obtained (i) using polynomial curve fitting of the experimental data and Method 1-\( f \), (ii) using polynomial fitting and Method 1-\( s \), (iii) using polynomial fitting and Method 2, (iv) using filtering of the experimental data and Method 1-\( f \), (v) using filtering and Method 1-\( s \), and (vi) using filtering and Method 2. In addition to the optimal values of material constants, Table 7.1 also list the corresponding minimum error used to identify the best fit, namely error \( f \) (defined in eq. (6.53) for Method 1-\( f \), error \( s \) (defined in eq. (6.54)) for Method 1-\( s \), and error \( h \) (defined in eq. (6.55)) for Method 2. For the Method 1-\( f \) cases the stress difference error \( s \) produced by the values of \( \lambda, \eta_p, \) and \( \eta_s \) which minimize the free surface profile error \( f \) is also computed and listed, and for Method 1-\( s \) list the accompanying error \( f \).
By observation the Table 7.1, all three methods, Methods 1-f, 1-s, and 2, employing either polynomial curve fitting or filtering of the experimental data, are successful in characterizing the relaxation time of the test fluid in each experiment. In each experiment the six predictions of $\lambda$ are quantitatively close. Further, the normal stress difference error $s$ in each Method 1-f calculation although not optimized, is none-the-less close to the minimum normal stress difference error $s$ as revealed by the Method 1-s calculation (e.g. in experiment 4-1-3 using polynomial fitting, $s = 9.77 \times 10^{-3}$ in Method 1-f, compared to the optimal value $s = 7.93 \times 10^{-3}$ found through Method 1-s, and using filtering, $s = 4.01 \times 10^{-3}$ in Method 1-f, compared to the optimal value $s = 3.10 \times 10^{-3}$ found through Method 1-s), the free surface profile error $f$ in each Method 1-s calculation is close to minimum normal stress difference error $f$ computed in the Method 1-f calculation (e.g. in experiment 4-1-3 using polynomial fitting, $f = 11.9 \times 10^{-5}$ in Method 1-s, compared to the optimal value $f = 11.1 \times 10^{-5}$ found through Method 1-f, and using filtering, $f = 5.94 \times 10^{-5}$ in Method 1-s, compared to the optimal value $f = 5.52 \times 10^{-5}$ found through Method 1-f), and the minimum stress difference error $h$ from the Method 2 calculation in each experiment is near the minimum stress difference error $h$ from the Method 1-s calculation (e.g. in experiment 4-2-3 using polynomial fitting, $h = 0.44 \times 10^{-3}$ in Method 2 and $s = 0.45 \times 10^{-3}$ in Method 1-s, and using filtering, $h = 0.13 \times 10^{-3}$ in Method 2 and $s = 0.13 \times 10^{-3}$ in Method 1-s).

Filtering of the experimental data is superior to polynomial curve fitting. Chapter 5 demonstrated through computation of spinline force and normal stress difference that filtering of the experimentally measured profile produces data that is more stable in numerical computations than that produced by polynomial curve fitting. The
study of Table 7.1 provide further indications of this: For the same method, filtering gives smaller errors than curve fitting (when using Method 1-f in fourteen of the eighteen experiments error $f$ is smaller with filtering; when using Method 1-s in fifteen of the eighteen experiments error $s$ is smaller with filtering; when using Method 2 in fourteen of the eighteen experiments error $h$ is smaller with filtering). The difference between the largest and smallest predictions of relaxation time from the three methods is smaller with filtering in fourteen of the eighteen experiments. Filtering gives better reproducibility: the differences in the predictions for $\lambda$ from the duplicate experiments 4-3-3 and 4-3-3x are smaller for Methods 1-f, 1-s, and 2.

There is no indication that any one of Methods 1-f, 1-s, or 2 produces characterizations that are superior or inferior to the other two: In the three pairs of experiments investigating sensitivity to the choice of $z_0 = L$ (4-1-3 and 4-1-3a, 4-6-3 and 4-6-3a, and 4-8-3 and 4-8-3a), in one Method 2 has the least sensitivity, in another it has the most sensitivity, and in the third pair it is in the middle when filtering; in the two pairs for which Method 2 is not the least sensitive, in one Method 1-f is best and in the other Method 1-s is best. Comparing the two methods which minimize the normal stress difference, in eight of the eighteen experiments error $s$ from Method 1-s is smaller than error $h$ from Method 2 when fitting, and $h$ is smaller in the other ten; when filtering the corresponding numbers are fourteen of eighteen and four of eighteen. It is worth noting that in the large majority of experiments the Method 2 prediction of relaxation time is between Method 1-f and 1-s predictions: When filtering, the Method 2 value is the smallest in six of the eighteen experiments, intermediate in twelve, and largest in none; Method 1-f is smallest in three and Method 1-s in nine. When fitting, the Method 2 value is the smallest in two experiments,
intermediate in fourteen, and largest in two; Method 1-f is smallest in eight and Method 1-s in the other eight.

While giving comparable characterizations to Methods 1-f and 1-s, Method 2 has the advantage of having significantly lower computational cost (approximately one-third the user time).

There is a strong dependence of relaxation time on flow conditions. The twelve-member family 4-*-3 of experiments examines the effect of varying windup rate, holding flow rate and filament length fixed; fig. 7.6 graphically displays the inverse relationship of relaxation time to windup rate. The five-member family 4-3-* of experiments examines the effect of varying filament length, holding flow rate and windup rate fixed; fig. 7.7 displays a proportional relationship of relaxation time to filament length. The five-member family *-3-3 of experiments examines the effect of varying flow rate, holding windup rate and filament length fixed; fig. 7.8 displays a nonmonotonic change of relaxation time with increasing flow rate.

In figs. 7.9 and 7.10 the ability of the single value of relaxation time averaged over all the experiments is investigated to predict the behavior of each experiment separately. One restricts to the 42 optimal values in Table 7.1 for the 14 experiments without the appendages "a" or "x" (denoting a different choice of $z = L$ and a reproduced case, respectively) returned using the three methods and the filtered data. The arithmetic average of the 14 values for each method are $\lambda_{\text{avg}} = 2.1684$ s for Method 1-f, $\lambda_{\text{avg}} = 2.1488$ s for Method 1-s, and $\lambda_{\text{avg}} = 2.1590$ s for Method 2. Fig. 7.9 displays, for each method, the difference between the optimal relaxation time in each experiment and the averaged relaxation time. Fig. 7.10 displays the errors $f$, $s$, and $h$ defined in eqs. (6.53)-(6.55), respectively, computed for each experiment.
with $\lambda = \lambda^{avg}$ and the predominant characterization $\eta_p = 0 \text{ Pa s}$ (see the discussion of viscosity immediately following), noting that this averaged set of material parameters is not optimum in any of the individual experiments. One observes that each error for the non-optimum average relaxation time is in general two orders of magnitude greater than the error produced in each experiment. Hence, the best single characterization of the entire set of experiments does a poor job of characterizing each experiment.

Figure 7.11 shows the measured free surface profile, profile computed from the coupled momentum and Oldroyd fluid-B constitutive equations with $\lambda = \lambda^{avg} = 2.1684 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$, and profile computed from the coupled momentum and Oldroyd fluid-B constitutive equations with $\lambda = \lambda^{opt} = 3.6907 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$ using Method 1-f for exp. 4-1-3. Figure 7.12 presents the normal stress differences computed from the momentum equation, the coupled momentum and Oldroyd fluid-B constitutive equations with $\lambda = \lambda^{avg} = 2.1488 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$, and the coupled momentum and Oldroyd fluid-B constitutive equations with $\lambda = \lambda^{opt} = 3.7328 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$ using Method 1-s for exp. 4-1-3. Figure 7.13 displays the normal stress differences computed from the momentum equation, the Oldroyd fluid-B constitutive equation with $\lambda = \lambda^{avg} = 2.1590 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$, and the Oldroyd fluid-B constitutive equation with $\lambda = \lambda^{opt} = 3.6849 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$ using Method 2 for exp. 4-1-3.

The observation that what is posited to be a material constant is found to be strongly flow dependent indicates that the Oldroyd fluid-B model is an inadequate framework under which to construct a general characterization of our test fluid. The
relaxation time obtained from each experiment is valid only for flows in the neighborhood of the experiment.

In the numerical optimizations for the constitutive parameters, fourteen of the eighteen experiments for Method 1-f, eleven of the eighteen experiments for Method 1-s, and fifteen of eighteen experiments for Method 2 using the polynomial curve fitting, and nine for Method 1-f, eight for Method 1-s, and seven for Method 2 using the filtering, the zero strain rate viscosity \( \eta_p \) of this test fluid is zero.

Recall that the stress \( T \) in an Oldroyd fluid-B can be decomposed as \( T = T^e + T^p \), where \( T^e = 2\eta_sD \) and \( T^p + \lambda \frac{D^t}{Df} T^p = 2\eta_pD \). The prevalent characterization of \( \eta_p = 0 \) indicates that in the elongational flow the stress in the polymer is predominately elastic.

In the following computation of categories filtering and Method 2 are adopted, since filtering of the experimental data, as shown in Chapter 5 and previous discussion, is superior to polynomial curve fitting, and Method 2 has the advantage of having significantly lower computational cost.

### 7.1.2 Numerical implementation and results of category \((F, a, GA, M2, Old1)\)

A real-valued genetic algorithm is employed to search a set of material constants of the single mode Oldroyd fluid-B model as follows:

1. The chromosome

\[
p = \{ \lambda, \eta_p \},
\]

(7.4)

with constraints \( 0 \leq \lambda \leq 5 \text{ s}, 0 \leq \eta_p \leq 11.3 \text{ Pa s} \), is defined, and the cost function eq. (6.55) is selected.
(2) An initial population of 200 chromosomes is generated randomly. The momentum equation (3.32) is integrated. A stress difference from the constitutive form is computed through eqs. (6.10) with a set of material constants (chromosome). The cost for each chromosome then is computed through eq. (6.55). The 200 costs and associated chromosomes are ranked from the lowest to highest, and only the first 24 chromosomes are survived.

(3) The 12 mothers and 12 fathers are randomly selected from 24 chromosomes and are randomly paired to 12 parents. A typical of parents is

\[
\begin{align*}
 p_{mother} & = [\lambda_{mom}, \eta_p^{mom}] \\
 p_{father} & = [\lambda_{dad}, \eta_p^{dad}]
\end{align*}
\]

The crossover point is selected randomly, and the rules for new parameters generated from the parents are either

\[
\begin{align*}
 \lambda_{child1} & = \lambda_{mom} - \gamma(\lambda_{mom} - \lambda_{dad}) \\
 \lambda_{child2} & = \lambda_{dad} + \gamma(\lambda_{mom} - \lambda_{dad}) \\
 \lambda_{child3} & = 0.5\lambda_{mom} + 0.5\lambda_{dad}
\end{align*}
\]

or

\[
\begin{align*}
 \eta_p^{child1} & = -1.5\eta_p^{mom} + 0.5\eta_p^{dad} \\
 \eta_p^{child2} & = 0.5\eta_p^{mom} - 1.5\eta_p^{dad} \\
 \eta_p^{child3} & = 0.5\eta_p^{mom} + 0.5\eta_p^{dad}
\end{align*}
\]

where \(\gamma\) is a random number between 0 and 1.

(4) Thirty-six \((12 \times 3)\) new children (chromosomes) are generated in the previous step.
The 60 (24 parents + 36 children) costs and associated chromosomes are ranked from lowest to highest, and only the first 24 chromosomes are survived.

(5) The values of the parameters are changed arbitrarily. The mutation rate \( \mu \) is 0.2.

(6) The process repeats step (3), (4), and (4) until 10 iterations.

The results are listed in Table 7.2.

Comparison between categories (F, a, MES, M2, Old1) and (F, a, GA, M2, Old1)

In comparison with categories (F, a, GA, M2, Old1), and (F, a, ES, M2, Old1), the errors using a genetic algorithm are slightly larger than those using a modified exhaustive search, and the optimal material constants produced by a genetic algorithm is close to those produced by a modified exhaustive search for all experiments. Six of fifteen experiments shows that \( \eta_p = 0 \), compared with seven of fifteen experiments by using a modified exhaustive search algorithm. It leads that genetic algorithms can be employed to other following categories with a high dimension search space confidently.

7.1.3 Numerical implementation and results of category (F, b, MES, M2, Old1)

In formulation b, as in formulation a, eq. (3.32) is integrated to produce the stress difference \( \tilde{T}^{mom}(\tilde{z}) - \tilde{T}^{mom}(\tilde{z}) \) along the filament necessary to satisfy the conservation of momentum, using the upstream value \( \tilde{T}_{zz}(0) - \tilde{T}_{rr}(0) \) deduced from the force measurement \( \delta \) through eq. (3.38).
Trial values of $\lambda$, $\eta_p$, and $\tilde{T}_{zz}(0)$ are proposed, $\tilde{T}_{rr}(0)$ is computed through $\tilde{T}_{zz}(0) - (\tilde{T}_{zz} - \tilde{T}_{rr})(0)$, and eqs. (6.10) are integrated to produce $\tilde{T}_{zz}^{\text{cons}}(\tilde{z})$ and $\tilde{T}_{rr}^{\text{cons}}(\tilde{z})$.

The order of this search space $\{\lambda, \eta_p, \tilde{T}_{zz}(0)\}$, where $\lambda \in [0, 5]$ s, $\eta_p \in [0, 11.3]$ Pa s, $\tilde{T}_{zz}(0) \in [0, 0.8]$, is three. A set of optimal material constants ($\lambda$, $\eta_p$, $\eta_s$) and a partition of the in-flow boundary stresses are obtained by using a modified exhaustive search procedure with coarse samplings of 0.1 s, 0.1 Pa s, and 0.1, respectively, followed by finest sampling increments 0.0001 s, 0.01 Pa s, and 0.01, respectively, and minimizing eq. (6.55); $\eta_s$ is then given by $\eta_s = \eta_0 - \eta_p$. The results are listed in Table 7.3.

Comparison between categories (F, a, MES, M2, Old1) and (F, b, MES, M2, Old1)

For the Oldroyd fluid-B model two formulations are adopted to determine the necessary in-flow boundary stresses: formulation a calculates the upstream stress values from the delivery tube deflection and the profile measurements at $z = 0$, and formulation b calculates the upstream stress values from the delivery tube deflection and the profile measurements along the entire spinline. Formulation b has a disadvantage to require iteration on a nonminimal set, however, that it includes formulation a as a special case. Table 7.3 presents that the formulation b always produces a final error at the end of iteration that is smaller than that arrived at via the formulation a.

7.1.4 Representation decomposing stress in solvent and polymer stresses

Recall that the total stress in Oldroyd fluid-B model can be decomposed into a solvent contribution $\tilde{T}^s$ and a polymer contribution $\tilde{T}^p$, $\tilde{T} = \tilde{T}^s + \tilde{T}^p$, $\tilde{T}^s = 2\eta_s D$,.
and $\dot{T}p + \lambda \frac{D}{Dt} \dot{T}p = 2\eta_p D$.

For a thin filament flow, the leading order equations in this representation are

$$T_{rr} = \dot{T}_{rr} + \dot{T}_{rr}, \quad T_{zz} = \dot{T}_{zz} + \dot{T}_{zz}$$

$$\dot{T}_{ss} = -S\dot{u}_{z,\dot{z}}, \quad \dot{T}_{zz} = 2S\dot{u}_{z,\dot{z}},$$

$$\dot{T}_{rr} + \Lambda (\ddot{u}_z T_{rr,\dot{z}} + \dot{u}_{z,\dot{z}} \ddot{T}_{rr}) = -Z\dot{u}_{z,\dot{z}},$$

$$\dot{T}_{zz} + \Lambda (\ddot{u}_z T_{zz,\dot{z}} + \dot{u}_{z,\dot{z}} \ddot{T}_{zz}) = 2Z\ddot{u}_{z,\dot{z}}. \quad (7.5)$$

The stress difference at $\dot{z} = 0$ of eq. (3.38) can be written as

$$\dot{T}_{zz}^p(0) - T_{rr}^p(0) = \frac{1}{\pi} + \frac{2}{BW} \left[ \phi_{nw}^{\text{nw}} \left( \frac{L_{nw}}{L} \right) - \phi_{nw}(0) \right] + \frac{1}{BF} \int_0^{\nuw} \phi_{nw}^3 \, d\dot{z}$$

$$+ \frac{1}{BW} \left( 2S\dot{u}_{z,\dot{z}}(0) + S\ddot{u}_{z,\dot{z}}(0) \right). \quad (7.6)$$

The inverse problem for material characterization in this representation employing formulation b is as follows: Equation (3.32) is integrated to produce the stress difference $\tilde{T}_{zz}^\text{mom}(\dot{z}) - \tilde{T}_{rr}^\text{mom}(\dot{z})$ along the filament necessary to satisfy the conservation of momentum using the upstream stress difference $(\dot{T}_{zz} - \dot{T}_{rr})(0)$ deduced from the force measurement $\delta$ through eq. (3.38). Trial values of $\lambda$, $\eta_p$ ($\eta_s = \eta_0 - \eta_p$), and $\dot{T}_{zz}^p(0)$ are selected, $\dot{T}_{rr}^p(0)$ is deduced from eq. (3.38), and eqs. (7.5)\textsubscript{5,6} are integrated to produce $\dot{T}_{zz}^p(\dot{z})$ and $\dot{T}_{rr}^p(\dot{z})$. $\dot{T}_{rr}^p(\dot{z})$ and $\dot{T}_{zz}^p(\dot{z})$ are computed from eqs. (7.5)\textsubscript{3,4} directly. One iterates on $\lambda$, $\eta_p$, and $\dot{T}_{zz}^p(0)$ and obtain the optimal values $\lambda$, $\eta_p$ by minimizing eq. (6.55); $\eta_s$ is obtained via $\eta_0 - \eta_p$. It can get the exactly same characterization as above since eqs. (7.5) are equivalent to eqs. (6.10) mathematically.
7.2 The single mode Giesekus model

Recall that the reformulated leading order dimensionless form of the Giesekus constitutive model (6.20) is

\[
\begin{align*}
\tilde{T}_{rr} & = \Lambda (\tilde{v}_z \tilde{T}_{rr,z} + \tilde{v}_{z,z} \tilde{T}_{rr}) + \frac{\lambda f_0}{r_0^2} \tilde{T}_{rr} + 2\Gamma \frac{\lambda f_0 S}{r_0^2} \tilde{v}_{z,z} \tilde{T}_{rr} \\
\tilde{T}_{zz} & = -\Lambda S (\tilde{v}_z \tilde{v}_{z,z} + \tilde{v}_{z,z}^2) - (S + Z) \tilde{v}_{z,z} - \Gamma \frac{\lambda f_0 S^2}{r_0^2} \tilde{v}_{z,z} \tilde{T}_{zz} \\
\end{align*}
\]

with

\[
\Lambda = \frac{\lambda v_0}{z_0}, \quad S = \frac{\eta_s r_0^2 v_0}{z_0 f_0}, \quad Z = \frac{\eta_p r_0^2 v_0}{z_0 f_0}, \quad \Gamma = \frac{\alpha}{\eta_p}.
\]

7.2.1 Numerical implementation and results of category (F, a, MES, M2, Gies1)

The stress difference \(\tilde{T}_{zz}(\bar{z}) - \tilde{T}_{rr}(\bar{z})\) is produced from the momentum equation (3.32) using the upstream value \(\tilde{T}_{zz}(0) - \tilde{T}_{rr}(0)\) deduced from eq. (3.38). Trial values of \(\lambda, \eta_s, \Gamma\) are proposed and the separated upstream values \(\tilde{T}_{zz}(0)\) and \(\tilde{T}_{rr}(0)\) are computed from eqs. (3.38), (3.39), and (6.20) evaluated at \(\bar{z} = 0\); using these values eqs. (6.19) are integrated to produce the normal stress components \(\tilde{T}_{zz}^{cons}\) and \(\tilde{T}_{rr}^{cons}\).

