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PREDICTION OF PERIODIC FORCED RESPONSE OF FRICTIONALLY
CONSTRAINED TURBINE BLADES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in Graduate
School of The Ohio State University

by

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*****

The Ohio State University
1999

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ABSTRACT

In turbine engine design, friction dampers are often employed to attenuate the turbine blade vibration and at the same time to increase aeroelastic stability of the turbine blades. The periodic forced response of turbine blades with shroud contacts and wedge dampers are investigated in this research. During the engine operation, the turbine blades may bend and twist to cause the friction contacts to experience friction constraint. The resulting constraint force can add friction damping as well as nonlinear spring force to the bladed disk system. When subjecting to periodic excitation, a 3D shroud contact model and a dual-interface friction model are employed to simulate the hysteresis loops of the constrained forces at the contact points of turbine bladed with shroud contacts or wedge dampers. The constrained forces can be considered as feedback forces that influence the response of the friction contacts. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions and this approach results in a set of nonlinear algebraic equations, which can be solved iteratively to yield the periodic response of blades having 3D nonlinear shroud constraint or wedge dampers.
It is shown that the resonant frequency shifts due to the additional spring constant introduced by the frictional constraint, and forced response is damped due to the additional friction damping introduced by frictional slip. In addition, the intermittent interface separation can cause multi-valued response that leads to jump phenomena. The predicted results are also compared with those of the direct time integration method so as to validate the proposed method, and the effect of super-harmonic components on the forced response and jump phenomenon is examined. It is shown that super-harmonic components may induce significant changes of the frictional characteristics such as the transitions between stick, slip and separation, and may affect the prediction of forced response and jump phenomena. It is demonstrated that the Multi-Harmonic Balance Method can predict accurately and efficiently for the periodic forced response by including the super-harmonic components in the analysis.
To My Father and My Wife
ACKNOWLEDGMENTS

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To my father, and my wife, Ling-Shang, I offer my sincere appreciation for their love, affection, and willingness to stand by me that make my entire career possible. Gratitude is also expressed to my sisters, and my in-laws for their constant support and encouragement.
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CHAPTER 1

INTRODUCTION

1.1 Background and Motivation

In turbine jet engine, turbine blades are positioned radially in the air flow field. During engine operation, the turbine blades can extract the power and compress the intake air flow from the gas stream, which in turn exerts cyclic loading effect on the rotating turbine blades. The leading cause of turbine blade failure is the high cycle fatigue of the blade material due to the cyclic loading effect. In order to prevent such failure, the resonant response needs to be attenuated to acceptable level. Since the turbine blades are in fact beam-like structures, they are very vulnerable to cyclic loading. To minimize the cyclic loading effect, design engineer can either make the blades strong enough to withstand the excitation well above detrimental level, or provide additional damping to attenuate the resonant response. In either case, design engineer must be able to accurately predict both the resonant frequencies and the resonant response of the bladed disk systems.
In turbine jet engine industry, dry friction is often employed to increase stiffness and provide damping [Griffin, 1980]. In practice, there are four types of dry friction damper, namely blade-to-ground, blade-to-blade, shrouds, and wedge damper [Yang, 1996]. These friction dampers are often employed in turbine design to attenuate the blade vibration and at the same time to increase aeroelastatic stability of the turbine. In the calculation of the forced response of a bladed disk system, which is a rotational periodic system [Thomas, 1979], a great simplification can be obtained by assuming that the bladed disk system is tuned, namely each blade of the system has exactly the same dynamic characteristics. In addition, the excitation of interest is that induced by the blades rotating through circumferential variations in the flow field. It can be shown that in effect each blade is exposed to a periodic excitation having the same amplitude but differing in phase by an amount which is proportional to the blade’s angular location on the disk. Furthermore, each blade is subjected to cyclic constraining friction force due to the vibration of the friction contacts. Therefore, the bladed disk system overall is cyclic symmetric, and the forced response of the bladed disk system can be predicted based on that of one blade in this case.

