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MODEL BASED SIGNAL PROCESSING FOR COMMUNICATIONS AND RADAR

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

Ashutosh Sabharwal, B.Tech., M.S.

* * * * *

The Ohio State University

1999

Dissertation Committee:

Lee C. Potter, Adviser
Randolph Moses
Urbashi Mitra

Approved by

Lee C. Potter
Adviser
Department of Electrical Engineering
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1999
ABSTRACT

In this dissertation, we report results on three problems in statistical signal processing: model selection, cell sectorization and receiver design for multirate direct-sequence code division multiple access (DS-CDMA) systems. All three problem formulations and their proposed solutions rely on parametric modeling of the systems under study.

First, we propose a computationally efficient procedure based on the Wald statistic for model order selection in nested models. The consistency of the proposed method is proven and performance is studied via Monte-Carlo simulations. Second, we propose a new criterion for cell sectorization in wireless cellular communication systems. The non-convex design problem is reparametrized into a convex problem, which allows computation of all the optimal solutions, using simple numerical procedures. The utility of the proposed technique to obtain lower bounds on antenna array sizes is demonstrated with examples. Third, DS-CDMA systems with multiple data rates, we derive linear receivers based on the minimum mean square error criterion. We show that, in general, the optimal linear receiver is cyclically time-varying, in contrast to time-invariant structures which have been widely studied in single rate DS-CDMA systems. Performance analysis is performed both analytically and via simulations.
To Human Resilience
ACKNOWLEDGMENTS

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VITA

1993 ........................................ B. Tech. Electrical Engineering
Indian Institute of Technology
New Delhi, India

1995 ........................................ M.S. Electrical Engineering
The Ohio State University
Columbus OH

PUBLICATIONS


FIELDS OF STUDY

Major Field: Electrical Engineering

Studies in:

  Topic 1  Statistics
  Topic 2  Mathematics
  Topic 3  Control Engineering
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CHAPTER 1

INTRODUCTION

Models are often used in scientific inquiry to capture the relevant features of the observed physical phenomenon. Models governed by parameters are often used, where a parameterization is usually chosen to capture relevant physical features of the data. Often, only a limited amount of finite precision, noise corrupted data is available, which seldom allows precise deterministic modeling. A widely used paradigm is to assume that the modeling error is an outcome of a random experiment. This leads to parametric statistical modeling.

Parametric statistical modeling has been widely used in social, physical and engineering sciences. The primary reason for such a widespread success of parametric modeling is its ability to compress large amounts of data into a convenient, parsimonious representation. Experimenters, equipped with physically motivated models, can use well understood methodologies to estimate and evaluate the unknown parameters of the models.

An ubiquitous example of parametric modeling is the sample mean. The sample mean is commonly reported instead of the entire data record with an aim to capture the trend in the data, a value representative of “most” of the data. The author believes that the reason for choosing a simple representation of the observed processes stems
from human belief\(^1\) that nature "does not play dice with humans"\(^2\). This belief is so strongly rooted that it is accepted as a logical principle, *Occam's Razor*, attributed to a fourteenth century philosopher, William of Ockham.

In this thesis, we restrict our attention to design and analysis of systems for data communications and radar processing. The three seemingly different problem areas considered here are components of a typical modern communication or radar processing system. Specifically, the three problems studied in this thesis are model selection, sectorization in cellular systems and receiver design for multiple data rate communication systems.

The common theme of the three studies is the reliance on parametric modeling to simplify design and mathematical analysis of systems. Even though simplifying assumptions are made throughout the sequel, the effectiveness of the resulting procedures is satisfying. The primary simplifying assumptions include models used for the signals of interest, distortions introduced due to channels, and models of data collection equipment (*e.g.*, power amplifiers, frequency response of antennae).

The thesis is organized as follows. In Sections 1.1-1.3, we provide a brief outline of the three reported studies. For each study, we start by motivating the problem followed by a discussion of the current trends and our contribution. For the convenience of the reader, the contributions of this thesis are summarized in Section 1.4. In Chapter 2, we present the proposed model order selection method, with complete proofs of results. In Chapter 3, the new criterion for cell sectorization and its convex reparameterization is presented; several design examples are also presented. In Chapter 4, we

\(^1\)A discussion to understand the reasons for this belief is out of the scope of this document.

\(^2\)An opinion held by Einstein [36].
present our results on minimum mean square error receiver design for multiple rate
direct sequence code division multiple access systems. Finally, conclusions with some
possible extensions to current work are given in Chapter 5.

1.1 Model Selection

Fisher [40–43], in his pioneering work, laid the mathematical foundations for a
branch of statistics commonly known as frequentist statistics. Fisher’s crowning
achievement was the convincing arguments for parametric modeling, which he de­
scribed as a convenient way of data summarization. His results regarding the math­
ematical accuracy of the maximum likelihood based procedures form the basis of the
frequentist statistics. The basic idea in maximum likelihood modeling is to use a
probability distribution function, \( p(x|\theta) \), which is believed to be the model of data
production. Given a finite number of observations, \( x = \{x_1, x_2, \ldots, x_N\} \), the objec­
tive is to obtain an estimate of the unknown fixed parameter, \( \theta \). Fisher suggested
to use as an estimate a value, \( \hat{\theta} \), which is most likely to have produced the observed
data. Mathematically, the parameter estimate \( \hat{\theta} \) is chosen by maximizing \( p(x|\theta) \) over
the set of feasible parameter values.

Fisher not only formalized the design of parameter estimators, but also empha­
sized the importance of performance analysis. In [43], he implicitly implied that an
estimator is not only a mapping from data to the parameter, but also comes with
guarantees regarding its performance. An important performance bound, the Crâmer-
Rao bound, for continuous parameter estimation was derived by extending Fisher’s ideas [94].
Typically, in modeling physical phenomena, the dimension of the parameter is also unknown. Motivating examples can be found in practically all physical system modeling, where it cannot be decided \textit{a priori} what subset of all possible effects were active during data production. Increasing the number of parameters in the model generally leads to a more accurate modeling of the data, but also increases the variability of parameter estimates. On the other hand, decreasing the dimension of the parameter leads to loss in modeling fidelity, thereby reducing the utility of estimates. If the unknown dimension is estimated using maximum likelihood parameter estimation, then usually the largest hypothesized dimension is chosen. The reason for such undesirable behaviour of maximum likelihood is that it only accounts for accuracy in modeling the observed data and ignores the effect of dimension on reliability.

The problem of accounting for the unknown dimension was first addressed by Akaike [2] in 1973. The proposed principle called \textit{Akaike Information Criterion} (AIC) was a generalization of the maximum likelihood principle via the Kullback-Liebler information [67]. The work by Akaike generated considerable interest in the statistical community. Soon, it was noted that AIC gives inconsistent results irrespective of the number of data samples. In 1978, Rissanen [97] and Schwarz [111] independently proposed new criteria with similar asymptotic forms, which provided consistent model estimates. The two criteria were derived with different philosophies.

Schwarz applied the well-known Bayesian principle and assumed regular exponential families to arrive at \textit{Bayesian Information Criterion}. Rissanen [97, 99, 100] proposed the \textit{minimum description length} principle which was motivated by Kolmogorov
complexity \[65\]^4. The work also has strong connections with universal coding and
prediction \[81,98\]. Rissanen proceeded by noting that any estimator can be viewed
as a data compression scheme. Following Kolmogorov^5, it was proposed to choose
the model that leads to shortest length codeword. Each parameter fixes a model for
the data, which leads to a source code. For the decoder to be able to reconstruct
the signal, not only the codeword but also the codebook is required. The resultant
codeword thus has two parts, one which specifies the codebook and the other specifies
the codeword in that codebook.

The minimum description length principle has been widely proposed \[13,75,110, 128\] in many applications where the parameter dimension is unknown. Several en­
couraging asymptotic results have been proved \[13,97,99,100\], but the implementa­
tion aspects of minimum description length based methods have received little atten­
tion. Computational complexity has impeded widespread use of minimum description
length methods. For instance, minimum description length based methods require \[33\]
maximum likelihood estimates of model parameters for each of the hypothesized mod­
els. However, numerous engineering problems involve nested models, where a simpler
model can be embedded in a more complex model to form a nesting. This fact has
been exploited in an ad hoc fashion \[58,135\] to avoid estimating parameters for all
hypothesized models.

The motivation of the method proposed in Chapter 2 comes from the differential
geometric structure of the hypothesized model class. Rao \[94\] in his landmark paper
not only provided the Crâmer-Rao bound but also laid the foundation for differential

\(^4\)A similar attempt was made in 1968 by Wallace and Boulton \[127\].

\(^5\)Kolmogorov complexity can be viewed as a mathematical formulation of Occam's Razor \[25\].
geometric methods in statistics. Rao showed that the Fisher information matrix is a Riemannian metric on the manifold of the parametric distributions [70]. But the first breakthrough in using differential geometry for design and analysis of estimators is due to Efron [34, 35]. The work by Efron generated considerable interest in the mathematical statistics community. For example, Efron's work was generalized to multidimensional parametric families [3, 5, 8, 10, 11, 62, 83] to study nonlinear regression [87, 88], hypothesis testing [27–29], sequential estimation [7], linear systems [6], ARMA models [96], and ancilliarity [4].

The geodesic statistic [29] has been proposed for hypothesis testing and we use its asymptotic approximation, the Wald statistic, to propose a model selection procedure for nested nonlinear model classes. The motivation is the computational simplicity of the Wald statistic [125] and its asymptotic equivalence to the likelihood ratio test [95, 125]. The model classes considered are sufficiently general to include many engineering models of interest. We restrict our attention to nested nonlinear superimposed models in Gaussian noise; examples include estimating multipath components in wireless communications [93], estimating the mechanisms in inverse scattering [51], and identification of new users in burst packet CDMA systems [134]. Use of the Wald statistic for model selection has been previously proposed in [18, 113, 135].

The advantage in employing the Wald statistic is that model order can be estimated from the parameter estimates of only the most complex model. The Wald statistic can be viewed as a "distance" of ML estimates from the parameter set of the simpler model. This interpretation is exact for linear models with affine parameter sets and is an asymptotic approximation for general nonlinear models [29].

The proposed methods can be extended to general nonlinear model classes.
To prove the consistency of the proposed order selection procedure, we extend the results in [60] to prove that ML parameter estimates are consistent when the Fisher information matrix computed at the overparameterized true parameter is not full rank. This consistency result is also applicable to the least-squares estimate for non-identically distributed non-Gaussian noise. We also discuss the inclusion of the proposed order estimate in a dynamic program to reduce the average complexity of the order selection procedure. Representative simulation results are provided for an inverse scattering application.

1.2 Cell Sectorization

The recent surge in the demand for the wireless communication has led to a need to develop advanced signal processing techniques to serve more users in a limited available spectrum. This growth has been furiously fueled by internet services, which have created a demand for high data rate, low delay services.

To increase the capacity of any multiuser system, either the spectral allocation or the transmitted power can be increased. Spectral resources are limited and government organizations generally regulate transmitted power\(^7\). With limitations on two obvious resources for increasing system capacity, exploiting spatial diversity, due to different physical location of the users, is being considered as the (last) most promising avenue [86].

Most modern wireless systems rely on the cellular concept, in which the service area is divided into small *cells* (Figure 1.1). Each cell is served by a base-station, which can simultaneously support several users. The division into small cells leads to

\(^7\)In USA, Federal Communications Commission (FCC) is responsible for standardization.
Figure 1.1: A typical cellular system. A base-station, B, receives signals from the desired user, T, and also interference from other active users.

a reduced power requirement for base-stations, cheaper and smaller handsets, and a drastic increase in the system capacity. In a typical cellular system, the base-station not only receives signal from the user of interest but also from other active users, both inside and outside the cell (Figure 1.1).

Cell sectorization has been widely proposed for improving system capacity in cellular systems [20,76,78]. The improvement in the system capacity is made possible by the reduced inter-cell and intra-cell interference. Every cell is divided into multiple sectors, and each sector is served by a dedicated antenna. With ideal sector antennas, the increase in system capacity for both voice and data networks is proportional to the number of sectors [59]. As an additional benefit, cell sectorization is reported to result in a considerable decrease in delay spread in the received signal [115]. However, perfect
sectorization is not realizable. Non-ideal radiation patterns reduce the capacity of sectorized cellular systems due to inter-sector interference [59, 73, 84].

Several approaches for sector beam synthesis have been proposed [17, 39, 66, 108]. Most published methods are extensions of digital filter design techniques [39, 66, 108]. However, sector beam synthesis using antenna arrays is considerably different from conventional filter design for the following reasons. First, the analogy between filter design and sector synthesis holds only for λ/2 inter-element spacing in uniform linear arrays with isotropic elements. Second, the phase of the synthesized beam is irrelevant for many array applications; in contrast, linear phase is an important requirement in many filter design problems. Third, the minimax criterion widely used in filter design is not well suited for multisector wireless applications because the criterion does not appear to be directly related to any significant communication performance parameter.

We propose a sector beam synthesis technique using phased antenna arrays to minimize the total inter-sector interference. We assume that each cell is divided into sectors of equal width and that each sector is served by an independent antenna array, as depicted in Figure 1 for the case of six sectors. In addition to synthesizing the desired pattern, the proposed technique allows us to quantify the relationships among sector size, beam efficiency, signal-to-interference ratio, and capacity.

The proposed approach minimizes the total inter-sector interference subject to a ripple constraint on the in-sector power pattern. This is an example of power synthesis [17, 79]. We show that the proposed non-convex problem can be converted into a convex fractional linear programming problem via a suitable reparametrization [133]. The convex reparameterization guarantees that a globally optimal solution can be
computed using a numerically efficient algorithm. The proposed approach accommodates non-isotropic antenna elements, arbitrary element spacing, and non-central sectors, but is limited to uniformly spaced linear (or planar) arrays.

1.3 Multirate DS-CDMA Receivers

The growth of wireless networks has resulted in systems offering heterogeneous services, such as voice, video and data. Several system architectures based on Direct Sequence Code Division Multiple Access (DS-CDMA) signalling have been proposed recently [1, 30, 91, 92]. These proposals include systems with multiple chipping rates [1], variable spreading gain with constant chip rate [1] and multicode CDMA [82]. Another instance of a multiple chipping rate system is a wideband DS-CDMA system operating in the presence of an existing narrowband DS-CDMA system.

Bursty data sources, like those encountered in ethernet, internet and file transfer applications, favour statistical multiplexing schemes [15]. It is well known that for (bursty) Markov sources, the throughput of the statistical multiplexing is higher [15] than any fixed assignment multiplexing schemes, such as time or frequency division multiple access. A packet-based DS-CDMA system [30, 53, 64, 68, 89, 114, 120] naturally allows statistical multiplexing, where multiple users can simultaneously transmit packets and the spreading gain diversity allows resolution of multiple packets (as compared to slotted ALOHA, where multiple packets lead to a collision).

Multiuser receivers for single rate DS-CDMA systems have been extensively studied following the seminal work by Verdú [121–123]. Both the implementation issues and performance analysis of single-rate systems have been widely studied (see

*The center of the sector is not normal to the plane of the array.*
for example [124] and references therein). In contrast, multiple data rate multiuser
detection is a relatively new problem area without maturity in system design and
performance analysis. Receivers which truly exploit the nature of multi-rate signals
have focussed on synchronous and pseudo-synchronous systems in non-multipath en­
vironments [19,21,22,82,109]. Designs which accomodate asynchronous or multipath
channels [116] do not fully take advantage of the multi-rate nature of the problem.

