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UMI
SPINOR GENERA UNDER FIELD EXTENSIONS FOR SKEW-HERMITIAN FORMS AND COHOMOLOGY.

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

By
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2000

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ABSTRACT

A.G. Earnest and J. Hsia proved an arithmetical version of Springer's theorem for lattices in certain quadratic spaces. This result depends on the so called norm principle. In this work, we prove a similar result for skew-hermitian lattices. We also introduce a cohomology theory for the group of units of a lattice, to give a cohomological interpretation of the relation between norm principle and Springer type results, which we generalize to arbitrary semi-simple groups.
To my parents
ACKNOWLEDGMENTS

I would like to thank Dr D. Shapiro and Dr P. Ponomarev for pointing out to me some obvious mistakes, and my adviser, Dr Hsia, by proof reading the original version of my paper.
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CHAPTER 1

INTRODUCTION.

One central matter in the theory of quadratic forms is that of classification. Over fields, nonsingular quadratic forms are classified by the cohomology set $H^1(k, G)$, where $G$ is the orthogonal group of some (any) given quadratic form, (see for example [8], example 3 in p.17). This corresponds to the fact that any two forms of equal dimension defined over a field $k$ become isometric over its algebraic closure, (if the characteristic is different from 2). On the other hand, a classical theorem of Springer, (see [18]), asserts that two non-isometric quadratic forms cannot become isometric under an extension of odd degree.

An arithmetical version of Springer's theorem was proved by A.E. Earnest and J.S. Hsia in [3]. Our main purpose in this work is to extend results of this type to the case of skew-hermitian forms over quaternion division algebras, i.e., to study the behavior of the spinor genera of such forms under field extensions of odd degree, in as much generality as possible.

This can be accomplish in essentially two ways; the first one is to find an integral cohomology theory, a theory that allows the study of lattices using cohomological exact sequences, in the same way that Galois cohomology allows us to study the

---

1For a definition of cohomology sets, see p. 12.
corresponding problem over fields. Since, as far as we know, a theory of this type is lacking in the current literature, we include it in part II of this work. The methods developed there are used at the end of part II to study a generalization of our main problem to arbitrary semi-simple groups. Our main result in this direction is proposition 9.6, (page 78). The second approach to the main problem deals with the theory of spinor norms and spinor class fields, (which is outlined in part I). This approach, which is a generalization of the ideas in [4] and [5], uses the more general concept of *spinor genera*.

Either approach reduces the problem to a *norm principle*, (see p. 75,138), which is a local condition, and can be verified directly, if the image of the local spinor map is known. However, we still do not have direct computations for all cases, (see proposition 14.4, p.143). The remaining *exceptional cases* are, however, rather specific, and it is likely that a result as general as that in [3] holds in this case.

**Integral cohomology.**

It is known, (see for example [7]), that if $G$ is an algebraic group defined over a base field $k$, which is defined as the set of automorphisms of a certain algebraic structure, (for instance, a quadratic or hermitian space, or an algebra over the base field), then the cohomology set $H^1(K/k, G_K)$ classifies the $K/k$-forms of $G$ (see page 14) , i.e., those algebraic structures of the same type, (in some sense), also defined over $k$, which, after extension to $K$, become isomorphic to the given structure.

It would be reasonable to expect, therefore, that a similar theory is available for
structures for which the corresponding automorphism group is not an algebraic group but a group of a more arithmetic nature, namely, an arithmetically defined subset of an algebraic group.

Such a theory is already hinted at in [14]. In this reference, two finiteness results are proven. The first one deals with the finiteness of the local cohomology set $H^1(\mathcal{G}_w, \Gamma_w)$, for an arithmetically defined group $\Gamma$, (where the notations are as in the reference). The second one deals with the finiteness of the kernel of the map

$$H^1(\mathcal{G}, \Gamma) \rightarrow \prod_{v \text{ place of } k} H(\mathcal{G}_{w(v)}, \Gamma_{w(v)}),$$

(where we have fixed a place $w(v)$ of the extension dividing each place $v$ of $k$). It is the proof of the second result which requires expressing the given kernel in terms of the set of double cosets, (see corollary 3.3 in [14])

$$G_k \backslash G_k / \prod_w \Gamma_w.$$

These double cosets are the same ones that classify the classes of lattices in a genus. This is the relation we want to pursue.

In part II of the present thesis, the following result is established. Let $K/k$ be a Galois extension of local or number fields, with Galois group $G = G_{K/k}$.

**Proposition 1.1.** Let $G$ be an arbitrary linear algebraic group acting on a vector space $V$, both defined over $k$. Let $L_K, \Lambda_K$ be $O_K$-lattices in $V_K$. If there is an element $\varphi \in G_K$ such that $\varphi(L_K) = \Lambda_K$, then $a_{\varphi} = \varphi^{\sigma} \varphi^{-1}$ is a well defined element of $H^1(\mathcal{G}, G_K)$. It is independent of the choice of a particular element $\varphi$, and depends only on the orbit of $L_K$ under $G_k$. The correspondence assigning, to a $G_k$-orbit of
lattices, an equivalence class of cocycles, is an injection. The image of this map corresponds to the kernel of the map

\[ H^1(G, G^A_K) \xrightarrow{i_*} H^1(G, G_K), \]

where \( i \) is the inclusion, (see prop. 6.1, p. 37). \(^2\)

The cocycles described above can be thought as lattices in the same space. We develop a more general concept of lattices in twisted spaces, at least for the case of stabilizers of a family of tensors, (see page 14 for the definition). In this context, the following result is obtained:

**Proposition 1.2.** Let \( G \) denote an algebraic group defined over \( k \), which is defined as the stabilizer of a family of tensors \( \Xi_n \) on \( V \), Then, the set \( H^1(G, G^A_K) \) is in one-to-one correspondence with the set of \( G_k \)-orbits of \( \mathcal{O}_K \)-lattices in all the spaces that are \( K/k \)-forms of \((V, \Xi_n)\). The kernel of the map

\[ H^1(G, G^A_K) \xrightarrow{i_*} H^1(G, G_K), \]

(where \( i \) is the inclusion), corresponds to those lattices that are in the same space as \( \Lambda_k \) (prop. 6.4, p. 38).

The relation between the set of cohomology classes and that of lattice classes is established by taking advantage of the long exact sequence, in cohomology, that arises from a short exact sequence of \( K \)-points of \( k \)-defined arithmetic groups.

\(^2\)See remark 2 in page 8 for the notations.
We analyze the cohomology of the general linear group and its relation to classification of lattices. In particular, one obtains that the set of $G_K$-orbits of free lattices corresponds to the kernel of

$$H^1(G_{K/k}, G^A_K) \longrightarrow H^1(G_{K/k}, GL^A_K(V)),$$

(see prop. 6.9, p. 42).

It is necessary to look at localizations to obtain a cohomological representation of the set of all lattices defined over $k$ \(^3\), (and not only the free ones). This is done in terms of the kernel of the map

$$H^1(G_{K/k}, G^A_K) \longrightarrow \prod_v H^1(G_{K_{v}/k_v}, GL^A_K(V)),$$

where $G_{K_{v}/k_v}$ is the Galois group in the localization (see prop. 6.19, p. 50).

Chapter 7 is devoted to the study of the relation between integral cohomology and genera. We concentrate on quadratic lattices and determine explicitly the set of lattices which can be described by cohomology, i.e., those lattices that become isometric over some extension. This is expressed in terms of the notion of $C$-genus, (which is introduced in chapter 7), and stated in the following form:

**Proposition 1.3.** For any lattice $\Lambda_k$ in a quadratic space, $(V_k, Q)$, we have that

$$C_{\text{gen}}(\Lambda_k) = \text{gen}(\Lambda_k). \quad (\text{prop. 7.6, p. 7.6}).$$

In the last section of chapter 7, we generalize the notion of $C$-genus and study the relation between cohomology and class number, (see prop. 7.17, p. 63).

\(^3\)A lattice is defined over $k$, if it is generated by its $k$-points. This is a stronger condition than $G_{K/k}$-invariance.
Chapter 8 specializes this theory to the case of skew-hermitian forms.

Chapter 9 focuses on the study of Springer type results for arbitrary groups.

Let $G$ be a semi-simple group with fundamental group $F$. The main result in chapter 9 is the following:

**Proposition 1.4.** Let extension $E/k$ be an arbitrary extension satisfying

$$(|F|, [E : k]) = 1.$$ 

Assume that $G$ is a semi-simple group such that

- $G$ is not a group of type $^2A_n$, $^3D_4$, $^6D_4$, $^2E_6$, nor $G$ is an adjoint group of type $^2D_{2k+1}$.
- $[G_k]_\infty$ is not compact.

Then, if the norm principle holds at any local place, not two classes in the same genus become equivalent over $E$. Furthermore, if $E/k$ is Galois, the same result holds without any restriction on the type of $G$, (see prop. 9.6, p. 78).

**Skew-hermitian forms.**

The construction of the spin group for skew-hermitian forms over quaternion division algebras was introduced for the first time in [16]. We review this construction at the end of part I. In part III, we develop a theory of lattices on skew-hermitian spaces, so that we have the arithmetical background on which to base our subsequent study of the image of the spinor norm map. Here, we point out some important differences between this case and that of quadratic forms.
Since the non-dyadic case has already been studied in [2], in this work, we concentrate on the dyadic case.

Important invariants for quadratic forms are the scale, norm and weight. The first one of these can be carried over with no significant changes to the case of skew-hermitian lattices, but that is not so for the other two. A notion of norm defined in terms of ideals would not carry much information for unimodular lattices because of the following basic fact: *All local unimodular skew-hermitian lattices are diagonalizable.* The correct analogy can be found using some filtration of $\mathcal{O}_k^-$ (but not $\mathcal{D}_k^-$) submodules of a maximal order $\mathcal{D}_k$ of the quaternion algebra.  

With this in mind we undertake a classification of hermitian forms in terms of structures of the Jordan components. We have paid little attention to the uniqueness of this representation. We are confronted with the following types of modular lattices.

\[
\begin{array}{c}
\text{modular} \\
\quad \text{unimodular} \begin{cases}
\quad \text{odd} \\
\quad \text{even}
\end{cases} \\
\quad \text{prime modular} \begin{cases}
\quad \text{diagonalizable} \\
\quad \text{non diagonalizable}
\end{cases}
\end{array}
\]

A lattice in the *non-diagonalizable case* can be shown to be sum of binary lattices (exactly as in the quadratic form case). The only hope to compute the image of the

---

4 This concept was inspired by Scharlau's definition of even forms (see [18]).

5 Notice that as rescaling can be done only by elements in the center, it is necessary to consider the cases of scales of odd and even valuation separately.
spinor norm for these lattices seems to be to show that the image is as big as possible for a sufficiently large family of those indecomposable binary lattices.

We summarize the results that we have found in the tables at the beginning of chapter 12. These results are used to study the behavior of spinor class numbers under extensions, (see proposition 14.4, p. 143).

**Remarks on notations.**

**Remark 1.** In all of this thesis, \( k, K, E \) denote number or local fields of characteristic 0, or algebraic extension of them. If \( k \) is a number field, \( \Pi(k) \) denotes the set of places of \( k \).

**Remark 2.** In all of the present thesis, an algebraic group means a linear algebraic group. All algebraic groups are assumed to be algebraic subgroups of the general linear group of a vector space \( V \), of finite dimension, over a sufficiently large algebraically closed field \( \Omega \) of characteristic 0. When we do not explicitly write the field (e.g. \( G, GL(V), SL_n(V) \)) we mean the \( \Omega \)-points of the group. When we work over a fixed local or number field \( k \), we say that \( G \) is defined over \( k \), if the equations defining \( G \) have coefficients in \( k \), (see section 2.1.1 in [13]). *This is the case for all groups considered here.* For any field \( E, k \subseteq E \subseteq \Omega \), we write \( G_E \) for the set of \( E \)-points of \( G \), (e.g. if \( G = GL(V) \), the set of \( k \) points is denoted \( GL_k(V) \)). The same conventions apply to spaces and algebras. All spaces and algebras are assumed to be finite dimensional.

\(^6\)In the sense that all localizations of number fields inject into \( \Omega \).
Exceptions to this rule are the multiplicative and additive groups. We denote $G_m = \Omega^*$, $G_a = \Omega$ when considered as algebraic groups. For the set of $k$-points we write $k^*, k$.

**Remark 3.** Unless otherwise stated, notation for a unitary group is as follows.

$$\mathcal{U}_n(\mathfrak{A}, h, V)$$

means that $\mathfrak{A}$ is an $\Omega$-algebra, $V$ a free $\mathfrak{A}$-module of rank $n$ and $h$ a hermitian or skew-hermitian form on $V$. If $\mathfrak{A}, V$ and $h$ are all defined over $k$, the set of $k$-points is written $\mathcal{U}_{n,k}(\mathfrak{A}, h, V)$. $V$ is omitted, if there is no risk of confusion. The orthogonal group of a quadratic form is written $\mathcal{O}_n(Q)$ or $\mathcal{O}_n(Q, V)$, where $n = \dim_{\Omega}(V)$, and same conventions apply.

**Remark 4.** The field on which a particular lattice is defined is always written as a subindex. If $K/k$ is an extension of local or number fields, and $\Lambda_k$ is a lattice in $V_k$, $\Lambda_K$ denotes the $\mathcal{O}_K$-lattice in $V_K$ generated by $\Lambda_k$.

**Remark 5.** If $G$ is an algebraic group acting on a space $V$, both defined over $k$, and $\Lambda_k$ is a $\mathcal{O}_k$-lattice on $V_k$, the stabilizer of $\Lambda_k$ in $G_k$ is denoted $G_k^{\Lambda_k}$. If $G = GL(V)$, this set is denoted $GL_k^{\Lambda}(V)$. Similar conventions apply to orthogonal or unitary groups.

**Remark 6.** Quaternion division algebras at finite places are written as $(\pi, \Delta)$ over $k$. Here $\pi$ is a uniformizing parameter of $k$ and $\Delta$ a unit in $k$ such that $k(\sqrt{\Delta})/k$ is unramified.

\footnote{See page 14 for definitions of tensors defined over $k$.}
i, j are assumed to be orthogonal skew-hermitian elements, (i.e., pure quaternions),
that satisfy $i^2 = i, j^2 = \Delta$. We also use the notation $\omega = \frac{i+i}{2} \in \mathcal{O}_{k(j)}$.

At real places we assume $i^2 = j^2 = -1$.

**Remark 7.** When denoting quadratic or skew-hermitian spaces or lattices, the fol­
lowing convention is adopted: If $(a_{i,j})$ is the Gram matrix, with respect to some
basis, we write $[a_{i,j}]$ for the space and $\langle a_{i,j} \rangle$ for the lattice generated by that basis.
In some examples, where the basis is not explicitly written, we denote its elements
by $(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1)$.

**Remark 8.** Whenever $K/k$ is a Galois extension of a number field $k$ and $v$ a place
of $k$, $w$ denotes a place of $K$ dividing $v$. We assume that one fixed such $w$ has
been chosen for every $v$. (This convention is also applied for infinite extension, e.g.
$K = k$). The only exception to this is the proof of proposition 9.6, (see the discussion
preceding the proof).

**Remark 9.** $G_{K/k}$ denotes the Galois group of the extension $K/k$. If there is no risk of
confusion, we write simply $G$. If $k$ is a number field, and localizations are considered,
we also use the notation $G_w = G_{K_w/k_w}$.

---

8Recall that $k$ is assumed to be of characteristic 0.
Part I

Background material.
CHAPTER 2

REVIEW OF GALOIS COHOMOLOGY

The main reference for the results in this chapter is [8], (chapter 1), or [13], (p. 13-26).

Definitions

Definition 2.1. Let $G$ be a finite group, $A$ a set provided with a $G$-action. The cohomology set $H^0(G, A)$ is defined as the set of fixed points $H^0(G, A) = A^G$. If $A$ has a group structure, $H^1(G, A)$ is defined as the quotient

$$H^1(G, A) = \{ \alpha : G \rightarrow A | \alpha(hg) = \alpha(h)\alpha(g)^h \} / \equiv,$$

where $\equiv$ is defined by

$$\alpha \equiv \beta \iff \text{there is a } a \in A \text{ such that } \alpha(g) = a^{-1}\beta(g)a^g \ \forall g \in G.$$

In what follows we write $\alpha_g$ instead of $\alpha(g)$.

In case that $A \subseteq B$ is a subgroup, there is a long exact sequence

$$0 \rightarrow A^g \rightarrow B^g \rightarrow (B/A)^g \rightarrow H^1(G, A) \rightarrow H^1(G, B),$$

and furthermore, under the obvious action of $B^g$ on $(B/A)^g$

$$(B/A)^g/B^g \cong \ker(H^1(G, A) \rightarrow H^1(G, B)). \tag{2.1}$$

$^1$This result is not found in [8], but can be found in [13] p.22, (see the paragraph following (1.11)).
If $A$ is normal in $B$, we have

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow (B/A)^G \longrightarrow H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, B/A).$$

In case $A$ is Abelian, (in particular, if it is central in $B$), the higher order cohomology groups are also defined and we have a long exact sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow (B/A)^G \longrightarrow H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, B/A) \longrightarrow H^2(G, A).$$

Finally, if $A$ and $B$ are both Abelian, this sequence extends to cohomology of all orders, (see for example [17] or [9]). All results of this section can be extended via direct limits to profinite groups acting continuously on discrete sets and groups (see for example [17] p.9 and p.42).

Examples.

The following is a known fact, (see for example [8], example 1, p.16).

**Proposition 2.2.** For any finite dimensional algebra $A$ defined over $k$, and any algebraic extension $K/k$ we have:

$$H^1(G_{K/k}, A_K^*) = 1.$$
Example 2.3. $GL_K(V) \cong (M_{n \times n}(K))^\ast$. Therefore, $H^1(G_{K/k}, GL_K(V)) = \{1\}$.  

Let $A$ be a finite dimensional central simple algebra, $E/k$ a Galois extension such that the reduced norm $N : A_E \longrightarrow E^\ast$ is surjective. The sequence

$$\{1\} \longrightarrow SL_1(A_E) \longrightarrow A_E^\ast \overset{N}{\longrightarrow} E^\ast \longrightarrow \{1\},$$

and 2.1 give

$$H^1(G_{E/k}, SL_1(A_E)) = k^\ast/N(A_E^\ast).$$

As a particular case, when $A_k = M_{n \times n}(k)$, we obtain the following result:

Example 2.4. For any field extension $K/k$ we have:

$$H^1(G_{K/k}, SL_{n,K}(V)) = \{1\}.$$

Tensors and $K/k$-forms.

In all of this thesis, a tensor of type $(l, m)$ on a space $V$, defined over $k$, means a $\Omega$-linear map $\tau : V^\otimes l \longrightarrow V^\otimes m$, where

$$V^\otimes r = \bigotimes_{i=1}^{r} V, \quad r \geq 1, \quad V^\otimes 0 = \Omega.$$ 

$\tau$ is said to be defined over $k$, if $\tau(V_k^\otimes l) \subseteq V_k^\otimes m$. $GL(V)$ acts on the set of tensors of type $(l, m)$ by $g(\tau) = g^\otimes m \circ \tau(g^\otimes l)^{-1}$. It makes sense, therefore, to speak about the stabilizer of a tensor.

\footnote{This result is known as Hilbert's theorem 90.}
Let $I$ be any set. By an $I$-family of tensors, we mean a map that associates to each element $i \in I$ a tensor $t_i$ of type $(n_i, m_i)$. $GL(V)$ acts on the set of all $I$-families by acting in each coordinate.

In all that follows, we say a family instead of an $I$-family unless the set of indices needs to be made explicit. Let $\mathcal{J}n$ be a family of tensors and $H = \text{Stab}_{GL(V)}(\mathcal{J}n)$. Then, $H$ is a linear algebraic group.

If $K/k$ is a Galois extension with Galois group $G = G_{K/k}$, we get an exact sequence

$$\{1\} \to H_K \to GL_K(V) \to X_K \to \{1\},$$

where $X_K$ is the $GL_K(V)$-orbit of $\mathcal{J}n$. It follows from 2.1 that

$$X_K^G/GL_K(V) \cong \ker(H^1(G, H_K) \to H^1(G, GL_K(V))) \cong H^1(G, H_K).$$

(Last line follows from Hilbert's theorem 90). The elements of the quotient in the left hand side can be thought of as isomorphism classes of pairs, $(W_k, \mathcal{J})$, which become isomorphic to $(V_k, \mathcal{J}n)$ when extended to $K$. These pairs are usually called $K/k$-forms of $(V, \mathcal{J}n)$, (or just $k$-forms, if $K = \bar{k}$).

**Example 2.5.** If $X$ is the set of quadratic forms on the space $V$, $Q \in X$, then $O_n(Q) = Stab_{GL(V)}(Q)$. Then equivalence classes of non-singular quadratic forms on $V_k$ are classified by $H^1(G_k, O_{n,k}(Q))$.  

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CHAPTER 3

ADELES AND IDELES

For an algebraic group $G$ defined over a number field $k$, we define the group of adelic points of $G$, which we denote by $G_{\mathbb{A}}^*$, by the formula

$$G_{\mathbb{A}}^* = \lim_{\to} \left( \prod_{v \in \Pi} G_{k_v} \times \prod_{v \in \Pi(k) - \Pi} G_{k_v}^\Lambda \right),$$

where $\Pi$ runs over the set of all finite subsets of $\Pi(k)$ containing all infinite places, and the connecting maps are inclusions. Notice that this definition does not depend on $\Lambda$ since any two such lattices are equal at almost all places, (see next section).

In particular, for the additive and multiplicative groups, we denote $A_k = (G_\alpha)_{\mathbb{A}}$, $J_k = (G_m)_{\mathbb{A}}$, and call them the adele group and the idele group of $k$.

In general, there is an inclusion

$$H^1(G_{K/k}, G_{\mathbb{A}_K}) \to \prod_v H^1(G_{K_v/k_v}, G_{K_v}),$$

where the image is the set of elements $x = (x_v)_v$ with $x_v$ trivial for almost all $v$, (see proposition 6.6. in [13]).

$G_{\mathbb{A}_K}$-action on the set of lattices.

For any lattice $\Lambda_k$ in $V_k$, its local component at $v$ is defined as its completion $\Lambda_{k_v}$ in $V_{k_v}$. A lattice is determined by its local components at all places. Two lattices
have equal local components at almost all places, and any set of local components satisfying this condition defines a lattice, (see [11] (81:14)).

If \( \Lambda_k \) is a lattice in a space \( V_k \), and \( \sigma \in G_{\mathbb{A}_k} \) is an adelic point of \( G \), we define \( \sigma \Lambda_k \) as the lattice satisfying the local relations

\[
(\sigma \Lambda_k)_v = \sigma_v \Lambda_{k_v}.
\]

We define the genus of \( \Lambda_k \) as its orbit under this action. The class of \( \Lambda_k \) is the set \( G_k \Lambda_k \), so that the set of classes contained in a genus is in one-to-one correspondence with the elements of the set of double cosets

\[
G_k \backslash G_{\mathbb{A}_k} / G^A_{\mathbb{A}_k}.
\]  \hspace{1cm} (3.1)

\( R \)-lattices.

Let \( R \) be a sub-ring of \( \text{End}_k(V) \), \( G \) a sub-group of \( GL_k(V) \). \( G \) is called an \( R \)-linear group, if \( G \subseteq C_{\text{End}_k(V)}(R) \). A lattice \( L_k \) is called a \( R \)-lattice if \( RL_k = L_k \). \( R \)-linear groups act on the set of \( R \)-lattices in a natural way. If \( R \) is a \( \mathcal{O}_k \)-algebra, we write \( R_k \) for \( R \) and define the localization \( R_{k_v} \) as the closure of \( R_k \) in \( \text{End}_{k_v}(V_{k_v}) \). \(^1\) Being a \( \mathcal{O}_k \)-algebra and a \( R_k \)-lattice are both local properties.

From here, it follows that if \( G \) is an \( R_k \)-linear group, and \( L_k \) an \( R_k \)-lattice, so is any other lattice in the \( G \)-genus of \( L_k \).

\(^1\)We also adopt the corresponding convention for algebraic extensions.
Spinor norm.

In all of this section, $K = \bar{k}$. Hence, $G = G_{k/k}, G_w = G_{\bar{k}/\bar{k}}$, (see remark 9 in page 10). For any semi-simple algebraic group $G$, with universal cover $\tilde{G}$, we have a short exact sequence

$$
\{1\} \rightarrow F \rightarrow \tilde{G} \rightarrow G \rightarrow \{1\}.
$$

This gives a long exact sequence in cohomology

$$
\{1\} \rightarrow F_k \rightarrow \tilde{G}_k \rightarrow G_k \rightarrow H^1(G, F).
$$

If $F = \mu_n$ is the set of $n$-th roots of 1, then

$$
H^1(G, F) = H^1(G, \mu_n) = k^*/(k^*)^n.
$$

This defines a map $\theta : G_k \rightarrow k^*/(k^*)^n$. This map is called the spinor norm on $G_k$.

**Example 3.1.** Notice that it does not suffice to require $F$ being cyclic, as we can see from the following example, which is a group of type $^2A_n$ in the notations of [8].

Let $G$ be the special unitary group of a hermitian form $h$ of dimension $n$ over a commutative two dimensional algebra $L$ such that $L_Q = \mathbb{Q}[i]$. Then $G_{\mathbb{Q}[i]}$ is isomorphic to the subgroup of $GL_{\mathbb{Q}[i]}(V) \times GL_{\mathbb{Q}[i]}(V)$ consisting of elements $(x, y)$ such that $x = y^{-1}$, and for which $\det(x) = \det(y) = 1$, (see [8], page 31), so the center $F$ of this group consist of $n$ roots of unity. In particular, it is cyclic of order $n$. If $n = 4$, the formula

$$
h(\lambda u, \lambda v) = N(\lambda)h(u, v)
$$

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shows that there are actually four of this elements defined over $\mathbb{Q}^2$, namely the four roots of unity in $L_\mathbb{Q}$. In particular, $G = G_{\mathbb{Q}/\mathbb{Q}}$ acts trivially on this set, so

$$H^1(G, F) \cong \text{Hom}(G, F) \cong \{ \text{cyclic extensions } K/\mathbb{Q}, \quad [K : \mathbb{Q}] = 4 \}.$$ 

These extensions do not correspond to extensions of the form $\mathbb{Q}(\sqrt[4]{\alpha})$ since $\mathbb{Q}$ do not contain the fourth roots of unity, e.g. $\mathbb{Q}(\sqrt[4]{2})$ is not normal over $\mathbb{Q}$. Therefore, there is no natural homomorphism between this cohomology set and $\mathbb{Q}^* / (\mathbb{Q}^*)^4$.