In turbine engine design, the receptance method along with Harmonic Balance Method have been widely used for dynamic analysis. It is observed that the forced response is periodic when subjected to periodic external excitation. By assuming that the forced response of the bladed disk system is also periodic and has the same fundamental
period as the excitation, the vibratory motions of the blades as well as the nonlinear constrained forces can be approximated by a series of harmonic functions and employed in Multi-Harmonic Balance Method to solve for the periodic response. When considering the fundamental harmonic only, approximate solution of the forced response is simply harmonic and has been derived in references [Menq and Griffin, 1985; Menq et al., 1986; Yang and Menq, 1997].

During the engine operation, the turbine blades rotate through the fluid flow with severe fluctuation, and as a result, the blades may bend and twist to experience friction constraint. This friction constraint increases the stiffness of the turbine blade and provides damping effect by dissipating vibratory energy, as a consequence of which the turbine blade's mode shapes change significantly from those of a blade without friction constraint. During a cycle of motion, a friction contact point experiences alternating stick-slip friction force that adds friction damping as well as additional spring force to the system [Cameron et al., 1990; Ferri, 1996; Griffin, 1980; Korkmaz, 1995]. This additional spring force increases the stiffness of the blade, as a consequence of which the blade's mode shapes may change significantly from the region where it is fully slipping, to the region where it is fully stuck [Menq and Griffin, 1985]. In general, in order to model such mode shape changes, a number of vibration modes of the blade are often necessary to synthesize a frictionally constrained vibration mode of the blade [Dowell and Schwartz, 1983a, 1983b; Menq and Griffin; 1985; Yang, 1996]. These vibration
modes can usually be calculated from modal analysis based on a finite element model of the blade without including the friction constraint. In this study, the resulting mode shapes are called free mode shapes.

Previously in predicting the resonant response of a frictionally constrained blade system, free mode shapes are used along with the Harmonic Balance Method and the resulting nonlinear algebraic equations can be solved numerically to predict the resonant response of a frictionally constrained blade system [Griffin, 1980; Menq and Griffin, 1985; Menq et al., 1986; Yang and Menq, 1997]. It is found that when the contact interface is near fully slipping, the resonant response can be predicted very well using a small number of free mode shapes. However, when the preload of the friction contact is relatively large and the interface is near fully stuck, a large number of free mode shapes are often necessary in order to correctly predict the resonant response, particularly for a shrouded blade system. For shrouded blades, high preload at the shroud contact is usually desirable. The location of shroud contact along with high preload often results in significant mode shape change, and consequently using free mode shapes for the prediction of the resonant response of a shrouded blade system becomes inefficient.

In this study, the concept of constrained mode shapes is employed to predict the resonant response of a frictionally constrained blade system. For a tuned bladed system, the constrained mode shapes can be calculated using a finite element model of a single
blade along with the cyclic symmetry constraint that simulates a fully stuck friction contact. The concept of quasi-comparison functions proposed by Meirovitch and Kwak [1991] along with the classical Rayleigh-Ritz method can also be used to calculate the constrained mode shapes and can usually yield better rate of convergence. The resulting constrained mode shapes of a tuned bladed system are often complex and can be used to obtain the constrained receptance of the frictionally constrained blade. By using receptances to represent the linear structures and impedances to characterize the nonlinear friction interfaces, it was shown that in predicting the forced response of a frictionally constrained system all the linear degrees of freedom can be condensed to receptance and the modeling of friction interface can be separated from the complex structural models [Menq and Griffin, 1985]. Similar concept was used to obtain reduced models of structures with local non-linearities [Friswell et al, 1996]. However, the focus of this work is to develop an approach in which the constrained mode shapes, when available, can be used to predict the resonant response of a frictionally constrained blade system even when the friction interface is not fully stuck. Its application to the design of shroud contact is of particular interest.

It can be shown that by examining each mode's contribution to the receptance at the friction contact point, the importance of each individual mode to the prediction of the resonant response of a frictionally constrained blade can be determined. Furthermore, by comparing the receptances calculated from free mode shapes and those from constrained
mode shapes, it is found that in the neighborhood of the fully slipping region, the prediction of resonant response requires fewer number of modes when using free mode shapes compared to using constrained mode shapes. On the other hand, in the neighborhood of the fully stuck region, it requires fewer number of modes if constrained mode shapes are used. Moreover, when free and constrained mode shapes are properly combined it is possible to yield very accurate prediction of the resonant response based on only very few vibration modes.

