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ON THE COHOMOLOGY OF THE HYPERELLIPTIC MAPPING CLASS GROUP

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

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Let $\Gamma_{g,k}^n$ be the mapping class group of an oriented surface $S_{g,k}^n$ of genus $g$, with $k$ boundary components, and $n$ punctures. We set $\Gamma_g = \Gamma_{g,0}^0$ and $\Gamma^n = \Gamma_{0,0}^n$. The focus of this dissertation is the hyperelliptic mapping class group, denoted by $\Delta_g$, which is defined as the normalizer of a special order two element in $\Gamma_g$. It is a particularly nice subgroup of $\Gamma_g$ to study, as it is a quotient of Artin's braid group $B_{2g+2}$, which leads to a graphical interpretation of $\Delta_g$ in terms of braids, which we can manipulate to obtain information about $\Delta_g$.

The dissertation is divided into four chapters; the first presents the necessary background information, the second discusses the relationship between braid groups and $\Delta_g$, the third concerns the Yagita invariant of $\Delta_g$ at the prime 2, and the last presents some cohomology computations.

The introduction is divided into five sections. The first section provides some background on Farrell cohomology. The next two sections define the mapping class groups and the Yagita invariant. The fourth section provides information on Riemann surfaces, which is needed to study $\Delta_g$ from a topological viewpoint. For an algebraic description, the fifth section discusses known presentations of mapping class groups.

Viewing the presentations of $\Delta_g$ and Artin's braid group $B_n$, one can readily see a classical map $B_{2g+2} \to \Delta_g$. In Chapter 2 we describe the kernel of this map, and then
get information about torsion and dihedral subgroups in $\Delta_g$ by studying elements of $B_{2g+2}$. This graphical tool is also used in Chapter 4 when we need some information about $\Gamma^\ddagger$.

The cohomology of $\Delta_g$ is $p$-periodic for all odd primes $p$, however it is never 2-periodic. This motivates a study in Chapter 3 of the Yagita invariant of $\Delta_g$ at the prime 2, which is a sort of generalized 2-period. We obtain complete results for every even genus $g$, and partial results for odd $g$. We also determine the 2-rank of $\Delta_g$.

Finally, in the last chapter we gather information about $\Delta_g$ from the topological viewpoint, toward the calculation of the Farrell cohomology of $\Delta_g$. We then combine this information along with the information and techniques of Chapter 2 to determine the $p$-part of the Farrell cohomology of $\Delta_g$ for $g = (p - 1)/2$ and $g = p - 1$, which are the first two cases of $\Delta_g$ containing $p$-torsion.
Dedicated to my parents,

and especially to my wife-to-be Yuka.
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CHAPTER 1
INTRODUCTION

1.1 Cohomology of groups

In this section, we will present only a few facts about group cohomology which are relevant to this dissertation. A comprehensive study of group cohomology can be found in [Bro82]. The content of this section can be found in that text.

Definition 1.1.1. Given a group $\Gamma$, we may regard $\mathbb{Z}$ as a trivial $\mathbb{Z}\Gamma$-module. Then the cohomological dimension of a group $\Gamma$, denoted $cd(\Gamma)$, is defined by

$$cd(\Gamma) = \text{proj dim}_{\mathbb{Z}\Gamma}(\mathbb{Z})$$

$$= \inf\{n : \mathbb{Z} \text{ admits a projective resolution of length } n\}$$

$$= \inf\{n : H^i(\Gamma, -) = 0 \text{ for } i > n\}$$

$$= \sup\{n : H^i(\Gamma, M) \neq 0 \text{ for some } \Gamma\text{-module } M\}$$

where $\text{proj dim}_R(M)$ denotes the projective dimension over the ring $R$ of an $R$-module $M$. The equivalence of the above definitions is shown in [Bro82]. (The first equality is by definition of projective dimension.)

Therefore, the cohomological dimension of a group measures the vanishing of the associated cohomology functor.
We mention two properties of cohomological dimension:

- If $\Gamma$ contains torsion, then $\text{cd}(\Gamma) = \infty$.

- (Serre's Theorem) If $[\Gamma : \Gamma'] < \infty$ and $\Gamma$ is torsion-free, then $\text{cd}(\Gamma') = \text{cd}(\Gamma)$.

There is also a virtual notion, which extends the idea of cohomological dimension to the case of groups containing torsion:

**Definition 1.1.2.** The virtual cohomological dimension of a group $\Gamma$, denoted $\text{vcd}(\Gamma)$, is defined to be the cohomological dimension of some torsion-free subgroup of finite index. (All such subgroups have the same cohomological dimension, which is implied by Serre's Theorem.)

Some groups of finite $\text{vcd}$ have a nice periodicity property in cohomology. This phenomenon is best viewed in Farrell cohomology rather than regular cohomology.

**Definition 1.1.3.** A group $\Gamma$ of finite $\text{vcd}$ is said to have periodic cohomology if for some $d \neq 0$ there is an element $u$ of $\tilde{H}^d(\Gamma, \mathbb{Z})$ which is invertible in the ring $\tilde{H}^*(\Gamma, \mathbb{Z})$. Cup product with $u$ then gives a periodicity isomorphism

$$\tilde{H}^i(\Gamma, M) \approx \tilde{H}^{i+d}(\Gamma, M)$$

for any $\Gamma$-module $M$ and any $i \in \mathbb{Z}$.

Similarly, we define the $p$-period:

**Definition 1.1.4.** Let $p$ be prime. A group $\Gamma$ of finite $\text{vcd}$ is said to have $p$-periodic cohomology if the $p$-primary component $\tilde{H}^d(\Gamma, \mathbb{Z})_{(p)}$ of $\tilde{H}^d(\Gamma, \mathbb{Z})$, contains an invertible element $u$ of non-zero degree $d$. We then have

$$\tilde{H}^i(\Gamma, M)_{(p)} \approx \tilde{H}^{i+d}(\Gamma, M)_{(p)}$$
for any $\Gamma$-module $M$ and any $i \in \mathbb{Z}$, and we define the $p$-period of $\Gamma$, denoted $p(\Gamma)$, to be the number $d$. We adopt the convention that the $p$-period of a $p$-torsion-free group is one (in this case, $\hat{H}^d(\Gamma, \mathbb{Z})_{(p)} = 0$).

To simplify the language, we will refer to the group $\Gamma$ as $p$-periodic when it has $p$-periodic cohomology.

There is a decomposition of rings

$$
\hat{H}^*(\Gamma, M)_{(p)} \approx \prod_p \hat{H}^*(\Gamma, M)_{(p)},
$$

where $p$ ranges over the primes such that $\Gamma$ has $p$-torsion. The number of such $p$'s is finite, which is implied by the finite $vcd$ of $\Gamma$. Therefore it is clear that $\Gamma$ has periodic cohomology if and only if $\Gamma$ has $p$-periodic cohomology for every prime $p$.

We have the following condition for $p$-periodicity:

**Theorem 1.1.1.** A group of finite $vcd$ is $p$-periodic if and only if it does not contain $\mathbb{Z}/p \times \mathbb{Z}/p$.

Finally we state a theorem, due to Brown, that is useful in calculating cohomology.

**Theorem 1.1.2.** [Bro82] If $\Gamma$ is a group of finite $vcd$ and is $p$-periodic, then

$$
\hat{H}^*(\Gamma)_{(p)} \approx \prod_{\pi \in S} \hat{H}^*(\pi)_{(p)},
$$

where $S$ is a set of representatives for the conjugacy classes of subgroups of order $p$. 

3
1.2 The Yagita invariant

The Yagita invariant, defined for a group $\Gamma$ of finite vcd along with a prime $p$, was defined in [Yag85] for finite groups, and extended to more general groups in [Tho89]. We shall denote this invariant by $Y_p(\Gamma)$. It is defined as follows.

Fixing a prime $p$, let $\Gamma$ be of finite vcd, and $\pi < \Gamma$ any subgroup of order $p$. Since $\Gamma$ is of finite vcd, there exists a torsion-free normal subgroup $\Delta$ of finite index in $\Gamma$. It is well-known that if $i : \mathbb{Z}/p \hookrightarrow G$ is an inclusion into a finite group, then the image of the induced map $i^*$ is non-trivial (see [Eve63]).

Since $\pi$ injects into the finite quotient $\Gamma/\Delta$, the image

$$\text{Im}\{H^k(\Gamma; \mathbb{Z}) \to H^k(\pi; \mathbb{Z})\}$$

of the restriction map in cohomology is non-zero for some degree $k > 0$. Reduction mod $p$ maps $H^*(\pi; \mathbb{Z})$ onto $\mathbb{F}_p[u] \subset H^*(\pi; \mathbb{F}_p)$, where $u$ is of degree 2. Therefore there is a maximal number $m = m(\pi, \Gamma)$ such that

$$\text{Im}\{H^k(\Gamma; \mathbb{Z}) \to H^k(\pi; \mathbb{F}_p)\} \subset \mathbb{F}_p[u^m] \subset H^*(\pi; \mathbb{F}_p).$$

Note that $m(\pi, \Gamma)$ is bounded by $m(\pi, \Gamma/\Delta)$, since the map $H^*(\Gamma/\Delta; \mathbb{Z}) \to H^*(\pi; \mathbb{F}_p)$ factors through $H^*(\Gamma; \mathbb{Z})$. Since $\Gamma/\Delta$ is finite, we conclude that $m(\pi, \Gamma)$ is bounded by a bound depending on $\Gamma$ only. The Yagita invariant $Y_p(\Gamma)$ is then defined as the least common multiple of the values $2m(\pi, \Gamma)$, where $\pi$ ranges over the subgroups of $\Gamma$ of order $p$. By convention, we set $Y_p(\Gamma) = 1$ when $\Gamma$ is $p$-torsion-free.

We mention two properties of the Yagita invariant:

- One sees from the definition that for $H < G$, we have $Y_p(H)|Y_p(G)$. 

• Given the short exact sequence $\Delta \to G \to G/\Delta$, where $\Delta$ is free, we have $Y_p(G)|Y_p(G/\Delta)$.

• The Yagita invariant is a generalization of the $p$-period to the case of groups which may not be $p$-periodic; that is, when $\Gamma$ is $p$-periodic, we have $p(\Gamma) = Y_p(\Gamma)$. A proof of this fact can be found in Yining Xia’s Ph.D. thesis of 1990 [Xia90].

1.3 Mapping class groups

Let $S_g$ denote an orientable surface of genus $g$, and more generally let $S^n_{g,k}$ denote a surface of genus $g$, with $n$ points and $k$ open disks removed. Let $\text{Diffeo}_+(S^n_{g,k})$ be the topological group of orientation-preserving diffeomorphisms of $S^n_{g,k}$, with the compact-open topology. Let $\text{Diffeo}_0(S^n_{g,k})$ be the connected component of the identity, or equivalently, the subgroup of diffeomorphisms isotopic to the identity. The mapping class group of $S^n_{g,k}$, denoted $\Gamma^n_{g,k}$, is defined to be the quotient $\text{Diffeo}_+(S^n_{g,k})/\text{Diffeo}_0(S^n_{g,k})$. An equivalent definition is to set $\Gamma^n_{g,k} = \pi_0(\text{Diffeo}_+(S^n_{g,k}))$, the group of connected components of $\text{Diffeo}_+(S^n_{g,k})$. We will write $\Gamma_g$ for the group $\Gamma^0_{0,0}$, and $\Gamma^n$ for $\Gamma^n_{0,0}$.

When studying mapping class groups, one usually assumes that $g > 1$, as $\Gamma_1 = \text{Sl}_2(\mathbb{Z})$, and thus is not an interesting case. In fact, in the literature this assumption is sometimes omitted. Throughout this dissertation we also will assume $g > 1$, as $\Delta_1 = \Gamma_1$.

We also have $\Delta_2 = \Gamma_2$, but for $g \geq 3$, $\Delta_g$ is neither normal or of finite index in $\Gamma_g$. (See [Coh93]).

The hyperelliptic diffeomorphism is the order two map which acts as a rotation of $\pi$
around an axis which passes through each hole of $S_g$, as shown in figure 1.1. The class of this diffeomorphism in $\Gamma_g$ will be denoted $C$, and will be called the hyperelliptic element. The group that is the main focus of this thesis is the hyperelliptic mapping class group, $\Delta_g$, which is defined as the normalizer of $\langle C \rangle \approx \mathbb{Z}/2$ in $\Gamma_g$.

We state here some relevant facts about the mapping class groups; recall we are assuming $g > 1$:

- $\Gamma_g$ is of finite vcd; precisely, $vcd(\Gamma_g) = 4g - 5$ [Har86]. This implies $\Delta_g$ is also of finite vcd.

- $\Gamma_g$ is never 2-periodic.

- For $p$ an odd prime, $\Gamma_g$ is $p$-periodic if and only if one of the following conditions holds (see [Mis94]):

  1. $g \not\equiv 1 \pmod{p}$
  2. $g$ is of the form $kp + 1$ with $k \not\equiv 0, -1 \pmod{p}$ and the interval $[(2k + 3)/p, (2k + 2)/(p - 1)]$ does not contain any integer.
The $p$-period of $\Gamma_g$ depends on $p$. It can be found in [Mis94].

