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Josephson Junction Arrays with Long-Range Coupling

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

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1999

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ABSTRACT

We investigate two Josephson junction array systems with long-range coupling. The first system is an array consisting of two layers of parallel superconducting wires. The wires interact via Josephson coupling with the wires of the opposing layer. The second system is a linear array of Josephson junctions within an electromagnetic cavity. The junctions interact with the photon mode of the cavity.

We develop a mean field theory for both systems. We predict the critical temperature of the wire array as a function of the magnetic field $B$ and the coupling strength $\lambda$ between the layers. For frustration $f = Ba^2/\Phi_0 = p/q$ for two integers $p$ and $q$, the critical temperature goes as $T_c \sim \lambda^{1/2}q^{-1/4}$. This dependence is in contrast to that of the conventional MFT which predicts $T_c \sim Nq^{-1/2}$.

For the junction array in the cavity, we find there is a threshold number $N_c$ of junctions required for junction coherence. For an array of $N$ junctions, for $N > N_c$, we have the energy of the photon mode goes as $E_{\text{photon}} \sim N^2$. Both the existence of a threshold number and the $N^2$ dependence of the photon energy above the threshold are indicators of coherence.

We also develop two dynamical models for the wire array, and predict the size and magnetic field dependence of the array critical current. The first dynamical model assumes each wire has a uniform phase along its length. We find in this model that
the critical current of the array goes as $I_c^\text{array} \approx I_c N/\sqrt{q}$. This is consistent with the conventional MFT $T_c$ dependence on frustration.

The second dynamical model accounts for phase variations along the wires and predicts a saturation value for the critical current: $I_c^\text{array} \sim 2I_c\sqrt{\lambda}$. This model also gives us a condition for which the uniform phase assumption of the first dynamical model should be valid: $N < \sqrt{\lambda}$. 
In memory of my father, John S. Harbaugh
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CHAPTER 1

INTRODUCTION

In this work, we will investigate the properties of two systems of Josephson junctions. Although the systems are constructed very differently, they both exhibit long-range interactions. That is, individual elements of each system are coupled to a large number of other elements within the system. Before going into the details, let us first review the fundamentals of Josephson junctions, which are integral to both systems under consideration.

1.1 Josephson Junctions

A Josephson junction is, at its simplest, merely the interface between two weakly-coupled superconducting regions. First derived by Josephson [1, 2], the superconducting regions' interaction is governed by the two equations,

\[ I_s = I_c \sin \Delta \theta, \]
\[ \Delta V = \frac{\hbar}{q} \frac{d}{dt} \Delta \theta, \]  \hspace{1cm} (1.1)

known as the Josephson relations. The supercurrent is \( I_s \), the critical current of the junction is \( I_c \), and \( \Delta \theta \) is the gauge-invariant phase difference between the two regions. The potential difference between the regions is \( \Delta V \), Planck's constant divided by \( 2\pi \).
is $\hbar$, the charge of the superconducting charge carriers is $q$, and $d/dt$ is the derivative with respect to time. These relations are derived in Appendix A. Their interpretation is as follows.

By definition, within the superconducting regions current may flow without resistance. This supercurrent is able to tunnel through the interface between the two regions. Josephson's current-phase relation quantifies the tunneling supercurrent. If there is no phase difference between the regions then there will be no supercurrent. At certain values of the phase difference, $\Delta \theta = (2m + 1)\pi/2$ for any integer $m$, the magnitude of the current is maximized. This maximum supercurrent is called the critical current.

Josephson's voltage-phase relation describes the time-evolution of the phase difference. For a given fixed voltage, the phase difference will increase linearly with time. From the current relation we can see that the current will then oscillate sinusoidally. Thus, a fundamental behavior of a Josephson junction is the conversion between a direct voltage and an alternating current.

The phase difference in the Eqs. (1.1) is the gauge-invariant phase difference. That is, the electromagnetic gauge has been taken into account. For a given choice of gauge,

$$\Delta \theta = \Delta \phi - \frac{2\pi}{\Phi_0} \int A \cdot ds,$$

(1.2)

where $\Delta \phi$ is a gauge-variant phase difference, $\Phi_0 = hc/q$ is the flux quantum, $A$ is the electromagnetic vector potential, and the path integral is integrated across the junction. This is also derived in Appendix A.
1.2 Circuit Model

Although the Josephson relations define the fundamental behavior of a Josephson junction, a more detailed model is often necessary. We model a more realistic junction as an ideal junction (one described only by the Josephson relations) in parallel with a resistor of resistance $R$ and a capacitor of capacitance $C$; see Fig. 1.1. This is known as the resistively, capacitively shunted junction (RCSJ) model. This model allows us to explore the dynamics of Josephson junctions, as explained below.

![Figure 1.1: Schematic diagram of a resistively, capacitively shunted junction. The model parameters are critical current $I_c$, resistance $R$ and capacitance $C$.](image)

1.3 Equation of Motion

The total external current through a Josephson junction using the RCSJ model is simply the sum of the different components of the current going in parallel: $I_{\text{ext}} =$
$I_s + I_{res} + I_{cap}$. Here $I_{res}$ is the resistive current and $I_{cap}$ is the capacitive current.

Using the Josephson relations, Eqs. 1.1, we have

$$
I_s = I_c \sin \theta,
$$
$$
I_{res} = \frac{\Delta V}{R} = \frac{\hbar}{qR} \frac{d}{dt} \theta,
$$
$$
I_{cap} = \frac{d}{dt} Q = C \frac{d}{dt} \frac{\Delta V}{q} = \frac{\hbar C}{q} \frac{d^2}{dt^2} \theta.
$$

Here we have written the gauge-invariant phase difference as $\theta$, and $Q$ is the instantaneous charge stored on one side of the capacitor ($-Q$ stored on the other side).

Thus, the equation of motion of the phase is

$$
\frac{\hbar C}{q} \frac{d^2}{dt^2} \theta(t) + \frac{\hbar}{qR} \frac{d}{dt} \theta(t) + I_c \sin \theta(t) - I_{ext}(t) = 0.
$$

(1.4)

By making the substitutions

$$
i_{ext} = \frac{I_{ext}}{I_c},
$$
$$
\beta = \frac{\hbar}{q I_c R dt},
$$

we have

$$
0 = \beta \ddot{\theta} + \dot{\theta} + \sin \theta - i_{ext}.
$$

(1.6)

The constant $\beta_c = qI_c CR^2/\hbar$ is called the McCumber parameter. It is an important intrinsic parameter of a Josephson junction.

1.4 Overdamped Junctions

In limiting case of small $\beta_c$, the first-order term predominates and the equation is overdamped. We also call junctions meeting this condition “overdamped”. Physically, the resistive term is large in this case and the system will reach equilibrium quickly.
For simplicity, overdamped junctions are often modeled without a capacitance, and therefore neglect the second-order term entirely.

\[ 0 = \dot{\theta} + \sin \theta - i_{\text{ext}}. \quad (1.7) \]

The equation of motion is first-order.

We can integrate Eq. (1.7) to find the time-averaged voltage,

\[ \frac{1}{I_{c} R} V_{\text{dc}} = \sqrt{i_{\text{ext}}^2 - 1}. \quad (1.8) \]

Thus, for an external current through the junction less than the critical current \( (i_{\text{ext}} < 1) \), the steady state voltage across the junction vanishes. If \( i_{\text{ext}} \geq 1 \), then the voltage varies smoothly from zero to \( V_{\text{dc}} = I_{\text{ext}} R \) at large currents. The voltage behavior of an overdamped changes from highly nonlinear (but continuous) at low currents to linear at large currents. Figure 1.2 displays Eq. (1.8) and two sample voltage waveforms.

Overdamped junctions are non-hysteretic. This means that no matter what state the junction starts, for any given \( i_{\text{ext}} \), there is only one steady-state solution which solves the equation of motion. However, as we will see in Section 2.2, arrays of overdamped junctions may be hysteretic when connected in certain geometries.

### 1.5 Underdamped Junctions

For limiting case of large \( \beta_{c} \), the second-order term predominates and Eq. (1.6) becomes underdamped. We also call the junction "underdamped" in this case. This means there is very little damping in the system, and the system will take a comparatively long time to reach a steady state. Again, to simplify the model, the resistance is often omitted and the first-order term is ignored. Of course, if this is done then \( \tau \)
Figure 1.2: Time-Averaged Voltage vs. External Current, for a single overdamped Josephson junction. Insets: the instantaneous voltage as a function of time across the single junction, at $I_{ext} = 1.8I_c$ and $I_{ext} = 4I_c$.

must be redefined in terms of the capacitance $C$ instead of the resistance $R$.

$$0 = \ddot{\theta} + \frac{1}{\sqrt{\beta_c}} \dot{\theta} + \sin \theta - i_{ext}$$  \hspace{1cm} (1.9)

Here we have used

$$\dot{\tau} = \frac{d}{d\tau} = \sqrt{\frac{\hbar C}{q I_c}} \frac{d}{dt}.$$ \hspace{1cm} (1.10)

And then, if $\beta_c \gg 1$, then the first-order becomes negligible. We are left with the second-order differential equation,

$$0 = \ddot{\theta} + \sin \theta - i_{ext}.$$ \hspace{1cm} (1.11)

Underdamped Josephson junctions are hysteretic; the state of the junction depends on its history. Let us assume a junction starts at zero external current, $i_{ext} = 0$. 

Then the phase remains constant. As the current is slowly increased, while the current is less than the critical current ($i_{\text{ext}} < 1$), the phase of the system can adjust so that the externally applied current flows without resistance. That is, $\sin \theta = i_{\text{ext}}$

As the current exceeds the critical current of the junction, the junction is forced into a finite voltage state, $V_{\text{dc}} = I_c R$. Note that at this point we cannot completely ignore the resistance of the model. The voltage across the junction jumps discontinuously to this value, unlike the overdamped junction whose voltage varies continuously. At larger currents, the junction approximately obeys Ohm's law.

The hysteresis comes as the current $i_{\text{ext}}$ is decreased from a large value. As $i_{\text{ext}}$ drops below unity, the junction remains in a finite voltage state. This is due to the lack of damping in Eq. (1.11); the junction continues in the finite-voltage state until the damping is sufficient to overcome the driving term $i_{\text{ext}}$. At a value of $i_{\text{ext}}$ determined by $\beta_c$, the junction will discontinuously drop back to the zero-voltage state. Thus, for a range of currents, an underdamped junction may be in one of two states, depending on the initial conditions.

1.6 Arrays

An array of Josephson junctions is, in general, any connected combination of junctions. Many array geometries have been considered [3, 4, 5, 6, 7, 8]. Arrays are constructed in order to amplify the properties of single junctions.

For instance, a single junction held at a fixed voltage $V_{\text{dc}}$ will radiate at frequency $\hbar V_{\text{dc}}/q$. An array of $N$ such junctions held at the same voltage will all radiate at the same frequency. If the phases of the junctions happen to coincide, then the radiation of the junctions will constructively interfere. An array in this state is called coherent,
and much effort has been put into finding the conditions under which arrays become coherent, e.g., [9, 10, 11]. Other properties of arrays that have been investigated include vortex dynamics [12, 13, 14], voltage plateaus [15, 16, 17], various phase transitions [18, 19, 20], for example.

1.7 Summary

Within this work, we will investigate the properties of two specific geometries of Josephson junction arrays. In Chapter 2 we will examine an array composed of superconducting wires that interact via Josephson coupling. We will calculate both dynamic and thermodynamic properties using several models, and we will determine the conditions under which those models are applicable.

In Chapter 3 we will consider an array of Josephson junctions within an electromagnetic cavity. We introduce a model to describe the interaction between the junctions and the cavity and calculate some thermodynamic quantities.
CHAPTER 2

JOSEPHSON-COUPLED WIRE ARRAYS

There has been experimental interest in arrays of superconducting wires, manufactured to interact via Josephson coupling [21, 22, 23, 24]. The goal is to construct a Josephson junction array such that the geometry will facilitate the global phase coherence of the array. Such arrays consist of two layers of superconducting wires as shown in Fig. 2.1. Typically, a constant in time (d.c.) “bias” current is inserted into and extracted from two different wires in the array. In experiments to date, two different boundary configurations have been used. In the “perpendicular” configuration, Fig. 2.1a, the bias current is inserted into the center wire of one layer and extracted from the center wire of the opposing layer. In the “parallel” configuration, Fig. 2.1b, the bias current is inserted into the wire at one edge of a layer and extracted from the wire at the other edge of the same layer. Since every wire interacts directly with each of the wires in the opposing layer, every wire is the nearest neighbor or next-nearest neighbor with every other wire in the array. We expect this geometry will lead to phase coherence within the array.

In addition to the externally supplied bias current, a magnetic field \( B \) may also be applied perpendicular to the array. In experiments, \( B \) is assumed both constant
Figure 2.1: Schematic diagrams of Josephson-coupled wire arrays in a) perpendicular configuration, and b) parallel configuration. Each layer is an array of parallel, equally-spaced superconducting wires, arranged perpendicularly to the wires of the opposing layer. At each intersection, as indicated by an x, the wires interact via Josephson coupling. The voltage is measured at, and bias current is applied to, the leads marked o. The wires are labeled by integers \(j\) and \(k\).

in time and uniform in space. The voltage across the array is measured between the current inputs, and is a function of the bias current, magnetic field, and time.

When biased by a d.c. current, the behavior of the array will depend on which, if any, junctions have a finite voltage difference. This will of course depend on the magnitudes of the bias current and the applied magnetic field. By examining the voltage across the inputs of the array, we can determine the critical current of the array \(J_c^{\text{array}}\), the largest bias current for which the d.c. voltage remains zero. This is one of the most important dynamical properties of the array. Sections 2.1 and 2.2 will
cover two dynamical models for arrays with this geometry. Both models are derived from the Josephson relations and Kirchhoff's circuit rules. The first is the simpler one, and we will use it to investigate how the critical current depends on the magnetic field. The second model will allow us to determine how the size of the array affects the critical current. We will discuss the strengths and limitations of each model in a later section.

We also investigate the thermodynamic properties of the array. The most obvious property is the critical temperature \( T_c \) at which the system has a transition from an ordered, zero-voltage state to an disordered state with finite voltage \([25, 26, 27, 28]\). In section 2.3 we develop a mean field theory and then look at several Monte-Carlo results with which the mean field theory may be compared.

2.1 Uniform Phase Model

We begin the investigation by producing a model for the dynamical behavior of the Josephson junction array. Then we look at the numerical methods for evaluating the model and the calculated results. Finally, we discuss the implications of our results.

2.1.1 Formalism

The first model naively assigns a single phase to each wire, \( \psi_j \) for the phases of the \( M \) wires of one layer and \( \phi_k \) for the phases of the \( N \) wires of the other layer, with integers \( j, k \) such that \( 1 \leq j \leq M \) and \( 1 \leq k \leq N \). The labeling pattern is indicated in Fig. 2.1. Using Kirchhoff's rule of current conservation, and assuming overdamped Josephson junctions (thus neglecting the second-order term), we find the following:

\[
I_B \delta_{jm} = \sum_{k=1}^{N} \left[ I_c \sin(\psi_j - \phi_k - A_{kj}) + \frac{\hbar}{qR} \frac{d}{dt} (\psi_j - \phi_k - A_{kj}) \right],
\]
\[ I_B \delta_{kn} = \sum_{j=1}^{M} \left[ I_c \sin(\psi_j - \phi_k - A_{kj}) + \frac{\hbar}{qR} \frac{d}{dt}(\psi_j - \phi_k - A_{kj}) \right]. \quad (2.1) \]

The constant \( A_{jk} \) is shorthand for the integral term of Eq. (1.2), and is explained below. We define the \( m \)th wire of one layer and the \( n \)th wire of the other as where the current is inserted and extracted, respectively. The first equation describes the interaction of \( \psi_j \) with all \( N \) wires of the second layer; and, likewise, the second equation describes the interaction of \( \phi_k \) with all \( M \) wires of the first layer. The boundary conditions (the left sides of the equations) are for the perpendicular configuration, Fig. 2.1a. Using Eqs. (1.5), we rewrite these equations as

\[ i_B \delta_{jm} = \sum_{k=1}^{N} \left[ \sin(\psi_j - \phi_k - A_{kj}) + \dot{\psi}_j - \dot{\phi}_k \right], \]
\[ i_B \delta_{kn} = \sum_{j=1}^{M} \left[ \sin(\psi_j - \phi_k - A_{kj}) + \dot{\psi}_j - \dot{\phi}_k \right]. \quad (2.2) \]

Or, equivalently,

\[ \dot{\phi}_k = \frac{1}{M} \left\{ \sum_{j=1}^{M} \left[ \psi_j + \sin(\psi_j - \phi_k - A_{kj}) \right] - i_B \delta_{kn} \right\}, \]
\[ \dot{\psi}_j = \frac{1}{N} \left\{ \sum_{k=1}^{N} \left[ \dot{\phi}_k - \sin(\psi_j - \phi_k - A_{kj}) \right] - i_B \delta_{jm} \right\}. \quad (2.3) \]

Combining the equations yields

\[ i_B = N \sum_{j=1}^{M} \dot{\psi}_j - M \sum_{k=1}^{N} \dot{\phi}_k + M \sum_{j=1}^{N} \sum_{k=1}^{N} \sin(\psi_j - \phi_k - A_{kj}), \quad (2.4) \]

which gives us

\[ \dot{\phi}_t = \frac{1}{N} \sum_{k=1}^{N} \dot{\phi}_k + \frac{1}{NM} \left[ i_B - \sum_{j=1}^{M} \sum_{k=1}^{N} \sin(\psi_j - \phi_k - A_{kj}) \right] - \frac{1}{M} \left[ i_B \delta_{kn} - \sum_{j=1}^{M} \sin(\psi_j - \phi_k - A_{kj}) \right], \]
\[ \dot{\psi}_t = \frac{1}{N} \sum_{k=1}^{N} \dot{\phi}_k + \frac{1}{N} \left[ i_B \delta_{tm} - \sum_{k=1}^{N} \sum_{l=1}^{N} \sin(\psi_t - \phi_k - A_{kl}) \right]. \quad (2.5) \]
Equations (2.5) leave one degree of freedom unrestricted: we may arbitrarily choose the origin of the phases. To fix this degree of freedom, it is typical to choose one of the phases to be zero. However, we have chosen instead the condition

\[ 0 = \frac{1}{N} \sum_{k=1}^{N} \dot{\phi}_k + \frac{1}{M} \sum_{j=1}^{M} \dot{\psi}_j. \] (2.6)

That is, we require that the instantaneous mean voltages of each layer to be equal and opposite. Combining Eqs. (2.4) and (2.6) yields

\[ \frac{1}{N} \sum_{k=1}^{N} \dot{\phi}_k = -\frac{1}{2NM} \left[ i_B - \sum_{j=1}^{M} \sum_{k=1}^{N} \sin(\psi_j - \phi_k - A_{kj}) \right]. \] (2.7)

Substituting Eq. (2.7) into Eqs. (2.5), we finally decouple the time derivatives of the phases:

\[ -\dot{\phi}_l = -\frac{1}{2NM} \left[ i_B - \sum_{j=1}^{M} \sum_{k=1}^{N} \sin(\psi_j - \phi_k - A_{kj}) \right] + \frac{1}{M} \left[ i_B \delta_{en} - \sum_{j=1}^{M} \sin(\psi_j - \phi_l - A_{kj}) \right], \]

\[ \dot{\psi}_l = -\frac{1}{2NM} \left[ i_B - \sum_{j=1}^{M} \sum_{k=1}^{N} \sin(\psi_j - \phi_k - A_{kj}) \right] + \frac{1}{N} \left[ i_B \delta_{em} - \sum_{k=1}^{N} \sin(\psi_l - \phi_k - A_{kl}) \right]. \] (2.8)

The symmetric form of these equations is the reason for our choice of condition (2.6). The form of Eqs. (8) also indicates that the evolution of a given phase in one layer depends only on the average of the phases in the opposing layer and on an overall average of the entire array. This suggests a mean-field approach which we will develop in Section 2.3.