Since turbine blades are often modeled as beam-like structures, the cyclic loading may excite the internal resonance of the blades [Ferri and Dowell, 1988]. When subjected to periodic excitation, the resulting relative motion at the friction contact as well as the constrained force are often periodic. In literature, there have been extensive studies of periodic forced response of beam-like structures subjected to friction contact with constant normal load experiencing one dimensional stick/slip motion [Guillen and Pierre, 1996; Ren and Beard, 1994; Wang and Chen, 1993; Cameron and Griffin, 1989; Ferri and Dowell, 1988; Pierre et al, 1985]. Since the friction models are one dimensional and obey Coulomb Friction Law, the periodic forced response is composed of odd number harmonic components. However, Yang and Menq [1998a] proposed a three dimensional friction contact model which can be applied in predicting the periodic forced response of structures having 3D frictional constraint. In this model, the relative motion of two contacting surfaces is resolved into two components: in-plane tangential
motion on the contact plane and normal component perpendicular to the plane. The in-plane tangential relative motion is two-dimensional, and it can induce stick-slip friction. On the other hand, the normal relative motion can cause variation of the contact normal load and, in extreme circumstances, separation of the two contacting surfaces. In this study, this 3D friction contact model is integrated with the Multi-Harmonic Balance Method in predicting the periodic forced response of shrouded blade systems. It is shown that both odd and even number harmonic components are present in the periodic forced response, due to the variation of the contact normal load. Furthermore, the internal resonance may be predicted as well.

When the variable normal load across the contact surface is taken into account, it will not only affect the stick-slip characteristic of the friction contact, but also directly impose nonlinear stiffness on the friction contact. This nonlinear stiffness is caused by the intermittent separation of the contact surface during the course of vibration. It is well known that this nonlinearity can induce multi-valued responses that lead to so called “jump phenomenon” [Den Hartog, 1931; Thomson, 1988]. In this study, the jump phenomenon of shrouded blade systems will be investigated.

Generally speaking, in the design of friction dampers such as blade-to-ground and blade-to-blade damper, and shroud contacts, they are characterized by friction force models involving only one friction interface. In these friction contacts, the evaluation of
friction force can be assumed as decoupled from the influence of other interfaces. On the other hand, a wedge damper is an example that involves two coupled interfaces. The wedge damper is fitted in a V-shape slot between the inclined platforms of the adjacent blades. During engine operation, the rotation of the bladed disk introduces the centrifugal force that pressed the wedge damper against the platforms. The vibration of the blades will introduce relative motions in those two coupled interfaces that result in very complicated contact kinematics including stick and slip. The resulting constraint force can add friction damping as well as nonlinear spring force to the bladed disk system [Cameron et al., 1990; Ferri, 1996; Griffin, 1980].

In this work, a wedge damper contact model proposed by Yang and Menq [1998b, 1998c] is employed to obtain the constrained force at the wedge damper contact of a bladed disk system. The bladed disk system is assumed to be tuned and the assumed blade motion has two components, namely tangential and radial components. When subjected to periodic excitation, the resulting relative motion at the wedge damper as well as the constrained force are assumed to be periodic. The constrained force can be considered as a feedback force that influences the response of the blade. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions. In the calculation of the nonlinear forced response of a blade, all the linear degrees of freedom can be condensed to receptance and the modeling of wedge damper can be separated from the complex
structure model. This approach results in a set of nonlinear algebraic equations which
can be solved iteratively to yield the periodic forced response of blades having wedge
damper constraint.

1.2 Research Objective and Scope

The objective of this research is to develop methodologies for prediction of
periodic forced response of frictionally constrained bladed disk system. The purpose is to
incorporate Multi-Harmonic Balance Method with receptance based on free and/or
constrained mode shapes in predicting the periodic forced response.

In the prediction of the forced response of a frictionally constrained mechanical
system, all the linear degrees of freedom can be condensed to receptance and the
modeling of friction interface can be separated from the complex structural models. The
constraining force can be seen as a feedback force that influences the response of the
system. By using Multi-Harmonic Balance Method along with Fast Fourier Transform, a
set of nonlinear algebraic equations can be obtained, and they can be solved iteratively for
the forced response. In this research, three types of representation for receptance are
investigated in the formulation of nonlinear algebraic equations:

(i) Complex mode shapes.
(ii) Constrained mode shapes.