- The \textit{Nielsen realization theorem}, proven by Kerckhoff [Ker83], states that any finite subgroup of $\Gamma_g$ can be realized by a finite group of maps in $\text{Diffeo}_+(S_g)$.

The following theorem of Birman and Hilden also deals with representing mapping classes by diffeomorphisms:

\textbf{Theorem 1.3.1.} [BH73] Let $h$ and $x$ be in $\Gamma_g$, where $x$ is of finite order $n$ and $h$ is in the normalizer of $(x)$, i.e., $hxh^{-1} = x^i$ for some $i$. Then $h$ and $x$ can be represented by topological mappings $\tilde{h}$ and $\tilde{x}$ satisfying $\tilde{x}^n = 1$ and $\tilde{h}\tilde{x}\tilde{h}^{-1} = \tilde{x}^i$.

By a result of Earle and Eells [EE67], the maps $\tilde{h}$ and $\tilde{x}$ may be taken to be in $\text{Diffeo}_+(S_g)$.

\section*{1.4 Riemann Surfaces}

We present here some basic background on the topic of finite group actions on Riemann surfaces.

Suppose $G$ is a finite group of orientation-preserving diffeomorphisms acting on $S_g$, the closed surface of genus $g$. Such diffeomorphisms must each have finitely many fixed points. These points will be called \textit{ramification points} for the action of $G$ on $S_g$. Their orbits under the action of $G$ have cardinality less than $|G|$.

The quotient $S_g/G$ is isomorphic to a closed surface of some genus $h$. There is
an associated branched covering \( \pi : S_g \to S_h \). That is, there is a finite set of points \( \{P_1, \ldots, P_b\} \), which are the images of the ramification points, such that the restriction

\[
S_g - \pi^{-1}\{P_1, \ldots, P_b\} \to S_h - \{P_1, \ldots, P_b\}
\]
is a covering in the traditional sense, with \(|G| \) sheets and transformation group \( G \). An alternate definition, then, of a ramification point is a point \( P \) such that \(|\pi^{-1}(\pi(P))| < |G|\).

The points \( \{P_1, \ldots, P_b\} \) are called branch points for the covering, with the order of a branch point \( P_i \) defined by

\[
ord(P_i) = |\text{stab}_G(Q_i)|
\]

where \( Q_i \) is any point in \( \pi^{-1}(P_i) \). Since \(|\pi^{-1}(\pi(P_i))| \) is the cardinality of the orbit of \( P_i \) under the action of \( G \), we have

\[
ord(P_i) = |G|/|\pi^{-1}(\pi(P_i))|.
\]

The list of orders \( ord(P_1), \ldots, ord(P_b) \) is called the branching data for the group action. According to the Riemann-Hurwitz equation, one has

\[
2g - 2 = |G|(2h - 2) + |G| \sum_{i=1}^{b} \left(1 - \frac{1}{n_i}\right),
\]

where \( n_i = ord(P_i) \). A theorem of Tucker (see [Tuc83]) implies that the order of a branch point must equal the order of some element of \( G \); in particular this implies the branch point orders cannot exceed the maximal order of group elements.

In particular, we can consider the action of a finite-order diffeomorphism on \( S_g \). This leads to the definition of the fixed point data as follows. Suppose \( f \in \text{Diffeo}_+(S_g) \).
has order $n$. Using the notation from above, we have a covering $\pi$ and a set $\{P_1, \ldots, P_b\}$ of branch points for this action of $\mathcal{Z}/n$. Again, let $n_i = \text{ord}(P_i)$. From each orbit $\pi^{-1}(P_i)$ we shall choose a representative $x_i$. Then $f^{n/m_i}$ generates $\text{stab}_{\mathcal{Z}/n}(x_i)$ and, with respect to a fixed Riemannian structure, the differential of $f^{n/m_i}$ acts as a rotation on the tangent space at $x_i$. Let $k_i$ be an integer such that $f^{k_i n/m_i}$ acts as a rotation through $2\pi/n_i$. The number $k_i$ is well-defined modulo $n_i$ and is prime to $n_i$. Therefore there is no loss of information by just considering $k_i/n_i \in \mathbb{Q}/\mathbb{Z}$ rather than $n_i$ and $k_i$ as separate integers. The fixed point data of $f$ is then defined as the collection

$$(g, n|k_1/n_1, \ldots, k_b/n_b).$$

and shall be denoted by $\sigma(f)$. The numbers $k_1/n_1, \ldots, k_b/n_b$ are unique up to order as elements of $\mathbb{Q}/\mathbb{Z}$. We may omit $g$ and $n$ from the data if they are clear from the context.

A classical theorem of Nielson [Nie37] states that two orientation-preserving diffeomorphisms of finite order are conjugate in $\text{Diffeo}_+(S^2)$ if and only if they have the same fixed point data. Furthermore, Symonds [Sym88] proved that the fixed point data of a finite order diffeomorphism depends only on its isotopy class. Since the Nielson realization theorem states that any finite order element $x \in \Gamma_g$ can be represented by a finite order diffeomorphism $f$, we can define fixed point data for torsion elements of $\Gamma_g$ by putting $\sigma(x) = \sigma(f)$. We conclude that two finite order elements of $\Gamma_g$ are conjugate if and only if they have the same fixed point data. This provides a method for counting conjugacy classes of subgroups of $\Gamma_g$ isomorphic to $\mathbb{Z}/n$.

In [GM87], Glover and Mislin determined all the possible fixed point data for
torsion in $\Gamma_g$. From their proof it follows that fixed point data always have an integral sum, that is

$$\sum_{i=1}^{b} \frac{k_i}{n_i} \in \mathbb{Z}.$$ 

1.5 Presentations

Presentations exist for some of the mapping class groups $\Gamma^n_{g,k}$. Although this endeavor was the work of many, over many years, a simplified presentation for $\Gamma^n_{g,1}$ was determined by Wajnryb for $k = 0$ or 1 [Waj83]. The presentations are given in terms of Dehn twists, which are diffeomorphisms associated to embedded circles in $S_g$. Given a simple closed curve $c$ in the oriented surface $S_g$, the Dehn twist about $c$ is defined to be the diffeomorphism which is the identity outside of a cylindrical neighborhood of $c$, and which acts on the cylindrical neighborhood so as to twist one end by $2\pi$, as shown in figure 1.2. Note that the twist is independent of the orientation of $c$, but does depend on the orientation of $S_g$.

The presentation given by Wajnryb has $2g + 1$ generators, representing Dehn twists about the curves $a_1, a_2, b_1, b_2, \ldots, b_g, w_1, w_2, \ldots, w_{g-1}$, as shown in figure 1.3 (Note $a_3, a_4, \ldots, a_g$ are unnecessary in Wajnryb's presentation). Also, the set of defining relations is finite. We will use the name of a curve also for the name of the corresponding Dehn twist.

The hyperelliptic mapping class group is equal to the subgroup of $\Gamma_g$ generated by $a_1, b_1, w_1, b_2, w_2, \ldots, w_{g-1}, b_g$, and $a_g$ [BH71]. To simplify its presentation, we will
give new names to the generators as follows:

\[
\begin{array}{cccccccc}
\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \ldots & \sigma_{2g-1} & \sigma_{2g} & \sigma_{2g+1} \\
\end{array}
\]

The presentation of \( \Delta_g \) is then on these generators, with the relations:

(i) \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) for \( 1 \leq i \leq 2g \),

(ii) \( \sigma_i \sigma_j = \sigma_j \sigma_i \) if \( |i - j| \geq 2 \),

(iii) \( z^{2g+2} = 1 \), where \( z = \sigma_1 \sigma_2 \ldots \sigma_{2g+1} \),

(iv) The element \( C = \sigma_1 \sigma_2 \ldots \sigma_{2g+1} \sigma_{2g+1} \ldots \sigma_2 \sigma_1 \) is in the center and satisfies \( C^2 = 1 \).

The element \( C \) is the hyperelliptic element as defined before (see [BH71]).

Note that the first two relations say that two Dehn twists about non-intersecting

![A Dehn Twist](image)

Figure 1.2: A Dehn Twist
curves commute, whereas Dehn twists about two curves $a$ and $b$ which intersect in one point satisfy the braid relation

$$aba = bab.$$  

This is true in general for Dehn twists on $S_g$.

As far back as 1934, Magnus [Mag34] determined a presentation for $\Gamma^n$. If, in the above presentation for $\Delta_g$, we replace $2g + 2$ with $n$, and replace relation (iv) with:

(iv)' $C = 1$,

then we obtain a slightly simplified form of Magnus’s presentation for $\Gamma^n$, on $n$ generators. (Here $C$ is the same as in (iv)).

Inspecting the presentations, we find a central extension

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \Delta_g \longrightarrow \Gamma^{2g+2} \longrightarrow 1.$$  \hspace{1cm} (1.1)

In fact, this non-split extension was determined by Birman and Hilden and used to determine their presentation of $\Delta_g$ from Magnus’s presentation of $\Gamma^n$ [BH71] (also see [BH73] for a simpler version of their proof).

Figure 1.3: Dehn twist generators for $\Gamma_g$
CHAPTER 2
BRAID CALCULATIONS IN $\Delta_q$

2.1 Artin's braid group

Artin's braid group on $n$ strings, $B_n$ (see [Art47] and [Bir71]), is defined via the presentation:

(i) There are generators $\sigma_1, \ldots, \sigma_{n-1},$

(ii) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| \geq 2,$ and

(iii) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n - 2.$

Alternate classical definitions of $B_n$ may be found in [Bir71]. We shall briefly describe one such definition.

The elements of $B_n$ can be represented by geometric braids on $n$ strings. A geometric braid on $n$ strings is defined as follows. We choose $n$ points $O_1, \ldots, O_n$ in $\mathbb{R}^2,$ that we will fix once and for all. Let $I = [0, 1].$ We then have $n$ points $P_1 = (O_1, 0), \ldots, P_n = (O_n, 0)$ and $n$ points $Q_1 = (O_1, 1), \ldots, Q_n = (O_n, 1)$ in $\mathbb{R}^2 \times I.$ We shall visualize $\mathbb{R}^2 \times I$ oriented with $\mathbb{R}^2 \times \{0\}$ a horizontal plane lying below the plane $\mathbb{R}^2 \times \{1\}.$ Then the points $P_i$ lie in the lower plane and the points $Q_i$ lie directly
above them. We “connect with braided strings” the points $P_1, \ldots, P_n$ to $Q_1, \ldots, Q_n$ (not necessarily connecting $P_i$ to $Q_i$). That is, we define curves

$$\gamma_i = (\alpha_i, \beta_i) : (I, \{0\}, \{1\}) \rightarrow (\mathbb{R}^2 \times I, P_i, Q_{\sigma(i)})$$

where $\sigma$ is a permutation of the array $(1, \ldots, n)$, and such that:

- $\beta_i(t_1) < \beta_i(t_2)$ for $0 \leq t_1 < t_2 \leq 1$ (the curves should move upwards), and
- $\alpha_i(t) \neq \alpha_j(t)$ for $i \neq j$ and all $t \in I$ (the curves should not intersect).

Let $A_i$ be the arc which is the image of $\gamma_i$ in $\mathbb{R}^2 \times I$. Their union $A = A_1 \cup \cdots \cup A_n$ is called a geometric braid. It is a subset of $\mathbb{R}^2 \times I$.

We define an equivalence relation as follows. If $A$ and $B$ are geometric braids, we shall write $A \sim B$ if one can be deformed to the other by an isotopic deformation

$$G_t : \mathbb{R}^2 \times I \rightarrow \mathbb{R}^2 \times I, \text{ for } t \in [0, 1]$$

of the ambient space such that

- $G_t$ is constant on the boundary $\mathbb{R}^2 \times \{0, 1\}$ for every $s$,
- $G_0(A) = A$ and $G_1(A) = B$, and
- each intermediate image $H_s(A)$ is also a geometric braid.

Finally, we may define (equivalent to the above) the braid group $B_n$ to consist of geometric braids, modulo this equivalence relation. Multiplication in $B_n$ is then performed by concatenation, that is, placing one geometric braid on top of the other.
and deforming vertically by a factor of one-half. In this thesis we shall adopt the convention that if $b_1$ and $b_2$ are braids, then $b_1b_2$ is represented with $b_1$ on top of $b_2$.

In short, we may represent elements of $B_n$ with braid diagrams, which consist of $n$ arcs connecting $n$ points on two parallel line segments. A braid diagram representing the braid group generator $\sigma_i$ is shown in figure 2.1.

![Braid Diagram](image)

Figure 2.1: A generator of $B_n$

If we define the braid group as in the geometric braid construction above, then the commutation relation (ii) in the presentation of $B_n$ obviously holds. Also, the braid relation (iii) is also clear, from figure 2.2.

A proof of the equivalence of our above descriptions of $B_n$, that is, a proof that the relations (i), (ii), and (iii) are a sufficient set of defining relations for the geometric braid defined group, can be found in [Bir71].

Note that if we add the relation

(iv) $\sigma_i^2 = 1$ for $1 \leq i \leq n - 1$
to the presentation of $B_n$ above, we obtain the classical presentation for the symmetric group on $n$ letters, $\Sigma_n$. We identify the element $\sigma_i$ with the transposition $(i \ i + 1)$.