The order of search space \(\{\lambda, \eta_s, \Gamma\}\), where \(\lambda \in [0, 5] \text{ s}, \eta_s \in [0, 11.3] \text{ Pa s},\) and \(\Gamma \in [10^{-10}, 10^{-6}] \cup \{0\}\), is three. A set of the optimal material constants \((\lambda, \alpha, \eta_p, \eta_s)\) is obtained by a modified exhaustive search procedure and minimizing eq.(6.55). The mobility parameter \(\alpha\) is calculated through \(\Gamma(\eta_0 - \eta_s).\) The results are listed in Table 7.3.
Comparison between categories (F, a, MES, M2, Gies1) and (F, a, MES, M2, Old1)

Ten of fifteen experiments for the Giesekus model using formulation a the mobility factor \( \alpha = 0 \) and achieve exactly same values of the optimal material constants for the Oldroyd fluid-B model using formulation a. For those experiments which \( \alpha \neq 0 \), the final error using the Giesekus form is always smaller (up to 10\% improvement) than that using the Oldroyd fluid-B form, since the Oldroyd fluid-B model is a special case of the Giesekus model with \( \alpha = 0 \).

7.3 The single mode FENE-P model

Recall that the reformulated leading order dimensionless form of the FENE-P constitutive model (6.28) is

\[
\begin{align*}
\dot{T}_{rr} &= \dot{T}^a_{rr} + \dot{T}^p_{rr}, \quad \dot{T}_{zz} = \dot{T}^a_{zz} + \dot{T}^p_{zz} \\
\dot{T}^a_{rr} &= -S\ddot{u}_{zz}, \quad \dot{T}^a_{zz} = 2S\ddot{u}_{zz} \\
\Phi\dot{T}^p_{rr} &= \Lambda(\ddot{u}_z\dot{T}^p_{zz,\dot{z}} + \ddot{u}_{zz,\dot{z}}\dot{T}^p_{rr}) - \Pi\frac{\Lambda f_0}{b\Phi r_0^2}\ddot{u}_z(\dot{T}^p_{zz,\dot{z}} + 2\dot{T}^p_{rr,\dot{z}}) \left[\dot{T}^p_{rr} + (1 - \xi b)\frac{Z}{\Lambda}\right] \\
&= - (1 - \xi b)Z\ddot{u}_{zz} \\
\Phi\dot{T}^p_{zz} &= \Lambda(\ddot{u}_z\dot{T}^p_{zz,\dot{z}} - 2\ddot{u}_{zz,\dot{z}}\dot{T}^p_{rr}) - \Pi\frac{\Lambda f_0}{b\Phi r_0^2}\ddot{u}_z(\dot{T}^p_{zz,\dot{z}} + 2\dot{T}^p_{rr,\dot{z}}) \left[\dot{T}^p_{zz} + (1 - \xi b)\frac{Z}{\Lambda}\right] \\
&= 2(1 - \xi b)Z\ddot{u}_{zz} \\
\end{align*}
\]

(7.7)

where

\[
\Phi = 1 + \frac{3}{b} \left[1 + \Pi\frac{f_0}{3r_0^2}(\dot{T}^p_{zz} + 2\dot{T}^p_{rr})\right], \quad \xi = \frac{2}{b(b + 2)}
\]

83
\[ \Lambda = \frac{\lambda v_0}{z_0}, \quad S = \frac{\eta_s \dot{\bar{u}}_s^2 v_0}{z_0 f_0}, \quad Z = \frac{\eta_p \dot{\bar{u}}_p^2 v_0}{z_0 f_0}, \quad \Pi = \frac{\lambda}{\eta_p} \]

7.3.1 Numerical implementation and results of category (F, b, MES, M2, FENE1)

The stress difference \( \bar{T}_{zz}^{mom}(\bar{z}) - \bar{T}_{rr}^{mom}(\bar{z}) \) is produced from the momentum equation (3.32) using the upstream value \((\bar{T}_{zz} - \bar{T}_{rr})(0)\) deduced from eq. (3.38). Trial values of \( \lambda, b, \Pi, (\eta_p = \lambda/\Pi, \eta_s = \eta_0 - \lambda/\Pi) \) and \( \bar{T}_{rr}^p(0) \) are proposed, \( \bar{T}_{rr}^p(0) \) is deduced from eqs. (3.38) and (6.28)1,2,3,4; eqs. (6.28)5,6 are integrated to produce \( \bar{T}_{rr}^p(\bar{z}) \) and \( \bar{T}_{zz}^p(\bar{z}) \). \( \bar{T}_{rr}^s(\bar{z}) \) and \( \bar{T}_{zz}^s(\bar{z}) \) are computed from eqs. (6.28)3,4 directly. The total stress components \( \bar{T}_{rr}^{cons} \) and \( \bar{T}_{zz}^{cons} \) from the constitutive equations are computed through \( \bar{T}_{rr}^{cons}(\bar{z}) = \bar{T}_{rr}^s(\bar{z}) + \bar{T}_{rr}^p(\bar{z}) \) and \( \bar{T}_{zz}^{cons}(\bar{z}) = \bar{T}_{zz}^s(\bar{z}) + \bar{T}_{zz}^p(\bar{z}) \).

The dimension of search space \( \{\lambda, b, \Pi, \bar{T}_{zz}(0)\} \), where \( \lambda \in [0, 5] \) s, \( b \in [10^5, 10^{10}] \), \( \Pi \in [0, 100] \) 1/Pa, and \( \bar{T}_{zz}(0) \in [0, 0.8] \), is four. A set of optimal material parameters \( (\lambda, b, \eta_p, \eta_s) \) and a partition of the in-flow boundary stresses are obtained by using a modified exhaustive search procedure, and minimizing eq. (6.55). \( \eta_p \) and \( \eta_s \) are then given by \( \lambda/\Pi \) and \( \eta_0 - \lambda/\Pi \), respectively. The results are listed in Table 7.3.

7.4 The modified single mode FENE-P model

Recall that the reformulated leading order dimensionless form of the modified FENE-P constitutive model (6.36) is

\[
\bar{T}_{rr} = \bar{T}_{rr}^s + \bar{T}_{rr}^p, \quad \bar{T}_{zz} = \bar{T}_{zz}^s + \bar{T}_{zz}^p \\
\bar{T}_{rr}^s = -S \bar{u}_{s,1}, \quad \bar{T}_{zz}^s = 2S \bar{u}_{s,2} \\
\Phi \bar{T}_{rr}^p + \Lambda(\bar{u}_{s,1}\bar{T}_{rr,1}^p + \bar{u}_{s,2}\bar{T}_{rr,2}^p) + \Pi \frac{\Lambda f_0}{b^2 \eta_p} \bar{u}_{s,1}(\bar{T}_{zz,1}^p + 2\bar{T}_{rr,1}^p) \left[ \bar{T}_{rr} - (1 - \xi b) \frac{Z}{\Lambda} \right]
\]

84
\[ - (1 - \xi b) Z \vec{u}_{z, \tilde{z}} \]

\[ \Phi \tilde{T}_{zz}^p + \Lambda (\vec{u}_{z, \tilde{T}_{zz}} - 2 \vec{u}_{z, \tilde{T}_{zz}}) + \Pi \frac{\Lambda f_0}{b \Phi r_0^2} \vec{u}_{z, \tilde{T}_{zz}} [\tilde{T}_{zz}^p - (1 - \xi b) Z \Lambda] \]

\[ = 2(1 - \xi b) Z \vec{u}_{z, \tilde{z}}, \quad (7.8) \]

where

\[ \Phi = 1 + \frac{3}{b} \left[ 1 - \frac{\Pi f_0}{3 r_0^2} (\tilde{T}_{zz}^p + 2 \tilde{T}_{rr}^p) \right], \quad \xi = \frac{2}{b(b + 2)} \]

\[ \Lambda = \frac{\lambda v_0}{z_0}, \quad S = \frac{\eta r_0^2 v_0}{z_0 f_0}, \quad Z = \frac{\eta r_0^2 v_0}{z_0 f_0}, \quad \Pi = \frac{\lambda}{\eta_p} \]

7.4.1 Numerical implementation and results of category (F, b, MES, M2, MFENE1)

The dimension of search space \( \{\lambda, b, \Pi, \tilde{T}_{zz}(0)\} \), where \( \lambda \in [0, 5] \) s, \( b \in [10^5, 10^{10}] \), \( \Pi \in [0, 100] \) 1/Pa, and \( \tilde{T}_{zz}(0) \in [0, 0.8] \), is four. A set of optimal material parameters \((\lambda, b, \eta_p, \eta_s)\) and a partition of the in-flow boundary stresses are obtained by using a modified exhaustive search procedure, and minimizing eq. (6.55). \( \eta_p \) and \( \eta_s \) are then given by \( \lambda/\Pi \) and \( \eta_0 - \lambda/\Pi \), respectively. The results are listed in Table 7.3.

Comparison between categories (F, b, MES, M2, MFENE1), (F, b, MES, M2, FENE1) and (F, b, MES, M2, Old1)

Twelve of fifteen experiment the final errors using the FENE-P model are larger than those using the Oldroyd fluid-B model. However, the final errors using the modified FENE-P model are always smaller than those using the Oldroyd fluid-B and FENE-P model. Fourteen of fifteen experiments a modified FENE-P model predicts that the polymer viscosities lies between 0.04 and 1.56 Pa s. On the contrary, the Oldroyd fluid-B model predicts that polymer viscosities are 0 Pa s for seven experiments, 11.3
Pa s for six experiments, and other values for the remaining experiments. Ten experiments the FENE-P model predicts that polymer viscosities lies between 0.02 and 0.36 Pa s, and five experiments the FENE-P model predicts that polymer viscosities lies between 10.40 and 11.25 Pa s.

7.5 The two-mode Oldroyd fluid-B model

Recall that the leading order dimensionless form of the two-mode Oldroyd fluid-B constitutive model (6.41) is

\[
\begin{align*}
\bar{T}_{rr} &= \bar{T}_{rr}^{*} + \bar{T}_{rr}^{p1} + \bar{T}_{rr}^{p2}, \quad \bar{T}_{zz} = \bar{T}_{zz}^{*} + \bar{T}_{zz}^{p1} + \bar{T}_{zz}^{p2}, \\
\bar{T}_{rr}^{*} &= -S\bar{u}_{z,z}, \quad \bar{T}_{zz}^{*} = 2S\bar{u}_{z,z}, \\
\bar{T}_{rr}^{p1} + \lambda_{k}(\bar{u}_{z}\bar{T}_{rr}^{p1} + \bar{u}_{z,z}\bar{T}_{rr}^{p2}) &= -Z_{k}\bar{u}_{z,z}, \quad k = 1, 2, \\
\bar{T}_{zz}^{p1} + \lambda_{k}(\bar{u}_{z}\bar{T}_{zz}^{p1} + 2\bar{u}_{z,z}\bar{T}_{zz}^{p2}) &= 2Z_{k}\bar{u}_{z,z}, \quad k = 1, 2
\end{align*}
\]

To integrate the last four equations of eq.(6.41), one needs the upstream boundary stress values \(\bar{T}_{rr}^{p1}\) and \(\bar{T}_{zz}^{p1}\), where \(k = 1, 2\). Formulation a is employed, and the measured quantities are inserted into the following equations:

\[
\begin{align*}
[\bar{T}_{zz}^{p1}(0) + \bar{T}_{zz}^{p2}(0)] - [\bar{T}_{rr}^{p1}(0) + \bar{T}_{rr}^{p2}(0)] &= \bar{T}_{zz}(0) - \bar{T}_{rr}(0) - \bar{T}_{zz}(0) + \bar{T}_{rr}(0), \quad (7.9)
\end{align*}
\]

where \(\bar{T}_{zz}(0) - \bar{T}_{rr}(0)\) is from eq.(3.38), \(\bar{T}_{zz}^{*}(0) = 2S\bar{u}_{z,z}(0)\), and \(\bar{T}_{rr}^{*}(0) = -S\bar{u}_{z,z}(0)\),

\[
\begin{align*}
[\bar{T}_{xx,\bar{z}}^{p1}(0) + \bar{T}_{xx,\bar{z}}^{p2}(0)] - [\bar{T}_{rr,\bar{z}}^{p1}(0) + \bar{T}_{rr,\bar{z}}^{p2}(0)] \\
= \bar{T}_{xx,\bar{z}}(0) - \bar{T}_{rr,\bar{z}}(0) - \bar{T}_{rr,\bar{z}}(0) + \bar{T}_{xx,\bar{z}}(0), \quad (7.10)
\end{align*}
\]

where \(\bar{T}_{xx,\bar{z}}(0) - \bar{T}_{rr,\bar{z}}(0)\) is from eq.(3.39), \(\bar{T}_{xx,\bar{z}}^{*}(0) = 2S\bar{u}_{z,z}(0)\), and \(\bar{T}_{rr,\bar{z}}^{*}(0) = -S\bar{u}_{z,z}(0)\),

\[
\begin{align*}
\bar{T}_{zz}^{p1}(0) + \lambda_{k}[(\bar{u}_{z}(0)\bar{T}_{zz,\bar{z}}^{p1}(0) + 2\bar{u}_{z,z}(0)\bar{T}_{zz,\bar{z}}^{p2}(0)] &= 2Z_{k}\bar{u}_{z,z}(0), \quad k = 1, 2
\end{align*}
\]
\[
\ddot{T}_{rr}^p(0) + \Lambda_k (\ddot{v}_z(0)\dddot{T}_{rr}^p(0) + \dddot{v}_z(0)\dddot{T}_{rr}^p(0)) = -Z_k \ddot{v}_z(0), \quad k = 1, 2. \tag{7.11}
\]

For any trial values \(\lambda_k, \eta_{pa}, (\eta_s = \eta_0 - \eta_{p1} - \eta_{p2}),\) eqs. (7.9) - (7.11) are only six equations but eight unknowns \(\dddot{T}_{rr}^p, \dddot{T}_{rr}^p, \dddot{T}_{rr}^p, \dddot{T}_{rr}^p, \dddot{T}_{rr}^p, \dddot{T}_{rr}^p, \dddot{T}_{rr}^p, \dddot{T}_{rr}^p\) \(k = 1, 2.\) One may make other two assumptions, such as \(\dddot{T}_{rr}^p(0) = \dddot{T}_{rr}^p(0) = 0\) or \(\dddot{T}_{rr}(0) = \dddot{T}_{rr}(0) = 0,\) so that this equation-unknown-system is closed. This method is not employed.