In the development of optimal linear receivers for multirate modulations, we ex­
plot the cyclostationarity of the modulation schemes. Cyclostationarity and its use
in communication can be largely attributed to Gardner [46-50]. The work by Gardner
clearly demonstrates the advantages of exploiting the cyclostationarity, or spectral re­
dundancy, of the communication signals. Most of Gardner’s work is dedicated to the
problem of signal separation, rather than data demodulation. In signal separation,
the complete waveform of the signal of interest is estimated in the presence of inter­
ference. The signal separation is thus in the same spirit as Wiener filtering [131], and
is not limited to communication systems. On the other hand, in data demodulation,
the data symbols, instead of the modulated signal, are of interest. In most practical
DS-CDMA systems, the data rate is less than the transmission bandwidth, which
implies that the two problems of data demodulation and signal separation are not
equivalent. This difference is also manifested in the linear least-squares solution for
the two problems. Consider, for example, a multiuser communication system where
each user transmits at the same data rate, and hence have the same period of cy­
clostationarity. For signal separation using the MMSE criterion, the optimal linear
receiver is cyclically time-varying [50]. On the other hand, the optimal linear receiver
for data demodulation is time-invariant [77,124].
In this thesis, we derive linear multiuser receivers based on the minimum mean squared error (MMSE) criterion, for multiuser systems where users with different symbol rates coexist. We first derive MMSE receivers without any causality constraints; this extends the work in [16, 46]. The non-causal structure proposed in this paper provides a basis for more practical causal realizations and also provides a baseline for comparison to other multi-rate receiver structures.

It is shown that the optimal linear MMSE receiver is, in general, time-varying. The same observation was also independently made in [19] for a dual rate synchronous DS-CDMA system; in [19], a non multipath environment with time-limited square pulses was assumed. The receiver structure developed here is sufficiently general to include the cases of arbitrary alphabet size, multipath channels and multiple bandwidth systems. Although the optimal linear demodulator for multirate systems is time-varying in general, the nature of periodicity is different from that seen in cyclic Wiener filtering [50].

In systems with multiple chipping rates, different user classes occupy different bandwidths. The choice of bandwidth of the front-end filter for the smaller bandwidth users determines the performance of their receivers, which is also intimately tied to the total system capacity. The front-end filter bandwidth determines the sampling rate which is one of the factors determining receiver cost for users with lesser spectral resources. We study the effect of front-end filter bandwidth on the performance of smaller bandwidth users.

1.4 Contributions

For the convenience of the reader, we summarize our contributions in this section.
1. **Model Order Selection** [105–107, 135]

- Propose a computationally efficient model order selection procedure for nested nonlinear models, based on the Wald statistic. The proposed algorithm extends our previous work on superimposed exponential model to general superimposed models.

- Extend the results on the consistency of maximum likelihood estimates to the case where the noiseless data can be generated by multiple parameters.

- Prove the consistency of the proposed model order selection procedure.

2. **Cell Sectorization** [101–103]

- Propose a new criterion for cell sectorization in cellular systems.

- Optimize the proposed non-convex criterion via a convex reparametrization and a numerically efficient linear programming technique.

- Demonstrate the utility of the proposed approach by calculating the lower bounds on array sizes for a DS-CDMA system.

3. **MMSE Receivers for Multirate Systems** [104]

- Demonstrate that the linear MMSE receiver, in general, is time-varying.

- Derive the optimal time-varying and time-invariant MMSE receivers and compare their MSE performance.

- Analytically characterize the effect of front-end filter bandwidth on system capacity.
CHAPTER 2

WALD STATISTIC FOR MODEL SELECTION IN NONLINEAR MODELS

2.1 Introduction

The maximum likelihood (ML) principle is commonly used to estimate unknown fixed parameters when the dimension of the model is known. In applications where the parameter dimension is also unknown, the minimum description length principle [97] has been widely proposed [13, 75, 110, 128]. Several encouraging asymptotic results have been proved [13, 97, 99, 100], but the implementation aspects of minimum description length based methods have received little attention. Computational complexity has impeded the widespread use of minimum description length methods. For instance, minimum description length based methods require [33] maximum likelihood estimates of model parameters for each of the hypothesized models. However, numerous engineering problems involve nested models, where a simpler model can be embedded in a more complex model to form a nesting. This fact has been exploited in an ad hoc fashion [58, 135] to avoid estimating parameters for all hypothesized models.
Motivated by the computational simplicity of the Wald statistic [125] and its asymptotic equivalence to the likelihood ratio test [95,125], we propose a model selection procedure based on the Wald statistic for nested nonlinear model classes. The model classes considered are sufficiently general to include many engineering models of interest. We restrict our attention to nested nonlinear superimposed models in Gaussian noise; examples include estimating multipath components in wireless communications [93], estimating the mechanisms in inverse scattering [51], and identification of new users in burst packet CDMA systems [134]. Use of the Wald statistic for model selection has been previously proposed for ARMA model [18], linear model [113], and undamped exponential model [135].

The advantage in employing the Wald statistic is that model order can be estimated from the parameter estimates of only the most complex model. The Wald statistic can be viewed as a "distance" between ML estimates and the parameter set of the simpler model. This interpretation is exact for linear models with affine parameter sets and is an asymptotic approximation for general nonlinear models [29].

To prove the consistency of the proposed order selection procedure, we extend the results in [60] to prove that ML parameter estimates are consistent when the Fisher information matrix at the overparameterized true parameter is not full rank. This consistency result is also applicable to the least-squares estimate for non-identically distributed non-Gaussian noise.

In Section 2.2, we formulate the model selection problem and highlight a rank property of the Fisher information matrix for models under consideration. In Section 2.3, the proposed order selection procedure is given. The consistency of the

\[^{9}\text{The proposed methods can be extended to general nonlinear model classes.}\]
proposed method is proven in Section 2.4. In Section 2.5, we discuss the inclusion of the proposed order estimate in a dynamic program to reduce the average complexity of the order selection procedure. Also, representative simulation results are provided for an inverse scattering application.

2.2 Nonlinear Superposition Models

We consider a weighted sum of parametric signals in noise,

\[ x(t) = g(t; \rho, \gamma) + \epsilon(t), \quad t = 1, 2, \ldots, N \]  
\[ = \sum_{i=1}^{p} a_i e^{j\phi_i} s(t; \theta_i) + \epsilon(t), \]  

where \( s(t; \theta) \) is a known parametric function of \( \theta \) and \( \epsilon(t) \) is zero-mean white circular Gaussian noise. The unknown parameters are \( \{p, \gamma\} \) where \( \gamma = \{(a_i, \phi_i, \theta_i)\}_{i=1}^{p} \). The choice of \( s(t; \theta) \) is application dependent; two examples are given below.

**Example 1 (Multipath propagation)**: For high speed communication, parsimonious channel modeling can be achieved by using the model in (2.1) in place of the commonly used FIR channel model. In (2.1), \( s(t; \theta_i) = s(t - \theta_i) \) where \( \theta_i \) denotes the time delay in receiving the \( i^{th} \) multipath component and \( s(t) \) is the known training waveform. For high definition television, where the effective delay spread can span several hundred symbols [61], the reduction in the number of unknown parameters is significant. A similar model is encountered in blind equalization based on second-order statistics for high-speed communication systems [61].
Example 2 (Radar scattering): For high frequency radar applications, the undamped exponential model is an accurate approximation to radar scattering [52, 90, Theorem 2]. For small relative bandwidths, the scattering model is \( s(t; \theta_i) = e^{-j\pi \theta_i t/c} \), where \( \theta_i \) is distance to the scattering object, \( t \) indexes frequency, and \( c \) is the speed of propagation.

We adopt the following terminology. Each parameter tuple \( \{a_i, \phi_i, \theta_i\} \) is referred to as a mode, with amplitude, phase and the location parameters, respectively. The unknown parameter \( p \) is referred to as the model order. The real-valued vectors \( a, \phi, \theta \) denote the amplitude, phase and location parameter subsets, respectively. The bounded parameter sets for each model order, \( p > 0 \), are defined as follows.

\[
\Gamma_k = \{ \gamma \in (0, a_{\text{max}})^k \times (-\pi, \pi)^k \times (\theta_{\text{min}}, \theta_{\text{max}})^k : I(\gamma) \text{ is full rank} \} \quad (2.3)
\]

where \( I(\gamma) \) is the Fisher information matrix (FIM). Note that the closure of \( \Gamma_k, \Gamma_k \), contains parameters for all the model orders less than or equal to \( k \). In the sequel, \( \Gamma_p, p < k \) is used to denote two representations of the same set: first, as a subset of \( \mathbb{R}^{3p} \), and second, as a subset of \( \Gamma_k \subset \mathbb{R}^{3k} \). Particular usage will be clear from the context.

We first characterize all the points in \( \Gamma_k \) which belong to a lower model order parameter set. We start by defining identifiability of a parameter.

**Definition 1 (Identifiability)** A parameter \( \gamma \) is identifiable if and only \( I(\gamma) \) is full rank.

The motivation for our terminology regarding identifiability of the modes comes from inverse function theorem [31], and was also used by Wald [126]. A differentiable
mapping is uniquely invertible at any point if only if the FIM is full rank. This implies if the FIM is not full rank then no unique inverse or equivalently no unique parameter estimate exists for certain noiseless data records.

Next, we compute the Fisher information matrix for $\gamma$. Let $s_R(t; \theta)$ and $s_I(t; \theta)$ denote the real and imaginary parts of the $s(t; \theta)$, respectively. For model order $k$, define the vectors,

$$
\begin{align*}
\mathbf{s}_R^T(\theta) &= [s_R(1; \theta) \ s_R(2; \theta) \ \cdots \ s_R(N; \theta)] \\
\mathbf{s}_I^T(\theta) &= [s_I(1; \theta) \ s_I(2; \theta) \ \cdots \ s_I(N; \theta)] \\
\mathbf{g}^T(\gamma) &= [g(1; \gamma) \ g(2; \gamma) \ \cdots \ g(N; \gamma)]
\end{align*}
$$

where $g(l; \gamma) = g(t; k, \gamma)|_{t=t_l}$. Stack the real and imaginary parts of $\mathbf{g}_t$ to form the following column vector.

$$
\mathbf{G}(\gamma) = \begin{bmatrix} \text{Real}(g(\gamma)) \\ \text{imag}(g(\gamma)) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} \mathbf{a} = \mathbf{B}\mathbf{a}
$$

where

$$
\begin{align*}
\mathbf{b}_i &= \begin{bmatrix} \cos(\phi_i)s_R(t_i; \theta_i) - \sin(\phi_i)s_I(t_i; \theta_i) \\
\sin(\phi_i)s_R(t_i; \theta_i) + \cos(\phi_i)s_I(t_i; \theta_i) \end{bmatrix} \\
\mathbf{a} &= \text{diag}(a_1, a_2, \ldots, a_k) = \text{diag}(\mathbf{a}^T)
\end{align*}
$$

The $2N \times 3k$ matrix defined as $D_{\gamma}\mathbf{G} = \left[ \frac{\partial \mathbf{G}(\gamma)}{\partial \gamma} \right]$ for $\gamma \in \mathbb{R}^{3k}$ is given by,

$$
\begin{align*}
D_{\gamma}\mathbf{G} &= \begin{bmatrix} \mathbf{B} & \mathbf{B}_\phi & \mathbf{B}_\theta \end{bmatrix} \mathbf{A} = \mathbf{D}\mathbf{A} \\
b_{\phi,i} &= \begin{bmatrix} -\sin(\phi_i)\mathbf{s}_R(t; \theta_i) - \cos(\phi_i)\mathbf{s}_I(t; \theta_i) \\
\cos(\phi_i)\mathbf{s}_R(t; \theta_i) - \sin(\phi_i)\mathbf{s}_I(t; \theta_i) \end{bmatrix} \\
b_{\theta,i} &= \begin{bmatrix} \cos(\phi_i)\mathbf{s}_R(t; \theta_i) - \sin(\phi_i)\mathbf{s}_I(t; \theta_i) \\
\sin(\phi_i)\mathbf{s}_R(t; \theta_i) + \cos(\phi_i)\mathbf{s}_I(t; \theta_i) \end{bmatrix} \\
\mathbf{A} &= \text{diag}(1, 1, \ldots, 1, \mathbf{a}^T, \mathbf{a}^T)
\end{align*}
$$
where $b_{\phi,i}$ is the $i^{th}$ column of $B_{\phi}$ and $b_{\theta,i}$ is the $i^{th}$ column of $B_{\theta}$. Further, $s'_{R}$ and $s'_{T}$ denote the derivatives of $s_{R}$ and $s_{R}$ with respect to $\theta$. The FIM is given by

$$I(\gamma) = \left(D_{2}G\right)^{T}\left(D_{2}G\right), \gamma \in \overline{\Gamma}_{k}.$$  

We first state the Proposition for the complex data case (2.1).

**Proposition 1 (FIM under overparameterization)** Assume that $s(t; \theta)$ is differentiable with respect to $\theta$, $\theta \in \Theta \subseteq \mathbb{R}$. Assume further that $s(t; \theta)$ gives linearly independent columns of $D$ for distinct $\{\theta_{1}, \theta_{2}, \ldots, \theta_{k}\}$. Also, let $N \geq 3k$. Then FIM for any $\gamma \in \overline{\Gamma}_{k} \subseteq \mathbb{R}^{3k}$ is full rank if and only if

(a) none of the amplitudes are zero, and

(b) $\theta_{i} \neq \theta_{j}$ for all $i \neq j$.

**Proof of Proposition 1:** Since $I(\gamma) = \left(D_{2}G\right)^{T}\left(D_{2}G\right)$, we have

$$\rho(I(\gamma)) = \rho(DA) = \rho\left(D_{2}G\right)$$

where $\rho(I(\gamma))$ denotes the rank of the matrix $I(\gamma)$. Suppose $\theta$ is such that $\theta_{i} \neq \theta_{j}$ for $i \neq j$. Then $\rho(I) = 3k - 2g$, where $g$ is the total number of zero elements of $a^{T}$.

Conversely, suppose $\theta$ is such that $\theta_{i} = \theta_{j}$ for some $i < j$. Then, $s(t; \theta_{i}) = s(t; \theta_{j})$ and $s'(t; \theta_{i}) = s'(t; \theta_{j})$. This implies that $\rho(I) \leq \rho(D) \leq 3k - 2$. The elimination of modes with nondistinct location parameters reduces the new set of modes to the previous case. ■

**Remark 1** The conditions in Proposition 1 are intuitively appealing: a parameter $\gamma \in \overline{\Gamma}_{k}$ belongs to $\overline{\Gamma}_{k-1}$ if and only if at least one mode has zero energy $(a_{i} = 0)$ or two modes have the same location parameter.
Remark 2 Let \( l \) be the number of unidentifiable modes, then for \( \gamma \in \Gamma_k \), \( 3(k - l) \leq \rho(I(\gamma)) < 3(k - l + 1) \).

Remark 3 (real data) If the data is real, i.e., \( \phi_i \equiv 0, a \in (-a_{\text{max}}, a_{\text{max}}) \), and \( s(t; \theta) \) is real for all \( t \) and \( \theta \), then \( D_2 G \) is given by

\[
D_2 G = \begin{bmatrix}
    s(\theta_1) & \cdots & s(\theta_k) & a_1 s'(\theta_1) & \cdots & a_k s'(\theta_k)
\end{bmatrix}
\]

where \( s^T(\theta) = [s(1; \theta) \ s(2; \theta) \ \cdots \ s(N; \theta)] \) and \( s'(\theta) \) is the derivative of \( s(\theta) \) with respect to \( \theta \). Also, \( A = \text{diag}(1, 1, \ldots, 1, a^T) \). Then, Proposition 1 applies for \( N \geq 2k \).

Furthermore, \( 2(k - l) \leq \rho(I(\gamma)) < 2(k - l + 1) \).

Remark 4 In [60, 69], the consistency of the least-squares estimate is proven by assuming the complete identifiability of the true parameter, i.e., the FIM is assumed to be full rank at the true parameter. If the model in (2.1) is overparametrized, then the FIM is rank deficient. Existing results are extended in Corollary 1 below to establish consistency for the case of singular FIM, thereby extending the results in [60, 69].

2.3 Model Order Selection

In this section, we present the proposed model order selection algorithm as an asymptotic implementation of the likelihood ratio test. We start by briefly outlining order selection based on the minimum description length (MDL) principle. Next, we appeal to the geometry of the nested model classes and propose geodesic distance as a statistic for order selection. Motivated by Wald's asymptotic approximations to geodesic
distance in [125], we propose an order selection algorithm using the generalized Wald statistic.