The dynamics of an array in the parallel configuration, Fig. 2.1b, can be simplified in an analogous manner. After a similar sequence of steps, we obtain for this case the
final set of equations:

\[
-\dot{\phi}_\ell = \frac{1}{2NM} \sum_{j=1}^{M} \sum_{k=1}^{N} \sin(\psi_j - \phi_k - A_{kj}) - \frac{1}{M} \sum_{j=1}^{M} \sin(\psi_j - \phi_\ell - A_{\ell j}),
\]

\[
\dot{\psi}_\ell = \frac{1}{2NM} \sum_{j=1}^{M} \sum_{k=1}^{N} \sin(\psi_j - \phi_k - A_{kj}) - \frac{1}{N} \sum_{k=1}^{N} \sin(\psi_\ell - \phi_k - A_{k\ell})
\]

\[
+ \frac{1}{N} i_B (\delta_{km} - \delta_{kn}).
\]

(2.9)

As is expected, this configuration has the same essential features, hinting at a mean-field approach, only with different boundary conditions.

Some explanation is needed for the vector potential term. We label the direction of the magnetic field, normal to the array, the z-direction. Therefore, \( B = B \hat{z} \); and, choosing a convenient gauge, the vector potential is \( A = B x \hat{y} \). The wire with phase \( \phi_k \) is in the x-direction as indicated in Fig. 2.1. The gauge-invariant phase \( \phi_{j,k}' \) of the wire at its intersection with wire \( j \) is

\[
\phi_{j,k}' = \phi_k - \frac{2\pi}{\Phi_0} \int_0^{x_a} B x \hat{y} \cdot dx = \phi_k. \quad (2.10)
\]

The line integral of the vector potential vanishes in this gauge. The wire with phase \( \psi_j \) is in the y-direction. The gauge-invariant phase \( \psi_{j,k}' \) of the wire at its intersection with wire \( k \) is

\[
\psi_{j,k}' = \psi_j - \frac{2\pi}{\Phi_0} \int_0^{x_a} B j a \hat{y} \cdot dy = \psi_j - 2\pi j k B a^2 / \Phi_0. \quad (2.11)
\]

Thus, the gauge-invariant phase difference across the junction at the intersection of wires \( j \) and \( k \) is \( \psi_j - \phi_k - 2\pi j k f \). We introduce \( f = B / (\Phi_0 / a^2) \). The factor \( f \) is known as the frustration and is a dimensionless measure of the magnetic field.
2.1.2 Numerical Methods

With the time derivatives of the phases solved for analytically, Eqs. (2.8), we can simply apply a Runge-Kutta algorithm to find the solution to the equations, see Appendix B. We use a fixed step-size algorithm. We set the dimensionless time \( \tau = t(qI_c R / \hbar) \) to zero, and assign a different random number \( r \) as the initial value of each phase \( (0 \leq \tau < 2\pi) \). For a given bias current, the equations of motion are integrated until \( \tau \geq 100 \) to allow the system to reach a steady state. We can determine that this is an adequate length of time to reach a steady state by comparing the state of the array at different times. We find that the state of the array is quantitatively similar after evolving 100 or 1000 dimensionless time units. A step-size \( \Delta \tau \) is chosen depending on the value of the bias current; \( 0.01 \leq \Delta \tau \leq 0.1 \).

By definition, the d.c. voltage across the array is
\[
\frac{1}{I_c R} V_{dc} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{d}{d\tau} (\Delta \theta) d\tau.
\]
\( \Delta \theta = \psi_m - \phi_n - 2\pi nm f \), which is the gauge-invariant phase difference between the current inputs. It is obviously impractical computationally to calculate the limit of \( T \to \infty \), as this would require integrating the equations of motion over an infinite time interval. Instead we make the approximation
\[
V_{dc} \approx \frac{1}{T} \int_0^T \Delta \theta d\tau = \frac{\Delta \theta(T) - \Delta \theta(0)}{T}.
\]
The larger the interval of integration \( T \) is, the better will be the approximation. We choose \( T = 100 \) for this model. After reaching a steady state, the dynamical equations are integrated for another 100 dimensionless time units; i.e., until \( \tau \geq 200 \). This last time interval, \( 100 < \tau < 200 \), is used to approximate the d.c. voltage. We have now calculated one point of a current voltage (IV) curve. The bias current is then changed.
by a small amount (but without reinitializing the phases), and the process is repeated for a wide range of bias currents.

2.1.3 Results

A sample $IV$ curve is shown in Fig. 2.2. The figure displays the d.c. voltage as a function of the bias current, for several values of the frustration, for an array of $18 \times 18$ wires in the parallel input wire configuration. The bias current is initially zero and is increased by steps $0.1I_c$. The $f = 0$ curve plots the $IV$ characteristic for the case of no magnetic field. From the figure, we can see that the array critical current is suppressed when the frustration is non-zero. By consulting many of these $IV$ curves, we can determine the how the critical current is affected by the frustration.

The critical currents in Fig. 2.3 have been calculated for an $18 \times 18$ array in the perpendicular configuration (top plot) and parallel configuration (bottom plot) for 1000 different frustration values. Bold tick marks on the upper axes indicate rational frustrations of the form $1/q$ and smaller tick marks signify frustrations $p/q$; $q < N - 1$ in all cases. The $\times$ symbols mark predictions to be explained later. Each data point has been calculated by slowly increasing the bias current until the average voltage becomes non-zero. This process is inherently noisy, because the system is very sensitive to numerical error when the bias current is near the critical current. Any small roundoff error can errantly knock the system into a finite voltage state. We will thus calculate a small d.c. voltage, even below the critical current. To give the data more clarity (i.e., to reduce these kinds of errors), it has been processed by a Savitzky-Golay filter [29]. This is a low-pass filter, in which each original data point is replaced by a weighted average over the original neighboring data points. The
Figure 2.2: DC Voltage vs. Bias Current, for various frustrations, for an 18 × 18 array, using the uniform-phase model.

average is weighted so that neither the height nor width of the peaks is modified, as would be the case if each neighboring data point were given equal weight (for which the height is reduced and the width increased). In Fig. 2.3, the averaging window is 0.005 units wide. Features of smaller scale are averaged out, while the actual peaks are preserved.

A better way to calculate the array critical current is by least squares fit to the form

\[
\frac{V_{dc}}{(I_c R)}^2 = m^2[(I_B/I_c)^2 - (I_{c\text{array}}/I_c)^2].
\]  

(2.14)
Figure 2.3: Array Critical Current vs. Frustration, for an 18 x 18 array, using the uniform phase model. The top plot shows data calculated using the perpendicular configuration; the bottom plot uses the parallel configuration. The data has been smoothed with a Savitzky-Golay filter. The x symbols indicate predictions from Eq. (2.17). The primary tick-marks indicate frustrations of the form $1/q$, and the minor tick-marks indicate frustrations of the form $p/q$.

This is the generic form of the d.c. voltage of a single Josephson junction, and from Fig. 2.2, this form is clearly approximately correct for our numerical data. We then calculate the d.c. voltage of the array for several values of the bias current, all slightly above the critical current. Figure 2.4 is the result of these calculations, for an 18 x 18 array (top plot) and a 14 x 14 array (bottom plot), for 2000 different frustration values. The x and tick marks have the same meanings as in Fig. 2.3. The data from this least squares fit extrapolation is much more precise than from the direct calculation.
of the critical current. There is still noise, albeit less, so again a Savitzky-Golay filter, this time with a 0.0025 wide window, has been applied to the original data.

![Array Critical Current vs. Frustration](image)

**Figure 2.4:** Array Critical Current vs. Frustration, using the uniform phase model and extrapolating the critical current via a least-squares fit. The top plot shows data calculated for an 18 x 18 array; the bottom plot is for a 14 x 14 array. The data has been smoothed with a Savitzky-Golay filter. The × symbols indicate predictions from Eq. (2.17). The primary tick-marks indicate frustrations of the form 1/q, and the minor tick-marks indicate frustrations of the form p/q.

### 2.1.4 Discussion

The frustration of a Josephson junction array is more than merely a measure of the magnetic field. Because the superconducting order parameter must be single-valued, the sum of the gauge-invariant phase differences around any loop must be equal to
the frustration, modulo $2\pi$. That is,

$$\left( f = \sum_{\text{loop}} \Delta \theta \right) \mod 2\pi. \tag{2.15}$$

Let us consider a loop that consists of a single plaquette. A plaquette is a loop that cannot be decomposed into smaller loops; in the array we are investigating, this is a single, small square of wires. If the magnetic field (and frustration) is zero, then the supercurrent, and thus the energy, is minimized when the magnitude of each phase difference around the plaquette is minimized. When the frustration is not zero, the energy of the plaquette cannot be minimized by individually minimizing the magnitudes of the phase differences. This is because, as indicated by Eq. (2.15), the sum of the phase difference cannot be zero in this case. This is why the quantity $f$ is known as the frustration: the individual junctions are "frustrated" from reaching their locally minimal energy.

We can add any multiple of $2\pi$ (e.g., $2n\pi$ for any integer $n$) to the sum of the phases around a plaquette and still satisfy Eq. (2.15). We can define the integer of multiplicity $n$,

$$n = f - \frac{1}{2\pi} \sum_{\text{loop}} \Delta \theta. \tag{2.16}$$

For a frustration $f = p/q$, if we were to calculate $n$ for every plaquette, we will notice a repetitive pattern. The pattern will have a period of $q$ in both the $x$- and $y$-directions. This pattern is called the flux lattice. A square of $q \times q$ plaquettes, if translated in the $x$- and $y$-directions by multiples of $q$ plaquettes, will reproduce the entire flux lattice. This primitive cell is called the flux lattice unit cell.

From the calculated data, it is obvious that the critical current of the array is a sensitive function of the frustration. At $f = 0$, the array is unfrustrated and the
Critical current is maximized near a rational frustration $f = p/q$, for positive integers $p$, $q$ with no common factors and $q < N - 1$, our numerical results suggest the relation

$$I_c^{\text{array}} \approx I_c P^{1/2} u^{1/4} (s/P^2)^{1/2}. \quad (2.17)$$

The number of $q \times q$ flux lattice unit cells that can be inscribed within the array is $u$, the sum of plaquettes contained in the $u$ unit cells is $s = uq^2$, and $P^2 = (N - 1)^2$ is the total number of plaquettes in the array.

We can interpret Eq. (2.17) as a measure of the commensurability between the array and the field. Because the array is finite in size, only a limited portion of the flux lattice can “fit” within the array. The better the fit, either by increasing the total number of unit cells, or by reducing the number of “extra” plaquettes, the higher will be $I_c^{\text{array}}$. The larger the commensurability, the more energy (and thus current) will be required to disrupt the flux lattice. For poorly commensurate fields, i.e., those with large $q$, the flux lattice unit cell will be larger than the array. In such a case, we expect that a very small current will be sufficient to disrupt the flux lattice.

Then, by noticing that $u = \lfloor P/q \rfloor^2$, where $\lfloor x \rfloor = n$, such that $n$ is the greatest integer satisfying $n \leq x$, we can simplify the relation (2.17) to

$$I_c^{\text{array}} \approx I_c P^{1/2} \left( \frac{q}{P} \right) \left[ \frac{P}{q} \right]^{3/2}. \quad (2.18)$$

In the limit of large $N$, the result becomes

$$I_c^{\text{array}} \approx I_c N/\sqrt{q}. \quad (2.19)$$

The values predicted by Eq. (2.17) are plotted in both Figs. 2.3 and 2.4 as $\times$ symbols. Comparing the results of the two figures, it is immediately clear that the least-squares extrapolation does a better job of finding the peaks. The direct calculation method
does a worse job at finding the peaks corresponding to high-denominator frustrations. In Fig. 2.4, there is a peak for every frustration with $q < P$.

2.2 Finite Inductance Model

Previously, we made the assumption that the phase of the superconducting order parameter does not vary along the length of the wires. But, perhaps, this is not an accurate assumption. In a long wire, there will be a kinetic inductance due to the fact that changes in the phase require a finite amount of time to propagate down the length of the wire. We will now examine a more general approach of modeling the wire array which will simulate a kinetic inductance within the wires. We will then determine what conditions will allow us to make the uniform phase assumption of Section 2.1.

2.2.1 Formalism

A simple way to allow the phase to vary along the wire is mentally to divide each wire into segments. For simplicity, each segment will have length $a$, the spacing constant between wires. In addition to the actual Josephson junctions coupling the wires of opposing layers, we model each segment of wire as if it contained a Josephson junction. Although only an approximation, this enables each segment of each wire to maintain a distinct phase. In Fig. 2.5, each node corresponds to the location of one independent phase variable. Each phase will interact directly with no more than three other phases: the phase along the wire in either direction (only one direction at the boundary) and the phase at the same location in the opposing layer.
We begin the formalism with the Kirchhoff circuit rules.

\[
I_{x,j,k} = I_{x,j+1,k} + I_{x,j+1,k},
\]
\[
I_{y,j,k+1} = I_{y,j,k} + I_{x,j,k+1},
\]

and

\[
0 = V_{x,j,k} - V_{x,j,k} - V_{x,j+1,k} - V_{y,j+1,k} \\
+ V_{x,j+1,k+1} + V_{x,j,k+1} - V_{x,j,k+1} + V_{y,j,k}.
\]

The current \( I_{x,j,k} \) is in the \( x \)-direction in the segment starting at the intersection of wires \( j \) and \( k \), and similarly for the currents in other directions; refer to Fig. 2.5. The potential \( V_{x,j,k} \) is across the same segment as the corresponding current. The index \( j \) runs from 1 to \( M \), and \( k \) from 1 to \( N \). The currents at the boundaries are \( I_{x,0,k} = I_{x,M,k} = I_{y,j,0} = I_{y,j,N} = 0 \), except for \( I_{x,0,n} = I_{y,m,N} = I_B \) for the perpendicular configuration, Fig. 2.1a, and \( I_{x,0,1} = -I_{x,M,1} = I_B \) for the parallel configuration, Fig. 2.1b.

We define the currents and potentials in terms of overdamped Josephson junctions:

\[
I_{x,j,k} = \frac{1}{R_x} \frac{\hbar}{q} \frac{d}{dt} (\phi_{j+1,k} - \phi_{j,k}) + I_{cxx} \sin (\phi_{j+1,k} - \phi_{j,k}),
\]
\[
I_{y,j,k} = \frac{1}{R_y} \frac{\hbar}{q} \frac{d}{dt} (\psi_{j,k+1} - \psi_{j,k}) + I_{cxy} \sin (\psi_{j,k+1} - \psi_{j,k}),
\]
\[
I_{z,j,k} = \frac{1}{R_z} \frac{\hbar}{q} \frac{d}{dt} (\psi_{j,k} - \phi_{j,k}) + I_{czz} \sin (\psi_{j,k} - \phi_{j,k}),
\]

and

\[
V_{x,j,k} = \frac{\hbar}{q} \frac{d}{dt} (\phi_{j+1,k} - \phi_{j,k}),
\]
\[
V_{y,j,k} = \frac{\hbar}{q} \frac{d}{dt} (\psi_{j,k+1} - \psi_{j,k}),
\]
\[
V_{z,j,k} = \frac{\hbar}{q} \frac{d}{dt} (\psi_{j,k} - \phi_{j,k}).
\]
The phase $\phi_{j,k}$ is the phase of the superconducting order parameter at the intersection of wires $j$ and $k$, in the first layer, and $\psi_{j,k}$ is phase at the same intersection in the other layer. The critical current of the actual junctions in the $z$-direction is $I_{c,z} = I_c$, and the critical currents of the wires are $I_{c,x} = I_{c,y} = \lambda I_c$. We also replace $R_z = R$, the resistance of the actual junctions, and $R_x = R_y = \mu R$. We shall call $\lambda$ the "coupling" parameter, because in this model it controls the degree of coupling between the layers relative to the coupling within the wires. Thus,

$$\lambda = \frac{I_{c,x}}{I_{c,z}}.$$  \hspace{1cm} (2.24)

Many of the properties of the array will depend on the value of $\lambda$. Typically, $\lambda \gg 1$ and $\mu \ll 1$, i.e., the critical current of the wires is much greater than that of the actual junctions and their resistance is much smaller than the resistance of the actual junctions.