7.5.1 Numerical implementation and results of category (F, b, MES, M2, Old2)

The stress difference \(\dddot{T}_{zz}^\text{mom}(\dddot{z}) - \dddot{T}_{rr}^\text{mom}(\dddot{z})\) is produced from the momentum equation (3.32) using the upstream value \((\dddot{T}_{zz} - \dddot{T}_{rr})(0)\) deduced from eq. (3.38). Trial values of \(\lambda_1, \lambda_2 (\lambda_1 < \lambda_2\) is assumed), \(\eta_{p1}, \eta_{p2} (\eta_s = \eta_0 - \eta_{p1} - \eta_{p2}),\) and \(\dddot{T}_{rr}^p(0), \dddot{T}_{rr}^p(0), \dddot{T}_{rr}^p(0)\) are selected, \(\dddot{T}_{rr}^p(0)\) is deduced from eqs. (3.38) and (6.41) and \(\dddot{T}_{rr}^p(0), \dddot{T}_{rr}^p(0)\) are integrated to produce \(\dddot{T}_{rr}^p(\dddot{z})\) and \(\dddot{T}_{rr}^p(\dddot{z}), \quad k = 1, 2.\)

\(\dddot{T}_{rr}^s(\dddot{z})\) and \(\dddot{T}_{zz}^s(\dddot{z})\) are computed from eqs. (6.41) directly. The total stress components \(\dddot{T}_{rr}^\text{cons} \) and \(\dddot{T}_{zz}^\text{cons}\) from the constitutive equations are computed through

\[
\dddot{T}_{rr}^\text{cons}(\dddot{z}) = \dddot{T}_{rr}^s(\dddot{z}) + \dddot{T}_{rr}^p(\dddot{z}) + \dddot{T}_{rr}^p(\dddot{z}) \quad \text{and} \quad \dddot{T}_{zz}^\text{cons}(\dddot{z}) = \dddot{T}_{zz}^s(\dddot{z}) + \dddot{T}_{zz}^p(\dddot{z}) + \dddot{T}_{zz}^p(\dddot{z}).
\]

The order of search space \(\{\lambda_1, \lambda_2, \eta_{p1}, \eta_{p2}, \dddot{T}_{rr}^p(0), \dddot{T}_{rr}^p(0), \dddot{T}_{rr}^p(0)\},\) where \(\lambda_1, \lambda_2 \in [0, 5]\) s with \(\lambda_1 < \lambda_2, \eta_{p1}, \eta_{p2} \in [0, 11.3] \) Pa s, \(\dddot{T}_{rr}^p(0), \dddot{T}_{rr}^p(0) \in [0, 0.8], \dddot{T}_{rr}^p(0) \in [-0.5, 0.5],\) is seven. A set of optimal material constants \((\lambda_1, \lambda_2, \eta_{p1}, \eta_{p2}, \eta_s)\) and a partition of the in-flow stresses are obtained by using a modified exhaustive search procedure and minimizing eq. (6.55). The results are listed in Table 7.4.
7.6 The three-mode Oldroyd fluid-B model

Recall that the leading order dimensionless form of the three-mode Oldroyd fluid-B constitutive model (6.43) is

\[
\begin{align*}
\ddot{T}_{rr} &= \ddot{T}_{rr}^s + \ddot{T}_{rr}^{p1} + \ddot{T}_{rr}^{p2} + \ddot{T}_{rr}^{p3}, \\
\ddot{T}_{zz} &= \ddot{T}_{zz}^s + \ddot{T}_{zz}^{p1} + \ddot{T}_{zz}^{p2} + \ddot{T}_{zz}^{p3}, \\
\ddot{T}_{rr}^s &= -S\ddot{u}_{zz}, \\
\ddot{T}_{zz}^s &= 2S\ddot{u}_{zz}, \\
\ddot{T}_{rr}^{p_k} + \Lambda_k(\ddot{v}_{zz} - \ddot{T}_{rr}^{p_k} + \ddot{v}_{zz} - \ddot{T}_{rr}^{p_k}) &= -Z_k\ddot{u}_{zz}, \quad k = 1, 3, \\
\ddot{T}_{zz}^{p_k} + \Lambda_k(\ddot{v}_{zz} - \ddot{T}_{zz}^{p_k} - 2\ddot{v}_{zz} + \ddot{T}_{zz}^{p_k}) &= 2Z_k\ddot{u}_{zz}, \quad k = 1, 3.
\end{align*}
\]

7.6.1 Numerical implementation and results of category \((F, b, GA, M2, Old3)\)

The stress difference \(\ddot{T}_{zz}^{mom}(\ddot{z}) - \ddot{T}_{zz}^{mom}(\ddot{z})\) is produced from the momentum equation (3.32) using the upstream value \(\ddot{T}_{zz}(0) - \ddot{T}_{rr}(0)\) deduced from eq. (3.38). Trial values of \(\lambda_1, \lambda_2, \lambda_3 (\lambda_1 < \lambda_2 < \lambda_3\) is assumed), \(\eta_{p1}, \eta_{p2}, \eta_{p3}\) \((\eta_s = \eta_0 - \eta_{p1} - \eta_{p2} - \eta_{p3})\), and \(\ddot{T}_{zz}^{p1}(0), \ddot{T}_{zz}^{p2}(0), \ddot{T}_{zz}^{p3}(0)\) are selected, \(\ddot{T}_{zz}^{p3}(0)\) is deduced from eqs. (6.43) and (6.43) are integrated to produce \(\ddot{T}_{zz}^{ph}(\ddot{z})\) and \(\ddot{T}_{zz}^{ph}(\ddot{z})\), \(k = 1, 2, 3. \ddot{T}_{rr}^{s}(\ddot{z})\) and \(\ddot{T}_{zz}^{s}(\ddot{z})\) are computed from eqs. (6.43) directly. The total stress components of \(\ddot{T}_{zz}^{cons}\) and \(\ddot{T}_{zz}^{cons}\) from the constitutive equations are computed through \(\ddot{T}_{zz}^{cons}(\ddot{z}) = \ddot{T}_{rr}^{s}(\ddot{z}) + \ddot{T}_{zz}^{s}(\ddot{z}) + \ddot{T}_{zz}^{p1}(\ddot{z})\) and \(\ddot{T}_{zz}^{cons}(\ddot{z}) = \ddot{T}_{zz}^{s}(\ddot{z}) + \ddot{T}_{zz}^{p1}(\ddot{z}) + \ddot{T}_{zz}^{p2}(\ddot{z}) + \ddot{T}_{zz}^{p3}(\ddot{z})\).

The order of search space \(\{\lambda_1, \lambda_2, \lambda_3, \eta_{p1}, \eta_{p2}, \eta_{p3}, \ddot{T}_{zz}^{p1}, \ddot{T}_{zz}^{p2}, \ddot{T}_{zz}^{p3}, \ddot{T}_{rr}^{p1}, \ddot{T}_{zz}^{p2}\}\) where \(\lambda_1, \lambda_2, \lambda_3 \in [0, 5] s\) with \(\lambda_1 < \lambda_2 < \lambda_3, \eta_{p1}, \eta_{p2}, \eta_{p3} \in [0, 11.3] Pa s, \ddot{T}_{zz}^{p1}(0), \ddot{T}_{zz}^{p2}(0), \ddot{T}_{zz}^{p3}(0) \in [0, 0.5] Pa, \ddot{T}_{rr}^{p1}(0), \ddot{T}_{zz}^{p2}(0) \in [-0.5, 0.5],\) is eleven. A set of optimal material constants \((\lambda_k, \eta_{pk}, \eta_s)\) and a partition of the in-flow stresses are obtained by using a genetic
algorithm and minimizing eq. (6.55). The results are listed in Table 7.5.

Comparison among categories (F, b, MEA, M2, Old1), (F, b, MEA, M2, Old2), and (F, b, GA, M2, Old3)

The final errors obtained from the two-mode model are smaller than those produced by the single mode model for all experiments. Only one experiment that the three-mode model using a genetic algorithm get a smaller error than that achieved from the two-mode model.

The relaxation time $\lambda$ obtained from the single mode model are between $\lambda_1$ and $\lambda_2$ for two-mode model, or between $\lambda_1$ and $\lambda_3$ for three-mode model. Figures 7.14, 7.15 and 7.16 display the optimal relaxation times of single mode and two-mode Oldroyd fluid-B models versus windup rate, filament length, and flow rate, respectively. Figures 7.17, 7.18 and 7.19 display the optimal relaxation times of three-mode Oldroyd fluid-B models versus windup rate, filament length, and flow rate, respectively. Figure 7.14 and 7.17 show the inverse relationship of largest relaxation time to windup rate; fig. 7.15 displays a proportional relationship of largest relaxation time to filament length; fig. 7.16 and 7.19 display a nonmonotonic change of largest relaxation time with increasing flow rate.

7.7 The modified two-mode FENE-P model

Recall that the reformulated leading order dimensionless form of a modified two-mode FENE-P constitutive model (6.44) is

\[
\dot{T}_{rr} = \dot{T}^a + \dot{T}^{p1} + \dot{T}^{p2}, \quad \dot{T}_{zz} = \dot{T}^a + \dot{T}^{p1} + \dot{T}^{p2},
\]
\[ \tilde{T}_{rr}^s = -S\tilde{u}_{\tilde{z},\tilde{z}}, \quad \tilde{T}_{zz}^s = 2S\tilde{u}_{\tilde{z},\tilde{z}}, \]

\[ \Phi_k \tilde{T}_{rr}^p + \Lambda_k (\tilde{u}_{\tilde{z}}\tilde{T}_{rr}^p + \tilde{u}_{\tilde{z},\tilde{z}} \tilde{T}_{rr}^p) + \Pi_k \frac{\Lambda_k f_0}{b_k \Phi_k r_0^2} \tilde{u}_{\tilde{z}} (\tilde{T}_{zz}^p + 2\tilde{T}_{rr}^p) . \]

\[ \left[ \tilde{T}_{rr}^p - (1 - \xi_k b_k) \frac{Z_k}{\Lambda_k} \right] = -(1 - \xi_k b_k) Z_k \tilde{u}_{\tilde{z},\tilde{z}}, \]

\[ \Phi_k \tilde{T}_{zz}^p + \Lambda_k (\tilde{u}_{\tilde{z}}\tilde{T}_{zz}^p - 2\tilde{u}_{\tilde{z},\tilde{z}} \tilde{T}_{zz}^p) + \Pi_k \frac{\Lambda_k f_0}{b_k \Phi_k r_0^2} \tilde{u}_{\tilde{z}} (\tilde{T}_{zz}^p + 2\tilde{T}_{rr}^p) . \]

\[ \left[ \tilde{T}_{zz}^p - (1 - \xi_k b_k) \frac{Z_k}{\Lambda_k} \right] = 2(1 - \xi_k b_k) Z_k \tilde{u}_{\tilde{z},\tilde{z}}, \]

\[ \Phi_k = 1 + \frac{3}{b_k} \left[ 1 - \Pi_k \frac{f_0}{3r_0^2} (\tilde{T}_{zz}^p + 2\tilde{T}_{rr}^p) \right], \]

with

\[ \Lambda_k = \frac{\lambda_k v_0}{z_0}, \quad S = \frac{\eta_p r_0^2 v_0}{z_0 f_0}, \quad Z_k = \frac{\eta_p r_0^2 v_0}{z_0 f_0}, \quad \Pi_k = \frac{\lambda_k}{\eta_p} . \]

### 7.7.1 Numerical implementation and results of category (F, b, GA, M2, MFENE2)

The stress difference \( \tilde{T}_{zz}^{mom}(\tilde{z}) - \tilde{T}_{rr}^{mom}(\tilde{z}) \) is produced from the momentum equation (3.32) using the upstream value \( \tilde{T}_{zz}(0) - \tilde{T}_{rr}(0) \) deduced from eq. (3.38). Trial values of \( \lambda_1, \lambda_2, b_1, b_2, \Pi_1, \Pi_2, (\eta_{pb} = \lambda_k / \Pi_k, \eta_s = \eta_0 - \eta_{p1} - \eta_{p2}), \) and \( \tilde{T}_{zz}^p(0), \tilde{T}_{rr}^p(0), \) and \( \tilde{T}_{zz}^{pi}(0) \) are selected, \( \tilde{T}_{zz}^{p2}(0) \) is deduced from eqs. (3.38) and (6.44)\_1,2,3,4; eqs. (6.44)\_5,6 are integrated to produce \( \tilde{T}_{rr}^{p3}(\tilde{z}) \) and \( \tilde{T}_{zz}^{p3}(\tilde{z}), k = 1, 2. \) \( \tilde{T}_{rr}^s(\tilde{z}) \) and \( \tilde{T}_{zz}^s(\tilde{z}) \) are computed from eqs. (6.44)\_3,4 directly. The total stress components of \( \tilde{T}_{zz}^{cons} \) and \( \tilde{T}_{zz}^{cons} \) from the constitutive equations are computed through

\[ \tilde{T}_{zz}^{cons}(\tilde{z}) = \tilde{T}_{rr}^s(\tilde{z}) + \tilde{T}_{rr}^{p3}(\tilde{z}) + \tilde{T}_{rr}^{p2}(\tilde{z}) \] and \( \tilde{T}_{zz}^{cons}(\tilde{z}) = \tilde{T}_{zz}^s(\tilde{z}) + \tilde{T}_{zz}^{p3}(\tilde{z}) + \tilde{T}_{zz}^{p2}(\tilde{z}). \)

The order of search space \( \{ \lambda_1, \lambda_2, b_1, b_2, \Pi_1, \Pi_2, \tilde{T}_{zz}^p(0), \tilde{T}_{zz}^{p2}(0), \tilde{T}_{rr}^{p3}(0) \} \), where \( \lambda_1, \lambda_2 \in [0, 5], b_1, b_2 \in [0, 10^7], \Pi_1, \Pi_2 \in [0, 100] \) 1/Pa, \( \tilde{T}_{zz}^p(0), \tilde{T}_{zz}^{p2}(0) \in [0, 0.5], \tilde{T}_{rr}^{p3}(0) \in [-0.5, 0.5] \) is nine. A set of optimal material constants \( (\lambda_k, b_k \eta_{pb}, \eta_s) \)
and a partition of the in-flow stresses are obtained by using a genetic algorithm and minimizing eq. (6.55); the \( \eta_p \) are then given by \( \eta_p = \lambda_k / \Pi_k \), \( \eta_s = \eta_0 - \eta_{p1} - \eta_{p2} \). The results are listed in Table 7.4

Comparison among categories (F, b, MES, M2, MFENE1) and (F, b, GA, M2, MFENE2)

Eight of fifteen experiments the errors \( h \) computed through the modified two-mode FENE-P model using a genetic algorithm are smaller than those from single-mode FENE-P model using a modified exhaustive search. Twelve experiments the relaxation times \( \lambda \) for the single mode model are between \( \lambda_1 \) and \( \lambda_2 \) for two-mode model except exp.4-7-3, 4-8-3 and 4-3-1. Figure 7.20, 7.21, and 7.22 display the optimal relaxation times of modified single mode FENE-P model and modified two-mode FENE-P model versus windup rate, filament length, and flow rate, respectively. Figure 7.20 shows the inverse relationship of largest relaxation time to windup rate; fig. 7.21 and 7.22 display a nonmonotonic change of largest relaxation time with increasing filament length and flow rate.

7.8 Comparison of stress ratios at \( \tilde{z} = 0 \)

Table 7.6 and 7.7 display the stress ratios for each experiment with different constitutive forms. In single mode models, all the stress ratios lie between -0.1157 and 0.0471 for the Oldroyd fluid-B and Giesekus models using formulation a, the stress ratios lie between -0.1967 and 0.2298 for the Oldroyd fluid-B model using formulation b, the stress ratios lie between -0.1617 and 0.1179 for the FENE-P model, and the stress ratios lie between -0.2101 and 0.1188 for the modified FENE-P model. Keunings
et al. [23] analyzed the profile development in continuous drawing of viscoelastic fluids and investigated that for UCM model and Phan-Thien and Tanner model the ratio lies between -0.5 and 0.

In multimode models, the polymer stress ratios are within a wide range, however, the stress ratios are between -0.1269 and 0.2635 for two-mode Oldroyd fluid-B model, the stress ratios are between -0.1745 and 0.2867 for three-mode Oldroyd fluid-B model, and the stress ratios are between -0.1627 and -0.0238 for a modified two-mode FENE-P model.

7.9 Summary and conclusions

In the inverse problem for material characterization applied to elongational free surface filament flow three methods of implementation have created,

- Method 1-f, which search for the smallest difference between the free surface profile that is a solution of the coupled momentum/constitutive equations and the measured profile,

- Method 1-s, which search for the smallest error between the normal stress difference produced by the coupled momentum/constitutive equations and the normal stress difference produced by inserting the measured profile into the momentum equation, and

- Method 2, which differentiates the measured profile to produce all kinematical quantities that appear in the momentum and constitutive equations, thereby decoupling these equations, and adjusts the constitutive equation to achieve the
best match of the stress predicted by the constitutive model to that that must be there to satisfy the momentum equation.

Two different techniques have been employed to remove the high-frequency noise introduced into fine scale free surface measurement by the video apparatus,

- Fitting of the profile in each digitized image to a polynomial function, followed by (in Method 2) analytical differentiation through two orders,

- Filtering of the experimental data, followed by (in Method 2) numerical differentiation through two orders.

Two different formulations have investigated to determine the in-flow boundary stress components,

- formulation a: calculation of upstream boundary stresses entirely from measurements at \( \tilde{z} = 0 \),

- formulation b: calculation of upstream boundary stresses using measurements along entire spinline.

Three optimal seeking algorithms have been employed to find a set of optimal material constants,

- exhaustive search algorithm,

- modified exhaustive search algorithm,

- genetic algorithm.

Seven constitutive forms also have been proposed to characterize this test fluid.
• single mode Oldroyd fluid-B model,
• two-mode Oldroyd fluid-B model,
• three-mode Oldroyd fluid-B model,
• single mode Giesekus model.
• single mode FENE-P model,
• modified single mode FENE-P model,
• modified two-mode FENE-P model.

There are total 168 different combinations to characterize the test fluid, and 14 categories are investigated in this study. The results of this dissertation research are:
(1) Filtering of the experimental data is better than fitting to continuous functions, in the sense of better reproducibility and less sensitivity to numerical noise.
(2) The decoupled inverse method (Method 2) produces at least as good a material characterization, with much less computational cost, as the standard approach of solving the coupled momentum/constitutive equations with trial coefficients until a best fit is obtained between the computed and experimentally measured profiles.
(3) Best single characterization averaged over the entire set of experiments for Oldroyd fluid-B model does a poor job of characterizing the experiments separately. (4) Formulation b is better than formulation a.
(5) Modified exhaustive search always gives smaller error when both modified exhaustive search algorithm and genetic algorithm are employed.
(6) In characterization, \( \alpha \) is found to be zero in the single mode Giesekus model.
(7) Twelve of fifteen experiment the errors produced by the single FENE-P model
are larger than those produced by the single mode Oldroyd fluid-B model.

(8) The modified single and two-mode FENE-P model are better constitutive models, in the sense of smaller error predictions, and less sensitivity to polymer viscosities predictions.