**Spinor norm and ideles.**

We start from the exact sequence:

$$0 \longrightarrow F \longrightarrow G \longrightarrow 0,$$

whence we obtain for any place $v \in \Pi(k)$, an exact sequence:

$$0 \longrightarrow F \longrightarrow \tilde{G}_v \longrightarrow G_v \longrightarrow 0.$$ 

From here, we get a long exact sequence in cohomology:

$$0 \longrightarrow F \longrightarrow \tilde{G}_v \longrightarrow G_v \overset{\Psi_v}{\longrightarrow} H^1(G_w, F),$$

and therefore, a map

$$\Pi_v \Psi_v : \prod_{v \in \Pi(k)} G_v \longrightarrow \prod_{v \in \Pi(k)} H^1(G_w, F).$$

\(^3\)Recall that determinant on the extension corresponds to reduced norm on the base field, and all roots of unity have norm 1.
If we assume that \( F \cong \mu_m \), so that we have an isomorphism:

\[
H^1(G_w, F) \cong (k_v^*)/(k_v^*)^m, \tag{3.2}
\]

it is known that the sequence

\[
\prod_{v \in \Omega(k)} \widetilde{G}_k \rightarrow \prod_{v \in \Omega(k)} G_k \rightarrow \prod_{v \in \Omega(k)} (k_v^*)/(k_v^*)^m
\]

can be restricted to adeles to get a sequence

\[
\widetilde{G}_{A_k} \rightarrow G_{A_k} \rightarrow J_k/J_k^m. \tag{3.3}
\]

The proof can be found in [12] (see lemma 13).

**Spinor norm and class number.**

The spinor norm provides us with a method to compute the class number of a lattice with respect to an algebraic group. \( G_k \backslash G_{A_k}/G_{A_k}^A \) is in general hard to compute, but if we apply to this quotient the spinor norm, we can get a partial classification. If we denote the kernel of the spinor norm by \( G'_{A_k} \), then, the quotient

\[
G_{A_k}/G_k G_{A_k}^A G'_{A_k},
\]

is in one-to-one correspondence with the quotient

\[
\Theta_{A_k}(G_{A_k})/\theta(G_k)\Theta_{A_k}(G_{A_k}^A), \tag{3.4}
\]

where \( \Theta_{A_k} \) is the spinor norm on the idele group, and \( \theta \) the spinor norm on the field. \( G_k G_{A_k}^A G'_{A_k} \)-orbits are called *spinor genera*, (see for instance [11], (102:7), for a deeper account of this subject).
Main Theorem in [7] is connected to this. It says that, if $G$ is simply connected, absolutely almost simple, and if $G_S$ is not compact, for a finite set $S$ of places of $k$, then $(G, S)$ has strong approximation, i.e.,

$$G_S G_k = G_{\mathbb{A}_k},$$

(3.5)

where $G_S = \prod_{p \in S} G_{k_p}$.

Assume that the universal cover $\tilde{G}$ of $G$ satisfies this condition for some $S$. Assume also that the kernel $F$ of the covering satisfies $F \cong \mu$, so that 3.3 applies. Then, we have the exact sequence

$$\tilde{G}_{\mathbb{A}_k} \xrightarrow{p} G_{\mathbb{A}_k} \xrightarrow{\theta} J_k/J_k^n.$$

If $\theta(g) = 1$, (i.e., $g\Lambda_k$ is in the spinor genus of $\Lambda_k$), then $g = p(h)$ for some $h \in \tilde{G}_{\mathbb{A}_k}$. If $h_n \xrightarrow{n \to \infty} h$, $h_n \in \tilde{G}_S \tilde{G}_k$, then $p(h_n) \in p(\tilde{G}_S \tilde{G}_k) \subseteq G_S G_k$ and $p(h_n) \xrightarrow{n \to \infty} p(h)$, so that $g \in \overline{G_S G_k}$.

In particular, since the stabilizer of a lattice is open and $G_S$ acts trivially on $S$-lattices, there is an element $g_k \in G_k$ such that $g\Lambda_k = g_k\Lambda_k$, or the spinor genus contains just one class.

In all of this thesis, we are concerned only with the case $S = \infty$. If $G$ has strong approximation with respect to $\infty$, one says that it has absolute strong approximation, (see for example [13] p.250).

Hasse principle for the spinor norm and localization.

The spinor norm satisfy the following Hasse principle:
Proposition 3.2. Let $k$ be a number field. If $G$ is a semi-simple algebraic group defined over $k$, and for $E = k, k_v$, $p_E : E^* \to E^*/(E^*)^2$ is the natural projection, $\theta_E : G_E \to E^*/(E^*)^2$ the spinor norm, we have

$$p_k^{-1}(\theta_k(G_k)) = \left( \prod_v p_k^{-1}(\theta_k(G_{k_v})) \right) \cap k.$$ 

Proof of proposition. The proof is a consequence of the Hasse principle for $G$ (see for example [8] theorem 1a, p.77). If fact, from the diagram:

\[
\begin{array}{ccc}
G_k & \longrightarrow & \prod_v G_{k_v} \\
\downarrow \theta_k & & \downarrow \prod_v \theta_{k_v} \\
H^1(G_k, \mu_n) & \xrightarrow{\phi} & \prod_v H^1(G_{k_v}, \mu_n) \\
\downarrow i_{\ast,k} & & \downarrow \prod_v i_{\ast,k_v} \\
H^1(G_k, \tilde{G}) & \xrightarrow{\chi} & \prod_v H^1(G_{k_v}, \tilde{G})
\end{array}
\]

If we suppose that $\phi(u) \in \text{im}(\prod_v \theta_{k_v})$, then we have

$$\prod_v i_{\ast,k_v}(\phi(u)) = 0 = \chi(i_{\ast,k}(u)),$$

but, by the Hasse principle, $\chi$ is injective, hence $i_{\ast,k}(u) = 0$, i.e., $u \in \text{im}(\theta_k)$ by exactness. This proves the claim. \]

Remark 3.3. It is proved in [8] that the spinor norm is surjective for finite places. Therefore, one must consider only archimedean places.
Spinor class field.

Introduction.

It is known, from class field theory, (see for example [10]), that there is a one-to-one correspondence,

\[ H \leftrightarrow L, \]

between open subgroups \( H \) of \( J_k \) containing \( k^* \) and finite Abelian extensions \( L/k \). The extension corresponding to \( k^*J_{k,\infty} \) is called the Hilbert class field of \( k \), (see for example [10]). It seemed desirable to have a corresponding interpretation for a more general algebraic group. In order to do this, we must replace the set of double cosets, (which in general has no additional structure), by a quotient of the idelic group \( J_k \), as in 3.4. Then, it is necessary to make sure that the resulting subgroup contains \( k^* \). This requires some additional adjustments in the image. This was done in [5], (section 2). We recall the main results in next section.

Construction of the spinor class field.

In this subsection, all notations are as in [5], except that we do not assume \( G \) to be an orthogonal group, but any group for which the spinor norm is defined, (see page 19). Let \( M \) be a lattice in \( V \). Let \( J_k^M \) be the subgroup of \( J_k \) defined by

\[ J_k^M = \{ j \in J_k | \forall \text{ finite } p, \exists \sigma \in G_k^M | \theta_p(\sigma) \}. \]
Furthermore, let $J_k(c)$ be the sets of ideles satisfying $^3 a \equiv 1 (\mod^* c)$ for a suitable $c$ divisible only by some infinite primes, (namely those at which the spinor norm is not surjective) $^4$. For any subset $X$ of $J_k$ we write $X(c) = J_k(c) \cap X$. Then, by Hasse principle, $\theta(G_k) = k^*(c)$. Notice that, because of the weak approximation theorem, we have $k^* J_k(c) = J_k$. Therefore, the third isomorphism theorem gives an isomorphism:

$$J_k/k^* = k^* J_k(c)/k^* \cong J_k(c)/k^* \cap J_k(c) = J_k(c)/\theta(G_k),$$

and there is a sequence of maps

$$J_k/k^* \xrightarrow{\cong} J_k(c)/\theta(G_k) \hookrightarrow J_k/\theta(G_k) \rightarrow J_k/\theta(G_k) J_k^M,$$

where the kernel of the composition is $k^* \theta(G_k^M)/k^*$. Furthermore, $J_k^M$ contains $\prod_{p \in \infty} \kappa_p$. Hence, the infinite coordinates can be arbitrarily adjusted. This shows $^5$ that the composed map is a surjection. Therefore, we have an isomorphism

$$J_k/k^* \theta(G_k^M) \cong J_k/\theta(G_k) J_k^M.$$

The group on the left-hand-side corresponds, via Artin map, to a field extension $\Sigma/k$. This is, by definition, the spinor class field of $M$. For the case of a quadratic form, see the reference [5].

$^3$See [10] for a definition of congruence mod* $c$.

$^4$See remark 3.3.

$^5$Since $J_k(c) J_k^M \supseteq J_k(c) \left( \prod_{p \in \infty} \kappa_p \right) \supseteq J_k$. 

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CHAPTER 4
SKEW-HERMITIAN FORMS, UNITARY GROUPS AND COVERINGS.

Reduced norm.

In all of this section we use the notation $x^y = yx^{-1}$. $N_{\mathbb{A}_k/k}$ denotes the reduced norm on the central simple algebra $\mathbb{A}_k$.

In this section we prove a standard formula to compute reduced norms on matrix rings over central simple algebras. Although we are just interested in the case of $2 \times 2$ matrices, this type of formulas can be generalized to matrices in any dimension. Let $\mathbb{A}_k$ be a central simple algebra defined over $k$. There is an extension $E/k$ such that $\mathbb{A}_E$ is a matrix algebra. The reduced norm is defined as the composition

$$\mathbb{A}_k \hookrightarrow \mathbb{A}_E \xrightarrow{\det} E.$$ 

It can be proved, (see for example [15], 2.3.34), that the image of the reduced norm is actually contained in the base field $k$.

**Proposition 4.1.** Suppose that we have a matrix $A \in M_{2 \times 2}(\mathbb{A}_k)$ given by

$$A = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix},$$

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where $q_1$ is invertible, then

$$N_{M_{2\times 2}(A_k)/k}(A) = N_{A_k/k}(q_1 q_4 - q_3^2 q_2).$$

If $q_4$ is invertible, we have

$$N_{M_{2\times 2}(A_k)/k}(A) = N_{A_k/k}(q_1 q_4 - q_2 q_3^2).$$

Proof of the proposition. This is immediate from the matrix equalities

$$\begin{pmatrix} 1 & 0 \\ -q_3 q_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} = \begin{pmatrix} q_1 & q_2 \\ 0 & -q_3 q_1^{-1} q_2 + q_4 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -q_2 q_4^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} = \begin{pmatrix} q_1 - q_2 q_4^{-1} q_3 & 0 \\ q_3 & q_4 \end{pmatrix},$$

since reduced norm is multiplicative and the extra factors have norm 1.]

Skew-hermitian forms over fields.

A spin group for skew-hermitian forms over a quaternion division algebra was already defined in [16]. A brief account is given in [2]. In this section, we list some of the properties that are used later in this work. See [2] for further details.

Let $(V, h)$ be a skew-hermitian $D$-vector space. Let $D = L \oplus jL$, where $L$ is a maximal commutative sub-algebra of $D$, $j \in D$, such that $\overline{j} = -j$, $\beta = j^2 \in O_k$ and conjugation by $j$ is the non trivial automorphism of $L$. Assume that $V, h, D, L, j$ are all defined over $k$. 

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Following this decomposition, we can write

\[ h(x, y) = f(x, y) + g(x, y)j, \]

where \( g \) is a bilinear form over \( L \). Then, one can define an algebra automorphism \( J \) on the even Clifford algebra of \( g \) by setting

\[ J(\lambda) = \bar{\lambda}, \ \lambda \in L, \]

\[ J(vw) = -\beta^{-1}(jv)(jw), \text{ for } v, w \in V, \]

for the generators and then show that the definition satisfy all the relations between the generators, and hence, it extends to an automorphism of the whole algebra.

The spinor group is defined by

\[ Spin_n(D, h) = \{ x \in Spin_{2n}(g) | Jx = x \}, \]

and it can be shown to be a double covering of the special unitary group \( SU_n(D, h) \).

The element \((s; \sigma)\) is defined by

\[ x \mapsto x - h(x, s)\sigma^{-1} s, \]

whenever \( s \in V_k \), and \( \sigma \in D_k^* \) satisfies \( \sigma - \bar{\sigma} = h(s, s) \). The elements of this type \((s; \sigma)\) (which are called simple rotations, of axis \( s \)) generate \( SU_{n,k}(D, h) \) for \( n \geq 2 \), and satisfy the formula

\[ \theta((s; \sigma)) = N(\sigma), \]

where \( \theta \) is the spinor norm and \( N \) the reduced norm on \( D_k \).
**Example 4.2.** We also have the formula

\[(s; \sigma)s = \sigma s.\]

Hence, in the case of rank 1, any element in the special unitary group has this form. (The proof reduces essentially to Hilbert’s theorem 90).

**Split-case.**

In case that \(D_k \cong M_{2 \times 2}(k)\), we can always assume

\[j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L_k = kP_1 \oplus kP_2,\]

where the matrices \(P_1\) and \(P_2\) are defined by

\[P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\]

so that \(j\) acts on \(L_k\) by interchanging the factors. If we write \(h\) in the form

\[h(x, y) = \begin{pmatrix} (x, y)_{1,1} & (x, y)_{1,2} \\ (x, y)_{2,1} & (x, y)_{2,2} \end{pmatrix},\]

then \((\ldots, \ldots)_{2,1}\) is a quadratic form satisfying \(^1\)

\[O_{2n,k}(\ldots, \ldots)_{2,1}, P_1 V) \cong U_{n,k}(D, h, V),\]

\(^1\)see lemma 4 in [2]. Note that \(\gamma\) in the reference is what we denote \((\ldots, \ldots)_{2,1}\).
and for \(x_1, x_2, y_1, y_2 \in P_1 V_k\), we have the formula \(^2\)

\[
h(x_1 + jx_2, y_1 + jy_2) = \begin{pmatrix} -(x_1, y_2)_{2,1} & (x_1, y_1)_{2,1} \\ -(x_2, y_2)_{2,1} & (x_2, y_1)_{2,1} \end{pmatrix}. \tag{4.1}
\]

Notice that if we set \(x_1 = y_1, x_2 = y_2, s = x_1 + jx_2\), this equation becomes

\[
h(s, s) = \begin{pmatrix} (x_1, x_1)_{2,1} & (x_2, x_1)_{2,1} \\ (x_1, x_2)_{2,1} & (x_2, x_2)_{2,1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The first factor is the Gram matrix of \((\ldots, \ldots)_{2,1}\) with respect to the basis \(\{x_1, x_2\}\).

**Hasse principle for the spinor norm.**

According to proposition 3.2, we have, for \(n \geq 2\):

\[
p_k^{-1}(\theta_k(\mathcal{U}_{n,k}(D, h))) = \{\rho \in k|\rho > 0 \forall \text{ real place } v \text{ with compact or non-split } \mathcal{U}_{n,k_v}(D, h)\}.
\]

In fact, we have that

\[
\prod_v p_{k_v}^{-1}(\text{im}(\theta_{k_v})) = \{\rho \in J_k|\rho_v > 0 \forall \text{ real place } v \text{ with compact or non-split } \mathcal{U}_{n,k_v}(D, h)\},
\]

\(^2\text{See (14) in [2]. For us, } \beta = 1, \gamma = (\ldots, \ldots)_{2,1}, \epsilon_1 = P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \epsilon_2 = P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
as can easily be seen from the fact that, for an infinite real place, the image is $\mathbb{R}^+$ for a definite quadratic form or for a skew-hermitian form over the standard quaternions.

Since all pure quaternions of the same norm are conjugate $^3$, and all positive real numbers are squares, it is not hard to see that any skew-hermitian real form is isometric to

$$\langle i \perp i \perp i \perp \ldots \perp i \rangle.$$

The image of the spinor norm is trivial, since it is spanned by elements of the form $N(\sigma)$, which are always positive. On the other hand, the set of vectors $v$, such that $h(v, v) = i$, is compact only in case $n = 1$, since otherwise it contains

$$te_1 + \sqrt{t^2 - 1}je_2$$

for arbitrarily large $t \in \mathbb{R}$. Since this set can be written as a continuous image of the unitary group, (by Witt’s theorem), the group itself cannot be compact. That is why both cases, compact and non-split, must be considered.

$^3$This follows immediately from Witt’s theorem if we consider the quaternion algebra as a four-dimensional quadratic space.
Spinor norm in hyperbolic planes.

A binary skew-hermitian space is called a hyperbolic plane, if it is isotropic. This is equivalent to having discriminant $^4 1$, or to be of the form

\[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix},
\]

(see for example [18]).

There is a simple way to compute spinor norm in hyperbolic planes without decomposing a rotation into simple rotations. To see this, we observe that the unitary group consist of those matrices satisfying

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{a} & \tilde{c} \\
\tilde{b} & \tilde{d}
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]

i.e.,

\[-b\tilde{a} + a\tilde{b} = 0, -c\tilde{d} + d\tilde{c} = 0, -b\tilde{c} + a\tilde{d} = 1.\]

From here we get $a\tilde{b}, d\tilde{c} \in k$, say $b = \lambda a, c = \mu d$, (the cases in which $a$ or $d$ is 0 can be handled in a similar way), and

\[
\left( \frac{1 - \lambda\mu}{N(d)} \right) a = d.
\]

Hence, we can write

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} q^{-1}
\]

\[^4\text{Discriminant of a skew-hermitian space is the reduced norm of its Gram matrix.}\]
for \( \alpha, \beta, \gamma, \delta \in k \) and a quaternion \( q \) satisfying 

\[
N(q) = \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},
\]

as can easily seen from the equation

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} q^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q^{-1} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \\
\begin{pmatrix} 0 & \frac{\alpha \delta - \gamma \beta}{N(q)} \\ -\frac{\alpha \delta - \gamma \beta}{N(q)} & 0 \end{pmatrix}.
\]

From the above expression we see that there exists a covering

\[
SL_2(V) \times SL_1(D) \longrightarrow \mathcal{U}_2(h, D),
\]

\((A, q) \mapsto Aq^{-1}, \) for \( \det(A) = 1, N(q) = 1, \)

so that any \( Aq^{-1}, A \in M_{2 \times 2}(k)^*, q \in D^*_k \) satisfying \( N(q) = \det(A), \) has the preimage

\[
\left( \frac{1}{\sqrt{\det(A)}} A, \frac{q}{\sqrt{N(q)}} \right),
\]

and therefore, its spinor norm is

\[
\theta(Aq^{-1}) = N(q)(k^*)^2 = \det(A)(k^*)^2.
\]
CHAPTER 5
MAXIMAL ORDERS AND LATTICES IN QUATERNION ALGEBRAS.

Maximal orders on quaternion algebras over local fields.

Let $k$ be a local field, $D$ a quaternion algebra defined over $k$. The reduced norm $q \mapsto N(q)$ defines a quadratic form on $D$.

It is known that for any local field, classes of $A$-maximal lattices for an ideal $A$ belong to only one class, (see for example [11] 91:2). From here, it follows, (by a reasoning in the space of pure quaternions) $^1$, that all maximal orders are conjugate.

For any local field, we have only two such quaternion algebras, namely, the (unique) quaternion division algebra, and the matrix algebra $M_{2\times 2}(k)$ $^2$.

In case $D_k$ is a division algebra, the quadratic form defined by the norm is anisotropic. Hence, (see for example [11], 91:1), there is a unique maximal lattice of

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$^1$There is a slight difficulty in the dyadic case with the elements of odd trace, but it is not too hard to solve.

$^2$This follows from the explicit description of the Brauer Group of a local field given, for example in [8], (theorem 2, p.50).

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integral scale, $D_k$. In fact, we can define an absolute value on $D_k$ by $|q|_{D_k} = |N(q)|_k$, so that $D_k$ is defined by

$$D_k = \{q \in D_k ||N(q)||_k \leq 1\}.$$ 

Notice that if $q_1, q_2 \in D_k$, then we have

$$|N(q_1q_2)|_k = |N(q_1)N(q_2)|_k = |N(q_1)|_k|N(q_2)|_k \leq 1,$$

hence $q_1q_2 \in D_k$, i.e., $D_k$ is an order.

If $K$ is a maximal subfield, which is unramified over $k$, and $i$ an element satisfying $i = -i$, $i^2 = \pi$, $\pi$ a uniformizing parameter of $O_k$, and $i\lambda i^{-1} = \bar{\lambda}$ for all $\lambda \in K$, then $D_k = K \oplus_k iK$ and we have

$$N(\lambda_1 + i\lambda_2) = N(\lambda_1) + \pi N(\lambda_2),$$

for $\lambda_1, \lambda_2 \in K$. Therefore, as $\nu_k(N(\lambda))$ is even for all $\lambda \in K$, we obtain, (by the dominance principle),

$$D_k = O_K \oplus iO_K.$$

Using this formulas, it is easy to check that $D_k$ is a principal local ring with uniformizing parameter $i$ and maximal ideal

$$M_k = m_K \oplus iO_K,$$

where $m_K$ is the maximal ideal of the ring of integers of the field $K$. 

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Lattices.

Let $V$ be a $\mathcal{D}$-vector space defined over $k$. A $\mathcal{O}_k$-lattice $L_k$ in $V_k$ is a $\mathcal{D}_k$-lattice if $\mathcal{D}_k L_k \subseteq L_k$ (see page 17).

If $\mathcal{D}_k$ is split, $L_k = P_1 L_k \oplus P_2 L_k$, (see [2] theorem 7). Therefore, if

$$s_1, \ldots, s_n, s_{n+1}, \ldots, s_{2n}$$

form a $\mathcal{O}_k$-basis for $P_1 L_k$, then

$$s_1 + js_{n+1}, \ldots, s_n + js_{2n}$$

is a $\mathcal{D}_k$-basis for $L_k$.

In the non-split case, existence of a basis follows from the fact that $\mathcal{D}_k$ is a principal left ideal domain\(^3\), whence any submodule of a free module is free, but if $e_1, \ldots, e_n$ is a basis of $V_k$, then

$$L_K \subseteq \pi^{-t} \bigoplus_{i=1}^n \mathcal{D}_k e_i$$

for $t$ big enough.

\(^3\)See [15], in particular 2.8.12 and 2.8.14.
Part II

Integral cohomology.
CHAPTER 6
LATTICES AND COHOMOLOGY.

Let \( k \) be a local or number field, \( K/k \) a Galois extension, \( G \subseteq GL(V) \) an algebraic group defined over \( k \), and \( \Lambda_k, L_k \) lattices on \( V_k \). Let \( G = G_{K/k} \).

**Proposition 6.1.** If there is an element \( \varphi \in G_K \) such that \( \varphi(L_K) = \Lambda_K \), then \( a_\varphi = \varphi^a \varphi^{-1} \) is a well defined element of \( H^1(G, G_K^A) \). It is independent of the choice of a particular element \( \varphi \), and depends only on the orbit of \( L_K \) under \( G_K \). The correspondence assigning, to a \( G_k \)-orbit of \( \mathcal{O}_K \)-lattices, an equivalence class of cocycles, is an injection. The image of this map is the kernel of the map

\[
H^1(G, G_K^A) \xrightarrow{i} H^1(G, G_K),
\]

where \( i \) is the inclusion.

**Proof.** Let \( X \) be the \( G_K \)-orbit of \( \Lambda_K \) in the set of lattices in \( V_K \). We have an exact sequence

\[
\{1\} \longrightarrow G_K^A \longrightarrow G_K \longrightarrow X \longrightarrow \{1\},
\]

whence by 2.1 we get

\[
X^G/G_k \cong \ker(H^1(G, G_K^A) \longrightarrow H^1(G, G_K)).
\]
Example 6.2. Using the fact that

\[ H^1(\mathcal{G}, GL_K(V)) = \{1\}, \]

we obtain that \( GL_k(V) \)-orbits of \( \mathcal{G} \)-invariant \( O_K \)-lattices are classified by the set \( H^1(\mathcal{G}, GL^A_K(V)) \).

If \( G \) is defined as the stabilizer of some extra structure, say a family of tensors, (as it is the case for a quadratic or hermitian form), we get a more precise result.

Recall that in page 14 we identified \( GL_k(V) \)-orbits of families of tensors with \( K/k \)-forms of \((V, Jn)\).

**Definition 6.3.** A lattice in \((V_K, Jn)\) is a pair \((\Lambda_K, Jn)\), where \( \Lambda_K \) is a lattice in \( V_K \). \( GL_K(V) \) acts on the set of pairs \((\Lambda_K, Jn)\), by acting on each component. Two lattices \( \Lambda_K, L_K \) are said to be in the same space if the pairs \((\Lambda_K, Jn) , (L_K, Jn)\) are in the same \( GL_K(V) \)-orbit.