(iii) Hybrid mode shapes.

In formulating the dynamic equation of a mechanical system, a finite element model is often employed in the analysis without including frictional constraints. In this case, real valued free mode shapes are derived from Modal Analysis, and they are used in the receptance representation. However, it can be shown that the mode shape can be complex valued when the mechanical system possesses cyclic symmetric structure. On the other hand, when the frictional constraints are cyclic symmetric, they can be included in the finite element model and the resulting complex constrained mode shapes can be employed in the receptance representation.

The focus of this research is to develop a methodology in which the concept of complex constrained mode shapes is employed to predict the forced response of a frictionally constrained bladed disk system. Moreover, by properly combining the free and constrained mode shapes, it is possible to accurately predict the forced response based on only very few vibration modes.

In this research, to demonstrate the predictive capability of Multi-Harmonic Balance Method with receptance representation, three types of frictionally constrained mechanical systems are investigated for the prediction of the periodic forced response:
(i) Mechanical structures having 3D frictional constraints.

(ii) Bladed disk systems having 3D nonlinear shroud constraints.

(iii) Bladed disk systems with wedge dampers.

In the 3D friction contact, it is shown that the normal load across the friction contact point may vary dynamically due to the 3D relative motion of the system, causing the contact point to stick, slip, and possibly separation. This will in effect impose nonlinear stiffness on the friction contact and result in intermittent separation of the friction contact surface. This nonlinearity can induce multi-valued responses that lead to jump phenomenon. In this work, the jump phenomenon will be investigated.

For the mechanical systems with moving components that are mutually constrained through frictional contacts, they can be characterized by friction force models involving only one friction interface. In these friction contacts, the evaluation of friction force is assumed to be decoupled from the influence of other interfaces. This class of problem can be found in the application of shrouded blade systems. In this research, the formulation is derived to predict the periodic forced response for frictionally constrained systems as well as the shrouded blade systems.
On the other hand, some mechanical systems may have moving component that involves two coupled interfaces and results in very complicated contact kinematics. One such application can be found in a bladed disk system having wedge dampers fitted between V-shape platforms of adjacent blades. In this research, the periodic forced response of a bladed disk system with wedge dampers are also studied.

Since the predicted periodic forced responses, which is based on the Multi-Harmonic Balance Method along with Fast Fourier Transform, are approximate solutions, it is necessary to validate the accuracy of these solutions. In this work, direct time integration method is employed to obtain the forced response of the frictionally constrained systems. The solutions obtained from time integration method are compared with the predicted forced response to examine the effectiveness of the proposed approach.

1.3 Literature Review

To explore the effect of frictional contact on structure dynamics, the forced response of a frictionally constrained bladed disk subjected to periodic excitation needs to be investigated. Figure 1.1 shows a shrouded blade constrained by a pair of shroud contacts. The equation of motion can be expressed as:

\[
\mathbf{m} \ddot{\mathbf{u}}(t) + \mathbf{c} \dot{\mathbf{u}}(t) + \mathbf{k} \mathbf{u}(t) = \mathbf{f}_E(t) - \mathbf{f}_N(u, \dot{u}, t)
\]  

(1.1)
Figure 1.1 A frictionally constrained shrouded blade.
where $m$, $c$, $k$ are the mass, damping, and stiffness matrices respectively, $f_e$ is the external excitation, and $f_y = \begin{bmatrix} f_r & f_t \end{bmatrix}^T$ is the nonlinear friction force that is a function of the displacement vector $u$. It is clear that the nonlinear aspect of this dynamic problem is embedded in the term $f_y$, whose relationship, namely impedance model, with the desired solution $u(t)$ has been established by Yang [1996].

There are at least three approaches in solving the motion equation (1.1). The first approach is the direct time integration method that employs numerical integration techniques, such as finite difference method [Bathe and Wilson, 1976; Menq and Griffin, 1983] and Runge-Kutta method [Gerald and Wheatley, 1984; Anderson and Ferri, 1985], that discretize the dynamic equations in the time domain and update the friction force at each time step. This results in a set of coupled linear algebraic equations that can be solved step-by-step to yield the solution. Though time integration method is easy to implement, it requires lengthy computational time to yield the desired steady state solution and the computational stability needs to be guaranteed when a complex structural model is used; as a result, it is seldom applied to practical applications. However, the solution of the direct time integration can prove useful in evaluating the accuracy of any other approximate methods.

