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EXISTENCE AND STABILITY OF TRAVELING WAVE SOLUTIONS OF NEURONAL NETWORK EQUATIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By
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ABSTRACT

In this thesis, based on some phenomena in neuronal networks, we derive certain system of integral-differential equations. Then we utilize various complicated analysis to demonstrate that the neuronal network equations possess traveling pulse solutions and these wave solutions are exponentially stable. We will use the shooting argument and the exchange lemma to prove the existence of wave solutions. We then use some fundamental analysis to characterize the asymptotic behavior of the solutions of some intermediate system as $z \to \pm \infty$. Then we use the method of variation of parameter to define solutions of the eigenvalue problem and we use the Evans function to find the eigenvalues of the operator. This is achieved by employing explicit solutions of the intermediate system to get accurate information of the Evans function. Since appropriate location of the spectrum of the operator implies linear stability and linear stability implies nonlinear stability in the sense of $L^\infty$-norm, we assert the nonlinear stability of the traveling wave.
Dedicated to my wife, Yan Zhang
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I believe in Jesus Christ. He give me great hint, love and wonderful life. Thank YOU very much, God!
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CHAPTER 1
INTRODUCTION AND PRELIMINARY LEMMAS

In this chapter we briefly review the development of the existence and stability theory of traveling waves. Then we motivate our integral-differential equations from biology. In the second section we list some technical lemmas.

1. Introduction

A. Traveling waves. The theory of nonlinear parabolic systems of partial differential equations is a center piece of modern applied mathematics, and such equations have a virtually ubiquitous presence as mathematical models in science and engineering. Nerve axon equations consist of a reaction-diffusion equation and several auxiliary ordinary differential equations. These equations, which include the Hodgkin-Huxley equations and the Fitzhugh-Nagumo equations, are paradigm examples of parabolic systems to which the maximum principle is difficult to apply. These equations have been central examples in reaction-diffusion equations. Traveling wave solutions play a central role in the study of reaction-diffusion equations. The most striking fact about these equations is their ability to support traveling wave solutions, also called solitary waves. They travel at constant speed, and their profile is time invariant. The information carried by a traveling wave is never changed or lost! These solutions are
often the centerpiece of a physical system because they represent the movement of information in a single direction and they play a key role in the characterization of the behavior of more general solutions. In a word, they are the most important nontrivial solutions of the mentioned partial differential equations. The nerve impulse, for example, is a particular traveling wave. In addition to the nerve impulse propagations, the traveling waves resemble many other propagation phenomena in the real world such as contagious diseases.

B. Existence. There are mainly two kinds of partial differential equations from the viewpoint of perturbation. The well-known Fitzhugh-Nagumo equations, the diffusive predator-prey equations are good examples of singular perturbation problems. The Korteweg-de Vries equations, the nonlinear Schrödinger equations are good examples of nonsingular perturbation problems. How to find traveling (solitary) wave solutions of these equations? This is not an easy question to answer. Generally speaking, scalar equations are easier to solve than systems of coupled equations. Some scalar equations possess explicit solitary wave solutions, such as the bistable equation, the Korteweg-de Vries equation, the nonlinear Schrödinger equation, the sine-Gordon equation, the Boussinesq equation, and so on. But relatively speaking, these equations are not very complicated. The shooting argument is an effective approach for proving the existence of traveling wave solutions. For parabolic systems, the Conley index and the theory of Leray-Schauder degree provides general frame of reference for the study of existence. The application of Leray-Schauder degree to traveling waves is not straightforward because of the lack of compactness of the associated operators due to the unbounded spatial domain. This difficulty can be
overcome either by working in suitably weighted function spaces or by truncating the spatial domain to a finite interval and by passing to a limit as the size of the domain becomes infinite. On the other hand, the index theory for isolated invariant sets of flows set forth a general program for measuring Morse decompositions in terms of a topological invariant. The principal illustrative examples in the early applications of the Conley index were connecting orbit problems derived from the traveling wave solutions for parabolic systems. Since then there has been continuing research, both generally and specifically, on the applications of these methods to the existence theory of traveling waves. The geometric singular perturbation method provides a powerful tool for verifying the existence of homoclinic and heteroclinic orbits of singularly perturbed problems. The geometric theory are obviously limited to certain ranges of parameters, often the region of parameter space where interesting behavior occurs. Homoclinic orbits are often related to singularly perturbed problems such as the well-known Fitzhugh-Nagumo equations, which is a simplified version of the Hodgkin-Huxley equations, describing the propagation of nerve impulses.

C. Stability. For a given traveling wave, it is of great importance to determine its stability relative to perturbations in the initial conditions for solutions of the model equations. Stable solutions are the most physically realistic since the external world provides enough perturbations that we can only expect to see waves which will dampen out these perturbations. The stability of wave solutions of the bistable scalar equation was solved in the important paper of Fife and McLeod. This result made essential use of the maximum principle, and its generalizations are limited systems which admit some kind of comparison principle. At about the same time, some
general abstract theorems on nonlinear stability based upon semigroup theory due to Sattinger and Henry appeared which required the underlying wave to be linearly stable. Since then several new ideas have emerged for studying the spectrum of linear operators arising in this manner which have permitted the application of the general theorems of Henry and Sattinger to many interesting problems. One key innovation was work by Evans on the stability of nerve impulses in neurophysiology, in which he defined a certain analytic function whose zeros coincide precisely with the eigenvalues of the linear operator. The Evans function was subsequently used by Jones in a rigorous proof of the stability of the impulse solution of the Fitzhugh-Nagumo system. Jones’s proof was based on geometric arguments; a more analytical approach to the same problem but also based on the Evans function was given by Yanagida. The Evans function has also been used in the study of the stability of wave solutions of other classes of equations such as the generalized Korteweg-de Vries equation, see Pego and Weinstein. The stability of a traveling wave of a system means if the initial data is not too far away from the traveling wave, then the solutions approach the traveling wave as time tends to infinity.

D. Technical tools. As well known, iteration technique is very useful in constructing uniformly convergent sequences in ordinary differential equations. This technique will be used in a few places for different purposes. The Fourier and the inverse Fourier transforms and the Green’s function for the heat equation will also be utilized in our discussions. It is very easy to deal with eigenvalues and eigenvectors of matrices by routine method and winding number computation. The stability of traveling wave solutions to nonlinear parabolic systems under small perturbations of the
wave can be characterized by the stability properties of a linear system of equations approximating the system about the traveling wave. The results include the resting case and thereby characterize the stability at rest in terms of a linear system with constant coefficients. We show that linear stability is equivalent to nonlinear stability of the traveling wave solution of the full nonlinear parabolic system. We also show that the resting states are exponentially stable relative to the linearized equation if and only if the spectrum of a family of matrices lies in a left half plane, uniformly bounded away from the imaginary axis.

E. Eigenvalue problems. It would be very convenient to rewrite the eigenvalue problem as a first order linear system of ordinary differential equations. By use of the spectral theory of linear operators, the stability of the traveling wave solutions under small perturbation of the initial conditions is shown to depend on the spectrum of a bounded linear differential operator. We use the properties of linear operators (in particular compact operators) and semigroups of operators to find the eigenvalues, possibly the eigenvectors as well, of certain linear operators relative to certain Banach spaces. We also use asymptotic behaviors of solutions to ordinary differential equations to define projection operators as well as to determine the dimensions of the null space and the range of these operators. More importantly, we derive the linearized criterion for stability of the traveling wave solution: except for the neutral eigenvalue $\lambda = 0$, the spectrum of the bounded linear differential operator lies in a left half plane, uniformly bounded away from the imaginary axis and $\lambda = 0$ is a simple eigenvalue.
F. Evans functions. Finally, we define a complex analytic function of a complex variable whose zeros are exactly the eigenvalues of the mentioned differential operator. By some delicate analysis, we show there is no zero of the Evans function in a right half plane, except for a simple zero $\lambda = 0$. This shows rigorously that the traveling wave solution is exponentially stable. Thus important insights into stability have been obtained. One should compare the methods and results for the nonlinear parabolic systems with those of a single reaction-diffusion equation, such as the bistable equation, to which the maximum principle is available, and with those of a nonlinear dispersive dissipative wave equation, such as the Korteweg-de Vries-Burgers equation. But to study various problems concerning the differential operators induced by the partial differential equations, we first need to give some definitions, notations and assumptions.

G. New biological models and results. Understanding neuronal function in spatially extended networks has immediate consequence in terms of practical applications. Magnetic resonance, as well as other brain imaging techniques are employed both clinically and in research to visualize spatio-temporal properties of activity in brain structures. Several disorders, such as cortical epilepsy and migraine are characterized by waves of activity spreading across the surface of the cortex. Recent work has focused on modeling neural systems in the spatial domain using large scale networks with equations based at the single cell or local population level. While certainly providing insight into the operation of these systems, the complexity of such models precludes a more complete understanding of the dynamics involved. Considering even simplified models post-synaptic, there has been very little formal analysis attempting
to understanding the mathematical systems describing spatially extended networks. Numerically, synaptically-coupled spatial networks have been shown to exhibit oscillating and propagating waves as well as both transient and persistent behaviors. More formally, the emergence of spatial structures has been established under certain conditions and the existence of traveling wave solutions have been verified under more general circumstances. Interactions between synaptically coupled neurons occur via action potentials. A single action potential evokes a voltage change, post-synaptic potential. For a series of $n$ action potentials, the voltage is given by a finite sum involving some weight function, a positive scaling function describing the strength of the synapse. Letting $n \to \infty$, we obtain some integral involving the Heaviside function. This is actually a convolution. The integral kernel, indicating the average instantaneous firing rate in a population of neurons. Synapse can be excitatory or inhibitory. With $K$ representing the average firing rate for a homogeneous population of neurons, the unit element is interpreted as the average voltage level across the population. In a network model, units act simultaneously as both presynaptic and elements so that the post-synaptic voltage also determines the presynaptic firing rate. In this thesis, based on some biological phenomena, we derive certain integral-differential equations. Then we utilize complicated analysis to demonstrate that the neuronal model equations possess traveling pulse solutions and these wave solutions are exponentially stable. We will use the shooting argument and the exchange lemma to prove the existence of wave solutions. We then use some standard method to construct solutions and to characterize the asymptotic behavior of the solutions of some intermediate system as $z \to \pm \infty$. Then we use the method of variation of parameter
to define solutions of the eigenvalue problem and we use the Evans function to find
the eigenvalues of the operator. This is achieved by employing explicit solutions of
the intermediate system to get accurate information of the Evans function. Since ap­
propriate location of the spectrum of the operator implies linear stability and linear
stability implies nonlinear stability in the sense of $L^\infty$-norm, we assert the nonlinear
stability of the traveling wave.

Consider the system of integral-differential equations

$$
\begin{align*}
  u_t &= f(u) - w + \alpha \int_{-\infty}^{\infty} K(x - y) H(u(y, t) - \theta) dy, \\
  w_t &= \varepsilon (u - \gamma w),
\end{align*}
$$

where $f(u) = u(1 - u)(u - a)$, $\alpha, \beta, \gamma, \varepsilon, \theta$ are all positive parameters, $0 < a <
\frac{1}{2}, \alpha > 0, 0 < \gamma < 4/(1 - a)^2, 0 < \varepsilon \ll 1, \rho_-(a) < \theta < \rho_+(a)$, i.e. $f'(\theta) > 0$, where
$\rho_{\pm}(a) = \frac{1}{3}[1 + a \pm \sqrt{1 - a + a^2}]$ are the knees of the cubic function $f$ and $\beta > 1$ is
the unique solution of the equation $f(x) + \alpha = 0$, so that $f'(\beta) < f'(0) = -a$. $K$ is
an even, nonnegative, completely continuous function such that

$$
\int_{-\infty}^{\infty} K(x) dx = 1, \quad \int_{-\infty}^{\infty} |K'(x)| dx < \infty.
$$

Moreover we assume that there are positive constants $C$ and $\rho > |a + f'(\beta)|$, such
that $K(x) \leq C \exp(-\rho|x|)$. for all $x \in R$. $H$ is the Heaviside step function: $H(x) = 1$
if $x > 0$, $H(x) = 0$ if $x < 0$. The derivative of the Heaviside function is the well
known Dirac delta impulse function, which is a generalized function (continuous linear
functional on $C_0^\infty$) defined by

$$
\int_{-\infty}^{\infty} \delta(x - y) f(y) dy = f(x), \quad \text{for all } f \in C_0^\infty(-\infty, \infty).
$$
The Heaviside function can be approximated by the continuous function

\[ H(\epsilon, x) = 0, \text{ if } x < -\epsilon; \quad \frac{1}{2} + \frac{x}{2\epsilon}, \text{ if } -\epsilon \leq x \leq \epsilon; \quad 1, \text{ if } x > \epsilon, \]

where \( \epsilon > 0 \) is a small parameter. The derivative of this function is

\[ H_x(\epsilon, x) = 0, \text{ if } x < -\epsilon; \quad \frac{1}{2\epsilon}, \text{ if } -\epsilon < x < \epsilon; \quad 0, \text{ if } x > \epsilon. \]

Note that the derivative \( H_x \) is discontinuous at \( |x| = \epsilon \), but the integral is convergent
\[ \int_{-\infty}^{\infty} H_x(\epsilon, x) dx = \int_{-\epsilon}^{\epsilon} H_x(\epsilon, x) dx = 1. \]
Moreover, for all \( f \in C^0(\mathbb{R}) \), we have
\[
\int_{-\infty}^{\infty} H_x(\epsilon, y)f(x-y)dy = \int_{-\epsilon}^{\epsilon} H_x(\epsilon, y)f(x-y)dy = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x-y)dy \rightarrow f(x) = \int_{-\infty}^{\infty} \delta(y)f(x-y)dy,
\]
as \( \epsilon \to 0 \). Thus we say \( H(\epsilon, x) \) approaches \( H \) "suitably". It is clear that \( H(\epsilon, x) \) approaches \( H(x) \) pointwisely (not uniformly) on \( \mathbb{R} \), as \( \epsilon \to 0 \). On the other hand, because \( |H(\epsilon, x) - H(x)| = 0, \text{ if } |x| > \epsilon \), one asserts that \( H(\epsilon, x) \) converges to \( H(x) \) uniformly outside the interval \((-\epsilon, \epsilon)\), as \( \epsilon \to 0 \). For all \( x \in \mathbb{R} \) and \( 0 < \epsilon < 1 \), we have \( H(\epsilon, x) \leq 1 \).

By Lebesgue's dominated convergence theorem one concludes that
\[
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} K(x-y)H(\epsilon, u(y) - \theta)dy = \int_{-\infty}^{\infty} K(x-y)H(u(y) - \theta)dy,
\]
\[
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} K'(x-y)H(\epsilon, u(y) - \theta)dy = \int_{-\infty}^{\infty} K'(x-y)H(u(y) - \theta)dy,
\]
uniformly on \( \mathbb{R} - (-\epsilon, \epsilon) \), pointwisely on \( \mathbb{R} \), where \( u \) is a real-valued function.

Let us state the main theorems of this thesis:
Theorem 1. Suppose that the conditions on the parameters and on the functions $K$ and $H$ are satisfied. Then there exists a locally unique traveling pulse solution $(\varphi(\varepsilon, z), \varphi(\varepsilon, z), v(\varepsilon))$, for all sufficiently small $\varepsilon > 0$, such that

$$v(\varepsilon)\phi_z = f(\phi) - \varphi + \alpha \int_{-\infty}^{\infty} K(z - y)H(\phi(y) - \theta)dy,$$

$$v(\varepsilon)\varphi_z = \varepsilon(\phi - \gamma \varphi),$$

$$\lim_{|z| \to \infty} (\varphi(\varepsilon, z), \varphi(\varepsilon, z)) = (0, 0), \lim_{\varepsilon \to 0} v(\varepsilon) = v(0).$$

Theorem 2. The unique traveling pulse solution is exponentially stable, relative to the full system of integral-differential equations, for all sufficiently small parameter $\varepsilon > 0$.

H. Notations. $R$ and $C$ will denote the set of real and complex numbers, respectively, and $R^+ = (0, \infty)$. $[x]$ will denote the greatest integer less than or equal to $x$. We will use $Rez$ and $Imz$ for the real and imaginary parts of the complex number $z$. We denote by $(X, \| \cdot \|)$ or $(Y, \| \cdot \|)$ any normed linear space, or Banach space, by $M(\subset X)$ any linear subspace, by $B(X \to Y)$ the set of all bounded linear operators, by $CC(X \to Y)$ the set of all completely continuous linear operators, by $X^*$ the dual space of $X$, by $A^*$ the adjoint operator of $A \in B(X \to Y)$, by $\rho(A)$ the resolvent set and by $\sigma(A)$ the spectral set of $A$.

2. Preliminary Lemmas

We will list many preliminary lemmas, some of the proofs will be given immediately. In the study of the existence and stability of the traveling waves, we will cite these lemmas. The Gronwall’s inequality are very useful.
**Lemma 1.** Let \( f, g \) and \( 0 < h \in L^1(0, \infty) \) satisfy the inequality

\[
g(t) \leq f(t) + \int_0^t g(s)h(s)ds,
\]

for all \( t > 0 \). Then

\[
g(t) \leq f(0) \exp \left( \int_0^t h(s)ds \right) + \int_0^t f'(s) \exp \left( \int_s^t h(r)dr \right) ds.
\]

**Proof.** Multiplying the given inequality by \( h(t) \exp(-\int_0^t h(s)ds) \) yields

\[
\left( g(t)h(t) - h(t) \int_0^t g(s)h(s)ds \right) \exp \left( -\int_0^t h(s)ds \right) \leq f(t)h(t) \exp \left( -\int_0^t h(s)ds \right).
\]

This inequality is equivalent to the following inequality

\[
\frac{d}{dt} \left[ \int_0^t g(s)h(s)ds \exp \left( -\int_0^t h(s)ds \right) \right] \leq -f(t) \frac{d}{dt} \left[ \exp \left( -\int_0^t h(s)ds \right) \right].
\]

Integrating it with respect to \( t \) over \([0, t]\) to reach

\[
\int_0^t g(s)h(s)ds \exp \left( -\int_0^t h(s)ds \right) \\
\leq f(0) - f(t) \exp \left( -\int_0^t h(s)ds \right) + \int_0^t f'(s) \exp \left( -\int_0^{s} h(r)dr \right) ds.
\]

Now multiplying the above by \( \exp \left( \int_0^t h(s)ds \right) \) gives

\[
f(t) + \int_0^t g(s)h(s)ds \leq f(0) \exp \left( \int_0^t h(s)ds \right) + \int_0^t f'(s) \exp \left( \int_s^t h(r)dr \right) ds.
\]

We now obtain

\[
g(t) \leq f(0) \exp \left( \int_0^t h(s)ds \right) + \int_0^t f'(s) \exp \left( \int_s^t h(r)dr \right) ds.
\]

The next lemma deals with several topics about square matrices. Most of them are highly related to the existence and stability of traveling wave solutions.
Lemma 2. (a) Let \(x = (x_1, ..., x_n)\) be a vector in \(n\)-dimensional space \(\mathbb{R}^n\) or \(\mathbb{C}^n\) and let \(A = (a_{ij}), B = (b_{ij}), C = (c_{ij})\) be any \(n \times n\) real or complex matrices. Their norms are defined by

\[
\|x\|^2 = \sum_{i=1}^{n} |x_i|^2, \quad \|A\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2.
\]

With this definition, we have the inequalities

\[
\|Ax\| \leq \|A\|\|x\|, \quad \|AB\| \leq \|A\|\|B\|.
\]

(b) For any \(n \times n\) matrices \(A, B, C\), it is easy to get

\[
\begin{pmatrix}
  I & A \\
  0 & I
\end{pmatrix}
\begin{pmatrix}
  ABC & 0 \\
  0 & B
\end{pmatrix}
\begin{pmatrix}
  0 & -I \\
  I & C
\end{pmatrix}
= \begin{pmatrix}
  AB & 0 \\
  B & BC
\end{pmatrix}.
\]

we obtain the estimates concerning the rank of matrix

\[
\text{rank}(AB) + \text{rank}(BC) \leq \text{rank}(ABC) + \text{rank}(B),
\]

\[
\text{rank}(A) + \text{rank}(B) \leq \text{rank}(AB) + n.
\]

(c) For any \(n \times n\) matrix \(A\), there exists an invertible matrix \(T\) such that \(T^{-1}AT = \text{diag}(J_1, J_2, ..., J_n)\), where \(J_k = \lambda_k I\) or \(J_k = \lambda_k I + C_k\) are the Jordan blocks and \(C_k\) is a matrix of the form

\[
\begin{pmatrix}
  0 & I_k \\
  0 & 0
\end{pmatrix}.
\]

(d) Let \(c\) be any vector in \(n\)-dimensional real space \(\mathbb{R}^n\) or complex space \(\mathbb{C}^n\). If for all vector \(r\) with \(rA = 0\), there holds \(rc = 0\), i.e. \(c \perp N(A')\), then the linear algebraic equation \(Ax = c\) has at least one solution. For all \(n \times n\) matrices \(A\) and \(B\), \(AB - BA\) is not invertible.
(e) For the determinant of matrix, we have the equality \( \det(AB) = \det A \det B \). Moreover, if \( AB = BA \), then

\[
\begin{pmatrix}
I & 0 \\
-B & A
\end{pmatrix}
\begin{pmatrix}
A & C \\
B & D
\end{pmatrix}
= 
\begin{pmatrix}
A & C \\
0 & AD - BC
\end{pmatrix}.
\]

Hence we obtain the equality, regardless if \( A \) is invertible

\[\det\begin{pmatrix}
A & C \\
B & D
\end{pmatrix} = \det(AD - BC).\]

Furthermore, because

\[
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
\begin{pmatrix}
\lambda I & B \\
A & I
\end{pmatrix}
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
= 
\begin{pmatrix}
I & A \\
B & \lambda I
\end{pmatrix},
\]

we obtain the identity \( \det(\lambda I - AB) = \det(\lambda I - BA) \).

The characteristic polynomial

\[f(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)\ldots(\lambda - \lambda_n) = \lambda^n - \text{Tr}A\lambda^{n-1} + \ldots + (-1)^n \det A.\]

Thus we have \( \text{Tr}A = \lambda_1 + \lambda_2 + \ldots + \lambda_n \) and \( \det A = \det A' = \lambda_1\lambda_2\ldots\lambda_n \). Moreover \( f(A) = 0 \). \( \text{Rank}(\lambda I - A) = n \)— the geometric multiplicity of \( \lambda \). The geometric multiplicity of an eigenvalue is less than or equal to the algebraic multiplicity of this eigenvalue. If for any eigenvalue \( \lambda \) of \( A \), the geometric multiplicity is equal to the algebraic multiplicity, then \( A \) is diagonalizable.

For any \( n \times n \) matrix \( A \) and any vectors \( \psi_1, \psi_2, \ldots, \psi_n \in \mathbb{C}^n \), we have the identity

\[
\sum_{i=1}^{n} \det(\psi_1, \ldots, \psi_{i-1}, A\psi_i, \psi_{i+1}, \ldots, \psi_n) = \text{Tr}A \det(\psi_1, \psi_2, \ldots, \psi_n).
\]
It is easy to see this if \( \psi_1, \psi_2, ..., \psi_n \) are the linearly independent generalized eigenvectors of \( A \). Other cases follow immediately.

(f) For the exponential matrix function \( \exp(A) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + ... \),
\[
\exp(A + B) = \exp(A)\exp(B) \text{ if and only if } AB = BA.
\]
In addition \( \exp(T^{-1}AT) = T^{-1}\exp(A)T \), where \( T \) is any invertible matrix. The determinant \( \det(\exp(A)) = \exp(\text{tr}(A)) = \exp(\sum_{i=1}^{n} \lambda_i) \), where \( \lambda_i \)'s are the eigenvalues of the matrix \( A \). On the other hand, if \( \det A \neq 0 \), then there exists an \( n \times n \) matrix \( B \), such that \( A = \exp(B) \).

In general if \( f \) is a polynomial then \( f(A)\xi = f(\lambda)\xi \).

(g) If \( A \) is an \( n \times n \) constant matrix, then the fundamental matrix of the linear homogeneous system \( \psi_z = A\psi \) is \( \exp(Az) \), the solution is given by \( \psi(z) = \exp(Az)c \), where \( c = (c_1, c_2, c_3, ...c_n)^T \) is a constant vector, or given by \( \psi(z) = \exp(\lambda z)\xi \), where \( \lambda \) is an eigenvalue of \( A \) and \( \xi \) is a corresponding eigenvector. Let \( f \) be a vector valued function defined on \( \mathbb{R} \), then the solution of the nonhomogeneous system \( \psi_z = A\psi + f(z) \) is given by \( \psi(z) = \exp(Az)c + \int_{z}^{\infty} \exp[A(z - s)]f(s)ds \). If \( A \) has \( r \) eigenvalues with positive real parts and \( s \) eigenvalues with negative real parts and \( r + s = n \), \( P_\pm \) is the matrix projections corresponding to the eigenvalues with \( \pm \) real parts, \( f \) is a vector valued bounded continuous function on \( \mathbb{R} \), then the unique bounded continuous solution of the nonhomogeneous continuous function on \( R \), then the unique bounded continuous solution of the nonhomogeneous system \( \psi_z = A\psi + f \) is
\[
\int_{-\infty}^{z} P_- \exp[A(z - x)]f(x)dx - \int_{z}^{\infty} P_+ \exp[A(z - x)]f(x)dx.
\]

(h) If \( A \) is an \( n \times n \) matrix valued function and \( X(z) \) is a fundamental matrix
of the system \( \psi_z = A(z)\psi \), then the general solution of the nonhomogeneous system \( \psi_z = A(z)\psi + f(z) \) is

\[
\psi(z) = X(z) \left( c + \int_0^z X(s)^{-1} f(s) ds \right).
\]

If there exist \( r \) linearly independent solutions \( \psi_i(z), i = 1, \ldots, r \) such that they approach zero exponentially fast as \( z \to -\infty \) and there exist \( s = n - r \) linearly independent solutions \( \psi_i(z), i = r+1, \ldots, n \) such that they approach zero exponentially fast as \( z \to +\infty \), define \( X(z) = (\psi_1(z), \ldots, \psi_n(z)) \). Suppose that \( \det X(z) \neq 0 \), for all \( z \in \mathbb{R} \). Then \( X(z) \) is a fundamental matrix. Let \( f \) be a vector valued bounded continuous function on \( \mathbb{R} \), then the unique bounded continuous solution of the nonhomogeneous system \( \psi_z = A(z)\psi + f \) is (we allow some parameters to appear here)

\[
\left( \psi_1(\lambda, \varepsilon, z), \ldots, \psi_n(\lambda, \varepsilon, z) \right) = 
\begin{pmatrix}
\vdots \\
c_1(\lambda, \varepsilon, z) \\
\vdots \\
c_n(\lambda, \varepsilon, z)
\end{pmatrix},
\]

where

\[
c_i(\lambda, \varepsilon, z) = \begin{cases}
- \frac{1}{E(\lambda, \varepsilon)} \int_z^\infty \exp\left[ - \int_0^s \text{Tr}A(\lambda, \varepsilon, r) \, dr \right] D_i(\lambda, \varepsilon, s) \, ds, & \text{for } 1 \leq i \leq r, \\
+ \frac{1}{E(\lambda, \varepsilon)} \int_\infty^z \exp\left[ - \int_0^s \text{Tr}A(\lambda, \varepsilon, r) \, dr \right] D_i(\lambda, \varepsilon, s) \, ds, & \text{for } r < i \leq n,
\end{cases}
\]

and

\[
D_i(\lambda, \varepsilon, s) = \det(\psi_1(\lambda, \varepsilon, s), \ldots, \psi_{i-1}(\lambda, \varepsilon, s), f(s), \psi_{i+1}(\lambda, \varepsilon, s), \ldots, \psi_n(\lambda, \varepsilon, s)), \\
E(\lambda, \varepsilon) = \exp\left[ - \int_0^z \text{Tr}A(\lambda, \varepsilon, s) \, ds \right] \det X(\lambda, \varepsilon, z).
\]
Moreover

\[
\exp[- \int_0^z \text{Tr}A(\lambda, \varepsilon, s)ds]D_1(\lambda, \varepsilon, z)
\]

is a continuous linear functional. By Riesz representation theorem, there exists a unique function \( \varphi_i(\lambda, \varepsilon, z) \), such that for all point \( f \in C^n \), the above linear functional \( = (f, \varphi_i(\lambda, \varepsilon, z)) \). A simple calculation shows that

\[
(f, \varphi_{iz}) = (f, \varphi_i)_z = -(Af, \varphi_i) = (f, -A'\varphi_i).
\]

Because \( f \) is an arbitrary vector in \( C^n \), we must have \( \varphi_{iz} + A'\varphi_i = 0 \), namely \( \varphi_i \) is the solution of the adjoint system, where \( i = 1, 2, ..., n \). Moreover, we have \( (\psi_i, \varphi_i) = E(\lambda, \varepsilon) \) and \( (\psi_j, \varphi_i) = 0 \), for all \( j \neq i \). In addition, \( c_i(z) = -\int_z^\infty (f(s), \varphi_i(s))ds \) for \( i \leq r \) and \( c_i(z) = \int_{-\infty}^z (f(s), \varphi_i(s))ds \) for \( i > r \), and all the products \( c_i(z)\psi_i(z), 1 \leq i \leq n \), are bounded on \( \mathbb{R} \).

(i) If \( X(z) \) is a matrix-valued continuously differentiable function and \( \det X(z) \neq 0 \), for all \( z \in \mathbb{R} \), then

\[
\frac{d}{dz}X(z)^{-1} = -X(z)^{-1}\left[\frac{d}{dz}X(z)\right]X(z)^{-1}.
\]

If \( X(z) \) is the fundamental matrix of \( \psi_z = A(z)\psi \), then \( (X(z))^{-1} \) is the fundamental matrix of the adjoint system \( \psi_z + A(z)'\psi = 0 \).

Let \( f(z) = \det X(z) \). It satisfies the equation \( f_z = \text{Tr}A(z)f \). It is clear to see \( \exp[- \int_0^z \text{Tr}A(s)ds] \det X(z) = \det X(0) \) is independent of \( z \).

The solution operator of the linear system \( \psi_z = A(z)\psi \) takes planes to planes. The normal vector to a plane solves the adjoint system \( \psi_z + A(z)'\psi = 0 \).
(j) Let \((x, y) = (x(z), y(z))\) be a solution of the first order nonlinear autonomous system of ordinary differential equations

\[
\frac{d}{dz} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix},
\]

where \(f\) and \(g\) are real-valued smooth functions of \(x\) and \(y\). Then

\[
\frac{d}{dz} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.
\]

Consider the first order linear system of ordinary differential equations

\[
\frac{d\psi}{dz} + \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \psi = 0.
\]

One solution of this system is given by

\[
\psi(z) = \exp \left\{ - \int_0^z \left[ f_x(x(s), y(s)) + g_y(x(s), y(s)) \right] ds \right\} \begin{pmatrix} +g(x, y) \\ -f(x, y) \end{pmatrix}.
\]

Remark. Let \((f, g)\) be a solution of the linear nonautonomous system

\[
\frac{d}{dz} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.
\]

Then

\[
\psi(z) = \exp \left\{ - \int_0^z \left[ a(s) + d(s) \right] ds \right\} \begin{pmatrix} +g \\ -f \end{pmatrix}
\]

is a solution of the adjoint system

\[
\frac{d\psi}{dz} + \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \psi = 0,
\]

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where \( a(z), b(z), c(z), d(z) \) are smooth functions. Consider the system \( x_z = f(x) \), where \( x = (x_1, x_2, x_3, \ldots, x_n) \), \( f = (f_1(x), f_2(x), f_3(x), \ldots, f_n(x)) \). Then \( f \) solves the linear nonautonomous equation

\[
\frac{df}{dt} = \left( \frac{\partial f_1}{\partial x_1} \right) f.
\]

(k) Let \( A \) be a matrix projection such that \( A^2 = A \). Then

\[
(I - A)^2 = I - A,
\]

\[
(A')^2 = A',
\]

\[
\text{rank} \, A = \text{Tr} \, A,
\]

\[
R(I - A) = N(A),
\]

\[
N(I - A) = R(A),
\]

\[
T^{-1}AT = \begin{pmatrix}
I_r & 0 \\
0 & 0
\end{pmatrix},
\]

for some invertible \( n \times n \) matrix \( T \).

(l) For the nonautonomous linear system \( \psi_z = A(z) \psi \), we can define a matrix projection \( P \). First define a linear space: \( V = \{ x_0 \in C^n \colon \text{the solution with initial data } \psi(0) = x_0 \text{ to the system } \psi_z = A(z) \psi \text{ approaches zero exponentially fast as } z \to +\infty \} \). Then define its complementary space by \( V \oplus W = C^n \). Now for any \( x \in C^n, x = y + z, \) such that \( y \in V \) and \( z \in W \), define \( Px = y \). Then \( P \) is the desired projection on \( (0, +\infty) \). If \( P \) and \( Q \) are the matrix projections of \( \psi_z = A(z) \psi \) on \( (-\infty, 0) \) and \( (0, \infty) \), respectively, then the matrix projections of the adjoint system \( \psi_z + A(z)' \psi = 0 \) are \( I - Q' \) and \( I - P' \), respectively. The subspace of initial values of bounded solutions of \( \psi_z = A(z) \psi \) is \( R(P) \cap N(Q) \), of \( \psi_z + A(z)' \psi = 0 \) is \( R(I - P') \cap R(Q') \).
(m) Let \( A \) be an \( n \times n \) constant matrix and let \( B(z) \) be an \( n \times n \) matrix valued function satisfying \( \|B(z)\| \leq C \exp(-\rho|z|) \), for some positive constants \( C \) and \( \rho \). We first consider solutions to the ordinary differential equations

\[
\frac{\partial \psi}{\partial z} - A \psi = 0, \quad \frac{\partial \psi}{\partial z} + A' \psi = 0.
\]

Then we construct related solutions to the perturbed differential equations

\[
\frac{\partial \psi}{\partial z} - [A + B(z)] \psi = 0, \quad \frac{\partial \psi}{\partial z} + [A + B(z)]' \psi = 0.
\]

Finally we use the method of variation of parameter to construct all solutions to the nonhomogeneous ordinary differential equations

\[
\frac{\partial \psi}{\partial z} - [A + B(z)] \psi = \alpha(z), \quad \frac{\partial \psi}{\partial z} + [A + B(z)]' \psi = \beta(z),
\]

where \( \alpha \) and \( \beta \) are bounded uniformly continuous functions on \( R \).

Suppose that \( \mu_1 < \ldots < \mu_r < 0 < \mu_{r+1} < \ldots < \mu_m \) are the distinct real parts of the eigenvalues of \( A \) and that \( n_k \) is the number of eigenvalues counting algebraic multiplicities with real part \( \mu_k \), for \( 1 \leq k \leq m \). There exists an invertible matrix \( T \) such that

\[
A_0 = T^{-1} A T = \begin{pmatrix}
J_1 & 0 & \ldots & 0 \\
0 & J_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & J_m 
\end{pmatrix},
\]
where

\[ J_k = \begin{pmatrix} 
\mu_k + i\nu_{k1} & * & \cdots & 0 \\
0 & \mu_k + i\nu_{k2} & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \mu_k + i\nu_{knk}
\end{pmatrix}, \]

is an \( n_k \times n_k \) matrix with \( \text{Re}(\mu_k + i\nu_{ki}) = \mu_k \) and \( * = 0 \) or 1.

Then the ordinary differential equations are reduced to

\[ \frac{\partial \psi}{\partial z} - A_0 \psi = 0, \quad \frac{\partial \psi}{\partial z} + A'_0 \psi = 0. \]

The fundamental matrices for these equations are \( \exp(+A_0z) \) and \( \exp(-A'_0z) \).

The following linearly independent solutions is a basis of the solution space of the above differential equations, respectively

\[ \exp(+A_0z)x_1, \ldots, \exp(+A_0z)x_n, \]
\[ \exp(-A'_0z)x_1, \ldots, \exp(-A'_0z)x_n. \]

Moreover

\[ \exp(+A_0z) = \begin{pmatrix} 
\exp(+J_1z) & 0 & \cdots & 0 \\
0 & \exp(+J_2z) & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \exp(+J_mz)
\end{pmatrix}, \]

\[ \exp(-A'_0z) = \begin{pmatrix} 
\exp(-J'_1z) & 0 & \cdots & 0 \\
0 & \exp(-J'_2z) & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \exp(-J'_mz)
\end{pmatrix} \]
If one of the Jordan blocks in $A_0$ takes the form $J = \mu I$, then $\exp(Jz) = \exp(\mu z)I$.

Suppose that one of the Jordan blocks in $A_0$ takes the form

$$J = \begin{pmatrix} 
\mu & 0 & \ldots & 0 \\
0 & \mu & \ldots & 0 \\
0 & 0 & \mu & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \mu \\
0 & 0 & \ldots & 0 \\
\end{pmatrix} = \mu I + C.$$

Then $C^m \neq 0$ but $C^{m+1} = 0$, and

$$\exp(Jz) = \exp(\mu z) \exp(Cz) = \exp(\mu z)(I + Cz + \frac{1}{2!}C^2z^2 + \ldots + \frac{1}{m!}C^mz^m)$$

There exist solutions $\{Y_l^k(z) = \exp(\pm A_0 z) x_l^k : 1 \leq l \leq n_k \text{ and } 1 \leq k \leq m\}$ to the ordinary differential equations $-A_0^l \psi = 0$ with limit

$$\lim_{|z| \to \infty} \frac{1}{z} \ln \|Y_l^k(z)\| = +\mu_k,$$

and solutions $\{Z_q^p(z) = \exp(\pm A_0'z) x_q^p : 1 \leq q \leq n_p \text{ and } 1 \leq p \leq m\}$ to the ordinary differential equations $\frac{\partial \psi}{\partial z} + A_0' \psi = 0$ with limit

$$\lim_{|z| \to \infty} \frac{1}{z} \ln \|Z_q^p(z)\| = -\mu_p.$$
If \( (k, l) = (p, q) \), then \((Y_i^k(z), Z_q^p(x)) = 1\), for all \(z, x \in R\). If \( k \neq p \), or if \( k = p \) but \( l < q \), then \((Y_i^k(z), Z_q^p(x)) = 0\), for all \(z, x \in R\).

Define

\[
X^k(z, x) = (Y_i^k(z), ..., Y_{n_k}^k(z))(Z_1^k(x), ..., Z_{n_k}^k(x))' = \exp(+A_0z) \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{n_k} & 0 \\ 0 & 0 & 0 \end{pmatrix} \exp(-A_0x)
\]

\[
= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \exp[J_k(z - x)] & 0 \\ 0 & 0 & 0 \end{pmatrix} = \exp[\mu_k(z - x)] \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

for \(1 \leq k \leq m\), where \(* = I_{nk}\) or \(* = \exp[C_k(z - x)]\) is an \(n_k \times n_k\) matrix, and \(* = I_{nk}\) if \(z = x\). Then

\[
X^k(z, x) = X^k(z - x, 0), \quad \exp[A_0(z - x)] = \sum_{k=1}^m X^k(z, x).
\]

All the linearly independent solutions of the ordinary differential equations \(\frac{\partial \psi}{\partial z} - A\psi = 0\) and \(\frac{\partial \psi}{\partial z} + A'\psi = 0\) are given by \(TY_i^k(z)\) and \((T')^{-1}Z_q^p(z)\), respectively. Without loss of generality, denote these solutions also by \(Y_i^k(z)\) and \(Z_q^p(z)\). These solutions satisfy the same conditions, i.e.

\[
(Y_i^k(z), Z_q^p(x)) = 1, \text{ for } (k, l) = (p, q); \quad (Y_i^k(z), Z_q^p(x)) = 0, \text{ for } k \neq p \text{ or } l < q,
\]

\[
X^k(z, x) = X^k(z - x, 0), \quad \exp[A(z - x)] = \sum_{k=1}^m X^k(z, x).
\]

Consider the perturbed differential equations

\[
\frac{\partial \psi}{\partial z} - [A + B(z)]\psi = 0, \quad \frac{\partial \psi}{\partial z} + [A + B(z)]'\psi = 0.
\]
There exist linearly independent solutions \( f^k(z) \) and \( g^p(z) \) to these ordinary differential equations, such that for some constant \( \rho > 0 \), one has the asymptotic behaviors
\[
\begin{align*}
f^k(z) - Y^k_l(z) &= \exp\left((+\mu_k - \rho)z\right)O(1), \text{ as } z \to +\infty, \text{ for } 1 \leq l \leq n_k, 1 \leq k \leq r, \\
f^k_l(z) - Y^k_l(z) &= \exp\left((+\mu_k + \rho)z\right)O(1), \text{ as } z \to -\infty, \text{ for } 1 \leq l \leq n_k, r < k \leq m, \\
g^p_q(z) - Z^p_q(z) &= \exp\left((-\mu_p + \rho)z\right)O(1), \text{ as } z \to -\infty, \text{ for } 1 \leq q \leq n_p, 1 \leq p \leq r, \\
g^p_q(z) - Z^p_q(z) &= \exp\left((-\mu_p - \rho)z\right)O(1), \text{ as } z \to +\infty, \text{ for } 1 \leq q \leq n_p, r < p \leq m.
\end{align*}
\]

It is clear that \( \frac{d}{dz}(f^k(z), g^p(z)) = 0 \), hence \((f^k(z), g^p(z))\) must be a constant. If \((k, l) = (p, q)\), then \((f^k(z), g^p(z)) = 1\). If \(k < p\) or \(k = p\) but \(l < q\), then \((f^k(z), g^p(z)) = 0\). Let us use Gram-Schmidt method to modify \( g^p_q \) so that \( f^k_l \) and \( g^p_q \) are mutually orthogonal. Define
\[
\begin{align*}
h^m_q(z) &= g^m_q(z) - \sum_{l=1+q}^{n_m} (f^m_l(z), g^m_q(z)) h^m_l(z), \text{ for } 1 \leq q \leq n_m, \\
h^p_q(z) &= g^p_q(z) - \sum_{k=1+p}^{m} \sum_{l=1}^{n_k} (f^k_l(z), g^p_q(z)) h^k_l(z) - \sum_{l=1+q}^{n_p} (f^p_l, g^p_q) h^p_l, \\
&\text{ for } 1 \leq q \leq n_p, 1 \leq p < m.
\end{align*}
\]

Then \( h^p_q(z) \) are linearly independent solutions to the adjoint equation \( \frac{\partial^2 \psi}{\partial z^2} + [A + B(z)]'\psi = 0 \) and for all \( 1 \leq q \leq n_p, 1 \leq p \leq m \), we have
\[
(f^k(z), h^p_q(z)) = \begin{cases} 1, & \text{if } (k, l) = (p, q), \\ 0, & \text{if } (k, l) \neq (p, q). \end{cases}
\]

Moreover for some constant \( \rho > 0 \), one has the asymptotic behaviors
\[
\begin{align*}
h^p_q(z) - Z^p_q(z) &= \exp(-\mu_p z)O(1), \text{ as } z \to -\infty, \text{ for } 1 \leq q \leq n_p, 1 \leq p \leq r, \\
h^p_q(z) - Z^p_q(z) &= \exp(-\mu_p z)O(1), \text{ as } z \to +\infty, \text{ for } 1 \leq q \leq n_p, r < p \leq m.
\end{align*}
\]
Define $Y^k(z, x) = (f^k_1(z), ..., f^k_n(z))(h^k_1(x), ..., h^k_n(x))'$. Then we have the estimates

$$Y^k(z, x) = \exp[\mu_k(z - x)]O(1),$$

as $z \to +\infty, x \to -\infty$ if $1 \leq k \leq r$ and as $z \to -\infty, x \to +\infty$ if $r < k \leq m$.

Moreover we have

$$\frac{\partial}{\partial z} Y^k(z, x) - [A + B(z)]Y^k(z, x) = 0,$$

$$\frac{\partial}{\partial x} Y^k(z, x) + Y^k(z, x)[A + B(x)] = 0.$$

By orthogonality, we have

$$Y^k(z, z)Y^k(z, z) = Y^k(z, z),$$

so

$$\frac{d}{dz} Y^k(z, z) = 0.$$

Therefore $Y^k(z, z)$ is a constant matrix. By the asymptotic analysis we see that $Y^k(z, z) = X^k(z, z)$. Hence

$$\sum_{k=1}^{m} Y^k(z, z) = I.$$

Now it is easy to see that

$$\Phi(z) = \sum_{k=1}^{r} \int_{-\infty}^{z} Y^k(z, x)\alpha(x)dx - \sum_{k=r+1}^{m} \int_{z}^{\infty} Y^k(z, x)\alpha(x)dx$$

is a bounded solution on $(-\infty, \infty)$ to the nonhomogeneous differential equation

$$\frac{\partial \psi}{\partial z} - [A + B(z)]\psi = \alpha(z).$$
If \( \psi \) is a solution to the nonhomogeneous equation, then there exist constants \( c^k_l \) such that

\[
\psi(z) = \Phi(z) + \sum_{k=1}^{m} \sum_{l=1}^{n_k} c^k_l f^k_l(z).
\]

If \( \psi \) is a bounded solution as \( z \to +\infty \), then for all \( 1 \leq q \leq n_p \) and \( r \leq p \leq m \), we must have

\[
\lim_{z\to+\infty} (\psi(z), h^p_q(z)) = 0.
\]

On the other hand, if for all \( 1 \leq q \leq n_p \) and \( r \leq p \leq m \), we have

\[
\lim_{z\to+\infty} (\psi(z), h^p_q(z)) = 0,
\]

then because \( \Phi(z) \) is a bounded solution to the nonhomogeneous differential equation, the fact

\[
\lim_{z\to+\infty} \Phi(z), h^p_q(z)) = 0,
\]

yields the limit

\[
c^p_q = \lim_{z\to+\infty} \left( \sum_{k=1}^{m} \sum_{l=1}^{n_k} c^k_l f^k_l(z), h^p_q(z) \right) = 0.
\]

Therefore

\[
\psi(z) = \Phi(z) + \sum_{k=1}^{r} \sum_{l=1}^{n_k} c^k_l f^k_l(z).
\]

Since for \( 1 \leq l \leq n_k \) and \( 1 \leq k \leq r \) and

\[
\lim_{z\to+\infty} f^k_l(z) = \lim_{z\to+\infty} \{ Y^k_l + \exp[(\mu_k - \rho)z]O(1) \} = 0,
\]

so \( \psi \) is a bounded solution as \( z \to +\infty \).

Therefore the solution \( \psi \) is bounded as \( z \to +\infty \) if and only if

\[
\lim_{z\to+\infty} (\psi(z), \varphi(z)) = 0,
\]
where $\varphi(z)$ is any solution of the adjoint differential equation
\[ \frac{\partial \varphi}{\partial z} + [A + B(z)]' \varphi = 0, \]
with
\[ \lim_{z \to +\infty} \varphi(z) = 0. \]

Similarly the solution $\psi$ is bounded as $z \to -\infty$ if and only if
\[ \lim_{z \to -\infty} (\psi(z), \varphi(z)) = 0, \]
where $\varphi(z)$ is any solution of the adjoint differential equation
\[ \frac{\partial \varphi}{\partial z} + [A + B(z)]' \varphi = 0, \]
with
\[ \lim_{z \to -\infty} \varphi(z) = 0. \]

Moreover, if the solution $\psi(z)$ of the nonhomogeneous equations is unbounded, then $\psi(z) = \exp(\rho z)$ as $z \to +\infty$ for some $\rho > 0$.