(9) Those constitutive forms are not good choices as general characterization of the test fluid over the range of elongational flow conditions studied (relaxation time, which under the constitutive forms construct should be a material constant, is revealed in these experiments to have strong flow dependence).
<table>
<thead>
<tr>
<th>exp.</th>
<th>method</th>
<th>$\lambda$ (s)</th>
<th>$\eta_p$ (Pa s)</th>
<th>$\eta_s$ (Pa s)</th>
<th>errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-1-3</td>
<td>1-/$f$, poly</td>
<td>3.6098</td>
<td>0.00</td>
<td>11.30</td>
<td>11.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.6666</td>
<td>0.00</td>
<td>11.30</td>
<td>11.9</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>3.6607</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-/$f$, filter</td>
<td>3.6907</td>
<td>0.00</td>
<td>11.30</td>
<td>5.52</td>
</tr>
<tr>
<td></td>
<td>1-/$s$, filter</td>
<td>3.7328</td>
<td>0.00</td>
<td>11.30</td>
<td>5.94</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>3.6849</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td>4-1-3a</td>
<td>1-/$f$, poly</td>
<td>3.5860</td>
<td>0.00</td>
<td>11.30</td>
<td>11.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.6343</td>
<td>0.00</td>
<td>11.30</td>
<td>11.7</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>3.6346</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-/$f$, filter</td>
<td>3.6768</td>
<td>0.00</td>
<td>11.30</td>
<td>5.11</td>
</tr>
<tr>
<td></td>
<td>1-/$s$, filter</td>
<td>3.7175</td>
<td>0.00</td>
<td>11.30</td>
<td>5.51</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>3.6849</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td>4-2-3</td>
<td>1-/$f$, poly</td>
<td>2.9665</td>
<td>0.00</td>
<td>11.30</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.1682</td>
<td>0.00</td>
<td>11.30</td>
<td>10.6</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>3.0758</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-/$f$, filter</td>
<td>3.0007</td>
<td>10.43</td>
<td>0.87</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>1-/$s$, filter</td>
<td>2.9995</td>
<td>6.50</td>
<td>4.80</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>2.9999</td>
<td>8.35</td>
<td>2.95</td>
<td>-</td>
</tr>
<tr>
<td>4-3-3</td>
<td>1-/$f$, poly</td>
<td>2.2400</td>
<td>0.00</td>
<td>11.30</td>
<td>7.36</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.1830</td>
<td>0.00</td>
<td>11.30</td>
<td>8.77</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>2.1688</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-/$f$, filter</td>
<td>2.2779</td>
<td>0.00</td>
<td>11.30</td>
<td>4.04</td>
</tr>
<tr>
<td></td>
<td>1-/$s$, filter</td>
<td>2.3329</td>
<td>0.00</td>
<td>11.30</td>
<td>5.22</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>2.2731</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td>4-4-3</td>
<td>1-/$f$, poly</td>
<td>2.0540</td>
<td>0.00</td>
<td>11.30</td>
<td>2.55</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.9327</td>
<td>9.28</td>
<td>2.02</td>
<td>10.1</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>1.9851</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-/$f$, filter</td>
<td>2.1101</td>
<td>0.00</td>
<td>11.30</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td>1-/$s$, filter</td>
<td>2.0984</td>
<td>9.30</td>
<td>2.00</td>
<td>0.56</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>2.0988</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 7.1: Optimal values of material constants $\lambda$, $\eta_p$, and $\eta_s$ of the single mode Oldroyd fluid-B constitutive model, as returned by all of the different methods (Methods 1-/$f$, 1-/$s$, and 2), using either polynomial curve fitting or filtering; corresponding errors (error $f$ for Method 1-/$f$, error $s$ for Method 1-/$s$, and error $h$ for Method 2 are minimized to identify the optimal material constants). Continued
<table>
<thead>
<tr>
<th>exp.</th>
<th>method</th>
<th>$\lambda$ (s)</th>
<th>$\eta_p$ (Pa s)</th>
<th>$\eta_s$ (Pa s)</th>
<th>$f$ ($10^{-5}$)</th>
<th>$s$ ($10^{-3}$)</th>
<th>$h$ ($10^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-5-3</td>
<td>1-f, poly</td>
<td>1.8423</td>
<td>0.00</td>
<td>11.30</td>
<td>2.55</td>
<td>0.15</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-s, poly</td>
<td>1.8525</td>
<td>0.00</td>
<td>11.30</td>
<td>2.60</td>
<td>0.14</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>1.8685</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>0.21</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>1.8154</td>
<td>8.56</td>
<td>2.74</td>
<td>0.45</td>
<td>0.51</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>1.7685</td>
<td>8.46</td>
<td>2.84</td>
<td>0.99</td>
<td>0.10</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>1.7823</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>0.11</td>
</tr>
<tr>
<td>4-6-3</td>
<td>1-f, poly</td>
<td>1.4027</td>
<td>0.00</td>
<td>11.30</td>
<td>2.58</td>
<td>6.51</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-s, poly</td>
<td>1.3329</td>
<td>7.21</td>
<td>4.09</td>
<td>7.65</td>
<td>1.90</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>1.3609</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>1.52</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>1.4657</td>
<td>7.40</td>
<td>3.90</td>
<td>2.99</td>
<td>6.48</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>1.4045</td>
<td>7.30</td>
<td>4.00</td>
<td>5.06</td>
<td>1.68</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>1.4367</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>2.02</td>
</tr>
<tr>
<td>4-6-3a</td>
<td>1-f, poly</td>
<td>1.4073</td>
<td>7.29</td>
<td>4.01</td>
<td>2.48</td>
<td>7.06</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-s, poly</td>
<td>1.3254</td>
<td>7.19</td>
<td>4.11</td>
<td>9.69</td>
<td>2.60</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>1.3586</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>1.55</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>1.4696</td>
<td>7.37</td>
<td>3.93</td>
<td>2.33</td>
<td>3.63</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>1.4196</td>
<td>7.30</td>
<td>4.00</td>
<td>4.05</td>
<td>1.30</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>1.4469</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>1.53</td>
</tr>
<tr>
<td>4-7-3</td>
<td>1-f, poly</td>
<td>1.0721</td>
<td>0.00</td>
<td>11.3</td>
<td>18.3</td>
<td>2.08</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-s, poly</td>
<td>1.0939</td>
<td>0.00</td>
<td>11.3</td>
<td>19.3</td>
<td>0.96</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>1.0525</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>2.29</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>1.1921</td>
<td>6.51</td>
<td>4.79</td>
<td>5.11</td>
<td>10.1</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>1.1391</td>
<td>6.49</td>
<td>4.81</td>
<td>9.01</td>
<td>4.93</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>1.1720</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>5.53</td>
</tr>
<tr>
<td>4-8-3</td>
<td>1-f, poly</td>
<td>0.9631</td>
<td>4.36</td>
<td>6.34</td>
<td>3.92</td>
<td>44.1</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-s, poly</td>
<td>0.7708</td>
<td>4.13</td>
<td>7.17</td>
<td>151</td>
<td>9.60</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>0.9350</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>8.53</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>0.9757</td>
<td>4.90</td>
<td>6.40</td>
<td>5.61</td>
<td>11.9</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>0.7381</td>
<td>5.00</td>
<td>6.30</td>
<td>290</td>
<td>4.35</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>0.9674</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>7.20</td>
</tr>
</tbody>
</table>
Table 7.1 continued

<table>
<thead>
<tr>
<th>exp.</th>
<th>method</th>
<th>λ (s)</th>
<th>ηp (Pa s)</th>
<th>ηs (Pa s)</th>
<th>V p</th>
<th>errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(10^{-5}) s (10^{-3}) h (10^{-3})</td>
<td></td>
</tr>
<tr>
<td>4-8-3a</td>
<td>1-f, poly</td>
<td>0.9759</td>
<td>0.00</td>
<td>11.3</td>
<td>2.14</td>
<td>11.7</td>
</tr>
<tr>
<td></td>
<td>1-s, poly</td>
<td>0.9085</td>
<td>4.14</td>
<td>7.16</td>
<td>13.0</td>
<td>1.73</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>0.9527</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>0.9910</td>
<td>4.95</td>
<td>6.35</td>
<td>3.67</td>
<td>4.20</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>0.8819</td>
<td>4.98</td>
<td>6.32</td>
<td>55.8</td>
<td>1.98</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>0.9861</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1-s,</td>
<td>poly</td>
<td></td>
<td></td>
<td></td>
<td>1.11</td>
<td></td>
</tr>
<tr>
<td>poly</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2.95</td>
<td></td>
</tr>
<tr>
<td>1-/,</td>
<td>filter</td>
<td></td>
<td></td>
<td></td>
<td>1.11</td>
<td></td>
</tr>
<tr>
<td>2,</td>
<td>filter</td>
<td></td>
<td></td>
<td></td>
<td>2.95</td>
<td></td>
</tr>
<tr>
<td>1-f,</td>
<td>poly</td>
<td>2.3860</td>
<td>0.00</td>
<td>11.30</td>
<td>2.73</td>
<td>2.15</td>
</tr>
<tr>
<td>1-s,</td>
<td>poly</td>
<td>2.3341</td>
<td>0.00</td>
<td>11.30</td>
<td>3.76</td>
<td>1.08</td>
</tr>
<tr>
<td>2,</td>
<td>poly</td>
<td>2.3525</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>2.3999</td>
<td>0.00</td>
<td>11.30</td>
<td>1.88</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>2.4288</td>
<td>0.00</td>
<td>11.30</td>
<td>2.19</td>
<td>0.19</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>2.3926</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1-s,</td>
<td>filter</td>
<td></td>
<td></td>
<td></td>
<td>0.38</td>
<td></td>
</tr>
<tr>
<td>2,</td>
<td>filter</td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>1-/,</td>
<td>poly</td>
<td>2.4695</td>
<td>0.00</td>
<td>11.30</td>
<td>0.67</td>
<td>3.35</td>
</tr>
<tr>
<td>1-s,</td>
<td>poly</td>
<td>2.4215</td>
<td>9.48</td>
<td>1.82</td>
<td>1.29</td>
<td>0.89</td>
</tr>
<tr>
<td>2,</td>
<td>poly</td>
<td>2.4368</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>2.5000</td>
<td>7.12</td>
<td>4.18</td>
<td>0.26</td>
<td>1.14</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>2.5103</td>
<td>0.00</td>
<td>11.30</td>
<td>0.27</td>
<td>1.05</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>2.5086</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1-s,</td>
<td>filter</td>
<td></td>
<td></td>
<td></td>
<td>0.66</td>
<td></td>
</tr>
<tr>
<td>2,</td>
<td>filter</td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>1-f,</td>
<td>poly</td>
<td>2.4503</td>
<td>0.00</td>
<td>11.30</td>
<td>6.59</td>
<td>39.9</td>
</tr>
<tr>
<td>1-s,</td>
<td>poly</td>
<td>2.2289</td>
<td>9.14</td>
<td>2.16</td>
<td>24.6</td>
<td>4.72</td>
</tr>
<tr>
<td>2,</td>
<td>poly</td>
<td>2.3239</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>2.5421</td>
<td>0.00</td>
<td>11.30</td>
<td>1.01</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>2.5285</td>
<td>9.66</td>
<td>1.64</td>
<td>1.05</td>
<td>0.67</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>2.5212</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1-s,</td>
<td>filter</td>
<td></td>
<td></td>
<td></td>
<td>0.58</td>
<td></td>
</tr>
<tr>
<td>2,</td>
<td>filter</td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>1-f,</td>
<td>poly</td>
<td>2.0947</td>
<td>0.00</td>
<td>11.30</td>
<td>5.04</td>
<td>35.0</td>
</tr>
<tr>
<td>1-s,</td>
<td>poly</td>
<td>2.4564</td>
<td>0.00</td>
<td>11.30</td>
<td>60.8</td>
<td>1.40</td>
</tr>
<tr>
<td>2,</td>
<td>poly</td>
<td>2.2451</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>2.1188</td>
<td>0.00</td>
<td>11.30</td>
<td>3.26</td>
<td>1.34</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>2.1611</td>
<td>0.00</td>
<td>11.30</td>
<td>4.10</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>2.1167</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1-s,</td>
<td>filter</td>
<td></td>
<td></td>
<td></td>
<td>0.58</td>
<td></td>
</tr>
<tr>
<td>2,</td>
<td>filter</td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>
Table 7.1 continued

<table>
<thead>
<tr>
<th>exp.</th>
<th>method</th>
<th>$\lambda$ (s)</th>
<th>$\eta_p$ (Pa s)</th>
<th>$\eta_s$ (Pa s)</th>
<th>errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$f$ (10^{-5})</td>
</tr>
<tr>
<td>4-3-1</td>
<td>1-f, poly</td>
<td>1.4844</td>
<td>0.00</td>
<td>11.30</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>1-s, poly</td>
<td>1.7375</td>
<td>0.00</td>
<td>11.30</td>
<td>16.0</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>1.6810</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>1.4642</td>
<td>0.00</td>
<td>11.30</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>1.4665</td>
<td>0.00</td>
<td>11.30</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>1.4607</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td>4-3-2</td>
<td>1-f, poly</td>
<td>2.0089</td>
<td>0.00</td>
<td>11.30</td>
<td>1.76</td>
</tr>
<tr>
<td></td>
<td>1-s, poly</td>
<td>2.1746</td>
<td>0.00</td>
<td>11.30</td>
<td>13.1</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>2.0884</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>2.0396</td>
<td>0.00</td>
<td>11.30</td>
<td>0.07</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>2.0570</td>
<td>0.00</td>
<td>11.30</td>
<td>0.08</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>2.0400</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td>4-3-4</td>
<td>1-f, poly</td>
<td>3.1427</td>
<td>10.44</td>
<td>0.86</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>1-s, poly</td>
<td>3.2501</td>
<td>0.00</td>
<td>11.30</td>
<td>2.88</td>
</tr>
<tr>
<td></td>
<td>2, poly</td>
<td>3.1664</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1-f, filter</td>
<td>3.1640</td>
<td>10.35</td>
<td>0.95</td>
<td>0.43</td>
</tr>
<tr>
<td></td>
<td>1-s, filter</td>
<td>3.1455</td>
<td>10.45</td>
<td>0.85</td>
<td>0.54</td>
</tr>
<tr>
<td></td>
<td>2, filter</td>
<td>3.1633</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
</tr>
<tr>
<td>exp.</td>
<td>$\lambda$ (s)</td>
<td>$\eta_p$ (Pa s)</td>
<td>$\eta_s$ (Pa s)</td>
<td>error $h$ ($10^{-4}$)</td>
<td></td>
</tr>
<tr>
<td>------</td>
<td>---------------</td>
<td>----------------</td>
<td>----------------</td>
<td>------------------</td>
<td></td>
</tr>
<tr>
<td>4-1-3</td>
<td>3.6873</td>
<td>0.00</td>
<td>11.30</td>
<td>47.1</td>
<td></td>
</tr>
<tr>
<td>4-2-3</td>
<td>3.0004</td>
<td>8.45</td>
<td>2.85</td>
<td>1.27</td>
<td></td>
</tr>
<tr>
<td>4-3-3</td>
<td>2.2731</td>
<td>0.00</td>
<td>11.30</td>
<td>1.29</td>
<td></td>
</tr>
<tr>
<td>4-4-3</td>
<td>2.0996</td>
<td>10.14</td>
<td>1.16</td>
<td>1.41</td>
<td></td>
</tr>
<tr>
<td>4-5-3</td>
<td>1.7816</td>
<td>8.64</td>
<td>2.66</td>
<td>1.07</td>
<td></td>
</tr>
<tr>
<td>4-6-3</td>
<td>1.4362</td>
<td>9.72</td>
<td>1.58</td>
<td>20.3</td>
<td></td>
</tr>
<tr>
<td>4-7-3</td>
<td>1.1732</td>
<td>3.20</td>
<td>8.10</td>
<td>55.3</td>
<td></td>
</tr>
<tr>
<td>4-8-3</td>
<td>0.9645</td>
<td>10.99</td>
<td>0.31</td>
<td>73.4</td>
<td></td>
</tr>
<tr>
<td>4-3-3x</td>
<td>2.3927</td>
<td>0.00</td>
<td>11.30</td>
<td>3.76</td>
<td></td>
</tr>
<tr>
<td>2-3-3</td>
<td>2.5208</td>
<td>8.10</td>
<td>3.20</td>
<td>5.97</td>
<td></td>
</tr>
<tr>
<td>1-3-3</td>
<td>2.5086</td>
<td>0.00</td>
<td>11.30</td>
<td>9.97</td>
<td></td>
</tr>
<tr>
<td>8-3-3</td>
<td>2.1274</td>
<td>0.00</td>
<td>11.30</td>
<td>12.1</td>
<td></td>
</tr>
<tr>
<td>4-3-1</td>
<td>1.4621</td>
<td>3.77</td>
<td>7.53</td>
<td>0.009</td>
<td></td>
</tr>
<tr>
<td>4-3-2</td>
<td>2.0405</td>
<td>0.00</td>
<td>11.30</td>
<td>0.61</td>
<td></td>
</tr>
<tr>
<td>4-3-4</td>
<td>3.1644</td>
<td>9.61</td>
<td>1.69</td>
<td>4.05</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.2: Optimal values of $\lambda$, $\eta_p$, $\eta_s$ for single mode Oldroyd fluid-B model, and the corresponding errors, using a Genetic Algorithm with 200 initial populations and mutation rate $\mu = 0.2$. 
<table>
<thead>
<tr>
<th>exp.</th>
<th>model</th>
<th>$\lambda$ (s)</th>
<th>$\eta_p$ (Pa.s)</th>
<th>$\eta_s$ (Pa.s)</th>
<th>$\alpha$</th>
<th>$b$</th>
<th>error $h$ ($10^{-4}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-1-3</td>
<td>Oldroyd-B&lt;sup&gt;a&lt;/sup&gt;</td>
<td>3.6849</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>47.0</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B&lt;sup&gt;b&lt;/sup&gt;</td>
<td>3.9575</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>5.64</td>
</tr>
<tr>
<td></td>
<td>Giesekus&lt;sup&gt;a&lt;/sup&gt;</td>
<td>3.6849</td>
<td>0.00</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>47.0</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sup&gt;b&lt;/sup&gt;</td>
<td>3.8058</td>
<td>11.13</td>
<td>0.17</td>
<td>-</td>
<td>$10^{10}$</td>
<td>15.4</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sub&gt;m&lt;/sub&gt;&lt;sup&gt;b&lt;/sup&gt;</td>
<td>3.5032</td>
<td>0.18</td>
<td>11.12</td>
<td>-</td>
<td>$10^{8}$</td>
<td>0.10</td>
</tr>
<tr>
<td>4-2-3</td>
<td>Oldroyd-B&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.9999</td>
<td>8.35</td>
<td>2.95</td>
<td>-</td>
<td>-</td>
<td>1.26</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.9965</td>
<td>7.15</td>
<td>4.15</td>
<td>-</td>
<td>-</td>
<td>1.25</td>
</tr>
<tr>
<td></td>
<td>Giesekus&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.9999</td>
<td>8.35</td>
<td>2.95</td>
<td>0.00</td>
<td>-</td>
<td>12.6</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sup&gt;b&lt;/sup&gt;</td>
<td>3.1184</td>
<td>0.19</td>
<td>11.11</td>
<td>-</td>
<td>$10^{6}$</td>
<td>3.39</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sub&gt;m&lt;/sub&gt;&lt;sup&gt;b&lt;/sup&gt;</td>
<td>3.0959</td>
<td>0.10</td>
<td>11.20</td>
<td>-</td>
<td>$10^{8}$</td>
<td>0.85</td>
</tr>
<tr>
<td>4-3-3</td>
<td>Oldroyd-B&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.2731</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>12.9</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.4578</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>0.87</td>
</tr>
<tr>
<td></td>
<td>Giesekus&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.2731</td>
<td>0.00</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>12.9</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.5028</td>
<td>0.02</td>
<td>11.28</td>
<td>-</td>
<td>$10^{7}$</td>
<td>5.17</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sub&gt;m&lt;/sub&gt;&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.4474</td>
<td>0.04</td>
<td>11.26</td>
<td>-</td>
<td>$10^{10}$</td>
<td>0.59</td>
</tr>
<tr>
<td>4-4-3</td>
<td>Oldroyd-B&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.0988</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>1.40</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.0788</td>
<td>8.30</td>
<td>3.00</td>
<td>-</td>
<td>-</td>
<td>1.20</td>
</tr>
<tr>
<td></td>
<td>Giesekus&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.0988</td>
<td>11.30</td>
<td>0.00</td>
<td>$8.85 \times 10^{-9}$</td>
<td>-</td>
<td>1.26</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.1087</td>
<td>0.07</td>
<td>11.28</td>
<td>-</td>
<td>$10^{7}$</td>
<td>1.54</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sub&gt;m&lt;/sub&gt;&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.1031</td>
<td>0.04</td>
<td>11.26</td>
<td>-</td>
<td>$10^{7}$</td>
<td>0.79</td>
</tr>
<tr>
<td>4-5-3</td>
<td>Oldroyd-B&lt;sup&gt;a&lt;/sup&gt;</td>
<td>1.7823</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>1.07</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B&lt;sup&gt;b&lt;/sup&gt;</td>
<td>1.7097</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>Giesekus&lt;sup&gt;a&lt;/sup&gt;</td>
<td>1.7833</td>
<td>11.30</td>
<td>0.00</td>
<td>$8.85 \times 10^{-9}$</td>
<td>-</td>
<td>1.06</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sup&gt;b&lt;/sup&gt;</td>
<td>1.7894</td>
<td>0.02</td>
<td>11.28</td>
<td>-</td>
<td>$10^{7}$</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sub&gt;m&lt;/sub&gt;&lt;sup&gt;b&lt;/sup&gt;</td>
<td>1.7978</td>
<td>0.04</td>
<td>11.26</td>
<td>-</td>
<td>$10^{7}$</td>
<td>0.08</td>
</tr>
</tbody>
</table>