**Proposition 6.4.** Let \( G \) denote an algebraic group defined over \( k \), which is defined as the stabilizer of a family of tensors \( Jn \) on \( V \). The set \( H^1(\mathcal{G}, G^A_K) \) is in one-to-one correspondence with the set of \( G_k \)-orbits of \( \mathcal{G} \)-invariant \( O_K \)-lattices in all the isomorphism classes of spaces that are \( K/k \)-forms of \((V, Jn)\). The kernel of the map

\[ H^1(\mathcal{G}, G^A_K) \overset{i}{\to} H^1(\mathcal{G}, G_K), \]

where \( i \) is the inclusion, correspond to those lattices that are in the same space as \( \Lambda_k \).
Proof. We have an action of $GL_K(V)$ on the set of all pairs $(L_K, \mathcal{I})$, where $L_K$ is a lattice and $\mathcal{I}$ a family of tensors, (with a fix index set, so that it is actually a set). If $T$ is the orbit of $(\Lambda_K, \mathcal{I}_n)$, we have a sequence
\[
\{1\} \longrightarrow G_K^\Lambda \longrightarrow GL_K(V) \longrightarrow T \longrightarrow \{1\},
\]
and the same argument as before applies. Last statement follows from the fact that spaces $(V_K, \mathcal{I})$ are classified by $H^1(\mathcal{G}_{K/\mathfrak{k}}, G_K)$, (see page 14 or [8] p.15). []

Now we try to classify families of tensors on the same lattice.

If we consider orbits under $GL_K^\Lambda(V)$ of a set $Y_K$, (which could be, for example, the set of families of tensors, on a fixed lattice $\Lambda_K$, that become isomorphic over $O_K$), we obtain an exact sequence
\[
\{1\} \longrightarrow G_K^\Lambda \longrightarrow GL_K^\Lambda(V) \longrightarrow Y_K \longrightarrow \{1\},
\]
(where
\[
G_K^\Lambda = Stab_{GL_K(V)}(y)
\]
for some $y \in Y$), whence we obtain that the set of orbits of $(Y_K)^G$ under $GL_K^\Lambda(V)$-action is indexed by
\[
\ker(H^1(\mathcal{G}, G_K^\Lambda) \longrightarrow H^1(\mathcal{G}, GL_K^\Lambda(V))).
\]

This applies in particular if $\Lambda_K = \Lambda_k \otimes_{O_k} O_K$ for suitable $\Lambda_k$. In this case, $L_K, \Lambda_K$ in the same $GL_K^\Lambda(V)$-orbit means that $L_k, \Lambda_k$ are isomorphic as $O_k$-modules.

Now we see a technique to obtain the intersection of both kernels in just one diagram. Let us start from the following commutative diagram with exact columns.
Here, $X_K$ is the $G_K$-class of $\Lambda_K$, and $Y_K$ the set of all lattices in $V_K$, isomorphic to $\Lambda_K$ as $\mathcal{O}_K$-modules.

Applying cohomology to this diagram, we get

$$\{1\} \rightarrow G_k^A \rightarrow GL_k^A(V) \rightarrow \cdots \rightarrow GL_k(V) \rightarrow X_K \rightarrow Y_K \rightarrow \{1\}$$

And from here, we can see that $G_k$-orbits of lattices, in the same space, that are
isomorphic to $\Lambda_k$ as $\mathcal{O}_k$-modules, and whose extensions to $K$ are in the same $G_K$-orbit as $\Lambda_K$, are in correspondence with the elements of the intersection of both kernels.

**Example 6.5.** If $\Lambda_k$ is free, the intersection of both kernels classifies free lattices on $V_k$ whose extensions to $K$ are in the same $G_K$-orbit.

**Remark 6.6.** Another way to obtain the same result is to observe that the definition, (see proposition 6.1), of the cocycle corresponding to a certain lattice $L_K$ as $a_\sigma = \varphi^\sigma \varphi^{-1}$, where $\varphi(L_K) = \Lambda_K$, does not depend on the algebraic group, in which $\varphi$ is contained. This approach has the advantage of working for any group $G$, whether or not it is defined as the stabilizer of a family of tensors.

We can put all this together in a proposition, if we make some definitions.

**Definition 6.7.** We say that a $\mathcal{O}_k$-lattice $\Lambda_k$ is **defined over** $k$, if $\Lambda_k \cong \mathcal{O}_k \otimes_{\mathcal{O}_k} \Lambda_k$ for some $\Lambda_k$. We say that it is a free lattice, (or more precisely, it is $k$-free), if $\Lambda_k$ can be chosen to be free.

**Definition 6.8.** Let

\[
\begin{align*}
\mathcal{L}_k(G, K/k) &= \{ a \in H^1(G, G_K^e) | a \text{ corresponds to a lattice defined over } k \}, \\
\mathcal{L}_{\text{free}}(G, K/k) &= \{ a \in \mathcal{L}_k(G, K/k) | a \text{ correspond to a free lattice} \}, \\
\mathcal{L}_V(G, K/k) &= \{ a \in H^1(G, G_K^e) | a \text{ correspond to a lattice in } (V_K, \mathfrak{n}) \}, \\
\mathcal{L}_{k, V}(G, K/k) &= \mathcal{L}_V(G, K/k) \cap \mathcal{L}_k(G, K/k), \\
\mathcal{L}_{\text{free}, V}(G, K/k) &= \mathcal{L}_V(G, K/k) \cap \mathcal{L}_{\text{free}}(G, K/k).
\end{align*}
\]
Let
\[ F_1 : H^1(G, G_K) \longrightarrow H^1(G, G_K), \]  
\[ F_2 : H^1(G, G_K^A) \longrightarrow H^1(G, GL_K^A(V)), \]
be the maps defined by the inclusions. Then we have the following proposition.

**Proposition 6.9.** Assume that \( \Lambda_k \) is free. The following identities hold:

\[ \mathcal{L}_V(G, K/k) = \ker F_1, \]
\[ \mathcal{L}_{\text{free}}(G, K/k) = \ker F_2, \]
\[ \mathcal{L}_{\text{free}, V}(G, K/k) = (\ker F_1) \cap (\ker F_2). \]

Later we give a similar interpretation to \( k \)-defined lattices.

**Example 6.10.**

\[ \ker(H^1(G_k/k, \mathcal{O}_{n_k}^\Lambda(Q)) \longrightarrow H^1(G_k/k, GL_k^A(V))) = \mathcal{L}_{\text{free}}(\mathcal{O}_n(Q), \overline{k}/k) \]

classifies free quadratic lattices that become isometric to \( \Lambda_k \) over some extension.

**Remark 6.11.** Notice that \( \mathcal{L}_V, \mathcal{L}_k, V, \mathcal{L}_{\text{free}, V} \) can still be defined, even if \( G \) is not the stabilizer of a family of tensors, as follows:

\[ \mathcal{L}_V(G, K/k) = \ker(H^1(G, G_K) \longrightarrow H^1(G, G_K)), \]
\[ \mathcal{L}_{k, V}(G, K/k) = \{ a \in \mathcal{L}_V(G, K/k) \mid \text{a corresponds to a lattice defined over } k \}, \]
\[ \mathcal{L}_{\text{free}, V}(G, K/k) = \{ a \in \mathcal{L}_V(G, K/k) \mid \text{a corresponds to a free lattice} \}. \]

In this case, the first and last identities of last proposition still hold. Notice that we can still interpret \( \mathcal{L}_V \) as lattices in the same space, because of proposition 6.1.
$H^1(\mathcal{G}, U_K)$ and the ideal group.

Assume, in all of this section, that $K/k$ is a finite Galois extension of local or number fields. $\mathcal{G} = \mathcal{G}_{K/k}$, $U_K = \mathcal{O}_K^*$.  

For any local or number field $E$, let $I_E$ be its group of fractional ideals, $P_E$ the subgroup of principal fractional ideals. There is a natural map $\alpha : I_k \rightarrow I_K$ defined by $\alpha(A) = A \otimes_{\mathcal{O}_k} \mathcal{O}_K$. Clearly, $\alpha(P_k) \subseteq P_K$, and we get a map $\alpha' : I_k/P_k \rightarrow I_K/P_K$. Notice also that we have the identities $P_E \cong E^*/U_E$, for $E = K, k$.

We start from the short exact sequence:

$$0 \rightarrow U_K \rightarrow K^* \rightarrow P_K \rightarrow 0,$$

whence we obtain a long exact sequence in cohomology:

$$0 \rightarrow U_k \rightarrow k^* \rightarrow (P_K)^G \rightarrow H^1(\mathcal{G}, U_K) \rightarrow H^1(\mathcal{G}, K^*) = 1.$$

The image of the map $k^* \rightarrow (P_K)^G$ is $\alpha(P_k)$. Therefore,

$$H^1(\mathcal{G}, U_K) \cong (P_K)^G / \alpha(P_k).$$

Now let us take an ideal $A = \prod_{p \in \Pi(k)} (\prod_{P | p} \mathcal{P}|^\beta(P))$. If $A$ is $\mathcal{G}$-invariant, all the powers $\beta(P)$, corresponding to prime divisors of the same prime of $k$, must be equal. In other words:

$$A = \prod_{p \in \Pi(k)} (\prod_{P | p} \mathcal{P})^\beta(p),$$

(6.3)

(where $\beta(p)$ is the common value of $\beta(P)$ for all $P$ dividing $p$), and this ideal is in $\alpha(I_k)$ if and only if $e_p$ divides $\beta(p)$ for all $p$. Hence, we have an exact sequence:

$$0 \rightarrow \ker \alpha' \rightarrow (P_K)^G / \alpha(P_k) \rightarrow \prod_{p \in \Pi(k)} (\mathbb{Z}/e_p),$$

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where the image of the last map correspond to those ideals, of the form (6.3), which are principal in $K$. In particular, since they all become principal in some extension, we can go to the algebraic closure, and its Galois group, and obtain a long exact sequence:

$$0 \longrightarrow I_k/P_k \longrightarrow H^1(G_{k/k}, U_k) \longrightarrow \prod_{p \in \Pi(k)} (\mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$ 

A refinement of this argument gives

$$H^1(G_{k/k}, U_k) \cong (I_k \otimes \mathbb{Q})/(P_k \otimes \mathbb{Z}) - Z.$$ 

**Special Linear Group and free lattices.**

Let $\mathcal{L}_k = \mathcal{L}_k(GL(V), K/k)$. Consider the first coordinate embedding, $\Phi : U_K \longrightarrow GL^K \Lambda(V)$, defined, for $u \in U_K$, by:

$$u \mapsto \begin{pmatrix} u & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

$\Phi$ induces a map in cohomology,

$$\Phi_* : H^1(\mathcal{G}, U_K) \longrightarrow H^1(\mathcal{G}, GL^K \Lambda(V)).$$

The image, under this map, of the subset $P_K \cap \alpha(I_k)$, in the notations of page 43, corresponds to $G_k$-orbits of lattices of the form:

$$A_K \oplus O_K \oplus \ldots \oplus O_{K},$$

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where $\mathcal{A}$ is defined over $k$. It is known, (see [11], (81:5)), that all lattices, over the ring of integers of a number (or local) field, have this form, whence we obtain the following result:

**Proposition 6.12.**

$$\mathcal{L}_k = \Phi_*(P_K \cap \alpha(I_k)).$$

Furthermore, $\Phi_*|_{P_K \cap \alpha(I_k)}$ is an injection.

However, a description of $\mathcal{L}_k$ in terms of kernels of maps would be a lot more useful. In order to do that, we need to be able to do localizations. We postpone it until the end of the chapter.

**Determinant class of a lattice.**

Let $[\mathcal{A}]$ be the $k^*$-orbit, of $\mathcal{O}_K$-ideals, corresponding to the $\mathcal{O}_K$-ideal $\mathcal{A}$.

In this subsection, we define an important invariant on the set of $\mathcal{G}$-invariant lattices. For this, we consider the following short exact sequence of $\mathcal{G}$-groups:

$$0 \longrightarrow SL^\mathcal{A}_{n,K}(V) \longrightarrow GL^\mathcal{A}_K(V) \xrightarrow{\det} U_K \longrightarrow 0,$$

which gives us the following long exact sequence in cohomology:

$$0 \longrightarrow SL^\mathcal{A}_{n,K}(V) \longrightarrow GL^\mathcal{A}_K(V) \longrightarrow U_K \longrightarrow H^1(\mathcal{G}, SL^\mathcal{A}_{n,K}(V))$$

$$\xrightarrow{\text{det.}} H^1(\mathcal{G}, GL^\mathcal{A}_K(V)) \xrightarrow{\text{det.}} H^1(\mathcal{G}, U_K).$$

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Actually, since it is easily seen that \( \det \circ \Phi = id \), we obtain that

\[
\det \circ \Phi = id = id,
\]

so that \( \det \) is surjective.

**Definition 6.13.** Let \( L_K \) be a \( \mathcal{G} \)-invariant lattice in \( V_K \), and let \( a = [a_{\sigma}]_\sigma \) the cocycle class defining the \( GL_k(V) \)-orbit of \( L_K \). Then we define the determinant class of \( L_K \), which we denote \( \det_\ast(L_K) \) by:

\[
\det_\ast(L_K) = \det_\ast(a) \in H^1(\mathcal{G}, U_K) \cong P_{K_f}^\mathcal{G}/\alpha(P_k),
\]

and we identify it with the corresponding ideal class.

**Example 6.14.** Consider the lattice

\[
L_K = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3 \oplus \ldots \mathcal{A}_n.
\]

If the ideal class of \( \mathcal{A}_i \) corresponds to the cocycle \([u_{i,\sigma}]_\sigma\), \( L_K \) corresponds to the cocycle defined by:

\[
a_\sigma = \begin{pmatrix}
    u_{1,\sigma} & 0 & 0 & \ldots & 0 \\
    0 & u_{2,\sigma} & 0 & \ldots & 0 \\
    0 & 0 & u_{3,\sigma} & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & u_{n,\sigma}
\end{pmatrix},
\]
whose determinant is given by:

\[
\det(a_\sigma) = \det \begin{pmatrix}
 u_{1,\sigma} & 0 & 0 & \ldots & 0 \\
 0 & u_{2,\sigma} & 0 & \ldots & 0 \\
 0 & 0 & u_{3,\sigma} & \ldots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & \ldots & u_{n,\sigma}
\end{pmatrix}
\]

\[= u_{1,\sigma} u_{2,\sigma} u_{3,\sigma} \ldots u_{n,\sigma},\]

which corresponds to the ideal class

\[[A] = [A_1 A_2 A_3 \ldots A_n].\]

**Example 6.15.** If \( k \) is a local field, with maximal ideal \( p \), and \( K/k \) is totally ramified of degree \( n \), \( K \) having maximal ideal \( \mathcal{P} \), then

\[
\det_*(\mathcal{P} \oplus \mathcal{P} \oplus \ldots \mathcal{P}) = [(\mathcal{P})^n] = [\alpha(p)] = 1,
\]

and also

\[
\det_*(\mathcal{O}_K \oplus \mathcal{O}_K \oplus \ldots \oplus \mathcal{O}_K) = 1,
\]

but the latter is defined over \( k \), and the first one is not. Therefore, \( \det_* \) is, in general, not injective.

However, we have the following result:

**Lemma 6.16.** \( \det_* \) is injective on \( \mathcal{L}_k \).
Proof of lemma. This is a direct consequence of proposition 6.12.]

Proposition 6.17. Suppose that $G \subseteq SL_{n,K}(V)$, then if

$$i_* : H^1(G, G^\Lambda_K) \rightarrow H^1(G, GL^\Lambda_K(V))$$

is the map induced by the inclusion, then

$$i_*^{-1}(\mathcal{L}_k) = \ker(i_*).$$

Proof of proposition. It is immediate from lemma 6.16 and the commutative diagram

$$
\begin{array}{ccc}
H^1(G, G^\Lambda_K) & \xrightarrow{i_*} & H^1(G, SL^\Lambda_{n,K}(V)) \\
\downarrow & & \downarrow \\
H^1(G, GL^\Lambda_K(V)) & \xrightarrow{\det} & H^1(G, U_K)
\end{array}
$$

that $\text{im}(i_*) \subseteq \ker(\det_*)$ and $\ker(\det_*) \cap \mathcal{L}_k = \{1\}$. []

In particular, such a group cannot identify a free lattice to a non-free $k$ defined lattice over any extension, (although it can identify a free lattice to a non-$k$-defined lattice, as the earlier example shows).

In this case, a description of $\mathcal{L}_{\text{free}}$ is equivalent to a description of $\mathcal{L}_k$, but the first problem has already been solved.

Localization.

Recall remarks 8 and 9 in page 10.
Notice that for an algebraic group $G$, there exist maps

$$F_v : H^1(G, G_K^A) \to H^1(G_w, G_{K_w}^A),$$

defined by inclusion and restriction. In relation to them, we have the following result:

**Lemma 6.18.** If $F_1 : H^1(G, G_K^A) \to H^1(G, G_K)$ is the map induced by the inclusion, and if it is known that

$$H^1(G, G_K) \to \prod_v H^1(G_w, G_{K_w})$$

is injective, then $\ker F_1 \supseteq \cap_v \ker F_v$.

**Proof of lemma.** Immediate from the following commutative diagram:

$$
\begin{array}{ccc}
H^1(G, G_K^A) & \xrightarrow{F_1} & H^1(G, G_K) \\
\downarrow \Pi_v F_v & & \downarrow r \\
\prod_v H^1(G_w, G_{K_w}^A) & \longrightarrow & \prod_v H^1(G_w, G_{K_w})
\end{array}
$$

[]

**Characterization of $L_k(G, K/k)$.**

$L_k(G, K/k)$ is the set of equivalence classes of lattices defined over $k$ that become isomorphic over $K$. A lattice $L_K$ is defined over $k$ if and only if it is generated by its $k$-points, i.e.,

$$L_K = \mathcal{O}_K(L_K \cap V_k).$$

This is a local property. On the other hand, for all localization $k_w$, any lattice defined over $k_w$ is free, i.e.,

$$L_{k_w}(G, K_w/k_w) = L_{free}(G, K_w/k_w).$$
Therefore, we obtain the following result:

**Proposition 6.19.**

\[ \mathcal{L}_k(G, K/k) = \ker \left( H^1(G, G_K^\text{ad}) \rightarrow \prod_v H^1(G_v, GL_{K_v}^\text{ad}(V)) \right). \]
CHAPTER 7
GENUS AND COHOMOLOGY.

C-genus for quadratic forms.

In this and next sections, we restrict ourselves to the case when $G$ is the orthogonal group of a quadratic form. Recall the definition of $L_k$, introduced in chapter 6, as the set of cocycle classes corresponding to classes of lattices defined over $k$. Recall also that $F_v$ was defined as the localization map

$$H^1(G, G^h_{K}) \rightarrow H^1(G_w, G^h_{K_w}),$$

where $G = G_{K/k}$, and $G_w = Stab_G(w)$, (see remark 9 in page 10), is the corresponding local Galois group.

In this section, we intend to interpret the cohomology of $G$, in terms of the genus of a quadratic lattice. For this, we introduce the following definition:

\textbf{Definition 7.1.} Let $L_k$ be a $O_k$-lattice that becomes isomorphic to $\Lambda_k$ over $K$. We say that $L_k$ is in the $(C, K)$-genus of $\Lambda_k$, (or the $C$-genus with respect to $K$), if the corresponding cocycle is in $L_k(G, K/k) \cap (\bigcap_v \ker F_v)$. We denote the set of such lattices by $C_{K-gen}(\Lambda_k)$ and call it the $(C, K)$-genus of $\Lambda_k$.

If the matrix of $\Lambda_k$ is $A$, this set is also denoted $C_{K-gen}(A)$. 

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**Proposition 7.2.** For any non-totally real extension \(^1\) \(K\),

\[ \text{spn}(\Lambda_k) \subseteq C_{K-\text{gen}}(\Lambda_k) \subseteq \text{gen}(\Lambda_k). \]

**Proof of proposition.** The second contention is clear, since the kernel of \(F_u\) contains exactly those forms which become isomorphic after the localization. The first contention follows from the fact that, if a lattice is in the same spinor genus as \(\Lambda_k\), the same holds for their extension to \(K\), and therefore, it becomes equivalent to \(\Lambda_K\) in this extension.

**Definition 7.3.** The absolute \(C\)-genus of \(k\) is defined by:

\[ C_{\text{gen}}(\Lambda_k) := C_{K-\text{gen}}(\Lambda_k). \]

The proposition applies clearly in this case.

**Lemma 7.4.** If \(K\) is non-totally-real, then the number of spinor genera contained in \(C_{K-\text{gen}}(\Lambda_k)\) is a power of 2.

**Proof of lemma.** To prove this we start by the observation that the set of spinor genera in a given genus is in one-to-one correspondence with the cosets of

\[ \mathcal{O}_{n,\Lambda_k}(Q) \mathcal{O}_{n,\Lambda_k}^\Lambda(Q) \mathcal{O}_{n,k}(Q) \]

in \(\mathcal{O}_{n,\Lambda_k}(Q)\). It can easily be checked that this correspondence is compatible with the injection

\[ \mathcal{O}_{n,\Lambda_k}(Q) \to \mathcal{O}_{n,\Lambda_K}(Q), \]

\(^1\)This is true, also, in the more general indefinite case.
in such a way that the spinor genera contained in $C_{K^{-\text{gen}}(A_k)}$ correspond to the elements in the kernel of the map

$$\frac{\mathcal{O}_{n,\Lambda_k}(Q)}{\mathcal{O}'_{n,\Lambda_k}(Q)\mathcal{O}^\Lambda_{n,\Lambda_k}(Q)\mathcal{O}_{n,\kappa}(Q)} \rightarrow \frac{\mathcal{O}_{n,\Lambda_k}(Q)}{\mathcal{O}'_{n,\Lambda_k}(Q)\mathcal{O}^\Lambda_{n,\Lambda_k}(Q)\mathcal{O}_{n,\kappa}(Q)},$$

which is a subgroup of the left hand side. As the order of the left hand side is a power of 2, this finishes the proof.[]

Example 7.5. The following example is taken from [1], (section 2.3): Consider the lattice given by the matrix

$$A_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 400 \end{pmatrix}.$$

Its genus contain four spinor genera, and twelve classes. Representatives of them are:

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 100 \end{pmatrix}, A_3 = \begin{pmatrix} 16 & 0 & -8 \\ 0 & 20 & 0 \\ -8 & 0 & 29 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 400 \end{pmatrix}, B_2 = \begin{pmatrix} 9 & -1 & 0 \\ -1 & 9 & 0 \\ 0 & 0 & 100 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 45 & -5 \\ 0 & -5 & 45 \end{pmatrix}, C_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 80 \end{pmatrix}.$$
Take a lattice with basis \{e_1, e_2, e_3\}, and define a quadratic form \(Q\) given by the matrix \(A_1\) with respect to this basis, i.e.,

\[
Q(e_1) = 1, Q(e_2) = 80, Q(e_3) = 100, B(e_i, e_j) = 0 (i \neq j),
\]

where \(B\) is the bilinear form obtained from \(Q\) by polarization.

Notice that \(Q(i e_2 + e_3) = -80 + 100 = 20\). Then, look for an element of the form \(\alpha e_2 + \beta e_3\), orthogonal to \(i e_2 + e_3\), i.e.,

\[
0 = B(i e_2 + e_3, \alpha e_2 + \beta e_3) = \alpha i Q(e_2) + \beta Q(e_3) = 80(\alpha i) + 100\beta
\]
or \(4\alpha i + 5\beta = 0\), which gives us \(\beta = 4i, \alpha = -5\). We observe that \(Q(-5 e_2 + 4 i e_3) = 2000 - 1600 = 400\), so the free lattice with basis \(\{e_1, i e_2 + e_3, -5 e_2 + 4 i e_3\}\) has matrix \(B_1\). A discriminant argument shows that the lattices are actually equal, hence \(A_2 \cong B_1\) over \(\mathbb{Z}[i] = \mathcal{O}_{\mathbb{Q}[i]}\). In particular, there are at least two spinor genera in \(\mathcal{G}_{\mathbb{Q}[i]}(A_2)\). (Notice that \(\mathbb{Q}[i]\) in non-totally-real).
Now consider a lattice with basis \( \{f_1, f_2, f_3\} \) and matrix \( C_2 \), i.e.,
\[
q(f_1) = 5, Q(f_2) = 16, Q(f_3) = 100, B(f_i, f_j) = 0 (i \neq j),
\]
and observe that \( Q(f_2 + (i\sqrt{3})f_1) = 1 \). The element \((5i\sqrt{3})f_2 - 16f_1\) is orthogonal to it, and satisfy
\[
Q(((5i\sqrt{3})f_2 - 16f_1) = (-75)16 + (256)5 = 80(-15 + 16) = 80,
\]
so that the lattice spanned by \( \{f_2 + (i\sqrt{3})f_1, (5i\sqrt{3})f_2 - 16f_1, f_3\} \) has matrix \( A_2 \), and therefore, we must have \( A_2 \cong C_2 \) over \( \mathbb{O}_{i\sqrt{3}} \supset \mathbb{Z}[i\sqrt{3}] \).

Now, we can apply proposition (7.2) to the non-totally real field \( \mathbb{Q}[i, \sqrt{3}] \) to show that \( C_{\mathbb{Q}[i, \sqrt{3}]-\text{gen}}(A_1) \) contains at least 3 spinor genera. Therefore, by lemma (7.4), we obtain
\[
C_{\mathbb{Q}[i, \sqrt{3}]-\text{gen}}(A_1) = \text{gen}(A_1).
\]
In particular, the absolute cohomological genus coincide with the classical genus, for a big enough extension of the base field. Our next goal is to prove that this is actually the case for any quadratic form.

**Absolute C-genus=Genus.**

Our Goal is to prove the following result:

**Proposition 7.6.** For any lattice \( \Lambda_k \), in a quadratic space \( (V_k, Q) \), we have
\[
C_{\text{gen}}(\Lambda_k) = \text{gen}(\Lambda_k).
\]
In order to do this we prove the following lemma:
Lemma 7.7. Let $\Lambda_k$ be a lattice in some quadratic space defined over $k$, and let $L_k$ be another lattice in the genus of $\Lambda_k$. Then, there exists an extension $K/k$ such that

$$L_K \in \text{spn}(\Lambda_K).$$

Proof of lemma. To prove this lemma, we recall that the spinor genera inside a particular genus correspond to the cosets in

$$J_k / D_k J_k^A,$$

where

$$D_k = \theta(O_{n,k}(Q)),$$

$$J_k^A = \{ i \in J_k | i_p \in \theta(O_{n,k_p}(Q)) \text{ for finite } p \},$$

and $\theta$ is the spinor norm.