The second approach is the analytical method proposed by Den Hartog [1931]. In this method two linear equations of motion are formulated, one for the case when the
interface sticks, the other for the case when the interface slips. By imposing appropriate boundary conditions for the transition between stick and slip, the steady state resonant response can be exactly solved. Den Hartog demonstrated this method by using a single degree of freedom system constrained by a friction interface with constant normal load, and Yeh [1964] extended it to a two-degree-of-freedom system. However, it will become impractical when faced with complex structures that result in a large number of governing equations of motion. Besides, this method fails to apply to friction interfaces that involve complicated contact kinematics, because the complex nonlinear friction phenomenon prohibits precise description of the boundary conditions for the transition among stick, slip, and separation.

The third approach is the so called Harmonic Balance Method [Caughey, 1960; Gelb and Vander, 1968; Menq and Griffin, 1983]. The harmonic balance method is developed based on the observation that, when the external excitation is harmonic, except for the case of subharmonic resonance, the displacement and the induced nonlinear friction force are also periodic and have the same fundamental period as the external excitation. Consequently, the displacement and the friction force can be approximated by infinite Fourier series. By assuming the displacement to have fundamental harmonic component only and truncating the Fourier series of the induced friction force after the fundamental harmonic term, these approximations are integrated with the linear structure to transform the nonlinear differential equation of motion into a set of nonlinear algebraic
equations that can be solved iteratively. It is shown that the Harmonic Balance Method is sufficiently accurate in friction damper design [Menq and Griffin, 1983].

In formulating the nonlinear algebraic equations, the receptance of the linear structures can be employed to integrate with Harmonic Balance Method [Earles and Williams, 1972; Dowell and Schwartz, 1983; Menq et al., 1986a]. The receptance, which relates the steady state response at point $i$ under unit harmonic excitation at point $j$, can be used to represent linear structures and it provides a feasible and realistic representation because it can be obtained by using either experimental methods or harmonic finite element analysis of the structures. When the excited linear structure is constrained by friction dampers, the induced friction forces at the contact points can in turn affect the response of the whole structure. Using the Harmonic Balance Method, these nonlinear friction forces can be approximated as functions of the contact points' displacements. They are then integrated with the linear structure, whose response is characterized by its receptances, to result in a set of nonlinear algebraic equations that involve all the unknown displacements of nodal points in the structure.

In the solution procedure, the modal information, such as mode shapes and modal frequencies obtained by modal analysis, is employed to compute the receptance. However, in turbine design, the blade's mode shapes may change significantly due to the frictional constraint. During a cycle of motion, a friction contact point experiences
alternating stick-slip friction force that adds friction damping as well as additional spring force to the system [Cameron et al, 1990; Ferri, 1996; Griffin, 1980; Korkmaz et al., 1995]. This additional spring force increases the stiffness of the blade, as a consequence of which the blade's mode shapes may change significantly from the region where it is fully slipping, to the region where it is fully stuck [Menq and Griffin, 1985]. In general, in order to model such mode shape changes, a number of vibration modes of the blade are often necessary to synthesize a frictionally constrained vibration mode of the blade [Dowell and Schwartz, 1983a, 1983b; Menq and Griffin, 1985; Yang, 1996]. These vibration modes can usually be calculated from modal analysis based on a finite element model of the blade without including the friction constraint. In this study, the resulting mode shapes are called free mode shapes.

The receptance based on the free mode shapes can be used along with the harmonic balance method and the resulting nonlinear algebraic equations can be solved numerically to predict the resonant response of a frictionally constrained blade system [Griffin, 1980; Menq and Griffin; 1985, Menq et al., 1986; Yang and Menq, 1997]. It is found that when the contact interface is near fully slipping, the resonant response can be predicted very well using a small number of free mode shapes. However, when the preload of the friction contact is relatively large and the interface is near fully stuck, a large number of free mode shapes are often necessary in order to correctly predict the resonant response, particularly for a shrouded blade system. For shrouded blades, high
preload at the shroud contact is usually desirable. The location of shroud contact along with high preload often results in significant mode shape change, and consequently using free mode shapes for the prediction of the resonant response of a shrouded blade system becomes inefficient.