Table 7.3: Optimal constitutive parameters of single mode Oldroyd fluid-B, Giesekus, FENE-P, and modified FENE-P model for each experiment and their corresponding error, subscript a: formulation a, b: formulation b, subscript m: modified. Continued
<table>
<thead>
<tr>
<th>exp.</th>
<th>model</th>
<th>$\lambda$ (s)</th>
<th>$\eta_p$ (Pa s)</th>
<th>$\eta_s$ (Pa s)</th>
<th>$\alpha$</th>
<th>$b$</th>
<th>error $h$ ($10^{-4}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-6-3</td>
<td>Oldroyd-B®</td>
<td>1.4367</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>20.2</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B®</td>
<td>1.3394</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>6.90</td>
</tr>
<tr>
<td></td>
<td>Giesekus®</td>
<td>1.4367</td>
<td>11.3</td>
<td>0.00</td>
<td>0.00</td>
<td>-</td>
<td>20.2</td>
</tr>
<tr>
<td></td>
<td>FENE-P®</td>
<td>1.4440</td>
<td>0.36</td>
<td>10.94</td>
<td>-</td>
<td>$10^6$</td>
<td>12.3</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>1.3299</td>
<td>0.04</td>
<td>11.26</td>
<td>-</td>
<td>$10^7$</td>
<td>0.82</td>
</tr>
<tr>
<td>4-7-3</td>
<td>Oldroyd-B®</td>
<td>1.1720</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>55.3</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B®</td>
<td>1.0979</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>26.5</td>
</tr>
<tr>
<td></td>
<td>Giesekus®</td>
<td>1.1727</td>
<td>11.30</td>
<td>0.00</td>
<td>8.85 x $10^{-9}$</td>
<td>-</td>
<td>55.1</td>
</tr>
<tr>
<td></td>
<td>FENE-P®</td>
<td>1.0952</td>
<td>10.95</td>
<td>0.35</td>
<td>-</td>
<td>$10^{10}$</td>
<td>26.0</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>1.2098</td>
<td>0.06</td>
<td>11.24</td>
<td>-</td>
<td>$10^7$</td>
<td>1.61</td>
</tr>
<tr>
<td>4-8-3</td>
<td>Oldroyd-B®</td>
<td>0.9675</td>
<td>11.3</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>72.0</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B®</td>
<td>0.8822</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>23.6</td>
</tr>
<tr>
<td></td>
<td>Giesekus®</td>
<td>0.9684</td>
<td>11.3</td>
<td>0.00</td>
<td>8.85 x $10^{-9}$</td>
<td>-</td>
<td>71.3</td>
</tr>
<tr>
<td></td>
<td>FENE-P®</td>
<td>0.9001</td>
<td>11.25</td>
<td>0.05</td>
<td>-</td>
<td>$10^{10}$</td>
<td>46.0</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>0.9072</td>
<td>0.09</td>
<td>11.21</td>
<td>-</td>
<td>$10^7$</td>
<td>5.71</td>
</tr>
<tr>
<td>4-3-3x</td>
<td>Oldroyd-B®</td>
<td>2.3926</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>3.78</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B®</td>
<td>2.4344</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>1.23</td>
</tr>
<tr>
<td></td>
<td>Giesekus®</td>
<td>2.3926</td>
<td>0.00</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>3.78</td>
</tr>
<tr>
<td></td>
<td>FENE-P®</td>
<td>2.4947</td>
<td>0.05</td>
<td>11.25</td>
<td>-</td>
<td>$10^7$</td>
<td>0.68</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>2.4802</td>
<td>0.31</td>
<td>10.99</td>
<td>-</td>
<td>$10^{6}$</td>
<td>0.32</td>
</tr>
<tr>
<td>2-3-3</td>
<td>Oldroyd-B®</td>
<td>2.5212</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>5.80</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B®</td>
<td>2.5006</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>5.77</td>
</tr>
<tr>
<td></td>
<td>Giesekus®</td>
<td>2.5219</td>
<td>11.30</td>
<td>0.00</td>
<td>8.85 x $10^{-9}$</td>
<td>-</td>
<td>5.80</td>
</tr>
<tr>
<td></td>
<td>FENE-P®</td>
<td>2.4970</td>
<td>10.40</td>
<td>0.90</td>
<td>-</td>
<td>$10^{10}$</td>
<td>5.75</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>2.5960</td>
<td>0.26</td>
<td>11.04</td>
<td>-</td>
<td>$10^{6}$</td>
<td>1.80</td>
</tr>
</tbody>
</table>

Continued
Table 7.3 continued

<table>
<thead>
<tr>
<th>exp.</th>
<th>model</th>
<th>$\lambda$ (s)</th>
<th>$\eta_p$ (Pa.s)</th>
<th>$\eta_s$ (Pa.s)</th>
<th>$\alpha$</th>
<th>$b$</th>
<th>error $h$ ($10^{-4}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-3-3</td>
<td>Oldroyd-B&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.5086</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>9.97</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.5413</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>9.06</td>
</tr>
<tr>
<td></td>
<td>Giesekus&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.5086</td>
<td>0.00</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>9.97</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.4998</td>
<td>0.17</td>
<td>11.13</td>
<td>-</td>
<td>$10^{10}$</td>
<td>10.7</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sub&gt;m&lt;/sub&gt;</td>
<td>2.5995</td>
<td>0.26</td>
<td>11.04</td>
<td>-</td>
<td>$10^6$</td>
<td>6.93</td>
</tr>
<tr>
<td>8-3-3</td>
<td>Oldroyd-B&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.1167</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>11.0</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.2266</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>3.13</td>
</tr>
<tr>
<td></td>
<td>Giesekus&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.1167</td>
<td>0.00</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>11.0</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.1910</td>
<td>10.95</td>
<td>0.35</td>
<td>-</td>
<td>$10^{10}$</td>
<td>4.18</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sub&gt;m&lt;/sub&gt;</td>
<td>2.4145</td>
<td>0.12</td>
<td>11.18</td>
<td>-</td>
<td>$10^6$</td>
<td>0.79</td>
</tr>
<tr>
<td>4-3-1</td>
<td>Oldroyd-B&lt;sup&gt;a&lt;/sup&gt;</td>
<td>1.4607</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>0.098</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B&lt;sup&gt;b&lt;/sup&gt;</td>
<td>1.4928</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>0.0052</td>
</tr>
<tr>
<td></td>
<td>Giesekus&lt;sup&gt;a&lt;/sup&gt;</td>
<td>1.4607</td>
<td>0.00</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>0.098</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sup&gt;b&lt;/sup&gt;</td>
<td>1.5084</td>
<td>0.04</td>
<td>11.26</td>
<td>-</td>
<td>$10^7$</td>
<td>0.011</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sub&gt;m&lt;/sub&gt;</td>
<td>1.4897</td>
<td>10.64</td>
<td>0.66</td>
<td>-</td>
<td>$10^7$</td>
<td>0.0047</td>
</tr>
<tr>
<td>4-3-2</td>
<td>Oldroyd-B&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.0400</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>0.61</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.0838</td>
<td>0.00</td>
<td>11.30</td>
<td>-</td>
<td>-</td>
<td>0.11</td>
</tr>
<tr>
<td></td>
<td>Giesekus&lt;sup&gt;a&lt;/sup&gt;</td>
<td>2.0400</td>
<td>0.00</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>0.61</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2.1095</td>
<td>0.06</td>
<td>11.24</td>
<td>-</td>
<td>$10^7$</td>
<td>0.12</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sub&gt;m&lt;/sub&gt;</td>
<td>2.1212</td>
<td>0.04</td>
<td>11.26</td>
<td>-</td>
<td>$10^7$</td>
<td>0.05</td>
</tr>
<tr>
<td>4-3-4</td>
<td>Oldroyd-B&lt;sup&gt;a&lt;/sup&gt;</td>
<td>3.1633</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>Oldroyd-B&lt;sup&gt;b&lt;/sup&gt;</td>
<td>3.1179</td>
<td>11.30</td>
<td>0.00</td>
<td>-</td>
<td>-</td>
<td>2.22</td>
</tr>
<tr>
<td></td>
<td>Giesekus&lt;sup&gt;a&lt;/sup&gt;</td>
<td>3.1633</td>
<td>11.30</td>
<td>0.00</td>
<td>0.00</td>
<td>-</td>
<td>4.00</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sup&gt;b&lt;/sup&gt;</td>
<td>3.2072</td>
<td>0.32</td>
<td>10.98</td>
<td>-</td>
<td>$10^6$</td>
<td>4.31</td>
</tr>
<tr>
<td></td>
<td>FENE-P&lt;sub&gt;m&lt;/sub&gt;</td>
<td>3.1166</td>
<td>1.56</td>
<td>9.74</td>
<td>-</td>
<td>$10^5$</td>
<td>1.01</td>
</tr>
<tr>
<td>Exp. no.</td>
<td>model</td>
<td>$\lambda_1$ (s)</td>
<td>$\lambda_2$ (s)</td>
<td>$\eta_{p1}$ (Pa s)</td>
<td>$\eta_{p2}$ (Pa s)</td>
<td>$\eta_s$ (Pa s)</td>
<td>$b_1, b_2$ ($10^6$)</td>
</tr>
<tr>
<td>---------</td>
<td>----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-------------------</td>
<td>-------------------</td>
<td>-----------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>4-1-3</td>
<td>Oldroyd</td>
<td>2.2845</td>
<td>4.7767</td>
<td>0.0</td>
<td>8.0</td>
<td>3.3</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>2.7258</td>
<td>4.5316</td>
<td>0.0037</td>
<td>0.0108</td>
<td>11.2855</td>
<td>3.1, 2.4</td>
</tr>
<tr>
<td>4-2-3</td>
<td>Oldroyd</td>
<td>0.9990</td>
<td>3.0034</td>
<td>0.0</td>
<td>0.7</td>
<td>10.6</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>2.8392</td>
<td>3.5753</td>
<td>0.0890</td>
<td>0.0912</td>
<td>11.1198</td>
<td>3.7, 8.8</td>
</tr>
<tr>
<td>4-3-3</td>
<td>Oldroyd</td>
<td>1.1295</td>
<td>2.6008</td>
<td>0.0</td>
<td>0.0</td>
<td>11.3</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>1.7108</td>
<td>3.4982</td>
<td>0.0208</td>
<td>0.0387</td>
<td>11.2405</td>
<td>9.4, 7.0</td>
</tr>
<tr>
<td>4-4-3</td>
<td>Oldroyd</td>
<td>0.2485</td>
<td>2.0871</td>
<td>10.5</td>
<td>0.1</td>
<td>0.7</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>1.9211</td>
<td>3.0139</td>
<td>0.0473</td>
<td>0.0440</td>
<td>11.2087</td>
<td>3.7, 4.8</td>
</tr>
<tr>
<td>4-5-3</td>
<td>Oldroyd</td>
<td>1.7019</td>
<td>1.8066</td>
<td>0.0</td>
<td>0.0</td>
<td>11.3</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>1.9550</td>
<td>2.1072</td>
<td>0.0419</td>
<td>0.0211</td>
<td>11.2162</td>
<td>5.0, 8.6</td>
</tr>
<tr>
<td>4-6-3</td>
<td>Oldroyd</td>
<td>1.3097</td>
<td>1.4595</td>
<td>0.0</td>
<td>0.0</td>
<td>11.3</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>1.1627</td>
<td>1.6972</td>
<td>0.0161</td>
<td>0.0822</td>
<td>11.2017</td>
<td>6.3, 5.9</td>
</tr>
<tr>
<td>4-7-3</td>
<td>Oldroyd</td>
<td>1.0571</td>
<td>1.1255</td>
<td>0.0</td>
<td>0.0</td>
<td>11.3</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>1.2984</td>
<td>1.6427</td>
<td>0.0320</td>
<td>0.04667</td>
<td>11.2214</td>
<td>8.3, 8.2</td>
</tr>
<tr>
<td>4-8-3</td>
<td>Oldroyd</td>
<td>0.8517</td>
<td>0.8949</td>
<td>0.0</td>
<td>0.0</td>
<td>11.3</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>1.2164</td>
<td>1.4055</td>
<td>0.0943</td>
<td>0.0173</td>
<td>11.1884</td>
<td>9.4, 7.7</td>
</tr>
<tr>
<td>4-3-3x</td>
<td>Oldroyd</td>
<td>2.3114</td>
<td>2.8505</td>
<td>8.9</td>
<td>0.0</td>
<td>2.4</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>2.3358</td>
<td>2.8052</td>
<td>0.0640</td>
<td>0.0352</td>
<td>11.2008</td>
<td>6.7, 8.6</td>
</tr>
<tr>
<td>2-3-3</td>
<td>Oldroyd</td>
<td>2.4695</td>
<td>2.5382</td>
<td>0.0</td>
<td>0.0</td>
<td>11.3</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>2.4619</td>
<td>3.0483</td>
<td>0.0631</td>
<td>0.1061</td>
<td>11.1308</td>
<td>3.8, 3.0</td>
</tr>
<tr>
<td>1-3-3</td>
<td>Oldroyd</td>
<td>1.6177</td>
<td>2.9861</td>
<td>0.0</td>
<td>0.0</td>
<td>11.3</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>2.0891</td>
<td>2.9657</td>
<td>0.0724</td>
<td>0.0961</td>
<td>11.1315</td>
<td>9.2, 9.7</td>
</tr>
<tr>
<td>8-3-3</td>
<td>Oldroyd</td>
<td>1.3139</td>
<td>2.7865</td>
<td>0.0</td>
<td>0.0</td>
<td>11.3</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>1.2949</td>
<td>2.9258</td>
<td>0.0644</td>
<td>0.0602</td>
<td>11.1754</td>
<td>3.7, 9.3</td>
</tr>
<tr>
<td>4-3-1</td>
<td>Oldroyd</td>
<td>0.3959</td>
<td>1.4997</td>
<td>1.1</td>
<td>10.2</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>1.5730</td>
<td>2.4263</td>
<td>0.2577</td>
<td>0.0338</td>
<td>11.0085</td>
<td>3.1, 9.2</td>
</tr>
<tr>
<td>4-3-2</td>
<td>Oldroyd</td>
<td>1.6550</td>
<td>2.2636</td>
<td>11.3</td>
<td>0.0</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>2.0278</td>
<td>2.9702</td>
<td>0.0950</td>
<td>0.0469</td>
<td>11.1581</td>
<td>2.3, 5.5</td>
</tr>
<tr>
<td>4-3-4</td>
<td>Oldroyd</td>
<td>2.8986</td>
<td>4.4270</td>
<td>7.0</td>
<td>0.0</td>
<td>4.3</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>FENE-P$_m$</td>
<td>2.3614</td>
<td>3.6472</td>
<td>0.1221</td>
<td>0.0423</td>
<td>11.1356</td>
<td>5.3, 2.9</td>
</tr>
</tbody>
</table>