Thus we obtain a surjection $B_n \twoheadrightarrow \Sigma_n$, where the image of a braid given as a braid diagram can be read off by following the strings from top to bottom: if a strand starts at position $k$ and ends at position $l$, then the associated permutation takes $k$ to $l$. In other words, the map $B_n \twoheadrightarrow \Sigma_n$ is given by “forgetting the under- and over-crossings” of a braid diagram.

### 2.2 $\Delta_g$ as a quotient of $B_{2g+2}$

Examining the presentation of $\Delta_g$, we find it is obtained from the presentation of $B_{2g+2}$, by adding the relations

1. $z^{2g+2} = 1$
2. $C^2 = 1$
\(3\) \([C, \sigma_i] = 1 \text{ for } i = 1, \ldots, 2g + 1\)

where we have defined

\[z = \sigma_1 \sigma_2 \cdots \sigma_{2g+1}\]

and

\[C = \sigma_1 \sigma_2 \cdots \sigma_{2g+1} \sigma_{2g+1} \cdots \sigma_2 \sigma_1\].

We shall alternately refer to the generators \(\sigma_i\) as elements of \(B_{2g+2}\) and \(\Delta_g\).

Therefore we obtain a surjection \(\rho : B_{2g+2} \twoheadrightarrow \Delta_g\), whose kernel is generated by the braids corresponding to \(z^{2g+2}, C^2\), and \([C, \sigma_i], i = 1, \ldots, 2g + 1\) (that is, braids whose words in \(B_{2g+2}\) are identical to the given words for these elements of \(\Delta_g\)). Similarly, \(\Gamma^n\) is a quotient of \(B_n\), and the set of generators for \(\ker(B_n \twoheadrightarrow \Gamma^n)\) is the same as \(\ker(\rho)\) but with \(C^2\) replaced by \(C\). Using the map \(\rho\), we may represent elements of \(\Delta_g\) with braids, keeping in mind that the braids are collected into equivalence classes.

In this dissertation, we shall frequently blur the distinction between elements in \(\Delta_g\) and their braid representatives. To get a feel for these braid representatives, we shall consider the kernel of \(\rho\).

The braids corresponding to the hyperelliptic element \(C\) and the order \(2g + 2\) element \(z\) are drawn in figure 2.3. For clarity, the braids are drawn with a specific number of strings, but for general genus these braids have the obvious similar form.

(In this dissertation, we will similarly draw braids in \(B_n\) for arbitrary \(n\) with a specific number of strings, when the general form is clear.)

One finds easily that the relations \([C, \sigma_i] = 1\) are automatically satisfied in the braid group for all \(i\) except \(i = 1\). Thus we need only three generators for the kernel.
of $B_{2g+2} \rightarrow \Delta_g$, namely $C^2$, $z^{2g+2}$, and $\sigma_1^{-1}C\sigma_1C^{-1}$. (We use $\sigma_1^{-1}C\sigma_1C^{-1}$ instead of $\sigma_1C\sigma_1^{-1}C^{-1}$, since the former has a simpler looking braid.) We will sketch these generators below.

We first consider the braid $z^{2g+2}$. More generally we shall sketch the braid $z^k$ for $1 \leq k \leq 2g+2$. Observe in Figure 2.3 a simplified form of $z^2$; in Figure 2.4 we extend this idea to illustrate $z^k$. Note then, that $z^{2g+2}$ is represented by a full twist of all $2g+2$ strings.

Figure 2.3: Braid representatives for $C$ and $z$

\[z^2 = \quad \ldots \ldots \ldots \ldots \ldots = \]

Figure 2.4: Equal braids representing $z^2$
We simplify the braid $\sigma_1^{-1}C\sigma_1C^{-1}$ in Figure 2.2.

Finally, adding the braid $C^2$, we collect the three generators of $\ker(\rho)$ together in Figure 2.7.
Figure 2.6: Equal braids representing $\sigma_1^{-1} C \sigma_1 C^{-1}$

Figure 2.7: Generators for $\ker\{B_{2g+2} \to \Delta_g\}$
2.3 The map $\Delta_g \to \Sigma_{2g+2}$

We shall describe commuting surjections as shown in the following diagram.

$$
\begin{array}{c}
B_{2g+2} \xrightarrow{p_3} \Delta_g \\
p_1 \downarrow \quad \phi \quad p_3 \\
\Sigma_{2g+2} \leftarrow \Gamma^{2g+2}
\end{array}
$$

Each of the groups in the diagram have $2g + 1$ generators, and the maps can be defined in the obvious way on these generators; some of these maps have already been observed. Since $\ker(p_2)$ consists of pure braids, the upper triangle of diagram 2.1 is commutative. Similarly, since $\ker(p_3) = \langle C \rangle$ has trivial image in $\Sigma_{2g+2}$, the lower triangle is also commutative.

We have described these maps purely in terms of group presentations; however, we can give geometric descriptions for $\phi$, $p_3$, and $p_4$ as follows.

To simplify the discussion, we may use the equivalent definition of $\Gamma^n$ as the mapping class group of the sphere with $2g + 2$ distinguished points, rather than punctures, which must remain fixed as a set by the representative diffeomorphisms. In [Bir71], Birman explains that $p_4$ corresponds to the action of a representative diffeomorphism on the $2g + 2$ distinguished points. The map $p_3$ can be interpreted geometrically as a map induced by the branched cover $S_g \to S_g/\langle C \rangle \approx S^2$, which takes the fixed points of $C$ injectively to the $2g + 2$ distinguished points of $S^2$. We may then reason that the map $\phi$ can be interpreted as being associated to the action of an element in $\Delta_g$ on the fixed points of $C$. 
This means that we can view the action of an element \( x \in \Delta_g \) on the fixed points of \( C \) by looking at a braid representative for \( x \). Similarly we can use braids to view the action of an element of \( \Gamma^n \) on the \( n \) distinguished points (or punctures).

### 2.4 Braid representatives for the hyperelliptic

In Propositions 2.4.1 and 2.4.2, we provide several braids which represent either the hyperelliptic or the identity.

**Proposition 2.4.1.** Let \( C_m \) be the braid as shown in Figure 2.8, with \( 1 \leq m \leq 2g+2 \). Then, as an element of \( \Delta_g \), \( C_m \) is equal to the hyperelliptic element \( C \).
Proof. Since $C$ is central in $\Delta_g$, the braid $bCb^{-1}$ shown in Figure 2.9 is equal to $C$ in $\Delta_g$. Also, the braids $C_m$ and $bCb^{-1}$ can be seen to be equal in $B_{2g+2}$, therefore $C_m = C$ in $\Delta_g$. □

Figure 2.10: The element $I_m$.

Proposition 2.4.2. Let $I_m$ be the braid in $B_{2g+2}$ consisting of side-by-side full twists of $m$ and $2g+2-m$ strings, respectively, where the twists have opposite orientations. (See Figure 2.10.) Then, as an element of $\Delta_g$,

$$I_m = \begin{cases} 1 & \text{if } m \text{ is even,} \\ C & \text{if } m \text{ is odd.} \end{cases}$$
Proof. Figure 2.11 illustrates the relation (in $\Delta_g$)

$$I_m C = I_{m-1},$$

or

$$I_m = C I_{m-1}. \quad (2.2)$$

Note that $I_0$ is a full twist of all $2g + 2$ strings, therefore $I_0 = 1$ in $\Delta_g$. The claim is then proven recursively, using relation 2.2.

Note that if we reverse the orientations of the twists in Figure 2.10, we obtain $I_m^{-1}$, which equals $I_m$ in $\Delta_g$, by the proposition.

2.5 Some torsion elements of $\Delta_g$

We may now present some braids representing elements of finite order in $\Delta_g$. We already have an element $z$ of order $2g + 2$, given in the presentation of $\Delta_g$. We use Proposition 2.4.1 to construct more torsion elements. This is done by finding braid elements whose $n^{th}$ power equals one of the braids $C_m$ as constructed in Proposition 2.4.1 and shown there to represent either 1 or $C$ in $\Delta_g$.

Consider the braids shown in figure 2.12, here drawn with a specific number of strings, but similarly defined for general genus. The first braid shown represents the order $2g + 2$ element $z$. The $2g + 1$ power of the second braid, and the $2g$ power of the third are both equal to $C_1$, that is, a full twist of $2g + 1$ strings next to a single string. (Note that successive powers of the second braid cause the strings to wind
Figure 2.11: $I_m C = I_{m-1}$
around the middle string, which is shown as a thick line.) The braid $C_1$ represents the hyperelliptic, therefore the elements shown are of orders $4g + 2$ and $4g$, respectively. (In fact, the order $4g + 2$ element is of maximal finite order in $\Gamma_g$; see [Mis94].)

Elements of prime order $p$ can be found using the order $4g$, $4g + 2$, and $2g + 2$ elements described above. That is, we can illustrate $p$-torsion in $\Delta_g$ whenever $p$ divides one of the successive integers $2g$, $2g + 1$, or $2g + 2$. We shall see later that this is a necessary condition for the existence of $p$-torsion in $\Delta_g$, and for $p$ an odd prime, there is a unique conjugacy class of $\mathbb{Z}/p$. Thus, up to conjugacy, the braids thus constructed represent all of the possible $p$-torsion in $\Delta_g$, for odd primes $p$.

2.6 A dihedral subgroup of $\Delta_g$.

Theorem 2.6.1. Let $g \geq 2$. $\Delta_g$ contains a subgroup isomorphic to $D_8$, the dihedral group of order 8, when either $g$ is even or $g \equiv 3 \pmod{4}$. For $g \equiv 1 \pmod{4}$, there is no such subgroup.
Figure 2.13: Equal braids representing $\tau$.

Figure 2.14: The element $\sigma$.

The proof will be provided by a series of lemmas, dealing with separate cases. We will show the existence of these subgroups in two ways; first, we will illustrate them directly from the viewpoint of braids, and then we will provide an alternate description which is more algebraic.

**Lemma 2.6.1.** For $g$ even, $\Delta_g$ contains a subgroup isomorphic to $D_8$.

*Proof.* Consider the elements of $\Delta_g$ which are represented by the braids $\tau$ and $\sigma$ as
shown in Figures 2.13 and 2.14. Note the two braids shown for $\tau$ are equal in the braid group. The square of $\tau$ is a full twist of all $2g + 2$ strings, and therefore is the identity in $\Delta_g$. Also, $\sigma^2$ is represented by side-by-side full twists of $g + 1$ strings, with opposite orientations. Since $g + 1$ is odd, this element represents $C$, as determined in Section 2.4. Therefore $\sigma$ is of order 4. Finally, we see in Figure 2.15 that $\tau\sigma\tau = \sigma^{-1}$, which completes the relations that define $D_8$.

\textbf{Lemma 2.6.2.} For any genus $g$, $\Delta_g$ contains a subgroup isomorphic to $D_{2g+2}$. When $g \equiv 3 \pmod{4}$, this implies $\Delta_g$ contains a subgroup isomorphic to $D_8$.

\textit{Proof.} Recall the element $z$ of order $2g + 2$ in $\Delta_g$, as defined in Section 2.2. Consider
also the half-twist element $\tau$ as in the previous lemma, which is of order two in $\Delta_g$.
For any genus $g$ we have the relation $z\tau z\tau = C$, or

$$\tau z\tau = Cz^{-1},$$

(2.3)
as illustrated in Figure 2.16. (Note $\tau = \tau^{-1}$). This implies

$$\tau z^2\tau = z^{-2}.$$  

(2.4)

Therefore $\tau$ and the order $g+1$ element $z^2$ generate a subgroup isomorphic to $D_{2g+2}$.

Suppose $g \equiv 3 \pmod{4}$. Then 8 divides $2g + 2$, and therefore

$$D_8 < D_{2g+2} < \Delta_g.$$
Explicitly, let $\zeta$ be the order four element $z^{(g+1)/2}$. Since $(g+1)/2$ is even, equation 2.3 implies

$$\tau \zeta \tau = \zeta^{-1}, \quad (2.5)$$

which means $\zeta$ and $\tau$ generate a group isomorphic to $D_8$. □

An alternate description of the subgroups found above can be provided using only the order two element $\tau$, the order $2g + 2$ element $z$, and relation 2.3. We provide this alternate description in the following.

**Lemma 2.6.3.** Let $z$ and $\tau$ be the elements of $\Delta_g$ as described above. Then the following subgroups are isomorphic to $D_8$:

1. $\langle \tau, z^{(g+1)/2} \rangle$, when $g \equiv 3 \pmod{4}$,
2. $\langle z^{g+1}, \tau z \rangle$, when $g$ is even,
3. $\langle \tau, z^{g+1} \tau \rangle$, when $g$ is even.

Moreover, the group (3) is equal to the group described in Lemma 2.6.1.