In this study, the concept of constrained mode shapes is employed to predict the resonant response of a frictionally constrained blade system. For a tuned bladed system, the constrained mode shapes can be calculated using a finite element model of a single blade along with the cyclic symmetry constraint that simulates a fully stuck friction contact. The concept of quasi-comparison functions proposed by Meirovitch and Kwak [1991] along with the classical Rayleigh-Ritz method can also be used to calculate the constrained mode shapes and can usually yield better rate of convergence. The resulting constrained mode shapes of a tuned bladed system are often complex and can be used to obtain the constrained receptance of the frictionally constrained blade. By using receptance to represent the linear structures and impedance to characterize the nonlinear friction interfaces, it was shown that in predicting the forced response of a frictionally constrained system all the linear degrees of freedom can be condensed to receptance and the modeling of friction interface can be separated from the complex structural models [Menq and Griffin; 1985]. Similar concept has been used to obtain reduced models of structures with local non-linearities [Friswell et al, 1996].
Since the receptance method can integrate the receptance formulation and the Harmonic Balance Method, it is able to handle complicated structures such as frictionally constrained structures. In addition, it provides an efficient computation algorithm and is widely employed in turbine jet engine industry for dynamic analysis of frictionally constrained turbine blades and friction damper design.

In previous studies of dry friction damper systems, according to Coulomb friction law, the friction coefficient at the contact interface is usually assumed to be constant, the relative motion across the friction contact point is often one-dimensional, and the system is subjected to constant normal load [Griffin, 1980]. It usually results in very simple contact kinematics and can be used to obtained analytical solutions for single-degree-of-freedom systems [Den Hartog, 1931; Wang, 1996], or to integrate with Multi-Harmonic Balance Method to yield approximate solutions for single-degree-of-freedom systems [Cameron and Griffin, 1989; Ren and Beards, 1994; Wang and Chen, 1993] and multiple-degree-of-freedom systems [Ferri and Dowell, 1988; Guillen and Pierre, 1996; Pierre et al, 1985].

Yang and Menq [1997] proposed a 2D version of shroud contact kinematics, in which the contact interface retains the normal component of the relative motion that causes normal load variation, while the in-plane tangential component of the relative motion degenerates into linear motion. In other words, the assumed blade motion has
only two components, namely axial and tangential components. To take the radial component into account, Yang et al. [1998] proposed a simplified three-dimensional shroud contact kinematics, where the two-dimensional in-plane tangential relative motion is assumed to be elliptical and is decomposed into two linear motions along the principal major and minor axes of the ellipse. A variable normal load friction force model [Yang and Menq, 1997] is then applied separately to each individual linear motion to estimate the equivalent stiffness and damping of the shroud contact. Yang and Menq [1998a] proposed a 3D shroud contact model, in which the joined effect of the 2D tangential relative motion and the normal relative motion is examined. They developed a set of analytical criteria to determine the transitions among stick, slip, and separation, when experiencing variable normal load. With these transition criteria, the constrained force can be predicted for any given 3D periodic relative motion across the contact interface.

Since a shrouded blade disk is often modeled as a rigid disk having the beam-like blades positioned radially on the periphery of the disk [Cottney and Ewins, 1974], the external force may excite the internal resonance of a shrouded blade. Whiteman and Ferri [1997] reported internal resonance phenomenon by using time integration method when studying a beam-like structure subjected to friction damper. However, in their study, the internal resonance cannot be predicted by using the Harmonic Balance Method that considers the fundamental harmonic component only. Ferri and Dowell [1988] employed the Multi-Harmonic Balance Method that takes the higher harmonic components into account.
account and predicted the internal resonance of a linear cantilever beam subjected to one
dimensional friction damper with constant normal load.