The selection of model order can be written as a multiple hypothesis testing problem

\[ H_k : p = k, \ k = 1, \ldots, K \] 

(2.19)

where \( K \) is the maximum candidate model order. The MDL and the maximum a posteriori probability (MAP) [33] are well-known solutions for the \( K \)-ary hypothesis testing problem in (2.19). The MDL and MAP estimates of \( p \) are of the form

\[ \hat{p}_{MDL} = \arg \min_{k=1,\ldots,K} \log f(x|\hat{\gamma}) + \frac{\alpha(k)}{2} \log N \] 

(2.20)

where \( \hat{\gamma} \in \Gamma_k \) is the maximum likelihood estimate of the parameter for model order \( k \). The parameter complexity term, \( \frac{\alpha(k)}{2} \log N \), depends on the model function \( g(t; \gamma) \): \( \alpha(k) \) is derived for several linear and nonlinear models in [33, 100]. Computation of the MDL cost in (2.20) requires the computation of the maximum likelihood estimate for each candidate model order.

To exploit the structure of nested classes under consideration, we appeal to the notional geometry in Figure 2.1. The nested parameter sets are denoted by \( \Gamma_j \) and \( \Gamma_k \) such that \( \Gamma_j \subset \Gamma_k \) (in our case, this is equivalent to \( j \leq k \)). The noiseless data is obtained by the transformation, \( g(t; \cdot) \). The noiseless data sequence, \( y = g(t; \gamma) \), lies in \( \mathbb{R}^N \) for real data and \( \mathbb{R}^{2N} \) for complex data. The least-squares estimate of the noiseless data, \( \hat{y} \), assuming that the true model order is \( k \), can be obtained by projecting the noisy measurement \( x(t) \) onto the manifold \( g(\Gamma_k) \). The least-squares estimate of the parameter, \( \hat{\gamma} \), is obtained by using the inverse image of \( g(t; \cdot) \) (Corollary 1 makes the inverse mapping precise). The shortest distance between any two points on the
manifold, \( g(\Gamma_k) \), is known as the *geodesic distance* [70]. Accordingly, we label the geodesic distance between \( g(\hat{\gamma}) \) and \( g(\Gamma_j) \) as the *geodesic statistic*, \( \mu(\hat{\gamma}, \Gamma_j) \).

To motivate the use of geodesic distance as a relevant statistic for model selection, we use a result proved in Section 2.4 regarding the consistency of the estimate, \( \hat{\gamma} \in \Gamma_k \). It is shown in Corollary 1 that if the FIM is continuous with respect to \( \gamma \), then as \( N \to \infty \),

\[
\inf_{\gamma \in \Gamma_k} \mu(\hat{\gamma}, \gamma) \rightarrow \begin{cases} 
0 & \text{if } j \geq \text{the true model order} \\
> 0 & \text{if } j < \text{the true model order}
\end{cases} \tag{2.21}
\]

Furthermore, the hypothesis test (2.19) can be rewritten as follows.

\[
H_k : \gamma \in \Gamma_k, \quad k = 1, \ldots, K \tag{2.22}
\]

Thus, detecting the correct model order is equivalent to finding the smallest parameter set of which the given parameter is a member. This implies that a test of the following form can be used to detect the correct model order.

\[
\mu(\hat{\gamma}, \Gamma_j) \begin{cases} 
\text{accept} & \text{if } \mu(\hat{\gamma}, \Gamma_j) \leq T_j(\gamma) \\
\text{reject} & \text{otherwise}
\end{cases} \tag{2.23}
\]

where the threshold, \( T_j(\gamma) \), is chosen based on the conditional distribution of \( \mu(\hat{\gamma}, \Gamma_j) \) given that the true parameter \( \gamma \) is an element of \( \Gamma_j \). The gain in using the geodesic statistic for order selection is that only one parameter estimate, \( \hat{\gamma} \in \Gamma_k \), is required for hypothesis testing as compared to the \( K \) estimates required for MDL test in (2.20). However, computation of the geodesic statistic is computationally intensive; in general, it involves solving a multidimensional boundary value problem. A computationally simple alternative to the geodesic statistic is the Wald statistic [125], which is a first order Taylor series approximation to the geodesic squared statistic [29].
The Wald statistic can be computed as follows. Assume that the following representation for the set $\Gamma_j$ is available for $k > j$:

$$\Gamma_j = \{ \gamma \in \Gamma_k : \tau(\gamma) = 0 \}$$

(2.24)

where $\tau(\cdot) \in \mathbb{R}^{(k-j)}$ is a differentiable restriction on the parameter set $\Gamma_k$. Note that $\tau(\cdot)$ characterizes the embedding of $\Gamma_j$ in $\Gamma_k$. Given the ML parameter estimate, $\hat{\gamma}$, the generalized Wald statistic [55] is given by,

$$W = \tau(\hat{\gamma})^T (R(\hat{\gamma})^T I(\hat{\gamma}) R(\hat{\gamma}))^{\dagger} \tau(\hat{\gamma})$$

(2.25)

where $(\cdot)^\dagger$ the Moore-Penrose pseudo inverse, and $R(\gamma)$ is the matrix of first derivatives of $\tau(\gamma)$ with respect to $\gamma$

$$R_{pq}(\gamma) = \frac{\partial \tau_p(\gamma)}{\partial \gamma_q}, \ p = 1, \ldots, (k-j), \ q = 1, \ldots, k.$$  

(2.26)

The Wald statistic is asymptotically similar\(^{10}\) to the likelihood ratio test statistic (LRT) [95]; this similarity was also noted in [18] to use the Wald statistic for order selection in ARMA models.

From Proposition 1, the embedding of $\Gamma_j$ in $\Gamma_k$ is equivalent to addition of $(k - j)$ unidentifiable modes to any parameter $\gamma \in \Gamma_j$. Furthermore, the identifiability conditions in Proposition 1 also provide the required restriction, $\tau(\cdot)$, on $\Gamma_k$ to identify the parameters in $\Gamma_j$. Note that the restrictions from Proposition 1 are in two parts. A mode is declared unidentifiable, if either

(i) its energy is zero, $a_i = 0$, or

(ii) its location parameter is identical to another mode, $\theta_i = \theta_j$, $j < i$ ($a_j = 0$).

\(^{10}\)Two statistics are asymptotically similar if their asymptotic distributions are identical.
A mode is unidentifiable if it satisfies at least one of the above criteria. The generalized Wald statistic (2.25) requires a unique embedding of the true parameter, \( \gamma \in \Gamma_j \), in \( \Gamma_k \). The bipartite nature of the restrictions leads to a non-unique embedding of the true parameter and hence does not allow a direct application of the Wald statistic as given in [55, 95, 125]. Motivated by the results in [135], the asymptotic equivalence of the generalized Wald statistic to the LRT [37] and the nature of the restrictions, \( r(\cdot) \), we propose the following order selection procedure.

1. Given the estimate of the parameter, \( \widehat{\gamma} \), or equivalently, the subparameters, \( \widehat{a} \), \( \widehat{\phi} \) and \( \widehat{\theta} \), form the order statistics, \( \widehat{a} \) and \( \widehat{\theta} \). The order statistic \( \widehat{a} \) is formed by reordering the elements of \( \widehat{a} \) such that \( \widehat{a}_i \geq \widehat{a}_{i+1} \) for all \( i = 1, \ldots, k - 1 \); the order statistic \( \widehat{\theta} \) is formed similarly. For the location parameter, form the order statistic of the first difference, \( \Delta \widehat{\theta} \), where \( \Delta \widehat{\theta}_i = \widehat{\theta}_i - \widehat{\theta}_{i+1} \). The modes are labeled to obtain two ordered sets, \( \widehat{m}^a \) and \( \widehat{m}^\theta \), corresponding to the order statistics, \( \widehat{a} \) and \( \Delta \widehat{\theta} \), respectively.

2. The restrictions on the amplitude and the location parameters are constructed as follows.

\[
A_k \widehat{a} = \begin{bmatrix} 0_{(k-k) \times k} & I_{k-k} \end{bmatrix} \widehat{a} = 0 \quad (2.27)
\]

\[
\Theta_k \Delta \widehat{\theta} = \begin{bmatrix} 0_{(k-k-1) \times k} & I_{k-k-1} \end{bmatrix} \Delta \widehat{\theta} = 0 \quad (2.28)
\]

where \( 0_{p \times q} \) is a \( p \times q \) matrix of zeros, and \( I_p \) is a \( p \times p \) identity matrix.

3. The generalized Wald statistic [55] can be written as follows,

\[
W^a_k = (A_k \widehat{a})^T \left( A_k^T I^r(\widehat{a}) A_k \right)^\dagger (A_k \widehat{a}) \quad (2.29)
\]

\[
W^\theta_k = \left( \Theta_k \Delta \widehat{\theta} \right)^T \left( \Theta_k^r I^r(\Delta \widehat{\theta}) \Theta_k \right)^\dagger \left( \Theta_k \Delta \widehat{\theta} \right) \quad (2.30)
\]
where $I(\cdot)$ is the FIM.

4. The order estimate is obtained as follows

$$\hat{p}^e = \arg \min_{k=0, \ldots, K} W_k^e + \frac{c(k)}{2} \log(N)$$ (2.31)

$$\hat{p}^d - 1 = \arg \min_{k=0, \ldots, K-1} W_k^d + \frac{c(k)}{2} \log(N)$$ (2.32)

$$\hat{p}_{\text{Wald}} = |\{\hat{m}_i^e : i = 1, \ldots, \hat{p}^e\} \cap \{\hat{m}_i^d : i = 1, \ldots, \hat{p}^d\}|$$ (2.33)

The idea underlying the proposed test is to reject unidentifiable modes, i.e., those with small energy or identical location parameters. The first test (2.31) rejects modes with small energies, and the second test (2.32) finds the subset of modes with unique location parameters. Only the modes which are accepted by both tests are retained.

The model order estimate, $\hat{p}_{\text{Wald}}$, is obtained by counting the number of accepted modes.

The penalty term in (2.31) and (2.32) is chosen to be identical to MDL (MAP), because the number of unknown parameters per mode is same in both tests. While the consistency of the order selection procedure is not critically dependent on this particular choice of the penalty term, best performance was observed in the non-asymptotic regime by choosing the same penalty term as in MDL (MAP).

Example: To clarify the steps of the proposed algorithm, consider the following example. Let $K = 4$ with the following parameter estimates.

$$\hat{m} = \left\{ \left(1.1, \frac{\pi}{3}, 0\right), \left(2, \frac{\pi}{4}, 1\right), \left(1, \frac{\pi}{4.5}, 1.1\right), \left(0.1, \frac{\pi}{3.3}, 2\right) \right\}$$ (2.34)

where $m_i$ is used to denote the 3-tuple of the parameters, $(a_i, \phi_i, \omega_i)$, of a mode.

Following the steps of the proposed algorithm, we obtain
1. Order statistics

\[
\bar{a} = [2 \ 1.1 \ 1 \ 0.1] 
\]  \hspace{1cm} (2.35)

\[
\bar{\theta} = [2 \ 1.1 \ 1 \ 0] 
\]  \hspace{1cm} (2.36)

\[
\Delta\bar{\theta} = [1 \ 0.9 \ 0.1] 
\]  \hspace{1cm} (2.37)

The corresponding relabeling of modes yields the following two ordered sets.

\[
\tilde{m}^a = \{m_2, m_1, m_3, m_4\} 
\]  \hspace{1cm} (2.38)

\[
\tilde{m}^\theta = \{(m_2, m_1), (m_3, m_4), (m_3, m_2)\} 
\]  \hspace{1cm} (2.39)

2. The restrictions are formed as in Step 2 of the proposed method. The order estimates, \( \hat{\rho}^a \) and \( \hat{\rho}^\theta \), are obtained following Steps 3 and 4 of the proposed method. Consider the case where \( \hat{\rho}^a = 3 \) and \( \hat{\rho}^\theta = 1 = 2 \). That is, mode \( m_4 \) is rejected by the amplitude test, and the location test rejects the pair, \( (m_3, m_2) \). Since both \( m_2 \) and \( m_3 \) are accepted by the amplitude test, any one of the two modes can be retained; we retain \( m_2 \) since its energy is more than \( m_3 \). Thus the final model order estimate is

\[
\hat{\rho}_{WALD} = |\{m_1, m_2, m_3\} \cap \{m_1, m_2, m_4\}| 
\]  \hspace{1cm} (2.40)

\[
= |\{m_1, m_2\}| 
\]  \hspace{1cm} (2.41)

\[
= 2 
\]  \hspace{1cm} (2.42)

2.4 Consistency of the Proposed Algorithm

In this section, we establish that the proposed order selection procedure, (2.31)-(2.33), is consistent. All asymptotics are in data length, but the results are also applicable to the i.i.d. case, in the limit as the noise variance vanishes. The three
steps in the proof are as follows. First, we prove that the least-squares estimate of the noiseless data, \( \hat{y} \in g(\Gamma_k) \), is a consistent estimator of \( y \). Second, we show that the least-squares estimate of the parameter, \( \hat{\gamma} \in \Gamma_k \), converges in the geodesic distance to the true parameter \( \gamma \in \Gamma_k \) (in the sense defined below). Third, equipped with the strong law of large numbers and the convergence of Taylor series, we obtain the desired consistency of \( \hat{\rho}_{WALD} \).

We first study the behaviour of the least-squares estimate (maximum likelihood if the noise is i.i.d. Gaussian) when the FIM is singular at the true parameter. We introduce some notation before stating the main result. Let \( g(\gamma) = (g(t; \gamma))_{t=1}^{\infty} \), and let \( \mathcal{Y} = g(\Gamma_k) \subset l_\infty(\mathbb{Z}) \) denote the set of noiseless data sequences generated by \( g \). The level sets of \( g \) define equivalence classes in the parameter space:

\[
\mathcal{E}(y) = \{ \gamma \in \Gamma_k : g(\gamma) = y \}
\]

The true parameter \( \gamma \in \Gamma_p \subset \Gamma_k \) belongs to the equivalence class, \( \mathcal{E} = \mathcal{E}(y) \). Complete identifiability of \( \gamma \) is equivalent to nonsingularity of FIM at \( \gamma \) (Proposition 1). Thus the true mode, when overparameterized, can only be identified up to the equivalence class, \( \mathcal{E} \). Any sequence \( z \) in \( \mathcal{Y} \) which minimizes

\[
Q^N(z) = \frac{1}{N} \sum_{t=1}^{N} |x(t) - z(t)|^2
\]  

will be called the least squares estimator, \( \hat{y}^N \), of \( g(\mathcal{E}) = y \). Under an assumption on the function \( g \), the consistency of the least squares estimator, \( \hat{y}^N \), can be proved by modifying a proof in [60].

**Theorem 1 (consistency in data space)** Let \( x(t) = y(t) + \epsilon(t) \), with zero mean and finite variance noise \( \epsilon(t) \). Let \( \hat{y}^N \) be the least squares estimate of the data sequence
\[ y = g(\mathcal{E}) \]. Assume that \( g(t; \gamma) \) is a continuous function of \( \gamma \), and that the tail cross product of the function \( g(\gamma) \) with itself exists and is non-zero. Further assume that

\[ Q(z) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} |y(t) - z(t)|^2 \quad (2.44) \]

has a unique minimum at \( z = y \), where \( y \in \mathcal{Y} \) is the true data sequence. Then, \( \{\hat{\gamma}^N\} \) and \( \hat{\sigma}_N^2 = Q^N(\hat{\gamma}^N) \) are strongly consistent estimators\(^{11}\) of \( y \) and the noise variance, \( \sigma^2 \).

Theorem 1 does not require the model \( g(t; \cdot) \) to be a superposition model and hence holds for more general nonlinear models.

Before proving Theorem 1, we briefly review the concept of tail product and tail cross products for complex valued sequences and complex valued random variables [69]; the case of real valued sequences was considered in [60]. Let \( u = (u(t)) \) and \( v = (v(t)) \) be two sequences of complex numbers and define

\[ [u, v]_N = \frac{1}{N} \sum_{t=1}^{N} u(t)v^*(t) \quad (2.45) \]

If \( \lim_{N \to \infty} [u, v]_N \) exists, its limit \([u, v]\) is called the tail product of \( u \) and \( v \). If \( u = v \), \( ||u|| = [u, u] \) is called the tail norm of \( u \). Let \( c \) and \( h \) be two sequences of complex valued functions on \( \overline{\Gamma}_k \), such that \([c(\gamma^a), h(\gamma^b)]_N\) converges uniformly to a complex number, \([c(\gamma^a), h(\gamma^b)]\), for all \( \gamma^a, \gamma^b \in \overline{\Gamma}_k \). If \((c, h)\) denotes the function on \( \overline{\Gamma}_k \times \overline{\Gamma}_k \to \mathbb{C} \) which takes \((\gamma^a, \gamma^b)\) into \([c(\gamma^a), h(\gamma^b)]\), then this function is called tail cross product of \( c \) and \( h \).

If \( c(t; \gamma) \) and \( h(t; \gamma) \) are continuous functions on \( \overline{\Gamma}_k \) and both \( c(t; \gamma) \) and \( h(t; \gamma) \) are uniformly bounded by a bounded function of \( t \), then the tail cross product of \( c \) and

\(^{11}\) A sequence of estimators, \( \hat{\gamma}^N \), is said to be strongly consistent for every \( \gamma \in \Gamma \) if \( P(\lim_{N \to \infty} \hat{\gamma}^N = \gamma) = 1 \).
h exists [69]. We restate a strong law of large numbers for real noise sequences [24], which directly applies to white circular complex noise sequences.

**Lemma 1** If \( \epsilon = (\epsilon(t)) \) is such that \( \mathbb{E}(\epsilon(t)) = 0 \) and \( \mathbb{E}(\epsilon^2(t)) = \sigma^2 < \infty \) for all \( t \), and if the tail norm of a sequence \( u \) of real numbers exists, then \( [u, \epsilon]_N \to 0 \) for almost every \( \epsilon \).

**Proof of Theorem 1** : To begin, from the strong law of large numbers (Lemma 1), we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} |\epsilon(t)|^2 = \sigma^2
\]  
(2.46)

Next, rewrite (2.43) to observe

\[
Q^N(z) = \frac{1}{N} \sum_{t=1}^{N} |g(t) + \epsilon(t) - z(t)|^2
\]  
(2.47)

\[
= \frac{1}{N} \sum_{t=1}^{N} [ |y(t) - z(t)|^2 + 2 \text{Re} \{ (y(t) - z(t))\epsilon^*_t \} + |\epsilon_t|^2 ]
\]  
(2.48)

By hypothesis, the tail cross product of \( g \) with itself exists. Hence, the second term in (2.48) goes uniformly to zero by Lemma 1. Therefore

\[
\lim_{N \to \infty} Q^N(z) = Q(z) + \sigma^2
\]  
(2.49)

uniformly for all \( z \in \mathcal{Y} \). Since \( g(t; \gamma) \in \mathbb{C} \) is a continuous function of \( \gamma \) for each \( t \), we have that \( g(\gamma) = (g(t; \gamma)) \) is also a continuous function of \( \gamma \). Since the set \( \bar{\Gamma}_k \subset \mathbb{R}^{2k} \) is bounded and closed, it is also compact. Continuity of \( g(\gamma) \) implies that the set \( \mathcal{Y} = g(\bar{\Gamma}_k) \) is a compact subset of \( l_{\infty}(\mathbb{Z}) \). Since \( \mathcal{Y} \) is compact, there exists a subsequence \( \tilde{y}^n \) of \( \tilde{y}^N \) such that \( \lim_{n \to \infty} \tilde{y}^n = \tilde{y} \), where \( \tilde{y} \) denotes the limit of \( \{\tilde{y}^n\}_{n=1}^{\infty} \).

Note that \( Q \) is continuous and \( Q^n \) converges uniformly to \( Q \) which implies

\[
\lim_{n \to \infty} Q^n(\tilde{y}^n) = Q(\tilde{y}) + \sigma^2
\]  
(2.50)
Since $\hat{y}^n$ is a least squares estimator, we obtain

$$Q^n(\hat{y}^n) \leq Q^n(y) = \frac{1}{n} \sum_{t=1}^{n} |e(t)|^2 = \sigma^2 \quad (2.51)$$

Therefore, as $n \to \infty$,

$$Q(\bar{y}) + \sigma^2 \leq \sigma^2 \quad (2.52)$$

which implies that $Q(\bar{y}) = 0$. Since $Q$ has a unique minimum at $y$, we obtain $\bar{y} = y$. Note that the above conclusion holds for all the limit points of the sequence $\hat{y}^N$. Thus $\hat{y}^N \to y$.

Using the same arguments as above, we obtain that $Q^N(\hat{y}^N) \to Q(y) + \sigma^2$. Since $Q(y) = 0$, we conclude that $Q^N(\hat{y}^N)$ is a consistent estimator of $\sigma^2$.

The model in Example 2 satisfies all the hypotheses of Theorem 1, and the model in Example 1 satisfies the hypotheses if $s(t)$ is a continuous function with non-zero tail cross product$^{12}$.

Next, we prove that the least squares estimator is also consistent in the parameter space in the following sense. If the least squares estimate of the data sequence, $\hat{y}^N$, is consistent, then the equivalence class defined by $\hat{y}^N, \tilde{E}^N$, comes arbitrarily close to $\mathcal{E}$. That is, if $Q(\hat{y}^N) \to 0$, then

$$\inf_{\gamma \in \mathcal{E}, \tilde{\gamma} \in \tilde{E}^N} \|\gamma - \tilde{\gamma}\| \to 0. \quad (2.53)$$

If the Fisher information matrix is continuous with respect to $\gamma$, then there exists a constant $C$ such that for any $\gamma \in \Gamma_K, \mu(\gamma, \Gamma_p) \leq C \inf_{\gamma \in \Gamma_p} \|\gamma - \tilde{\gamma}\|$. Thus, the above convergence of $\tilde{E}^N$ in 2-norm also implies the convergence in the geodesic distance.

$^{12}$Non-zero tail cross product implies that $s(t)$ is a finite power signal. In communication applications, $s(t)$ is generally a finite energy continuous waveform, like a raised cosine pulse; in this case, Theorem 1 holds for a fixed $N$ as $\sigma \to 0$. 
Corollary 1 (consistency in parameter space) Under the assumptions of Theorem 1, $\hat{E}^N$ is a singleton set with probability 1 and as $N \to \infty$ converges almost surely to some element of $E$.

Proof of Corollary 1: The Fisher information, $I(\gamma)$, is by hypothesis full rank at all $\gamma \in \Gamma_k$. Also, the set $\overline{\Gamma}_k \setminus \Gamma_k$ has a measure zero. Thus, by inverse function theorem [31], the inverse of $g$ is unique a.e. on $\overline{\Gamma}_k$, i.e., $\hat{E}^N$ is a singleton set with probability 1.

Since $\overline{\Gamma}_k$ is compact, $g$ is uniformly continuous on $\Gamma_k$. Further, since $g$ is one-to-one on $\Gamma_k$, $g^{-1}$ is uniformly continuous and unique almost everywhere on $\overline{\Gamma}_k$. Since $\hat{y}^N \to y$, the sequence $\hat{y}^N$ is Cauchy. From the uniform continuity of $g^{-1}$, it follows that $\hat{\gamma}^N$ is Cauchy and hence converges. Assume $\hat{\gamma} \to \gamma \not\in E$. Continuity of $g$ implies $g(\hat{\gamma}) = \hat{d}^N \to g(\gamma) \neq y$, which is a contradiction. Hence, $\hat{\gamma}$ converges to a point in $E$.

Thus the equivalence class estimates, $\hat{E}^N$, are almost always singletons, and the estimated parameter vector converges to an element in the true equivalence class, $E$. All parameters in the equivalence class $E$ produce the same data $y$ and hence are indistinguishable using observations $x(t), t = 1, \ldots, N$ for all $N$.

The limit point, $\lim_{N \to \infty} \hat{E}^N \in E$ for a particular noise realization is algorithm dependent. For example, in our simulations, we found that parameter estimates computed using subspace based techniques for superimposed undamped exponential models tend to converge to a point with $p$ modes at the true locations and $(K - p)$ modes used to model the noise; the amplitudes of the extraneous modes converge to zero such that no two location parameters are the same. On the other hand, the
limit points of a FFT-initialized method [135] converge to closely spaced equal energy modes.

Since the least-squares estimate of the parameter converges to an element in the equivalence class, $E$ (Corollary 1), the expected value of $\hat{E}^N$ may not converge to a single parameter value. This implies that when the equivalence class $E$ is not a singleton, the asymptotic distribution of $\hat{E}^N$ does not exist. Thus, the results pertaining to the asymptotic distribution of Wald statistic [125] are not applicable. Instead, we rely on convergence of the Taylor series expansion and the law of large numbers to prove the consistency of the proposed model order estimate.

**Theorem 2 (model order consistency)** Let the true model order be denoted by $p$ and the hypothesized maximum model order by $K \geq p$. Also assume that the conditions for Proposition 1 and Theorem 1 hold. Further assume that $f(x|\gamma)$ is at least twice continuously differentiable with respect to $\gamma$. Finally, assume that the Fisher information matrix satisfies the following rate conditions\(^{13}\) for $\tau^a(N), \tau^\theta(N) > \log(N),

$$
\lim_{N \to \infty} \frac{1}{\tau^a(N)} I(a) = \Sigma^a
$$
$$
\lim_{N \to \infty} \frac{1}{\tau^\theta(N)} I(\theta) = \Sigma^\theta
$$

for nonnegative definite matrices, $\Sigma^a$ and $\Sigma^\theta$. Then, $\hat{p}_{WALD} \to p$.

\(^{13}\)These rate conditions are similar to the Assumption B in [132].
Proof of Theorem 2: Since \( f(x|\gamma) \) is assumed to be at least twice differentiable, the loglikelihood, \( l(\gamma) = \log f(x|\gamma) \), can be expanded in the following Taylor series [129]

\[
l(\gamma) = l(\hat{\gamma}) + (\hat{\gamma} - \gamma)^T \frac{\partial l(\gamma)}{\partial \gamma} \bigg|_{\gamma = \hat{\gamma}} + (\hat{\gamma} - \gamma)^T H(\hat{\gamma})(\hat{\gamma} - \gamma) + \text{higher order terms}
\]

where \( H(\gamma) \) is the Hessian matrix with entries \( H_{ij}(\gamma) = \frac{\partial^2 l(\gamma)}{\partial \gamma_i \partial \gamma_j} \). By hypothesis, \( l(\gamma) \) is continuous and \( \hat{\gamma} \to \gamma \) which implies \( (\hat{\gamma} - \gamma)^T H(\hat{\gamma})(\hat{\gamma} - \gamma) \to 0 \) [129]. From the strong law of large numbers [118], it follows that \( H(\hat{\gamma}) \to I(\gamma) \). Also, from the assumed continuity of FIM, \( I(\gamma) \), we obtain that \( I(\hat{\gamma}) \to I(\gamma) \). Combining the above facts, we obtain

\[
(\hat{\gamma} - \gamma)^T I(\hat{\gamma})(\hat{\gamma} - \gamma) \to 0 
\]  
(2.54)

Note that the Taylor series also converges for any bounded transform of the parameters. This implies that \( \gamma \) and \( \hat{\gamma} \) in (2.54) can be replaced with \( \alpha \) and \( \tilde{\alpha} \), respectively. The transform \( \Theta_k \) is linear with bounded entries, and hence is a bounded transform. Thus the convergence result in (2.54) can likewise be obtained with \( \Delta \tilde{\theta} \) replacing \( \gamma \).

We first demonstrate that asymptotically, \( \widehat{p}_{\text{WALD}} \leq p \). First consider the amplitude test. From Proposition 1, it follows that there are \( (K - p^a) \) modes in \( \gamma \) with zero amplitudes, where \( p^a \) is the the number of modes with non-zero amplitude. For \( k \geq p^a \), \( A_k \alpha = 0 \) and from (2.29) we have

\[
W_k^a = (A_k \tilde{\alpha} - A_k \alpha)^T I(A_k \tilde{\alpha}) (A_k \tilde{\alpha} - A_k \alpha) 
\]  
(2.55)

From the convergence of Taylor series (2.54), we get \( \lim_{N \to \infty} W_k^a = 0 \), for \( k \geq p^a \). Thus,

\[
\lim_{N \to \infty} \widehat{p}^a = \lim_{N \to \infty} \left[ \arg \min_{k=0,\ldots,K} W_k^a + \frac{\alpha(k)}{2} \log(N) \right] \leq p^a 
\]  
(2.56)

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Thus, all modes with zero amplitudes are rejected by the amplitude test (2.31). Following the same arguments, the location test (2.32) rejects mode pairs having zero location differences. Thus, all the unidentifiable modes are rejected in (2.33) and maybe more, i.e.,

\[
\lim_{N \to \infty} \tilde{p}_{\text{WALD}} \leq p \tag{2.57}
\]

We next show that none of the identifiable modes are rejected, i.e., \( \lim_{N \to \infty} \tilde{p}_{\text{WALD}} \geq p \). From the structure of the FIM and Proposition 1, it follows that if \( p \) modes are identifiable, then the FIMs for amplitudes and location subparameters of the identifiable modes are also full rank. That is, \( I(a_p) \) and \( I(\theta_p) \) are full rank where \( a_p \) and \( \theta_p \) are parameters of the identifiable modes.

Again consider the amplitude test first. Let \( \lim_{N \to \infty} \tilde{a} = a \). Identifiability of each subparameter implies that for all \( k \leq p \), \( \rho(\Sigma_k^a) = b \geq 1 \), where \( \Sigma_k^a = \lim_{N \to \infty} \frac{I(A_k\tilde{a})}{\tau^a(N)} \).

Consider the singular value decomposition of \( \Sigma_k^a = U_k^T \Lambda_k U_k \). Let \( L_k^T = U_k^T \Lambda_k^{1/2} \) and observe \( \rho(L_k) = \rho(\Sigma_k^a) \). The matrix \( \Sigma_k^a \) is FIM for the linearized model, \( L_k A_k a + \epsilon \) where \( \epsilon \) is i.i.d. Gaussian. Thus,

\[
\lim_{N \to \infty} \frac{1}{\tau^a(N)} W_k^a = (A_k\tilde{a})^T L_k^T L_k (A_k\tilde{a}) = \|L_k A_k \tilde{a}\|^2 \geq 0 \tag{2.58}
\]

with equality iff \( A_k \tilde{a} \) is in the null space of \( L_k \), which contradicts the fact that the linearized model has non-zero energy. Thus \( \lim_{N \to \infty} \frac{1}{\tau^a(N)} W_k^a \) is bounded away from zero. Combining the above with the Taylor series convergence (2.54), we obtain that \( W_k^a \sim O(\tau^a(N)) \) for \( k \leq p \). Since \( \tau^a(N) > \log(N) \), we immediately obtain \( \lim_{N \to \infty} \tilde{p}^a \geq p \). Using similar arguments, we obtain \( \lim_{N \to \infty} \tilde{p}^\theta \geq p \). Thus, asymptotically all the identifiable modes are retained, which implies,

\[
\lim_{N \to \infty} \tilde{p}_{\text{WALD}} \geq p \tag{2.59}
\]
Combining (2.57) and (2.59), we obtain \( \lim_{N \to \infty} \hat{\rho}_{\text{Wald}} = p. \)

Note that the model classes discussed in Examples 1 and 2 satisfy all the conditions of Theorem 2. The rate of growth of submatrices of FIM for Example 2 can be found in [117], where it was shown that \( r_A(N) = N \) and \( r^\theta(N) = N^3 \).

## 2.5 Implementation Issues and Simulation Results

In this section, we briefly discuss some implementation issues and present simulation results for the nested model class discussed in Example 2.

### 2.5.1 Dynamic program

Rather than estimate \( \hat{\gamma} \) for the maximum model order \( K, \hat{\gamma} \in \Gamma_j \), may be sequentially estimated for \( j \leq k \) to minimize the expected computational cost. This is conveniently achieved by embedding the proposed model order estimate in a dynamic program [38,45]. Based on the \( \text{a priori} \) probability distributions of model order and model parameters, and the cost of computing estimates for different candidate models, a sequential procedure is obtained to selectively estimate candidate model parameters. The complete dynamic program can be intractable since it requires distribution of \( \hat{\rho}_{\text{Wald}} \); a useful asymptotic approximation can be obtained by using the consistency of \( \hat{\rho}_{\text{Wald}} \). Thus, the asymptotic dynamic program (ADP) is obtained by using

\[
\hat{\rho}_{\text{Wald}}(\hat{\gamma} \in \Gamma_{k}) = \begin{cases} 
\delta_{p} & p < k, \\
\delta_{k} & p \geq k, \quad \forall \; \gamma \in \Gamma_{k}
\end{cases}
\]

where \( \delta_{m} \) is Kronecker delta.
Having obtained \( p_{\text{wald}} \), the parameters of the modes retained in (2.33) provide an estimate for the unknown parameters of the modes; denote the parameter estimate so obtained as \( \gamma = \{ (\alpha_i, \phi_i, \theta_i) \}_{i=1}^{p_{\text{wald}}} \). An alternate parameter estimate, \( \gamma_p \), is obtained by minimizing the cost (2.43) over the parameter set \( \Gamma_{p_{\text{wald}}} \). The variance of the parameter estimate, \( \gamma \), is higher than the variance of the parameter estimate \( \gamma_p \), because \( \gamma \) is a subset of a larger parameter \( \gamma \in \Gamma_k, k \geq p_{\text{wald}} \). For large differences, \( K - p_{\text{wald}} \), the variance of \( \gamma \) may be undesirably high. On the other hand, a complete reestimation of the parameters to obtain \( \gamma_p \) adds to computation. A low complexity alternative is to use a gradient descent method with \( \gamma \) as initial estimates [135].

### 2.5.2 Simulation results

The detection performance and computational complexity of the proposed method is compared to MDL and MAP [33]. The penalty terms for the MDL and MAP estimates are \( \alpha(k) = 3k \) and \( \alpha(k) = 5k \), respectively. The difference in the penalty terms for the two rules arises from different asymptotic analyses. The asymptotic rule for MDL is derived by considering i.i.d. data collection or equivalently as the noise power vanishes. On the other hand, the penalty term for the MAP rule is obtained as data length, \( N \), increases unboundedly; for the model in Example 2, this implies independent but non-identically distributed data.

For the simulation results, a uniform distribution on model order for \( k = [1, \ldots, 7] \) was assumed. The maximum hypothesized model order was chosen to be \( K = 7 \), and for each experiment, 1000 trials were performed. The ADP chooses\(^{14} \) to compute only the ML estimate for \( K = 7 \).

\(^{14}\)For \( K = 15 \), for example, ADP computes ML estimate for \( k = 13 \). If \( p_{\text{wald}} < 13 \), the procedure stops, else the ML estimate for \( k = 15 \) is computed. For large maximum model order \( K \), the ADP can significantly reduce computation.
In Figure (2.2), the detection performance of $\hat{p}_{\text{WALD}}$ is compared to $\hat{p}_{\text{MDL}}$ and $\hat{p}_{\text{MAP}}$ for $N = 64$ samples. The probability of correct detection is shown as a function of SNR/mode/Fbin, where the Fourier bin is defined as $\text{Fbin} = 1/N$.

In Figure (2.3) the detection performance is shown as a function of number of samples, $N$, with $\text{SNR/mode} = -15 \text{ dB}$. The ratio of the average computational cost of MDL and MAP to the average computational cost of $\hat{p}_{\text{WALD}}$ is shown in Figure (2.4) as a function of the data length. The average computational cost of MDL and MAP is more than three times the proposed method, with similar detection performance. Note that the computational savings increase as either maximum model order, $K$, or the data length, $N$, increases.

### 2.6 Conclusions

In this chapter, we presented a model order estimation procedure for superimposed models, which are frequently encountered in engineering applications. The proposed technique is consistent and is computationally more efficient than order selection using the minimum description length principle. The consistency proof requires the consistency of the overparameterized maximum likelihood estimates, for which the existing results were extended to the case where the noiseless data can be generated by multiple parameters.
Figure 2.1: Notional geometry of the nested classes.
Figure 2.2: Probability of correct detection of model order as a function of SNR

Figure 2.3: Probability of correct detection of model order as a function of data length at SNR/mode = -15 dB
Figure 2.4: Ratio of average computational complexity \( \left( \frac{C_{MDL/MAP}}{C_{Wald}} \right) \) as a function of data length at SNR/mode = -15 dB
SECTOR BEAM SYNTHESIS FOR CELLULAR SYSTEMS
USING PHASED ANTENNA ARRAYS

3.1 Introduction

Cell sectorization has been widely proposed for improving system capacity in cellular systems [20, 76, 78]. Every cell is divided into multiple sectors, and each sector is served by a dedicated antenna. With ideal sector antennas, the increase in system capacity for both voice and data networks is proportional to the number of sectors [59]. As an additional benefit, cell sectorization is reported to result in a considerable decrease in delay spread in the received signal [115]. However, perfect sectorization is not realizable. Non-ideal radiation patterns reduce the capacity of sectorized cellular systems due to inter-sector interference [59, 73, 84].

In this chapter, we propose a sector beam synthesis technique using phased antenna arrays to minimize the total inter-sector interference. We assume that each cell is divided into sectors of equal width and that each sector is served by an independent antenna array, as depicted in Figure 3.1. In addition to synthesizing the desired pattern, the proposed technique allows us to quantify the relationships among sector size, beam efficiency, signal-to-interference ratio, and capacity.
Several approaches for sector beam synthesis have been proposed [17,39,66,108]. Most published methods are extensions of digital filter design techniques [39,66,108]. However, sector beam synthesis using antenna arrays is considerably different from conventional filter design for the following reasons. First, the analogy between filter design and sector synthesis holds only for $\lambda/2$ inter-element spacing in uniform linear arrays (ULAs) with isotropic elements. Second, the phase of the synthesized beam is irrelevant for many array applications; in contrast, linear phase is an important requirement in many filter design problems. Third, the minimax criterion widely used in filter design is not well suited for multisector wireless applications because the criterion does not appear to be directly related to any significant communication performance parameter.

In contrast, the proposed approach minimizes the total inter-sector interference subject to a ripple constraint on the in-sector power pattern. This is an example of power synthesis [17,79]. We show that the proposed non-convex problem can be converted into a convex fractional linear programming problem with linear constraints, via a suitable reparametrization [133]. The convex reparameterization guarantees that a globally optimal solution can be computed using a numerically efficient algorithm. The proposed approach accommodates non-isotropic antenna elements, arbitrary element spacing, and non-central sectors\(^{15}\), but is limited to uniformly spaced linear (or planar) arrays.

\(^{15}\) The center of the sector is not normal to the plane of the array.
In Section 3.2, we motivate the design criterion of maximum beam efficiency subject to a constraint on in-sector ripple. The optimization problem is solved by fractional linear programming by using a suitable reparametrization to yield convex constraints. Some numerical results from the algorithm are presented in Section 3.3. To illustrate the utility of proposed results, we compute the minimum size of the antenna arrays required for the sectorized CDMA system analysed in [59].

A joint optimization involving all the cell antenna elements as a single array would lead to a smaller inter-sector interference. However, we avoid a joint optimization for two reasons. First, a design dividing the array elements to a number of disjoint groups, one group for each sector antenna, allows distributed implementation of the hardware for each sector. This simplifies the architecture of the base station. Second, the commonly used antenna arrays employ directional antenna elements, and hence the loss in a distributed optimization is small.

3.2 Problem Formulation

3.2.1 The linear array pattern

Consider a linear array of \( K \) antennas at locations \( x_0, x_1, \ldots, x_{K-1} \in \mathbb{R} \). The radiation pattern of each antenna element is \( Q_i(\theta) \), \( i = 0, \ldots, K - 1 \). A plane wave with wavelength \( \lambda \) is incident from direction \( \theta \) and propagates across the array (which we assume for simplicity does not change the field). The angle \( \theta \) is measured with respect to the normal of the array. The \( K \) signal outputs are scaled by complex weights \( w_i \) and summed to give the linear array beam pattern \( W(\theta) \)

\[
W(\theta) = \sum_{i=0}^{K-1} w_i Q_i(\theta) \exp \left( -\frac{2\pi}{\lambda} x_i \sin(\theta) \right) = \sum_{i=0}^{K-1} w_i g_i(\theta). \quad (3.1)
\]
The weights, \( w = [w_0, w_1, \ldots, w_{K-1}]^T \), are chosen to achieve a desired beam pattern \( W(\theta) \). The power density emitted by the antenna array is given by \( P(\theta) = |W(\theta)|^2 \).

### 3.2.2 Inter-sector interference and beam efficiency

A terminal located in any sector (e.g., terminal T1 in Sector 1, Figure 3.1) is subjected to interference generated by the base station transmitters driving the remaining \( N-1 \) sectors. Assuming that all \( N \) transmitters are equal in power and emit uncorrelated signals, then the total interference power density suffered by a terminal located at angle \( \theta \) is

\[
J(\theta) = \sum_{n=1}^{N-1} P \left( \theta + n \frac{2\pi}{N} \right)
\]

where \( J(\theta) \) and \( P(\theta) \) are, in this case, expressed in Watt/radian\(^{16} \). Integrating \( J(\theta) \) over one sector, we obtain the total interference power received by the sector from other sectors in the same cell. With no loss of generality, we consider the sector \([0, \frac{2\pi}{N}]\).

\[
J_s = \int_0^{\frac{2\pi}{N}} J(\theta) d\theta = \int_0^{\frac{2\pi}{N}} \sum_{n=1}^{N-1} P \left( \theta + n \frac{2\pi}{N} \right) d\theta = \int_0^{\frac{2\pi}{N}} P(\theta) d\theta
\]

The beam efficiency, \( \eta \), is defined as the ratio of the power transmitted in the sector of interest to the total power radiated by the array, i.e.,

\[
\eta = \frac{\int_0^{\frac{2\pi}{N}} P(\theta) d\theta}{\int_0^{2\pi} P(\theta) d\theta}
\]

\[
= \frac{\int_0^{2\pi} P(\theta) d\theta - \int_0^{\frac{2\pi}{N}} P(\theta) d\theta}{\int_0^{2\pi} P(\theta) d\theta} = 1 - \frac{J_s}{P}
\]

\(^{16}\)The antenna arrays considered allow beam pattern shaping only in the horizontal plane, thereby reducing the problem to a two-dimensional problem.
where $P$ is the power transmitted by one transmitter. Hence, $(1 - \eta)P$ represents the total interference power radiated within the boundaries of one sector. Thus, $\eta$ is an important parameter in sectorized systems.

### 3.2.3 Maximizing beam efficiency

Consider a sector of interest $\Theta_s = [\theta_l, \theta_u], -\pi \leq \theta_l \leq \theta_u \leq \pi$. The beam efficiency is given by

$$
\eta = \frac{\int_{\theta_l}^{\theta_u} P(\theta) d\theta}{\int_{-\pi}^{\pi} P(\theta) d\theta}
$$

(3.7)

The power transmitted by the antenna array to the sector spanning $\Theta_s$ is

$$
P_{\Theta_s} = \int_{\theta_l}^{\theta_u} P(\theta) d\theta
$$

$$
= \int_{\theta_l}^{\theta_u} \left( \sum_{i=0}^{K-1} w_i g_i(\theta) \right) \left( \sum_{k=0}^{K-1} w_k g_k(\theta) \right) d\theta
$$

$$
= \sum_{i=0}^{K-1} \sum_{k=0}^{K-1} w_i \bar{w}_k \int_{\theta_l}^{\theta_u} g_i(\theta) \bar{g}_k(\theta) d\theta
$$

$$
= \mathbf{w}^H \mathbf{A} \mathbf{w}
$$

(3.8)

where $A_{ik} = \int_{\theta_l}^{\theta_u} g_i(\theta) \bar{g}_k(\theta) d\theta$. By substituting $\theta_l = -\pi$ and $\theta_u = \pi$, the total power radiated by the array can be computed as

$$
P_{\Theta} = \mathbf{w}^H \mathbf{M} \mathbf{w}
$$

(3.9)

where $M_{ik} = \int_{-\pi}^{\pi} g_i(\theta) \bar{g}_k(\theta) d\theta$. The matrix $A$ is Hermitian and positive definite for all values of $-\pi \leq \theta_l \leq \theta_u \leq \pi$. Furthermore, $M$ is a symmetric and positive definite matrix. The matrix $M$ is real if $Q_i(\theta)$ is symmetric about $\theta = 0$ for all $i = 0, \ldots, K-1.$
Using Equations (3.8) and (3.9), efficiency can be written as

\[
\eta = \frac{w^H A w}{w^H M w} = \frac{v^H M^{-1/2} A M^{-1/2} v}{v^H v} = \frac{v^H T v}{v^H v}
\]  

(3.11)

where \( v = M^{1/2} w \) and \( M^{1/2} \) is the matrix square root of \( M \). From Equation (3.10), it is evident that maximizing beam efficiency in absence of any constraints requires finding an eigenvector corresponding to the maximum eigenvalue of the matrix \( T \). The matrix \( T \) depends on the design variables, \( K, \frac{d}{\lambda}, Q_i(\theta) \) and \( \Theta_s \), but is independent of the weight vector \( w \). Thus, fixing the design variables fixes the maximally efficient pattern (or the subspace of patterns if the maximum eigenvalue of \( T \) is non-unique).

A pattern thus obtained has a large in-sector peak-to-peak ripple, especially at the end points of sectors. As an example, a maximally efficient pattern for a 30 degree central sector is shown in Figure 3.2 (for the array described in Section 3.3). The high in-sector ripple, which is characteristic of the maximally efficient patterns, is undesirable in cellular applications, for it leads to a wide distribution of the user powers. This motivates a constraint on in-sector peak-to-peak ripple, \( \rho \),

\[
\rho = 10 \log_{10} \frac{\max_{\theta \in \Theta_s} P(\theta)}{\min_{\theta \in \Theta_s} P(\theta)}
\]  

(3.12)

A ripple constraint, \( \rho \leq \epsilon \) for \( 0 < \epsilon < \infty \), can be expressed as a set constraint, \( \mathcal{C}_\epsilon \), on the array weights, \( w \).

\[
\mathcal{C}_\epsilon = \{ w : \rho \leq \epsilon \}
\]  

(3.13)

For any \( w \in \mathcal{C}_\epsilon \) with \( \min_{\theta \in \Theta_s} P(\theta) = \gamma \), the ripple constraint \( \rho \leq \epsilon \) implies

\[
\gamma \leq P(\theta) \leq 10^{\epsilon/10} \gamma, \quad \theta \in \Theta_s.
\]  

(3.14)
We arbitrarily assign $\gamma = 1$, thereby fixing a scale for $w$, and define
\begin{equation}
D_c = \{w : 1 \leq P(\theta) \leq 10^{4/10}, \quad \theta \in \Theta_s, \quad \text{and} \quad P(\theta) \geq 0, \quad \theta \in \Theta \backslash \Theta_s\}.
\end{equation}

where $\Theta = [-\pi, \pi]$ and $\Theta \backslash \Theta_s = \{\theta : \theta \in \Theta \text{ and } \theta \notin \Theta_s\}$. The positivity constraint, $P(\theta) \geq 0$, always holds since $P(\theta)$ is the power of the pattern $W(\theta)$; the importance of explicitly requiring positivity will become apparent in the next subsection. From the definitions of $\rho$ and the set $D_c$, we conclude that if $w \in C_c$, then $\alpha w \in D_c$ for some $\alpha > 0$. Because efficiency is invariant to scaling of $w$, we can use $D_c$ to constrain the in-sector ripple without any loss of generality. Thus, the proposed optimization problem is
\begin{equation}
\max_{w \in D_c} \eta
\end{equation}

The optimization problem in (3.16) is an example of power synthesis, in which no phase requirements are prescribed [17]; therefore, all available degrees of freedom in $w$ are utilized to optimize the power pattern. Power synthesis generally leads to a non-convex optimization problem (e.g. (3.16)), and is considered more complicated than traditional convex design approaches [17]. However, power synthesis is very important because pattern specifications for radar and communication systems are often defined in terms of magnitude only [17, 63, 79]. Using the ideas in [133], we show that for uniform linear arrays, a reparametrization of the power synthesis task in (3.16) yields a convex optimization problem.

3.2.4 Reparametrization (ULA)

Consider the following reparametrization of $P(\theta)$ assuming a uniform linear array (ULA), i.e., $Q_i(\theta) = Q(\theta), \quad x_i = i\Delta$, for all $i = 0, \ldots, K-1$. The array power
pattern may be written

\[ P(\theta) = |W(\theta)|^2 \]  

(3.17)

\[ = |Q(\theta)|^2 \sum_{i=0}^{K-1} w_i \exp \left( -j \frac{2\pi \Delta}{\lambda} i \sin(\theta) \right) \sum_{k=0}^{K-1} \bar{w}_k \exp \left( j \frac{2\pi \Delta}{\lambda} k \sin(\theta) \right) \]  

(3.18)

\[ = |Q(\theta)|^2 \sum_{i=0}^{K-1} \sum_{k=0}^{K-1} w_i \bar{w}_k \exp \left( j \frac{2\pi \Delta}{\lambda} (k-i) \sin(\theta) \right) \]  

(3.19)

\[ = \sum_{m=-(K-1)}^{K-1} p_m \exp \left( j \frac{2\pi \Delta}{\lambda} m \sin(\theta) \right) |Q(\theta)|^2 = d_H^T p \]  

(3.20)

where

\[ p_m = \sum_{i=-(K-1)}^{K-1} w_i \bar{w}_{i+m}, \quad m = -(K-1), \ldots, K-1, \]  

(3.21)

\[ d_{\theta,m} = \exp \left( j \frac{2\pi \Delta}{\lambda} m \sin(\theta) \right) |Q(\theta)|^2 \]  

(3.22)

The constraint set \( \mathcal{D}_c \) can be rewritten in terms of \( p \) as

\[ \mathcal{D}_c = \{ p : 1 \leq d_H^T p \leq 10^{\theta_{10}} \forall \theta \in \Theta_s, \text{ and} \]  

(3.23)

\[ d_H^T p \geq 0 \forall \theta \in \Theta \setminus \Theta_s \} \]

The positivity constraint on power, \( P(\theta) = d_H^T p \geq 0 \), was redundant in (3.15) but is required in (3.23) so that the optimization may be performed over the new parameterization, \( p \). This ensures that we consider only those \( p \) which represent a valid autocorrelation sequence for some \( \mathbf{w} \in \mathbb{C}^K \). The constraint set in (3.23) is an infinite set of linear inequalities in \( p \). Further, the beam efficiency can also be rewritten in terms of \( p \),

\[ \eta = \frac{a_H^T p}{b_H^T p} \]  

(3.24)

where for \( m = -(K-1), \ldots, K-1 \)

\[ a_m = \int_{\Theta_s} |Q(\theta)|^2 \exp \left( -j \frac{2\pi \Delta}{\lambda} m \sin(\theta) \right) d\theta \]  

(3.25)

\[ b_m = \int_{-\pi}^{\pi} |Q(\theta)|^2 \exp \left( -j \frac{2\pi \Delta}{\lambda} m \sin(\theta) \right) d\theta \]  

(3.26)

48
Thus, the constrained optimization problem in (3.16) can be stated as a semi-infinite fractional linear program (FLP) in terms of the new variable, $p$.

\[(F): \quad p^* = \arg \max_{p \in \mathbb{D}_x} \frac{a^H p}{b^H p} \quad (3.27)\]

Using the methods of [26], we show that the FLP in (3.27) can be converted to the following linear program.

\[(L): \quad (u^*, t^*) = \arg \max_{(u, t) \in \mathcal{K}_e} a^H u \quad (3.28)\]

The set $\mathcal{K}_e$ is given by

\[\mathcal{K}_e = \{ (u, t) : t \leq d^H_0 u \leq t 10^{1/10}, \theta \in \Theta_s; \quad d^H_0 u \geq 0, \theta \in \Theta \setminus \Theta_s; \quad t > 0; \quad b^H u = 1 \} \quad (3.29)\]

where $u = tp$ and $t = (b^H u)^{-1}$. The equivalence of (F) and (L) can be demonstrated as follows. If $p \in \mathbb{D}_e$, then with $u = tp$, $t > 0$, we obtain

\[1 \leq d^H_0 p \Rightarrow t \leq d^H_0 u \quad (3.30)\]

\[d^H_0 p \leq 10^{1/10} \Rightarrow d^H_0 u \leq t 10^{1/10} \quad (3.31)\]

\[d^H_0 p \geq 0 \Rightarrow d^H_0 u \geq 0 \quad (3.32)\]

\[t = (b^H p)^{-1} \Rightarrow b^H u = 1 \quad (3.33)\]

which implies $(u, t) \in \mathcal{K}_e$. Similarly, if $(u, t) \in \mathcal{K}_e$, $t > 0$, then the converse is true in (3.30)-(3.32). This implies that $p = \frac{1}{t} u \in \mathbb{D}_e$. Lastly,

\[\frac{a^H p}{b^H p} = \frac{a^H u}{b^H u} = a^H u \quad (3.34)\]

Thus, we have established that if $(u^*, t^*)$ is an optimal solution for Problem (L), then

$\frac{1}{t^*} u^*$ is an optimal solution for Problem (F) and vice versa.
Problem (L) is a semi-infinite linear program in $4K - 1$ real variables, and is therefore a convex optimization problem. For convex optimization problems, a globally optimal solution can be computed using numerically efficient methods.

A simple method to approximately solve (3.28) is to discretize the constraint set $\mathcal{K}_c$ [54]. For instance, select a set of angles, $-\pi \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_L \leq \pi$ and replace $\mathcal{K}_c$ with the set

$$\mathcal{K}_c^N = \{(u, t) : t \leq \overrightarrow{d_{\theta_i}u} \leq t10^{-10}, \theta_i \in \Theta_i;$$

$$\overrightarrow{d_{\theta_i}u} \geq \delta, \theta_i \in \Theta \setminus \Theta_i;$$

$$t > 0; \overrightarrow{b^H u} = 1\}.$$ (3.35)

where $\delta > 0$ is a small number chosen so that the spectral factorization condition is met by the solution to the discretized problem. The discretized constrained problem is an outer approximation [54] to the original semi-infinite problem and can be efficiently solved in practice. Further, as the discretization becomes finer, the solution of the discretized problem approaches the solution of (3.28); a standard rule of thumb is to choose $L \approx 15K$ for discretizing semi-infinite programs [133], but we use $L \approx 225K$ in our computed results for improved accuracy.

Having obtained the globally optimal solution, $p^*$, a corresponding optimal array weight vector $w^*$ can be obtained via spectral factorization [9]. There exist up to $2^M$ excitation vectors, $w^*$, with autocorrelation $p^*$, where $2M$ is the number of zeros of $p^*$ with magnitude not equal to one. This provides an added degree of freedom in the choice of the optimal weight vector. For example, from among all spectral factors

17 There exists an $w \in \mathbb{C}^K$ such that $p$ is the autocorrelation of $w$ if and only if $\overrightarrow{d_{\theta_i}p} \geq 0$ for all $\theta \in [-\pi, \pi]$ [9].

18 The parameter $\delta$ is introduced to ensure that $\overrightarrow{d_{\theta_i}p} \geq 0$, for all $\theta \in (\theta_i, \theta_{i+1})$, $i = 1, \ldots, L - 1$, and hence $p \in \mathcal{K}_c$; $\delta$ can be increased until the spectral factorization condition is met [133].

19 We use the Mathworks, Inc. MATLAB routine Ip to solve the discretized problem (3.35).
of $p^*$, an excitation vector with least difference between the power fed to adjacent elements would reduce strong mutual coupling effects in the array.

### 3.2.5 Extensions: constrained SIR

The worst case signal-to-interference ratio (SIR) experienced by any terminal is $\min_{\theta \in [\theta_i, \theta_u]} \frac{P(\theta)}{J(\theta)}$. This will typically happen at the sector boundaries, i.e., for $\theta = \theta_l$ or $\theta = \theta_u$, and will be less than 1 (see the numerical results in the next section). In some applications, it may be desirable to keep the worst case SIR above a critical value, $\beta$, for most of the sector.

$$\text{SIR}(\theta) = \frac{P(\theta)}{J(\theta)} \geq \beta, \quad \theta \in \Theta_t \subset \Theta_s$$

(3.36)

We assume that $\Theta_t$ is a strict subset of $\Theta_s$ and does not include the end points of the sector. The semi-infinite constraint on SIR is also a linear constraint, for any $\theta \in \Theta_s$, as shown below.

$$\frac{P(\theta)}{J(\theta)} \geq \beta$$

(3.37)

$$d_0^H p \geq \beta \sum_{n=1}^{N-1} d_{(\theta+n\frac{2\pi}{N})}^H p$$

(3.38)

$$\left( d_\theta - \beta \sum_{n=1}^{N-1} d_{(\theta+n\frac{2\pi}{N})} \right)^H p \geq 0$$

(3.39)

Thus, the linear program (L) in (3.28)-(3.29) can be readily extended to include the constraint in (3.39) limiting worst-case SIR.

### 3.3 Numerical Results

In this section, we present some numerical results obtained using the proposed synthesis technique. Specifically, we show how efficiency varies as a function of the
size of the array and sector size; both central and non-central sectors are considered. Finally, we present an example to demonstrate the use of the proposed method in calculating antenna array sizes for the DS-CDMA system analyzed in [59].

For all results in this section, we use $\epsilon = 3 \text{ dB}$ and a single element radiation pattern, $Q(\theta)$, that is an analytical approximation to the measured radiation pattern of a prototype array element built by Lucent Technologies (see Figure 3.3).

### 3.3.1 Beam efficiency versus sector width

In Figure 3.4, beam efficiency as a function of sector width is shown for central sectors. For a fixed array and a limit on the peak to peak ripple, the beam efficiency decreases as the sector width decreases. This implies that the average signal-to-interference ratio, $10 \log_{10} \frac{\eta}{1-\eta}$, decreases with a finer sectorization. In other words, to maintain a desired SIR, a finer sectorization requires a larger array with an increased cost. The same conclusions can be drawn for the non-central sectors, spanning $[0, \theta_u]$, as shown in Figure 3.5. Because of the specific $Q(\theta)$ in Figure 3.3, very large non-central sectors cannot be generated as efficiently as the smaller sectors. In Tables 1 and 2, the corresponding average signal-to-interference ratios are shown.

In Figure 3.6, we compare two 90 degree central sectors generated using 16-element arrays with $\frac{\Delta}{\lambda} = 0.5$ and $\frac{\Delta}{\lambda} = 0.55$. Finally, in Figure 3.7, two 45 degree non-central sectors are generated using a single 16-element antenna array to cover a full 90 degree sector with $\frac{\Delta}{\lambda} = 0.55$; this example demonstrates how finer sectorization can be obtained without employing additional antenna arrays.
3.3.2 Array size calculations

To illustrate the utility of the proposed sector synthesis technique, we compute the antenna array requirements for a sectorized DS-CDMA cellular system considered in [59] for voice communication. Similar analysis can be performed for throughput of data networks using the results in [59].

In [59], the imperfection in sectorization was modeled by considering an ideal antenna with a beam covering the sector \([0, \frac{2\pi}{N} + \nu]\) where \(\nu\) is the overlap angle between sectors. Capacity, throughput, and delay analysis was performed by varying the overlap angle, \(\nu\). The efficiency for such a pattern is \(\eta = \frac{2\pi}{2\pi + \nu N}\). For overlap angle \(\nu = 0\), the efficiency is \(\eta = 1\). As \(\nu\) increases, the efficiency of the beam decreases. Also, for a fixed \(\nu\), the efficiency of the sector beam decreases as the sectorization is made finer.

For \(N = 6\) sectors, and using \(Q(\theta)\) in Figure 3.3, a 5-element array leads to 90% efficiency (for a central sector with \(\varepsilon = 3\text{dB}\) and \(\frac{\lambda}{\lambda} = 0.5\)), which implies a 6.66 degree overlap between the sectors. Using the results in [59], \(\nu = 6.66\) degree leads to 16% loss in the capacity (in terms of the number of users) when compared against the system with perfect sectorization (but with no voice activity monitoring). This loss can be reduced to below 10% if the antenna array size is increased to 12 elements, which corresponds to an efficiency of roughly 95%. We note that the above size calculations should be considered as lower bounds on the antenna array size for a given set of system parameters.
3.4 Conclusions

We presented a new method to synthesize sector beams using phased antenna arrays. To realize the full capacity benefits of cell sectorization, it is important that the sector antennas are designed to minimize the inter-sector interference. We presented a simple procedure to maximize beam efficiency, or equivalently, to minimize the inter-sector interference, with an in-sector ripple constraint on the power density. The proposed technique allows the designer to quantify the tradeoffs involved in finer sectorization, antenna array size, and system capacity.
Figure 3.1: A 90 degree cell sectorization with four antenna arrays serving one sector each.
Figure 3.2: Maximally efficient beam pattern for $K = 16, \frac{\lambda}{\lambda} = 0.5$. The in-sector peak-to-peak ripple is 38.61 dB and the efficiency of the beam pattern is 99.94%.
Figure 3.3: Radiation pattern of the antenna element used in simulation.
Figure 3.4: Beam efficiency as a function of central sector width. The in-sector peak-to-peak ripple is 3 dB and $\frac{A}{\lambda} = 0.5$. 
Figure 3.5: Beam efficiency as a function of non-central sector width ($\Theta_s = [0, \theta_u]$). The in-sector peak-to-peak ripple is 3 dB and $\frac{4}{\lambda} = 0.5$. 
Table 3.1: Average signal-to-interference ratio \(= 10 \log_{10} \left(\frac{n}{(1-\eta)}\right)\) for a 3 dB in-sector peak-to-peak ripple and \(\frac{\Delta}{\lambda} = 0.5\).

<table>
<thead>
<tr>
<th>Central sector width (in degrees)</th>
<th>Average SIR (in dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(K = 4)</td>
</tr>
<tr>
<td>15</td>
<td>-0.29</td>
</tr>
<tr>
<td>30</td>
<td>5.52</td>
</tr>
<tr>
<td>45</td>
<td>6.47</td>
</tr>
<tr>
<td>60</td>
<td>8.91</td>
</tr>
<tr>
<td>90</td>
<td>10.22</td>
</tr>
</tbody>
</table>

Table 3.2: Average signal-to-interference ratio \(= 10 \log_{10} \left(\frac{n}{(1-\eta)}\right)\) for a 3 dB in-sector peak-to-peak ripple and \(\frac{\Delta}{\lambda} = 0.5\).

<table>
<thead>
<tr>
<th>Non-central sector width (in degrees) ((\Theta_x = [0, \theta_u]))</th>
<th>Average SIR (in dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(K = 4)</td>
</tr>
<tr>
<td>15</td>
<td>-0.21</td>
</tr>
<tr>
<td>30</td>
<td>5.48</td>
</tr>
<tr>
<td>45</td>
<td>5.93</td>
</tr>
<tr>
<td>60</td>
<td>6.04</td>
</tr>
<tr>
<td>75</td>
<td>-8.54</td>
</tr>
</tbody>
</table>
Figure 3.6: 90 degree central sectors for $\frac{\Delta \lambda}{\lambda} = 0.5, 0.55$ and $K = 16$. 
Figure 3.7: Two 45 degree non-central sectors for $\frac{\Delta}{\lambda} = 0.55$ generated using a single 16-element array.
CHAPTER 4

MMSE RECEIVERS FOR MULTIRATE DS-CDMA SYSTEMS

4.1 Introduction

The growth of wireless networks has resulted in systems offering heterogeneous services, such as voice, video and data. Several system architectures based on Direct Sequence Code Division Multiple Access (DS-CDMA) signalling have been proposed recently [1, 30, 91, 92]. These proposals include systems with multiple chipping rates [1], variable spreading gain with constant chip rate [1] and multicode CDMA [82]. Another instance of a multiple chipping rate system is a wideband DS-CDMA system operating in the presence of an existing narrowband DS-CDMA system.

 Receivers which truly exploit the nature of multi-rate signals have focussed on synchronous and pseudo-synchronous systems in non-multipath environments [19, 21, 22, 82, 109]. Designs which accomodate asynchronous or multipath channels [116] do not fully take advantage of the multi-rate nature of the problem.

 In this chapter, we derive linear multiuser receivers based on the minimum mean squared error (MMSE) criterion, for multiuser systems where users with different symbol rates coexist. We first derive MMSE receivers without any causality constraints; this extends the work in [16, 46]. The proposed non-causal structure provides a basis
for more practical causal realizations and also provides a baseline for comparison to other multi-rate receiver structures. We show that the optimal linear MMSE receiver is, in general, time-varying. The same observation was also independently made in [19] for a dual rate synchronous DS-CDMA system; a non multipath environment with time-limited square pulses was assumed. The receiver structure developed herein is sufficiently general to include the cases of arbitrary alphabet size, multipath channels and multiple bandwidth systems. As the work in [19] and the current work both exploit the cyclostationarity of the multirate signals, there is a similarity in some of our conclusions.

In systems with multiple chipping rates, different user classes occupy different bandwidths. The choice of bandwidth of the front-end filter for the smaller bandwidth users determines the performance of their receivers. This performance, and hence total bandwidth, is a determining function in the total system capacity. Also, the front-end filter bandwidth determines the sampling rate which is one of the factors determining receiver cost for users with lesser spectral resources.

This chapter is organized as follows. In Section 4.2, the system model is described. The time-varying MMSE receiver is developed in Section 4.3. The effect of front-end filter bandwidth on the performance of smaller bandwidth users is studied in Section 4.4. The optimal time-invariant MMSE receiver is derived in Section 4.5 and its mean-squared error is compared to that of the time-varying MMSE receiver. Simulation results for a three class system are provided in Section 4.6; to gain insight, only a non-multipath environment is considered. The analytical predictions on relative performance and the effect of filter bandwidth are verified by the numerical results.
4.2 Problem Formulation

We consider multiuser multiple rate transmission. For simplicity of presentation, we assume that all users have the same carrier frequency. It is noted that for practical multiple bandwidth systems, some form of frequency planning and thus multiple carriers will improve the efficiency of the system [116]. The methods considered in the current work are easily extended to multiple carrier systems.

The received baseband signal is

$$x(t) = \sum_{k=1}^{C} \sum_{i=1}^{P_k} A_{ik} r_{ik}(t) + n(t) \quad (4.1)$$

$$r_{ik}(t) = \sum_{l=-\infty}^{\infty} b_{ik}(l) s_{ik}(t - lT_k - \tau_{ik}) \quad (4.2)$$

where $C$ is the number of service classes. There are $P_k$ users in class $k$; each user in a particular class transmits at the same rate, $\frac{1}{T_k}$, and employs the spreading gain of $L_{ik}$. Users are indexed by two variables: $k$ indicates the rate/class and $i$ indicates the user number within rate $k$. The received signal for user $ik$ is denoted by $r_{ik}(t)$. The corresponding received amplitude is $A_{ik}$. Each user $ik$ is received after a propagation delay of $\tau_{ik}$. The additive noise process, $n(t)$, is assumed to be white and Gaussian with zero mean and power $\frac{N_0}{2}$. The information stream for user $ik$ is denoted by $b_{ik}(l)$. For simplicity of presentation, binary phase shift keying (BPSK) modulation is assumed. The information bits are independent from user to user and in time. The spreading waveform is denoted by $s_{ik}(t)$ and is formed by modulating a pseudorandom noise sequence by the pulse shape $\phi_k(t)$

$$s_{ik}(t) = \sum_{n=1}^{L_{ik}} a_{ik}(n) \phi_k(t - nT_{ch}) \quad (4.3)$$
where $a_{ik}(n)$ is the pseudo-random code for user $ik$. The pulse $\phi_k(t)$ is the convolution of the pulse shaping filter (e.g., raised cosine) and the channel response. Arbitrary pulse shapes are considered herein. In particular, we wish to consider bandwidth efficient pulses. Such bandlimitation is essential to our examination of the effect of the bandwidth of the front-end filter in smaller bandwidth user performance in Section 4.4.

To motivate the methods employed, we first investigate the second order statistics of a single spread spectrum signal in the absence of the noise. Consider a CDMA signal with symbol period $T$ and spreading gain $L$, $r(t) = A \sum_l b(l)s(t - lT)$, where $s(t)$ is the symbol waveform; we assume periodic modulation. The covariance function of $r(t)$, $R_r(t, u)$, is

$$
E \left\{ A \sum_{l=-\infty}^{\infty} b(l)s(t - lT)A \sum_{k=-\infty}^{\infty} b(k)s(u - kT) \right\} \tag{4.4}
$$

$$
= A^2 \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} E \{ b(l)b(k) \} s(t - lT)s(u - kT) \tag{4.5}
$$

$$
= A^2 \sum_{m=-\infty}^{\infty} R_b(mT) \sum_{l=-\infty}^{\infty} s(t - lT)s(u - lT - mT) \tag{4.6}
$$

where $R_b(mT)$ is the correlation of the stationary data sequence. For uncoded data, $R_b(mT) = \delta_m$, where $\delta_m$ is the Dirac delta. The summation in $l$ is periodic with period $T$. If the support of $s(t)$ is time limited to $[0, T']$, then $R(t, u)$ is zero for $|t - u| > T'$. Since $R(t, u)$ is a periodic function with period $T$, its Fourier series expansion exists with fundamental frequency as $1/T$. For the raised cosine pulse, $\phi(t)$, the non-zero cycle frequencies of $r(t)$ are $0, 1/T, \ldots, L/T$ [47].


\footnote{Rectangular pulse shapes and thus very wide bandwidth front end filters are assumed in [19].}
Given the prior discussion, we can determine the autocorrelation of the received multirate signal, $x(t)$ in (4.1)

$$R_x(t, u) = \sum_{k=1}^{C} \sum_{i=1}^{P_k} A_{ik}^2 R_{ik}(t, u) + R_n(t - u). \quad (4.7)$$

The periodicity of $R_x(t, u)$ depends on the ratios of individual symbol periods, $T_k$. If any ratio $\frac{T_k}{T_1}$ is an irrational number, the function $R_x(t, u)$ is not periodic. For the rest of the chapter, we confine our attention to the case where all ratios $\frac{T_k}{T_1}$ are rational. This implies that $R_x(t, u)$ is periodic with period $pT_1$ where $p$ is the smallest integer such that $\frac{pT_1}{T_k} \in \mathbb{Z}^+$ for all $1 < k \leq C$.

Based on the results in [16,44] for single rate systems, the optimal linear MMSE receiver will be periodic with period $pT_1$. This observation was also made in [19]. This is because the cyclostationary process $x(t)$ can be converted into a vector stationary process (for bandlimited systems), with vector lengths a multiple of $pT_1$. Thus for symbol-by-symbol MMSE detection, the linear filter will be time-varying.

### 4.3 MMSE Receiver

Without loss of generality, assume that the user of interest has symbol period $T_1$. Further, we assume that the covariance is periodic with period $pT_1$ where $p$ is the smallest integer such that $\frac{pT_1}{T_k} \in \mathbb{Z}^+$ for all $1 < k \leq C$. Since the symbols for each user are uncorrelated, the signal of interest can be decomposed into $p$ virtual users (Figure 4.2), each with symbol period $pT_1$. Similarly, each of the interfering users can be decomposed into $\frac{pT_1}{T_k}$ virtual users. This effectively maps the original multirate problem onto a single rate problem, and we need to find $p$ optimal MMSE receivers, one for each virtual user of interest. The optimal MMSE receiver for each of the $p$ virtual user is time-invariant and can be computed using the results in [16].
as follows. The desired linear receiver structure is shown in Figure 4.1, where $h(t)$ is the front-end filter.

Let $T = pT_1$ and define $v_k = \frac{T}{T_k}$; each user in class $k$ can be decomposed into $v_k$ virtual users. The received signal for user $ik$ is written as

$$r_{ik}(t) = \sum_{z=0}^{v_k-1} \sum_{l=-\infty}^{\infty} b(lv_k + z)s_{ik}(t - zT_k - lT - \tau_{ik}). \tag{4.8}$$

Note that the original multirate problem has $\sum_{k=1}^{C} P_k$ users and is equivalent to a single rate problem with $P = \sum_{k=1}^{C} P_k v_k$ virtual users. The only difference is that the symbol period of each virtual user is longer than its original symbol period. Define

$$S_{ik}(f) = \begin{bmatrix} H(f)S_{ik}(f) \\ H(f)S_{ik}(f)e^{-j2\pi f T_k} \\ \vdots \\ H(f)S_{ik}(f)e^{-j2\pi f (v_k-1)T_k} \end{bmatrix} \tag{4.9}$$

where $S_{ik}(f)$ is the Fourier transform of the spreading waveform $s_{ik}(t - \tau_{ik})$ and $H(f)$ is the Fourier transform of the front end filter $h(t)$. Further define

$$Q_k^T(f) = [S_{ik}^T(f) \ldots S_{P_k k}^T(f)] \tag{4.10}$$

$$Q^T(f) = [Q_1^T(f) \ldots Q_C^T(f)] \tag{4.11}$$

and the normalized matrix

$$R(f) = \frac{1}{T}A \sum_{n=-\infty}^{\infty} \frac{Q(f - \frac{n}{T}) Q^H(f - \frac{n}{T})}{|H(f - \frac{n}{T})|^2 N_0/2} + I \tag{4.12}$$

where $A$ is a diagonal matrix with powers of each virtual user and $I$ is an identity matrix. For each of the $p$ virtual users of interest, we define a column vector $B_z$ with only one non-zero entry at the $z^{th}$ location, equal to the power of the $z^{th}$ virtual user, $\frac{A_{ik}^2}{p}$. With the above notation, the time-invariant optimal linear filter for each of the
where \( g_z(f) = R^{-1}(f)B_z \) and we have assumed that all \( P \) virtual users have linearly independent signaling waveforms. Further simplification of (4.13) yields the structure of the optimal linear receiver for \( z^{th} \) virtual user in Figure 4.3, and the complete time-varying optimal solution is seen in Figure 4.4. The filter \( K_{ik}(f) \) is defined as follows,

\[
K_{z,ik}(f) = \sum_{i=0}^{v_k-1} g_{z, M+i}(f)e^{-j2\pi f T_k}
\]  

where \( M = (i-1)P_k + \sum_{h=0}^{k-1} P_h v_h \). The receiver for virtual user \( z \) is comprised of the front end matched filter, followed by a bank of filters matched to each user’s spreading waveform, a bank of interference suppression filters and then combining.

### 4.4 Effect of Front-End Filter Bandwidth

In a system with multiple chipping rates, users with smaller bandwidth will potentially be provided with low complexity (i.e. low cost) receivers. This motivates us to investigate the effect of front-end filter bandwidth on the receiver performance, since the sampling rate is an important factor in the cost of the receiver. Furthermore, the same analysis will also indicate the effect of the front-end filter on system capacity in terms of the number of active users. Note that the subsequent analysis allows us to draw conclusions without having to resort to any assumptions about the correlation between user codes; for the same reason, the conclusions only provide trends.

Consider a multirate system with two user classes with symbol periods, \( T_1 > T_2 \), chipping periods, \( T_{c_1} > T_{c_2} \) and spreading gains, \( L_1 \) and \( L_2 \). Let the number of users in each class be \( P_1 \) and \( P_2 \), respectively.
First consider detection of users in class 2. The observation interval is \( T_2 \) and bandwidth of the front end filter is \( B = \frac{1}{2T_2} \). The approximate number of dimensions for the low rate users is \( L'_1 = \left\lfloor \frac{T_2}{T_1} \right\rfloor \) \[112\] and for the high rate users is \( L_2 = \frac{1}{2\pi c_2} - T \) \[112\]. A sufficient condition to guarantee identifiability of all users is that the total number of users is less than the total number of dimensions per symbol. Thus, the number of users in the system is governed by

\[
\frac{P_2}{L_2} + \frac{P_1}{L'_1} \leq 1 \tag{4.15}
\]

For demodulating the low rate users, the observation interval is \( T_1 \) and \( B \) is such that \( \frac{1}{2T_1} \leq B \leq \frac{1}{2T_2} \). The approximate number of dimensions for each low rate user is \( 2^{\frac{1}{2T_1}} T = L_1 \). Each high rate user contributes at most \( g = \left[ \frac{T_2}{T_1} \right] + 1 \) virtual users and each virtual user has approximately \( L'_2 = \left\lfloor L_2 2BT_2 \right\rfloor \) dimensions. Thus,

\[
\frac{P_1}{L_1} + \frac{gP_2}{L'_2} \leq 1 \tag{4.16}
\]

Therefore, the total number of dimensions increases linearly with bandwidth. This implies that the interference suppression capabilities for the low rate users increases linearly with bandwidth, \( B \).

### 4.5 Comparison with Time-invariant MMSE Receiver

In this section, we derive and study the performance of the time-invariant MMSE (TI-MMSE) receiver for the user of interest. As shown in Section 4.3, the optimal linear MMSE receiver may be time-varying for the multirate DS-CDMA signal (4.1). For the cases where the optimal linear receiver is time-varying, the time-varying receiver has higher computational complexity compared to a time-invariant structure. Further, if a wideband DS-CDMA system is being overlaid on an existing narrowband
system, then the analysis of the time-invariant receivers is required to quantify the
loss in performance for the existing users employing time-invariant MMSE receivers\(^{21}\).

By inspection of (4.8), it is clear the optimal linear receiver will be time-invariant
for class \(k\) if and only if \(v_k = 1\). For the general case, the time-invariant receiver
can be found as follows. Again consider demodulation of user II. Since the received
signal is wide-sense cyclostationary, the symbols \(b_{11}(lv_1)\) and \(b_{11}(lv_1 + k)\) for a fixed
\(k \in [0, \ldots, v_1]\), encounter the same interference. Since the receiver is time-invariant,
it has to suppress the interference encountered by each of the \(v_1\) virtual users of
interest. In other words, even though the interference environment is periodically
time-varying, a time-invariant processing requires simultaneous suppression of inter­
ference encountered by all virtual users. Hence the effective symbol periods of each
interfering virtual user is \(T_1\) instead of \(T\). Equivalently, we have a single rate problem
with covariance of the received signal as \(R(t, u) = \sum_{i=0}^{v_1-1} R(t + iT_1, u + iT_1)\), which is
periodic with period \(T_1\). Define
\[
\begin{align*}
\bar{Q}_1^T(f) &= [S_{11}(f) \cdots S_{R_1}(f)] \\
\bar{Q}^T(f) &= [\bar{Q}_1^T(f) \bar{Q}_2^T(f) \cdots \bar{Q}_k^T(f)] 
\end{align*}
\] (4.17) (4.18)
The TI-MMSE can be found as
\[
\bar{U}(f) = \bar{Q}^H(f) \frac{\bar{G}(f)}{|H(f)|^2 N_o/2}
\] (4.19)
where \(\bar{G}(f) = \bar{R}^{-1}(f)\bar{B}_1\), where the first entry \(\bar{B}_1\) is the power of user 11 and the
rest are all zero. The matrix \(\bar{R}(f)\) is obtained by replacing \(T\) with \(T_1\) and \(Q(f)\)

\(^{21}\)Note that the need for time-varying receivers can be completely obviated by appropriate spectral
shaping of the user pulses to resemble an FDMA system [116]. In a multirate FDMA system,
where users with different symbol periods occupy disjoint portions of the spectrum, time-invariant
receivers are optimal linear MMSE receivers. This is because interference from other rate users can
be completely suppressed by a front bandpass filter followed by a time-invariant receiver to suppress
users with same symbol periods.
with $Q(f)$ in the expression for $R(f)$. The mean-squared error (MSE) for the two receivers, assuming $E\{|b_{1l}|^2\} = 1$, is given by

$$MSE_{TV} = 1 - \sum_{z=1}^{v_1} \int_{-\infty}^{\infty} \frac{G_T^T(f)Q(f)Q_H(f)B_z}{|H(f)|^2N_0/2}$$

(4.20)

$$MSE_{TI} = 1 - \int_{-\infty}^{\infty} \frac{G_T^T(f)Q(f)Q_H(f)B_1}{|H(f)|^2N_0/2}$$

(4.21)

Note that MMSE receiver designed for the $z^{th}$ virtual user has the lowest MSE among all linear receivers, i.e., $\frac{1}{v_1}MSE_{TI} \geq MSE_{TV}$ for all $z = 1, \ldots, v_1$. This implies, that $MSE_{TI} \geq \sum_{z=1}^{v_1} MSE_{TV} = MSE_{TV}$. The TI-MMSE is designed to suppress more users than the TV-MMSE, and hence achieves a higher MSE.