**Proof.** The group (1) has already been shown to be isomorphic to $D_8$ in lemma 2.6.2.

For the group (2), note that $z^{g+1}$ is of order two. Relation 2.3 implies $\tau z \tau z = C$, so $\tau z$ is of order 4. Also, since $g + 1$ is odd, we have, using relation 2.3,

$$z^{g+1}(\tau z)z^{g+1} = (z^{g+1}\tau)zz^{g+1} = (C\tau z^{-1})zz^{g+1} = C\tau z = (\tau z)^2(\tau z) = (\tau z)^3,$$
which completes the relations that define $D_8$.

For the group (3), $\tau$ is of order two, and since $g + 1$ is odd, relation 2.3 implies

$$z^{g+1}z^{g+1} = z^{g+1}(\tau z^{g+1})$$
$$= z^{g+1}(Cz^{-(g+1)})$$
$$= C,$$

so $z^{g+1}\tau$ is of order four. Finally, we check the last relation defining $D_8$:

$$\tau(z^{g+1}\tau) = \tau z^{g+1}$$
$$= Cz^{-(g+1)}\tau$$
$$= (z^{-(g+1)}\tau)^2z^{-(g+1)}\tau$$
$$= (z^{-(g+1)}\tau)^3.$$

To prove the last claim, that the group (3) is equal to the group described in Lemma 2.6.1, we observe in Figure 2.17 that $\sigma = z^{g+1}\tau$. Note the form of $z^{g+1}$ is as described in Section 2.2. □

To complete the proof of Theorem 2.6.1, we must show that $\Delta_g$ does not contain a $D_8$ when $g \equiv 1$ (mod 4). We will need the following two lemmas.

Lemma 2.6.4. Let $g \equiv 1$ (mod 4), and let $x \in \Delta_g$ be an order four element whose square is not the hyperelliptic element. Then $x$ has no fixed points.

Proof. Let $N_i$ denote the number of branch points of order $i$ for the action of $\langle x \rangle$ on
Figure 2.17: Expressing $\sigma$ in terms of $z$ and $\tau$. 
Sg. There are branch points of orders two and four. The Riemann-Hurwitz equation reads

\[ 2g - 2 = 4(2h - 2) + 4 \left( \frac{3}{4} N_4 + \frac{1}{2} N_2 \right). \]

Letting \( g = 4k + 1 \), this simplifies to

\[ 8k + 8 - 2N_2 = 8h + 3N_4. \]

Recall from Lemma 3.1.2 that for \( g \) odd, an order two element in \( \Delta_g \) not equal to the hyperelliptic element must have zero or four fixed points. Thus \( N_2 \), which is the number of fixed points of \( x^2 \), must be either zero or four. In either case, the above equation reduces mod 8 to

\[ N_4 \equiv 0 \pmod{8}. \]

But \( x \) cannot have more fixed points than \( x^2 \), so \( N_4 \leq 4 \), which implies \( N_4 \) must be zero. This means \( x \) is fixed-point free. \( \Box \)

**Lemma 2.6.5.** Suppose \( \Delta_g \) contains a subgroup \( G \) of order \( 2^n \), \( n \geq 0 \), and that the hyperelliptic element \( C \) is not contained in \( G \). Then the order of \( G \) must divide \( 2g + 2 \).

**Proof.** Considering the group \( \langle C \rangle \times G \) as a group of diffeomorphisms, we see that \( G \) acts on the \( 2g + 2 \) fixed points of \( C \). Suppose an element \( x \in G \) of order \( 2^m \) has a fixed point in common with \( C \). This implies the group \( \langle x \rangle \times \langle C \rangle \approx \mathbb{Z}/2^m \times \mathbb{Z}/2 \) has a branch point of order \( |\langle x \rangle \times \langle C \rangle| = 2^{m+1} \), which is impossible, as the order of a branch point cannot exceed the maximal order for group elements, which is \( 2^m \). Therefore \( G \) must act on the fixed points of \( C \) freely. This implies the \( 2g + 2 \) fixed
points of $C$ are partitioned into orbits of cardinality $|G|$, which means $|G|$ must divide $2g + 2$. □

Lemma 2.6.6. $\Delta_g$ does not contain a subgroup isomorphic to $D_8$ when $g \equiv 1 \pmod{4}$.

Proof. Let $g = 4k + 1$, and suppose there is a subgroup isomorphic to $D_8$ in $\Delta_g$. There are two order four elements of this subgroup, one we will call $\sigma$, the other is $\sigma^{-1}$.

We will consider two cases; either $\sigma^2 = C$ or $\sigma^2 \neq C$.

CASE 1: $\sigma^2 \neq C$. Then $D_8$ cannot contain $C$, as $\sigma^2$ is the only central order two element of $D_8$. Using Lemma 2.6.5, we find the order of $D_8$, which is 8, must divide $2g + 2$, which implies $g \equiv 3 \pmod{4}$, contrary to our assumption.

CASE 2: $\sigma^2 = C$. By analyzing fixed point data, we will show that an element of order four in $\Gamma_g$ cannot be conjugate to its inverse (hence the same is true in $\Delta_g$). Since the relations which define $D_8$ imply that $\sigma$ and $\sigma^{-1}$ are conjugate, there can be no such subgroup in $\Delta_g$.

We will first count branch points for the action of $\langle \sigma \rangle \approx \mathbb{Z}/4$ on $S_g$. Recall $C$ has $2g + 2$ fixed points. Let $\sigma$ have $f$ fixed points. These are fixed points of $\sigma^2 = C$. We then have

$$
\begin{align*}
N_4 &= f \\
N_2 &= \frac{2g + 2 - f}{2},
\end{align*}
$$

where $N_i$ denotes the number of branch points of order $i$. The Riemann-Hurwitz equation for this data is

$$
2g - 2 = 4(2h - 2) + 4 \left[ \frac{3}{4} f + \frac{1}{2} \left( \frac{2g + 2 - f}{2} \right) \right],
$$

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which simplifies to

\[ 4 = 8h + 2f. \]

The only non-negative integer solution is \( h = 0, f = 2 \). Therefore,

\[
N_4 = 2 \\
N_2 = \frac{2g+2-2}{2} = g.
\]

The fixed point data for the element \( \sigma \) must then be of the form

\[
\left( \frac{i_1}{4}, \frac{i_2}{4}, \frac{1}{2}, \ldots, \frac{1}{2} \right)_{g},
\]

where \( i_1 \) and \( i_2 \) are either 1 or 3. Since the fixed point data must have an integral sum, and since \( g \) is odd, the only possibilities are \((i_1, i_2) = (1, 1)\) or \((i_1, i_2) = (3, 3)\). If \( \sigma \) has fixed point data corresponding to \((i_1, i_2) = (1, 1)\), then its inverse \( \sigma^3 \) will have \((i_1, i_2) = (3, 3)\), and vice versa. Since fixed point data determines conjugacy classes, \( \sigma \) cannot be conjugate to its inverse.

\[ \square \]
CHAPTER 3
THE YAGITA IN Variant OF $\Delta_g$ AT THE PRIME 2

For $g > 1$ the group $\Delta_g$ is never 2-periodic, since we can always find a subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, as we will see in section 3.2. A generalization of the $p$-period is the Yagita invariant $Y_p$, which we calculate in this chapter for $p = 2$. We shall determine the Yagita invariant of $\Delta_g$ at the prime 2 for all $g$ even, and provide some partial results for odd genus. We also determine the 2-rank of $\Delta_g$, that is, the maximal $n$ such that $\Delta_g$ contains a cartesian product of $n$ copies of $\mathbb{Z}/2$. The results are as follows.

Theorem 3.0.1. The 2-rank of $\Delta_g$ is 2 when $g$ is even, and 3 when $g$ is odd.

Theorem 3.0.2. Let $Y_2(\Delta_g)$ denote the Yagita invariant of $\Delta_g$ at the prime 2. Then:

- $Y_2(\Delta_g) = 4$ when $g$ is even.
- $Y_2(\Delta_g) = 4$ when $g \equiv 3 \pmod{8}$.
- $Y_2(\Delta_g)$ is either 2 or 4 when $g \equiv 5 \pmod{8}$.
- $Y_2(\Delta_g)$ is either 4 or 8 when $g \equiv 7 \pmod{8}$

The best we have for $g \equiv 1 \pmod{8}$ is to use the following to obtain upper bounds:
Theorem 3.0.3. Assume that \( g = l2^\alpha + 1 \) with \( l \) an odd integer and \( \alpha \geq 0 \). Then \( Y_2(\Delta_g) \) divides \( 2^{\alpha+2} \).

Proof. This is a corollary to Theorem 3 in [Xia98], which concerns \( Y_2(\Gamma_g) \), but here restated to give an upper bound on \( Y_2(\Delta_g) \). \qed

3.1 Some lemmas on subgroups of order 2

In the following two lemmas, we consider a subgroup \( \pi \) of order two lying in either \( \Gamma_g \) or \( \Delta_g \). Suppose the associated branched covering \( S_g \to S_g/\pi \) has \( n \) branch points, and quotient space of genus \( h \). Note \( n \) is both the number of fixed points for an order two diffeomorphism generating \( \pi \) and the number of branch points for the covering. The associated Riemann-Hurwitz equation is then \( 2g - 2 = 2(2h - 2) + n \).

The following lemma is Proposition 4.3 of [GMX94], there stated for subgroups \( \pi \) of odd prime order \( p \). The proof is based on an action of \( \Gamma_g \) on a certain space \( X_\infty \), depending on \( \pi \), which is a subspace of the Teichmüller space \( T_g \). The assumption \( p > 2 \) was made to avoid the case of \( \pi = \langle C \rangle \) acting on a genus two surface, in which case \( X_\infty \) is empty and the argument fails. However, for \( g > 2 \), \( X_\infty \) is non-empty for any subgroup \( \pi \) of prime order, as stated earlier in the paper, and the proof carries through verbatim. Thus for groups \( \pi \) of order two we may state the proposition as follows.

Lemma 3.1.1. Let \( g > 2 \) and \( \pi \subset \Gamma_g \) be a subgroup of order 2, with associated Riemann-Hurwitz equation \( 2g - 2 = 2(2h - 2) + n \). Then there exists a cohomology element \( e \in H^{6(g-h)-2n}(\Gamma_g; \mathbb{Z}) \) whose restriction to \( H^{6(g-h)-2n}(\pi; \mathbb{Z}) \) is nontrivial.
Lemma 3.1.2. Let $\pi$ be a subgroup of order 2 in $\Delta_g$, with associated Riemann-Hurwitz equation $2g - 2 = 2(2h - 2) + n$. Also assume $\pi$ is not generated by the hyperelliptic element. Then if $g$ is odd, $n$ equals 0 or 4, and if $g$ is even, $n$ is 2.

Proof. For $g$ odd, let $g = 2k + 1$, and the Riemann-Hurwitz equation reads

$$2(2k + 1) - 2 = 2(2h - 2) + n,$$

or

$$4k = 4(h - 1) + n.$$ 

Reducing this mod 4 we obtain

$$n \equiv 0 \pmod{4}. \quad (3.1)$$

For $g$ even, a similar calculation gives us

$$n \equiv 2 \pmod{4}. \quad (3.2)$$

The group $\pi \times \langle C \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ acts on $S_g$, with quotient space $S^2$. All branch points for the associated cover are of order 2, since that is the maximal order in $\mathbb{Z}/2 \times \mathbb{Z}/2$. The $2g + 2$ fixed points of $C$ give $g + 1$ branch points. Let there be $b$ additional branch points. Then the Riemann-Hurwitz equation reads

$$2g - 2 = 4(-2) + 4(g + 1 + b) \left(1 - \frac{1}{2}\right)$$

which implies $b = 2$. These two additional branch points correspond to 4 ramification points in $S_g$; these are fixed points of order 2 elements of $\pi \times \langle C \rangle$. 

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When $g$ is odd, Equation 3.1 implies the four ramification points must be fixed points of a single order 2 element. There are two order 2 elements other than $C$ in $\pi \times \langle C \rangle$; we see, therefore, that one of them has four fixed points, and the other has none.

Similarly, when $g$ is even, Equation 3.2 implies the two order 2 elements in $\pi \times \langle C \rangle$ different from $C$ must each have two fixed points.

From this one readily obtains

**Corollary 3.1.1.** The hyperelliptic element $C$ is the unique finite-order element of $\Delta_g$ which can be represented by a finite-order diffeomorphism with $2g + 2$ fixed points.

**Proof.** Let $x$ be represented by such a diffeomorphism. One can easily determine from the Riemann-Hurwitz equation that $x$ must be of order two. Then the above lemma implies that $x$ must equal $C$. \qed

### 3.2 The 2-rank of $\Delta_g$

In this section, we will prove Theorem 3.0.1. We will use the following lemma to provide a lower bound on the 2-rank.

**Lemma 3.2.1.** $\Delta_g$ contains a subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ when $g$ is even, and a subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ when $g$ is odd.

**Proof.** Recall the element $z$ of order $2g + 2$, and the order two element $\tau$ represented by a half-twist braid, cf. Section 2.6. The element $\zeta = z^{g+1}$ is of order two. Examining braid representatives of $\tau$, $\zeta$, and $C$, one finds they have different images under the
map \( B_g \rightarrow \Sigma_{2g+2} \), which factors through \( \Delta_g \). Thus \( \tau, \zeta, \) and \( C \) are distinct elements in \( \Delta_g \). Furthermore, the element \( C \) is central, and \( \zeta \) commutes with \( \tau \) if and only if \( g \) is odd, by the relation 2.3. Therefore we have

\[
\mathbb{Z}/2 \times \mathbb{Z}/2 \cong \langle \tau \rangle \times \langle C \rangle \subset \Delta_{2k},
\]

and

\[
\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \cong \langle \tau \rangle \times \langle \zeta \rangle \times \langle C \rangle \subset \Delta_{2k+1}.
\]

In fact, we already knew that \( \mathbb{Z}/2 \times \mathbb{Z}/2 \) is contained in \( \Delta_{2k} \), as it is a subgroup of \( D_8 \), which was shown in section 2.6 to be contained in \( \Delta_{2k} \). Indeed, the \( \mathbb{Z}/2 \times \mathbb{Z}/2 \) described above is in the \( D_8 \) described there. □

**Proof of Theorem 3.0.1.** Suppose \( (\mathbb{Z}/2)^n \) is contained in \( \Delta_g \). We will assume one \( \mathbb{Z}/2 \) is generated by the hyperelliptic involution, \( C \), since this must be the case if \( n \) is maximal.

Lemma 2.6.5 implies \( 2^{n-1} \), the order of \( (\mathbb{Z}/2)^n / \langle C \rangle \), divides \( 2g + 2 \). That is, \( 2^{n-2} \) divides \( g + 1 \).

When \( g \) is even, this implies \( n \) can be at most two. The above lemma then implies the 2-rank of \( \Delta_g \) is 2 for \( g \) even.

Now suppose \( g \) is odd, and consider the branched cover associated to the action of \( (\mathbb{Z}/2)^n \) on \( S_g \). All branch points must be of order 2, since that is the maximal order for group elements. There are \( (g+1)/2^{n-2} \) order 2 branch points which are the image of the fixed points of \( C \). Suppose there are \( b \) additional branch points. Note
that since $S_g/(C) \approx \mathbb{S}^2$, the quotient of the $(\mathbb{Z}/2)^n$ action must also have genus zero.

The Riemann-Hurwitz equation then reads

$$2g - 2 = 2^n(-2) + 2^n \left( \frac{1}{2} \right) \left( \frac{g+1}{2^{n-2}} + b \right),$$

which one solves to obtain

$$b = \frac{4(2^{n-1} - 1)}{2^{n-1}},$$

from which we see the maximal $n$ is 3. The above lemma then implies the 2-rank of $\Delta_g$ is 3 for $g$ odd. □

### 3.3 The Yagita invariant of $\Delta_{2k}$ at the prime 2

**Lemma 3.3.1.** Let $g > 0$ be even. Then $Y_2(\Delta_g) = 4$.

*Proof.* Xia proved $Y_2(\Gamma_g) = 4$ for even genus $g > 0$ [Xia98]. This provides an upper bound for $Y_2(\Delta_g)$. Recall from Section 2.6 that $\Delta_g$ contains a subgroup isomorphic to $D_8$ for any even genus $g$. Since $Y_2(D_8) = 4$ (See [Yag85]), this provides a lower bound and completes the proof. □

### 3.4 Partial results on $Y_2(\Delta_g)$ for odd genus

**Lemma 3.4.1.** Let $g \equiv 3, 7 \pmod{8}$. Then 4 divides $Y_2(\Delta_g)$.

*Proof.* This follows immediately from Lemma 2.6.1, since $Y_2(D_8) = 4$. □

**Lemma 3.4.2.** Let $g \equiv 7 \pmod{8}$. Then $Y_2(\Delta_g)$ divides 8.
Proof. If \( g \equiv 7 \pmod{8} \), then we have
\[
g = 8k + 7 = 2(4k + 3) + 1,
\]
or \( g = 2l + 1 \) where \( l \) is odd. The claim then follows from Theorem 3.0.3. \( \square \)

Lemma 3.4.3. Let \( g \equiv 3, 5 \pmod{8} \). Then \( Y_2(\Delta_g) \) is either 2 or 4.

Proof. We will assume \( g \equiv 3 \pmod{8} \). The case \( g \equiv 5 \pmod{8} \) is proven similarly.

Let \( \pi = \langle x \rangle \) be a group of order 2 in \( \Delta_g \), with associated Riemann-Hurwitz equation \( 2g - 2 = 2(2h - 2) + n \). We have seen \( n \) must be either 0, 4, or \( 2g + 2 \).

To obtain an upper bound on the numbers \( m(\pi) \) as in the definition of the Yagita invariant, we look for a cohomology element \( e(\pi) \in H^{2k}(\Delta_g; \mathbb{Z}) \) whose restriction to \( H^{2k}(\pi; \mathbb{Z}) \) is non-trivial, where \( l \) is odd. It then follows that \( m(\pi) \mid 2^{k-1} \).

Case 1: \( n = 0 \). From the Riemann-Hurwitz equation we obtain \( h = (g + 1)/2 \).

Recall from Lemma 3.1.1 that there is a cohomology class \( e \in H^d(\Gamma_g; \mathbb{Z}) \) whose restriction to \( H^d(\pi; \mathbb{Z}) \) is non-trivial, where \( d = 6(g - h) - 2n \). By restriction, we obtain a class \( e' \in H^d(\Delta_g; \mathbb{Z}) \) which also restricts non-trivially to \( H^d(\pi; \mathbb{Z}) \). Here
\[
d = 6 \left( g - \frac{g + 1}{2} \right) - 2(0) = 3(g - 1) \equiv 6 \pmod{8}.
\]

Then \( d = 2j \), \( j \) odd, which implies \( m(\pi) = 1 \).

Case 2: \( n = 4 \). Here, we have \( h = (g - 1)/2 \), and the class \( e' \) as defined in the previous case has degree
\[
d = 6 \left( g - \frac{g - 1}{2} \right) - 2(4) = 3(g + 1) - 8 \equiv 4 \pmod{8}.
\]

Then \( d = 4j \), \( j \) odd, which implies \( m(\pi) \mid 2 \).
Case 3: \( \tilde{n} = 2g + 2 \). In this case, \( \tilde{x} \) must be the hyperelliptic element \( C \), and \( \tilde{h} = 0 \). The class \( e' \) has degree \( 2(g - 2) \), and \( g - 2 \) is odd; hence \( m(\pi) = 1 \).

The Yagita invariant is defined to be the least common multiple of the numbers \( 2m(\pi) \), where \( \pi \) ranges over the subgroups of order 2 in \( \Delta_g \). The cases above show that \( m(\pi)|2 \) for all \( \pi \), thus \( Y_2(\Delta_g) \) must be either 2 or 4.

Combining Lemmas 3.4.1 and 3.4.3, we obtain

**Corollary 3.4.1.** \( Y_2(\Delta_g) = 4 \) for \( g \equiv 3 \) (mod 8).

Combining Lemmas 3.4.1 and 3.4.2, we obtain

**Corollary 3.4.2.** \( Y_2(\Delta_g) = 4 \) or \( 8 \) for \( g \equiv 3 \) (mod 8).
CHAPTER 4
THE FARRELL COHOMOLOGY OF $\Delta_g$

4.1 Branching data for $\langle C \rangle \times \mathbb{Z}/p \subset \Delta_g$

Given an element $x$ of odd prime order $p$ in $\Delta_g$, we have $|Cx| = 2p$. In sections 4.3 and 4.4 we will be interested in fixed point data for $Cx$. First we must determine branching data.

We can realize both $C$ and $x$ by periodic diffeomorphisms $S_g \to S_g$, also denoted by $C$ and $x$, which commute (we apply the Nielsen Realization Theorem to the order $2p$ element $Cx$). $C$ permutes the fixed points of $x$ and vice versa. We know $C$ has $2g + 2$ fixed points, and $S_g/\langle C \rangle \approx S^2$. The map $x : S_g \to S_g$ covers a map $\overline{x} : S_g/\langle C \rangle \to S_g/\langle C \rangle$, also of order $p$. The branched covering $S_g \to S_g/\langle Cx \rangle$ is equivalent to the composition of branched coverings

$$
S_g \xrightarrow{\pi_1} S_g/\langle C \rangle \xrightarrow{\pi_2} (S_g/\langle C \rangle)/\langle \overline{x} \rangle.
$$

The covering $S_g \to S_g/\langle Cx \rangle$ has branch points of orders 2, $p$, and $2p$. Let the number of branch points of order $i$ be denoted $N_i$. We determine $N_2$, $N_p$, and $N_{2p}$ as follows.

Let $P_1, P_2, \ldots, P_{2g+2}$ be the fixed points of $C$ acting on $S_g$. These map to distinct
points \( \pi_1(P_1), \ldots, \pi_1(P_{2g+2}) \) in \( S_g/\langle C \rangle \). Some of these points will be fixed by \( \bar{x} \), the rest must be permuted among themselves, which follows from the fact that \( x \) permutes the \( P_i \)'s. From the Riemann-Hurwitz equation, we know the action of \( \langle \bar{x} \rangle \approx \mathbb{Z}/p \) on \( S_g/\langle C \rangle \approx S^2 \) has exactly two fixed points. So 0, 1, or 2 of the points \( \pi_1(P_1), \ldots, \pi_1(P_{2g+2}) \) are fixed by \( \bar{x} \). A point \( \pi_1(P_i) \) that is fixed by \( \bar{x} \) determines a branch point \( \pi_2\pi_1(P_i) \) of order \( 2p \) for the covering \( S_g \to S_g/\langle C x \rangle \). The rest, which are permuted by \( \bar{x} \), identify \( p \)-fold to branch points of order 2 in \( S_g/\langle C x \rangle \).

Thus, if \( N_{2p} = 0 \), then \( \langle \bar{x} \rangle \approx \mathbb{Z}/p \) permutes the points \( \pi_1(P_1), \ldots, \pi_1(P_{2g+2}) \), without fixing any, which implies \( p \mid 2g + 2 \), and that \( N_2 = \frac{2g+2}{p} \). Similarly

\[
N_{2p} = 1 \Rightarrow p \mid 2g + 1 \quad \text{and} \quad N_2 = \frac{2g+1}{p}
\]

and

\[
N_{2p} = 2 \Rightarrow p \mid 2g \quad \text{and} \quad N_2 = \frac{2g}{p}.
\]

We see that if \( p \) does not divide \( 2g \), \( 2g + 1 \), or \( 2g + 2 \), then \( \mathbb{Z}/p \not\subseteq \Delta_g \). Also, note that an odd prime \( p \) will divide only one of these numbers.

Finally, we determine \( N_p \) from the Riemann-Hurwitz equation. Then the branching data for the action of \( \langle C x \rangle \) on \( S_g \) can be summarized as follows:

\[
\begin{align*}
p \mid 2g & \quad \Rightarrow \quad N_2 = \frac{2g}{p}, \quad N_p = 0, \quad N_{2p} = 2 \\
p \mid 2g + 1 & \quad \Rightarrow \quad N_2 = \frac{2g+1}{p}, \quad N_p = 1, \quad N_{2p} = 1 \\
p \mid 2g + 2 & \quad \Rightarrow \quad N_2 = \frac{2g+2}{p}, \quad N_p = 2, \quad N_{2p} = 0.
\end{align*}
\]

Note that we can determine the number of fixed points for the order \( p \) element \( x \) from the branching data above. For example, when \( p \mid 2g + 1 \), the pre-image in \( S_g \) of the
branch point of order $p$ consists of two fixed points of $x$, and the pre-image of the branch point of order $2p$ gives us one more fixed point, for a total of three. Thus we have:

\[
p|2g \quad \Rightarrow \quad f = 2,
\]

\[
p|2g + 1 \quad \Rightarrow \quad f = 3,
\]

\[
p|2g + 2 \quad \Rightarrow \quad f = 4,
\]

where $f$ denotes the number of fixed points for $\mathbb{Z}/p$.

### 4.2 Prime-power torsion in $\Delta_g$.

We have shown above that $\Delta_g$ has $p$-torsion whenever $p$ divides $2g$, $2g + 1$, or $2g + 2$. In this section we will prove the following more general result.

**Theorem 4.2.1.** Let $p$ be an odd prime. $\Delta_g$ contains a subgroup isomorphic to $\mathbb{Z}/p^\alpha$ if and only if $p^\alpha$ divides $2g$, $2g + 1$, or $2g + 2$.

For a partial result in the case $p = 2$, see Lemma 2.6.5.

**Proof.** We have exhibited elements in $\Delta_g$ of orders $2g$, $2g + 1$, and $2g + 2$, thus the converse statement is true.