Unlike other platform dampers such as blade-to-ground and blade-to-blade
dampers, the shroud contact constrains the blade along the normal direction of the contact
surface. Consequently, the vibration that is not parallel to the contact surface will cause
the normal load across the friction contact to vary dynamically. A 3D shroud contact
model proposed by Yang and Menq [1998a] is employed to obtain the constrained force
at the shroud contact. In the three dimensional relative motion of the shroud contact, it
has an in-plane component and the other component perpendicular to the contact plane.
The in-plane tangential relative motion induces stick-slip friction and the normal relative
motion can cause variation of the contact normal load and, in extreme circumstances,
separation of the two contacting shroud surfaces. When subjected to periodic excitation,
the resulting three dimensional relative motion at the shroud contact as well as the
constrained force are assumed to be periodic. The constrained force can be considered as
a feedback force that influences the response of the shrouded blade. By using the Multi-
Harmonic Balance Method [Ren and Beards. 1994; Guillen and Pierre, 1996] along with
Fast Fourier Transform, the constrained force can be approximated by a series of
harmonic functions and in the calculation of the nonlinear forced response of a shrouded
blade, all the linear degrees of freedom can be condensed to receptance and the modeling
of shroud contact can be separated from the complex structure model. This approach
results in a set of nonlinear algebraic equations which can be solved iteratively to yield the periodic forced response of blades having 3D nonlinear shroud constraints.

Since the relative motion of shrouded blades vibration is often three dimensional, which can be decomposed into an in-plane component and a normal component, the normal relative motion can cause variable normal load across the contact surfaces and, in extreme cases, separation of the shroud contact surfaces. The variable normal load will not only affect the stick-slip characteristic of the friction contact, but also directly impose nonlinear stiffness on the friction contact. This nonlinear stiffness is caused by the intermittent separation of the shroud contact surface during the course of vibration. It is well known that this nonlinearity can induce multi-valued responses that lead to so called “jump phenomenon” [Thomson, 1988; Nayfeh and Mook, 1979; Stoker, 1950]. The multi-valued resonant response can be solved by using the standard continuation technique [Allgower and Georg, 1990].

In turbine engine design, friction dampers are often employed in attenuating the blade vibration as well as increasing aeroelastic stability of the turbine. Examples of the application include platform dampers such as blade-to-ground and blade-to-blade damper, and shroud contacts [Griffin, 1990]. However, the platform dampers and shroud contacts are characterized by friction force models involving only one friction interface. In these friction contacts, the evaluation of friction force can be assumed as decoupled from the
influence of other interfaces. On the other hand, a wedge damper is an example that involves two coupled interfaces, in which the wedge damper is fitted in a V-shape slot between the inclined platforms of the adjacent blades. During engine operation, the rotation of the bladed disk introduces the centrifugal force that pressed the wedge damper against the platforms. The vibration of the blades will introduce relative motions in those two coupled interfaces that result in very complicated contact kinematics including stick and slip. The resulting constraint force can add friction damping as well as nonlinear spring force to the bladed disk system [Cameron et al., 1990; Ferri, 1996; Griffin, 1980].

In practice, the wedge damper contact interface can be curved shaped or flat. In a curved wedge damper, the contact kinematics allow rolling motion of the damper [Pfeiffer and Hajek, 1992; Sexto et al., 1997; Csaba and Andersson, 1997]. On the other hand, Yang and Menq [1998b, 1998c] proposed a two-dimensional flat shaped wedge damper model, which is employed to obtain the constrained force at the friction contacts. The assumed blade motion is periodic and has two components, namely tangential and radial components. When subjected to periodic excitation, the resulting relative motion at the wedge damper as well as the constrained force are also periodic. The constrained force can be considered as a feedback force that influences the response of the blade. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions. In the calculation of the nonlinear forced response of a blade, all the linear degrees of freedom
can be condensed to receptance and the modeling of wedge damper can be separated from the complex structure model. This approach results in a set of nonlinear algebraic equations which can be solved iteratively to yield the periodic forced response of blades having wedge damper constraint.

1.4 Dissertation Overview

The first chapter of this dissertation provides the background information and the research objective and scope for the subject of this research. Chapter 2 explores the forced response of bladed disk system using receptance representation based on free and constrained mode shapes. Chapter 3 investigates the periodic forced response of mechanical systems with frictional constrained components, in which the constraining force is simulated by using a 3D friction contact model. Chapter 4 describes the modeling of a shroud contact by using the 3D friction contact model presented in Chapter 3, and investigates the periodic forced response of a shrouded blade system. