CONVERGENCE OF AVERAGES IN ERGODIC THEORY

DISSERTATION

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By

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ABSTRACT

Von Neumann’s Mean Ergodic Theorem and Birkhoff’s Pointwise Ergodic Theorem lie in the foundation of Ergodic Theory. Over the years there have been many generalizations of the two, most recently a version of pointwise ergodic theorem for measure-preserving actions of amenable groups due to Elon Lindenstrauss. In the first chapter, we extend some of Lindenstrauss’ results to measure-preserving actions of countable left-cancellative amenable semigroups and to averaging along more general types of Følner sequences.

In the next three chapters, we study convergence of Cesàro averages of the form
\[
\frac{1}{|L_n|} \sum_{g \in L_n} \prod_{i=1}^s T_{g(i)}^i f_i
\]
for measure-preserving actions of countable amenable groups, where \((L_n)\) is a Følner sequence. We extend some of the results obtained by D. Berend and V. Bergelson for joint properties of \(\mathbb{Z}\)-actions to joint properties of actions of countable amenable groups. In particular, we obtain a criterion for joint ergodicity of actions of countable amenable groups by automorphisms of a not necessarily abelian compact group.

In the last chapter of this dissertation, we investigate convergence of Cesàro averages for two non-commuting measure-preserving transformations along a regular sequence of intervals, i.e. we study averages of the form
\[
\frac{1}{b_n - a_n} \sum_{k=a_n}^{b_n} T^k f \cdot S^k g,
\]
(1)
where $b_n - a_n \to \infty$. Mean convergence of this expression depends both on the transformations $T$ and $S$ and the regular sequence of intervals $[a_n, b_n]_{n=1}^{\infty}$. We investigate the relationship between convergence of the expressions (2) and certain combinatorial properties of the regular sequences of intervals $[a_n, b_n]_{n=1}^{\infty}$. 
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\[
\frac{1}{b_n - a_n} \sum_{k=a_n}^{b_n} T^k f \cdot S^k g,
\]  

(2)

where \(b_n - a_n \to \infty\). Mean convergence of this expression depends both on the transformations \(T\) and \(S\) and the regular sequence of intervals \([a_n, b_n]_{n=1}^{\infty}\). We investigate the relationship between convergence of the expressions (2) and certain combinatorial properties of the regular sequences of intervals \([a_n, b_n]_{n=1}^{\infty}\).
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CHAPTER 1

POINTWISE ERGODIC THEOREMS FOR AMENABLE SEMIGROUPS

1.1 Definitions and Preliminary Results

In this chapter, a semigroup means a countable semigroup. A semigroup \( S \) is called left cancellative if for all \( x, a, b \in S \) one has

\[ xa = xb \implies a = b. \]

Let \( A \) and \( B \) be subsets of a semigroup \( S \). We define

\[ A^{-1}B \overset{\text{def}}{=} \{ x \mid Ax \cap B \neq \emptyset \}. \tag{1.1} \]

Note that in a semigroup expressions involving \( A^{-1} \) are in general not associative, e.g.

\[ (A^{-1}B)c \neq A^{-1}(Bc). \]

**Example 1.1.** Consider the additive semigroup \( \mathbb{N} \). We have:

\[ (-[90, 100] + [1, 10]) + 100 = \emptyset, \]

but

\[ -[90, 100] + ([1, 10] + 100) = [1, 20]. \]
Since we will be using expressions of the form $A^{-1}(Bc)$ quite often, we define

$$A^{-1}Bc \overset{\text{def}}{=} A^{-1}(Bc) = \{ x \mid Ax \cap Bc \neq \emptyset \}. \quad (1.2)$$

We study amenable semigroups. A semigroup $S$ is called left amenable if there exists a left invariant mean $M$ on the space of all bounded complex-valued functions on $S$. See Appendix B for the definition of invariant mean and a brief review of some of the properties of amenable groups and semigroups.

**Definition 1.2.** By a left Følner sequence $(F_n)$ in $S$, we mean a sequence of finite subsets of $S$ such that for every $x \in S$

$$\lim_{n \to \infty} \frac{|F_n \Delta xF_n|}{|F_n|} = 0.$$ 

Since we only consider left amenable semigroups and left Følner sequences, the word “left” will sometimes be omitted. We shall denote the set of all left Følner sequences in $S$ as $F(S)$.

**Proposition 1.3.** A left cancellative semigroup is left amenable iff it contains a left Følner sequence.

A proof can be found in Appendix B.

**Remark 1.4.** In this chapter, we only consider amenable semigroups that are left cancellative since we want to make use of Følner sequences as defined above. See Remark 1.22 for more discussion on this issue.

**Definition 1.5.** An action $T$ of a semigroup $S$ on a probability measure space $(X, \mathcal{B}, \mu)$ is a homomorphism $g \mapsto T_g$ from $S$ into the semigroup of measure-preserving transformations of $X$ onto itself.
A dynamical system is the quadruple \((X, \mathcal{B}, \mu, T)\) such that \(T\) is an action of a semigroup \(S\) on a probability measure space \((X, \mathcal{B}, \mu)\).

**Definition 1.6.** A dynamical system \((X, \mathcal{B}, \mu, T)\) is *ergodic* if for every \(A \in \mathcal{B}\) with \(0 < \mu(A) < 1\) there exists \(g \in S\) such that \(\mu(T_g^{-1}A \triangle A) > 0\).

For each action \(T\) of a semigroup \(S\) on \((X, \mathcal{B}, \mu)\), we can define a right unitary representation (that we shall also denote by \(T\)) of \(S\) on the space \(H = L^2(X)\) by \((T_gf)(x) = f(T_gx)\).

The following theorem was first proven for \(\mathbb{Z}\)-actions in [vN32] and later generalized to more general groups and semigroups by other authors.

Let \(H\) be a Hilbert space and let \(T\) be a right unitary representation of a left cancellative left amenable semigroup \(S\) in \(H\). A vector \(v \in H\) is called \(T\)-invariant if \(T_gv = v\) for all \(g \in S\).

**Theorem 1.7 (Mean Ergodic Theorem).** Let \(T\) be a right unitary representation of a left cancellative left amenable semigroup \(S\) in a Hilbert space \(H\), let \((F_n)\) be a Følner sequence in \(S\), and let \(P : H \to H\) be the orthogonal projection of \(H\) onto the subspace of the vectors that are invariant with respect to the representation \(T\). Then for every \(f \in H\)

\[
\frac{1}{|F_n|} \sum_{g \in F_n} T_g f \to Pf
\]

in \(H\)-norm.

The first pointwise ergodic theorem was established in [Bir31]. We give a modern formulation of it.

**Theorem 1.8 (Pointwise Ergodic Theorem for \(\mathbb{Z}\)-actions).** Let \(T\) be a measure-preserving transformation of a probability measure space \((X, \mathcal{B}, \mu)\). Then for every
f ∈ L^1(X) the sequence
\[
\frac{1}{N+1} \sum_{k=0}^{N} f(T^k x)
\]
converges a.e.

Since Theorem 1.7 holds for every Følner sequence in \( \mathbb{Z} \) and, in particular, for every sequence of intervals \([a_n, b_n]\) with \( b_n - a_n \to \infty \), one naturally asks the question: along which sequences of intervals in \( \mathbb{Z} \) does the pointwise ergodic theorem hold? This question turns out to be non-trivial. A satisfactory answer to it has been finally given in 1992 in [RW92] (see Theorem 1.13).

Another question that inspired a great deal of research is whether one can prove some pointwise ergodic theorems for more general amenable groups and semigroups. Since even in \( \mathbb{Z} \) there exist Følner sequences along which pointwise ergodic theorem does not hold (see Example 1.14 below), one can not possibly hope to prove a pointwise ergodic theorem for every Følner sequence in a general group (since it is not true even in \( \mathbb{Z} \).)

A positive answer to this question for a certain class of nested rectangles in \( \mathbb{Z}^d \) was given by Norbert Wiener [Wie39]. For more general groups and semigroups, some positive results were obtained in [Cal53], [Tem67], [Eme74], [OW83]. The latter results made use of Følner sequences satisfying the so-called Tempelman’s Condition:

**Definition 1.9 (Tempelman’s Condition).** A Følner sequence \((F_n)\) in an amenable semigroup \( S \) is said to satisfy Tempelman’s Condition if

1. \( F_1 \subset F_2 \subset F_3 \subset \ldots \)

2. There exists a constant \( C \) such that for all \( n \) and \( a \in S \), one has

\[
|F_n^{-1} F_n a| \leq C |F_n|
\] (1.3)
Remark 1.10. If $S$ is a group, then the condition (1.3) may be written as

$$\left| F_n^{-1}F_n \right| \leq C|F_n| \quad (1.4)$$

There exist several weaker versions of Tempelman’s Condition. The following condition is given in [Tem92] and [RW92].

Definition 1.11 (Weak Tempelman’s Condition). A Følner sequence in an amenable semigroup is said to satisfy Weak Tempelman’s Condition if there exists a constant $C$ such that

$$\left| \bigcup_{k=1}^{n} F_k^{-1}F_n a \right| < C|F_n| \quad (1.5)$$

for all $n > 0$ and $a \in S$.

Remark 1.12. If $S$ is a group, then the condition (1.5) may be written as

$$\left| \bigcup_{k=1}^{n} F_k^{-1}F_n \right| < C|F_n| \quad (1.6)$$

In [RW92] Rosenblatt and Wierdl proved the following

Theorem 1.13. Let $(I_n)$ be a sequence of intervals in $\mathbb{Z}$ such that $|I_n|$ is increasing. Then pointwise ergodic theorem holds along the sequence $(I_n)$ iff it satisfies Weak Tempelman’s Condition.

It is interesting to consider a few examples of sequences of intervals in $\mathbb{Z}$ that satisfy and do not satisfy Weak Tempelman’s Condition.

Example 1.14.

- $I_n = [0,n]$ is good. ($C = 2$)
• Every nested sequence of intervals is good. \((C = 2)\)

• \(I_n = [n^2, n^2 + n]\) is bad.

• \(I_n = [4^n, 4^n + 2^n]\) is still bad.

• \(I_n = [2^{2^n}, 2^{2^n} + \sqrt{2^{2^n}}]\) is good.

It is easy to see that every sequence of intervals \([a_n, b_n]\) in \(\mathbb{Z}\), such that \(b_n - a_n \to \infty\), contains a subsequence satisfying Weak Tempelman’s Condition. (This is a special case of Theorem 1.20 below.)

The following two questions still remain open.

**Question 1.15.** Is there a necessary and sufficient condition on a Følner sequence in \(\mathbb{Z}\) that guarantees that the pointwise ergodic theorem holds along it?

**Question 1.16.** Is there an analogue of Theorem 1.13 for non-abelian groups or for abelian groups of infinite rank (such as the multiplicative group of positive rationals)?

**Remark 1.17.** Unfortunately, existence of Følner sequences satisfying Weak Tempelman’s Condition has only been proven for amenable groups of polynomial growth. An example of a finitely-generated amenable group in which there is no Følner sequence satisfying Weak Tempelman’s condition is given in [Lin00]. The following finer condition was proposed in [Shu88].

**Definition 1.18 (Shulman’s Condition).** A Følner sequence in a group is called *tempered* if there exists a constant \(C\) such that

\[
|\bigcup_{k=1}^{n-1} F_{k}^{-1} F_n| < C |F_n| \tag{1.7}
\]

for all \(n\).
Remark 1.19. It is not trivial to come up with a suitable form of Shulman’s Condition for semigroups. This issue shall be discussed in Section 1.6.

Theorem 1.20 ([Lin99]). Every Følner sequence in an amenable group has a tempered subsequence. In particular, every amenable group has a tempered Følner sequence.

In [Shu88] a pointwise ergodic theorem has been proven for tempered Følner sequences and for functions in $L^2$. In [ST00] this result is extended to $L^\alpha$ for $\alpha > 1$. However, the decisive step in the quest for a pointwise ergodic theorem for amenable groups has been made in [Lin99].

Theorem 1.21 ([Lin99], [Lin00]). Let $G$ be an amenable group acting on a measure space $(X, \mathcal{B}, \mu)$ and let $(F_n)$ be a tempered Følner sequence. Then for every $f \in L^1(X)$ the limit
\[
\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(T_g x)
\]
exists a.e.

In Section 1.2 we develop machinery that can be used to reduce a certain class of pointwise ergodic theorems for semigroup actions to some covering lemmas for these semigroups.

In the Sections 1.3–1.6, we consider four different covering lemmas, in increasing difficulty.

In Section 1.3, we prove a version of Wiener’s Covering Lemma that is used to prove a pointwise ergodic theorem along Følner sequences satisfying Tempelman’s Condition. Although this result is well-known, we present it here since it makes it much easier to understand the more difficult covering lemmas presented later. We also
make the following modest contribution. Namely, we treat the case of a nested Følner sequence that does not satisfy Tempelman’s Condition and consider the question of pointwise convergence for expressions of the form

$$\frac{1}{\max_{a \in S} |F_n^{-1} F_n a|} \sum_{g \in F_n} f(T_g x)$$

(Theorem 1.30, Corollary 1.31).

In Section 1.4, we prove a covering lemma that we call Rosenblatt-Wierdl Covering Lemma. It is used to prove a pointwise ergodic theorem for Følner sequences satisfying Weak Tempelman’s Condition. It also plays a key role in the study of pointwise convergence of the expressions of the form

$$\frac{1}{\max_{a \in S} |\bigcup_{i=1}^n F_i^{-1} F_n a|} \sum_{g \in F_n} f(T_g x)$$

(Theorem 1.37, Corollary 1.38).

The next two sections are devoted to generalizations of Theorem 1.21.

In Section 1.5, we prove a version of Covering Lemma by Lindenstrauss that we use to prove a pointwise ergodic theorem for amenable groups along not necessarily tempered Følner sequences for the expressions of the form

$$\frac{1}{\max_{a \in S} |\bigcup_{i=1}^{n-1} F_i^{-1} F_n a|} \sum_{g \in F_n} f(T_g x)$$

(Theorem 1.44, Corollary 1.45).

In Section 1.6, we generalize Theorem 1.21 to left cancellative left amenable semigroups. We introduce a notion of tempered Følner sequences for left cancellative semigroups (Definition 1.52). We prove a pointwise ergodic theorem along tempered Følner sequences in such semigroups. However, we have not been able to prove a pointwise ergodic theorem along non-tempered Følner sequences for the expressions
of the form
\[
\frac{1}{\max_{a \in S} |\bigcup_{i=1}^{n-1} F_{i}^{-1} F_{n} a|} \sum_{g \in F_{n}} f(T_{g}x).
\]
Also, we have not been able to obtain an analogue of Theorem 1.20 for semigroups.
See Section 1.6 for more discussion on these issues.

In Section 1.7, we discuss some open problems.

**Remark 1.22.** We only consider left cancellative amenable semigroups. The reason is that a non-left-cancellerative amenable semigroup may not possess a Følner sequence (in the sense of Definition 1.2). It always satisfies Følner Condition (Definition B.10), but this is not enough to prove Mean Ergodic Theorem (Theorem 1.7) or Transfer Principle (Proposition 1.27).

### 1.2 Reduction of Main Theorem

In this section, we reduce a pointwise ergodic theorem for an action of an amenable semigroup to a purely combinatorial statement pertaining to the semigroup itself.

This procedure has been used by many authors, for instance, in [Pet83], [RW95], [Tem92], to prove certain statements about pointwise convergence of averages. For now, let us fix a countable left cancellative left amenable semigroup $S$. Let a Følner sequence $(F_{n})$ in $S$, and probability measure space $(X, \mathcal{B}, \mu)$, on which $S$ acts be given. Let $(q_{n})$ be a sequence of positive numbers. Let $f \in L^{1}(X)$. We consider convergence of the following ergodic averages:

\[
\frac{1}{q_{n}} \sum_{g \in F_{n}} T_{g} f.
\]
We shall be primarily interested in the case when $q_{n} = |F_{n}|$, but we shall consider the general case as well.
The following notation shall be used:

\[ A_n(F_n, q_n)[f] = \frac{1}{q_n} \sum_{g \in F_n} T_g f. \]

\[ M_t(F_n, q_n)[f](x) = \max_{n \leq t} A_n[|f|](x) \]

\[ M(F_n, q_n)[f](x) = \sup_n A_n[|f|](x). \]

In most cases, the sequences \((F_n)\) and \((q_n)\) are fixed and we shall use the abbreviated notation \(A_n[f]\), \(M_t[f]\), and \(M[f]\) instead of \(A_n(F_n, q_n)[f]\), \(M_t(F_n, q_n)[f](x)\), and \(M(F_n, q_n)[f](x)\) respectively.

We consider the left action\(^1\) \(L\) of \(S\) on itself given by \(L_g x = gx\). This action induces a right action on the set of the functions \(\varphi : S \to \mathbb{C}\) with finite support. We define

\[ \tilde{A}_n[\varphi] = \frac{1}{q_n} \sum_{g \in S} L_g \varphi. \]

\[ \tilde{M}_t[\varphi](x) = \max_{n \leq t} \tilde{A}_n[|\varphi|](x) \]

\[ \tilde{M}[\varphi](x) = \sup_n \tilde{A}_n[|\varphi|](x). \]

Let a Følner sequence \((F_n)\) and a sequence of factors \((q_n)\) be fixed.

We formulate the following three types of statements (that may or may not be true):

- **Pointwise Convergence (PWC).** For every \(f \in L^1(X)\) the limit

\[
\lim_{n \to \infty} A_n[f](x)
\]

\(^1\)Here, by action we mean a group action in purely algebraic sense, not an action in the sense of Definition 1.5. There is no measure involved.
exists a.e.

- **Maximal Inequality (MAX IE).** There exists \( c > 0 \), depending on the sequence \((F_n)\), but not on \( X \), such that for every \( f \in L^1(X) \), one has

\[
\mu\{x \mid M[f](x) > \lambda\} \leq \frac{c}{\lambda} ||f||_1.
\]

(1.8)

for all \( \lambda > 0 \).

- **Maximal Inequality on \( S \) (MAX IE on \( S \)).** There exists a constant \( c \) (depending on \((F_n)\)) such that for every function \( \varphi : S \rightarrow \mathbb{C} \) with finite support and every \( \lambda > 0 \),

\[
\left|\{a \in S \mid \tilde{M}[\varphi](a) > \lambda\}\right| \leq \frac{c}{\lambda} \|\varphi\|_1.
\]

(1.9)

Note that it is enough to prove MAX IE and MAX IE on \( S \) for the case \( \lambda = 1 \).

We shall prove that

\[
\text{MAX IE on } S \implies \text{MAX IE}
\]

We shall also prove that, if \( \lim_{n \to \infty} |F_n|/q_n \) exists and is finite, then

\[
\text{MAX IE} \implies \text{PWC}.
\]

**Lemma 1.23.** Let a left cancellative left amenable semigroup \( S \) act on a probability measure space \( X \). Then \( L^2(X) = H_{inv} \oplus H_{erg} \), where

\[
H_{inv} = \{f \in L^2(X) \mid T_g f = f \ \forall g \in S\}
\]

\[
H_{erg} = \text{Span}\{h - T_g h \mid h \in L^\infty(X), g \in S\}.
\]
Proof. Assume that \( f \perp H_{\text{erg}} \). Then for all \( g \in S \) and all \( h \in L^\infty(X) \) we have:

\[
(f, h - T_g h) = 0 \\
(f, h) = (f, T_g h) \\
(f, h) = (T_g^* f, h), \quad \text{where } T_g^* \text{ is the adjoint operator.}
\]

In particular, for all \( A \in \mathcal{B} \), we have:

\[
\int_A f \, d\mu = \int_A T_g^* f \, d\mu
\]

Hence, \( f = T_g^* f \) a.e. for all \( g \in S \). In view of Lemma F.9, it follows that \( f = T_g f \) a.e. for all \( g \in S \) and, therefore, \( f \in H_{\text{inv}} \).

Reversing the steps, we see that if \( f \in H_{\text{inv}} \), then \( f \perp H_{\text{erg}} \). Thus, \( H_{\text{inv}} \) is the orthogonal complement of the closed subspace \( H_{\text{erg}} \). Hence,

\[
L^2(X) = H_{\text{inv}} \oplus H_{\text{erg}}.
\]

\[\square\]

**Proposition 1.24.** Assume that \( \lim_{n \to \infty} |F_n|/q_n \) exists and is finite. Then

\[
\text{MAX IE } \implies \text{ PWC.}
\]

**Remark 1.25.** This fact is well-known for the case when \( q_n = |F_n| \). The proof can be found, for example, in [Pet83]. The proof of the more general fact that we give here is very similar to that classical proof. Of course, the case when \( q_n \neq |F_n| \), but \( 0 < \lim_{n \to \infty} |F_n|/q_n < \infty \) is not important. The case when \( \lim_{n \to \infty} |F_n|/q_n = 0 \) is important. As we shall see below, it is possible that for a given Følner sequence \( (F_n) \), PWC does not hold for \( q_n = |F_n| \), but holds for a sequence of denominators \( (q_n) \) such that \( \lim_{n \to \infty} |F_n|/q_n = 0 \).
Proof. Let

\[ C = \{ f \in L^1(X) \mid A_n[f] \text{ converges a.e.} \} \]

\[ D = \{ f + (h_1 - T_a h_1) + \cdots (h_m - T_a h_m) \mid f \in L^1(X), T_g f = f \forall g \in S, \]

\[ h_1, \ldots, h_m \in L^\infty(X), g_1, \ldots, g_m \in S \}. \]

We need to prove that \( C = L^1(X) \). We claim that \( D \subset C \).

Namely, let us first consider the case when \( q_n = |F_n| \). If \( f = T_g f \) for all \( g \in S \), then, clearly, \( f \in C \). If \( f = h - T_a h \) for some \( h \in L^\infty(X) \) and \( a \in S \), then for all \( x \in X \),

\[
\left| \frac{1}{|F_n|} \sum_{g \in F_n} T_g(h - T_a h)(x) \right| = \left| \frac{1}{|F_n|} \sum_{g \in F_n} (T_h g - T_a h)(x) \right| \\
\leq \frac{1}{|F_n|} \sum_{g \in F_n, \triangle a F_n} |T_h g(x)| \leq \frac{|F_n \triangle a F_n|}{|F_n|} \|h\|_\infty \xrightarrow{n \to \infty} 0,
\]

since \( (F_n) \) is a left Følner sequence. So, if \( q_n = |F_n| \), then \( D \subset C \).

The fact the \( D \subset C \) in the case, when \( q_n \neq |F_n| \), but \( \lim_{n \to \infty} |F_n|/q_n \) exists and is finite, follows from the following simple observation. Let \((a_n)\) be a sequence of complex numbers such that the limit

\[
\lim_{n \to \infty} \frac{a_n}{|F_n|}
\]

exists. Assume that

\[
\lim_{n \to \infty} \frac{|F_n|}{q_n} = \alpha < \infty.
\]

Then

\[
\lim_{n \to \infty} \frac{a_n}{q_n} = \lim_{n \to \infty} \frac{a_n}{|F_n|} \cdot \lim_{n \to \infty} \frac{|F_n|}{q_n} = \alpha \lim_{n \to \infty} \frac{a_n}{|F_n|}.
\]

Thus, \( D \subset C \) and it is enough to prove that:
(1) $D$ is dense in $L^1$

(2) $C$ is closed.

Indeed:

(1) By Lemma 1.23, $L^2 \cap D$ is dense in $L^2$ and, therefore, in $L^1$. Hence, $D$ is dense in $L^1$.

(2) Let $f_k \in C$, $k = 1, 2, \ldots$, and $f_k \to f$ in $L^1$ as $k \to \infty$. We will show that the sequence $A_n[f]$ is fundamental a.e. Indeed,

$$|A_m[f] - A_n[f]| \leq |A_m[f] - A_n[f_k]| + |A_m[f_k]| + |A_n[f_k]| \quad \forall k.$$ 

The first term on the right approaches 0 a.e. as $m, n \to \infty$. Hence, for every $\lambda > 0$,

$$\mu\{\limsup_{m,n \to \infty} |A_m[f] - A_n[f]| > \lambda\} \leq \mu\{2\sup_n |A_n[f_k]| > \lambda\} \leq \mu\{2\sup_n |f_k| > \lambda\} \leq \frac{2c}{\lambda} \|f - f_k\|_1 \quad \text{(by MAX IE)}.$$ 

Since $\|f - f_k\| \to 0$, the sequence $A_n[f]$ is fundamental a.e and, therefore, $\lim_{n \to \infty} A_n[f](x)$ exists a.e.

\[ \square \]

**Lemma 1.26.** Let $S$ be a left cancellative left amenable semigroup and $g_1, \ldots, g_n \in S$.

Then for every $\varepsilon > 0$ there exists a finite set $K \subset S$ such that

$$|g_1 K \cap \cdots \cap g_n K| > (1 - \varepsilon)|K|.$$
Proof. Let \((F_n)\) be a Følner sequence. Let \(\varepsilon > 0\) be given. There exists an index \(r\) such that \(|F_r \triangle g_k F_r| < (\varepsilon/n)|F_r|\) for \(1 \leq k \leq n\). Let \(K = F_r\). Then

\[
|g_1 K \cap \cdots \cap g_n K| \geq |K \cap g_1 K \cap \cdots \cap g_n K| \\
\geq |K| - |K \triangle g_1 K| - \cdots - |K \triangle g_n K| \\
> |K| - n(\varepsilon/n)|K| \\
= (1 - \varepsilon)|K|.
\]

\(\square\)

**Proposition 1.27.** MAX IE on \(S \Rightarrow\) MAX IE.

Proof. Let \(f \in L^1(X)\), and \(\lambda > 0\) be given. It is enough to prove that for all \(t\)

\[
\mu\{x \in X \mid M_t[f](x) > \lambda\} \leq \frac{c}{\lambda} \|f\|_1.
\]

Let \(t > 0\) and \(x \in X\) be given. Let \(K \subset S\) be a “big” finite set to be chosen later. We define

\[
\varphi(a) = \begin{cases} f(T_a x), & \text{if } a \in K, \\ 0, & \text{otherwise} \end{cases}
\]

Note that

\[
M_t[f](T_a x) = \max_{n \leq t} \frac{1}{q_n} \sum_{g \in F_n} f(T_{ga} x) = \tilde{M}_t[\varphi](a),
\]

provided that \(ga \in K\) for all \(g \in \bigcup_{n \leq t} F_n\). In other words, \(a\) must be in \(K'\), where

\[
K' = \bigcap_{g \in \bigcup_{n \leq t} F_n} g^{-1} K.
\]

By Lemma 1.26, for a fixed \(t\) and \(\varepsilon\), we can choose \(K\) so that \(|K'| > (1 - \varepsilon)|K|\). Now, let

\[
P = \{y \in X \mid M_t[f](y) > \lambda\}
\]

\[
\Pi = \{a \in S \mid \tilde{M}_t[\varphi](a) > \lambda\}
\]

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Then
\[
\sum_{a \in K'} 1_P(T_a x) = \sum_{a \in K'} 1_{\Pi}(a) \leq |\Pi|
\leq \frac{c}{\lambda} \|\varphi\|_1 \quad \text{(by MAX IE on } S)\\
= \frac{c}{\lambda} \sum_{g \in K} |f(T_g x)|
\]
Taking integral from both sides of the above inequality, we obtain
\[
|K'| \cdot \mu(P) \leq \frac{c}{\lambda} |K| \|f\|_1.
\]
When \(\varepsilon \to 0, |K|/|K'| \to 1\). Thus, we have
\[
\mu\{x \in X \mid M_t[f](x) > \lambda\} = \mu(P) \leq \frac{c}{\lambda} \|f\|_1.
\]
Making \(t \to \infty\), we conclude that
\[
\mu\{x \in X \mid M[f](x) > \lambda\} \leq \frac{c}{\lambda} \|f\|_1.
\]

Remark 1.28. Proposition 1.27 is often referred to as “Transfer Principle”. While it appears implicitly in [Wie39] and works by other authors, it was Calderón who first noticed its importance and stated it explicitly in [Cal68].

1.3 Wiener’s Covering Lemma and Its Applications

The Maximal Inequality on \(S\) may be proven using a covering lemma of this or another kind. For a Følner sequence satisfying Tempelman’s Condition (Definition 1.9) one uses the following covering lemma due to Wiener [Wie39] generalized by Tempelman in [Tem67]. Here we present an even more general version of it that allows us to prove a maximal inequality for \(q_n = \max_{a \in S} |F_n^{-1} F_n a|\) even if the Følner sequence does not satisfy Tempelman’s Condition.
Lemma 1.29 (Wiener’s Covering Lemma). Let $S$ be a discrete semigroup. Let $(F_n)$ be a nested Følner sequence. Let $E \subseteq S$ be a finite set and let for each $a \in E$ a positive integer $n(a)$ be given. Then there exists a subset $E' \subseteq E$ such that

1. $F_{n(a)}a \cap F_{n(b)}b = \emptyset \quad \forall a, b \in E', \ a \neq b$,

2. $E \subseteq \bigcup_{a \in E'} F_{n(a)}^{-1} F_n(a)a$.

Proof. Let $a_1 \in E$ be an element such that $n(a_1)$ is maximal and let

$$C_1 = F_{n(a_1)}^{-1} F_{n(a_1)}a_1.$$ 

Note that $a_1 \in C_1$. Let $E_1 = E \setminus C_1$.

If $E_1$ is non-empty, choose an $a_2 \in E_1$ so that $n(a_2)$ is maximal and let

$$C_2 = F_{n(a_2)}^{-1} F_{n(a_2)}a_2.$$ 

Note that $a_2 \in C_2$. Let $E_2 = E_1 \setminus C_2$.

Keep doing it, i.e. set

$$C_i = F_{n(a_i)}^{-1} F_{n(a_i)}a_i$$ \hspace{1cm} (1.10)

and $E_i = E_{i-1} \setminus C_i$. Since $E$ is finite, this procedure will terminate and we set $E' = \{a_1, a_2, \ldots \}$. Condition (2) is satisfied by construction. We claim that condition (1) is satisfied as well.

Indeed, let $a, b \in E'$ and $a \neq b$. We may assume that $b$ has been chosen before $a$. Then $n(a) \leq n(b)$. Suppose that $F_{n(a)}a \cap F_{n(b)}b \neq \emptyset$, This means that there exist elements $x \in F_{n(a)} \subseteq F_{n(b)}$ and $y \in F_{n(b)}$ such that $xa = yb$. Then $a = x^{-1}yb$, i.e. $a \in F_{n(b)}^{-1} F_{n(b)}b$. But according to our construction, the set $F_{n(b)}^{-1} F_{n(b)}b$ is deleted from $E$ before $a$ is chosen. Contradiction.

So, both conditions (1) and (2) hold and we are done. \qed
Theorem 1.30. Let $S$ be a left cancellative left amenable semigroup acting on a probability measure space $(X, \mathcal{B}, \mu)$. Let $(F_n)$ be a nested left Følner sequence in $S$ and $f \in L^1(X)$. Let

$$q_n = \max_{a \in S} |F_n^{-1} F_n a|.$$ 

Then

$$\mu(M(F_n, q_n)[f](x) > \lambda) \leq \frac{1}{\lambda} \|f\|_1. \tag{1.11}$$

Proof. In view of Proposition 1.27, it is enough to prove that the MAX IE on $S$ holds for every $\varphi \in l_1(S)$ with constant $c = 1$. We may assume that $\varphi$ is real-valued and non-negative. Also, may assume that $\lambda = 1$. Let

$$E = \{a \in S \mid \hat{M}[\varphi](a) > 1\}.$$

Since $\varphi$ has finite support, $E$ is a finite set. Notice that $a \in E$ iff there exists $n > 0$ such that

$$q_n < \sum_{g \in F_n} \varphi(ga) = \sum_{g \in F_n, a} \varphi(g). \tag{1.12}$$

Let $n(a)$ be the smallest $n$ such that the above holds. By virtue of Wiener’s Covering Lemma, we can choose $E' \subset E$ so that the sets $F_{n(a)} \cdot a$ for $a \in E'$ are pairwise disjoint and

$$|E| \leq \left| \bigcup_{a \in E'} F_{n(a)}^{-1} F_n a \right| \leq \sum_{a \in E'} \left| F_{n(a)}^{-1} F_n a \right| \leq \sum_{a \in E'} q_n(a). \tag{1.13}$$

Taking sum of (1.12) over $E'$, we obtain

$$\sum_{a \in E'} q_n(a) \leq \sum_{a \in E'} \sum_{g \in F_n(a)} \varphi(g). \tag{1.14}$$
Since the sets $F_{n(a)}a$ are pairwise disjoint for $a \in E'$, it follows that

$$
\sum_{a \in E'} \sum_{g \in F_{n(a)}a} \varphi(g) \leq \sum_{g \in S} \varphi(g) = \|\varphi\|_1. \tag{1.15}
$$

Combining (1.13), (1.14), and (1.15), we see that

$$
|E| < \|\varphi\|_1. \tag{1.16}
$$

Thus, MAX IE on $S$ holds and we are done.

\begin{proof}
\end{proof}

\textbf{Corollary 1.31.} If, in addition to the assumptions of Theorem 1.30, the limit

$$
\lim_{n \to \infty} \frac{|F_n|}{q_n}
$$

exists and is finite, then

$$
\frac{1}{q_n} \sum_{g \in F_n} f(T_g x)
$$

converges a.e.

\textbf{Corollary 1.32.} Let $G$ be an amenable group acting on a probability measure space $(X, \mathcal{B}, \mu)$. Let $(F_n)$ be a nested left Følner sequence in $S$ such that

$$
\lim_{n \to \infty} \frac{|F_n|}{|F_n^{-1}F_n|}
$$

exists and is finite and let $f \in L^1(X)$. Then

$$
\frac{1}{|F_n^{-1}F_n|} \sum_{g \in F_n} f(T_g x)
$$

converges a.e.

\textbf{Proof.} In a group, $|F_n^{-1}F_n a| = |F_n^{-1}F_n|$. \end{proof}

\textbf{Remark 1.33.} In order to have $|F_n^{-1}F_n a| = |F_n^{-1}F_n|$, it is not enough to assume that $S$ is cancellative or even embeddable into a group, as Example 1.1 in the beginning of this chapter shows.
**Corollary 1.34.** Let $S$ be a left cancellative left amenable semigroup acting on a probability measure space $(X, \mathcal{B}, \mu)$. Let $(F_n)$ be a left Følner sequence satisfying Tempelman’s condition and let $f \in L^1(X)$. Then

$$\frac{1}{|F_n|} \sum_{g \in F_n} f(T_g x)$$

converges a.e.

**Proof.** According to Theorem 1.30, the MAX IE holds for $q_n = \max_{a \in S} |F_n^{-1}F_n a|$. Since $q_n \leq C|F_n|$, when we replace $q_n$ by $q'_n = |F_n|$, MAX IE still holds. By Proposition 1.24, it follows that

$$\frac{1}{|F_n|} \sum_{g \in F_n} f(T_g x)$$

converges a.e. \hfill \Box

**Remark 1.35.** Corollary 1.34 has been proven in [Tem67]. As far as I know, Lemma 1.29 and Theorem 1.30 has only been proven for Følner sequences satisfying Tempelman’s Condition.

### 1.4 Rosenblatt-Wierdl’s Covering Lemma and Its Applications

It is possible to generalize the results obtained in the previous section by dropping the assumption that the Følner sequence $(F_n)$ is nested. Then we have to make use of expressions of the form $\bigcup_{i=1}^n F_i^{-1}F_n$ instead of $F_n^{-1}F_n$. The results that we present below have been obtained by Rosenblatt and Wierdl in [RW92] for the case $S = \mathbb{Z}$. The book [Tem92] contains some of these results for the case of general semigroup actions as well.
The proofs are very similar to the proofs given in the previous section. The main new ingredient is that in order to make the sets \(C_i\) disjoint, we have to set

\[
E_i = \bigcup_{i=1}^{n(a_i)} F_i^{-1} F_{n(a_i)} a_i
\]

instead of

\[
E_i = F_i^{-1} F_{n(a_i)} a_i.
\]

**Lemma 1.36 (Rosenblatt-Wierdl’s Covering Lemma).** Let \(S\) be a countable semigroup and let \((F_n)\) be a Følner sequence. Let \(E \subset S\) be a finite set and let for each \(a \in E\) a positive integer \(n(a)\) be given. Then there exists a subset \(E' \subset E\) such that

1. \(F_{n(a)} a \cap F_{n(b)} b = \emptyset\) \(\forall a, b \in E', a \neq b,\)
2. \(E \subset \bigcup_{a \in E'} \bigcup_{i=1}^{n(a)} F_i^{-1} F_{n(a)} a.\)

**Proof.** Let \(a_1 \in E\) be an element such that \(n(a_1)\) is maximal and let

\[
C_1 = \bigcup_{i=1}^{n(a_1)} F_i^{-1} F_{n(a_1)} a_1.
\]

Note that \(a_1 \in C_1\). Let \(E_1 = E \setminus C_1\).

If \(E_1\) is non-empty, choose an \(a_2 \in E_1\) so that \(n(a_2)\) is maximal and let

\[
C_2 = \bigcup_{i=1}^{n(a_2)} F_i^{-1} F_{n(a_2)} a_2.
\]

Note that \(a_2 \in C_2\). Let \(E_2 = E_1 \setminus C_2\).

Keep doing it, i.e. set

\[
C_k = \bigcup_{i=1}^{n(a_k)} F_i^{-1} F_{n(a_k)} a_k\] (1.17)
and \( E_k = E_{k-1} \setminus C_k \). Since \( E \) is finite and at each step we remove at least one element from it, this procedure will terminate and we set \( E' = \{ a_1, a_2, \ldots \} \). Condition (2) is satisfied by construction. We claim that condition (1) is satisfied as well.