It is easy to see that this correspondence is natural with respect to the inclusion of fields, hence it suffices to see that there exists an extension $K$, such that $J_k \subseteq D_K J_k^A$. It suffices to check that there is an extension $K$, such that $D_K J_k^A$ contains $^2 k J_k, oo$, since we can always replace $k$ by its Hilbert class field $k^H$, and it is known that $J_k \subseteq k^H J_k, oo$.

Now, since there are only finitely many classes in (7.1), and we have that

$$\theta(O_{n,k_0}(Q)) = O_{k_0}^*(k_v^*)^2/(k_v^*)^2,$$

for almost all $v$, we can choose the representatives in $k J_k, oo / D_k J_k^A$ of the form $\alpha i$

where

$$\alpha \in k, \ i \in \{ j \in J_k, oo \mid j_v = 1 \text{ for almost all } v \}. $$

$^2$Here, as usual $J_k, oo = \{ i \in J \mid ||i_v||_v = 1 \text{ for finite } v \}$. 

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Therefore, we can take an extension $K$, such that $\alpha$ is a square on $K$, and $\iota_v$ is a square in $O_{K_v}$ for all $v$, (since we only need to fix it at only finitely many places), so that
\[ \alpha \iota \in (K^*)^2 J_{K,\infty}^2 \subseteq D_K J_K^A. \]

Proof of proposition 7.6. So far, we have proven that for any lattice of the form $\sigma \Lambda_k$ for $\sigma \in O_{n,A_k}(Q)$, $\sigma \Lambda_K$ is in the same spinor genus as $\Lambda_K$ for some extension $K$. By going to a suitable non-totally real extension of $K$, it becomes an element of the same class. Therefore, it corresponds to an element in the cohomology set.

Lattices represented by cocycles.

In this section, $k$ is a number field.

The results of last sections have an important consequence. If we want to classify those lattices represented by cocycles, (i.e., those lattices that become isomorphic to the given lattice over some extension), it suffices to study the corresponding problem at all localizations. We state that as a lemma.

Lemma 7.8. Let $(V,Q),(V',Q')$ be quadratic spaces defined over $k$. Let $L_k$ be a lattice in $(V_k,Q)$, $\Lambda_k$ a lattice in $(V'_k,Q')$. Suppose that there is an extension $K/k$ so that $(V_K,Q) \cong (V'_K,Q')$, and $L_{K_P} \cong \Lambda_{K_P}$ for all places $P$ in $K$, then there is an extension $E/k$, such that $L_E \cong \Lambda_E$. 

\[ \]
In other words, it suffices to characterize the lattices that become locally isomorphic at all places for some extension. By going to an extension, we can assume that they are in the same space. Since two lattices in the same space are equal at almost all places, it suffices to require that, for each place, there is one local extension, over which the lattices become isometric.

Lemma 7.9. If \( v \) is a non-dyadic place, any two unimodular quadratic lattices became isometric on the unique unramified quadratic extension.

Proof of lemma. Let \( L_{k_v}, \Lambda_{k_v} \) be unimodular lattices. The hypothesis implies that the orthogonal groups of these lattices have good reduction, and its reductions are isometric on the unique quadratic extension of the residue field. In other words, if \( E = k_v(\sqrt{\Delta}) \) for a non-square unit \( \Delta, m_E = m_{k_v} \otimes_{k_v} \mathcal{O}_E \), then there is an isometry \( \phi \) between \( L_E/m_EL_E \) and \( \Lambda_E/m_E\Lambda_E \). Let \( \{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n\} \) be an orthogonal basis of \( L_E/m_EL_E \), which lifts to a basis \( \{e_1, e_2, \ldots, e_n\} \) of \( L_E \). Then, since \( \phi \) is an isometry, \( \{\phi(\tilde{e}_1), \phi(\tilde{e}_2), \ldots, \phi(\tilde{e}_n)\} \) is a basis of \( \Lambda_E/m_E\Lambda_E \), and therefore, it lifts to a basis \( \{f_1, f_2, \ldots, f_n\} \) of \( \Lambda_E \). Now \( Q(e_1) \equiv Q(f_1) \pmod{m_E} \). As both sides are units, we can apply the local square theorem, and obtain \( Q(e_1) = \alpha^2 Q(f_1) \), where \( \alpha \) can be chosen congruent to 1 \pmod{m_E} \). Replacing \( f_1 \) by \( \alpha f_1 \), we can assume \( Q(e_1) = Q(f_1) \). Now we replace \( f_i \) for \( i = 2, \ldots, n \) by \( f_i - bQ(f_1, f_i)Q(f_1)^{-1}f_1 \), (notice that \( bQ(f_1, f_i)Q(f_1)^{-1}f_1 \) is in \( m_E\Lambda_E \)), and \( e_i \) by \( \tau(e_i) \), where \( \tau \) is a rotation chosen \(^3\)

\(^3\)Which exists by Witt's theorem.
so that

\[ \tau(e_1) = f_1. \]

Now, we finish the proof by induction.\[\]

**Lemma 7.10.** Let \( v \) be a non-dyadic place. If \( L_{k_v} \) and \( \Lambda_{k_v} \) have Jordan components of the same dimension, i.e., if

\[
L_{k_v} = \bigoplus_{i=1}^{n} L_{i,k_v}, \quad \Lambda_{k_v} = \bigoplus_{i=1}^{n} \Lambda_{i,k_v}, \quad \dim_{k_v}(k_vL_{i,k_v}) = \dim_{k_v}(k_v\Lambda_{i,k_v}),
\]

where \( s(L_{i,k_v}) = s(\Lambda_{i,k_v}) = m_{k_v}^i \), then they become isomorphic over the unique unramified quadratic extension of \( k_v \).

**Proof of lemma.** Trivial from last lemma.\[\]

**Proposition 7.11.** Let \( v \) be a non-dyadic place. Two lattices become isometric on some extension, if and only if, they have Jordan components of the same dimension for corresponding scales.

**Proof.** Sufficiency comes from last lemma, necessity is clear.\[\]

To study the case of dyadic places, we need to look at theorem 93:28 in [11], which essentially says that, if two lattices have the same invariants, they are in the same class provided that certain conditions, ((i)-(iii)), involving representations over fields, are satisfied. This conditions are easily proved to be trivial over any sufficiently large extension. Therefore, it is enough to study the behavior of the fundamental invariants

\[ t, \dim L_{i,k_v}, s_i, w_i, a_i \]
under algebraic extensions, (see the reference for the definitions).

It is clear that \( t, \dim L_{i,k_u}, \xi_i \), must remain invariant under algebraic extensions. Any two elements, of the same size, became equal up to squares of units in a suitable extension. Hence, it suffices to know the ideal generated by the element \( a_i \), which is, by definition, the norm \( n_i \).

Therefore, to completely describe the set of lattices represented by cocycles, we must know the behavior of \( m_i, n_i \) under extensions.

For the norm, the following explicit formula is known, (see [11], p.227).

\[ n_i = \sum_j Q(x_j)\mathcal{O}_{k_u} + 2s_i, \]

where \( x_1, \ldots x_n \) is a basis of \( L_{i,k_u} \). From here, it follows that the norm is preserved under extensions.

Now, we have the following weight formula, (ref. [11], p.280).

\[ m_i = \sum_j b\mathfrak{d}\left(\frac{Q(x_j)}{b}\right) + 2s_i, \]

where \( b = Q(x_m) \) is chosen so as to make \( Q(x_m)\mathcal{O}_{k_u} \) largest, and \( \mathfrak{d} \) denotes the quadratic defect. \(^4\)

By taking a sufficiently large extension, \( K_w/k_w \), we can assume that

\[ \mathfrak{d}\left(\frac{Q(x_j)}{a}\right) = 0, \]

\(^4\)The quadratic defect is defined as follows. Any element \( \xi \) has at least one expression of the form \( \xi = \eta^2 + \alpha \), \( \eta, \alpha \in k_w \), then

\[ \mathfrak{d}(\xi) = \bigcap_{\xi = \eta^2 + \alpha} a\mathcal{O}_{k_u}. \]

\( \mathfrak{d}(\xi) \) is a fractional ideal or 0. \( \mathfrak{d}(\xi) = 0 \) if and only if \( \xi \in (k_w^\times)^2 \).
for all \( j \), hence \( w_i = 2s_i \), so that we can state the following proposition:

**Proposition 7.12.** Let \( v \) be a dyadic place. Two lattices become isometric on some extension, if and only if, they have Jordan components of the same dimension, and the corresponding scales and norms are equal.

\[
\prod_u F_v : H^1(G, G^A_K) \rightarrow \prod_{v \in \Pi(k)} H^1(G_v, G^A_{K_v}),
\]

where \( G = G_{K/k}, G_v = G_{K_v/K_v} \), (see remark 9 in page 10).

**Definition 7.13.** With the notations as above, we define

\[ WC_{K-gen}(\Lambda_k, G) = \ker(\prod_u F_v). \]

If \( G \) is defined as the stabilizer of a family of tensors, we define,

\[ C_{K-gen}(\Lambda_k, G) = WC_{K-gen}(\Lambda_k, G) \cap L_k(G, K/k), \]

\[ FC_{K-gen}(\Lambda_k, G) = WC_{K-gen}(\Lambda_k, G) \cap \text{free}(G, K/k). \]

We call these three sets, *wide C-genus*, *C-genus*, and *free C-genus*, respectively.
Proposition 7.14. For any linear algebraic group $G$, which is defined as the stabilizer of a family of tensors, we have

$$WC_{K-gen}(\Lambda_k, G) = C_{K-gen}(\Lambda_k, G).$$

Proof. In fact, it suffices to show that

$$WC_{K-gen}(\Lambda_k, G) \subseteq \mathcal{L}_k(G, K/k),$$

but this follows from proposition 6.19 and the commutative diagram

$$\begin{array}{ccc}
H^1(G, G^\Lambda_K) & \longrightarrow & H^1(G, GL^\Lambda_K(V)) \\
\downarrow & & \downarrow \\
\prod_{u \in \Pi(k)} H^1(G_w, G^\Lambda_{K_w}) & \longrightarrow & \prod_{u \in \Pi(k)} H^1(G_w, GL^\Lambda_{K_w}(V))
\end{array}$$

Definition 7.15. We define the $VC$-genus of $\Lambda_k$, for any algebraic group $G$, by the formula

$$VC_{K-gen}(\Lambda_k, G) = \ker \left( H^1(G, G^\Lambda_K) \longrightarrow H^1(G, G_K) \times \prod_{u \in \Pi(k)} H^1(G_w, G^\Lambda_{K_w}) \right).$$

If $G$ is defined as the stabilizer of a family of tensors, we have

$$VC_{K-gen}(\Lambda_k, G) = C_{K-gen}(\Lambda_k, G) \cap \mathcal{L}_V(G, K/k).$$

The $VC$-genus correspond to a set of equivalence classes of lattices in the same space. Therefore, its elements correspond \(^5\) to a subset of the set of double cosets

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\(^5\)This is true even if $G$ is not the stabilizer of a family of tensors.
In particular, it follows that, if some group $G$ has class number one, for some lattice $\Lambda$, the map

$$H^1(G, G^\Lambda) \rightarrow H^1(G, G_K) \times \prod_{w \in \mathcal{O}(k)} H^1(G_w, G^\Lambda_{K_w})$$

has trivial kernel.

This proves the following proposition:

**Proposition 7.16.** If $G$ has class number 1, with respect to a lattice $\Lambda_k$, then 7.2 has trivial kernel, for every Galois extension $K/k$. (Compare with corollary 4 on p. 491 of [13]).

This, in particular, applies to a group having absolute strong approximation $^6$. However, a lot more is true. In fact, we have the following result:

**Proposition 7.17.** If $G$ has absolute strong approximation over $k$, then the map 7.2 is injective.

**Proof.** If $\Lambda_K, L_K$ are two $G$-invariant $\mathcal{O}_K$-lattices, in the same space, that are locally in the same $G_{K_v}$-orbit for all $v$, we can choose elements $\sigma_v \in G_{K_v}$ such that $\sigma_v \Lambda_{K_v} = L_{K_v}$ for all $v$ and $\sigma_v = 1$ at almost all places. Now, any global element $\sigma$, close enough to $\sigma_v$ at all finite places where $\sigma_v \neq 1$, and stabilizing $\Lambda_{K_w} = L_{K_w}$ at the remaining finite places, satisfies $\sigma \Lambda_K = L_K$ as claimed. []

$^6$See the discussion following 3.5.
For the particular case of quadratic forms, proposition 7.6 gives a converse to proposition 7.16. More generally, whenever the hypothesis of lemma 6.18 is satisfied, (as is the case for quadratic forms), only wide $C$-genus needs to be defined, since we have the following result:

**Proposition 7.18.** If $G$ satisfy the hasse principle over the field $k$, (i.e., if the condition of lemma 6.18 is satisfied), then

$$VC_{K-gen}(\Lambda_k, G) = WC_{K-gen}(\Lambda_k, G).$$

In particular, if $G$ is the stabilizer of a family of tensors,

$$VC_{K-gen}(\Lambda_k, G) = C_{K-gen}(\Lambda_k, G).$$

Almost faithful actions.

In this sub-section, $k$ is a number field. $G$ is a linear algebraic group defined over $k$.

Proposition 7.17 refers to lattices in the space $V$, where $G \subseteq GL(V)$, i.e., the action of $G$ on $V$ is faithful. In the applications, we need to consider the case in which $G \rightarrow GL(V)$ has finite kernel.

**Definition 7.19.** Let $G$ be a linear algebraic group acting on a space $V$. We say that the action of $G$ on $V$ is almost faithful, if the kernel of the map $G \rightarrow GL(V)$ is finite. We also say that $V$ is an almost faithful $G$-module. It is called a faithful $G$-module, if the action is faithful. It is said to be defined over $k$, if the map $G \rightarrow GL(V)$ is defined over $k$. 
Proposition 7.20. Let $V$ be an almost faithful $G$-module, defined over $k$. $\Lambda_k$ a lattice in $V_k$. Then, there exists a faithful $G$-module $\widetilde{V}$, and a lattice $\widetilde{\Lambda}_k$ in $\widetilde{V}_k$, such that

$$G^\Lambda_k = G^\widetilde{\Lambda}_k.$$ 

Proof. Let $W$ be any faithful $G$-module, defined over $k$. Let $L_k$ be a fixed lattice in $W_k$. Let $\phi : G \to GL(V)$ be the map defining the module. Then, $\phi, \phi^{-1}$ have integral coefficients with respect to basis $\mathcal{B}$ of $\Lambda_k, L_k$, i.e., $G^\Lambda_{k_v} = G^L_{k_v}$, for almost all places $v$. At the remaining places, $G^\Lambda_{k_v}$ is compact, hence it is contained in a maximal compact subgroup $B_v$. From lemma 6 in [12], (p.555), there exists a local lattice $N_v$, satisfying $G^N_{k_v} = B_v$.

Let $N_k$ be the lattice defined by the following local conditions:

$$N_{k_v} = \begin{cases} L_{k_v} & \text{if } G^\Lambda_{k_v} = G^L_{k_v} \\ N_v & \text{otherwise} \end{cases}.$$ 

It follows that $G^N_k \supseteq G^\Lambda_k$.

Now, define $\widetilde{V} = V \oplus W$, ($G$ acting on each component), $\widetilde{\Lambda}_k = \Lambda_k \oplus N_k$. 

Proposition 7.21. Assume that $G$ satisfy the hasse principle over fields, and it has absolute strong approximation over $k$. If $\Lambda_K$ is a lattice in an almost faithful $G$-module $V$, then the map

$$H^1(G, G^\Lambda_K) \to \prod_{v \in \Pi(k)} H^1(G_v, G^\Lambda_{K_v})$$

is injective.

---

It could be necessary to replace $\Lambda_k, L_k$ by free lattices, but since any two lattices are equal at almost all places, it is irrelevant.
Proof. Because of last proposition, we can assume $V$ is faithful. From proposition 7.18, we see that the kernel of 7.2 equals the kernel of 7.3. Now we apply proposition 7.17. This implies that, the kernel is trivial. As this is true for all lattices, injectivity follows. 

Spinor norm and genera.

Let $G \subseteq GL(V)$ be a semi-simple group whose fundamental group is $F = \mu_n$. Then, the spinor norm is defined for $G$.

Let $\Lambda_k$ be any lattice in $V_k$. The proof of proposition 7.6, (p. 55), applies, virtually unchanged, to prove the following result:

Proposition 7.22. Let $K = \bar{k}$. Let $\text{gen}(\Lambda_k, G)$ denote the classical genus of $\Lambda_k$ with respect to $G$. Then, $VC_{K_{\text{gen}}}(\Lambda_k, G) = \text{gen}(\Lambda_k, G)$.

This result allows us to use cohomology to study the genus of any such group.
CHAPTER 8
INTEGRAL SKEW-HERMITIAN FORMS AND
COHOMOLOGY.

Now we study the cohomology of $U^h_{n,k}(D, h)$, and its relation with the genus.

In all of this chapter, $K/k$ is a Galois extension. Recall the conventions in remark 9 in page 10.

Notice that if two lattices are in the same genus, the same is true over any extension. Choosing an extension that split $D$, and applying the corresponding result for quadratic forms, we obtain the following consequence.

Proposition 8.1. Two skew-hermitian lattices in the same genus become isometric over some quadratic extension, i.e., $C$-genus is equivalent to genus for skew-hermitian lattices, (see chapter 7, p. 61).

In particular, they are always represented by cocycles, and the theory developed here applies.
Failure of the Hasse principle over fields.

It is proved in [8], (p.138), that the canonical map,

\[ H^1(G, \mathcal{U}_{n,K}(\mathcal{D}, h)) \rightarrow \prod_{\nu \in \bar{\Pi}(k)} H^1(G_{\nu}, \mathcal{U}_{n,K_{\nu}}(\mathcal{D}, h)), \]

is not in general injective, whence we cannot determine a skew-hermitian form entirely from its local behavior. For this reason the set of classes in

\[ G_k \backslash G_{\Lambda_k} / G_{\Lambda_{\bar{k}}}, \]

for \( G = \mathcal{U}_n(\mathcal{D}, h) \), do not correspond to all the lattice that are locally isomorphic to \( \Lambda_k \), but only to those that are contained in the same global space. In particular, lemma 6.18 cannot be applied, so that if \( K = \bar{k} \), the kernel of the map

\[ H^1(G, G_{\Lambda_k}^A) \rightarrow H^1(G, G_K) \]

needs also be considered when computing the cohomology set corresponding to classes in the same genus, (i.e., \( VC \)-genus does not equal \( C \)-genus, see page 61). The cohomology set defined as the kernel of the map

\[ H^1(G, G_{\Lambda_k}^A) \rightarrow \prod_{\nu \in \bar{\Pi}(k)} H^1(G_{\nu}, G_{\Lambda_{\nu}}^A) \]

gives a slightly different notion of genera. On the other hand, If we restrict ourselves to proper equivalence of skew-hermitian forms, (which makes no difference for extensions which do not split the quaternion algebra, and therefore, for extensions of odd degree), Hasse principle does hold at the field level, and again the distinction between the two concepts of class number become meaningless, ( see for example [8], end of p.135 ).

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The unitary group as a stabilizer of tensors.

Let $K/k$ be a Galois extension. $\mathcal{O}_K, \mathcal{O}_k$ the corresponding rings of integers, $\mathcal{D}$ a quaternion division algebra defined over $k$, $\mathcal{D}_k$ a maximal order in $\mathcal{D}_k$, and $\mathcal{D}_K = \mathcal{O}_K \mathcal{D}_k$. Let $(V_k, \mathcal{D}, h)$ be a skew-hermitian space over $\mathcal{D}$, defined over $k$, and let $\Lambda_k$ be a free $\mathcal{O}_k$-lattice in $V_k$. Let $L_K$ be a $\mathcal{G}$-invariant $\mathcal{O}_k$-lattice in $V_K$.

Notice that since $E$ is the center of $\mathcal{D}_E$, (for $E = k, K$), for any element $\alpha \in \mathcal{D}_E$, $v \mapsto \alpha v$ is a $E$ linear map on $V_E$. Therefore, $\mathcal{D}$ can be considered as a sub-ring of $\text{Hom}(V)$, defined over $k$. In particular, it can be regarded as a set of tensors. The unitary group is defined by the relations

$$U_n(\mathcal{D}, h) = \{g \in GL(V)|g^*(h) = h, g^*(q) = q, \forall q \in \mathcal{D}\}.$$  

Actually, it suffices to consider generators $^1 i, j$, (defined over $k$), of $\mathcal{D}$. They can be regarded as tensors of type $(1, 1)$. Under this identification, $h$ is a map from $V^{\otimes 2}$ to $\text{End}(V)$, or equivalently, a map from $V^{\otimes 3}$ to $V$, i.e., a tensor of type $(3, 1)$.

Hence, $U_n(\mathcal{D}, h)$ is an algebraic group defined as the stabilizer of a family of tensors, so that, proposition 6.4 applies, and we may state the following proposition:

**Proposition 8.2.** Let $G = U_n(\mathcal{D}, h)$ . Then, the set $H^1(G, G^n)$ is in one-to-one correspondence, with the set of $G_k$-orbits of $G$-invariant $\mathcal{O}_k$-lattices in all the skew-hermitian spaces that become isometric to $(V, \mathcal{D}, h)$ on $K$. The kernel of the map

$$H^1(G, G^K) \xrightarrow{\text{res}} H^1(G, G_K),$$

$^1$Notice that $\text{Stab}_{GL(V)}(i,j) = GL(\mathcal{D}, V)$ is the group of units of an algebra. Therefore, it has trivial cohomology. It follows that the pair $(V, \mathcal{D})$ has no non-trivial $K/k$-forms.
where i is the inclusion, correspond to those lattices that are in the same space as $\Lambda_k$. 

Notice that $SU_n(D, h)$ is a connected, semi-simple, algebraic group, so that, it satisfies the Hasse principle over fields. The following result follows:

**Proposition 8.3.** The map

$$H^1(G, SU_{n,K}^\Lambda(D, h)) \longrightarrow \prod_{v \in \Omega(k)} H^1(G_w, SU_{n,K_w}^\Lambda(D, h)),$$

classifies the lattices in the same genus, (over $k$), that become isometric over $K$.

**Genera and proper genera.**

In case $D_K$ is non-split, $U_{n,K}(D, h) = SU_{n,K}(D, h)$, whence

$$H^1(G, U_{n,K}^\Lambda(D, h)) = H^1(G, SU_{n,K}^\Lambda(D, h)).$$

This is also true at the local, non-split places.

At local split places, the unitary group of any lattice contains a reflection, (see [11], 91:4). Therefore, we have a short exact sequence

$$\{1\} \longrightarrow SU_{n,K_w}^\Lambda(D, h) \longrightarrow U_{n,K_w}^\Lambda(D, h) \stackrel{N}{\longrightarrow} \mu_2 \longrightarrow \{1\},$$

where $N$ is the reduced norm, whence we get a long exact sequence in cohomology,

$$\begin{array}{ccc}
U_{n,K_w}^\Lambda(D, h) & \stackrel{N}{\longrightarrow} & \mu_2 \\
\downarrow & & \downarrow \\
H^1(G_w, SU_{n,K_w}^\Lambda(D, h)) & \longrightarrow & H^1(G_w, \mu_2) \\
\end{array}$$

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Again, \( N \) is surjective, whence we obtain that

\[
H^1(G_w, SU_{n,k}(D, h)) \rightarrow H^1(G_w, U_{n,k}(D, h))
\]

has trivial kernel. Since this is true for all lattices, it is injective.

Because of these observations, to study the behavior of class number of skew-hermitian lattices under Galois extensions, it suffices to study the behavior of proper class number.

In next chapter, we develop a theory of behavior of class number for arbitrary connected, semi-simple groups \( G \). In particular, it will be applicable to \( G = SU_n(D, h) \).
CHAPTER 9

NORM PRINCIPLE FOR ARBITRARY SEMI-SIMPLE GROUPS.

In all of this chapter, $k$ is a number field, $K/k$ an Galois extension of odd degree, $V$ a vector space defined over $k$, $\Lambda_k$ a lattice in $V_k$.

The Hasse principle for simply connected, semi-simple groups, (see [13], p. 286), reduced the study of $VC$-genus, (which is the subset of the genus that become in the same class under some extension), to that of wide $C$-genus, (which is the kernel of the localization map). The results presented here are, however, independent of the Hasse principle, since we only require the contention

$$VC_{K-gen}(\Lambda_k, G) \subseteq WC_{K-gen}(\Lambda_k, G),$$

which is satisfied for any algebraic group $G$.

Galois extensions.

Let $G$ be a connected, semi-simple algebraic group, defined over $k$. We want to study the conditions, under which, the map,

$$H^1(G, G^A_K) \longrightarrow \prod_{v \in \Omega(k)} H^1(G_v, G^A_{K_v}),$$
is injective.

The short exact sequence

\[ \{1\} \longrightarrow F \longrightarrow \tilde{G} \overset{p_K}{\longrightarrow} G \longrightarrow \{1\}, \]

gives us an exact sequence

\[ 1 \longrightarrow F_K \longrightarrow \tilde{G}_K \overset{p_K}{\longrightarrow} G_K \overset{\theta_K}{\longrightarrow} H^1(G_{R/K}, F). \]

If we define the action of \( \tilde{G} \) on \( V \) via \( g.v = p(g)(v) \), \( \tilde{G} \) acts almost freely on \( V \).