Table 7.4: Optimal constitutive parameters of two-mode Oldroyd fluid-B and modified FENE-P model for each experiment and their corresponding error. Note that the optimal values of FENE-P model obtained from Genetic Algorithm with 10000 initial populations and mutation rate 0.2.
<table>
<thead>
<tr>
<th>Exp. No.</th>
<th>$\lambda_1$ (s)</th>
<th>$\lambda_2$ (s)</th>
<th>$\lambda_3$ (s)</th>
<th>$\eta_{lp_1}$ (Pa s)</th>
<th>$\eta_{lp_2}$ (Pa s)</th>
<th>$\eta_{lp_3}$ (Pa s)</th>
<th>$\eta_s$ (Pa s)</th>
<th>error ($10^{-4}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-1-3</td>
<td>2.0602</td>
<td>3.7500</td>
<td>4.9163</td>
<td>3.88</td>
<td>0.15</td>
<td>5.69</td>
<td>1.58</td>
<td>0.88</td>
</tr>
<tr>
<td>4-2-3</td>
<td>1.8617</td>
<td>2.7309</td>
<td>3.5110</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>11.30</td>
<td>1.52</td>
</tr>
<tr>
<td>4-3-3</td>
<td>1.1844</td>
<td>2.4280</td>
<td>3.0387</td>
<td>2.24</td>
<td>0.00</td>
<td>0.92</td>
<td>8.14</td>
<td>0.20</td>
</tr>
<tr>
<td>4-4-3</td>
<td>1.9607</td>
<td>2.2294</td>
<td>2.6061</td>
<td>2.36</td>
<td>1.38</td>
<td>0.00</td>
<td>7.56</td>
<td>1.48</td>
</tr>
<tr>
<td>4-5-3</td>
<td>1.2345</td>
<td>1.7235</td>
<td>2.1644</td>
<td>1.04</td>
<td>0.00</td>
<td>1.62</td>
<td>8.64</td>
<td>0.89</td>
</tr>
<tr>
<td>4-6-3</td>
<td>0.8256</td>
<td>1.2471</td>
<td>2.0712</td>
<td>2.27</td>
<td>0.75</td>
<td>0.00</td>
<td>8.28</td>
<td>11.8</td>
</tr>
<tr>
<td>4-7-3</td>
<td>0.9871</td>
<td>1.0523</td>
<td>2.0369</td>
<td>0.25</td>
<td>0.53</td>
<td>2.26</td>
<td>8.26</td>
<td>30.7</td>
</tr>
<tr>
<td>4-8-3</td>
<td>0.7296</td>
<td>0.8390</td>
<td>1.0922</td>
<td>5.44</td>
<td>1.31</td>
<td>0.00</td>
<td>4.55</td>
<td>30.0</td>
</tr>
<tr>
<td>4-4-3x</td>
<td>1.2183</td>
<td>2.4182</td>
<td>2.7928</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>11.30</td>
<td>3.19</td>
</tr>
<tr>
<td>2-3-3</td>
<td>1.7403</td>
<td>2.1934</td>
<td>3.3517</td>
<td>0.61</td>
<td>1.62</td>
<td>2.84</td>
<td>6.23</td>
<td>8.07</td>
</tr>
<tr>
<td>1-3-3</td>
<td>0.1500</td>
<td>0.7800</td>
<td>2.5311</td>
<td>1.12</td>
<td>0.00</td>
<td>0.00</td>
<td>10.18</td>
<td>8.70</td>
</tr>
<tr>
<td>8-3-3</td>
<td>1.6236</td>
<td>2.0653</td>
<td>2.6274</td>
<td>1.33</td>
<td>1.37</td>
<td>0.07</td>
<td>8.53</td>
<td>3.21</td>
</tr>
<tr>
<td>4-3-1</td>
<td>0.7790</td>
<td>2.8963</td>
<td>3.3995</td>
<td>3.26</td>
<td>0.00</td>
<td>0.00</td>
<td>8.04</td>
<td>0.05</td>
</tr>
<tr>
<td>4-3-2</td>
<td>1.0470</td>
<td>1.2014</td>
<td>2.2361</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>11.30</td>
<td>0.35</td>
</tr>
<tr>
<td>4-3-4</td>
<td>0.7673</td>
<td>2.5789</td>
<td>3.3387</td>
<td>2.90</td>
<td>0.00</td>
<td>4.10</td>
<td>4.30</td>
<td>7.90</td>
</tr>
</tbody>
</table>

Table 7.5: Optimal constitutive parameters of three-mode Oldroyd fluid-B model for each experiment and their corresponding error. Note that the optimal values obtained from Genetic Algorithm with 10000 initial populations and mutation rate 0.2.
<table>
<thead>
<tr>
<th>Exp. No.</th>
<th>Oldroyd-B(^a)</th>
<th>Oldroyd-B(^b)</th>
<th>Giesekus(^a)</th>
<th>FENE-P(^g)</th>
<th>FENE-P(^b)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(T_{xx}(0)) (T_{rr}(0))</td>
<td>(T_{xx}(0)) (T_{rr}(0))</td>
<td>(T_{xx}(0)) (T_{rr}(0))</td>
<td>(T_{xx}(0)) (T_{rr}(0))</td>
<td>(T_{xx}(0)) (T_{rr}(0))</td>
</tr>
<tr>
<td>4-1-3</td>
<td>-0.0448</td>
<td>-0.1967</td>
<td>-0.0448</td>
<td>-0.1322</td>
<td>-0.0460</td>
</tr>
<tr>
<td>4-2-3</td>
<td>-0.0730</td>
<td>-0.0704</td>
<td>-0.0730</td>
<td>-0.1354</td>
<td>-0.0815</td>
</tr>
<tr>
<td>4-3-3</td>
<td>-0.0203</td>
<td>-0.1781</td>
<td>-0.0202</td>
<td>-0.0018</td>
<td>-0.1633</td>
</tr>
<tr>
<td>4-4-3</td>
<td>-0.1087</td>
<td>-0.0885</td>
<td>-0.0270</td>
<td>-0.1146</td>
<td>-0.0818</td>
</tr>
<tr>
<td>4-5-3</td>
<td>-0.1157</td>
<td>-0.0601</td>
<td>-0.1156</td>
<td>-0.1355</td>
<td>-0.0668</td>
</tr>
<tr>
<td>4-6-3</td>
<td>-0.0819</td>
<td>0.0552</td>
<td>-0.0819</td>
<td>-0.0359</td>
<td>-0.0357</td>
</tr>
<tr>
<td>4-7-3</td>
<td>-0.0519</td>
<td>0.0815</td>
<td>-0.0518</td>
<td>0.0871</td>
<td>-0.0229</td>
</tr>
<tr>
<td>4-8-3</td>
<td>0.0471</td>
<td>0.2298</td>
<td>0.0472</td>
<td>0.1179</td>
<td>0.1188</td>
</tr>
<tr>
<td>4-3-3x</td>
<td>-0.0499</td>
<td>-0.0841</td>
<td>-0.0499</td>
<td>-0.1337</td>
<td>-0.1015</td>
</tr>
<tr>
<td>2-3-3</td>
<td>-0.0242</td>
<td>0.0000</td>
<td>-0.0241</td>
<td>-0.0032</td>
<td>-0.0531</td>
</tr>
<tr>
<td>1-3-3</td>
<td>-0.0422</td>
<td>-0.0823</td>
<td>-0.0422</td>
<td>-0.0339</td>
<td>-0.0911</td>
</tr>
<tr>
<td>8-3-3</td>
<td>-0.0778</td>
<td>-0.1697</td>
<td>-0.0778</td>
<td>-0.1501</td>
<td>-0.2101</td>
</tr>
<tr>
<td>4-3-1</td>
<td>-0.0479</td>
<td>-0.0646</td>
<td>-0.0479</td>
<td>-0.0976</td>
<td>-0.0631</td>
</tr>
<tr>
<td>4-3-2</td>
<td>-0.0976</td>
<td>-0.1004</td>
<td>-0.0976</td>
<td>-0.1617</td>
<td>-0.1243</td>
</tr>
<tr>
<td>4-3-3</td>
<td>-0.0800</td>
<td>-0.0467</td>
<td>-0.0800</td>
<td>-0.0880</td>
<td>-0.0382</td>
</tr>
</tbody>
</table>

Table 7.6: Stress ratios at \(\dot{\varepsilon} = 0\) of single mode models: Oldroyd fluid-B, Giesekus, FENE-P, and modified FENE-P for each experiment.
<table>
<thead>
<tr>
<th>Exp.</th>
<th>model</th>
<th>$T_x^1(0)$</th>
<th>$T_x^2(0)$</th>
<th>$T_x^3(0)$</th>
<th>$T_{xx}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-1-3</td>
<td>Oldroyd-B</td>
<td>0.0250</td>
<td>0.0000</td>
<td>-</td>
<td>0.0113</td>
</tr>
<tr>
<td>3 Oldroyd-B</td>
<td>1.0166</td>
<td>-0.5971</td>
<td>-0.3268</td>
<td>-0.0831</td>
<td></td>
</tr>
<tr>
<td>2 FENE-P</td>
<td>0.2839</td>
<td>-0.4053</td>
<td>-</td>
<td>-0.0704</td>
<td></td>
</tr>
<tr>
<td>4-2-3</td>
<td>Oldroyd-B</td>
<td>-0.4667</td>
<td>0.0353</td>
<td>-</td>
<td>0.0007</td>
</tr>
<tr>
<td>3 Oldroyd-B</td>
<td>0.7846</td>
<td>-0.0757</td>
<td>-0.2652</td>
<td>-0.0087</td>
<td></td>
</tr>
<tr>
<td>2 FENE-P</td>
<td>-0.0176</td>
<td>-0.3556</td>
<td>-</td>
<td>-0.0660</td>
<td></td>
</tr>
<tr>
<td>4-3-3</td>
<td>Oldroyd-B</td>
<td>0.0000</td>
<td>0.0000</td>
<td>-</td>
<td>-0.0015</td>
</tr>
<tr>
<td>3 Oldroyd-B</td>
<td>2.8826</td>
<td>-0.2045</td>
<td>-1.7652</td>
<td>-0.1255</td>
<td></td>
</tr>
<tr>
<td>2 FENE-P</td>
<td>0.1690</td>
<td>-1.2233</td>
<td>-</td>
<td>-0.1095</td>
<td></td>
</tr>
<tr>
<td>4-4-3</td>
<td>Oldroyd-B</td>
<td>0.0000</td>
<td>-0.0059</td>
<td>-</td>
<td>-0.0055</td>
</tr>
<tr>
<td>3 Oldroyd-B</td>
<td>0.5908</td>
<td>-1.1804</td>
<td>-2.2610</td>
<td>-0.0905</td>
<td></td>
</tr>
<tr>
<td>2 FENE-P</td>
<td>0.1882</td>
<td>-1.4695</td>
<td>-</td>
<td>-0.0960</td>
<td></td>
</tr>
<tr>
<td>4-5-3</td>
<td>Oldroyd-B</td>
<td>-0.1667</td>
<td>0.9420</td>
<td>-</td>
<td>-0.0609</td>
</tr>
<tr>
<td>3 Oldroyd-B</td>
<td>4.0838</td>
<td>-0.9317</td>
<td>-1.2413</td>
<td>-0.1019</td>
<td></td>
</tr>
<tr>
<td>2 FENE-P</td>
<td>0.3261</td>
<td>-3.4268</td>
<td>-</td>
<td>-0.1395</td>
<td></td>
</tr>
<tr>
<td>4-6-3</td>
<td>Oldroyd-B</td>
<td>-0.7394</td>
<td>2.9788</td>
<td>-</td>
<td>0.0530</td>
</tr>
<tr>
<td>3 Oldroyd-B</td>
<td>3.1503</td>
<td>-0.1969</td>
<td>-0.9763</td>
<td>0.0886</td>
<td></td>
</tr>
<tr>
<td>2 FENE-P</td>
<td>1.5497</td>
<td>-0.4082</td>
<td>-</td>
<td>-0.1188</td>
<td></td>
</tr>
<tr>
<td>4-7-3</td>
<td>Oldroyd-B</td>
<td>-0.1000</td>
<td>0.9153</td>
<td>-</td>
<td>0.1285</td>
</tr>
<tr>
<td>3 Oldroyd-B</td>
<td>0.8747</td>
<td>-0.1082</td>
<td>-5.0869</td>
<td>0.1347</td>
<td></td>
</tr>
<tr>
<td>2 FENE-P</td>
<td>0.0592</td>
<td>-0.3593</td>
<td>-</td>
<td>-0.1271</td>
<td></td>
</tr>
<tr>
<td>4-8-3</td>
<td>Oldroyd-B</td>
<td>0.1667</td>
<td>0.4525</td>
<td>-</td>
<td>0.2635</td>
</tr>
<tr>
<td>3 Oldroyd-B</td>
<td>0.9383</td>
<td>-0.9680</td>
<td>-0.1148</td>
<td>0.2867</td>
<td></td>
</tr>
<tr>
<td>2 FENE-P</td>
<td>0.5813</td>
<td>-1.9018</td>
<td>-</td>
<td>-0.1416</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.7: Stress ratios at $z = 0$ of two-mode Oldroyd fluid-B, three-mode Oldroyd fluid-B, and modified two-mode FENE-P for each experiment. Continued
Table 7.7 continued