Making the usual change to dimensionless variables using Eqs. (1.5), we have

$$i_{x_{j,k}} = \frac{1}{\mu} (\dot{\phi}_{j+1,k} - \dot{\phi}_{j,k}) + \lambda \sin(\phi_{j+1,k} - \phi_{j,k}),$$
\[ i_{y,j,k} = \frac{1}{\mu} (\dot{\psi}_{j,k+1} - \dot{\psi}_{j,k}) + \lambda \sin(\psi_{j,k+1} - \psi_{j,k}), \]
\[ i_{z,j,k} = \dot{\psi}_{j,k} - \dot{\phi}_{j,k} + \sin(\psi_{j,k} - \phi_{j,k}). \] (2.25)

With our choice of phase variables, Eq. (2.21) is identically true;

\[ 0 = \dot{\psi}_{j,k} - \dot{\phi}_{j,k} - \dot{\phi}_{j+1,k} + \dot{\phi}_{j,k} - \dot{\psi}_{j+1,k} + \dot{\psi}_{j+1,k} + \dot{\psi}_{j+1,k+1} + \dot{\psi}_{j+1,k+1} - \dot{\phi}_{j+1,k+1} + \dot{\phi}_{j+1,k+1} - \dot{\psi}_{j,k+1} + \dot{\psi}_{j,k+1} - \dot{\psi}_{j,k}. \] (2.26)

Including the boundary conditions, we have the following $2NM$ coupled first-order differential equations. For all $1 \leq k \leq N$, we have for $j = 1$,

\[ (1 + \mu) \dot{\phi}_{1,k} - \mu \dot{\psi}_{1,k} - \dot{\phi}_{2,k} = \mu \sin(\psi_{1,k} - \phi_{1,k}) + \lambda \mu \sin(\phi_{2,k} - \phi_{1,k}) \]
\[ - \mu \dot{\psi}_{1,0,k}; \] (2.27)

for $1 < j < M$,

\[ (2 + \mu) \dot{\phi}_{j,k} - \mu \dot{\psi}_{j,k} - \dot{\phi}_{j-1,k} - \dot{\psi}_{j+1,k} = \mu \sin(\psi_{j,k} - \phi_{j,k}) + \lambda \mu \sin(\phi_{j-1,k} - \phi_{j,k}) \]
\[ + \lambda \mu \sin(\phi_{j+1,k} - \phi_{j,k}); \] (2.28)

and finally for $j = M$,

\[ (1 + \mu) \dot{\phi}_{M,k} - \mu \dot{\psi}_{M,k} - \dot{\phi}_{M-1,k} = \mu \sin(\psi_{M,k} - \phi_{M,k}) + \lambda \mu \sin(\phi_{M-1,k} - \phi_{M,k}) \]
\[ + \mu \dot{\psi}_{M,0,k}. \] (2.29)

Furthermore, for all $1 \leq j \leq M$, and for $k = 1$,

\[ (1 + \mu) \dot{\psi}_{j,1} - \mu \dot{\phi}_{j,1} - \dot{\psi}_{j,2} = \mu \sin(\phi_{j,1} - \psi_{j,1}) + \lambda \mu \sin(\psi_{j,2} - \psi_{j,1}) \]
\[ - \mu \dot{\psi}_{j,0}; \] (2.30)
for $1 < k < N$,

$$(2 + \mu)\dot{\psi}_{j,k} - \mu\dot{\phi}_{j,k} - \dot{\psi}_{j,k-1} - \dot{\psi}_{j,k+1} = \mu \sin(\phi_{j,k} - \psi_{j,k}) + \lambda \mu \sin(\psi_{j,k-1} - \psi_{j,k})$$

$$+ \lambda \mu \sin(\psi_{j,k+1} - \psi_{j,k}); \quad (2.31)$$

and, at last, for $k = N$,

$$(1 + \mu)\dot{\psi}_{j,N} - \mu\dot{\phi}_{j,N} - \dot{\psi}_{j,N-1} = \mu \sin(\phi_{j,N} - \psi_{j,n}) + \lambda \mu \sin(\psi_{j,N-1} - \psi_{j,N})$$

$$+ \mu i_{y,j,N}. \quad (2.32)$$

### 2.2.2 Numerical Methods

The first-order terms of this set of equations, Eqs. (2.27)–(2.32), are not easily decoupled. There is still the phase degree of freedom to be chosen; however, there is no simple condition that will simplify these equations as there is for Eqs. (2.8). The coefficients of the first-order terms constitute a symmetric, positive-definite matrix, $M$, of order $2NM$. The system of equations then can be written as a matrix equation,

$$M\phi = s. \quad (2.33)$$

The vector $\phi$ has as its $2NM$ components the variables $\{\dot{\psi}_{j,k}, \dot{\phi}_{j,k}\}$. The vector $s$ is a function of the phases and the bias current as described by Eqs. (2.27)–(2.32).

We have not included the vector potential terms, but it is easy to do so. However, we find that including the terms vastly increases the time needed to numerically integrate the equations, because the terms inhibit (frustrate) the system from reaching a steady state. Our results for this model are entirely at the limit of zero applied magnetic field.

Once the matrix equation (2.33) has been solved, we can explicitly, if only numerically, calculate the values of the time derivatives of the phases. The matrix $M$
need only be inverted once, since it is not a function of time, unlike \( s \). As well as for the uniform phase model, Section 2.1.2, we use a Runge-Kutta algorithm, but in this case, we choose an adaptive step-size algorithm, for greater efficiency. The phases are randomly initialized, and the system is again integrated for \( 0 \leq \tau \leq 100 \) so that a steady state may be reached. For this system, we find that we need to average the voltage over 1000 dimensionless time units in order to calculate consistent values. A longer measurement time is needed for this model because of the larger number of degrees of freedom.

2.2.3 Results and Discussion

We have fixed the frustration at zero and so explore the behavior of the array as a function of its size. Figure 2.6 plots the d.c. voltage of the array as a function of the bias current, for array sizes \( N = 2 \) to \( N = 19 \), from left to right, respectively. The coupling parameter, Eq. (2.24), is fixed at \( \lambda = 100 \) for these curves. We fix the resistance parameter to \( \mu = 0.01 \) for all of our dynamical simulations. The bias current starts at zero and is slowly swept up with increments of \( 0.1I_c \) to a large maximum value and swept down with decrements of \( 0.01I_c \). It can be seen immediately that the array exhibits hysteresis. Computationally, the hysteresis is possible because we use the final state of one bias current value as the initial state of the next bias current value.

Hysteresis

For every size array, we find that there is a range (depending on the array size) of bias currents for which there are two states. These are visible on the graph for \( N \leq 7 \),
Figure 2.6: DC Voltage vs. Bias Current, for array sizes $N = 2$ to $N = 19$, from left to right, respectively; using the finite-inductance model. The coupling parameter is $\lambda = 100$, and the resistance parameter is $\mu = 0.01$. The lower branches of the hysteresis loops correspond to increasing current; the upper branches to decreasing current.

and exist at higher currents for the larger arrays. The lower voltage branch corresponds to the bias current being swept up; the higher voltage branch corresponds to the bias current being swept down. These hystereses occur even though the individual junctions and wires are non-hysteretic. The transition to the upper voltage state can be reliably reproduced; however, the transition to the lower states is sensitive to the rate of decrease of the bias current. Figure 2.7 plots the upper transition current $I_t$ as function of the array size. The dashed line is a plot of

$$I_t = N^2 I_c$$

(2.34)
and quite accurately predicts the transition current.

Figure 2.7: Array Transition Current vs. Array Size, using the finite-inductance model. The computational data is marked by + symbols. The dashed line graphs $N^2I_c$ for comparison.

This value for the upper transition current is readily explained as the total of the critical currents of all the actual junctions. This suggests that the transition corresponds to a global redistribution of the currents flowing through the array. The simulation allows us to examine the current and voltage at every point in the array. The following pattern emerges: when the bias current is less than the array critical current, there is, by definition, no voltage across the array inputs. Moreover, there are no potential differences across any of the junctions. The bias current is distributed almost equally among the junctions. In Fig. 2.8 the area of each circle is
proportional to the current flowing through the corresponding junction. Empty circles represent currents with no voltage drop, and filled circles indicate a finite voltage drop. Figure 2.8a shows the pattern for the array below its critical current.

Figure 2.8: Schematic diagrams of the array current patterns: a) $I_B < I_c^{\text{array}}$ (superconducting state), b) $I_c^{\text{array}} < I_B < I_t$ (lower hysteretic branch), c) $I_t < I_B$ (upper hysteretic branch). There is a symbol corresponding to each junction in the array. The area of the symbol is proportional to the current through the junction. Empty circles $\circ$ represent supercurrents. Filled disks $\bullet$ indicate there is a finite voltage across the junction.

When the bias current is between the array critical current and the transition current, there is a voltage across the array. More specifically, there is a voltage between the input wires. Investigating the details of the simulation, we find that the potential difference is across only the input wires; there is no difference across the others. This pattern is displayed in Fig. 2.8b. Most of the current goes through the center junction, where the input wires intersect. The junctions not on the input wires carry only supercurrents.

The intermediate pattern reaches its limit of stability as the bias current reaches $N^2 I_c$, in the state at which all of the off-center junctions are nearing their critical
currents $I_c$. It then becomes energetically more favorable for the currents to distribute the potential difference more equitably. The array switches to the pattern in Fig. 2.8c. All of the junctions have a potential difference, with the greatest difference between the input/output wires. As the bias current increases beyond the transition current, our model does not find any other current states.

The array of Josephson junctions behaves approximately as an array of resistors. Using Kirchhoff's rules, a resistor network in this geometry (ignoring the resistance of the wires themselves) is

$$R_N = \frac{2N-1}{N^2} R.$$ (2.35)

When the bias current is sufficiently larger than $I_c^{array}$, the current and voltage across the array approximately obeys Ohm's law, $V_{dc} \approx I_B R_N$. Below the transition, the voltage is approximately

$$V_{dc} \approx \frac{2}{N+1} R \sqrt{I_B^2 - (I_c^{array})^2}. \quad (2.36)$$

This is the same form of a single junction with critical current $I_c^{array}$ and shunt resistance $2R/(N + 1)$. We can derive this resistance by finding the resistance of a network of wires and resistors in the pattern indicated by Fig. 2.8b. We substitute each junction with a finite voltage by a resistance $R$. The remaining junctions have no potential difference across them.

Along with Eq. (2.34), we estimate the voltage discontinuity at the transition current:

$$\Delta V_t \approx \frac{N - 1}{N + 1} I_c R. \quad (2.37)$$
And if $N$ is large, this is simply $\Delta V_t \approx I_c R$. Although the value of the voltage discontinuity is not explicitly measured by Sohn et al., their data seems to show voltage jumps of approximately this magnitude.

**Critical Current Saturation**

From Fig. 2.6, we can see that the array critical current does not increase uniformly; i.e., as the size of the array increases, the critical currents increase by successively smaller amounts. In Fig. 2.9, we have plotted the array critical current as a function of the array size, for several values of the coupling parameter. The critical current increases linearly for small values of $N$; as can be seen by comparison with the dashed line of slope one. As the size of the array increases to larger values, the critical current fails to increases linearly. In fact, the critical current saturates at a maximum value. This maximum value is approximately $I_{c_{\text{array}}} \approx 2I_c \sqrt{\lambda}$. The first significant deviation from linearity occurs near $N \approx (3/2) \sqrt{\lambda}$.

The saturation of the array critical current is, of course, due to our inclusion of an inductance in the wires. This inductance is modeled by allowing the phase to vary along the length of the wire. The length of the wire is effectively limited by allowing the phase variation. We can use this as a guideline to explain how well the uniform-phase assumption works. If the size of the array satisfies the condition

$$N \leq \sqrt{\lambda} = \sqrt{I_{c_{\text{wire}}}/I_c},$$

(2.38)

where $I_{c_{\text{wire}}}$ is the critical current of the wire, then it is reasonable to use the uniform-phase model. In Section 2.3.4 we will discuss the limits of the uniform-phase model further.
2.3 Mean Field Theories

We can also investigate the thermodynamic properties of the array. The most obvious property is the critical temperature $T_c$ at which the system crosses from an ordered, non-voltage state to an unordered, voltage state. We will first develop two mean field theories (MFT) and then compare these to a small number of Monte-Carlo results.

Within typical mean field theories, one considers the interaction between a single degree of freedom and an average over all of the degrees of freedom. In the MFT, the various degrees of freedom are directly coupled only to this mean. In general, mean
field theories best describe systems where each degree of freedom interacts with many other degrees of freedom.

In contrast, our MFT considers the interaction between a single degree of freedom (in our case, a single phase variable) and an average over only the phases of the opposing layer. The interactions within a layer are treated exactly; the interactions between layers are treated in the mean field approximation. Because each wire does in fact interact with all of the wires in the opposing layer, we can expect our MFT to work well. From the results of Section 2.1.1, see especially Eqs. (2.8), we expect that the MFT will be exact, in the limit of uniform phases in the wire [30, 23]. We will find the conditions for which the uniform phase limit is valid [22].

2.3.1 Formalism

The MFT for Josephson junction array with this geometry has already been investigated [30, 28], but previously it has been assumed that the phase of the superconducting order parameter is uniform along each wire. We will now consider the possibility of phase differences developing along the wires. We will use the same layout as the dynamics with finite wire inductance in Section 2.2.1. Each segment of wire will be modeled as a Josephson junction with coupling energy $\lambda E_J$, with $E_J = \hbar I_c / q$ being the coupling energy of the actual junctions connecting the wires. The ratio of the coupling energies $\lambda$, we call the coupling parameter of the system. This is equivalent to the coupling parameter identified in Section 2.2.1.

We will develop the MFT first for the simpler case of no magnetic field. Then we consider the case with a finite magnetic field and finally assume that the magnetic field is rational. In each case we assume the array is infinite in size, so that we do
not need to consider boundary conditions. This corresponds to the “thermodynamic limit”, i.e. $N \to \infty$.

**Zero Magnetic Field**

We first consider the system with no magnetic field. The classical Hamiltonian is, in the case of no magnetic field,

$$H = -\lambda E_J \sum_{j,k} \cos(\phi_{j,k} - \phi_{j+1,k}) - \lambda E_J \sum_{j,k} \cos(\psi_{j,k} - \psi_{j,k+1})$$

$$- E_J \sum_{j,k} \cos(\phi_{j,k} - \psi_{j,k}).$$

(2.39)

The first two summations describe the interactions within the two separate layers. The last summation represents the interaction between the layers. We have used the same phase notation as before, and the sums run over all wire intersections. The MFT order parameters are

$$\eta_{\phi_{j,k}} = \langle \cos \phi_{j,k} \rangle = \int d\Gamma \cos \phi_{j,k} \exp(-\beta H) \int d\Gamma \exp(-\beta H),$$

$$\eta_{\psi_{j,k}} = \langle \cos \psi_{j,k} \rangle = \int d\Gamma \cos \psi_{j,k} \exp(-\beta H) \int d\Gamma \exp(-\beta H).$$

(2.40)

The exponential coefficient is the usual inverse temperature $\beta = 1/(k_B T)$, and the integral is over all phases,

$$\int d\Gamma = \prod_{j,k} \int_{-\pi}^{\pi} d\phi_{j,k}/(2\pi) \int_{-\pi}^{\pi} d\psi_{j,k}/(2\pi).$$

(2.41)

With no magnetic field, the order parameters will be identical: $\eta_{\phi_{j,k}} = \eta_{\psi_{j,k}} = \eta$. For the MFT approximation, we replace

$$\cos(\phi_{j,k} - \psi_{j,k}) = \cos \phi_{j,k} \cos \psi_{j,k} + \sin \phi_{j,k} \sin \psi_{j,k}$$

$$\approx \cos \phi_{j,k} \cos \psi_{j,k} + \sin \phi_{j,k} \sin \psi_{j,k}$$

$$\approx \eta \cos \phi_{j,k},$$

(2.42)
taking the average over the variables \( \{\psi_{j,k}\} \). The rest of the Hamiltonian is treated exactly, giving the MFT form

\[
H_{\text{mf}} = H_x + H_z = -E_J \sum_{j,k} [\lambda \cos(\phi_{j,k} - \phi_{j+1,k}) + \eta \cos \phi_{j,k}].
\]

The terms with coefficient \( \lambda \) are included in \( H_x \) and represent the interactions along the wires. The remaining terms are grouped into the term \( H_z \), the interactions between the layers. The self-consistency equation for the order parameter in this MFT is

\[
\eta \approx \int d\Gamma \phi \cos \phi_{j,k} \exp(-\beta H_{\text{mf}}) \int d\Gamma \phi \exp(-\beta H_{\text{mf}}),
\]

integrating only over the phases of one layer (only the \( \phi \) variables). Near the critical temperature \( T_c \), the order parameter will be small \( |\eta| \ll 1 \). Expanding the exponential and keeping only terms of zeroth and first order in \( \eta \), we have

\[
\eta \approx \int d\Gamma \phi \cos \phi_{j,k} \exp(-\beta H_x) \left( 1 + \beta E_J \eta \sum \cos \phi_{j',k'} \right) \int d\Gamma \phi \exp(-\beta H_x) \left( 1 + \beta E_J \eta \sum \cos \phi_{j',k'} \right).
\]

The term in \( \eta^0 \) vanishes, as does the second term of \( \cos \phi_{j,k} \cos \phi_{j',k'} = [\cos(\phi_{j,k} - \phi_{j',k'}) + \cos(\phi_{j,k} + \phi_{j',k'})]/2 \). This leaves us with

\[
\eta = \eta \beta E_J \int d\Gamma \phi \sum_{j'} \cos(\phi_{j,k} - \phi_{j',k}) \exp(-\beta H_x) \int d\Gamma \phi \exp(-\beta H_x).
\]

The cosine can be represented as

\[
\cos(\phi_{j,k} - \phi_{j',k}) = \frac{1}{2} \text{Re} \left[ \exp i(\phi_{j,k} - \phi_{j',k}) \right] = \frac{1}{2} \text{Re} \prod_{\ell=j}^{j'-1} \exp i(\phi_{\ell,k} - \phi_{\ell+1,k}).
\]
Introducing $\theta_{\ell,k} = \phi_{\ell,k} - \phi_{\ell+1,k}$, the self-consistency equation (2.46) can be systematically simplified:

\[
1 = \frac{1}{2} \beta E_f \Re \sum_{j'=-\infty}^{\infty} \prod_{\ell=j}^{j'-1} \left[ \int d\Gamma_{\phi} \exp i\theta_{\ell,k} \prod_{j'',k''} \exp(\beta \lambda E_f \cos \theta_{j'',k''}) \right] \\
= \frac{1}{2} \beta E_f \Re \sum_{j'=-\infty}^{\infty} \prod_{\ell=j}^{j'-1} \int_{-\pi}^{\pi} d\theta_{\ell,k} \exp i\theta_{\ell,k} \exp(\beta \lambda E_f \cos \theta_{\ell,k}) \\
= \frac{1}{2} \beta E_f \Re \sum_{j'=-\infty}^{\infty} \left[ \int_{-\pi}^{\pi} d\theta \exp i\theta \exp(\beta \lambda E_f \cos \theta) \right]^{1-j'|j'|}. \quad (2.48)
\]

We use the integral definition of the modified Bessel function [31],

\[
I_n(z) = I_{-n}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \exp(\pm ni\theta) \exp(z \cos \theta). \quad (2.49)
\]

And we define the function:

\[
\Lambda(x) = \frac{I_1(x)}{I_0(x)}. \quad (2.50)
\]

And we also find the closed form for the value of the series,

\[
\cdots + q^2 + q + 1 + q + q^2 + \cdots = \sum_{j=-\infty}^{\infty} q^{|j|} = \frac{1 + q}{1 - q}. \quad (2.51)
\]

We then simplify Eq. (2.48) to

\[
1 = \frac{1}{2} \beta E_f \frac{1 + \Lambda(\lambda \beta E_f)}{1 - \Lambda(\lambda \beta E_f)}. \quad (2.52)
\]

The value of $\beta$ which solves this equation gives $T_c$ within this MFT at zero magnetic field. We will immediately extend this theory by considering a finite magnetic field.
Finite Magnetic Field

For the same system with a finite magnetic field, we use the same Hamiltonian, but include the vector potential.

\[
H = -\lambda E_J \sum_{j,k} \cos(\phi_{j,k} - \phi_{j+1,k}) - \lambda E_J \sum_{j,k} \cos(\psi_{j,k} - \psi_{j,k+1}) - E_J \sum_{j,k} \cos(\phi_{j,k} - \psi_{j,k} - A_{k,j}),
\]

(2.53)

using the same vector potential term as used for the dynamics, \(A_{k,j} = 2\pi B a^2 j k / \Phi_0\).