Now, we restrict this sequence to stabilizers of \( \Lambda \). Exactness is preserved, because of the way the action was defined. This gives

\[ 1 \longrightarrow F_K \longrightarrow \tilde{G}^\Lambda_K \overset{p_K}{\longrightarrow} G^\Lambda_K \overset{\theta_K}{\longrightarrow} H^1(G_{R/K}, F), \]

which breaks into 2 subsequences,

\[ 1 \longrightarrow F_K \longrightarrow \tilde{G}^\Lambda_K \longrightarrow \text{im}(p_K) \longrightarrow 1, \]

and

\[ 1 \longrightarrow \text{im}(p_K) \longrightarrow G^\Lambda_K \overset{\theta_K}{\longrightarrow} \text{im}(\theta_K) \longrightarrow 1. \]

Now, from the short exact sequence

\[ \{1\} \longrightarrow F_K \longrightarrow \tilde{G}^\Lambda_K \longrightarrow \text{im}(p_K) \longrightarrow \{1\}, \]
we obtain the commutative diagram

\[
\begin{array}{ccccccccc}
H^1(G, F_K) & \overset{l_1}{\longrightarrow} & \prod_{v \in \Pi(k)} H^1(G_w, F_{K_w}) \\
\downarrow & & \downarrow \\
H^1(G, \tilde{G}_K^\lambda) & \overset{l_2}{\longrightarrow} & \prod_{v \in \Pi(k)} H^1(G_w, \tilde{G}_{K_w}^\lambda) \\
\downarrow & & \downarrow \\
H^1(G, \text{im}(p_K)) & \overset{l_3}{\longrightarrow} & \prod_{v \in \Pi(k)} H^1(G_w, \text{im}(p_{K_w})) \\
\downarrow & & \downarrow \\
H^2(G, F_K) & \overset{l_4}{\longrightarrow} & \prod_{v \in \Pi(k)} H^2(G_w, F_{K_w})
\end{array}
\]

Assume now that \((|F|, [K : k]) = 1\). Then, because of corollary 10.2 in [9],
\[H^m(G, F_K) = H^m(G_w, F_{K_w}) = 1\] for \(m = 1, 2\). Hence, we obtain a commutative diagram \(^1\).

\[
\begin{array}{ccccccccc}
H^1(G, \tilde{G}_K^\lambda) & \overset{\cong}{\longrightarrow} & H^1(G, \text{im}(p_K)) \\
\downarrow l_2 & & \downarrow l_3 \\
\prod_{v \in \Pi(k)} H^1(G_w, \tilde{G}_{K_w}^\lambda) & \overset{\cong}{\longrightarrow} & \prod_{v \in \Pi(k)} H^1(G_w, \text{im}(p_{K_w})).
\end{array}
\]

So, one map is injective, if and only if, so is the other. Now, according to the main result at the beginning of page 188 in [7], if \(G\) is an almost simple simply connected group, such that \([G_k]_\infty\) is non-compact, (i.e., \(G_k\) is of non-compact type) , then \(G\) has absolute strong approximation, so that proposition 7.21 applies.

\(^1\)Since this argument work for all \(G\)-invariant \(O_K\)-lattices, the kernel is trivial for all lattices, and therefore, the map is injective.
Again by corollary 10.2 in [9], we obtain \( H^1(G, \text{im}(\theta_K)) = \{1\} \), so we obtain a commutative diagram
\[
\begin{array}{ccc}
\frac{\text{im}(\theta_k)g}{\text{im}(\theta_k)} & \xrightarrow{\iota_5} & \prod_{v \in S} \frac{\text{im}(\theta_{K_v})^{\sigma_w}}{\text{im}(\theta_{h_v})} \\
\downarrow & & \downarrow \\
H^1(G, \text{im}(p_K)) & \xrightarrow{\iota_5} & \prod_{v \in \Pi(k)} H^1(G_w, \text{im}(p_{K_w})) \\
\downarrow & & \downarrow \\
H^1(G, G^h_K) & \xrightarrow{\eta} & \prod_{v \in \Pi(k)} H^1(G_w, G^h_{K_w}) \\
\downarrow & & \downarrow \\
\{1\} & & \{1\}.
\end{array}
\]

Hence, if we knew that \( l_5 \) is surjective, we would obtain that \( l_7 \) has trivial kernel.

Therefore, we can state the following result:

**Proposition 9.1.** If \( G \) is a semi-simple algebraic group of non-compact type, then no two classes of lattices in the same proper genus can become isometric over a Galois extension of degree relatively prime to \(|F|\), provided that the following condition is satisfied at all places:

\[
[\text{im}(\theta_{K_w})^{\sigma_w} = \text{im}(\theta_{h_w}).
\]

Notice than the condition, of \( G_k \) being of non-compact type, also implies that every spinor genus contains a single class.

We claim that this condition is a version of the norm principle, which we define now.
Definition 9.2. If $E/k$ is a finite algebraic extension and $G$ is a semi-simple algebraic group, $G \subseteq GL(V)$, $\Lambda_k$ a lattice in $V_k$. If $\theta_X = G_X^\Lambda \rightarrow H^1(G_{\overline{X}/X}, F)$ is the spinor norm, for $X = E, k$, (or any localization), we say that the norm principle is satisfied if

$$N(\text{im}(\theta_{E_u})) \subseteq \text{im}(\theta_{k_v})$$

where $v$ is the restriction of $u$ to $k$, for any place $u$ of $E$.

Notice that the reverse contention is trivial, in case $(|F|, [K : k]) = 1$.

In fact, we have the following result:

Proposition 9.3. Let $G$ be a finite group. If $M$ is a $G$-module, such that $m \mapsto |G|m$ is invertible on $M$, then $N(M) = M^G$, where $N$ is the norm element in $\mathbb{Z}[G]$.

Proof. From the short exact sequence

$$\{0\} \rightarrow \ker N \rightarrow \text{Coind}_{1}^{G}(M) \rightarrow N \rightarrow M \rightarrow \{0\},$$

we obtain, since $(\text{Coind}_{1}^{G}(M))^G \cong M$,

$$M \rightarrow M^G \rightarrow H^1(G, \ker N) = \{0\},$$

where we use corollary 10.2 in [9].

Remark 9.4. Proposition 9.1 extends the norm principle to semi-simple groups for which the spinor class field is not defined. However, it has the additional restriction of a Galois condition. As we will see in next section, this condition can be removed for some algebraic groups. In particular, this is the case for any group for which $F = \mu_n$.  

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Non-Galois Extensions.

Notations are as before. $E/k$ is an extension satisfying $(|F|, [E : k]) = 1$, and $K/k$ its Galois closure.

In all of this section, we will assume that $F$ satisfies the following two conditions:

(i) $H^1(G, F_K) \to \prod_{v \in S} H^1(G_w, F_{K_w})$ is surjective, for every finite set $S \subseteq \Pi(k)$.

(ii) $H^2(G, F_K) \to \prod_{v \in \Pi(k)} H^2(G_w, F_{K_w})$ is injective.

A quick look at the table, in [13], p. 332, shows that this is true, except maybe, in the following cases:

(a) $G$ is of type $^2A_n, ^3D_4, ^6D_4, ^2E_6$.

(b) $G$ is an adjoint $^2$ group of type $^2D_{2k+1}$.

The proof of this fact is essentially weak approximation on $G_m$, (or $R_{L/k}(G_m)$), for (i), and the Hasse principle for central simple algebras for (ii).

Lemma 9.5. If (i), (ii) are satisfied, and $[G_k]_{\infty}$ is not compact, then the map,

$$H^1(G, \text{im}(p_K)) \to \prod_{v \in \Pi(k)} H^1(G_w, \text{im}(p_{K_w})),$$

is injective.

\[ ^2\text{Notice that SU}_n(D, h) is not adjoint, (and in fact } F = \mu_2, \text{ so that, it is not an exception.} \]
Proof. It is an easy diagram chasing argument, using the diagram,

$$
\begin{array}{c}
H^1(G, F_k) \xrightarrow{l_1} \prod_{w \in S} H^1(G_w, F_{K_w}) \\
\downarrow \\
H^1(G, \tilde{G}_k^A) \xrightarrow{l_2} \prod_{w \in \Pi(k)} H^1(G_w, \tilde{G}_{K_w}^A) \\
\downarrow \\
H^1(G, \text{im}(p_K)) \xrightarrow{l_3} \prod_{w \in \Pi(k)} H^1(G_w, \text{im}(p_{K_w})) \\
\downarrow \\
H^2(G, F_k) \xrightarrow{l_4} \prod_{w \in \Pi(k)} H^2(G_w, F_{K_w}) \\
\end{array}
$$

where $S$ is the finite set outside of which $H^1(G_w, \tilde{G}_{K_w}^A) = \{1\}$, (see [13], corollary in p. 294), and the observation that it holds for any lattice. []

Now, we are ready to generalize the results in last section to non-Galois extensions.

Let $\mathcal{H}$ be the Galois group of $K/E$. For any place $u$ of $E$, we will denote $v$ for its restriction to $k$.

**Proposition 9.6.** Let extension $E/k$ be an arbitrary extension satisfying

$$(|F|, [E : k]) = 1.$$

Assume that $G$ is a semi-simple group such that

- $G$ is not a group of type $^2A_n$, $^3D_4$, $^6D_4$, $^2E_6$, nor is an adjoint group of type $^2D_{2k+1}$.
- $[G_{k}]_\infty$ is not compact.

Then, if $N(\text{im}(\theta_{E_u})) = \text{im}(\theta_{K_u})$ for any $u \in \Pi(E)$, not two classes in the same genus become equivalent over $E$. Furthermore, if $E/k$ is Galois, the same result holds without any restriction on the type of $G$. 

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Recall that we have an isomorphism:

\[ H^1(G_w, G^K_w) \cong H^1(G, \prod_{w|u} G^K_w), \]

given by Shapiro's lemma, (see [13], p. 25).

Since this proof contains heavy use of restriction maps, we use the second version, contrary to our custom.

**Proof.** For the last statement, see the preceding section. We prove the first one.

Consider the following diagram:

\[
\begin{array}{c}
\begin{array}{ccc}
\text{im}(\theta_K) & \xrightarrow{\iota_3} & \text{im}(\theta_{Kw}) \\
\text{im}(\theta_K) & \xrightarrow{\iota_2} & \Pi_{w \in \Pi(K)} \text{im}(\theta_{Kw}) \\
H^1(G, \text{im}(p_K)) & \xrightarrow{\iota_1} & H^1(G, \prod_{w|u} \text{im}(p_{Kw})) \\
\phi_3 & & \phi_2 \\
H^1(G, G^K_K) & \xrightarrow{\iota_r} & H^1(G, \prod_{w|u} G^K_{Kw}) \\
\phi_1 & & \phi_1 \\
H^1(G, \text{im}(\theta_K)) & & \\
\end{array}
\end{array}
\]

and the corresponding one for \( \mathcal{H} \)-action

\[
\begin{array}{c}
\begin{array}{ccc}
\text{im}(\theta_K) & \xrightarrow{\iota_3} & \text{im}(\theta_{Kw}) \\
\text{im}(\theta_K) & \xrightarrow{\iota_2} & \Pi_{w \in \Pi(E)} \text{im}(\theta_{Kw}) \\
H^1(\mathcal{H}, \text{im}(p_K)) & \xrightarrow{\iota_1} & H^1(\mathcal{H}, \prod_{w|u} \text{im}(p_{Kw})) \\
\phi_3 & & \phi_2 \\
H^1(\mathcal{H}, G^K_K) & \xrightarrow{\iota_r} & H^1(\mathcal{H}, \prod_{w|u} G^K_{Kw}) \\
\phi_1 & & \phi_1 \\
H^1(\mathcal{H}, \text{im}(\theta_K)) & & \\
\end{array}
\end{array}
\]

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We have restriction maps \( \text{res}^G \) mapping each set in the first diagram to the corresponding set in the second diagram. We denote \( \text{res}^G(x) = x|_\mathcal{H} \) in all cases. This restriction maps commute with the maps in each diagram, in the following sense: for \( f = l_i, \psi_i, \phi_i \), we obtain \( \tilde{f}(x|_\mathcal{H}) = f(x)|_\mathcal{H} \).

Let \( b \in \ker(l_7) \), be such that, \( b|_\mathcal{H} = 1 \). Then, \( \psi_3(b)|_\mathcal{H} = 1 \), i.e., \( \psi_3(b) = 1 \), (see [9], 10.4, p. 85). By exactness, there exists \( a \in H^1(\mathcal{G}, \text{im}(p_K)) \), such that, \( \psi_2(a) = b \). If we can prove \( a \in \text{im}(\psi_1) \), it follows that \( b = 1 \).

Let \( \rho_k, \rho'_k \) be the canonical projections,

\[
\rho_k : [\text{im}(\theta_K)]^G \rightarrow \frac{[\text{im}(\theta_K)]^G}{\text{im}(\theta_k)} ,
\]

\[
\rho'_k : \left[ \prod_{w \in \Pi(K)} \text{im}(\theta_K) \right]^G \rightarrow \frac{\left[ \prod_{w \in \Pi(K)} \text{im}(\theta_K) \right]^G}{\prod_{w \in \Pi(K)} \text{im}(\theta_k)} .
\]

\( \rho_E, \rho'_E \) are defined in a similar way. Notice that, for \( x \in [\text{im}(\theta_K)]^G \subseteq [\text{im}(\theta_K)]^K \), we have \( \rho_k(x)|_\mathcal{H} = \rho_E(x) \). Same applies to localizations.

As \( \phi_2 l_5(a) = 1 \), there exists \( m \in \left[ \prod_{w \in \Pi(K)} \text{im}(\theta_K) \right]^G \), such that, \( \phi_1(\rho'_k(m)) = l_5(a) \). Since \( b|_\mathcal{H} = 1 \), we obtain \( a|_\mathcal{H} = \tilde{\phi}_1(\rho_E(e)) \), for some \( e \in [\text{im}(\theta_K)]^K \). Hence,

\[
\tilde{\phi}_1 l_5(\rho_E(e)) = \tilde{l}_5(a|_\mathcal{H}) = l_6(a)|_\mathcal{H} = \phi_1(\rho'_k(m))|_\mathcal{H} = \tilde{\phi}_1(\rho_k(m)|_\mathcal{H}) .
\]

As \( \tilde{\phi}_1 \) is injective, \( \tilde{l}_5(\rho_E(e)) = \rho_k(m)|_\mathcal{H} \).

We claim that the norm principle implies \( l_5(\rho_k(e^{tN})) = \rho'_k(m) \). Assuming the claim, \( l_6(\psi_1(\rho_E(e)^{tN})) = l_6(a) \). As \( l_6 \) is injective, by the previous lemma, \( a = \psi_1(e^{tN}) \).

Now, we prove the claim. Consider the functions

\[
g : [\text{im}(\theta_K)]^G \rightarrow \left[ \prod_{w \in \Pi(K)} \text{im}(\theta_K) \right]^G .
\]
\[ \tilde{g}: [\text{im}(\theta_K)]^\mathcal{H} \longrightarrow \left[ \prod_{\omega \in \Omega(K)} \text{im}(\theta_{K_{\omega}}) \right]^\mathcal{H}. \]

Then, \( \tilde{l}_5(\rho_E(e)) = \rho_k(m)|_{\mathcal{H}} \rho_E(m) \), which means \( \rho_E(\tilde{g}(e)m^{-1}) = 1 \), i.e., \( \tilde{g}(e)m^{-1} \in \prod_{\omega \in \Omega(E)} \text{im}(\theta_{E_{\omega}}) \). By the norm principle, \( N(\tilde{g}(e)m^{-1}) \in \prod_{v \in \Omega(k)} \text{im}(\theta_{k_v}) \). Chose \( t \) such that \( t[E : k] = 1 \ (\text{mod } |F'|) \), then

\[ \tilde{g}(N(e)t)m^{-1} = g(N(e)t)m^{-1} \in \prod_{v \in \Omega(k)} \text{im}(\theta_{k_v}). \]

In other words, \( l_5(\rho_k(N(e)t)) = \rho_k(m) \). \( \square \)
Part III

Integral quaternions and Skew-hermitian lattices.
CHAPTER 10
EVEN SKEW-HERMITIAN ELEMENTS.

In all of this chapter, $D$ denotes a quaternion division algebra, defined over the local field $k$, and $D_k$ its unique maximal order.

One call an element $q$ of $D_k$, skew-hermitian, (or a pure quaternion), if $\bar{q} = -q$, and write $q \in D_k^0$. Now assume $q \in D_k^0 = D_k^0 \cup D_k$. $q$ is called even if there is an element $h \in D_k$ such that $q = h - \bar{h}$. If 2 is a unit, every skew-hermitian element is even, since it suffices to take $h = \frac{q}{2}$.

**Proposition 10.1.** Let $q$ be a unit in $D_k$, such that $k(q)/k$ is unramified. Then, $q$ is even.

**Proof.** There is nothing to prove, unless $k$ is dyadic. In this case, $k(q)$ must contain a unit of trace 1, since the trace is surjective for an extension of finite fields, and Henssel's lemma applies. If $\eta$ is such a unit, it must satisfy $\bar{\eta} = -\eta + 1$. Hence, $\eta - \bar{\eta} = 2\eta - 1$, which is a skew-hermitian unit, so it must be of the form $uq$ for $u \in O_k^*$, but then

$$u^{-1}\eta - \bar{u^{-1}}\eta = q.$$
Corollary. If $k$ is a local field, $D_k$ the unique non-split quaternion division algebra over $k$, $D_k$ a maximal order in $D_k$, then $D_k$ contains an even unit.

Proof. $D_k$ contains an unramified extension.

Definition 10.2. We call a skew-hermitian element odd, if it is not even.

Proposition 10.3. In the same notations, if $k$ is dyadic, and $q^2 = \pi$ is a uniformizing parameter, then $q$ is odd.

Proof. Observe first that for any $h \in D_k$, $k(h) = k(h - \overline{h})$. Any element of $O_k(q)$ is of the form $\alpha + q\beta$, with $\alpha, \beta \in O_k$, so that any even skew-hermitian element is of the form $2q\beta$. 

Lemma 10.4. The set $E$, of even elements, is an additive subgroup.

Proof. Trivial.

In all that follows, $i, j$ are orthogonal skew-hermitian elements, (i.e., $ij = -ji$), such that $i^2 = \pi, j^2 = \Delta$, (see remark 6 in page 9).

Proposition 10.5. The set of even elements is

$$E = 2O_{k(j)i} \oplus O_{kj}.$$ 

Proof. Since we know it contains $O_{k(j)i}$, it suffices to find those elements in $O_{k(j)i}$ that are even, but this is clear from the proof of proposition 10.3.
Remark: \( \mathcal{D}_k/\mathcal{E} \) is a \( \mathcal{O}_{k(j)}/2\mathcal{O}_{k(j)} \)-module. In particular, if \( k/Q_2 \) is unramified, it is a \( \mathbb{F}_{k(j)} \)-vector space of dimension 1.

More generally, there exist a natural filtration

\[
\mathcal{D}_k^0 = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \mathcal{E}_2 \supset \ldots \supset \mathcal{E}_{e-1} \supset \mathcal{E}_e = \mathcal{E},
\]

where \( \mathcal{E}_r = \pi^r \mathcal{O}_{k(j)}i \oplus \mathcal{O}_{k,j} \), and \( (\pi)^e = (2) \).

Remark:

\[
\mathcal{E}_r = \mathcal{E} + [((\pi^r) \cap \mathcal{D}_k^0)].
\]

In particular, \( \mathcal{E}_r \) does not depend on the choice of \( i,j \), and it is invariant by conjugation.

Definition 10.6. We define the oddity defect of a skew-hermitian element by

\[
\nu_k(q) = \max \{ i | q \in \mathcal{E}_i \}.
\]

We call an element \( q \) totally odd if \( \nu_k(q) = 0 \).

Algebraic properties of odd and even elements.

Lemma 10.7. Any uniformizing parameter of \( \mathcal{D}_k \) is totally odd.

Proof. This is immediate from the formula

\[
\mathcal{E}_r = \pi^r \mathcal{O}_{k(j)}i \oplus \mathcal{O}_{k,j}.
\]

The next lemma is useful when studying lattices.
Lemma 10.8. All even units are congruent.

Proof. It suffices to show that they are all congruent to $j$. Now, notice that

$$(\alpha + i\beta)j(\bar{\alpha} - i\beta) = (N(\alpha) + \pi N(\beta))j + 2i(j\beta\bar{\alpha}).$$

An arbitrary even skew-hermitian unit can be written as $\varepsilon j + 2i\gamma$, where $\varepsilon \in \mathbb{O}_k^*$, so it suffices to solve, in $k(j)$,

$$\beta\bar{\alpha} = j^{-1}\gamma, \quad N(\alpha) + \pi N(\beta) = \varepsilon,$$

i.e., $N(\beta)N(\bar{\alpha}) = -\Delta^{-1}N(\gamma)$, $N(\alpha) + \pi N(\beta) = \varepsilon$. This means that $N(\alpha), \pi N(\beta)$ are roots of

$$x^2 - \varepsilon x - \pi \Delta^{-1}N(\gamma) = 0,$$

whose roots are

$$x = \frac{\varepsilon \pm \sqrt{\varepsilon^2 + 4\pi \Delta^{-1}N(\gamma)}}{2}.$$

Now, $\delta = \varepsilon^2 + 4\pi \Delta^{-1}N(\gamma)$ is a square, by the local squares theorem. Let $\delta = \varepsilon^2(1+\rho)^2$, then

$$2\rho + \rho^2 = 4\pi \Delta^{-1}N(\gamma)e^{-1}.$$

If $|\rho|_{\mathcal{O}_k} > |2|_{\mathcal{O}_k}$ the dominance principle gives us a contradiction. Therefore, both roots are integers. Considerations on the sum and the product show that exactly one is unit. Hence, $N(\alpha), N(\beta)$ can be determined. Now, we choose any $\alpha$ with the suitable norm, (which exists, since all units are norms), and find $\beta$ from the equation $\beta\bar{\alpha} = j^{-1}\gamma$.]

We have a similar result for other skew-hermitian elements. For this we need a lemma.
Lemma 10.9. Any two pure quaternions $a_1, a_2$, with the same norm are conjugate, i.e., $\exists \lambda \in \mathcal{D}_k$ such that $a_1 = \lambda a_2 \lambda^{-1}$.

Proof. Because of Witt's theorem for spaces, there is an element $\sigma$ in the orthogonal group $\mathcal{O}_{3,k}(N, \mathcal{D}^0)$ such that $\sigma(a_1) = a_2$. Since $(\mathcal{D}^0_k, N)$ is anisotropic, $a_1 k \oplus a_2 k$ is non-singular, so that, multiplying by a reflection on $(a_1 k \oplus a_2 k)^\perp$, we can assume $\sigma \in \mathcal{S}\mathcal{O}_{3,k}(N, \mathcal{D}^0)$. On the other hand, $\mathcal{D}_k^*$ acts on $\mathcal{D}^0_k$ by conjugation, preserving the norm. Comparing dimensions, we see that over the algebraic closure, \footnote{One way to do this is to do it by Lie algebra computations over the complex numbers, and then use the fact that the theory of fields of characteristic 0 is complete.} the map

$$\mathcal{D}_k^* \rightarrow \mathcal{S}\mathcal{O}_{3,k}(N, \mathcal{D}^0)$$

is surjective, so there is a short exact sequence

$$1 \rightarrow \mathbb{k}^* \rightarrow \mathcal{D}_k^* \rightarrow \mathcal{S}\mathcal{O}_{3,k}(N, \mathcal{D}^0) \rightarrow 0,$$

which gives, upon applying cohomology,

$$1 \rightarrow k^* \rightarrow \mathcal{D}_k^* \rightarrow \mathcal{S}\mathcal{O}_{3,k}(N, \mathcal{D}^0) \rightarrow H^1(G, k^*) = 1.$$

Therefore, $\mathcal{D}_k^* \rightarrow \mathcal{S}\mathcal{O}_{3,k}(N, \mathcal{D}^0)$ is a surjection.[]

Proposition 10.10. Any two skew-hermitian integers $a_1, a_2$, with the same norm, that are not of the form $\pi^r u$ for an even unit $u$, are conjugate over $\mathcal{D}_k$.\footnote{One way to do this is to do it by Lie algebra computations over the complex numbers, and then use the fact that the theory of fields of characteristic 0 is complete.}
Proof. By last lemma, $\exists \lambda \in D_k$ such that $a_2 = \lambda a_1 \lambda^{-1}$. By the condition on $a_1$, $k(a_1)/k$ ramifies. Hence, there is an element $\xi$ in $k(a_1)$ such that $|\xi|_{D_k} = |\lambda|_{D_k}$. Therefore,

$$a_2 = (\lambda \xi^{-1})a_1(\lambda \xi^{-1})^{-1} = (\lambda \xi^{-1}) \in D_k^*.$$

Commutators.

Lemma 10.11. Assume that $a$ is an odd skew-hermitian unit, and let $r$ be any integral quaternion such that

$$ra - ar \equiv 0 \pmod{M^i_k}.$$

Then, there exists an element $r_0 \in k(a)$ such that

$$r \equiv r_0 \pmod{M^{k-\nu_p_k(2)}_k},$$

and for all $\alpha_1, \alpha_2 \in k(a)$, we have

$$N(\alpha_1 + \alpha_2 r) \equiv N(\alpha_1 + \alpha_2 r_0) \pmod{M^{2-\nu_p_k(4)}_k(\alpha_2^2)}.$$

Furthermore, it can be assumed that $r - r_0$ is a skew-hermitian element orthogonal to $a$.

Proof. There exists a skew hermitian unit $b$, such that $ab + ba = 0$, (since $a$ is odd). Let $r = \gamma_1 + b\gamma_2$, $\gamma_1, \gamma_2 \in k(a)$. An easy computation shows that

$$ra - ar = 2b\gamma_2 a.$$
In other words, if $r_0 = \gamma_1$, then

$$|ra - ar|_{D_k} = |2(a(r - r_0)|_{D_k},$$

i.e., $r \equiv r_0 \pmod{\mathcal{M}_{k}^{(-\nu D_k)}(2)}$. Also,

$$N(\alpha_1 + \alpha_2 r) = N(\alpha_1 + \alpha_2 \gamma_1) + N(b)N(\alpha_2 \gamma_2),$$

so, it suffices to see that $|N(\alpha_2 \gamma_2)|_{D_k} = |\alpha_2 (r - r_0)\|_{D_k}$. The last statement is clear, since $b\gamma_2$ is orthogonal to $a$. \(\square\)
CHAPTER 11
SKEW-HERMITIAN FORMS ON LOCAL LATTICES.

In all of this chapter, $k$ is a local field, $\mathcal{D}$ a quaternion algebra defined over $k$, $(V, h)$ a skew-hermitian space, over $\mathcal{D}$, defined over $k$. $\mathcal{O}_k$ denotes a maximal order of $\mathcal{D}_k$.