4.6 Simulation Results

First, we study the performance of the proposed receivers for the following three class system. The symbol periods of the three classes are assumed to be $T_k = k, k = 1, 2, 3$. All users in all three classes employ the same spreading gain, $L = 16$. There are three users in class 1 and four each in classes 2 and 3, i.e., $P_1 = 3, P_2 = 4, P_3 = 4$. Further, $A_{1k} = 1$ for $k = 1, 2, 3$ and $A_{ik} = 3$ for $i = 2, \ldots, P_k, k = 1, 2, 3$. The codes for all users were chosen at random and a half-cosine pulse was chosen as $\phi_k(t)$ for simplicity. Finally, a pseudo-synchronous situation is assumed, i.e., $\tau_{ik} = 0$ for $i, k$.

The probability of error as a function of SNR for the matched filter, time-invariant MMSE (TI-MMSE) and the proposed time-varying MMSE (TV-MMSE) receivers is studied. In Figures 4.5, 4.6, 4.7, the three receivers are compared for users 11, 12, 13 respectively; the single user bound is also shown. For probability of error equal to $10^{-3}$, the loss incurred by the TI-MMSE receiver as compared to the TV-MMSE varies from 1 dB (user 13) to 4 dB (user 12). Finally, the user with smallest bandwidth,
user 13, has the poorest performance as compared to the single user bound. These trends were also observed in [116].

The effect of varying front-end filter bandwidth on probability of error for class 3 users is shown in Figure 4.8; a $\frac{T}{4}$ sampling of the input signal was used. From Figure 4.8, a linear decrease in probability of error as a function of front-end filter bandwidth is observed. Furthermore, if the front-end filter for user 13 is matched to its own bandwidth, the loss in performance is approximately 8 dB for an error probability of $10^{-3}$.

4.7 Conclusions

In this chapter, the optimal linear MMSE receiver and the optimal linear time-invariant receiver for multirate DS-CDMA systems were derived and compared. It was shown that the optimal linear MMSE filter is periodically time-varying and its period depends on the symbol periods of the interfering users. Using simulations, it was shown that time-invariant MMSE can incur a large loss in probability of error performance compared to the optimal time-varying MMSE receiver. Also, TV-MMSE receivers for users with more bandwidth had a lower probability of error as compared to the coexisting smaller bandwidth users.

The effect of the front-end bandwidth on smaller bandwidth user’s receiver performance was studied both analytically and via simulations. It was observed that a loss in total system capacity and a degradation in probability of error is incurred if the sampling frequency is less than the Nyquist frequency of any of the interfering users.
Figure 4.1: Receiver structure.

User 11
0 \[ T \]
1 \[ 2T \]
2 \[ 3T \]

User 12
0 \[ T \]

Decomposition into virtual users

Virtual User 1
0 \[ T \]

Virtual User 2
0 \[ T \]

Virtual User 3
0 \[ T \]

User 12
0 \[ T \]

Figure 4.2: Decomposition of users into virtual users for a two class case, each with one user. User 11 of class 1 is decomposed into 3 virtual users, each with symbol period \( T \), and User 12 is left undecomposed.
Figure 4.3: MMSE receiver for each virtual user.
Figure 4.4: Complete MMSE receiver for a user in the presence of multirate interference.
Figure 4.5: Probability of error as a function of SNR for user 11.
Figure 4.6: Probability of error as a function of SNR for user 12.
Figure 4.7: Probability of error as a function of SNR for user 13.
Figure 4.8: Probability of error as a function of front-end filter bandwidth for user 13.
CHAPTER 5

FUTURE WORK

In this concluding chapter, we suggest directions for future work. In Section 5.1, we examine the challenges of parametric modeling. We conclude this thesis with possible extensions in Section 5.2.

5.1 Critique of Parametric Modeling

In Chapter 1, the concept of parametric modeling was promoted on the grounds of simplicity and widespread applicability. A significant research effort in engineering, statistics and mathematics is geared towards developing techniques by assuming that the hypothesized model is exact, an assumption which is, more often than not, suspect. Parametric techniques, in general, perform poorly under slight perturbation in the assumed model [57, 130]. Even though no formal analysis is available, the author believes that the techniques developed using the higher order differential geometric structures [5, 11, 62] are increasingly sensitive to the accuracy of the assumptions. To the best of author's knowledge, most of the research effort in analysis with modeling errors, barring the work by White [130], is for linear models. Furthermore, the modeling errors are also assumed to be linear in most statistical research.
Nonlinear parametric models are also fraught with numerical problems. Most estimators for nonlinear parameters require solving a computationally hard non-convex problem (for example, computation of maximum likelihood estimates in Examples 1 and 2, Chapter 2).

In the light of the above discussion, the utility of the results proved in Chapters 2-4 becomes questionable. For example, the definition of the parameter and model order consistency relies on the fact that the true model belongs to the set of hypothesized models. The concerns regarding the utility of the usually accepted statistical performance measures (consistency, efficiency, unbiasedness etc) and attempts to address them are not new [12,99,130]. Although significant progress has been made in universal modeling and coding [12], these generalizations come at a price. In practice, the rate of convergence of the estimators to their asymptotic values is significantly slower than the case where the true distribution is assumed [25]. This is expected since lesser knowledge is assumed regarding the observed process. The result of slow convergence rate is that the universal coding and modeling techniques perform poorly for practical sample sizes [25].

This leads to a dilemma in the choice of concepts, for they affect the resulting procedures²². That is, conceptually appealing schemes may perform poorly in practice and practically useful schemes seem to rely on questionable assumptions.

5.2 Possible Extensions

In this section, we suggest potential extensions to the results reported in Chapters 2-4. We choose to continue our future efforts primarily in the context of classical

²²A possible solution is to choose a philosophy which “works.” Then the theoretician and practitioner would differ on the definition of what they think is “working.”
statistics and information theory, i.e., the hypothesized model is assumed to exactly represent the true distribution.

5.2.1 Model selection

In Chapter 2, no connections with universal coding [25,32] were explored. If the composite source model is assumed to be nested, then the proposed scheme also provides a universal coding scheme in the same spirit as MDL [12,98]. Optimal rates of convergence for universal coding procedures have been reported in [12]. Since both MDL and the proposed Wald statistic procedure are consistent, it remains to be determined if they perform differently in terms of rate of convergence. The author conjectures that for nested classes, the proposed procedure has the same rate of convergence, and hence is a lower complexity procedure that performs as well as MDL. Note that MDL is applicable for all model classes, nested or non-nested, but the Wald statistic based procedure applies to only nested model classes.

The cost of computation can be directly included into the problem formulation to yield a constrained optimization problem; the optimization is performed over all procedures with a bounded average cost. This possibility has been explored [38,45] and generally leads to dynamic programming based procedures. But the resultant procedures are generally intractable for most practical cases [38,106]. Nonetheless, the study of optimal procedures is useful to provide lower bounds on the achievable performance. A useful problem formulation is to study universal coding and modeling with complexity constraints. The constrained universal coding problem becomes more meaningful since new results in time and memory constrained Kolmogorov complexity [74] have been reported\textsuperscript{23}.

\textsuperscript{23}Recall that the motivation for MDL was the \textit{unconstrained} Kolmogorov complexity.
In [61], MDL was proposed to estimate the number of multipaths with the channel model given in Example 1, Chapter 2, for equalization in HDTV. In digital modems, complexity of equalization forms the bulk of total computation. Thus an accurate determination of the model order is important to reduce the average complexity and guarantee a minimum desired performance level. Furthermore, as the data rate increases in multipath rich environments, the cost of the equalization also increases. Also, typically large data lengths \( N > 10^4 \) and large maximum model orders \( K > 15 \) are encountered in high speed communications. Thus, the computational savings of the proposed Wald statistic over MDL can be significant to allow the design of equalizers with reasonable complexity.

Statistical signal processing is extensively used in the physical layer of communication systems. Yet, very little published work has aimed to establish explicit connections between physical layer performance bounds (e.g., Cramér-Rao bound) and information theoretic quantities (e.g., channel capacity). These connections are particularly relevant for time-varying channels. For instance, choice of equalization method (blind vs. non-blind) and choice of training sequence length, can potentially be made analytically if the role of statistical bounds is made explicit in the communication bounds.

We outline a procedure to determine information theoretically optimal training sequence. Consider a block fading channel [85], where the fading coefficients are assumed to be constant for every block of fixed size \( T \); the following scheme can be easily extended to frequency selective channels. Each block suffers an independent fade. Assume the common packet transmission scheme, where a training sequence is

\footnote{Average complexity directly affects the battery life.}
transmitted in the header of each packet. The training sequence is used to estimate the channel, which is then used in data detection. To compute the capacity of such a scheme, the inaccuracy of the channel estimates needs to be modeled. The Cramér-Rao bound provides a resolution bound and thus can be used to define a compound channel as follows. Let \( \hat{h} \) denote the channel estimate, and \( J_L(\hat{h}) \) denote the Cramér-Rao bound computed at \( \hat{h} \) for a training sequence of length \( L \). Note that for fading channels, \( J_L(\hat{h}) \) depends only on the length of the training sequence. The compound channel can be defined as

\[
C_L = \left\{ h : \hat{h} \pm kJ_L(\hat{h}), k = 1, 2, \ldots, U \right\}
\]  

(5.1)

where \( U \) is chosen such that \( \text{Prob}(|\hat{h} + UJ_L(\hat{h}) - h_o| < \epsilon) > 1 - \epsilon \). The capacity of the above compound channel can be found using the results in [71, 72]; denote it by \( C_L \). As \( L \) increases, the reliability of the channel estimates increases, which implies that the volume of the compound channel, \( C_L \), decreases. The decrease in the volume of the compound channel implies that \( C_L \) should increase. But increasing \( L \) also implies that the effective channel usage decreases, since only \((T - L)\) of the transmitted symbols per block are information bearing symbols. Hence, \( C_L \) is not monotonically increasing. We conjecture that it will be concave in \( L \) and the extremal point provides the information theoretically optimum value of the training sequence.

5.2.2 Cell sectorization

The algorithm for cell sectorization given in Chapter 3 was specifically designed for uniform linear arrays. Non-uniformly spaced arrays (circular, cylindrical) and two dimensional planar arrays have also been proposed for wireless cellular and satellite systems. Although the criterion proposed in Chapter 3 is applicable for an arbitrary
antenna array, the design technique is applicable only for uniformly spaced linear arrays. The key difficulty in using the reparametrization in Chapter 3 appears to be unavailability of the spectral factorization theorem for the non-uniformly sampled and multidimensional linear systems. To the authors' best knowledge, only tools from algebraic geometry, Gröbner basis [14] in particular, seem to provide some hope in this direction.

5.2.3 Receivers for multirate systems

The development and analysis of receivers for multiple data rate systems is still in its infancy. The specific issues of future interest are

- **Suboptimal Receivers**: In Chapter 4, it was shown that the optimal linear MMSE receiver is in general cyclically time-varying with a period $p$. Thus the implementation of the optimal time-varying receiver requires $p$ time-invariant receivers of the same length. For multicarrier systems, the period $p$ may be prohibitively large, thus creating the need for (suboptimal) low complexity receivers. A possible method of reducing the complexity is as follows.

Since the optimal receiver is periodic, it admits a Fourier series expansion.

\[
Y(f, l) = Y(f, l + p) \quad (5.2)
\]

\[
= \sum_{n=0}^{p-1} Y^{(n)}(f)e^{j2\pi n l} \quad (5.3)
\]

where $Y(f, l) = Y_i(f)e^{-j2\pi ft_{i}}, l = kp + i, 0 \leq i < p$, and $l$ is the symbol number to be demodulated. Note that the alternate representation of the optimal receiver also requires $p$ time-invariant filters. The output of each of the $p$ filters is sampled every $T_1$ seconds and weighted by an appropriate Fourier coefficient. A
low-complexity suboptimal receiver can be designed by choosing a time-varying receiver with \( l \) Fourier coefficients, where \( l \ll p \).

- **Choice of Suboptimal Receivers**: For a given choice of \( l \), there are \( \binom{p}{l} \) choices of suboptimal receivers, with the same computational complexity. To choose the best suboptimal receiver from \( \binom{p}{l} \) possible suboptimal receivers, we suggest the following procedure. The mean square error of the optimal receiver admits the following series representation,

\[
\text{MSE}_{\text{optimal}} = 1 - \sum_{n=1}^{p} E^{(n)}
\]  

where \( E^{(n)} \) is the contribution due to the \( n^{th} \) Fourier coefficient. The \( l \) coefficients can be chosen such that the increase in MSE of the resulting filter is minimum. Alternately, the filter coefficients can be ordered based on their \( E^{(n)} \) values and \( l \) coefficients with the highest \( E^{(n)} \) values can be retained.

- **Code Design**: Code design for multirate systems can be done to allow the use of suboptimal receivers with minimal performance loss as compared to the optimal receiver. Again, we resort to the Fourier series expansion to heuristically explain the problem of code design. The zeroth coefficient of the Fourier series represents the "DC" component of the optimal time-varying filter. For the cases where the interference does not vary significantly over \( p \) symbols of the user of interest, the zeroth Fourier coefficient is dominant. This implies that the performance loss in using the time-invariant receiver is small as compared to the optimal receiver.

- **Frequency Planning and Power Control**: One of the main objectives of a cellular system designer is to accommodate as many users as possible, in a fixed
spectrum, while maintaining a minimum level of quality of service. This objective can be achieved by adapting both the transmission rates and powers of the users. An information theoretically optimal power and rate control for a fixed number of users was presented in [56,119]. But for practical systems, the scheme in [56,119] provides guidelines, at best, for practical system design. The assumption regarding complete channel knowledge at both transmitter and receiver is unrealistic and generally not applicable. For the same reason, the increase in capacity due to additional transmit and receive antennae are unrealistic, since they do not account for inaccuracies in channel estimation. For a more realistic capacity analysis for fading channels, the recent work by Marzetta and Hochwald [80] is more relevant to practical systems. An information theoretic analysis under the following setup will be a desirable next step.

1. Users can occupy different bandwidths.

2. At first, flat fading is assumed for simplicity. A more realistic assumption is to model the channels as multipath channels. It is worth mentioning that fading channels are commonly assumed in information theoretic analysis because they lead to convex multiuser rate regions [119], while the same is not true for multipath channels [23]. The convexity of the rate regions is exploited to obtain simple capacity achieving power control methods in [119].

3. The fading coefficients are assumed to be constant over a fixed size block of data (block fading channel [85]).
4. Both transmitter and receiver are assumed to have no knowledge of the fading coefficients and no ad-hoc training schemes are assumed.

5. Both transmitter and receiver are equipped with one antenna; extensions to multiple antenna case should follow from the results in [80].

Based on the capacity analysis for the above system, the effect of a specific frequency plan can be better understood. Further, optimal power and rate control methods can be derived in a more realistic setting.
BIBLIOGRAPHY