Indeed, let \( a, b \in E' \) and \( a \neq b \). We may assume that \( b \) has been chosen before \( a \). Then \( n(a) \leq n(b) \). Suppose that \( F_{n(a)}a \cap F_{n(b)}b \neq \emptyset \). This means that there exist elements \( x \in F_{n(a)} \) and \( y \in F_{n(b)} \) such that \( xa = yb \). Then \( a = x^{-1}yb \), i.e.

\[
a \in F_{n(a)}^{-1}F_{n(b)}b \subset \bigcup_{i=1}^{n(b)} F_{i}^{-1}F_{n(b)}b
\]

that is supposed to be deleted from \( E \) before \( a \) is chosen. Contradiction.

Thus, both conditions (1) and (2) hold and we are done. \( \square \)

**Theorem 1.37.** Let \( S \) be a left cancellative left amenable semigroup acting on a probability measure space \((X, \mathcal{B}, \mu)\). Let \( (F_n) \) be a left Følner sequence in \( S \) and \( f \in L^1(X) \). Let

\[
q_n = \max_{a \in S} \left| \bigcup_{i=1}^{n} F_{i}^{-1} F_n a \right|.
\]

Then

\[
\mu(M(F_n, q_n)[f](x) > \lambda) \leq \frac{1}{\lambda} \| f \|_1.
\] (1.18)

**Proof.** In view of Proposition 1.27, it is enough to prove that the MAX IE on \( S \) holds for every \( \varphi \in l_1(S) \) with constant \( c = 1 \). We may assume that \( \varphi \) is real-valued and non-negative. Also, may assume that \( \lambda = 1 \). Let

\[
E = \{ a \in S \mid \tilde{M}[\varphi](a) > 1 \}.
\]

Since \( \varphi \) has finite support, \( E \) is a finite set. Notice that \( a \in E \) iff there exists \( n > 0 \) such that

\[
q_n < \sum_{g \in F_n} \varphi(ga) = \sum_{g \in F_n, a} \varphi(g).
\] (1.19)
Let \( n(a) \) be the smallest \( n \) such that the above holds. By virtue of Tempelman’s Covering Lemma, we can choose \( E' \subset E \) so that the sets \( F_{n(a)} \cdot a \) for \( a \in E' \) are pairwise disjoint and

\[
|E| \leq \left| \bigcup_{a \in E'} \bigcup_{i=1}^{n(a)} F_i^{-1} F_{n(a)} a \right| \leq \sum_{a \in E'} \left| \bigcup_{i=1}^{n(a)} F_i^{-1} F_{n(a)} a \right| \leq \sum_{a \in E'} q_n(a). \tag{1.20}
\]

Taking sum of (1.19) over \( E' \), we obtain

\[
\sum_{a \in E'} q_n(a) \leq \sum_{a \in E'} \sum_{g \in F_{n(a)} a} \varphi(g) \tag{1.21}
\]

Since the sets \( F_{n(a)}a \) are pairwise disjoint for \( a \in E' \), it follows that

\[
\sum_{a \in E'} \sum_{g \in F_{n(a)} a} \varphi(g) \leq \sum_{g \in S} \varphi(g) = \| \varphi \|_1. \tag{1.22}
\]

Combining (1.20), (1.21), and (1.22), we see that

\[
|E| < \| \varphi \|_1.
\]

Thus, MAX IE on \( S \) holds and we are done. \( \square \)

**Corollary 1.38.** If, in addition to the assumptions of Theorem 1.37, the limit

\[
\lim_{n \to \infty} \frac{|F_n|}{q_n}
\]

exists and is finite, then

\[
\frac{1}{q_n} \sum_{g \in F_n} f(T_g x)
\]

converges a.e.

**Corollary 1.39.** Let \( S \) be an amenable group acting on a probability measure space \((X, \mathcal{B}, \mu)\). Let \( (F_n) \) be a left Følner sequence in \( S \) such that

\[
\lim_{n \to \infty} \frac{|F_n|}{\left| \bigcup_{i=1}^{n} F_i^{-1} F_n \right|}
\]

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exists and is finite and let $f \in L^1(X)$. Then

$$\frac{1}{|\bigcup_{i=1}^{n} F_{i-1} F_n|} \sum_{g \in F_n} f(T_g x)$$

converges a.e.

**Corollary 1.40.** Let $S$ be a left cancellative left amenable semigroup acting on a probability measure space $(X, \mathcal{B}, \mu)$. Let $(F_n)$ be a left Følner sequence satisfying Weak Tempelman’s condition and let $f \in L^1(X)$. Then

$$\frac{1}{|F_n|} \sum_{g \in F_n} f(T_g x)$$

converges a.e.

**Remark 1.41.** Corollary 1.40 has been proven in [Tem92]. For the case $S = \mathbb{Z}$, Lemma 1.36 can be found in [RW92].

### 1.5 Lindenstrauss’ Covering Lemma and Its Applications

As we have already noted in Section 1.1, Weak Tempelman’s Condition is too strong. We would like to obtain analogues of results from the previous section for Følner sequences satisfying the weaker Shulman’s Condition (Definition 1.18). The only difference between Weak Tempelman’s and Shulman’s Conditions is the range of summation. Namely, we use the expression $\bigcup_{i=1}^{n} F_{i-1} F_n$ in Shulman’s Condition as opposed to $\bigcup_{i=1}^{n} F_{i-1} F_n$ in Weak Tempelman’s Condition. This difference is significant. Indeed, as we noted above (Remark 1.17), there may be no Følner sequences satisfying Weak Tempelman’s Condition in a general amenable group. However, according to Theorem 1.20, in every amenable group, a Følner sequence satisfying Shulman’s Condition always exists.
This difference also leads to an obstacle in proving a Covering Lemma. To prove Lemma 1.36, we need to choose disjoint sets of the form \( F_{n(a)}a \). Since it might well happen that \( n(a) = n(b) \) for \( a \neq b \), to be sure that \( F_{n(a)}a \cap F_{n(b)}b = \emptyset \), we have to subtract \( \bigcup_{i=1}^{n(a)} F_i^{-1}F_{n(a)}a \), not just \( \bigcup_{i=1}^{n(a)-1} F_i^{-1}F_{n(a)}a \), from \( E \). How can we achieve the desired result by subtracting \( \bigcup_{i=1}^{n(a)} F_i^{-1}F_{n(a)}a \)? One possibility is to arrange for all \( n(a) \)'s to be distinct. It is not clear at all how to achieve that. Another possibility is to group all \( a \)'s with same value of \( n(a) \) into sets, call them \( A \)'s, and then at each step work with the whole set \( A \) instead of just one element \( a \). But how to do it so that the sets \( F_{n(a)}a \) with \( a \) from same set \( A \) are disjoint? Having all this in mind, we can better appreciate the following covering lemma due to Lindenstrauss [Lin00]. Here, we present a slightly generalized version of this lemma suitable for proving the maximal inequality with denominators \( q_n = |\bigcup_{i=1}^{n-1} F_i^{-1}F_n| \) even if the Følner sequence does not satisfy Shulman’s Condition.

Let us make some notational remarks before we formulate the covering lemma. We consider a countable amenable group \( G \). We introduce set-valued random variables \( B(a) \). These are functions from some measure space, which we denote as \( \Omega \), into the class of all finite subsets of \( G \). Let \( \mathcal{F} \) be the probability measure on \( \Omega \). If \( X \) is a random variable and \( P \) is an event, i.e. a measurable subset of \( \Omega \), then \( \mathcal{E}(X) \) denotes the expectation and \( \mathcal{E}(X \mid P) \) denotes the conditional expectation with respect to the event \( P \).

**Lemma 1.42 (Lindenstrauss’ Covering Lemma).** Let \( G \) be a countable group, \((F_n)\) a Følner sequence, \( E \subset G \) a finite set, and let for each \( a \in E \) a positive integer \( n(a) \) be chosen. Let \( \delta > 0 \) be given. Then it is possible to find set valued random variables \( B(a) \) for \( a \in E \) such that
(1) \(B(a)\) is either \(F_n(a)\) or \(\emptyset\).

(2) If we set \(\Lambda : G \to \mathbb{N}\) to be the random function \(\Lambda(g) = \sum_{a \in E} 1_{B(a)}(g)\), then

\[
\mathcal{E}(\Lambda(g) \mid \Lambda(g) > 0) < 1 + \delta.
\]

(3) For some \(\gamma > 0\) that depends on \(\delta\),

\[
\mathcal{E}\left(\sum_{a \in E} \left| \bigcup_{i=1}^{n(a)-1} F_i^{-1} B(a) \right| \right) > \gamma |E|.
\]

The proof makes use of the following simple lemma.

**Lemma 1.43.** Let \(Y = \sum_{i=1}^{n} X_i\), where the \(X_i\)'s are i.i.d.\(^2\) random variables which are 1 with probability \(p\) and 0 with probability \(1 - p\). Then

\[
\mathcal{E}(Y \mid Y \geq 1) \leq 1 + (n - 1)p.
\]

**Proof.** We use induction to prove this lemma. For \(n = 1\) the above inequality holds trivially. To prove the general case, we note that

\[
\mathcal{E}(Y \mid Y \geq 1) = \mathbb{P}(X_1 = 1)\mathcal{E}(Y \mid X_1 = 1 \text{ and } Y \geq 1))
\]

\[
+ \mathbb{P}(X_1 = 0)\mathcal{E}(Y \mid X_1 = 0 \text{ and } Y \geq 1))
\]

(1.23)

Observe that

\[
\mathcal{E}(Y \mid X_1 = 1 \text{ and } Y \geq 1)) = 1 + (n - 1)p
\]

(1.24)

and

\[
\mathcal{E}(Y \mid X_1 = 0 \text{ and } Y \geq 1)) \leq 1 + (n - 2)p
\]

(1.25)

\(^2\)“independent and identically distributed”
by induction hypothesis.

Combining (1.23), (1.24), and (1.25), we obtain

\[ E(Y \mid Y \geq 1) \leq \mathbb{P}(X_1 = 1)(1 + (n - 1)p) + \mathbb{P}(X_1 = 0)(1 + (n - 2)p) \]
\[ = p + (n - 1)p^2 + (1 - p) + (1 - p)(n - 2)p = 1 + (n - 2)p + p^2 \leq 1 + (n - 1)p. \]

Proof of Lindenstaruss’ Covering Lemma. Let \( N = \max_{a \in E} n(a) \). For \( 1 \leq j \leq N \) we let

\[ A_j = \{ a \in E \mid n(a) = j \}. \]

We start from \( j = N \). For all \( a \in A_N \) we let (independently from all other \( b \in A_N \))

\[ B(a) = \begin{cases} F_Na & \text{with probability } p_N = \frac{\delta}{|F_N|}, \\ \emptyset & \text{with probability } 1 - p_N \end{cases} \tag{1.26} \]

Assume that we have already defined \( B(a) \) for all \( a \in A_k, j < k \leq N \). Then for each \( a \in A_j \) we do the following (independently from all other \( b \in A_j \))

- If \( F_ja \) is disjoint from all \( B(a') \) for all \( a' \in A_k, j < k \leq N \), then
  \[ B(a) = \begin{cases} F_ja & \text{with probability } p_j = \frac{\delta}{|F_j|}, \\ \emptyset & \text{with probability } 1 - p_j \end{cases} \tag{1.27} \]

- Otherwise \( B(a) = \emptyset \).

It is enough to consider the case when \( \delta \) is a rational number. In this case, the random variables \( B(a) \) defined above may be implemented by means of a finite set \( \Omega \) with normalized counting measure. Therefore, every subset of \( \Omega \) is measurable.

We first show that

\[ E(\Lambda(g) \mid \Lambda(g) > 0) < 1 + \delta. \]

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Let $\Lambda_j$ be the random function
\[
\Lambda_j(g) = \sum_{a \in A_j} 1_{B(a)}(g).
\]
We remark that for $j \neq j'$ the events $\Lambda_j(g) > 0$ and $\Lambda_{j'}(g) > 0$ are mutually exclusive.

We now define for every $g \in G$ and $j$
\[
A(j, g) = \left( A_j \setminus \bigcup_{j'+1 \leq j} \bigcup_{a' \in A_{j'}} F_{j'}^{-1}B(a') \right) \cap F_j^{-1}g.
\]
This represents the set of all $a \in A_j$ such that $g \in F_ja$ and such that $F_ja$ is disjoint from $B(a')$ for all $a'$. By our construction, $|A(j, g)|$ is the number of sets $B(b)$, with $b \in A_j$, such that the probability to contain the element $g$ is positive. Clearly, $|A(j, g)| \leq |F_j|$. Using Lemma 1.43, we see that
\[
\mathbb{E}(\Lambda_j(g) \mid \Lambda_j(g) > 0) = \mathbb{E}(\Lambda_j(g) \mid \Lambda_j(g) \geq 1) \leq 1 + p_j |A(j, g)| \leq 1 + p_j |F_j| = 1 + \delta.
\]
Hence,
\[
\mathbb{E}(\Lambda(g) \mid \Lambda(g) > 0) \leq 1 + \delta.
\]

We use induction to show that
\[
\mathbb{E} \left( \sum_{a \in E} \left| \bigcup_{i=1}^{n(a)-1} F_i^{-1}B(a) \right| \right) > \gamma |E| \tag{1.28}
\]
where
\[
\gamma = \frac{\delta}{\delta + 1}. \tag{1.29}
\]

If $N = 1$,
\[
\mathbb{E} \left( \sum_{a \in A_1} \left| \bigcup_{i=1}^{n(a)-1} F_i^{-1}B(a) \right| \right) = \mathbb{E} \left( \sum_{a \in A_1} |B(a)| \right) = \sum_{a \in A_1} \mathbb{E} \left( \sum_{g \in F} 1_{B(a)}(g) \right)
\]
\[
= |A_1||F_1| \frac{\delta}{|F_1|} = \delta |A_1| = \delta |E| > \gamma |E|.
\]

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and thus (1.28) is satisfied.

Now, assume that (1.28) holds for $\tilde{N} = N - 1$. Note that the distribution on $B(a)$ for $a \in A_j$, $j < N$ is the same as one would obtain for $\tilde{N} = N - 1$ and

$$\tilde{A}_j = A_j \setminus \bigcup_{a \in A_N} F^{-1}_j B(a).$$

By the induction hypothesis, for all $\omega \in \Omega$,

$$\mathbb{E} \left( \sum_{j=1}^{N-1} \sum_{a \in A_j} \left| \bigcup_{i=1}^{j-1} F^{-1}_i B(a) \right| \right) \geq \gamma \left| \bigcup_{j=1}^{N-1} \tilde{A}_j \right|.$$

Note that

$$\left| \bigcup_{j=1}^{N-1} \tilde{A}_j \right| = \left| \bigcup_{j=1}^{N-1} A_j \setminus \bigcup_{a \in A_N} \bigcup_{j=1}^{N-1} F^{-1}_j B(a) \right| \geq \left| \bigcup_{j=1}^{N-1} A_j \right| - \left| \bigcup_{a \in A_N} \bigcup_{j=1}^{N-1} F^{-1}_j B(a) \right|. \quad (1.30)$$

It follows that

$$\mathbb{E} \left( \sum_{j=1}^{N-1} \sum_{a \in A_j} \left| \bigcup_{i=1}^{j-1} F^{-1}_i B(a) \right| \right) \geq \gamma \left| \bigcup_{j=1}^{N-1} A_j \right| - \gamma \left| \bigcup_{a \in A_N} \bigcup_{j=1}^{N-1} F^{-1}_j B(a) \right|. \quad (1.31)$$

Adding $\sum_{a \in A_N} \left| \bigcup_{i=1}^{N-1} F^{-1}_i B(a) \right|$ to both sides of (1.31) and taking expectation, we have:

$$\mathbb{E} \left( \sum_{a \in E} \left| \bigcup_{i=1}^{n(a)-1} F^{-1}_i B(a) \right| \right) \geq \gamma \left| \bigcup_{j=1}^{N-1} A_j \right| + (1 - \gamma) \mathbb{E} \left( \sum_{a \in A_N} \left| \bigcup_{i=1}^{N-1} F^{-1}_i B(a) \right| \right). \quad (1.32)$$

Clearly,

$$\mathbb{E} \left( \sum_{a \in A_N} \left| \bigcup_{i=1}^{N-1} F^{-1}_i B(a) \right| \right) \geq \sum_{a \in A_N} \mathbb{E} \left( |B(a)| \right) = |A_N| \frac{\delta}{|F_N|} |F_N| = \delta |A_N|.

Because of (1.29), $\delta (1 - \gamma) = \gamma$. Hence, combining (1.32) and (1.33), we have

$$\mathbb{E} \left( \sum_{a \in E} \left| \bigcup_{i=1}^{n(a)-1} F^{-1}_i B(a) \right| \right) \geq \gamma \left| \bigcup_{j=1}^{N-1} A_j \right| + |A_N| = \gamma \left| \bigcup_{j=1}^{N-1} A_j \right| = \gamma |E|. \quad \square$$
Theorem 1.44. Let $G$ be an amenable group acting on a probability measure space $(X, \mathcal{B}, \mu)$. Let $(F_n)$ be a left Følner sequence in $G$ and let

$$q_n = \left| \bigcup_{i=1}^{n-1} F_i^{-1} F_n \right|.$$ 

There exists a constant $c > 0$ such that for all $f \in L^1(X)$,

$$\mu(M(F_n, q_n)[f](x) > \lambda) \leq \frac{c}{\lambda} \|f\|_1. \tag{1.34}$$

Proof. The proof is quite similar to the proof of Proposition 1.37.

In view of Proposition 1.27, it is enough to prove that the MAX IE on $G$ holds for every $\varphi \in l_1(G)$. We may assume that $\varphi$ is real-valued and non-negative. Also, may assume that $\lambda = 1$. Let

$$E = \{a \in G \mid \tilde{M}[\varphi](a) > 1\}.$$ 

Since $\varphi$ has finite support, $E$ is a finite set. Notice that $a \in E$ iff there exists $n > 0$ such that

$$q_n < \sum_{g \in F_n} \varphi(ga) = \sum_{g \in F_n a} \varphi(g). \tag{1.35}$$

Let $n(a)$ be the smallest $n$ such that the above holds. We apply Lindenstrauss’ Covering Lemma with $\delta = 1$. There exist a positive number $\gamma$ and set-valued random variables $B(a)$ for $a \in E$ such that:

1. $B(a)$ is either $F_{n(a)} a$ or $\emptyset$,

2. $\mathcal{E}(1_{B(a)}(g)) \leq 2$,

3. $\mathcal{E} \left( \sum_{a \in E} \left| \bigcup_{i=1}^{n(a)-1} F_i^{-1} B(a) \right| \right) > \gamma |E|$. 

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Then
\[
\mathcal{E} \left( \sum_{a \in E} \sum_{g \in G} \varphi(g) 1_{B(a)}(g) \right) = \sum_{g \in G} \varphi(g) \mathcal{E} \left( \sum_{a \in E} 1_{B(a)}(g) \right) \leq 2 \sum_{g \in G} \varphi(g) = 2 \| \varphi \|_1. \tag{1.36}
\]

On the other hand,
\[
\mathcal{E} \left( \sum_{a \in E} \sum_{g \in G} \varphi(g) 1_{B(a)}(g) \right) = \mathcal{E} \left( \sum_{a \in E} \sum_{g \in B(a)} \varphi(g) \right). \tag{1.37}
\]

If \( B(a) = \emptyset \), then
\[
\sum_{g \in B(a)} \varphi(g) = 0 = \bigcup_{i=1}^{n(a)-1} F_i^{-1} B(a) = 0.
\]

If \( B(a) = F_{n(a)} a \), then by virtue of (1.35),
\[
\sum_{g \in B(a)} \varphi(g) > q_n(a) = \bigcup_{i=1}^{n(a)-1} F_i^{-1} F_{n(a)} = \bigcup_{i=1}^{n(a)-1} F_i^{-1} B(a).
\]

So, in either case,
\[
\sum_{g \in B(a)} \varphi(g) \geq \bigcup_{i=1}^{n(a)-1} F_i^{-1} B(a). \tag{1.38}
\]

Thus,
\[
\mathcal{E} \left( \sum_{a \in E} \sum_{g \in B(a)} \varphi(g) \right) \geq \mathcal{E} \left( \sum_{a \in E} \bigcup_{i=1}^{n(a)-1} F_i^{-1} B(a) \right) > \gamma |E|. \tag{1.39}
\]

Combining (1.36), (1.37), and (1.39), we see that
\[
|E| < \frac{2}{\gamma} \| \varphi \|_1.
\]

\[\blacksquare\]

**Corollary 1.45.** If, in addition to the assumptions of Theorem 1.44, the limit
\[
\lim_{n \to \infty} \frac{|F_n|}{q_n}
\]
exists and is finite, then
\[ \frac{1}{q_n} \sum_{g \in F_n} f(T_g x) \]
converges a.e.

**Corollary 1.46.** Let $G$ be an amenable group acting on a probability measure space $(X, \mathcal{B}, \mu)$. Let $(F_n)$ be a left Følner sequence satisfying Shulman’s condition (Definition 1.18) and let $f \in L^1(X)$. Then
\[ \frac{1}{|F_n|} \sum_{g \in F_n} f(T_g x) \]
converges a.e.

**Remark 1.47.** As we can see, for the ergodic averages of the form
\[ \frac{1}{q_n} \sum_{g \in F_n} f(T_g x) \]  
(1.40)
there exist several alternatives:

(1) If $q_n = \left| \bigcup_{i=1}^{n-1} F_i^{-1} F_n \right|$ and $\lim_{n \to \infty} |F_n|/q_n = \alpha > 0$, then, according to Corollary 1.45, (1.40) converges a.e. to a function $\overline{f}$:
\[ \overline{f}(x) = \lim_{n \to \infty} \frac{1}{q_n} \sum_{g \in F_n} (T_g f)(x). \]

Taking integral of both sides, we obtain
\[ \int_X \overline{f} \, d\mu = \lim_{n \to \infty} \frac{|F_n|}{q_n} \cdot \int_X f \, d\mu = \alpha \int_X f \, du. \]

(2) If $q_n = |\bigcup_{i=1}^{n-1} F_i^{-1} F_n| \leq C|F_n|$, but $\lim_{n \to \infty} |F_n|/q_n$ does not exist, then (1.40) might not converge. For example, consider a constant function $f$ and the sequence $(q_n)$ given by
\[ q_n = \begin{cases} |F_n|, & \text{if } n \text{ is odd} \\ 2|F_n|, & \text{if } n \text{ is even} \end{cases} \]

Then clearly (1.40) does not converge.
If \( q_n = |\bigcup_{i=1}^{n-1} F_i^{-1} F_n| \) and  
\[
\lim_{n \to \infty} \frac{|F_n|}{q_n} = 0, 
\]
then (1.40) converges a.e. to 0. Indeed, (1.40) converges to a function \( \overline{f} \) a.e. by Corollary 1.46. Let \( \overline{g} \) denote the operation of taking the limit as \( n \to \infty \) of \( A_n(F_n, q_n)[g] \). It is clear that if \( h \) is a bounded function, then \( \overline{h}(x) = 0 \) for all \( x \). Let \( f \in L^1(X) \) be given and let \( (h_m) \) be a sequence of bounded functions that converges to \( f \) in \( L^1 \). Then  
\[
\overline{f} = (\overline{f - h_m}) + h_m. 
\]
and  
\[
\|\overline{f}\|_1 \leq \|(f - h_m)\|_1 + \|h_n\|_1.  \tag{1.41}
\]
Note that  
\[
\int_X \left| \frac{1}{q_n} \sum_{g \in F_n} T_g(h_m - f) \right| d\mu \leq \frac{1}{q_n} \sum_{g \in F_n} \int_X |T_g(h_m - f)| d\mu.  \tag{1.42}
\]
We know that \( q_n > |F_n| \) and  
\[
\int_X |T_g(h_m - f)| d\mu = \int_X |h_m - f| d\mu \xrightarrow{m \to \infty} 0. 
\]
Taking (1.41) and (1.42) into account, we conclude that \( \|\overline{f}\|_1 = 0 \).

(4) What happens if  
\[
\lim_{n \to \infty} \frac{1}{q_n} \left| \bigcup_{i=1}^{n-1} F_i^{-1} F_n \right| = \infty,  \tag{1.43}
\]
we do not know in general. It is shown, in [RW92], that when \( G = \mathbb{Z} \) and \( (F_n) \) consists of intervals in \( \mathbb{Z} \) one has the following result: If (1.43) holds, then for
every \( \mathbb{Z} \)-action on a non-atomic measure space \((X, \mathcal{B}, \mu)\), there exists a function \( f \in L^1(X) \) such that

\[
\limsup_{n \to \infty} |A_n(F_n, q_n) f(x)| = \infty \quad \text{for a.e. } x \in X.
\]

It is possible to obtain a similar result for \( \mathbb{Z}^d \)-actions. However, we do not have yet a machinery to produce such negative results for actions by more general amenable groups.

(5) Even more complicated is the case when

\[
\limsup_{n \to \infty} \frac{1}{q_n} \left| \bigcup_{i=1}^{n-1} F_{i+1}^{-1} F_n \right| = \infty, \quad (1.44)
\]

but

\[
\liminf_{n \to \infty} \frac{1}{q_n} \left| \bigcup_{i=1}^{n-1} F_{i+1}^{-1} F_n \right| < \infty, \quad (1.45)
\]

In this case our ergodic averages may or may not converge depending on the thickness of the subsequence of \( \mathbb{N} \) along which convergence in (1.45) occurs.

**Example 1.48.** The following example is given in [RW92]. Let a non-atomic probability measure space \((X, \mathcal{B}, \mu)\) be fixed. Consider the additive group \( \mathbb{Z} \) and the Følner sequence \((F_n)\) given by

\[ F_n = [2^n, 2^n + n]. \]

Then

\[ \left| \bigcup_{i=1}^{n-1} (F_n - F_i) \right| \geq \frac{n^2}{2}. \]

Since \( |F_n|/n^2 \to 0 \), in view of Remark 1.47,

\[
\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=2^n}^{2^{n+1}} f(T^k x) = 0 \quad \text{for a.e. } x
\]

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for all \( f \in L^1(X) \) and every aperiodic measure-preserving transformation \( T \) of \( X \). On the other hand (by the result of Rosenblatt and Wierdl mentioned in part (4) of Remark 1.47), for every aperiodic measure-preserving transformation \( T \) of \( X \) and every \( \delta > 0 \), there exists \( f \in L^1(X) \) such that

\[
\limsup_{n \to \infty} \left| \frac{1}{n^{2-\delta}} \sum_{k=2^n}^{2^{n+1}} f(T^k x) \right| = \infty \quad \text{a.e.}
\]

However, it is possible to find a subsequence of \((F_n)\), so that the ergodic averages \( A_n(F_n, |F_n|) \) converge. Consider

\[
G_n = [2^{a_n}, 2^{a_n} + a_n],
\]

where the sequence of natural numbers \((a_n)\) is defined recursively as follows: \( a_1 = 1 \) and \( a_{n+1} = 2^a_n, \ n = 1, 2, \ldots \). Then

\[
\bigcup_{i=1}^{n-1} (G_n - G_i) \subset [2^{a_n} - 2^{a_{n-1}} - a_{n-1}, 2^{a_n} + a_n]
\]

and

\[
\left| \bigcup_{i=1}^{n-1} (G_n - G_i) \right| \leq a_n + 2^{a_{n-1}} + a_{n-1} = 2^{a_{n-1}} + 2^{a_{n-1}} + a_{n-1} \leq 3 \cdot 2^{a_{n-1}} \leq 3a_n = 3|G_n|.
\]

Hence, according to Theorem 1.44, for every non-atomic dynamical system \((X, \mathcal{B}, \mu, \mathbb{Z})\),

\[
\lim_{n \to \infty} \frac{1}{a_n} \sum_{k=2^{a_n}}^{2^{a_{n}+a_n}} f(T^k x) = 0 \quad \text{for a.e.} \ x.
\]

This also follows from [RW92, Theorem 1.7].

**Example 1.49.** Consider the multiplicative group of positive rational numbers \( \mathbb{Q}_+^* \).

It can be also thought of as \( \bigoplus \mathbb{Z} \). This is an abelian (and, therefore, amenable), infinitely generated group.

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We consider the following Følner sequence in $\mathbb{Q}_+^\times$:

$$F_n = \left\{ p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n} \mid -n \leq k_1, k_2, \ldots, k_n \leq n \right\},$$

(1.46)

where $p_1, p_2, \ldots$ are the prime numbers ordered by their magnitude. It is easy to see that $(F_n)$ is a Følner sequence in $\mathbb{Q}_+^\times$. Note that $|F_n| = (2n + 1)^n$.

Let us evaluate

$$q_n = \left| \bigcup_{i=1}^{n-1} F_i^{-1} F_n \right|.$$

Since our Følner sequence is nested,

$$q_n = |F_{n-1}^{-1} F_n|$$

$$= \left| \left\{ p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n} \mid -2n + 1 \leq k_1, k_2, \ldots, k_{n-1} \leq 2n - 1, \ 0 \leq k_n \leq n \right\} \right|$$

$$= (4n - 1)^{n-1} (2n + 1) < 2^{n-1} |F_n|.$$  

Hence, according to Theorem 1.44, for every dynamical system $(X, \mathcal{B}, \mu, \mathbb{Q}_+^\times)$ and every $f \in L^1(X)$, the sequence

$$\frac{1}{2^{n-1}|F_n|} \sum_{g \in \mathbb{Q}_+^\times} f(T_g x)$$

converges a.e. In view of Theorem 1.20, we should be able to find a subsequence of $(F_n)$ along which the ergodic averages $A(F_n, |F_n|)$ would converge a.e. We can take the Følner sequence $(G_k)$ defined as $G_k = F_{2^k}$. Then

$$|G_k| = (2 \cdot 2^k + 1) 2^{2^k}$$

and

$$|G_{k-1}^{-1} G_k| = (2 \cdot 2^{2^k} + 2 \cdot 2^{2^{k-1}} + 1)^{2^{2^k-1}} (2^{2^k} + 1)^{2^{2^k-2^{k-1}}}.$$
Let \( n = 2^{2^{k-1}} \). Then \( 2^k = n^2 \) and

\[
\frac{|G_{k-1}^{-1}G_k|}{|G_k|} = \frac{(2 \cdot 2^k + 2 \cdot 2^{2^{k-1}} + 1)2^{2^{k-1}}}{(2 \cdot 2^k + 1)2^{2k}}
\]
\[
= \frac{(2n^2 + 2n + 1)(2n^2 + 1)^n}{(2n^2 + 1)^n}
\]
\[
= \left(1 + \frac{2n}{2n^2 + 1}\right)^n \to e.
\]

Thus, according to Corollary 1.46, one has a regular pointwise convergence theorem along the Følner sequence \((G_k)\).

**Example 1.50.** Another interesting example is the so called Lamplighter Group. Let

\[ G = \mathbb{Z} \ltimes \bigoplus_{\mathbb{Z}} \mathbb{Z}_2. \]

with the group operation defined as

\[(i, a) \cdot (j, b) = (i + j, \sigma^i a + b),\]

where \( \sigma \) is the left shift:

\[(\sigma x)_k = x_{k+1}.\]

Let \( \tilde{G} = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2 \) and let \( e_k \in \tilde{G} \) be defined as

\[ e_k = (\ldots 0 \ldots 1 \ldots 0 \ldots). \]

Let

\[ F_n = \left\{ (j, b) \in G \mid |j| \leq n, b = \sum_{k=-2n}^{2n} \beta_k e_k, \text{ where } \beta_k = 0 \text{ or } 1 \right\}. \]

We claim that \( (F_n) \) is a left Følner sequence in \( G \). Let \( g \in G \) be an arbitrary element. Then \( g = (i, a) \) for some \( i \in \mathbb{Z} \) and \( a = \sum_{k=-m}^{m} \alpha_k e_k \). Let \( f \in F_n \). Then \( f = (j, b) \),
where \( b = \sum_{k=-2n}^{2n} \beta_k e_k \). So, we have:

\[
gf = (i, a)(j, b) = (i + j, \sigma^j a + b) = \left( i + j, \sum_{k=-m-j}^{m-j} \alpha_k e_k + \sum_{k=-2n}^{2n} \beta_k e_k \right).
\]

If \( n > m \), then \(-2n < -m - j < m - j < 2n\). Hence,

\[
\sum_{k=-m-j}^{m-j} \alpha_k e_k + \sum_{k=-2n}^{2n} \beta_k e_k = \sum_{k=-2n}^{2n} \gamma_k e_k,
\]

where \( \gamma_k \) is either 0 or 1. It follows that if \( n > m \), then \( gf \in F_n \) unless \(|i + j| > n\).

For a fixed \( g \) and \( F_n \), the number of elements \( gf \) with \( f \in F_n \) such that \(|i + j| > n\) is no greater than \(|i| \cdot 2^{4n+1}\). Hence,

\[
|g F_n \triangle F_n| \leq |i| \cdot 2^{4n+1},
\]

whereas

\[
|F_n| = (2n + 1) 2^{4n+1}.
\]

Therefore,

\[
\frac{|g F_n \triangle F_n|}{|F_n|} \leq \frac{|i|}{2n + 1} \quad n \to \infty.
\]

So, \((F_n)\) is a left \( \varphi \)olner sequence in \( G \). One can easily see that \((F_n)\) does not satisfy Tempelman’s Condition. Let us see whether it satisfies Shulman’s Condition. Let us write a formula for \( F_m^{-1} \).

If \( g = (i, a) \), where \( a = \sum_{k=-2m}^{2m} \alpha_k e_k \), then

\[
g^{-1} = (i, a)^{-1} = \left( -i, \sum_{k=-2m+i}^{2m+i} \alpha_k e_k \right).
\]

Thus,

\[
F_m^{-1} = \left\{ (i, a) \mid |i| \leq m, a = \sum_{k=-2m+i}^{2m+i} \alpha_k e_k, \text{ where } \alpha_k = 0 \text{ or } 1 \right\}
\]

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and
\[
F^{-1}_m F_n = \left\{ (i + j, a + b) \mid |i| \leq m, |j| \leq n, a = \sum_{k=-2m+i-j}^{2m+i-j} \alpha_k e_k, b = \sum_{k=-2n}^{2n} \beta_k e_k \right\}.
\]

Note that in (1.47), \(2m + i - j < 3m + n\) and \(-2m + i - j > -3m - n\). It follows that, if \(n \geq 3m\), then
\[
\sum_{k=-2m+i-j}^{2m+i-j} \alpha_k e_k + \sum_{k=-2n}^{2n} \beta_k e_k = \sum_{k=-2n}^{2n} \gamma_k e_k.
\]

and
\[
F^{-1}_m F_n = \left\{ (l, c) \mid |l| \leq m + n, c = \sum_{k=-2n}^{2n} \gamma_k e_k \right\}.
\]

Consequently,
\[
|F^{-1}_m F_n| = (2n + 2m + 1)2^{4n+1}
\]
and
\[
\frac{|F^{-1}_m F_n|}{|F_n|} = 1 + \frac{2m}{2n + 1}
\]

Let \(G_k = F_{n_k}\), where \(n_k = 3^k\). Then \(n_k = 3n_{k-1}\). Hence, to estimate \(G^{-1}_m G_n\) with \(n > m\), we can make use of (1.49). Then
\[
\left| \bigcup_{i=1}^{k-1} G^{-1}_i G_k \right| = \left| \frac{G^{-1}_k G_k}{|G_k|} \right| = \left| \frac{F^{-1}_m F_{n_k}}{|F_n|} \right| = 1 + \frac{2 \cdot 3^{k-1}}{2 \cdot 3^k + 1} < 2.
\]
Thus, the Følner sequence \((G_k)\) satisfies Shulman’s Condition and we have the regular \((q_k = |G_k|)\) pointwise ergodic theorem along \((G_k)\).

1.6 A Version of Lindenstrauss’ Covering Lemma for Semigroups and Its Applications

Since Tempelman’s covering lemma (Lemma 1.36) holds not only for a group but for an arbitrary semigroup as well, it is natural to try to prove Lindenstrauss’
Covering Lemma for semigroups. However, here we face some serious difficulties. Let us consider the following example.

**Example 1.51.** Assume that we have a badly non-cancellative semigroup. For instance, let the semigroup $S$ consist of all finite subsets of $\mathbb{Z}$ together with the empty set with the semigroup operation being the set intersection. Further, assume that for all $a \in E$, $n(a) = 1$. Let $F_1$ contain only sets from the compliment of the union of the sets belonging to $E$. Then for all $a \in E$, $F_1a = \{\emptyset\}$. If an analogue of Lindenstrauss’ Covering Lemma held for this situation, it would have been possible to define such probability distribution on the sets $B(a)$ that expectation of their intersections would be less than $1 + \delta$ and union is bigger than some fixed constant times $|E|$. To make it so that

$$\mathcal{E}(\Lambda(g) \mid \Lambda(g) > 0) < 1 + \delta,$$

we have to let

$$B(a) = \begin{cases} \{\emptyset\}, & \text{with probability } \delta/|E|, \\ \emptyset, & \text{otherwise} \end{cases}$$

Then

$$\mathcal{E}\left(\sum_{a \in E}|B(a)|\right) = |E|\frac{\delta}{|E|} = \delta.$$

But we want

$$\mathcal{E}\left(\sum_{a \in E}|B(a)|\right) > \gamma|E|,$$

for some fixed constant $\gamma$. Impossible.

So, to be able to use Lindenstrauss’ approach, we have to assume some degree of cancellatvity. We were able to prove the Covering Lemma below under assumption that $S$ is left-cancellative. Note that without left cancellativity assumption some other
results needed for a pointwise convergence theorem such as the Transfer Principle (Proposition 1.27) would not hold as well.

We were not able to obtain the analogues of the results from the previous section for the general sequence of factors \( q_n \). We only consider the case when \( q_n = |F_n| \) and impose an analogue of Shulman’s Condition (Definition 1.18) on \((F_m)\) right from the beginning.

**Definition 1.52 (Shulman’s Condition for Semigroups).** A Følner sequence in a left-cancellative semigroup \( S \) is called *tempered* if there exists a constant \( C \) such that for all \( n \)

\[
|\bigcup_{k=1}^{n-1} F_k^{-1} F_n a| < C|F_n| \quad \text{for all } a \in S
\]

**Remark 1.53.** It is easy to see that for groups the above definition is equivalent to Definition 1.18.

**Lemma 1.54 (Lindenstrauss’ Covering Lemma for Semigroups).** Let \( S \) be a left-amenable left-cancellative semigroup, \((F_n)\) a tempered left Følner sequence, \( E \subset S \) a finite set, and let for each \( a \in E \) a positive integer \( n(a) \) be chosen. Let \( \delta > 0 \) be given. Then it is possible to find set valued random variables \( \Phi(a) \) for \( a \in E \) such that

1. \( \Phi(a) \) is either \( F_{n(a)} \) or \( \emptyset \).
2. Let \( B(a) = \Phi(a)a \) and \( \Lambda : G \to \mathbb{N} \) be the random function

\[
\Lambda(g) = \sum_{a \in E} 1_{B(a)}(g).
\]

Then

\[
\mathcal{E}(\Lambda(g) \mid \Lambda(g) > 0) < 1 + \delta.
\]
(3) For some $\gamma > 0$ that depends on $\delta$,

$$E \left( \sum_{a \in E} |\Phi(a)| \right) > \gamma |E|. \quad (1.50)$$

**Proof of the Lemma.** Let $N = \max_{a \in E} n(a)$. For $1 \leq j \leq N$ we let

$$A_j = \{ a \in E \mid n(a) = j \}.$$

We start from $j = N$. For all $a \in A_N$ we let (independently from all other $b \in A_N$)

$$\Phi(a) = \begin{cases} 
F_N a & \text{with probability } p_N = \frac{\delta}{|F_N|}, \\
\emptyset & \text{with probability } 1 - p_N
\end{cases} \quad (1.51)$$

Assume that we have already defined $\Phi(a)$ for all $a \in A_k$, $j < k \leq N$. Then for each $a \in A_j$ we do the following (independently from all other $b \in A_j$)

- If $F_j a$ is disjoint from all $\Phi(a')$ for all $a' \in A_k$, $j < k \leq N$, then

$$\Phi(a) = \begin{cases} 
F_j a & \text{with probability } p_j = \frac{\delta}{|F_j|}, \\
\emptyset & \text{with probability } 1 - p_j
\end{cases} \quad (1.52)$$

- Otherwise $\Phi(a) = \emptyset$.

We first show that

$$E(\Lambda(g) \mid \Lambda(g) > 0) < 1 + \delta.$$

Let $\Lambda_j$ be the random function

$$\Lambda_j(g) = \sum_{a \in A_j} 1_{B(a)}(g).$$

We remark that for $j \neq j'$ the events $\Lambda_j(g) > 0$ and $\Lambda_{j'}(g) > 0$ are mutually exclusive.

We now define for every $g \in S$ and $j$

$$A(j, g) = \left( A_j \setminus \bigcup_{j' = j+1}^N \bigcup_{a' \in A_{j'}} F_{j'}^{-1} B(a') \right) \cap F_j^{-1} g.$$
(where $F^{-1}_j g := \{ a \in S \mid g \in F_j a \}$). This represents the set of all $a \in A_j$ such that $g \in F_j a$ and such that $F_j a$ is disjoint from $B(a')$ for all $a'$. By our construction, the quantity $|A(j, g)|$ is the number of sets $B(b)$, with $b \in A_j$, such that the probability to contain the element $g$ is positive. Clearly, $|A(j, g)| \leq |F^{-1}_j g|$. By left-cancellativity, $|F^{-1}_j g| \leq |F_j|$. Hence, $|A(j, g)| \leq |F_j|$. Using Lemma 1.43, we see that

$$E(\Lambda_j(g) \mid \Lambda_j(g) > 0) = E(\Lambda_j(g) \mid \Lambda_j(g) \geq 1) \leq 1 + p_j |A(j, g)| \leq 1 + p_j |F_j| = 1 + \delta.$$  

Hence,

$$E(\Lambda(g) \mid \Lambda(g) > 0) \leq 1 + \delta.$$  

We use induction to show that

$$E \left( \sum_{a \in E} |\Phi(a)| \right) > \gamma |E|  \tag{1.53}$$  

for

$$\gamma = \frac{\delta}{1 + C\delta},  \tag{1.54}$$  

where $C$ is the constant from Definition 1.52.

If $N = 1$,

$$E \left( \sum_{a \in A_1} |\Phi(a)| \right) = \sum_{a \in A_1} E \left( \sum_{g \in F} 1_{\Phi(a)}(g) \right) = |A_1||F_1| \frac{\delta}{|F_1|} = \delta |A_1| = \delta |E| > \gamma |E|.  \tag{1.53}$$

and thus (1.53) is satisfied.

Now, assume that (1.28) holds for $\tilde{N} = N - 1$. Note that the distribution on $\Phi(a)$ for $a \in A_j$, $j < N$ is the same as one would obtain for $\tilde{N} = N - 1$ and

$$\tilde{A}_j = A_j \setminus \bigcup_{a \in A_N} F_j^{-1} B(a).$$
By the induction hypothesis,

$$
E \left( \sum_{j=1}^{N-1} \sum_{a \in A_j} |\Phi(a)| \right) \geq \gamma \left| \bigcup_{j=1}^{N-1} A_j \right|.
$$

Since \((F_n)\) is tempered,

$$
\left| \bigcup_{j=1}^{N-1} A_j \right| = \left| \bigcup_{j=1}^{N-1} A_j \right| - \left| \bigcup_{a \in A_N} \bigcup_{j=1}^{N-1} F_j^{-1} B(a) \right| \\
\geq \left| \bigcup_{j=1}^{N-1} A_j \right| - C \sum_{a \in A_N} |\Phi(a)|. \quad (1.55)
$$

It follows that

$$
E \left( \sum_{j=1}^{N-1} \sum_{a \in A_j} |\Phi(a)| \right) \geq \gamma \left| \bigcup_{j=1}^{N-1} A_j \right| - \gamma C \sum_{a \in A_N} |\Phi(a)|. \quad (1.56)
$$

Adding \(\sum_{a \in A_N} |\Phi(a)|\) to both sides of (1.56) and taking expectation, we have:

$$
E \left( \sum_{a \in E} |\Phi(a)| \right) \geq \gamma \left| \bigcup_{j=1}^{N-1} A_j \right| + (1 - \gamma C)E \left( \sum_{a \in A_N} |\Phi(a)| \right). \quad (1.57)
$$

Clearly,

$$
E \left( \sum_{a \in A_N} |\Phi(a)| \right) = \sum_{a \in A_N} E(|\Phi(a)|) = |A_N| \frac{\delta}{|F_N|} |F_N| = \delta |A_N|. \quad (1.58)
$$

Hence,

$$
E \left( \sum_{a \in E} |\Phi(a)| \right) \geq \gamma \left| \bigcup_{j=1}^{N-1} A_j \right| + (1 - \gamma C)\delta |A_N|. \quad (1.59)
$$

Because of (1.54), \(\delta (1 - \gamma C) = \gamma\). Hence, combining (1.57) and (1.58), we have

$$
E \left( \sum_{a \in E} |\Phi(a)| \right) \geq \gamma \left( \left| \bigcup_{j=1}^{N-1} A_j \right| + |A_N| \right) = \gamma \left| \bigcup_{j=1}^{N} A_j \right| = \gamma |E|.
$$

\(\blacksquare\)
Theorem 1.55. Let \( S \) be a left-amenable left-cancellative semigroup acting on a probability measure space \((X, \mathcal{B}, \mu)\). Let \((F_n)\) be a tempered left Følner sequence in \( S \). There exists a constant \( c \) such that for all \( f \in L^1(X) \),

\[
\mu(\mathcal{M}(F_n)[f](x) > \lambda) \leq \frac{c}{\lambda} ||f||_1.
\] (1.60)

Proof. In view of Proposition 1.24 and Proposition 1.27, it is enough to prove that the MAX IE on \( S \) holds for every \( \varphi \in l_1(S) \). We may assume that \( \varphi \) is real-valued and non-negative. Also, may assume that \( \lambda = 1 \). Let

\[
E = \{ a \in S \mid \tilde{M}[\varphi](a) > 1 \}.
\]

Since \( \varphi \) has finite support, \( E \) is a finite set. Notice that \( a \in E \) iff there exists \( n > 0 \) such that

\[
|F_n| < \sum_{g \in F_n} \varphi(ga) = \sum_{g \in F_n} \varphi(g).
\] (1.61)

Let \( n(a) \) be the smallest \( n \) such that the above holds. We apply Lemma 1.54 with \( \delta = 1 \). There exist a positive number \( \gamma \) and set-valued random variables \( \Phi(a) \) for \( a \in E \) such that:

1. \( \Phi(a) \) is either \( F_{n(a)} \) or \( \emptyset \),

2. \( E(1_{\Phi(a)}(g)) \leq 2 \),

3. \( E \left( \sum_{a \in E} |\Phi(a)| \right) > \gamma |E| \).

Then

\[
E \left( \sum_{a \in E} \sum_{g \in S} \varphi(g)1_{\Phi(a)}(g) \right) = \sum_{g \in S} \varphi(g)E \left( \sum_{a \in E} 1_{\Phi(a)}(g) \right) \leq 2 \sum_{g \in S} \varphi(g) = 2||\varphi||_1.
\] (1.62)
On the other hand,

\[
\mathbb{E}\left(\sum_{a \in E} \sum_{g \in S} \varphi(g)1_{\Phi(a)\sigma}(g)\right) = \mathbb{E}\left(\sum_{a \in E} \sum_{g \in \Phi(a)\sigma} \varphi(g)\right). \tag{1.63}
\]

If \(\Phi(a) = \emptyset\), then

\[
\sum_{g \in \Phi(a)\sigma} \varphi(g) = 0 = |\Phi(a)| = 0.
\]

If \(\Phi(a) = F_{n(a)}\), then by virtue of (1.61),

\[
\sum_{g \in \Phi(a)\sigma} \varphi(g) > |F_{n(a)}| = |\Phi(a)|.
\]

So, in either case,

\[
\sum_{g \in B(a)} \varphi(g) \geq |\Phi(a)|. \tag{1.64}
\]

Thus,

\[
\mathbb{E}\left(\sum_{a \in E} \sum_{g \in \Phi(a)} \varphi(g)\right) \geq \mathbb{E}\left(\sum_{a \in E} |\Phi(a)|\right) > \gamma|E|. \tag{1.65}
\]

Combining (1.62), (1.63), and (1.65), we see that

\[
|E| < \frac{2}{\gamma} \|\varphi\|_1.
\]

\(\Box\)

**Corollary 1.56.** Let \(S\) be an left amenable left cancellative semigroup acting on a probability measure space \((X, \mathcal{B}, \mu)\). Let \((F_n)\) be a left Følner sequence in \(S\) satisfying Shulman’s condition and let \(f \in L^1(X)\). Then

\[
\frac{1}{|F_n|} \sum_{g \in F_n} f(T_g x)
\]

converges a.e.
Example 1.57. Consider the multiplicative semigroup $\mathbb{N}^\times$. A left Følner sequence $(F_n)$ is given by

$$F_n = \{ p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n} | -n \leq k_1, k_2, \ldots, k_n \leq n \},$$

Similarly to Example 1.49, we can see that $(F_n)$ is not tempered Følner sequence, but a subsequence of it is.

This example can be generalized. Let $S$ be a cancellative (from both sides) left amenable semigroup. Then, according to Theorem C.3, it is embeddable into an amenable group $G$ that is generated by $S$. Moreover, according to Proposition C.5, every Følner sequence in $S$ is a Følner sequence in $G$ as well. Hence, for cancellative left amenable semigroups we can use all the results that we obtained for groups, in particular, Theorem 1.20. Thus, we obtain

**Theorem 1.58.** Every Følner sequence in a cancellative left amenable semigroup has a tempered subsequence. In particular, every cancellative left amenable semigroup has a tempered Følner sequence.

1.7 Open Problems

**Question 1.59.** Does an analogue of Theorem 1.58 hold for left cancellative, but not right cancellative semigroups? One example of such semigroup is “Lamplighter Semigroup” described in Example B.9. As one can check, a natural Følner sequence in this semigroup satisfies Tempelman’s Condition, so the pointwise ergodic theorem trivially holds. What happens for other left cancellative semigroups?

Rosenblatt and Wierdl [RW92] proved a pointwise ergodic theorem along sequences of intervals in $\mathbb{Z}$ under a condition that is identical to the condition used
by Lindenstrauss. Their result only holds for sequences of intervals in \( \mathbb{Z} \) (not for general Følner sequences). According to [RW92], this result also holds for sequences of cubes in \( \mathbb{Z}^d \). So, it is a quite limited result compared to Theorem 1.44. However, Rosenblatt and Wierdl’s result is stronger in the following sense.

**Question 1.60.** Rosenblatt and Wierdl showed that for a sequence of intervals the condition of being tempered is, in certain sense, necessary and sufficient for having a pointwise ergodic theorem along this sequence. Can this result be extended? Unfortunately, as a simple example shows, even in \( \mathbb{Z} \) one may find a non-tempered Følner sequence along which the pointwise ergodic theorem holds. Thus, we must refine Shulman’s condition (1.7) to make it necessary and sufficient in general case. However, this seems to be a very hard problem.

**Question 1.61.** Another possibility is to consider some analogues of intervals in more general groups and try to see whether Shulman’s condition is a necessary condition for pointwise convergence in this class of sequences. Of course, this class has to be natural enough for this investigation to make sense. One possibility is to consider the class \( \mathcal{R} \) of Følner sequences such that one can tile the group by shifts of an element of this Følner sequence. This is exactly the class for which Rokhlin’s Lemma [Pet83, cf. p. 48] holds. Is Shulman’s Condition necessary and sufficient for Følner sequences from the class \( \mathcal{R} \)?
CHAPTER 2

JOINT ERGODICITY OF GROUP ACTIONS

2.1 Definitions and Preliminary Results

Let $G$ be a countable amenable group and $(X, \mathcal{B}, \mu)$ a probability measure space. We are interested in studying joint ergodic properties of several $G$-actions $T^{(1)}, \ldots, T^{(s)}$ on $X$. By this we mean studying the convergence of the ergodic averages of the form

$$\frac{1}{|L_n|} \sum_{g \in L_n} \prod_{i=1}^{s} f_i(T_g x)$$

for a Følner sequence $(L_n)$ and $x \in X$. For brevity, we shall often just write $\prod_{i=1}^{s} T_g f_i$ meaning that

$$\left(\prod_{i=1}^{s} T_g f_i\right) (x) = \prod_{i=1}^{s} f_i(T_g x).$$

The symbol $T^{(0)}$ shall denote the identity $G$-action,

$$T^{(0)}_g \overset{\text{def}}{=} \text{Id} \quad \text{for all } g \in G.$$

**Definition 2.1.** Let $G$ be a countable amenable group and $T^{(1)}, \ldots, T^{(s)}$ be measure-preserving $G$-actions on $X$. The $G$-actions $T^{(1)}, \ldots, T^{(s)}$ are called
(1) **weakly jointly ergodic** (w.j.e.) if for all sets $A_0, \ldots, A_s \in \mathcal{B}$ and for every Følner sequence $(L_n)$ in $G$

$$\lim_{n \to \infty} \frac{1}{|L_n|} \sum_{g \in L_n} \mu \left( \bigcap_{i=0}^{s} T_g(i)(A_i) \right) = \prod_{i=0}^{s} \mu(A_i) \quad (2.1)$$

(2) **jointly ergodic** (j.e.) if for all sets $A_i, B_j \in \mathcal{B}$, $1 \leq i, j \leq s$ and for every Følner sequence $(L_n)$ in $G$

$$\lim_{n \to \infty} \frac{1}{|L_n|^2} \sum_{g,h \in L_n} \mu \left( \bigcap_{i=1}^{s} T_g(i)(A_i) \cap T_h(i)(B_i) \right) = \prod_{i=1}^{s} \mu(A_i) \mu(B_i) \quad (2.2)$$

The following theorem gives more justification to the word “weak” in Definition 2.1.

**Theorem 2.2.** $T^{(1)}, \ldots, T^{(s)}$ are

(1) w.j.e. iff

$$\lim_{n \to \infty} \frac{1}{|L_n|} \sum_{g \in L_n} \prod_{i=1}^{s} T_g(i) f_i \xrightarrow{w.a.s.-L^2} \prod_{i=1}^{s} \int_X f_i \, d\mu \quad (2.3)$$

(2) j.e. iff

$$\lim_{n \to \infty} \frac{1}{|L_n|} \sum_{g \in L_n} \prod_{i=1}^{s} T_g(i) f_i \xrightarrow{L^2} \prod_{i=1}^{s} \int_X f_i \, d\mu \quad (2.4)$$

for all $f_1, \ldots, f_s \in L^\infty(X, \mathcal{B}, \mu)$ and any Følner sequence of sets $(L_n)$.

**Proof.** For simplicity, assume that all functions are real.

(1) “If” part. Let the condition (2.3) hold and let $A_0, A_1, \ldots, A_s \in \mathcal{B}$ be given. Then one has:

$$\lim_{n \to \infty} \left( \frac{1}{|L_n|} \sum_{g \in L_n} \prod_{i=1}^{s} T_g(i) f_i \cdot 1_{A_0} \right) = \lim_{n \to \infty} \frac{1}{|L_n|} \sum_{g \in L_n} \int_X \left( \prod_{i=1}^{s} T_g(i) 1_{A_i} \cdot 1_{A_0} \right) d\mu$$

$$= \prod_{i=1}^{s} \int_X 1_{A_i} \, d\mu \cdot \int_X 1_{A_0} \, d\mu$$

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Therefore, (2.1) holds.

"Only if" part. Let $T^{(1)}, \ldots T^{(s)}$ be w.j.e. Clearly, (2.3) holds for characteristic functions of measurable sets. These functions form a dense subset in $L^\infty(X)$. A routine check shows that for every $j$ with $1 \leq j \leq s$ and for fixed $f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_s$, the set of functions $f_j$ for which (2.3) holds is closed in $L^\infty$. Therefore, (2.3) holds for all $f_1, \ldots, f_s \in L^\infty$.

(2) "If" part. Let the condition (2.4) hold. Let $A_i, B_i \in \mathcal{B}, 1 \leq i \leq s$, be given and let $f_i = 1_{A_i}, \tilde{f}_i = 1_{B_i}$. Then one has:

$$\frac{1}{|L_N|} \sum_{g \in L_N} \prod_{i=1}^s T_g^{(i)} f_i \overset{L^2}{\underset{N \to \infty}{\longrightarrow}} \prod_{i=1}^s \int_X f_i \, d\mu$$

$$\frac{1}{|L_N|} \sum_{g \in L_N} \prod_{i=1}^s T_g^{(i)} \tilde{f}_i \overset{L^2}{\underset{N \to \infty}{\longrightarrow}} \prod_{i=1}^s \int_X \tilde{f}_i \, d\mu$$

Therefore,

$$\frac{1}{|L_N|^2} \left( \sum_{g \in L_N} \prod_{i=1}^s T_g^{(i)} f_i, \sum_{h \in L_N} \prod_{i=1}^s T_h^{(i)} \tilde{f}_i \right) \overset{N \to \infty}{\longrightarrow} \prod_{i=1}^s \int_X f_i \, d\mu \prod_{i=1}^s \int_X \tilde{f}_i \, d\mu$$

and (2.2) follows.

"Only if" part. One has:

$$\left\| \frac{1}{|L_N|} \sum_{g \in L_N} \prod_{i=1}^s T_g^{(i)} f_i - \prod_{i=1}^s \int_X f_i \, d\mu \right\|^2 = \frac{1}{|L_N|^2} \sum_{g,h \in L_N} \int_X \prod_{i=1}^s T_g^{(i)} f_i \cdot T_h^{(i)} f_i \, d\mu$$

$$- \frac{2}{|L_N|} \sum_{g \in L_N} \int_X \left( \prod_{i=1}^s T_g^{(i)} f_i \right) d\mu \cdot \prod_{i=1}^s \int_X f_i \, d\mu + \left( \prod_{i=1}^s \int_X f_i \, d\mu \right)^2$$

$$= \frac{1}{|L_N|^2} \sum_{g,h \in L_N} \int_X \prod_{i=1}^s T_g^{(i)} f_i \cdot T_h^{(i)} f_i \, d\mu - \left( \prod_{i=1}^s \int_X f_i \, d\mu \right)^2$$

$$- 2 \left( \prod_{i=1}^s \int_X f_i \, d\mu \right) \cdot \left( \frac{1}{|L_N|} \sum_{g \in L_N} \int_X \left( \prod_{i=1}^s T_g^{(i)} f_i \right) d\mu - \prod_{i=1}^s \int_X f_i \, d\mu \right). \quad (2.5)$$
This proves the condition (2.4) for characteristic functions if we set \( f_i = 1_{A_i} \). The general case follows by using the density argument.

\[ \square \]

**Remark 2.3.** Theorem 2.2 also holds if we replace \( L^2 \)-convergence by \( L^p \)-convergence and consider functions \( f_1, \ldots, f_s \) from \( L^p(X) \) instead of \( L^\infty(X) \) for \( p \geq 2(s + 1) \).

**Corollary 2.4.** If \( G \)-actions \( T^{(i)}, 1 \leq i \leq s \), are j.e., then they are w.j.e.

**Remark 2.5.** It is clear that if several actions are w.j.e., then each of them is ergodic.

In [BB84], [Ber85b], and [BB86], D. Berend and V. Bergelson studied joint ergodic properties of several \( \mathbb{Z} \)-actions. They have obtained several criteria for weak and strong joint ergodicity of such actions. We prove analogues of their results in Sections 2.4 and 2.5. In [BB86], it has been shown that the notions of j.e. and w.j.e. are equivalent for \( \mathbb{Z} \)-actions. In Section 2.4, we prove this result for actions of countable abelian groups and discuss whether this result holds for non-abelian groups. In Section 2.6 we extend notions of joint functions and sets introduced by D. Berend in [Ber85b] for \( \mathbb{Z} \)-actions to general amenable groups and discuss their properties. In Section 2.7, we investigate connections between notions of joint ergodicity, total ergodicity, and total joint ergodicity.

The notions of joint ergodicity in context of commuting actions of general groups were first introduced by V. Bergelson and J. Rosenblatt in [BR88a]. However, their notion of mutual ergodicity is slightly different from the notion of joint ergodicity that we study here. We shall make use of some of the results obtained in [BR88a] below.

In Sections 2.2 and 2.3 we present some technical results that will be needed later.
2.2 Some Spectral Properties of Measurable Actions

Here we present some spectral properties of measurable actions. If $G$ is an abelian group, then $\hat{G}$ denotes the dual group (the group of characters) of $G$. The reader may consult Appendix E for a review of basic results from character $g$ theory that are used below.

**Definition 2.6.** A function $f \in L^2(X)$ is called an eigenfunction of $G$-action $T$ if there exists a character $\lambda$ of $G$ such that $T_g f = \lambda(g) f$ for all $g \in G$. The character $\lambda$ is called an eigenvalue of $T$.

The set of all eigenvalues of an action is called its spectrum. If the action is ergodic, then its spectrum is a subgroup of the unit circle.

**Definition 2.7.** Let $G$ be a countable abelian group. We say that $G$-actions $T^{(1)}, \ldots, T^{(s)}$ have mutually disjoint spectra if the following condition is satisfied. If eigenvalues $\lambda_i$ of $T^{(i)}$, $1 \leq i \leq s$, are such that $\prod_{i=1}^{s} \lambda_i = 1$, then $\lambda_i = 1$ for all $1 \leq i \leq s$.

In case of two $G$-actions $T$ and $S$, this just means that $T$ and $S$ have no common eigenvalues except 1. If actions $T^{(i)}$, $1 \leq i \leq s$, have mutually disjoint spectra, then, in particular, every pair of transformations $T^{(i)}$, $T^{(j)}$ with $1 \leq i < j \leq s$ have no common eigenvalues except 1.

We shall be using direct products of several measurable actions. On the unitary level, a direct product corresponds to the tensor product of the actions’ unitary representations. For that reason, we shall normally be using the symbol $\otimes$ as opposed to $\times$ to denote the direct product of actions.
Proposition 2.8. Let $G$ be a countable abelian group. Let $T^{(i)}$, $1 \leq i \leq s$, be $G$-actions on $(X, \mathcal{B}, \mu)$. The action $\bigotimes_{i=1}^{s} T^{(i)}$ is ergodic iff each $T^{(i)}$ is ergodic and they have mutually disjoint spectra.

Proof. “Only if” direction is easy. If $T^{(1)} \otimes \cdots \otimes T^{(s)}$ is ergodic then, clearly, each $T^{(i)}$ is ergodic. If they do not have mutually disjoint spectra, i.e. if for $1 \leq i \leq s$ there exist $\lambda_i \in \hat{G}$, not all equal to 1, and square-integrable functions $f_i$ such that $T^{(i)}_g f_i = \lambda_i(g) f_i$ for all $g \in G$ and $\prod_{i=1}^{s} \lambda_i = 1$, then the function $f = f_1 \otimes f_2 \otimes \cdots \otimes f_s$ is a non-constant invariant function of $T^{(1)} \otimes \cdots \otimes T^{(s)}$.

To prove the “if” part, we first consider the case $s = 2$. Assume that our hypothesis is satisfied. Let $f \in L^2(X \times X)$ be a $T^{(1)} \otimes T^{(2)}$-invariant function. By Lemma F.10, $f$ can be expanded in $L^2$ as

$$f = \sum_n u_n \otimes v_n$$

so that $T^{(1)}_g u_n = \lambda_n(g) u_n$, $T^{(2)}_g v_n = \lambda'_n(g) v_n$ for all $g \in G$, and $\lambda_n \lambda'_n = 1$ for all $n$. Since $T^{(1)}$ and $T^{(2)}$ have mutually disjoint spectra, $\lambda_n = 1$ and $\lambda'_n = 1$ for all $n$. So, all functions $u_n$ and $v_n$ are constants. Therefore, $f = \text{const}$ as well and, consequently, $T^{(1)} \otimes T^{(2)}$ is ergodic.

The general case follows by using induction on $s$. Namely, let us have $s$ ergodic $G$-actions $T^{(1)}, \ldots, T^{(s)}$ with mutually disjoint spectra, $s > 2$. If the actions $T^{(i)}$, $1 \leq i \leq s - 1$, have eigenfunctions $f_i$ and eigenvalues $\lambda_i$, such that $\prod_{i=1}^{s-1} \lambda_i = 1$, then all $\lambda_i = 1$ since otherwise we set $f_s$ to be a constant and $\lambda_s = 1$ and get a contradiction with $T^{(1)}, \ldots, T^{(s)}$ having mutually disjoint spectra. So, by the induction hypothesis, $T^{(1)} \otimes \cdots \otimes T^{(s-1)}$ is ergodic. $T^{(s)}$ is also ergodic. It remains to show that they have mutually disjoint spectra. Assume that there exists $\lambda \in \hat{G}$ that is an eigenvalue
for both \( T_g^{(1)} \otimes \cdots \otimes T_g^{(s-1)} \) and \( T^{(s)} \). In view of Remark F.11, \( \lambda = \prod_{i=1}^{s-1} \lambda_i \), where \( \lambda_1, \ldots, \lambda_{s-1} \) are eigenvalues of the actions \( T^{(1)}, \ldots, T^{(s-1)} \). Since \( T^{(1)}, \ldots, T^{(s)} \) have mutually disjoint spectra, it follows that \( \lambda_1 = \cdots = \lambda_{s-1} = 1 \). Thus, the actions \( T^{(1)} \otimes \cdots \otimes T^{(s-1)} \) and \( T^{(s)} \) have mutually disjoint spectra. Applying the case \( s = 2 \), we conclude that \( T^{(1)}, \ldots, T^{(s)} \) is ergodic.

Lemma 2.9. If \( T^{(1)}, \ldots, T^{(s)} \) are weakly jointly ergodic actions of a countable abelian group \( G \), then they have mutually disjoint spectra.

Proof. Assume they do not. Then they have eigenfunctions \( f_i \) and corresponding to them eigenvalues \( \lambda_i \), \( 1 \leq i \leq s \), such that \( \prod_{i=1}^{s} \lambda_i = 1 \) and at least one of the eigenvalues, say, \( \lambda_j \neq 1 \). Then \( f_j \neq \text{const} \) and \( \int_X f_j \, d\mu = 0 \). So, the right side of (2.3) is zero. On the other hand, the left side of (2.3) does not converge to zero. Contradiction.

Corollary 2.10. If actions \( T^{(1)}, \ldots, T^{(s)} \) of a countable abelian group are weakly jointly ergodic, then their direct product is ergodic.

Is there a generalization of the Corollary above for the case when \( G \) is not abelian? The answer is positive if the actions \( T^{(1)}, \ldots, T^{(s)} \) commute.

Lemma 2.11. If \( T^{(1)}, \ldots, T^{(s)} \) are weakly jointly ergodic commuting actions of a countable group \( G \), then the action \( \bigotimes_{i=1}^{s} T^{(i)} \) is ergodic.

Proof. See [BR88a, Theorem 2.6].

2.3 Van der Corput’s Inequality for Amenable Groups

The following proposition can be found in [BR88a].
Lemma 2.12. Let $G$ be a countable amenable group and let $(L_n)$ be a left Følner sequence in $G$. Let $A : G \to \mathcal{H}$ be a function where $\mathcal{H}$ is a Hilbert space. Assume that $A(G)$ is bounded. Let

$$S_1(N) = \frac{1}{|L_N|} \sum_{g \in L_N} A(g)$$

and

$$S_2(N, H) = \frac{1}{|L_H|} \sum_{h \in L_H} \frac{1}{|L_N|} \sum_{g \in L_N} A(hg)$$

for $N, H \in \mathbb{N}$. Then for all $H$

$$\lim_{N \to \infty} \|S_1(N) - S_2(N, H)\| = 0$$

Proof. Assume that $A : G \to \mathcal{H}$ is a bounded function, $\alpha = \max_{g \in G} A(g)$. Let a Følner sequence $(L_n)$, a number $H \in \mathbb{N}$, and $\varepsilon > 0$ be given. By definition of Følner sequence, there exists $N_0 \in \mathbb{N}$ such that $|L_N \triangle hL_N| < (\varepsilon / \alpha)|L_N|$ for all $N > N_0$ and $h \in L_H$. Therefore,

$$\|S_1(N) - S_2(N, H)\| = \left\| \frac{1}{|L_N|} \sum_{g \in L_N} A(g) - \frac{1}{|L_H|} \sum_{h \in L_H} \frac{1}{|L_N|} \sum_{g \in L_N} A(hg) \right\|$$

$$\leq \left\| \frac{1}{|L_N|} \sum_{g \in L_N} \frac{1}{|L_H|} \sum_{h \in L_H} A(g) - \frac{1}{|L_H|} \sum_{h \in L_H} \frac{1}{|L_N|} \sum_{g \in L_N} A(hg) \right\|$$

$$= \left\| \frac{1}{|L_N|} \frac{1}{|L_H|} \sum_{h \in L_H} \sum_{g \in L_N} (A(g) - A(hg)) \right\|$$

$$\leq \frac{1}{|L_N|} \frac{1}{|L_H|} \sum_{h \in L_H} \sum_{g \in L_N} \|A(g)\| < \frac{1}{|L_N|} \frac{1}{|L_H|} \frac{\varepsilon}{\alpha} |L_N| \alpha = \varepsilon$$

for all $N > N_0$.

The following Lemma is an abstract analogue of the well-known Van der Corput’s inequality.
Lemma 2.13 (Van der Corput’s Inequality). Let $G$ be a countable amenable group and let $(L_n)$ be a left Følner sequence in $G$. Let $A : G \to \mathcal{H}$ be a bounded function where $\mathcal{H}$ is a Hilbert space. For every $H \in N$ and $\varepsilon > 0$ there exists $N_0 \in N$ such that for all $N > N_0$

$$\left\| \frac{1}{|L_N|} \sum_{g \in L_N} A(g) \right\|^2 \leq \frac{1}{|L_N|} \sum_{g \in L_N} \frac{1}{|L_H|^2} \sum_{u,v \in L_H} (A(ug), A(vg)) + \varepsilon \quad (2.6)$$

Proof. First, observe that in view of Lemma 2.12,

$$\lim_{N \to \infty} \left\| \frac{1}{|L_N|} \sum_{g \in L_N} A(g) - \frac{1}{|L_N|} \sum_{g \in L_N} \frac{1}{|L_H|} \sum_{h \in L_H} A(hg) \right\| = 0.$$

By Cauchy-Schwartz inequality,

$$\left\| \frac{1}{|L_N|} \sum_{g \in L_N} \frac{1}{|L_H|} \sum_{h \in L_H} A(hg) \right\|^2 \leq \frac{1}{|L_N|} \sum_{g \in L_N} \frac{1}{|L_H|^2} \sum_{h \in L_H} A(hg)^2$$

$$= \frac{1}{|L_N|} \sum_{g \in L_N} \frac{1}{|L_H|^2} \sum_{u \in L_H} \sum_{v \in L_H} (A(ug), A(vg)) .$$

\[\square\]

2.4 Main Theorem on Joint Ergodicity

Theorem 2.14. Let $T^{(1)}, \ldots, T^{(s)}$ be actions of a countable abelian group $G$ on a probability space $(X, \mathcal{B}, \mu)$. The following conditions are equivalent:

(1) $T^{(1)}, \ldots, T^{(s)}$ are jointly ergodic.

(2) $T^{(1)}, \ldots, T^{(s)}$ are weakly-jointly ergodic.

(3) (a) $T^{(1)} \otimes \cdots \otimes T^{(s)}$ is ergodic.

(b) $\frac{1}{|L_N|} \sum_{g \in L_N} \int_X \prod_{i=1}^s T^{(i)}_g f_i \, d\mu \xrightarrow{N \to \infty} \prod_{i=1}^s \int_X f_i \, d\mu$

for every $f_1, f_2, \ldots, f_s \in L^\infty(X)$ and every Følner sequence of sets $(L_N)$ in $G$. 

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Proof. The implications $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3b)$ are trivial. The implication $(2) \Rightarrow (3a)$ follows from Corollary 2.10.

$(3) \Rightarrow (1)$: We need to show that given $f_1, f_2, \ldots, f_s \in L^\infty(X)$ and a Følner sequence $(L_N)$ one has

$$\frac{1}{|L_N|} \sum_{g \in L_N} \prod_{i=1}^{s} T_{g}^{(i)} f_i \stackrel{L^2}{\longrightarrow} \prod_{i=1}^{s} \int_X f_i d\mu$$

It may be assumed that each $f_i$ is a real function bounded by 1 in absolute value and that $\int_X f_i d\mu = 0$. Let $A(g) = \prod_{i=1}^{s} T_{g}^{(i)} f_i, \ g \in G$. Then $A : G \rightarrow L^2(X)$ is a bounded function because

$$\|A(g)\|_2 \leq \prod_{i=1}^{s} \|T_{g}^{(i)} f_i\|_{\infty} = \prod_{i=1}^{s} \|f_i\|_{\infty} \leq 1.$$

Let us fix a positive integer $H$ and $\varepsilon > 0$. By Lemma 2.13, there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$ one has

$$\left\| \frac{1}{|L_N|} \sum_{g \in L_N} \prod_{i=1}^{s} T_{g}^{(i)} f_i \right\|_2^2 \leq \frac{1}{|L_N|} \sum_{g \in L_N} \frac{1}{|L_H|^2} \sum_{u,v \in L_H} \int_X \prod_{i=1}^{s} T_{g+u}^{(i)} f_i \cdot T_{g+v}^{(i)} f_i d\mu + \varepsilon$$

$$= \frac{1}{|L_H|^2} \sum_{u,v \in L_H} S(N, u, v) + \varepsilon,$$

where

$$S(N, u, v) = \frac{1}{|L_N|} \int_X \prod_{i=1}^{s} T_{g+u}^{(i)} f_i \cdot T_{g+v}^{(i)} f_i d\mu.$$

From the condition (3b), we see that

$$S(N, u, v) \xrightarrow{N \rightarrow \infty} \prod_{i=1}^{d} \int_X T_{u}^{(i)} f_i \cdot T_{v}^{(i)} f_i d\mu$$

Let $(X^s, B^s, \mu^s, T) = \prod_{i=1}^{d} (X, B, \mu, T^{(i)})$. Let $F \in L^\infty(X^s)$ be defined by

$$F = f_1 \otimes \cdots \otimes f_s.$$
Holding $H$ fixed, it now follows from (2.7) and (2.8) that
\[
\lim_{N \to \infty} \left\| \frac{1}{|L_N|} \sum_{g \in L_N} \prod_{i=1}^s T_g^{(i)} f_i \right\|_2^2 \leq \frac{1}{|L_H|^2} \sum_{u,v \in L_H} \int_{X^s} T_u F \cdot T_v F d\mu^s + \varepsilon. \tag{2.9}
\]

Since $T$ is ergodic,
\[
\frac{1}{|L_H|^2} \sum_{u,v \in L_H} \int_{X^s} T_u F \cdot T_v F d\mu^s = \int_{X^s} \left( \frac{1}{|L_H|} \sum_{h \in L_H} T_h f \right)^2 d\mu^s \xrightarrow{H \to \infty} 0. \tag{2.10}
\]

From (2.9) and (2.10), we conclude that
\[
\frac{1}{|L_N|} \sum_{g \in D_N} \prod_{i=1}^s T_g^{(i)} f_i \xrightarrow{L^2} \sum_{i=1}^s \int_{X^s} f_i d\mu.
\]

\[\square\]

Remark 2.15. For the case of $\mathbb{Z}$-actions, Theorem 2.14 has been proven by V. Bergelson and D. Berend in [BB86].

Remark 2.16. In the proof of Theorem 2.14 we used the fact that $G$ is abelian only once — when we quoted Corollary 2.10. If the group $G$ is non-abelian, but $T^{(1)}, \ldots, T^{(s)}$ commute, the 3 conditions of Theorem 2.2 are still equivalent since we can use Lemma 2.11 instead of Corollary 2.10. Another case when our 3 conditions are equivalent is when ergodicity of the tensor product of $T^{(1)}, \ldots, T^{(s)}$ can be taken for granted. If $X$ is a compact group and $T^{(1)}, \ldots, T^{(s)}$ are actions of continuous group-automorphisms, then weak joint ergodicity of $T^{(1)}, \ldots, T^{(s)}$ implies ergodicity of $\bigotimes_{i=1}^s T^{(i)}$. Namely, if $T^{(1)}, \ldots, T^{(s)}$ are w.j.e, then each of $T^{(1)}, \ldots, T^{(s)}$ is ergodic. As we shall see later (Proposition 4.10), each action $T^{(1)}, \ldots, T^{(s)}$ is weakly mixing and, in view of Theorem 3.3, it follows that $\bigotimes_{i=1}^s T^{(i)}$ is ergodic. So, for this case (actions by continuous automorphisms of a compact group) the three types of convergence — strong, weak, and even weaker type of convergence (3b) — are equivalent.
2.5 Commutative Case

Now, let us consider the case when the $G$-actions $T^{(1)}, \ldots, T^{(s)}$ commute. By that we mean that $T^{(i)}_g T^{(j)}_g = T^{(j)}_g T^{(i)}_g$ for all $1 \leq i < j \leq s$ and $g \in G$. Then one can prove the following generalization of the results obtained in [BB86]

**Lemma 2.17.** If $T^{(1)}, \ldots, T^{(s)}$ are commuting actions on $(X, \mathcal{B}, \mu)$ of a countable abelian group $G$ such that $\bigotimes_{i=1}^{s} T^{(i)}$ is ergodic and each $T^{(i)} T^{(1)^{-1}}$, $1 \leq i \leq s$, is ergodic, then $\bigotimes_{i=2}^{s} (T^{(i)} T^{(1)^{-1}})$ is ergodic.

**Proof.** Assume, to the contrary, that $\bigotimes_{i=2}^{s} (T^{(i)} T^{(1)^{-1}})$ is non-ergodic. Then, according to Proposition 2.8, there exist $\lambda_i \in \hat{G}$, $2 \leq i \leq s$, not all of which are 1, with $\prod_{i=2}^{s} \lambda_i = 1$, and functions $f_i \in L^2(X)$ such that

$$T^{(i)}_g T^{(1)^{-1}} f_i = \lambda_i(g) f_i, \quad i = 2, 3, \ldots, s. \quad g \in G$$

Since $T^{(1)}$ and $T^{(i)} T^{(1)^{-1}}$ commute and latter is ergodic, by Lemma F.7, $f_i$ forms an eigenfunction of $T^{(1)}$, say

$$T^{(1)}_g f_i = \eta_i(g) f_i \quad \forall g \in G, \quad i = 2, 3, \ldots, s.$$ 

Consequently,

$$T^{(i)}_g f_i = \lambda_i \eta_i f_i, \quad i = 2, 3, \ldots, s.$$ 

It follows that the function $F \in L^2(X^s)$ defined by

$$F(x_1, x_2, \ldots, x_s) = \frac{f_2(x_2) f_3(x_3) \cdots f_s(x_s)}{f_2(x_1) f_3(x_1) \cdots f_s(x_1)}, \quad (x_1, x_2, \ldots, x_s) \in X^s$$

is a non-constant $\bigotimes_{i=1}^{s} T^{(i)}$-invariant function. \qed

**Theorem 2.18.** Let $T^{(1)}, \ldots, T^{(s)}$ be commuting actions of a countable abelian group $G$ on a probability space $(X, \mathcal{B}, \mu)$. The following conditions are equivalent:
(1) $T^{(1)}, \ldots, T^{(s)}$ are w.-j.e.

(2) $T^{(1)}, \ldots, T^{(s)}$ are j.e.

(3) $T^{(1)} \otimes \cdots \otimes T^{(s)}$ is ergodic, and $T^{(i)}$, $T^{(j)}$ are j.e. for every $i \neq j$.

(4) $T^{(1)} \otimes \cdots \otimes T^{(s)}$ is ergodic, and all $T^{(i)}T^{(j)^{-1}}$, $i \neq j$, are ergodic.

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (4) are trivial in view of Theorem 2.14.

(4) $\Rightarrow$ (1): We shall use induction on $s$. The case $s = 1$ is trivial. If commuting $G$-actions $T^{(1)}, \ldots, T^{(s)}$, $s > 1$, are given, such that $T^{(1)} \otimes \cdots \otimes T^{(s)}$ is ergodic, and all $T^{(i)}T^{(j)^{-1}}$, $i \neq j$, are ergodic, then, according to Lemma 2.17, $\bigotimes_{i=2}^{s} (T^{(i)}T^{(1)^{-1}})$ is ergodic. Then, by the induction hypothesis, $T^{(2)}T^{(1)^{-1}}, T^{(3)}T^{(1)^{-1}}, \ldots, T^{(s)}T^{(1)^{-1}}$ are jointly ergodic. But the latter condition, since $T^{(1)}, \ldots, T^{(s)}$ commute, is equivalent to the condition (3b) of Theorem 2.14. Hence, $T^{(1)}, \ldots, T^{(s)}$ are j.e. \hfill $\Box$

2.6 Other Approaches to Joint Ergodicity

Following D. Berend [Ber85b, p. 261–262], we introduce the following notions.

Definition 2.19. Given a a set of $G$-actions $T^{(1)}, \ldots, T^{(s)}$ on a probability measure-space $(X, \mathcal{B}, \mu)$, a joint function is an $s$-tuple of measurable functions from $X$ to $\mathbb{C}^*$.

A joint invariant function is a joint function such that

$\prod_{i=1}^{s} T^{(i)}_g f_i = \prod_{i=1}^{s} f_i$ for all $g \in G$.

A constant joint function is a joint function $(f_1, \ldots, f_s)$ such that $f_i = \text{const}$, $1 \leq i \leq s$. 61
Definition 2.20. Let a set of $G$-actions $T^{(1)}, \ldots, T^{(s)}$ on $(X, \mathcal{B}, \mu)$ be given. We say that this set has property:

(J0) if $T^{(1)}, \ldots, T^{(s)}$ are j.e.

(J1) if for every $A_0, A_1, \ldots, A_s \in \mathcal{B}$ with $\prod_{i=0}^s \mu(A_i) > 0$ there exists $g \in G$ such that $\mu(A_0 \cap T_g^{(1)} A_1 \cap \cdots \cap T_g^{(s)} A_s) > 0$ (this may be called “joint transitivity”)

(J2) if the only joint invariant functions for $T^{(1)}, \ldots, T^{(s)}$ are constants.

In the case of a single $G$-action ($s = 1$), it is well known that the three conditions above are equivalent (compare with Theorem F.3). When $s > 1$, the situation is more complicated.

Proposition 2.21.

\[
(J0) \implies (J1) \implies T_1 \times \cdots \times T_s \text{ is ergodic} \implies (J2)
\]

Proof. The implication $(J0) \implies (J1)$ is trivial.

$(J0) \implies (J2)$: Let $(f_1, \ldots, f_s)$ be a non-constant joint invariant function for $T^{(1)}, \ldots, T^{(s)}$. Let

\[
f_0 = \frac{1}{\prod_{i=1}^s f_i}.
\]

Assume first that each $f_i$ takes only positive values. We claim that $f_i \in L^1(X)$, $0 \leq i \leq s$. For, otherwise we can take functions $h_i \in L^\infty(X)$, $0 \leq i \leq s$, with $h_i \leq f_i$ and

\[
\prod_{i=1}^s \int_X h_i \, d\mu > 1.
\]

By Theorem 2.2, for every left Følner sequence $(F_n)$,

\[
\frac{1}{|F_n|} \sum_{g \in F_n} \int_X \prod_{i=0}^s T_g^{(i)} f_i \, d\mu \geq \frac{1}{|F_n|} \sum_{g \in F_n} \int_X \prod_{i=0}^s T_g^{(i)} h_i \, d\mu \xrightarrow{n \to \infty} \prod_{i=1}^s \int_X h_i \, d\mu > 1.
\]
which is a contradiction. Since \((f_0^\delta, f_1^\delta, \ldots, f_s^\delta)\) is also a joint invariant function for \(T^{(1)}, \ldots, T^{(s)}\) for every \(\delta > 0\), this implies that \(f_i \in L^p(X)\) for every \(p > 0\) and each \(i\). Therefore, in view of Remark 2.3, we may apply Theorem 2.2 to functions \(f_0, f_1, \ldots, f_s\) and obtain

\[
1 = \frac{1}{|F_n|} \sum_{g \in F_n} \prod_{i=0}^s T_g^{(i)} f_i \, d\mu \to \prod_{i=1}^s \int_X f_i \, d\mu. \tag{2.11}
\]

Substituting \(f_i^2\) for \(f_i\), \(0 \leq i \leq s\), we similarly get

\[
1 = \prod_{i=1}^s \int_X f_i^2 \, d\mu. \tag{2.12}
\]

Combining (2.11) and (2.12), we have

\[
1 = \prod_{i=1}^s \int_X f_i^2 \, d\mu = \prod_{i=1}^s \left( \int_X f_i \, d\mu \right)^2. \tag{2.13}
\]

Since

\[
\int_X f_i^2 \, d\mu \geq \left( \int_X f_i \, d\mu \right)^2, \quad 0 \leq i \leq s,
\]

it follows that

\[
\int_X f_i^2 \, d\mu = \left( \int_X f_i \, d\mu \right)^2, \quad 0 \leq i \leq s,
\]

so that \(f_i = \text{const}, 0 \leq i \leq s\).

Now, let us consider the general case. If \((f_1, \ldots, f_s)\) is a joint invariant function for \(T^{(1)}, \ldots, T^{(s)}\), then so is \(|f_1|, \ldots, |f_s|\), and by the preceding part each \(|f_i|\) is a constant. Hence, we may assume that \(|f_i| = 1, 1 \leq i \leq s\). The same reasoning as before, shows that

\[
1 = \int_X |f_i| \, d\mu = \left| \int_X f_i \, d\mu \right|, \quad 0 \leq i \leq s,
\]

and so \(f_i = \text{const}, 1 \leq i \leq s\).
(J1) \implies T_1 \times \cdots \times T_s \text{ is ergodic: It is clear that if (J1) holds, then all actions } T_i, 1 \leq i \leq s, \text{ are ergodic. Assume that } T_1 \times \cdots \times T_s \text{ is not ergodic. Then, according to Proposition 2.8, there exist non-constant functions } f_i, 1 \leq i \leq s, \text{ such that }
\prod_{i=1}^{s} T_g^{(i)} f_i = \prod_{i=1}^{s} f_i, \quad g \in G. \tag{2.14}

The } s \text{-tuple } f_1, \ldots, f_s \text{ forms a joint invariant function for } (T^{(1)}, \ldots, T^{(s)}).

(J2) \implies T_1 \times \cdots \times T_s \text{ is ergodic: It is clear that if (J2) holds, then all actions } T_i, 1 \leq i \leq s, \text{ are ergodic. Assume that } T_1 \times \cdots \times T_s \text{ is not ergodic. Then, according to Proposition 2.8, there exist non-constant eigenfunctions } f_i \text{ of actions } T^{(i)}, 1 \leq i \leq s \text{ such that }
\prod_{i=1}^{s} T_g^{(i)} f_i = \prod_{i=1}^{s} f_i, \quad g \in G. \tag{2.15}

We may assume that } |f_i(x)| = 1 \text{ for } x \in X \text{ and } 1 \leq i \leq s. \text{ (Since } T^{(i)} \text{ is ergodic, } |f_i| = \text{const and so can not be } 0. \text{ Thus, we may divide by } |f_i|. )

Set } f_0 = \prod_{i=1}^{s} f_i. \text{ Define measures } \nu_i \text{ on the unit circle in the complex plane by }
\nu_i = \mu \circ f_i^{-1}, \quad i = 0, 1, \ldots, s.

For each } 0 \leq i \leq s \text{ pick a point } \alpha_i \text{ in the support of } \nu_i. \text{ If } f_i \text{ is non-constant, then } \nu_i \text{ is not a point mass, and hence we may assume that } \alpha_0 \neq \prod_{i=1}^{s} \alpha_i. \text{ We can find neighborhoods } I_i \text{ of } \alpha_i, 1 \leq i \leq s, \text{ such that if } \beta_i \in I_i, 0 \leq i \leq s, \text{ then } \beta_0 \neq \prod_{i=1}^{s} \beta_i.

Setting } A_i = f_i^{-1}(I_i) \text{ we have } \prod_{i=0}^{s} \mu(A_i) = \prod_{i=0}^{s} \nu_i(I_i) > 0. \text{ However, for all } g \in G,
\bigcap_{i=0}^{s} T_g^{(i)-1} A_i = \left\{ x : \prod_{i=1}^{s} f_i(x) \in I_0, \ T_g^{(1)} f_1(x) \in I_1, \ldots, T_g^{(s)} f_s(x) \in I_s \right\} = \emptyset.

\square

More can be said in the case when } T^{(1)}, \ldots, T^{(s)} \text{ commute.
Theorem 2.22. If $T^{(1)}, \ldots, T^{(s)}$ are commuting actions of a countable abelian group $G$, then the conditions (J0), (J1), and (J2) are equivalent.

Proof. In view of Theorem 2.18, it is enough to show that any of the conditions (J1) and (J2) implies that $T^{(i)}T^{(j)^{-1}}$ are ergodic for all $i \neq j$.

If (J1) holds, then for every $A, B \in \mathcal{B}$ with $\mu(A)\mu(B) > 0$ we have

$$\mu(T^{(i)}_gT^{(j)}_g^{-1}A \cap B) = \mu(T^{(i)}_g A \cap T^{(j)}_g B)$$

$$= \mu(X \cap \cdots \cap X \cap T^{(i)}_g A \cap X \cap \cdots \cap X \cap T^{(j)}_g B \cap X \cap \cdots \cap X) > 0$$

for some $g \in G$. Hence, $T^{(i)}T^{(j)^{-1}}$ is ergodic.

Now, let (J2) holds. Assume that for some $i \neq j$, $T^{(i)}T^{(j)^{-1}}$ is not ergodic. Then there exists a non-trivial invariant set $A$ for $T^{(i)}T^{(j)^{-1}}$. Let $f(x) = 1_A(x) + 1$ and $h(x) = 1/f(x)$. Then the $s$-tuple

$$(1, \ldots, 1, f, 1, \ldots, 1, h, \ldots, 1)$$

is a non-constant joint invariant function for $(T^{(1)}, \ldots, T^{(s)})$. \hfill \Box

Remark 2.23. In the non-commutative case, (J1) $\not\Rightarrow$ (J0) in general. In Chapter 5, we shall give an example (Example 5.6) of two transformations, $S$ and $T_E$, for which the Cesàro averages converge to the right limit along a Følner sequence in $\mathbb{Z}$ and, therefore, satisfy the condition (J1), but do not converge to the right limit along another Følner sequence and, therefore, are not j.e.

It remains an open question whether any of the implications (J1) $\Rightarrow$ (J2), (J2) $\Rightarrow$ (J1), or (J2) $\Rightarrow$ (J0) holds in non-commutative case.
Remark 2.24. One can define joint invariant sets in the following way: An $s$-tuple of measurable sets $(A_1, \ldots, A_s)$ is called \textit{joint invariant set} for $T^{(1)}, \ldots, T^{(s)}$ if
\[
\bigcap_{i=1}^{s} T^{(i)}_g A_i = \bigcap_{i=1}^{s} A_i \pmod{\mu} \quad \text{for all } g \in G.
\]
(2.16)
Now, we say that a set of $G$-actions $T^{(1)}, \ldots, T^{(s)}$ satisfies the condition
(J3) if $T^{(1)}, \ldots, T^{(s)}$ have no joint invariant sets $A_1, \ldots, A_s$ with
\[
0 < \prod_{i=1}^{s} \mu(A_i) < 1.
\]

It is clear that (J1) $\implies$ (J3) and (J2) $\implies$ (J3). However, nothing else is known about the relation of the condition (J3) to the other conditions even in the commutative case.

Remark 2.25. For the case of $\mathbb{Z}$-actions, most of the results of this section can be found in [Ber85b].

2.7 Total Ergodicity

Throughout this section $G$ will be a (not necessarily abelian) countable group. By cosets of a subgroup of $G$ we shall always understand its left cosets.

Definition 2.26. Let $T$ be an action of a group $G$. Let $\Gamma$ be a subgroup of $G$. The \textit{subaction} $T_\Gamma$ is the restriction of the map $T$ to the subgroup $\Gamma$.

Definition 2.27. A $G$-action $T$ is called \textit{totally ergodic} if for every normal subgroup $\Gamma \subset G$ of a finite index the subaction $T_\Gamma$ is ergodic. $T$ is called \textit{strongly totally ergodic} if for every infinite subgroup $\Gamma \subset G$ the subaction $T_\Gamma$ is ergodic.
Since \( \mathbb{Z} \) does not contain any subgroups of infinite index, in the case of \( \mathbb{Z} \)-actions, the notions of total ergodicity and strong total ergodicity are equivalent (and coincide with the well-known definition of total ergodicity for a single transformation). This, however, is not the case for the actions of more general groups.

**Example 2.28.** Let \( A \) be any invertible totally ergodic transformation of a measure space \((X, \mathcal{B}, \mu)\). Consider the \( \mathbb{Z}^2 \)-action \( T \) on \((X, \mathcal{B}, \mu)\) generated by \( A \) and \( I \), the identity transformation. In other words, \( T_{(m,n)} = A^m \). Then \( T \) is ergodic since there are no non-trivial sets invariant with respect to \( A \) and, therefore, there are no non-trivial sets invariant with respect to the action \( T \). Let \( \Gamma \subset \mathbb{Z}^2 \) be a subgroup of finite index. It contains an element \( g = (m_0, n_0) \) with \( n_0 \neq 0 \). Then \( T_g = A^{n_0} \). Since \( A \) is totally ergodic, the transformation \( A^{n_0} \) is ergodic. Hence, \( T_{\Gamma} \) is ergodic and so, \( T \) is totally ergodic. On the other hand, the infinite subgroup of \( \mathbb{Z}^2 \) generated by \((1,0)\) acts on \( X \) by identity transformations and, therefore, is not ergodic. Hence, \( T \) is not strongly totally ergodic.

Of course, one can ask whether there exist examples of ergodic but not totally ergodic actions. The simplest example is a \( \mathbb{Z} \)-action given by the transformation \( A \) on a space consisting of two points interchanging the two points. Then \( A \) is ergodic, but \( A^2 \) is not. A more interesting example is given below.

**Example 2.29.** Let \( X = \prod_{n=2}^{\infty} \mathbb{Z}_n \) and let the transformation \( A \) be given by

\[
(A(x))_n = x_n + 1 \pmod{n}.
\]

Then \( T \) is ergodic. However, not only \( T \) is not totally ergodic, but \( T_{\Gamma} \) is not ergodic for every proper subgroup of \( \mathbb{Z} \).
We would like to obtain some equivalent definitions of total ergodicity. If $G$ is abelian, the following theorem gives two such definitions.

**Theorem 2.30.** Let $T$ be a measurable action by a countable abelian group $G$. The following conditions are equivalent:

1. $T$ is totally ergodic.

2. For every non-constant function $f \in L^2(X)$ the set $\{T_gf\}_{g \in G}$ is infinite.

3. If $\lambda \in \hat{G}$ is an eigenvalue of the unitary action $T$, $\lambda \neq 1$, then $\{\lambda(g)\}_{g \in G}$ is an infinite subset of $C$.

**Proof.** The implication (2) $\Rightarrow$ (3) is trivial. To show that (1) $\Rightarrow$ (2) assume that for a non-constant function $f$ the set $\{T_gf\}_{g \in G}$ is finite. Then the set

$$\Gamma = \{g \in G \mid T_gf = f\}$$

is a subgroup of $G$ of a finite index, so $T$ is not totally ergodic.

(3) $\Rightarrow$ (1): Let $T$ be non-totally ergodic. Then there exists a subgroup $\Gamma \subset G$ with $[G : \Gamma] = n < \infty$ and a function $f \neq \text{const}$ such that $T_\gamma f = f$ for all $\gamma \in \Gamma$. Let $g_1\Gamma, g_2\Gamma, \ldots, g_n\Gamma$ be the cosets of $\Gamma$, where $g_1 = e$. Look at the finite-dimensional Hilbert space $\mathcal{H}$ generated by the functions $T_{g_k}f$, $2 \leq k \leq n$. The space $\mathcal{H}$ is an invariant subspace for the unitary operators $T_{g_2}, \ldots, T_{g_n}$. The operators $T_{g_2}|_{\mathcal{H}}, \ldots, T_{g_n}|_{\mathcal{H}}$ have a common basis in $\mathcal{H}$ made out of their eigenfunctions. Let $F$ be one of these eigenfunctions, $F \neq \text{const}$. Then $T_{g_k}F = \lambda_k F$ for some $\lambda_k \in C$, $2 \leq k \leq n$. Let the function $\lambda : G \to C$ be defined as $\lambda(g) = \lambda_k$ if $g \in g_k\Gamma$ where we set $\lambda_1 = 1$. It is easy to see that so defined $\lambda$ in an eigenvalue of $T$. Clearly, the set $\{\lambda(g)\}_{g \in G}$ contains no more than $n$ elements. \qed
There exists a generalization of the above theorem for the case when $G$ is not necessarily abelian. Let $L^2_0(X)$ denote the orthogonal complement of the subspace of constant-functions in $L^2$.

**Theorem 2.31.** Let $T$ be a measurable action by a countable group $G$. The following conditions are equivalent:

(1) $T$ is totally ergodic.

(2) For every function $f \in L^2_0(X)$ the set $\{T_gf\}_{g \in G}$ is infinite.

(3) For every finite-dimensional $T$-invariant subspace $\mathcal{H}$ in $L^2_0(X)$ the set of the unitary operators $\{T_g|_{\mathcal{H}}\}_{g \in G}$ is infinite.

*Proof.* The implications $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are clear.

$(3) \Rightarrow (1)$: Let $T$ be non-totally ergodic. Then there exists a normal subgroup $\Gamma \subset G$ with $[G : \Gamma] = n < \infty$ and a function $f \neq \text{const}$ such that $T_\gamma f = f$ for all $\gamma \in \Gamma$. Let $g_1\Gamma, g_1\Gamma, \ldots, g_n\Gamma$ be the cosets of $\Gamma$, where $g_1 = e$. Look at the finite-dimensional Hilbert space $\mathcal{H}$ generated by the functions $T_{g_0}f$, $0 \leq k \leq n - 1$. The space $\mathcal{H}$ is an invariant subspace for the unitary operators $T_{g_2}, \ldots, T_{g_n}$.

If $g \in G$, then $g = g_i\gamma$ for some $\gamma \in \Gamma$ and some $i, 1 \leq i \leq n$. Since $\Gamma$ is a normal subgroup of $G$, it follows that

$$T_gT_{g_k}f = T_{g\gamma g_k}f = T_{g_k\gamma}f = T_{g_k}f = T_{g_i}T_{g_k}f.$$  

(where $\gamma' \in H$).

Therefore, for every $g \in G$, one has $T_{g_i}|_{\mathcal{H}} = T_{g_k}|_{\mathcal{H}}$ for some $i, 1 \leq i \leq n$. Thus, the set $\{T_{g_i}|_{\mathcal{H}}\}_{g \in G}$ contains no more than $n$ distinct elements. $\square$
A simple sufficient condition for total ergodicity of arbitrary group actions is given by the following theorem.

**Theorem 2.32.** If the subaction $T_H$ for an infinite subgroup $H \subset G$ is totally ergodic, then $T$ is totally ergodic.

**Proof.** Assume $T$ is not totally ergodic, i.e. there exists a normal subgroup $\Gamma \subset G$ of a finite index such that $T_\Gamma$ is non-ergodic. Look at $H \cap \Gamma$. This is a normal subgroup of $H$. $H \cap \Gamma$ has a finite index as a subgroup of $H$ because

$$
H/H \cap \Gamma \sim H\Gamma/\Gamma \subset G/\Gamma.
$$

However, $T_{H\cap\Gamma}$ is non-ergodic as a subaction of $T_\Gamma$. This is a contradiction with the assumption that $T_H$ is totally ergodic. \qed

**Corollary 2.33.** If the transformation $T_h$ for an element $h \in G$ of infinite order is totally ergodic (as a $\mathbb{Z}$-action), then $T$ is totally ergodic.

**Proof.** Take $H = \{h^n|n \in \mathbb{Z}\}$ and apply Theorem 2.32. \qed

**Remark 2.34.** The condition of Theorem 2.32 is not only sufficient, but also necessary: take the group $G$ itself as $H$. On the other hand, the condition of the Corollary is not necessary as the example below shows.

**Example 2.35.** This is an example of a totally ergodic $\mathbb{Z}^2$-action $T$ such that no transformation $T_{(m,n)}$ is ergodic. Let $T^2$ be the 2-dimensional torus and let $\tau_1$, $\tau_2$ be the measurable transformations of $T^2$ given by $\tau_1(x, y) = (x + \sqrt{2}, y + 2\sqrt{2}) \mod 1$ and $\tau_2(x, y) = (x + 2\sqrt{2}, y + \sqrt{2}) \mod 1$. 

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Let \((m,n) \in \mathbb{Z}^2\) with \((m,n) \neq 0\). We claim that \(T_{(m,n)}\) is non-ergodic. Namely, 
\[T_{(m,n)}(x) = (x + (m + 2n)\sqrt{2}, y + (2m + n)\sqrt{2}).\]
Since \((m + 2n)\sqrt{2}\) and \((2m + n)\sqrt{2}\) are rationally dependent, \(T_{(m,n)}\) is non-ergodic.

On the other hand, assume that the \(\mathbb{Z}^2\)-action \(T\) is non-ergodic. Then there exists a non-constant function \(f \in L^2(T^2)\) such that \(T_{(m,n)}f = f\) for all \((m,n) \in \mathbb{Z}^2\). In particular, \(\tau_1 f = T_{(1,0)}f = f\) and \(\tau_2 f = T_{(0,1)}f = f\). Note that \(f\) can be written as
\[f(x,y) = \sum_{(a,b) \in \mathbb{Z}^2} C_{a,b} e^{2\pi i (ax + by)}.\]

The condition that \(\tau_1 f = f\) and \(\tau_2 f = f\) implies that \(a + 2b = 0\) and \(2a + b = 0\) which means that \(a = b = 0\), so \(f = \text{const}\). The contradiction proves that \(T\) is ergodic.

Analogously, one can show that \(T\) is totally ergodic.

**Definition 2.36.** Let \(T^{(1)}, \ldots, T^{(s)}\) be actions of a countable group \(G\) on a measurable space \((X, \mathcal{B}, \mu)\). The actions \(T^{(1)}, \ldots, T^{(s)}\) are called **totally jointly ergodic** if for every subgroup of a finite index \(\Gamma \subset G\) the actions \(T^{(1)}_{\Gamma}, \ldots, T^{(s)}_{\Gamma}\) are j.e.

In [Ber85b] it has been proved that if commuting \(\mathbb{Z}\)-actions \(T^{(1)}, \ldots, T^{(s)}\) are j.e., then they are totally jointly ergodic. However, for more general group actions, total joint ergodicity does not follow from joint ergodicity.

**Example 2.37.** Let \(X\) and \(A\) be as in Example 2.29. Let us consider two \(\mathbb{Z}^2\)-actions \(S\) and \(S'\) defined as follows. The action \(S\) is generated by the identity transformation \(I\) and the transformation \(A\), i.e. \(S_{(n,m)} = A^m\). The action \(S'\) is generated by \(A\) and \(I\), i.e. \(S'_{(n,m)} = A^n\). Then, obviously, \(S\) and \(S'\) are ergodic. We claim, moreover, that \(S\) and \(S'\) are j.e. Namely, by Theorem 2.18, it is enough to show that \(S \times S'\) and \(SS'^{-1}\) are ergodic. The action \(SS'^{-1}\) given by \(SS'^{-1}_{(n,m)}x = A^{m-n}x\) is ergodic.
Moreover, the action $S \times S'$ given by $S \times S'_{(n,m)}(x,y) = (A^mx, A^n y)$ is ergodic by Proposition 2.8. However, neither $S$ nor $S'$ is totally ergodic (for $S$ take the subgroup $\Gamma = \{(m, 2n) \mid n, m \in \mathbb{Z} \} \subset \mathbb{Z}^2$).

It might still be possible to prove total joint ergodicity of totally jointly ergodic commuting actions for general abelian groups if one assumes, in addition, that one of the actions is totally ergodic. The proof of this statement would be more complicated than the one given in [Ber85b] for $\mathbb{Z}$-actions.
CHAPTER 3

JOINT WEAK AND STRONG MIXING

3.1 Definitions and Preliminary Results

Let $G$ be a countable amenable group and $\Delta \subset G$. If a Følner sequence $\mathcal{L} = (L_n)$ is given, then the upper density of the set $\Delta$ with respect to the sequence $\mathcal{L}$ is

$$d_\mathcal{L}(\Delta) = \lim_{n \to \infty} \frac{|\Delta \cap L_n|}{|L_n|}.$$  \hfill (3.1)

The density of the set $\Delta$ with respect to the sequence $\mathcal{L}$ is

$$d_\mathcal{L}(\Delta) = \lim_{n \to \infty} \frac{|\Delta \cap L_n|}{|L_n|}$$ \hfill (3.2)

(if the limit exists). The upper density of the set $\Delta$ is

$$\bar{d}(\Delta) = \sup_{\mathcal{L} \in \mathcal{F}(G)} d_\mathcal{L}(\Delta).$$ \hfill (3.3)

If $d_\mathcal{L}(\Delta)$ exists for all $\mathcal{L} \in \mathcal{F}(G)$ and does not depend on $\mathcal{L}$, we call that number the density of the set $\Delta$, and write $d(\Delta)$.

Let $G$ be a countable amenable group and let $(a_g)$ be a sequence in a Banach space $B$ indexed by elements of $G$.

We say that $\lim_{g \to \infty} a_g = g$ if for all $\varepsilon > 0$ the set $S = \{ g \in G : \|a_g - a\| < \varepsilon \}$ is finite.
We say that $\text{D-lim}_{g \to \infty} a_g = a$ if for all $\varepsilon > 0$ the set $S = \{g \in G : \|a_g - a\| < \varepsilon\}$ is of density 1.

The following lemma is a generalization of a fact that is well-known for numerical sequences indexed by natural numbers (see, for example, [Pet83, Lemma 6.2]).

**Lemma 3.1.** Let $G$ be a countable amenable group and let $(a_g)$ be a bounded sequence in a Banach space $B$ indexed by elements of $G$. The following conditions are equivalent:

(1) $\text{D-lim}_{g \to \infty} a_g = a$.

(2) For every Følner sequence $(L_n)$ one has

$$\lim_{n \to \infty} \frac{1}{|L_n|} \sum_{g \in L_n} \|a_g - a\| = 0.$$ (3.4)

(3) For every $L \in \mathcal{F}(G)$ there exists a subset $E_L \subset G$ with $d_L(E_L) = 0$ and such that $\lim_{g \to \infty, g \notin E_L} a_g = a$.

**Proof.** (1) $\Rightarrow$ (2): Let $\varepsilon > 0$ be given and let $(L_n)$ be a Følner sequence in $G$. If

$$S = \{g \in G : \|a_g - a\| < \varepsilon\},$$

then, by the definition of D-lim, one has

$$\lim_{n \to \infty} \frac{|S \cap L_n|}{|L_n|} = 1.$$ 

Let $\alpha$ be an upper bound for $\|a_g - a\|$. One has:

$$\frac{1}{|L_n|} \sum_{g \in L_n} \|a_g - a\| \leq \frac{|S \cap L_n|}{|L_n|} \cdot \varepsilon + \frac{|(G \setminus S) \cap L_n|}{|L_n|} \cdot \alpha$$

$$\leq \varepsilon + \left(1 - \frac{|S \cap L_n|}{|L_n|}\right) \alpha \leq 2\varepsilon$$

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for big enough $n$. Hence,

$$\lim_{n \to \infty} \frac{1}{|L_n|} \sum_{g \in L_n} \|a_g - a\| = 0.$$  

(2) $\Rightarrow$ (3): Let a Følner sequence $\mathcal{L} = (L_n)$ be given and let

$$E_m = \left\{ g \in G \mid \|a_g - a\| > \frac{1}{m} \right\}.$$  

Then $E_1 \subset E_2 \subset \ldots$, and each $E_m$ has density 0 because of the assumption. Therefore, one can choose a sequence of natural numbers $i_1 < i_2 < \cdots < i_{m-1} < i_m < \ldots$ such that

$$\frac{|E_m \cap L_n|}{|L_n|} < \frac{1}{m} \text{ for all } n > i_{m-1}.$$  

Let

$$E_{\mathcal{L}} = E_0 \cup \bigcup_{m=2}^{\infty} E_m \cap \left( \bigcup_{k=i_{m-1}+1}^{i_m} L_k \setminus \bigcup_{k=1}^{i_{m-1}} L_k \right),$$  

where $E_0 = G \setminus \bigcup_{n=1}^{\infty} L_n$.

Clearly,

$$\lim_{g \to \infty, g \notin E_{\mathcal{L}}} a_g = 0.$$  

On the other hand, if $i_{m-1} < n \leq i_m$, then

$$\frac{|E_{\mathcal{L}} \cap L_n|}{|L_n|} \leq \frac{|E_m \cap L_n|}{|L_n|} < \frac{1}{m}.$$  

Hence, $d_\mathcal{L}(E_{\mathcal{L}}) = 0$.  

(3) $\Rightarrow$ (1): Let $\varepsilon > 0$ and a Følner sequence $\mathcal{L} = (L_n)$ be given and let $E_{\mathcal{L}} \subset G$ be a set satisfying

$$d_\mathcal{L}(E_{\mathcal{L}}) = 0 \quad \text{and} \quad \lim_{g \to \infty, g \notin E_{\mathcal{L}}} a_g = a.$$  

Then there exists a finite set $\Phi \subset G$ such that $|a_g - a| < \varepsilon$ for all $g \in G \setminus (E_{\mathcal{L}} \cup \Phi)$.

It is clear that $S = \{ g \in G : |a_g - a| < \varepsilon \} \supset G \setminus (E_{\mathcal{L}} \cup \Phi)$ and, therefore,

$$d_\mathcal{L}(S) \geq d_\mathcal{L}(G \setminus (E_{\mathcal{L}} \cup \Phi)) = 1 - d_\mathcal{L}(E_{\mathcal{L}} \cup \Phi) = 1.$$  

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Since this is true for every $L \in F(G)$, we can conclude that $d(S) = 1$.

**Definition 3.2.** Let $T$ be a measurable action of a countable amenable group $G$ on a measure space $(X, \mathcal{B}, \mu)$. The action $T$ is called *weakly mixing* if for all $A, B \in \mathcal{B}$ one has:

$$\text{D-lim}_{g \to \infty} \mu(T_g A \cap B) = \mu(A)\mu(B)$$

**Theorem 3.3 ([BR88b]).** Let $T$ be a measure-preserving action of a countable amenable group $G$ on a probability measure space $(X, \mathcal{B}, \mu)$. The following statements about the action $T$ are equivalent:

1. $T$ is weakly mixing.

2. For all $f_1, f_2 \in L^2_0(X)$ one has $\text{D-lim}_{g \to \infty} (T_g f_1, f_2) = 0$.

3. For all $f_1, f_2 \in L^2_0(X)$ and for every $(L_n) \in F(G)$ one has:

$$\lim_{n \to \infty} \frac{1}{|L_n|} \sum_{g \in L_n} |(T_g f_1, f_2)| = 0.$$ 

4. $T \otimes T$ is weakly mixing.

5. $T \otimes T$ is ergodic.

6. $T \otimes S$ is ergodic for every ergodic $G$-action $S$.

7. When considered as a representation of the group $G$ in $L^2(X)$, $T$ has no nontrivial finite-dimensional subrepresentations (or, in the case when $G$ is abelian, $T$ has no non-constant eigenfunctions).

**Remark 3.4.** In [BR88b], weakly mixing actions of groups that are not necessarily amenable are considered. (The case of amenable groups has already been investigated
in [Dye65].) We are not interested in the non-amenable case. This is why we have formulated Theorem 3.3 only for amenable groups.

**Proposition 3.5 ([BR88a]).** Let $S$ be a weakly mixing action of an amenable group $G$ and $T$ be an ergodic $G$-action on a non-atomic separable probability space $(X, \mathcal{B}, \mu)$. If $T$ and $S$ commute, then $T$ is also weakly mixing.