The MFT order parameters are

\[
\eta_{\phi;j,k} = \langle \exp i \phi_{j,k} \rangle = \int d\Gamma \exp i \phi_{j,k} \exp(-\beta H) \int d\Gamma \exp(-\beta H),
\]

\[
\eta_{\psi;j,k} = \langle \exp i \psi_{j,k} \rangle = \int d\Gamma \exp i \psi_{j,k} \exp(-\beta H) \int d\Gamma \exp(-\beta H).
\]

(2.54)

The MFT Hamiltonian is

\[
H_{\text{mf}} = H_x + H_z
\]

\[
= -E_J \sum_{j,k} \left[ \lambda \cos(\phi_{j,k} - \phi_{j+1,k}) - \langle \cos(\phi_{j,k} - \psi_{j,k} - A_{k,j}) \rangle \right].
\]

(2.55)

The second term \(H_z\) is averaged over only the \(\psi\) variables; this is the MFT approximation. It may be rewritten as

\[
H_z = -E_J \text{Re} \sum_{j,k} \langle \exp i \psi_{j,k} \exp i(A_{k,j} - \phi_{j,k}) \rangle \psi
\]

\[
= -E_J \text{Re} \sum_{j,k} \eta_{\psi;j,k} \exp i(A_{k,j} - \phi_{j,k}).
\]

(2.56)

The self-consistency equation for the order parameter in this MFT is

\[
\eta_{\phi;j,k} = \int d\Gamma \exp i \phi_{j,k} \exp(-\beta H_{\text{mf}}) \int d\Gamma \exp(-\beta H_{\text{mf}})
\]

\[
\approx \beta E_J \text{Re} \sum_{j'k} \eta_{\psi;j',k} \int d\Gamma \exp i \phi_{j,k} \exp i(A_{k,j'} - \phi_{j',k}) \exp(-\beta H_z)
\]

\[
\int d\Gamma \exp(-\beta H_{\text{mf}}).
\]

(2.57)
We have made the same small-\( \eta \) approximation, following a similar course as the no magnetic field case. Continuing in the same manner, we come to

\[
\eta_{\varphi;j,k} = \frac{1}{2} \beta E_J \text{Re} \sum_{j'=-\infty}^{\infty} \eta_{\varphi;j',k} \exp i A_{k;j'} \prod_{t=j}^{j'-1} \left[ \int \text{d} \Gamma_{\varphi} \exp i \theta_{t,k} \prod_{j'',k''} \exp(\beta \lambda E_J \cos \theta_{j'',k''}) \right] \]

\[
\int \text{d} \Gamma_{\varphi} \prod_{j'',k''} \exp(\beta \lambda E_J \cos \theta_{j'',k''}),
\]

which ends as

\[
\eta_{\varphi;j,k} = \frac{1}{2} \beta E_J \sum_{j'=-\infty}^{\infty} \eta_{\varphi;j',k} \exp(i A_{k;j'}) \Lambda^{j-j'}(\lambda \beta E_J),
\]

\[
\eta_{\varphi;j,k} = \frac{1}{2} \beta E_J \sum_{k'=-\infty}^{\infty} \eta_{\varphi;j,k'} \exp(-i A_{k';j}) \Lambda^{k-k'}(\lambda \beta E_J).
\]

Where we have also written the analogously derived equation for the other MFT order parameter. Substituting one equation into the other decouples the order parameters by layer.

\[
\eta_{\varphi;j,k} = \frac{1}{4} (\beta E_J)^2 \sum_{j'=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \eta_{\varphi;j',k'} \exp[i2\pi f j'(k - k')] \Lambda^{j-j'}|k-k'|(\lambda \beta E_J),
\]

\[
\eta_{\varphi;j,k} = \frac{1}{4} (\beta E_J)^2 \sum_{j'=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \eta_{\varphi;j',k'} \exp[-i2\pi f (j - j') k'] \Lambda^{j-j'}|k-k'|(\lambda \beta E_J).
\]

In general, this is an infinite set of coupled equations. If, however, the frustration is a rational fraction; i.e., \( f = p/q \) with \( p \) and \( q \) positive integers with no common factors, then it is reasonable to assume that the order parameters have a period of \( q \) in each of their indices.

\[
\eta_{j+mq,k+nq} = \eta_{j,k} \quad \text{for all integers } m, n
\]

This is equivalent to saying that the flux lattice forms a \( q \times q \) unit cell, as in conventional square Josephson arrays. A unit cell of this size is also plausible because the
exponential in the sum has the same periodicity as the unit cell. We can express the periodicity in the manner:

$$\eta_{j,k} = \sum_{j'=0}^{q-1} \sum_{k'=0}^{q-1} \eta_{j',k'} \delta_{j'=j'} \delta_{k'=k'} \mod q \delta_{k=k'} \mod q. \quad (2.62)$$

Upon substituting into Eqs. (2.60), we obtain

$$\eta_{\phi;j,k} = \frac{1}{4}(\beta E_J)^2 \sum_{j'=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \sum_{j''=0}^{q-1} \sum_{k''=0}^{q-1} \eta_{\phi;j'',k''} \delta_{(j'=j'')} \mod q \delta_{(k'=k'')} \mod q \exp[i2\pi f j'(k - k')] \Lambda^{|j-j'|+|k-k'|}(\lambda \beta E_J). \quad (2.63)$$

By noting that the value of the exponential does not change when \(j'\) and \(k'\) replace \(j''\) and \(k''\), respectively, and if we then rearrange the sums, we find

$$\eta_{\phi;j,k} = \frac{1}{4}(\beta E_J)^2 q^{q'} \sum_{j'=0}^{q-1} \sum_{k'=0}^{q-1} \eta_{\phi;j'',k''} \exp[i2\pi f j''(k - k'')] \times \left[ \sum_{j''=-\infty}^{\infty} \Lambda^{j-j'}(\lambda \beta E_J) \delta_{(j'=j'')} \mod q \right] \times \left[ \sum_{k''=-\infty}^{\infty} \Lambda^{k-k'}(\lambda \beta E_J) \delta_{(k'=k'')} \mod q \right]. \quad (2.64)$$

We make use of the identity

$$\cdots + q^{n-2N} + q^{n-N} + q^n + q^{n+N} + q^{n+2N} + \cdots = \sum_{j=-\infty}^{\infty} q^{j|\delta_{(j-n)} \mod N} = \frac{q^n + q^{N-n}}{1 - q^N}, \quad (2.65)$$

which holds for \(0 \leq q < 1\), and any integers \(n, N\) satisfying \(0 < n < N\). Equation (2.64) simplifies to the final form

$$\eta_{\phi;j,k} = \sum_{j'=0}^{q-1} \sum_{k'=0}^{q-1} C_{j'} C_{k'} \eta_{\phi;j',k'} \exp[i2\pi f j'(k - k')], \quad (2.66)$$

where

$$C_j = \frac{1}{2} \beta E_J(\Lambda^j + \Lambda^{q-j})/(1 - \Lambda^q). \quad (2.67)$$
2.3.2 Numerical Methods

Thus there is one coupled MFT equation for each the $q^2$ order parameters of each layer. We can represent these equations as the matrix equation

$$\eta = C(\beta E_J)\eta.$$ \hspace{1cm} (2.68)

The matrix equation is an eigenvalue equation of $C$, for which the desired eigenvalue is unity.

The matrix $C$ is asymmetric and complex (i.e., non-Hermitian). Although the eigenvalues are transcendental functions of the temperature, they can be found numerically. As it turns out, the eigenvalues have the form $\alpha_{n,k} = |\alpha_n| \exp(i2\pi k/q)$ for integers $0 < n \leq q$ and $0 \leq k < q$; i.e., there are $q$ distinct magnitudes $|\alpha_n|$, each magnitude being $q$-fold degenerate.

When $\beta E_J$ is small, the eigenvalue magnitudes are also small (this can be seen from the form of $C_j$). As $\beta E_J$ is increased, at some point the largest $|\alpha_n|$ will reach unity. This process corresponds to lowering the temperature until the critical temperature is reached, and Eq. (2.68) is satisfied. Thus, to find the critical temperature $T_c$, we must find the value of $\beta E_J$ for which the largest eigenvalue of $C$ has a magnitude of precisely one. Numerically, we look for an eigenvalue within $10^{-9}$ of unity.

2.3.3 Results

Figure 2.10 shows the results of the critical temperature $T_c$ as a function of the coupling parameter $\lambda$ for various values of the frustration $f$. Numerically, to an excellent approximation, the critical temperature does not depend on the numerator $p$ of the frustration $f = p/q$; it depends solely on $q$. Presumably, this behavior is
implied by the self-consistency equations (2.60), although we have not been able to show this analytically.

\[
\begin{align*}
\frac{1}{0} & = \frac{1}{2} \\
\frac{1}{0} & = \frac{1}{3} \\
\frac{1}{0} & = \frac{1}{4} \\
\frac{1}{0} & = \frac{1}{5} \\
\frac{1}{0} & = \frac{1}{6} \\
\frac{1}{0} & = \frac{1}{7} \\
\frac{1}{0} & = \frac{1}{8} \\
\frac{1}{0} & = \frac{1}{9} \\
\frac{1}{0} & = \frac{1}{10} \\
\frac{1}{0} & = \frac{1}{20}
\end{align*}
\]

Coupling Parameter \(A\)

Figure 2.10: MFT Critical Temperature vs. Coupling Parameter, for various frustrations.

We then look at how the MFT critical temperature varies as a function of the frustration in Fig. 2.11. We calculate \(T_c\) normalized by the zero-field critical temperature, varying the frustration for all distinct values of \(p/q\) with \(q \leq 30\). We have chosen the particular value of \(\lambda = 1000\) for this data. It is clear that \(p\) does not affect the value of \(T_c\). Looking at how \(T_c\) varies in both Figs. 2.10 and 2.11, we can conclude

\[
k_B T_c / E_J \sim \lambda^{1/2} q^{-1/4},
\]

(2.69)

to an excellent approximation.

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The form of the equation implies that as \( q \) becomes large, the critical temperature approaches zero. In the case of an irrational frustration, i.e., in the limit of \( q \to \infty \), \( T_c \) is precisely zero. Thus \( T_c(f) \) is an everywhere discontinuous function of the frustration. This is an artifact of our MFT, which is in the thermodynamic limit. The critical temperature of a finite array would exhibit peaks of finite width. As the flux lattice unit cell in a finite array increases in size with \( q \), one would expect that the energy required to disrupt the ordered state should become smaller and smaller. Our MFT thus has the correct qualitative properties.
We compare our results with experiment in Fig. 2.12. This figure shows critical temperatures measured by Sohn et al. [24], the critical temperatures as calculated within a conventional MFT [23], and by our MFT which allows for phase variations along the wires. The conventional MFT predicts a critical temperature of the form

\[ k_B T_c/E_J \sim Nq^{-1/2}, \]  

which clearly fails in the limit of large \( N \). As the size of the array increases, one must eventually recognize the variations in phase along the wires. A comparison with the experimental data reveals the MFT with phase variation is closer to the actual data than the conventional MFT. The critical temperature of the conventional MFT falls off too quickly for large \( q \). Our MFT, despite the obvious discrepancy, does do better. This is to be expected, since the data was taken for large arrays, \( N \geq 600 \), and so is beyond the conventional MFT.

With the results of the MFT, we can estimate the how large the array can be before we need to account for the phase variations, and therefore need to go beyond the conventional MFT. The root-mean-square (rms) phase variation along one wire segment, \( \delta \theta_1(T) \) is of the order \( \sqrt{k_B T_c/(\lambda E_J)} \). Along \( k \) segments, this variation is

\[ \delta \theta_k \sim \sqrt{\sum_{j=1}^{k} \delta \theta_1} \sim \delta \theta_1 \sqrt{k}. \]  

Let us define the array screening length \( \zeta \) as the length of the wire such that the rms phase variation is of order unity. If \( \zeta \) is \( n \) segments long, i.e. \( \zeta = na \), then \( \delta \theta_n \sim 1 \). At the critical temperature (2.69), the screening length will be

\[ \zeta \sim a \frac{\lambda E_J}{k_B T_c} \sim a \frac{\lambda E_J}{E_J \sqrt{\lambda}} \sim a \sqrt{\lambda}. \]  

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Figure 2.12: Comparison of Experiment with MFT Critical Temperature. Data is from Sohn et al. [24]. The MFT with phase variations is presented in this work, and the MFT with uniform phase is described in, e.g., Ref. [23].

We should not expect the array to maintain an ordered state with a unit cell larger than the screening length. That is, for

\[ q > \sqrt{\lambda}, \]  

(2.73)

the system will be unable to maintain the \( q \times q \) flux lattice unit cell. This is evident in Fig. 2.10, where for small values of \( \lambda \) and large values of \( q \), the critical temperature increasingly deviates from the simple power law \( T_c \sim \sqrt{\lambda} \) behavior.

Significantly, the array screening length we have derived here from the MFT follows the same power law as the size limit for the uniform phase model, Eq. (2.38).
Both quantities are a measure of the length scale over which phase variations become non-negligible.

### 2.4 Monte Carlo

We have also used Monte Carlo methods using the original Hamiltonian, Eq. (2.39), to calculate several thermodynamic properties of the system with no applied magnetic field ($f = 0$). We use a conventional Monte Carlo Metropolis algorithm, each step considering the variation of one of the $2N^2$ phase variable. For typical calculations presented herein, we have used a $10 \times 10$ array and periodic boundary conditions, and allowed the algorithm to evolve the array for on the order of $> 10^8$ steps.

We first consider the specific heat of the system,

$$\frac{1}{k_B}C_V = \frac{\langle H^2 \rangle - \langle H \rangle^2}{(Nk_BT)^2}. \tag{2.74}$$

The angle brackets $\langle \cdots \rangle$ denote a canonical average. The calculations of the specific heat for several values of the coupling constant $\lambda$ are plotted in Fig. 2.13.

In the limit of large values of $\lambda$, the specific heat approaches the independent wire limit. That is, for a single, independent long wire, with Hamiltonian

$$H_{\text{wire}} = E_J \sum_j \cos \phi_j, \tag{2.75}$$

the specific heat has the closed form:

$$C_{V,\text{wire}}/k_B = (\beta E_J)^2 - \beta E_J \Lambda (\beta E_J) - (\beta E_J)^2 \Lambda^2 (\beta E_J). \tag{2.76}$$

Equation (2.76) is plotted as the bold curve in Fig. 2.13. It is not unexpected that the array in the large-$\lambda$ limit closely matches the independent wire limit. In this
Figure 2.13: Specific Heat vs. Temperature, for a $10 \times 10$ array at $f = 0$, for several values of the coupling parameter $\lambda$. Solid curve indicates the exact specific heat of single, very long wire; Eq. (2.76).

limit, the coupling between the wires decreases as $\lambda$ increases. The calculated values for $\lambda \geq 10^2$ do not fall exactly on the curve of $C_{V,\text{wire}}$ due to finite size effects of our simulation.

The specific heat of the array, for the smaller values of $\lambda$, has an increasingly large peak at the phase transition. As $\lambda$ decreases, the coupling between the layer grows stronger, and the phase $\psi_{j,k}$ and $\phi_{j,k}$ are constrained to the same value. In this limit, the array behaves as a conventional Josephson-junction square lattice; the peak corresponds to the Kosterlitz-Thouless transition [32], with $k_B T_c \sim 0.90E_J$ [33].
We also calculate the helicity modulus [34] of the array. The helicity modulus is a measure of the "stiffness" of the phase-ordering within the array and is proportional to the superfluid density. It has the form [35]

$$\gamma = \lambda \left\langle \sum_{j,k} \cos \theta_{j,k} \right\rangle - \frac{\lambda E_J}{k_B T_c} \left[ \left\langle \left( \sum_{j,k} \sin \theta_{j,k} \right)^2 \right\rangle - \left\langle \sum_{j,k} \sin \theta_{j,k} \right\rangle^2 \right]. \quad (2.77)$$

Here, $\sum_{j,k}$ denotes a sum over all nearest-neighbor phase differences $\theta_{j,k}$. A finite helicity modulus indicates an ordered, superconducting state. The helicity modulus vanishes in disordered states. Figure 2.14 displays the helicity modulus for several values of $\lambda$.

![Figure 2.14: Helicity Modulus vs. Temperature, for a 10 x 10 array at $f = 0$, for several values of the coupling parameter $\lambda$.](image)

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The helicity modulus vanishes at the critical temperature $T_c$. For small values of $\lambda$, the helicity modulus has a continuous transition. At larger values of the coupling parameter, $\gamma$ appears to have a discontinuity at $T_c$, consistent with a Kosterlitz-Thouless transition.

2.5 Summary

We can summarize the results of our investigation of Josephson-coupled wire arrays. The wires of these arrays exhibit strong coupling among the wires. This is because each wire directly interacts with one-half of the other wires; i.e., all of the wires in the opposing layer. Thus, in the array as a whole, every wire is a nearest-neighbor or next-nearest-neighbor to all of the other wires. This property is evident from the results of both the dynamics and thermodynamics.

In an array with long wires, such that the phase variation along the lengths of the wires is non-negligible, we have produced a model which effectively predicts the size-dependent behavior of the array. The critical current of the array saturates at a finite value due to the limiting effect of phases variations on the length of the wires. The saturation value is

$$I_{c}^{\text{array}} \approx 2\sqrt{I_{c}^{\text{wire}}I_{c}}.$$  \hspace{1cm} (2.78)

The size of the array for which this saturation begins to have a noticeable effect is

$$N \approx \left(\frac{3}{2}\right)\sqrt{I_{c}^{\text{wire}}/I_{c}}.$$  \hspace{1cm} (2.79)

This value, calculated with the data from the dynamical model, is consistent with the value found via MFT. We thus have a good measure of the conditions under which the uniform-phase approximation may be made for either dynamics or thermodynamics.
We have also discovered hysteretic behavior in the array. This is surprising since the individual elements of the array are not hysteretic. We can thus surmise that the hysteresis is a global property of the array. Careful examination of the simulated array indicates that the current-voltage patterns within the array undergo a discontinuous transition.