In this chapter, we prove the results we need on the classification theory of $\mathcal{O}_k$-integral lattices on $(V, h)$. This results are the analogs to the classical results on quadratic lattices over local fields, which can be found, for example, in [11], (see (93:15)).

Non-split case.

In all of this section, $\mathcal{D}_k$ is a quaternion division algebra. $\mathcal{O}_k$ is the unique maximal order of $\mathcal{D}_k$, $\mathcal{M}_k$ its unique maximal ideal.

Definition 11.1. For a $\mathcal{O}_k$-integral lattice $\Lambda_k$, and a vector $s \in \Lambda_k$, we define

$$G(s) = \max_{b \in \Lambda_k} |h(s, b)|_{\mathcal{O}_k},$$

and call it the height of $s$ in $\Lambda_k$. 

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Lemma 11.2. Let $\Lambda_k$ be a $\mathcal{D}_k$-integral lattice on the skew-hermitian space $(V, h)$. Let $A^*$ be a "Sk-integral lattice on the skew-hermitian space $(V, h)$.

Let $\widetilde{h} : \Lambda_k \rightarrow \text{Hom}_{\mathcal{D}_k}(\Lambda_k, \mathcal{D}_k)$ be the function defined for $\widetilde{h}(b)(s) = h(s, b)$. Let $M_k$ be a sub-lattice of $\Lambda_k$, such that

$$(\text{Res}^A_M \circ \widetilde{h})(M_k) = (\text{Res}^A_M \circ \widetilde{h})(\Lambda_k),$$

where $\text{Res}^A_M$ is the restriction map. Then, $\Lambda_k = M_k \perp M_k^\perp$.

Proof. Let $b \in \Lambda_k$, by hypotheses, there is an element $b' \in M_k$, such that $(\text{Res}^A_M \circ \widetilde{h})(b) = \widetilde{h}(b')$, i.e., $h(s, b) = h(s, b')$, for all $s \in M_k$, but this means that

$$b = b' + (b - b') \in M_k \perp M_k^\perp,$$

as required.\[]

Proposition 11.3. Any $\mathcal{D}_k$-integral lattice $\Lambda_k$, in $(V, h)$, is orthogonal sum of lattices of dimensions 1 and 2.

Proof. Let $s \in \Lambda_k$ be such that $\Theta(s)$ is maximal, and $|h(s, s)|_{\mathcal{D}_k}$ maximal under this condition. If $|h(s, s)|_{\mathcal{D}_k} = \Theta(s)$, this implies, by the above proposition, that $\Lambda_k = \mathcal{D}_k s \perp (\mathcal{D}_k s)^\perp$, and induction follows. Otherwise, there exists an element $t \in \Lambda_k$, such that, $|h(s, t)|_{\mathcal{D}_k}$ is maximal and,

\[ |h(t, t)|_{\mathcal{D}_k} \leq |h(s, s)|_{\mathcal{D}_k} < |h(s, t)|_{\mathcal{D}_k} \]

1This lemma holds in a much more general setting, (see [18]).

2Since a left ideal of $\mathcal{D}_k$ is completely determined by the absolute value of a generator.
By rescaling, (by an element of the center), we can assume $|h(s, t)|_{\mathcal{D}_k} = 1$ or $|s|_{\mathcal{D}_k}$. Let $b \in \Lambda_k$, then $|h(s, b)|_{\mathcal{D}_k}, |h(t, b)|_{\mathcal{D}_k}$ are at most $|h(s, t)|_{\mathcal{D}_k}$. To complete the proof, we need to find an element $\alpha s + \beta t$ in $\mathcal{D}_k s \oplus \mathcal{D}_k t$, such that

$$h(s, \alpha s + \beta t) = h(s, b),$$

$$h(t, \alpha s + \beta t) = h(t, b),$$

i.e.,

$$h(s, s)\alpha + h(s, t)\beta = h(s, b),$$

$$h(t, s)\alpha + h(t, t)\beta = h(t, b).$$

For this, we need a little lemma from the theory of linear equations over local division algebras, (essentially a non-Abelian analog to the fact that, a matrix is invertible, if its determinant is a unit).

**Lemma 11.4.** If $q_2, q_3$ are elements of $\mathcal{D}_k^*$, and $q_1, q_4$ are in $\mathcal{M}_k$, then any system of equations

$$q_1 x + q_2 y = r_1,$$

$$q_3 x + q_4 y = r_2,$$

with $r_1, r_2$ in $\mathcal{D}_k$ has solutions $x, y \in \mathcal{D}_k$.

**Proof of lemma.** Rewrite the equations in matrix form

$$\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix},$$

and use the formulas at the beginning of chapter 4 to check that the reduce norm of the matrix is a unit, so that it is invertible in $\mathbb{M}_{2 \times 2}(\mathcal{D}_k)$. 

Applying this lemma, there exist elements \( \alpha, \beta \), such that

\[
\xi^{-1}h(s, s)\overline{\alpha} + \xi^{-1}h(s, t)\overline{\beta} = \xi^{-1}h(s, b),
\]

\[
\xi^{-1}h(t, s)\overline{\alpha} + \xi^{-1}h(t, t)\overline{\beta} = \xi^{-1}h(t, b),
\]

where \( \xi \) is 1 or \( i \). Therefore, \( D_k s \oplus D_k t \) satisfy the conditions of lemma 11.2, so that,

\[\Lambda_k = (D_k s \oplus D_k t) \perp (D_k s \oplus D_k t)^\perp.\]

Once more, induction applies. 

**Proposition 11.5.** If \( k \) is non-dyadic, any \( D_k \)-lattice has an orthogonal basis.

**Proof.** This assertion is proved in [2], (see (Lemma 10 in the reference)). 

**Definition 11.6.** Let \( S(\Lambda_k) = \max_{s \in \Lambda_k} S(s) \), and let

\[s(\Lambda_k) = \{ q \in D_k | ||q||_{D_k} \leq S(\Lambda_k) \}.\]

We call this ideal the scale of \( \Lambda_k \). If the discriminant of the lattice equals

\[N(s(\Lambda_k))^{\rho(\Lambda_k)},\]

(where \( \rho(\Lambda_k) \) is the \( D_k \)-rank of the lattice, and \( N \) is the reduced norm), the lattice is called modular.

It is customary, when studying quadratic forms, the use of rescaling to reduce all proofs about modular lattices to the case when the scale is (1). In the case of skew-hermitian forms over \( D_k \), it is only possible to "re-scale" by elements in the center. Since the \( D_k \)-valuation of elements in the center is always even, this rescaling can only reduce the problem to two fundamental cases, which turn out to be essentially different, as next proposition, and the example that follows it, show.
Proposition 11.7. Any modular skew-hermitian lattice, with a scale of even valuation, has an orthogonal basis.

Proof. It suffices to prove it for a binary lattice. By re-scaling, we can assume that that the scale is (1). It suffices to show it represents a unit. By the assumption, there is a basis \( \{s, t\} \), in which the lattice takes the form

\[
\begin{pmatrix}
\rho_1 & 1 \\
-1 & \rho_2
\end{pmatrix}
\]

If one of the \( \rho_m \)'s is a unit, there is nothing to prove. Suppose this is not the case. Then, a straightforward computation shows that

\[
h(s + \alpha t, s + \alpha t) = \rho_1 + \alpha \rho_2 \bar{\alpha} + (\bar{\alpha} - \alpha),
\]

so it suffices to choose \( \alpha \), such that \((\alpha - \bar{\alpha})\) is a unit. \[ \]

Example 11.8. Any element represented by the lattice \(^3\)

\[
\begin{pmatrix}
0 & i \\
i & 0
\end{pmatrix}
\]

is of the form \(^4\)

\[
q[(\alpha i + \beta)i + i(-\alpha i + \bar{\beta})]q = q[\pi(\alpha - \bar{\alpha}) + 2\beta i]q,
\]

for \( q \in \mathcal{D}_k \), \( \alpha, \beta \in \mathcal{O}_{k(U)} \). Hence, its norm at most \( |\pi|_{\mathcal{D}_k} = |i^2|_{\mathcal{D}_k} \).

\[^3\text{See remark 6 in page 9.}\]

\[^4\text{i.e., } h(s, s) \text{ for a vector } s \text{ of the form } q[(\alpha i + \beta)s + t] \text{ or } q[s + (\alpha i + \beta)t], \text{ for } q \text{ an } (\alpha i + \beta) \text{ in } \mathcal{D}_k. \text{ Any vector can be written in one of those forms.}\]
Jordan form.

Let us consider any decomposition,

$$\Lambda_k \cong \Lambda_{1,k} \perp \Lambda_{2,k} \perp \Lambda_{3,k} \perp \ldots \Lambda_{n,k},$$

in which each $\Lambda_{m,k}$ has rank 1 or 2, and is modular. A Jordan form can be defined in the same way as it was done in [11], (section 91c), for quadratic forms, i.e., by collecting lattices in this decomposition having the same scale $s$. The orthogonal sum of these sub-lattices is the Jordan component corresponding to the scale $s$. The uniqueness of the dimension of the Jordan components can be established as in the case of quadratic forms $^5$. We are not concerned with uniqueness in this work.

**Example: The prime-hyperbolic plane.**

**Definition 11.9.** We call a lattice a *prime-hyperbolic plane*, if it is isometric to the lattice

$$\mathcal{H}(i) \cong \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

for a prime element $i$. It is clear that this definition is independent on the choice of $i$.

**Proposition 11.10.** If $k$ is a 2-adic field $^6$, any non-diagonalizable, prime-modular lattice, on a hyperbolic plane, is a prime-hyperbolic plane.


$^6$i.e., 2 is a prime.
Proof. It is clear that the lattice must be isometric to

\[
\begin{pmatrix}
0 & i \\
i & a
\end{pmatrix},
\]

for some \(a\) such that \(|a|_{\mathcal{D}_k} < |i|_{\mathcal{D}_k}\), but then \(|a|_{\mathcal{D}_k} \leq |2|_{\mathcal{D}_k}\).

If \(s, t\) is a basis that gives us the above representation, then we can replace \(t\) by \(t + \lambda s\), which replaces \(a\) by

\[
h(t + \lambda s, t + \lambda s) = a + \lambda i + i\lambda.
\]

By setting \(\lambda = \alpha + i\beta\), it can be seen that

\[
h(t + \lambda s, t + \lambda s) = a + 2\alpha i + \pi(\beta - \beta).
\]

Setting \(a = a_1 + a_2 i, |a_1|_{\mathcal{D}_k}, |a_2|_{\mathcal{D}_k} \leq |2|_{\mathcal{D}_k} = |\pi|_{\mathcal{D}_k}\), we obtain

\[
h(t + \lambda s, t + \lambda s) = (a_2 + 2\alpha)i + (a_1 + \pi(\beta - \beta)).
\]

We see that this expression can be chosen to be 0, since \(a_1 \in k(j)\), so \(a_1^{-1}\) is even.\(\square\)

**Example: Indecomposable anisotropic binary lattices.**

In this section, we discuss the structure theory of indecomposable lattices, in a binary anisotropic space. Recall that such a space must have a discriminant different from 1.

Any such lattice can be written in the form

\[\Lambda_k = \begin{pmatrix} a & i \\ i & b \end{pmatrix},\]
and any skew-hermitian element represented by it must be divisible by at least \( \pi \).

Define
\[
L_k = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \Lambda_k.
\]

The identity
\[
\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} a & i \\ i & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} a & -\pi \\ \pi & -ibi \end{pmatrix},
\]
proves that \( L_k \) is a unimodular lattice, on which \( \Lambda_k \) has index \( |F_k(j)| \).

Let \( G = U_2(D, h) \) be the unitary group of the skew-hermitian form. For use in next chapter, we want to study the index \( [G_k^{L_k} : Stab_{G_k}(\Lambda_k)] \), or equivalently, the number of elements of the orbit \( G_k^{L_k} \Lambda_k \). We show this index to be 1, whenever \( k \) is unramified over \( \mathbb{Q}_2 \).

Recall that \( \mathcal{M}_k \) denotes the maximal ideal of \( D_k \). One dimensional subspaces of \( L_k/\mathcal{M}_k L_k \) correspond to maximal sub-lattices of \( L_k \). This lattices have scale smaller than 1, if and only if, the corresponding space is isotropic.

Any skew-hermitian (or hermitian) regular space in characteristic 2 is isometric to
\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

There are exactly
\[
\frac{t^2 - 1}{t - 1} = t + 1,
\]
(where \( t = |F_k| \)), isotropic subspaces, since they are given by slopes \( z \) satisfying
$z \bar{z} = 1$. However, some subspaces must be disregarded, as the following example shows.

**Example 11.11.** Let

$$L_k = \begin{pmatrix} j & 0 \\ 0 & j + i \end{pmatrix},$$

then an easy computation shows that $h((1,1),(1,1)) = 2j + i$, so that the lattice $\mathcal{D}_k(1,1) + \mathcal{M}_k L_k$ is diagonalizable.

Therefore, we must study the condition under which the lattice is not diagonalizable. Take a unimodular lattice

$$L_k = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \quad q_1 \equiv q_2 \equiv 1 \pmod{\mathcal{M}_k}.$$

We want to know how many lattices $\Lambda_k$ can there be, so that

$$L_k \supset \Lambda_k \supset \mathcal{M}_k L_k, \quad [L_k : \Lambda_k] = [\Lambda_k : \mathcal{M}_k L_k] = t^2 = |\mathbb{F}_q(j)|,$$

and, $h(v,v) \equiv 0 \pmod{\mathcal{M}_k^2}$, for all $v \in \Lambda_k$.

Any such lattice is of the form $\Lambda_k = \mathcal{M}_k L_k + \mathcal{D}_k v$, where

$$h(v,v) \equiv 0 \pmod{\mathcal{M}_k^2},$$

and for all $w \in L, \lambda \in \mathcal{D}_k$, we have

$$h(\lambda v + iw, \lambda v + iw) \equiv 0 \pmod{\mathcal{M}_k^2}.$$

An easy computation shows that it suffices to require

$$ih(w,v)\bar{\lambda} \equiv \lambda h(v,w)i \pmod{\mathcal{M}_k^2},$$

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i.e.,

\[ h(w, v) \lambda \equiv i^{-1} h(v, w) i \pmod{M_k}, \]

which is trivial, since any element in \( D_k \) is congruent to an element of \( O_{k(i)} \) and \( i \) acts on \( O_{k(i)} \) by conjugation. Therefore, we can drop the last condition, and just require that \( h(v, v) \equiv 0 \pmod{M_k^2} \).

Assume \( v = (x, y) \), so that this equation becomes

\[ xq_1 x + yq_2 y \equiv 0 \pmod{M_k^2}, \]

i.e.,

\[ xq_1 x \equiv yq_2 y \pmod{M_k^2}, \]

since 2 is divisible by \( i^2 \). To get a nontrivial solution, we must assume that \( x, y \) are both units.

Up to changing the generator, we can always assume that \( x = 1 \). The equation becomes

\[ q_1 \equiv yq_2 y \pmod{M_k^2}. \]

If we let \( q_1 = 1 + \rho_1 \), this becomes

\[ \rho_1 \equiv (N(y) - 1) + y \rho_2 y \pmod{M_k^2}. \]

\( N(y) \) is in the center \( k \), \( N(y) \equiv 1 \pmod{M_k} \), and \( M_k^2 \cap k = M_k \cap k \), so that the equation becomes

\[ \rho_1 \equiv y \rho_2 y \pmod{M_k^2}. \]
Let $\rho_l = ir_l$, we obtain $r_1 \equiv i^{-1}yir_2\bar{y} \pmod{\mathbb{M}_k}$ or, since $i$ acts by conjugation on the residue field,

$$r_1 \equiv r_2y^2 \pmod{\mathbb{M}_k}.$$

This shows that, if at least one of the $r_l$'s is a units, there is at most one possible value for $y$ in the residue field, (since square roots in characteristic 2 are unique), and therefore, at most one maximal non-diagonalizable sub-lattice.

Assume now that 2 is a prime of $k$. If we write $q_l = \alpha_l + i\beta_l$ with $\alpha_l, \beta_l \in k(j)$, we see that the condition, that $r_l$ is not a unit, implies that $\beta_l$ is not a unit, and hence, $q_1, q_2$ are both even skew-hermitian elements, contradicting the fact that the space is anisotropic. 7

From here, the following result follows easily.

**Proposition 11.12.** Assume that 2 is a prime of $k$. For any skew-hermitian, indecomposable binary lattice $\Lambda_k$, over $\mathcal{D}_k$, of scale (i), there is a unimodular lattice $L_k$, in the same space, such that

$$\mathcal{U}_2^L(\mathcal{D}, h) \subseteq \mathcal{U}_2^\Lambda(\mathcal{D}, h).$$

This is used later to reduce the computation of spinor norms to the diagonalizable case.

7Recall that all even units are congruent, and have, therefore, the same norm, up to squares.
Split case.

In all of this sub-subsection, $k$ is a local field, and $\mathcal{D}_k \cong \mathbb{M}_{2 \times 2}(k)$ is a split quaternion algebra, defined over $k$. $\mathcal{D}_k \cong \mathbb{M}_{2 \times 2}(\mathcal{O}_k)$ is a maximal order in $\mathcal{D}_k$, (which is not unique in this case).

**Proposition 11.13.** *Any skew-hermitian $\mathcal{D}_k$-lattice has a diagonal basis.***

**Proof.** By known results in the theory of quadratic forms, (see, for example, [11], (92:1) and (93:15)), the $\mathcal{O}_k$-lattice $P_1 \Lambda_k$, for

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

has a decomposition,

$$P_1 \Lambda_k = \Lambda_{1,k} \perp \Lambda_{2,k} \perp \Lambda_{3,k} \perp \Lambda_{4,k} \perp \ldots \Lambda_{m,k},$$

into lattices of rank 2, and hence,

$$\Lambda_k = (\Lambda_{1,k} \perp j\Lambda_{1,k}) \perp (\Lambda_{2,k} \perp j\Lambda_{2,k}) \perp \ldots (\Lambda_{m,k} \perp j\Lambda_{m,k}).$$

It suffices, therefore, to recall that, if $L$ is a $\mathcal{O}_k$-lattice on $P_1 V_k$, then $(L \perp jL)$ is a $\mathbb{M}_{2 \times 2}(\mathcal{O}_k)$-lattice, and we have

$$\text{rank } \mathcal{O}_k(L) = 2\text{rank } \mathbb{M}_{2 \times 2}(\mathcal{O}_k)(L \perp jL).$$

In fact, if

$$\Lambda_{m,k} = \mathcal{O}_k x_m \oplus \mathcal{O}_k y_m,$$

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we observe that

\[ \Lambda_{m,k} + j\Lambda_{m,k} = \mathcal{O}_k(x_m + jy_m), \]

since we have the following rules for computation, (where \(x, y \in P_1V_k\)):

\[
\begin{bmatrix}
  1 & 0 \\
  0 & 0
\end{bmatrix}
(x + jy) = x,
\]

\[
\begin{bmatrix}
  0 & 1 \\
  0 & 0
\end{bmatrix}
(x + jy) = y,
\]

\[
\begin{bmatrix}
  0 & 0 \\
  1 & 0
\end{bmatrix}
(x + jy) = jx,
\]

\[
\begin{bmatrix}
  0 & 0 \\
  0 & 1
\end{bmatrix}
(x + jy) = jy.
\]

(11.1)
Part IV

Skew-hermitian lattices and spinor norm computations.
CHAPTER 12
IMAGE OF THE SPINOR NORM FOR
SKEW-HERMITIAN LATTICES IN THE NON-SPLIT CASE.

In all of this chapter, $\mathcal{D}_k$ is a quaternion division algebra over a non-archimedean local field $k$, and $\mathcal{O}_k$ the unique maximal order.

Definition 12.1. (See [2] p.178). If $\Lambda_k$ is a $\mathcal{D}_k$-lattice in a skew-hermitian space over $\mathcal{D}_k$, we define $H(\Lambda_k) \subseteq k^*$, by the relation

$$H(\Lambda_k)/(k^*)^2 = \Theta(U_{n,k}(\mathcal{D}, h)),$$

where $\Theta$ denotes the spinor norm. A similar convention is adopted for the space $V_E$, for any field extension $E/k$.

Remark: If $L_k$ is a sub-lattice splitting $\Lambda_k$\(^1\), it is clear that

$$H(L_k) \subseteq H(\Lambda_k).$$

Definition 12.2. We also use $H'(\Lambda_k)$, defined by

$$H'(\Lambda_k)/(k^*)^2 = \text{Span}\{\Theta((s : \sigma))|s \in \Lambda_k, (s : \sigma)(\Lambda_k) = \Lambda_k\}.$$

\(^1\)i.e., an orthogonal summand of $\Lambda_k$.
Tables for local computations.

The tables in this section contain the results that are proved later in this chapter. For completeness sake, we summarize also the archimedean and non-dyadic cases, which are needed in what follows.

We need to know how to compute the image in the following cases:

Complex places.
In this case, the spinor norm is trivial, since \((\mathbb{C}^*)^2 = \mathbb{C}^*\).

Real places.
Non-split case. In this case, the unitary group is generated by elements of the form \((s : \sigma)\), whose spinor norm is \(N(\sigma)\), which is always positive for all \(\sigma \in \mathcal{D}\). Therefore, the image of the spinor norm must be \(\mathbb{R}^+/(\mathbb{R}^*)^2 = \{1\}\).

Split case. This case is a direct application of formula 4.1. In fact, it is enough to solve the corresponding problem for the quadratic form \((\cdots, \cdots)_{2,1}\), (see the discussion preceding 4.1). Therefore, to compute the image of the spinor norm, we just need to know the signature of \((\cdots, \cdots)_{2,1}\), (see also [2], p.177). The image of the spinor norm, in each case, is given by table 12.1.
If the form \((\cdots, \cdots)_{2,1}\) is \(H(V_{\mathbb{R}}) = \cdots\)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>positive definite,</td>
<td>(\mathbb{R}^+)</td>
</tr>
<tr>
<td>negative definite,</td>
<td>(\mathbb{R}^+)</td>
</tr>
<tr>
<td>indefinite,</td>
<td>(\mathbb{R}^*)</td>
</tr>
</tbody>
</table>

Table 12.1: Split real places.

**Non-archimedean non-split places.**

**Non-dyadic case.** This was already studied in [2], (theorems 3-5). We summarize this information in table 12.2. Assume that we have a decomposition,

\[
\Lambda_k = \mathcal{D}_k s_1 \perp \mathcal{D}_k s_2 \perp \mathcal{D}_k s_3 \perp \cdots \mathcal{D}_k s_n, \quad a_m = h(s_m, s_m).
\]

**Dyadic case.** We have only a partial understanding of this case. Let us summarize the information, that we have so far, in the tables.

Assume that we have a decomposition,

\[
\Lambda_k = \Lambda_{1,k} \perp \Lambda_{2,k} \perp \Lambda_{3,k} \perp \cdots \Lambda_{n,k},
\]

into indecomposable lattices. If \(\Lambda_{m,k}\) has rank 1, we write \(\Lambda_{m,k} = \mathcal{D}_k s_m, \quad h(s_m, s_m) = a_m\). For the unimodular case, see table 12.3. The results for non-modular lattices are summarized in tables 12.4–12.6.
If the numbers $N(a_m)$'s...

<table>
<thead>
<tr>
<th>All have even valuation,</th>
<th>$H(\Lambda_k) =$...</th>
</tr>
</thead>
<tbody>
<tr>
<td>(O_k^<em>(k^</em>)^2)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>All have different odd valuations, and are in the same quadratic class,</th>
<th>$N(k(a_m)^*)$</th>
</tr>
</thead>
</table>

| Are in any other case,      | $k^*$            |

Table 12.2: Non-dyadic places.

In the rest of this chapter, we assume that $k$ is dyadic.

Image on indecomposable lattices.

According to example 4.2, the unitary group in the case of dimension 1 is the norm 1 torus of the extension $k(h(s, s))/k$, where $s$ is a non-zero vector in the space. Since all elements of norm 1 are units, all of them stabilize any lattice. Therefore, the image of the spinor norm is $N(k(h(s, s))^*)$. We state this as a proposition.

**Proposition 12.3.** The image of the spinor norm for the 1-dimensional lattice $\mathcal{D}_k s$ is $H(\mathcal{D}_ks) = N(k(a)^*)$, where $a = h(s, s)$.  

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If the $\Lambda_{m,k}$’s are... & $H(\Lambda_k) =$
\hline
all of rank 1, all $a_m$’s are even units up to rescaling, & $\mathcal{O}_k^*(k^*)^2$
\hline
all of rank 1, not all $a_m$’s are even units up to rescaling, and $n \geq 2$, & $k^*$
\hline
all of rank 1, $n = 1$, & $N(k(a_1)^*)$
\hline
not all diagonalizable, 2 is a prime, & $k^*$
\hline
not all diagonalizable, 2 is not a prime, & unknown
\hline

Table 12.3: Unimodular dyadic places.
<table>
<thead>
<tr>
<th>In case that...</th>
<th>$H(\Lambda_k) =$...</th>
</tr>
</thead>
<tbody>
<tr>
<td>two $\Lambda_{m,k}$'s have rank 1, with the norm of the length of the generators in different quadratic classes,</td>
<td>$k^*$</td>
</tr>
<tr>
<td>some Jordan component of dimension at least 2 in the second or fourth cases of the above table,</td>
<td>$k^*$</td>
</tr>
<tr>
<td>all the Jordan components are re-scalings of even lattices, (i.e., all diagonal elements have norm $-\Delta$),</td>
<td>$\mathcal{O}_k^<em>(k^</em>)^2$</td>
</tr>
<tr>
<td>all Jordan components have rank 1 and the $a_m$'s have norms in the same quadratic class ($\neq -\Delta$), which is a unit and the difference of two consecutive valuations is more that $\nu\Delta_k(64)$,</td>
<td>$N(k(a_m)^*)$</td>
</tr>
</tbody>
</table>

Table 12.4: Dyadic places, general case I.
<table>
<thead>
<tr>
<th>In case that...</th>
<th>$H(A_k) =$...</th>
</tr>
</thead>
<tbody>
<tr>
<td>all Jordan components have rank 1, and the $a_m$'s have norms in the same quadratic class, $(\neq -\Delta)$, which is a unit, and the difference of some two consecutive valuations is at most $\nu_{D_k}(64)$,</td>
<td>unknown</td>
</tr>
<tr>
<td>all Jordan components have rank 1, and the $a_m$'s have norms in the same quadratic class, which is a prime element, and for some $m$, $</td>
<td>a_m/a_{m+1}</td>
</tr>
<tr>
<td>All Jordan components have rank 1 and the $a_m$'s have norms in the same quadratic class $(\neq -\Delta)$, which is a prime element, and we are not in the above case...</td>
<td>unknown</td>
</tr>
</tbody>
</table>

Table 12.5: Dyadic places, general case II.
In case that...

| some of the Jordan components are not diagonalizable, but 2 is not a prime and 1-dimensional $\Lambda_{m,k}$'s have the same norm up to squares... | unknown |

Table 12.6: Dyadic places, general case III.