**Definition 3.6.** Let $T^{(1)}, \ldots, T^{(s)}$ be actions of a countable amenable group $G$ on a measure space $(X, \mathcal{B}, \mu)$. The actions $T^{(1)}, \ldots, T^{(s)}$ are called

1. weak-jointly weakly mixing (w-j.w.m.) – if
   
   $$\lim_{g \to \infty} \mu \left( \bigcap_{i=0}^{s} T^{(i)}_g A_i \right) = \prod_{i=0}^{s} \mu(A_i), \quad A_0, A_1, \ldots, A_s \in \mathcal{B};$$

2. jointly weakly mixing (j.w.m.) – if
   
   $$\lim_{g,h \to \infty} \mu \left( \bigcap_{i=1}^{s} \left( T^{(i)}_g A_i \cap T^{(i)}_h A_i \right) \right) = \prod_{i=1}^{s} (\mu(A_i)\mu(B_i)), \quad A_1, B_1, \ldots, A_s, B_s \in \mathcal{B}.$$

**Proposition 3.7.** If $T^{(1)}, \ldots, T^{(s)}$ are (weak-)jointly weakly mixing, then they are (weakly) jointly ergodic.

**Proof.** This follows from the definition of joint ergodicity and Lemma 3.1. □

**Theorem 3.8.** Let $T^{(1)}, \ldots, T^{(s)}$ be actions of a countable amenable group $G$ on a probability measure space $(X, \mathcal{B}, \mu)$. The following conditions are equivalent:

1. The $G$-actions $T^{(1)}, \ldots, T^{(s)}$ on $(X, \mathcal{B}, \mu)$ are w-j.w.m. (resp. j.w.m.)

2. The $G$-actions
   
   $$T^{(1)} \otimes T^{(1)} \otimes T^{(2)} \otimes T^{(2)} \otimes \cdots \otimes T^{(s)} \otimes T^{(s)}$$

   on $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu)$ are w-j.w.m. (resp. j.w.m.)
(3) For every measure space \((Y, \mathcal{B}, \nu)\) and all w.j.e. (resp. j.e.) \(G\)-actions \(S^{(1)}, \ldots, S^{(s)}\) on \((Y, \mathcal{B}, \nu)\), the \(G\)-actions

\[ T^{(1)} \otimes S^{(1)}, T^{(2)} \otimes S^{(2)}, \ldots, T^{(s)} \otimes S^{(s)} \]

on \((X \times Y, \mathcal{B} \times \mathcal{B}, \mu \times \nu)\) are w.j.e. (resp. j.e.)

(4) The \(G\)-actions

\[ T^{(1)} \otimes T^{(1)}, T^{(2)} \otimes T^{(2)}, \ldots, T^{(s)} \otimes T^{(s)} \]

on \((X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu)\) are w.j.e. (resp. j.e.)

**Proof.** We shall prove only the “weak-joint” part of the statements. The proof of the “joint” part is analogous.

\((1) \Rightarrow (2)\): Let \(A_0, B_0, A_1, B_1, \ldots, A_s, B_s \in \mathcal{B}\) be given. Then

\[
\text{D-lim}_{g \to \infty} \mu \left( \bigcap_{i=0}^{s} T^{(i)}_g A_i \right) = \prod_{i=0}^{s} \mu(A_i) \quad \text{and} \quad \text{D-lim}_{g \to \infty} \mu \left( \bigcap_{i=0}^{s} T^{(i)}_g B_i \right) = \prod_{i=0}^{s} \mu(B_i).
\]

By Lemma 3.1, for every Følner sequence \(L = (L_n)\) there exist \(E_1, E_2 \subset G\) with \(d_L(E_1) = 0, d_L(E_2) = 0\) and such that

\[
\lim_{g \to \infty} \mu \left( \bigcap_{i=0}^{s} T^{(i)}_g A_i \right) = \prod_{i=0}^{s} \mu(A_i)
\]

and

\[
\lim_{g \to \infty} \mu \left( \bigcap_{i=0}^{s} T^{(i)}_g B_i \right) = \prod_{i=0}^{s} \mu(B_i).
\]

Let \(E = E_1 \cup E_2\). Then \(d_L(E) = 0\) and

\[
\lim_{g \to \infty} (\mu \times \mu) \left( \bigcap_{i=0}^{s} (T^{(i)} \otimes T^{(i)})_g (A_i \times B_i) \right) = \lim_{g \to \infty} \mu \left( \bigcap_{i=0}^{s} T^{(i)}_g A_i \right) \cdot \lim_{g \to \infty} \mu \left( \bigcap_{i=0}^{s} T^{(i)}_g B_i \right) = \prod_{i=0}^{s} \mu(A_i) \cdot \prod_{i=0}^{s} \mu(B_i)
\]

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Hence, $T^{(1)} \otimes T^{(1)}, T^{(2)} \otimes T^{(2)}, \ldots, T^{(s)} \otimes T^{(s)}$ are w.j.w.m.

(2) $\Rightarrow$ (1): Trivial.

(1) $\Rightarrow$ (3): Let weakly jointly ergodic actions $S^{(1)}, \ldots, S^{(s)}$ on $(Y, \mathcal{B}, \nu)$ be given. In order to show that $T^{(1)} \otimes S^{(1)}, T^{(2)} \otimes S^{(2)}, \ldots, T^{(s)} \otimes S^{(s)}$ are weakly jointly ergodic, it is enough to show that for all $A_0, A_1, \ldots, A_s \in \mathcal{B}$ and $B_0, B_1, \ldots, B_s \in \mathcal{B}$ and every Følner sequence $(L_n)$ one has

$$\lim_{n \to \infty} \frac{1}{|L_n|} \sum_{g \in L_n} (\mu \times \nu) \left( \bigcap_{i=0}^{s} (T^{(i)} \otimes S^{(i)})_g (A_i \times B_i) \right) = \prod_{i=1}^{s} \mu(A_i) \nu(B_i). \quad (3.5)$$

But the left-hand side of (3.5) is

$$\lim_{n \to \infty} \frac{1}{|L_n|} \sum_{g \in L_n} \mu \left( \bigcap_{i=0}^{s} T^{(i)}_g A_i \right) \nu \left( \bigcap_{i=0}^{s} S^{(i)}_g B_i \right)$$

$$= \lim_{n \to \infty} \frac{1}{|L_n|} \sum_{g \in L_n} \left\{ \prod_{i=1}^{s} \mu(A_i) \cdot \nu \left( \bigcap_{i=0}^{s} S^{(i)}_g B_i \right) \right\} + \left[ \mu \left( \bigcap_{i=0}^{s} T^{(i)}_g A_i \right) - \prod_{i=1}^{s} \mu(A_i) \right] \cdot \nu \left( \bigcap_{i=0}^{s} S^{(i)}_g B_i \right) \right\} \quad (3.6)$$

Since $S^{(1)}, \ldots, S^{(s)}$ are w.j.e., the first term of (3.6) tends to $\prod_{i=1}^{s} \mu(A_i) \nu(B_i)$. The second term is dominated by

$$\frac{1}{|L_n|} \left| \mu \left( \bigcap_{i=0}^{s} T^{(i)}_g A_i \right) - \prod_{i=1}^{s} \mu(A_i) \right|,$$

which tends to 0 as $n \to \infty$ because $T^{(1)}, \ldots, T^{(s)}$ are w.j.w.m.

(3) $\Rightarrow$ (4): Trivial.
(4) ⇒ (1): If \( A_0, A_1, \ldots, A_s \in \mathcal{B} \) and \((L_n)\) is a Følner sequence, then

\[
\frac{1}{|L_n|} \sum_{g \in L_n} \left[ \mu \left( \bigcap_{i=0}^{s} T_g^{(i)} A_i \right) - \prod_{i=1}^{s} \mu(A_i) \right]^2 \\
= \frac{1}{|L_n|} \sum_{g \in L_n} \left[ \mu \left( \bigcap_{i=0}^{s} T_g^{(i)} A_i \right) \right]^2 \\
- 2 \prod_{i=1}^{s} \mu(A_i) \cdot \frac{1}{|L_n|} \sum_{g \in L_n} \mu \left( \bigcap_{i=0}^{s} T_g^{(i)} A_i \right) + \left( \prod_{i=1}^{s} \mu(A_i) \right)^2 \\
= \frac{1}{|L_n|} \sum_{g \in L_n} (\mu \times \mu) \left( \bigcap_{i=0}^{s} (T^{(i)} \otimes T^{(i)})_g (A_i \times A_i) \right) \\
- 2 \prod_{i=1}^{s} \mu(A_i) \cdot \frac{1}{|L_n|} \sum_{g \in L_n} (\mu \times \mu) \left( \bigcap_{i=0}^{s} (T^{(i)} \otimes T^{(i)})_g (A_i \times X) \right) + \left( \prod_{i=1}^{s} \mu(A_i) \right)^2.
\]

Since \( T^{(1)} \otimes T^{(1)}, T^{(2)} \otimes T^{(2)}, \ldots, T^{(s)} \otimes T^{(s)} \) are w.j.e., as \( n \to \infty \) this expression tends to

\[
\prod_{i=1}^{s} (\mu \times \mu)(A_i \times A_i) - 2 \prod_{i=1}^{s} \mu(A_i) \cdot \prod_{i=1}^{s} (\mu \times \mu)(A_i \times X) + \left( \prod_{i=1}^{s} \mu(A_i) \right)^2 \\
= \left( \prod_{i=1}^{s} \mu(A_i) \right)^2 - 2 \left( \prod_{i=1}^{s} \mu(A_i) \right)^2 + \left( \prod_{i=1}^{s} \mu(A_i) \right)^2 = 0. \quad (3.7)
\]

In view of Lemma 3.1, it follows that \( T^{(1)}, \ldots, T^{(s)} \) are w-j.w.m.

**Corollary 3.9.** Let \( T^{(1)}, \ldots, T^{(s)} \) be actions of a countable abelian group \( G \). The actions \( T^{(1)}, \ldots, T^{(s)} \) are weak-jointly weakly mixing iff they are jointly weakly mixing.

**Proof.** This follows from Theorem 3.8 and Theorem 2.14.

**Corollary 3.10.** Let \( T^{(1)}, \ldots, T^{(s)} \) be commuting actions of a countable amenable group \( G \). The actions \( T^{(1)}, \ldots, T^{(s)} \) are weak-jointly weakly mixing iff they are jointly weakly mixing.
Corollary 3.11. Let $T^{(1)}, \ldots, T^{(s)}$ be actions of a countable amenable group $G$ by continuous automorphisms of a compact group $X$. The actions $T^{(1)}, \ldots, T^{(s)}$ are weak-jointly weakly mixing if and only if they are jointly weakly mixing.

Corollaries 3.10 and 3.11 follow from Theorem 3.8 in view of Remark 2.16.

3.2 Abelian Case, Joint Eigenfunctions

Theorem 3.12. Let $T^{(1)}, \ldots, T^{(s)}$ be commuting actions of a countable abelian group $G$ on a non-atomic separable measure space $(X, \mathcal{B}, \mu)$. The following conditions are equivalent:

1. $T^{(1)}, \ldots, T^{(s)}$ are w-j.w.m.

2. $T^{(1)}, \ldots, T^{(s)}$ are j.w.m.

3. $T^{(i)}$ is weakly mixing for every $i$, and every pair of actions $T^{(i)}$, $T^{(j)}$ is j.w.m. for every $i \neq j$.

4. $T^{(i)}$ is weakly mixing for every $i$, and $T^{(i)}T^{(j)-1}$ is weakly mixing for every $i \neq j$.

5. $T^{(1)}, \ldots, T^{(s)}$ are j.e., and there exists a weakly-mixing $G$-action $T$ commuting with $T^{(i)}$ for some $i$.

Proof. The equivalence of (1) and (2) is established in Corollary 3.9. The fact that (2), (3), and (4) are equivalent follows from Theorem 2.18, and part (4) of Theorem 3.8 (Joint weak mixing of $T^{(i)}$’s is equivalent to joint ergodicity of $T^{(i)} \otimes T^{(i)}$’s. Therefore, we can make use of the criteria of joint ergodicity.). The implication (4) $\Rightarrow$ (5) is trivial. The implication (5) $\Rightarrow$ (4) follows from Proposition 3.5, Theorem 2.18, and the fact that our actions are commuting. \hfill \Box
Definition 3.13. Let $T^{(1)}, \ldots, T^{(s)}$ be an action of a countable group $G$ on a probability space $(X, \mathcal{B}, \mu)$. Joint function $(f_1, f_2, \ldots, f_s)$ is called a joint eigenfunction of $T^{(1)}, \ldots, T^{(s)}$, if there exists $\lambda \in \hat{G}$ (which is called a joint eigenvalue of $T^{(1)}, \ldots, T^{(s)}$), such that
\[
\prod_{i=1}^{s} T^{(i)}_{g} f_{i} = \lambda(g) \prod_{i=1}^{s} f_{i}, \quad g \in G.
\]

Proposition 3.14. Let $T^{(1)}, \ldots, T^{(s)}$ be actions of a countable amenable group $G$. If $T^{(1)}, \ldots, T^{(s)}$ are j.w.m., then they have no joint eigenfunctions except constants.

Proof. Suppose $(f_1, f_2, \ldots, f_s)$ is a $(T^{(i)})^{s}_{i=1}$-joint eigenfunction. For $1 \leq i \leq s$ define $F_i : X \times X$ by $F_i(x, y) = f_i(x)/f_i(y)$. It is clear that $(F_1, F_2, \ldots, F_s)$ is a $(T^{(i)} \otimes T^{(i)})^{s}_{i=1}$-joint invariant function, so by Proposition 2.21 it is a constant. Hence, $(f_1, f_2, \ldots, f_s)$ is constant as well. \hfill \Box

Theorem 3.15. Let $T^{(1)}, \ldots, T^{(s)}$ be commuting actions of a countable abelian group $G$. The actions $T^{(1)}, \ldots, T^{(s)}$ are j.w.m. iff they have no joint eigenvalues except constants.

Proof. In view of Proposition 3.14, we need to prove the “if” part only. If the actions $T^{(1)}, \ldots, T^{(s)}$ have no non-constant joint eigenfunctions, then by Theorem 2.22, $T^{(1)}, \ldots, T^{(s)}$ are j.e. Assume that $T^{(1)}, \ldots, T^{(s)}$ are not j.w.m. Then by Theorem 3.12 $T^{(1)}$ is not weakly mixing. By Theorem 3.3, $T_1$ has a non-constant eigenfunction, say, $f_1$. Let $f_i = 1$ for $2 \leq i \leq s$. Then $(f_1, f_2, \ldots, f_s)$ is a non-constant $(T^{(i)})^{s}_{i=1}$-joint eigenfunction. \hfill \Box

Remark 3.16. In [Ber85b], D. Berend introduced the notions of weak joint weak mixing, joint weak mixing, and joint eigenfunctions for $\mathbb{Z}$-actions. He proved Theorem 3.12 and Theorem 3.15 for that case.
3.3 Joint Strong Mixing

An action $T$ of a countable group $G$ is called (strongly) mixing if $\lim_{g \to \infty} \mu(T_g A \cup B) = \mu(A)\mu(B)$ for all $A, B \in \mathcal{B}$.

**Definition 3.17.** The actions $T^{(1)}, \ldots, T^{(s)}$ of a countable group $G$ are called weak-jointly strongly mixing (w-j.s.m.) if for all $A_0, A_1, \ldots, A_s \in \mathcal{B}$ one has

$$\lim_{g \to \infty} \mu\left( \bigcap_{i=0}^{s} T_g^{(i)} A_i \right) = \prod_{i=1}^{s} \mu(A_i).$$

A set $\Delta \subseteq G^2$ is of eventually bounded fibers if there exists $C > 0$ such that $|\{h \in G \mid (g, h) \in \Delta\}| \leq C$ for all $g \in G$ outside a finite set. Let $(a_{g,h})$ be a sequence in a Banach space $B$ indexed by the elements of $G^2$. We shall write EC-$\lim_{g,h} a_{g,h} = a$ if for every $\varepsilon > 0$ there exists a set $\Delta$ of eventually bounded fibers such that $\|a_{g,h} - a\| < \varepsilon$ for all $(g, h) \in G^2 \setminus \Delta$.

**Definition 3.18.** The actions $T^{(1)}, \ldots, T^{(s)}$ of a countable group $G$ are called jointly strongly mixing (j.s.m.) if for all $A_1, B_1, \ldots, A_s, B_s \in \mathcal{B}$ one has

$$\text{EC-} \lim_{g,h} \mu\left( \bigcap_{i=1}^{s} (T_g^{(i)} A_i \cap T_h^{(i)} B_i) \right) = \prod_{i=1}^{s} (\mu(A_i)\mu(B_i)).$$

**Proposition 3.19.** $T^{(1)}, \ldots, T^{(s)}$ are

1. weak jointly strongly mixing iff

$$\lim_{g \to \infty} \int_X \prod_{i=0}^{s} T_g^{(i)} f_i \, d\mu = \prod_{i=0}^{s} \int_X f_i \, d\mu, \quad f_0, f_1, \ldots, f_s \in L^\infty(X);$$

2. jointly strongly mixing iff

$$\text{EC-} \lim_{g,h} \int_X \prod_{i=1}^{s} (T_g^{(i)} f_i \cdot T_h^{(i)} g_i) \, d\mu = \prod_{i=0}^{s} \left( \int_X f_i \, d\mu \cdot \int_X g_i \, d\mu \right), \quad f_1, g_1, \ldots, f_s, g_s \in L^\infty(X).$$

**Proof.** The proof is routine. $\square$
CHAPTER 4

JOINT PROPERTIES OF ACTIONS BY AUTOMORPHISMS OF COMPACT GROUPS

4.1 Preliminary Results

In this chapter, we shall be interested in the case when a countable (possibly non-abelian) group $G$ acts on a compact group by its automorphisms. In this case, the action $T$ is a homomorphism $T : G \to \text{Aut}(X)$. A number of important facts about ergodicity and mixing of such actions has been established (see, for example [Sch95]). We give a brief summary of some of those results below.

We consider finite-dimensional continuous unitary representations of the compact group $X$. If finite-dimensional continuous unitary representations $\tau_1$ and $\tau_2$ are unitarily equivalent, we write $\tau_1 \sim \tau_2$. A review of theory of continuous unitary representations of compact group can be found in Appendix E.

**Theorem 4.1.** Let $T$ be an action of a countable group $G$ by automorphisms of a compact group $X$. The action $T$ is ergodic iff for every non-trivial continuous irreducible unitary representation $\tau$ of $X$ on a Hilbert space $\mathcal{H}$ the group

$$G_\tau = \{ g \in G \mid T_g \tau \sim \tau \}$$

has infinite index in $G$. 

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Here \( T_g\tau \) denotes the unitary representation of \( X \) defined as \((T_g\tau)(x) = \tau(T_gx)\).

**Remark 4.2.** This theorem has been proven in [Ber85a] under weaker assumptions than in our version. In the case of \( \mathbb{Z} \)-actions, this result is due to Halmos [Hal43] in the case of abelian \( X \) and Kaplansky [Kap49] in the general case.

**Theorem 4.3 ([Sch95]).** Let \( T \) be an action of a countable group \( G \) by automorphisms of a compact group \( X \). The action \( T \) is mixing iff for every non-trivial, continuous, irreducible, unitary representation \( \tau \) of \( X \) on a Hilbert space \( \mathcal{H} \) the group

\[
G_\tau = \{ g \in G \mid T_g\tau \sim \tau \}
\]

is finite.

The following fact is curious.

**Proposition 4.4.** Let \( T \) be an action of a countable group \( G \) by group-automorphisms of a compact group \( X \). The action \( T \) is ergodic iff it is totally ergodic. \( T \) is mixing iff it is strongly totally ergodic.

This follows directly from the two above theorems and Definition 2.27.

Theorem 4.1 and Theorem 4.3 admit simpler formulations in the case when \( X \) is abelian. We shall denote the dual group of \( X \) as \( \hat{X} \) and the dual right action of \( T \) (defined in Appendix E) on \( \hat{X} \) as \( \hat{T} \). Since a function \( \gamma \) can be thought of as element of both \( L^2(X) \) and \( \hat{X} \) we shall use notation \( T\gamma \) when we think about \( \gamma \) as an element of \( L^2(X) \) and \( \hat{T}\gamma \) when we think about \( \gamma \) as an element of \( \hat{X} \). Since \( X \) is compact, \( \hat{X} \) is a discrete countable group.

**Corollary 4.5.** Let \( T \) be an action of a countable group \( G \) by automorphisms of a compact abelian group \( X \). The action \( T \) is ergodic iff for every non-trivial character
\[ \gamma \in \hat{X} \text{ the group} \]
\[ G_\gamma = \left\{ g \in G \mid \hat{T}_g \gamma = \gamma \right\} \]
has infinite index in \( G \).

**Corollary 4.6.** Let \( T \) be an action of a countable group \( G \) by automorphisms of a compact abelian group \( X \). The action \( T \) is mixing iff for every non-trivial character \( \gamma \in \hat{X} \) the group
\[ G_\gamma = \left\{ g \in G \mid \hat{T}_g \gamma = \gamma \right\} \]
is finite.

**Remark 4.7.** Since every infinite subgroup of \( \mathbb{Z} \) has finite index, for \( \mathbb{Z} \)-actions by automorphisms of compact groups ergodicity is equivalent to mixing. Since already the group \( \mathbb{Z}^2 \) has infinite subgroups of infinite index, this fact does not hold for general group actions.

**Example 4.8.** Let \( X = \mathbb{T}^2 \) be the 2-dimensional torus. Let us consider the \( \mathbb{Z}^2 \)-action \( T \) on \( X \) generated by two automorphisms of \( X \), \( I \) and \( A \), where \( I \) is the identity transformation and \( A \) is an ergodic (as a \( \mathbb{Z} \)-action) automorphism: \( T_{(m,n)} = A^n \). One can easily see that so defined action \( T \) is ergodic but not mixing.

A natural question to ask is how weak mixing of the actions under consideration is related to their ergodicity and (strong) mixing. The following result is quite surprising (considering Example 4.8).

**Proposition 4.9.** Let \( T \) be an action of a countable group \( G \) by automorphisms of a compact abelian group \( X \). The action \( T \) is ergodic iff it is weakly mixing.
Proof. Assume that $T$ is not weakly mixing. Then, by Theorem 3.3, $T \times T$ is not ergodic. By Corollary 4.5 and because the dual group of $X \times X$ is $\hat{X} \times \hat{X}$, there exists $\gamma \in \hat{X} \times \hat{X}$ such that the group

$$G_\gamma = \left\{ g \in G \mid (\hat{T} \times \hat{T}) \gamma = \gamma \right\}$$

is of finite index in $G$. But $\gamma = (\gamma_1, \gamma_2)$ for some $\gamma_1, \gamma_2 \in \hat{X}$. Therefore, $G_{\tau_1} = \left\{ g \in G \mid \hat{T}_g \gamma_1 = \gamma_1 \right\} \supset G_\gamma$. Hence, the action $T$ is not ergodic. \qed

The same fact holds for the general case when $X$ is a compact group, not necessarily abelian.

**Proposition 4.10.** Let $T$ be an action of a countable group $G$ by automorphisms of a compact group $X$. The action $T$ is ergodic iff it is weakly mixing.

Proof. Assume that $T$ is not weakly mixing. Then by Theorem 3.3, $T \otimes T$ is not ergodic. By Theorem 4.3, there exists a non-trivial, continuous, irreducible, unitary representation $\tau$ of $X \times X$ on $L^2(X \times X) = L^2(X) \otimes L^2(X)$ such that the group

$$G_\tau = \left\{ g \in G \mid (T_g \otimes T_g)\tau \sim \tau \right\}$$

has a finite index in $G$. By Theorem E.5, there exist continuous irreducible unitary representations of $X$ on $L^2(X)$, $\tau_1$ and $\tau_2$, such that $\tau$ is unitarily equivalent to the $X \times X$-representation given by $\tau_1 \otimes \tau_2$. If $(T_g \otimes T_g)\tau$ is unitarily equivalent to $\tau$ for some $g \in G$, then there exists a unitary operator $U_g$ from $L^2(X \times X)$ to itself such that

$$(T_g \otimes T_g)(\tau_1(x_1) \otimes \tau_2(x_2))U_g = U_g(\tau_1(x_1) \otimes \tau_2(x_2)), \quad \text{for all } x_1, x_2 \in X.$$ 

Hence, $(T_g \tau_1)U_g = U_g \tau_1$ and $(T_g \tau_2)U_g = U_g \tau_2$. It follows that

$$G_{\tau_1} = \left\{ g \in G \mid T_g \tau_1 \sim \tau_1 \right\} \supset G_\tau.$$ 

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Therefore, the action $T$ is not ergodic.

Remark 4.11. Proposition 4.9 is a special case of Proposition 4.10. Since the proof of Proposition 4.9 is somewhat simpler than the proof of Proposition 4.10, we presented both proofs.

4.2 Joint Ergodicity of Actions by Group Automorphisms

Proposition 4.12. Let $T^{(1)}, \ldots, T^{(s)}$ be actions of a countable amenable group $G$ by automorphisms of a compact abelian group $X$. The actions $T^{(1)}, \ldots, T^{(s)}$ are j.e. iff they are w.j.e.

Proof. See Remark 2.16

Definition 4.13. Let $(a_g)_{g \in G}$ be a sequence in a set $A$ indexed by elements of an amenable countable group $G$. Denote $S_a = \{g \mid a_g = a\}$ for $a \in A$. The sequence $(a_g)$ is of density 0 multiplicity if $d(S_a) = 0$, i.e.

$$\lim_{n \to \infty} \frac{|S_a \cap L_n|}{|L_n|} = 0 \quad \text{for every } (L_n) \in \mathcal{F}(G) \text{ and all } a \in G.$$

Theorem 4.14. Let $T^{(1)}, \ldots, T^{(s)}$ be actions of a countable amenable group $G$ by automorphisms of a compact abelian group $X$. The actions $T^{(1)}, \ldots, T^{(s)}$ are jointly ergodic iff for all $\gamma_1, \gamma_2, \ldots, \gamma_s \in \hat{X}$, not all of them 0, the set $\left(\sum_{i=1}^{s} \hat{T}_g^{(i)} \gamma_i\right)_{g \in G}$ is of density 0 multiplicity.

Proof. By Proposition 4.12, $T^{(1)}, \ldots, T^{(s)}$ are j.e. iff they are w.j.e. Since linear combinations of characters are dense in $L^2(X)$ and in view of Theorem 2.2, $T^{(1)}, \ldots, T^{(s)}$ are weakly jointly ergodic iff for every $\gamma_1, \gamma_2, \ldots, \gamma_s \in \hat{X}$, not all 0, every $\gamma_0 \in \hat{X}$, and
every \((L_n) \in F(G)\) we have
\[
\left( \frac{1}{|L_n|} \sum_{g \in L_n} \prod_{i=1}^{s} T_g^{(i)} \gamma_i, \gamma_0 \right) \xrightarrow{L^2(X)} \frac{1}{|L_n|} \left\{ g \in L_n \left| \sum_{i=1}^{s} \hat{T}_g^{(i)} \gamma_i = \gamma_0 \right. \right\} \xrightarrow{n \to \infty} 0,
\]
because \(\prod_{i=1}^{s} T_g^{(i)} \gamma_i\) is \(\sum_{i=1}^{s} \hat{T}_g^{(i)} \gamma_i\) when thought of as an element of \(\hat{X}\). This means that \(\left( \sum_{i=1}^{s} \hat{T}_g^{(i)} \gamma_i \right)_{g \in G}\) is of density 0 multiplicity. \(\square\)

**Remark 4.15.** In the statement of Theorem 4.14 it is enough to consider only such \(\gamma_1, \ldots, \gamma_s \in \hat{X}\) that \(\sum_{i=1}^{s} \gamma_i = 0\). Namely, if \(\sum_{i=1}^{s} \hat{T}_g^{(i)} \gamma_i = \sum_{i=1}^{s} \hat{T}_{g_0}^{(i)} \gamma_i\) for some \(g_0 \in G\) and all \(g \in \Delta \subseteq G\), then for \(\gamma'_i = \gamma_i - \hat{T}_{g_0}^{(i)} \gamma_i\), \(1 \leq i \leq s\), one has \(\sum_{i=1}^{s} \gamma'_i = 0\) and \(\sum_{i=1}^{s} \hat{T}_g^{(i)} \gamma'_i = 0\) for all \(g \in \Delta\). Hence, we can formulate an equivalent condition for joint ergodicity:

**Corollary 4.16.** Let \(T^{(1)}, \ldots, T^{(s)}\) be actions of a countable amenable group \(G\) by automorphisms of a compact abelian group \(X\). The actions \(T^{(1)}, \ldots, T^{(s)}\) are jointly ergodic iff for all \(\gamma_1, \gamma_2, \ldots, \gamma_s \in \hat{X}\), not all of them 0, the set
\[
\left\{ g \in G \left| \sum_{i=1}^{s} \hat{T}_g^{(i)} \gamma_i = 0 \right. \right\}
\]
has density 0.

For the case when \(s = 2\) the condition becomes even simpler.

**Corollary 4.17.** Let \(T\) and \(S\) be actions of a countable amenable group \(G\) by automorphisms of a compact abelian group \(X\). The actions \(T\) and \(S\) are jointly ergodic iff for all \(\gamma \in \hat{X}, \gamma \neq 0\), the set
\[
\left\{ g \in G \left| \hat{T}_g = \hat{S}_g \gamma \right. \right\}
\]
has density 0.
Remark 4.18. For $\mathbf{Z}$-actions, Theorem 4.14 has been established in [Ber85b] and, in more convenient form, in [BB86].

Now, we would like to extend this theorem to the case when $X$ is a compact, not necessarily abelian group. Let $\mathcal{H}_0$ be a fixed infinite-dimensional Hilbert space. We shall consider continuous unitary representations $\tau$ of $X$ in a finite-dimensional subspace $\mathcal{H}$ of $\mathcal{H}_0$. (The reader may consult Appendix E for a brief review of representation theory of compact groups.) If we fix a basis in $\mathcal{H}$, then $\tau$ is given by a $n \times n$ unitary matrix $(t_{k,l})$, whose elements $t_{k,l}$ are continuous complex-valued functions on $X$. We shall denote by $U$ the set of all continuous unitary finite-dimensional representations of $X$ in $\mathcal{H}_0$. Let $\Gamma$ be the dual object of $X$, i.e. the set of all equivalence classes of representations in $U$. If $\tau \in \gamma \in \Gamma$, then the dimension of the representation space of $\tau$ is denoted by $d_\tau$ or $d_\gamma$. If $\tau \in U$ and $T$ is a $G$-action by automorphisms of the group $X$, then $T_g \tau$ is another irreducible continuous unitary representation of $X$ in $\mathcal{H}$ given by $(T_g \tau)(x) = \tau(T_g x)$. Thus, the action $T$ gives rise to a dual right action on $U$ which induces, in turn, an action on $\Gamma$ which shall also be denoted by $T$. If $\tau_1 \sim (t_{k,j}^1)_{k,j=1,n_1}$ and $\tau_2 \sim (t_{k,j}^2)_{k,j=1,n_2}$ are two unitary representations of $X$ in finite-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then we can define their tensor product $\tau_1 \otimes \tau_2$ as the representation of $X$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ with the matrix $(t_{k,j_1,j_2})$ where $t_{k_1,l_1,j_2} = t_{k_1,l_1}^1 t_{k_2,l_2}^2$, $1 \leq k_1, l_1 \leq n_1$; $1 \leq k_2, l_2 \leq n_2$. Similarly, one can define the tensor product $\bigotimes_{i=1}^s \tau_i$ of $s$ representations $\tau_1, \ldots, \tau_s$ of $X$ in finite-dimensional Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_s$ as a representation of $X$ in the Hilbert space $\bigotimes_{i=1}^s \mathcal{H}_i$.

Let $\tau \in \gamma \in \Gamma$ and $\tau' \in \gamma' \in \Gamma$. The unitary representation $\tau \otimes \tau'$ is not irreducible in general. However, in view of the Theorem E.9, it can be decomposed as

$$\tau \otimes \tau' = m_1 \tau_1 \oplus \cdots \oplus m_r \tau_r.$$
for some irreducible representations $\tau_i$ with multiplicities $m_i$, $1 \leq i \leq r$. We define a binary operation $\times$ on $\Gamma$ as follows:

$$\gamma \times \gamma' \overset{\text{def}}{=} \{\tau_1, \ldots, \tau_r\}.$$  

(See Appendix E for more details.)

Let us consider the regular left representation $\tau$ of $X$ in $L^2(X)$. By Peter-Weyl Theorem (Theorem E.12) $L^2(X)$ can be represented as a direct sum of $\tau$-invariant finite-dimensional Hilbert spaces $\mathcal{H}_r$, $r = 1, 2, \ldots$, so that $\tau$ is decomposed onto a direct sum of finite-dimensional irreducible representations $\tau_r$, $r = 1, 2, \ldots$. Let us fix a basis in $L^2(X)$ and let representations $\tau_r$ have matrices $(t_{r,k,l})_{k,l=1,d_r}$ with respect to that basis, $d_r = d_{\tau_r}$. According to Theorem E.12, the set of functions $\{t_{r,k,l}\}$ spans a dense linear subspace in $L^2(X)$.

We shall need the following definition.

**Definition 4.19.** Let $\{A_g \mid g \in G\}$ be a family of non-empty subsets of a set $A$ indexed by elements of a countable amenable group $G$. We say that the family of sets $(A_g)_{g \in G}$ is *weakly disjoint* if for every $a \in A$ the set $\{g \in G \mid a \in A_g\}$ has density 0 in $G$.

In other words, the sets $A_g$ are weakly disjoint if for every subset $P$ of $G$ of positive density, the set $\bigcap_{g \in P} A_g$ is empty.

Now, let $T^{(1)}, \ldots, T^{(s)}$ be actions of a countable amenable group $G$ by automorphisms of $X$. Let $\gamma_1, \ldots, \gamma_s \in \Gamma$ and let $\tau_i \in \gamma_i$, $1 \leq i \leq s$. Fix a $g \in G$ and consider the continuous unitary representation $\tau = \bigotimes_{i=1}^s T_g^{(i)} \tau_i$. The representation $\tau$ does not have to be irreducible, but, in view of Theorem E.9, it can be represented as a direct
sum of irreducible representations:

\[
\tau = k_1 \psi_1 \oplus k_2 \psi_2 \oplus \cdots \oplus k_m \psi_m
\]  

(4.1)

for some \( \psi_i \in \gamma'_i \in \Gamma \) with multiplicities \( k_i > 0, 1 \leq i \leq m \), where \( m \) is an integer number depending on \( g \). On the level of the dual object \( \Gamma \), we have:

\[
T_g^{(1)} \gamma_1 \times \cdots \times T_g^{(s)} \gamma_s = \{ \gamma'_1, \ldots, \gamma'_m \}.
\]

**Theorem 4.20.** Let \( T^{(1)}, \ldots, T^{(s)} \) be actions of a countable amenable group \( G \) by automorphisms of a compact group \( X \). The actions \( T^{(1)}, \ldots, T^{(s)} \) are jointly ergodic iff for all \( \gamma_1, \ldots, \gamma_s \in \Gamma \), not all of them trivial, the family of sets

\[
(T_g^{(1)} \gamma_1 \times \cdots \times T_g^{(s)} \gamma_s)_{g \in G}
\]

is weakly disjoint.

**Proof.** “If” part. In view of Proposition 4.12, it is enough to prove that the actions \( T^{(1)}, \ldots, T^{(s)} \) are weakly jointly ergodic. In view of Theorem 2.2 and because the functions \( t_{r_i}^{k_i,l_i}, 1 \leq k, l \leq d, r = 1, 2, \ldots, \) span a dense set in \( L^2(X) \), \( T^{(1)}, \ldots, T^{(s)} \) are weakly jointly ergodic if for all natural numbers \( r_0, k_0, l_0, r_1, k_1, l_1, \ldots, r_s, k_s, l_s \) such that \( 1 \leq k_i, l_i \leq d \) and not all \( t_{r_i}^{k_i,l_i} \) are constants, and every Følner sequence \( (L_n) \in F(G) \) one has

\[
\left( \frac{1}{|L_n|} \sum_{g \in L_n} \prod_{i=1}^{s} T_g^{(i)} t_{r_i}^{k_i,l_i} t_{k_0,l_0}^{r_0} \right)_{L^2(X)} \xrightarrow{n \to \infty} 0.
\]  

(4.2)

Let us fix \( g \in G \) and let \( t_g = \prod_{i=1}^{s} T_g^{(i)} t_{r_i}^{k_i,l_i} \). Note that \( t_g \) is a coordinate function of the representation \( \tau = \bigotimes_{i=1}^{s} T_g^{(i)} r_{r_i} \). Since \( \tau = k_1 \psi_1 \oplus k_2 \psi_2 \oplus \cdots \oplus k_m \psi_m \) for some irreducible unitary representations \( \psi_1, \ldots, \psi_m \), it follows that \( t_g \) is a linear
combination of coordinate functions of $\psi_1, \psi_2, \ldots, \psi_{m_g}$. Because of the orthogonality
conditions (Theorem E.12, equation (E.5)), if none of the representations $\psi_i, 1 \leq i \leq m_g$, is unitarily equivalent to $\tau_{r_0}$, then $(t_g, t_{k_0,l_0}^{r_0}) = 0$. If $\tau_{r_0}$ is unitarily equivalent to
some $\psi_i, 1 \leq i \leq m(g)$, then $\gamma_{r_0} \in T_g^{(i)} \gamma_1 \times \cdots \times T_g^{(s)} \gamma_s$, where $\gamma_{r_0} \in \tau_{r_0}$. But the sets $T_g^{(i)} \gamma_1 \times \cdots \times T_g^{(s)} \gamma_s, g \in G$, are weakly disjoint. Therefore, $(t_g, t_{k_0,l_0}^{r_0}) = 0$ only for a
set of $g$ of density 0. Hence, (4.2) holds and $T^{(1)}, \ldots, T^{(s)}$ are w.j.e.