As the bias current is slowly increased, the transition occurs at $I_B = N^2 I_c$, the total of the critical currents of the array. The decreasing-current transition does not occur at a well-defined bias current. Instead, the current at which the transition to the lower voltage occurs depends on the rate of the current’s decrease. This is analogous to a super-cooling. The meta-stable high-voltage state becomes increasingly unstable as the current decreases. At some point, a small variation (numerical error in our simulation, but thermal variation in an hypothetical experiment) will induce the entire array to discontinuously jump to the low-voltage state.

This hysteretic behavior is an excellent example of an emergent property. That is, although the junctions are non-hysteretic, their collective behavior becomes sufficiently complex to cause the array to produce unexpected results.

We consider also the effects of a finite magnetic field on the array. Within the MFT with phase variations, the array critical temperature depends on the field as

$$k_B T_c \sim E_J q^{-1/4}.$$  \hspace{1cm} \text{(2.80)}$$

This is in contrast to the $k_B T_c \sim E_J N q^{-1/2}$ prediction made by the “conventional” MFT with uniform phases [23]. The strength of our MFT is that it predicts a finite $T_c$ in the thermodynamic limit of $N \to \infty$; the conventional MFT $T_c$ increases without bound.
Furthermore, also different from to the conventional MFT, ours predicts \( T_c \) is an everywhere-discontinuous function of the field. As discussed previously, this is due to the thermodynamic limit we have taken. There is no reason to expect such behavior from a finite array. The conventional MFT, which models the thermodynamical properties of a finite array with uniform phases in each wire, predicts that \( T_c \) is a continuous function of the frustration.

Finally, we look at the behavior as the field varies of the array in the limit where the phases does not vary along the wires. We find the commensurability between the finite array and the flux lattice to be an excellent predictor of the critical current of the array. Specifically, for large arrays of size \( N \),

\[
I_{c}^{\text{array}} \approx I_c N/\sqrt{q}.
\]  

(2.81)

Not coincidentally, this \( q^{-1/2} \) behavior is the same as the dependence of \( T_c \) in the MFT with uniform phases. Both quantities depend on the same averaged interactions between the phases of the wires of opposing layers. As either the current or the temperature increases the beyond their respective critical value, the array undergoes a transition from an ordered to an unordered state.

The Josephson-coupled wire arrays display a range of interesting phenomena, both dynamic and thermodynamic. The global dynamics show fundamentally different behavior from the local dynamics. Furthermore, in an experimentally realizable limit, the dynamics and MFT thermodynamics predict similar results: the size of the array is restricted by the phase variations within the wires. These properties are not found in many systems, and thus these arrays are an excellent system for the investigation of both mean field theories and global ordering.
Researchers have long sought to cause Josephson junction arrays to radiate coherently [36, 37, 38, 39]. To achieve this goal, a standard approach is to inject a d.c. current into an overdamped array. If this current is sufficiently large, it generates an a.c. voltage $V_{ac}$ across the junctions, of frequency $\omega_J = 2eV_{dc}/\hbar$, where $V_{dc}$ is the time-averaged voltage across the junction [1]. Each junction then radiates, typically at microwave frequencies. If the junctions are coherently phase-locked, the radiated power $P \sim N^2$, where $N$ is the number of phase-locked junctions. This $N^2$ proportionality is a hallmark of phase coherence. But many difficulties inhibit phase coherence in practice. For example, the junctions always have a disorder-induced spread in critical currents, which produces a distribution of Josephson frequencies and makes phase locking difficult [40, 41, 10, 11]. Furthermore, in small-capacitance (and underdamped) Josephson junctions, quantum phase fluctuations inhibit phase locking [42, 43, 44, 45, 46, 47]. Thus, until recently, the most efficient coherent emission was found in two-dimensional arrays of overdamped Josephson junctions, where quantum fluctuations are minimal.

Recently, Barbara et al. have reported a remarkable degree of coherent emission in arrays of underdamped junctions [48, 49]. Their arrays were placed in a microwave
cavity, so as to couple each junction to a resonant mode of the cavity. If the mode has a suitable frequency and is coupled strongly enough to the junction, it can be excited by a Josephson current through the junction. The power in this mode then feeds back into the other junctions, causing the array to phase-lock and inducing a total power $P \sim N^2$. For a given coupling, Barbara et al. found that there is a threshold number of junctions $N_c$ below which no emission was observed. The coupling, and hence $N_c$, could be varied by moving the array relative to the cavity walls.

Barbara et al. interpreted their results by analogy with the Jaynes-Cummings model [50, 51] of two-level atoms interacting with a radiation field in a single-mode resonant cavity. In this case, each Josephson junction acts as a two-level atom; the coupling between the “atoms” is provided by the induced radiation field. A dynamical calculation based on a model similar to that of Jaynes and Cummings has been carried out by Bonifacio and collaborators [52, 53] for Josephson junction arrays in a cavity. Their model does produce spontaneous emission into the cavity above a threshold junction number, provided that the Heisenberg equations are treated in a certain semi-classical limit appropriate to large numbers of photons in the cavity.

In this work, we present a simple model for the onset of phase locking and coherent emission by an underdamped Josephson junction array in a resonant cavity. We also calculate the threshold for the onset of phase coherence, using a form of mean-field theory. Our model derives from more conventional models of Josephson junction arrays, but treats the interaction with the radiation field quantum-mechanically. Within the mean field theory, we find that for any strength of that coupling, there exists a threshold number of junctions $N_c$ in a linear array above which the array is coherent. Above that threshold, the energy in the photon field is quadratic in the
number of junctions, as found experimentally [48]. The model is easily generalized
to two-dimensional arrays, and can readily be extended to include disorder. Furthermore,
the coupling constant between junctions and radiation field can, in principle,
be explicitly calculated, given the geometry of the array and the resonant cavity.

3.1 Formalism

We consider a Josephson junction array containing $N$ junctions in series, placed
in a resonant cavity and arranged in a geometry shown schematically in Fig. 3.1.
The array is held at a fixed potential $\Phi$, applied by an external voltage source. The
Hamiltonian for this array is taken as the sum of four terms:

$$H = H_J + H_C + H_{\text{photon}} + H_{\text{int}}.$$  (3.1)

Here $H_J = -\sum_{j=1}^{N} E_J \cos \phi_j$ is the Josephson coupling energy, where $\phi_j$ is the gauge-
invariant phase difference across the $j$th junction and $E_J = \hbar I_c/q$. The critical
current of the Josephson junction is $I_c$, and $q$ is the magnitude of the charge of
superconducting charge carriers. The capacitive energy of the array is $H_C$, which we
assume can be written in the form $H_C = E_C \sum_{j=1}^{N} n_j^2$, where $E_C = q^2/(2C)$, $C$ is
the junction capacitance, and $n_j$ is the difference in the number of charge carriers
between the two grains connected by the $j$th junction.

The field energy may be written as $H_{\text{photon}} = \hbar \Omega (a^\dagger a + 1/2)$, where $\Omega$ is the
frequency of the cavity resonant mode (assumed to be the only mode supported by
the cavity), and $a^\dagger$ and $a$ are the usual photon creation and annihilation operators,
satisfying the commutation relations $[a, a^\dagger] = 1; [a, a] = [a^\dagger, a^\dagger] = 0$. We assume that
the number operator $n_j$ and phase $\phi_k$ have commutation relations $[n_j, \exp(\pm i\phi_k)] =
\pm \exp(\pm i\phi_j) \delta_{jk}$, which implies that $n_j$ can be represented as $-i\partial/\partial\phi_j$. 

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Figure 3.1: Schematic of the geometry used in our calculations, consisting of a linear array of underdamped Josephson junctions coupled to a resonant cavity, and subjected to an applied voltage $\Phi$. The array consists of $N$ grains, represented by the dots, coupled together by Josephson junctions, represented by the crosses. Nearest-neighbor grains $j$ and $j + 1$ are connected by a Josephson junction, and the cavity is assumed to support a single resonant photonic mode.

The crucial term in the Hamiltonian for phase locking is the interaction term $H_{\text{int}}$. We write this in the form $H_{\text{int}} = (1/c) \int J \cdot A d^3x$, where $J$ is the Josephson current density, $A$ is the vector potential corresponding to the electric field of the cavity mode, $c$ is the speed of light, and the integral is carried out over the cavity volume. Since $J$ is comprised of the Josephson currents $I_c \sin \phi_j$ passing through the junctions, we may write this last term as

$$H_{\text{int}} = E_j \sum_{j=1}^{N} A_j \sin \phi_j,$$

(3.2)
where

\[
A_j = \frac{(2\pi/\Phi_0)}{\int_{j}^{(j+1)}Ax} A \cdot ds
\]  

(3.3)

is the integral across the \(j\)th junction and \(\Phi_0 = \hbar c/q\) is the flux quantum. Alternatively, the Josephson coupling energy in the presence of a vector potential is \(-\frac{1}{2} \sum_j E_j \cos(\theta_j - A_j)\). By expanding the cosine to first order in the \(A_j\)'s, we obtain Eq. (3.2), as well.

The vector potential \(A\) may be expressed in terms of the creation and annihilation operators for the photon quanta [54] as

\[
A = \sqrt{\frac{\hbar c^2}{\Omega}} E_m(a - a^\dagger).
\]  

(3.4)

The electric field cavity mode \(E_m\) satisfies the normalization integral over the cavity volume \(V\),

\[
1 = \int_V |E_m|^2 d^3x.
\]  

(3.5)

And then, by substituting into Eq. (3.3), we find

\[
A_j = \sqrt{\frac{\hbar c^2}{\Omega}} i(a - a^\dagger)\alpha_j,
\]  

(3.6)

with the mode coupling constant,

\[
\alpha_j = \frac{(2\pi/\Phi_0)}{\int_{jAx}^{(j+1)Ax} E_m \cdot ds}.
\]  

(3.7)

Finally, we need to impose boundary conditions on this linear array. For our assumed form of capacitive energy, the voltage \(\Phi_j\) across the \(j\)th junction may be written \(\Phi_j = q\langle n_j \rangle/C\), where \(\langle \ldots \rangle\) denotes a quantum-statistical average. We impose the constant-voltage boundary condition by using the method of Lagrange multipliers,
adding to the Hamiltonian a term of the form \( \mu \sum_{j=1}^{N} n_j \), where the constant \( \mu \) will be determined later by specifying the total voltage drop \( \Phi = \sum_{j=1}^{N} \Phi_j \) across the linear array. Then assuming for simplicity that all the constants \( \alpha_j \) have the same value \( \alpha \) for our linear array, and introducing the notation

\[
\hbar g / \sqrt{V} = E_J \alpha \sqrt{\hbar c^2 / (2\Omega)},
\]

where \( V \) is the cavity volume, we may express the quantity to be minimized in the form

\[
H' = H + \mu \sum_{j=1}^{N} n_j
= \hbar \Omega \left( a^\dagger a + \frac{1}{2} \right)
+ \sum_{j=1}^{N} \left[ -E_J \cos \phi_j + E_C n_j^2 + \frac{\hbar g}{\sqrt{V}} i(a - a^\dagger) \sin \phi_j + \mu n_j \right].
\]

The eigenstates of \( H' \) are many-body wave functions, depending on the phase variables \( \phi_j \) and \( n_j \), and the photon coordinates \( a \) and \( a^\dagger \). We estimate the ground state using a mean-field approximation. To define this approximation, we express \( H' \) in the form

\[
H' = H_{\text{phase}} + H_{\text{photon}} + H_{\text{int}},
\]

where \( H_{\text{phase}} = \sum_{j=1}^{N} (-E_J \cos \phi_j + E_C n_j^2 + \mu n_j) \), and \( H_{\text{int}} = i(\hbar g / \sqrt{V})(a - a^\dagger) \sum_{j=1}^{N} \sin \phi_j \). The mean-field approximation consists of writing

\[
H_{\text{int}} \approx \frac{i \hbar g}{\sqrt{V}} \left( \langle a - a^\dagger \rangle \sum_{j=1}^{N} \sin \phi_j + (a - a^\dagger) \sum_{j=1}^{N} \langle \sin \phi_j \rangle \\
+ (a - a^\dagger) \sum_{j=1}^{N} \langle \sin \phi_j \rangle \right).
\]

This approximation is equivalent to retaining terms only through first order in fluctuations about the mean. Specifically, if \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are operators depending respectively
on the photon and phase variables, and if \( \delta O_j = O_j - \langle O_j \rangle \), then the approximation retains all terms in product of the form \( O_1 O_2 \) through first order in the \( \delta O_j \)'s (hence eliminating the term \( \delta O_1 \delta O_2 \)).

With this approximation, \( H' \) is decomposed into a sum of one-body terms, each of which depends only on the photon variables or on the phase variables of one junction, plus a constant term. The eigenstates of \( H' \), in this approximation, are of the form

\[
\Psi(a, a^\dagger; \{\phi_j\}) = \psi_{\text{photon}}(a, a^\dagger) \prod_{j=1}^N \psi_j(\phi_j),
\]

where \( \psi_{\text{photon}} \) and the \( \psi_j \)'s are one-body wave functions of the photon and phase variables, respectively.

That part of \( H' \) which depends on photon variables may be written

\[
H'_{\text{photon}} = H_{\text{photon}} + i(\hbar g/\sqrt{V})(a - a^\dagger) \sum_{j=1}^N (\sin \phi_j),
\]

where \( (\sin \phi_j) \) denotes a quantum-mechanical expectation value with respect to \( \psi_j(\phi_j) \). Henceforth, we assume that \( \psi_j \) and hence \( (\sin \phi_j) \) are independent of \( j \). With the definition \( \lambda = \langle \exp i\phi_j \rangle \), \( H'_{\text{photon}} \) takes the form

\[
H'_{\text{photon}} = \hbar \Omega \left( a^\dagger a + \frac{1}{2} \right) - \frac{N\hbar g\bar{\lambda}}{\sqrt{V}}(a - a^\dagger),
\]

where \( \bar{\lambda} = (\sin \phi_j) = (\lambda - \lambda^*)/(2i) \). This is the Hamiltonian of a displaced harmonic oscillator; its ground state energy eigenvalue \( E_{\text{photon};0} \) is readily found by completing the square with \( b^\dagger = a^\dagger + iNg\bar{\lambda}/(\Omega\sqrt{V}) \).

\[
H'_{\text{photon}} = \hbar \Omega \left( b^\dagger b + \frac{1}{2} \right) - \frac{N^2 g^2 \hbar \bar{\lambda}^2}{\Omega V}.
\]

The resulting ground state energy of \( H'_{\text{photon}} \) is

\[
E_{\text{photon};0} = \frac{1}{2} \hbar \Omega - N^2 E_{J0} \bar{\lambda}^2,
\]

where we have written

\[
E_{J0} = \hbar g^2/(\Omega V).
\]
Similarly, in the ground state, since \( \langle b^\dagger \rangle = 0 \),

\[
\langle a^\dagger \rangle = -i \frac{N g \tilde{\lambda}}{\Omega \sqrt{V}}.
\]  

(3.17)

The wave function \( \psi_j(\phi_j) \) is an eigenstate of the effective single-particle Schrödinger equation \( H_j \psi_j = E_j \psi_j \), where

\[
H_j = -E_j \cos \phi_j + E_C (n_j^2 + \mu n_j / E_C) + \frac{\hbar g}{\sqrt{\Omega}} i \sin \phi_j (a - a^\dagger),
\]  

(3.18)

and \( (a - a^\dagger) = 2i N g \tilde{\lambda} / (\Omega \sqrt{V}) \) from Eq. (3.17). Using this expression and completing the square, we can write

\[
H_j = E_C (n_j - \bar{n})^2 - E_C \bar{n}^2 + 2N \tilde{\lambda} E_{j0} \sin \phi_j - E_j \cos \phi_j,
\]  

(3.19)

where \( \bar{n} = -\mu / (2E_C) \). Introducing the notation

\[
E_{\alpha}^2 = E_j^2 + E_{\text{int}}^2,
\]  

(3.20)

where

\[
E_{\text{int}} = 2N \tilde{\lambda} E_{j0},
\]  

(3.21)

and \( \phi_\alpha = \tan^{-1}(E_{\text{int}} / E_j) \), we obtain

\[
H_j = -E_\alpha \cos(\phi_j - \phi_\alpha) + E_C (n_j - \bar{n})^2 - E_C \bar{n}^2.
\]  

(3.22)

Although Eq. (3.22) can be transformed into Mathieu's equation by a suitable change of variables, it is more convenient to solve it directly. Because \( H_j \) is periodic in \( \phi_j \) with period \( 2\pi \), its eigenstates must be Bloch functions. But since \( \phi_j \) and \( \phi_j + 2\pi \) are the same physical state, we seek particular Bloch states which are also \( 2\pi \)-periodic. The most general such solution may be written as a Fourier series:
\psi_j(\phi_j) = \sum_{\ell=-\infty}^{\infty} c_\ell \exp(\imath \ell \phi_j). \] The energies \( E_{\text{phase}} \) and coefficients \{c_\ell\} can then be calculated as the eigenvalues and eigenvectors of a Hermitian matrix \( A \), such that

\[ A_{\ell,\ell'} = E_C(\ell - \bar{n})^2 \delta_{\ell,\ell'} + \frac{1}{2} E_\alpha [\delta_{\ell,\ell+1} \exp(\imath \phi_\alpha) + \delta_{\ell,\ell-1} \exp(-\imath \phi_\alpha)]. \] (3.23)

The total ground-state energy of the coupled system is \( E_{\text{total}} = N E_{\text{phase};0} + E_{\text{photon};0} - E_d \). Here \( E_{\text{phase};0} \) is the lowest eigenvalue of \( A \), or equivalently, of the Schrödinger equation

\[ H_j \psi_j(\phi_j) = E_{\text{phase}} \psi_j(\phi_j). \] (3.24)

Note that \( E_{\text{phase};0} \) is also a function of \( \bar{\lambda} \). \( E_d \) is a "double-counting correction" to compensate for having included the interaction energy in both \( H_{\text{photon}} \) and the \( H_j \)'s; it is given by the last term on the right-hand side of Eq. (3.11), i.e.,

\[ E_d = \frac{\hbar g}{\sqrt{V}} \langle a - a^\dagger \rangle \sum_{j=1}^N \langle \sin \phi_j \rangle = -2 N^2 \bar{\lambda}^2 E_J 0. \] (3.25)

Hence, the total ground-state energy is

\[ E_{\text{total}}(\bar{\lambda}) = N E_{\text{phase};0}(\bar{\lambda}) + E_{\text{photon};0}(\bar{\lambda}) + E_d(\bar{\lambda}) \]
\[ = N E_{\text{phase};0}(\bar{\lambda}) + \frac{1}{2} \hbar \Omega + N^2 E_J 0 \bar{\lambda}^2. \] (3.26)

The actual ground-state energy is found from this expression by minimizing \( E_{\text{total}} \) with respect to \( \bar{\lambda} \), holding \( \bar{n} \) fixed.