(n) Let $A(z)$ be an $n \times n$ matrix valued continuous function. The system $\dot{\psi}_z = A(z)\psi$ with fundamental matrix $X(z)$ has an exponential dichotomy on some interval if there is a matrix projection $P$ and positive constants $C$ and $\alpha$, such that
\[ ||X(z)PX(x)^{-1}|| \leq C \exp[-\alpha(z - x)], \quad \text{for } z \geq x, \]
\[ ||X(z)(I - P)X(x)^{-1}|| \leq C \exp[-\alpha(x - z)], \quad \text{for } z \leq x. \]

Define the matrix projection $Q(z) = X(z)PX(z)^{-1}$. It is the solution of the matrix equation $X'(z) = A(z)X(z) - X(z)A(z)$. For all $x$ and $z$, we have
\[ X(z)PX(x)^{-1} = X(z)X(x)^{-1}Q(s) = Q(t)X(z)X(s)^{-1}, \]
\[ X(z)(I - P)X(x)^{-1} = X(z)X(x)^{-1}[I - Q(x)]X(z)X(x)^{-1}. \]

the stable space

\[ \{ \xi \in \mathbb{C}^n : \sup_{t\geq 0} \|X(z)X(0)^{-1}\xi\| < \infty \}. \]

the unstable space

\[ \{ \xi \in \mathbb{C}^n : \sup_{t\leq 0} \|X(z)X(0)^{-1}\xi\| < \infty \}. \]

(o) Let \( A(z) \) be an \( n \times n \) bounded, uniformly continuous matrix valued function defined on \( \mathbb{R} \), such that the following system

\[ \frac{d\psi}{dz} = A(z)\psi \quad (1.3) \]

has a solution which approaches zero exponentially fast as \( |z| \to \infty \). Then the linear operator \( L : C^1(\mathbb{R}, \mathbb{R}^n) \to C(\mathbb{R}, \mathbb{R}^n) \) defined by \( L\psi = \frac{d\psi}{dz} - A(z)\psi \) is Fredholm and \( f \in R(L) \) if and only if

\[ \int_{-\infty}^{\infty} g(z)'f(z)dz = 0, \quad (1.4) \]

for all bounded solutions \( g(z) \) of the adjoint system

\[ \frac{d\psi}{dz} + A(z)\psi = 0. \]

Moreover if \( V \) and \( W \) are the stable and unstable subspaces for \( \psi' = A\psi \), then the index of \( L \) equals

\[ \dim V \cap W - \dim (V^\perp \cap W^\perp) = \dim V + \dim W - n. \]

Proof. Let \( P \) and \( Q \) be the projections on \((-\infty, 0)\) and \((0, +\infty)\), respectively. Suppose that in both cases the associated fundamental matrix satisfy \( X(0) = I \).
Then the adjoint system has fundamental matrix \((X(z))^{-1}\). The adjoint system has a solution which approaches zero exponentially fast as \(|z| \to \infty\) with projection \(I - P'\) on \((0, \infty)\) and \(I - Q'\) on \((-\infty, 0)\). Obviously \(\dim N(L) = \dim (V \cap W)\).

Now the subspace of bounded solutions is \(V \cap W = R(P) \cap N(Q)\) and for \(\psi_z + A(z)\psi = 0\) it is \(V^\perp \cap W^\perp = R(I - P') \cap N(Q')\).

Let \(f \in R(L)\), then there exists \(h \in C^1(R, R^n)\) such that \(h' - Ah = f\). Now if \(g\) is a bounded solution of the adjoint system then

\[
\int_{-\infty}^{\infty} g(z)' f(z) dz = \int_{-\infty}^{\infty} [g(z)' h'(z) - g(z)' A(z) h(z)] dz = \int_{-\infty}^{\infty} [g(z)' h'(z) + g'(z) h(z)] dz = (g(z) h(z)) \bigg|_{-\infty}^{\infty} = 0.
\]

Conversely, suppose that \(f \in C^1(R, R^n)\) and that the assumption \(\int g(z)' f(z) dz = 0\), for all bounded solutions of the adjoint system. If a vector satisfies \(V'(P + Q - I) = 0\), define a bounded solution of the adjoint system as follows:

\[
g(z) = (X')^{-1}(z)(I - P') V, \quad \text{for } z \geq 0,
\]

\[
g(z) = (X')^{-1}(z)(Q') V, \quad \text{for } z \leq 0.
\]

It follows that

\[
V' \int_{-\infty}^{0} QX^{-1}(z)f(z) dz + V' \int_{0}^{\infty} (I - P)X^{-1}(z)f(z) dz = 0.
\]

This means that the linear algebraic equation

\[
(P + Q - I) \xi = \int_{-\infty}^{0} QX^{-1}(z)f(z) dz + \int_{0}^{\infty} (I - P)X^{-1}(z)f(z) dz,
\]

has a solution \(\xi\). Define \(h(z)\) on the positive and negative real axis as follows

\[
X(z)P \xi + \int_{z}^{s} X(z)PX^{-1}(s)f(s) ds - \int_{z}^{\infty} X(z)(I - P)X^{-1}(s)f(s) ds,
\]

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\[ X(z)(I - Q)x + \int_{-\infty}^{z} X(z)QX^{-1}(s)f(s)ds - \int_{z}^{0} X(z)(I - Q)X^{-1}(s)f(s)ds. \]

Then \( h \in C^1(R, R^n) \). It is a bounded solution of the nonhomogeneous linear system
\[
\frac{d\psi}{dz} = A(z)\psi + f(z).
\]

Thus \( L\psi = f \) and so \( f \in R(L) \), as desired.

To show that \( R(L) \) is closed and has finite codimension, note that each bounded solution \( f(z) \) of the adjoint system defines a bounded linear functional on \( C^1(R, R^n) \) through
\[
f \rightarrow \int_{-\infty}^{\infty} g(z)'f(z)dz
\]
and this correspondence gives an isomorphism between \( V \perp \cap W \perp \) and a finite dimensional subspace of the dual space \( C(R, R^n)^* \). \( R(L) \) is the subspace of \( C(R, R^n)^* \) annihilated by this finite-dimensional subspace of \( C(R, R^n)^* \), so \( R(L) \) is closed and \( \text{codim} R(L) = \dim(V \perp \cap W \perp) \). Hence \( L \) is Fredholm, as asserted. Now the index of \( L \) is equal to
\[
\dim N(L) - \text{codim} R(L)
\]
\begin{align*}
= \dim(V \cap W) - \dim(V \perp \cap W \perp) \\
= \dim(V \cap W) - [n - \dim(V + W)] \\
= \dim(V \cap W) - [n - (\dim V + \dim W - \dim(V \cap W))] \\
= \dim V + \dim W - n.
\end{align*}

(p) Consider the matrix valued initial value problems
\[
M_t = MA, \quad M(x, 0) = M_0, \\
N_t = -AN, \quad N(x, 0) = N_0,
\]

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where $A = (a_{ij}(x, t))$ is an $n \times n$ matrix. If $M_0N_0 = N_0M_0 = I$, then both $M$ and $N$ are invertible.

Proof. Since

$$(MN)_t = M_tN + MN_t = MAN - MAN = 0, \quad M(0)N(0) = I,$$

so $M(t)N(t) = I$ and $M(t)$ and $N(t)$ are always invertible.

Let $A = (a_{ij})$ and $B(t) = (b_{ij}(t))$ be $n \times n$ matrices, such that

$$||\exp(At)|| \leq C e^{-\alpha t}, \quad ||B(t)|| \leq \frac{\alpha - \sigma}{4C},$$

where $C > 0$ and $\alpha > \sigma > 0$ are constants.

Consider the initial value problem of the matrix-valued equation

$$M_t = AM + B(t)M, \quad M(x, 0) = I.$$

For all $t > s > 0$, we have the estimate

$$||M(t)M^{-1}(s)|| \leq C \exp[-\frac{3\alpha + \sigma}{4}(t - s)].$$

Proof. Multiplying the equation by $\exp(-At)$ and integrating with respect to $t$, we obtain

$$\exp(-At)M(t) = \exp(-As)M(s) + \int_s^t \exp(-Ar)B(r)M(r)dr.$$ 

Clearly we have

$$M(t)M^{-1}(s) = \exp[A(t - s)] + \int_s^t \exp[A(t - \tau)]B(\tau)M(\tau)M^{-1}(s)d\tau.$$
Hence it is not hard to get the following estimates

\[
\|M(t)M^{-1}(s)\| \leq \|\exp[A(t - s)]\| + \int_s^t \|\exp[A(t - r)]B(r)M(r)M^{-1}(s)\|dr
\]

\[
\leq C \exp[-\alpha(t - s)] + \frac{\alpha - \sigma}{4} \int_s^t \exp[-\alpha(t - r)]\|M(r)M^{-1}(s)\|dr.
\]

Equivalently

\[
\exp[\alpha(t - s)]\|M(t)M^{-1}(s)\| \leq C + \frac{\alpha - \sigma}{4} \int_s^t \exp[\alpha(r - s)]\|M(r)M^{-1}(s)\|dr.
\]

By Gronwall’s inequality, we have

\[
\exp[\alpha(t - s)]\|M(t)M^{-1}(s)\| \leq C \exp(\frac{\alpha - \sigma}{4}(t - s)).
\]

Now the desired estimate can be established.

1. Let \(A, B, C, D\) be any matrices. The tensor product or Kronecker product is defined by

\[
A \otimes B = (a_{ij}B)_{1 \leq i, j \leq n}.
\]

Then we have

\[
(A \otimes B)' = A' \otimes B', \quad A \otimes B = \overline{A} \otimes \overline{B},
\]

\[
(A \otimes C)(B \otimes D) = AB \otimes CD,
\]

\[
(A \otimes B)^{-1} = A^{-1} \otimes B^{-1},
\]

\[
det(A \otimes B) = (det A)^n(det B)^n,
\]

\[
\text{rank}(A \otimes B) = \text{rank}A\text{rank}B.
\]

If \(A\) is a matrix or a vector, denote the transpose by \(A'\). The following simple observation plays a role.
(s) Let $\xi(\eta)$ be the eigenvector of the matrix $A(A')$ corresponding to the eigenvalue $\lambda(\mu)$. If $\lambda \neq \mu$, then $\eta'\xi = 0$, i.e. eigenvectors of $A$ and $A'$ corresponding to different eigenvalues are mutually perpendicular or orthogonal.

(t) Let $\lambda$ be an algebraically simple eigenvalue of $A$ and let $\xi$ and $\eta$ satisfy the equations $A\xi = \lambda \xi$ and $A'\eta = \lambda \eta$, respectively, then $\eta'\xi \neq 0$.

(u) Let $a, b, c, d$ be complex numbers such that the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two nonzero eigenvalues with different real parts:

$$\lambda_{\pm} = \frac{1}{2} \left[ a + d \pm \sqrt{(a - d)^2 + 4bc} \right],$$

$$(\lambda - a)(\lambda - d) = bc,$$

$$\lambda_{+} + \lambda_{-} = a + d,$$

$$\lambda_{+} - \lambda_{-} = \sqrt{(a - d)^2 + 4bc}.$$ 

The inverse matrix of $A$ is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$ 

In general if $A = (a_{ij})$ is an $n \times n$ matrix, then

$$A^{-1} = \frac{1}{\text{det } A} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}.$$ 

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The solution to the linear algebraic equations \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix} \) is

\[
\frac{1}{ad - bc} \begin{pmatrix} dr - bs \\ as - cr \end{pmatrix}.
\]

Suppose that \( b \neq 0 \). An eigenvector of \( A \) is \( \begin{pmatrix} b \\ \lambda_\pm - a \end{pmatrix} \) and an eigenvector of \( A' \) is \( \begin{pmatrix} \lambda_\pm - d \\ b \end{pmatrix} \). Projections corresponding to \( \lambda_\pm \) are

\[
\frac{1}{2\lambda_\pm - a - d} \begin{pmatrix} \lambda_\pm - d & b \\ c & \lambda_\pm - a \end{pmatrix}.
\]

If \( c \neq 0 \) one can show the projections are defined in the same way. Define

\[
P_+ = \frac{1}{\sqrt{(a - d)^2 + 4bc}} \begin{pmatrix} \lambda_+ - d & b \\ c & \lambda_+ - a \end{pmatrix},
\]

\[
P_- = \frac{1}{\sqrt{(a - d)^2 + 4bc}} \begin{pmatrix} d - \lambda_- & -b \\ -c & a - \lambda_- \end{pmatrix}.
\]

Then they are mutually orthogonal projection operators, i.e. \( P_+^2 = P_+ \) and \( P_+P_- = P_-P_+ = 0 \). More importantly, there hold

\[
P_+ + P_- = I,
\]

\[
A = \lambda_+ P_+ + \lambda_- P_-,
\]

\[
\exp(Az) = \exp(\lambda_+ z)P_+ + \exp(\lambda_- z)P_-.
\]
If \( b = c = 0 \), then \( \lambda_+ = a \) and \( \lambda_- = d \), we choose \( P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \).

(v) Let the constant matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) have one eigenvalue with positive real part and one eigenvalue with negative real part. Let \( f(z) \) and \( g(z) \) be bounded continuous complex-valued functions. The unique bounded solution to the linear differential equations

\[
\frac{d}{dz} \begin{pmatrix} \phi \\ \psi \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},
\]

is given by

\[
\int_{-\infty}^{z} \exp[\lambda_-(z - x)] P_- \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} dx - \int_{z}^{\infty} \exp[\lambda_+(z - x)] P_+ \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} dx
\]

\[
= -\frac{1}{\sqrt{(a - d)^2 + 4bc}} \begin{pmatrix} \lambda_- - d & b \\ c & \lambda_- - a \end{pmatrix} \int_{-\infty}^{z} \exp[\lambda_-(z - x)] \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} dx
\]

\[
-\frac{1}{\sqrt{(a - d)^2 + 4bc}} \begin{pmatrix} \lambda_+ - d & b \\ c & \lambda_+ - a \end{pmatrix} \int_{z}^{\infty} \exp[\lambda_+(z - x)] \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} dx.
\]

This result is crucial in proving that the normal spectrum of a linear differential operator consists of isolated eigenvalues rather than densely populated eigenvalues. See Evans [E3] for details.

(w) Consider the reaction-diffusion equations

\[
\begin{align*}
    u_t &= u_{xx} + f(u, w), \\
    w_t &= e w_{xx} + g(u, w).
\end{align*}
\]
Assume that $f(0,0) = g(0,0) = 0$, i.e. $(0,0)$ is a fixed point of the system. Suppose that the linearization about the fixed point is

$$u_t = u_{xx} - au - bw,$$

$$w_t = ew_{xx} + cu - dw.$$

After formally applying the Fourier transform one obtains

$$\frac{d}{dt} \begin{pmatrix} \hat{u} \\ \hat{w} \end{pmatrix} = - \begin{pmatrix} a + \xi^2 & b \\ -c & d + e\xi^2 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{w} \end{pmatrix} = -A(\xi^2) \begin{pmatrix} \hat{u} \\ \hat{w} \end{pmatrix}.$$

Suppose that $a, b, c, d$ are positive constants with $a > d, (a - d)^2 \geq 4bc$ and $0 \leq e \leq 1$. This assumption is true for the Fitzhugh-Nagumo, the generalized Fitzhugh-Nagumo and the diffusive predator-prey equations. Since

$$\det[\lambda I - A(\xi^2)] = \lambda^2 - [a + d + (1 + e)e\xi^2]\lambda + (a + \xi^2)(d + e\xi^2) + bc,$$

$A(\xi^2)$ possesses two eigenvalues

$$\lambda_{\pm}(\xi^2) = \frac{1}{2} \left\{ a + d + (1 + e)e\xi^2 \pm \sqrt{(a - d + (1 - e)e\xi^2)^2 - 4bc} \right\}.$$

If $e = 1$, then $\lambda_{\pm}(\xi^2) = \xi^2 + \frac{1}{2}[a + d \pm \sqrt{(a - d)^2 - 4bc}]$ and $\text{Re}\lambda_{\pm}(\xi^2) > 0$. If $e < 1$, because of the inequality

$$\sqrt{(a - d + (1 - e)e\xi^2)^2 - 4bc} \geq a - d + (1 - e)e\xi^2 - \sqrt{4bc},$$

we obtain the following estimates

$$a + \xi^2 - \sqrt{bc} < \lambda_{\pm}(\xi^2) < a + \xi^2,$$
\begin{align*}
d + e\xi^2 &< \lambda_-(\xi^2) < d + e\xi^2 + \sqrt{bc}, \\
\frac{bc}{a - d + (1 - e)\xi^2 - \sqrt{bc}} &< \lambda_+(\xi^2) - a - \xi^2 < 0, \\
0 &< \lambda_-(\xi^2) - d - e\xi^2 < \frac{bc}{a - d + (1 - e)\xi^2 - \sqrt{bc}}, \\
\lim_{\xi^2 \to \infty} \lambda_+(\xi^2) - a - \xi^2 &\to 0, \\
\lim_{\xi^2 \to \infty} \lambda_-(\xi^2) - d - e\xi^2 &\to 0.
\end{align*}

Moreover we have

\[\exp[-A(\xi^2)t] = \exp[-\lambda_+(\xi^2)t]P_+(\xi^2) + \exp[-\lambda_-(\xi^2)t]P_-(\xi^2).\]

Therefore the fixed point \((0,0)\) is exponentially stable relative to the reaction-diffusion equations, by a theorem proved by Evans \([E2]\).

This result is closely related to the location of the real parts of the essential spectrum of certain operators.

Remark. If the matrix

\[A(\xi^2) = \begin{pmatrix} a + \xi^2 & -b \\ c & d + e\xi^2 \end{pmatrix},\]

or if \(d > a, (d-a)^2 \geq 4bc\) and \(e \geq 1\), then the same results hold.

On the other hand if

\[A(\xi^2) = \begin{pmatrix} a + \xi^2 & b \\ c & d + e\xi^2 \end{pmatrix}\]

and if \(ad < bc\), then the essential spectrum crosses the imaginary axis and the traveling wave solution is exponentially unstable.
Consider a system which has no diffusion effect but involves nonlocal effect (convolution of some kernel with a nonlinear function or a Heaviside step function). This kind of equations appear in neuronal networks, chemical reactions and combustions.

Suppose that \((0,0)\) is a fixed point. The linearization of this system about the point is

\[
\frac{d}{dt} \begin{pmatrix} u \\ w \end{pmatrix} = - \begin{pmatrix} a & b \\ -c & d \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix},
\]

where \(a, b, c, d\) are positive constants. The eigenvalues of the matrix are \(\frac{1}{2}(a + d \pm \sqrt{(a - d)^2 - 4bc})\), so the real parts are negative numbers.

Remark. If the matrix

\[
A(\xi^2) = \begin{pmatrix} a + \xi^2 & -b \\ c & d + e\xi^2 \end{pmatrix},
\]

or if \(d > a, (d - a)^2 \geq 4bc\) and \(e \geq 1\), then the same results hold.

On the other hand if

\[
A(\xi^2) = \begin{pmatrix} a + \xi^2 & b \\ c & d + e\xi^2 \end{pmatrix}
\]

and if \(ad < bc\), then the essential spectrum crosses the imaginary axis and the traveling wave solution is exponentially unstable.

The eigenvalue equations concerning the stability problem of traveling wave solutions can always be written as a first order linear system of ordinary differential equations. Furthermore the asymptotic system as \(z \to \pm\infty\) is a constant coefficient system. By projectivizing such an autonomous linear system, one obtains a flow on \(CP^n\).
Suppose that \( \Omega \subset C^n \times R \) is a set of the form

\[
\Omega = \bigcup_{z \in I} \Omega(z) \times \{z\},
\]

where \( I \) is an open interval in \( R \) and \( \Omega(z) \) is a family of neighborhoods in \( C^n \) such that the boundary \( \partial \Omega(z) \) varies smoothly. \( \Omega \) is positively invariant relative to \( I \) if for any solution \( \beta(z) \) of a system of differential equations with initial data \( \beta(z_0) \in \Omega(z_0) \) for some \( z_0 \in I \), then \( \beta(z) \in \Omega(z) \), for all \( z \geq z_0 \) for which \( z \in I \).

The following results are highly due to the uniform hyperbolicity of fixed points of the frozen projectivized systems, where the traveling wave solution is near the slow manifolds.

(i) Let \( A = (a_{ij}) \) be an \( (n+1) \times (n+1) \) matrix and let \( y = y(z) \) be a continuously differentiable solution of the autonomous differential equations

\[
\frac{dy}{dz} = Ay, \quad \frac{dy_i}{dz} = \sum_{j=0}^{n} a_{ij} y_j.
\]

If \( y_0 \neq 0 \), let \( (\beta_1, \beta_2, \ldots, \beta_n) = (y_1/y_0, y_2/y_0, \ldots, y_n/y_0) \), then we have

\[
\frac{d\beta_i}{dz} = a_{i0} + \sum_{j=1}^{n} a_{ij} \beta_j - \beta_i \left( a_{00} + \sum_{j=1}^{n} a_{0j} \beta_j \right).
\]

In vector form we have

\[
\frac{d}{dz} \begin{pmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = A \begin{pmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} - \begin{pmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \begin{pmatrix} a_{00}, a_{01}, a_{02}, \ldots, a_{0n} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix},
\]

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or in another form

\[
\frac{d}{dz} \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix} = \begin{pmatrix}
a_{10} \\
a_{20} \\
\vdots \\
a_{n0}
\end{pmatrix} + \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix} \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix} - a_{00} \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix} - \sum_{j=1}^{n} a_{0j} \beta_j \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix}.
\]

Let

\[
F(\beta) = \begin{pmatrix}
a_{10} \\
a_{20} \\
\vdots \\
a_{n0}
\end{pmatrix} + \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix} \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix} - a_{00} \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix} - \sum_{j=1}^{n} a_{0j} \beta_j \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix}.
\]

Then

\[
\frac{dF}{d\beta} = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix} - (a_{00} + \sum_{j=1}^{n} a_{0j} \beta_j) I - \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix} (a_{01}, a_{02}, \ldots, a_{0n}).
\]

If \( a_{00} \) is a simple eigenvalue of \( A \), i.e. \( a_{0j} = 0 \) for \( j = 1, 2, \ldots, n \), then

\[
\frac{d}{dz} \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix} = \begin{pmatrix}
a_{10} \\
a_{20} \\
\vdots \\
a_{n0}
\end{pmatrix} + \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix} - a_{00} I \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix},
\]

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Therefore \( Y \) is an eigenvector of the matrix \( A \) if and only if \( F(\tilde{Y}) = 0 \). Moreover if \( Y \) is an eigenvector corresponding to a simple eigenvalue \( \mu_i \), then the eigenvalues of \( (dF/d\beta)(\tilde{Y}) \) are \( \mu_j - \mu_i \). In particular if \( \mu_1 \) is a simple eigenvalue of largest positive real part, then \( \tilde{Y} \) is an attracting fixed point of the projectivized system. See also Jones [J1] and Gardner and Jones [GJ].

However, the projectivized system of eigenvalue equations are usually nonautonomous, because it depends in some way on the traveling wave, which is a function of the independent variable \( z \). For these nonautonomous systems, to characterize the behavior of their solutions, we need to construct positively invariant subsets relative to an open interval \( I \subset \mathbb{R} \). Then it is natural to consider the frozen system.

(y) Consider the autonomous system

\[
\frac{\partial \beta}{\partial z} = F(\beta, \gamma).
\]

It has a family of fixed points \( \beta_0 = \beta_0(\gamma) \), depending smoothly on \( \gamma \in I \). We have

\[
F(\beta, \gamma) - F(\beta_0, \gamma) = \left( \int_0^1 \frac{\partial F^i}{\partial \beta_j} (s \beta + (1-s) \beta_0, \gamma) ds \right)_{1 \leq i, j \leq n} (\beta - \beta_0),
\]

\[
F(\beta, \gamma) - F(\beta_0, \gamma) - \frac{\partial F}{\partial \beta}(\beta_0, \gamma)(\beta - \beta_0)
\]

\[
= \left( \int_0^1 \int_0^1 s \sum_{k=1}^n \frac{\partial^2 F^i}{\partial \beta_j \partial \beta_k} (st \beta + (1-st) \beta_0, \gamma)(\beta_k - \beta_0) ds dt \right)_{1 \leq i, j \leq n} (\beta - \beta_0).
\]
Therefore

\[ \frac{d}{dz}(\beta - \beta_0) = \frac{\partial F}{\partial \beta}(\beta_0, \gamma)(\beta - \beta_0) \]

\[ + \left( \int_0^1 \int_0^1 s \sum_{k=1}^n \frac{\partial^2 F^i}{\partial \beta_j \partial \beta_k} (st, \beta + (1-st)\beta_0, \gamma)(\beta_k - \beta_{0k}) ds dt \right) \]

\( 1 \leq i, j \leq n \)

For most projectivized systems obtained from eigenvalue equations arising from stability analysis of traveling wave solutions of partial differential equations, the coefficients \( c_{ijk} = \frac{\partial^2 F^i}{\partial \beta_j \partial \beta_k} \) are constants for all \( i, j, k = 1, 2, 3, ..., n \). Hence we obtain

\[ \frac{d}{dz}(\beta - \beta_0) = \frac{\partial F}{\partial \beta}(\beta_0, \gamma)(\beta - \beta_0) + \frac{1}{2} \left( \sum_{k=1}^n c_{ijk}(\beta_k - \beta_{0k}) \right) \]

\( 1 \leq i, j \leq n \)

We would like to transform this system into a compact form so that it can be dealt with easily. Let \( T = T(\gamma) \) be a suitable invertible matrix and let

\[ w = T^{-1}(\beta - \beta_0), \]

\[ B = T^{-1} \frac{\partial F}{\partial \beta}(\beta_0, \gamma) T, \]

\[ M = T^{-1} \left( \sum_{k=1}^n c_{ijk}(\beta_k - \beta_{0k}) \right) \]

\( 1 \leq i, j \leq n \)

Then

\[ \frac{dw}{dz} = Bw + \frac{1}{2} Mw. \]

Under certain circumstances (e.g. when the traveling wave is near the slow manifolds in singular perturbation problems), we find that \( B = B_1 + B_2 \) with \( B_1 \) a diagonal matrix with negative entries, uniformly bounded away from zero, and \( B_2 = O(\varepsilon) \). There is a constant \( a > 0 \), independent of the eigenvalue parameter \( \lambda \), the small parameter \( \varepsilon \) and the independent variable \( z \), such that for all \( w \in C^n \),

\[ \Re(w, Bw) \leq -a\|w\|^2. \]

We also find that if the eigenvalue equations are properly
written, then \( a_{01} = 1, a_{00} = a_{02} = \ldots = a_{0n} = 0 \), so that \( F^i(\beta) = a_{i0} + \sum_j a_{ij} \beta_j - \beta_1 \beta_i \),

where \( a_{ij} \) are independent of \( \beta \), hence

\[
    c_{ijk} = \frac{\partial^3 F^i}{\partial \beta_j \partial \beta_k} = -\frac{\partial^2 (\beta_1 \beta_i)}{\partial \beta_j \partial \beta_k} = -\delta_{ij}, \text{ if } k = 1;
\]

\[
    = -\delta_{ij}, \text{ if } k = i; \text{ and } = 0, \text{ otherwise},
\]

\[
    \sum_{k=1}^{n} c_{ijk} (\beta_k - \beta_{0k}) = -\delta_{ij} (\beta_1 - \beta_{01}) - \delta_{ij} (\beta_i - \beta_{0i}),
\]

\[
    \left( \sum_{k=1}^{n} c_{ijk} (\beta_k - \beta_{0k}) \right)_{1 \leq i,j \leq n} = -(\beta_1 - \beta_{01}) I - \begin{pmatrix}
    \beta_1 - \beta_{01} & 0 & \ldots & 0 \\
    \beta_2 - \beta_{02} & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    \beta_n - \beta_{0n} & 0 & \ldots & 0
\end{pmatrix}.
\]

Eventually, we have

\[
    M = -(\beta_1 - \beta_{01}) I - T^{-1} \begin{pmatrix}
    \beta_1 - \beta_{01} \\
    \beta_2 - \beta_{02} \\
    \vdots \\
    \beta_n - \beta_{0n}
\end{pmatrix} (1, 0, \ldots, 0)^T
\]

\[
    = -(\beta_1 - \beta_{01}) I - w(1, 0, \ldots, 0)^T.
\]

If the first row of \( T(\gamma) \) is controlled by a positive constant \( C \) independent of \( \lambda, \varepsilon, z \) and \( w \), actually \( C = 1 \) or \( C = 2 \), then we have the estimate

\[
    \frac{d}{dz} ||w||^2 = 2Re(w, Bw) + Re(w, Mw) \leq -2a||w||^2 + C||w||^3.
\]

If \( ||w|| > 0 \) then this inequality is equivalent to

\[
    \frac{d}{dz} \frac{1}{||w||} - \frac{a}{||w||} + \frac{C}{2} \geq 0.
\]
This is further equivalent to the following

\[
\frac{d}{dz} \left[ \exp(-az) \frac{1}{\|w\|} - \exp(-az) \frac{C}{2a} \right] \geq 0.
\]

Integrating in \( z \) yields

\[
\exp(-az) \frac{1}{\|w(z)\|} - \exp(-az) \frac{C}{2a} \geq \exp(-ax) \frac{1}{\|v(x)\|} - \exp(-ax) \frac{C}{2a},
\]

for all \( z \geq x \). If \( \|w(x)\| \leq 2a/C \), then

\[
\|w(z)\| \leq \frac{1}{\left\{ \frac{C}{2a} + \exp[a(z - x)] \left[ \frac{1}{\|w(x)\|} - \frac{C}{2a} \right] \right\}} \leq \frac{2a}{C}.
\]

Certainly if we increase \( C \) or decrease \( a \) in the above discussions, we have a similar result. Thus we have proved the following important assertion.

There is a positive constant \( \eta_0 = 2a/C \), independent of \( \lambda, \varepsilon, z \) and \( w \), such that the set \( \hat{\Omega} = \{ (\beta, \gamma) : \beta \in CP^n, \|T^{-1}(\gamma)[\beta - \beta_0(\gamma)]\| < \eta, \gamma \in I \} \) is positively invariant, for all \( 0 < \eta < \eta_0 \).

Compared to the papers by Jones [J1] and by Gardner and Jones [GJ] in Transactions of AMS, our construction only involves one parameter \( \eta \), but their construction depends on additional parameters. The new lemma will simplify the construction of the invariants about slow manifolds for most singularly perturbed problems.

(z) Let \( A = A(\lambda, z) \) be an \( n \times n \) matrix and let \( \psi_1, \psi_2, \psi_3, ..., \psi_n \) be linearly independent solutions of the linear system \( \psi_z = A(\lambda, z)\psi \). Then \( \psi_1 \wedge \psi_2 \wedge ... \wedge \psi_k \) is a solution of the system \( \psi_z = A^{(k)}(\lambda, z)\psi \), where \( A^{(k)}(\lambda, z) \) is an \( m \times m \) matrix, where

\[
m = \binom{n}{k}.
\]

The eigenvalues of \( A^{(k)}(\lambda, z) = \lim_{z \to \pm \infty} A^{(k)}(\lambda, z) \) are the sums of any
$k$-tuple of eigenvalues of $A_{\pm}(\lambda) = \lim_{z \to \pm \infty} A(\lambda, z)$. If $A_{\pm}(\lambda)$ has $k$ eigenvalues with positive real part, say $\mu_1(\lambda), \mu_2(\lambda), \mu_3(\lambda), \ldots, \mu_k(\lambda)$, then the number

$$\alpha(\lambda) = \sum_{i=1}^{k} \mu_i(\lambda)$$

is the eigenvalue of $A^{(k)}_{\pm}(\lambda)$ with largest positive real part. There exist $m$ eigenvalues of $A^{(k)}_{\pm}(\lambda)$, counting algebraic multiplicities, so there is only one way to obtain the eigenvalue of $A^{(k)}_{\pm}(\lambda)$ with largest positive real part $\alpha(\lambda) = \sum_{i=1}^{k} \mu_i(\lambda)$. The $2n$-th order tensor product of the same matrix $A \otimes \ldots \otimes A = Tr A = \sum_{i} \lambda_i$ will be useful.
CHAPTER 2
EXISTENCE

This chapter contains three sections. The first section is concerned with the existence of solutions of the initial value problem. The second section deals with the existence of traveling front solutions and the last section deals with the existence of the pulse solutions. In the first section we consider a general system of integral-differential equations which contains system (1.1-1.2). We allow the existence of a second order diffusion with nonnegative coefficient in one or more scalar equations of the general system. Thus the system we will consider includes the nerve axon equations and (1.1-1.2). Our analysis is essentially independent of the diffusion; if there is no diffusion in any of these equations, then similar arguments and results hold. For simplicity we consider the case the diffusion coefficients equal one.

1. Existence and Uniqueness

The existence, uniqueness and continuous dependence on initial values of solutions of the partial differential equations will be established in this section.

The partial differential equations under consideration consist of a reaction-diffusion equation and several subsidiary ordinary differential equations. This is a system of
partial and ordinary differential equations:

\[ W_t = f(x, t, W(x, t)) + W_{xx}^0, \quad W(x, 0) = \varphi(x). \quad (2.1) \]

These equations are equivalent to the following system of integral equations

\[ \begin{align*}
W^0(x, t) &= \varphi^0(\cdot) * G(\cdot, t) + \int_0^t f^0(\cdot, s, W(\cdot, s)) * G(\cdot, t - s)ds, \\
W^i(x, t) &= \varphi^i(x) + \int_0^t f^i(x, s, W(x, s))ds, \quad 1 \leq i \leq n.
\end{align*} \]

If there exists a bounded, uniformly continuous solution to the integral equations, then so does to the partial and ordinary differential equations.

**Theorem A.** Let the initial data \( \varphi \in X \) such that the first component transversally crosses the line \( \varphi^1 = \theta \) precisely twice, namely there exists precisely two points \( z_1 \) and \( z_2 \), such that \( \varphi^1(z_1) = \varphi^1(z_2) = \theta, \) but \( \varphi'(z_i) \neq 0. \) Then there exists a unique bounded, uniformly continuous solution to problem (1.1-1.2) with \( \|W(t)\|_\infty \leq e^{\ell T} \|\varphi\|_\infty. \) If in addition \( \varphi' \in X \) and

\[ \max_{i} \sup_{(x,t)} \left[ |f^i(x, t, W(x, t))| + |f^i_x(x, t, W(x, t))| \right] < \infty, \]

then there exists a unique bounded, uniformly continuous solution to problem (1.1-1.2), such that

\[ W(x, t), \quad W_x(x, t), \quad W^i_t(x, t) \in L^\infty(0, T; X), \quad \text{for all } i = 1, ..., n. \]

Moreover \( W^0_{xx}(x, t) \) and \( W^0_t(x, t) \) exist and are continuous.

Proof. With the hypothesis, one can make use of the iteration technique (also known as successive approximation) to construct a uniformly convergent sequence. Let

\[ W^0_0(x, t) = \varphi^0(\cdot) * G(\cdot, t), \]
\[ W^i(x, t) = \varphi^i(x), \quad 1 \leq i \leq n, \]
\[ W^0_i(x, t) = \varphi^0(\cdot) * G(\cdot, t) + \int_0^t f^0(\cdot, s, W^0_0(\cdot, s)) * G(\cdot, t - s)ds, \]
\[ W^i(x, t) = \varphi^i(x) + \int_0^t f^i(x, s, W^0_0(x, s))ds, \quad 1 \leq i \leq n. \]

Then we have the elementary estimates
\[
|W^0_i(x, t) - W^0_0(x, t)| \leq \int_0^t \|f^0(\cdot, s, W^0_0(\cdot, s))\|_\infty ds \leq \rho \|\varphi\|_\infty t,
\]
\[
|W^i(x, t) - W^0_i(x, t)| \leq \int_0^t \|f^i(\cdot, s, W^0_0(\cdot, s))\|_\infty ds \leq \rho \|\varphi\|_\infty t,
\]
\[
\|W_i(t) - W_0(t)\|_\infty \leq \rho \|\varphi\|_\infty t.
\]

Let us present the iteration procedure
\[
W^0_{m+1}(x, t) = \varphi^0(\cdot) * G(\cdot, t) + \int_0^t f^0(\cdot, s, W^0_m(\cdot, s)) * G(\cdot, t - s)ds, \]
\[
W^i_{m+1}(x, t) = \varphi^i(x) + \int_0^t f^i(x, s, W^0_m(x, s))ds, \quad 1 \leq i \leq n. \]

Then the difference \(W^0_{m+1}(x, t) - W^0_m(x, t)\) solves the equations
\[
W^0_{m+1}(x, t) - W^0_m(x, t) = \int_0^t [f^0(\cdot, s, W^0_m(\cdot, s)) - f^0(\cdot, s, W^0_{m-1}(\cdot, s))] * G(\cdot, t - s)ds, \]
\[
W^i_{m+1}(x, t) - W^i_m(x, t) = \int_0^t [f^i(x, s, W^0_m(x, s)) - f^i(x, s, W^0_{m-1}(x, s))]ds.
\]

Let us estimate these differences.
\[
|W^0_{m+1}(x, t) - W^0_m(x, t)| \leq \int_0^t \|f^0(\cdot, s, W^0_m(\cdot, s)) - f^0(\cdot, s, W^0_{m-1}(\cdot, s))\|_\infty ds \leq \rho \int_0^t \|W_m(s) - W_{m-1}(s)\|_\infty ds,
\]
\[
|W^i_{m+1}(x, t) - W^i_m(x, t)| \leq \rho \int_0^t \|W_m(s) - W_{m-1}(s)\|_\infty ds,
\]
\[
\|W_{m+1}(t) - W_m(t)\|_\infty \leq \rho \int_0^t \|W_m(s) - W_{m-1}(s)\|_\infty ds.
\]

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By mathematical induction method we can easily obtain the estimates

\[
\|W_{m+1}(t) - W_m(t)\|_\infty \leq \frac{(Lt)^{m+1}}{(m+1)!} \|\varphi\|_\infty,
\]

\[
\sum_{m=0}^{\infty} \|W_{m+1}(t) - W_m(t)\|_\infty \leq \sum_{m=0}^{\infty} \frac{(Lt)^{m+1}}{(m+1)!} \|\varphi\|_\infty \leq e^{Lt} \|\varphi\|_\infty.
\]

\[
\|W_m(t)\|_\infty \leq e^{Lt} \|\varphi\|_\infty.
\]

Clearly \(W_m(x, t)\) is a bounded, uniformly continuous function on \(R \times [0, T]\), for all integer \(m \geq 0\). Moreover, \(W_m(x, t)\) converges uniformly to a bounded, uniformly continuous function \(W(x, t)\) in \(R \times [0, T]\). Hence the system of integral equations, or equivalently the system of partial and ordinary differential equations, has a unique bounded, uniformly continuous global solution \(W(x, t)\). Moreover

\[
\|W(t)\|_\infty \leq e^{Lr} \|\varphi\|_\infty.
\]

Now let \(\varphi\) and \(\varphi' \in X\). Differentiating the integral equations about \(x\) yields

\[
W^0_x(x, t) = \varphi^0_x(\cdot) * G(\cdot, t) + \int_0^t f^0_x(\cdot, s, W(\cdot, s)) * G(\cdot, t - s)ds
\]

\[
+ \sum_{j=0}^{n} \int_0^t (f^j_W(\cdot, s, W(\cdot, s))W^j_2(\cdot, s)) * G(\cdot, t - s)ds,
\]

\[
W^i_x(x, t) = \varphi^i_x(x) + \int_0^t f^i_x(x, s, W(x, s))ds
\]

\[
+ \sum_{j=0}^{n} \int_0^t f^j_W(x, s, W(x, s))W^j_2(x, s)ds.
\]

Let

\[
C = \max_{(x,t)} \left[ |f^i(x, t, W(x, t))| + |f^j_x(x, t, W(x, t))| \right].
\]

Then \(0 \leq C < \infty\) and

\[
\|W^0_x(t)\|_\infty \leq \|\varphi^0_x\|_\infty + Ct + \rho \int_0^t \|W^i_x(s)\|_\infty ds,
\]

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By Gronwall's inequality, we obtain the global estimate

$$||W_z(t)||_\infty \leq ||\varphi_z||_\infty + Ct + \rho \int_0^t ||W_z(s)||_\infty ds,$$

$$||W_z(t)||_\infty \leq ||\varphi_z||_\infty + Ct + \rho \int_0^t ||W_z(s)||_\infty ds.$$ 

Moreover, by the ordinary differential equations we have

$$|W^i_t(x, t)| \leq |f^i(x, t, W(x, t))| \leq C.$$

It is easy to see that $W^0_{xx}(x, t)$ and $W^0_t(x, t)$ exist and are continuous. This is because

$$W^0_{xx}(x, t) = \varphi^0_x(\cdot) * H(\cdot, t) + \int_0^t f^0_x(\cdot, s, W(\cdot, s)) * H(\cdot, t - s)ds$$

$$+ \sum_{j=0}^n \int_0^t (f^0_{Wj}(\cdot, s, W(\cdot, s))W^j_x(s)) * H(\cdot, t - s)ds.$$ 

$$W^0_t = f^0_t(x, t, W(x, t)) + W^0_{xx}.$$ 

where

$$H(x, t) = G_x(x, t) = -\frac{x}{4t\sqrt{\pi t}} \exp \left( -\frac{x^2}{4t} \right).$$

Let $\phi$ and $\psi \in X$, let $U$ and $V$ be the solutions of problem (1) corresponding to the initial values $U(x, 0) = \phi(x)$ and $V(x, 0) = \psi(x)$, respectively. Set $\varphi(x) = \phi(x) - \psi(x)$ and $W(x, t) = U(x, t) - V(x, t)$. Then

$$W^0(x, t) = \varphi^0(\cdot) * G(\cdot, t) + \int_0^t [f^0(\cdot, s, U(\cdot, s)) - f^0(\cdot, s, V(\cdot, s))] * G(\cdot, t - s)ds,$$

$$W^i_t(x, t) = \varphi^i(x) + \int_0^t [(f^i(x, s, U(x, s)) - f^i(x, s, V(x, s)]ds, \quad 1 \leq i \leq n.$$
Because $f$ is Lipschitz continuous, i.e. there is a constant $\rho > 0$, such that

\[ |f(x, s, U) - f(x, s, V)| \leq \rho |U - V|, \]

for all $U$ and $V$, we now obtain

\[
\begin{align*}
\|W^0(t)\|_\infty & \leq \|\phi^0\|_\infty + \rho \int_0^t \|W(s)\|_\infty ds, \\
\|W^1(t)\|_\infty & \leq \|\phi^1\|_\infty + \rho \int_0^t \|W(s)\|_\infty ds, \\
\|W(t)\|_\infty & \leq \|\phi\|_\infty + \rho \int_0^t \|W(s)\|_\infty ds.
\end{align*}
\]

Gronwall's inequality yields the estimate

\[ \|U(t) - V(t)\|_\infty \leq e^{\rho t} \|\phi - \psi\|_\infty. \]

We see the solution of problem (1.1-1.2) depends continuously on the initial value.

Consider the initial value problems

\[
\frac{\partial}{\partial t} \begin{pmatrix} U(z, t) \\ W(z, t) \end{pmatrix} + v(z) \frac{\partial}{\partial z} \begin{pmatrix} U(z, t) \\ W(z, t) \end{pmatrix} = \begin{pmatrix} f'(\phi(z)) & -1 \\ \varepsilon & -\varepsilon \gamma \end{pmatrix} \begin{pmatrix} U(z, t) \\ W(z, t) \end{pmatrix},
\]

\[
\begin{pmatrix} U(z, 0) \\ W(z, 0) \end{pmatrix} = \begin{pmatrix} U_0(z) \\ W_0(z) \end{pmatrix}.
\]

By the method of successive approximation, for any bounded uniformly continuous function $(U_0(z), W_0(z))$, there exists a unique bounded uniformly continuous global solution $(U(z, t), W(z, t))$ to the initial value problem. Choose two linearly independent initial functions $(U_{01}(z), W_{01}(z)) \equiv (1, 0)$ and $(U_{02}(z), W_{02}(z)) \equiv (0, 1)$ in the linear space $X$ consisting of bounded and uniformly continuous functions.
defined on \( R \). Thus the global solutions \((U_1(z, t), W_1(z, t))\) and \((U_2(z, t), W_2(z, t))\) are linearly independent since the equation is linear in \((U, W)\). Therefore we have \( \det \begin{pmatrix} U_1(z, t) & U_2(z, t) \\ W_1(z, t) & W_2(z, t) \end{pmatrix} \neq 0 \). Now given any initial data \((U_0(z), W_0(z)) \in X\), we have \((U_0(z), W_0(z)) = U_0(z)(U_{01}(z), W_{01}(z)) + W_0(z)(U_{02}(z), W_{02}(z))\). We use the method of variation of parameter to find the solutions of (1.1-1.2). Suppose that

\[
\begin{pmatrix} u(z, t) \\ w(z, t) \end{pmatrix} = \begin{pmatrix} U_1(z, t) & U_2(z, t) \\ W_1(z, t) & W_2(z, t) \end{pmatrix} \begin{pmatrix} c_1(z, t) \\ c_2(z, t) \end{pmatrix}
\]

is a solution of the initial value problem for the nonhomogeneous equations

\[
\frac{\partial}{\partial t} \begin{pmatrix} U(z, t) \\ W(z, t) \end{pmatrix} + v(\varepsilon) \frac{\partial}{\partial z} \begin{pmatrix} U(z, t) \\ W(z, t) \end{pmatrix} = \begin{pmatrix} f'(\phi(z)) & -1 \\ \varepsilon & -\varepsilon \gamma \end{pmatrix} \begin{pmatrix} U(z, t) \\ W(z, t) \end{pmatrix}
\]

\[
+ \begin{bmatrix} \frac{\alpha}{\phi'(0)}K(z)U(0, t) - \frac{\alpha}{\phi'(z_0)}K(z - z_0)U(z_0, t) \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Then applying product rule gives

\[
\begin{pmatrix} U_1(z, t) & U_2(z, t) \\ W_1(z, t) & W_2(z, t) \end{pmatrix}_t \begin{pmatrix} c_1(z, t) \\ c_2(z, t) \end{pmatrix} + v(\varepsilon) \begin{pmatrix} U_1(z, t) & U_2(z, t) \\ W_1(z, t) & W_2(z, t) \end{pmatrix} \begin{pmatrix} c_1(z, t) \\ c_2(z, t) \end{pmatrix}_t
\]

\[
+ v(\varepsilon) \begin{pmatrix} U_1(z, t) & U_2(z, t) \\ W_1(z, t) & W_2(z, t) \end{pmatrix}_z \begin{pmatrix} c_1(z, t) \\ c_2(z, t) \end{pmatrix} + v(\varepsilon) \begin{pmatrix} U_1(z, t) & U_2(z, t) \\ W_1(z, t) & W_2(z, t) \end{pmatrix}_z \begin{pmatrix} c_1(z, t) \\ c_2(z, t) \end{pmatrix}
\]

\[
= \begin{pmatrix} f'(\phi(z)) & -1 \\ \varepsilon & -\varepsilon \gamma \end{pmatrix} \begin{pmatrix} U_1(z, t) & U_2(z, t) \\ W_1(z, t) & W_2(z, t) \end{pmatrix} \begin{pmatrix} c_1(z, t) \\ c_2(z, t) \end{pmatrix}
\]

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we get

\[ \begin{pmatrix} U_1(z, t) & U_2(z, t) \\ W_1(z, t) & W_2(z, t) \end{pmatrix} + v(\varepsilon) \begin{pmatrix} c_1(z, t) \\ c_2(z, t) \end{pmatrix} t = \begin{pmatrix} c_1(z, t) \\ c_2(z, t) \end{pmatrix} \]

Since \((U_i(z, t), W_i(z, t))\) is a solution of the homogeneous equations, where \(i = 1, 2\), we get

\[ \begin{pmatrix} U_1(z, t) & U_2(z, t) \\ W_1(z, t) & W_2(z, t) \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\phi'(0)} K(z) U(0, t) - \frac{\alpha}{\phi'(0)} K(z - z_0) U(z_0, t) \\ \frac{\alpha}{\phi'(0)} K(z) U(0, t) - \frac{\alpha}{\phi'(0)} K(z - z_0) U(z_0, t) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

The coefficient matrix is invertible, for all \((x, t) \in \mathbb{R} \times \mathbb{R}^+\). Actually

\[ \begin{pmatrix} U_1(z, t) & U_2(z, t) \\ W_1(z, t) & W_2(z, t) \end{pmatrix} = \exp[\varepsilon t - f'((\phi(z)) t] \begin{pmatrix} +W_2(z, t) & -U_2(z, t) \\ -W_1(z, t) & +U_1(z, t) \end{pmatrix}. \]

Hence

\[ \begin{pmatrix} c_1(z, t) \\ c_2(z, t) \end{pmatrix} + v(\varepsilon) \begin{pmatrix} c_1(z, t) \\ c_2(z, t) \end{pmatrix} t = \begin{pmatrix} \frac{\alpha}{\phi'(0)} K(z) U(0, t) - \frac{\alpha}{\phi'(0)} K(z - z_0) U(z_0, t) \\ \frac{\alpha}{\phi'(0)} K(z) U(0, t) - \frac{\alpha}{\phi'(0)} K(z - z_0) U(z_0, t) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ \begin{pmatrix} c_1(z, t) \\ c_2(z, t) \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\phi'(0)} K(z) U(0, t) - \frac{\alpha}{\phi'(0)} K(z - z_0) U(z_0, t) \\ \frac{\alpha}{\phi'(0)} K(z) U(0, t) - \frac{\alpha}{\phi'(0)} K(z - z_0) U(z_0, t) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ \times \exp[\varepsilon t - f'((\phi(z)) t] \begin{pmatrix} +W_2(z, t) & -U_2(z, t) \\ -W_1(z, t) & +U_1(z, t) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ = \begin{pmatrix} \frac{\alpha}{\phi'(0)} K(z) U(0, t) - \frac{\alpha}{\phi'(0)} K(z - z_0) U(z_0, t) \end{pmatrix} \exp[\varepsilon t - f'((\phi(z)) t] \begin{pmatrix} +W_2(z, t) \\ -W_1(z, t) \end{pmatrix}. \]
Integrating in $t$ yields

$$
\begin{pmatrix}
  c_1(z,t) \\
  c_2(z,t)
\end{pmatrix} = \begin{pmatrix}
  U_0(z) \\
  W_0(z)
\end{pmatrix}
$$

$$
+ \int_0^t \left[ \frac{\alpha}{\phi'(0)} K(z_0) U(0, s) - \frac{\alpha}{\phi'(z_0)} K(z - z_0) U(z_0, s) \right] \exp[\varepsilon \gamma s - f'(\phi(z))s] \begin{pmatrix}
  +W_2(z, s) \\
  -W_1(z, s)
\end{pmatrix} ds.
$$

Therefore we obtain the solution

$$
\begin{pmatrix}
  U(z, t) \\
  W(z, t)
\end{pmatrix} = \begin{pmatrix}
  U_1(z, t) & U_2(z, t) \\
  W_1(z, t) & W_2(z, t)
\end{pmatrix} \begin{pmatrix}
  U_0(z) \\
  W_0(z)
\end{pmatrix}
$$

$$
+ \begin{pmatrix}
  U_1(z, t) & U_2(z, t) \\
  W_1(z, t) & W_2(z, t)
\end{pmatrix} \int_0^t \left[ \frac{\alpha}{\phi'(0)} K(z) U(0, s) - \frac{\alpha}{\phi'(z_0)} K(z - z_0) U(z_0, s) \right]
$$

$$
\times \exp[\varepsilon \gamma s - f'(\phi(z))s] \begin{pmatrix}
  +W_2(z, s) \\
  -W_1(z, s)
\end{pmatrix} ds.
$$

The initial condition satisfies $(c_1(z, 0), c_2(z, 0)) = (U_0(z), W_0(z))$. There exists a unique bounded uniformly continuous solution $(c_1(z, t), c_2(z, t))$ to this equation. Therefore the existence and uniqueness of global solution of (1.1-1.2) in $X$ is guaranteed.

### 2. Existence of Traveling Wave Fronts

Consider traveling wave solutions of the scalar integral-differential equation

$$
u_t(x, t) = f(u(x, t)) + \alpha \int_{-\infty}^{\infty} K(x - y) H(u(y, t) - \theta)dy.$$
The traveling wave, if it exists, solves the following ordinary differential equation and boundary conditions as $z \to \pm \infty$,

\[ v \frac{\partial \phi}{\partial z}(z) = f(\phi) + \alpha \int_{-\infty}^{\infty} K(z - y) H(\phi(y) - \theta) dy, \]

\[ \lim_{z \to -\infty} (\phi(z), \phi_z(z)) = (0, 0), \quad \lim_{z \to \infty} (\phi(z), \phi_z(z)) = (\beta, 0). \]

Multiplying the traveling wave equation by $\phi_z$ and integrating in $z$ over $\mathbb{R}$ gives

\[ v \int_{-\infty}^{\infty} (\phi_z)^2 dz = \int_0^\beta f(x) dx + \int_{-\infty}^{\infty} \phi_z \int_{-\infty}^{z} K(x) dx dz. \]

Hence the traveling wave speed is positive if the wave exists. We need a traveling front satisfying $\phi(0) = \theta, \phi'(0) > 0$ and $\phi'(z) \geq 0$ for all $z \in \mathbb{R}$. A simple observation shows that the solution of the following initial value problem

\[ v \frac{\partial \phi}{\partial z} = f(\phi) + \alpha \int_{-\infty}^{z} K(x) dx, \quad \phi(0) = \theta, \]

is a candidate of the desired traveling wave solution. However this equation is not autonomous. Let $k$ be a small positive constant and let $z = \frac{1}{2k} [\ln(1 + \tau) - \ln(1 - \tau)]$ so that $\tau = [\exp(2kz) - 1]/[\exp(2kz) + 1] = \tanh(kz)$. Then $\tau(0) = 0$ and we get an autonomous system

\[ \frac{d\phi}{dz} = \frac{f(\phi)}{v} + \frac{\alpha}{v} \int_{-1}^{\tau} \frac{1}{k(1 - s^2)} K \left( \frac{1}{2k} \ln \frac{1 + s}{1 - s} \right) ds, \]

\[ \frac{d\tau}{dz} = k(1 - \tau^2), \quad \tau(0) = 0. \]

Define

\[ h(\tau) = \int_{-1}^{\tau} \frac{1}{k(1 - s^2)} K \left( \frac{1}{2k} \ln \frac{1 + s}{1 - s} \right) ds. \]
The improper integral is absolutely convergent, since \(0 \leq K \in L^1(R)\). Then \(h\) is well defined and \(h(-1) = 0\) and \(h(1) = 1\).

The fixed points of this system are \((\tau, \phi) = (-1, 0), (-1, a), (-1, 1), (1, \beta)\). The linearization of the above system about the fixed points \((-1, 0)\) and \((1, \beta)\) are given by

\[
\begin{align*}
\frac{d\phi}{dz} &= -\frac{a\phi}{v} \frac{d\tau}{dz} = 2k\tau, \\
\frac{d\phi}{dz} &= \frac{f'(\beta)\phi}{v} \frac{d\tau}{dz} = -2k\tau.
\end{align*}
\]

Therefore \((-1, 0)\) is a saddle point and \((1, \beta)\) is an attracting point. The traveling wave of interest is a trajectory connecting the point \((-1, 0)\) at \(\tau = -1\) to the point \((1, \beta)\) at \(\tau = 1\). Consider then the initial value problem

\[
\begin{align*}
\frac{d\phi}{d\tau} &= \frac{1}{k(1 - \tau^2)} \left[ \frac{f(\phi)}{v} + \alpha \int_{-1}^{\tau} \frac{1}{k(1 - s^2)} K \left( \frac{1}{2k} \ln \frac{1 + s}{1 - s} \right) ds \right], \\
\phi(0) &= \theta, \quad \frac{d\phi}{d\tau}(0) = \frac{\alpha + 2f(\theta)}{2vk} > 0.
\end{align*}
\]

Here we have implicitly used some restrictions on \(\alpha\), namely \(\alpha + 2f(\theta) > 0\) and \(\alpha + f(x) > 0\) for all \(0 < x < 1\). Notice this is not that much of a restriction.

The following observations are important in proving the existence of the traveling wave. (1) The derivative \(d\phi/d\tau\) never vanishes and (2) \(\phi(\tau) < \beta\) in the interval \((0, 1)\) so long as the wave solution exists. In fact it is easy to compute

\[
\begin{align*}
\frac{d^2\phi}{d\tau^2} &= \frac{2\tau}{1 - \tau^2} \frac{d\phi}{d\tau} \\
&+ \frac{1}{k(1 - \tau^2)} \left[ \frac{f'(\phi)}{v} \frac{d\phi}{d\tau} + \frac{\alpha}{v} \frac{1}{k(1 - \tau^2)} K \left( \frac{1}{2k} \ln \frac{1 + \tau}{1 - \tau} \right) \right].
\end{align*}
\]
Hence if \( \frac{d\phi}{d\tau} = 0 \) at some point \(-1 < \tau_0 < 1\), then
\[
\frac{d^2\phi}{d\tau^2} = \frac{\alpha}{v \left[ k(1 - \tau^2) \right]^2} K \left( \frac{1}{2k} \ln \frac{1 + \tau}{1 - \tau} \right) > 0.
\]
Contradiction. Hence \( \frac{d\phi}{d\tau} > 0 \). To see (2), if \( \phi = \beta \) for some \( 0 < \tau_0 < 1 \), then
\[
f(\beta) + \alpha \int_{-1}^{\tau_0} \frac{1}{k(1 - s^2)} K \left( \frac{1}{2k} \ln \frac{1 + s}{1 - s} \right) ds < f(\beta) + \alpha = 0,
\]
so \( \frac{d\phi}{d\tau} < 0 \) at \( \tau_0 \). Contradiction. These observations also hold in the interval \((-1, 0)\).

There exists a unique solution in at least a small neighborhood of the point \((\tau, \phi) = (0, \theta)\). Let \( s = \sup\{0 < \tau \leq 1: \text{the solution } \phi(\tau, v) \text{ exists}\}\).

Rewrite the differential equation as an integral equation, using the initial data at \( \tau = \tau_1 \), where \( \tau_1 \) is such that the solution exists on \([0, \tau_1]\),
\[
\phi(\tau, v) = \phi(\tau_1, v) + \int_{\tau_1}^{\tau} \frac{1}{k(1 - \xi^2)} \left[ f(\phi(\xi, v)) + \alpha \int_{-1}^{\xi} \frac{1}{k(1 - s^2)} K \left( \frac{1}{2k} \ln \frac{1 + s}{1 - s} \right) ds \right] d\xi.
\]
Let \([\tau_1, \tau_2] \subset (0, 1)\). Define a nonlinear mapping \( A \) from a Banach space \( X = C[\tau_1, \tau_2] \) to itself by
\[
A\psi = \phi(\tau_1, v) + \int_{\tau_1}^{\tau} \frac{1}{k(1 - \xi^2)} \left[ f(\psi(\xi, v)) + \alpha \int_{-1}^{\xi} \frac{1}{k(1 - s^2)} K \left( \frac{1}{2k} \ln \frac{1 + s}{1 - s} \right) ds \right] d\xi.
\]
For all \( \psi_1 \) and \( \psi_2 \in X \), we have
\[
A\psi_1 - A\psi_2 = \int_{\tau_1}^{\tau} \frac{1}{vk(1 - \xi^2)} \left[ f(\psi_1(\xi, v)) - f(\psi_2(\xi, v)) \right] d\xi.
\]
It suffices to show \( A \) is a contraction to verify the local existence of the solution. Notice that \( f \) is locally Lipschitz continuous, i.e. there is a positive constant \( B \) such that
\[
|f(x) - f(y)| \leq B|x - y|,
\]
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for all \(-2 \leq x, y \leq 2\). By this estimate, we get

\[
|A\psi_1 - A\psi_2| \leq \frac{B}{uk} \int_{\tau_1}^{\tau_2} \frac{1}{1 - \xi^2} \sup_{[\tau_1, \tau_2]} |\psi_1(\tau) - \psi_2(\tau)|. 
\]

We have \(B \int_{\tau_1}^{\tau_2} \frac{1}{1 - \xi^2} d\xi < uk\) if \(|\tau_2 - \tau_1|\) is sufficiently small, so that \(A\) is a contraction. Since \(X\) is a complete metric space, there exists a unique continuous solution \(\phi = \phi(\tau) \in X\), by Banach fixed point theorem. The above argument shows the set \(\{\tau: \text{the solution exists at } \tau\}\) is both open and closed in \([0, 1)\). We can iteratively extend the solution to the interval \([0, 1)\). So \(s = 1\). If \(\lim_{\tau \to 1} \phi(\tau, v) < \beta\), then \(\lim_{\tau \to 1} \frac{d\phi}{d\tau}(\tau, v) = +\infty\) and \(\lim_{\tau \to 1} \phi(\tau, v) = +\infty\). Contradiction. Therefore \(\lim_{\tau \to 1} \phi(\tau, v) = \beta\).

We now consider the following initial value problem in the interval \([-1, 0]\)

\[
\frac{d\phi}{d\tau} = \frac{1}{k(1 - \tau^2)} \left[ f(\phi) + \frac{\alpha}{v} \int_{-1}^{\tau} \frac{1}{k(1 - s^2)} K \left( \frac{1}{2k} \ln \frac{1 + s}{1 - s} \right) ds \right],
\]

\[
\phi(-1) = 0, \quad \frac{d\phi}{d\tau}(-1) = 0.
\]

There exists a unique solution in at least a small neighborhood of the point \((\tau, \phi) = (-1, 0)\). As before we can iteratively extend the unique solution to the interval \([-1, 0]\).

Let \(v_1\) and \(v_2\) be two positive constants and let \(\phi(\tau, v_i)\) be the corresponding solutions of the initial value problem. Setting \(\phi(\tau) = \phi(\tau, v_1) - \phi(\tau, v_2)\) and \(h(\tau) = \frac{1}{k(1 - \tau^2)} \int_0^1 f'(s\phi(\tau, v_1) + (1 - s)\phi(\tau, v_2)) ds\). Then by using the equations

\[
v_1 \frac{d\phi}{d\tau} = \frac{1}{k(1 - \tau^2)} \left[ f(\phi) + \alpha \int_{-1}^{\tau} \frac{1}{k(1 - s^2)} K \left( \frac{1}{2k} \ln \frac{1 + s}{1 - s} \right) ds \right],
\]

we get

\[
v_2 \frac{d\phi}{d\tau} - h(\tau)\phi = (v_2 - v_1) \frac{d\phi}{d\tau}(\tau, v_1).
\]
For simplicity, let $v_2 = 1$. Multiplying this equation by the integrating factor 
\[ \exp \left[ -\int_b^\tau h(s) ds \right] \] yields 
\[
\frac{d}{d\tau} \left\{ \exp \left[ -\int_b^\tau h(s) ds \right] \phi \right\} = (v_2 - v_1) \exp \left[ -\int_b^\tau h(s) ds \right] \frac{d\phi}{d\tau}(\tau, v_1),
\]
where $b > -1$ is a negative number. Integrating in $\tau$ over the interval $[b, 0]$ yields 
\[
\phi(0) \exp \left[ -\int_b^0 h(s) ds \right] - \phi(b) = (v_2 - v_1) \int_b^0 \exp \left[ -\int_b^\tau h(s) ds \right] \frac{d\phi}{d\tau}(\tau, v_1) d\tau.
\]
This is equivalent to the integral equation 
\[
\phi(0) = \exp \left[ \int_b^0 h(s) ds \right] \phi(b) + (v_2 - v_1) \int_b^0 \exp \left[ \int_b^\tau h(s) ds \right] \frac{d\phi}{d\tau}(\tau, v_1) d\tau.
\]
It is easy to see $h(\tau) \to -\infty$ as $\tau \to -1$, since $\phi(\tau, v_i) \to 0$ as $\tau \to -1$. Now we obtain the equation from the above equation by letting $b \to -1$, 
\[
\phi(0, v_1) - \phi(0, v_2) = (v_2 - v_1) \int_{-1}^0 \exp \left[ \int_b^{\tau} h(s) ds \right] \frac{d\phi}{d\tau}(\tau, v_1) d\tau.
\]
Thus $\phi(\tau = 0, v)$ depends continuously on $v$. By continuity, we see there exists a unique $v > 0$ such that $\phi(\tau = 0, v) = \theta$.

Moreover by construction we see the unique traveling wave front converges to zero and $\beta$ exponentially fast as $z \to -\infty$ and $z \to +\infty$ respectively.

For some $\varphi \in (0, f(\rho_+(a)) + \alpha)$, using the same argument as above we can show there exists a traveling wave solution to the scalar equation 
\[
u_t(x, t) = f(u(x, t)) - \varphi + \alpha \int_{-\infty}^{\infty} K(x - y) H(u(y, t) - \theta) dy,
\]
with the same wave speed as the front, but now $\phi_z \leq 0$ for all $z \in \mathbb{R}$. Actually, there are three constants 
\[
\phi_+(a) = \frac{2 + 2a}{3}, \quad \phi_-(a) = \frac{2 + 2a}{3} - \beta, \quad w(a) = \alpha + \frac{2}{27} (1 + a)(1 - 2a)(2 - a),
\]
such that

\[
\alpha + f(\phi(a)) = f(\phi(a)) = w(a),
\]
\[
f\left(\frac{2 + 2a}{3}\right) = f\left(\frac{2 + 2a}{3} - \beta\right) + f(\beta),
\]
\[
f'\left(\frac{2 + 2a}{3} - \beta\right) = f'(\beta),
\]
\[
f'\left(\frac{2 + 2a}{3}\right) = f'(0),
\]
\[
\rho_-(a) < \theta < \rho_+(a) < \frac{2 + 2a}{3}.
\]

There exists a unique number \( z_0 \in \mathbb{R} \), such that \( \phi(z_0) = \theta \) and \( \beta'(z_0) < 0 \). Hence for the pulse solution \( \phi(z, \varepsilon) \) there exists a smooth function \( z_0(\varepsilon) \), such that \( z_0(\varepsilon) \rightarrow z_0 \) as \( \varepsilon \rightarrow 0 \) and \( \phi(z_0(\varepsilon), \varepsilon) = \theta \) and \( \phi_\varepsilon(z_0(\varepsilon), \varepsilon) < 0 \).

3. Existence of Traveling Pulse Solutions

In recent years there has been increasing interest in analyzing the behaviors of nonlinear neuronal networks. The use of nonlinear neuronal networks will assist in revealing the operation of the human brain. These networks are believed to mimic the behavior of real masses of neurons. A neuron is a cell which is specialized to process and transmit information. Like other cells of an organism, it has a cell body with a nucleus and is surrounded by a membrane which can support a voltage difference. Unlike other cells, it has two systems of elongated extensions: the dendrites and the axon. There are analogies between chemical reactors and certain biological systems. One such analogy is rather obvious: a single living organism is a dynamic structure built
of molecules and ions, many of which react and diffuse. Another analogy is the electric potential of a membrane which can diffuse like a chemical and can interact with real chemical species (ions) which are transported through the membrane. In the biological and chemical systems, scientists have derived many differential equations. They are also called reaction-diffusion equations. Traveling fronts, pulses, standing patterns, and other spatial structures have been analytically demonstrated to exist as wave solutions of these equations. We employ the tools and techniques used in the study of reaction-diffusion equations to extend the theoretical understanding of synaptically coupled neuronal networks on a one-dimensional spatial domain. Using singularly perturbed geometric theory and other methods, traveling wave solutions are successfully constructed and the spatially distributed inhibition is discussed, leading to various issues about the stability of the wave solution.

In this section we give a geometric construction of homoclinic and heteroclinic orbits for singularly perturbed differential equations. By using methods from invariant manifold theory, we show that transversal intersection of singular stable and singular unstable manifolds of the slow reduced problem implies the existence of transversal homoclinic or heteroclinic orbits of the singularly perturbed problem. Szmolyan derived analytical conditions for transversality. He showed how these results can be used to prove the existence of homoclinic and heteroclinic orbits for singularly perturbed differential equations which depend on additional parameters. Jones and Kopell and others established a more general tool, called the exchange lemma, to prove existence and local uniqueness of multipulse solutions for very general nonlinear autonomous systems of differential equations. The only hypothesis is the transversal intersection.
One can of course use Szmolyan’s result to achieve this point. One can also calculate the scalar product of a tangent vector to the singular unstable manifold and a normal vector to the singular stable manifold, along the intersection orbit, to show the transversality.

Let $m \geq 1$ and $n \geq 1$ be integers, let $M \subset \mathbb{R}^{m+n}$ be an open subset (smooth manifold) and let $(x, y) \in M$. Suppose that $\varepsilon_0 > 0$ is sufficiently small and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. We consider the nonlinear, autonomous, singularly perturbed system of ordinary differential equations

$$
\varepsilon \frac{dx}{dt} = f(x, y, \varepsilon), \quad \frac{dy}{dt} = g(x, y, \varepsilon),
$$

where $f$ and $g$ are sufficiently smooth, i.e. $(f, g) \in C^r(M \times (-\varepsilon_0, \varepsilon_0) \to R^{m+n})$ for $r \geq 2$. We consider $x$ and $y$ as functions of the independent variable $t$. The main feature is the existence of two different time scales: the slow time scale $t$ and the fast time scale $\tau = t/\varepsilon$. For $\varepsilon \neq 0$ these equations define a smooth dynamical system on $M$. Problems of this form arise frequently in applications. One of the main sources for singularly perturbed problems is traveling wave solutions for reaction-diffusion equations or for viscous approximations of hyperbolic conservation laws. By transforming the slow time scale to the fast time scale $\tau$, we obtain the equivalent fast system

$$
\frac{dx}{d\tau} = f(x, y, \varepsilon), \quad \frac{dy}{d\tau} = \varepsilon g(x, y, \varepsilon).
$$

The variable $x$ is usually called the fast variable and the variable $y$ is usually
called the slow variable. By setting $\varepsilon = 0$ in these systems we obtain two essentially different problems, the slow reduced problem

$$0 = f(x, y, 0), \quad \frac{dy}{dt} = g(x, y, 0),$$

and the fast reduced problem

$$\frac{dx}{d\tau} = f(x, y, 0), \quad \frac{dy}{d\tau} = 0.$$

Let $S = \{(x, y) \in M : f(x, y, 0) = 0\}$ be the slow manifold. It is a manifold on which the slow reduced problem defines a dynamical system. On the other hand, $S$ is a manifold of equilibria for the fast reduced problem. The slow reduced system essentially captures the slow dynamics and the fast reduced problem essentially captures the fast dynamics. By appropriately combining results on the dynamics of these two limiting problems one obtains results on the dynamics of the singularly perturbed problem for small values of $\varepsilon$. A particularly elegant and useful approach in understanding the relationships between the singularly perturbed problem and the limiting problems is furnished by the theory of invariant manifolds for singularly perturbed problems. The global center, center-stable, and center-unstable manifolds are constructed in [F4]. Recently methods from homoclinic bifurcation theory have been used in the investigation of transition layers for singularly perturbed problems. Szmolyan’s analysis of the existence of homoclinic and heteroclinic orbits is based on the combined use of invariant manifold theory and methods from homoclinic and heteroclinic bifurcation theory. He built on the invariant manifold approach to prove the existence of homoclinic and heteroclinic orbits. More specifically he developed a
method to prove transversal intersection of the singular stable and singular unstable manifolds of normally hyperbolic invariant manifolds. The normally hyperbolic invariant manifolds arise from the dynamics of the slow reduced problems as hyperbolic fixed points or hyperbolic periodic orbits of the slow reduced problem. The connecting orbits are provided by the fast dynamics described by the fast reduced problem. This result is an extension of an earlier result where conditions for the existence of orbits heteroclinic to hyperbolic fixed points have been given. The analytical conditions for the necessary transversality are of Melnikov integral type and they depend only on the slow reduced problem and the fast reduced problem.

The fast system and the trivial equation \( d\varepsilon/d\tau = 0 \) gives

\[
\frac{d}{d\tau} \begin{pmatrix} x \\ y \\ \varepsilon \end{pmatrix} = \begin{pmatrix} f(x, y, \varepsilon) \\ \varepsilon g(x, y, \varepsilon) \\ 0 \end{pmatrix}.
\]

The linearization of this system about the point \( (x, y, 0) \in S \times (-\varepsilon_0, \varepsilon_0) \) is given by

\[
\frac{d}{d\tau} \begin{pmatrix} x \\ y \\ \varepsilon \end{pmatrix} = \begin{pmatrix} f_x(x, y, 0) & f_y(x, y, 0) & f_\varepsilon(x, y, 0) \\ 0 & 0 & g(x, y, 0) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \varepsilon \end{pmatrix}.
\]

The characteristic polynomial of this matrix is equal to

\[
\det \begin{pmatrix} f_x(x, y, 0) - \lambda I_m & f_y(x, y, 0) & f_\varepsilon(x, y, 0) \\ 0 & -\lambda I_n & g(x, y, 0) \\ 0 & 0 & -\lambda \end{pmatrix} = (-\lambda)^{n+1} \det[f_x(x, y, 0) - \lambda I_m].
\]
\( \lambda = 0 \) is a trivial eigenvalue of algebraic multiplicity \( n + 1 \). The eigenvalues of \( f_x(x, y, 0) \) are called nontrivial eigenvalues. We assume that the numbers of nontrivial eigenvalues in the left half plane, on the imaginary axis, and in the right half plane are \( m^s, m^c \) and \( m^u \). The corresponding stable, center, and unstable eigenspaces \( E^s, E^c \) and \( E^u \) have dimensions \( m^s, m^c + n + 1 \) and \( m^u \), respectively.

Let

\[
S_R = \{(x, y) \in S : \text{rank} \left( \frac{\partial f^i}{\partial x^j}(x, y, 0) \right)_{1 \leq i, j \leq m} = m \}.
\]

\( S_R \) is the open set where the nontrivial eigenvalues are nonzero. We can parametrize \( S_R \) locally by solving the equation \( f(x, y, 0) = 0 \), according to the implicit function theorem. Therefore \( x = x(y) \) is locally well defined.

Let \( S_H \subset S_R \) be the open subset where all the nontrivial eigenvalues have nonzero real parts, i.e. compact subsets \( K \subset S_H \) are normally hyperbolic invariant manifolds of the fast reduced problem. Near the slow manifolds there exists uniform hyperbolicity.

**Definition.** Let \( M_1 \) and \( M_2 \) be submanifolds of a manifold \( M \). The manifold \( M_1 \) and \( M_2 \) intersect transversally at a point \( p \in M_1 \cap M_2 \) if and only if

\[
T_p M = T_p M_1 + T_p M_2
\]

holds, where \( T_p M \) denotes the tangent space of the manifold \( M \).

**Definition.** Let \( N_1, N_2 \) and \( N_3 \) be invariant manifolds of a dynamical system. The orbit of a point \( p \) is heteroclinic to \( N_1 \) and \( N_2 \) if \( p \) lies in the unstable manifold of \( N_1 \) and lies in the stable manifold of \( N_2 \). The orbit of a point \( p \) is homoclinic to \( N \) if \( p \) lies in both the unstable and the stable manifolds of \( N \). The homoclinic or
heteroclinic orbit is called transversal if the stable and unstable manifolds intersect transversally.

If two manifolds intersect trivially, namely nontransversally, then at each intersection point the product of the normal vector of one manifold and the tangent vector of another manifold is zero.

The existence of homoclinic and heteroclinic orbits can be based on constructing singular homoclinic and heteroclinic orbits. A singular orbit consists of orbits of the slow reduced problem and orbits of the fast reduced problem and connects invariant manifolds of the reduced problem. The slow manifold $S_H$ may consist of several branches, two of which are

$$S_1 = \{(x(y), y) \in S_H : y \in U_1 \subset \mathbb{R}^n\}, \quad S_2 = \{(x(y), y) \in S_H : y \in U_2 \subset \mathbb{R}^n\}.$$

We assume that $U_1 \cap U_2$ is nonempty. Let $N_1 \subset S_1$ and $N_2 \subset S_2$ denote two normally hyperbolic invariant manifolds of the slow reduced problem. The existence of the singular homoclinic or heteroclinic orbit is equivalent to the nonempty intersection of the manifolds

$$N_1^u = \bigcup_{p \in W_1^u} F^u(p), \quad N_2^s = \bigcup_{p \in W_2^s} F^s(p)$$

which we call the singular unstable manifold of $N_1$ and the singular stable manifold of $N_2$, respectively. If $\varepsilon = 0$, the unstable and stable fibers $F^u$ and $F^s$ at the point $p = (x(y), y) \in S$ are the unstable and stable manifolds of the hyperbolic fixed point $x(y)$ of the equation

$$\frac{dx}{d\tau} = f(x, y, 0).$$
Lemma 3. Consider the nonlinear autonomous singularly perturbed systems

\[
\frac{d}{d\tau} \begin{pmatrix}
x \\
y \\
\mu
\end{pmatrix} = \begin{pmatrix}
f(x, y, \mu, \varepsilon) \\
\varepsilon g(x, y, \mu, \varepsilon) \\
0
\end{pmatrix},
\]

where \( x \in \mathbb{R}^m, y \in \mathbb{R}^n, m = k + l \). Assume that for each \( \mu \) and \( \varepsilon > 0 \), there is a locally unique hyperbolic fixed point \( P(\mu) \) with \( k \) unstable directions and \( l + n \) stable directions of the eigenvalues associated with the latter, \( n \) tend to zero with \( \varepsilon > 0 \). Let \( \{S^i\}, i = 0, \ldots, N \) denote a family of slow manifolds for the \( \varepsilon = 0 \) equation (with the fixed point for \( \varepsilon > 0 \) in \( S^0 \)) and assume that for each \( i \), \( S^i \) is normally hyperbolic with splitting \( k \) stable and \( l \) unstable. Assume further that there is a singular homoclinic orbit, with finitely many jumps, each from \( S^i \) to \( S^{i+1} \) for some \( i \) (where the \( \{S^i\} \) are not necessarily disjoint, so the singular orbit may visit the same slow manifold more than once). Finally assume that in \( (x, y, \mu) \) space the following transversality conditions hold for the \( \varepsilon = 0 \) system: \( W^u(S^0)|_{[P(\mu), \mu]} \cap W^s(S^1) \) and \( W^u(S^i)|_{\text{singular orbit}} \cap W^s(S^{i+1}) \), where \([P(\mu), \mu]\) is the graph as \( \mu \) is varied of the \( \varepsilon = 0 \) limit of the \( \varepsilon > 0 \) fixed point. Let \( S^0 = S^{N+1} \). Then for \( \varepsilon > 0 \) sufficiently small, there is a locally unique homoclinic orbit to the system near the singular homoclinic orbit.

The traveling wave equations are not autonomous. We introduce a new variable such that the new equations are autonomous. Define

\[
\tau(\varepsilon, z) = -1 + 4/\left[1 + (2 - \varepsilon)\exp(-2kz) + \exp(-\varepsilon z) + \varepsilon \exp\left(\frac{1}{\ln\varepsilon} z\right)\right].
\]

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Then $\tau(\varepsilon, 0) = 0$ and approximately $\tau(\varepsilon, \ln \frac{1}{\varepsilon}) = -1 + 4/[2 + 1/e]$, $\tau(\varepsilon, \frac{1}{\varepsilon}) = -1 + 4/[1 + 1/e]$, $\tau(\varepsilon, \exp(\frac{1}{\varepsilon})) = 3$. Moreover

$$
\lim_{z \to -\infty} \tau(\varepsilon, z) = -1,
$$

$$
\lim_{z \to +\infty} \tau(\varepsilon, z) = 3,
$$

$$
\frac{3 - \tau}{1 + \tau} = (2 - \varepsilon) \exp(-2kz) + \exp(-\varepsilon z) + \varepsilon \exp(\frac{1}{\ln \varepsilon} - z).
$$

Hence we have

$$
\frac{4}{(1 + \tau)^2} \frac{d\tau}{dz} = 2k(2 - \varepsilon) \exp(-2kz) + \varepsilon \exp(-\varepsilon z) - \varepsilon \exp(\frac{1}{\ln \varepsilon} - z) > 0.
$$

Thus $\tau$ is a strictly increasing function in $z$. By the inverse function theorem, one can obtain $z = z(\varepsilon, \tau)$ so that

$$
\frac{d\tau}{dz} = (1 + \tau)(3 - \tau) \rho(\varepsilon, \tau),
$$

where $\rho(\varepsilon, \tau) > 0$, for all $\tau \in (-1, 3)$, is a smooth function in $\tau$.

As $\varepsilon \to 0$, the $\tau$ equation reduces to

$$
\tau(z) = -1 + 4[2 + 2 \exp(-2kz)]^{-1} = \frac{\exp(2kz) - 1}{\exp(2kz) + 1}.
$$

This is the same equation as the one we used to solve the front or the back.

Define

$$
h(\varepsilon, \tau, \phi) = \int_{-\infty}^{\infty} K(z(\varepsilon, \tau) - y) H(\phi(y) - \theta) dy.
$$

Since $H$ is the Heaviside step function, $h$ depends on $\phi$ weakly. Actually if $\phi$ crosses the line $\phi = \theta$ for finitely many times, say there are $z_i \in \mathbb{R}$ such that $\phi(z_i) = \theta$ and $\phi'(z_i) \neq 0$, then

$$
\frac{\partial h}{\partial z} = \sum_{i=1}^{N} \frac{K(z - z_i)}{\phi'(z_i)}.
$$
We are interested in a single pulse solution, so there must be exactly two such points. Here the points \( z_i \) are functions of \( \phi \). Motivated by the above equation and the analysis, we have to include the equations satisfied by various parameters in the traveling wave system. If \( \phi(z) \) approaches zero as \( z \to \pm \infty \), then \( h(-1) = h(3) = 0 \). This information will help us determine the fixed points of the traveling wave system.

The traveling wave equations for the possible pulse solution are

\[
\begin{pmatrix}
  v\phi \\
  \tau \\
  h \\
  v\varphi \\
  v \\
  a_1 \\
  a_2 \\
  z_1 \\
  z_2
\end{pmatrix} =
\begin{pmatrix}
  f(\phi) - \varphi + \alpha h(\varepsilon, \tau, \phi) \\
  (1 + \tau)(3 - \tau)\rho(\varepsilon, \tau) \\
  \frac{1}{a_1}K(z - z_1) - \frac{1}{a_2}K(z - z_2) \\
  \varepsilon(\phi - \gamma \varphi) \\
  0 \\
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}.
\]

Then we have to make sure that \( \phi(z_1) = \phi(z_2) = \theta, \phi'(z_1) = a_1, \phi'(z_2) = a_2 \) and \( \phi(z) \neq \theta \) for all \( z \neq z_1 \) and \( z \neq z_2 \). We have to prove several transversality conditions regarding these parameters. The space of these parameters will be specified later.

The only fixed point of this system (neglecting the \( \tau \)-coordinate) in \( \mathbb{R}^3 \) is \((0, 0, v)\), but if we include the \( \tau \)-coordinate, then the fixed points in \( \mathbb{R}^4 \) would be \((0, 0, -1, v)\) and \((0, 0, 3, v)\). The desired traveling wave solution is a trajectory connecting \((0, 0, -1)\) and \((0, 0, 3)\) for some positive wave speed \( v(\varepsilon) \) close to \( v \). Namely, the wave solution
need to satisfy the boundary conditions

\[
\lim_{z \to -\infty} (\phi(\varepsilon, z), \varphi(\varepsilon, z), \tau(\varepsilon, z)) = (0, 0, -1), \quad \lim_{z \to +\infty} (\phi(\varepsilon, z), \varphi(\varepsilon, z), \tau(\varepsilon, z)) = (0, 0, 3).
\]

The above system is just the fast system. Here \(z\) is the fast time scale and \(t = \varepsilon z\) is the slow time scale. Moreover \(x = (\phi, \tau)\) is the fast variable and \(y = (\varphi, v)\) is the slow variable. The slow system is

\[
\frac{\partial}{\partial t}
\begin{pmatrix}
\varepsilon \nu \phi \\
\varepsilon \tau \\
\nu \varphi \\
v
\end{pmatrix}
= \begin{pmatrix}
f(\phi) - \varphi + \alpha h(\varepsilon, \tau) \\
(1 + \tau)(3 - \tau)\rho(\varepsilon, \tau) \\
\phi - \gamma \varphi \\
0
\end{pmatrix}.
\]

The fast and the slow reduced systems are given by, respectively

\[
\frac{\partial}{\partial z}
\begin{pmatrix}
\nu \phi \\
\tau \\
\nu \varphi \\
v
\end{pmatrix}
= \begin{pmatrix}
f(\phi) - \varphi + \alpha h(0, \tau) \\
k(1 - \tau^2) \\
0 \\
0
\end{pmatrix},
\]

and

\[
\frac{\partial}{\partial t}
\begin{pmatrix}
0 \\
0 \\
\nu \varphi \\
v
\end{pmatrix}
= \begin{pmatrix}
f(\phi) - \varphi + \alpha h(0, \tau) \\
k(1 - \tau^2) \\
\phi - \gamma \varphi \\
0
\end{pmatrix}.
\]

The slow manifolds are given by

\[
S_1 = \{(\phi, f(\phi), -1, v) : \phi \in \mathbb{R}\}, \quad S_2 = \{(\phi, f(\phi), 3, v) : \phi \in \mathbb{R}\}.
\]
Of particular interest are the left branch and the right branch, and we stay away from the turning point of the cubic function, namely we need the parts \( \phi < \rho_-(a) \) and \( \phi > \rho_+(a) \), these slow manifolds are uniformly normally hyperbolic.

We need to construct a singular homoclinic orbit and show that the singular unstable manifold intersect the singular stable manifold transversally. Then by the exchange lemma proved by Jones and Kopell or by the persistence and smoothness of invariant manifold theorem proved by Fenichel, we can show the existence of the unique smooth pulse solution, for all sufficiently small \( \varepsilon > 0 \). To do this we need to construct the front, the back, the left and the right of the singular homoclinic orbit, by setting \( \varepsilon = 0 \) in different time scales, namely we use the fast and the slow reduced systems to construct the four piece of smooth curves. Define the singular homoclinic orbit \( \Gamma_0 \) in \( R^4 \) which consists of the following four piece of smooth curves:

The front \( \Gamma_1 : \{(\phi(z), \tau(z), 0, v) : z \in \mathbb{R}\} \),

The right \( \Gamma_2 : \{(\phi, +1, \alpha + f(\phi), v) : \phi > \rho_+(a)\} \),

The back \( \Gamma_3 : \{(\phi(z), \tau(z), w(a), v) : z \in \mathbb{R}\} \),

The left \( \Gamma_4 : \{(\phi, -1, f(\phi), v) : \phi < \rho_-(a)\} \).

\( \Gamma_0 \) is called singular because \( \Gamma_2 \) and \( \Gamma_4 \) consist of fixed points of the fast reduced system.

Let \( C_R^u \) be the singular unstable manifold of the right singular curve and let \( W_L^s \) be the stable manifold of the left singular curve. Then \( \dim C_R^u = 3 \). Let \( p_0 \) be a point on the back \( \Gamma_3 \) of the singular pulse solution. The singular unstable manifold of the right singular curve and the stable manifold of the left singular curve can be
expressed as graphs of smooth functions $\phi = f(\tau, \varphi)$ and $\phi = g(\tau, \varphi)$ near the point $p_0$, respectively. It is easy to check that near $p_0$ the derivatives $f_{\varphi}(\tau, \varphi) < 0$ and $g_{\varphi}(\tau, \varphi) > 0$. At $p_0$ there are two vectors $p = (1, -c, 0)$ and $q = (g_{\varphi}(a, \varphi), 0, 1)$ in the three-dimensional space $R^3$. They are tangent to $W^s \Gamma_3$. More explicitly $p$ is tangent to $\Gamma_3$ and $q \perp \Gamma_3$. Note that $p \times q = -(c, 1, -cg_{\varphi}(a, \varphi))$ and any normal vector to $W^s \Gamma_3$ at $p_0$ is parallel to $p \times q$, where $c > 0$.

Let $N(0)$ be the unit normal vector to $W^s \Gamma_3$ at $p_0$ with positive $\varphi$-component. The intersection $\{(\tau, f(\tau, \varphi), \varphi) : \varphi \in (0, f(p_+(a)) + \alpha)\}$ is a smooth curve passing through $p_0$. Let $T(0)$ be the unit tangent vector to this curve at $p_0$ with positive $\varphi$-component. The unit normal vector and the unit tangent vector are therefore given by $N(0) = \alpha(c, 1, -cg_{\varphi})$ and $T(0) = \beta(0, f_{\varphi}, 1)$, where $\alpha$ and $\beta$ are positive constants. Now we obtain $(T(0), N(0)) = \alpha\beta[f_{\varphi}(p_0) - g_{\varphi}(p_0)]$. Since $f_{\varphi}(p_0) < 0$ and $g_{\varphi}(p_0) > 0$, we have $(T(0), N(0)) < 0$. This proves the intersection is transversal. The one-dimensional differentiable manifold $\{(\phi, \tau, \nu) : \phi = \theta, \tau = 0, \nu > 0\}$ has a unit tangent vector $(0, 0, 1)$. A normal vector of the two-dimensional smooth manifold, $\{(\phi, \tau, \nu) : \phi = \phi(\tau, \nu)\}$ is a solution of the traveling wave equation connecting the point $(-1, 0)$ to $(1, \beta)$, for each $\nu > 0$, is given by the cross product of two tangent vectors $(\phi_{\tau}, 1, 0)$ and $(\phi_{\nu}, 0, 1)$, namely $(1, -\phi_{\tau}, -\phi_{\nu})$. The scalar product of the unit tangent vector and the normal vector is $-\phi_{\nu} > 0$. Thus these two manifolds intersect transversally. Similarly the two manifolds intersect transversally.

Remark. Notice that we can fix the phase of the wave at any point $z = c$ by $\phi(\varepsilon, c) = \theta$, where $c \in R$. There are infinitely many traveling wave solutions satisfying
the same boundary conditions at $z = \pm \infty$ but with different wave speed. On the other hand it is unique up to the restriction $\phi(\varepsilon, 0) = \theta$. 
CHAPTER 3
STABILITY

In this chapter we prove that some structure of the spectrum of certain operator implies the linear stability. Linear stability implies nonlinear stability and then we deal with the spectrum of the operator.

1. Equivalence of Linear and Nonlinear Stabilities

For a large class of differential equations, e.g. nonlinear parabolic system of partial differential equations, linear stability is equivalent to nonlinear stability of traveling waves. Linear stability is concerned with the convergence of solutions of the initial value problem of the linear system to some multiple of the derivative of the traveling wave as time approaches infinity. Nonlinear stability is concerned with the convergence of solutions of the initial value problem of the nonlinear system to some translation of the traveling wave as time approaches infinity. We will show that the system of integral-differential equations have the same property.

Suppose that there is a traveling wave solution \( \phi(z) = \phi(x + vt) \) to the partial differential equations (1.1-1.2), with an appropriate propagation speed \( v \), such that

\[
\lim_{|z| \to \infty} \phi(z) = \lim_{|z| \to \infty} \phi_z(z) = 0.
\]
The traveling wave solution solves the ordinary differential equations

\[ g(\phi) - v\phi_z + \alpha \int_{-\infty}^{\infty} K(z - y)H(\phi(y) - \theta)dy = 0. \]  \tag{3.1}

It is easy to see that if \( \phi \) is bounded and \( v \neq 0 \), then \( \phi_z \) and \( \phi_{zz} \) are bounded.

Let \( U(z, t) = W(z - vt, t) \). Then \( U_z = W_x, U_t = W_t - vW_x = W_t - vU_z \). Thus

\[ U_t = g(U) - vU_z + \alpha \int_{-\infty}^{\infty} K(z - y)H(U(y, t) - \theta)dy, U(z, 0) = \varphi(z). \]  \tag{3.2}

The traveling wave solution \( \phi \) is a stationary solution of this system. The linearization of equations (3) about the traveling wave solution \( \phi \) is

\[ V_t = g'(\phi)V - vV_z + \frac{\alpha}{\phi'(0)}K(z)V(0, t)e - \frac{\alpha}{\phi'(z_0)}K(z - z_0)V(z_0, t)e, V(z, 0) = \psi(z). \]  \tag{3.3}

Obviously \( V = \phi_z(z) \) is a bounded, time-independent solution of this problem.

We need solutions of the form \( V(z, t) = e^{\lambda t}\psi(z) \) of (4), where \( \lambda \in \mathbb{C} \) is a complex constant. This leads to the eigenvalue problem \( L\psi = \lambda\psi \).

Solutions to systems (2, 3, 4) with the corresponding given initial data satisfy the following equations, respectively

\[
\begin{align*}
U^i(z, t) &= \varphi^i(z - vt) + \int_0^t g^i(U(z - vt + vs, s))ds \\
&\quad + \alpha \int_0^t \int_{-\infty}^{\infty} K(z - y)H(U(y, s) - \theta)dyds, \\
\phi^i(z) &= \phi^i(z - vt) + \int_0^t g^i(\phi(z - vt + vs))ds \\
&\quad + \alpha \int_0^t \int_{-\infty}^{\infty} K(z - y)H(\phi(y) - \theta)dyds, \\
V^i(z, t) &= \psi^i(z - vt) \\
&\quad + \int_0^t \sum_{j=0}^{n} g_{iW^j}(\phi(z - vt + vs))V^j(z - vt + vs, s)ds
\end{align*}
\]
\[ + \frac{\alpha}{\phi'(0)} K(z) \int_0^t V(0, s) ds - \frac{\alpha}{\phi'(z_0)} K(z - z_0) \int_0^t V(z_0, s) ds. \]

**Definition.** System (3) is exponentially stable at the traveling wave solution \( \phi \) if there exist positive constants \( C, M \) and \( \alpha \) such that for any solution \( U \) to (3) with the initial data \( \phi \in X \) and \( \| \phi - \phi \|_\infty \leq C \), there exists a constant \( h \) with \( |h| \leq M \| \phi - \phi \|_\infty \), such that \( \| U(t) - \phi_h \|_\infty \leq M \| \phi - \phi \|_\infty e^{-\alpha t} \), for all \( t \geq 0 \). Let \( \| \phi_z \|_\infty < \infty \). System (4) is exponentially stable at the derivative \( \phi_z \) of the traveling wave solution if there exist positive constants \( M \) and \( \alpha \) such that for any solution \( V \) to (4) with the initial data \( \psi \in X \), there exists a constant \( h \) with \( |h| \leq M \| \psi \|_\infty \), such that \( \| V(t) - h \phi_z \|_\infty \leq M \| \psi \|_\infty e^{-\alpha t} \), for all \( t \geq 0 \).

**Lemma 4.** If there is a time-independent constant \( M > 0 \) such that the solutions of problem (3) or (4) satisfy
\[ \sup_{t \geq 0} \| U(t) - \phi \|_\infty \leq M, \quad \text{or} \quad \sup_{t \geq 0} \| V(t) \|_\infty \leq M, \tag{3.4} \]
then for all \( t > 0 \),
\[ \| U(t) - \phi - V(t) \|_\infty \leq \| \phi - \psi \|_\infty e^{Lt} + \frac{1}{2} \sigma M^2 \left( e^{Lt} - 1 \right). \tag{3.5} \]

**Proof.** The difference \( U(z, t) - \phi(z) - V(z, t) \) satisfies the system
\[
U^i(z, t) - \phi^i(z) - V^i(z, t)
= \varphi^i(z - vt) - \phi^i(z - vt) - \psi^i(z - vt) \\
+ \int_0^t \left[ g^i(U(z - vt + vs, s)) - g^i(\phi(z - vt + vs)) \right] ds
\]

\[
- \sum_{j=0}^n g^j V^j(\phi(z - vt + vs))V^j(z - vt + vs, s) ds.
\]

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Therefore by triangle inequality we have

\[ \|U^i(t) - \phi^i - V^i(t)\|_\infty \]
\[ \leq \|\varphi^i(\cdot) - \phi^i(\cdot) - \psi^i(\cdot)\|_\infty \]
\[ + \int_0^t \|g^i(U(\cdot, s)) - g^i(\phi(\cdot)) - \sum_{j=0}^n g_{Wj}(\phi(\cdot))V^j(\cdot, s)\|_\infty ds. \]

By Taylor's formula, one can derive

\[ g^i(U(z, s)) - g^i(\phi(z)) - \sum_{j=0}^n g^j_{Wj}(\phi(z))V^j(z, s) \]
\[ = g^i(U(z, s)) - g^i(\phi(z) + V(z, s)) \]
\[ + g^i(\phi(z) + V(z, s)) - g^i(\phi(z)) - \sum_{j=0}^n g^j_{Wj}(\phi(z))V^j(z, s) \]
\[ = \sum_{j=0}^n g^i_{Wj}(\xi^j_i(U^j(z, s) - \phi^j(z) - V^j(z, s)) \]
\[ + \frac{1}{2} \sum_{j=0}^n \sum_{k=0}^n g^i_{WjWk}(\xi^j_i^k)V^j(z, s)V^k(z, s), \]

if \( \sup_{t \geq 0} \|V(t)\|_\infty \leq M; \) and

\[ g^i(U(z, s)) - g^i(\phi(z)) - \sum_{j=0}^n g^j_{Wj}(\phi(z))V^j(z, s) \]
\[ = g^i(U(z, s)) - g^i(\phi(z)) - \sum_{j=0}^n g^j_{Wj}(\phi(z))(U^j(z, s) - \phi^j(z)) \]
\[ + \sum_{j=0}^n g^i_{Wj}(\phi(z))(U^j(z, s) - \phi^j(z)) - \sum_{j=0}^n g^j_{Wj}(\phi(z))V^j(z, s) \]
\[ = \sum_{j=0}^n g^i_{Wj}(\phi(z))(U^j(z, s) - \phi^j(z) - V^j(z, s)) \]
\[ + \frac{1}{2} \sum_{j=0}^n \sum_{k=0}^n g^i_{WjWk}(\eta^j_ik)(U^j(z, s) - \phi^j(z))(U^k(z, s) - \phi^k(z)), \]

if \( \sup_{t \geq 0} \|U(t) - \phi\|_\infty \leq M. \)
Both cases will lead to the estimate

\[
\|U(t) - \phi - V(t)\|_\infty \leq \|\varphi(\cdot) - \phi(\cdot) - \psi(\cdot)\|_\infty + \frac{1}{2} \sigma M^2 t + \rho \int_0^t \|U(s) - \phi - V(s)\|_\infty ds.
\]

Gronwall’s inequality now implies

\[
\|U(t) - \phi - V(t)\|_\infty \leq \|\varphi - \phi - \psi\|_\infty e^{Lt} + \frac{1}{2\rho} \sigma M^2 (e^{Lt} - 1).
\]

**Lemma 5.** If there exist positive constants \(C, M\) and \(T\), independent of the initial data \(\varphi\), such that the condition \(\|\varphi - \phi\|_\infty \leq C\) implies there exists a constant \(h\) with \(|h| \leq M\|\varphi - \phi\|_\infty\), such that \(\|U(T) - \phi_h\|_\infty \leq \frac{1}{2}\|\varphi - \phi\|_\infty\), then (3) is exponentially stable at \(\phi\). If there exist positive constants \(M\) and \(T\), such that \(\psi \in X\) implies there exists a constant \(h\) with \(|h| \leq M\|\psi\|_\infty\), such that \(\|V(T) - h\phi_z\|_\infty \leq \frac{1}{2}\|\psi\|_\infty\), then (4) is exponentially stable at \(\phi_z\).

**Proof.** Let the initial data \(\varphi\) of (3) satisfy \(\|\varphi - \phi\|_\infty \leq C\). Then there is a constant \(h_1\) with \(|h_1| \leq M\|\varphi - \phi\|_\infty\) such that

\[
\|U(T) - \phi_{h_1}\|_\infty \leq \frac{1}{2}\|\varphi - \phi\|_\infty.
\]

Therefore there is a constant \(h_2\) with \(|h_2| \leq \frac{1}{2} M\|\varphi - \phi\|_\infty\) such that

\[
\|U(2T) - \phi_{h_1 + h_2}\|_\infty \leq \frac{1}{4}\|\varphi - \phi\|_\infty.
\]

Moreover there is a constant \(h_3\) with \(|h_3| \leq \frac{1}{4} M\|\varphi - \phi\|_\infty\) such that

\[
\|U(3T) - \phi_{h_1 + h_2 + h_3}\|_\infty \leq \frac{1}{8}\|\varphi - \phi\|_\infty.
\]
By mathematical induction method we obtain a sequence of real numbers \( \{h_k\} \) with \( |h_k| \leq \frac{1}{2^{k-1}} M \| \varphi - \phi \|_\infty \) such that

\[
\| U(kT) - \phi_{h_1+h_2+\ldots+h_k} \|_\infty \leq \frac{1}{2^k} \| \varphi - \phi \|_\infty.
\]

Let \( h = \sum h_k \), then \( |h| \leq 2M \| \varphi - \phi \|_\infty \). We now have the estimates

\[
\begin{align*}
\| U(kT) - \phi_h \|_\infty &\leq \| U(kT) - \phi_{h_1+\ldots+h_k} \|_\infty + \| \phi_{h_1+\ldots+h_k} - \phi_h \|_\infty \\
&\leq \frac{1}{2^k} (1 + 2M \| \phi_z \|_\infty) \| \varphi - \phi \|_\infty,
\end{align*}
\]

\[
\begin{align*}
\| U(t + kT) - \phi_h \|_\infty &\leq e^{\alpha t} \| U(kT) - \phi_h \|_\infty \\
&\leq 2^{-k} e^{\alpha t} (1 + 2M \| \phi_z \|_\infty) \| \varphi - \phi \|_\infty,
\end{align*}
\]

\[
\begin{align*}
\| U(t) - \phi_h \|_\infty &\leq 2^{-k} e^{\alpha t} (1 + 2M \| \phi_z \|_\infty) \| \varphi - \phi \|_\infty \\
&\leq 2 e^{\alpha t} (1 + 2M \| \phi_z \|_\infty) \| \varphi - \phi \|_\infty e^{-\alpha t}.
\end{align*}
\]

\[
\alpha = \ln 2/T, \quad kT \leq t \leq (k + 1)T.
\]

Let the initial data \( \psi \in \mathcal{X} \). Then there is a constant \( h_1 \) with \( |h_1| \leq M \| \psi \|_\infty \) such that

\[
\| V(T) - h_1 \phi_z \|_\infty \leq \frac{1}{2} \| \psi \|_\infty.
\]

Therefore there is a constant \( h_2 \) with \( |h_2| \leq \frac{1}{2} M \| \psi \|_\infty \) such that

\[
\| V(2T) - h_1 \phi_z - h_2 \phi_z \|_\infty \leq \frac{1}{4} \| \psi \|_\infty.
\]

Moreover there is a constant \( h_3 \) with \( |h_3| \leq \frac{1}{4} M \| \psi \|_\infty \) such that

\[
\| V(3T) - h_1 \phi_z - h_2 \phi_z - h_3 \phi_z \|_\infty \leq \frac{1}{8} \| \psi \|_\infty.
\]
By mathematical induction method we obtain a sequence of real numbers \( \{h_k\} \) with \( |h_k| \leq \frac{1}{2^k-1} M \|\psi\|_\infty \) such that

\[
\|V(kT) - h_1\phi_z - \ldots - h_k\phi_z\|_\infty \leq \frac{1}{2^k} \|\psi\|_\infty.
\]

Let \( h = \sum h_k \), then \( |h| \leq 2M\|\psi\|_\infty \).

\[
\|V(kT) - h\phi_z\|_\infty \leq \|V(kT) - (h_1 + \ldots + h_k)\phi_z\|_\infty \\
+ \| (h_1 + \ldots + h_k)\phi_z - h\phi_z\|_\infty \\
\leq \frac{1}{2^k}(1 + 2M\|\phi_z\|_\infty)\|\psi\|_\infty,
\]

\[
\|V(\tau + kT) - h\phi_z\|_\infty \leq e^{\rho T}\|V(kT) - h\phi_z\|_\infty \\
\leq 2^{-k}e^{\rho T}(1 + 2M\|\phi_z\|_\infty)\|\psi\|_\infty,
\]

\[
\|V(t) - h\phi_z\|_\infty \leq 2^{-k}e^{\lambda T}(1 + 2M\|\phi_z\|_\infty)\|\psi\|_\infty \\
\leq 2e^{\lambda T}(1 + 2M\|\phi_z\|_\infty)\|\psi\|_\infty e^{-\alpha t};
\]

\[\alpha = \ln 2/T, \quad kT \leq t \leq (k + 1)T.\]

**Theorem B.** Let \( v \neq 0 \) and let \( \phi \) be a traveling wave solution of (1.1-1.2). Then the system (3) is exponentially stable at \( \phi \) if and only if the system (4) is exponentially stable at \( \phi_z \).

**Proof.** Suppose that system (3) is exponentially stable at \( \phi \). Let \( \|\psi\|_\infty \leq C \) and \( T > 0 \) such that

\[
8M \leq e^{\alpha T}, \quad 4M^2\|\phi_{zz}\|_\infty\|\psi\|_\infty \leq 1,
\]

\[
4\sigma M^2(1 + \|\phi_z\|_\infty)^2\|\psi\|_\infty(e^{\lambda T} - 1) \leq \rho.
\]
Let $U$ and $V$ be the solutions of (3) and (4) with initial data $\phi + \psi$ and $\psi$, respectively. By definition there is a constant $h$ with $|h| \leq M\|\psi\|_\infty$, such that

$$
\|U(t) - \phi_h\|_\infty \leq M\|\psi\|_\infty e^{-at},
$$
$$
\|U(t) - \phi\|_\infty \leq \|U(t) - \phi_h\|_\infty + \|\phi_h - \phi\|_\infty
\leq M\|\psi\|_\infty e^{-at} + |h|\|\phi_z\|_\infty
\leq M\|\psi\|_\infty(1 + \|\phi_z\|_\infty).
$$

Lemma 4.1 implies that

$$
\|U(t) - \phi - V(t)\|_\infty \leq M^2(1 + \|\phi_z\|_\infty)^2\|\psi\|_\infty^2\sigma\rho^{-1}(e^{Lt} - 1).
$$

Then

$$
\|V(T) - h\phi_z\|_\infty \leq \|U(T) - \phi - V(T)\|_\infty
+ \|U(T) - \phi_h\|_\infty + \|\phi_h - \phi - h\phi_z\|_\infty
\leq \frac{1}{2}\|\psi\|_\infty.
$$

By Lemma 4.2 we see system (4) is exponentially stable at $\phi_z$.

Suppose that system (4) is exponentially stable at $\phi_z$. Let $\varphi \in X$ and $T > 0$ such that

$$
8M \leq e^{\alpha T},
4M^2\|\phi_z\|_\infty\|\varphi - \phi\|_\infty \leq 1,
4\sigma M^2(1 + \|\phi_z\|_\infty)^2\|\varphi - \phi\|_\infty(e^{Lt} - 1) \leq \rho.
$$

Let $U$ and $V$ be the solutions of (3) and (4) with initial data $\varphi$ and $\varphi - \phi$, respectively. By definition there is a constant $h$ with $|h| \leq M\|\varphi - \phi\|_\infty$, such that

$$
\|V(t) - h\phi_z\|_\infty \leq M\|\varphi - \phi\|_\infty e^{-at}.
$$
Therefore we get

\[ \|V(t)\|_\infty \leq M(1 + \|\phi_2\|_\infty)\|\varphi - \phi\|_\infty, \]

\[ \|V(T) - h\phi_2\|_\infty \leq \frac{1}{3}\|\varphi - \phi\|_\infty. \]

Lemma 4.1 implies that

\[ \|U(t) - \phi - V(t)\|_\infty \leq M^2(1 + \|\phi_2\|_\infty)^2\|\varphi - \phi\|_\infty^2 \sigma \rho^{-1}(e^{Lt} - 1). \]

Therefore one obtains the estimates

\[ \|U(T) - \phi_h\|_\infty \leq \|U(T) - \phi - V(T)\|_\infty \]

\[ + \|V(T) - h\phi_2\|_\infty + \|\phi_h - \phi - h\phi_z\|_\infty \]

\[ \leq \frac{1}{2}\|\varphi - \phi\|_\infty. \]

By Lemma 4.2 we see system (3) is exponentially stable at \( \phi \).

2. Linear Stability

The necessary and sufficient conditions for linear stability will be given in this section. All the discussions are processed relatively to the Banach space \( X \) or \( D \).

Let \( \Lambda_t \psi \) and \( \Gamma_t \psi \) be the solutions of systems (1) and (1) with \( \Lambda_0 \psi = \psi \) and \( \Gamma_0 \psi = \psi \), respectively. Then \( \Lambda_t \) and \( \Gamma_t \) are the semigroup of operators on \( X \) with infinitesimal generator \( L(\varepsilon) \) and \( T \), respectively. Moreover for all \( \psi \in X \)

\[ \|\Lambda_t \psi\|_\infty \leq e^{Lt}\|\psi\|_\infty, \quad \|\Gamma_t \psi\|_\infty \leq M\|\psi\|_\infty e^{-at}. \]

Here is the strategy of this section. For all \( t \geq 0 \), the operator \( \Lambda_t \) will be splitted into two parts: a projection from \( X \) to span \( \{\phi_z\} \) and its remainder. The remainder
will decay exponentially to zero as \( t \to +\infty \). The stability at \( \phi_z \) of the linearized system (1) is closely related to the spectrum of \( \Lambda_t \), which is related to that of \( L(\varepsilon) \).

Thus we want to find the relationship between \( \sigma(\Lambda_t) \) and \( \sigma(L(\varepsilon)) \). We establish a correspondence between \( \{ \lambda \in \sigma(\Lambda_t) : |\lambda| > e^{-\alpha t} \} \) and \( \{ \lambda \in \sigma(L(\varepsilon)) : \text{Re} \lambda > -\alpha \} \). To accomplish this we study \( L(\varepsilon) \) as a perturbation of \( T \) and \( \Lambda_t \) as a perturbation of \( \Gamma_t \).

The results depend in an essential manner on the fact that \( P(\phi(z)) \to 0 \) as \( |z| \to \infty \).

Let

\[
N_{\lambda,t} = \{ \psi \in X : (\Lambda_t - \lambda I)\psi = 0 \}, \quad N_{\lambda,t}^* = \{ \nu^* \in X^* : (\Lambda_t - \lambda I)^*\nu^* = 0 \}.
\]

The next lemma explores the essential spectrum of the linear differential operator \( L(\varepsilon) \).

**Lemma 6.** The spectrum of the linear operator \( T \) relative to the space \( X \) is given by

\[
\sigma(T) = \sigma_p(T) = \{ \lambda \in \mathbb{C} : \exists \xi \in \mathbb{R}, \text{ such that } i\xi \in \sigma_p(A(\lambda, \varepsilon)) \}.
\]

Moreover we have

\[
\sup_{\lambda \in \sigma(T)} \text{Re} \lambda \leq -\omega, \quad \{ \lambda \in \mathbb{C} : \text{Re} \lambda > -\omega \} \subset \rho(T).
\]

**Proof.** Let \( C_I = \{ \lambda \in \mathbb{C} : A(\lambda, \varepsilon) \text{ has a purely imaginary eigenvalue} \} \), let \( C_N = \{ \lambda \in \mathbb{C} : A(\lambda, \varepsilon) \text{ has no purely imaginary eigenvalue} \} \). It is easy to see that \( C_I \subset \sigma_p(T) \).

Let \( \lambda \in C_N \), let \( P^+ \) be the projection associated with the eigenvalues with positive real part and let \( P^- = I - P^+ \). Let \( \psi \in X \) and let \( \Gamma(z) = \)

\[
\int_{-\infty}^{z} P^- \exp[(z - x)A(\lambda, \varepsilon)]\psi(x)dx - \int_{z}^{\infty} P^+ \exp[(z - x)A(\lambda, \varepsilon)]\psi(x)dx.
\]
Then $\Gamma$ is the unique bounded continuous solution in $D$ to $T_1 \Gamma = \psi$. In fact

$$T_\lambda \Gamma = \frac{\partial}{\partial z} \Gamma(z) - A(\lambda, \varepsilon) \Gamma(z)$$

$$= P^{-}_\lambda \psi(z) + \int_{-\infty}^{z} P^{-}_\lambda A(\lambda, \varepsilon) \exp[(z-x)A(\lambda, \varepsilon)]\psi(x)dx$$

$$+ P^+_\lambda \psi(z) - \int_{z}^{\infty} P^+_\lambda A(\lambda, \varepsilon) \exp[(z-x)A(\lambda, \varepsilon)]\psi(x)dx$$

$$- \int_{-\infty}^{z} A(\lambda, \varepsilon) P^{-}_\lambda \exp[(z-x)A(\lambda, \varepsilon)]\psi(x)dx$$

$$+ \int_{z}^{\infty} A(\lambda, \varepsilon) P^+_\lambda \exp[(z-x)A(\lambda, \varepsilon)]\psi(x)dx = \psi(z),$$

and all other solution is not bounded.

Therefore $T_\lambda$ is 1-1 and onto and

$$T^{-1}_\lambda \psi(z) = \int_{-\infty}^{z} P^{-}_\lambda \exp[(z-x)A(\lambda, \varepsilon)]\psi(x)dx - \int_{z}^{\infty} P^+_\lambda \exp[(z-x)A(\lambda, \varepsilon)]\psi(x)dx.$$

Since

$$\|T^{-1}_\lambda \psi(z)\| \leq C\||\psi||,$$

one knows that $T^{-1}_\lambda \in B(X \to X)$ is a bounded linear operator and $C_N \subset \rho(T)$. Now $C_I \subset \sigma_p(T) \subset \sigma(T) = C - \rho(T) \subset C_I$. Thus $\sigma(T) = \sigma_p(T) = C_I$.

Differentiating equation (1) with respect to $z$, we see $L\phi_z = 0$, thus $0 \in \sigma_p(L(\varepsilon))$ and $\phi_z$ is a corresponding eigenfunction.

Let $\Omega = \{\lambda \in \rho(T): \Re \lambda > -\omega\}$. This is an open, unbounded, simply connected region in the complex plane.

**Lemma 7.** Relatively to the Banach space $X$, we have

(1) For all $\lambda \in \rho(T)$, $T^{-1}_\lambda P$ is a compact operator.

(2) For all $\lambda \in \rho(T)$ with $\Re \lambda > L$, there exists the inverse $(\lambda I - L(\varepsilon))^{-1}$ and

$$(\lambda I - L(\varepsilon))^{-1} = \int_{0}^{\infty} e^{-\lambda t} \Lambda t dt.$$
(3) Let $\lambda \in \rho(T)$, then
\[ \lambda \in \sigma(L(\epsilon)) \iff \lambda \in \sigma_p(L(\epsilon)) \iff N_\lambda \neq \{0\}. \]

(4) Let $\lambda \in \rho(T)$, then
\[ \dim N_\lambda = \dim N^*_\lambda. \]

(5)
\[ \sigma(T) \subset \sigma(L(\epsilon)), \quad \sigma(L(\epsilon)) = \sigma_p(L(\epsilon)). \]

(6) Let $\lambda \in \rho(T)$, then $\dim N_\lambda \leq$ the number of $\mu_i \in \sigma_p(A(\lambda, \epsilon))$ with $\text{Re}\mu_i > 0$ and $\dim N_\lambda \leq$ the number of $\mu_j \in \sigma_p(A(\lambda, \epsilon))$ with $\text{Re}\mu_j < 0$, counting multiplicities.

(7) Let $\lambda \in \Omega$, then $\dim N_\lambda = \dim N^*_\lambda = \dim N^*_\lambda \leq 1.

(8) Any compact subset of $\Omega$ contains only a finite number of points of $\sigma(L(\epsilon))$.

Proof. (1) Let $E \subset X$ be a bounded subset, $\|\psi\|_\infty \leq M$, for all $\psi \in E$. Direct calculation shows
\[ (T^{-1}_\lambda \psi)_z = -\frac{1}{\nu}[\psi + A(\lambda, \epsilon)T^{-1}_\lambda \psi]. \]

Now we have the estimates
\[ \|T^{-1}_\lambda \psi\|_z \leq (\|\psi\|_\infty + \|A(\lambda, \epsilon)\| \|T^{-1}_\lambda\| \|\psi\|_\infty)/\nu, \]
\[ \|T^{-1}_\lambda P\psi\|_z \leq (\|P\| \|\psi\|_\infty + \|A(\lambda, \epsilon)\| \|T^{-1}_\lambda\| \|P\| \|\psi\|_\infty)/\nu \leq C, \]
where $C > 0$ is independent of $z \in R$ and $\psi \in E$. This illustrates that $T^{-1}_\lambda P\psi$ is equicontinuous. In addition
\[ T^{-1}_\lambda () \]
\[ = \int_0^\infty P^-_\lambda \exp[A(\lambda, \epsilon)x]() \ dx \]
\[ - \int_{-\infty}^0 P^+_\lambda \exp[A(\lambda, \epsilon)x]() \ dx, \]

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hence
\[ \|T^{-1}_\lambda P\psi\| \leq \|T^{-1}_\lambda (\cdot)\| \]
\[ \leq \int_0^\infty \|P^\tau_x \exp[A(\lambda, \varepsilon)x](\cdot)\| \, dx \]
\[ + \int_{-\infty}^0 \|P^\tau_x \exp[A(\lambda, \varepsilon)x](\cdot)\| \, dx. \]

Since
\[ \|P(\phi(x))\| \leq C \exp(-\rho|z|), \]
for some positive constants $C$ and $\rho$, it is not hard to see the right hand side is controlled by a function which tends to zero as $|z| \to \infty$. By Arzela-Ascoli’s theorem, $T^{-1}_\lambda P$ is compact.

(2) For all $\lambda \in \rho(T)$ with $\text{Re} \lambda > L$, we have $L(\varepsilon) - \lambda I = T_\lambda (I + T^{-1}_\lambda P)$. Recall that $T^{-1}_\lambda P$ is compact. If $-1 \in \sigma_p(T^{-1}_\lambda P)$, then there is a bounded uniformly continuous eigenfunction $\psi \in X$, such that $(I + T^{-1}_\lambda P)\psi = 0$ or $L(\varepsilon)\psi = \lambda \psi$, so that $\Lambda_t \psi = e^{\lambda t} \psi$ is a solution of system (1). It satisfies $e^{\text{Re} \lambda t}\|\psi\| = \|\Lambda_t \psi\| \leq e^{\lambda t}\|\psi\|$, which yields $\text{Re} \lambda \leq L$, a contradiction. Therefore $-1 \in \rho(T^{-1}_\lambda P)$ and $(\lambda I - L(\varepsilon))^{-1}$ exists and is bounded. Moreover, since $L(\varepsilon)$ is the infinitesimal generator of the semigroup $\{\Lambda_t : t \geq 0\}$, we have for all $\psi \in X$,

\[ L(\varepsilon) \int_0^t e^{-\lambda s} \Lambda_s \psi ds = \int_0^t e^{-\lambda s} \frac{d}{ds} (\Lambda_s \psi) ds \]
\[ = e^{-\lambda t} \Lambda_t \psi - \psi + \lambda \int_0^t e^{-\lambda s} \Lambda_s \psi ds. \]

Thus
\[ (L(\varepsilon) - \lambda I) \int_0^t e^{-\lambda s} \Lambda_s \psi ds = e^{-\lambda t} \Lambda_t \psi - \psi. \]
Because \( \|e^{-\lambda t}\psi\| \leq \exp[(L - Re\lambda)t]\|\psi\| \to 0 \) as \( t \to +\infty \), one obtains

\[
(L(\varepsilon) - \lambda I) \int_0^\infty e^{-\lambda t}\Delta_t\psi dt = -\psi,
\]
i.e.

\[
(\lambda I - L(\varepsilon))^{-1} = \int_0^\infty e^{-\lambda t}\Delta_t dt.
\]

(3) For all \( \lambda \in \rho(T) \cap \sigma(L(\varepsilon)) \), since \( L(\varepsilon) - \lambda I = T\lambda(I + T\lambda^{-1}P) \) and \( T\lambda^{-1}P \) is compact, one asserts that \(-1 \in \sigma_p(T\lambda^{-1}P)\), then there is a bounded uniformly continuous eigenfunction \( \psi \in X \), such that \((I + T\lambda^{-1}P)\psi = 0 \) or \( L(\varepsilon)\psi = \lambda\psi \), so that \( \lambda \in \sigma_p(L(\varepsilon)) \).

(4) For all \( \lambda \in \rho(T), \) if \( \nu \in N'_\lambda \), then

\[
\frac{\partial}{\partial z} \bar{\nu} - [B(\lambda, \varepsilon) + P_2(\phi(z))]\bar{\nu} = 0,
\]
and by asymptotic property \( \nu \in L^1 \). Then for all \( \psi \in D \), we have

\[
((L(\varepsilon) - \lambda I)\nu^*)(\psi) = \nu^*((L(\varepsilon) - \lambda I)\psi) \\
= \int_{-\infty}^\infty ((L(\varepsilon) - \lambda I)\psi(z), \nu(z))dz = \int_{-\infty}^\infty (\psi(z), (L(\varepsilon) - \lambda I)\nu(z))dz = 0.
\]

This means

\[
(L(\varepsilon) - \lambda I)\nu^* = 0, \quad (I + T\lambda^{-1}P)^* T\lambda^* \nu^* = 0.
\]

Therefore \( T\lambda^* \nu^* \in N((I + T\lambda^{-1}P)^*) \). Thus we can define a mapping from \( N'_\lambda \) to \( N((I + T\lambda^{-1}P)^*) \) by \( \phi\nu = T\lambda^* \nu^* \). If \( \phi\nu_1 = \phi\nu_2 \), i.e. \( T\lambda^* \nu_1^* = T\lambda^* \nu_2^* \), then for all \( \psi \in D \), we have

\[
\int_{-\infty}^\infty (T\lambda^* \psi(z), \nu_1(z) - \nu_2(z))dz = 0.
\]
Hence taking \( \psi = T_\lambda^{-1}(\nu_1 - \nu_2) \) we get \( \nu_1 = \nu_2 \) and \( \phi \) is a 1-1 mapping.

If \( \nu_1, \ldots, \nu_k \) are linearly independent in \( N_\lambda' \), then \( T_\lambda^*\nu_1', \ldots, T_\lambda^*\nu_k' \) are linearly independent in \( N((I + T_\lambda^{-1}P)^*) \).

Since
\[
N_\lambda = N(I + T_\lambda^{-1}P),
\]
we see
\[
\dim N_\lambda = \dim N(I + T_\lambda^{-1}P) = \dim((I + T_\lambda^{-1}P)^*) \geq \dim N_\lambda'.
\]
Similarly we get \( \dim N_\lambda' \geq \dim N_\lambda \). Hence \( \dim N_\lambda = \dim N_\lambda' \).

(5) Let \( \lambda \in \sigma(T) \), so that \( A(\lambda, \varepsilon) \) has a purely imaginary eigenvalue \( i\mu \). Let \( v \) be a corresponding eigenvector. Then \( T_\lambda(e^{im\varepsilon}v) = 0 \).

Hence we have
\[
i\xi v(\varepsilon) + A(\lambda, \varepsilon)v = 0.
\]
Let \( f_m(z) = 1 - \exp(-\frac{z^2}{m^2}) \). Then \( \|f_m\|_\infty = 1 \) and
\[
\lim_{m \to \infty} \|(L(\varepsilon) - \lambda I)f_m(z)e^{im\varepsilon}\|_\infty = 0.
\]

So \( \lambda \in \sigma_a(L) \). Thus we have \( \sigma(T) \subset \sigma(L(\varepsilon)) \). Similarly \( \sigma(T') \subset \sigma(L(\varepsilon)') \).

Let \( \lambda \in \sigma(L(\varepsilon)) \). If \( \lambda \in \sigma(T) = \sigma(T') \), then \( \lambda \in \sigma(L(\varepsilon)') \). If \( \lambda \in \rho(T) \), then \( N_{\lambda} \neq 0 \). By (4) \( \dim N_{\lambda}' = \dim N_{\lambda} \geq 1 \), so \( \lambda \in \sigma(L(\varepsilon)') \). Hence \( \sigma(L(\varepsilon)) \subset \sigma(L(\varepsilon)') \).

Similarly, \( \sigma(L(\varepsilon)') \subset \sigma(L(\varepsilon)) \). Therefore \( \sigma(L(\varepsilon)) = \sigma(L(\varepsilon)') \).

(6) Let \( \lambda \in \rho(T) \). Then \( A(\lambda, \varepsilon) \) has no purely imaginary eigenvalue.

Let \( N_{\lambda}^{0\pm} = \{ \psi : \psi \text{ is a solution of the equation (1) such that } \psi(z) \text{ is bounded as } z \to \pm\infty \} \), respectively.
Let $N^\pm_\lambda = \{ \psi : \psi \text{ is a solution of the equation (1) such that } \psi(z) \text{ is bounded as } z \to \pm \infty \}$, respectively.

Recall that $\|P(z)\| \leq C \exp(-\rho|z|)$. If $A(\lambda, \varepsilon)$ has no purely imaginary eigenvalue then the dimension of the subspace of solutions of (1) with $\psi$ bounded as $z \to +\infty$ is equal to the dimension of the subspace of solutions to the unperturbed equation (1) with $\psi$ bounded as $z \to +\infty$. Notice any bounded solutions to these equations go to zero exponentially fast as $z \to \pm \infty$, respectively. Then $\dim N_\lambda \leq \dim N^\pm_\lambda = \dim N^{\text{rel}}_\lambda$ = the number of eigenvalues of $A(\lambda, \varepsilon)$ with $\text{Re}\mu < 0$ and $\text{Re}\mu > 0$, respectively.

(7) If $\lambda \in \Omega$, by Lemma 5.5, the matrix $A(\lambda, \varepsilon)$ has only one eigenvalue with positive real part. By (1), the assertion is true.

(8) Notice that if $\lambda \in \Omega$, then
\[ L(\varepsilon) - \lambda I = T_\lambda(I + T_\lambda^{-1}P). \]

Hence $\lambda \in \sigma(L(\varepsilon)) \cap \Omega$ if and only if $I + T_\lambda^{-1}P$ has no inverse. The set
\[ \{ T_\lambda^{-1}P : \lambda \in \Omega \} \]
is a homomorphic family of compact operators. If $\text{Re}\lambda > L$, then $I + T_\lambda^{-1}P$ has an inverse. So any compact subset of $\Omega$ contains only a finite number of $\lambda \in \sigma(L(\varepsilon))$.

**Lemma 8.** For the system of integral-differential equations, $K_t = \Lambda_t - \Gamma_t$ is a compact operator, for all fixed $t > 0$.

**Proof.** Let $E \subset X$ be a bounded subset, namely $\|\psi\|_\infty \leq M$, for all $\psi \in E$. For fixed $t > 0$, the solution of equation (1) with initial data $\psi \in E$ meets the estimates
\[ \|\Lambda_t\psi\|_\infty \leq \|\psi\|_\infty e^{Lt} \leq Me^{Lt}. \]
There exist positive constants $C$ and $\rho$, such that

$$\|P(\phi(z))\| \leq C \exp(-\rho|z|).$$

$$\frac{dK_t}{dt} = \frac{d\Lambda_t}{dt} - \frac{d\Gamma_t}{dt} = L(\varepsilon)\Lambda_t - T\Gamma_t = TK_t + P\Lambda_t,$$

$$K_t = \Gamma_t \int_0^t \Gamma_s^{-1} P\Lambda_s ds.$$

We continue our investigation of the relationship between the spectrum of $\Lambda_t$ and the spectrum of $L(\varepsilon)$ relative to the Banach space $X$.

If $\mu \in \sigma_p(L(\varepsilon)) \cap \Omega$, then $e^{\mu t}$ is an eigenvalue of $\Lambda_t$, i.e. $e^{\mu t} \in \sigma(\Lambda_t)$, for all $t > 0$. A partial inverse of this fact is given in the following lemmas.

**Lemma 9.** Let $t > 0$ be a fixed number and let $\lambda \in \sigma(\Lambda_t)$ and $|\lambda| > Me^{-\alpha t}$.

Then

1. $\lambda \in \sigma_p(\Lambda_t)$ and $1 \leq k = \dim N_{\lambda,t} < \infty$.
2. There exist distinct eigenvalues $\mu_1, \ldots, \mu_k \in \sigma_p(L(\varepsilon)) \cap \Omega$, such that $e^{\mu_1 t} = \ldots = e^{\mu_k t} = \lambda$.
3. Let $0 \neq \psi_i \in N_{\mu_i}$ and $0 \neq \nu_i \in N_{\mu_i}'$, where $i = 1, 2, \ldots, k$, then $\psi_1, \ldots, \psi_k$ is a basis for $N_{\lambda,t}$ and $\nu_1^*, \ldots, \nu_k^*$ is a basis for $N_{\lambda,t}^*$.
4. $(\Lambda_t - \lambda I)X$ is closed and $\text{codim}(\Lambda_t - \lambda I)X = \dim N_{\lambda,t}^* = \dim N_{\lambda,t}$.
5. There exists the direct sum decomposition

$$N_{\lambda,t} = \bigoplus_{i=1}^k N_{\mu_i}, \quad N_{\lambda,t}^* = \bigoplus_{i=1}^k N_{\mu_i}^*. $$

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Proof. Because $|\lambda| > Me^{-\alpha t} \geq \|\Gamma_t\|$, $\lambda I - \Gamma_t$ has a bounded linear inverse. Since

$$
\lambda I - \Lambda_t = \lambda I - \Gamma_t - K_t = (\lambda I - \Gamma_t) \left( I - (\lambda I - \Gamma_t)^{-1}K_t \right),
$$

and $K_t$ is a compact linear operator, $(\lambda I - \Gamma_t)^{-1}$ is a bounded linear operator, one sees $(\lambda I - \Gamma_t)^{-1}K_t$ is also a compact operator. Thus either $1 \in \rho((\lambda I - \Gamma_t)^{-1}K_t)$, or $1 \in \sigma_p((\lambda I - \Gamma_t)^{-1}K_t)$. If $1 \in \rho((\lambda I - \Gamma_t)^{-1}K_t)$, then $\lambda I - \Lambda_t$ has a bounded linear inverse. Contradiction. Hence $1 \in \sigma_p((\lambda I - \Gamma_t)^{-1}K_t)$. There exists at least a nontrivial function $\psi \in X$, such that

$$
(I - (\lambda I - \Gamma_t)^{-1}K_t)\psi = 0,
$$

and

$$
(\lambda I - \Lambda_t)\psi = 0.
$$

Therefore $\lambda \in \sigma_p(\Lambda_t)$. Moreover, the null space of $I - (\lambda I - \Gamma_t)^{-1}K_t$ is finite-dimensional, $1 \leq k = \dim N(I - (\lambda I - \Gamma_t)^{-1}K_t) < \infty$. Since $N(I - (\lambda I - \Gamma_t)^{-1}K_t) = N(\lambda I - \Lambda_t) = N_{\lambda,t}$, so $1 \leq k = \dim N_{\lambda,t} < \infty$. Let $\psi_1, ..., \psi_k \in N_{\lambda,t}$ be a basis, then

$$
\Lambda_t\psi_i = \lambda\psi_i, \quad \text{for } i = 1, ..., k.
$$

We want to convert the semigroup of operators $\{\Lambda_t : t \geq 0\}$ with infinitesimal generator $L(\varepsilon)$ into another semigroup of operators with infinitesimal generator a diagonalizable matrix. For all $s \geq 0$, we have

$$
\Lambda_t\Lambda_s\psi_i = \Lambda_s\Lambda_t\psi_i = \Lambda_s(\lambda\psi_i) = \lambda\Lambda_s\psi_i,
$$

so $\Lambda_s\psi_i \in N_{\lambda,t}$. There exist $a_{i1}(s), a_{i2}(s), ..., a_{ik}(s)$, such that

$$
\Lambda_s\psi_i = a_{i1}(s)\psi_1 + a_{i2}(s)\psi_2 + ... + a_{ik}(s)\psi_k.
$$
Thus

\[ \Lambda_s \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_k \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) & \ldots & a_{1k}(s) \\ a_{21}(s) & a_{22}(s) & \ldots & a_{2k}(s) \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}(s) & a_{k2}(s) & \ldots & a_{kk}(s) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_k \end{pmatrix}. \]

Moreover

\[ \Lambda_r \Lambda_s \psi_i = \Lambda_r \left( a_{i1}(s) \psi_1 + a_{i2}(s) \psi_2 + \ldots + a_{ik}(s) \psi_k \right) \]
\[ = a_{i1}(s) \Lambda_r \psi_1 + a_{i2}(s) \Lambda_r \psi_2 + \ldots + a_{ik}(s) \Lambda_r \psi_k \]
\[ = \left( a_{i1}(s), a_{i2}(s), \ldots, a_{ik}(s) \right) \begin{pmatrix} \Lambda_r \psi_1 \\ \Lambda_r \psi_2 \\ \vdots \\ \Lambda_r \psi_k \end{pmatrix} \]
\[ = \left( a_{i1}(s), a_{i2}(s), \ldots, a_{ik}(s) \right) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_k \end{pmatrix}. \]

Hence

\[ \Lambda_r \Lambda_s \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_k \end{pmatrix} = \left( a_{ij}(s) \right) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_k \end{pmatrix}. \]
On the other hand,

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k
\end{pmatrix}
\Lambda_r \Lambda_s
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k
\end{pmatrix}
= \Lambda_{r+s}
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k
\end{pmatrix}
= (a_{ij}(r+s))
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k
\end{pmatrix}
\]

Therefore \((a_{ij}(r+s)) = (a_{ij}(r))(a_{ij}(s))\), for all \(r, s \geq 0\). Moreover

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k
\end{pmatrix}
\Lambda_0
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k
\end{pmatrix}
= (a_{ij}(0))
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k
\end{pmatrix}
= (a_{ij}(0))
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k
\end{pmatrix},
\]

implies \((a_{ij}(0)) = I\). Hence \((a_{ij}(s))\) is a semigroup of operators.

There exists a unique matrix \(C = (c_{ij})\), such that

\[(a_{ij}(s)) = \exp(Cs).\]

Moreover, there exists an invertible matrix \(T\), such that

\[
T^{-1}CT = \begin{pmatrix}
J_1 & 0 & \ldots & 0 \\
0 & J_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & J_m
\end{pmatrix} = D,
\]

where

\[
J_k = \begin{pmatrix}
\mu_k & 0 & \ldots & 0 \\
0 & \mu_k & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \mu_k
\end{pmatrix},
\]

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or

\[
J_k = \begin{pmatrix}
\mu_k & 1 & 0 & \ldots & 0 & 0 \\
0 & \mu_k & 1 & \ldots & 0 & 0 \\
& \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \mu_k & 1 \\
0 & 0 & 0 & \ldots & 0 & \mu_k
\end{pmatrix}.
\]

Now

\[T^{-1}(a_{ij}(s))T = T^{-1}\exp(Cs)T = \exp(T^{-1}CTs) = \exp(Ds).\]

But

\[
\lambda \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k
\end{pmatrix} = \Lambda_t \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k
\end{pmatrix} = (a_{ij}(t)) \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k
\end{pmatrix},
\]

so \((a_{ij}(t)) = \lambda I\) and \(\exp(Dt) = T^{-1}\lambda IT = \lambda I\). This means the \(J_k\)'s take the form

\[
J_k = \begin{pmatrix}
\mu_k & 0 & \ldots & 0 \\
0 & \mu_k & \ldots & 0 \\
& \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \mu_k
\end{pmatrix}, \text{ for } k = 1, \ldots, m,
\]

i.e. the matrix \(C\) is diagonalizable. Let

\[
T^{-1}CT = \begin{pmatrix}
\mu_1 & 0 & \ldots & 0 \\
0 & \mu_2 & \ldots & 0 \\
& \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \mu_k
\end{pmatrix}.
\]
One easily obtains the following equations from the above analysis

\[
\Lambda, T^{-1} = \exp(C_s) I = \exp(C_s) \Lambda
\]

Since \( T \) is invertible, \( T^{-1} \) is also invertible. Thus

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k \\
\end{pmatrix} = T^{-1} \exp(C_s) \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\mu_1 s & 0 & \cdots & 0 \\
0 & \mu_2 s & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mu_k s \\
\end{pmatrix} = T^{-1} \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_k \\
\end{pmatrix}
\]
is also a basis for $N_{\lambda,t}$. For this basis, we have the equations

$$
\Lambda_s \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_k \end{pmatrix} = \exp \begin{pmatrix} \mu_1 s & 0 & \ldots & 0 \\ 0 & \mu_2 s & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_k s \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_k \end{pmatrix},
$$

$$
\Lambda_t \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_k \end{pmatrix} = \Lambda_t T^{-1} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_k \end{pmatrix} = T^{-1} \Lambda_t \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_k \end{pmatrix},
$$

thus we obtain the important equations

$$
\Lambda \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_k \end{pmatrix} = \exp \begin{pmatrix} \mu_1 t & 0 & \ldots & 0 \\ 0 & \mu_2 t & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_k t \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_k \end{pmatrix},
$$

$$
\lambda I = \exp \begin{pmatrix} \mu_1 t & 0 & \ldots & 0 \\ 0 & \mu_2 t & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_k t \end{pmatrix}.
$$
Hence $A = \exp\{f, it) for i = 1, ..., k. But |\lambda| > Me^{-\alpha t}$, so $\Re\mu_i > -\alpha$ and $\mu_i \in \Omega$. Below by the explicit expression of $(L(\varepsilon) - \mu I)^{-1}$, we find a partial relationship between $\sigma(\Lambda_i)$ and $\sigma(L(\varepsilon))$. Since $\Lambda_i \phi_i = e^{\mu_i s} \phi_i$ and $\Re(\mu_i + \alpha + L) > L$, we have

\[
((\mu_i + \alpha + L)I - L(\varepsilon))^{-1} = \int_0^\infty \exp[-(\mu_i + \alpha + L)s] \Lambda_s ds.
\]

\[
((\mu_i + \alpha + L)I - L(\varepsilon))^{-1} \phi_i = \int_0^\infty \exp[-(\mu_i + \alpha + L)s] e^{\mu_i s} \phi_i ds
\]

\[
= \int_0^\infty \exp[-(\alpha + L)s] \phi_i ds
\]

\[
= \frac{1}{\alpha + L} \exp[-(\alpha + L)s] \phi_i|_0^\infty = \frac{1}{\alpha + L} \phi_i;
\]

\[
((\mu_i + \alpha + L)I - L(\varepsilon)) \phi_i = (\alpha + L) \phi_i,
\]

\[
L(\varepsilon) \phi_i = \mu_i \phi_i,
\]

\[
\mu_i \in \sigma_p(L(\varepsilon)),
\]

\[
\mu_i \in \sigma_p(L(\varepsilon)) \cap \Omega,
\]

and $\phi_i$ is the corresponding eigenfunction and dim $N_{\mu_i} = 1$. If $\mu_i = \mu_j$ for some $i \neq j$, then dim $N_{\mu_i} \geq 2$, a contradiction. So $\mu_i \neq \mu_j$, for all $i \neq j$. Obviously we have the direct sum decomposition

\[
N_{\lambda, t} = \bigoplus_{i=1}^k N_{\mu_i}.
\]

Moreover, let $0 \neq \nu_i \in N_{\mu_i}'$, then $\nu_1, \nu_2, ..., \nu_k$ are linearly independent and

\[
(\Lambda_s \nu_i^*)(\psi) = \nu_i^*(\Lambda_s \psi) = \int_{-\infty}^\infty (\Lambda_s \psi(z), \nu_i(z)) dz,
\]

\[
\frac{d}{ds}(\Lambda_s \nu_i^*)(\psi) = \frac{d}{ds} \int_{-\infty}^\infty (\Lambda_s \psi(z), \nu_i(z)) dz = \int_{-\infty}^\infty (L(\varepsilon) \Lambda_s \psi(z), \nu_i(z)) dz
\]

\[
= \int_{-\infty}^\infty (\Lambda_s \psi(z), L \nu_i(z)) dz = \int_{-\infty}^\infty (\Lambda_s \psi(z), \mu_i \nu_i(z)) dz
\]

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\[
\mu_\alpha \int_{-\infty}^{\infty} (\Lambda_{s} \psi(z), \nu_i(z)) dz = \mu_i \nu_i^*(\Lambda_{s} \psi) = \mu_i (\Lambda_{s}^* \nu^*_i)(\psi),
\]

\[
\Lambda_0^* = I^*, \quad \Lambda_0^* \nu_i^* = e^{\mu_i} \nu_i^*,
\]

\[
\Lambda_i^* \nu_i^* = \lambda \nu_i^*, \quad \nu_i^* \in N\lambda^*_{\lambda, t}.
\]

It is clear that \(\nu_1^*, ..., \nu_k^*\) are linearly independent. Therefore we have

\[
\bigoplus_{i=1}^{k} N_{\mu_i}^* \subset N_{\lambda, t}^*.
\]

We have already proved earlier that \(\dim N_{\mu_i} = \dim N_{\mu_i}^* = \dim N_{\mu_i}^* = 1\), so

\[
k = \dim \bigoplus_{i=1}^{k} N_{\mu_i}^* \leq \dim N_{\lambda, t}^*.
\]

Because \((\lambda I - \Gamma_t)^{-1} K_t\) is a compact operator, \(((\lambda I - \Gamma_t)^{-1} K_t)^* = K_t^* (\lambda I^* - \Gamma_t^*)^{-1}\).

Thus

\[
\dim N(I - (\lambda I - \Gamma_t)^{-1} K_t) = \dim N(I^* - K_t^* (\lambda I^* - \Gamma_t^*)^{-1}).
\]

\[
\dim N(I - (\lambda I - \Gamma_t)^{-1} K_t) = \dim N(\lambda I - \Lambda_t),
\]

\[
\dim N(I^* - K_t^* (\lambda I^* - \Gamma_t^*)^{-1}) = \dim N(\lambda I^* - \Lambda_t^*),
\]

\[
\dim N(\lambda I - \Lambda_t) = \dim N(\lambda I^* - \Lambda_t^*) = k.
\]

\[
N_{\lambda, t}^* = \bigoplus_{i=1}^{k} N_{\mu_i}^*.
\]

**Lemma 10.** Let \(t > 0\) be a fixed number and let \(\lambda \in \sigma(\Lambda_t)\) and \(|\lambda| > e^{-\alpha t}\). Then

1. \(\lambda \in \sigma_p(\Lambda_t)\) and \(1 \leq l = \dim N_{\lambda, t} < \infty\).
2. There exist distinct eigenvalues \(\mu_1, ..., \mu_l \in \sigma_p(L(\epsilon)) \cap \Omega\), such that \(e^{\mu_1 t} = ... = e^{\mu_l t} = \lambda\).
3. Let \(0 \neq \psi_i \in N_{\mu_i}\) and \(0 \neq \nu_i \in N_{\mu_i}^*\), where \(i = 1, 2, ..., l\), then \(\psi_1, ..., \psi_l\) is a basis for \(N_{\lambda, t}\) and \(\nu_1^*, ..., \nu_l^*\) is a basis for \(N_{\lambda, t}^*\).
(4) \((\Lambda_t - \lambda I)X\) is closed and \(\text{codim}(\Lambda_t - \lambda I)X = \dim N^*_{\lambda,t} = \dim N_{\lambda,t}\).

(5) There exists the direct sum decomposition

\[
N_{\lambda,t} = \bigoplus_{i=1}^l N_{\mu_i}, \quad N^*_{\lambda,t} = \bigoplus_{i=1}^l N^*_{\mu_i}.
\]

Proof. Let \(|\lambda| > e^{-\alpha t}\) and \(\lambda \in \sigma(\Lambda_t)\), then there exists an integer \(m\) such that \(|\lambda^m| > M \exp(-\alpha m t)\). Since \(\lambda^m \in \sigma(\Lambda_{mt})\), we have \(\lambda^m \in \sigma_p(\Lambda_{mt})\) and \(1 \leq k = \dim N_{\lambda^m, mt} < \infty\). Moreover, there exist distinct eigenvalues \(\mu_1, ..., \mu_k \in \sigma_p(L(\varepsilon)) \cap \Omega\), such that \(\lambda^m = e^{\mu_imt}\). Let \(\psi_i \in N_{\mu_i}\) and \(\nu_i \in N_{\mu_i}'\), where \(i = 1, ..., k\), then

\[
L(\varepsilon)\psi_i = \mu_i \psi_i, \quad \Lambda_t \psi_i = \exp(\mu_i t)\psi_i,
\]

and

\[
L(\varepsilon)'\nu_i = \mu_i \nu_i, \quad \Lambda^*_t \nu_i = \exp(\mu_i t)\nu_i^*.
\]

Because

\[
X \supset (\lambda I - \Lambda_t)X \supset (\lambda^m I - \Lambda_{mt})X
\]

\[
= (\lambda^m I - \Gamma_{mt})(I - (\lambda^m I - \Gamma_{mt})^{-1} K_{mt})X,
\]

and \((\lambda^m I - \Lambda_{mt})X\) is closed, and

\[
1 \leq \text{codim}(\lambda^m I - \Lambda_{mt})X = \dim N(\lambda^m I^* - \Lambda^*_{mt})
\]

\[
= \dim N(\lambda^m I - \Lambda_{mt}) = k < \infty,
\]

so the quotient space \(X/(\lambda^m I - \Lambda_{mt})X\) is of finite dimension. Moreover

\[
\frac{(\Lambda I - \Lambda_t)X}{(\lambda^m I - \Lambda_{mt})X} \subset \frac{X}{(\lambda^m I - \Lambda_{mt})X}
\]
is a linear subspace in a finite dimensional space, thus it is a closed linear subspace.

Because the mapping defined by

$$\phi : X \to \frac{X}{(\lambda^m I - \Lambda_{lt})X}$$

is continuous, one can show directly the preimage

$$(\lambda I - \Lambda_{lt})X = \phi^{-1}\left(\frac{(\lambda I - \Lambda_{lt})X}{(\lambda^m I - \Lambda_{mt})X}\right)$$

in $X$ is also closed. Thus we have $\text{codim}(\lambda I - \Lambda_{lt})X = \dim N(\lambda I^* - \Lambda_{lt}^*) = \dim N_{l,t}^*.$

For all $\psi \in N_{\lambda,t} \subset N_{\lambda^m,mt}$, we have $\Lambda_{lt}\psi = \lambda\psi,$ there exist $a_1, \ldots, a_k,$ such that

$$\psi = a_1\psi_1 + \ldots + a_k\psi_k.$$  Thus

$$\lambda a_1\psi_1 + \ldots + \lambda a_k\psi_k = a_1e^{\mu_1t}\psi_1 + \ldots + a_ke^{\mu_kt}\psi_k.$$  Thus

$$a_i(\lambda - \exp(\mu_it)) = 0.$$  If $a_i \neq 0,$ then $\lambda = \exp(\mu_it).$

Let $l$ be the number of eigenvalues $\mu_i's$ such that $\lambda = \exp(\mu_it).$ Without loss of generality, we assume that this is true for $i = 1, \ldots, l.$ Then $\Lambda_{lt}\psi_i = \lambda\psi_i.$ So $\dim N_{\lambda,t} \geq l.$ If $\dim N_{\lambda,t} > l,$ then there exists a nontrivial $\psi \in N_{\lambda,t} \subset N_{\lambda^m,mt},$ which is linearly independent of $\psi_1, \ldots, \psi_l.$ There exist $b_1, \ldots, b_k$ and for some $j > l, b_j \neq 0,$ such that $\psi = b_1\psi_1 + \ldots + b_k\psi_k.$ According to our above analysis, $\lambda_j = \exp(\mu_jt).$ Contradiction. Therefore $\dim N_{\lambda,t} = l.$ Similarly, $\dim N_{\lambda,t}^* = l.$ Therefore $\text{codim}(\lambda I - \Lambda_{lt})X = \dim N_{\lambda,t} = \dim N_{\lambda,t}^*.$

If $(\lambda I - \Lambda_{lt})X = X$, then $\dim N_{\lambda,t} = 0$ and $\lambda I - \Lambda_{lt}$ is 1-1. Therefore $\lambda \in \rho(\Lambda_{lt}).$ This is a contradiction. Thus $(\lambda I - \Lambda_{lt})X \neq X$ and $\dim N_{\lambda,t} \geq 1.$ Hence there is a nontrivial $\psi \in X,$ such that $\Lambda_{lt}\psi = \lambda\psi,$ or $\lambda \in \sigma_p(\Lambda_{lt}).$
Finally we have the decomposition

\[ N_{\lambda,t} = \bigoplus_{i=1}^{l} N_{\mu_i}, \quad N_{\lambda,t}^* = \bigoplus_{i=1}^{l} N_{\mu_i}^*. \]

**Lemma 11.** Let \(|\lambda| > e^{-\alpha t}\) for fixed \(t > 0\). Then

(1) There holds the relationships

\[
\begin{align*}
\{e^{\mu t} : \mu \in \sigma(L(\varepsilon)) \cap \Omega, t > 0\} &\subset \\
\{\lambda \in \sigma(A_t) : |\lambda| > e^{-\alpha t}, t > 0\} &\subset \{e^{\mu t} : \mu \in \sigma_p(L(\varepsilon)) \cap \Omega : t > 0\}, \\
\{e^{\mu t} : \mu \in \sigma(L(\varepsilon)) \cap \Omega, t > 0\} &\subset \{\lambda \in \sigma(A_t) : |\lambda| > e^{-\alpha t}, t > 0\}.
\end{align*}
\]

(2) For all \(\sigma : 0 < \sigma < \alpha\), there exist at most finite number of \(\lambda \in \sigma(A_t)\) with \(|\lambda| > \exp(-\sigma t)\) and finite number of eigenvalues \(\lambda \in \sigma(L(\varepsilon)) \cap \Omega\) with \(\text{Re}\lambda > -\sigma\).

Proof. By Lemma 6.5, one can prove (1) immediately. For all \(\sigma : 0 < \sigma < \alpha\), let \(\sigma_1 = \frac{1}{2}(\sigma + \alpha)\), then \(0 < \sigma < \sigma_1 < \alpha\). Let \(m \geq 1\) such that

\[ \exp(-\sigma t) > M^{1/m} \exp(-\sigma_1 t). \]

Let

\[ O = \{\lambda \in \mathbb{C} : |\lambda| > M \exp(-\alpha m t)\}. \]

Then \(O\) is a connected open subset in the complex plane. The subset of bounded linear operators

\[ \{(\lambda I - \Gamma_{mt})^{-1}K_{mt} : \lambda \in O\} \]

is a homomorphic family of compact operators. \(I - (\lambda I - \Gamma_{mt})^{-1}K_{mt}\) has an inverse for all \(\lambda\) with \(|\lambda| > M \exp(L mt)\). Obviously

\[ \{\lambda \in \mathbb{C} : M \exp(-\sigma_1 m t) \leq |\lambda| \leq M \exp(L mt)\} \subset O \]

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is a compact subset. It contains at most a finite number of distinct \( \lambda \) such that
\[ I - (\lambda I - \Gamma_{mt})^{-1} K_{mt} \] or \( \lambda I - \Lambda_{mt} \) is not invertible.

We now have
\[
\{ \lambda \in \mathbb{C} : \lambda \in \sigma(\Lambda_t), |\lambda| > \exp(-\sigma t) \}
\subset \{ \lambda \in \mathbb{C} : \lambda \in \sigma(I_t), |\lambda| > M^{1/m} \exp(-\sigma_1 t) \}
\subset \{ \lambda^m \in \mathbb{C} : \lambda^m \in \sigma(I_t^m), |\lambda^m| > M \exp(-\sigma_1 mt) \}.
\]

Hence there exist at most a finite number of \( \lambda \in \sigma(\Lambda_t) \) such that \( |\lambda| > \exp(-\sigma t) \). The number of \( \lambda \in \sigma(L(\varepsilon)) \cap \Omega \) with \( \Re \lambda > -\sigma \) is at most finite. This also asserts that the eigenvalues of \( \Lambda_t \) and \( L(\varepsilon) \) are isolated points in \( \Omega \).

In a sufficiently small neighborhood of the origin, except for \( \lambda = 0 \), there is no other eigenvalue of \( L(\varepsilon) \). Similarly in a sufficiently small neighborhood of the point \( \lambda = 1 \), except for \( \lambda = 1 \), there is no other eigenvalue of \( \Lambda_t \).

Since \( L(\varepsilon) \phi_2 = 0, \Lambda_t(\phi_2) = \phi_2, 1 \in \sigma_p(\Lambda_t), \dim N_0 = \dim N_0' = \dim N_0^* = 1 \), therefore \( \lambda = 0 \) is a simple eigenvalue of \( L(\varepsilon), L(\varepsilon)' \) and \( L(\varepsilon)^* \). Let \( 0 \neq \nu \in N_0' \). Then \( \nu^*(L(\varepsilon)\psi) = 0, L(\varepsilon)^*\nu^* = 0, \Lambda_t^*\nu^* = \nu^* \), for all \( t \geq 0 \).

Let \( S\psi = \nu^*(\psi)\phi_2 \). Then \( SL(\varepsilon) = L(\varepsilon)S = 0 \).

If \( \nu^*(\phi_2) \neq 0 \), let \( \nu^*(\phi_2) = 1 \). Then \( S(\phi_2) = \phi_2 \) and \( S^2 = S \). Thus \( S \) is a projection and compact operator. If \( \nu^*(\phi_2) = 0 \), then \( S(\phi_2) = 0 \) and \( S^2 = 0 \).

If there is a function \( \psi_0 \in D \), such that \( L(\varepsilon)\psi_0 = \phi_2 \), then \( \Lambda_t\psi_0 = \psi_0 + t\phi_2 \) is a solution of (1), so that it is not exponentially stable at \( \phi_2 \).
Theorem C. (1) If \( \nu^*(\phi_z) = 0 \), then there is a bounded uniformly continuous function \( \psi_0 \in D \) such that \( L(\varepsilon)\psi_0 = \phi_z \) and

\[
\frac{1}{2\pi i} \int_{\Gamma_0} (xI - L(\varepsilon))^{-1}\psi_0 \, dx = \psi_0, \quad (3.6)
\]

\[
\frac{1}{2\pi i} \int_{\Gamma_1} (xI - \Lambda_t)^{-1}\psi_0 \, dx = \psi_0. \quad (3.7)
\]

where \( \Gamma_i \) is a sufficiently small circle about \( i = 0, 1 \).

If \( \nu^*(\phi_z) = 1 \), then there is no bounded uniformly continuous function \( \psi \in D \) such that \( L(\varepsilon)\psi = \phi_z \), and

\[
\frac{1}{2\pi i} \int_{\Gamma_0} (xI - L(\varepsilon))^{-1} \, dx = S, \quad (3.8)
\]

\[
\frac{1}{2\pi i} \int_{\Gamma_1} (xI - \Lambda_t)^{-1} \, dx = S, \quad (3.9)
\]

provided there is no \( 0 \neq \mu \in \sigma(L(\varepsilon)) \cap \Omega \) with \( e^{\mu t} = 1 \) for this \( t > 0 \), i.e. \( \dim N_{1,t} = 1 \).

(2) System (1) is exponentially stable at \( \phi_z \) if and only if \( \nu^*(\phi_z) = 1 \) and \( \max\{\text{Re}\lambda : \lambda \in \sigma(L(\varepsilon)) \setminus \{0\}\} \leq -\sigma < 0 \).

Proof. (1) Recall \( \dim N((I + T^{-1}P)^*) = \dim N((I + T^{-1}P) = \dim N(L(\varepsilon)) = 1 \) and \( T^*\nu^* \) is the unique function in \( N((I + T^{-1}P)^*) \). There exists a \( \psi_0 \in D \) such that \( L(\varepsilon)\psi_0 = \phi_z \) if and only if \( (I + T^{-1}P)\psi_0 = T^{-1}\phi_z \) if and only if \( T^{-1}\phi_z \in (I + T^{-1}P)X = N((I + T^{-1}P)^*)^\perp \) if and only if \( (T^*\nu^*)(T^{-1}\phi_z) = 0 \) if and only if \( \nu^*(\phi_z) = 0 \).

Suppose that \( \nu^*(\phi_z) = 0 \), i.e. there is a bounded uniformly continuous function \( \psi_0 \in D \) such that \( L(\varepsilon)\psi_0 = \phi_z \). Let \( \Gamma_i \) be a sufficiently small circle about \( i = 0, 1 \), it is easily seen

\[
\frac{1}{2\pi i} \int_{\Gamma_1} (xI - \Lambda_t)^{-1}\psi_0 \, dx = \frac{1}{2\pi i} \int_{\Gamma_1} \left( \frac{\psi_0}{x - 1} + \frac{t\phi_z}{(x - 1)^2} \right) \, dx = \psi_0.
\]
Now suppose that $\nu^*(\phi_z) = 1$, i.e. there is no bounded uniformly continuous function $\psi \in D$ such that $L(\epsilon)\psi = \phi_z$. Clearly $(L(\epsilon) + S)^{-1}$ exists if and only if $N((L(\epsilon) + S)) = \{0\}$ and $(L(\epsilon) + S)\psi = 0$ if and only if $L(\epsilon)\psi = -\nu^*(\psi)\phi_z$, where $\psi \in D$. From the last equation we see $\nu^*(\psi) = 0$, so $L(\epsilon)\psi = 0$, which implies $\psi = \alpha \phi_z$. The assumption $\nu^*(\phi_z) = 1$ yields $\alpha = \nu^*(\alpha \phi_z) = \nu^*(\psi) = 0$. Therefore we have $\psi = 0$ and $(L(\epsilon) + S)^{-1}$ exists. Below we want to use $(zI - L(\epsilon) - S)^{-1}$ and $S$ to represent $(zI - L(\epsilon))^{-1}$. For all $\psi \in D$ and for all $z \in C$, we have

$$S(zI - L(\epsilon))\psi = (zI - L(\epsilon))S\psi = zS\psi.$$  

Thus for all nonzero complex number $z \in \rho(L(\epsilon))$,

$$S(zI - L(\epsilon))^{-1} = (zI - L(\epsilon))^{-1}S = S/z.$$  

By this relationship, we can deduce

$$(zI - L(\epsilon) - S) \left[ (zI - L(\epsilon))^{-1} - \frac{S}{z(1 - z)} \right]$$

$$= \left[ (zI - L(\epsilon))^{-1} - \frac{S}{z(1 - z)} \right] (zI - L(\epsilon) - S) = I.$$  

Hence for all $z \in \rho(L(\epsilon))$ with $z \neq 0, 1$, we obtain

$$(zI - L(\epsilon))^{-1} = (zI - L(\epsilon) - S)^{-1} + \frac{S}{z(1 - z)}.$$  

Obviously we have

$$L(\epsilon) + S = T(I + T^{-1}P + T^{-1}S),$$

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$T^{-1}P + T^{-1}S$ is compact, and $I + T^{-1}P + T^{-1}S$ is 1-1, thus $I + T^{-1}P + T^{-1}S$ has a bounded linear inverse. Therefore $L(\varepsilon) + S$ has a bounded linear inverse. For all sufficiently small $|x|$, we have
\[
(xI - L(\varepsilon) - S)^{-1} = -\sum_{n=0}^{\infty} x^n(L(\varepsilon) + S)^{-1}.
\]
Thus
\[
\frac{1}{2\pi i} \int_{\gamma_0} (xI - L(\varepsilon))^{-1} dx = \frac{1}{2\pi i} \int_{\gamma_0} (xI - L(\varepsilon) - S)^{-1} + \frac{S}{x(1-x)} dx = \frac{1}{2\pi i} \int_{\gamma_0} \sum_{n=0}^{\infty} x^n(L(\varepsilon) + S)^{-1} dx + \frac{1}{2\pi i} \int_{\gamma_0} \frac{S}{x(1-x)} dx = S.
\]
For all $\psi \in X$ and for all $t \geq 0$, we have
\[
S\Lambda_t \psi = \nu^*(\Lambda_t \psi) \phi_z = (\Lambda_t \nu^*)(\psi) \phi_z = \nu^*(\psi) \phi_z = S\psi.
\]
and
\[
\Lambda_t S\psi = \Lambda_t (\nu^*(\psi) \phi_z) = \nu^*(\psi) \Lambda_t \phi_z = \nu^*(\psi) \phi_z = S\psi,
\]
hence $S\Lambda_t = \Lambda_t S = S$. As before, $(I-\Lambda_t + S)^{-1}$ exists if and only if $N(I-\Lambda_t + S) = \{0\}$ and $(I-\Lambda_t + S)\psi = 0$ if and only if $(I-\Lambda_t)\psi = -S\psi$. The equations $S(I-\Lambda_t) = 0$ and $S^2 = S$ yield $S\psi = -S(I-\Lambda_t)\psi = 0$, which implies $(I-\Lambda_t)\psi = 0$. Since $\dim N_{1,t} = 1$, there is a constant $\alpha$, such that $\psi = \alpha \phi_z$. As before, $\alpha \phi_z = S\psi = 0$ implies $\alpha = 0$. Hence $\psi = 0$. Therefore $(I - \Lambda_t + S)^{-1}$ exists. Now we want to use $(zI - \Lambda_t + S)^{-1}$ and $S$ to represent $(zI - \Lambda_t)^{-1}$. For all $\psi \in X$ and for all $z \in C$, we have
\[
S(zI - \Lambda_t) = (zI - \Lambda_t)S = (z-1)S.
\]
It follows that for all \( I \neq z \in \rho(\Lambda_t) \), one has

\[
S(zI - \Lambda_t)^{-1} = (zI - \Lambda_t)^{-1}S = \frac{S}{z - 1}.
\]

Then

\[
(zI - \Lambda_t + S) \left[ (zI - \Lambda_t)^{-1} + \frac{S}{z(1 - z)} \right] = (zI - \Lambda_t + S)(zI - \Lambda_t + S) = I.
\]

Finally for all \( 0, 1 \neq z \in \rho(\Lambda_t) \),

\[
(zI - \Lambda_t)^{-1} = (zI - \Lambda_t + S)^{-1} - \frac{S}{z(1 - z)}.
\]

For all \( t \) with \( e^{\alpha t} > M \), we have

\[
I - \Lambda_t + S = I - \Gamma_t + S - K_t = (I - \Gamma_t)(I + (I - \Gamma_t)^{-1}(S - K_t)),
\]

\((I - \Gamma_t)^{-1}(S - K_t)\) is a compact operator, and \( I + (I - \Gamma_t)^{-1}(S - K_t) \) is \(1\)-\(1\), thus \( I + (I - \Gamma_t)^{-1}(S - K_t) \) has a bounded linear inverse. Therefore \( I - \Lambda_t + S \) has a bounded linear inverse. For all sufficiently small \(|1 - x|\), we have

\[
(xI - \Lambda_t + S)^{-1} = \sum_{n=0}^{\infty} (1 - x)^n (I - \Lambda_t + S)^{-n-1}.
\]

Now we obtain

\[
\frac{1}{2\pi i} \int_{\Gamma_1} (xI - \Lambda_t)^{-1} dx = \frac{1}{2\pi i} \int_{\Gamma_1} \sum_{n=0}^{\infty} (1 - x)^n (I - \Lambda_t + S)^{-n-1} dx - \frac{1}{2\pi i} \int_{\Gamma_1} \frac{S}{x(1 - x)} dx = S.
\]

(2) If system (1) is exponentially stable at \( \phi_z \), then \( \nu^*(\phi_z) = 1 \). System (1) is stable at \( \phi_z \) implies that \( \text{Re}\lambda < 0 \), for all \( \lambda \in \sigma(L(\varepsilon)) \cap \Omega \). By Lemma 6.6,
the number of eigenvalues \( \lambda \in \sigma(L(\varepsilon)) \) with \( \text{Re}\lambda > -\alpha/2 > -\alpha \) is at most finite. Therefore \( \max\{\text{Re}\lambda : \lambda \in \sigma(L(\varepsilon)) - \{0\}\} < 0 \). On the other hand, let \( \nu^*(\phi_z) = 1 \) and \( \max\{\text{Re}\lambda : \lambda \in \sigma(L(\varepsilon)) - \{0\}\} \leq -\sigma < 0 \). Then for all fixed \( t > 0 \) and for all \( \lambda \in \sigma(\Lambda_t) \), either \( |\lambda| \leq e^{-\alpha t} \leq \exp(-\sigma t) \), or \( |\lambda| > e^{-\alpha t} \). By Lemma 6.5, there is an eigenvalue \( \mu \in \sigma(L(\varepsilon)) \cap \Omega \), such that \( \lambda = e^{\mu t} \). By assumption \( \text{Re}\mu < -\sigma \). Thus \( |\lambda| \leq \exp(-\sigma t) \). Therefore for all \( t > 0 \), except for the eigenvalue \( \lambda = 1 \), the spectrum \( \sigma(\Lambda_t) \) is surrounded by a circle centered at origin with radius \( \exp(-\sigma t) \), where \( \sigma > 0 \). Let \( \Gamma_0 \) be the circle with radius \( \exp(-\sigma t) \) about the origin and \( \Gamma_1 \) be a sufficiently small circle about the point \( \lambda = 1 \). Outside \( \Gamma_0 \cup \Gamma_1 \), we have the estimate 
\[
\| (zI - \Lambda_t + S)^{-1} \| \leq C(z) < \infty.
\]
For all natural number \( m \geq 1 \).

\[
\Lambda_m = \Lambda_t^m = \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} z^m (zI - \Lambda_1)^{-1} dz
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} z^m \left[ (zI - \Lambda_1 + S)^{-1} - \frac{S}{z(1 - z)} \right] dz
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} z^m (zI - \Lambda_1 + S)^{-1} dz + \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} \frac{z^{m-1}}{z - 1} S dz
\]
\[
= \frac{1}{2\pi} \int_{|z| = e^{-\sigma}} z^m (zI - \Lambda_1 + S)^{-1} dz + S = U_m + S,
\]
\[
\leq \frac{1}{2\pi} \int_{|z| = e^{-\sigma}} \| z^m (zI - \Lambda_1 + S)^{-1} \| |dz|
\]
\[
\leq \frac{e^{-m\sigma}}{2\pi} \int_{|z| = e^{-\sigma}} \| (zI - \Lambda_1 + S)^{-1} \| |dz|
\]
\[
\leq \frac{Me^{-m\sigma}}{2\pi}.
\]

For all \( t > 0 \), let \( m = \lfloor t \rfloor \) and \( r = t - m \). Then \( t = r + m \) and

\[
\Lambda_t = \Lambda_r \Lambda_m = \Lambda_r (U_m + S) = \Lambda_r U_m + S,
\]

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so that
\[ \| \lambda_t - S \| = \| A_r U_m \| \leq \| A_r \| \| U_m \| \leq M e^{L_r + \sigma t} \exp(-\sigma t). \]

Moreover, \( |\nu^*(\psi)| \leq \| \psi \|_\infty \| \nu \|_{L^1}. \) Theorem C is proved.

The Evans function is a powerful tool in analyzing stability of traveling waves. It is useful because it is complex analytic and it is real valued on the real line. Moreover the eigenvalues of the linearized operator about the traveling wave corresponds to the zeros of the Evans function and the algebraic multiplicity of an eigenvalue is equal to the order of the zero of the Evans function. To study the stability of traveling wave solutions of some equations, one rewrites the differential equations in moving coordinate and linearizes this equation about the traveling. Thus one can define a linear operator. The Evans function may depend on additional parameters, for example for singularly perturbed problems it also depend on the singular perturbation parameter. For most reaction-diffusion equations, the Evans function is zero at \( \lambda = 0. \) There are a few known methods to define the Evans function. Evans himself (1975) is the first to give the rigorous definition of such a function. It is defined as the scalar product of two complex analytic functions, both are candidates of eigenfunctions of the operator. In 1990, Alexander, Gardner and Jones defined the Evans function for a semilinear parabolic system, by using the determinant of some “fundamental” matrix. Here each column of the matrix is a candidate of eigenfunction of the operator. In this paper, because the interested equations involve a nonlocal term, we define the Evans function in a way different from that of Evans and Alexander et al. We also use a determinant to define the Evans function. But this is quite different from the way Alexander, Gardner and Jones did. Each column or row does not corresponds
to a possible eigenfunction. In general it is very difficult to find the explicit Evans functions. Even for a scalar reaction-diffusion equation, such as the bistable equation, the explicit Evans function is extremely hard to find. Up to now only the Korteweg-de Vries equation possesses an “almost” explicit Evans function. In our paper we expose where and how the eigenvalue parameter appears in the determinant. In this sense one could say the Evans function is semi-explicit.

3. Sketch of Proof

To solve the stability problem of the traveling wave solution, one first rewrites the system of integral-differential equations in moving coordinates. Since the linear stability implies the nonlinear stability, one then linearizes the new equation about the traveling wave solution and then finds the eigenvalue problem by seeking for special solutions of the linearized system. One can rewrite the eigenvalue problem as a first order linear system and find the asymptotic system. Then one finds the eigenvalues of the matrix in the asymptotic system and then defines a complex analytic function of an intermediate system. One then uses the method of variation of parameter to find the solutions of the original eigenvalue system involving the nonlocal term. Meanwhile we can define a complex analytic function, called the Evans function, whose zeroes coincide with eigenvalues of the operator $L(\epsilon)$. By the asymptotic behavior of the Evans function one shows that there is no eigenvalue of the linear differential operator outside a large circle whose radius is independent of $\epsilon$. By using the particular form of the Evans function one demonstrates that any complex number with
Re$\lambda > 0$ is not an eigenvalue. Also by some delicate analysis of the Evans function one shows that there is no eigenvalue except for the neutral one $\lambda = 0$ on the imaginary axis. Our method is different from Jones, because by constructing an positively invariant subset, he demonstrates that there is no eigenvalue inside a uniform curve and outside a sufficiently small circle, whose radius is of order $\varepsilon$. Then by the stability result of the front and the back and by homotopy argument he finds there are exactly two eigenvalues inside the small circle. Then by the transversality of intersection of singular unstable manifold and singular stable manifold and also by the exchange lemma, he asserts the derivative of the Evans function at zero is positive. Thus the critical eigenvalue is real and negative. Let us compare the equations. Our equation has no diffusion but involves a nonlocal term, which makes the traveling wave equation nonautonomous. The Fitzhugh-Nagumo equations have a second order diffusion term in the first equation, but involves no nonlocal term. Moreover the first equation enjoys the maximum principle when $w$ is fixed.

In this chapter we use $C, K$ or $M$ to denote various positive constants; they may be different from line to line and may depend on some parameters. Furthermore, $\lambda$ represents the eigenvalue parameter, $\varepsilon$ represents the singular perturbation parameter, $\psi$ represents candidates of eigenfunctions, $A^T$ represents the transposed matrix or vector of $A$. Other notations are as usual.

**Lemma 12.** Suppose that $\phi$ is a wave front solution to a scalar equation or a traveling pulse solution to a singularly perturbed system of integral-differential equations, such that there are finitely many (say $N$) points $z_i$, such that $\phi(z_i) = \theta$ and $\phi'(z_i) \neq 0$, i.e. the wave solution passes through the threshold for $N$ times. Then
the “linearization” of the integral

\[ \int_{-\infty}^{\infty} K(z - y)H(u(y, t) - \theta)dy \]

about another integral

\[ \int_{-\infty}^{\infty} K(z - y)H(\phi(y) - \theta)dy \]

is given explicitly by

\[ \sum_{i=1}^{N} \frac{1}{|\phi'(z_i)|} K(z - z_i)[u(z_i, t) - \phi(z_i)]. \]

Proof. By Lebesgue’s dominated convergence theorem, we get

\[
\int_{-\infty}^{\infty} K(z - y)H(u(y, t) - \theta)dy - \int_{-\infty}^{\infty} K(z - y)H(\phi(y) - \theta)dy
\]

\[
= \lim_{\varepsilon \to 0} \left[ \int_{-\infty}^{\infty} K(z - y)H(\varepsilon, u(y, t) - \theta)dy - \int_{-\infty}^{\infty} K(z - y)H(\varepsilon, \phi(y) - \theta)dy \right]
\]

\[
= \lim_{\varepsilon \to 0} \left[ \int_{-\infty}^{\infty} K(z - y)H(\varepsilon, \phi(y) - \theta)(u(y, t) - \phi(y))dy \right]
\]

\[
= \lim_{\varepsilon \to 0} \left[ \frac{1}{2\varepsilon} \int_{y: |\phi(y) - \theta| < \varepsilon} K(z - y)(u(y, t) - \phi(y))dy \right]
\]

\[
= \lim_{\varepsilon \to 0} \left[ \frac{1}{2\varepsilon} \sum_{i=1}^{N} \int_{y: |y - z_i| < C_i \varepsilon} K(z - y)(u(y, t) - \phi(y))dy \right]
\]

\[
= \sum_{i=1}^{N} \frac{1}{|\phi'(z_i)|} K(z - z_i)[u(z_i, t) - \phi(z_i)].
\]

4. Scalar Equations

To study the stability of a homoclinic solution, it is necessary to talk about the stability of the front and the back of the homoclinic solution. We will first consider the scalar integral differential equation

\[ u_t = f(u) + \alpha \int_{-\infty}^{\infty} K(x - y)H(u(y, t) - \theta)dy. \]
In moving coordinate $z = x + vt$, $U(z, t) = u(z - vt, t)$, this equation can be written as

$$U_t + vU_z = f(U) + \alpha \int_{-\infty}^{\infty} K(z - y) H(U(y, t) - \theta) dy.$$ 

The traveling wave solution (wave front) is a stationary solution of this equation. To study the stability of the traveling wave, we linearize the scalar equation about the wave

$$U_t + vU_z = f'(\phi)U + \frac{\alpha}{\phi'(0)} K(z)U(0, t).$$

The global solutions of the initial value problems for this linear equation exist and are tangent to the traveling wave. Of particular importance are bounded smooth solutions defined on $R$, for each fixed $t$. Thus one looks for solutions of the form $e^{\lambda t} \psi(z)$. This leads to the following eigenvalue problem $\lambda \psi + v\psi_z = f'(\phi)\psi + \frac{\alpha}{\phi'(0)} K(z)\psi(0)$. To investigate the eigenvalues and eigenvectors we define a bounded linear differential operator $L : L\psi = -v\psi_z + f'(\phi)\psi + \frac{\alpha}{\phi'(0)} K(z)\psi(0)$. Differentiating the traveling wave equation with respect to $z$ gives $v\phi_{zz} = f'(\phi)\phi_z + \alpha K(z)$. Thus $\lambda = 0$ is an eigenvalue of the operator $L(\varepsilon)$. It would be very convenient to discuss the eigenvalues and eigenvectors of the auxiliary operator $L_0 : L_0\psi = -v\psi_z + f'(\phi)\psi$. The essential spectrum of this operator consists of two vertical lines $\lambda = -a + iv\xi$ and $\lambda = f'(\beta) + iv\xi$, where $\xi \in R$. Define $\Omega = \{\lambda : \text{Re}\lambda > -a\}$. This is an unbounded, open, simply connected subset in the complex plane. We also consider the differential equations associated with the eigenvalue problem of the auxiliary operator

$$v\psi_z + [\lambda - f'(\phi)]\psi = 0, \quad v\psi_z = [\lambda - f'(\phi)]\psi.$$
The asymptotic equation of $v \psi_z + [\lambda - f'(\phi)]\psi = 0$ as $z \to +\infty$ is

$$v \psi_z + [\lambda - f'(\beta)]\psi = 0.$$  

Obviously $\exp[-\frac{1}{v}(\lambda - f'(\beta))z]$ is a complex analytic solution of the asymptotic equation. There exists a unique, complex analytic solution $\psi_1(\lambda, z)$ to the equation $v \psi_z + [\lambda - f'(\phi)]\psi = 0$ and a positive constant $\delta(\lambda)$, such that as $z \to +\infty$, we have

$$\psi_1(\lambda, z) - \exp[-\frac{1}{v}(\lambda - f'(\beta))z] = \exp[-\frac{1}{v}(\lambda - f'(\beta) + \delta(\lambda))z]O(1).$$

Rewriting the equation $v \psi_z + [\lambda - f'(\phi)]\psi = 0$ gives

$$\frac{\partial}{\partial z} \left\{ \exp[-\frac{1}{v}(\lambda - f'(\beta))z] \psi(\lambda, z) \right\} = \frac{1}{v} \exp[-\frac{1}{v}(\lambda - f'(\beta))z] [f'(\phi(z)) - f'(\beta)] \psi(\lambda, z).$$

Integrating in $z$ over $(z, \infty)$ and using the asymptotic behavior of $\psi_1$ as $z \to \infty$ yields

$$1 - \exp[-\frac{1}{v}(\lambda - f'(\beta))z] \psi(\lambda, z) = \frac{1}{v} \int_z^\infty \exp[-\frac{1}{v}(\lambda - f'(\beta))r] [f'(\phi(r)) - f'(\beta)] \psi(\lambda, r) dr.$$

Setting

$$g(\lambda, z) = \exp[-\frac{1}{v}(\lambda - f'(\beta))z] \psi(\lambda, z).$$

Then

$$1 - g(\lambda, z) = \frac{1}{v} \int_z^\infty [f'(\phi(r)) - f'(\beta)] g(\lambda, r) dr.$$

Therefore

$$g(\lambda, z) = \exp[-\frac{1}{v} \int_z^\infty f'(\beta) - f'(\phi(r)) dr],$$

and

$$\psi(\lambda, z) = \exp[-\frac{1}{v}(\lambda - f'(\beta))z] \exp[-\frac{1}{v} \int_z^\infty f'(\beta) - f'(\phi(r)) dr].$$
Explicitly this solution satisfies

\[ \psi_1(\lambda, z) = \exp[-\frac{\lambda z}{v}]\psi_1(0, z). \]

Similarly the asymptotic equation of \( v\psi_z = [\lambda - f'(\phi)]\psi \) as \( z \to -\infty \) is

\[ v\psi_z = (\lambda + a)\psi. \]

As before \( \exp[\frac{1}{v}(\lambda + a)z] \) is an analytic solution of this asymptotic equation.

Similarly there exists a unique, complex analytic solution \( \psi_2(\lambda, z) \) to the equation

\( v\psi_z = [\lambda - f'(\phi)]\psi \) and a positive constant \( \delta(\lambda) \), such that as \( z \to -\infty \), we have

\[ \psi_2(\lambda, z) - \exp[\frac{1}{v}(\lambda + a)z] = \exp[\frac{1}{v}(\lambda + a + \delta(\lambda))z]O(1). \]

Solving the above equation and using the asymptotic behavior as \( z \to -\infty \) yields

\[ \psi(\lambda, z) = \exp[\frac{1}{v}(\lambda + a)z] \exp[-\frac{1}{v} \int_{-\infty}^{z} a + f'(\phi(r))dr]. \]

Explicitly this solution satisfies

\[ \psi_2(\lambda, z) = \exp[\frac{\lambda z}{v}]\psi_2(0, z). \]

The intermediate complex analytic Evans function \( D(\lambda) \) is defined to be the scalar product of these functions

\[ D(\lambda) = \psi_1(\lambda, z)\psi_2(\lambda, z) = \psi_1(0, z)\psi_2(0, z) \]

\[ = \exp[\frac{1}{v}(a + f'(\beta))z] \exp[-\frac{1}{v} \int_{-\infty}^{z} a + f'(\phi(r))dr + \frac{1}{v} \int_{z}^{\infty} f'(\beta) - f'(\phi(r))dr] \neq 0, \]

which is independent of \( z \in \mathbb{R} \) and \( \lambda \in \Omega \). This implies that there exists no eigenvalue of \( L_0 \) in the region \( \Omega \). See Evans [E4]. This also implies that any
nontrivial solution of the original eigenvalue problem can not vanish at \( z = 0 \).

Now suppose that \( \psi(\lambda, z) = c(\lambda, z)\psi_1(\lambda, z) \) is a solution of the eigenvalue problem
\[
L(\varepsilon)\psi = -\nu \psi_z + f'(\phi)\psi + \frac{\partial}{\partial \phi(0)} K(z)\psi(0) = \lambda\psi.
\]
Then \( c \) solves the linear equation
\[
u\psi_1(\lambda, z)\frac{\partial c}{\partial z} = \frac{\partial}{\partial \phi(0)} K(z)\psi(\lambda, 0).
\]
By using the Evans function \( D(\lambda) \), we see
\[
\frac{\partial \psi}{\partial z}(\lambda, z) = \frac{\alpha}{vD(\lambda)\phi'(0)} K(z)\psi_2(\lambda, z)\psi(\lambda, 0).
\]
Therefore we have the solution by integrating about \( z \)
\[
c(\lambda, z) = c(\lambda) + \frac{\alpha}{vD(\lambda)\phi'(0)} \int_{-\infty}^{\infty} K(r)\psi_2(\lambda, r)d\psi(\lambda, 0).
\]

Hence the solution of the original eigenvalue problem is given by
\[
\psi(\lambda, z) = c(\lambda)\psi_1(\lambda, z) + \frac{\alpha}{vD(\lambda)\phi'(0)} \int_{-\infty}^{\infty} K(r)\psi_2(\lambda, r)d\psi(\lambda, 0)\psi_1(\lambda, z).
\]

Here \( c(\lambda) \) is to be determined by the compatibility condition of \( \psi(\lambda, 0) \) at \( z = 0 \).

The compatibility condition
\[
\psi(\lambda, 0) = c(\lambda)\psi_1(\lambda, 0) + \frac{\alpha}{vD(\lambda)\phi'(0)} \int_{-\infty}^{0} K(z)\psi_2(\lambda, z)dz\psi(\lambda, 0)\psi_1(\lambda, 0),
\]
implies that
\[
c(\lambda) = \frac{\psi(\lambda, 0)}{\psi_1(\lambda, 0)} \left[ 1 - \frac{\alpha}{vD(\lambda)\phi'(0)} \int_{-\infty}^{0} K(z)\psi_2(\lambda, z)dz\psi_1(\lambda, 0) \right].
\]

It is not hard to convince oneself that
\[
\lim_{z \to \pm\infty} \int_{-\infty}^{z} K(r)\psi_2(\lambda, r)d\psi_1(\lambda, z) = 0,
\]
\[
\lim_{z \to \pm\infty} c(\lambda)\psi_1(\lambda, z) = 0,
\]
\[
\lim_{z \to -\infty} ||c(\lambda)\psi_1(\lambda, z)|| = +\infty,
\]

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if \( c(\lambda) \neq 0 \). Therefore \( \lambda \in \Omega \) is an eigenvalue of \( L(\varepsilon) \) if and only if \( c(\lambda) = 0 \). Recall that \( \lambda = 0 \) is an eigenvalue of the operator \( L(\varepsilon) \). So \( c(0) = 0 \), which is equivalent to

\[
\frac{\alpha}{vD(0)\phi'(0)} \int_{-\infty}^{0} K(z)\psi_2(0, z)dz\psi_1(0, 0) = 1.
\]

We now define a complex analytic function in \( \Omega \), called the Evans function,

\[
E(\lambda) = 1 - \frac{\alpha}{vD(\lambda)\phi'(0)} \int_{-\infty}^{0} K(z)\psi_2(\lambda, z)dz\psi_1(\lambda, 0).
\]

It is not hard to see \( E(\lambda) = 0 \) if and only if \( \lambda \) is an eigenvalue of the operator \( L(\varepsilon) \). For all \( \text{Re} \lambda \geq 0 \) but \( \lambda = 0 \), we have

\[
|\int_{-\infty}^{0} K(z)\psi_2(\lambda, z)dz| < \int_{-\infty}^{0} K(z)\psi_2(0, z)dz.
\]

Hence

\[
|E(\lambda)| > 1 - \frac{\alpha}{vD(\lambda)\phi'(0)} \int_{-\infty}^{0} K(z)\psi_2(0, z)dz\psi_1(0, 0) = 0.
\]

So \( E(\lambda) \neq 0 \) for all \( \text{Re} \lambda \geq 0 \) except for \( \lambda = 0 \). Indeed by the dominated convergence theorem we see as \( |\lambda| \to \infty \), the Evans function \( E(\lambda) \to 1 \). Thus there exists a positive constant \( M \), such that every complex number \( \lambda \in \Omega \) with \( |\lambda| > M \) is not an eigenvalue of \( L(\varepsilon) \). For all \( \lambda \in \Omega \) with \( |\lambda| \leq M \), by integration by parts, we have

\[
\int_{-\infty}^{0} K(z)\exp[-\frac{\lambda z}{v}]\psi_2(0, z)dz = \int_{-\infty}^{0} \exp[-\frac{\lambda z}{v}] \frac{\partial}{\partial z} \left( \int_{-\infty}^{z} K(r)\psi_2(0, r)dr \right) dz
\]

\[
= \int_{-\infty}^{0} K(z)\psi_2(0, z)dz - \frac{\lambda}{v} \int_{-\infty}^{0} \left( \int_{-\infty}^{z} K(r)\psi_2(0, r)dr \right) \exp[-\frac{\lambda z}{v}]dz.
\]

Notice that \( D(\lambda) = D(0) \neq 0 \). It is easy to see \( E(\lambda) \neq 0 \) if \( \lambda \in \Omega \) and \( \lambda \neq 0 \). The transversality of intersection of the stable and unstable manifolds implies that
the neutral eigenvalue $\lambda = 0$ is simple. Another way to show $\lambda = 0$ is simple is to consider the nonhomogeneous differential equation

$$v \frac{\partial \psi}{\partial z} + \phi_z = f'(\phi) \psi + \frac{\alpha}{\phi'(0)} K(z) \psi(0).$$

We use the method of variation of parameter to find the solutions of this equation. Recall that $\psi_1(0, z)$ is the solution of the intermediate equation $v \psi_z = f'(\phi) \psi$. Suppose that $\psi(z) = c(z) \psi_1(0, z)$ is a solution of the nonhomogeneous equation. Then $c$ satisfies the equation

$$v \psi_1(0, z) \frac{\partial c}{\partial z} + \phi_z = \frac{\alpha}{\phi'(0)} K(z) \psi(0).$$

The intermediate Evans function yields

$$\frac{\partial c}{\partial z} = \frac{1}{vD(0)} \left[ \frac{\alpha}{\phi'(0)} K(z) \psi(0) - \phi_z \right] \psi_2(0, z).$$

If we integrate in $z$, we get

$$c(z) = c + \frac{\alpha}{vD(0) \phi'(0)} \int_{-\infty}^{z} K(r) \psi_2(0, r) dr \psi(0) - \frac{1}{vD(0)} \int_{-\infty}^{z} \phi_z(r) \psi_2(0, r) dr.$$

Thus the solution is given by

$$\psi(z) = c \psi_1(0, z) + \frac{\alpha}{vD(0) \phi'(0)} \int_{-\infty}^{z} K(r) \psi_2(0, r) dr \psi(0) \psi_1(0, z) - \frac{1}{vD(0)} \int_{-\infty}^{z} \phi_z(r) \psi_2(0, r) dr \psi_1(0, z).$$

The compatibility condition of $\psi(z)$ at $z = 0$ is

$$\psi(0) = c \psi_1(0, 0) + \frac{\alpha}{vD(0) \phi'(0)} \int_{-\infty}^{0} K(z) \psi_2(0, z) dz \psi(0) \psi_1(0, 0) - \frac{1}{vD(0)} \int_{-\infty}^{0} \phi_z(z) \psi_2(0, z) dz \psi_1(0, 0).$$
Collecting terms yields the equation

\[ \psi(0)[1 - \frac{\alpha}{vD(0)\phi'(0)} \int_{-\infty}^{0} K(z)\psi_2(0, z)dz\psi_1(0, 0)] = c\psi_1(0, 0) - \frac{1}{vD(0)} \int_{-\infty}^{0} \phi_{z}(z)\psi_2(0, z)dz\psi_1(0, 0). \]

The left hand side is zero since \( E(0) = 0 \). Since \( \psi_1(0, 0) \neq 0 \), we have

\[ c = \frac{1}{vD(0)} \int_{-\infty}^{0} \phi_{z}(z)\psi_2(0, z)dz \neq 0. \]

This implies that the solution \( \psi(z) = c(z)\psi_1(0, z) \) is unbounded as \( z \to -\infty \). Thus there exists no bounded continuous solution to the equation (1) on \( R \). Hence the traveling wave solution is exponentially stable.

5. A System of Integral-Differential Equations

The development of this section is parallel to the previous one but is much more complicated and difficult. Consider the system of integral-differential equations

\[ u_t = f(u) - w + \alpha \int_{-\infty}^{\infty} K(x - y)H(u(y, t) - \theta)dy, \]
\[ w_t = \varepsilon(u - \gamma w), \]

where \( f(u) = u(1 - u)(u - a) \) is a cubic function, \( 0 < 2a < 1, \alpha > 0, \rho_{-}(a) < \theta < \rho_{+}(a), \rho_{\pm}(a) = \frac{1}{3}[1 + a \pm \sqrt{1 - a + a^2}], 0 < \varepsilon \ll 1, 0 < \gamma < 4/(1 - a)^2, (u, w) = (u(x, t), w(x, t)) \in R^2 \) is a vector-valued function of the spatial variable \( x \) and the temporal variable \( t \).

To consider the stability of the traveling wave solutions, it is natural to rewrite
the system in moving coordinate. Let \( z = x + v(\varepsilon)t \) and let \((U(z, t), W(z, t)) = (u(z - v(\varepsilon)t, t), w(z - v(\varepsilon)t, t))\), we have

\[
U_t + v(\varepsilon)U_z = f(U) - W + \alpha \int_{-\infty}^{\infty} K(z - y)H(U(y, t) - \theta)dy,
\]

\[
W_t + v(\varepsilon)W_z = \varepsilon(U - \gamma W).
\]

Obviously the traveling wave solution is a time-independent solution of this system. To consider the linear stability, one needs to linearize the system about the wave solution

\[
U_t + v(\varepsilon)U_z = f'(\phi)U - W + \frac{\alpha}{\phi'(0)} K(z)U(0, t) - \frac{\alpha}{\phi'(z_0)} K(z - z_0)U(z_0, t),
\]

\[
W_t + v(\varepsilon)W_z = \varepsilon(U - \gamma W).
\]

The global solutions of the initial value problems for the linear and nonlinear systems exist. In addition the solutions of the linear system are tangent to the underlying traveling wave solutions. In another word, solutions of this system lie in the tangent space of the wave solution. Seeking for particular solutions of the form \((U(z, t), W(z, t)) = e^{\lambda t} (\psi^1(z), \psi^2(z))\), where \( \lambda \) is a complex number, leads to the eigenvalue problem

\[
\lambda \psi^1 + v(\varepsilon)\psi^1_z = f'(\phi)\psi^1 - \psi^2
\]

\[+\frac{\alpha}{\phi'(0)} K(z)\psi^1(\lambda, \varepsilon, 0) - \frac{\alpha}{\phi'(z_0)} K(z - z_0)\psi^1(\lambda, \varepsilon, z_0),
\]

\[
\lambda \psi^2 + v(\varepsilon)\psi^2_z = \varepsilon(\psi^1 - \gamma \psi^2).
\]

Define \( BC^k(R, C^2) = \{ \psi : \psi \) is a complex vector-valued function defined on \( R \), such that all its derivatives up to order \( k \) are bounded and uniformly continuous
on $\mathbb{R}$. To study the eigenvalues and eigenfunctions, we define a bounded linear differential operator $L(\varepsilon) : BC^1(\mathbb{R}, \mathbb{R}^2) \rightarrow BC^0(\mathbb{R}, \mathbb{R}^2)$ by

$$L(\varepsilon) \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} -\nu(\varepsilon)\psi_z^1 + f'(\phi)\psi^1 - \psi^2 + N \\ -\nu(\varepsilon)\psi_z^2 + \varepsilon(\psi^1 - \gamma\psi^2) \end{pmatrix},$$

where

$$f(z) = \frac{\alpha}{\phi'(0)}K(z) > 0, \quad g(z) = -\frac{\alpha}{\phi'(z_0)}K(z - z_0) > 0,$$

$$N = f(z)\psi^1(\lambda, \varepsilon, 0) + g(z)\psi^1(\lambda, \varepsilon, z_0).$$

We are abusing our notation on $f$, but it should be easy to distinguish the function $f(z)$ from the cubic function $f(u)$. The eigenvalue problem can then be written in a compact form $L(\varepsilon)\psi = \lambda\psi$. If $\lambda \in \Omega$ is an eigenvalue of $L(\varepsilon)$ associated with the eigenfunction $\psi(\lambda, \varepsilon, z)$, then $\lambda$ is also an eigenvalue associated with the eigenfunction $\overline{\psi}(\lambda, \varepsilon, z)$. We will show that if $\lambda$ is an eigenvalue then it must be real, since complex eigenvalues appear in conjugate pairs. The stability of the traveling wave solutions is determined completely by the structure of the spectrum of the operator $L(\varepsilon)$. The spectrum is divided into two parts: the essential spectrum consists of distinct smooth curves and the normal spectrum consists of a few isolated eigenvalues with finite algebraic multiplicities. The essential spectrum consists of those complex number $\lambda$ such that certain matrix $A(\lambda, \varepsilon)$ defined below has a purely imaginary eigenvalue $\im \xi$. To guarantee the resting state $(0,0)$ is stable, the essential spectrum must be in the left half plane and be uniformly bounded away from the imaginary axis. For our
problem the essential spectrum of the operator $L(\varepsilon)$ consists of two vertical lines and is given by

$$\left\{ iv(\varepsilon)\xi - \frac{a + \varepsilon \gamma \pm \sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}}{2} : \xi \in \mathbb{R} \right\}.$$ 

We now define an open, unbounded, simply connected subset

$$\Omega = \left\{ \lambda : \text{Re}\lambda > -\frac{a + \varepsilon \gamma - \sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}}{2} \right\}.$$ 

Then inside this region $\Omega$, all the spectrum of $L(\varepsilon)$ are eigenvalues with finite algebraic multiplicity. To see the normal spectrum is not an empty set, differentiating the traveling wave equation to get

$$v(\varepsilon)\phi_{zz} = f'(\phi)\phi_z - \phi_z + \alpha[K(z) - K(z - z_0)],$$

$$v(\varepsilon)\varphi_{zz} = \varepsilon(\phi_z - \gamma \varphi_z).$$

This implies that $\lambda = 0$ is an eigenvalue inside $\Omega$ of $L(\varepsilon)$. If this neutral eigenvalue is simple and if there exists no other eigenvalue inside the region $\text{Re}\lambda \geq 0$ of $L(\varepsilon)$, then the wave solution is exponentially stable. We will define and use some Evans function to achieve this.

Rewrite the eigenvalue problem as a first order linear system of differential equations

$$v(\varepsilon) \frac{\partial}{\partial z} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} + \begin{pmatrix} \lambda - f'(\phi) & 1 \\ -\varepsilon & \lambda + \varepsilon \gamma \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} \mathcal{N} \\ 0 \end{pmatrix}.$$ (3.10)

We will also consider the following homogeneous auxiliary system

$$v(\varepsilon) \frac{\partial}{\partial z} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} + \begin{pmatrix} \lambda - f'(\phi) & 1 \\ -\varepsilon & \lambda + \varepsilon \gamma \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = 0.$$ (3.11)
The asymptotic system of both (3.10) and (3.11), as \(|z| \to \infty\), is

\[
\psi(z) \frac{\partial}{\partial z} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} + \begin{pmatrix} \lambda + a & 1 \\ -\varepsilon & \lambda + \varepsilon \gamma \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = 0.
\] (3.12)

The eigenvalues of the matrix \(A(\lambda, \varepsilon) \equiv \begin{pmatrix} \lambda + a & 1 \\ -\varepsilon & \lambda + \varepsilon \gamma \end{pmatrix}\) are

\[
\mu_{\pm}(\lambda, \varepsilon) = \lambda + a + \varepsilon \gamma \pm \frac{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}}{2}.
\]

Define

\[
\omega_{\pm}(\varepsilon) = \frac{a + \varepsilon \gamma \pm \sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}}{2}.
\]

Then \(\mu_{\pm}(\lambda, \varepsilon) = \lambda + \omega_{\pm}(\varepsilon)\) and we have the estimates \(a - \frac{2\varepsilon}{a} < \omega_{+}(\varepsilon) < a\) and \((\gamma + 1/a)\varepsilon < \omega_{-}(\varepsilon) < (\gamma + 2/a)\varepsilon\). Moreover \([\omega_{+}(\varepsilon) - a][\omega_{-}(\varepsilon) - a] = \varepsilon\). Define

\[
T(\varepsilon) = \begin{pmatrix} 1 & 1 \\ \omega_{+}(\varepsilon) - a & \omega_{-}(\varepsilon) - a \end{pmatrix}.
\]

Then

\[
T(\varepsilon)^{-1} = \frac{1}{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \begin{pmatrix} a - \omega_{-}(\varepsilon) & 1 \\ \omega_{+}(\varepsilon) - a & -1 \end{pmatrix},
\]

and

\[
T(\varepsilon)^{-1} A(\lambda, \varepsilon) T(\varepsilon) = \begin{pmatrix} \mu_{+}(\lambda, \varepsilon) & 0 \\ 0 & \mu_{-}(\lambda, \varepsilon) \end{pmatrix},
\]

and then

\[
T(\varepsilon)^{-1} \exp[A(\lambda, \varepsilon)(z - x)] T(\varepsilon) = \begin{pmatrix} \exp[\mu_{+}(\lambda, \varepsilon)(z - x)] & 0 \\ 0 & \exp[\mu_{-}(\lambda, \varepsilon)(z - x)] \end{pmatrix}.
\]
\[ = \exp[\lambda(z - x)] \begin{pmatrix} \exp[\omega_+(\varepsilon)(z - x)] & 0 \\ 0 & \exp[\omega_-(\varepsilon)(z - x)] \end{pmatrix}. \]

The projections \( P_+(\varepsilon) \) and \( P_-(\varepsilon) \) associated with the matrix \( A(\lambda, \varepsilon) \) are

\[
P_+(\varepsilon) = \frac{1}{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \begin{pmatrix} \omega_+(\varepsilon) - \varepsilon \gamma & 1 \\ -\varepsilon & \omega_+(\varepsilon) - a \end{pmatrix},
\]

\[
P_-(\varepsilon) = \frac{1}{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \begin{pmatrix} \varepsilon \gamma - \omega_-(\varepsilon) & -1 \\ \varepsilon & a - \omega_-(\varepsilon) \end{pmatrix}.\]

By these projections we have the expressions

\[
\exp[A(\lambda, \varepsilon)(z - x)] = \exp[\lambda(z - x)] \exp[A(0, \varepsilon)(z - x)],
\]

and

\[
\exp[A(0, \varepsilon)(z - x)] = \exp[\omega_+(\varepsilon)(z - x)]P_+(\varepsilon) + \exp[\omega_-(\varepsilon)(z - x)]P_-(\varepsilon)
\]

\[
= \frac{1}{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},
\]

where

\[
a_{11} = [\omega_+(\varepsilon) - \varepsilon \gamma] \exp[\omega_+(\varepsilon)(z - x)] + [\varepsilon \gamma - \omega_-(\varepsilon)] \exp[\omega_-(\varepsilon)(z - x)],
\]

\[
a_{12} = \exp[\omega_+(\varepsilon)(z - x)] - \exp[\omega_-(\varepsilon)(z - x)],
\]

\[
a_{21} = -\varepsilon \exp[\omega_+(\varepsilon)(z - x)] + \varepsilon \exp[\omega_-(\varepsilon)(z - x)],
\]

\[
a_{22} = [\omega_+(\varepsilon) - a] \exp[\omega_+(\varepsilon)(z - x)] + [a - \omega_-(\varepsilon)] \exp[\omega_-(\varepsilon)(z - x)].
\]
6. The Intermediate Evans Function

In the open, unbounded, simply connected region $\Omega$, the matrix $A(\lambda, \varepsilon)$ has two eigenvalues with positive real parts and no eigenvalue with negative real part. The eigenvectors corresponding to these eigenvalues are $Y_{\pm}(\varepsilon) = (1, \omega_{\pm}(\varepsilon) - a)^T$, independent of $\lambda$. Keep in mind that $\omega_{\pm}(\varepsilon) - a < 0$ and that $\omega_{+}(\varepsilon) - a = O(\varepsilon)$. Clearly $\exp[-\frac{1}{v(\varepsilon)}\mu_{\pm}(\lambda, \varepsilon)z]Y_{\pm}(\varepsilon)$ is an analytic solution of the asymptotic system. Evans [E4] set up a standard argument which shows that there exists a unique complex analytic solution $\psi_i(\lambda, \varepsilon, z)$ to the system (3.11), and there is a positive constant $\delta_i(\lambda, \varepsilon)$, where $i = 1, 2$, such that as $z \to +\infty$,

$$
\psi_i(\lambda, \varepsilon, z) = \exp[-\frac{1}{v(\varepsilon)}\mu_i(\lambda, \varepsilon)z]Y_i(\varepsilon) = \exp[-\frac{1}{v(\varepsilon)}\mu_i(\lambda, \varepsilon)z - \delta_i(\lambda, \varepsilon)z]O(1).
$$

By equation (3.11) we have

$$
v(\varepsilon) \frac{\partial}{\partial z} \left\{ \exp[-\frac{1}{v(\varepsilon)}A(\lambda, \varepsilon)z]\psi(\lambda, \varepsilon, z) \right\}
$$

$$
= [a + f'(\phi(z))] \exp[-\frac{1}{v(\varepsilon)}A(\lambda, \varepsilon)z] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \psi(\lambda, \varepsilon, z).
$$

Integrating in $z$ over the interval $[z, +\infty)$, noticing the asymptotic behavior of $\psi_i(\lambda, \varepsilon, z)$ as $z \to +\infty$, we get

$$
\psi_i(\lambda, \varepsilon, z) = \exp[-\frac{1}{v(\varepsilon)}\mu_i(\lambda, \varepsilon)z]Y_i(\varepsilon)
$$

$$
-\frac{1}{v(\varepsilon)} \int_z^{\infty} [a + f'(\phi(r))] \exp[-\frac{1}{v(\varepsilon)}A(\lambda, \varepsilon)(z - r)] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \psi_i(\lambda, \varepsilon, r) dr.
$$

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Since \( \mu_i(\lambda, \varepsilon) = \lambda + \omega_i(\varepsilon) \), and \( Y_i(\varepsilon) \) is independent of \( \lambda \), by uniqueness we must have the equation \( \psi_i(\lambda, \varepsilon, z) = \psi_i(0, \varepsilon, z) \exp[-\frac{\lambda z}{v(\varepsilon)}] \), for all \( z \in \mathbb{R} \) and \( i = 1, 2 \). On the other hand, direct calculation shows that if \( \psi_i(0, \varepsilon, z) \) is the solution of the equation (3.11) for \( \lambda = 0 \), then \( \psi_i(\lambda, \varepsilon, z) = \psi_i(0, \varepsilon, z) \exp[-\frac{\lambda z}{v(\varepsilon)}] \) is the solution of (3.11), for all \( \lambda \in \Omega \). We will modify \( \psi_2 \) if necessary such that as \( |z| \to +\infty \). we have

\[
\psi_i(\lambda, \varepsilon, z) = \exp[-\frac{1}{v(\varepsilon)} \mu_+(\lambda, \varepsilon) z] Y_+(\varepsilon) O(1),
\]

\[
\psi_2(\lambda, \varepsilon, z) = \exp[-\frac{1}{v(\varepsilon)} \mu_- (\lambda, \varepsilon) z] Y_-(\varepsilon) O(1).
\]

If there is a \( z_1 \in \mathbb{R} \) such that \( \psi^1(\lambda, \varepsilon, z_1) = \psi^2(\lambda, \varepsilon, z_1) = 0 \), then a simple estimate shows \( \psi^1(\lambda, \varepsilon, z) = \psi^2(\lambda, \varepsilon, z) = 0 \), for all \( z \in \mathbb{R} \). This is a trivial solution. So the two components of any nontrivial solution can not vanish simultaneously. If \( \psi^1(\lambda, \varepsilon, z_1) = 0 \) but \( \psi^2(\lambda, \varepsilon, z_1) \neq 0 \), then \( \psi^1(0, \varepsilon, z_1) = 0 \) and \( \psi^2(0, \varepsilon, z_1) \neq 0 \). We will show this is impossible either. In addition there exists no \( z_1 \in \mathbb{R} \), such that \( \psi^1(\lambda, \varepsilon, z_1) \neq 0 \) and \( \psi^2(\lambda, \varepsilon, z_1) = 0 \). Therefore none of the components vanishes at any finite point.

Let \( \beta = \psi^2/\psi^1 \). Then the projectivized equation is given by

\[
v(\varepsilon) \frac{\partial \beta}{\partial z} = \beta^2 - [\varepsilon \gamma + f'(\phi(z))] \beta + \varepsilon.
\]

This equation is independent of \( \lambda \) ! If \( \beta(\lambda, \varepsilon, z_1) = 0 \) for some \( z_1 \in \mathbb{R} \), then \( v(\varepsilon) \beta_z(\lambda, \varepsilon, z_1) = \varepsilon > 0 \). But this contradicts the fact \( \beta(z) \) converges to some negative constant as \( z \to +\infty \). Thus \( \beta(\lambda, \varepsilon, z) < 0 \), for all \( z \in \mathbb{R} \). It is not hard to convince oneself that \( \psi^1(\lambda, \varepsilon, z) \neq 0 \) for all \( z \in \mathbb{R} \). Hence \( \psi^2(\lambda, \varepsilon, z) \neq 0 \), for all \( z \in \mathbb{R} \). Therefore \( \text{Re} \psi^1(\lambda, \varepsilon, z) > 0 \) and \( \text{Re} \psi^2(\lambda, \varepsilon, z) < 0 \), for all \( z \in \mathbb{R} \). It is easy to see if \( z \) is such that
$-2\sqrt{\varepsilon} < \varepsilon \gamma + f'(\phi(z)) < 2\sqrt{\varepsilon}$, then $\beta_2(z) > 0$. If $z$ is such that $\varepsilon \gamma + f'(\phi(z)) > 0$, then $\beta_2(z) \neq 0$, hence $\beta_2(z) > 0$, so in the interval $\{z \in \mathbb{R} : \varepsilon \gamma + f'(\phi(z)) > -2\sqrt{\varepsilon}\}$ there is no maximum or minimum. In another word, if there is a maximum or minimum, it must occur at the interval $\{z \in \mathbb{R} : \varepsilon \gamma + f'(\phi(z)) \leq -2\sqrt{\varepsilon}\}$. By uniqueness, we have $\beta_1(\varepsilon, z) \neq \beta_2(\varepsilon, z)$, for all $z \in \mathbb{R}$, more explicitly,

$$\beta_1(\varepsilon, z) = \frac{\psi_2^2(\lambda, \varepsilon, z)}{\psi_1^2(\lambda, \varepsilon, z)} > \frac{\psi_2^2(\lambda, \varepsilon, z)}{\psi_1^2(\lambda, \varepsilon, z)} = \beta_2(\varepsilon, z),$$

for all $z \in \mathbb{R}$.

Consider the adjoint system of (3.11) and its asymptotic system

$$v(\varepsilon) \frac{\partial}{\partial z} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \lambda - f'(\phi) & -\varepsilon \\ 1 & \lambda + \varepsilon \gamma \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (3.13)$$

$$v(\varepsilon) \frac{\partial}{\partial z} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \lambda + \varepsilon & -\varepsilon \\ 1 & \lambda + \varepsilon \gamma \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (3.14)$$

The eigenvalues of this matrix are the same as $\mu_\pm(\lambda, \varepsilon)$, but the eigenvectors are different: $Z_\pm(\varepsilon) = (\omega_\pm(\varepsilon) - \varepsilon \gamma, 1)^T$, again independent of $\lambda$. Keep in mind that $\omega_\pm(\varepsilon) - \varepsilon \gamma > 0$ and that $\omega_-(\varepsilon) - \varepsilon \gamma = O(\varepsilon)$. As before exp[$-\frac{1}{v(\varepsilon)}\mu_\pm(\lambda, \varepsilon)z]Z_\pm(\varepsilon)$ is an analytic solution of the asymptotic system (5). There exists a unique complex analytic solution $\psi_i(\lambda, \varepsilon, z)$ to the homogeneous system (4), and there is a positive constant $\delta_i(\lambda, \varepsilon)$, where $i = 3, 4$, such that as $z \to -\infty$,

$$\psi_i(\lambda, \varepsilon, z) - \exp[-\frac{1}{v(\varepsilon)}\mu_i(\lambda, \varepsilon)z]Z_i(\varepsilon) = \exp[-\frac{1}{v(\varepsilon)}\mu_i(\lambda, \varepsilon)z + \delta_i(\lambda, \varepsilon)z]O(1).$$

As before, $\mu_i(\lambda, \varepsilon) = \lambda + \omega_i(\varepsilon)$, and $Z_i(\varepsilon)$ is independent of $\lambda$, by uniqueness we must have $\psi_i(\lambda, \varepsilon, z) = \psi_i(0, \varepsilon, z) \exp[-\frac{\Delta z}{v(\varepsilon)}]$, for all $z \in \mathbb{R}$ and $i = 3, 4$. Let us
define a complex analytic function, which we call the intermediate Evans functions, defined by $D_1(\lambda, \varepsilon) = (\psi_1(\lambda, \varepsilon, z), \psi_3(\lambda, \varepsilon, z))$ or $D_2(\lambda, \varepsilon) = (\psi_2(\lambda, \varepsilon, z), \psi_4(\lambda, \varepsilon, z))$. Without loss of generality, let $D_1(\lambda, \varepsilon) = D_2(\lambda, \varepsilon)$, otherwise modify the amplitude of $\psi_2$ if necessary. We define two more complex analytic functions $D_3(\lambda, \varepsilon) = (\psi_1(\lambda, \varepsilon, z), \psi_4(\lambda, \varepsilon, z))$ and $D_4(\lambda, \varepsilon) = (\psi_2(\lambda, \varepsilon, z), \psi_3(\lambda, \varepsilon, z))$. Without loss of generality, assume that $(\psi_1, \psi_4) = 0$ and $(\psi_2, \psi_3) = 0$. Otherwise replace $\psi_2$ and $\psi_4$ by $\psi_2 - \frac{\psi_2(\varepsilon)}{\psi_3(\varepsilon)} \psi_3$ and $\psi_4 - \frac{\psi_4(\varepsilon)}{\psi_1(\varepsilon)} \psi_1$, respectively. Moreover as $z \to -\infty$, we have the asymptotic behavior $\psi_2(\lambda, \varepsilon, z) = \exp[-\frac{1}{\psi(\varepsilon)} \int \mu_-(\lambda, \varepsilon) z\gamma Y_-(\varepsilon) O(1)]$ and as $z \to +\infty$ we have the asymptotic behavior $\psi_4(\lambda, \varepsilon, z) = \exp[+\frac{1}{\psi(\varepsilon)} \int \mu_-(\lambda, \varepsilon) z\gamma Z_-(\varepsilon) O(1)]$. It is standard to show that $\lambda$ is an eigenvalue of the intermediate operator $L_0 : L_0 \psi = \left( -v(\varepsilon)\psi_1 + f'(\phi)\psi - \psi^2 \right) \text{ if and only if } \lambda$ is a zero of the complex analytic functions $D_1(\lambda, \varepsilon)$. Moreover, the order of a zero $\lambda$ of the Evans function is equal to the algebraic multiplicity of the eigenvalue. See also Evans [E4]. It is easy to see $D(\lambda, \varepsilon) \neq 0$, for all $\lambda \in \Omega$. This implies that if $\psi^i(\lambda, \varepsilon, 0) = \psi^i(\lambda, \varepsilon, z_0) = 0$ for the original eigenvalue problem, then $\lambda$ is not an eigenvalue of $L(\varepsilon)$. Another way to define these solutions $\psi_i(\lambda, \varepsilon, z)$, where $i = 3, 4$, is to apply Lemma 1. One obtains immediately

\[
\psi_3(\lambda, \varepsilon, z) = \exp[\frac{1}{\psi(\varepsilon)} \int_0^z 2\lambda + \varepsilon \gamma - f'(\phi(r)) dr] \begin{pmatrix} +\psi_2^2(\lambda, \varepsilon, z) \\ -\psi_2^1(\lambda, \varepsilon, z) \end{pmatrix},
\]

\[
\psi_4(\lambda, \varepsilon, z) = \exp[\frac{1}{\psi(\varepsilon)} \int_0^z 2\lambda + \varepsilon \gamma - f'(\phi(r)) dr] \begin{pmatrix} -\psi_1^2(\lambda, \varepsilon, z) \\ +\psi_1^1(\lambda, \varepsilon, z) \end{pmatrix}.
\]
Then using the properties of $\psi_1(\lambda, \varepsilon, z)$ and $\psi_2(\lambda, \varepsilon, z)$ we see

$$
\psi_3(\lambda, \varepsilon, z) = \exp \left[ \frac{\lambda z}{v(\varepsilon)} \right] \psi_3(0, \varepsilon, z) = R(\lambda, \varepsilon, z) \begin{pmatrix} +\psi_2^0(0, \varepsilon, z) \\ -\psi_2^1(0, \varepsilon, z) \end{pmatrix},
$$

$$
\psi_4(\lambda, \varepsilon, z) = \exp \left[ \frac{\lambda z}{v(\varepsilon)} \right] \psi_4(0, \varepsilon, z) = R(\lambda, \varepsilon, z) \begin{pmatrix} -\psi_2^0(0, \varepsilon, z) \\ +\psi_2^1(0, \varepsilon, z) \end{pmatrix},
$$

where

$$
R(\lambda, \varepsilon, z) = \exp \left[ \frac{\lambda z}{v(\varepsilon)} \right] \exp \left[ \frac{1}{v(\varepsilon)} \int_0^z \varepsilon \gamma - f'(\phi(r))dr \right].
$$

One observes that as $z \to -\infty$, the solutions $\psi_3(\lambda, \varepsilon, z)$ and $\psi_4(\lambda, \varepsilon, z)$ approach zero exponentially fast, at the rate $\exp \left[ -\frac{1}{v(\varepsilon)} \mu_+(\lambda, \varepsilon) z \right]$ and $\exp \left[ -\frac{1}{v(\varepsilon)} \mu_-(\lambda, \varepsilon) z \right]$, respectively.

Moreover $(\psi_1(\lambda, \varepsilon, z), \psi_4(\lambda, \varepsilon, z)) = (\psi_2(\lambda, \varepsilon, z), \psi_3(\lambda, \varepsilon, z)) = 0$. Explicitly, we can now calculate the intermediate Evans function $D(\lambda, \varepsilon)$ as follows

$$
\begin{align*}
&= (\psi_1(\lambda, \varepsilon, z), \psi_3(\lambda, \varepsilon, z)) = (\psi_2(\lambda, \varepsilon, z), \psi_4(\lambda, \varepsilon, z)) \\
&= (\psi_1(0, \varepsilon, z), \psi_3(0, \varepsilon, z)) = (\psi_2(0, \varepsilon, z), \psi_4(0, \varepsilon, z)) \\
&= \exp \left[ \frac{1}{v(\varepsilon)} \int_0^z 2\lambda + \varepsilon \gamma - f'(\phi(r))dr \right] \det \begin{pmatrix} \psi_1^1(\lambda, \varepsilon, z) & \psi_2^1(\lambda, \varepsilon, z) \\ \psi_1^2(\lambda, \varepsilon, z) & \psi_2^2(\lambda, \varepsilon, z) \end{pmatrix} \\
&= \exp \left[ \frac{1}{v(\varepsilon)} \int_0^z \varepsilon \gamma - f'(\phi(r))dr \right] \det \begin{pmatrix} \psi_1^1(0, \varepsilon, z) & \psi_2^1(0, \varepsilon, z) \\ \psi_1^2(0, \varepsilon, z) & \psi_2^2(0, \varepsilon, z) \end{pmatrix} \\
&= \exp \left[ \frac{1}{v(\varepsilon)} \int_0^z \varepsilon \gamma - f'(\phi(r))dr \right] \det \left( \exp \left[ -\frac{\omega_+(\varepsilon) z}{v(\varepsilon)} \right] Y_+(\varepsilon) + \exp \left[ -\frac{\omega_-(\varepsilon) z}{v(\varepsilon)} - \delta z \right] O(1), \right. \\
&\quad \left. \exp \left[ -\frac{\omega_-(\varepsilon) z}{v(\varepsilon)} \right] Y_-(\varepsilon) + \exp \left[ -\frac{\omega_-(\varepsilon) z}{v(\varepsilon)} - \delta z \right] O(1) \right) \\
&= \exp \left[ -\frac{1}{v(\varepsilon)} \int_0^z \omega_+ f'(\phi(r))dr \right] \det \left( Y_+(\varepsilon) + e^{-\delta z} O(1), Y_-(\varepsilon) + e^{-\delta z} O(1) \right)
\end{align*}
$$
\[
\begin{align*}
&= \lim_{z \to +\infty} \exp\left[-\frac{1}{v(\varepsilon)} \int_0^z a + f'(\phi(r))dr\right] \det \left(Y_+(\varepsilon) + e^{-\delta z}O(1), Y_-(\varepsilon) + e^{-\delta z}O(1)\right) \\
&= \exp\left[-\frac{1}{v(\varepsilon)} \int_0^\infty a + f'(\phi(z))dz\right] \det (Y_+(\varepsilon), Y_-(\varepsilon)) \\
&= -\exp\left[-\frac{1}{v(\varepsilon)} \int_0^\infty a + f'(\phi(z))dz\right]\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon} < 0,
\end{align*}
\]

This result is independent of \( \lambda \). The above analysis also shows

\[
\det(\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) = \exp\left[-\frac{1}{v(\varepsilon)} \int_0^z 2\lambda + \varepsilon \gamma - f'(\phi(r))dr\right] D(\lambda, \varepsilon).
\]

Let \( \psi^T \) denote the transpose of \( \psi \). Since

\[
\begin{pmatrix}
\psi_3^T(\lambda, \varepsilon, z) \\
\psi_4^T(\lambda, \varepsilon, z)
\end{pmatrix}
(\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) =
\begin{pmatrix}
D(\lambda, \varepsilon) & 0 \\
0 & D(\lambda, \varepsilon)
\end{pmatrix},
\]

we have

\[
(\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z))^{-1} = \frac{1}{D(\lambda, \varepsilon)}
\begin{pmatrix}
\psi_3^T(\lambda, \varepsilon, z) \\
\psi_4^T(\lambda, \varepsilon, z)
\end{pmatrix}
\]

\[
= \frac{1}{D(\lambda, \varepsilon)} \exp\left[\frac{1}{v(\varepsilon)} \int_0^z 2\lambda + \varepsilon \gamma - f'(\phi(r))dr\right]
\begin{pmatrix}
+\psi_2^2(\lambda, \varepsilon, z) & -\psi_1^2(\lambda, \varepsilon, z) \\
-\psi_1^2(\lambda, \varepsilon, z) & +\psi_1^1(\lambda, \varepsilon, z)
\end{pmatrix}
\]

\[
= \frac{1}{D(\lambda, \varepsilon)} R(\lambda, \varepsilon, z)
\begin{pmatrix}
+\psi_2(0, \varepsilon, z) & -\psi_1(0, \varepsilon, z) \\
-\psi_1(0, \varepsilon, z) & +\psi_1(0, \varepsilon, z)
\end{pmatrix}.
\]

We now consider particularly the solutions of the homogeneous system for \( \lambda = 0 \).

This solution is closely related to \((\phi_z, \varphi_z)\). The latter solves the equations

\[
v(\varepsilon) \frac{\partial}{\partial z}
\begin{pmatrix}
\phi_z \\
\varphi_z
\end{pmatrix}
+ \begin{pmatrix}
-f'(\phi) & 1 \\
-\varepsilon & \varepsilon \gamma
\end{pmatrix}
\begin{pmatrix}
\phi_z \\
\varphi_z
\end{pmatrix}
= \alpha
\begin{pmatrix}
K(z) - K(z - z_0) \\
0
\end{pmatrix}.
\]

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Recall that \((\phi_z, \varphi_z)\) is bounded and uniformly continuous on \((-\infty, +\infty)\). Moreover, it approaches zero exponentially fast as \(|z| \to \infty\). It can be represented as a "linear" combination of \(\psi_1(0, \varepsilon, z)\) and \(\psi_2(0, \varepsilon, z)\), namely there exist two smooth functions \(c_1(0, \varepsilon, z)\) and \(c_2(0, \varepsilon, z)\), such that

\[
\begin{pmatrix}
\phi_z(\varepsilon, z) \\
\varphi_z(\varepsilon, z)
\end{pmatrix} = (\psi_1(0, \varepsilon, z), \psi_2(0, \varepsilon, z)) \begin{pmatrix}
c_1(0, \varepsilon, z) \\
c_2(0, \varepsilon, z)
\end{pmatrix}.
\]

Let

\[
S(\varepsilon, z) = \alpha \frac{1}{v(\varepsilon)D(0, \varepsilon)} \exp\left[ \frac{1}{v(\varepsilon)} \int_0^z \varepsilon \gamma - f'(\phi(\varepsilon)) d\varepsilon \right].
\]

By the method of variation of parameter, it is easy to see \((c_1, c_2)\) solves the equations

\[
\frac{\partial}{\partial z} \begin{pmatrix}
c_1(0, \varepsilon, z) \\
c_2(0, \varepsilon, z)
\end{pmatrix} = S(\varepsilon, z) \begin{pmatrix}
+\psi_2^2(0, \varepsilon, z) & -\psi_1^1(0, \varepsilon, z) \\
-\psi_2^1(0, \varepsilon, z) & +\psi_1^2(0, \varepsilon, z)
\end{pmatrix} \begin{pmatrix}
K(z) - K(z - z_0) \\
0
\end{pmatrix}.
\]

Solving this system one gets

\[
\begin{pmatrix}
c_1(0, \varepsilon, z) \\
c_2(0, \varepsilon, z)
\end{pmatrix} = \begin{pmatrix}
c_1(\varepsilon) \\
c_2(\varepsilon)
\end{pmatrix} + \int_{-\infty}^z S(\varepsilon, r) \begin{pmatrix}
+\psi_2^2(0, \varepsilon, r) \\
-\psi_2^1(0, \varepsilon, r)
\end{pmatrix} [K(r) - K(r - z_0)] dr.
\]

Since \((\phi_z, \varphi_z)\) is bounded and approaches zero as \(|z| \to +\infty\), we must choose \(c_1(\varepsilon) = c_2(\varepsilon) = 0\). Hence we have

\[
\begin{pmatrix}
\phi_z \\
\varphi_z
\end{pmatrix} = (\psi_1(0, \varepsilon, z), \psi_2(0, \varepsilon, z)) \int_{-\infty}^z S(\varepsilon, r) \begin{pmatrix}
+\psi_2^2(0, \varepsilon, r) \\
-\psi_2^1(0, \varepsilon, r)
\end{pmatrix} [K(r) - K(r - z_0)] dr.
\]

It is easy to see the right hand side approaches zero as \(z \to \pm \infty\). The rate of convergence could be algebraically fast instead of exponentially fast. More accurately by L'hospital rule, this rate depends on the function \(K\).
Using the above equation (1), we can obtain the relationship

\[
(\psi_1(0, \varepsilon, z), \psi_2(0, \varepsilon, z))^{-1} \begin{pmatrix} \phi_z \\ \varphi_z \end{pmatrix} = \int_{-\infty}^{\infty} S(\varepsilon, r) \begin{pmatrix} +\psi_2^2(0, \varepsilon, r) \\
-\psi_1^2(0, \varepsilon, r) \end{pmatrix} [K(r) - K(r - z_0)] dr.
\]

Multiplying it by \((\psi_1^1(0, \varepsilon, 0), \psi_2^1(0, \varepsilon, 0))\) gives

\[
(\psi_1^1(0, \varepsilon, 0), \psi_2^1(0, \varepsilon, 0))(\psi_1(0, \varepsilon, z), \psi_2(0, \varepsilon, z))^{-1} \begin{pmatrix} \phi_z \\ \varphi_z \end{pmatrix}
\]

\[
= (\psi_1^1(0, \varepsilon, 0), \psi_2^1(0, \varepsilon, 0)) \int_{-\infty}^{\infty} S(\varepsilon, r) \begin{pmatrix} +\psi_2^2(0, \varepsilon, r) \\
-\psi_1^2(0, \varepsilon, r) \end{pmatrix} [K(r) - K(r - z_0)] dr.
\]

Hence we get

\[
\int_{-\infty}^{\infty} S(\varepsilon, r) \varphi^2(0, \varepsilon, r)[K(r) - K(r - z_0)] dr
\]

\[
= (\psi_1^1(0, \varepsilon, 0), \psi_2^1(0, \varepsilon, 0)) \int_{-\infty}^{\infty} S(\varepsilon, r) \begin{pmatrix} +\psi_2^2(0, \varepsilon, r) \\
-\psi_1^2(0, \varepsilon, r) \end{pmatrix} [K(r) - K(r - z_0)] dr
\]

\[
= (\psi_1^1(0, \varepsilon, 0), \psi_2^1(0, \varepsilon, 0))(\psi_1(0, \varepsilon, z), \psi_2(0, \varepsilon, z))^{-1} \begin{pmatrix} \phi_z \\ \varphi_z \end{pmatrix}
\]

\[
= \frac{1}{D(\lambda, \varepsilon)} R(\lambda, \varepsilon, z)(\psi_1(0, \varepsilon, 0), \psi_2(0, \varepsilon, 0)) \begin{pmatrix} +\psi_2^2(0, \varepsilon, z) \\
-\psi_1^2(0, \varepsilon, z) \end{pmatrix} \begin{pmatrix} \phi_z \\ \varphi_z \end{pmatrix}
\]

\[
= \frac{1}{D(\lambda, \varepsilon)} R(\lambda, \varepsilon, z)(\varphi^2(0, \varepsilon, z), -\varphi^1(0, \varepsilon, z)) \begin{pmatrix} \phi_z \\ \varphi_z \end{pmatrix}.
\]

Similarly we get

\[
\int_{-\infty}^{\infty} S(\varepsilon, r) \varphi^4(0, \varepsilon, r)[K(r) - K(r - z_0)] dr
\]

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\[
\frac{1}{D(\lambda, \varepsilon)} R(\lambda, \varepsilon, z) (\varphi^4(0, \varepsilon, z), -\varphi^3(0, \varepsilon, z)) \begin{pmatrix} \phi_z \\ \varphi_z \end{pmatrix}.
\]

7. The Evans Function for the Eigenvalue Problem (3.10)

We now make use of the intermediate Evans function \(D(\lambda, \varepsilon)\) to find solutions of the original eigenvalue problem. First these solutions must be well defined. Each of them must be unique, complex analytic. In addition, the asymptotic behavior as \(|z| \to \infty\), as \(|\lambda| \to +\infty\) and as \(\varepsilon \to 0\) must be clearly studied. We need to pay close attention to see where the singular perturbation parameter \(\varepsilon\) appear. By using the method of variation of parameter to solve the nonhomogeneous differential equations, we can find two linearly independent, complex analytic solutions \(\psi_i(\lambda, \varepsilon, z)\), where \(i = 5, 6\), such that

\[
\lim_{z \to \infty} \psi_i(\lambda, \varepsilon, z) = 0.
\]

All other complex analytic solutions can be written as the linear combination of these two analytic solutions.

For any smooth function \(c(\lambda, \varepsilon, z)\) and \(i = 1, 2\), \(c(\lambda, \varepsilon, z)\psi_i(\lambda, \varepsilon, z)\) will not be a solution of (3.10). One must use the fundamental matrix \((\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z))\) to construct a solution of (3.10). Suppose that

\[
\psi(\lambda, \varepsilon, z) = c_1(\lambda, \varepsilon, z)\psi_1(\lambda, \varepsilon, z) + c_2(\lambda, \varepsilon, z)\psi_2(\lambda, \varepsilon, z)
\]

\[
= (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \begin{pmatrix} c_1(\lambda, \varepsilon, z) \\ c_2(\lambda, \varepsilon, z) \end{pmatrix}.
\]
is a solution of the nonhomogeneous system (3.10). By product rule, we have

\[
v(\varepsilon)(\psi_1, \psi_2) \frac{\partial}{\partial z} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + v(\varepsilon) \frac{\partial}{\partial z} (\psi_1, \psi_2) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \lambda - f'(\phi) & 1 \\ -\varepsilon & \lambda + \varepsilon \gamma \end{pmatrix} (\psi_1, \psi_2) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} N \\ 0 \end{pmatrix}.
\]

Notice that

\[
v(\varepsilon) \frac{\partial}{\partial z} (\psi_1, \psi_2) + \begin{pmatrix} \lambda - f'(\phi) & 1 \\ -\varepsilon & \lambda + \varepsilon \gamma \end{pmatrix} (\psi_1, \psi_2) = 0.
\]

Therefore we get

\[
v(\varepsilon)(\psi_1, \psi_2) \frac{\partial}{\partial z} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} N \\ 0 \end{pmatrix}.
\]

Let

\[
S(\lambda, \varepsilon, z) = \frac{1}{v(\varepsilon)D(\lambda, \varepsilon)} \exp[\frac{\lambda z}{v(\varepsilon)}] \exp[\frac{1}{v(\varepsilon)} \int_0^z \varepsilon \gamma - f'(\phi(r))dr].
\]

For all fixed \(\varepsilon\) and \(z\), \(S\) is a scalar complex analytic function in \(\lambda\). Independent of \(\psi_1, \psi_2, \psi_3, \psi_4\). As \(|z| \to +\infty\), we have

\[
S(\lambda, \varepsilon, z) = \frac{1}{v(\varepsilon)D(\lambda, \varepsilon)} \exp[\frac{\lambda + a + \varepsilon \gamma z}{v(\varepsilon)}] \exp[\frac{1}{v(\varepsilon)} \int_0^z a + f'(\phi(r))dr].
\]

By using the properties of the intermediate Evans function, it is easy to get

\[
\frac{\partial}{\partial z} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{v(\varepsilon)D(\lambda, \varepsilon)} \begin{pmatrix} \psi^T_2(\lambda, \varepsilon, z) \\ \psi^T_4(\lambda, \varepsilon, z) \end{pmatrix} \begin{pmatrix} N(\lambda, \varepsilon, z) \\ 0 \end{pmatrix}
\]
\[
S(A, \varepsilon, z) = S(X, \varepsilon, z) \begin{pmatrix} +\psi_2^2(0, \varepsilon, z) & -\psi_1^1(0, \varepsilon, z) \\ -\psi_1^2(0, \varepsilon, z) & +\psi_1^1(0, \varepsilon, z) \end{pmatrix} \begin{pmatrix} N(\lambda, \varepsilon, z) \\ 0 \end{pmatrix}
\]

\[
= S(\lambda, \varepsilon, z) \begin{pmatrix} +\psi_2^2(0, \varepsilon, z) \\ -\psi_1^2(0, \varepsilon, z) \end{pmatrix} N(\lambda, \varepsilon, z)
\]

\[
= F(\lambda, \varepsilon, z) \psi^1(\lambda, \varepsilon, 0) + G(\lambda, \varepsilon, z) \psi^1(\lambda, \varepsilon, z_0),
\]

where

\[
F(\lambda, \varepsilon, z) = S(\lambda, \varepsilon, z) f(z) \begin{pmatrix} +\psi_2^2(0, \varepsilon, z) \\ -\psi_1^2(0, \varepsilon, z) \end{pmatrix},
\]

\[
G(\lambda, \varepsilon, z) = S(\lambda, \varepsilon, z) g(z) \begin{pmatrix} +\psi_2^2(0, \varepsilon, z) \\ -\psi_1^2(0, \varepsilon, z) \end{pmatrix}.
\]

\[
F(\lambda, \varepsilon, z) = \exp\left[\frac{\lambda z}{\nu(\varepsilon)}\right] F(0, \varepsilon, z) \quad \text{and} \quad G(\lambda, \varepsilon, z) = \exp\left[\frac{\lambda z}{\nu(\varepsilon)}\right] G(0, \varepsilon, z)
\]

depends on \(\psi_1\) and \(\psi_2\). They are complex analytic functions in \(\lambda\) for fixed \(\varepsilon\) and \(z\), and they approach zero exponentially fast as \(z \to -\infty\). More precisely, we have the asymptotic behavior as \(|z| \to +\infty\),

\[
F(\lambda, \varepsilon, z) = \begin{pmatrix} \exp[\mu(+\lambda z)/\nu(z)] \\ \exp[\mu(-\lambda z)/\nu(z)] \end{pmatrix} f(z) O(1), \int_{-\infty}^{z} F(\lambda, \varepsilon, r) dr = \begin{pmatrix} \exp[\mu(+\lambda z)/\nu(z)] \\ \exp[\mu(-\lambda z)/\nu(z)] \end{pmatrix} o(1),
\]

\[
G(\lambda, \varepsilon, z) = \begin{pmatrix} \exp[\mu(+\lambda z)/\nu(z)] \\ \exp[\mu(-\lambda z)/\nu(z)] \end{pmatrix} g(z) O(1), \int_{-\infty}^{z} G(\lambda, \varepsilon, r) dr = \begin{pmatrix} \exp[\mu(+\lambda z)/\nu(z)] \\ \exp[\mu(-\lambda z)/\nu(z)] \end{pmatrix} o(1),
\]

so that

\[
\lim_{z \to +\infty} \begin{pmatrix} \exp[-\mu(+\lambda z)/\nu(z)] & 0 \\ 0 & \exp[-\mu(-\lambda z)/\nu(z)] \end{pmatrix} F(\lambda, \varepsilon, z) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

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\[
\lim_{z \to \pm \infty} \begin{pmatrix}
\exp\left(-\frac{\mu_\infty(\lambda, \varepsilon)}{v(\varepsilon)} z\right) & 0 \\
0 & \exp\left(-\frac{\mu_\infty(\lambda, \varepsilon)}{v(\varepsilon)} z\right)
\end{pmatrix}
G(\lambda, \varepsilon, z) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Now recall that \((c_1(\lambda, \varepsilon, z), c_2(\lambda, \varepsilon, z))\) satisfies the following equations

\[
\frac{\partial}{\partial z} \begin{pmatrix}
c_1(\lambda, \varepsilon, z) \\
c_2(\lambda, \varepsilon, z)
\end{pmatrix} = F(\lambda, \varepsilon, z)\psi^1(\lambda, \varepsilon, 0) + G(\lambda, \varepsilon, z)\psi^1(\lambda, \varepsilon, z_0).
\]

Solving these equations yields

\[
\begin{pmatrix}
c_1(\lambda, \varepsilon, z) \\
c_2(\lambda, \varepsilon, z)
\end{pmatrix} = \begin{pmatrix}
c_1(\lambda, \varepsilon) \\
c_2(\lambda, \varepsilon)
\end{pmatrix} + \int_{-\infty}^{\infty} F(\lambda, \varepsilon, r) dr \psi^1(\lambda, \varepsilon, 0) + \int_{-\infty}^{\infty} G(\lambda, \varepsilon, r) dr \psi^1(\lambda, \varepsilon, z_0).
\]

Therefore we obtain the possibly analytic solutions

\[
\psi(\lambda, \varepsilon, z)
= (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \begin{pmatrix}
c_1(\lambda, \varepsilon, z) \\
c_2(\lambda, \varepsilon, z)
\end{pmatrix} \\
= (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \begin{pmatrix}
c_1(\lambda, \varepsilon) \\
c_2(\lambda, \varepsilon)
\end{pmatrix} \\
+ (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \int_{-\infty}^{\infty} F(\lambda, \varepsilon, r) dr \psi^1(\lambda, \varepsilon, 0) \\
+ (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \int_{-\infty}^{\infty} G(\lambda, \varepsilon, r) dr \psi^1(\lambda, \varepsilon, z_0).
\]

If \((\psi^1(\lambda, \varepsilon, 0), \psi^1(\lambda, \varepsilon, z_0))\) and \((c_1(\lambda, \varepsilon), c_2(\lambda, \varepsilon))\) are complex analytic functions in \(\lambda\) for all fixed \(\varepsilon\), then \(\psi(\lambda, \varepsilon, z)\) is also analytic in \(\lambda\), for fixed \(z\) and \(\varepsilon\). Clearly we have the limits

\[
\lim_{z \to \pm \infty} (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \int_{-\infty}^{\infty} F(\lambda, \varepsilon, r) dr = 0,
\]

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\[
\lim_{z \to +\infty} \psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) | G(\lambda, \varepsilon, r) dr = 0, \\
\lim_{z \to +\infty} (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \begin{pmatrix} c_1(\lambda, \varepsilon) \\ c_2(\lambda, \varepsilon) \end{pmatrix} = 0, \\
\lim_{z \to +\infty} \| (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \begin{pmatrix} c_1(\lambda, \varepsilon) \\ c_2(\lambda, \varepsilon) \end{pmatrix} \| = +\infty.
\]

if \((c_1(\lambda, \varepsilon), c_2(\lambda, \varepsilon)) \neq (0, 0)\). It is therefore clear to see

\[
\lim_{z \to +\infty} (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \begin{pmatrix} c_1(\lambda, \varepsilon, z) \\ c_2(\lambda, \varepsilon, z) \end{pmatrix} = 0, \\
\lim_{z \to +\infty} \| (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \begin{pmatrix} c_1(\lambda, \varepsilon) \\ c_2(\lambda, \varepsilon) \end{pmatrix} \| = +\infty,
\]

if \((c_1(\lambda, \varepsilon), c_2(\lambda, \varepsilon)) \neq (0, 0)\). So \(\psi(\lambda, \varepsilon, z) = c_1(\lambda, \varepsilon, z)\psi_1(\lambda, \varepsilon, z) + c_2(\lambda, \varepsilon)z)\psi_2(\lambda, \varepsilon, z)\)
is a good candidate of eigenfunctions of the eigenvalue problem (3.10). The complex number \(\lambda \in \Omega\) is an eigenvalue of \(L(\varepsilon)\) if and only if \((c_1(\lambda, \varepsilon), c_2(\lambda, \varepsilon)) = (0, 0)\). Motivated by the work of Evans, Jones and others, we would like to relate the eigenvalues of the operator \(L(\varepsilon)\) to zeroes of some analytic function. If one can define such a function and investigate its zeroes, then the necessary information of stability would be obtained. Because of the presence of the nonlocal term, the Evans function for the equation (3.10) must be defined in an unusual way. We need two linearly independent, complex analytic solutions which approach zero exponentially fast as \(z \to +\infty\). The above vector \(\begin{pmatrix} c_1(\lambda, \varepsilon) \\ c_2(\lambda, \varepsilon) \end{pmatrix}\) is not arbitrary, it will be determined by the compatibility condition of \(\psi(\lambda, \varepsilon, z)\) at \(z = 0\) and \(z = z_0\). It is by solving two linearly
independent vectors \[ \begin{pmatrix} c_1(\lambda, \varepsilon) \\ c_2(\lambda, \varepsilon) \end{pmatrix} \] that we can find two linearly independent, complex analytic solutions \( \psi_5(\lambda, \varepsilon, z) \) and \( \psi_6(\lambda, \varepsilon, z) \). To solve the vector \( (c_1(\lambda, \varepsilon), c_2(\lambda, \varepsilon)) \), we start from the following component equations

\[
\psi^1(\lambda, \varepsilon, 0) = (\psi_1^1(\lambda, \varepsilon, 0), \psi_2^1(\lambda, \varepsilon, 0)) \begin{pmatrix} c_1(\lambda, \varepsilon) \\ c_2(\lambda, \varepsilon) \end{pmatrix} \\
+ (\psi_1^1(\lambda, \varepsilon, 0), \psi_2^1(\lambda, \varepsilon, 0)) \int_{-\infty}^{0} F(\lambda, \varepsilon, z) dz \psi^1(\lambda, \varepsilon, 0) \\
+ (\psi_1^1(\lambda, \varepsilon, 0), \psi_2^1(\lambda, \varepsilon, 0)) \int_{-\infty}^{0} G(\lambda, \varepsilon, z) dz \psi^1(\lambda, \varepsilon, 0).
\]

\[
\psi^1(\lambda, \varepsilon, z_0) = (\psi_1^1(\lambda, \varepsilon, z_0), \psi_2^1(\lambda, \varepsilon, z_0)) \begin{pmatrix} c_1(\lambda, \varepsilon) \\ c_2(\lambda, \varepsilon) \end{pmatrix} \\
+ (\psi_1^1(\lambda, \varepsilon, z_0), \psi_2^1(\lambda, \varepsilon, z_0)) \int_{-\infty}^{z_0} F(\lambda, \varepsilon, z) dz \psi^1(\lambda, \varepsilon, 0) \\
+ (\psi_1^1(\lambda, \varepsilon, z_0), \psi_2^1(\lambda, \varepsilon, z_0)) \int_{-\infty}^{z_0} G(\lambda, \varepsilon, z) dz \psi^1(\lambda, \varepsilon, 0).
\]

These equations are too long. We need a compact form of equations. Define

\[
P_1(\lambda, \varepsilon) = (\psi_1^1(\lambda, \varepsilon, 0), \psi_2^1(\lambda, \varepsilon, 0)) \int_{-\infty}^{0} F(\lambda, \varepsilon, z) dz,
\]

\[
Q_1(\lambda, \varepsilon) = (\psi_1^1(\lambda, \varepsilon, 0), \psi_2^1(\lambda, \varepsilon, 0)) \int_{-\infty}^{0} G(\lambda, \varepsilon, z) dz,
\]

\[
P_2(\lambda, \varepsilon) = (\psi_1^1(\lambda, \varepsilon, z_0), \psi_2^1(\lambda, \varepsilon, z_0)) \int_{-\infty}^{z_0} F(\lambda, \varepsilon, z) dz,
\]

\[
Q_2(\lambda, \varepsilon) = (\psi_1^1(\lambda, \varepsilon, z_0), \psi_2^1(\lambda, \varepsilon, z_0)) \int_{-\infty}^{z_0} G(\lambda, \varepsilon, z) dz.
\]

Rearranging terms and using the above notation give the equations

\[
(\psi_1^1(\lambda, \varepsilon, 0), \psi_2^1(\lambda, \varepsilon, 0)) \begin{pmatrix} c_1(\lambda, \varepsilon) \\ c_2(\lambda, \varepsilon) \end{pmatrix} = [1 - P_1(\lambda, \varepsilon)] \psi^1(\lambda, \varepsilon, 0) - Q_1(\lambda, \varepsilon) \psi^1(\lambda, \varepsilon, z_0),
\]

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\[
(\psi_1^1(\lambda, \varepsilon, z_0), \psi_2^1(\lambda, \varepsilon, z_0)) \begin{pmatrix}
c_1(\lambda, \varepsilon) \\
c_2(\lambda, \varepsilon)
\end{pmatrix} = [1 - Q_2(\lambda, \varepsilon)]\psi^1(\lambda, \varepsilon, z_0) - P_2(\lambda, \varepsilon)\psi^1(\lambda, \varepsilon, 0).
\]

These equations are still complicated. Define

\[M(\lambda, \varepsilon) = \begin{pmatrix}
P_1(\lambda, \varepsilon) - 1 & Q_1(\lambda, \varepsilon) \\
P_2(\lambda, \varepsilon) & Q_2(\lambda, \varepsilon) - 1
\end{pmatrix}.
\]

Thus in matrix notation we get

\[
\begin{pmatrix}
\psi_1(\lambda, \varepsilon, 0) & \psi_2(\lambda, \varepsilon, 0) \\
\psi_1(\lambda, \varepsilon, z_0) & \psi_2(\lambda, \varepsilon, z_0)
\end{pmatrix}
\begin{pmatrix}
c_1(\lambda, \varepsilon) \\
c_2(\lambda, \varepsilon)
\end{pmatrix} = -M(\lambda, \varepsilon)
\begin{pmatrix}
\psi^1(\lambda, \varepsilon, 0) \\
\psi^1(\lambda, \varepsilon, z_0)
\end{pmatrix}.
\]

Since \(\psi_1(\lambda, \varepsilon, z)\) and \(\psi_2(\lambda, \varepsilon, z)\) grow at different rate

\[
\det \begin{pmatrix}
\psi_1^1(\lambda, \varepsilon, 0) & \psi_2^1(\lambda, \varepsilon, 0) \\
\psi_1^1(\lambda, \varepsilon, z_0) & \psi_2^1(\lambda, \varepsilon, z_0)
\end{pmatrix} = \exp\left[-\frac{1}{\nu(\varepsilon)}\lambda z_0\right] \det \begin{pmatrix}
\psi_1^1(0, \varepsilon, 0) & \psi_2^1(0, \varepsilon, 0) \\
\psi_1^1(0, \varepsilon, z_0) & \psi_2^1(0, \varepsilon, z_0)
\end{pmatrix} \neq 0,
\]

we will give a more rigorous proof of this fact later, by using some projectivized equation. One can easily solve the above equations to obtain \((c_1(\lambda, \varepsilon), c_2(\lambda, \varepsilon))\), provided the prescribed complex vector \((\psi^1(\lambda, \varepsilon, 0), \psi^1(\lambda, \varepsilon, z_0))\) is given. Because we are investigating analytic solutions, we would choose \((\psi^1(\lambda, \varepsilon, 0), \psi^1(\lambda, \varepsilon, z_0))\) to be analytic functions of \(\lambda\) and smooth functions of \(\varepsilon\). Hence \((c_1(\lambda, \varepsilon), c_2(\lambda, \varepsilon))\) would be also analytic. This in turn shows that \(\psi(\lambda, \varepsilon, z)\) is complex analytic in \(\lambda\), for fixed \(z \in \mathbb{R}\) and \(\varepsilon > 0\). Letting \((\psi^1(\lambda, \varepsilon, 0), \psi^1(\lambda, \varepsilon, z_0)) = (1, 0)\) and \((\psi^1(\lambda, \varepsilon, 0), \psi^1(\lambda, \varepsilon, z_0)) = (0, 1)\), respectively, we obtain two linearly independent, complex analytic solutions \(\psi_5(\lambda, \varepsilon, z)\) and \(\psi_6(\lambda, \varepsilon, z)\). Any other choose of \((\psi^1(\lambda, \varepsilon, 0), \psi^1(\lambda, \varepsilon, z_0))\) would give a well defined solution of (3.10). Actually all the solution of (3.10) with \(\lim_{z \to +\infty} \psi(\lambda, \varepsilon, z) = \)
0 can be written in the assumed form. By the way, any nontrivial solution of the original eigenvalue problem cannot vanish at \( z = 0 \) and \( z = z_0 \) simultaneously. We now define a complex analytic function, called the Evans function, by

\[
E(\lambda, \varepsilon) = \det M(\lambda, \varepsilon).
\]

This function is well defined. If \( \lambda \in \Omega \) is such that \( E(\lambda, \varepsilon) = 0 \), then the vectors

\[
\begin{pmatrix}
P_1(\lambda, \varepsilon) - 1 \\
P_2(\lambda, \varepsilon)
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
Q_1(\lambda, \varepsilon) \\
Q_2(\lambda, \varepsilon) - 1
\end{pmatrix}
\]

are parallel to each other, there must be some complex numbers \( \psi^1(\lambda, \varepsilon, 0) \) and \( \psi^1(\lambda, \varepsilon, z_0) \), analytic in \( \lambda \) and smooth in \( \varepsilon \), such that

\[
\begin{pmatrix}
\psi^1_1(\lambda, \varepsilon, 0) & \psi^1_2(\lambda, \varepsilon, 0) \\
\psi^1_1(\lambda, \varepsilon, z_0) & \psi^1_2(\lambda, \varepsilon, z_0)
\end{pmatrix}
\begin{pmatrix}
c_1(\lambda, \varepsilon) \\
c_2(\lambda, \varepsilon)
\end{pmatrix}
= -M(\lambda, \varepsilon)
\begin{pmatrix}
\psi^1(\lambda, \varepsilon, 0) \\
\psi^1(\lambda, \varepsilon, z_0)
\end{pmatrix}
= 0.
\]

hence \( (c_1(\lambda, \varepsilon), c_2(\lambda, \varepsilon)) = (0, 0) \) and \( \lambda \) is an eigenvalue. On the other hand, if \( \lambda \) is an eigenvalue, then

\[
M(\lambda, \varepsilon)
\begin{pmatrix}
\psi^1(\lambda, \varepsilon, 0) \\
\psi^1(\lambda, \varepsilon, z_0)
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

This implies that \( E(\lambda, \varepsilon) = 0 \), since \( (\psi^1(\lambda, \varepsilon, 0), \psi^1(\lambda, \varepsilon, z_0) \neq (0, 0) \). The above analysis shows that \( \lambda \in \Omega \) is an eigenvalue of \( L(\varepsilon) \) if and only if \( E(\lambda, \varepsilon) = 0 \). Obviously \( \text{rank}M(\lambda, \varepsilon) = 1 \) if \( E(\lambda, \varepsilon) = 0 \); \( \text{rank}M(\lambda, \varepsilon) = 2 \) if \( E(\lambda, \varepsilon) \neq 0 \). Define the null spaces \( NS(\lambda, \varepsilon) = \{ v \in C^2 : M(\lambda, \varepsilon)v = 0 \} \). Thus the dimension \( \dim NS(\lambda, \varepsilon) = 1 \) if \( E(\lambda, \varepsilon) = 0 \); \( \dim NS(\lambda, \varepsilon) = 0 \) if \( E(\lambda, \varepsilon) \neq 0 \). Consequently we have the geometric multiplicity or the dimension \( \dim N(L(\varepsilon) - \lambda I) = 1 \) if \( \lambda \in \Omega \) is an eigenvalue and \( \dim N(L(\varepsilon) - \lambda I) = 0 \) if \( \lambda \in \Omega \) is not an eigenvalue.

Let us investigate the asymptotic behavior of \( E(\lambda, \varepsilon) \) as \( |\lambda| \to \infty \).
Lemma 13. Let the bounded smooth function $f$ satisfy $f' \in L^1(a, b)$, where $(a, b)$ is a finite interval or $a = -\infty$ or $b = \infty$. Then we have the limit

$$\lim_{|\xi| \to \infty} \int_a^b \exp(ia\xi)f(x)dx = 0.$$

Proof. By integration by parts, one can complete the proof.

By Lebesgue's dominated convergence theorem and Lemma 13, we have

$$\lim_{\lambda \in \Omega, |\lambda| \to \infty} P_1(\lambda, \varepsilon) = \lim_{\lambda \in \Omega, |\lambda| \to \infty} Q_1(\lambda, \varepsilon) = 0, \quad \lim_{\lambda \in \Omega, |\lambda| \to \infty} E(\lambda, \varepsilon) = 1,$$

uniformly with respect to $\varepsilon$. Thus there exists a positive constant $M$, independent of $\lambda, \varepsilon$ and $z$, such that every $\lambda \in \Omega$ with $|\lambda| > M$ is not an eigenvalue of the operator $L(\varepsilon)$. Now let us search for eigenvalues inside the circle $|\lambda| = M$. $E(\lambda, \varepsilon) = 0$ if and only if $\mu = 1$ is an eigenvalue of the matrix $M(\lambda, \varepsilon) + I$, if and only if there exists a nonzero eigenvector \( \begin{pmatrix} x \\ y \end{pmatrix} \in C^2 \), such that

$$\begin{pmatrix} P_1(\lambda, \varepsilon) & Q_1(\lambda, \varepsilon) \\ P_2(\lambda, \varepsilon) & Q_2(\lambda, \varepsilon) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

More explicitly we have

\[
(\psi_1(\lambda, \varepsilon, 0), \psi_2(\lambda, \varepsilon, 0)) \int_{-\infty}^0 S(\lambda, \varepsilon, z)[xf(z) + yg(z)] \begin{pmatrix} \psi_2^2(0, \varepsilon, z) \\ -\psi_1^2(0, \varepsilon, z) \end{pmatrix} dz = x, \\
(\psi_1(\lambda, \varepsilon, z_0), \psi_2(\lambda, \varepsilon, z_0)) \int_{-\infty}^0 S(\lambda, \varepsilon, z)[xf(z) + yg(z)] \begin{pmatrix} +\psi_2^2(0, \varepsilon, z) \\ -\psi_1^2(0, \varepsilon, z) \end{pmatrix} dz = y.
\]
Actually if $\lambda \in \Omega$ is an eigenvalue, then
\[
\begin{pmatrix}
Q_1(\lambda, \varepsilon) \\
Q_2(\lambda, \varepsilon) - 1
\end{pmatrix} = -\frac{\varepsilon}{y} \begin{pmatrix}
P_1(\lambda, \varepsilon) - 1 \\
P_2(\lambda, \varepsilon)
\end{pmatrix}.
\]
In particular $(x, y) = (\phi'(0), \phi'(z_0))$ if $\lambda = 0$. Then $-\frac{\varepsilon}{y} = |\phi'(0)/\phi'(z_0)| > 0$. This equation will simplify the calculation below.

8. Explicit Solutions of the Intermediate System

To find the eigenvalues of the operator $L(\varepsilon)$, we need the “explicit expression” of the Evans function. This requires us to calculate the $P_i(\lambda, \varepsilon)$ and $Q_i(\lambda, \varepsilon)$ and find their asymptotic behavior as $|\lambda| \to \infty$, $i = 1, 2$. Let us first study the asymptotic behavior as $z \to \pm \infty$ of the following determinants

\[
\varphi^1(\lambda, \varepsilon, z) = \begin{vmatrix}
\psi^1_1(\lambda, \varepsilon, 0) & \psi^2_1(\lambda, \varepsilon, 0) \\
\psi^1_2(\lambda, \varepsilon, z) & \psi^2_2(\lambda, \varepsilon, z)
\end{vmatrix},
\varphi^2(\lambda, \varepsilon, z) = \begin{vmatrix}
\psi^1_1(\lambda, \varepsilon, 0) & \psi^2_1(\lambda, \varepsilon, 0) \\
\psi^1_2(\lambda, \varepsilon, z) & \psi^2_2(\lambda, \varepsilon, z)
\end{vmatrix},
\varphi^3(\lambda, \varepsilon, z) = \begin{vmatrix}
\psi^1_1(\lambda, \varepsilon, z_0) & \psi^2_1(\lambda, \varepsilon, z_0) \\
\psi^1_2(\lambda, \varepsilon, z) & \psi^2_2(\lambda, \varepsilon, z)
\end{vmatrix},
\varphi^4(\lambda, \varepsilon, z) = \begin{vmatrix}
\psi^1_1(\lambda, \varepsilon, z_0) & \psi^2_1(\lambda, \varepsilon, z_0) \\
\psi^1_2(\lambda, \varepsilon, z) & \psi^2_2(\lambda, \varepsilon, z)
\end{vmatrix}.
\]

Let $\varphi_1(\lambda, \varepsilon, z) = (\varphi^1(\lambda, \varepsilon, z), \varphi^2(\lambda, \varepsilon, z))$ and $\varphi_2(\lambda, \varepsilon, z) = (\varphi^3(\lambda, \varepsilon, z), \varphi^4(\lambda, \varepsilon, z))$. It is very easy to see $\varphi^1(\lambda, \varepsilon, 0) = 0, \varphi^2(\lambda, \varepsilon, 0) = \det(\psi_1(\lambda, \varepsilon, 0), \psi_2(\lambda, \varepsilon, 0)) = D(\lambda, \varepsilon) \neq 0$, similarly we have $\varphi^3(\lambda, \varepsilon, z_0) = 0, \varphi^4(\lambda, \varepsilon, z_0) = \det(\psi_1(\lambda, \varepsilon, z_0), \psi_2(\lambda, \varepsilon, z_0)) = D(\lambda, \varepsilon) \exp\left[-\frac{1}{v(\varepsilon)} \int_{z_0}^z 2\lambda + \varepsilon \gamma - f'(\phi(x)) dx\right] \equiv D(\lambda, \varepsilon, z_0) \neq 0$. It is easy to see $\varphi_1(\lambda, \varepsilon, z)$ and $\varphi_2(\lambda, \varepsilon, z)$ are linearly independent. Actually by definition, we have

\[
(\varphi_1(\lambda, \varepsilon, z), \varphi_2(\lambda, \varepsilon, z)) = (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \begin{pmatrix}
-\psi_2^1(\lambda, \varepsilon, 0) & -\psi_2^1(\lambda, \varepsilon, z_0) \\
+\psi_1^1(\lambda, \varepsilon, 0) & +\psi_1^1(\lambda, \varepsilon, z_0)
\end{pmatrix}.
\]
Therefore
\[
\det(\varphi_1(\lambda, \varepsilon, z), \varphi_2(\lambda, \varepsilon, z)) = \det(\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \det \begin{pmatrix}
-\psi_2^1(\lambda, \varepsilon, 0) & -\psi_2^1(\lambda, \varepsilon, z_0) \\
+\psi_1^1(\lambda, \varepsilon, 0) & +\psi_1^1(\lambda, \varepsilon, z_0)
\end{pmatrix}
\]
\[
= \det(\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \det \begin{pmatrix}
\psi_1^1(\lambda, \varepsilon, 0) & \psi_2^1(\lambda, \varepsilon, 0) \\
\psi_1^1(\lambda, \varepsilon, z_0) & \psi_2^1(\lambda, \varepsilon, z_0)
\end{pmatrix}.
\]

Since they are linearly independent solutions of the intermediate system, we can use them to define a new intermediate Evans function by
\[
\det(\varphi_1(\lambda, \varepsilon, z), \varphi_2(\lambda, \varepsilon, z)) \exp[\frac{1}{v(\varepsilon)} \int_0^\varepsilon 2\lambda + \varepsilon \gamma - f'(\phi(r))dr] = \det(\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \exp[\frac{1}{v(\varepsilon)} \int_0^\varepsilon 2\lambda + \varepsilon \gamma - f'(\phi(r))dr] \\
\times \det \begin{pmatrix}
\psi_1^1(\lambda, \varepsilon, 0) & \psi_2^1(\lambda, \varepsilon, 0) \\
\psi_1^1(\lambda, \varepsilon, z_0) & \psi_2^1(\lambda, \varepsilon, z_0)
\end{pmatrix} = D(\lambda, \varepsilon)\varphi^1(\lambda, \varepsilon, z_0) \neq 0.
\]

In the light of \(\varphi^2(0, \varepsilon, z)\) and \(\varphi^4(0, \varepsilon, z)\), we can write \(P_1(\lambda, \varepsilon)\) and \(Q_1(\lambda, \varepsilon)\) in a compact form
\[
P_1(\lambda, \varepsilon) = \int_{-\infty}^0 S(\lambda, \varepsilon, z)f(z)\varphi^2(0, \varepsilon, z)dz, \quad Q_1(\lambda, \varepsilon) = \int_{-\infty}^0 S(\lambda, \varepsilon, z)g(z)\varphi^2(0, \varepsilon, z)dz,
\]
\[
P_2(\lambda, \varepsilon) = \int_{-\infty}^{\infty} S(\lambda, \varepsilon, z)f(z)\varphi^4(0, \varepsilon, z)dz, \quad Q_2(\lambda, \varepsilon) = \int_{-\infty}^{\infty} S(\lambda, \varepsilon, z)g(z)\varphi^4(0, \varepsilon, z)dz.
\]

To find \(\varphi_1(\lambda, \varepsilon, z)\) explicitly, we start from the equations
\[
v(\varepsilon) \frac{\partial}{\partial z} \begin{pmatrix}
\psi_1^1(\lambda, \varepsilon, z) & \psi_2^1(\lambda, \varepsilon, z) \\
\psi_1^2(\lambda, \varepsilon, z) & \psi_2^2(\lambda, \varepsilon, z)
\end{pmatrix} = \begin{pmatrix}
-\psi_2^1(\lambda, \varepsilon, 0) \\
+\psi_1^1(\lambda, \varepsilon, 0)
\end{pmatrix}
\]

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one shows that \( \psi_1(\lambda, \varepsilon, z) \) satisfies the equation

\[
\psi_1(\lambda, \varepsilon, z) = 0.
\]

Rewriting this system as

\[
v(\varepsilon) \frac{\partial \varphi}{\partial z} + \begin{pmatrix} \lambda - f'(\phi) & 1 \\ -\varepsilon & \lambda + \varepsilon \gamma \end{pmatrix} \varphi = 0.
\]

Multiplying it by the integrating factor \( \exp\left[\frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) z\right] \) gives

\[
v(\varepsilon) \frac{\partial \varphi}{\partial z} \left\{ \exp\left[\frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) z\right] \varphi(\lambda, \varepsilon, z) \right\} = \left[ a + f'(\phi) \right] \exp\left[\frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) z\right] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi(\lambda, \varepsilon, z).
\]

Integrating in \( z \) over \([z, 0]\) yields

\[
v(\varepsilon) \varphi(\lambda, \varepsilon, 0) - v(\varepsilon) \exp\left[\frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) z\right] \varphi(\lambda, \varepsilon, z)
= \int_z^0 \left[ a + f'(\phi(r)) \right] \exp\left[\frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) r\right] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi(\lambda, \varepsilon, r) dr.
\]

Therefore we obtain the explicit solution

\[
\varphi(\lambda, \varepsilon, z) = \exp\left[-\frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) z\right] \varphi(\lambda, \varepsilon, 0)
- \frac{1}{v(\varepsilon)} \int_z^0 \left[ a + f'(\phi(r)) \right] \exp\left[-\frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) (z - r)\right] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi(\lambda, \varepsilon, r) dr.
\]

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To partially get rid of the burden of notation, below we will only be concerned with the case \( \lambda = 0 \). For the cases \( \lambda \neq 0 \), we replace \( \omega_+(\varepsilon) \) and \( \omega_-(\varepsilon) \) with \( \mu_+(\lambda, \varepsilon) \) and \( \mu_-(\lambda, \varepsilon) \), respectively. The two components solve the equations

\[
\varphi^1(\lambda, \varepsilon, z) = \frac{D(\lambda, \varepsilon)}{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \left\{ \exp\left[ -\frac{1}{v(\varepsilon)} \omega_+(\varepsilon) z \right] - \exp\left[ -\frac{1}{v(\varepsilon)} \omega_-(\varepsilon) z \right] \right\} \\
- \frac{1}{v(\varepsilon)} \sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon} \int^0_z \left[ a + f'(\phi(r)) \right] dr \\
\times \left\{ \left[ \omega_+(\varepsilon) - \varepsilon \gamma \right] \exp\left[ -\frac{1}{v(\varepsilon)} \omega_+(\varepsilon) (z - r) \right] \right\} \varphi^1(\lambda, \varepsilon, r) dr \\
+ \left[ \omega_-(\varepsilon) - \varepsilon \gamma \right] \exp\left[ -\frac{1}{v(\varepsilon)} \omega_-(\varepsilon) (z - r) \right] \right\} \varphi^1(\lambda, \varepsilon, r) dr,
\]

\[
\varphi^2(\lambda, \varepsilon, z) = \frac{D(\lambda, \varepsilon)}{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \left\{ \left[ \omega_+(\varepsilon) - a \right] \exp\left[ -\frac{1}{v(\varepsilon)} \omega_+(\varepsilon) z \right] \right\} \\
+ \left[ a - \omega_-(\varepsilon) \right] \exp\left[ -\frac{1}{v(\varepsilon)} \omega_-(\varepsilon) z \right] \right\} \varphi^1(\lambda, \varepsilon, r) dr \\
+ \frac{\varepsilon}{v(\varepsilon)} \sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon} \int^0_z \left[ a + f'(\phi(r)) \right] dr \\
\times \left\{ \exp\left[ -\frac{1}{v(\varepsilon)} \omega_+(\varepsilon) (z - r) \right] - \exp\left[ -\frac{1}{v(\varepsilon)} \omega_-(\varepsilon) (z - r) \right] \right\} \varphi^1(\lambda, \varepsilon, r) dr.
\]

If \( \varphi^1(\lambda, \varepsilon, z) \) is positive in a small neighborhood, say \((-\delta, 0)\), then one can easily construct a contradiction. So it is negative in at least a small interval \((-\delta, 0)\). Setting \( \pi^1(\lambda, \varepsilon, z) = \varphi^1(\lambda, \varepsilon, z)/D(\lambda, \varepsilon) \), then \( \pi^1(\lambda, \varepsilon, z) > 0 \) in this interval. Similarly \( \varphi^1(\lambda, \varepsilon, z) > 0 \) and \( \pi^1(\lambda, \varepsilon, z) < 0 \) in at least a small interval \((0, \delta)\), where \( \delta > 0 \) is a finite number or \( \delta = \infty \). If \( \beta \geq 0 \) in the interval \( \{ z : \varepsilon \gamma + f'(\phi(z)) < 0 \} \), then

\[
v(\varepsilon) \beta_z = \beta^2 - [\varepsilon \gamma + f'(\phi(z))] \beta + \varepsilon \geq \varepsilon,
\]

so \( \beta \) is always increasing and this will force \( \beta \) to be negative for sufficiently large \( z \), where \( z \) is negative. Contradiction. Hence there exists a negative number \( z_1 \) such
that \( \beta(z_1) = 0 \) and \( \beta(z_1) = \frac{\varepsilon}{\nu(\varepsilon)} > 0 \). If there exists another number \( z_2 < z_1 \) such that \( \beta(z_2) = 0 \), then we must have \( \beta'(z_2) \leq 0 \). This contradicts the fact \( \beta'(z_2) = \frac{\varepsilon}{\nu(\varepsilon)} > 0 \).

Therefore \( z_1 \) is the only point such that \( \beta(z_1) = 0 \). Consequently \( \beta(z) < 0 \) for all \( z < z_1 \) and \( \beta(z) > 0 \) if \( z > z_1 \). To determine how \( z_1 \) depends on \( \varepsilon \), let us suppose that \( \beta = \beta_1 + \frac{1}{u} \) is a solution of the equation \( v(\varepsilon)\beta_z = \beta^2 - [\varepsilon\gamma + f'(\phi)]\beta + \varepsilon \). Then \( u \) solves the first order linear equation

\[
v(\varepsilon)u_z + [2\beta_1 - \varepsilon\gamma - f'(\phi)]u + 1 = 0.
\]

The solution of this equation satisfies

\[
u(z) \exp\left[\frac{1}{v(\varepsilon)} \int_{z_1}^{z} 2\beta_1(s) - \varepsilon\gamma - f'(\phi(s))ds\right]
\]

\[= c - \frac{1}{v(\varepsilon)} \int_{z_1}^{z} \exp\left[\frac{1}{v(\varepsilon)} \int_{z_1}^{s} 2\beta_1(r) - \varepsilon\gamma - f'(\phi(r))dr\right]ds.
\]

Equivalently we have

\[
\frac{1}{\nu(z)} = \exp\left[\frac{1}{v(\varepsilon)} \int_{z_1}^{z} 2\beta_1(s) - \varepsilon\gamma - f'(\phi(s))ds\right]
\times \left\{c - \frac{1}{v(\varepsilon)} \int_{z_1}^{z} \exp\left[\frac{1}{v(\varepsilon)} \int_{z_1}^{s} 2\beta_1(r) - \varepsilon\gamma - f'(\phi(r))dr\right]ds\right\}^{-1}.
\]

Recall that \( \beta(z) = \beta_1(z) + \frac{1}{u(z)} \to +\infty \) as \( z \to 0^- \), so

\[
c = \frac{1}{v(\varepsilon)} \int_{z_1}^{0} \exp\left[\frac{1}{v(\varepsilon)} \int_{z_1}^{s} 2\beta_1(s) - \varepsilon\gamma - f'(\phi(s))ds\right]dz.
\]

Now we obtain the solution of the equation \( v(\varepsilon)\beta_z = \beta^2 - [\varepsilon\gamma + f'(\phi)]\beta + \varepsilon \)

\[
\beta(z) = \beta_1(z) + \exp\left[\frac{1}{v(\varepsilon)} \int_{z_1}^{z} 2\beta_1(s) - \varepsilon\gamma - f'(\phi(s))ds\right]
\]

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\[ \times \left\{ \frac{1}{v(\varepsilon)} \int_{z_1}^{0} \exp\left[ \frac{1}{v(\varepsilon)} \int_{z_1}^{z} 2\beta_1(s) - \varepsilon \gamma - f'(\phi(s))ds \right]dz \right. \\
- \frac{1}{v(\varepsilon)} \int_{z_1}^{v} \exp\left[ \frac{1}{v(\varepsilon)} \int_{z_1}^{\varepsilon} 2\beta_1(r) - \varepsilon \gamma - f'(\phi(r))dr \right]ds \right\}^{-1}. \]

On the other hand, we know \( \beta(z_1) = 0 \), so we get the equation

\[ \int_{z_1}^{0} \exp\left[ \frac{1}{v(\varepsilon)} \int_{z_1}^{z} 2\beta_1(s) - \varepsilon \gamma - f'(\phi(s))ds \right]dz = 1 - \frac{v(\varepsilon)}{\beta_1(z_1)} > 1. \]

Let \( z = z_1 + \ln \frac{t}{z_1} \), then \( \frac{dz}{dt} = \frac{t}{z_1} \). Now we get

\[ \int_{-\infty}^{0} S(\lambda, \varepsilon, z)f(z)\varphi^2(0, \varepsilon, z)dz = \int_{z_1}^{0} S(\lambda, \varepsilon, z)f(z)\varphi^2(0, \varepsilon, z)dz + \int_{-\infty}^{z_1} S(\lambda, \varepsilon, z)f(z)\varphi^2(0, \varepsilon, z)dz \]
\[ = \int_{z_1}^{0} S(\lambda, \varepsilon, z)f(z)\varphi^2(0, \varepsilon, z)dz + \int_{z_1}^{0} \frac{1}{z}S(\lambda, \varepsilon, z_1 + \ln \frac{z}{z_1})f(z_1 + \ln \frac{z}{z_1})\varphi^2(0, \varepsilon, z_1 + \ln \frac{z}{z_1})dz \]
\[ = \int_{z_1}^{0} S(\lambda, \varepsilon, z)f(z)\varphi^2(0, \varepsilon, z) + \frac{1}{z}S(\lambda, \varepsilon, z_1 + \ln \frac{z}{z_1})f(z_1 + \ln \frac{z}{z_1})\varphi^2(0, \varepsilon, z_1 + \ln \frac{z}{z_1})dz. \]

It is to see the integrand is positive.

By the projectivized equation \( v(\varepsilon) \frac{\partial}{\partial z} = -\varepsilon \beta^2 + [\varepsilon \gamma + f'(\phi(z))] \beta - 1 \), where \( \beta(\varepsilon, z) = \varphi^1(\varepsilon, z) / \varphi^2(\varepsilon, z) \), we conclude that \( z = 0 \) is the only point such that \( \varphi^1(\lambda, \varepsilon, 0) = 0 \) and \( z = z_0 \) is the only point such that \( \varphi^3(\lambda, \varepsilon, z_0) = 0 \). Therefore \( \delta = \infty \). One would be interested in a more accurate estimate. Let \( g(\lambda, \varepsilon, z) = \pi^1(\lambda, \varepsilon, z) \exp\left[ \frac{1}{v(\varepsilon)} \omega_+(\varepsilon)z \right] \).

From the first equation, we now get

\[ g(\lambda, \varepsilon, z) = \frac{1}{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \left\{ 1 - \exp\left[ \frac{1}{v(\varepsilon)} \sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon z} \right] \right\} \]
\[ - \frac{1}{v(\varepsilon) \sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \int_{z_1}^{0} [a + f'(\phi(r))] \]
\[ \times \left\{ \omega_+(\varepsilon) - \varepsilon \gamma + [\varepsilon \gamma - \omega_-(\varepsilon)] \exp\left[ \frac{1}{v(\varepsilon)} \sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon (z - r)} \right] \right\} g(\lambda, \varepsilon, r)dr. \]
Let us derive some preliminary estimates to control the integrand. There exists a positive constant $\rho$ such that $0 \leq \phi(z) \leq \exp \left[ \frac{1}{v(\epsilon)} \rho z \right]$ for all $z \leq 0$. Then $0 < a + f'(\phi(z)) = (2 + 2a - 3\phi)\phi \leq \exp \left[ \frac{1}{v(\epsilon)} \rho z \right]$ for $z < 0$. Moreover for all $z < r < 0$, we have

\[ [\omega_+ (\epsilon) - \epsilon \gamma] \exp \left[ \frac{1}{v(\epsilon)} \omega_+ (\epsilon)(z - r) \right] + [\epsilon \gamma - \omega_- (\epsilon)] \exp \left[ - \frac{1}{v(\epsilon)} \omega_- (\epsilon)(z - r) \right] \]

\[ = [\omega_+ (\epsilon) - \omega_- (\epsilon)] \exp \left[ - \frac{1}{v(\epsilon)} \omega_+ (\epsilon)(z - r) \right] + [\omega_- (\epsilon) - \epsilon \gamma] \exp \left[ - \frac{1}{v(\epsilon)} \omega_- (\epsilon)(z - r) \right] \]

\[ + [\epsilon \gamma - \omega_- (\epsilon)] \exp \left[ - \frac{1}{v(\epsilon)} \omega_- (\epsilon)(z - r) \right] \]

\[ \geq \sqrt{(a - \epsilon \gamma)^2 - 4\epsilon} \exp \left[ - \frac{1}{v(\epsilon)} \omega_+ (\epsilon)(z - r) \right]. \]

We need to derive some bounds of $g$. First let us consider the upper bound.

\[ g(\lambda, \epsilon, z) \leq \frac{1}{\sqrt{(a - \epsilon \gamma)^2 - 4\epsilon}} - \frac{1}{v(\epsilon)} \int_z^0 [a + f'(\phi(r))] \exp \left[ - \frac{1}{v(\epsilon)} \omega_+ (\epsilon)(r - z) \right] \]

Applying Lemma 2.1 yields

\[ g(\lambda, \epsilon, z) \leq \frac{1}{\sqrt{(a - \epsilon \gamma)^2 - 4\epsilon}} \exp \left[ - \frac{1}{v(\epsilon)} \int_z^0 a + f'(\phi(r)) \right]. \]

Secondly let us derive the lower bound of $g$. Clearly

\[ -g(\lambda, \epsilon, z) \leq - \frac{1}{\sqrt{(a - \epsilon \gamma)^2 - 4\epsilon}} \left\{ 1 - \exp \left[ - \frac{1}{v(\epsilon)} \sqrt{(a - \epsilon \gamma)^2 - 4\epsilon z} \right] \right\} \]

\[ - \frac{\omega_+ (\epsilon) - \epsilon \gamma}{v(\epsilon) \sqrt{(a - \epsilon \gamma)^2 - 4\epsilon}} \int_z^0 [a + f'(\phi(r))] \left[ -g(\lambda, \epsilon, r) \right] \]

Generalized Gronwall’s inequality gives the estimate

\[ -g(\lambda, \epsilon, z) \leq - \frac{1}{v(\epsilon)} \int_z^0 \exp \left[ \frac{1}{v(\epsilon)} \sqrt{(a - \epsilon \gamma)^2 - 4\epsilon z} \right]. \]
\[ x \exp\left[ -\frac{\omega_+(\varepsilon) - \varepsilon y}{v(\varepsilon) \sqrt{(a - \varepsilon y)^2 - 4\varepsilon}} \int_s^a + f'(\phi(r))dr \right]ds. \]

Hence \( g(\lambda, \varepsilon, z) \) has a positive lower bound. This implies that \( \pi^1(\lambda, \varepsilon, z) \) never vanishes in \((-\infty, 0)\). For all \( z < 0 \), by the above analysis, we have

\[ \pi^1(0, \varepsilon, z) \leq \frac{1}{\sqrt{(a - \varepsilon y)^2 - 4\varepsilon}} \left\{ \exp\left[ -\frac{1}{v(\varepsilon)} \omega_+(\varepsilon)z \right] - \exp\left[ -\frac{1}{v(\varepsilon)} \omega_-(\varepsilon)z \right] \right\}. \]

Recall \( D(\lambda, \varepsilon) = D(0, \varepsilon) < 0 \), so we have \( \varphi^1(\lambda, \varepsilon, z) > 0, \pi^1(\lambda, \varepsilon, z) < 0 \) if \( z > 0 \) and \( \varphi^1(\lambda, \varepsilon, z) < 0, \pi^1(\lambda, \varepsilon, z) > 0 \) if \( z < 0 \). Therefore

\[
\varphi^2(\lambda, \varepsilon, z) = \frac{D(\lambda, \varepsilon)}{\sqrt{(a - \varepsilon y)^2 - 4\varepsilon}} \left\{ [\omega_+(\varepsilon) - a] \exp\left[ -\frac{1}{v(\varepsilon)} \omega_+(\varepsilon)z \right] + [a - \omega_- (\varepsilon)] \exp\left[ -\frac{1}{v(\varepsilon)} \omega_- (\varepsilon)z \right] \right\} \\
+ \frac{D(\lambda, \varepsilon)\varepsilon}{v(\varepsilon) \sqrt{(a - \varepsilon y)^2 - 4\varepsilon}} \int_z^0 [a + f'(\phi(r))] \\
\times \left\{ \exp\left[ -\frac{1}{v(\varepsilon)} \omega_+(\varepsilon)(z - r) \right] - \exp\left[ -\frac{1}{v(\varepsilon)} \omega_- (\varepsilon)(z - r) \right] \right\} \pi^1(\lambda, \varepsilon, r)dr \\
eq D(\lambda, \varepsilon) \pi^2(\lambda, \varepsilon, z), \pi^2(\lambda, \varepsilon, z) > 0, \text{ for all } z < 0, \\
\varphi^2(\lambda, \varepsilon, z) \equiv I_1(\lambda, \varepsilon, z) + I_2(\lambda, \varepsilon, z) + I_3(\lambda, \varepsilon, z). \\

It is easy to derive the following inequality about \( I_1(\lambda, \varepsilon, z) \) and \( I_3(\lambda, \varepsilon, z) \)

\[
\frac{1}{\sqrt{(a - \varepsilon y)^2 - 4\varepsilon}} [a - \omega_+(\varepsilon)] \exp\left[ -\frac{1}{v(\varepsilon)} \omega_+(\varepsilon)z \right] \\
> \frac{\varepsilon}{v(\varepsilon) \sqrt{(a - \varepsilon y)^2 - 4\varepsilon}} \int_z^0 [a + f'(\phi(r))] \\
\times \left\{ \exp\left[ -\frac{1}{v(\varepsilon)} \omega_+(\varepsilon)(z - r) \right] - \exp\left[ -\frac{1}{v(\varepsilon)} \omega_- (\varepsilon)(z - r) \right] \right\} \pi^1(\lambda, \varepsilon, r)dr,
\]

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at least for $z$ negatively large. Therefore $I_1(\lambda, \varepsilon, z) + I_3(\lambda, \varepsilon, z) > 0$ and is of order $\varepsilon$, for fixed $\lambda$ and $z$, but $I_2(\lambda, \varepsilon, z) < 0$ and is of order $1$. In the same fashion, we have

$$\varphi^3(\lambda, \varepsilon, z) = \frac{D(\lambda, \varepsilon, z_0)}{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \{\exp[-\frac{1}{v(\varepsilon)}\omega_+(\varepsilon)(z - z_0)] - \exp[-\frac{1}{v(\varepsilon)}\omega_-(\varepsilon)(z - z_0)]\}$$

$$- \frac{1}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \int_z^{z_0} [a + f'(\phi(r))]$$

$$\times \{[\omega_+(\varepsilon) - \varepsilon \gamma] \exp[-\frac{1}{v(\varepsilon)}\omega_+(\varepsilon)(z - r)]$$

$$+ [\varepsilon \gamma - \omega_-(\varepsilon)] \exp[-\frac{1}{v(\varepsilon)}\omega_-(\varepsilon)(z - r)]\}dr$$

$$\equiv D(\lambda, \varepsilon, z_0)\pi^3(\lambda, \varepsilon, z),$$

$$\varphi^4(\lambda, \varepsilon, z) = \frac{D(\lambda, \varepsilon, z_0)}{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \{(\omega_+(\varepsilon) - a] \exp[-\frac{1}{v(\varepsilon)}\omega_+(\varepsilon)(z - z_0)]$$

$$+ [a - \omega_-(\varepsilon)] \exp[-\frac{1}{v(\varepsilon)}\omega_-(\varepsilon)(z - z_0)]\}$$

$$+ \frac{D(\lambda, \varepsilon, z_0)\varepsilon}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \int_z^{z_0} [a + f'(\phi(r))]$$

$$\times \{\exp[-\frac{1}{v(\varepsilon)}\omega_+(\varepsilon)(z - r)] - \exp[-\frac{1}{v(\varepsilon)}\omega_-(\varepsilon)(z - r)]\}\pi^3(\lambda, \varepsilon, r)dr$$

$$\equiv D(\lambda, \varepsilon, z_0)\pi^4(\lambda, \varepsilon, z), \pi^4(\lambda, \varepsilon, z) > 0, \text{ for all } z < z_0,$$

$$\varphi^4(\lambda, \varepsilon, z) \equiv J_1(\lambda, \varepsilon, z) + J_2(\lambda, \varepsilon, z) + J_3(\lambda, \varepsilon, z).$$

Same inequalities hold for $J_i(\lambda, \varepsilon, z)$. These estimates will be very useful in seeking for critical eigenvalues. Keep in mind that $\pi^3(\lambda, \varepsilon, z) > 0$ if $z < z_0$ and $\pi^3(\lambda, \varepsilon, z) < 0$ if $z > z_0$. In addition,

$$\exp[-\frac{1}{v(\varepsilon)}\lambda z_0]D(0, \varepsilon, z_0) = \exp[-\frac{1}{v(\varepsilon)}(\lambda + a + \varepsilon \gamma) z_0] \exp[\frac{1}{v(\varepsilon)} \int_0^{z_0} a + f'(\phi(z)) dz]D(\lambda, \varepsilon).$$

By the fast-slow structure of the traveling wave, namely, along the slow manifolds the traveling wave solution spends a time which is of order $\frac{1}{\varepsilon}$, along the transition
layers it spends a time of order 1. Moreover, $f'(\beta) < f'(1) < f'(0) = -a < 0$. and recall the definition of $z_0(\varepsilon)$, i.e. $\phi'(z_0) = \theta$ and $\phi'(z_0) < 0$, it is not hard to verify
\[
\exp\left[\frac{1}{v(\varepsilon)} \int_{0}^{z_0} a + f'(\phi(z))dz\right] = \exp\left[\frac{\kappa}{\varepsilon}\right]O(1),
\]
for some positive constant $\kappa$. Now we obtain the estimates

\[
P_1(\lambda, \varepsilon) = \int_{-\infty}^{0} S(\lambda, \varepsilon, z)f(z)D(\lambda, \varepsilon)\pi^2(0, \varepsilon, z)dz
= \frac{1}{v(\varepsilon)} \int_{-\infty}^{0} \exp\left[-\frac{1}{v(\varepsilon)}(\lambda + a + \varepsilon \gamma)z\right]
\times \exp\left[\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r))dr\right]f(z)\pi^2(0, \varepsilon, z)dz
= O(1),
\]

\[
Q_1(\lambda, \varepsilon) = \int_{-\infty}^{0} S(\lambda, \varepsilon, z)g(z)D(\lambda, \varepsilon)\pi^2(0, \varepsilon, z)dz
= \frac{1}{v(\varepsilon)} \int_{-\infty}^{0} \exp\left[-\frac{1}{v(\varepsilon)}(\lambda + a + \varepsilon \gamma)z\right]
\times \exp\left[\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r))dr\right]g(z)\pi^2(0, \varepsilon, z)dz
= \exp\left[-\frac{\kappa}{\varepsilon}\right]O(1),
\]

\[
P_2(\lambda, \varepsilon) = \int_{-\infty}^{z_0} S(\lambda, \varepsilon, z)f(z)D(0, \varepsilon, z_0)\pi^4(0, \varepsilon, z)dz
= \frac{1}{v(\varepsilon)} \int_{-\infty}^{z_0} \exp\left[-\frac{1}{v(\varepsilon)}(\lambda + a + \varepsilon \gamma)(z - z_0)\right]
\times \exp\left[\frac{1}{v(\varepsilon)} \int_{z}^{z_0} a + f'(\phi(r))dr\right]f(z)\pi^4(0, \varepsilon, z)dz
= \frac{1}{v(\varepsilon)} \int_{-\infty}^{0} \exp\left[-\frac{1}{v(\varepsilon)}(\lambda + a + \varepsilon \gamma)z\right]
\times \exp\left[\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(z_0))dr\right]f(z + z_0)\pi^4(0, \varepsilon, z + z_0)dz
= O(1),
\]

\[
Q_2(\lambda, \varepsilon) = \int_{-\infty}^{z_0} S(\lambda, \varepsilon, z)g(z)D(0, \varepsilon, z_0)\pi^4(0, \varepsilon, z)dz
\]
\[
\frac{1}{v(\varepsilon)} \int_{-\infty}^{z_0} \exp\left[ \frac{1}{v(\varepsilon)} (\lambda + \varepsilon \gamma)(z - z_0) \right] \\
\times \exp\left[ \frac{1}{v(\varepsilon)} \int_{z}^{z_0} a + f'(\phi(r))dr \right] g(z)\pi^4(0, \varepsilon, z)dz \\
= \frac{1}{v(\varepsilon)} \int_{-\infty}^{0} \exp\left[ \frac{1}{v(\varepsilon)} (\lambda + \varepsilon \gamma)z \right] \\
\times \exp\left[ \frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r + z_0))dr \right] g(z + z_0)\pi^4(0, \varepsilon, z + z_0)dz \\
= \exp\left[ -\frac{\kappa}{\varepsilon} \right] O(1).
\]

We only emphasize the estimates depending on \( \varepsilon \) here. In the next section we will emphasis the estimates depending on \( \lambda \).

9. Bounds of \( P_i(\lambda, \varepsilon) \) and \( Q_i(\lambda, \varepsilon) \)

With the help of the explicit solutions \( \varphi_i(\lambda, \varepsilon, z) \), we now come to estimate \( P_i(\lambda, \varepsilon) \) and \( Q_i(\lambda, \varepsilon) \). First let us derive some preliminary estimates.

**Lemma 14.** Let \( a \) be a positive number and let \( f \) be a bounded smooth function such that

\[
\lim_{t \to -\infty} f(t)e^{at} = 0.
\]

Then we have

\[
\int_{-\infty}^{z} [f'(t) + af(t)]e^{at}dt = f(z)e^{az}.
\]

**Lemma 15.** Let \( a \) be a complex number with positive real part and let \( f \) be a positive, bounded, continuously differentiable function defined on \((-\infty, 0)\), such that \( f' \in L^1(-\infty, 0) \). Define

\[
u(a, z) = ae^{-az} \int_{-\infty}^{z} f(t)e^{at}dt.
\]

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Then \( u \) is uniformly bounded and smooth and satisfies
\[
\frac{\partial u}{\partial z} = af(z) - au(a, z),
\]
\[
u(a, 0) = a \int_{-\infty}^{0} f(t)e^{at}dt.
\]

In addition
\[
\int_{-\infty}^{z} f(t)e^{at}dt = \frac{1}{a}e^{az}u(a, z),
\]
\[
\lim_{z \to -\infty} u(a, z) = \lim_{z \to -\infty} f(z),
\]
\[
\lim_{a \to 0^+} u(a, z) = \lim_{z \to -\infty} f(z),
\]
\[
|u(a, z)| \leq C, \text{ for all } a, z,
\]
\[
u(a, z) \neq 0, \text{ for all } a, z.
\]

Proof. Since \( f \) is continuously differentiable, using integration by parts
\[
u(a, z) = e^{-az} \int_{-\infty}^{z} f(t)[e^{at}]'dt = f(z) - e^{-az} \int_{-\infty}^{0} f'(t)e^{at}dt.
\]

It is easy to see \(|u(a, z)| \leq |f(z)| + \int_{-\infty}^{0} |f'(z)|dz \leq C\), independent of \( a \) and \( z \). In addition, letting \( a \to 0^+ \), we see
\[
\lim_{a \to 0^+} u(a, z) = \lim_{z \to -\infty} f(z).
\]

By L’Hospital’s rule, the limit
\[
\lim_{z \to -\infty} u(a, z) = \lim_{z \to -\infty} ae^{-az} \int_{-\infty}^{z} f(t)e^{at}dt = \lim_{z \to -\infty} f(z).
\]

Now we use these preliminary estimates to derive bounds of \( P_i(\lambda, \varepsilon) \) and \( Q_i(\lambda, \varepsilon) \).

First of all, since \([P_1(0, \varepsilon) - 1][Q_2(0, \varepsilon) - 1] = P_2(0, \varepsilon)Q_1(0, \varepsilon)\) and \( P_i(0, \varepsilon) > 0, Q_i(0, \varepsilon) > \)
0, \( Q_1(0,\varepsilon) = \exp[-\frac{\varepsilon}{\varepsilon}]O(1) \), we have \( P_1(0,\varepsilon) \leq 1 \). Secondly for all complex number with \( \text{Re}\lambda > 0 \) and all \( \varepsilon > 0 \), there holds

\[
|P_1(\lambda,\varepsilon)| < P_1(0,\varepsilon), \quad |Q_1(\lambda,\varepsilon)| < Q_1(0,\varepsilon) < 1,
|P_2(\lambda,\varepsilon)| < P_2(0,\varepsilon), \quad |Q_2(\lambda,\varepsilon)| < Q_2(0,\varepsilon) < 1.
\]

To see roughly how \( P_i(\lambda,\varepsilon) \) and \( Q_i(\lambda,\varepsilon) \) depend on \( \lambda \), let us consider a special function \( K(z) = \frac{k}{2\pi i} \exp[-\frac{k}{u(i)}|z|] \), for all \( z \in R \), where \( k > 0 \) is a constant. First

\[
P_1(\lambda,\varepsilon) = \int_{-\infty}^{0} S(\lambda,\varepsilon, z) f(z) \varphi^2(0,\varepsilon, z) dz
= \int_{-\infty}^{0} S(\lambda,\varepsilon, z) f(z) I_1(0,\varepsilon, z) dz + \int_{-\infty}^{0} S(\lambda,\varepsilon, z) f(z) I_2(0,\varepsilon, z) dz
= \int_{-\infty}^{0} S(\lambda,\varepsilon, z) f(z) I_3(0,\varepsilon, z) dz \equiv I_1(\lambda,\varepsilon) + I_2(\lambda,\varepsilon) + I_3(\lambda,\varepsilon).
\]

Let us employ Lemma 15 to estimate these terms one by one.

\[
\frac{\omega_+(\varepsilon) - a}{v(\varepsilon)\sqrt{(a - \varepsilon\gamma)^2 - 4\varepsilon}} \int_{-\infty}^{0} \exp\left[\frac{\mu_-(\lambda,\varepsilon) + k}{v(\varepsilon)}z\right] \exp\left[\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r)) dr\right] dz
= \frac{\omega_+(-\varepsilon) - a}{\mu_-(\lambda,\varepsilon) + k}\sqrt{(a - \varepsilon\gamma)^2 - 4\varepsilon} p_{11}(-\lambda,\varepsilon),
\]

\[
\frac{a - \omega_-(\varepsilon)}{v(\varepsilon)\sqrt{(a - \varepsilon\gamma)^2 - 4\varepsilon}} \int_{-\infty}^{0} \exp\left[\frac{\mu_+(\lambda,\varepsilon) + k}{v(\varepsilon)}z\right] \exp\left[\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r)) dr\right] dz
= \frac{a - \omega_-(-\varepsilon)}{\mu_+(\lambda,\varepsilon) + k}\sqrt{(a - \varepsilon\gamma)^2 - 4\varepsilon} p_{12}(-\lambda,\varepsilon),
\]

\[
\frac{\varepsilon}{v(\varepsilon)^2\sqrt{(a - \varepsilon\gamma)^2 - 4\varepsilon}} \int_{-\infty}^{0} \exp\left[\frac{\mu_-(\lambda,\varepsilon) + k}{v(\varepsilon)}z\right] \exp\left[\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r)) dr\right] dz
= \frac{\varepsilon}{v(\varepsilon)(\mu_-(\lambda,\varepsilon) + k)\sqrt{(a - \varepsilon\gamma)^2 - 4\varepsilon}} p_{13}(-\lambda,\varepsilon).
\]
For $Q_1(\lambda, \varepsilon) = \int_{-\infty}^{0} S(\lambda, \varepsilon, z)g(z)\varphi^2(0, \varepsilon, z)dz$, the following estimates hold

$$\frac{\omega_+(\varepsilon) - \lambda}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[-\frac{kz_0}{v(\varepsilon)}\right]$$

$$\times \int_{-\infty}^{0} \exp\left[\frac{\mu-(\lambda, \varepsilon) + k}{v(\varepsilon)} z\right] \exp\left[-\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r))dr\right]dz$$

$$= \frac{\omega_+(\varepsilon) - \lambda}{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[-\frac{kz_0}{v(\varepsilon)}\right]p_{11}(\lambda, \varepsilon),$$

$$\times \int_{-\infty}^{0} \exp\left[\frac{\mu_+(\lambda, \varepsilon) + k}{v(\varepsilon)} z\right] \exp\left[-\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r))dr\right]dz$$

$$= \frac{\mu_+(\lambda, \varepsilon) + k}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[-\frac{kz_0}{v(\varepsilon)}\right]p_{12}(\lambda, \varepsilon),$$

$$\times \int_{-\infty}^{0} \exp\left[\frac{\mu-(\lambda, \varepsilon) + k}{v(\varepsilon)} z\right] \exp\left[-\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r))dr\right]dz$$

$$= \frac{\mu-(\lambda, \varepsilon) + k}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[-\frac{kz_0}{v(\varepsilon)}\right]p_{13}(\lambda, \varepsilon).$$

Overall, $Q_1(\lambda, \varepsilon) = \exp[-\frac{\varepsilon}{\varepsilon}]O(1)$, for all $\lambda \in \Omega$ with $|\lambda| \leq M$.