“Only If” part. Assume that there exist $\gamma_1, \gamma_2, \ldots, \gamma_s$, not all of them trivial, such
that the sets $T_g^{(1)} \gamma_1 \times \cdots \times T_g^{(s)} \gamma_s, g \in G$, are not weakly disjoint. This means that there
exists $\gamma_0 \in \Gamma$ and a subset $S$ of $G$ of positive density such that $\gamma_0 \in T_g^{(1)} \gamma_1 \times \cdots \times T_g^{(s)} \gamma_s$
for all $g \in S$. Let us consider the character of the representation $\tau = \bigotimes_{i=1}^s T_g^{(i)} \tau_i$:

$$\chi(\tau) = \prod_{i=1}^s \chi(T_g^{(i)} \tau_i) = \prod_{i=1}^s T_g^{(i)} \chi(\tau_i).$$

(4.3)

Let $f_i = \chi(\tau_i), 1 \leq i \leq s$. Then the functions $f_i$ are continuous. Note that at least
one of $\tau_i$’s, $1 \leq i \leq s$, is non-trivial. Let it be $\tau_1$. Then $f_1$ is not a constant. Moreover,
since $\tau_1 \not\sim 1$, $\int_X f_1 \ d\mu = 0$ (by virtue of Theorem E.9).

On the other hand, in view of (4.1), one has:

$$\chi(\tau) = \sum_{i=1}^{m_g} k_i \chi(\psi_i).$$

(4.4)

If $g \in S$, then at least one of $\psi_i$’s, say, $\psi_{l_g} \in \gamma_0$, so if we let $f_0 = \chi(\gamma_0)$, then
$\int_X \chi(\tau) f_0 \ d\mu = k_g \geq 1$, where $k_g \overset{\text{def}}{=} k_{l_g}$ is the multiplicity of $\psi_{l_g}$ in the direct sum
decomposition (4.1) of $\tau$. Combining (4.3) and (4.4), we obtain:

$$\lim_{n \to \infty} \frac{1}{|L_n|} \sum_{g \in L_m} \left( \prod_{i=1}^s T_g^{(i)} f_i, f_0 \right)_{L^2(X)} \geq d_\mathcal{L}(S).$$

Choosing a Følner sequence $\mathcal{L}$ such that $d_\mathcal{L}(S) = d(S) > 0$ we get a contradiction.
Remark 4.21. We can modify the condition “the family of sets

\[ (T_1^{(1)} \gamma_1 \times \cdots \times T_s^{(s)} \gamma_s)_{g \in G} \]

is weakly disjoint” by observing that, in view of Corollary E.11,

\[ \gamma_0 \in T_1^{(1)} \gamma_1 \times \cdots \times T_s^{(s)} \gamma_s \iff 1 \in \bar{\gamma}_0 \times T_1^{(1)} \gamma_1 \times \cdots \times T_s^{(s)} \gamma_s, \]

where 1 denotes the equivalence class of the trivial representation. Hence, an equivalent criterion is: “for all \( \gamma_0, \gamma_1, \ldots, \gamma_s \in \Gamma \), not all of them trivial, the set

\[ \left\{ g \in G \mid 1 \in \bigotimes_{i=0}^s T_i^{(i)} \gamma_i \right\} \]

has density 0”.

Remark 4.22. D. Berend in [Ber88] proved a spatial case of Theorem 4.20 for commuting \( \mathbb{Z} \)-actions. For that case he was able to obtain a simpler criterion of ergodicity of the actions \( T_1^{(1)}, \ldots, T_s^{(s)} \). Namely, he proved that \( T_1^{(1)}, \ldots, T_s^{(s)} \) are j.e. iff the only solution of the equation

\[ T_n^{(i)} \gamma = T_n^{(j)} \gamma \]

with \( 0 \leq i < j \leq s \), \( \gamma \in \Gamma \), and \( n \in \mathbb{Z} \), is \( \gamma = 1 \).

4.3 Joint Weak Mixing of Actions by Group Automorphisms

Theorem 4.23. Let \( T_1^{(1)}, \ldots, T_s^{(s)} \) be actions of a countable amenable group \( G \) by automorphisms of a compact abelian group. The actions \( T_1^{(1)}, \ldots, T_s^{(s)} \) are jointly weakly mixing iff they are weakly jointly ergodic.

Proof. By Proposition 3.7, if \( T_1^{(1)}, \ldots, T_s^{(s)} \) are j.w.m., then they are j.e. and, therefore, w.j.e. It remains to prove validity of the converse statement.
Assume that $T^{(1)}, \ldots, T^{(s)}$ are not jointly weakly mixing. In view of Theorem 3.8 and Theorem 4.14, there exist $\gamma_1', \gamma_2', \gamma_2'', \ldots, \gamma_s', \gamma_s'' \in \hat{X}$, not all of them 0, a subset $S$ of $G$ of positive density, and $\gamma_0', \gamma_0'' \in \hat{X}$ such that

$$ (\gamma_0', \gamma_0'') = \sum_{i=1}^{s} \left( T_{g}^{(i)} \times T_{g}^{(i)} \right)(\gamma_i', \gamma_i'') $$

for all $g \in S$. WLOG one of $\gamma_i'$, $1 \leq i \leq s$, is non-zero. Since

$$ \gamma_0' = \sum_{i=1}^{s} T_{g}^{(i)} \gamma_i' $$

for all $g \in S$, it follows that $T^{(1)}, \ldots, T^{(s)}$ are not j.e. Contradiction. \hfill \Box

A generalization of this theorem exists for the case when $X$ is a non-abelian compact group. The proof is somewhat more complicated.

**Theorem 4.24.** Let $T^{(1)}, \ldots, T^{(s)}$ be actions of a countable amenable group $G$ by automorphisms of a compact group $X$. The actions $T^{(1)}, \ldots, T^{(s)}$ are jointly weakly mixing iff they are weakly jointly ergodic.

**Proof.** If $T^{(1)}, \ldots, T^{(s)}$ are j.w.m., then they are w.j.e. It remains to prove the converse statement. Let $T^{(1)}, \ldots, T^{(s)}$ be not jointly weakly mixing. In view of Theorem 3.8 and Theorem 4.20, there exist $\gamma_1, \ldots, \gamma_s \in \Gamma(X \times X)$, a subset $S$ of $G$ of positive density, and $\gamma_0 \in \Gamma(X \times X)$ such that $\gamma_0 \in X_{i=1}^{s} (T_{g}^{(i)} \otimes T_{g}^{(i)}) \gamma_i$ for all $g \in S$. Note that there exist $\gamma_1', \gamma_2', \gamma_2'', \ldots, \gamma_s', \gamma_s'' \in \Gamma(X)$ such that $\gamma_i = \gamma_i' \otimes \gamma_i''$, $1 \leq i \leq s$. Let

$$ \gamma = \bigotimes_{i=1}^{s} (T^{(i)} \otimes T^{(i)}) g \gamma_i = \left( \bigotimes_{i=1}^{s} T_{g}^{(i)} \gamma_i' \right) \otimes \left( \bigotimes_{i=1}^{s} T_{g}^{(i)} \gamma_i'' \right). \tag{4.5} $$

Note that we deal with two quite different types of tensor product in (4.5). The operation on the dual object denoted by the symbol $\bigotimes$ corresponds to the tensor
product of two or more representations of $X$ considered as a representation of $X$. The operation $\otimes$ corresponds to the tensor product of two representations of $X$ considered as a representation of $X \times X$.

Let $X^s_{i=1} T^{(i)}_g \gamma'_i = \{\delta'_1, \ldots, \delta'_{m_g'}\}$ and $X^s_{i=1} T^{(i)}_g \gamma''_i = \{\delta''_1, \ldots, \delta''_{m_g''}\}$, where $\psi'_j \in \gamma'_j \in \Gamma(X)$ and $\psi''_j \in \gamma''_j \in \Gamma(X)$, $1 \leq p \leq m'_g$, $1 \leq q \leq m''_g$. Then

$$\gamma = \{\delta'_p \otimes \delta''_q | p = 1, \ldots, m'_g, q = 1, \ldots, m''_g\} \quad (4.6)$$

Note that $\delta'_p \otimes \delta''_q$ is an element of $\Gamma(X \times X)$. We know that $\gamma_0 \in \bigotimes_{i=1}^s (T^{(i)}_g \otimes T^{(i)}_g) \gamma_i$ for all $g \in S$. This means that for given $g$ there exist $p$, $1 \leq p \leq m'_g$, and $q$, $1 \leq q \leq m''_g$ such that $\gamma'_p \otimes \gamma''_q = \gamma_0$. But this means that there exists $\gamma_0' \in \Gamma(X)$ such that for each $g \in S$ one has $\gamma'_p \in \gamma_0'$ (where $p$ depends on $g$). Hence, by Theorem 4.20, $T^{(1)}, \ldots, T^{(s)}$ are not j.e.

\[ \square \]

Remark 4.25. Equivalence of joint ergodicity and joint weak mixing for $\mathbb{Z}$-actions by automorphisms of abelian groups has been established in [Ber85b].

### 4.4 Joint Strong Mixing of Actions by Group Automorphisms

**Definition 4.26.** In the setup of Definition 4.13 the sequence $(a_g)$ is of

1. finite multiplicity — if $|S_a| < \infty$, $a \in A$,

2. bounded multiplicity — if $\sup_{a \in A} |S_a| < \infty$.

**Theorem 4.27.** Let $T^{(1)}, \ldots, T^{(s)}$ be actions of a countable amenable group $G$ by automorphisms of a compact abelian group $X$. The actions $T^{(1)}, \ldots, T^{(s)}$ are

1. w.j.s.m. iff the sequence $\left(\sum_{i=1}^s \hat{T}^{(i)}_g \gamma_i\right)_{g \in G}$ is of finite multiplicity;
(2) j.s.m. iff the sequence \( \left( \sum_{i=1}^{s} \hat{T}_{g}^{(i)} \gamma_i \right)_{g \in G} \) is of bounded multiplicity.

The proof is routine.

It is possible to obtain a similar condition for actions by automorphisms of non-abelian groups along the lines of Theorem 4.20.

**Question 4.28.** Is it true that if the \( G \)-actions \( T^{(1)}, \ldots, T^{(s)} \) by automorphisms of an abelian compact group \( X \) are w-j.s.m., then they are j.s.m.? It is not even known for \( \mathbb{Z} \)-actions.

### 4.5 Examples and Some Open Problems

**Example 4.29.** Let \( X = (\mathbb{Z}_2)^\mathbb{Z}_2 \) and let \( T, S \in \text{Aut}(\mathbb{Z}_2, X) \) be \( \mathbb{Z}_2 \)-actions on \( X \) defined as follows: \( (T_{(p,q)}(x))_{m,n} = x_{m+p,n+q} \) and \( (S_{(p,q)}(x))_{m,n} = x_{m+p,n+q} \). We claim that so defined \( \mathbb{Z}_2 \)-actions \( T \) and \( S \) are mixing, jointly ergodic, but not w.-j.s.m.

Indeed, the group \( \hat{X} \) is isomorphic to \( \bigoplus_{\mathbb{Z}_2} \mathbb{Z}_2 \). The dual actions \( \hat{T} \) and \( \hat{S} \) act on \( \hat{X} \) by shifts. Let \( \gamma \in \hat{X}, \gamma \neq 1 \). It is clear that \( \gamma \) can not be a periodic point for either \( \hat{T} \) or \( \hat{S} \). Hence, both \( T \) and \( S \) are mixing according to Theorem 4.3.

On the other hand,

\[
\hat{T}_{g}\gamma = \hat{S}_{g}\gamma \iff g \in H = \{(p,q) \in \mathbb{Z}_2^2 \mid q = 0\}.
\]

Since \( H \) is infinite, but of 0 density, \( T \) and \( S \) are j.e., but not w.-j.s.m.

**Example 4.30.** It is quite hard to come up with a not-trivial example of an ergodic automorphism (or an ergodic action of a countable group by automorphisms) of a non-abelian compact group.

The first example that comes to mind is to consider the group \( X = Y^G \) where \( G \) is a countable group and \( Y \) is a non-abelian compact group. For example, \( X = (S_3)^\mathbb{Z}_2 \).
We can consider shift automorphisms of $X$. However, they are not different from shifts on, say, $(\mathbb{Z}_6)^2$. Another family of automorphisms of $X$ is given by inner automorphisms of the base group $Y$, namely $A_h : Y \to Y$ with $A_h(y) = h^{-1}yh$, where $h$ is a fixed element of $Y$. Then we can define the automorphism $B$ on $X$ given by $(B(x))_g = (hgh^{-1})_g$ for all $g \in G$ and use it, together with a shift automorphism, to define a $\mathbb{Z}^2$-action on $X$. However, on the level of the dual object of $Y$, $B$ acts exactly the same as the identity operator. Thus, using shifts and inner automorphisms only, one can not define a non-trivial example of the kind that we need.

More interesting possibility stems from the so called Ledrappier’s example. In [Led78], F. Ledrappier considered the shift invariant subgroup $X$ of $(\mathbb{Z}_2)^2$ given by

$$X = \left\{ x = (x_{m,n}) \in (\mathbb{Z}_2)^2 \mid x_{m,n} + x_{m+1,n} + x_{m,n+1} = 0 \; \forall (m,n) \in \mathbb{Z}^2 \right\}. \quad (4.7)$$

and the $\mathbb{Z}^2$-actions by shifts on $X$. This action has very interesting dynamical properties. See [Sch95] where this and similar dynamical systems studied in great detail.

One can consider a non-abelian analogue of (4.7)

$$Z = \left\{ x = (x_{m,n}) \in (S_3)^2 \mid x_{m,n} \cdot x_{m+1,n} \cdot x_{m,n+1} = 1 \; \forall (m,n) \in \mathbb{Z}^2 \right\},$$

and study shift actions on it. (Of course, one can consider another compact non-abelian group instead of $S_3$.) In order to apply to $Z$ the methods developed earlier in this chapter, it would be necessary to determine the structure of the dual object of $Z$.

Nevertheless, this example shows that there exist natural examples of group actions by automorphisms of non-abelian groups with non-trivial dynamical properties.

In this chapter, we studied joint properties of group actions by a special class of transformations — group automorphisms. We were able to get some convenient
criteria for joint dynamic properties of such actions. There exist other classes of actions for which it might be possible to obtain similar results. Let us mention a few of them. Unfortunately, it was not possible for the author to study these cases in this work. All of them are good directions for further research.

(1) Actions by **group rotations**. We consider a compact group $X$ and the action by a countable group $G$ given by $T_g x = x \cdot b_g$, where $b$ is a homomorphism from $X$ to $G$ and we write $b_g = b(g)$. The abelian case is not hard. Let we have $s$ rotation actions of an abelian group $X$ given by $T_g^{(i)} x = x + b_g^{(i)}$, $1 \leq i \leq s$. One can show that $T^{(1)}, \ldots, T^{(s)}$ are jointly ergodic iff for all $\gamma_1, \ldots, \gamma_s \in \hat{X}$, not all of them 0, there exists $g \in G$ such that

$$\gamma_1 (b_g^{(1)}) + \cdots + \gamma_s (b_g^{(s)}) \neq 0.$$  

The non-abelian case is more complex and worth studying.

(2) Actions by **affine transformations**. An action by affine transformations of a compact group $X$ is a transformation $T$ given by $Tx = A_g x \cdot b_g$, where $A_g$ is an automorphism of $X$ and $b_g \in X$ for all $g \in G$. In [Hah63] and [Tho72] the single $\mathbb{Z}$-actions by affine transformations of abelian and non-abelian groups respectively have been studied. As far as we know, affine actions by more general groups have never been studied as neither been studied joint ergodic properties of such actions. Unfortunately, the criteria for joint ergodicity of affine actions turn out to be quite complicated to be practical, even in abelian case.

(3) **Automorphisms on nil-manifolds** have been studied extensively (see, for example, [Aus70]). The problem of characterizing ergodic and joint ergodic
properties of group actions by such automorphisms is a natural one to study.
Here one has readily available interesting non-abelian examples.
CHAPTER 5

JOINT ERGODICITY ALONG SEQUENCES OF INTERVALS

5.1 Definitions and Preliminary Results

Suppose that we have a Lebesgue probability space \((X, \mathcal{B}, \mu)\). Let \(L^2_0(X)\) be the orthogonal complement in \(L^2(X)\) of the one-dimensional subspace of the constant functions. Also, \(L^\infty_0(X) \overset{\text{def}}{=} L^\infty(X) \cap L^2_0(X)\).

We consider sequences of intervals \([a_n, b_n] = \{a_n, a_n+1, \ldots, b_n\}, n = 1, 2, \ldots, \) in \(\mathbb{N}\), where \((a_n)\) and \((b_n)\) are sequences of natural numbers, \(b_n \geq a_n\). By a regular sequence of intervals we mean a sequence of intervals \([a_n, b_n]_{n=1}^\infty\) in \(\mathbb{N}\) such that \(b_n - a_n \to \infty\) as \(n \to \infty\).

One way to characterize an ergodic measure-preserving transformation is as follows:

**Fact 5.1.** A measure-preserving transformation \(T\) is ergodic iff for every \(f \in L^2_0(X)\) one has

\[
\frac{1}{n} \sum_{k=0}^{n} T^k f \xrightarrow{n \to \infty} 0. \tag{5.1}
\]
But one can also fix an arbitrary regular sequence of intervals \([a_n, b_n]_{n=1}^\infty\) instead of the sequence \([0, n]_{n=1}^\infty\) and obtain the following:

**Fact 5.2.** A measure-preserving transformation \(T\) is ergodic iff for a fixed regular sequence of intervals \([a_n, b_n]_{n=1}^\infty\) and every \(f \in L^2_0(X)\) one has

\[
\frac{1}{b_n - a_n} \sum_{k=a_n}^{b_n} T^k f \xrightarrow{L^2} 0.
\]

(5.2)

To see that Fact 5.1 and Fact 5.2 indeed characterize ergodicity one uses von Neumann Mean Ergodic Theorem (Theorem 1.7). One can use either of them as a definition of an ergodic transformation.

In Chapter 2, we defined joint ergodicity of several group actions. In this chapter, we would like to study some aspects of joint ergodicity of two \(\mathbb{N}\)-actions, i.e. two not necessarily invertible, measure-preserving transformations. We shall define joint ergodicity for pairs of measure-preserving actions in terms of functions from \(L^2_0(X)\).

This is consistent with Definition 2.1, in view of Theorem 2.2 and Remark 2.3. We are interested in convergence of joint ergodic averages along regular sequences of intervals. In Remark 5.27 we shall discuss how our results change if one considers convergence along Følner sequences of general type.

**Definition 5.3.** A pair of measure-preserving transformations \((T, S)\) is called jointly ergodic along a sequence of intervals \([a_n, b_n]_{n=1}^\infty\) if for all functions \(f, g \in L^\infty_0(X)\) one has:

\[
\frac{1}{b_n - a_n} \sum_{k=a_n}^{b_n} T^k f \cdot S^k g \xrightarrow{L^2} 0.
\]

(5.3)
**Definition 5.4.** A pair of measure-preserving transformations \((T, S)\) is called *uniformly jointly ergodic* if for all functions \(f, g \in L^\infty_0(X)\) and every regular sequence of intervals \([a_n, b_n]_{n=1}^{\infty}\) one has:

\[
\frac{1}{b_n - a_n} \sum_{k=a_n}^{b_n} T^k f \cdot S^k g \xrightarrow{L^2, n \to \infty} 0.
\]  

(5.4)

**Remark 5.5.** In Chapter 2, we used the term *jointly ergodic* as opposed to *uniformly jointly ergodic* to refer to a notion of uniform joint ergodicity along all Følner sequence. The reason was that we were not interested in the more subtle issue of convergence along individual Følner sequences. Since, in this chapter, we are going to investigate how convergence of joint ergodic averages (5.4) depends on the choice of a regular sequence of intervals, we shall be using the terms *uniformly jointly ergodic* and *jointly ergodic along a sequence of intervals* as defined above.

Are Definition 5.4 and Definition 5.3 equivalent? In other words, is Definition 5.3 invariant with respect to the choice of the regular sequence of intervals \([a_n, b_n]_{n=1}^{\infty}\)? The answer is positive if \(T\) and \(S\) commute and are invertible. Repeating the arguments that have been used in the proof of Theorem 2.18, one can show that a pair of commuting invertible transformations \(T\) and \(S\) are jointly ergodic along a regular sequence of intervals iff the transformations \(T \times S\) and \(TS^{-1}\) are both ergodic. Since this criterion is independent of the choice of the regular sequence of intervals, the notion is independent of this choice as well. Invertibility of \(T\) and \(S\) is not important since one can consider the natural extensions \(T'\) and \(S'\) of the transformations \(T\) and \(S\) and prove that \(T\) and \(S\) are jointly ergodic along a sequence of intervals iff \(T' \times S'\) and \(T'S'^{-1}\) are ergodic as it is done in [BB84].

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As we shall see below, when $T$ and $S$ do not commute, Definition 5.3 is not invariant with respect to the choice of the sequence of intervals $[a_n, b_n]_{n=1}^\infty$.

Before we begin to discuss what happens in the non-commutative case, let us introduce some more definitions. By $\mathbb{N}$ we denote the sequence of intervals $[0, n]_{n=1}^\infty$. Let $L$ be the set of all regular sequences of intervals in $\mathbb{N}$. Given $E \subset \mathbb{N}$ the upper density of $E$ along a regular sequence of intervals $L = [a_n, b_n]_{n=1}^\infty$ is

$$\overline{d}_L(E) = \lim_{n \to \infty} \frac{|E \cap [a_n, b_n]|}{b_n - a_n}.$$

Of course, $\overline{d}_L(E)$ is just the familiar notion of upper density of $E$ which is customarily denoted as $\overline{d}(E)$.

The upper Banach density

$$d^*(E) \overset{\text{def}}{=} \sup_{L \in L} \overline{d}_L(E).$$

Finally, if $(c_n)$ is a sequence of complex numbers and $L = [a_n, b_n]_{n=1}^\infty \in L$, then we define upper Cesàro limit of $(c_n)$ along $L$ as

$$C_L - \lim_{n \to \infty} c_n = \lim_{n \to \infty} \frac{1}{b_n - a_n} \sum_{n \in L_n} c_n.$$

Clearly, if $E \subset \mathbb{N}$, then

$$C_L - \lim_{n \to \infty} 1_E(n) = \overline{d}_L(E).$$

The Cesàro limit of $(c_n)$ along $L$ is defined as

$$C_L - \lim_{n \to \infty} c_n = \lim_{n \to \infty} \frac{1}{b_n - a_n} \sum_{n \in L_n} c_n,$$

if the limit exits.
5.2 A Universal Counterexample

**Example 5.6.** Let $E \subset \mathbb{N}$ be given. Let $D = E \cup (-E)$. So defined set $D$ is a symmetric subset of $\mathbb{Z}$. We define two automorphisms, $S$ and $T_E$, of the abelian compact group $X = (\mathbb{Z}_2)^\mathbb{Z}$ as follows. The transformation $S$ is the left shift on $X$, that is

$$(Sx)_k = x_{k+1}, \quad x \in X, \ k \in \mathbb{Z}.$$ 

Define a permutation $\pi$ on $\mathbb{Z}$ by

$$\pi(k) = \begin{cases} k, & k \in D \\ -k, & k \notin D. \end{cases}$$

Let $T_E$ be the automorphism of $X$ given by

$$(T_E x)_k = x_{\pi(\pi(k)+1)}.$$ 

Then

$$(T^m_E x)_k = x_{\pi(\pi(k)+n)}, \quad n \in \mathbb{Z}.$$ 

Therefore,

$$(T^m_E x)_k = \begin{cases} x_{k-n}, & \text{if } k \notin D \text{ and } -k + n \notin D, \\ x_{-k+n}, & \text{if } k \notin D \text{ and } -k + n \in D, \\ x_{-k-n}, & \text{if } k \in D \text{ and } k + n \notin D, \\ x_{k+n}, & \text{if } k \in D \text{ and } k + n \in D \end{cases}$$

So, the transformation $T_E$ is “asymptotically” the right shift if $E$ is “small”. It is easy to see that so defined measure-preserving transformations $S$ and $T_E$ are both ergodic. Let $\mathcal{L} = [a_n, b_n]_{n=1}^\infty$ be a regular sequence of intervals. In view of (the proof of) Corollary 4.17, the $\mathbb{Z}$-actions $T_E$ and $S$ are j.e. along $\mathcal{L}$ iff for every $0 \neq \gamma \in \hat{X}$ the set

$$P_\gamma = \{ n \in \mathbb{Z} \mid \hat{S}^n \gamma = \hat{T}_E^n \gamma \}$$

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satisfies the condition $d_{L}(P_{\gamma}) = 0$. The dual space $\hat{X}$ is the set of all sequences with elements in $\mathbb{Z}_2$ with only finitely many non-zero entries. The dual actions of $S$ and $T_E$ are given by

$$
(\hat{S}\gamma)_k = \gamma_{k-1},
$$

$$
(\hat{T}_E\gamma)_k = \gamma_{\pi(\pi(k)-1)},
$$

for $\gamma \in \hat{X}$ and $k \in \mathbb{Z}$.

Hence, $(\hat{T}_E^n\gamma)_k = \gamma_{\pi(\pi(k)-n)}$.

For all $\gamma \in \hat{X}$ let

$$
F_\gamma = \{k \in \mathbb{Z} \mid \gamma_k = 1\},
$$

$$
b_\gamma = \max\{|k| \mid k \in F_\gamma\}.
$$

It is easy to check that

$$
F_{\hat{S}n\gamma} \subseteq [-b_\gamma + n, b_\gamma + n], \quad n \in \mathbb{N}
$$

and that if $n$ is such that $\pm F_\gamma - n \cap D \neq \emptyset$, then

$$
F_{\hat{T}_E^n\gamma} \subseteq [-b_\gamma - n, b_\gamma - n], \quad n \in \mathbb{N}.
$$

If $d_{L}(E) = 0$, then the set

$$
H_\gamma = \{n \in \mathbb{N} \mid \pm F_\gamma - n \cap D \neq \emptyset\}
$$

has density 0 along the sequence $L$. If $\gamma \neq 0$, then

$$
\{n > b_\gamma \mid \hat{S}^n\gamma = \hat{T}_E^n\gamma\} \subseteq \{n > b_\gamma \mid F_{\hat{S}n\gamma} \cap F_{\hat{T}_E^n\gamma} \neq \emptyset\} \subseteq H_\gamma.
$$

So, if $d_{L}(E) = 0$, then $S$ and $T_E$ are j.e. along $L$.

Assume now that $\bar{d}_{L}(E) > 0$. Let

$$
\gamma = (\ldots, -2, -1, 0, 1, 0, 0, \ldots).
$$

Then $\hat{S}^n\gamma = \hat{T}_E^n\gamma$ for all $n \in E$. Therefore, $S$ and $T_E$ are not j.e. along $L$. 

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Let us summarize our results as a proposition.

**Proposition 5.7.** Let \( E \subset \mathbb{N} \) and let \((X, \mathcal{B}, \mu)\) be a Lebesgue probability space. There exist two measure-preserving transformations \( T \) and \( S \) of \( X \) such that given a regular sequence of intervals \( \mathcal{L} \), \( T \) and \( S \) are j.e. along \( \mathcal{L} \) iff \( \bar{d}_E(E) = 0 \).

**Proof.** Every Lebesgue space is metrically isomorphic to the space \((\mathbb{Z}_2)^2\) with Haar measure. The transformations \( S \) and \( T_E \) described in Example 5.6 are jointly ergodic along a regular sequence of intervals iff their images under this isomorphism are.

**Remark 5.8.** Our universal example is based on the example given by D. Berend and V. Bergelson in [BB86]. They considered sets \( E \subset \mathbb{N} \) such that \( \bar{d}_N(E) = 0 \), but \( d^*(E) > 0 \) (for instance, one can take \( E = \bigcup_{n=1}^{\infty} [n^3, n^3 + n] \)). For such a set, the transformations \( S \) and \( T_E \) are jointly ergodic along \( \mathbb{N} \) ("jointly ergodic" according to the terminology of [BB86]), but not uniformly jointly ergodic.

Our example is more general. For instance, we can construct two transformations that are j.e. along a regular sequence of intervals, but not j.e. along \( \mathbb{N} \). Namely, it is enough to find a set \( E_2 \subset \mathbb{N} \) such that \( \bar{d}_N(E_2) > 0 \) but \( \bar{d}_L(E_2) = 0 \) for another regular sequence of intervals \( \mathcal{L} \). Let

\[
E_2 = \bigcup_{n=1}^{\infty} [n^2 - n + 1, n^2],
\]

If we represent the set \( E_2 \) by a sequence of 1’s and 0’s, where 1’s correspond to the elements of \( \mathbb{N} \) that belong to \( E_2 \), then we have

\[
E_2 \sim 10110011100011110000\ldots
\]

Clearly, \( \bar{d}_N(E_2) = \frac{1}{2} \). Now, let

\[
\mathcal{L}_2 = [n^2 + 1, n^2 + n]_{n=1}^{\infty}.
\]
Since $E_2$ is disjoint from all intervals forming the sequence $L_2$, it follows that

$$
\bar{d}_{L_2}(E_2) = 0.
$$

### 5.3 A Partial Ordering on $L$

Let us fix a Lebesgue space $X$. Let $JE([a_n, b_n]_{n=1}^\infty)$ be the set of all pairs of measure-preserving transformations of $X$ that are jointly ergodic along $[a_n, b_n]_{n=1}^\infty$. As we have seen in the previous section, the set $JE([a_n, b_n]_{n=1}^\infty)$ depends on $[a_n, b_n]_{n=1}^\infty$. However, it is clear that $JE([0, n]_{n=1}^\infty) = JE([0, 2n]_{n=1}^\infty)$. What about $JE([0, n]_{n=1}^\infty)$ and $JE([0, n^2]_{n=1}^\infty)$? $JE([0, 2n]_{n=1}^\infty)$?

To address this question, we would like to find a partial ordering on the set of sequences of intervals corresponding to the partial ordering among $JE([a_n, b_n]_{n=1}^\infty)$ given by inclusion. In other words, we would like to find a relation “$>$” on the set of all sequences of intervals such that $JE([a_n, b_n]_{n=1}^\infty) \subseteq JE([a'_n, b'_n]_{n=1}^\infty)$ iff $[a_n, b_n]_{n=1}^\infty > [a'_n, b'_n]_{n=1}^\infty$. The following natural relation happens to work.

**Definition 5.9.** We say that a regular sequence of intervals $L$ is **stronger** than another regular sequence of intervals $L'$ and write $L > L'$, if for every set $E \subset \mathbb{N}$ one has:

$$
\bar{d}_L(E) = 0 \implies \bar{d}_{L'}(E) = 0,
$$

(5.5)

We say that two sequences of intervals, $L$ and $L'$, are **equivalent**, denoted as $L \sim L'$, if we have both $L > L'$ and $L' > L$.

**Lemma 5.10.** Let $L, L' \in L$, $L > L'$. Then for every bounded sequence $(c_n)$ of non-negative real numbers

$$
\lim_{n \to \infty} c_n = 0 \implies \lim_{n \to \infty} c_n = 0
$$

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Proof. Let \((c_n)\) be a bounded sequence of positive real numbers. It can be approximated in \(l_\infty\)-norm from below by finite linear combinations of indicator functions of subsets of \(\mathbb{N}\), i.e. for every \(\varepsilon > 0\) there exist \(r \in \mathbb{N}\), \(E_k \subset \mathbb{N}\), and positive real numbers \(\alpha_k\), \(1 \leq k \leq r\), such that \(c_n = \sum_{k=1}^{r} \alpha_k 1_{E_k}(n) + \varepsilon_n\), where \(0 \leq \varepsilon_n < \varepsilon\). Since \(c_n \geq \alpha_k 1_{E_k}(n)\) for \(1 \leq k \leq r\), it follows that \(d_{\mathcal{L}}(E_k) = C_{\mathcal{L}} \lim_{n \to \infty} 1_{E_k}(n) = 0\) for \(1 \leq k \leq r\). Therefore, because of our assumption, \(d_{\mathcal{L}'}(E_k) = 0\) for all \(1 \leq k < r\).

Now,

\[
C_{\mathcal{L}'} \lim_{n \to \infty} c_n = C_{\mathcal{L}'} \lim_{n \to \infty} \sum_{k=1}^{r} \alpha_k 1_{E_k}(n) + \varepsilon \leq \sum_{k=1}^{r} \alpha_k d_{\mathcal{L}'}(E_k) + \varepsilon = \varepsilon.
\]

Since \(\varepsilon\) is an arbitrary positive number, we are done. \(\square\)

Proposition 5.11. \(\text{JE} (\mathcal{L}) \subseteq \text{JE} (\mathcal{L}')\) iff \(\mathcal{L} \succ \mathcal{L}'\).

Proof. Let \(\mathcal{L} \succ \mathcal{L}'\) and let \((T,S) \in \text{JE} (\mathcal{L})\). To prove that \((T,S) \in \text{JE} (\mathcal{L}')\), it is enough to show that for all \(f,g \in L_0^\infty(X)\) the equation (5.3) holds. According to [BB86, Theorem 2.1], it is enough to prove weak convergence for (5.3) to hold (for invertible transformations it follows form Theorem 2.2). Thus, it remains to show that for all positive real-valued \(f,g \in L_0^2(X)\) and \(h \in L^2(X)\), one has

\[
C_{\mathcal{L}'} \lim_{n \to \infty} \int_X T^m f S^n g h d\mu = 0.
\]

We set

\[
c_n = \int_X T^m f S^n g h d\mu
\]

and apply Lemma 5.10.

The other direction follows from Proposition 5.7. \(\square\)

Remark 5.12. In particular, the notions of joint ergodicity along \(\mathcal{L}\) and \(\mathcal{L}'\) are equivalent iff \(\mathcal{L} \sim \mathcal{L}'\).
5.4 Examples

Example 5.13. Let $\mathcal{L} = [0, 2n]_{n=1}^\infty$. Let us show that $\mathcal{L} \sim \mathbb{N}$. Namely, let $E \subset \mathbb{N}$ with $\bar{d}_\mathbb{N}(E) = 0$. Then $\bar{d}_\mathcal{L}(E) = 0$ because $\mathcal{L}$ is a subsequence of $\mathbb{N}$. Conversely, let $E \subset \mathbb{N}$ with $\bar{d}_\mathcal{L}(E) = 0$. Note that for all $n$ one has

$$\frac{|E \cap [0, n]|}{n} \leq 2 \frac{|E \cap [0, 2n]|}{2n}$$

which implies that $\bar{d}_\mathbb{N}(E) \leq 2\bar{d}_\mathcal{L}(E)$. Hence, $\bar{d}_\mathbb{N}(E) = 0$ and $\mathcal{L} \sim \mathbb{N}$.

Example 5.14. Let $\mathcal{L} = [0, n^2]_{n=1}^\infty$. Then $\mathcal{L} \preceq \mathbb{N}$ since $\mathcal{L}$ is a subsequence of $\mathbb{N}$. Let us show that $\mathcal{L} \succeq \mathbb{N}$ as well. Namely, let $E \subset \mathbb{N}$ with $\bar{d}_\mathcal{L}(E) = 0$. Let $\varepsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ one has

$$\frac{|E \cap [0, n^2]|}{n^2} < \varepsilon.$$

Let $n > \max(n_0, 2)$. If $n^2 \leq m \leq (n + 1)^2$, then

$$\frac{|E \cap [0, m]|}{m} \leq \frac{|E \cap [0, (n + 1)^2]|}{m} < 2 \frac{|E \cap [0, (n + 1)^2]|}{(n + 1)^2} < 2\varepsilon.$$

So, $\mathcal{L} \sim \mathbb{N}$.

Using same arguments, one can show that $[0, n^k]_{n=1}^\infty \sim \mathbb{N}$ and $[0, k^n]_{n=1}^\infty \sim \mathbb{N}$ for every natural number $k \geq 2$. However, it does not work if $\mathcal{L}$ is a very fast growing subsequence of $\mathbb{N}$ as the next example shows.

Example 5.15. Let $\mathcal{L} = [0, n!]_{n=1}^\infty$. It is clear that $\mathcal{L} \preceq \mathbb{N}$. We claim that $\mathcal{L} \not\succeq \mathbb{N}$. Namely, let $E = \bigcup_{n=1}^\infty [n!, 2 \cdot n!]$. To show that $\bar{d}_\mathbb{N}(E) > 0$ it is enough to find a subsequence $\mathcal{M}$ of $\mathbb{N}$ such that $\bar{d}_\mathcal{M}(E) > 0$. Let $\mathcal{M} = [0, 2 \cdot n!]_{n=1}^\infty$. Then $\bar{d}_\mathcal{M}(E) = \frac{1}{2}$ and, therefore, $\bar{d}_\mathbb{N}(E) \geq \frac{1}{2}$. On the other hand,

$$\frac{|E \cap [0, (n + 1)!]|}{(n + 1)!} \leq \frac{|[0, 2 \cdot n!] \cap [0, (n + 1)!]|}{(n + 1)!} = \frac{2n!}{(n + 1)!} \xrightarrow{n \to \infty} 0.$$
Hence, $d_{\mathcal{L}}(E) = 0$.

**Example 5.16.** Let $\mathcal{L} = [n, n + \lceil \sqrt{n} \rceil]_{n=1}^{\infty}$. We claim that $\mathcal{N} \not\sim \mathcal{L}$. Indeed, let

$$E = \bigcup_{n=1}^{\infty} [n!, n! + \sqrt{n!}].$$

Then $d_{\mathcal{L}}(E) = 1$, but $d_{\mathcal{N}}(E) = 0$. Hence, $\mathcal{N} \not\sim \mathcal{L}$.