It is left to compute $H(\Lambda_k)$ for a lattice, $\Lambda_k$, of the type

$$\begin{pmatrix} \alpha & i \\ i & \beta \end{pmatrix}.$$  

We can only compute a few particular cases. Fortunately, These include all non-diagonalizable lattices, when 2 is a prime.

**Example 12.4.** Consider the lattice

$$\Lambda_k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

with Gram matrix

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}.$$
Since the elements in the unitary group, of the hyperbolic plane, are of the form
\[ Aq^{-1}, A \in \mathbb{M}_{2 \times 2}(k), q \in \mathcal{D}^*, \] (see page 31), those of the unitary group of \( \Lambda_k \) must be of the form
\[
\begin{pmatrix}
    i & 0 \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    Aq^{-1} & i^{-1} & 0 \\
    0 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
    \alpha iq^{-1}i^{-1} & \beta iq^{-1} \\
    \gamma iq^{-1}i^{-1} & \delta q^{-1}
\end{pmatrix}.
\]

This element stabilizes the lattice, if and only if, it has integral coefficients, i.e., \( \alpha q^{-1}, \beta iq^{-1}, \gamma q^{-1}i^{-1}, \delta q^{-1} \) are all integers. By a rescaling, we can assume that \( q \) is either a unit or a prime element.

I) \( q \) is a unit, then \( \alpha, \beta, \delta, \gamma \) are integers and \( \gamma \) must be in \( \mathcal{m}_k \). By choosing \( \gamma = 0 \), we see that any unit is in the image of the spinor norm. \( (N(q) \) can be chosen arbitrarily).

II) \( q \) is a prime element. Then, \( \beta \) is an integer, and all the others are in the maximal ideal. Choosing \( \beta \) to be a unit, \( \gamma \) a prime element, \( (\mathcal{O}_k) \), \( \det(A) \) is a prime element, which can be chosen equal to \( N(q) \). Hence, the image of the spinor norm contains a prime element.

From (I) and (II) we conclude that
\[ H(\Lambda_k) = \mathbb{k}. \]

**Example 12.5.** Assume 2 is a prime of \( k \), and \( \Lambda_k \) is an anisotropic binary lattice, Then as we proved in proposition 11.12, there exist a unimodular lattice \( L_k \) containing

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A_k, such that \( \mathcal{U}_{2,k}(D, h) \subseteq \mathcal{U}_{2,k}(D, h) \). By the structure results for unimodular lattices, we know \( L_k \) must be of the form

\[
\begin{pmatrix}
q_1 & 0 \\
0 & q_2
\end{pmatrix},
\]

where, since the space is anisotropic, \( q_1, q_2 \) must have norms in different quadratic classes. This shows that the spinor norm is surjective for \( L_k \), and therefore, also for \( \Lambda_k \).

Because of the two preceding examples, (and since it was proved, in propositions 11.10 and 11.12, that they cover all indecomposable binary lattices, whenever \( k / \mathbb{Q}_2 \) is unramified), we can state the following important result.

**Proposition 12.6.** If \( k / \mathbb{Q}_2 \) is unramified, any binary indecomposable lattice \( \Lambda_k \) satisfies

\[
H(\Lambda_k) = k^*.
\]

**Some elementary generalizations.**

In the general case, we have an orthogonal decomposition of the type

\[
\Lambda_k = \Lambda_{1,k} \perp \Lambda_{2,k} \perp \Lambda_{3,k} \perp \Lambda_{4,k} \perp \ldots \Lambda_{m,k},
\]

where each lattice \( \Lambda_{r,k} \) has rank 1 or 2. It is clear that, we have the following inclusion

\[
H(\Lambda_k) \supseteq \prod_{m=1}^{m} H(\Lambda_{m,k}) \supseteq \prod_{m=1}^{m} H'(\Lambda_{m,k}),
\]

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which can be used to find some elementary conditions, under which, we can ensure that $H(\Lambda_k) = k^*/(k^*)^2$. In fact, if we define

$$H'(\Lambda_k; \Lambda_{1,k}, \Lambda_{2,k}, \ldots, \Lambda_{m,k}) = \prod_{m=1}^{m} H'(\Lambda_{m,k}),$$

$$H''(\Lambda_k; \Lambda_{1,k}, \Lambda_{2,k}, \ldots, \Lambda_{m,k}) = \prod_{m=1}^{m} H(\Lambda_{m,k}),$$

(notice that they depend on the decomposition), it suffices to check that for some decomposition

$$H'(\Lambda_k; \Lambda_{1,k}, \Lambda_{2,k}, \ldots, \Lambda_{m,k}) = k^*,$$

or

$$H''(\Lambda_k; \Lambda_{1,k}, \Lambda_{2,k}, \ldots, \Lambda_{m,k}) = k^*.$$

Examples of this are the following results.

**Proposition 12.7.** If in the decomposition

$$\Lambda_k = \Lambda_{1,k} \perp \Lambda_{2,k} \perp \Lambda_{3,k} \perp \ldots \perp \Lambda_{m,k},$$

we have two components of rank 1, say $D_k s_1, D_k s_2$, for which

$$N(h(s_1, s_1)), N(h(s_2, s_2))$$

belong to different classes in $k^*/(k^*)^2$, then $H(\Lambda_k) = k^*$.

**Proof.** Trivial from proposition 12.3.[4]

**Proposition 12.8.** If $k/\mathbb{Q}_2$ is unramified, and in the decomposition

$$\Lambda_k = \Lambda_{1,k} \perp \Lambda_{2,k} \perp \Lambda_{3,k} \perp \ldots \perp \Lambda_{m,k},$$

we have a component of rank 2, then $H(\Lambda_k) = k^*$. 

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Proof. Trivial from proposition 12.6.]

Unimodular case.

Oddity norm.

We start by defining the analogous concept of norm, for unimodular skew-hermitian lattices.

**Definition 12.9.** Assume \( \Lambda_k \) is a unimodular lattice, (and therefore, diagonalizable). Then, we define the oddity norm by

\[
\mathcal{I}(A f_{\text{fe}}) = \min \{ n(h(s,a)) | s \in \Lambda_k \}.
\]

Now we can define even and odd lattices.

**Definition 12.10.** A skew-hermitian lattice is even if it represents only even skew-hermitian elements, i.e., \( n(\Lambda_k) = \mathcal{E}_{\nu_k(2)} \).

Since we know, that all even units are congruent, all even lattices must be of the form

\[
\langle j \rangle \perp \langle j \rangle \perp \ldots \perp \langle j \rangle.
\]

In particular, note that

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \cong \langle j \rangle \perp \langle j \rangle.
\]

**Definition 12.11.** A lattice \( \Lambda_k \) is odd if it is not even. It is totally odd if

\[
n(\Lambda_k) = \mathcal{D}_k^0.
\]
Image for even lattices.

In all of this sub-section, assume

\[ \Lambda_k = \langle j \rangle \perp \langle j \rangle \perp \ldots \perp \langle j \rangle. \]

Let \( s \) be a primitive vector, in the lattice, that defines a rotation \((s : \sigma)\). As \( s \) is primitive, \( \sigma \) must satisfy the condition \( |\sigma|_{D_k} \geq \Theta(s) = 1 \), on the other hand, the skew-hermitian part of \( \sigma \) is even, so we must have

\[ \sigma = \alpha + 2i\beta, \; \alpha, \beta \in \mathcal{O}_k(j), \]

where the term \( \alpha \) is dominant, and hence, \( N(\sigma) \) must be in \( \mathcal{O}_k^+(k^*)^2 \). On the other hand \( h(s, s) \) is even, and hence, congruent to \( j \), so

\[ H(D_k s) = H(\langle j \rangle) = N(k(j)^* \rangle = \mathcal{O}_k^+(k^*)^2. \]

In other words, we have proved that \( H'(\Lambda_k) = \mathcal{O}_k^+(k^*)^2 \), which implies \( H(\Lambda_k) \supset \mathcal{O}_k^+(k^*)^2 \).

Now, we study the opposite contention. It suffices to prove that the orthogonal group is generated by simple rotations. In order to do this, we prove some general lemmas about the action of simple rotations on elements representing \( j \).

**Lemma 12.12.** Let \((V, h)\) be a skew-hermitian space, \( t, u \in V_k \), such that

\[ h(u, u) = h(t, t) = a, \; u = rt + t_0, \; t_0 \in t^\perp. \]

If we set \( s = t - u \), \( \sigma = h(t, t - u) \), then \( \sigma = a(1 - \tilde{r}) \), and \((s ; \sigma)\) is a well defined simple rotation.

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Proof.

\[ \sigma = h(t, t - u) = h(t, t)(1 - \bar{r}) = a - a\bar{r}. \]

As \( h(t, t) = h(u, u) = a \), then \( a = ra\bar{r} + h(t_0, t_0) \), or \( h(t_0, t_0) = a - ra\bar{r} \). Now,

\[ h(s, s) = (1 - r)h(t, t)(1 - \bar{r}) + h(t_0, t_0) = a - a\bar{r} - ra + ra\bar{r} + (a - ra\bar{r}) = \]

\[ = (a - a\bar{r}) - (a - a\bar{r}) = \sigma - \sigma. \]

Therefore, \( (s : \sigma) \) is a well defined simple rotation. \( \Box \)

Definition 12.13. A simple rotation \( (s : \sigma) \), where \( \sigma \) is a unit, is called primary.
Observe that primary simple rotations have spinor norm in \( \mathcal{O}_k^* (k^*)^2/(k^*)^2 \).

Lemma 12.14. Let \( L_k \) be an arbitrary skew-hermitian lattice of integral scale, \( t, u \in L_k \). If \( h(t, t) = h(u, u) = j \), with \( |1 - h(t, u)|_{\mathcal{D}_k} = 1 \), then there is a primary simple rotation sending \( t \) to \( u \).

Proof. Let \( u = rt + t_0, t_0 \in t^\perp \). Let \( s, \sigma \) be as in lemma 12.12, for \( a = j \), so that \( (s : \sigma) \) stabilizes the lattice, if and only if,

\[ |\sigma|_{\mathcal{D}_k} = |1 - \bar{r}|_{\mathcal{D}_k} \geq \mathcal{S}(s) = \max \{|1 - \bar{r}|_{\mathcal{D}_k}, \mathcal{S}(t_0)\}. \]

Therefore, it fixes the lattice unless \( |1 - r|_{\mathcal{D}_k} < \mathcal{S}(t_0) \). Now we observe that this cannot hold if

\[ |1 - r|_{\mathcal{D}_k} = |1 - r + (1 - j)r|_{\mathcal{D}_k} = |1 - h(t, u)|_{\mathcal{D}_k} = 1, \]

which also proves that \( (s : \sigma) \) is primary. \( \Box \)
Lemma 12.15. Let $L_k$ be an arbitrary skew-hermitian lattice of integral scale, $t, u \in L_k$. If $h(u, u) = h(t, t) = j$, there is a product of at most two simple rotations sending $t$ to $u$. Furthermore, we can assume that one is a rotation with axis $t$ and the other is primary.

Proof. By last lemma, we know that if $u = rt + t_0, t_0 \in t^\perp$, with $|1 - r|_D = 1$ there is nothing to prove, so we assume that $r \equiv 1 \pmod{M_k}$.

We just need to replace $t$ by $\xi t$, where $\xi \in k(j)$, is such that $\xi j \xi = N(\xi)j = j$, (which can be done for such a $\xi$ by mean of a simple rotation of the one-dimensional sub-lattice $D_k t$), i.e., it suffices to check that there exists a unit $\xi \in k(j)$ such that $N(\xi) = 1$, but $\xi$ is not congruent to 1 modulo $M_k$. For this, it suffices to choose $\xi = \chi$, where $\chi \in k(j)$ is a unit such that $\chi, \overline{\chi}$ are not congruent modulo $M_k$. But this is always possible, since the extension $\mathbb{F}_{k(j)}/\mathbb{F}_k$ is non-trivial.[]

From here, it is very easy to establish the general result for even lattices.

Proposition 12.16. For all even lattice,

$$\Lambda_k = \langle j \rangle \perp \ldots \perp \langle j \rangle,$$

we have $H(\Lambda_k) = \mathcal{O}^*_k(k^*)^2$.

This follows immediately from the following lemma:

Lemma 12.17. The orthogonal group of an even unimodular lattice with orthogonal basis $\{s_1, \ldots, s_n\}$ is generated by simple rotations with axis $s_m$, and primary simple rotations.
Proof. For \( n = 1 \), any element is a rotation. Assume \( n \geq 2 \).

Let \( \phi \) be a rotation, \( t = \phi(s_1) \), in particular \( h(s_1, s_1) = h(t, t) = j \), so by lemma 12.15, there are simple rotations \( \tau_1 \) and \( \tau_2 \), satisfying the conditions, such that \( \tau_1 \tau_2 \phi \) fixes \( s_1 \), and induction applies. \( \square \)

We have one more consequence of lemma 12.15.

**Proposition 12.18.** If \( M \) and \( N \) are lattices of integral scale, then

\[
M \perp \langle j \rangle \cong N \perp \langle j \rangle \Rightarrow M \cong N.
\]

Proof. By 12.15, there is a rotation taking any element representing \( j \) to any other. Hence, the same applies to their orthogonal complements. \( \square \)

**Remark** To see that there is not a completely general cancellation law, we can use the example:

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \perp \langle i \rangle \cong \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix} \perp \langle i \rangle,
\]

where the basis are \( u, v, s \) and \( u + s, v - iv + s \).

To see this, observe that

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cong \langle j \rangle \perp \langle j \rangle
\]

is even, and

\[
\begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix} \cong \langle j + i \rangle \perp \langle -j - i \rangle,
\]

(where the basis are \( x, y \) and \( x + \omega y, x + (-i + \bar{\omega})y \)), is odd.
Image for odd lattices.

In this section, we try to compute the image of the spinor norm, for an odd unimodular lattice.

Let

$$\Lambda_k \cong (a_1) \perp (a_2) \perp (a_3) \perp \ldots \perp (a_n),$$

if $k(a_m) \not\cong k(a_l)$ for some $m, l$ we obtain, by a previous result, that $H(\Lambda_k) = k^*$. Therefore, we only need to study the case in which they are all isomorphic.

Assume $k(a_m) \cong k(a_l)$ for every $m, l$, in particular $k(a_1) \cong k(a_2)$, so that $[a_1] \perp [a_2]$ is an hyperbolic plane, and hence, the lattice is of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix},$$

where $y$ is an odd unit.

**Proposition 12.19.** For any odd, unimodular, skew-hermitian lattice $\Lambda_k$, of rank at least 2, $H(\Lambda_k) = k^*$.

**Proof.** It suffices to consider

$$\begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix},$$

(with basis $s, t$). As $y$ is can be modified by an even unit, we can assume that $y = j + i\chi, \chi \in O_{k(j)}, |\chi|_{\mathcal{D}_k} > |2|_{\mathcal{D}_k}$. Notice that

$$N(j + i\chi) = -\Delta - \pi N(\chi).$$
Any norm from \( k(j + i \chi) \) is contained in the image of the spinor norm. It also contains the element

\[
\theta((\omega s + t : \frac{1}{2} h(\omega s + t, \omega s + t))) = -\pi N(\chi) (\text{mod } (k^*)^2),
\]

since \( h(\omega s + t, \omega s + t) = i \chi \), and therefore,

\[
|h(\omega s + t, \omega s + t)|_{\mathcal{O}_k} = |i \chi|_{\mathcal{O}_k} \geq |2|_{\mathcal{O}_k}.
\]

We claim that \( -\pi N(\chi) \) is not a norm from \( k(j + i \chi) \). In fact, it suffices to check that

\[
[1] \perp [-\Delta - \pi N(\chi)] \perp [\pi N(\chi)]
\]

is anisotropic. But, using the relation

\[
[a] \perp [b] = [a + b] \perp [(a + b)ab],
\]

we see that this is equivalent to say that

\[
[1] \perp [-\Delta] \perp \left(1 + \frac{\pi}{\Delta} N(\chi)\right) \pi N(\chi)
\]

is anisotropic, which is clear since \( [1] \perp [-\Delta] \) represents only elements of even valuation. So, it represents \( (1 + \frac{\pi}{\Delta} N(\chi)) \) and \( N(\chi) \) but not \( \pi.\)

\[2\]Here \( \omega - \bar{\omega} = j \). See remark 6 in page 9.

\[3\]Because this is the set of norms for an unramified extension.
Generalizations for non-modular lattices.

Proposition 12.20. For all lattices with a Jordan decomposition

\[ \Lambda_k = \bigoplus_{i=1}^{n} \Lambda_{m,i}, \]

in which every \( \Lambda_{m,i} \) is a rescaling of an even unimodular lattice, we have \( H(\Lambda_k) = \mathcal{O}_k^+(k^*)^2 \).

Proof. Repeat the proof of lemma 12.17 for a lattice

\[ \mathcal{D}_k s_1 \perp \mathcal{D}_k s_2 \perp \ldots \perp \mathcal{D}_k s_n, \quad h(s_m, s_m) = u_m j, u_m \in k, \]

and \( |u_1|_{\mathcal{D}_k} \geq |u_2|_{\mathcal{D}_k} \geq \ldots \geq |u_n|_{\mathcal{D}_k} \). Notice from the proof that it suffices to consider simple rotations \((s; \sigma)\) where

\[ s = (1 - r)s_m + s_0, \quad s_0 \in (\mathcal{D}_k s_{m+1} \perp \ldots \perp \mathcal{D}_k s_n), \quad h(s_m, s_m)(1 - r), \]

and \( |1 - r|_{\mathcal{D}_k} = 1 \), but then \( \sigma \) has even valuation, i.e., \( N(\sigma) \in \mathcal{O}_k^+(k^*)^2 \). [ ]

Now, we prove a lemma concerning lattices for which all the diagonal components have the same norm, up to squares, and this norm is a unit different from \(-\Delta\). We can also assume, that all the diagonal components have different valuations, since otherwise the spinor norm is the whole field of scalars, (see proposition 12.19).

Lemma 12.21. Assume that

\[ \Lambda_k = \bigoplus_{i=1}^{n} \mathcal{D}_k s_m, \]

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where all $h(s_m, s_m)$ have equal norm up to squares, and

$$|2h(s_r, s_r)|_{D_k} \leq |h(s_1, s_1)|_{D_k}$$

for $r < l$. Assume also that all $h(s_m, s_m)$ are rescaling of units. Then, we claim that the unitary group $U_{n,k}^A(D, h)$ of the lattice is generated by simple rotations with axis $s_m$, and simple rotations of the form $(s; \sigma)$, where $|\sigma|_{D_k} \geq |2h(s_m, s_m)|_{D_k}$, with $s$ having the form $s = (1 - r)s_m - s_0$, for some

$$s_0 \in \bigcup_{r=1}^{n-m} D_k s_m + r,$$

and $\sigma = h(s_m, s_m)(1 - \tau)$.

**Proof.** Let $\phi \in U_{n,k}^A(D, h)$, with $\phi(s_1) = rs_1 + s_0, s_0 \in s_1^\perp$. Then the rotation $(s_1 - \phi(s_1) : \sigma)$, where $^4$

$$\sigma = h(s_1, s_1 - \phi(s_1)) = h(s_1, s_1)(1 - \tau),$$

stabilizes the lattice, unless

$$|1 - \tau|_{D_k} < \mathcal{G}(s_0) \leq |2h(s_1, s_1)|_{D_k},$$

so that $r \equiv 1 \pmod{2i}$, but this condition can easily be avoided by multiplying our basis vector by an element of the form $\lambda = \frac{\chi}{\bar{\chi}}$, where $\chi$ is chosen so that $\chi$ and $\bar{\chi}$ are not congruent modulo $(2i)$. This can always be done, since $k(h(s_1, s_1))$ contains a

---

$^4$See lemma 12.12.
skew-hermitian unit $\chi$, which must be of the form $cj + ai$, for a unit $c \in \mathcal{O}_k^*$, therefore $\chi - \bar{\chi} = 2cj + 2ai$, which is not congruent to 0 modulo $(2i)$. Induction applies. All conditions on $s$ and $\sigma$ are now straightforward computations. 

The image of the spinor norm is generated by

$$N(k(h(s_1, s_1))^*)(k^*)^2/(k^*)^2,$$

(12.1)

and the images of elements of the form $N(1 - r)$, being $r$ such that $h(s_m, s_m) - rh(s_m, s_m)$ is represented by $\mathcal{D}_k s_{m+1} \perp \ldots \perp \mathcal{D}_k s_n$, and $r \neq 1 \pmod{2i}$. As 12.1 have index 2, there are exactly two possibilities for the spinor norm, it can be either this group or the whole of $k^*/(k^*)^2$. Our next goal is to determine under which conditions we obtain one or the other.

So far, we have only a partial understanding on this subject. As the following lemma shows, we get 12.1 whenever the distance between consecutive valuation of scales of different Jordan components is big enough.

Lemma 12.22. If

$$\left| \frac{h(s_{m+1}, s_{m+1})}{h(s_m, s_m)} \right|_{\mathcal{D}_k} < 64|_{\mathcal{D}_k}, \ m = 1, \ldots, n - 1,$$

then

$$H \left( \begin{array}{c} s_1 \\ \vdots \\ s_n \end{array} \right) = N(k(h(s_1, s_1))^*)(k^*)^2.$$

Proof. Assume

$$\left| \frac{h(s_{m+1}, s_{m+1})}{h(s_m, s_m)} \right|_{\mathcal{D}_k} \leq \pi^+|_{\mathcal{D}_k}, \ m = 1, \ldots, n - 1.$$
By last lemma, it is enough to consider rotations \((s; \sigma)\), where

\[ s = rs_m + s_0, \quad s_0 \in (D_k s_{m+1} \perp \ldots \perp D_k s_n), \]

and \(N(\sigma) = N(h(s_m, s_m))N(1 - r)\|\sigma\|_{D_k} \geq |2h(s_m, s_m)|_{D_k}\) (in particular \(|1 - r|_{D_k} \geq |2|_{D_k}\)).

Now, let \(a_m = h(s_m, s_m)\) for \(m = 0, 1, \ldots, n\). Then \(a_0 = r a_m \tau - a_m\). We can assume \(m = 1\) and, by a rescaling, \(a_1\) is a unit, so that

\[ N(a_1) \equiv N(r)^2 N(a_1) \pmod{\pi^t}, \]

i.e., \(N(r)^2 \equiv 1 \pmod{\pi^t}\).

From here it follows that \(\pi^t\) divides \((1 - N(r)^2)\). By the pigeon hole principle, \(i^t\) divides either \(1 - N(r)\) or \(1 + N(r)\).

In the first case, the congruence \(ra_{1} \tau - a_1 \equiv 0 \pmod{\pi^t}\), implies \(\tau \equiv r^{-1} N(r) \equiv r^{-1} \pmod{\pi^t}\), hence

\[ ra_m - a_m \tau \equiv 0 \pmod{i^t a_m}. \]

By lemma 10.11, \(\exists r_0 \in k(a_1)\) with \(r_0 \equiv r \pmod{M_k^{2t - \nu_{D_k}(2)}}\), and \((r - r_0)\) skew-hermitian and orthogonal to \(a_1\), so that

\[ N(1 - r_0) \in N(k(a_1)^*)(k^*)^2, \]

and also

\[ N(1 - r) \equiv N(1 - r_0) \pmod{M_k^{2t - \nu_{D_k}(4)}). \]

If

\[ |i^{2t - \nu_{D_k}(4)}|_{D_k} < |4N(1 - r_0)|_{D_k} = |4N(1 - r)|_{D_k} \]

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the result follows immediately from the local squares theorem. Now, recall that we assumed that $|1 - r| \geq |2|$, hence $|N(1 - r)| \geq |4|$, so it suffices to have

$$2t - \nu_{D_k}(4) > \frac{1}{2} \nu_{D_k}(4 \times 4),$$

i.e.,

$$t > \frac{\nu_{D_k}(64)}{2} = \nu_k(64).$$

In the second case, $N(r) \equiv -1 \pmod{(i^t)}$, i.e.,

$$ra_1 \equiv -a_1 r \pmod{(i^ta_1)},$$

so that if $b_1$ is a skew hermitian unit orthogonal to $a_1$, then there exists an element $r_0 \in k(a_1)$ such that

$$r \equiv b_1 r_0 \pmod{\mathfrak{M}_{k}^{t - \nu_{D_k}(2)}}.$$

But in this case

$$N(1 - r) \equiv N(1 - b_m r_0) = 1 + N(b_m) N(r_0)$$

$$\equiv 1 + N(r) \equiv 1 + (-1) \equiv 0 \pmod{\mathfrak{M}_{k}^{t - \nu_{D_k}(2)}}.$$

Since we know that the generators satisfy $|N(1 - r)| \geq |4|$, we get a contradiction provided $t - \nu_{D_k}(2) > \nu_{D_k}(4)$, i.e., $t > \nu_{D_k}(8) = \nu_k(64)$. []

The same computation shows the following lemma.

**Lemma 12.23.** Assume that $u$ is an element of the center, $|64| \leq |u| \leq |2|$. Then for any odd unit $a$,

$$H \left( \begin{array}{cc} a & 0 \\ 0 & u a \end{array} \right) = k^*/(k^*)^2.$$
if and only if, there is a unit \( r \) such that \( ra - a \) is represented by \((ua)\), and \( N(1-r) \not\in N(k(a)^*) \).