<table>
<thead>
<tr>
<th>Exp.</th>
<th>model</th>
<th>$T_{\eta_1}(0)$</th>
<th>$T_{\eta_2}(0)$</th>
<th>$T_{\eta_3}(0)$</th>
<th>$T_{\eta_4}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-3-3x</td>
<td>2 Oldroyd-B</td>
<td>-0.645</td>
<td>1.5524</td>
<td>-</td>
<td>-0.0949</td>
</tr>
<tr>
<td></td>
<td>3 Oldroyd-B</td>
<td>1.0606</td>
<td>-0.9759</td>
<td>-0.0850</td>
<td>-0.0410</td>
</tr>
<tr>
<td></td>
<td>2 FENE-P</td>
<td>0.1034</td>
<td>-0.645</td>
<td>-</td>
<td>-0.1084</td>
</tr>
<tr>
<td>2-3-3</td>
<td>2 Oldroyd-B</td>
<td>0.0000</td>
<td>0.0000</td>
<td>-</td>
<td>-0.0016</td>
</tr>
<tr>
<td></td>
<td>3 Oldroyd-B</td>
<td>-2.0370</td>
<td>0.3870</td>
<td>0.8299</td>
<td>0.0826</td>
</tr>
<tr>
<td></td>
<td>2 FENE-P</td>
<td>0.1963</td>
<td>-1.5499</td>
<td>-</td>
<td>-0.0438</td>
</tr>
<tr>
<td>1-3-3</td>
<td>2 Oldroyd-B</td>
<td>-0.6746</td>
<td>1.1416</td>
<td>-</td>
<td>0.1509</td>
</tr>
<tr>
<td></td>
<td>3 Oldroyd-B</td>
<td>2.0503</td>
<td>-0.7530</td>
<td>-0.0332</td>
<td>0.1063</td>
</tr>
<tr>
<td></td>
<td>2 FENE-P</td>
<td>1.2008</td>
<td>-1.5304</td>
<td>-</td>
<td>-0.0238</td>
</tr>
<tr>
<td>8-3-3</td>
<td>2 Oldroyd-B</td>
<td>0.0000</td>
<td>-0.0426</td>
<td>-</td>
<td>-0.0243</td>
</tr>
<tr>
<td></td>
<td>3 Oldroyd-B</td>
<td>0.8004</td>
<td>-2.1543</td>
<td>1.2680</td>
<td>-0.1140</td>
</tr>
<tr>
<td></td>
<td>2 FENE-P</td>
<td>0.1013</td>
<td>-0.2929</td>
<td>-</td>
<td>-0.0996</td>
</tr>
<tr>
<td>4-3-1</td>
<td>2 Oldroyd</td>
<td>-0.050</td>
<td>-0.0033</td>
<td>-</td>
<td>-0.0061</td>
</tr>
<tr>
<td></td>
<td>3 Oldroyd-B</td>
<td>0.5218</td>
<td>-0.3166</td>
<td>-0.8358</td>
<td>-0.0615</td>
</tr>
<tr>
<td></td>
<td>2 FENE-P</td>
<td>0.3047</td>
<td>-0.2282</td>
<td>-</td>
<td>-0.1033</td>
</tr>
<tr>
<td>4-3-2</td>
<td>2 Oldroyd-B</td>
<td>0.0000</td>
<td>-0.1788</td>
<td>-</td>
<td>-0.1269</td>
</tr>
<tr>
<td></td>
<td>3 Oldroyd-B</td>
<td>27.441</td>
<td>-46.108</td>
<td>-0.3740</td>
<td>-0.1745</td>
</tr>
<tr>
<td></td>
<td>2 FENE-P</td>
<td>0.3016</td>
<td>-4.1573</td>
<td>-</td>
<td>-0.1627</td>
</tr>
<tr>
<td>4-3-3</td>
<td>2 Oldroyd-B</td>
<td>-0.2250</td>
<td>1.9111</td>
<td>-</td>
<td>-0.0009</td>
</tr>
<tr>
<td></td>
<td>3 Oldroyd-B</td>
<td>5.0919</td>
<td>-1.3003</td>
<td>-0.2603</td>
<td>0.0422</td>
</tr>
<tr>
<td></td>
<td>2 FENE-P</td>
<td>-2.3365</td>
<td>0.6003</td>
<td>-</td>
<td>-0.1498</td>
</tr>
</tbody>
</table>
Figure 7.1: Exps. 4-1-3, 4-2-3, and 4-3-3: Best fits to Oldroyd fluid-B using Method 1-f (best fit between measured and computed free surface profile) with polynomial curve fitting (- - -) and filtering (-----), measured profile (—) and computed profiles (top set is exp. 4-1-3, middle set is exp. 4-2-3, and bottom set is exp. 4-3-3).
Figure 7.2: Exps. 4-1-3, 4-2-3, and 4-3-3: Best fits to Oldroyd fluid-B using Method 1-f (best fit between measured and computed free surface profile) with polynomial curve fitting (-----) and filtering (•••), normal stress difference computed from the momentum equation by the measured profile (—) and from coupled momentum and constitutive equations (top set is exp. 4-3-3, middle set is exp. 4-2-3, and bottom set is exp. 4-1-3).
Figure 7.3: Exps. 4-1-3, 4-2-3, and 4-3-3: Best fits to Oldroyd fluid-B using Method 1-s (best fit between stress difference computed from the momentum equation, and coupled momentum and constitutive equations) with polynomial curve fitting (---) and filtering (•••), measured profile (—) and computed profiles (top set is exp. 4-1-3, middle set is exp. 4-2-3, and bottom set is exp. 4-3-3).
Figure 7.4: Exps. 4-1-3, 4-2-3, and 4-3-3: Best fits to Oldroyd fluid-B using Method 1-s (best fit between stress difference computed from the momentum equation and coupled momentum and constitutive equations) with polynomial curve fitting (---) and filtering (···), normal stress difference computed from the momentum equation by the measured profile (——) and from coupled momentum and constitutive equations (top set is exp. 4-3-3, middle set is exp. 4-2-3, and bottom set is exp. 4-1-3).
Figure 7.5: Exps. 4-1-3, 4-2-3, and 4-3-3: Best fits to Oldroyd fluid-B using Method 2 of the normal stress difference computed from the momentum equation (— —) and the constitutive equations with polynomial curve fitting (— --) and filtering (· · ·) (top set is exp. 4-3-3, middle set is exp. 4-2-3, and bottom set is exp. 4-1-3).
Figure 7.6: Optimal relaxation time $\lambda$ versus windup rate using filtering (top plot) and polynomial curve fitting (bottom plot): Method 1-$f$ (o), Method 1-s (x), and Method 2 (+). Note the lesser spread with filtering.
Figure 7.7: Optimal relaxation time $\lambda$ versus nominal filament length: Method 1-$f$ (o), Method 1-s (x), and Method 2 (+) using filtering.
Figure 7.8: Optimal relaxation time $\lambda$ versus flow rate: Method 1- $f$ (o), Method 1- $s$ (x), and Method 2 (+) using filtering.
Figure 7.9: $\Delta \lambda = \lambda^{\text{opt}} - \lambda^{\text{avg}}$ for the 14 experiments, where $\lambda^{\text{opt}}$ is the optimal relaxation time for the experiment and a given method and $\lambda^{\text{avg}}$ is the average value of the 14 optimal relaxation times for that method: Method 1-f ($\circ$): $\lambda^{\text{avg}} = 2.1684$ s, Method 1-s ($\times$): $\lambda^{\text{avg}} = 2.1488$ s, Method 2 ($+$): $\lambda^{\text{avg}} = 2.1590$ s.
Figure 7.10: Errors $f$ (o), $s$ (x), and $h$ (+) for the 14 experiments when $\lambda = \lambda^{\text{avg}}$ and $\eta_p = 0 \text{ Pa s.}$
Figure 7.11: Exp 4-1-3: Measured free surface profile (--), profile computed from the coupled momentum and single mode Oldroyd fluid-B constitutive equations with $\lambda = \lambda^{\text{avg}} = 2.1684$ s, $\eta_p = 0$ Pa s, and $\eta_s = 11.3$ Pa s (---), and profile computed from the coupled momentum and single mode Oldroyd fluid-B constitutive equations with $\lambda = \lambda^{\text{opt}} = 3.6907$ s, $\eta_p = 0$ Pa s, and $\eta_s = 11.3$ Pa s using Method 1-f (- - -).
Figure 7.12: Exp. 4-1-3: Normal stress differences computed from the momentum equation (---), the coupled momentum and single mode Oldroyd fluid-B constitutive equations with \( \lambda = \lambda^{avg} = 2.1488 \) s, \( \eta_p = 0 \) Pa s, and \( \eta_s = 11.3 \) Pa s (---), and the coupled momentum and single mode Oldroyd fluid-B constitutive equations with \( \lambda = \lambda^{opt} = 3.7328 \) s, \( \eta_p = 0 \) Pa s, and \( \eta_s = 11.3 \) Pa s using Method 1-s (---).
Figure 7.13: Exp. 4-1-3: Normal stress differences computed from the momentum equation (---), the single mode Oldroyd fluid-B constitutive equation with $\lambda = \lambda^{avg} = 2.1590 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$ (- - -), and the single mode Oldroyd fluid-B constitutive equation with $\lambda = \lambda^{opt} = 3.6849 \text{ s}$, $\eta_p = 0 \text{ Pa s}$, and $\eta_s = 11.3 \text{ Pa s}$ using Method 2 (- - -).
Figure 7.14: The optimal relaxation time $\lambda$ ($+$) of single mode Oldroyd fluid-B model and the optimal relaxation times $\lambda_1$ ($*$) and $\lambda_2$ (o) of two-mode Oldroyd fluid-B model versus wind-up rate. Note that $\lambda_1 < \lambda_2$. 
Figure 7.15: The optimal relaxation time $\lambda$ (+) of single mode Oldroyd fluid-B model and the optimal relaxation times $\lambda_1$ (x) and $\lambda_2$ (o) of two-mode Oldroyd fluid-B model versus filament length. Note that $\lambda_1 < \lambda_2$. 
Figure 7.16: The optimal relaxation time $\lambda$ (+) of single mode Oldroyd fluid-B model and the optimal relaxation times $\lambda_1$ (×) and $\lambda_2$ (o) of two-mode Oldroyd fluid-B model versus flow rate $Q$. Note that $\lambda_1 < \lambda_2$. 
Figure 7.17: The optimal relaxation times $\lambda_1 (\circ)$, $\lambda_2 (\ast)$ and $\lambda_3 (\circ)$ of three-mode Oldroyd fluid-B model versus wind-up rate. Note that $\lambda_1 < \lambda_2 < \lambda_3$. 
Figure 7.18: The optimal relaxation times $\lambda_1$ (o), $\lambda_2$ (★) and $\lambda_3$ (o) of three-mode Oldroyd fluid-B model versus filament length. Note that $\lambda_1 < \lambda_2 < \lambda_3$. 
Figure 7.19: The optimal relaxation times $\lambda_1 (\circ)$, $\lambda_2 (\star)$, and $\lambda_3 (\diamond)$ of three-mode Oldroyd fluid-B model versus flow rate $Q$. Note that $\lambda_1 < \lambda_2 < \lambda_3$. 
Figure 7.20: The optimal relaxation time \( \lambda (+) \) of modified single mode FENE-P model and the optimal relaxation times \( \lambda_1 (★) \) and \( \lambda_2 (o) \) of modified two mode FENE-P model versus wind-up rate. Note that \( \lambda_1 < \lambda_2 \).
Figure 7.21: The optimal relaxation time $\lambda$ (+) of modified single mode FENE-P model and the optimal relaxation times $\lambda_1$ (*) and $\lambda_2$ (o) of modified two mode FENE-P model versus filament length. Note that assumed that $\lambda_1 < \lambda_2$. 
Figure 7.22: The optimal relaxation time $\lambda$ (\(+\)) of modified single mode FENE-P model and the optimal relaxation times $\lambda_1$ (\(\ast\)) and $\lambda_2$ (\(\circ\)) of modified two mode FENE-P model versus flow rate $Q$. Note that $\lambda_1 < \lambda_2$. 
CHAPTER 8

INVERSE FORMULATION AND IMPLEMENTATION OF MATERIAL CHARACTERIZATION PROCEDURE II

A goal for elongational characterization is to find the measurement of material response, for instance viscosity and relaxation time, in stretching-type flows. In previous chapter three inverse methods have been employed to determine a set of material constants for an assumed constitutive form that relates the stress at any point to the strain or the strain rate. In this chapter a new characterization method is proposed to deduce a constitutive functional form, as well as the values of coefficients in this form; the only assumption is that the stress depends only on strain and strain rate. Section 8.1 describes how a constitutive functional form is constructed by this new characterization method. Explicitly, a constitutive form as well as the values of the coefficients in this form are deduced by computing the paths of different experiment of a fluid in elongational viscosity/strain/strain-rate space, and then fitting these paths to a surface in this space. The Trouton or elongational viscosity [7, 31, 40] is defined as:

\[ \overline{\eta} = \frac{T_{zz} - T_{rr}}{\dot{\varepsilon}}. \]  

(8.1)

The numerical implementation and result are presented in section 8.2.
8.1 Elongational viscosity as a function of strain $\varepsilon$ and strain rate $\dot{\varepsilon}$

With fluid density $\rho$, surface tension $\sigma$, gravity $g$, and its slope $\phi, z$ known from the measurement as well as the characteristic scales $r_0, z_0, f_0$ selected, the stress difference $T_{zz} - T_{rr} = f_0^2 (\bar{T}_{zz} - \bar{T}_{rr})$ is integrated along the filament entirely from the momentum consideration

$$\left(\bar{T}_{zz} - \bar{T}_{rr}\right)\ddot{z} + \frac{1}{BW} \frac{\ddot{\phi} z}{\phi^2} + 2\left(\bar{T}_{zz} - \bar{T}_{rr}\right) \frac{\ddot{\phi} z}{\phi} + \frac{1}{BF} = 0,$$

with

$$\frac{1}{BW} = \frac{\sigma r_0}{f_0}, \quad \frac{1}{BF} = \frac{\rho gr_0^2 z_0}{f_0}.$$  \hfill (8.3)

The extensional Hencky strain $\varepsilon(z)$ along the filament is deduced directly through

$$\varepsilon(z) = \ln \tilde{u}_z(z) = \ln \frac{1}{\phi^2(z)}.$$  \hfill (8.4)

The strain rate $\dot{\varepsilon}$ is computed from $\dot{u}_{sz} = \frac{\dot{z} \tilde{u}_s}{\tilde{\phi} z}$ where $\tilde{u}_s, \tilde{\phi}$ is obtained by differentiating a global continuous function

$$\tilde{p}(\tilde{z}) = \tilde{p}_0 + \tilde{p}_1 \tilde{z} + \tilde{p}_2 \tilde{z}^2 + \tilde{p}_3 \tilde{z}^3,$$  \hfill (8.5)

where $\tilde{p}_k, k = 0, 1, 2, 3$, are computed from all data points of $\tilde{u}_s, \tilde{\phi}$.

A constitutive functional form with its coefficients for a single fluid is deduced by fitting the paths of many different flows of this fluid to a surface in stress/strain/strain-rate space. The only assumption is that stress depends only on strain and strain rate. The proposed constitutive functional form is

$$\log \eta = (a_0 + a_1 \varepsilon)(b_0 + b_1 \dot{\varepsilon} + b_2 \dot{\varepsilon}^2 + b_3 \dot{\varepsilon}^3),$$  \hfill (8.6)

where $a_0, a_1, b_0, b_1, b_2, b_3$ are constants to be determined.
8.2 Numerical implementation and results

The strains, strain rates, and stress differences are computed for exp. 4-1-3, 4-2-3, 4-3-3, 4-4-3, 4-5-3, 4-6-3, 4-7-3, 4-8-3. The behavior such as strain rates and stress differences for the remaining experiments are within the range of above experiments.

The values of coefficients $a_i$ and $b_i$ are obtained by using a modified exhaustive search algorithm and minimizing

$$\Upsilon = \frac{1}{N_{\text{total}}} \sum_{k=1}^{N_{\text{total}}} \left[ \log_{10}(\eta)_k - (a_0 + a_1\varepsilon_k)(b_0 + b_1\dot{\varepsilon}_k + b_2\dot{\varepsilon}_k^2 + b_3\dot{\varepsilon}_k^3) \right]^2,$$

(8.7)

where $N_{\text{total}} = \sum_{j=\text{(j-th experiment)}} \left[ P_j + (a_j-1)+400 \right]/2$.

The optimal surface is $\log_{10}\eta = (1.39 + 0.10\varepsilon)(2.29 + 1.89\dot{\varepsilon} - 0.95\dot{\varepsilon}^2 + 0.013\dot{\varepsilon}^3)$, and the error $\Upsilon$ is $9.30 \times 10^{-3}$.

Figure 8.1 displays 8 paths of experiments in strain/strain-rate plane. Fig. 8.2 presents 8 paths of experiments in viscosity/strain-rate plane. Fig. 8.3 demonstrates 8 paths of experiments in viscosity/strain plane and the stress differences in logarithm scale are proportional to strains. Fig. 8.4 shows 8 paths of experiments in viscosity/strain/strain-rate space. Fig. 8.5 is a surface fitted to the paths of the 8 experiments, the figure shows that this constitutive form predicts elongational viscosity well in a lower strain/strain-rate region.

8.3 Conclusion

Characterization procedure II can produce a characterization of the extensional response of the fluid, both constitutive from and values of coefficients in this from, over

133
a wide range of flow conditions using the momentum equation and flow measurements only.
Figure 8.1: Paths of 8 experiments in strain/strain-rate plane
Figure 8.2: Paths of 8 experiments in elongational viscosity/strain-rate plane
Figure 8.3: Paths of 8 experiments in elongational viscosity/strain plane
Figure 8.4: Paths of 8 experiments in elongational viscosity/strain/strain-rate space
Figure 8.5: Surface fitting of 8 experiments in elongational viscosity/strain/strain-rate space
Figure A.1: Exp. 4-0-3x: measured free surface profile $\phi^{nw}$. 
Figure A.2: Exp. 2-0-3: measured free surface profile $\phi^{nw}$. 
Figure A.3: Exp. 1-0-3: measured free surface profile $\phi_{nw}$. 
Figure A.4: Exp. 8-0-3: measured free surface profile $\phi^w$. 
Figure A.5: Exp. 4-0-1: measured free surface profile $\phi^\text{nw}$.
Figure A.6: Exp. 4-0-2: measured free surface profile $\phi^{nw}$. 
Figure A.7: Exp. 4-0-4: measured free surface profile $\phi^{nw}$. 
Figure A.8: Exp. 4-2-3: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.9: Exp. 4-2-3: piecewise continuous free surface slope $\phi_{zz}^{\text{poly}}(z)$ (---) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{zz}^{\text{fit}}(z_k)$ (---) generated by numerical differentiation of the filtered profile $\phi_{zz}^{\text{fit}}(z_k)$. 
Figure A.10: Exp. 4-2-3: axial velocity $v_z^{exp}(z_k)$ generated by $Q/(\pi \phi^{exp}(z_k))$ and $v_z^{fil}(z_k)$ generated by $Q/(\pi \phi^{fil}(z_k))$. 
Figure A.11: Exp. 4-2-3: piecewise continuous velocity gradient $v_{z,x}^{\text{poly}}(z)$ (---) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z,x}^{\text{global}}(z)$ (· · ·), discrete velocity gradient $v_{z,x}^{\text{filt}}(z_k)$ (——) generated by the numerical differentiation of the filtered velocity $v_{z,x}^{\text{filt}}(z_k)$, and refiltered velocity gradient $v_{z,x}^{\text{filt,filt}}(z_k)$ (· · ·) generated by $v_{z,x}^{\text{filt}}(z_k)$. 

150
Figure A.12: Exp. 4-2-3: continuous second gradient $v_{z,zz}^{\text{global}}(z)$ of velocity (—- ) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,zz}^{\text{global}}(z)$, and discrete second gradient $v_{z,zz}^{\text{refiltered}}(z_k)$ of velocity (—-) generated from the numerical differentiation of refiltered velocity gradient $v_{z,k}^{\text{refiltered}}(z_k)$. 
Figure A.13: Exp. 4-2-3: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
Figure A.14: Exp. 4-3-3: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{filt}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.15: Exp. 4-3-3: piecewise continuous free surface slope $\phi_{xz}^{pol}(z)$ (---) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{z}(z_k)$ (- - -) generated by numerical differentiation of the filtered profile $\phi_{z}^{fil}(z_k)$. 
Figure A.16: Exp. 4-3-3: axial velocity $v_z^{exp}(z_k)$ generated by $Q/(\pi \phi^{exp}(z_k))$ and $v_z^{fli}(z_k)$ generated by $Q/(\pi \phi^{fli}(z_k))$. 
Figure A.17: Exp. 4-3-3: piecewise continuous velocity gradient $v_{z,x}^{poly}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z,x}^{global}(z)$ (···), discrete velocity gradient $v_{z,x}^{fui}(z_k)$ (——) generated by the numerical differentiation of the filtered velocity $v_{z,x}^{fui}(z_k)$, and refiltered velocity gradient $v_{z,x}^{fui/fui}(z_k)$ (··· - · ·) generated by $v_{z,x}^{fui}(z_k)$. 
Figure A.18: Exp. 4-3-3: continuous second gradient $v_{s,zz}^{global}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v_{s,z}^{global}(z)$, and discrete second gradient $v_{s,zz}^{fil}(z_k)$ of velocity (—) generated from the numerical differentiation of refiltered velocity gradient $v_{s,z}^{fil}(z_k)$. 
Figure A.19: Exp. 4-3-3: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
Figure A.20: Exp. 4-4-3: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.21: Exp. 4-4-3: piecewise continuous free surface slope $\phi_{z,x}^{\text{poly}}(z)$ (---) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{z,x}^{\text{ill}}(z_k)$ (—) generated by numerical differentiation of the filtered profile $\phi_{z,x}^{\text{ill}}(z_k)$. 
Figure A.22: Exp. 4-4-3: axial velocity $v_{z}^{\text{exp}}(z_k)$ generated by $Q/(\pi \phi^{\text{exp}}(z_k))$ and $v_{z}^{\text{fit}}(z_k)$ generated by $Q/(\pi \phi^{\text{fit}}(z_k))$. 
Figure A.23: Exp. 4-4-3: piecewise continuous velocity gradient $v_{z, z}^{\text{poly}}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z, z}^{\text{global}}(z)$ (· · ·), discrete velocity gradient $v_{z, z}^{\text{discrete}}(z_k)$ (——) generated by the numerical differentiation of the filtered velocity $v_{z}^{\text{filtered}}(z_k)$, and refiltered velocity gradient $v_{z, z}^{\text{refiltered}}(z_k)$ (· · · ·) generated by $v_{z, z}^{\text{filtered}}(z_k)$. 
Figure A.24: Exp. 4-4-3: continuous second gradient $v_{z,z}^{\text{global}}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,z}^{\text{global}}(z)$, and discrete second gradient $v_{z,z}^{\text{filter}}(z_k)$ of velocity (——) generated from the numerical differentiation of refiltered velocity gradient $v_{z,z}^{\text{filter}}(z_k)$. 
Figure A.25: Exp. 4-4-3: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
Figure A.26: Exp. 4-5-3: measured free surface profile $\phi_{\text{exp}}(z_k)$ and filtered profile $\phi_{\text{fil}}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.27: Exp. 4-5-3: piecewise continuous free surface slope $\phi_{\text{poly}}(z)$ (— —) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{\text{fit}}(z_k)$ (—) generated by numerical differentiation of the filtered profile $\phi_{\text{fit}}(z_k)$. 
Figure A.28: Exp. 4-5-3: axial velocity $v_{z}^{\text{exp}}(z_k)$ generated by $Q/(\pi \phi^{\text{exp}}(z_k))$ and $v_{z}^{\text{fib}}(z_k)$ generated by $Q/(\pi \phi^{\text{fib}}(z_k))$. 
Figure A.29: Exp. 4-5-3: piecewise continuous velocity gradient $u_{zz}^{\text{poly}}(z)$ (---) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $u_{zz}^{\text{global}}(z)$ (· · ·), discrete velocity gradient $u_{zz}^{f\tilde{i}}(z_k)$ (——) generated by the numerical differentiation of the filtered velocity $u_{zz}^{f\tilde{i}}(z_k)$, and refiltered velocity gradient $u_{zz}^{f\tilde{i}f\tilde{i}}(z_k)$ (· · · ·) generated by $u_{zz}^{f\tilde{i}}(z_k)$. 
Figure A.30: Exp. 4-5-3: continuous second gradient $v^{\text{global}}_{z,z}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v^{\text{global}}_{z,z}(z)$, and discrete second gradient $v^{\text{fit}}_{z,z}(z_k)$ of velocity (-----) generated from the numerical differentiation of refiltered velocity gradient $v^{\text{fit}}_{z,z}(z_k)$. 