For the forward implication, we assume the existence of a $\mathbb{Z}/p^\alpha$ in $\Delta_g$. Then we have $\mathbb{Z}/2 \times \mathbb{Z}/p^\alpha = \mathbb{Z}/2p^\alpha < \Gamma_g$, where the order two element is the hyperelliptic element $C$. Also, we may represent this subgroup as a group of diffeomorphisms acting on $S_g$. Recall that $C$ has $2g + 2$ fixed points. Those which are not fixed by any
other element of $\mathbb{Z}/2p^\alpha$ will map to branch points of order two in the associated cover $S_g \to S_g/(\mathbb{Z}/2p^\alpha)$. Also, each branch point of order $2p^i$ will have $2p^\alpha/2p^i = p^{\alpha-i}$ points in its pre-image, which are of the fixed points of $C$. Thus we count the order-two branch points as

$$N_2 = \frac{2g + 2 - \sum_{i=1}^{\alpha} p^{\alpha-i} N_{2p^i}}{p^\alpha},$$  \hspace{1cm} (4.1)

where $N_m$ denotes the number of branch points of order $m$. Note that the quotient $S_g/(\mathbb{Z}/2p^\alpha)$ is of genus zero since the quotient by the action of $(C)$ alone is of genus zero. The Riemann-Hurwitz equation for this action of $\mathbb{Z}/2p^\alpha$ on $S_g$ is then as follows:

$$2g - 2 = 2p^\alpha(-2) + 2p^\alpha \left[ \left(1 - \frac{1}{2}\right)^{2g+2-\sum_{i=1}^{\alpha} p^{\alpha-i} N_{2p^i}} \right. \left. + \sum_{i=1}^{\alpha} \left(1 - \frac{1}{2p^i}\right) N_{2p^i}\right].$$

One simplifies this equation to obtain

$$4(p^\alpha - 1) = \sum_{i=1}^{\alpha} 2p^{\alpha-i}(p^i - 1)(N_{2p^i} + N_{p^i}).$$  \hspace{1cm} (4.2)

Reducing this equation modulo $p$, the only term in the sum which survives is at $i = \alpha$. and we obtain

$$-4 \equiv 2(-1)(N_{2p^\alpha} + N_{p^\alpha}) \pmod{p},$$

or

$$2 \equiv N_{2p^\alpha} + N_{p^\alpha} \pmod{p}.$$  

Because $p$ is odd, this means $N_{2p^\alpha} + N_{p^\alpha}$ is at least two, and therefore the last term $2(p^\alpha - 1)(N_{2p^\alpha} + N_{p^\alpha})$ of the sum in equation 4.2 is at least $4(p^\alpha - 1)$. But this equals
the left hand side of the equation, so the remaining terms of the sum must all be zero, which implies $N_{2p^i} = N_{p^i} = 0$ for $1 \leq i \leq (\alpha - 1)$. Therefore the action of $\mathbb{Z}/2p^\alpha$ on $S_g$ has branch points only of orders $2$, $p^\alpha$, and $2p^\alpha$.

Equation 4.1 now reads

$$N_2 = \frac{2g + 2 - N_{2p^\alpha}}{p^\alpha},$$

and the above argument implies that $N_{2p^\alpha}$ must be at most 2. Since $N_2$ is an integer, this means $p^\alpha$ must divide $2g$, $2g + 1$, or $2g + 2$. \hfill \Box

### 4.3 Conjugacy classes of $\mathbb{Z}/p$ in $\Delta_g$

Toward the calculation of the Farrell cohomology of $\Delta_g$, we will determine the number of conjugacy classes of $\mathbb{Z}/p$ in $\Delta_g$, with $p$ an odd prime, by computing fixed point data. In the next section we will determine the index of the centralizer of a $\mathbb{Z}/p$ in its normalizer. We remark that these results remain true if we substitute $\mathbb{Z}/p^\alpha$ in place of $\mathbb{Z}/p$, with similar proofs. For simplicity, however, we will be concerned only with $\mathbb{Z}/p$, which suffices to obtain the subsequent results on cohomology.

We will use the notation $N_G(H)$ and $C_G(H)$ to denote the normalizer and centralizer, respectively, of $H$ in $G$.

**Lemma 4.3.1.** Let $G_1,G_2$, and $G$ be finite subgroups of $\Delta_g$, which do not contain the hyperelliptic element $C$. Then

(i) $G_1$ is conjugate to $G_2$ in $\Delta_g$ if and only if $\langle C \rangle \times G_1$ and $\langle C \rangle \times G_2$ are conjugate in $\Gamma_g$.  

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(ii) \( N_{\Delta_g}(G) = N_{\Gamma_g}((C) \times G) \).

(iii) \( C_{\Delta_g}(G) = C_{\Gamma_g}((C) \times G) \).

Proof. The third statement is clear from the definitions, as is the forward implications for both (i) and (ii).

For the converse in (i), assume there is an \( h \in \Gamma_g \) such that conjugation by \( h \) is an isomorphism \( (C) \times G_1 \xrightarrow{\sim} (C) \times G_2 \). The element \( hCh^{-1} \in \Delta_g \) can be represented by an order two diffeomorphism with \( 2g+2 \) fixed points, since the conjugacy class of an order two element in \( \Gamma_g \) is determined by the number of its fixed points. Therefore, by Corollary 3.1.1, \( h^{-1}Ch \) must equal \( C \), that is, \( h \) commutes with \( C \). Then by definition, \( h \) is in \( \Delta_g \). So \( G_1 \) and \( G_2 \) are conjugate in \( \Delta_g \).

To prove the converse in (ii), we set \( G_1 = G_2 = G \) and use the same argument as above.

\[ \square \]

So, to count conjugacy classes of \( \mathbb{Z}/p \) in \( \Delta_g \), we will count the conjugacy classes of \( (C) \times \mathbb{Z}/p \) in \( \Gamma_g \). The result is as follows.

**Theorem 4.3.1.** Let \( p \) be an odd prime. Then \( \Delta_g \) contains a subgroup isomorphic to \( \mathbb{Z}/p \) if and only if \( p \) divides \( 2g, 2g + 1, \) or \( 2g + 2 \). Moreover, any two subgroups of order \( p \) in \( \Delta_g \) are conjugate in \( \Delta_g \).

*Proof.* Our work in the previous section, along with the torsion elements exhibited in section 2.5, proves the first assertion.
To prove the second assertion, we will study the fixed point data for an element $Cx$ of order $2p$, where $x$ is of order $p$ in $\Delta_g$. In general, the fixed point data for an element of order $2p$ in $\Gamma_g$ are of the form
\[
\left( \frac{1}{2}, \frac{i}{2}, \frac{1}{p}, \ldots, \frac{\beta_{N_p}}{p}, \frac{\gamma_1}{2p}, \ldots, \frac{\gamma_{N_{2p}}}{2p} \right)
\]
where $N_i$ denotes the number of branch points of order $i$. Thus, $\frac{1}{2}$ is repeated $N_2$ times in the fixed point data. The fixed point data always have an integral sum, thus
\[
\frac{1}{2} + \cdots + \frac{1}{2} + \frac{\beta_1}{p} + \cdots + \frac{\gamma_{N_{2p}}}{2p} \in \mathbb{Z}
\]
or, equivalently,
\[
pN_2 + 2\beta_1 + \cdots + 2\beta_{N_p} + \gamma_1 + \cdots + \gamma_{N_{2p}} \equiv 0 \pmod{2p}. \tag{4.3}
\]
We will determine the fixed point data for $Cx$ in three cases.

**CASE 1:** $p|2g$. We have $N_2 = \frac{2g}{p}$, $N_p = 0$, and $N_{2p} = 2$. Note $N_2$ is even. Putting these into Equation 4.3, we obtain
\[
\gamma_1 + \gamma_2 \equiv 0 \pmod{2p}. \tag{4.4}
\]
Also, note $\gamma_1$ and $\gamma_2$ are relatively prime to $2p$. The possible fixed point data for $Cx$ are then
\[
\left( \frac{1}{2}, \ldots, \frac{i}{2}, \frac{1}{2p}, \frac{-i}{2p} \right), \quad i = 1, 3, 5, \ldots, 2p - 1.
\]
In interpreting these fixed point data, recall the fixed point data are considered as elements of $\mathbb{Q}/\mathbb{Z}$, thus the numbers $i$ are defined modulo $2p$. 

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Without loss of generality, we may assume

$$\sigma(Cx) = \left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2p}, \frac{1}{2} \right).$$

Then

$$\sigma((Cx)^i) = \left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{i}{2p}, \frac{-i}{2p}\right),$$

for $i = 1, 3, 5, \ldots, 2p - 1$, so all the possible fixed point data are realized by a single group $(C) \times \mathbb{Z}/p$. Recall the fixed point data of an finite-order element determines its conjugacy class in $\Gamma_g$. Since in the above argument $x$ was an arbitrary element of order $p$, this means there is only one conjugacy class of $(C) \times \mathbb{Z}/p$ in $\Gamma_g$, which implies there is only one conjugacy class of $\mathbb{Z}/p$ in $\Delta_g$, by the above lemma.

**Case 2:** $p|2g + 1$. We have $N_2 = \frac{2g + 1}{p}$, $N_p = 1$, and $N_{2p} = 1$. Putting these into Equation 4.3, we obtain

$$pN_2 + 2\beta_1 + \gamma_1 \equiv 0 \pmod{2p}$$

or, since $N_2$ is odd,

$$2\beta_1 + \gamma_1 \equiv p \pmod{2p}. \quad (4.5)$$

Thus, if $\beta_1 = i$, then $\gamma_1 = p - 2i$. The possible fixed point data for $Cx$ are then

$$\left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{i}{2p}, \frac{p - 2i}{2p}\right), \quad i = 1, 2, \ldots, p - 1.$$

So there are $p - 1$ possible fixed point data, realized by the powers $(Cx)^i$ of some element $Cx$, where $i$ is prime to $2p$. As in the above case, we obtain one conjugacy class of $(C) \times \mathbb{Z}/p$ in $\Gamma_g$, and thus only one conjugacy class of $\mathbb{Z}/p$ in $\Delta_g$. 

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CASE 3: $p|2g + 2$. We have $N_2 = \frac{2g+2}{p}$, $N_p = 2$, and $N_{2p} = 0$. Putting these into Equation 4.3, we obtain

$$2\beta_1 + 2\beta_2 \equiv 0 \pmod{2p}$$

or

$$\beta_1 + \beta_2 \equiv 0 \pmod{p}.$$ 

This means the possible fixed point data of $C_x$ are

$$\left(\frac{1}{2}, \ldots, \frac{i}{2}, p, -\frac{i}{p}\right), \quad i = 1, 2, \ldots, \frac{p - 1}{2}.$$ 

which correspond to powers $(C_x)^i$ of some element $C_x$, where $i$ is prime to $2p$. As before we obtain one conjugacy class of $\mathbb{Z}/p$ in $\Delta_g$. □

4.4 The index $[N(\mathbb{Z}/p) : C(\mathbb{Z}/p)]$

**Theorem 4.4.1.** Let $x$ be an element of $\Delta_g$ of odd prime order $p$. Then

$$[N_{\Delta_g}((x)) : C_{\Delta_g}((x))] = \begin{cases} 2 & \text{if } p \text{ divides } 2g \text{ or } 2g + 2 \\ 1 & \text{if } p \text{ divides } 2g + 1. \end{cases}$$

**Proof.** Let $x$ be of odd prime order $p$. It is well-known that for a cyclic subgroup $\langle a \rangle$ of order $n$ in a group $G$, the index $[N_G(\langle a \rangle) : C_G(\langle a \rangle)]$ is equal to the number of $i$'s between 1 and $n - 1$ such that $a$ is conjugate to $a^i$ in $G$. Also, by Lemma 4.3.1,

$$[N_{\Delta_g}((x)) : C_{\Delta_g}((x))] = [N_{\Gamma_g}((C_x)) : C_{\Gamma_g}((C_x))].$$
so we can determine this index, which we shall abbreviate by \([N : C]\), by examining fixed point data to see when \((Cx)^i\) is conjugate to \(Cx\). We return to the fixed point data determined within the proof of Lemma 4.3.1.

**CASE 1:** \(p|2g\). In Lemma 4.3.1, we found the fixed point data

\[
\sigma((Cx)^i) = \left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{i}{2p}, \frac{-i}{2p}\right), \quad i = 1, 3, 5, \ldots, 2p - 1,
\]

for \(x\) a suitable generator for the order \(p\) subgroup. From this we see \(Cx\) is conjugate to \((Cx)^i\) only for \(i = -1\). (That is, \(i = 2p - 1\); the numbers \(i\) are identified modulo \(p\).) Thus the index \([N : C]\) is 2.

**CASE 2:** \(p|2g + 1\). Here the fixed point data are

\[
\sigma((Cx)^i) = \left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{i}{p}, \frac{p - 2i}{2p}\right), \quad i = 1, 2, \ldots, p - 1,
\]

from which we clearly see \(Cx\) is never conjugate to \((Cx)^i\) for \(i \neq 1\). The index \([N : C]\) is therefore 1.

**CASE 3:** \(p|2g + 1\). Here the fixed point data are

\[
\sigma((Cx)^i) = \left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{i}{p}, \frac{-i}{p}\right), \quad i = 1, 2, \ldots, p - 1.
\]

Again, we find \(Cx\) is conjugate to \((Cx)^i\) only for \(i = -1\), and the index \([N : C]\) is therefore 2.

\[\square\]

### 4.5 The \(p\)-period of \(\Delta_g\)

We can now determine the \(p\)-period of \(\Delta_g\) for \(p\) an odd prime and for all \(g \geq 2\), using our work in the previous sections. The result is as follows.
Theorem 4.5.1. \( \Delta_g \) is \( p \)-periodic for every odd prime \( p \). Moreover, the \( p \)-period of \( \Delta_g \) is as follows:

\[
p(\Delta_g) = \begin{cases} 
4 & \text{if } p \text{ divides } 2g \text{ or } 2g + 2 \\
2 & \text{if } p \text{ divides } 2g + 1, \\
1 & \text{otherwise.}
\end{cases}
\]

Recall that the \( p \)-period of a \( p \)-torsion free group is 1, hence the last statement about the \( p \)-period is just a restatement of a previous finding.