Finally, the constant \( \mu \) is determined by imposing a fixed applied voltage. In our approximation for the capacitive energy, we have

\[ \Phi = \frac{q}{C} \sum_{j=1}^N \langle n_j \rangle = \frac{q}{C} \langle \frac{\partial}{\partial \mu} H' \rangle = \frac{q}{C} \frac{\partial}{\partial \mu} \langle H' \rangle, \] (3.27)

where \( \langle O \rangle \) denotes an average of a quantity \( O \) with respect to the ground-state wave function, and it is understood that the wave function is obtained at the value of \( \bar{\lambda} \).
which minimizes $E_{\text{total}}$. The last equality in Eq. (3.27) can be proven as follows. First, note that $H'$ is of the form $H' = A + \mu B$, where both $A$ and $B$ ($= \sum_{j=1}^{N} n_j$) are Hermitian operators, and $\mu$ is real. Hence $\langle B \rangle = \langle \partial H' / \partial \mu \rangle = \int d\Gamma \Psi^\dagger (\partial H'/\partial \mu) \Psi$, where $\Psi$ is the ground state wave function and $d\Gamma$ denotes an integration over the relevant phase space. But if $E_{\text{total}}(\mu)$ is an eigenvalue of $H'$, then

$$
\frac{\partial}{\partial \mu} E_{\text{total}} = \frac{\partial}{\partial \mu} \int d\Gamma \Psi^\dagger H' \Psi \\
= \int d\Gamma \Psi^\dagger B \Psi + \int d\Gamma \left[ \left( \frac{\partial}{\partial \mu} \Psi^\dagger \right) H' \Psi + \Psi^\dagger H' \frac{\partial}{\partial \mu} \Psi \right]. \tag{3.28}
$$

Using the fact that $H'$ is Hermitian, and that $E_{\text{total}}$ satisfies $H' \Psi = E_{\text{total}}(\mu) \Psi$, we can rewrite this last integral as

$$
E_{\text{total}}(\mu) \int d\Gamma \left[ \left( \frac{\partial}{\partial \mu} \Psi^\dagger \right) \Psi + \Psi^\dagger \frac{\partial}{\partial \mu} \Psi \right] = E_{\text{total}}(\mu) \frac{\partial}{\partial \mu} \int d\Gamma \Psi^\dagger \Psi = 0, \tag{3.29}
$$

since $\Psi$ is normalized so that $\int d\Gamma \Psi^\dagger \Psi = 1$. Hence,

$$
\frac{\partial}{\partial \mu} \langle H' \rangle = \langle B \rangle = \langle \frac{\partial}{\partial \mu} H' \rangle, \tag{3.30}
$$

and the last equality in Eq. (3.27) is valid as stated. Thus, the value of $\mu$, or equivalently $\bar{n} = -\mu/(2E_C)$, is now determined for a given $\Phi$ by the condition $\partial E_{\text{total}} / \partial \mu = C \Phi / q$.

### 3.2 Numerical Methods

For a specified value of $\bar{n}$, we solve Eq. (3.24) numerically, truncating the matrix $A$ at a finite size. We then use this solution to calculate the ground-state energy $E_{\text{total}}$, Eq. (3.26), and to minimize it numerically with respect to $\lambda$. We find that a $25 \times 25$ matrix is sufficient to find the minimum energy with an error of less than $\pm 10^{-8} E_C$. In our model, $\bar{n}$ is related to the voltage drop $\Phi$ by the implicit relation (3.27).
3.3 Results

In Fig. 3.2, we plot $\tilde{\lambda}_0$, the energy-minimizing value of $\tilde{\lambda}$, as a function of $N$, assuming $\bar{n} = 0$. Two curves are plotted. The full curve shows $\tilde{\lambda}_0$ for the case $E_J = 0$, i.e., no direct Josephson coupling. The dashed curve in Fig. 3.2 shows the minimizing $\tilde{\lambda}$ but for a finite direct Josephson coupling. In both cases, there is clearly a threshold array size $N_c$, below which $\tilde{\lambda}_0 = 0$. For $N > N_c$, $\tilde{\lambda}_0 > 0$. Since $\tilde{\lambda} = \langle \sin \phi_j \rangle$, the Josephson array has a net supercurrent, expressed as current on the voltage leads, when the number of junctions is greater than the threshold. As $N$ increases, $\tilde{\lambda}_0$ approaches unity, which corresponds to complete phase-locking. The value of $N_c$ is larger for finite Josephson coupling than for zero direct coupling; thus, it appears, paradoxically, that the finite direct coupling actually impedes the transition to coherence. This point will be discussed further below.

For $E_J = 0$ and $\bar{n} = 0$, $N_c$ may easily be found analytically. Near the threshold, $|\tilde{\lambda}| \ll 1$, and the lowest eigenvalue $E_{\text{phase};0}$ of Eq. (3.24) is approximately $-(E_C/2)(2N\tilde{\lambda}E_J/E_C)^2$. This eigenvalue follows from the properties of Mathieu functions [55]. The value of $\tilde{\lambda}$ becomes nonzero when $\partial^2 E_{\text{total}}(\tilde{\lambda})/\partial \tilde{\lambda}^2 = 0$, or

$$2NE_{J0} = E_C.$$  \hspace{1cm} (3.31)

This value agrees quite well with our numerical results (cf. Fig. 3.2). Note that, for any nonzero value of the coupling $E_{J0}$, no matter how small, there always exists a threshold value of $N$, above which phase coherence becomes established.

The inset to Fig. 3.2 shows the total energy in the photon mode, $E_{\text{photon}} = \hbar \Omega (\langle a^\dagger a \rangle + 1/2)$, plotted as a function of $N$ for $\bar{n} = 0$. From Eq. (3.26), we find that $E_{\text{photon}} = \hbar \Omega/2 + N^2\tilde{\lambda}^2E_{J0}$, which is the quantity plotted in the inset. As is
Figure 3.2: Minimizing Coherence Order Parameter vs. Number of Junctions, for two values of the direct Josephson coupling energy $E_J$. The value of the order parameter which minimizes $E_{\text{total}}(\bar{\lambda})$ is $\bar{\lambda}_0$. Other parameters are $\hbar g/\sqrt{V} = 0.3E_C$, $\hbar\Omega/2 = 2.6E_C$, and $\bar{n} = 0$. The coupling parameter $E_{J0}$, Eq. (3.16), is given by $E_{J0} \approx 0.017E_C$. Inset: Total Photon Energy vs. the Square of the Number of Junctions, for the same parameters and the same two values of $E_J$.

It is evident from the plot, $E_{\text{photon}}$ varies approximately linearly with $N^2$ all the way from the coherence threshold to large values of $N$, where $\bar{\lambda} \to 1$. This $N^2$ dependence is a hallmark of phase coherence.

The voltage $\Phi$ across the array, Eq. (3.27), is not easily fixed as an independent variable within the present approach. Instead, we consider $\bar{n}$ (or equivalently $\mu$) as an independent variable and plot several derived quantities of interest; the results are shown in Figs. 3.3 and 3.4. In Fig. 3.3, we plot $\bar{\lambda}_0$ as a function of $\bar{n}$ for several array
sizes at fixed coupling constants $E_{J0}$, $E_C$, and $E_J$. As expected, $\tilde{\lambda}_0$ is periodic in $\bar{n}$ with a period of unity. This periodicity follows directly from the phase part of the Hamiltonian, Eq. (3.22), or equivalently, from Eq. (3.23), which indicates that the two matrices $A(\bar{n} + 1)$ and $A(\bar{n})$ are identical.

Figure 3.3 shows that, for any given $E_{J0}$ and $N$, the order parameter $\tilde{\lambda}_0$ achieves its maximum value when $\bar{n}$ has a half-integer value, i.e., the array is most easily made coherent at such values of $\bar{n}$. In particular, an array whose size is slightly below the threshold value at integer values of $\bar{n}$ can be made to become coherent, with a finite $\tilde{\lambda}_0$, when $\bar{n}$ is increased—that is, when a suitable voltage is applied. On the other hand, for values of $N$ far above the threshold, $\tilde{\lambda}_0$ is little affected by a change in $\bar{n}$.

In Fig. 3.4, we show the quantity $\langle n_j \rangle$ as a function of $\bar{n}$, for several values of $N$ and fixed value of the coupling constant ratios $E_{J0}/E_C$ and $E_J/E_C$. This quantity is related to the voltage drop across one junction, in our model, by $\Phi/N = q\langle n_j \rangle/C$. For sufficiently large arrays, $\langle n_j \rangle \approx \bar{n}$ and the voltage drop is nearly linear in $\bar{n}$ in this mean-field approximation. For arrays closer to the coherence threshold, $\langle n_j \rangle$ is a highly nonlinear function of $\bar{n}$. However, the deviation from linearity, $\langle n_j \rangle - \bar{n}$, is, once again, a periodic function of $\bar{n}$ with period unity. The discontinuous jumps in $\bar{n}$ as a function of $\langle n_j \rangle$ are not related to the voltage plateaus seen experimentally. Instead, they represent regions of incoherence ($\tilde{\lambda}_0 = 0$), whereas the regions in which $\langle n_j \rangle$ is a smooth function of $\bar{n}$ are regimes of phase coherence.

In Fig. 3.5, we again plot $\tilde{\lambda}_0(N)$ for two fixed ratios $E_J/E_C$, but this time for $\bar{n} = 1/2$. From Fig. 3.3, we expect this choice of $\bar{n}$ to maximize the tendency for phase coherence and thus to reduce the threshold array size for the onset of phase coherence. Indeed, in the absence of direct Josephson coupling, this threshold is
Figure 3.3: Minimizing Coherence Order Parameter vs. Voltage Parameter, for several values of the array size $N$. The voltage parameter is $\bar{n} = \mu / (2E_C)$. Other parameters are $\hbar g / \sqrt{V} = 0.3E_C$, $\hbar \Omega / 2 = 2.6E_C$, $E_J = 0$, and $E_{J0} \approx 0.017E_C$.

reduced to below unity (that is, $\bar{\lambda}_0$ remains finite, even at $N = 1$, for our choice of $E_{J0}$). In fact, for this value of $\bar{n}$, only an infinitesimal coupling to the resonant mode is required to induce phase coherence in this model. Once again (cf. Fig. 3.5), the addition of a finite direct Josephson coupling actually increases the threshold number for phase coherence at $\bar{n} = 1/2$ as it does at $\bar{n} = 0$.

3.4 Discussion

Although the present work is only a mean-field approximation, we expect that it will be quite accurate for large $N$. The reason for this accuracy is that in this model
Figure 3.4: Voltage vs. Voltage Parameter, for several values of the array size $N$. The voltage is $\langle n_j \rangle = \Phi C/(qN)$, where $\Phi/N$ is the voltage drop across one junction, and the voltage parameter is $\bar{n} = \mu/(2E_C)$. Other parameters are $\hbar g/\sqrt{V} = 0.3E_C$, $\hbar \Omega/2 = 2.6E_C$, $E_J = 0$, and $E_{J0} \approx 0.017E_C$. Numerically, for $N = 20$, many values of $\bar{n}$ produce the same $\langle n_j \rangle$.

the one photonic degree of freedom is coupled to every phase difference, and thus experiences an environment which is very close to the mean, whatever the state of the individual junctions. Such small fluctuations are necessary in order for a mean-field approach to be accurate. In fact, this approach is not unlike that used in Section 2.3 to describe Josephson-coupled wire arrays.

It may appear surprising that a finite, direct Josephson coupling actually increases the threshold array size for coherence. But upon closer examination this behavior is reasonable. If there is no direct coupling ($E_J = 0$), the phase difference across each
Figure 3.5: Minimizing Coherence Order Parameter vs. Number of Junctions, for two values of the direct Josephson coupling energy $E_J$. All parameters are identical to those in Fig. 3.2, except for $\tilde{n} = 1/2$.

A junction evolves independently, except for the global coupling to the resonant photon mode. When the array exceeds its critical size, this coupling produces coherence. If the same array now has a finite $E_J$, there are two coupling terms. These are not simply additive, but are $\pi/2$ out of phase: the direct coupling favors $\phi_j = 0$, while the photonic one favors $\phi_j = \pi/2$. For a large enough array, the coupling to the photon field still predominates and produces global phase coherence, but this occurs at a higher threshold, in our model, than in the absence of direct coupling.

It is worthwhile to explain briefly what is meant by the approximation $E_J = 0$, since the coupling constant $g$ would also vanish if $E_J$ were strictly zero. But from
the definition of $E_\alpha$, Eq. (3.20), it is clear that this approximation is sufficient so long as the inequality $|E_J| \ll |E_{\text{int}}|$ is valid. From our definitions of these quantities, Eqs. (3.16) and (3.21), we can reexpress this condition as the inequality

$$E_J \ll 2N^\lambda \frac{\hbar g^2}{\Omega V}. \quad (3.32)$$

This condition may well be satisfied at large $N$ even if it is less well obeyed at small numbers of junctions. Thus, it may be numerically reasonable to consider the limit $E_J = 0$ in our mean-field approximation, as we have described above.

A striking feature of our results is the very low coherence threshold ($N = 1$) when $\bar{n} = 1/2$. In fact, for any $N$, only an infinitesimal coupling to the cavity mode is required to induce phase coherence at this value of $\bar{n}$. The reason for this low threshold is that, in the absence of coupling, junction states with $n_J = 0$ and $n_J = 1$ are degenerate. Any coupling is therefore sufficient to break the degeneracy and produce phase coherence. A similar effect has been noted previously in studies of more conventional Josephson junction arrays in the presence of an offset voltage [56, 57].

One aspect of the experiment which still remains to be accounted for in this model is the resonance condition seen experimentally, where it is found that the voltage drop across the array takes on discrete step-like values which are integer multiples of $\hbar \Omega / q$. It is possible that these voltage steps are beyond the reach of our Hamiltonian model, even if we were able to go beyond the mean-field approximation. We believe that the steps may only appear in a model which includes both a fixed-bias-current boundary condition and some damping. Another possibility is that the steps would emerge from a suitable dynamical model.

The present work can be readily extended in a number of ways. Clearly, we can generalize it to include more complex (i.e., two dimensional) geometries and
junction-photon interactions which incorporate the fact that the cavity field varies from one junction to another. It is also straightforward to include the dynamical effects mentioned above, by writing out the Heisenberg equations of motions for the Hamiltonian (3.9). That is, from

\[ \dot{\mathcal{O}} = \frac{1}{i\hbar}[\mathcal{O}, H], \quad (3.33) \]

for a relevant operator \( \mathcal{O} \), the equations of motion of the Hamiltonian are

\[ \ddot{a} = -\Omega^2 a + \frac{g\Omega}{\sqrt{V}} \sum_{j=1}^{N} \sin \phi_j, \]

\[ \dot{n}_j = -\frac{E_j}{\hbar} \sin \phi_j + 2\frac{g}{\sqrt{V}} a \cos \phi_j, \]

\[ \dot{\phi}_j = 2\frac{E_C}{\hbar} n_j + \frac{\mu}{\hbar}. \quad (3.34) \]

We define \( \bar{a} = (a - a^\dagger)/(2i) \), so that all of the operators are purely real.

In the limit of large numbers of photons, these equations can be treated classically, as done in Ref. [53] for the Jaynes-Cummings [50] model in one dimension. We believe that this approach would be a promising way to extend the present work by explicitly including the effects of driving currents and damping. Finally, one can extend the present approach to finite temperatures, where the mean-field approximation should lead to a finite-temperature transition between incoherence and coherence, as the temperature is reduced at fixed array parameters (i.e., at fixed \( N, E_J, E_C, \) and \( E_J \)).
CHAPTER 4

CONCLUSION

We have considered two systems. The first is an array of superconducting wires, divided into two layers so that each wire interacts with every wire in the opposing layer. The second is an array of Josephson junctions, each of which interacts with the photon mode of the surrounding cavity. Although the two systems are geometrically dissimilar, they actually exhibit very similar behavior.

Both are systems in which a mean field theory is very applicable. In either system, each degree of freedom interacts with many other degrees of freedom; this is the basis for calling the interactions long-range. In the wire array, the interaction is a direct Josephson coupling between a wire in one layer and all of the wires in the opposing layer. In the photon-coupled junction array, there is an indirect interaction with all of the junctions, via coupling with the photon mode. In either case, there are other interactions: within a single wire, or direct Josephon coupling between neighboring junctions.

In both systems, the MFT behavior predominates over the other interactions, but the other interactions do limit the range of the MFT. For the wire arrays, we find the condition (2.73) limits the effective size of the array in terms of the coupling between the layers and the magnetic field. Arrays larger than this size cannot reach
a coherent state for incommensurate fields. The dynamics of the system confirm this result, Eq. (2.38).

The junction array within the cavity requires a threshold number of junctions to become coherent. This threshold is increased by the direct Josephson coupling within the array. The effect of the direct coupling is only overcome at larger array sizes, as discussed in Section 3.4.

In conclusion, this work has investigated the behavior of two Josephson systems whose long-range interactions allow mean field theories to effectively explain the behavior of the systems, under realizable conditions. We have considered the thermodynamics of both systems, within a mean field theory for each. We have investigated the dynamics of the first system, considering two models.
APPENDIX A

JOSEPHSON RELATIONS

The Josephson relations (1.1) are the fundamental equations governing Josephson coupled systems, and can be derived from the phenomenological Ginzburg-Landau (GL) theory of superconductivity. See references [2, 58] for more complete treatments. In this theory, the superconducting state is described by the complex scalar order parameter $\psi(x, t)$. The GL theory is valid for temperatures near $T_c$ and for small spatial variations in $\psi$ [58].