169
Figure A.31: Exp. 4-5-3: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
Figure A.32: Exp. 4-6-3: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.33: Exp. 4-6-3: piecewise continuous free surface slope $\phi^{\text{poly}}_{zz}(z)$ (---) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi^{\text{fil}}_{zz}(z_k)$ (—) generated by numerical differentiation of the filtered profile $\phi^{\text{fil}}(z_k)$. 
Figure A.34: Exp. 4-6-3: axial velocity $v_z^{exp}(z_k)$ generated by $Q/(\pi \phi^{exp^2}(z_k))$ and $v_z^{fit}(z_k)$ generated by $Q/(\pi \phi^{fit^2}(z_k))$. 
Figure A.35: Exp. 4-6-3: piecewise continuous velocity gradient $v_{z, z}^{\text{poly}}(z)$ (---) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z, z}^{\text{global}}(z)$ (···), discrete velocity gradient $v_{z, z}^{\hat{u}}(z_k)$ (---) generated by the numerical differentiation of the filtered velocity $v_{z, z}^{\hat{u}}(z_k)$, and refiltered velocity gradient $v_{z, z}^{\hat{u} \hat{u}}(z_k)$ (----) generated by $v_{z, z}^{\hat{u}}(z_k)$. 
Figure A.36: Exp. 4-6-3: continuous second gradient $v_{z,zz}^{\text{global}}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,zz}^{\text{global}}(z)$, and discrete second gradient $v_{z,zz}^{\text{filt}}(z_k)$ of velocity (——) generated from the numerical differentiation of refiltered velocity gradient $v_{z,zz}^{\text{filt}}(z_k)$. 
Figure A.37: Exp. 4-6-3: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
Figure A.38: Exp. 4-7-3: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.39: Exp. 4-7-3: piecewise continuous free surface slope $\phi^{\text{poly}}_x(z)$ (---) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi^{fu}_x(z_k)$ (—) generated by numerical differentiation of the filtered profile $\phi^{fu}(z_k)$.  

178
Figure A.40: Exp. 4-7-3: axial velocity $u_z^{exp}(z_k)$ generated by $Q/(\pi \phi^{exp^2}(z_k))$ and $u_z^{in}(z_k)$ generated by $Q/(\pi \phi^{in^2}(z_k))$. 
Figure A.41: Exp. 4-7-3: piecewise continuous velocity gradient $v_{z,x}^{pol}(z)$ (---) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z,x}^{global}(z)$ (···), discrete velocity gradient $v_{z,x}^{d}(z_k)$ (——) generated by the numerical differentiation of the filtered velocity $v_{z,x}^{f}(z_k)$, and refiltered velocity gradient $v_{z,x}^{f/d}(z_k)$ (----) generated by $v_{z,x}^{f}(z_k)$.

180
Figure A.42: Exp. 4-7-3: continuous second gradient $\nu_{\text{global}}^{\text{global}}(z)$ of velocity (—) generated from analytical differentiation of the globally smoothed polynomial fit $\nu_{\text{global}}^{\text{global}}(z)$, and discrete second gradient $\nu_{\text{refiltered}}^{\text{global}}(z_k)$ of velocity (—) generated from the numerical differentiation of refiltered velocity gradient $\nu_{\text{refiltered}}(z_k)$. 
Figure A.43: Exp. 4-7-3: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
Figure A.44: Exp. 4-8-3: measured free surface profile $\phi^{\exp}(z_k)$ and filtered profile $\phi^{\text{filt}}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.45: Exp. 4-8-3: piecewise continuous free surface slope \( \phi_{ax}^{poly}(z) \) (---) generated by analytical differentiation of the polynomial fit, and discrete free surface slope \( \phi_{z}^{fil}(z_k) \) (—) generated by numerical differentiation of the filtered profile \( \phi_{z}^{fil}(z_k) \).
Figure A.46: Exp. 4-8-3: axial velocity $u_z^{exp}(z_k)$ generated by $Q/(\pi \phi^{exp1}(z_k))$ and $u_z^{f\tilde{u}}(z_k)$ generated by $Q/(\pi \phi^{f\tilde{u}}(z_k))$. 
Figure A.47: Exp. 4-8-3: piecewise continuous velocity gradient $v^{poly}_x(z)$ (---) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v^{global}_x(z)$ (---), discrete velocity gradient $v^{fid}_x(z_k)$ (---) generated by the numerical differentiation of the filtered velocity $v^{lid}_x(z_k)$, and refiltered velocity gradient $v^{fifid}_x(z_k)$ (---) generated by $v^{lid}_x(z_k)$. 
Figure A.48: Exp. 4-8-3: continuous second gradient $v_{zz}^{\text{global}}(z)$ of velocity ($- -$) generated from analytical differentiation of the globally smoothed polynomial fit $v_{zz}^{\text{global}}(z)$, and discrete second gradient $v_{zz}^{\text{fil/fil}}(z_k)$ of velocity (-----) generated from the numerical differentiation of refiltered velocity gradient $v_{zz}^{\text{fil}}(z_k)$.
Figure A.49: Exp. 4-8-3: spinline force calculated from eq. (3.33) using filtering (——), and calculated from eq. (3.34) using filtering (−−−). In the profile the two forces are indistinguishable.
Figure A.50: Exp. 4-3-3x: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{fil}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.51: Exp. 4-3-3x: piecewise continuous free surface slope $\phi_{\text{poly}}^x(z)$ (- - -) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{\text{fil}}^x(z_k)(- - -)$ generated by numerical differentiation of the filtered profile $\phi_{\text{fil}}^x(z_k)$. 
Figure A.52: Exp. 4-3-3x: axial velocity $v_x^{exp}(z_k)$ generated by $Q/(\pi \phi^{exp}(z_k))$ and $v_x^{ff}(z_k)$ generated by $Q/(\pi \phi^{ff}(z_k))$. 
Figure A.53: Exp. 4-3-3x: piecewise continuous velocity gradient $v_{z,z}^{\text{poly}}(z)$ (---) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z,z}^{\text{global}}(z)$ (· · ·), discrete velocity gradient $v_{z,z}^{\text{filt}}(z_k)$ (—) generated by the numerical differentiation of the filtered velocity $v_{z,z}^{\text{filt}}(z_k)$, and refiltered velocity gradient $v_{z,z}^{\text{filt/filt}}(z_k)$ (· · · ·) generated by $v_{z,z}^{\text{filt}}(z_k)$. 
Figure A.54: Exp. 4-3-3x: continuous second gradient $v_{zz}^{global}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v_{zz}^{global}(z)$, and discrete second gradient $v_{zz}^{filt}(z_k)$ of velocity (— ) generated from the numerical differentiation of refiltered velocity gradient $v_{zz}^{filt}(z_k)$. 
Figure A.55: Exp. 4-3-3x: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
Figure A.56: Exp. 2-3-3: measured free surface profile $\phi^{\text{exp}}(z_k)$ and filtered profile $\phi^{\text{fil}}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.57: Exp. 2-3-3: piecewise continuous free surface slope $\phi_{poly}(z)$ (---) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{fl}(z_k)$ (—) generated by numerical differentiation of the filtered profile $\phi_{fl}(z_k)$. 

196
Figure A.58: Exp. 2-3-3: axial velocity $v^{exp}_z(z_k)$ generated by $Q/(\pi\phi^{exp2}(z_k))$ and $v^{fil}_z(z_k)$ generated by $Q/(\pi\phi^{fil2}(z_k))$. 
Figure A.59: Exp. 2-3-3: piecewise continuous velocity gradient $v_{z}^{\text{poly}}(z)$ (---) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z}^{\text{global}}(z)$ (···), discrete velocity gradient $v_{z}^{\text{fil}}(z_k)$ (----) generated by the numerical differentiation of the filtered velocity $v_{z}^{\text{f}}(z_k)$, and refiltered velocity gradient $v_{z}^{\text{filf}}(z_k)$ (-----) generated by $v_{z}^{\text{f}}(z_k)$. 
Figure A.60: Exp. 2-3-3: continuous second gradient $v^{\text{global}}_{z,z}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v^{\text{global}}_{z,z}(z)$, and discrete second gradient $v^{\text{fit}}_{z,z}(z_k)$ of velocity (---) generated from the numerical differentiation of refiltered velocity gradient $v^{\text{fit}}_{z,z}(z_k)$. 
Figure A.61: Exp. 2-3-3: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
Figure A.62: Exp. 1-3-3: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{fit}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.63: Exp. 1-3-3: piecewise continuous free surface slope $\phi_{z}^{poly}(z)$ (---) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{z}^{fil}(z_{k})(\cdots)$ generated by numerical differentiation of the filtered profile $\phi^{fil}(z_{k})$. 
Figure A.64: Exp. 1-3-3: axial velocity $v_z^e(z_k)$ generated by $Q/(\pi\phi^e(z_k))$ and $v_z^{i2}(z_k)$ generated by $Q/(\pi\phi^{i2}(z_k))$. 
Figure A.65: Exp. 1-3-3: piecewise continuous velocity gradient $v_{z,z}^{\text{poly}}(z)$ (-----) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z,z}^{\text{global}}(z)$ (···), discrete velocity gradient $v_{z,z}^{\text{ref}}(z_k)$ (---) generated by the numerical differentiation of the filtered velocity $v_{z,z}^{\text{ref}}(z_k)$, and refiltered velocity gradient $v_{z,z}^{\text{ref/ref}}(z_k)$ (· · ·) generated by $v_{z,z}^{\text{ref}}(z_k)$. 

204
Figure A.66: Exp. 1-3-3: continuous second gradient $v_{z,z}^{\text{global}}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,z}^{\text{global}}(z)$, and discrete second gradient $v_{z,z}^{\text{fil/fil}}(z_k)$ of velocity (—) generated from the numerical differentiation of refiltered velocity gradient $v_{z,z}^{\text{fil}}(z_k)$. 
Figure A.67: Exp. 1-3-3: spinline force calculated from eq. (3.33) using filtering (---), and calculated from eq. (3.34) using filtering (-----). In the profile the two forces are indistinguishable.
Figure A.68: Exp. 8-3-3: measured free surface profile $\phi^{exp}(z_k)$ and filtered profile $\phi^{filt}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.69: Exp. 8-3-3: piecewise continuous free surface slope $\phi_{poly}^x(z)$ (-----) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{fl}^x(z_k)$ (-----) generated by numerical differentiation of the filtered profile $\phi_{fl}^x(z_k)$. 
Figure A.70: Exp. 8-3-3: axial velocity $v_{z}^{\text{exp}}(z_k)$ generated by $Q/(\pi \phi^{\text{exp}}(z_k))$ and $v_{z}^{f}(z_k)$ generated by $Q/(\pi \phi^{(z_k)})$. 
Figure A.71: Exp. 8-3-3: piecewise continuous velocity gradient $v_{z,z}^{poly}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z,z}^{global}(z)$ (· · ·), discrete velocity gradient $v_{z,z}^{fil}(z_k)$ (-----) generated by the numerical differentiation of the filtered velocity $v_{z}^{fil}(z_k)$, and refiltered velocity gradient $v_{z,z}^{fil,fil}(z_k)$ (···) generated by $v_{z,z}^{fil}(z_k)$.
Figure A.72: Exp. 8-3-3: continuous second gradient $v_{z,zz}^{global}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,zz}^{global}(z)$, and discrete second gradient $v_{z,zz}^{dis}(z_k)$ of velocity (—) generated from the numerical differentiation of refiltered velocity gradient $v_{z,zz}^{dis}(z_k)$. 
Figure A.73: Exp. 8-3-3: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
Figure A.74: Exp. 4-3-1: measured free surface profile $\phi^{\text{exp}}(z_k)$ and filtered profile $\phi^{\text{fil}}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.75: Exp. 4-3-1: piecewise continuous free surface slope $\phi^{\text{poly}}(z)$ (—) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi^{\text{fil}}(z_k)$ (---) generated by numerical differentiation of the filtered profile $\phi^{\text{fil}}(z_k)$. 
Figure A.76: Exp. 4-3-1: axial velocity $v_z^{exp}(z_k)$ generated by $Q/(\pi \phi^{exp2}(z_k))$ and $v_z^{fui}(z_k)$ generated by $Q/(\pi \phi^{fui2}(z_k))$. 
Figure A.77: Exp. 4-3-1: piecewise continuous velocity gradient $v_{zz}^{\text{poly}}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{zz}^{\text{global}}(z)$ (···), discrete velocity gradient $v_{zz}^{\text{fin}}(z_k)$ (—) generated by the numerical differentiation of the filtered velocity $v_{zz}^{\text{fil}}(z_k)$, and refiltered velocity gradient $v_{zz}^{\text{fin}^{\text{fil}}}(z_k)$ (···) generated by $v_{zz}^{\text{fil}}(z_k)$. 
Figure A.78: Exp. 4-3-1: continuous second gradient $v_{z,z}^{global}(z)$ of velocity (-- -- ) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,z}^{global}(z)$, and discrete second gradient $v_{z,z}^{fil}(z_k)$ of velocity (—) generated from the numerical differentiation of refiltered velocity gradient $v_{z,z}^{fil}(z_k)$. 
Figure A.79: Exp. 4-3-1: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
Figure A.80: Exp. 4-3-2: measured free surface profile $\phi^{\text{exp}}(z_k)$ and filtered profile $\phi^{\text{fil}}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.81: Exp. 4-3-2: piecewise continuous free surface slope $\phi_{x}^{pol}(z)$ (-----) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\phi_{x}^{fit}(z_k)$ (----) generated by numerical differentiation of the filtered profile $\phi_{x}^{fit}(z_k)$. 
Figure A.82: Exp. 4-3-2: axial velocity $v_{z}^{exp}(z_k)$ generated by $Q/(\pi \phi^{exp}(z_k))$ and $v_{z}^{fu}(z_k)$ generated by $Q/(\pi \phi^{fu}(z_k))$. 
Figure A.83: Exp. 4-3-2: piecewise continuous velocity gradient $v_{z,z}^{poly}(z)$ (- - -) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{z,z}^{global}(z)$ (・・・), discrete velocity gradient $v_{z,z}^{fil}(z_k)$ (—) generated by the numerical differentiation of the filtered velocity $v_{z}^{fil}(z_k)$, and refiltered velocity gradient $v_{z,z}^{fil/fil}(z_k)$ (- - -) generated by $v_{z,z}^{fil}(z_k)$. 
Figure A.84: Exp. 4-3-2: continuous second gradient $v_{global}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v_{global}(z)$, and discrete second gradient $v_{z,z}^{fil,fil}(z_k)$ of velocity (—) generated from the numerical differentiation of refiltered velocity gradient $v_{z,z}^{fil}(z_k)$. 
Figure A.85: Exp. 4-3-2: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
Figure A.86: Exp. 4-3-4: measured free surface profile $\phi_{\text{exp}}(z_k)$ and filtered profile $\phi_{\text{fil}}(z_k)$. In the profile the discrete functions are indistinguishable.
Figure A.87: Exp. 4-3-4: piecewise continuous free surface slope $\dot{\phi}_{x z}^{\text{poly}}(z)$ (---) generated by analytical differentiation of the polynomial fit, and discrete free surface slope $\dot{\phi}_{x z}^{f\ell}(z_k)$ (—) generated by numerical differentiation of the filtered profile $\phi_{x z}^{f\ell}(z_k)$.
Figure A.88: Exp. 4-3-4: axial velocity $v_z^{exp}(z_k)$ generated by $Q/\left(\pi \phi^{exp^2}(z_k)\right)$ and $v_z^{iu}(z_k)$ generated by $Q/\left(\pi \phi^{iu^2}(z_k)\right)$.
Figure A.89: Exp. 4-3-4: piecewise continuous velocity gradient $v_{\text{poly}}^z(z)$ (---) generated by analytical differentiation of the polynomial fit, globally smoothed velocity gradient $v_{\text{global}}^z(z)$ (···), discrete velocity gradient $v_{\text{filt}}^{z,k}(z_k)$ (——) generated by the numerical differentiation of the filtered velocity $v_z^{\text{filt}}(z_k)$, and refiltered velocity gradient $v_{\text{refilt}}^{z,k}(z_k)$ (· · ·) generated by $v_{\text{refilt}}^{z,k}(z_k)$.
Figure A.90: Exp. 4-3-4: continuous second gradient $v_{z,zz}^{global}(z)$ of velocity (---) generated from analytical differentiation of the globally smoothed polynomial fit $v_{z,zz}^{global}(z)$, and discrete second gradient $v_{z,zz}^{filt}(z_k)$ of velocity (——) generated from the numerical differentiation of refiltered velocity gradient $v_{z,zz}^{filt}(z_k)$. 

229
Figure A.91: Exp. 4-3-4: spinline force calculated from eq. (3.33) using filtering (—), and calculated from eq. (3.34) using filtering (— — —). In the profile the two forces are indistinguishable.
BIBLIOGRAPHY


233