The remainder of this section will be devoted to proving the above theorem.

To prove the first assertion, we will show that \( \Delta_g \) does not contain a subgroup isomorphic to \( \mathbb{Z}/p \times \mathbb{Z}/p \). Theorem 1.1.1, due to Brown, then implies \( \Delta_g \) is \( p \)-periodic.

Recall from section 1.5 the short-exact sequence:

\[
1 \to \mathbb{Z}/2 \to \Delta_g \to \Gamma^{2g+2} \to 1,
\]

where \( \Gamma^{2g+2} \) denotes the mapping class group of the sphere with \( 2g + 2 \) punctures. Suppose there is a subgroup \( \mathbb{Z}/p \times \mathbb{Z}/p \lt \Delta_g \). Then the above sequence implies that there is a subgroup \( \mathbb{Z}/p \times \mathbb{Z}/p \lt \Gamma^{2g+2} \). Any branch points in the associated cover must be of order \( p \). The Riemann-Hurwitz equation for this action is

\[
-2 = p^2(2h - 2) + p^2 \left( 1 - \frac{1}{p} \right) n,
\]

where \( h \) is the genus of the quotient, and \( n \) the number of branch points. Reducing this equation modulo \( p \), we obtain

\[
-2 \equiv 0 \pmod{p},
\]

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which is impossible for an odd prime. Hence $\Delta_g$ does not contain a $\mathbb{Z}/p \times \mathbb{Z}/p$, and is therefore $p$-periodic.

To finish the proof of Theorem 4.5.1, we will determine the $p$-period. Recall $\Delta_g$ contains $p$-torsion if and only if $p$ divides $2g$, $2g+1$, or $2g+2$. In most of these cases $\Gamma_g$ is $p$-periodic. Recall $\Gamma_g$ is $p$-periodic whenever $g \not\equiv 1 \pmod{p}$. For a prime $p \geq 5$, $p$ dividing $2g$, $2g+1$, or $2g+2$ implies $g \not\equiv 1 \pmod{p}$. Also, Xia proved in [Xia92a] that $\Gamma_g$ is $3$-periodic if and only if $g \not\equiv 1 \pmod{3}$. If $3|2g$ or $3|2g+2$, then $g \not\equiv 1 \pmod{3}$. However, if $3|2g+1$, then

$$2g+1 = 3k \Rightarrow g = 3 \left( \frac{k-1}{2} \right) + 1 \Rightarrow g \equiv 1 \pmod{3},$$

and $\Gamma_g$ is therefore not $3$-periodic.

So, in the cases when $\Delta_g$ has $p$-torsion and is $p$-periodic, $\Gamma_g$ is $p$-periodic also, except for when $p = 3$ divides $2g+1$.

As we have seen, there is at most one conjugacy class of $\mathbb{Z}/p$ in $\Delta_g$. Therefore Brown's decomposition theorem implies

$$\tilde{H}^*(\Delta_g)_{(p)} \approx \tilde{H}^*(N(\mathbb{Z}/p))_{(p)}, \quad (4.6)$$

whenever $p$ divides $2g$, $2g+1$, or $2g+2$. Since $\Delta_g$ is $p$-periodic, $N(\mathbb{Z}/p)$ is $p$-periodic, and therefore by Lemma 3.1 of [GMX92], we know the $p$-period of $N(\mathbb{Z}/p)$, which is the $p$-period of $\Delta_g$, has the form $2[N(\mathbb{Z}/p) : C(\mathbb{Z}/p)]p^n$, where $\alpha \geq 0$ is some integer. Glover, Mislin, and Xia [GMX92] proved the $p$-period of a $p$-periodic mapping class group $\Gamma_g$ is not divisible by $p$, and hence in these cases the $p$-period of $\Delta_g$ is also not divisible by $p$. Thus the $p$-period of $\Delta_g$ is $2[N(\mathbb{Z}/p) : C(\mathbb{Z}/p)]$, except possibly in the
case \( p = 3, 3|2g + 1 \). Our calculation in Theorem 4.4.1 of the index \([N(\mathbb{Z}/p) : C(\mathbb{Z}/p)]\) provides us the \( p \)-period for all odd primes \( p \), except in this one case.

For the case \( 3|2g + 1 \) we can show the 3-period of \( \Delta_g \) is also \( 2[N(\mathbb{Z}/p) : C(\mathbb{Z}/p)] \), using the following.

**Lemma 4.5.1.** Let \( y \) be an element of order \( n \) in \( \Gamma_g \). Suppose also that \( y \) may be represented by a diffeomorphism of order \( n \) with exactly one fixed point. Then there is an injection \( I : N_{\Gamma_g}((y)) \hookrightarrow \Gamma^1_g \).

**Proof.** Let \( h \in N_{\Gamma_g}((y)) \). Then \( hyh^{-1} = y^k \), for some \( k \) prime to \( n \). By theorem 1.3.1, we may represent \( h \) and \( y \) by diffeomorphisms, which we name \( f \) and \( z \), satisfying

\[
\begin{align*}
    fz^{-1} &= z^k \\
    z^n &= 1.
\end{align*}
\]

By assumption, \( y \) may be represented by a diffeomorphism with exactly one fixed point. This implies any diffeomorphism of order \( n \) representing \( y \) must have exactly one fixed point. Let \( P \) be the fixed point of \( z \). Then

\[
z^kf(P) = fz(P) = f(P)
\]

which implies \( zf(P) = f(P) \), hence \( f(P) \) is a fixed point of \( z \), implying \( f(P) = P \). So \( f \in \text{Diffeo}_+(S_g, P) \). Therefore we view \( f \) as an element of \( \Gamma^1_g \) and set \( I(h) = f \).

(i) \( I \) is well defined: Suppose \( h_1 = h_2 \) in \( N_{\Gamma_g}((y)) \), i.e., \( h_1 \) and \( h_2 \) are isotopic in \( \text{Diffeo}_+(S_g) \), and \( h_1yh_1^{-1} = y^k = h_2yh_2^{-1} \). Represent \( h_1 \) and \( h_2 \) by \( f_1 \) and \( f_2 \) in \( \text{Diffeo}_+(S_g, P) \) as defined above. The maps \( f_1 \) and \( f_2 \) are isotopic in \( \text{Diffeo}_+(S_g) \).
since, as elements of $\Gamma_g$, we have $f_1 = h_1 = h_2 = f_2$. Also, $f_1$ and $f_2$ satisfy $f_1zf_1^{-1} = z^* = f_2zf_2^{-1}$. Therefore, by Birman and Hilden, there is a homotopy $H : S_g \times [0,1] \to S_g$ with $H_0 = f_1$, $H_1 = f_2$, and $H_s z H_s^{-1} = z^{k_s}$ for all $s \in [0,1]$. Therefore $H_s$ fixes the point $P$, for all $s \in [0,1]$, which means $H$ is an isotopy through $\text{Diffeo}_+(S_g, P)$. Therefore $f_1 = f_2$ as elements of $\Gamma_g^1$. So $I$ is well-defined.

(ii) $I$ is injective: Suppose $f_1 = I(h_1) = I(h_2) = f_2$. Then $f_1$ and $f_2$ are isotopic through $\text{Diffeo}_+(S_g, P)$, which means they are isotopic through $\text{Diffeo}_+(S_g)$. So, as elements of $\Gamma_g$ we have $h_1 = f_1 = f_2 = h_2$, implying $I$ is one-to-one. □

We now return to the case $3|2g + 1$. Although $p = 3$ is the only case we need, the following argument holds for any odd prime $p$ dividing $2g + 1$. We have already observed (Theorem 4.3.1) that there is, up to conjugacy, a unique subgroup of order $p$ in $\Delta_g$. Let $x$ be an element of order $p$ in $\Delta_g$. The element $C_x$ of order $2p$ has exactly one fixed point, cf. section 4.1. Therefore, by the above lemma, $N_{\Gamma_g}(\langle C_x \rangle) = N_{\Delta_g}(\langle x \rangle)$ injects into $\Gamma_g^1$. In her Ph.D. thesis [Lu98], Qin Lu has shown that $\Gamma_g^1$ has $p$-period 2 for every odd prime $p$. Therefore the $p$-period of $N_{\Delta_g}(\langle x \rangle)$, which is the $p$-period of $\Delta_g$, divides 2. Since $p(\Delta_g) = 2[\mathbb{N}(\mathbb{Z}/p) : C(\mathbb{Z}/p)]p^\alpha$, we have $\alpha = 0$ and $[\mathbb{N}(\mathbb{Z}/p) : C(\mathbb{Z}/p)] = 1$. This agrees with our above assertion, and now provides us the 3-period when $3|2g + 1$. This last case finishes the proof of Theorem 4.5.1.
4.6 The case $g = \frac{(p-1)}{2}$

The first case of $\Delta_g$ containing $p$-torsion is at $g = (p-1)/2$, which is also the first case of $p$-torsion in $\Gamma_g$. In this section we will study the Farrell cohomology of $\Delta_{(p-1)/2}$. The main result is as follows.

**Theorem 4.6.1.** Let $p$ be an odd prime. Then

$$\hat{H}^*(\Delta_{(p-1)/2}; \mathbb{Z})(p) \approx \hat{H}^*(\mathbb{Z}/p; \mathbb{Z}) = \mathbb{F}_p[u, u^{-1}],$$

where $u$ is a cohomology class of degree two.

We will need the following lemma. Recall from section 2.3 the existence of a map from $\Delta_g$ onto $\Sigma_{2g+2}$.

**Lemma 4.6.1.** Suppose $x \in \Delta_g$ is an element of prime order $q \geq 2$, not equal to the hyperelliptic element. Then the image of $x$ under the map $\phi : \Delta_g \to \Sigma_{2g+2}$ is non-trivial.

**Proof.** Recall that the image of a finite-order element under $\phi$ is determined by its action on the $2g + 2$ fixed points of the hyperelliptic element $C$. If $x$ is of order two, then Lemma 3.1.2 implies $x$ has at most four fixed points. Also we know odd prime order elements have at most four fixed points, cf. section 4.1. Therefore the number of fixed points for $x$ is always less than $2g + 2$, so $x$ must act on the fixed points of $C$ non-trivially, and hence has non-trivial image in $\Sigma_{2g+2}$. \[\square\]

Since we will use Brown's decomposition theorem to compute cohomology, we are interested in normalizers of order $p$ subgroups of $\Delta_{(p-1)/2}$. The following lemma will make the calculation of cohomology easy.
Lemma 4.6.2. Let $p$ be an odd prime, and let $\pi < \Delta_{(p-1)/2}$ be a subgroup of order $p$. Then the normalizer of $\pi$ is equal to $\langle C \rangle \times \pi \approx \mathbb{Z}/2p$, where $C$ is the hyperelliptic element.

Proof. Let $N_\Delta$ and $C_\Delta$ denote the normalizer and centralizer, respectively, of $\pi$. We have determined that $N_\Delta = C_\Delta$ whenever $p$ divides $2g + 1$, as in this case. Furthermore, Xia determined in [Xia92c] that the normalizer of a $\mathbb{Z}/p$ in $\Gamma_{(p-1)/2}$ is finite, therefore $N_\Delta$ is also finite.

Note $\mathbb{Z}/p \times \mathbb{Z}/p$ is not contained in $N_\Delta$, as $\Delta_\pi$ is $p$-periodic for all odd primes $p$. Nor is $\mathbb{Z}/p^2$ contained in $N_\Delta$, since $p^2$ does not divide $2g$, $2g + 1$, or $2g + 2$, cf. Theorem 4.2.1. Therefore, the only $p$-torsion in $N_\Delta$ is $\pi$ itself.

Let $q \neq p$ be prime, $q \geq 2$, and suppose $y \neq C$ is of order $q$ in $N_\Delta = C_\Delta$. Then $y$ commutes with $x$. Recall the map $\phi$, here it is from $\Delta_{(p-1)/2}$ to $\Sigma_{p+1}$. Under $\phi$, $x$ and $y$ map to commuting non-trivial elements, by the above lemma. But $\phi(x)$ is a product of $p$-cycles, and $\phi(y)$ a product of $q$-cycles, which must be disjoint since they commute. But this is impossible in $\Sigma_{p+1}$, as $p + q \geq p + 1$. Therefore $N_\Delta$ does not contain any prime order torsion except for $\pi$ and $\langle C \rangle$, and as $N_\Delta$ is finite, we must therefore have $N_\Delta = \langle C \rangle \times \pi$.