Under the gauge transformation of the electromagnetic potentials,

$$
\Phi' = \Phi - \frac{1}{c} \frac{\partial}{\partial t} \chi,
$$

$$
A' = A + \nabla \chi,
$$

the order parameter transforms as [2]

$$
\psi' = \psi \exp[iq\chi/(\hbar c)].
$$

where $\chi(x, t)$ is a scalar function chosen for a particular gauge. From the general form of the complex order parameter $\psi = |\psi| \exp(i\phi)$, the gauge transformation of the parameter is

$$
|\psi'| \exp(i\phi') = |\psi| \exp[i(\phi + 2\pi \chi/\Phi_0)].
$$
Or,
\[ \phi' = \phi + \frac{2\pi}{\Phi_0} \chi, \]  
(A.4)

while the magnitude of order parameter remains constant under the transformation; 
\[ |\psi'| = |\psi|. \]  
We have introduced the constant \( \Phi_0 = hc/q, \) the flux quantum.

It is often useful to transform to a gauge in which the vector potential vanishes; 
\[ A' = 0. \]  
That is, we choose
\[ \nabla \chi = -A, \]  
(A.5)

so that
\[ \phi' = \phi - \frac{2\pi}{\Phi_0} \int A \cdot ds. \]  
(A.6)

This transformation allows us to directly account for the effects of an applied magnetic 
on the phase of the order parameter.

### A.1 Voltage Relation

For a superconductor in equilibrium, one might posit that the order parameter 
\( \psi \) is constant in time. However, this cannot be the case, since under certain gauge 
transformations time variation can be introduced via \( \chi. \) We can exclude any variation 
of \( \chi \) with respect to position, because, by Eq. (A.1), this would imply \( \Phi \) is also 
a function of position. But within a superconductor the scalar potential must be 
uniform. So, the general form of the order parameter of a superconductor in a steady 
state [2] is

\[ \psi(x, t) = \psi(x) \exp[i\alpha(t)], \]  
(A.7)
where \( \alpha(t) \) is a function of time.

Let us suppose we have two similar superconducting regions with order parameters \( \psi_1 = |\psi_1| \exp(i\phi_1) \) and \( \psi_2 = |\psi_2| \exp(i\phi_2) \). The regions are weakly coupled together and in equilibrium, so that we can assume the magnitudes are the same; i.e., \( |\psi_1| = |\psi_2| \). In other words, the charge carrier density \( n_s = |\psi|^2 \) is the same in both regions. But, the phases of the complex order parameters may be different. Let us choose a gauge such that we have

\[
\psi_1(x, t) = \psi_1(x), \\
\psi_2(x, t) = \psi_2(x) \exp[i\alpha(t)]. \tag{A.8}
\]

An in this gauge, we have potentials \( \Phi_1 = \Phi_1 + \Delta V \). Then, applying a gauge transformation with \( \chi = ct\Delta V \), we obtain \( \Phi'_1 = \Phi_1 - \Delta V \) and \( \Phi'_2 = \Phi_1 \), and hence

\[
\psi'_1(x, t) = \psi_1(x) \exp(iq\Delta V t/\hbar), \\
\psi'_2(x, t) = \psi_2(x) \exp\{i[\alpha(t) + q\Delta V t/\hbar]\}. \tag{A.9}
\]

After the gauge transformation, if we interchange indices 1 \( \leftrightarrow \) 2 and replace \( \Delta V \) by \( -\Delta V \), then the order parameters and potentials will the same as with those we started. Therefore, we must have \( \alpha(t) = q\Delta V t/\hbar \). If we note that \( \alpha(t) \) is the gauge-invariant phase difference between the two regions, we have

\[
\Delta V = \frac{\hbar}{q} \frac{d}{dt} \Delta \theta. \tag{A.10}
\]

This is the Josephson voltage relation.

### A.2 Current Relation

Starting with the same two superconducting regions, let us define \( L \) as the width of the separation between the regions and \( A \) as the cross-sectional area of the separation.
The GL equation, assuming that the z-direction is normal to the interface, is

\[ 0 = \xi^2 \frac{d^2 \psi}{dz^2} + \psi - \psi^3. \]  

(A.11)

where \( \xi \) is the GL coherence length [58]. If \( L \ll \xi \), we can approximate Eq. (A.11) as \( d^2\psi/dz^2 = 0 \). The general solution to the GL equation in this approximation, with boundary conditions (A.8) for the interface between the regions is

\[ \psi = \left( 1 - \frac{z}{L} \right) + \left( \frac{z}{L} \right) \exp[i\alpha(t)]. \]  

(A.12)

The current through the interface will be

\[ I_s = -\frac{q\hbar A}{2m} \left( \psi^* \frac{d}{dz} \psi - \psi \frac{d}{dz} \psi^* \right) = \frac{q\hbar A}{mL} \sin \alpha(t). \]  

(A.13)

Again noting that \( \alpha \) is the gauge-invariant phase difference between the regions and substituting \( I_c = q\hbar A/(mL) \), we arrive at the Josephson current relation

\[ I_s = I_c \sin \Delta \theta. \]  

(A.14)

### A.3 Energy Relation

The energy associated with a Josephson junction can be easily derived from the dynamical Josephson relations. The energy of the junction is the work needed to change the phase difference across the junction from zero to \( \Delta \theta \),

\[ E = \int dt I_s V = \int dt I_c \sin \phi \frac{\hbar}{q} \frac{d}{dt} \phi, \]  

(A.15)

which becomes

\[ E(\Delta \theta) = \frac{\hbar}{q} I_c \int_0^{\Delta \theta} d\phi \sin \phi = K - E_J \cos \Delta \theta. \]  

(A.16)
The coupling energy of Josephson junction is thus $E_J = \hbar I_c/q$, and $K$ is an arbitrary constant of integration. The junction energy $E$ is minimized when the phase difference across it vanishes.
APPENDIX B

RUNGE-KUTTA METHOD

A first-order differential equation,

\[ \frac{dx}{dt} = F(t, x), \quad (B.1) \]

can be solved numerically with the approximation

\[ x_{n+1} = x_n + \Delta t F(t_n, x_n), \quad (B.2) \]

known as the Euler method. We start with an initial value of \((t_0, x_0)\), and at each discrete step in time \((t_n, x_n)\), use the derivative function \(F\) to calculate the next step \((t_{n+1}, x_{n+1})\). For a small enough step-size \(\Delta t\), the values of \(x_n\) will approach the exact solution to Eq. (B.1). It can be shown that the error inherent to this method goes as \(\Delta t\); i.e., this is a first-order approximation. Thus, halving the step-size, halves the error.

More sophisticated algorithms exist [29]. The most commonly used are Runge-Kutta algorithms. These algorithms, instead of using one estimate for each step, use multiple estimates for each step. Several calculations are made for each step, and a weighted average of these estimates is used to advance to the next step. Although
the multiple estimates require more computation, they have the advantage of having inherent errors of higher orders, so that better accuracy is achieved per unit of computation time.

There are many varieties of the basic Runge-Kutta integration algorithm. The one used for most of the dynamic results calculated in this research is a fourth-order method with an embedded fifth-order calculation. By comparing the fourth- and fifth-order calculations, we can make an estimate of the error inherent to the fourth-order calculation. This allows the algorithm to adjust its step-size of integration and find a good compromise between minimizing the number of steps and minimizing the error. The algorithm ultimately provides a fourth-order solution, with a fifth-order error estimate. An outline of the algorithm follows.

The equations to be solved are written as a set of first-order differential equations (written in vector notation for conciseness),

\[
\frac{d}{dt} \mathbf{x} = \mathbf{F}(t, \mathbf{x}). \quad (B.3)
\]

We then calculate six estimates for the next step using the current step \((t_n, \mathbf{x}_n)\) and step-size \(\Delta t_n\):

\[
\begin{align*}
F_1 &= \Delta t_n \mathbf{F}(t_n + c_{11} \Delta t_n, \mathbf{x}_n), \\
F_2 &= \Delta t_n \mathbf{F}(t_n + c_{21} \Delta t_n, \mathbf{x}_n + c_{21} F_1), \\
F_3 &= \Delta t_n \mathbf{F}(t_n + c_{31} \Delta t_n, \mathbf{x}_n + c_{31} F_1 + c_{32} F_2), \\
F_4 &= \Delta t_n \mathbf{F}(t_n + c_{41} \Delta t_n, \mathbf{x}_n + c_{41} F_1 + c_{42} F_2 + c_{43} F_3), \\
F_5 &= \Delta t_n \mathbf{F}(t_n + c_{51} \Delta t_n, \mathbf{x}_n + c_{51} F_1 + c_{52} F_2 + c_{53} F_3 + c_{54} F_4), \\
F_6 &= \Delta t_n \mathbf{F}(t_n + c_{61} \Delta t_n, \mathbf{x}_n + c_{61} F_1 + c_{62} F_2 + c_{63} F_3 + c_{64} F_4 + c_{65} F_5). \quad (B.4)
\end{align*}
\]
We then calculate an error estimate for the next step,\n\[ \epsilon_n = \frac{1}{\Delta t_n} \max(c_{t_1} F_1 + c_{t_2} F_2 + c_{t_3} F_3 + c_{t_4} F_4 + c_{t_5} F_5 + c_{t_6} F_6). \] \hfill (B.5)\n
Here, \( \max(x) \) is the magnitude of the component of \( x \) with the largest magnitude. We compare this with the desired error \( \epsilon_0 \). If the \( \epsilon_n > \epsilon_0 \) then the step is recalculated with a smaller \( \Delta t_n \). Otherwise we accept the step. In either case the expression
\[ \lambda \Delta t_n \left| \frac{\epsilon_0}{\epsilon_n} \right|^{1/4} \] \hfill (B.6)
gives us either the retry value for \( \Delta t_n \) or the next step size \( \Delta t_{n+1} \). The constant \( \lambda \) with \( 0 < \lambda \leq 1 \) is a safety factor chosen to keep the step-size choice conservative.

After accepting a given step, we have
\[ t_{n+1} = t_n + \Delta t_n, \]
\[ x_{n+1} = x_n + c_{t_1} F_1 + c_{t_2} F_2 + c_{t_3} F_3 + c_{t_4} F_4 + c_{t_5} F_5 + c_{t_6} F_6. \] \hfill (B.7)

Thus, the time steps to the next value, and the corresponding \( x \) is equal to the previous value plus a weighted mean of the estimates. The process continues on for the following steps. The algorithm, so far, as been stated in terms of a plethora of constants. The exact values of these constants depend on which version of the algorithm is used. We have used the coefficients as derived by Cash and Carp [59].

\[
c_{t_1} = 0, \quad c_{t_2} = \frac{1}{4}, \quad c_{t_3} = \frac{3}{8}, \quad c_{t_4} = \frac{12}{13}, \quad c_{t_5} = 1, \quad c_{t_6} = \frac{1}{2}, \quad c_{t_7} = \frac{1}{4}, \quad c_{t_8} = \frac{3}{32}, \quad c_{t_9} = \frac{9}{32}, \quad c_{t_{10}} = \frac{1932}{2197}, \quad c_{t_{11}} = \frac{7200}{2197}, \quad c_{t_{12}} = \frac{7296}{2197}, \quad c_{t_{13}} = \frac{439}{216}, \quad c_{t_{14}} = -8, \quad c_{t_{15}} = \frac{3680}{513}, \quad c_{t_{16}} = \frac{845}{4104}.\n\]
The three zeroes in total among the coefficients for $\epsilon_n$ and $x_{n+1}$ allow for more efficient calculation than other sets of coefficients.
APPENDIX C

SAMPLE COMPUTER CODE

We have written many programs to compute the various numerical quantities of this work. In this Appendix, we will annotate one of the programs used to calculate the results of Section 2.2. This program is representative of the codes used. The program jjlrp computes the average voltage across a Josephson-coupled wire array, using the "finite-inductance" model. The voltage is calculated for a range of bias currents.

The program is written in C++ for a DEC UNIX platform and is divided into three files. We will look at each file individually, describing the code, and then we will discuss the overall operation of the code. The actual files can be downloaded from "http://www.physics.ohio-state.edu/~jkenth/dissertation.html".

C.1 Definition File

The first file is jjlrp_base.h, which contains all of the basic code definitions. After each segment of code there will be a short description of its function.

// jjlrp_base.h 01/30/1997 Kent Harbaugh
//
// Josephson Junction Array with Long-Range Interactions
// with non-constant Phase

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This block contains the function declarations for the vector-pipelining routines from the Fortran DXML library. These library functions allow our code to take full advantage of DEC Alpha CPU.

typedef double real;

const real Pi=M_PI, TwoPi=2.0*Pi;

// in jjlrp_base.cpp
void PrintBanner(void);
void GetParameters(void);
void AllocateMemory(void);
void Initialize(void);
void Function(real Output[]);
void PrintTiming(void);
void InvertMatrix(void);
int IndexPlus(int I, int J);
int IndexMinus(int I, int J);
real &Matrix(int I, int J);
real RandomUniform(void);
real Round(real X, real Y);
int GetInt(const char Prompt[]);
long GetLong(const char Prompt[]);
real GetReal(const char Prompt[]);
void Integrate(void);

// in main file
int main(int argc, char *argv[]);
void GetExtraParameters(void);
void InitialMeasurement(void);
void Measurement(void);
void FinalMeasurement(void);
void SaveData(const char FileName[]);

extern int Nx, Ny, NxNy, Nn, Nn1, Nn, Nx1, Ny1, Nx1p, NxNy1, NxNy1mNx,
  PhaseSeed, NumPt;
extern long MaxStep, Count1, Count2, Timing;
extern size_t VectSz;
extern real AccError, BoundaryCurrent, CriticalCurrent, DeltaTime, IntTime,
  MagneticField, MaxTime, MinTime, Resistance, StepAdj,
  StepError, TotalError, UseDeltaTime, Time;

#endif

In the remainder of this file are declarations for the various functions and global variables to be defined in the other two files.
The second file is `jjlrp_base.cpp` which contains the basic functions of the program. These functions are mostly mathematical or are of generic utility.

```cpp
#include <math.h>
#include <stdio.h>
#include <stdlib.h>
#include <time.h>

#include "jjlrp_base.h"

int Nx, Ny, NxNy, Nn, Nn1, Nm, Nx1, Ny1, Nxp1, NxNy1, NxNy1mNx, PhaseSeed, NumPt;
long MaxStep, Count1, Count2, Timing;
size_t VectSz;
real AccError, BoundaryCurrent, CriticalCurrent, DeltaTime, IntTime,
    MagneticField, MaxTime, MinTime, Resistance, StepAdj, StepError,
    TotalError, UseDeltaTime, Time;
```

Here we include the first file and define the basic global variables. We will describe the utility of these variables as they are used in the code.

```cpp
void PrintBanner(void)
    { puts("\n"
        "# Josephson Junction Array with Long-Range Interactions\n"
```
void GetParameters()
{
Nx=GetInt("Number of junctions in x-dir");
Ny=GetInt("Number of junctions in y-dir");
CriticalCurrent=GetReal("Critical current in xy-plane");
Resistance=GetReal("Resistance in xy-plane");
MagneticField=GetReal("Magnetic field");
BoundaryCurrent=GetReal("Current at boundary");
PhaseSeed=GetInt("Random seed for phase initialization");
MaxStep=GetLong("Maximum number of steps");
MinTime=GetReal("Start-measurement time");
MaxTime=GetReal("Integration time");
DeltaTime=GetReal("Initial step-size");
TotalError=GetReal("Total Error");
StepAdj=GetReal("Step-size adjustment factor");
NumPt=GetInt("Number data points");

GetExtraParameters();

NxNy=Nx*Ny; Nn=NxNy*2; Nn1=Nn-1; Nm=(Nn1*Nn)/2;
Nx1=Nx-1; Ny1=Ny-1; Nxp1=Nx+1; NxNy1=NxNy-1; NxNy1mNx=NxNy-1-Nx;
VectSz=Nn*sizeof(real);
}

The first of these functions prints program identification and copyright. The
second function accepts input for the many parameters necessary to execute the code
properly. Let us look at these in detail.

Nx is the number of junctions along the x-direction; i.e., the number of wires
parallel to the y-axis. It is equivalent to \( M \) in our model.

Ny is the number of junctions along the y-direction; i.e., the number of wires
parallel to the x-axis. It is equivalent to \( N \) in our model.
Critical Current is the normalized critical current of the wires. It is equivalent to \( \lambda \) in our model.

Resistance is the normalized resistance of the wires. It is equivalent to \( \mu \) in our model.

Magnetic Field is the normalized magnetic field applied to the system. It is equivalent to \( f \) in our model.

Boundary Current is the normalized bias current applied to the system. It is equivalent to \( I_B/I_c \) in our model.

Phase Seed is an integer used to initialize the random number generator used to initialize the phases.

Max Step is the maximum number of Runge-Kutta steps the program is allowed to execute before halting.

Min Time is the normalized time the system is given to reach a steady state.

Max Time is the total normalized time the system is integrated over.

DeltaTime is the initial step-size of the Runge-Kutta algorithm.

Total Error is the goal total error to be allowed to accumulate while integrating the equations of motion.

Step Adj is a dimensionless parameter used to keep the adaptive step-sizes conservative. Equivalent to \( \lambda \) in the Runge-Kutta algorithm.