For $P_2(\lambda, \varepsilon) = \exp[-\frac{\lambda z_0}{v(\varepsilon)}] \int_{-\infty}^{\infty} S(\lambda, \varepsilon, z)f(z)\varphi^4(0, \varepsilon, z)dz$, the following estimates hold

$$\frac{\omega_+(\varepsilon) - \lambda}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[\frac{\lambda + a + \varepsilon \gamma + k}{v(\varepsilon)} z_0\right]$$

$$\times \int_{-\infty}^{\infty} \exp\left[\frac{\mu-(\lambda, \varepsilon) + k}{v(\varepsilon)} (z - z_0)\right] \exp\left[-\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r))dr\right]dz$$

$$= \frac{\omega_+(\varepsilon) - \lambda}{[\mu-(\lambda, \varepsilon) + k]\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[-\frac{kz_0}{v(\varepsilon)}\right]p_{21}(\lambda, \varepsilon),$$

$$\times \int_{-\infty}^{\infty} \exp\left[\frac{\mu_+(\lambda, \varepsilon) + k}{v(\varepsilon)} (z - z_0)\right] \exp\left[-\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r))dr\right]dz$$

$$= \frac{\mu_+(\lambda, \varepsilon) + k}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[-\frac{kz_0}{v(\varepsilon)}\right]p_{22}(\lambda, \varepsilon),$$

$$\times \int_{-\infty}^{\infty} \exp\left[\frac{\mu-(\lambda, \varepsilon) + k}{v(\varepsilon)} (z - z_0)\right] \exp\left[-\frac{1}{v(\varepsilon)} \int_{z}^{0} a + f'(\phi(r))dr\right]dz$$

$$= \frac{\mu-(\lambda, \varepsilon) + k}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[-\frac{kz_0}{v(\varepsilon)}\right]p_{23}(\lambda, \varepsilon).$$
\[
\frac{a - \omega_-(\varepsilon)}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[\frac{\lambda + a + \varepsilon \gamma + k}{v(\varepsilon)z_0}\right] \\
\times \int_{-\infty}^{z_0} \exp\left[\frac{\mu_-(\lambda, \varepsilon) + k}{v(\varepsilon)}(z - z_0)\right] \exp\left[\frac{1}{v(\varepsilon)} \int_0^a + f'(\phi(r))dr\right]dz \\
= \frac{\varepsilon}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[\frac{\lambda + a + \varepsilon \gamma + k}{v(\varepsilon)z_0}\right]p_{22}(\lambda, \varepsilon), \\
\times \int_{-\infty}^{z_0} \exp\left[\frac{\mu_-(\lambda, \varepsilon) + k}{v(\varepsilon)}(z - z_0)\right] \exp\left[\frac{1}{v(\varepsilon)} \int_0^a + f'(\phi(r))dr\right]dz \\
= \frac{\varepsilon}{v(\varepsilon)\mu_-(\lambda, \varepsilon) + k]\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[\frac{\lambda + a + \varepsilon \gamma + k}{v(\varepsilon)z_0}\right]p_{23}(\lambda, \varepsilon).
\]

For \( Q_2(\lambda, \varepsilon) = \exp[-\frac{A_0}{v(\varepsilon)}] \int_{-\infty}^{z_0} S(\lambda, \varepsilon, z)g(z)\varphi^4(0, \varepsilon, z)dz \), the following estimates hold

\[
\frac{\omega_+(\varepsilon) - a}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[\frac{\lambda + a + \varepsilon \gamma}{v(\varepsilon)z_0}\right] \\
\times \int_{-\infty}^{z_0} \exp\left[\frac{\mu_-(\lambda, \varepsilon) + k}{v(\varepsilon)}(z - z_0)\right] \exp\left[\frac{1}{v(\varepsilon)} \int_0^a + f'(\phi(r))dr\right]dz \\
= \frac{\omega_+(\varepsilon) - a}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[\frac{\lambda + a + \varepsilon \gamma}{v(\varepsilon)z_0}\right]p_{21}(\lambda, \varepsilon), \\
\times \int_{-\infty}^{z_0} \exp\left[\frac{\mu_-(\lambda, \varepsilon) + k}{v(\varepsilon)}(z - z_0)\right] \exp\left[\frac{1}{v(\varepsilon)} \int_0^a + f'(\phi(r))dr\right]dz \\
= \frac{a - \omega_-(\varepsilon)}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[\frac{\lambda + a + \varepsilon \gamma}{v(\varepsilon)z_0}\right]p_{22}(\lambda, \varepsilon), \\
\times \int_{-\infty}^{z_0} \exp\left[\frac{\mu_-(\lambda, \varepsilon) + k}{v(\varepsilon)}(z - z_0)\right] \exp\left[\frac{1}{v(\varepsilon)} \int_0^a + f'(\phi(r))dr\right]dz \\
= \frac{\varepsilon}{v(\varepsilon)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[\frac{\lambda + a + \varepsilon \gamma}{v(\varepsilon)z_0}\right]p_{23}(\lambda, \varepsilon).
\]
\[
\frac{\varepsilon}{v(\varepsilon)[\mu_-(\lambda, \varepsilon) + k]\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \exp\left[\frac{\lambda + a + \varepsilon \gamma}{v(\varepsilon)} z_0\right] p_{23}(\lambda, \varepsilon).
\]

As before \( Q_2(\lambda, \varepsilon) = \exp[-\frac{\varepsilon}{\varepsilon}]O(1) \) for all \( \lambda \in \Omega \) with \(|\lambda| \leq M \). In the above estimates, the \( p \)'s depend weakly on \( \lambda \) and \( \varepsilon \). They can essentially be regarded as constants, independent of \( \lambda \) and \( \varepsilon \).

There is an inner relationship among the \( p \)'s if \( \lambda \) is an eigenvalue, in particular \( \lambda = 0 \) is an eigenvalue. To derive the equations governing the \( p \)'s, recall that

\[
\frac{\phi'(0)}{\alpha} \left( \begin{array}{c}
P_1(0, \varepsilon) - 1 \\
P_2(0, \varepsilon)
\end{array} \right) + \frac{\phi'(z_0)}{\alpha} \left( \begin{array}{c}
Q_1(0, \varepsilon) \\
Q_2(0, \varepsilon) - 1
\end{array} \right) = 0.
\]

So we obtain the equations for \( \lambda = 0 \),

\[
\begin{align*}
\frac{a - \omega_-(\varepsilon)}{[\mu_+(\lambda, \varepsilon) + k]\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} & \left[1 + \exp\left(-\frac{kz_0}{v(\varepsilon)}\right)\right] p_{12}(\lambda, \varepsilon) \\
+ & \frac{1}{[\mu_-(\lambda, \varepsilon) + k]\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \left[1 + \exp\left(-\frac{kz_0}{v(\varepsilon)}\right)\right] \\
\times & \left[(\omega_+(\varepsilon) - a)p_{11}(\lambda, \varepsilon) + \frac{\varepsilon}{v(\varepsilon)} p_{13}(\lambda, \varepsilon)\right] = \frac{\phi'(0)}{\alpha}, \\
\frac{a - \omega_-(\varepsilon)}{[\mu_+(\lambda, \varepsilon) + k]\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} & \left[1 + \exp\left(-\frac{kz_0}{v(\varepsilon)}\right)\right] \\
\times & \exp\left[\frac{\lambda + a + \varepsilon \gamma + k}{v(\varepsilon)} z_0\right] p_{22}(\lambda, \varepsilon) \\
+ & \frac{1}{[\mu_-(\lambda, \varepsilon) + k]\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \left[1 + \exp\left(-\frac{kz_0}{v(\varepsilon)}\right)\right] \\
\times & \left[(\omega_+(\varepsilon) - a)p_{21}(\lambda, \varepsilon) + \frac{\varepsilon}{v(\varepsilon)} p_{23}(\lambda, \varepsilon)\right] \exp\left[\frac{\lambda + a + \varepsilon \gamma + k}{v(\varepsilon)} z_0\right] \\
= & \frac{\phi'(z_0)}{\alpha} \exp\left[\frac{1}{v(\varepsilon)} \int_{z_0}^\infty \lambda + \varepsilon \gamma - f'(\phi(z)) \, dz\right].
\end{align*}
\]

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10. Completion of Proof of the Stability

To show the traveling wave solution is stable, we must show that there exists some positive constant $c(\varepsilon)$, such that the only simple eigenvalue in the region $\text{Re}\lambda > -c(\varepsilon)$ is $\lambda = 0$. Because there is no eigenvalue outside a uniformly large circle which is independent of $\varepsilon$, it suffices to show that there exists no eigenvalue in the subset $\text{Re}\lambda \geq 0$ except for a simple eigenvalue $\lambda = 0$. Let us prove a preliminary result here.

**Lemma 16.** Let $f$ be a positive $L^1$-integrable function defined on the real line. Then for all interval $(a, b)$ and all nonzero real number $\xi$, where $-\infty < a < b < \infty$ or $a = -\infty$ or $b = +\infty$, we have

$$| \int_a^b e^{ix\xi} f(x) dx | < \int_a^b f(x) dx.$$

Proof. To show for all nonzero number $\xi$ and all interval $(a, b)$ the above estimate holds, it suffices to show there exists a small interval $(\alpha, \beta) \subset (a, b)$, such that

$$| \int_\alpha^\beta e^{ix\xi} f(x) dx | < \int_\alpha^\beta f(x) dx.$$

Suppose that for all $(\alpha, \beta) \subset (a, b)$, there holds

$$| \int_\alpha^\beta e^{ix\xi} f(x) dx | = \int_\alpha^\beta f(x) dx.$$

Then we have the identity

$$| \int_\alpha^\beta \cos(x\xi) f(x) dx |^2 + | \int_\alpha^\beta \sin(x\xi) f(x) dx |^2 = | \int_\alpha^\beta f(x) dx |^2.$$

Differentiating this identity about $\beta$ gives

$$2 \int_\alpha^\beta \cos(x\xi) f(x) dx \cos(\beta\xi) f(\beta) + 2 \int_\alpha^\beta \sin(x\xi) f(x) dx \sin(\beta\xi) f(\beta) = 2 \int_\alpha^\beta f(x) dx f(\beta).$$
Since \( f > 0 \) on the real line, we have
\[
\int_\alpha^\beta \cos(x\xi)f(x)dx \cos(\beta \xi) + \int_\alpha^\beta \sin(x\xi)f(x)dx \sin(\beta \xi) = \int_\alpha^\beta f(x)dx.
\]

Now differentiating about \( \alpha \) yields
\[
\cos(\alpha \xi) \cos(\beta \xi)f(\alpha) + \sin(\alpha \xi) \sin(\beta \xi)f(\alpha) = f(\alpha).
\]

Again cancelling \( f(\alpha) \) we get
\[
\cos(\alpha \xi) \cos(\beta \xi) + \sin(\alpha \xi) \sin(\beta \xi) = 1.
\]

This is equivalent to \( \cos[(\alpha - \beta)\xi] = 1 \) or \( (\alpha - \beta)\xi = 2n\pi \) for some integer \( n \). But \( \alpha \) and \( \beta \) are arbitrary, so we get a contradiction. Thus this implies there exists at least a small interval such that \( |\int_\alpha^\beta e^{ixf(x)dx}| < \int_\alpha^\beta f(x)dx \).

Let us first show that there exists no eigenvalue on the imaginary axis except for \( \lambda = 0 \). Suppose that \( \xi \neq 0 \). Then
\[
|P_1(i\xi, \varepsilon)| \leq P_1(0, \varepsilon), \quad |Q_1(i\xi, \varepsilon)| \leq Q_1(0, \varepsilon),
\]
\[
|P_2(i\xi, \varepsilon)| \leq P_2(0, \varepsilon), \quad |Q_2(i\xi, \varepsilon)| \leq Q_2(0, \varepsilon),
\]
where at least one \( < \) holds, thus
\[
|E(i\xi, \varepsilon)| = |[P_1(i\xi, \varepsilon) - 1][Q_2(i\xi, \varepsilon) - 1] - P_2(i\xi, \varepsilon)Q_1(i\xi, \varepsilon)|
\geq |[P_1(i\xi, \varepsilon) - 1][Q_2(i\xi, \varepsilon) - 1] - P_2(i\xi, \varepsilon)Q_1(i\xi, \varepsilon)|
> [1 - P_1(0, \varepsilon)][1 - Q_2(0, \varepsilon)] - P_2(0, \varepsilon)Q_1(0, \varepsilon) = 0,
\]
and \( E(i\xi, \varepsilon) \neq 0 \), for all \( \xi \neq 0 \).
**Lemma 17.** There exists no eigenvalue on the imaginary axis, except for $\lambda = 0$.

Secondly let us show that there exists no eigenvalue in the region $\text{Re}\lambda > 0$. If $K$ has compact support, then there is a positive number $K_0$, such that if $|z| > M_0$, then $K(z) = 0$. Recall that $z_0(\varepsilon) = O(1/\varepsilon)$, so $K(z - z_0) = 0$, for all $z \leq 0$. This means $Q_1(\lambda, \varepsilon) = 0$, for all $\lambda \in \Omega$ and all sufficiently small $\varepsilon > 0$. For all $\lambda \in \Omega$ with $\text{Re}\lambda > 0$, we have $|E(\lambda, \varepsilon)| = |P_1(\lambda, \varepsilon) - 1||Q_1(\lambda, \varepsilon) - 1| > |P_1(0, \varepsilon) - 1||Q_1(0, \varepsilon) - 1| = 0$. Therefore $\lambda$ is not an eigenvalue. Since $E(\lambda, \varepsilon)$ is analytic, there are at most finitely many eigenvalues of $L(\varepsilon)$ inside the region $\{\lambda \in \Omega : |\lambda| \leq M\}$. Hence any candidate has necessarily a negative real part. Therefore the traveling wave solution is stable, if $\lambda = 0$ is a simple eigenvalue.

Now we consider general cases of $K(z)$. For all $\lambda \in \Omega$ with $\text{Re}\lambda > 0$, we have

$$|E(\lambda, \varepsilon)| = |P_1(\lambda, \varepsilon) - 1||Q_2(\lambda, \varepsilon) - 1| - P_2(\lambda, \varepsilon)Q_1(\lambda, \varepsilon)|$$

$$\geq |P_1(\lambda, \varepsilon) - 1||Q_2(\lambda, \varepsilon) - 1| - |P_2(\lambda, \varepsilon)Q_1(\lambda, \varepsilon)|$$

$$> |P_1(0, \varepsilon) - 1||Q_2(0, \varepsilon) - 1| - |P_2(0, \varepsilon)Q_1(0, \varepsilon)|$$

$$= [1 - P_1(0, \varepsilon)][1 - Q_2(0, \varepsilon)] - P_2(0, \varepsilon)Q_1(0, \varepsilon) = 0,$$

hence $\lambda$ is not an eigenvalue.

**Lemma 18.** There exists no eigenvalue in the region $\text{Re}\lambda > 0$.

To see how close the critical eigenvalue to the origin is, let us start from the eigenvalue equation $(\lambda + \varepsilon\gamma)\psi^2 + v(\varepsilon)\psi_x^2 = \varepsilon\psi^1$, we get the estimate

$$(\lambda + \varepsilon\gamma)^2 \int_{-\infty}^{\infty} |\psi^2|^2 dz \leq (\lambda + \varepsilon\gamma)^2 \int_{-\infty}^{\infty} |\psi^2|^2 dz + (\varepsilon)^2 \int_{-\infty}^{\infty} |\psi_x^2|^2 dz = \varepsilon^2 \int_{-\infty}^{\infty} |\psi^1|^2 dz.$$
Setting
\[ C^2 = \int_{-\infty}^{\infty} |\psi'|^2 dz / \int_{-\infty}^{\infty} |\psi|^2 dz. \]

Then \(|\lambda + \varepsilon \gamma| \leq C\varepsilon\). This implies the critical eigenvalue is of order \(\varepsilon\). Same arguments as before shows the traveling wave is stable, if the eigenvalue \(\lambda = 0\) is simple.

Indeed it is easy to see the derivative of the Evans function at \(\lambda = 0\) is positive. Recall that \((\phi, \varphi)\) is the traveling wave and \((\psi_1(0, \varepsilon, z), \psi_2(0, \varepsilon, z))\) is the fundamental matrix of the intermediate system if \(\lambda = 0\). Consider the nonhomogeneous differential equations
\[
v(\varepsilon) \frac{\partial}{\partial z} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} + \begin{pmatrix} -f'(\phi) & 1 \\ -\varepsilon & \varepsilon \gamma \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} f(z)\psi^1(0) + g(z)\psi^1(z_0) \\ 0 \end{pmatrix} - \begin{pmatrix} \phi_z \\ \varphi_z \end{pmatrix}.
\]

Let us use the method of variation of parameter to find the solutions of this system. Suppose that
\[
c_1(\varepsilon, z)\psi_1(0, \varepsilon, z) + c_2(\varepsilon, z)\psi_2(0, \varepsilon, z) = (\psi_1(0, \varepsilon, z), \psi_2(0, \varepsilon, z)) \begin{pmatrix} c_1(\varepsilon, z) \\ c_2(\varepsilon, z) \end{pmatrix}
\]
is a solution. Then \((c_1(\varepsilon, z), c_2(\varepsilon, z))\) satisfies the equations
\[
v(\varepsilon)(\psi_1(0, \varepsilon, z), \psi_2(0, \varepsilon, z)) \frac{\partial}{\partial z} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} f(z)\psi^1(0) + g(z)\psi^1(z_0) \\ 0 \end{pmatrix} - \begin{pmatrix} \phi_z \\ \varphi_z \end{pmatrix}.
\]

The intermediate Evans function gives
\[
\frac{\partial}{\partial z} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = S(0, \varepsilon, z) \begin{pmatrix} +\psi_2^2(0, \varepsilon, z) & -\psi_1^2(0, \varepsilon, z) \\ -\psi_2^1(0, \varepsilon, z) & +\psi_1^1(0, \varepsilon, z) \end{pmatrix} \begin{pmatrix} f(z)\psi^1(0) + g(z)\psi^1(z_0) \\ 0 \end{pmatrix}
\]

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$$-S(0, \varepsilon, z) \begin{pmatrix} +\psi_2^0(0, \varepsilon, z) & -\psi_1^0(0, \varepsilon, z) \\ -\psi_1^0(0, \varepsilon, z) & +\psi_1^0(0, \varepsilon, z) \end{pmatrix} \begin{pmatrix} \phi_z \\ \varphi_z \end{pmatrix}.$$ Integrating in $z$ yields the equation

$$\begin{pmatrix} c_1(\varepsilon, z) \\ c_2(\varepsilon, z) \end{pmatrix} = \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix} + \int_{-\infty}^{z} S(0, \varepsilon, r) \begin{pmatrix} +\psi_2^0(0, \varepsilon, r) & -\psi_1^0(0, \varepsilon, r) \\ -\psi_1^0(0, \varepsilon, r) & +\psi_1^0(0, \varepsilon, r) \end{pmatrix} \begin{pmatrix} f(r)\psi_1^0(0) + g(r)\psi_1^0(z_0) \\ 0 \end{pmatrix} dr$$

$$- \int_{-\infty}^{z} S(0, \varepsilon, r) \begin{pmatrix} +\psi_2^0(0, \varepsilon, r) & -\psi_1^0(0, \varepsilon, r) \\ -\psi_1^0(0, \varepsilon, r) & +\psi_1^0(0, \varepsilon, r) \end{pmatrix} \begin{pmatrix} \phi_z(\varepsilon, r) \\ \varphi_z(\varepsilon, r) \end{pmatrix} dr.$$

By the definition of $\varphi_1$, we have

$$\begin{pmatrix} \psi_1^0(0, \varepsilon, 0), \psi_2^0(0, \varepsilon, 0) \end{pmatrix} \int_{-\infty}^{0} S(0, \varepsilon, r) \begin{pmatrix} +\psi_2^0(0, \varepsilon, r) & -\psi_1^0(0, \varepsilon, r) \\ -\psi_1^0(0, \varepsilon, r) & +\psi_1^0(0, \varepsilon, r) \end{pmatrix} \begin{pmatrix} \phi_z(\varepsilon, r) \\ \varphi_z(\varepsilon, r) \end{pmatrix} dr$$

$$= \int_{-\infty}^{0} S(0, \varepsilon, r) (\varphi^2(0, \varepsilon, r), -\varphi^1(0, \varepsilon, r)) \begin{pmatrix} \phi_z(\varepsilon, r) \\ \varphi_z(\varepsilon, r) \end{pmatrix} dr \equiv T(0),$$

$$\begin{pmatrix} \psi_1^0(0, \varepsilon, z_0), \psi_2^0(0, \varepsilon, z_0) \end{pmatrix} \int_{-\infty}^{z_0} S(0, \varepsilon, r) \begin{pmatrix} +\psi_2^0(0, \varepsilon, r) & -\psi_1^0(0, \varepsilon, r) \\ -\psi_1^0(0, \varepsilon, r) & +\psi_1^0(0, \varepsilon, r) \end{pmatrix} \begin{pmatrix} \phi_z(\varepsilon, r) \\ \varphi_z(\varepsilon, r) \end{pmatrix} dr$$

$$= \int_{-\infty}^{z_0} S(0, \varepsilon, r) (\varphi^4(0, \varepsilon, r), -\varphi^3(0, \varepsilon, r)) \begin{pmatrix} \phi_z(\varepsilon, r) \\ \varphi_z(\varepsilon, r) \end{pmatrix} dr \equiv T(z_0).$$

It is easy to see $T(0) = O(1)$ and $T(z_0) = \exp[-\frac{t}{\varepsilon}]O(1)$. The compatibility condition of $\psi^1(0, \varepsilon, z)$ at $z = 0$ and $z = z_0$ yields

$$\psi^1(0, \varepsilon, 0) = (\psi_1^0(0, \varepsilon, 0), \psi_2^0(0, \varepsilon, 0)) \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix}$$
\[
\psi^1(0, \varepsilon, z_0) = (\psi^1_1(0, \varepsilon, z_0), \psi^1_2(0, \varepsilon, z_0)) \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix} + P_1(0, \varepsilon)\psi^1(0, \varepsilon, 0) + Q_1(0, \varepsilon)\psi^1(0, \varepsilon, z_0) - T(0),
\]

Collecting terms gives
\[
(\psi^1_1(0, \varepsilon, 0), \psi^1_2(0, \varepsilon, 0)) \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix} = [1 - P_1(0, \varepsilon)]\psi^1(0, \varepsilon, 0) - Q_1(0, \varepsilon)\psi^1(0, \varepsilon, z_0) + T(0),
\]

\[
(\psi^1_1(0, \varepsilon, z_0), \psi^1_2(0, \varepsilon, z_0)) \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix} = [1 - Q_2(0, \varepsilon)]\psi^1(0, \varepsilon, z_0) - P_2(0, \varepsilon)\psi^1(0, \varepsilon, 0) + T(z_0).
\]

In matrix form we have
\[
\begin{pmatrix}
\psi^1_1(0, \varepsilon, 0) & \psi^1_1(0, \varepsilon, 0) \\
\psi^1_1(0, \varepsilon, z_0) & \psi^1_2(0, \varepsilon, z_0)
\end{pmatrix}
\begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix} = M(0, \varepsilon)
\begin{pmatrix}
\psi^1(0, \varepsilon, 0) \\
\psi^1(0, \varepsilon, z_0)
\end{pmatrix} +
\begin{pmatrix} T(0) \\ T(z_0) \end{pmatrix}.
\]

The algebraic system
\[
M(0, \varepsilon)
\begin{pmatrix} \psi^1(0, \varepsilon, 0) \\ \psi^1(0, \varepsilon, z_0) \end{pmatrix} =
\begin{pmatrix} T(0) \\ T(z_0) \end{pmatrix},
\]

has no solution, because
\[
\det M(0, \varepsilon) = E(0, \varepsilon) = 0,
\]

and the null space \( \{ v \in \mathcal{C}^2 : vM(0, \varepsilon) = (0, 0) \} \) is 1-dimensional. Actually the null space is \( \{ cw : c \in \mathcal{C} \} \), where \( w = (1, Q_1(0, \varepsilon)/[1 - Q_2(0, \varepsilon)]) = (1, \exp[-\pi/\varepsilon]O(1)) \). The
scalar product of the two vectors \( w \) and \((T(0), T(z_0))\) is \( T(0) + T(z_0) \exp[-\frac{x}{\varepsilon}]O(1) \neq 0\), since \( T(z_0) = \exp[-\frac{x}{\varepsilon}]O(1) \). So the right hand side is not a zero vector. Solving this equation we see \((c_1(\varepsilon), c_2(\varepsilon)) \neq (0, 0)\). Hence any solution to the nonhomogeneous system is unbounded as \( z \to -\infty \). We have obtained the following result.

**Lemma 19.** There is no bounded smooth solution to the equations

\[
\begin{align*}
  v(\varepsilon)\psi_1^1 + \phi_z &= f'(\phi)\psi^1 - \psi^2 + \frac{\alpha}{\phi'(0)} K(z)\psi^1(0) - \frac{\alpha}{\phi'(z_0)} K(z - z_0)\psi^1(z_0), \\
  v(\varepsilon)\psi_2^2 + \varphi_z &= \varepsilon(\psi^1 - \gamma \psi^2).
\end{align*}
\]

This means the neutral eigenvalue \( \lambda = 0 \) is simple. The locally unique traveling wave solution is exponentially stable.

**Remark.** We are also interested in the explicit \( P_1(\lambda, \varepsilon) \) and \( Q_1(\lambda, \varepsilon) \). Let us consider a special case \( K(z) = \frac{1}{2} \exp(-|z|) \). For this case there holds

\[
\begin{pmatrix}
  P_1(\lambda, \varepsilon) \\
  P_2(\lambda, \varepsilon)
\end{pmatrix}
= \exp(z_0) \begin{pmatrix}
  \frac{\phi'(z_0)}{\phi'(0)} \\
  \phi'(0)
\end{pmatrix}
\begin{pmatrix}
  Q_1(\lambda, \varepsilon) \\
  Q_2(\lambda, \varepsilon)
\end{pmatrix},
\]

for all \( \lambda \) and \( \varepsilon \). Suppose that \( \lambda \in \Omega \) is an eigenvalue of \( L(\varepsilon) \) and that \((\xi, \eta)\) is such that

\[
M(\lambda, \varepsilon) \begin{pmatrix}
  \xi \\
  \eta
\end{pmatrix} = 0.
\]

Therefore we obtain

\[
\begin{align*}
  P_1(\lambda, \varepsilon) &= \frac{\xi \phi'(z_0) \exp(z_0)}{\xi \phi'(z_0) \exp(z_0) + \eta \phi'(0)}, \\
  Q_1(\lambda, \varepsilon) &= \frac{\xi \phi'(z_0)}{\xi \phi'(z_0) \exp(z_0) + \eta \phi'(0)}, \\
  P_2(\lambda, \varepsilon) &= \frac{\eta \phi'(0) \exp(z_0)}{\xi \phi'(z_0) \exp(z_0) + \eta \phi'(0)}, \\
  Q_2(\lambda, \varepsilon) &= \frac{\eta \phi'(0)}{\xi \phi'(z_0) \exp(z_0) + \eta \phi'(0)}.
\end{align*}
\]
In particular for \( \lambda = 0 \) we have

\[
\begin{align*}
P_1(\lambda, \varepsilon) &= \frac{\exp(z_0)}{\exp(z_0) + 1}, Q_1(\lambda, \varepsilon) = \frac{1}{\exp(z_0) + 1}, \\
P_2(\lambda, \varepsilon) &= \frac{\exp(z_0)}{\exp(z_0) + 1}, Q_2(\lambda, \varepsilon) = \frac{1}{\exp(z_0) + 1}.
\end{align*}
\]

These are beautiful expressions.

11. The Evans Function for Particular Function \( K \)

In general the explicit Evans function is very difficult to find. Let us look at the special case \( K(z) = \frac{k}{2} \exp(-k|z|) \), where \( k > 0 \) is a constant. Now the Evans function \( E(\lambda, \varepsilon) \) can be calculated explicitly as follows

\[
\begin{align*}
E(\lambda, \varepsilon) &= \frac{P_1(\lambda, \varepsilon) - 1}{P_2(\lambda, \varepsilon)} - \frac{Q_1(\lambda, \varepsilon)}{Q_2(\lambda, \varepsilon)} \\
&= 1 - P_1(\lambda, \varepsilon) - Q_2(\lambda, \varepsilon) \\
&= 1 - k(\lambda)O(\varepsilon) + \exp[-\frac{k}{\varepsilon}]O(1).
\end{align*}
\]
Let us consider the case $K$ has compact support. Let $K(z) = \frac{1}{2k}$ if $|z| \leq k$ and $K(z) = 0$ if $|z| > k$, where $k$ is a positive constant. Then $E(\lambda, \varepsilon) = [P_1(\lambda, \varepsilon) - 1][Q_2(\lambda, \varepsilon) - 1]$, where $P_1(\lambda, \varepsilon) = 1$ if $\lambda = 0$. If $\lambda \neq 0$, then

\[
\int_{-\infty}^{0} S(\lambda, \varepsilon, z)f(z)\varphi^2(0, \varepsilon, z)dz = \frac{\alpha[a - \omega_-(\varepsilon)]}{2k\varphi(0)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \int_{-k}^{0} \exp\left[-\frac{1}{\varepsilon} (\lambda + \omega_+(\varepsilon))z\right]dz - C(\lambda, \varepsilon)\varepsilon
\]

\[
= \frac{\alpha[a - \omega_-(\varepsilon)]}{2k\varphi(0)[\lambda + \omega_+(\varepsilon)]\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \left\{1 - \exp\left[-\frac{1}{\varepsilon} (\lambda + \omega_+(\varepsilon))k\right]\right\} - C(\lambda, \varepsilon)\varepsilon
\]

\[
= \frac{\omega(\varepsilon)}{\lambda + \omega_+(\varepsilon)} \left\{1 - \exp\left[-\frac{1}{\varepsilon} (\lambda + \omega_+(\varepsilon))k\right]\right\} - C(\lambda, \varepsilon)\varepsilon
\]

where

\[
\omega(\varepsilon) = \frac{\alpha[a - \omega_-(\varepsilon)]}{2k\varphi(0)\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} = [1 + C(0, \varepsilon)\varepsilon\omega_+(\varepsilon)]/\left\{1 - \exp\left[-\frac{1}{\varepsilon} \omega_+(\varepsilon)k\right]\right\}.
\]

In both cases, the critical eigenvalue is negatively very small.

Remark. We conjecture that the neutral eigenvalue is essentially due to the front and the critical eigenvalue is essentially due to the back.

Remark. We hope to show by homotopy argument that, as $K$ changes continuously in $L^1$ norm, the critical eigenvalues will vary continuously, but never crosses the imaginary axis. This result is an analogy of the Fitzhugh-Nagumo equations.

Remark. If we approach the Evans function from another aspect, i.e. by calculation

\[
P_1(\lambda, \varepsilon) = (\psi_1^1(\lambda, \varepsilon, 0), \psi_2^1(\lambda, \varepsilon, 0)) \int_{-\infty}^{0} F(\lambda, \varepsilon, z)dz
\]

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\[ \begin{align*}
&= (\psi_1(0, \epsilon, 0), \psi_2(0, \epsilon, 0)) \int_{-\infty}^{0} \exp \left[ \frac{\lambda z}{v(\epsilon)} \right] F(0, \epsilon, z) \, dz \\
&= \int_{-\infty}^{0} \exp \left[ \frac{\lambda z}{v(\epsilon)} \right] \frac{\partial}{\partial z} \left[ \int_{-\infty}^{z} S(0, \epsilon, r)f(r)\varphi^2(\lambda, \epsilon, r) \, dr \right] \, dz \\
&= \int_{-\infty}^{0} S(0, \epsilon, z)f(z)\varphi^2(\epsilon, z) \, dz \\
&- \frac{\lambda}{v(\epsilon)} \int_{-\infty}^{0} \left[ \int_{-\infty}^{z} S(0, \epsilon, r)f(r)\varphi^2(\lambda, \epsilon, r) \, dr \right] \exp \left[ \frac{\lambda z}{v(\epsilon)} \right] \, dz \\
&\equiv P_1(0, \epsilon) + P_{11}(\lambda, \epsilon).
\end{align*} \]

Similarly we have
\[
Q_1(\lambda, \epsilon) = Q_1(0, \epsilon) + Q_{11}(\lambda, \epsilon),
\]
\[
P_2(\lambda, \epsilon) = P_2(0, \epsilon) + P_{21}(\lambda, \epsilon),
\]
\[
Q_2(\lambda, \epsilon) = Q_2(0, \epsilon) + Q_{21}(\lambda, \epsilon).
\]

The definition of the Evans function \( E(\lambda, \epsilon) \) yields now
\[
\begin{vmatrix}
P_1(\lambda, \epsilon) - 1 & Q_1(\lambda, \epsilon) \\
P_2(\lambda, \epsilon) & Q_2(\lambda, \epsilon) - 1
\end{vmatrix}
= \begin{vmatrix}
P_1(0, \epsilon) + P_{11}(\lambda, \epsilon) - 1 & Q_1(0, \epsilon) + Q_{11}(\lambda, \epsilon) \\
P_2(0, \epsilon) + P_{21}(\lambda, \epsilon) & Q_2(0, \epsilon) + Q_{21}(\lambda, \epsilon) - 1
\end{vmatrix}
\]
\[
= \begin{vmatrix}
P_1(0, \epsilon) - 1 & Q_1(0, \epsilon) \\
P_2(0, \epsilon) & Q_2(0, \epsilon) - 1
\end{vmatrix}
+ \begin{vmatrix}
P_1(0, \epsilon) - 1 & Q_{11}(\lambda, \epsilon) \\
P_2(0, \epsilon) & Q_{21}(\lambda, \epsilon)
\end{vmatrix}
\]
\[
+ \begin{vmatrix}
P_{11}(\lambda, \epsilon) & Q_1(0, \epsilon) \\
P_{21}(\lambda, \epsilon) & Q_2(0, \epsilon) - 1
\end{vmatrix}
\]

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Our conjecture is that for any $K$, satisfying the conditions given earlier, there are two positive constants $C_1$ and $C_2$, such that the above $= C_1 \lambda^2 + C_2 \lambda \epsilon$. Then we see the eigenvalues of $L(\epsilon)$ in $\Omega$ are $\lambda = 0$ and $\lambda = -c(\epsilon)\epsilon$.

If $\lambda = \xi + i\eta$ is an eigenvalue of $L(\epsilon)$, then $\lambda = \xi - i\eta$ is also an eigenvalue. Thus $\lambda[(\lambda - \xi)^2 + \eta^2]$ is an analytic factor of the Evans function. But this contradicts the expression of the Evans function. Hence all the eigenvalues in $\Omega$ are real. Also $0 = |E(\lambda, \epsilon)| > E(\xi, \epsilon) \geq 0$ if $\eta \neq 0$, this is impossible.

In this paper we showed the solution $\psi(\lambda, \epsilon, z)$ of the original eigenvalue problem is well defined but is not an eigenfunction if $\lambda \neq 0$ and possibly $-c(\epsilon)\epsilon$, for some positive constant $c(\epsilon)$. The transversal intersection of the singular stable and unstable
manifolds implies that this neutral eigenvalue $\lambda = 0$ is algebraically simple. Hence the traveling pulse is exponentially stable.

12. Generalization

The explicit solutions $\varphi_1(\lambda, \varepsilon, z)$ and $\varphi_2(\lambda, \varepsilon, z)$ of the auxiliary system are special examples of the following function. Define a vector-valued, complex analytic function $\varphi$ as follows

$$\varphi(\lambda, \varepsilon, s, z) = \left(\begin{array}{c}
\psi_1^1(\lambda, \varepsilon, z) & \psi_1^2(\lambda, \varepsilon, z) \\
-\psi_2^1(\lambda, \varepsilon, s) & +\psi_1^1(\lambda, \varepsilon, s)
\end{array}\right) = \left(\begin{array}{c}
\psi_1^1(\lambda, \varepsilon, s) & \psi_1^2(\lambda, \varepsilon, s) \\
-\psi_2^1(\lambda, \varepsilon, s) & +\psi_1^1(\lambda, \varepsilon, s)
\end{array}\right).
$$

Obviously $\varphi_1(\lambda, \varepsilon, z) = \varphi(\lambda, \varepsilon, 0, z)$ and $\varphi_2(\lambda, \varepsilon, z) = \varphi(\lambda, \varepsilon, z_0, z)$.

Then $\varphi$ satisfies the differential equations

$$u(\varepsilon) \frac{\partial \varphi}{\partial z} + \begin{pmatrix} \lambda - f'(\phi) & 1 \\ -\varepsilon & \lambda + \varepsilon \gamma \end{pmatrix} \varphi = 0,$$

with the initial data

$$\varphi(\lambda, \varepsilon, s, s) = \det(\psi_1(\lambda, \varepsilon, s), \psi_2(\lambda, \varepsilon, s))(0, 1)^T = D(\lambda, \varepsilon) \exp\left[-\frac{1}{u(\varepsilon)} \int_0^s 2\lambda + \varepsilon \gamma - f'(\phi(r)) dr\right](0, 1)^T \equiv D(\lambda, \varepsilon, s)(0, 1)^T.$$
We are interested in the explicit solution of this problem. This system can be written as

\[ v(\varepsilon) \frac{\partial \varphi}{\partial z} + \begin{pmatrix} \lambda + a & 1 \\ -\varepsilon & \lambda + \varepsilon \gamma \end{pmatrix} \varphi = [a + f'(\phi)] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi. \]

Multiplying this system by the integrating factor \( \exp\left[ \frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) z \right] \) yields an equivalent system

\[ v(\varepsilon) \frac{\partial}{\partial z} \left\{ \exp\left[ \frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) z \right] \varphi(\lambda, \varepsilon, s, z) \right\} = [a + f'(\phi)] \exp\left[ \frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) z \right] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi(\lambda, \varepsilon, s, z). \]

Integrating in \( z \), we have

\[ v(\varepsilon) \exp\left[ \frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) s \right] \varphi(\lambda, \varepsilon, s, s) - v(\varepsilon) \exp\left[ \frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) z \right] \varphi(\lambda, \varepsilon, s, z) \]

\[ = \int_z^s [a + f'(\phi(r))] \exp\left[ \frac{1}{v(\varepsilon)} A(\lambda, \varepsilon) r \right] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi(\lambda, \varepsilon, s, r) dr. \]

Rearranging terms gives

\[ \varphi(\lambda, \varepsilon, s, z) = \exp\left[ -\frac{1}{v(\varepsilon)} A(\lambda, \varepsilon)(z - s) \right] \varphi(\lambda, \varepsilon, s, s) \]

\[ - \frac{1}{v(\varepsilon)} \int_z^s [a + f'(\phi(r))] \exp\left[ -\frac{1}{v(\varepsilon)} A(\lambda, \varepsilon)(z - r) \right] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varphi(\lambda, \varepsilon, s, r) dr. \]

To partially get rid of the burden of notation, below we will only be concerned with the case \( \lambda = 0 \). For the cases \( \lambda \neq 0 \), we replace \( \omega_+(\varepsilon) \) and \( \omega_-(\varepsilon) \) with \( \mu_+(\lambda, \varepsilon) \) and \( \mu_-(\lambda, \varepsilon) \) respectively. Writing the two components out explicitly, one obtains

\[ \varphi^1(\lambda, \varepsilon, s, z) = \frac{D(\lambda, \varepsilon, s)}{\sqrt{(a - \varepsilon \gamma)^2 - 4\varepsilon}} \left\{ \exp\left[ -\frac{1}{v(\varepsilon)} \omega_+(\varepsilon)(z - s) \right] - \exp\left[ -\frac{1}{v(\varepsilon)} \omega_-(\varepsilon)(z - s) \right] \right\} \]

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\[ - \frac{1}{v(\varepsilon) \sqrt{(a - \varepsilon \gamma)^2 - 4 \varepsilon}} \int_s^z [a + f'(\phi(r))] \]
\[ \times \{[\omega_+(\varepsilon) - \varepsilon \gamma] \exp[-\frac{1}{v(\varepsilon)} \omega_+(\varepsilon)(z - r)] \]
\[ + [\varepsilon \gamma - \omega_-(\varepsilon)] \exp[-\frac{1}{v(\varepsilon)} \omega_-(\varepsilon)(z - r)]\} \, dr \]
\[ \equiv D(\lambda, \varepsilon, s) \pi(A, \varepsilon, s, z), \]
\[ \varphi^2(\lambda, \varepsilon, s, z) = \frac{D(\lambda, \varepsilon, s)}{\sqrt{(a - \varepsilon \gamma)^2 - 4 \varepsilon}} \{[\omega_+(\varepsilon) - a] \exp[-\frac{1}{v(\varepsilon)} \omega_+(\varepsilon)(z - s)] \]
\[ + [a - \omega_-(\varepsilon)] \exp[-\frac{1}{v(\varepsilon)} \omega_-(\varepsilon)(z - s)]\} \]
\[ + \frac{D(\lambda, \varepsilon, s) \varepsilon}{v(\varepsilon) \sqrt{(a - \varepsilon \gamma)^2 - 4 \varepsilon}} \int_s^z [a + f'(\phi(r))] \]
\[ \times \{\exp[-\frac{1}{v(\varepsilon)} \omega_+(\varepsilon)(z - r)] - \exp[-\frac{1}{v(\varepsilon)} \omega_-(\varepsilon)(z - r)]\} \pi(A, \varepsilon, s, r) \, dr \]
\[ \equiv D(\lambda, \varepsilon, s) \pi^2(\lambda, \varepsilon, s, z), \]
\[ \pi^2(\lambda, \varepsilon, s, z) \equiv I_1(\lambda, \varepsilon, s, z) + I_2(\lambda, \varepsilon, s, z) + I_3(\lambda, \varepsilon, s, z). \]

By the projectivized equation \( v(\varepsilon) \frac{\partial^2}{\partial z^2} = -\varepsilon \beta^2 + [\varepsilon \gamma + f'(\phi(z))] \beta - 1 \), where \( \beta(\varepsilon, z) = \frac{\varphi^1(\varepsilon, z)}{\varphi^2(\varepsilon, z)} \), we conclude that \( z = s \) is the only point such that \( \varphi^1(\lambda, \varepsilon, s, s) = 0 \).

For all \( \lambda \in \Omega \) and \( \varepsilon > 0 \), we have \( \pi^1(\lambda, \varepsilon, s, z) > 0 \) if \( z < s \) and \( \pi^1(\lambda, \varepsilon, s, z) < 0 \) if \( z > s \). Moreover, \( \pi^2(\lambda, \varepsilon, s, z) > 0 \) for all \( z < s \). Clearly \( I_1(\lambda, \varepsilon, s, z) + I_3(\lambda, \varepsilon, s, z) < 0 \) and is of order \( \varepsilon \), for fixed \( \lambda \) and \( z \), but \( I_2(\lambda, \varepsilon, s, z) < 0 \) and is of order 1.

By definition, we have
\[ (\varphi(\lambda, \varepsilon, s, z), \varphi(\lambda, \varepsilon, t, z)) = \begin{pmatrix} \psi_1^1(\lambda, \varepsilon, z) & \psi_2^1(\lambda, \varepsilon, z) \\ \psi_1^2(\lambda, \varepsilon, z) & \psi_2^2(\lambda, \varepsilon, z) \end{pmatrix} \begin{pmatrix} -\psi_2^1(\lambda, \varepsilon, s) & -\psi_1^2(\lambda, \varepsilon, t) \\ +\psi_1^1(\lambda, \varepsilon, s) & +\psi_2^1(\lambda, \varepsilon, t) \end{pmatrix}. \]

Hence
\[ \det(\varphi(\lambda, \varepsilon, s, z), \varphi(\lambda, \varepsilon, t, z)) = \det(\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \]
Since they are linearly independent solutions of the intermediate system, we can use them to define new intermediate Evans functions by

\[
\begin{align*}
    \det(\varphi(\lambda, \varepsilon, s, z), \varphi(\lambda, \varepsilon, t, z)) & \exp\left[\frac{1}{v(\varepsilon)} \int_0^2 2\lambda + \varepsilon \gamma - f'(\phi(r)) dr\right] \\
    = & \det(\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \exp\left[\frac{1}{v(\varepsilon)} \int_0^2 2\lambda + \varepsilon \gamma - f'(\phi(r)) dr\right] \varphi^1(\lambda, \varepsilon, s, t) \\
    = & D(\lambda, \varepsilon) \varphi^1(\lambda, \varepsilon, s, t).
\end{align*}
\]

The two solutions \(\varphi(\lambda, \varepsilon, s, z)\) and \(\varphi(\lambda, \varepsilon, t, z)\) of the auxiliary system are linearly independent if \(s \neq t\).

13. Remarks on the Evans Function

The vector-valued analytic function \(\varphi(\lambda, \varepsilon, s, z)\) defined in the previous section is just a column of the following matrix-valued function. Define a matrix-valued, complex analytic function \(M(\lambda, \varepsilon, s, z) =\)

\[
\begin{align*}
    \begin{pmatrix}
        \psi_1^1(\lambda, \varepsilon, z) & \psi_2^1(\lambda, \varepsilon, z) \\
        \psi_1^2(\lambda, \varepsilon, z) & \psi_2^2(\lambda, \varepsilon, z)
    \end{pmatrix} & \begin{pmatrix}
        +\psi_2^1(\lambda, \varepsilon, s) & -\psi_2^2(\lambda, \varepsilon, s) \\
        -\psi_1^2(\lambda, \varepsilon, s) & +\psi_1^1(\lambda, \varepsilon, s)
    \end{pmatrix} \\
    = & (\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z))(\psi_1(\lambda, \varepsilon, s), \psi_2(\lambda, \varepsilon, s))^{-1} \det(\psi_1(\lambda, \varepsilon, s), \psi_2(\lambda, \varepsilon, s))^{-1}.
\end{align*}
\]
\[
\begin{pmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{pmatrix},
\]

where

\[
\begin{align*}
m_{11} &= \begin{vmatrix}
\psi_1^1(\lambda, \varepsilon, z) & \psi_2^1(\lambda, \varepsilon, z) \\
\psi_1^2(\lambda, \varepsilon, s) & \psi_2^2(\lambda, \varepsilon, s)
\end{vmatrix}, \\
m_{12} &= \begin{vmatrix}
\psi_1^1(\lambda, \varepsilon, s) & \psi_2^1(\lambda, \varepsilon, s) \\
\psi_1^2(\lambda, \varepsilon, z) & \psi_2^2(\lambda, \varepsilon, z)
\end{vmatrix}, \\
m_{21} &= \begin{vmatrix}
\psi_1^2(\lambda, \varepsilon, z) & \psi_2^2(\lambda, \varepsilon, z) \\
\psi_1^2(\lambda, \varepsilon, s) & \psi_2^2(\lambda, \varepsilon, s)
\end{vmatrix}, \\
m_{22} &= \begin{vmatrix}
\psi_1^1(\lambda, \varepsilon, s) & \psi_2^1(\lambda, \varepsilon, s) \\
\psi_1^2(\lambda, \varepsilon, z) & \psi_2^2(\lambda, \varepsilon, z)
\end{vmatrix}. 
\end{align*}
\]

This matrix satisfies the ordinary differential equations

\[
u(\varepsilon) \frac{\partial M}{\partial z} + \begin{pmatrix}
\lambda - f'(\phi) & 1 \\
-\varepsilon & \lambda + \varepsilon \gamma
\end{pmatrix} M = 0.
\]

Notice that if \( z = s \) then \( M(\lambda, \varepsilon, s, s) = \det(\psi_1(\lambda, \varepsilon, s), \psi_2(\lambda, \varepsilon, s))I \). It is actually the fundamental matrix of the auxiliary system. Moreover

\[
\det M(\lambda, \varepsilon, s, z) = \det(\psi_1(\lambda, \varepsilon, z), \psi_2(\lambda, \varepsilon, z)) \det(\psi_1(\lambda, \varepsilon, s), \psi_2(\lambda, \varepsilon, s)).
\]
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