Notice that, by lemma 12.21, we can always assume that \( r \equiv 1 \pmod{2} \).

**Open question 1.** Is this always the case with the given restriction on \(|u|_k^2\)?

If not, is there a value \( a \), such that, the spinor norm is surjective, if and only if, \(|u|_k > a^2\)?

**Diagonalizable prime-modular lattices.**

By redefining \( i, j \), we can assume that \( i \) is an arbitrary prime element.

Since any two elements with the same norm are conjugate, we just need to study lattices of the form

\[
\begin{pmatrix}
i & 0 \\
0 & u i
\end{pmatrix}, \quad u \in \mathcal{O}_k^*.
\]

Conjugating by elements in \( k(i) \), (i.e., modifying by norms from \( k(i) \)), we can assume \( u = -1 \) or \( u = -\Delta \). \(^5\) Now, \( ji j = \Delta i \). Hence, it suffices to consider

\[
\begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix}.
\]

Now, for \( \alpha_1, \alpha_2 \in k(j) \), we have

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 \\
i & 0 \\
0 & -i
\end{pmatrix}
\begin{pmatrix}
\bar{\alpha}_1 \\
\bar{\alpha}_2
\end{pmatrix} = \alpha_1 i \bar{\alpha}_1 - \alpha_2 i \bar{\alpha}_2
\]

\(^5\)Since \( \Delta \) is not a norm from \( k(i) \), as follows from \( D_k = \left( \frac{5\Delta}{k} \right) \).

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(a_1^2 - a_2^2)i = (a_1 - a_2)(a_1 + a_2)i.

So, if $a_1 = i^1_2, a_2 = i^1_2$, (which are integers), we obtain that the lattice represents $ji$, which has different norm as $i$, (and the element representing it is part of a diagonal basis). Hence, we have proved the following lemma.

**Lemma 12.24.** The spinor norm of any lattice whose Jordan form contains a diagonalizable component with scale of odd valuation of dimension ≥ 2, is the whole of $k^*/(k^*)^2$.

Therefore, we know the spinor norm for all diagonalizable prime-modular lattices.

**Non-diagonalizable lattices.**

The results we have obtained for indecomposable binary lattices, allow us to state the following general result, (see proposition 12.6).

**Proposition 12.25.** If $\Lambda_k$ is non-diagonalizable, and $k/Q_2$ is unramified, we have $H(\Lambda_k) = k^*$. 

This result implies that we can always assume that the lattice is diagonalizable, when working with a field $k$ satisfying the hypothesis.
Generalizations of the prime case.

Example 12.26. Now consider the lattice

\[
\begin{pmatrix}
i & 0 \\
0 & ui \\
\end{pmatrix},
\]

where \( u \) is in the center, \( \nu_k(u) = \frac{1}{2} \nu_{D_k}(u) \) is odd and less than \( \nu_k(4) \). Then, if \( \alpha_1, \alpha_2 \in k(j) \),

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 \\
0 & -i \\
\end{pmatrix} \begin{pmatrix}
i & 0 \\
0 & -i \\
\end{pmatrix} \begin{pmatrix}
\bar{\alpha}_1 \\
\bar{\alpha}_2 \\
\end{pmatrix} = (\alpha_1^2 + u\alpha_2^2)i,
\]

whose norm is \( -N(\alpha_1^2 + u\alpha_2^2)\pi \) so it suffices to find \( \alpha_1, \alpha_2 \in k(j) \) such that \( N(\alpha_1^2 + u\alpha_2^2) \) is not a square. Set \( \alpha_1 = 1 \), then

\[
N(1 + u\alpha_2^2) = 1 + u \operatorname{tr}(\alpha_2^2) + u^2 N(\alpha_2)^2.
\]

Hence, it suffices to find \( \alpha_2 \in \mathcal{O}_{k(j)} \) such that \( \operatorname{tr}(\alpha_2^2) \) is not congruent to 0 modulo \( (\pi) \), but \( \mathbb{F}_{k(j)}/\mathbb{F}_k \) is non-trivial, and \( \mathbb{F}_{k(j)} \) is perfect, so this can always be satisfied.

This allow us to state the following result.

Proposition 12.27. If

\[
\Lambda_k \cong \bigcap_{m=1}^{n} \langle u_m i \rangle,
\]

and for some \( m \) we have \( |u_{m+1}/u_m|_k = |\pi^4|_k \), \( t \) odd, \( t \leq \nu_k(4) \), then \( H(\Lambda_k) = k^* \).
Open question 2. This is as much as we can do in this case, the image of the spinor norm for general lattices of the form

$$\Lambda_k \cong \bigoplus_{m=1}^{n} \langle u_{m} \rangle$$

remains unknown.
CHAPTER 13

IMAGE OF THE SPINOR NORM IN THE SPLIT CASE

FOR NON-DYADIC PRIMES.

In all of this chapter, \( k \) is a non-dyadic local field, \( \pi \) its uniformizing parameter. \( \mathcal{D}_k \cong M_{2 \times 2}(k), \mathcal{D}_k \cong M_{2 \times 2}(\mathcal{O}_k) \). We prove some results that are used in next chapter to study the norm principle. We use heavily the fact that under the isomorphism between the unitary group of a skew-hermitian form \( h \) and the orthogonal group of \((\ldots,\ldots)_{2,1}\), (see the discussion preceding 4.1), the stabilizer of \( \Lambda_k \) in the special unitary groups maps to the stabilizer of \( P_1\Lambda_k \) in the orthogonal group, (see [2], theorem 7, p.181).

The notation \( H(\Lambda_k) \) is defined as at the beginning of chapter 12.

Image on lattices of rank 1.

Assume first that \( \Lambda_k = M_{2 \times 2}(\mathcal{O}_k)s \), then if

\[
h(s, s) = \begin{pmatrix}
\xi_{1,1} & \xi_{1,2} \\
\xi_{2,1} & -\xi_{1,1}
\end{pmatrix},
\]

according to the formula 4.1, for \( s = x + jy \) we get

\[
\xi_{1,1} = -(x, y)_{1,2}, \xi_{1,2} = (x, x)_{1,2}, \xi_{2,1} = -(y, y)_{1,2},
\]

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Observe that by 11.1 the lattice $P_1 \Lambda_k$ can be written as $P_1 \Lambda_k = O_kx \oplus O_ky$, and this lattice has Gram matrix

\[
\begin{pmatrix}
\xi_{1,2} & -\xi_{1,1} \\
-\xi_{1,1} & -\xi_{2,1}
\end{pmatrix}
= \begin{pmatrix}
\xi_{1,1} & \xi_{1,2} \\
\xi_{2,1} & -\xi_{1,1}
\end{pmatrix} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix},
\tag{13.1}
\]

whence the discriminant of this form is

\[
\det \begin{pmatrix}
\xi_{1,2} & -\xi_{1,1} \\
-\xi_{1,1} & -\xi_{2,1}
\end{pmatrix}
= \det \begin{pmatrix}
\xi_{1,1} & \xi_{1,2} \\
\xi_{2,1} & -\xi_{1,1}
\end{pmatrix} \det \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} = N(h(s, s)).
\]

**Definition 13.1.** If $-N(h(s, s))$ is a square, i.e., the corresponding binary quadratic lattice is hyperbolic, we call the lattice $D_k$ unary-hyperbolic. Otherwise, we call it non-unary-hyperbolic. In last case, $k(h(s, s))$ is a field. If $O_k(h(s, s)) \subseteq D_k$, we say that the lattice is in the good case.

In case that $k(h(s, s))$ is a field, essentially the same analysis, as in the non-split case, holds.

**Lemma 13.2.** If $D_k$ is non-unary-hyperbolic, then

\[H(\Lambda_k) \subseteq N(k(h(s, s))^*).\]

*If we are in the good case, they are equal.*

**Proof.** The group of units of this lattice is the set of units of norm 1 in the field $k(h(s, s))$ that are contained in $D_k$, i.e., $D_k \cap O_k^*(h(s, s))$. But all elements in $O_k^*(h(s, s))$ are of the form $\sigma^{-1}\tilde{\sigma}$ for $\sigma \in k(h(s, s))^*$, so that they have norm

\[N(\sigma) \in N(k(h(s, s))^*).\]
Unary-hyperbolic lattices.

As all non-dyadic quadratic lattices are diagonalizable, if $\Lambda_k$ is unary-hyperbolic, the lattice $\Pi \Lambda_k$ must be of the form $\langle x \rangle \perp \langle -\alpha^2 x \rangle$. By [6], (theorem 3), if $\alpha$ is not a unit, the image of the spinor norm is $O_k^+(k^*)^2/(k^*)^2$ if $-1$ is not a square and trivial otherwise. If $\alpha$ is a unit, we can assume that the lattice is

$$\langle x \rangle \perp \langle -x \rangle \cong \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}.$$ 

Therefore, the special orthogonal group consists of matrices of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

where $\lambda$ is a unit. We claim that

$$\theta \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \lambda.$$ 

In fact,

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix} = \tau_{e_1+e_2} \tau_{e_1+\lambda e_2},$$

where $\{e_1, e_2\}$ is a basis of the lattice, hence

$$\theta \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = Q(e_1 + e_2)Q(e_1 + \lambda e_2) = 4x^2 \lambda \equiv \lambda (mod(k^*)^2).$$

Therefore, the image of the spinor norm is $O_k^+(k^*)^2/(k^*)^2$.

**Definition 13.3.** We say that $\langle x \rangle \perp \langle -x \rangle$ is the *good case*. The case when $\alpha$ is not a unit and $-1$ is not a square is called *quasi-good*, and the remaining case *bad*. 

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Elementary generalizations.

This results are used in chapter 14 to prove the norm principle for non-dyadic fields. Recall that, because of [11], (92:4), the orthogonal group of a non-dyadic lattice is generated by products of two reflections.

**Proposition 13.4.** If \( \Lambda_k \) has an orthogonal splitting,

\[
\Lambda_k = \Lambda_{1,k} \perp \Lambda_{2,k} \perp \Lambda_{3,k} \perp \ldots \Lambda_{n,k},
\]

into components of rank 1, so that \( \Lambda_{m,k} = M_{2 \times 2}(\mathcal{O}_k)s_m \) with \( h(s_m, s_m) \) a unit in \( M_{2 \times 2}(\mathcal{O}_k) \), then \( H(\Lambda_k) = \mathcal{O}_k^*(k^*)^2 \).

**Proof.** According to the results in last section, the corresponding \( \mathcal{O}_k \)-lattice has an orthogonal decomposition into 2-dimensional unimodular lattices, (see 13.1). Therefore, it is unimodular, and its orthogonal group has good reduction.

**Proposition 13.5.** If \( \Lambda_k \) has an orthogonal splitting,

\[
\Lambda_k = \Lambda_{1,k} \perp \Lambda_{2,k} \perp \Lambda_{3,k} \perp \ldots \Lambda_{n,k},
\]

into components of rank 1, so that \( \Lambda_{m,k} = M_{2 \times 2}(\mathcal{O}_k)s_m \), for \( m = 1, \ldots, n \), are unary-hyperbolic with \( h(s_m, s_m) \)'s in the same maximal Abelian sub-algebra, and if the valuations

\[
\nu_k(\det(h(s_m, s_m))), \ m = 1, \ldots, n,
\]

are all congruent modulo 4, then \( H(\Lambda_k) = \mathcal{O}_k^*(k^*)^2 \). Furthermore, if at least one of the \( h(s_m, s_m) \)'s is in the good or quasi-good cases, we have equality.
Proof. As in the end of last section, all $P_1 \Lambda_{m,k}$ are of the form $\beta_m(\langle x \rangle \perp \langle -\alpha^2 x \rangle)$, and by the hypotheses on determinants $\nu_k(\beta_m), m = 1, \ldots, n$ have the same parity. Therefore, all Jordan components have scales with valuations of the same parity. Hence, the result follows from [6], (theorem 3). \]

**Proposition 13.6.** Assume $\Lambda_k$ has an orthogonal splitting,

$$\Lambda_k = \Lambda_{1,k} \perp \Lambda_{2,k} \perp \Lambda_{3,k} \perp \ldots \Lambda_{n,k},$$

into components of rank 1 so that $\Lambda_{m,k} = M_{2 \times 2}(O_k)s_m$, with $h(s_m, s_m) \in M_{2 \times 2}(O_k)$. Assume also that $\nu_k[\det(h(s_m, s_m))]$, for $m = 1, \ldots, n$, is even, congruent modulo 4 and the lattices $\Lambda_{m,k}, m = 1, \ldots, n$ are all in the good case. Then,

$$H(\Lambda_k) = O_k^*(k^*)^2.$$

**Proof.** As they are in the good case we can write $h(s_m, s_m) = \pi^{n_m}U_m$, where $U_m$ is a unit, and $n_m$'s have the same parity. Hence, all Jordan components of $P_1 \Lambda_k$ have scales with valuations of the same parity. Therefore, the result follows from [6], (theorem 3).\]

**Proposition 13.7.** If $\Lambda_k$ has an orthogonal splitting,

$$\Lambda_k = \Lambda_{1,k} \perp \Lambda_{2,k} \perp \Lambda_{3,k} \perp \ldots \Lambda_{n,k},$$

into components of rank 1, where $\Lambda_{m,k} \cong \langle (-\pi)^{t_m} \rangle$, $t_m + 1 \geq t_m + 2$, $N(i) = -\pi$, and $i$ is in the good case, then the image of the spinor norm is $N(k(i)^*)$. 

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Proof. By hypotheses and Witt’s theorem,

\[ i \cong \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix}, \]

hence \( P_1 \Lambda_{m,k} \cong (-\pi)^{l_m}(1 \perp (-\pi)) \). It follows that \( P_1 \Lambda_k \) has all its Jordan components of the form \((-\pi)^t\), so that, (by [6], theorem 3), the image of the spinor norm is

\[ \{(k^*)^2, (-\pi)(k^*)^2\} = N(k(i)^*). \]

Example 13.8. Let \( i \in M_{2 \times 2}(O_k) \) be as in last proposition, and consider the lattice \( \Lambda_\kappa = \langle i \rangle \perp \langle \pi i \rangle \). Then,

\[ P_1 \Lambda_\kappa \cong \langle 1 \rangle \perp (-\pi) \perp \langle \pi \rangle \perp (-\pi^2), \]

which is easily seen, (by the relation

\[ \langle \pi \rangle \perp (-\pi) \cong (\Delta \pi) \perp (-\Delta \pi), \]

and [6], theorem 3), to have full spinor norm.]
In this chapter, we apply the spinor norm computations we have done, to the study of the behavior of spinor class number under extensions.

Example of computation.

Let $\mathcal{D}_\mathbb{Q} = \left( \frac{-1}{\mathbb{Q}} \right)$ be the ring of standard quaternions, with generators $i, j$ satisfying $i^2 = j^2 = -1$, $ij = -ji$. Let

$$L_\mathbb{Q} = \begin{pmatrix} j & 0 \\ 0 & 16j \end{pmatrix},$$

which is unimodular at all $p \neq 2, \infty$, so that the image of the spinor norm at these places is $\mathbb{Z}_p^*(\mathbb{Q}_p^*)^2$. At $p = 2$, lemma 12.22 applies, so that the image is

$$N(\mathbb{Q}_2(j)^*) = \text{Span}\{2, 5\}(\mathbb{Q}_2^*)^2.$$

Finally, at $\infty$ the image is $\mathbb{R}^+$, since $\mathcal{D}_\mathbb{R}$ does not split. Therefore, the spinor class field is the class field corresponding to

$$H = \mathbb{Q}^*(\mathbb{R}^+ \times N(\mathbb{Q}_2(j)^*) \times \prod_{p \neq 2, \infty} \mathbb{Z}_p^*(\mathbb{Q}_p^*)^2).$$
Let $\sigma$ be the idele defined by $\sigma_2 = 3, \sigma_p = 1$ for $p \neq 2$. Then, $\sigma \in H$ if and only if there exists $q \in \mathbb{Q}^+$ such that

$$3q \in N(\mathbb{Q}_2(j)^*), \quad q \in \mathbb{Z}_p^*(\mathbb{Q}_p^2) \forall p \neq 2.$$ 

Setting $q = p_1^{a_1} \cdots p_n^{a_n}$ and comparing valuations shows that $q = 2$ or $1$ up to squares, but the condition at $2$ rules out both cases, i.e., $\sigma \notin H$. It follows that the spinor class field $\Sigma$ ramifies at $2$. It is immediate that it does not ramify at any other place, except maybe $\infty$. Hence, $\Sigma \subseteq \mathbb{Q}[^1, \sqrt{2}]$. Now, looking at the localization at $2$ yields $\Sigma = \mathbb{Q}[^1]$. []

Growth of class number under extensions and spinor class field.

It is known, (see for example [5], section 3, or Theorem 4.1 in [3] ), that the following norm principle for absolute spinor norms holds, for a quadratic forms $g$, over a number field extension $E/k$ on which either $2$ is a prime on $k$ or $(2, \text{disc}(g)) = 1$.

$$N_{E/k}(\theta(\mathcal{O}_{n,A_E}^A(g))) \subseteq \theta(\mathcal{O}_{n,A_k}^A(g)).$$

(Compare with p. 75). Since this is a local condition, it follows immediately from this result that the corresponding statement holds for skew-hermitian forms, in the case of an extension $E/k$ which splits completely at the finitely many places that do not split the quaternion algebra. (Since the condition is trivially satisfied at those places). In order to get rid of these restriction, it is necessary to undertake a more careful study of the norm principle at the non-split cases.
In fact, it suffices to prove that, if $D_k$ does not split at $v$, then
\[ N_{E_v/k_v}(\theta(U_{n,E_v}^A(D, h))) \subseteq \theta(U_{n,k_v}^A(D, h)). \]

In case that the norm principle holds for all places, it follows, (see [5]), that the spinor class field $\Sigma(A_E)$ of $A_E$ contains $E\Sigma(A_k)$. If, furthermore, $E$ and $\Sigma(A_k)$ are linearly disjoint, we obtain a surjection
\[ G_{\Sigma(A_E)/E} \to G_{\Sigma(A_k)/k}, \]
given by the restriction map. This can be reinterpreted in terms of spinor class number as follows: The number of spinor genera in a genus is non-decreasing. Furthermore, the composition of the natural map,
\[ G_{\Sigma(A_k)/k} \cong \frac{J_k}{k^*\theta(G_k^A)} \to \frac{J_E}{k^*\theta(G_E^A)} \cong G_{\Sigma(A_E)/E}, \]
and the restriction map, is the map $a \mapsto a^{[E:k]}$. Hence, if the extension is odd, we obtain the following fact: no two proper spinor genera in the genus of $A_k$ are identified over $E$.

Remark 14.1. This spinor class field approach has the advantage, over the cohomological approach, that it uses the more general concept of spinor genera, and hence, it does not require a non-compactness condition. However, it cannot be extended to algebraic groups for which $F \neq \mu_n$.

Non-dyadic non-split places.

In all of this sub-subsection, $k$ is a non-dyadic non-archimedean local field. $D$ is a quaternion algebra, defined over $k$, which does not split on $k$. $D_k$ is the unique maximal order of $D_k$. We write $a_m = h(s_m, s_m)$. 

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If \( k \subset K \subset E \) are fields, then \( N_{K/k} \circ N_{E/K} = N_{E/k} \). Therefore, if we have extensions \( L/k, E/k \), it follows from the diagram,

\[
\begin{array}{ccc}
E & \rightarrow & L \\
E & \downarrow & L \\
& k & \\
\end{array}
\]

that

\[ N_{L/k}(L^*) \subseteq N_{E/k}(N_{EL/E}((EL^*))). \]  \hspace{1cm} (14.1)

In the first two cases in table 12.2, (page 107), the spinor norm has the form \( N_{L/k}(L^*) \), for a suitable quadratic extension \( L \), so that the norm principle is immediate for extensions of odd degree.

We claim that it also holds for even degree extensions, in almost all cases. Notice that by Witt’s theorem applied to the maximal order, (regarded as a quadratic lattice), all elements with equal norm, (up to squares), can be assumed to lie in the same maximal subfield. We assume this in all that follows.

Case 1. All \( a_m \)'s have even valuation. In this case we have

\[ \theta(\mathcal{U}_{n,k}^{\Delta}(\mathcal{D}, h)) = \mathcal{O}_k^*(k^*)^2/(k^*)^2. \]

If \( E \) does not split \( k(\sqrt{\Delta}) \), we claim that Proposition 13.6 applies. Actually we just need to prove that we are in the good case for each of the diagonal components, \(^1\)

\(^1\)See remark 6 in page 9.
(but if $\kappa$ generates $\mathcal{O}_{k(a_m)}$ over $\mathcal{O}_k$ for $\kappa^2$ a non-square unit in $\mathcal{O}_k$, the same is true over $E$), and the congruence condition, (which is trivial since $E$ contains a ramified quadratic extension of $k$). Now, the proof goes as for odd degree extensions (since $N_{k(\sqrt{\Delta})/k}(k(\sqrt{\Delta})^*) = \mathcal{O}_k^*(k^*)^2$). Therefore, we can assume that $k(\sqrt{\Delta}) \subseteq E$.

In this case

$$N_{E/k}(\theta(\mathcal{U}_{n,E}^\Delta(D, h))) \subseteq N_{E/k}(E^*)$$

$$= N_{k(\sqrt{\Delta})/k}(N_{E/k(\sqrt{\Delta})}(E^*))$$

$$\subseteq N_{k(\sqrt{\Delta})/k}(k(\sqrt{\Delta})^*)$$

$$= \mathcal{O}_k^*(k^*)^2. \quad (14.2)$$

\[
\]

Case 2. All $a_m$ have different odd valuation and all are in the same quadratic class.

In this case, we have

$$\theta(\mathcal{U}_{n,k}^\Delta(D, h)) = N(k(a_m)^*)/(k^*)^2.$$

Assume first that $E(a_m)$ is a field.

If $e(E/k) = 1$, we are in the exceptional case (see below).

If $e(E/k)$ is odd > 1, (but $[E : k]$ is even), all units of $k$ become squares in $E$, the difference between consecutive valuation in $E$ is a multiple of $e(E/k)$, (hence bigger than 1), and we are in the good case since the lattice is defined over $k$. Therefore, proposition 13.7 applies, and this case is similar to that of odd extensions.

It follows that we can assume $e(E/k)$ to be even. Hence, all the norms of the diagonal elements have even, congruent modulo 4, valuations, so that proposition
13.6 applies, and the image of the spinor norm over \( E \) is contained in \( \mathcal{O}_E^*(E^*)^2 / (E^*)^2 \). However, since \( e(E/k) \) is even, \( N_{E/k}(\mathcal{O}_E^*(E^*)^2) \subseteq (k^*)^2 \).

Finally, assume \( k(a_m) \) injects in \( E \), i.e., \( 2 k(\sqrt{a_m^2}) \subseteq E \). This case follows as 14.2.

Case 3. In any other case

\[
\theta(\mathcal{U}_{n,k}^h(D, h)) = k^*/(k^*)^2,
\]

so the condition is vacuously true. []

Let us state the result we have just proved.

**Proposition 14.2.** The norm principle holds for all local lattices over non-dyadic fields, except maybe in the exceptional case.

[]

**Exceptional case.** Assume \( e(E/k) = 1 \), so that by example 13.8, if \( N(i) = -\pi \), the lattice

\[
\begin{pmatrix}
i & 0 \\
0 & \pi i
\end{pmatrix}
\]

has full spinor norm over \( E \), but \( N_{E/k}(E^*) = \mathcal{O}_E^*(k^*)^2 \), which is not contained in \( N(k(a_m)^*) \).

---

\(^2\)Note that \( k(a_m) \) is a sub-algebra of \( D_k \) while \( k(\sqrt{a_m^2}) \) is a field extension of \( k \).
Dyadic non-split cases, odd extensions.

Reasoning as in the beginning of last sub-subsection, (i.e., by applying 14.1), one obtain the corresponding result, at least for all the cases in which the spinor norm have been explicitly determined. The explicit formulae in tables 12.4-12.6, (p. 109-111), show that the norm principle holds except maybe in the following cases:

- 2 is not a prime at the bottom field and the lattice is not diagonalizable.
- The lattice is diagonalizable, all Jordan components have rank 1, all the diagonal elements are rescaling of odd units with equal norms, (up to squares), and the difference of the valuations of two consecutive diagonal elements is less or equal to \( \nu_{P_k}(64) \).
- The lattice is diagonalizable, all Jordan components have rank 1, all the diagonal elements are rescaling of prime elements with equal norms, (up to squares), and the difference of the \( D_k \)-valuations, of any two consecutive diagonal elements, is divisible by 4, or is more than \( \nu_{P_k}(4) \).

Let us state this as a proposition.

Proposition 14.3. The norm principle holds for all local lattices over dyadic fields, under extensions of odd degree, except maybe in the three cases mentioned above.

Last proposition, together with theorem 4.1 in [3], gives the following result.

Proposition 14.4. If \( \Lambda_k \) is a \( D_k \)-lattice in the skew-hermitian space \( (V,h) \), and \( E/k \) is any odd degree field extension, then the distinct proper spinor genera in the genus

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of $\Lambda_k$ remain distinct when lifted to $E$, provided that for every dyadic local place $v$ of $k$, one of the following holds:

- $D_{k_v}$ splits, and either $k_v/\mathbb{Q}_2$ is unramified, or $\Lambda_{k_v}$ is unimodular of rank $\geq 2$.
- $D_{k_v}$ does not split, and $\Lambda_{k_v}$ is not in any of the three cases mentioned above.
- $v$ splits completely in $E/k$.

**Proof.** It suffices to see that the norm principle holds in any of these cases. The first case follows from theorem 4.1 in [3], since the unitary group in the split case can be identified with an orthogonal group via theorem 7 in [2], (p. 181). The second case follows from last proposition. The last one is trivial. []

**Corollary.** If $\Lambda_k$ is a $D_k$-lattice in the skew-hermitian space $(V, h)$, and $E/k$ is any odd degree field extension, then the distinct proper spinor genera in the genus of $\Lambda_k$ remain distinct when lifted to $E$, provided that $\Lambda_k$ is unimodular at every dyadic place. []
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