Num Pt is the number of data points that will be recorded.
Additional parameters are set within the third file.

```c
void AllocateMemory(void)
{ Boundary = new real[Nn]; F1 = new real[Nn]; F2 = new real[Nn]; F3 = new real[Nn];
  F4 = new real[Nn]; F5 = new real[Nn]; F6 = new real[Nn]; Finput = new real[Nn];
  Fluxes = new real[Nn]; InitPhases = new real[Nn]; _Matrix = new real[Nn];
  Phases = new real[Nn]; PtXs = new real[NumPt]; PtYs = new real[NumPt];
  Vect1 = new real[Nn]; Vect2 = new real[Nn]; Vect3 = new real[Nn];
  Vect4 = new real[Nn]; Vect5 = new real[Nn]; Vect6 = new real[Nn];
  Zero = new real[Nn];
  if (Boundary == NULL || F1 == NULL || F2 == NULL || F3 == NULL || F4 == NULL ||
      F5 == NULL || F6 == NULL || Finput == NULL || InitPhases == NULL ||
      _Matrix == NULL || Phases == NULL || PtXs == NULL || PtYs == NULL ||
      Vect1 == NULL || Vect2 == NULL || Vect3 == NULL || Vect4 == NULL ||
      Vect5 == NULL || Vect6 == NULL || Zero == NULL)
    { puts("Insufficient memory"); exit(EXIT_FAILURE); }
  Vect2o = Vect2 + 1;
}

void Initialize(void)
{ int one = 1; real zero = 0.0;
  dset(&Nm, &zero, _Matrix, &one);
  real r2 = Resistance + 2.0, r1 = Resistance + 1.0, rm = -Resistance;
  for (int j = 1; j <= Ny; j++)
    { Matrix(IndexPlus(1, j), IndexPlus(1, j)) = r1;
      Matrix(IndexPlus(1, j), IndexMinus(1, j)) = rm;
      Matrix(IndexPlus(1, j), IndexPlus(i + 1, j)) = -1.0;
      for (int i = 2; i <= Nx; i++)
        { Matrix(IndexPlus(i, j), IndexPlus(i - 1, j)) = -1.0;
          Matrix(IndexPlus(i, j), IndexPlus(i, j)) = r2;
          Matrix(IndexPlus(i, j), IndexMinus(i, j)) = rm;
          Matrix(IndexPlus(i, j), IndexPlus(i + 1, j)) = -1.0;
          Matrix(IndexPlus(Nx, j), IndexPlus(Nx - 1, j)) = -1.0;
          Matrix(IndexPlus(Nx, j), IndexPlus(Nx, j)) = r1;
          Matrix(IndexPlus(Nx, j), IndexMinus(Nx, j)) = rm;
        }
    }
  for (int i = 1; i <= Nx; i++)
    { Matrix(IndexMinus(i, 1), IndexMinus(i, 1)) = r1;
      Matrix(IndexMinus(i, 1), IndexPlus(i, 1)) = rm;
      Matrix(IndexMinus(i, 1), IndexMinus(i, 1 + 1)) = -1.0;
      for (int j = 2; j <= Ny; j++)
        { Matrix(IndexMinus(i, j), IndexMinus(i, j - 1)) = -1.0;
          Matrix(IndexMinus(i, j), IndexMinus(i, j)) = r2;
```
Matrix(IndexMinus(i,j),IndexPlus(i,j))=rm;
Matrix(IndexMinus(i,j),IndexMinus(i,j+1))=-1.0;
}
Matrix(IndexMinus(i,Ny),IndexMinus(i,Ny-1))=-1.0;
Matrix(IndexMinus(i,Ny),IndexMinus(i,Ny))=r1;
Matrix(IndexMinus(i,Ny),IndexPlus(i,Ny))=rm;
}
InvertMatrix();
dset_(&Mn, &zero, Zero, &one);
memcpy(Boundary, Zero, VectSz);
Boundary[IndexPlus(1,(Ny+1)/2)]=BoundaryCurrent*Resistance;
Boundary[IndexMinus((Nx+1)/2,Ny)]=BoundaryCurrent*Resistance;
memcpy(Fluxes, Zero, VectSz);
for(int i=1; i<=Nx; i++)
{real flux=TwoPi*MagneticField*(i-1);
 for(int j=1; j<=Ny; j++) Fluxes[IndexMinus(i,j)]=flux;
}
srandom(PhaseSeed);
InitPhases[0]=0.0;
for(int i=1; i<Nn; i++) InitPhases[i]=TwoPi*RandomUniform();
dset_(&NumPt, &zero, PtXs, &one); dset_(&NumPt, &zero, PtYs, &one);
StepError=TotalError/MaxTime;
Count1=0; Count2=0; Timing=0; AccError=0.0; IntTime=0.0;
}

These functions allocate the memory necessary for the program and initialize the variables. The matrix initialization is the most complex. First, the elements of the matrix are calculated. These elements are the coefficients of the first-order terms of Eqs. (2.27)–(2.32). The matrix is then inverted. This, in effect, decouples the first-order terms.
The boundary conditions are also determined. The currents at all of the boundaries are set appropriately; in this case, for the perpendicular configuration. The vector potential terms are calculated for our particular gauge.

The phases are initialized to "random" values. Although the specific pseudo-random function we use is not of particularly high quality, it is sufficient for our purpose. This is the only point in the code which makes use of a random-number generator.

```c
void Function(real Output[]) {
memcpy(Vectl, Zero, VectSz); memcpy(Vect2, Zero, VectSz);
memcpy(Vect3, Zero, VectSz);
int index=0;
for(int j=1; j<=Ny; j++)
    {Vect2[index]=Finput[index+NxNy]-Finput[index];
     Vect3[index]=Finput[index+1]-Finput[index];
     index++;
     for(int i=2; i<=Nx1; i++)
         {Vect1[index]=Finput[index-1]-Finput[index];
          Vect2[index]=Finput[index+NxNy]-Finput[index];
          Vect3[index]=Finput[index+1]-Finput[index];
          index++;
         }
     Vect1[index]=Finput[index-1]-Finput[index];
     Vect2[index]=Finput[index+NxNy]-Finput[index];
     index++;
    }
for(int i=1; i<=Nx; i++)
    {index=i+NxNy1;
     Vect2[index]=Finput[index-NxNy]-Finput[index];
     Vect3[index]=Finput[index+Nx]-Finput[index];
     index+=Nx;
     for(int j=2; j<=Ny1; j++)
         {Vect1[index]=Finput[index-Nx]-Finput[index];
          Vect2[index]=Finput[index-NxNy]-Finput[index];
          Vect3[index]=Finput[index+Nx]-Finput[index];
          index+=Nx;
         }
     Vect1[index]=Finput[index-Nx]-Finput[index];
     Vect2[index]=Finput[index-NxNy]-Finput[index];
}
```

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The function Function computes the values of derivatives of the phases, as a function of the current value of the phases. All of the gauge-invariant phase differences are computed first. Then, the sines of these values are calculated, and terms are added together. The result is the vector $s$ of Eq. (2.33), which is then multiplied by the previously-computed inverse of the matrix $M$. This function has been optimized to make use of the DXML library.

```c
void PrintTiming(void)
{
    printf("\nCount1:=%ld; Count2:=%ld; Timing:=%ld;\n", Count1, Count2, Timing);
    printf("Error:=%g;\n", AccError); putchar('\n');
    real time=Timing/(real)CLOCKS_PER_SEC;
    if(time!=0.0)
    { long s=(long)(time+0.5), m=s/60, h=m/60, d=h/24; h%=24; m%=60; s%=60;
        printf("# Total time = %.3g sec", time);
        if(d || h || m) printf(" "); if(d) printf("%ld", d);
        if(d || h || m) printf("%ld:%02ld:%02ld", h, m, s);
        printf("", Steps per secs = %.3g\n", Count2/time);
    }
    if(Count2!=0)
    { printf("# Step recalculation = %.3g%%, Mean step-size = %.3g\n", 
               (1.0-Count2/(real)Count1)*100.0, IntTime/Count2);
        const long O=1, Omul=0, Odiv=2*0, Osin=7*0, Opow=10*0, Omem=10*0, Oabs=0;
        long Ofunc=(Nn+9*Nn)+O+(Nn+10*Nn)*Omul+3*Nn+Osin+3*Omem,
        Ofunc2=(Nn+9*Nn)+O+(Nn+10*Nn)*Omul+3*Nn+Osin+3*Omem,
```

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This routine prints timing information that has been accumulated during the program's execution. Primarily, this can be used for fine-tuning the Runge-Kutta algorithm.

```c
void InvertMatrix(void)
{
    int info=0;
    dpptrf("u", &Nn1, _Matrix, &info);
    if(info) {puts("Cannot factor matrix"); exit(EXIT_FAILURE);}
    dpptri("u", &Nn1, _Matrix, &info);
    if(info) {puts("Cannot invert matrix"); exit(EXIT_FAILURE);}
}
```

This function makes use of two DXML routines to efficiently compute the inverse of the matrix.

```c
int IndexPlus(int I, int J)
// I=x index=1..Nx, J=y index=1..Ny
// returns matrix index
{return I-Nx+1+J*Nx;}
```

```c
int IndexMinus(int I, int J)
// I=x index=1..Nx, J=y index=1..Ny
// returns matrix index
{return I+Nx*Ny+1+J*Nx;}
```

```c
real &Matrix(int I, int J)
// I=row=1..Nn1, J=column=1..Nn1
{static real error=0.0;
 if(I<1 || Nn1<I || J<1 || Nn1<J) return error;
 else
 {if(J<I) {int tmp=I; I=J; J=tmp;}}
```

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return _Matrix[I+(J*(J-1))/2-1];
}
}

The purpose of these functions is merely to simplify index bookkeeping by keeping all of the index calculations in one place.

real RandomUniform(void)
{const real norm=1.0/(1UL<<31);
 return (random()&norm+random())*norm;
}

real Round(real X, real Y)
{return floor(X/Y+0.5)*Y;
}

These are two simple mathematical functions that we use in one or two places. RandomUniform returns a pseudo-random number \( x \), such that \( 0 < x < 1 \). Round rounds the parameter \( X \) to the nearest multiple of \( Y \).

int GetInt(const char Prompt[])
{return (int)GetLong(Prompt);
}

long GetLong(const char Prompt[])
{fputs("# ",stdout); fputs(Prompt,stdout); fputs(" ",stdout);
 const int bufsz=256;
 char *buf=new char[bufsz];
 gets(buf, bufsz, stdin);
 long x=atol(buf);
 delete buf;
 printf("%ld\n", x);
 return x;
}

real GetReal(const char Prompt[])
{fputs("# ",stdout); fputs(Prompt,stdout); fputs(" ",stdout);
 const int bufsz=256;

char *buf=new char[bufsz];
fgets(buf, bufsize, stdin);
real x=atof(buf);
delete buf;
printf("%g\n", x);
return x;
}

These three functions print a prompt and then convert the user’s input into the appropriate data type.

void Integrate(void)
{
    // #define FEHLBERG
    #ifdef FEHLBERG
        // Fehlberg coefficients
        real // c2t=1.0/4.0, c3t=3.0/8.0, c4t=12.0/13.0, c6t=1.0/2.0,
            c21=1.0/4.0,
            c31=3.0/32.0, c32=9.0/32.0,
            c41=1932.0/2197.0, c42=-7200.0/2197.0, c43=7296.0/2197.0,
            c51=439.0/216.0, c52=-8.0, c53=3680.0/513.0, c54=-845.0/4104.0,
            c61=-8.0/27.0, c62=2.0, c63=-3544.0/2565.0, c64=1859.0/4104.0,
            c65=-11.0/40.0,
        cx1=16.0/135.0, cx3=6656.0/12825.0, cx4=28561.0/56430.0, cx5=-9.0/50.0,
        cx6=2.0/55.0,
        ce1=1.0/360.0, ce3=-128.0/4275.0, ce4=-2197.0/75240.0, ce5=1.0/50.0,
        ce6=2.0/55.0;
    #endif
    // Cash-Karp coefficients
    real // c2t=1.0/5.0, c3t=3.0/10.0, c4t=3.0/5.0, c6t=7.0/8.0,
        c21=1.0/5.0,
        c31=3.0/40.0, c32=9.0/40.0,
        c41=3.0/10.0, c42=-9.0/10.0, c43=6.0/5.0,
        c51=-11.0/54.0, c52=5.0/2.0, c53=-70.0/27.0, c54=35.0/27.0,
        c61=1631.0/55296.0, c62=175.0/512.0, c63=575.0/13824.0,
        c64=44275.0/110592.0, c65=253.0/4096.0,
        cx1=37.0/378.0, cx3=250.0/621.0, cx4=125.0/594.0, cx6=512.0/1771.0,
        ce1=-277.0/64512.0, ce3=6925.0/370944.0, ce4=-6925.0/202752.0,
        ce5=-277.0/14336.0, ce6=277.0/7084.0;
    #endif
    int one=1;
UseDeltaTime=DeltaTime; Time=0.0;

InitialMeasurement();

while(Time<MaxTime)
    {clock_t t0=clock();
     TopOfLoop: if(++Count1>MaxStep) {puts("Step overload"); break;}
     memcpy(Finput, Phases, VectSz);
     Function(F1);
     dzaxpy_(&Nn, &c21, F1, &one, Phases, &one, Finput, &one);
     Function(F2);
     dzaxpy_(&Nn, &c31, F1, &one, Phases, &one, Finput, &one);
     daxpy_(&Nn, &c32, F2, &one, Finput, &one);
     Function(F3);
     dzaxpy_(&Nn, &c41, F1, &one, Phases, &one, Finput, &one);
     daxpy_(&Nn, &c42, F2, &one, Finput, &one);
     daxpy_(&Nn, &c43, F3, &one, Finput, &one);
     Function(F4);
     dzaxpy_(&Nn, &c51, F1, &one, Phases, &one, Finput, &one);
     daxpy_(&Nn, &c52, F2, &one, Finput, &one);
     daxpy_(&Nn, &c53, F3, &one, Finput, &one);
     daxpy_(&Nn, &c54, F4, &one, Finput, &one);
     Function(F5);
     dzaxpy_(&Nn, &c61, F1, &one, Phases, &one, Finput, &one);
     daxpy_(&Nn, &c62, F2, &one, Finput, &one);
     daxpy_(&Nn, &c63, F3, &one, Finput, &one);
     daxpy_(&Nn, &c64, F4, &one, Finput, &one);
     daxpy_(&Nn, &c65, F5, &one, Finput, &one);
     Function(F6);
     dvcal_(&Nn, &ce1, F1, &one, Finput, &one);
     daxpy_(&Nn, &ce3, F3, &one, Finput, &one);
     daxpy_(&Nn, &ce4, F4, &one, Finput, &one);
     daxpy_(&Nn, &ce5, F5, &one, Finput, &one);
     daxpy_(&Nn, &ce6, F6, &one, Finput, &one);
     real_error=damax_(&Nn, Finput, &one),
     stepError=UseDeltaTime=stepError*UseDeltaTime;
     if(error>stepError) {UseDeltaTime=0.5; goto TopOfLoop;}
}
Count2++;  
daxpy_(&Nn, &cx1, F1, &one, Phases, &one);  
daxpy_(&Nn, &cx3, F3, &one, Phases, &one);  
daxpy_(&Nn, &cx4, F4, &one, Phases, &one);  
#endif  
daxpy_(&Nn, &cx5, F5, &one, Phases, &one);  
#endif  
daxpy_(&Nn, &cx6, F6, &one, Phases, &one);  
Phases[0]=0.0;  
Time+=UseDeltaTime; AccError+=error;  
Measurement();  
if(error==0.0) UseDeltaTime*=2.0;  
else UseDeltaTime*=StepAdj*pow(stepError_UseDeltaTime/error,0.25);  

clock_t t1=clock(); Timing+=t1-t0;  
}  

IntTime+=Time;  

FinalMeasurement();  
}  

The Integrate routine is the most important piece of code. It implements the Runge-Kutta algorithm explained in Appendix B. The code uses the Cash-Karp coefficients [59], although another set of coefficients can be used instead. The vector math is done by vector-pipelined library calls. The error is estimated and compared to the nominal value, and the step-size is adjusted accordingly.

The code also checks to see if sufficient time has passed for the time-averaging measurement may begin. Clock timing and acceptance counts are also computed.

C.3 Main Code File

The final of the three files is jjlrp.cpp. It is responsible for input and output, and for properly running the mathematical routines of the second file.
// jjlrp.cpp 01/30/1997 Kent Harbaugh

// Josephson Junction Array with Long-Range Interactions
// with non-constant Phase

// Varying the input current

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// All Rights Reserved

#include <math.h>
#include <stdio.h>
#include <stdlib.h>
#include <time.h>

#include "jjlrp_base.h"

real MaxCurrent, MinCurrent;
int Unmeasured;
real mTime, Volt0, Voltage;

int main(int argc, char *argv[]) {
    PrintBanner();
    GetParameters();
    AllocateMemory();
    Initialize();
    memcopy(Phases, InitPhases, VectSz);

    real deltaCurrent=(MaxCurrent-MinCurrent)/(NumPt-1);
    for(int i=0; i<NumPt; i++)
        {if(Count1>MaxStep) break;
         real current=MinCurrent+deltaCurrent*i;
         Boundary[IndexPlus(1,(Ny+1)/2)]=current*Resistance;
         Boundary[IndexMinus((Nx+1)/2,Ny)]=current*Resistance;
         Integrate();
         PtXs[i]=current; PtYs[i]=Round(Voltage, TotalError);
        }

    PrintTiming();
    SaveData(argc>1?argv[1]:"data.out");

    return EXIT_SUCCESS;
}
The function main orchestrates the program. It calls the appropriate routines to read the parameters, allocate memory, and initialize the variables. The function then steps through the range of currents, and records the current and voltage of each data point. Finally, it calls the functions that output the results.

```c
void GetExtraParameters(void)
{
    MinCurrent=GetReal("Minimum current");
    MaxCurrent=GetReal("Maximum current");
}
```

This function reads in the extra parameters needed by this file. MinCurrent is the initial value of the bias current and MaxCurrent is the final value. Despite their names, MinCurrent does not have to be less than MaxCurrent. This allows the system to start at a large bias current and decrease.

```c
void InitialMeasurement(void)
{
    Unmeasured=1;
}
```

```c
void Measurement(void)
{
    if(Unmeasured && Time>=MinTime)
    {
        Unmeasured=0; mTime=Time;
        Volt0=Phases[IndexPlus(1,(Ny+1)/2)]-Phases[IndexMinus((Nx+1)/2,Ny)];
    }
}
```

```c
void FinalMeasurement(void)
{
    Voltage=(Unmeasured? 0.0: (Phases[IndexPlus(1,(Ny+1)/2)]-
                           Phases[IndexMinus((Nx+1)/2,Ny)]-Volt0)/(Time-mTime));
}
```

These are the functions which do the actual measurements. When the Time first passes MinTime, the phase difference between the input and output wires is recorded. Nothing more is measured until the integration is complete. Then, the first phase
difference is subtracted from the final phase difference and divided by the elapsed time. This quantity is $V_{dc}/(I_cR)$ from the model.

```c
void SaveData(const char FileName[])
{
    FILE *fp=fopen(FileName,"w");
    for(int i=0; i<NumPt; i++)
        fprintf(fp, "%15.12g %15.12g\n", PtXs[i], PtYs[i]);
    fclose(fp);
}
```

And the last function outputs the recorded data into a file.

### C.4 Overall Operation

This program has been divided into three parts for convenience. The functions of the second file, jjlrp_base.cpp, are those that are needed no matter what quantities are being measured in the simulation. As it stands, the code calculates $IV$ curves; for instance, Fig. 2.6. The third file can easily be modified to look at a different aspect of the model, without any reprogramming either of the first two files. For example, we have changed the third file so that it can precisely calculate the critical current of the array for a wide range of array sizes; see Fig. 2.9. Once the code has been changed to calculate the desired quantities, it is a simple matter to vary the input parameters to calculate a range of curves. This program is written for one particular model, but by changing Function routine, other dynamical systems may be integrated.
BIBLIOGRAPHY


[55] Handbook of Mathematical Functions ed. M. Abramowitz and I. A. Stegun, Dover (1965); Eqs. (20.3.1) and (20.3.15).