To show that $\mathcal{N} \prec \mathcal{L}$, let $E \subset \mathcal{N}$ with $\bar{d}_{\mathcal{N}}(E) > 0$. We need to show that $\bar{d}_{\mathcal{L}}(E) > 0$. Indeed, since $d_{\mathcal{N}}(E) > 0$, there exists $c > 0$ such that

$$\left| \left[ 0, m \right] \cap E \right| > c$$

for an unbounded increasing sequence of numbers $m$. For every $m$ one can represent the interval $[0, m]$ as a union of disjoint intervals of the form $[n_k, n_k + \lceil \sqrt{n_k} \rceil]$:

$$[1, m] = [n_1, n_1 + \lceil \sqrt{n_1} \rceil] \cup \cdots \cup [n_r, n_r + \lceil \sqrt{n_r} \rceil].$$

A simple estimate shows that the number of the intervals on the right $r(m) \leq 3\sqrt{m}$.

Since

$$\left| \left[ 0, m \right] \cap E \right| = \left| \left[ n_1, m_{11} + \lceil \sqrt{n_1} \rceil \right] \cap E \right| \cdots + \left| \left[ n_r, m_{1r} + \lceil \sqrt{n_r} \rceil \right] \cap E \right| > c,$$

there exists an index $n_k(m)$ such that

$$\left| \left[ n_k(m), n_k(m) + \lceil \sqrt{n_k(m)} \rceil \right] \cap E \right| > \frac{c}{3\sqrt{m}}$$

and, therefore,

$$\left| \left[ n_k(m), n_k(m) + \lceil \sqrt{n_k(m)} \rceil \right] \cap E \right| > \frac{c}{3}. $$

It is clear that $\lim_{m \to \infty} k(m) = \infty$. Thus, we obtained a subsequence

$$[n_k(m), n_k(m) + \lceil \sqrt{n_k(m)} \rceil]_{m=1}^{\infty}.$$
of \( L \) such that
\[
\frac{|[n_{k(m)}, n_{k(m)} + \sqrt{n_{k(m)}}] \cap E|}{\sqrt{n_{k(m)}}} \geq \frac{|[n_{k(m)}, n_{k(m)} + \sqrt{n_{k(m)}}] \cap E|}{\sqrt{m}} > \frac{c}{3},
\]
which means that \( \overline{d}_L(E) > 0 \).

**Example 5.17.** Let us consider the subsequence \([n^2, n^2 + n]_{n=1}^{\infty}\) of \([n, n + \sqrt{n}]_{n=1}^{\infty}\).

It is easy to see that \([n^2, n^2 + n]_{n=1}^{\infty} \not\preccurlyeq [0, n]_{n=1}^{\infty}\) and \([n^2, n^2 + n]_{n=1}^{\infty} \not\succcurlyeq [0, n]_{n=1}^{\infty}\).

Let us summarize our examples below:

\[
[0, n]_{n=1}^{\infty} \sim [0, n^2]_{n=1}^{\infty} \sim [0, 2^n]_{n=1}^{\infty} \not\prec [0, n]_{n=1}^{\infty}.
\]

\[
[0, n]_{n=1}^{\infty} \not\succcurlyeq [n, n + \sqrt{n}]_{n=1}^{\infty}.
\]

\[
[0, n]_{n=1}^{\infty} \perp [n^2, n^2 + n]_{n=1}^{\infty}.
\]

**Remark 5.18.** The example of \([n, n + \sqrt{n}]_{n=1}^{\infty}\) is especially interesting since this is a regular sequence of intervals that is universally bad for the pointwise ergodic theorem [AdJ75, RW92]. We shall discuss connections between our results and pointwise convergence of Cesàro averages for a single transformation later in this chapter.

### 5.5 Some Properties of \((L, \prec)\)

**Proposition 5.19.** Let \( \mathcal{L} \in L \). Then there exist \( \mathcal{L}', \mathcal{L}'' \in L \) such that

\[
\mathcal{L}' \not\succcurlyeq \mathcal{L} \not\preccurlyeq \mathcal{L}''.
\]

**Proof.** Let \( \mathcal{L} = (L_n) \in L \) be given. Let us first construct \( \mathcal{L}' \not\succcurlyeq \mathcal{L} \). Namely, for each \( n \) let \( L'_n \) be any subinterval of \( L_n \) such that \(|L'_n| = \left\lceil \frac{1}{2} |L_n| \right\rceil \). Then the resulting sequence \( \mathcal{L}' = (L'_n) \) is not equivalent to \( \mathcal{L} \). Namely, for \( E = \bigcup_{n=1}^{\infty} L_n \setminus \bigcup_{n=1}^{\infty} L'_n \) one has \( \overline{d}_L(E) = 1/2 \) whereas \( \overline{d}_{\mathcal{L}'}(E) = 0 \). On the other hand, for every \( E \subset \mathbb{N} \) and \( \varepsilon > 0 \)

\[
\frac{|L'_n \cap E|}{|L'_n|} \leq \frac{|L_n \cap E|}{|L_n|} \leq (2 + \varepsilon) \frac{|L_n \cap E|}{|L_n|} \quad \text{for } n \text{ big enough}
\]

\[
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\]
and, therefore, $\bar{d}_{L'}(E) \leq 2\bar{d}_E(E)$. So, $L \not\sim L'$.

To construct $L''$, let us first construct an auxiliary regular sequence of intervals $M = (M_n)$. Let $n_1 = 1$ and $M_1 = L_{n_1}$. Let $n_2$ be the smallest index such that $|L_{n_2}| \geq 8$ and such that $L_{n_2} \setminus L_{n_1}$ contains an interval of length 2. Let $M_2$ to be such a subinterval. If $M_1, \ldots, M_{k-1}$ are already chosen, let $n_k$ be the smallest index such that $|L_{n_k}| \geq k^3$ and $L_{n_k} \setminus \bigcup_{i=1}^{n_k-1} L_i$ contains a subinterval of length $k$. Let $M_k$ be such a subinterval. Then, as one can easily check, $\bar{d}_L(\bigcup_{k=1}^{\infty} M_k) = 0$, whereas, certainly, $\bar{d}_M(\bigcup_{k=1}^{\infty} M_k) = 1$ and so $M \not\preceq L$. Now, we define $L''$ by setting

$$L''_n = \begin{cases} L_k, & \text{if } n = 2k, \\ M_k, & \text{if } n = 2k + 1 \end{cases}$$

Then $L'' \preceq L$ and $L'' \preceq M$. Because $M \not\preceq L$, it follows that $L'' \not\preceq L$ and, therefore, $L'' \not\sim L$.

The method used to build $L''$ from $L$ and $M$ can be generalized. Let $L', L'' \in L$. We define $L' + L''$ to be the regular sequence $L$ with elements

$$L_n = \begin{cases} L'_k, & \text{if } n = 2k, \\ L''_k, & \text{if } n = 2k + 1 \end{cases}$$

So defined binary operation “$+$” is commutative and associative up to equivalence $\sim$. It is clear that $L' + L'' \succ L', L''$. Moreover, it is the weakest sequence that has this property.

**Proposition 5.20.** Let $L', L'' \in L$. Then

1. $L' + L'' \succ L'$ and $L' + L'' \succ L''$,

2. if $L \succ L'$ and $L \succ L''$, then $L \succ L' + L''$.
Proof. (1) Let \( E \subset \mathbb{N} \) be such that \( \bar{d}_{\mathcal{L}'+\mathcal{L}''}(E) = 0 \). Then \( E \) has 0 density along every subsequence of \( \mathcal{L}' + \mathcal{L}'' \), in particular, along \( \mathcal{L}' \) and \( \mathcal{L}'' \).

(2) Let \( \bar{d}_{\mathcal{L}(E)} = 0 \). Then \( \bar{d}_{\mathcal{L}'}(E) = 0 \) and \( \bar{d}_{\mathcal{L}''}(E) = 0 \), i.e. for every \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \) one has

\[
\frac{|E \cap L'_n|}{|L'_n|} < \varepsilon \quad \text{and} \quad \frac{|E \cap L''_n|}{|L''_n|} < \varepsilon.
\]

Let \( \mathcal{L}' + \mathcal{L}'' = (M_n) \). Since for all \( n \) either \( M_n = L'_n \) or \( M_n = L''_n \), it follows that for all \( n > n_0 \)

\[
\frac{|E \cap M_n|}{|M_n|} < \varepsilon.
\]

\[\square\]

**Corollary 5.21.** For all \( \mathcal{L}', \mathcal{L}'' \in \mathcal{L} \), one has

\[
\text{JE} (\mathcal{L}' + \mathcal{L}) = \text{JE} (\mathcal{L}') \cap \text{JE} (\mathcal{L}'').
\]

One can similarly define the sum \( \sum_{k=1}^{\infty} \mathcal{L}^{(k)} \) of a countable set of regular sequences \( \mathcal{L}^{(k)}, k \in \mathbb{N} \). Namely, if

\[
\mathcal{L}^{(1)} = (L^{(1)}_1, L^{(1)}_2, L^{(1)}_3, \ldots)
\]

\[
\mathcal{L}^{(2)} = (L^{(2)}_1, L^{(2)}_2, L^{(2)}_3, \ldots)
\]

\[
\mathcal{L}^{(3)} = (L^{(3)}_1, L^{(3)}_2, L^{(3)}_3, \ldots)
\]

\[
\vdots
\]

we set \( \sum_{k=1}^{\infty} \mathcal{L}^{(k)} = (L^{(1)}_1, L^{(1)}_2, L^{(1)}_3, L^{(2)}_1, L^{(2)}_2, L^{(2)}_3, \ldots) \). It is easy to see that

\[
\sum_{k=1}^{\infty} \mathcal{L}^{(k)} \succ \mathcal{L}^{(k)}
\]

for all \( k = 1, 2, \ldots \).

We have established some important facts about the set \( \mathcal{L} \).

**Theorem 5.22.** (1) The set \( \mathcal{L} \) has neither minimal nor maximal element with respect to the partial ordering \( \prec \).
(2) The factor set $\mathcal{L}/\sim$ is uncountable.

**Proof.** (1) follows from Proposition 5.19. (2) follows from the following observation. If $\mathcal{L}^{(k)}, k \in \mathbb{N}$ contains representatives of all elements of $\mathcal{L}/\sim$, then $\sum_{k=1}^{\infty} \mathcal{L}^{(k)}$ is another regular sequence of intervals that is not equivalent to any of $\mathcal{L}^{(k)}$. \qed

We have seen that for every finite or countable set $A \subset \mathcal{L}$ there exists $\mathcal{L} \in \mathcal{L}$ that is stronger than all elements of $A$. Interestingly, one can not always find an element of $\mathcal{L}$ that is weaker than two given elements of $\mathcal{L}$.

**Example 5.23.** Let

$$\mathcal{L}' = (L'_n) = [n^2, n^2 + n]_{n=1}^{\infty} \quad \text{and} \quad \mathcal{L}'' = (L''_n) = [n^2 + n + 1, n^2 + 2n]_{n=1}^{\infty}.$$  

Let

$$E' = \bigcup_{n=1}^{\infty} L'_n \quad \text{and} \quad E'' = \bigcup_{n=1}^{\infty} L''_n.$$  

Then $\tilde{d}_{\mathcal{L}'}(E) = 0$ for every $E \subset E'$ and $\tilde{d}_{\mathcal{L}''}(E) = 0$ for every $E \subset E''$. So, if $\mathcal{L} \prec \mathcal{L}', \mathcal{L}''$, then one would have $\tilde{d}_{\mathcal{L}}(E) \leq \tilde{d}_{\mathcal{L}}(E \cap E') + \tilde{d}_{\mathcal{L}}(E \cap E'') = 0$ for every $E \subset \mathbb{N}$, which is impossible. Therefore, there is no $\mathcal{L}$ such that $\mathcal{L} \prec \mathcal{L}', \mathcal{L}''$.

**Remark 5.24.** Example 5.23 is an example of two sequences of intervals $\mathcal{L}'$ and $\mathcal{L}''$ for which there is no regular sequence of intervals $\mathcal{L}$ with $JE(\mathcal{L}') \cup JE(\mathcal{L}'') \subset JE(\mathcal{L})$.

### 5.6 Other Types of Convergence

The partial ordering $\prec$ and the equivalence relation $\sim$ on the set of all sequences of intervals that we defined above were induced by the inclusion relation between classes of systems that jointly ergodic along the sequences of intervals.
If we consider types of convergence different from (5.3), we might obtain different types of partial ordering and equivalence relations on $L$. In view of Lemma 5.10, one would expect that our equivalence relation “$\sim$” is the finest among this type of equivalence relation, i.e. the equivalence classes defined by a different type of convergence of ergodic averages would be unions of the equivalence classes defined by the relation “$\sim$”. Of special interest is pointwise convergence of Cesàro averages for a single transformation.

**Definition 5.25.** An invertible measure-preserving transformation $T$ of a Lebesgue space $X$ is called pointwise ergodic along a regular sequence of intervals $L$ (in short, $T \in \text{PWE}(L)$) if for all functions $f \in L^1_0(X)$ one has:

$$
\frac{1}{b_n - a_n} \sum_{k=a_n}^{b_n} T^k f(x) \xrightarrow[n \to \infty]{} 0 \text{ for a.e. } x.
$$

(5.6)

The relation “$\text{PWE}(L) \subseteq \text{PWE}(L')$” also defines a partial ordering and an equivalence relation on the set of all sequences of intervals which we will denote as $\succ_p$ and $\sim_p$ respectively.

In view of Birkhoff’s Pointwise Ergodic Theorem (Theorem 1.8), among these equivalence classes there exists a global minimal element — the equivalence class corresponding to the sequence of intervals $N$. Interestingly enough, there is also a global maximal element — the equivalence class corresponding to the sequence $[n, n + \lceil \sqrt{n} \rceil]_{n=1}^\infty$. This is because for every invertible transformation there exists $f \in L^1(X)$ such that (5.6) does not converges a.e. along this sequence ([AdJ75]).
The natural question to ask: Is there anything between these two? The short answer is “almost nothing”. Roughly speaking, PWE (\(L\)) is non-empty iff the regular sequence of intervals \(L\) satisfies Shulman’s Condition.

More precisely, according to [RW92], if \([a_n, b_n]_{n=1}^\infty\) satisfies Shulman’s condition, then PWE \(([a_n, b_n]_{n=1}^\infty)\) contains all ergodic transformations. If a regular sequence of intervals \([a_n, b_n]_{n=1}^\infty\) does not satisfy Shulman’s Condition and the sequence of numbers \((b_n - a_n)\) is increasing, then PWE \(([a_n, b_n]_{n=1}^\infty)\) is empty. If the sequence \(b_n - a_n\) is not increasing, but \([a_n, b_n]_{n=1}^\infty\) satisfies a stronger condition:

\[
\lim_{n \to \infty} \frac{[a_n, b_n] - \bigcup_{i=1}^{n-1} [a_i, b_i]}{b_n - a_n} = \infty,
\]

then PWE \(([a_n, b_n])\) is also empty.

The question of how big the set PWE \(([a_n, b_n]_{n=1}^\infty)\) can be for those sequences of intervals that fall into neither of the three above categories remains open, as far as we know.

Let us note, however, that if we consider only sequences of intervals with increasing length, then the set \(L/\sim_p\) contains only two elements — the maximal and the minimal elements.

### 5.7 Other Types of Sequences

We studied convergence along sequences of regular intervals of non-negative integers. However, one may ask: why not to consider convergence along different (not necessarily wider) classes of sequences of finite sets. Let us mention a few of the possibilities:

1. all Følner sequences in \(N\)
(2) sequences of intervals or Følner sequences in \( \mathbb{Z} \).

(3) nested sequences of finite sets of the form \( \{ p_1, p_2, \ldots, p_k \} \) where \( p_1 < p_2 < \cdots < p_k \).

(4) all sequences of finite subsets of \( \mathbb{N} \)

(5) all Følner sequences in a countable amenable group \( G \)

In this section, we shall discuss how replacing “all sequences of intervals” by “all Følner sequences in \( \mathbb{N} \)” changes our results. We shall also mention, what happens if we consider sequences of intervals in \( \mathbb{Z} \) instead of \( \mathbb{N} \). Extending our results to classes of sequences (3) – (6) is very interesting, but beyond the scope of this work.

Let \( F \) be the set of all Følner sequences in \( \mathbb{N} \). Clearly, \( L \subset F \). As one can easily see, all of the facts that we have proved about \( L \) (Lemma 5.10 – Theorem 5.22) hold true for \( F \) without any changes. However, there remains two important questions:

(1) Are \( L/\sim \) and \( F/\sim \) identical? In other words, is it true that for every Følner sequence \( \mathcal{F} \) there exits a regular sequence of intervals \( \mathcal{L} \) such that \( \mathcal{L} \sim \mathcal{F} \)?

(2) Do the classes of uniformly convergent objects change if we move from sequences of intervals to all Følner sequences? For instance: Is the set of all pairs of transformations that are jointly ergodic along every regular sequence of intervals identical to the set of all pairs of transformations that are jointly ergodic along all Følner sequences?

It turns out that the answer to the first question is negative, but the answer to the second question is positive, which means that for most practical purposes sequences of intervals are as good as Følner sequences in \( \mathbb{Z} \).
To address these two questions, we need a characterization of all Følner sequences in $\mathbb{N}$.

Let $H \subset \mathbb{N}$ be a finite set. Then $H$ consists of a finite number of non-adjacent intervals, so that one can write

$$H = [a_1, b_1] \cup \cdots \cup [a_n, b_n]$$

and this decomposition is unique. Let

$$\ell(H) = \min_{k=1,\ldots,n} (b_k - a_k). \quad (5.7)$$

We call the number $\ell(H)$, the *footprint* of the set $H$.

A sequence of finite sets $(F_n)$ in $\mathbb{N}$ is a Følner sequence iff it can be represented as $F_n = G_n \cup H_n$ so that

1. $\ell(G_n) \to \infty$ as $n \to \infty$,
2. $|H_n|/|F_n| \to 0$ as $n \to \infty$.

(For a proof, see Appendix D, Lemma D.1.)

Now we are in a position to prove the following theorem.

**Theorem 5.26.** For every $\mathcal{F} \in \mathcal{F}$ there exists $\mathcal{L} \in \mathcal{L}$ such that $\mathcal{L} \succ \mathcal{F}$.

**Proof.** Let $\mathcal{F} = (F_n) \in \mathcal{F}$. In view of Lemma D.1, there exist sequences $\mathcal{G} = (G_n)$ and $\mathcal{H} = (H_n)$ of finite subsets of $\mathbb{N}$ such that $(F_n) = (G_n) \cup (H_n)$ and

1. $\ell(G_n) \to \infty$ as $n \to \infty$,
2. $|H_n|/|F_n| \to 0$ as $n \to \infty$. 

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Each set $G_n$ consists of a finite number, say $r(n)$, of non-adjacent intervals. Let us denote them as $I_{n,1}, I_{n,2}, \ldots, I_{n,r(n)}$. For every $E \in \mathbb{N}$ with $\bar{d}_F(E) > 0$ we are going to give a method of how to choose one of these intervals, we shall call this interval $I_{n,k_E(n)}$, out of each set $G_n$, so that the resulting regular sequence of intervals $L_E = (I_{n,k_E(n)})_{n=1}^\infty$ is such that $\bar{d}_{L_E}(E) > 0$. Namely, let $E \in \mathbb{N}$ with $\bar{d}_F(E) > 0$ be given. WLOG we may assume that $d_F(E) > 0$ (just go to a subsequence). Then there exist $n_0 \in \mathbb{N}$ and $\alpha > 0$ such that $|E \cap G_n|/|G_n| > \alpha$ for all $n > n_0$. We claim that for each $n > n_0$ there exists an index $k_E(n)$ such that $|E \cap I_{n,k_E(n)}|/|I_{n,k_E(n)}| > \alpha$. Namely, if not, then $|E \cap I_{n,k}| |I_{n,k}| < \alpha$ for some $n > n_0$ and $1 \leq k \leq r(n)$. But then

$$\frac{|E \cap G_n|}{|G_n|} = \frac{|E \cap I_{n,1}| + \cdots + |E \cap I_{n,r(n)}|}{|I_{n,1}| + \cdots + |I_{n,r(n)}|} < \alpha.$$}

Contradiction. This finishes the construction of the sequence $L_E$. It is a regular sequence of intervals with $b_n - a_n \to \infty$ because of the property (1) of the sequence $\mathcal{G}$.

Now, we are ready to construct the sequence $L$ we are looking for. Take

$$L = I_{1,1}, I_{1,2}, \ldots, I_{1,r(1)}, \ldots, I_{n,1}, I_{n,2}, \ldots, I_{n,r(n)}, \ldots$$

From what we have just proved it follows that for every $E \in \mathbb{N}$ with $\bar{d}_F(E) > 0$ there exists a subsequence $n_k$ such that $\lim_{k \to \infty} |E \cap L_{n_k}|/|L_{n_k}| > 0$. Therefore, $\lim_{n \to \infty} |E \cap L_n|/|L_n| > 0$. \hfill \Box

Remark 5.27. Theorem 5.26 gives a positive answer to our second question on the connection between $F$ and $L$. Namely, it shows that for every Følner sequence $\mathcal{F}$, there exists a regular sequence of intervals along which convergence is at least as hard as along $\mathcal{F}$. Therefore, a sequence convergent along all Følner sequences is no
better than a sequence convergent along all sequences of intervals. In particular, if one defines a new “uniform” density as

\[ d_F(E) = \sup_{L \in F} \bar{d}_L(E), \]

then for \( E \subseteq \mathbb{Z} \) one has \( d_F(E) = 0 \) iff \( d^*(E) = 0 \). (The question of whether always \( d_F(E) = d^*(E) \) is more subtle.)

Using the method employed in the proof of Theorem 5.26, one can similarly show that for all \( F \subseteq \mathbb{F} \) and every \( E \subseteq \mathbb{N} \) there exists \( L'_E \in \mathbb{L} \) such that \( \bar{d}_{L'_E}(E) \leq \bar{d}_F(E) \).

But one can not use it to construct a uniform sequence \( L' \in \mathbb{L} \) such that \( L' \prec F \).

Example 5.28. Let \( F_n = \bigcup_{m=1}^n I_{n,m} \), where \( I_{n,m} = [n^5 + mn^3, n^5 + mn^3 + n^2]_{n=1}^\infty \). Then \( F = (F_n) \) is a Følner sequence. We claim that there is no \( L \in \mathbb{L} \) such that \( L \prec F \).

Namely, assume \( L = (L_n) \) is such a sequence. Let \( A = \bigcup_{n=1}^\infty F_n \). Since \( L \prec F \), it follows that for \( k \) big enough \( L_k \cap A \neq \emptyset \) and, moreover, \( |L_k \cap A|/|L_k| \rightarrow 1 \) as \( k \rightarrow \infty \).

For big \( k \) the distance between consecutive intervals forming the sets \( F_k \) is bigger than the length of a single interval and, therefore, for big \( k \) each \( L_k \) intersects one and only one of the sets \( I_{n,m} \), say, \( I_{n(k),m(k)} \), and one has \( \lim_{k \rightarrow \infty} |L_k \cap I_{n(k),m(k)}|/|L_k| = 1 \). Now, let us choose a subsequence \( (L_{s_k}) \) of \( L \) such that \( (L_{s_k}) \) intersects each set \( F_n \) no more than once. Then \( |L_{s_k}|/|F_n(k)| \rightarrow 0 \) as \( k \rightarrow \infty \) and, consequently, \( \bar{d}_F(\bigcup_{k=1}^\infty L_{s_k}) = 0 \).

On the other hand, obviously, \( d_L(\bigcup_{k=1}^\infty L_{s_k}) = 1 \). Contradiction.

Remark 5.29. This examples gives a negative answer to our first question — there exists \( F \in \mathbb{F} \) that is not equivalent to any regular sequence of intervals.

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Finally, let us discuss how our results may be generalized for sequences of intervals in \( \mathbb{Z} \). One can suggest two definitions for partial ordering of regular sequences of intervals in \( \mathbb{Z} \).

1. For two regular sequence of intervals \( L \) and \( L' \) in \( \mathbb{Z} \) we shall write \( L \succ_z L' \), if for every set \( E \subset \mathbb{Z} \) one has:

\[
\bar{d}_L(E) = 0 \implies \bar{d}_{L'}(E) = 0,
\]

2. We say that a regular sequence of intervals \( L \) in \( \mathbb{Z} \) is \textit{stronger} than another regular sequence of intervals \( L' \) and write \( L \succ_s L' \), if for every symmetric set \( E \subset \mathbb{Z} \) one has:

\[
\bar{d}_L(E) = 0 \implies \bar{d}_{L'}(E) = 0.
\]

Which of the two definitions is the “right” one?

If we use the first definition, then we can prove an analogue of Lemma 5.10 and, therefore,

\[
[a_n, b_n]_{n=1}^\infty \succ_z [a'_n, b'_n]_{n=1}^\infty \implies \text{JE} ([a_n, b_n]_{n=1}^\infty) \subseteq \text{JE} ([a'_n, b'_n]_{n=1}^\infty). \quad (5.8)
\]

On the other hand, if we use the second definition, then we can make use of our universal counterexample and, therefore,

\[
[a_n, b_n]_{n=1}^\infty \succ_s [a'_n, b'_n]_{n=1}^\infty \iff \text{JE} ([a_n, b_n]_{n=1}^\infty) \subseteq \text{JE} ([a'_n, b'_n]_{n=1}^\infty). \quad (5.9)
\]

It remains an open question whether the inverse implications for (5.8) or (5.9) hold.
APPENDIX A

SOME FACTS FROM THE THEORY OF SEMIGROUPS

In this appendix, we give some definitions and results from semigroups theory and about embedding semigroups into groups. Most of these results can be found in either [Lja63] or [CP61].

Definition A.1. A semigroup is a non-empty set $S$, in which for every ordered pair of elements $x, y \in S$ there is another element called their product $u = xy \in S$, so that for all $x, y, z \in S$ we have

$$(xy)z = x(yz).$$

Definition A.2. Let $S_1$ and $S_2$ be semigroups. A mapping $\psi : S_1 \to S_2$ is called a homomorphism if $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in S_1$. A 1-1 homomorphism is called an isomorphism.

Definition A.3. A semigroup $S$ is called a group if for all $a, b \in S$ there exist $x$ and $y$ in $S$, such that

$$xa = b, \quad ay = b.$$ 

Definition A.4. An equivalence relation $\sim$ on $S$ is called a congruence if for all $x_1, x_2, y_1, y_2 \in S$, one has:

$$x_1 \sim x_2, \quad y_1 \sim y_2 \implies x_1 y_1 \sim x_2 y_2.$$
Proposition A.5. An equivalence relation $\sim$ in a semigroup $S$ is a congruence iff for all $a, b, s \in S$ one has:

$$a \sim b \implies as \sim bs \text{ and } sa \sim sb.$$  \hspace{1cm} (A.1)

Proof. Assume that $\sim$ is a congruence in $S$. Let $a \sim b$ and $s \in S$. Since $s \sim s$, then by definition, $as \sim bs$ and $sa \sim sb$.

Conversely, assume that for all $a, b, s \in S$, (A.1) holds. Let $a \sim b$ and $c \sim d$. Then $ac \sim bc$ and $bc \sim bd$. By transitivity of $\sim$, it follows that $ac \sim bd$. \hfill \Box

Proposition A.6. Let $\sim$ be a congruence in $S$. Then the factor-set $\overline{S} = S/\sim$ is a semigroup homomorphic to $S$.

Proof. Let $X, Y, Z \in \overline{S}$. We claim that

$$XY \cap Z \neq \emptyset \implies XY \subset Z,$$  \hspace{1cm} (A.2)

where multiplication of two subsets $A, B$ of $S$ is defined as

$$AB \overset{\text{def}}{=} \{ab \mid a \in A, b \in B\}.$$ 

Indeed, assume that $x_1 \in X$, $y_1 \in Y$, and $x_1y_1 \in Z$. Let $x_2 \in X$ and $y_2 \in Y$ be some other elements. We know that $x_1 \sim x_2$ and $y_1 \sim y_2$. By the definition of congruence, it follows that $x_1y_1 \sim x_2y_2$ and, therefore, $x_2y_2 \in Z$.

Since the equivalence classes are pairwise disjoint, it follows from (A.2) that for every $X, Y \in \overline{S}$ there exists a unique $Z \in \overline{S}$ such that $XY \subset Z$.

Let us define an operation $\circ$ in $\overline{S}$ as follows:

$$X \circ Y = Z \text{ if } XY \subset Z.$$ 

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As the above discussion shows, this operation is well-defined. It is trivial to see that the operation \( \circ \) is associative.

The natural projection \( h : S \to \overline{S} \) is a homomorphism with respect to so defined semigroup operation in \( \overline{S} \). Indeed, if \( Z = h(xy) \), then \( xy \in Z \) and so \( h(x) \circ h(y) = Z \).

**Definition A.7.** A nonempty set \( I \) is a right ideal of a semigroup \( S \) if \( IS \subset I \). A nonempty set \( I \) is a left ideal of a semigroup \( S \) if \( SI \subset I \).

**Definition A.8.** A semigroup \( S \) is said to be left (right) reversible if every pair of right (left) ideals have a non-empty intersection.

In other words, \( S \) is left reversible if for all \( a, b \in S \) there exist \( u, v \in S \) such that \( au = bv \).

**Definition A.9.** A semigroup \( S \) is called left cancellative if for all \( x, a, b \in S \) one has

\[ xa = xb \implies a = b. \]

The semigroup is called right cancellative if for all \( x, a, b \in S \) one has

\[ ax = bx \implies a = b. \]

**Proposition A.10.** Let \( S \) be a left reversible semigroup and let \( \approx \) be the relation on \( S \) defined by setting \( x \approx y \) if there exists \( s \in S \) for which \( xs = ys \). Then \( \approx \) is a congruence on \( S \) and the semigroup \( \overline{S} = S/\approx \) is right cancellative.

*Proof.* It is clear that the relation \( \approx \) is reflexive and symmetric. To show that it is transitive, assume that \( a \approx b \) and \( b \approx c \). Then there exist \( s, r \in S \) such that \( as = bs \) and \( br = cr \). Since \( S \) is left reversible, there exists \( u, v \in S \) such that \( au = bv \) and \( br = cr \) so

\[ as = bs = auu^{-1}bu = avv^{-1}bv = ar = cr = bs. \]

Thus \( a \approx c \) and \( \approx \) is transitive. Therefore, \( \approx \) is a congruence on \( S \). The semigroup \( \overline{S} = S/\approx \) is right cancellative since for all \( x, a, b \in \overline{S} \) one has

\[ xa = xb \implies x \approx y \implies a = b. \]
and $br = cr$. Since $S$ is left reversible, there exist $u, v \in S$ such that $su = rv$. Then $asu = bsu = brv = crv = csu$ and thus $a \approx c$.

To show that $\approx$ is a congruence, according to Proposition A.5, it is enough to show that for all $s \in S$

$$a \approx b \implies as \approx bs \text{ and } sa \approx sb.$$ 

Let $a \approx b$. This means that $ar = br$ for some $r \in S$. It follows that $sar = sbr$ for all $s \in S$ and so $sa \approx sb$. Now, by left reversibility, there exist $u, v \in S$ such that $ru = sv$. Then $asv = aru = bru = bsr$. Therefore, $as \approx bs$. So, $\approx$ is a congruence in $S$ and, by Proposition A.6, the factor set $\overline{S} = S/\approx$ is a semigroup.

Assume that for some $X_1, X_2$, and $Y$ in $\overline{S}$, one has $X_1Y = X_2Y$. Then for every $x_1 \in X$ and $y \in Y$ there exists $x_2 \in X$ and $y' \in Y$ such that $x_1y = x_2y'$. Since $y \approx y'$, there exists $s \in S$ such that $ys = y's$ and so $x_1ys = x_2y's = x_1ys$. Hence, $x_1 \approx x_2$. It follows that $X_1 = X_2$. Consequently, the semigroup $\overline{S}$ is right cancellative. □

Now, we define so called partial transformations that will be used in our subsequent constructions. Let $\Omega$ be a set and let $A, B \subseteq \Omega$. Then a mapping $h$ from $A$ onto $B$ is called a partial transformation or partial mapping of the set $\Omega$. We use the following notation: $\text{Dom } h = A$ and $\text{Im } h = B$. If $h_1$ and $h_2$ are two partial transformations of $\Omega$, we may define their product to be the partial transformation $h = h_1h_2$ with $\text{Dom } h = \{ x \in \text{Dom } h_2 \mid h_1(x) \in \text{Dom } h_1 \}$ and such that $h(x) = h_1(h_2(x))$. This operation defines a semigroup structure among partial transformations.

Now, let $\mathcal{H}$ be a semigroup of one-to-one partial mapping of some set $\Omega$ such that for every $h \in \mathcal{H}$, the inverse transformation $h^{-1}$ is in $\mathcal{H}$ as well. Note that $h^{-1}$ is an inverse transformation and does not represent an inverse element to $h$ with respect
to the semigroup multiplication operation. It is easy to see that $H$ does not have to be a group. In what follows we construct a factor-group of the semigroup $H$.

First, we define a partial ordering in $H$. We say that $h_1 \prec h_2$ if $\text{Dom } h_1 \subset \text{Dom } h_2$ and $h_1(x) = h_2(x)$ for all $x \in \text{Dom } h_1$.

Now, we define another relation in $H$: we write that $h_1 \sim h_2$ if there exists $t \in H$ such that $t \prec h_1$ and $t \prec h_2$. It is clear that so defined relation $\sim$ is reflexive and symmetric. It is also transitive. Namely, let $h_1 \sim h_2$ and $h_2 \sim h_3$. Then there exist $t, s \in H$ such that $t \prec h_1$, $t \prec h_2$, $s \prec h_2$, and $s \prec h_3$. Let $r = ts^{-1}s$. Then $\text{Dom } r \subset \text{Dom } t$ and $\text{Dom } r \subset \text{Dom } s$. Also, $\forall x \in \text{Dom}(r)$, $r(x) = t(x) = h_2(x) = s(x)$. Hence, $r \prec t \prec h_1$ and $r \prec s \prec h_3$. Therefore, $h_1 \sim h_3$.

Thus, the relation $\sim$ is an equivalence relation. Moreover, as one can easily check, it is a congruence. Hence, by Proposition A.6, $\tilde{H} = H/\sim$ is a semigroup.

We claim that $\tilde{H}$ is actually a group. Indeed, let $a, b \in \tilde{H}$. We need to show that there exist $x, y \in \tilde{H}$ such that $xb \sim a$ and $yb \sim a$, which would imply that $\tilde{x}b = \tilde{a}$ and $\tilde{y}b = \tilde{a}$. Indeed, let $x = ab^{-1}$ and $y = b^{-1}a$. Then $xb = ab^{-1}b < a$ and $by = bb^{-1}a < a$. Thus, according to Definition A.3, $\tilde{H}$ is a group.

Let us summarize our results in a lemma.

**Lemma A.11.** Let $H$ be a semigroup of one-to-one partial mappings of a set $\Omega$ such that for every $h \in H$, the inverse transformation $h^{-1}$ is also in $H$. Let the equivalence relation $\sim$ in $H$ be defined as described above. Then $H/\sim$ is a group.

We use this lemma to prove the following theorem due to Ore [Ore31].

**Theorem A.12.** Let $S$ be a left reversible semigroup with two-sided cancellation. Then $S$ is embeddable in a group.
Proof. Let \( S \) be the semigroup of all partial one-to-one transformations of \( S \). For every \( a \in S \) we define \( s_a \in S \) to be the partial transformation such that
\[
\text{Dom } s_a = S, \quad s_a(x) = ax \quad (\forall x \in S).
\]
The transformation \( s_a \) is one-to-one since the semigroup \( S \) is cancellative.

Let us consider the subsemigroup \( H \) of \( S \), generated by all \( s_a \) and \( s_a^{-1} \) for all \( a \in S \). A general element of \( H \) has the form
\[
h = h_1 h_2 \ldots h_m, \quad \text{where } h_i = s_{a_i} \text{ or } h_i = s_{a_i}^{-1}, \ i = 1, \ldots, m.
\]
Since the partial transformation \( h^{-1} = h_m^{-1} \ldots h_1^{-1} \) also belongs to \( H \), we may apply Lemma A.11 and conclude that the factor semigroup \( \tilde{H} = H/\sim \) is a group.

We need one more fact about the semigroup \( \mathcal{K} \) to finish the proof. Namely, we need to know that for every \( h \in \mathcal{K} \), \( \text{Dom } h \neq \emptyset \). We present the proof of this fact as Lemma A.15 below. Note that this is the only place where we use left reversibility.

Let \( \eta : \mathcal{K} \rightarrow \tilde{H} \) be the natural homomorphism. Let \( \psi : S \rightarrow \mathcal{K} \) be the homomorphism defined as \( \psi(a) = s_a \). Consider the homomorphism \( \xi = \eta \psi \). This is a homomorphism from the semigroup \( S \) into the group \( \tilde{H} \). We claim that \( \xi \) is one-to-one. Indeed, suppose that for some \( a, b \in S, a \neq b \) we have \( \xi(a) = \xi(b) \). This means that \( s_a \sim s_b \), i.e. there exists a partial transformation \( t \in \mathcal{K} \) such that \( t \prec s_a \) and \( t \prec s_b \). Take some \( z \in \text{Dom } t \) (such an element exists by Lemma A.15). Then
\[
s_az = tz, \quad s_bz = tz.
\]
But
\[
s_az = az, \quad s_bz = bz.
\]
Thus \( az = bz \), which is a contradiction since \( S \) is cancellative. Thus, \( \psi \) is one-to-one, i.e. it is an isomorphism from \( S \) into the group \( \tilde{H} \). \( \square \)
Remark A.13. It can be shown that the condition of left-reversibility is not only necessary, but also a sufficient condition for a cancellative semigroup to be embeddable into a group (see [CP61, Theorem1.24]). For an example of a cancellative semigroup that is not embeddable into a group see Mal’cev’s Example in AppendixC.

Corollary A.14. A cancellative commutative semigroup is embeddable into a group.

Lemma A.15. Let $\mathcal{H}$ be as in the proof of Theorem A.12. Then for all $h \in \mathcal{H}$, $\text{Dom } h$ and $\text{Im } h$ are right ideals of the semigroup $S$.

Proof. Let $h \in \mathcal{H}$. Then

$$h = h_1 \ldots h_m, \quad \text{where } h_i = s_{a_i} \text{ or } h_i = s_{a_i}^{-1}, \quad i = 1, \ldots, m.$$ 

We shall use induction by $m$.

First, consider the case $m = 1$. Then either $h = s_a$, in which case $\text{Dom } h = S$ and $\text{Im } h = aS$, or $h = s^{-1}a$ and then $\text{Im } h = S$ and $\text{Dom } h = aS$. The sets $S$ and $aS$ are clearly right ideals.

Now, let us consider the case $m > 1$. By symmetry, it is enough to prove that $\text{Dom } h$ is a right ideal of $S$. First, let us show that $\text{Dom } h \neq \emptyset$. Indeed, by induction hypothesis, there exist some $a \in \text{Im } h_2 \ldots h_m$ and $b \in \text{Dom } h_1$. By left irreversibility, there exist $u, v \in S$ such that

$$au = bv = w \quad \text{ (for some } w \in S).$$

Since $\text{Im}(h_2 \ldots h_m)$ and $\text{Dom } h_1$ are right ideals, $w \in \text{Im}(h_2 \ldots h_m)$ and $w \in \text{Dom } h_1$. Since $w \in \text{Im}(h_2 \ldots h_m)$, it follows that there exists $c \in S$ such that $h_2 \ldots h_m(c) = w$. On the other hand, since $w \in \text{Dom } h_1$, it follows that $c \in \text{Dom}(h_1h_2 \ldots h_m) = \text{Dom } h$, which shows that $\text{Dom } h$ is non-empty.
It remains to show that Dom $h$ is a right ideal. Let $x \in \text{Dom } h$ and let $y \in S$. We need to show that $xy \in \text{Dom } h$. Clearly, $x \in \text{Dom} (h_2 \ldots h_m)$. Since $\text{Dom} (h_2 \ldots h_m)$ is a right ideal, $xy \in \text{Dom} (h_2 \ldots h_m)$. Note that

$$(h_2 \ldots h_m)(xy) = ((h_2 \ldots h_m)(x)) \cdot y$$

(this is not obvious, but can be easily checked by induction). Since $(h_2 \ldots h_m)(x) \in \text{Dom } h_1$, it follows that

$$(h_2 \ldots h_m)(xy) = ((h_2 \ldots h_m)(x)) \cdot y \in \text{Dom } h_1,$$

i.e. $xy \in \text{Dom } h$. This finishes the proof. \hfill \Box
APPENDIX B

SOME FACTS ABOUT AMENABLE GROUPS AND SEMIGROUPS

In this appendix, we present definitions and some of major results from the theory of amenable groups and semigroups. Main references on the subject are [Pat88], [Gre69], and [Pie84]. We only consider discrete countable semigroups and groups. The more general case of \( \sigma \)-compact locally-compact semigroups is quite similar.

Let \( S \) be a countable semigroup. Let \( B(S) \) be the Banach space of all bounded complex-valued functions on \( S \) equipped with sup norm \( \| f \|_\infty \).

**Definition B.1.** A linear functional \( M : B(S) \to \mathbb{C} \) is a mean if

1. \( M(f) = \bar{M}(f) \quad \forall f \in B(S) \)
2. \( \inf_{x \in S} f(x) \leq M(f) \leq \sup_{x \in S} f(x) \) for all real-valued \( f \in B(S) \)

**Definition B.2.** A mean \( M \) is called left invariant if

\[
\forall f \in B(S) \forall a \in S \quad M(a f) = M(f),
\]

where \( a f(x) = f(ax) \). \( M \) is called right invariant if

\[
\forall f \in B(S) \forall a \in S \quad M(f a) = M(f)
\]

where \( f_a(x) = f(xa) \).
**Definition B.3.** A semigroup $S$ is called *left (right) amenable* if there exists a left (right) invariant mean $M$ on $B(S)$. By an *amenable* semigroup we mean a semigroup that is either left or right amenable.

**Remark B.4.** If $S$ is a group, then it is left amenable iff it is right amenable. Indeed, if $M$ is a left invariant mean on $B(S)$ we can define another mean $\tilde{M}(f) = M(f^*)$, where $f^*(x) = f(x^{-1})$. It is easy to see that $\tilde{M}$ is a right invariant mean. Thus, for groups we shall use the term *amenable*. For semigroups, the situation is more complicated. There exist examples of semigroups that are left, but not right amenable and vice versa.

**Theorem B.5.** An abelian group is amenable.

A proof can be found in [Pat88, p. 14].

**Theorem B.6.** If $S$ is a left (right) amenable semigroup and $\pi$ is a homomorphism from $S$ onto another semigroup $H$, then $H$ is left (right) amenable.

**Proof.** Let $M$ be the left (right) invariant mean on $B(G)$. Let $\tilde{M}$ be the functional on $B(H)$ defined as $\tilde{M}(f) = M(f \circ \pi)$. It is clear that if $f \in B(H)$, then $f \circ \pi \in B(G)$. It is easy to check that so defined functional $\tilde{M}$ is a left (right) invariant mean on $H$. \hfill \square

**Lemma B.7.** A left (right) amenable semigroup is left (right) reversible (see Definition A.8).

**Proof.** We consider the “left amenable” case. The other case is similar. Let $I$ and $J$ be two disjoint right ideals in $S$ and let $a \in I$ and $b \in J$. If $S$ is left amenable, then
there exists a left invariant mean $M$ on $B(S)$. Since $1_S(x) \geq 1_I(x) + 1_J(x)$ for all $x \in S$, we have

$$M(1_S) \geq M(1_I) + M(1_J) \geq M(1_{aS}) + M(1_{bS}) = M(1_S) + M(1_S),$$

which is impossible as $M(1_S) = 1$. \hfill \Box

Thus, in an amenable semigroup, we may define a relation $a \approx b$ if there exists $s \in S$ such that $as = bs$ which, according to Proposition A.10, is a congruence and the factor-semigroup $\overline{S} = S/\approx$ is right cancellative.

**Proposition B.8.** A left reversible semigroup $S$ is left amenable iff $S/\approx$ is left amenable.

A proof can be found in [Pat88, p. 35].

**Example B.9.** Let us consider the Lamplighter Semigroup. It is defined as follows.

Let

$$S = \mathbb{N} \ltimes \bigoplus_{\mathbb{N}} \mathbb{Z}_2,$$

with the semigroup operation defined as

$$(i, a) \cdot (j, b) = (i + j, \sigma^i a + b),$$

where $\sigma$ is the left shift. It is a left-cancellative left-amenable semigroup.

Because of loss of information due to left shift, this semigroup is left cancellative, but not right cancellative. It is easy to see that it is left reversible. In fact, if we have two elements of $S$, $s_1 = (i, a)$ and $s_2 = (j, b)$, there always exist big enough natural numbers $k$ and $l$ so that

$$(i, a) \cdot (k, 0) = (j, b) \cdot (l, 0) = (i + k, 0).$$
Now, consider again two arbitrary elements of $S$, $(i,a)$ and $(j,b)$. It is clear that $(i,a) \approx (j,b)$ iff $i = j$. Indeed, we can always choose a big enough natural number $k$ such that

$$(i,a) \cdot (k,0) = (i,a) \cdot (k,0) = (i+k,0).$$

Thus, the factor semigroup $S/\approx$ is isomorphic to the additive semigroup $\mathbb{N}$. Since $\mathbb{N}$ is left amenable, it follows that $S$ is left amenable as well.

A very important feature of amenable semigroups is the presence of a summing sequence of sets, a so called Følner sequence.

**Definition B.10.** We say that a semigroup $S$ satisfies Følner Condition if for every $\varepsilon > 0$ and every finite set $K \subset S$ there exists a finite set $F$ such that

$$\frac{|F \setminus xF|}{|F|} < \varepsilon \quad \forall x \in K \quad (B.1)$$

**Definition B.11.** We say that a semigroup $S$ satisfies Strong Følner Condition if for every $\varepsilon > 0$ and every finite set $K \subset S$ there exists a finite set $F$ such that

$$\frac{|F \triangle xF|}{|F|} < \varepsilon \quad \forall x \in K \quad (B.2)$$

**Theorem B.12 (cf. [Pat88, p. 131]).** If a semigroup $S$ is left amenable, then it satisfies Følner Condition.

**Theorem B.13 (cf. [Pat88, p. 145]).** If a semigroup $S$ satisfies Strong Følner Condition, then it is left amenable.

**Theorem B.14 (cf. [Pat88, p. 145]).** A left cancellative semigroup $S$ is left amenable iff it satisfies Strong Følner Condition.
Remark B.15. Because we want to be able to make use of Strong Følner Condition, we normally consider left cancellative semigroups only.

By a left Følner sequence \((F_n)\) in \(S\), we mean a sequence of finite subsets of \(S\) such that for every \(x \in S\)

\[
\lim_{n \to \infty} \frac{|F_n \triangle xF_n|}{|F_n|} = 0.
\]

Remark B.16. The word “left” will often be omitted and we would say just “Følner sequence” meaning “left Følner sequence”.

Proposition B.17. A left cancellative semigroup is left amenable iff it contains a left Følner sequence.

Proof. This is an easy consequence of Theorem B.14. \(\square\)
APPENDIX C

EMBEDDING AMENABLE SEMIGROUPS INTO AMENABLE GROUPS

The main result presented in this appendix is Theorem C.3 due to Wilde and Witz [WW67] stating that every amenable cancellative semigroup is embeddable into an amenable group generated by this semigroup.

Definition C.1. Let $G$ be a group and let $S$ be a subsemigroup of $G$. We say that $S$ generates $G$ if $G$ is generated by the set $S \cup S^{-1}$, where $S^{-1}$ is the set of inverse elements (with respect to the group operation in $G$) of elements of $S$.

Lemma C.2. If a discrete countable group $G$ is generated by an amenable semigroup $S$, then $G$ is amenable.

Proof. We consider the case when $S$ is left amenable. Let $M$ be a left invariant mean on $B(S)$. We define the left invariant mean $	ilde{M}$ on $B(G)$ as follows: $	ilde{M}(f) = M(f|_S)$. Clearly, $	ilde{M}$ is a mean. Let us check that $	ilde{M}$ is left invariant. Indeed, for all $a \in S$,

$$
\tilde{M}(f(ax)) = M(f(ax)|_S) = M(f|_S) = \tilde{M}(f)
$$

and

$$
\tilde{M}(f(a^{-1}x)) = M(f(a^{-1}x)|_S) = M(f(aa^{-1}x)|_S) = M(f|_S) = M(f).
$$
Since $S$ generates $G$, it follows that for all $b \in G$, $\tilde{M}(f(bx)) = \tilde{M}(f)$, i.e. that $\tilde{M}$ is left invariant. \hfill \Box

**Theorem C.3.** Every countable cancellative amenable semigroup $S$ is embeddable into a countable amenable group that is generated by $S$.

**Proof.** We consider left amenable case. By Lemma B.7, for all $a, b \in S$, there exist $x, y \in S$ such that $ax = by$. Then by Theorem A.12, the cancellative semigroup $S$ is embeddable into a group $G$. We need to show that $G$ is amenable. In the proof of Theorem A.12 we constructed the group $\tilde{\mathcal{H}}$ and the isomorphism $\xi$ from $S$ into $\tilde{\mathcal{H}}$. The semigroup $\xi(S)$ is an amenable subsemigroup of $\tilde{\mathcal{H}}$. From the way $\tilde{\mathcal{H}}$ was constructed, it is clear that $\xi(S)$ generates $\tilde{H}$. Thus, by Lemma C.2, $\tilde{H}$ is amenable.

The right amenable case is completely symmetric. We would need to use a “left” analogue of Theorem A.12, left ideals instead of right ideals, etc. \hfill \Box

**Remark C.4.** To prove that the group $G$ is amenable, we could argue a bit differently. Namely, since $S, S^{-1} \subset G$, then even if $G$ if not generated by $S$ (which is the case, but not clear from formulation of Theorem A.12), still there is a subgroup in $G$ that is generated by $S$ that is amenable by Lemma B.7.

Now, assume that $(F_n)$ is a Følner sequence in a left amenable semigroup $S$ and we constructed an amenable group $G$ such that $S \cup S^{-1}$ generate $G$. Is $(F_n)$ a left Følner sequence in $G$? The answer is positive.

**Proposition C.5.** Let a semigroup $S$ generate a group $G$ and let $(F_n)$ be a left Følner sequence in $S$. Then $(F_n)$ is a left Følner sequence in $G$.  

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Proof. Since \((F_n)\) is left Følner sequence in \(S\), then for all \(x \in S\),

\[
\frac{|xF_n \triangle F_n|}{|F_n|} \to 0. \tag{C.1}
\]

Then

\[
\frac{|x^{-1}F_n \triangle F_n|}{|F_n|} = \frac{|F_n \triangle xF_n|}{|F_n|} \to 0. \tag{C.2}
\]

Every element \(g \in G\) can be represented as \(g = x_1x_2 \ldots x_m\), where \(x_i \in S \cup S^{-1}\). To prove that for all \(g \in G\),

\[
\frac{|gF_n \triangle F_n|}{|F_n|} \to 0. \tag{C.3}
\]

we use induction by \(m\). The case \(m = 1\) is clear by virtue of (C.1) and (C.2). Assume that we have proven (C.3) for some \(m\). Then

\[
\frac{|x_1x_2 \ldots x_mF_n \triangle F_n|}{|F_n|} = \frac{|x_2 \ldots x_mF_n \triangle x_1^{-1}F_n|}{|F_n|} \tag{C.4}
\]

\[
\leq \frac{|x_2 \ldots x_mF_n \triangle F_n|}{|F_n|} + \frac{|x_1^{-1}F_n \triangle F_n|}{|F_n|} \to 0. \tag{C.5}
\]

by induction hypothesis. \qed

**Mal’cev’s Example.** The condition that the semigroup is amenable is essential in Theorem C.3. There exist examples of cancellative semigroups not embeddable into groups. First such example belongs to Mal’cev [Mal37]. Examples of this kind are constructed in the following way. First, one comes up with a certain property that always holds in a group, but might not hold in a cancellative semigroup. Mal’cev used the following property:

**Condition M.** \(ax = by, cx = dy, au = bv \implies cu = dv\).

Then one constructs a cancellative semigroup in which this property does not hold. Mal’cev considered the free semigroup \(S\) on 8 generators, denoted as \(a, b, c,\)
$d$, $x$, $y$, $u$, and $v$, with relations $ax = by$, $cx = dy$, and $au = bv$. $S$ is a cancellative semigroup. However, $cu \neq dv$, and so Condition M is not satisfied. Thus, $S$ is not embeddable into a group.
APPENDIX D

A CHARACTERIZATION OF FØLNER SEQUENCES IN \( \mathbb{Z} \)

In this appendix, we give a convenient characterization of Følner sequences in \( \mathbb{Z} \). The proposed characterization in this form seems to be new.

We use the following notation. Let \( H \subset \mathbb{Z} \). Then \( H \) consists of a finite number of non-adjacent intervals, so that one can write

\[
H = [a_1, b_1] \cup \cdots \cup [a_n, b_n]
\]

and this decomposition is unique. Let

\[
\ell(H) = \min_{k=1, \ldots, n} (b_k - a_k).
\]  \hspace{1cm} (D.1)

We call the number \( \ell(H) \), the footprint of the set \( H \).

**Lemma D.1.** A sequence of finite sets \( (F_n) \) in \( \mathbb{Z} \) is a Følner sequence iff it can be represented as \( F_n = G_n \cup H_n \) so that

1. \( \ell(G_n) \rightarrow \infty \) as \( n \rightarrow \infty \),

2. \( |H_n|/|F_n| \rightarrow 0 \) as \( n \rightarrow \infty \).
Proof. "If" part. Let $F_n = G_n \cup H_n$ that satisfy the conditions (i) and (ii). Let $r \in \mathbb{Z}$ be given. Because of our assumptions, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ one has $\ell(G_n) > 1/\varepsilon$ and $|H_n|/|F_n| < \varepsilon$. Note that

$$|F_n \triangle (F_n + r)| \leq |G_n \triangle (G_n + r)| + 2|H_n|.$$  

Since $\ell(G_n) > 1/\varepsilon$, the set $G_n$ contains no more than $\varepsilon|G_n|$ gaps between its intervals. Therefore, for all $n > n_0$

$$\frac{|F_n \triangle (F_n + r)|}{|F_n|} \leq \frac{|G_n \triangle (G_n + r)|}{|F_n|} + \frac{2|H_n|}{|F_n|} \leq \frac{2\varepsilon r|G_n|}{|F_n|} + 2\varepsilon \leq (2 + 2r)\varepsilon.$$  

Since $\varepsilon$ is arbitrary, the first part of lemma is proved.

"Only if" part. Assume the statement is not true, i.e. there exists a Følner sequence $(F_n)$ for which the decomposition above is impossible. This means that $\exists \varepsilon > 0 \ \forall n_0 > 0 \ \exists n > n_0$ such that $F_n$ can not be written as $F_n = H_n \cup G_n$ with $\ell(G_n) > 1/\varepsilon$ and $|H_n|/|F_n| < \varepsilon$. Hence, $F_n$ contains a lot of small intervals. Namely, let $F_n'$ be the union of all intervals of length less than $1/2\varepsilon$. Then $|F_n'|/|F_n| > \varepsilon$ (otherwise, we could let $H_n = F_n'$, $G_n = F_n \setminus F_n'$, and this would be the decomposition that we assumed does not exist). It follows that the set $F_n'$ (and, therefore, the set $F_n$ as well) contains at least $\varepsilon^2/2 |F_n|$ intervals. Hence,

$$\frac{|F_n \triangle F_n + 1|}{|F_n|} \geq \frac{\varepsilon^2}{2}.$$  

Contradiction. \(\Box\)

The result contained in Lemma D.1 can be extended to more general amenable groups.
APPENDIX E

BASIC RESULTS OF CHARACTER AND REPRESENTATION THEORY

E.1 Character Theory

Here we present some of the basic results of Character Theory. Proofs can be found, for example, in [HR60a].

Let $X$ be a locally compact abelian group. Let $\hat{X}$ be the collection of all continuous homomorphisms from $X$ into the unit circle $T$. The members of $\hat{X}$ are characters of $X$. Under the operation of pointwise multiplication of functions $\hat{X}$ is an abelian group. With the compact open topology $\hat{X}$ becomes a locally compact abelian group. The group $\hat{X}$ is called the dual group of $X$.

One can show that $\hat{T}^n = \mathbb{Z}^n$ and each $\gamma \in \hat{T}^n$ is of the form:

$$\gamma(x_1, \ldots, x_n) = e^{2\pi i (p_1 + \cdots + p_n)} \text{ for some } (p_1, \ldots, p_n) \in \mathbb{Z}^n.$$

More generally, for a compact group $X$, the dual group $\hat{X}$ is discrete.

The following facts hold.

Theorem E.1. $\hat{X_1 \times X_2} = \hat{X_1} \times \hat{X_2}$. 

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**Theorem E.2.** Let $X_i, i \in I$, be a non-void family of locally compact abelian groups. Then

$$\prod_{i \in I} X_i = \bigoplus_{i \in I} \hat{X}_i.$$ 

In particular, the dual group for $\mathbb{Z}_p^\mathbb{Z}$ is isomorphic to $\bigoplus_{i=-\infty}^{\infty} \mathbb{Z}_p$.

**Theorem E.3.** If $X$ is a compact abelian group, then the elements of $\hat{G}$ form an orthonormal basis for $L^2(X, \mu)$, where $\mu$ is the Haar measure.

If $A : X \rightarrow X$ is an automorphism, we can define the dual automorphism $\hat{A} : \hat{X} \rightarrow \hat{X}$ by $\hat{A}\gamma(x) = \gamma(Ax), \ x \in X$, where $\gamma \in \hat{X}$. Then $\hat{A}$ is an automorphism of $\hat{X}$.

### E.2 Representation Theory of Compact Groups

Now we present some of the basic facts of the theory of unitary representations of compact groups. Proofs can be found in either [HR60b] or [NŠ82].

Let $X$ be a compact group. A (continuous) unitary representation $\tau$ of $X$ in a complex Hilbert space $\mathcal{H}$ is a homomorphism from $X$ to the group of all continuous unitary operators of $\mathcal{H}$ so that the resulting map $X \times \mathcal{H} \rightarrow \mathcal{H}$ is continuous. For each $g \in X$, $\tau(g)$ is a unitary operator on $\mathcal{H}$. If $\mathcal{H}$ is finite-dimensional, then, with respect to a fixed basis, a unitary representation can be defined by a unitary matrix $(t_{k,l}, 1 \leq k, l \leq d_\tau)$, where $d_\tau = \dim \mathcal{H}$, and $t_{k,l}$ are continuous complex-valued functions on $X$. The functions $t_{k,l}$ are called coordinate functions of the representation $\tau$. The number $d_\tau$ is called the dimension of the representation $\tau$. A representation $\tau$ is called finite-dimensional if $d_\tau < \infty$.

A subspace $\mathcal{H}'$ of $\mathcal{H}$ is called $\tau$-invariant if for all $g \in X$, $\tau_g \mathcal{H}' \subset \mathcal{H}'$. A representation $\tau$ is called irreducible if it has no closed invariant subspaces other than $\{0\}$ and all $\mathcal{H}$.
**Theorem E.4.** Every irreducible continuous unitary representation of a compact group $X$ is finite-dimensional.

Two unitary representations of $X$, $\tau$ on $\mathcal{H}$ and $\tau'$ on $\mathcal{H}'$, are called *(unitarily) equivalent* if there exists a bounded unitary operator $U : \mathcal{H} \to \mathcal{H}'$ such that

$$\tau'(g)U = U\tau(g) \quad \text{for all } g \in X.$$ 

We write $\tau \sim \tau'$.

Let $\mathcal{U}(X)$ be the set of all continuous irreducible unitary representations of $X$ (in some fixed infinite-dimensional Hilbert space) and let $\Gamma(X)$ be the set of all equivalence classes of $\mathcal{U}(X)$. The set $\Gamma(X)$ is called the *dual object* of $X$. The symbol $\mathbb{1}$ will sometimes denote the trivial irreducible representation given by the identity operator on 1-dimensional Hilbert space.

A *tensor product* $\tau_1 \otimes \tau_2$ of two finite-dimensional unitary representations $\tau_1$ and $\tau_2$ on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively is the unitary representation on the space $\mathcal{H}_1 \otimes \mathcal{H}_2$ defined as

$$(\tau_1 \otimes \tau_2(g)) (v_1 \otimes v_2) = \tau_1(v_1) \otimes \tau_2(v_2), \quad \text{for all } v_1 \in \mathcal{H}_1, v_2 \in \mathcal{H}_2.$$ 

If $\tau_1 \sim (t^1_{k,l})_{k,l=1,n_1}$ and $\tau_2 \sim (t^2_{k,l})_{k,l=1,n_2}$, then $\tau_1 \otimes \tau_2$ is given by the matrix $(t_{k_1,k_2,l_1,l_2})$, where $t_{k_1,l_1,k_2,l_2} = t^1_{k_1,l_1} t^2_{k_2,l_2}$ for $1 \leq k_1, l_1 \leq n_1, 1 \leq k_2, l_2 \leq n_2$. One can show that $\tau \otimes \tau' \sim \tau' \otimes \tau$. Similarly, one defines a tensor product of finitely many representations.

**Theorem E.5.** Let $X_i$ be a non-void family of compact groups and let $X = \prod_{i \in I} X_i$. Let $\{i_1, \ldots, i_m\}$ be a finite subset of $I$ and let $\tau_{i_k}$ be irreducible unitary representations of $X_{i_k}$, $1 \leq k \leq m$. Then the mapping

$$(g_i) \mapsto \tau_{i_1}(g_{i_1}) \otimes \cdots \otimes \tau_{i_m}(g_{i_m})$$

(E.1)
is an irreducible unitary representation of $X$. Moreover, every irreducible unitary representation of $X$ is equivalent to a representation of the form (E.1).

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}_i$, $i \in I$, be a family of pairwise orthogonal non-zero closed linear subspaces of $H$ whose union spans a dense linear subspace of $\mathcal{H}$. Then $\mathcal{H}$ is isometric to $\bigoplus_{i \in I} \mathcal{H}_i$. Let $\tau$ be a representation of a compact group $X$ on $\mathcal{H}$ such that each subspace $H_i$ is invariant under $\tau$. Let $\tau_i(g)$ be the linear operator $\tau(g)$ with its domain restricted to $\mathcal{H}_i$. Then $\tau_i$ is a unitary representation of $X$ in $H_i$ for each $i$. We say that $\tau$ is the direct sum of the representations $\tau_i$:

$$\tau = \bigoplus_{i \in I} \tau_i.$$  

(E.2)

If $\{\sigma_j\}_{j \in J}$ is the family of mutually inequivalent representations $\tau_i$ and if $m_j$ is the multiplicity of $\sigma_i$ in the decomposition (E.2), then we also write:

$$\tau = \bigoplus_{j \in J} m_j \sigma_j.$$

Let $\tau$ be a finite-dimensional representation of $X$ with a matrix elements $(t_{k,l})$, $1 \leq k, l \leq n$, with respect to a fixed basis. The conjugate representation $\bar{\tau}$ is a unitary representation of $X$ with matrix elements $((\bar{t}_{k,l}))$. If $\gamma \in \Gamma(X)$ and $\tau \in \gamma$, then $\bar{\gamma}$ denotes the equivalence class containing the representation $\bar{\tau}$.

Let $\tau$ be a finite-dimensional representation of a compact group $X$. The character $\chi_\tau$ of the representation $\tau$ is the function on $X$ defined as

$$\chi_\tau(g) = \text{tr} \tau(g).$$

The notion of “character of a representation” differs from the one of “character of a group”. Every character of a group is the character of the 1-dimensional representation defined by it. However, the character of a representation of a compact group is a character of the group iff the representation is 1-dimensional.
**Theorem E.6.** Let $\tau$ and $\tau'$ be representations of a compact group $X$. One has:

1. $\chi_{\bar{\tau}} = \overline{\chi_\tau}$,
2. $\chi_{\tau \oplus \tau'} = \chi_\tau + \chi_{\tau'}$,
3. $\chi_{\tau \otimes \tau'} = \chi_\tau \chi_{\tau'}$.

**Theorem E.7.** Two continuous unitary finite-dimensional representations $\tau$ and $\tau'$ of a compact group $X$ are equivalent iff

$$\chi_\tau = \chi_{\tau'}.$$ 

**Theorem E.8.** Every continuous unitary representation of a compact group $X$ can be represented as a direct sum of irreducible finite-dimensional unitary representations.

**Theorem E.9.** Every finite-dimensional continuous unitary representation $\tau$ of a compact group $X$ can be represented as a direct sum of irreducible unitary representations of $X$:

$$\tau = m_1 \tau_1 \oplus \cdots \oplus m_r \tau_r. \quad (E.3)$$

The decomposition (E.3) is unique up to a permutation of indexes. Let $\sigma$ be an irreducible unitary representation of $X$. Then one has:

$$\int_X \chi_\tau \overline{\chi_\sigma} \, d\mu = \begin{cases} m_j, & \text{if } \sigma \sim \sigma_j \ (j = 1, 2, \ldots, r) \\ 0, & \text{otherwise} \end{cases} \quad (E.4)$$

Let $\gamma, \gamma' \in \Gamma(X)$ and let $\tau \in \gamma$ and $\tau' \in \gamma'$. Consider the unitary representation $\tau \otimes \tau'$ of $X$. This representation is not irreducible in general. However, in view of the previous theorem, it can be decomposed as

$$\tau \otimes \tau' = m_1 \tau_1 \oplus \cdots \oplus m_r \tau_r.$$
We define $\gamma \times \gamma' \overset{\text{def}}{=} \{\tau_1, \ldots, \tau_r\}$.

If $B \subset \Gamma$, then

$$\gamma \times B \overset{\text{def}}{=} \bigcup_{\gamma' \in B} \gamma \otimes \gamma'.$$

**Theorem E.10.** Let $\gamma_1, \gamma_2 \in \Gamma(X)$. Then $1 \in \gamma_1 \times \gamma_2$ iff $\gamma_1 = \bar{\gamma}_2$.

**Corollary E.11.** Let $\gamma \in \Gamma(X)$ and $B \subset \Gamma(X)$. Then $1 \in \gamma \times B$ iff $\bar{\gamma} \in B$.

The left regular representation of a compact group $X$ is the unitary representation $\tau$ of $X$ in the Hilbert space $L^2(X, \mu)$, given by $\tau(g)f(x) = f(g^{-1}x)$. (The measure $\mu$ is the left-invariant Haar measure.)

**Theorem E.12 (Peter-Weyl).** Let $\tau$ be the left regular representation of a compact group $X$. Let $\{\tau_i, i \in I\}$ be the set of all mutually non-equivalent irreducible representations of $X$ with matrix elements $t_{k,l}^i$, $1 \leq k, l \leq d_i = d_{\tau_i}, i \in I$, with respect to some basis. Let $\mathcal{H}_i$ be the subspace of $L^2(X)$ spanned by the functions $t_{k,l}^i$, $1 \leq k, l \leq d_i$. Then $L^2(X) = \bigoplus_{i \in I} \mathcal{H}_i$, each space $\mathcal{H}_i$ is invariant with respect to the representation $\tau$, and the restriction of $\tau$ onto $\mathcal{H}_i$ is equivalent to $d_i \tau_i$ (the direct sum of $d_i$ copies of the representation $\tau_i$). In other words,

$$\tau = \bigoplus_{i \in I} d_i \tau_i.$$

Also,

$$\int_X t_{k,l}^i \overline{t_{k',l'}^{i'}} d\mu = \begin{cases} 1/d_i, & \text{if } i = i', k = k', \text{and } l = l' \\ 0, & \text{otherwise} \end{cases}$$

(E.5)

The functions $(\sqrt{d_i} t_{k,l}^i)$, $1 \leq k, l \leq d_i, i \in I$, form an orthonormal basis for $L^2(X)$.

The equation (E.5) is often referred to as orthogonality conditions.
Let $T$ be an automorphism of a compact group $X$ and let $\tau$ be an irreducible finite-dimensional unitary representation of $X$. Then we have another irreducible unitary representation that we denote as $\tau T$ defined as $(\tau T)(g) = \tau(Tg)$. Thus, $T$ defines an operator on $\mathcal{U}(X)$ and $\Gamma(X)$. 
APPENDIX F

SOME FACTS FROM ERGODIC THEORY AND
FUNCTIONAL ANALYSIS

Here we present some basic facts from ergodic theory. For some of these results, we give exact references. Others (at least for $\mathbb{Z}$-actions) may be found in textbooks such as [Pet83] or [CFS82].

Let $S$ be a countable semigroup.

**Definition F.1.** An action $T$ of $S$ on a probability measure space $(X, \mathcal{B}, \mu)$ is a homomorphism $g \mapsto T_g$ from $S$ into the semigroup $\mathcal{M}(X)$ of measure-preserving transformations of $X$ into itself.

A dynamical system is the quadruple $(X, \mathcal{B}, \mu, T)$ such that $T$ is an action of a semigroup $S$ on a probability measure space $(X, \mathcal{B}, \mu)$.

The simplest case of an action is a $\mathbb{N}$- or $\mathbb{Z}$- action given by a single measure-preserving transformation $T$ (that has to be invertible in the case of $\mathbb{Z}$-action).

**Definition F.2.** A dynamical system $(X, \mathcal{B}, \mu, T)$ is ergodic if for every $A \in \mathcal{B}$ with $0 < \mu(A) < 1$ there exists $g \in S$ such that $\mu(T_g A \Delta A) > 0$.

A set $A \in \mathcal{B}$ is called an invariant set for the action $T$ if $T_g A = A \mod \mu$ for all $g \in G$. A measurable function $f$ is called an invariant function for the action $T$ if $f(Tx) = f(x)$ a.e. for all $g \in G$. 

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Theorem F.3. The following statements are equivalent.

(1) $T$ is ergodic.

(2) $T$ has no non-constant measurable invariant functions.

(3) $T$ has no invariant set $A \in \mathcal{B}$ with $0 < \mu(A) < 1$.

(4) For every $A \in \mathcal{B}$ with $\mu(A) > 0$ one has

$$\mu \left( \bigcup_{g \in S} (T_g)^{-1} A \right) = 1.$$ 

(5) For every $A, B \in \mathcal{B}$ with $\mu(A) > 0$, $\mu(B) > 0$ there exists $g \in S$ with

$$\mu((T_g)^{-1} A \cap B) > 0.$$ 

For each action $T$ of a semigroup $S$ on $(X, \mathcal{B}, \mu)$, we can define a right unitary representation (that we shall also denote by $T$) of $S$ on the space $H = L^2(X)$ by $(T_g f)(x) = f(T_g x)$.

Let $H$ be a Hilbert space and let $T$ be a right unitary representation of a left cancellative left amenable semigroup $S$ in $H$. A vector $v \in H$ is called $T$-invariant if $T_g v = v$ for all $g \in S$.

Theorem F.4 (Mean Ergodic Theorem). Let $T$ be a right unitary representation of a left cancellative left amenable semigroup $S$ in a Hilbert space $H$, let $(F_n)$ be a Følner sequence in $S$, and let $P : H \to H$ be the orthogonal projection of $H$ onto the subspace of the vectors that are invariant with respect to the representation $T$. Then for every $f \in H$

$$\frac{1}{|F_n|} \sum_{g \in F_n} T_g f \to Pf$$

in $H$-norm.
Lemma F.5. Let a left cancellative left amenable semigroup \( S \) act on a probability measure space \( X \). Then \( L^2(X) = H_{\text{inv}} \oplus H_{\text{erg}} \), where

\[
H_{\text{inv}} = \{ f \in L^2(X) \mid T_g f = f \ \forall g \in S \}
\]
\[
H_{\text{erg}} = \text{Span}\{ h - T_g h \mid h \in L^\infty(X), g \in S \}.
\]

Remark F.6. Usually, one uses a somewhat more obvious splitting with

\[
H_{\text{inv}} = \{ f \in L^2(X) \mid T_g f = f \ \forall g \in S \}
\]
\[
H_{\text{erg}} = \text{Span}\{ h - T_g h \mid h \in L^2(X), g \in S \}.
\]

Let \( T \) be a unitary operator on a Hilbert space \( \mathcal{H} \). A non-zero vector \( v \in \mathcal{H} \) is called an eigenvector of \( T \) if \( Tv = \lambda v \) for some \( \lambda \in \mathbb{C} \).

Lemma F.7. Let \( T \) be an ergodic transformation on a probability measure space \( (X, \mathcal{B}, \mu) \) and let \( f \in L^2(X) \) be its eigenfunction. Let \( T' \) be another transformation on same space commuting with \( T \). Then \( f \) is an eigenfunction of \( T' \) as well.

Proof. Let \( Tf = \lambda f, f \neq 0 \). Since \( T \) is ergodic, \( |f| \) is a non-zero constant a.e. Since \( T \) and \( T' \) commute, \( T(T'f) = \lambda T'f \). Let

\[
g(x) = \frac{T'f(x)}{f(x)}.
\]

Then \( Tg = g \). Since \( T \) is ergodic, \( g \) is a constant a.e., say, \( g(x) = c \). Then \( T'f = cf \) a.e. \( \square \)

Definition F.8. Let \( T \) be a unitary operator in a Hilbert space \( a H \). The adjoint operator is the unitary operator \( T^* \) in \( H \) having the property that for all \( u, v \in H \) one has

\[
(Tu, v) = (u, T^*v).
\]
Lemma F.9 (cf. [Kre85, Lemma 1.2, p. 3]). Let $T$ be a unitary operator in a Hilbert space $H$ and $u \in H$. Then $u = Tu$ holds iff $u = T^* u$.

Lemma F.10 (cf. [Fur81, Lemma 4.18]). Let $T$ and $T'$ be unitary operators on Hilbert spaces $H$ and $H'$ respectively and let $w \in H \otimes H'$ be an eigenvector of $U \otimes U'$:

$$(U \otimes U')w = \lambda w.$$ 

Then

$$w = \sum_n c_n u_n \otimes u'_n,$$

where $Uu_n = \lambda_n u_n$, $U'u'_n = \lambda'_n u'_n$ and the sequences $\{u_n\}$, $\{u'_n\}$ are orthonormal.

Remark F.11. It follows that the eigenvalues of $U \otimes U'$ are given by $\lambda \lambda'$, where $\lambda$ and $\lambda'$ are eigenvectors of $U$ and $U'$ respectively. Similarly, one can show that the eigenvalues of $U_1 \otimes \cdots \otimes U_s$, where $U_i$ are unitary operators, are given by $\prod_{i=1}^s \lambda_i$, where $\lambda_i$ is an eigenvalue of the operator $U_i$, $1 \leq i \leq s$.

If we have two dynamical systems $(X, \mathcal{B}, \mu, S, T)$ and $(Y, \mathcal{B}, \nu, T')$, we can define their direct product $(X \times Y, \mathcal{B} \times \mathcal{B}, \mu \times \nu, S, T \times T')$ in a natural way. On unitary level, this defines tensor product $T \otimes T'$ of the unitary representations $T$ and $T'$. 

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