Proof of Theorem 4.6.1. Recall there is exactly one conjugacy class of $\mathbb{Z}/p$ in $\Delta_{(p-1)/2}$. Therefore Brown’s decomposition theorem combined with the above lemma implies

$$\widetilde{H}^*(\Delta_{(p-1)/2}; \mathbb{Z})_{(p)} \approx \widetilde{H}^*(N_\Delta; \mathbb{Z})_{(p)} \approx \widetilde{H}^*(\mathbb{Z}/2p; \mathbb{Z})_{(p)} \approx \widetilde{H}^*(\mathbb{Z}/p; \mathbb{Z}).$$

□
4.7 The case \( g = p - 1 \).

The next case in which \( \Delta_g \) contains \( p \)-torsion is at \( g = p - 1 \). In this section, we will calculate the \( p \)-part of the Farrell cohomology of \( \Delta_{p-1} \). The result is as follows.

**Theorem 4.7.1.** Let \( p \) be an odd prime. Then

\[
\tilde{H}^i(\Delta_{p-1}; \mathbb{Z}_p) = \begin{cases} 
\mathbb{Z}/p & i \equiv 0, 1 \pmod{4} \\
0 & i \equiv 2, 3 \pmod{4}.
\end{cases}
\]

The work in this section will be to determine the normalizer of a subgroup isomorphic to \( \mathbb{Z}/p \) in \( \Delta_{p-1} \). A result of Xia will then easily complete the proof of the above theorem.

**Lemma 4.7.1.** The normalizer of \( \pi \) contains a subgroup isomorphic to \( D_8 \).

*Proof.* Recall that there is a unique conjugacy class of \( \mathbb{Z}/p \) in \( \Delta_{p-1} \). Therefore, we will not lose generality as we focus on a specific subgroup \( \pi \cong \mathbb{Z}/p \). Recall the element \( z \) of order \( 2g + 2 \); here it is of order \( 2p \). Let \( \pi = \langle z^2 \rangle \) be our model of \( \mathbb{Z}/p \).

Also recall from Section 2.6 the construction of two subgroups isomorphic to \( D_8 \) in \( \Delta_g \) for even genus. We may use either one of them, as they both normalize \( \pi = \langle z^2 \rangle \). Specifically, recall \( \langle \tau, z^{g+1}\tau \rangle \cong D_8 \), and the relation

\[
\tau z \tau = C z^{-1},
\]

which implies

\[
\tau z^2 \tau = (C z^{-1})^2 = z^{-2}.
\]
Then \( \tau \) normalizes \( \pi \), taking a generator to its inverse. Also,

\[
z^{g+1} \tau(z^2)(z^{g+1} \tau)^{-1} = z^{g+1} \tau(z^2)\tau^{-1}z^{-(g+1)}
\]

\[
= z^{g+1}z^{-2}z^{-(g+1)}
\]

\[
= z^{-2},
\]

so \( z^{g+1} \tau \) normalizes \( \pi \), also by taking a generator to its inverse. Thus \( (\tau, z^{g+1} \tau) \approx D_8 \) is contained in the normalizer.

In the above lemma, we may have used our other example \( \langle z^{g+1}, \tau z \rangle \approx D_8 \), since it also normalizes \( \pi \). In addition, one may show that these two subgroups isomorphic to \( D_8 \) are conjugate by an infinite order element. This will necessarily follow from our work in this section, but it can be shown directly with braid elements.

To determine the structure of the normalizer of \( \pi \) in \( \Delta_g \), we will make use of the results of Xia in [Xia92b], where he calculated normalizers in \( \Gamma_{p-1} \). By the Riemann-Hurwitz formula we know \( x \) must have exactly four fixed points. (In fact, in Section 4.1 we have seen this is true for \( \mathbb{Z}/p < \Delta_g \), \( p \) dividing \( g + 1 \).) Xia uses this fact to construct the top exact sequence in the following diagram:

\[
\begin{array}{ccc}
\mathbb{Z}/p & \hookrightarrow & N_{\Gamma_{p-1}}(\mathbb{Z}/p) \xrightarrow{j} \Gamma^4_0 \\
\mathbb{Z}/p & \hookrightarrow & N_{\Delta_{p-1}}(\mathbb{Z}/p) \xrightarrow{j} \Delta^4_0 \\
\end{array}
\]

Here we define \( \Gamma^4_0 \) as the mapping class group of the sphere with four punctures, where the punctures are allowed to be permuted. Let \( I = j(C) \), the image in \( \Gamma^4_0 \) of the hyperelliptic element. We complete the diagram above by defining \( \Delta^4_0 \) to be the centralizer of \( I \).
Lemma 4.7.2. $\text{im}(\bar{j}) = \text{im}(j) \cap \Delta_0^4$.

Proof. Clearly $\text{im}(\bar{j}) \subseteq \text{im}(j) \cap \Delta_0^4$. Conversely, let $f \in \text{im}(j) \cap \Delta_0^4$. Then $fI = I f$, and there exists an $\bar{f} \in \text{N}_{\Gamma_{p-1}}(\mathbb{Z}/p)$ with $j(\bar{f}) = f$. We need to show $\bar{f}$ is in $\Delta_{p-1}$, that is, $\bar{f}C = C \bar{f}$. Since

$$j(\bar{f}\bar{C}\bar{f}^{-1}C) = ff^{-1}I = 1,$$

we have

$$\bar{f}\bar{C}\bar{f}^{-1}C \in \ker(j) = \mathbb{Z}/p = \langle x \rangle,$$

and therefore $\bar{f}\bar{C}\bar{f}^{-1}C = x^m$ for some $m$, or $\bar{f}\bar{C}\bar{f}^{-1} = Cx^m$. Since $\bar{f}\bar{C}\bar{f}^{-1}$ is of order two, we have

$$\bar{f}\bar{C}\bar{f}^{-1} = (\bar{f}\bar{C}\bar{f}^{-1})^p = (Cx^m)^p = C$$

or $\bar{f}C = C\bar{f}$. □

There are two conjugacy classes of $\mathbb{Z}/p$ in $\Gamma_{p-1}$, which one determines from fixed point data. Xia determined the image of $j$ for both classes. We have the following diagram with exact rows:

$$\xymatrix{ K_4 \ar[r] & \Gamma_0^4 \ar[r] & \Sigma_4 \ar[u] \ar[r] & D_8 \ar[r] & G \ar[u] \ar[r] & \text{im}(\bar{j}) \ar[r] & \text{im}(j) \ar[r] & F \ar[u] }$$

(4.8)
where the second row is Xia’s calculation for the conjugacy class considered here. The map $\Gamma_0^4 \to \Sigma_4$ takes an element to the permutation corresponding to its action on the four punctures. The kernel $K_4$ is called the pure mapping class group of the sphere with four punctures. It is isomorphic to $\mathbb{Z} \ast \mathbb{Z}$.

We need to determine the subgroups $F$ and $G$. From Lemma 4.7.1 we know that $D_8$ must be in $\text{im}(j)$, since the kernel of $j$ is $\mathbb{Z}/p$, and $D_8$ contains no $p$-torsion. Therefore the group $G$ in the diagram above must be $D_8$, since the kernel of the map from $\text{im}(j)$ is free.

Now we must determine $F$. Since $\text{im}(j) = \text{im}(j) \cap \Delta_0^4$, we have $F = K_4 \cap \Delta_0^4$. That is, $F$ is the part of $K_4$ invariant under conjugation by $I$. We will use braid diagrams to determine $F$.

Recall that $\Gamma_0^4$ is a quotient of $B_4$. If an element of $\Gamma_0^4$ is represented by a braid, we can view the action of the element on the punctures by looking at the image of the braid in the symmetric group $\Sigma_4$. For example, consider the braids shown in Figure 4.1; they represent order two elements in $\Gamma_0^4$, since their squares are a full
twist or side-by-side opposite twists, which are the identity in \( \Gamma_0^4 \). Observing their image in \( \Sigma_4 \), we find the first two of these elements transpose the four punctures in pairs, whereas the third element fixes two punctures.

Recall from the branching data computed in Section 4.1 that the hyperelliptic element \( C \) in \( \Delta_{p-1} \) fixes none of the order \( p \) element's fixed points. Therefore, the image \( j(C) = I \) in \( \Gamma_0^4 \) fixes none of the four punctures. So \( I \) must be represented by a braid such as the first two shown in Figure 4.1. Let \( J \) be the first of these two braids.

First we will find the elements of \( \Gamma_4 < \Gamma_0^4 \) which are invariant under conjugation by \( J \), and then show that these are the same as the invariants under conjugation by \( I \).

**Lemma 4.7.3.** \( (K_4)^{(J)} = K_4 \).

**Proof.** We will view elements of \( \Gamma_0^4 \) with braid representatives, and blur the distinction between these elements and their braids. The subgroup \( K_4 \) in \( \Gamma_0^4 \) consists of elements whose representatives are pure braids. That is, we have the following commutative diagram, where \( P_4 \) represents the pure braid group on four strings.

\[
P_4 \hookrightarrow B_4 \twoheadrightarrow \Sigma_4
\]

\[
| \quad | \quad |
K_4 \hookrightarrow \Gamma_0^4 \twoheadrightarrow \Sigma_4
\]

The pure braid group on four strings, \( P_4 \), is generated by six elements, shown in Figure 4.2 [Bir71]. Therefore, these braids may be taken to be representatives of generators for \( K_4 \). We need to show conjugation by \( J \) fixes each generator.
It is clear that \( J A_{12} J = A_{12} \) and \( J A_{34} J = A_{34} \). (In fact, \( A_{12} = A_{34} \), since \( A_{12} A_{34}^{-1} \) consists of side-by-side opposite twists, hence is the identity.) Figure 4.3 shows that \( J A_{23} J = A_{14} \), and therefore \( J A_{14} J = A_{23} \).

But Figure 4.4 shows that \( A_{23} = A_{14} \). So, thus far, the action of \( J \) has been shown to be trivial on all generators except \( A_{13} \) and \( A_{24} \). Finally, Figure 4.5 shows that

\[
A_{13} = A_{23} A_{12}^{-1},
\]

and a similar braid calculation shows

\[
A_{24} = A_{34} A_{23}^{-1}.
\]

Thus \( A_{13} \) and \( A_{24} \) are products of the other generators, hence also fixed under conjugation by \( J \).

We have determined that \( J \) acts trivially on \( K_4 \), but to determine the group \( F \), we need to determine the action of \( I \) on \( K_4 \). However, the following lemma implies that \( I \) and \( J \) are conjugate in \( \Gamma_0^4 \).
**Lemma 4.7.4.** Let $x$ be any order two element in $\Gamma_0^4$ which does not fix any of the four punctures, and let $J$ be the order two element as above. Then $x$ is conjugate to $J$ in $\Gamma_0^4$.

**Proof.** We have the exact sequence

$$K_4 \hookrightarrow \Gamma_0^4 \xrightarrow{p} \Sigma_4$$

Let $p(J) = \sigma$, and $p(x) = \sigma'$. Both of these are permutations of order two fixing no letters; that is, they are both products of two disjoint 2-cycles. Such permutations are conjugate in $\Sigma_4$. Therefore let $\alpha$ be in $\Sigma_4$ with $\alpha \sigma' \alpha^{-1} = \sigma$. Since $p$ is onto, there exists an $f$ with $p(f) = \alpha$. Then

$$p(fxf^{-1}J) = \alpha \sigma' \alpha^{-1} \sigma = \sigma^2 = 1,$$

which implies $fxf^{-1}J$ equals some element $\lambda$ in $K_4$, or

$$fxf^{-1} = \lambda J.$$
\[ A_{23} = B \Rightarrow A_{23} = B^{-1}. \]

\[ B = A_{14}^{-1} \Rightarrow A_{23} = B^{-1} = (A_{14}^{-1})^{-1} = A_{14}. \]

Figure 4.4: Showing \( A_{23} = A_{14} \) with a braid calculation.
But using the fact that $J$ acts trivially on $K_4$ we obtain

$$1 = (fxf^{-1})^2 = (\lambda J)^2 = \lambda J \lambda J = \lambda^2,$$

which implies $\lambda = 1$, as $K_4$ is a free group. Therefore

$$fxf^{-1} = J,$$

so $x$ is conjugate to $J$.

Since a pure braid remains pure under conjugation by any braid, the subgroup $K_4$ remains fixed as a set under conjugation. Therefore, by an inner automorphism on $\Gamma_0^4$, we may replace $J$ with $I$, and $K_4$ will still be represented by $P_4$. since $J$ acts trivially on $K_4$, the action of $I$ must then also be trivial. Therefore $F$ equals all of $K_4$. \[\square\]
Since we have shown the groups $F$ and $G$ of diagram 4.8 to be $K_4$ and $D_8$, respectively, we therefore have

$$\text{im}(j) = \text{im}(\overline{j}),$$

which implies

$$N_{\Delta_{p-1}}(\pi) = N_{\Gamma_{p-1}}(\pi).$$

Also, Brown's decomposition theorem implies

$$\widehat{H}^*(\Delta_{p-1}; \mathbb{Z})_p \approx \widehat{H}^*(N_{\Delta_{p-1}}(\pi); \mathbb{Z})_p.$$ 

Since Xia [Xia92b] calculated $\widehat{H}^*(N_{\Gamma_{p-1}}(\pi); \mathbb{Z})_p$, we may appeal to his result to complete the proof of Theorem 4.7.1.
BIBLIOGRAPHY


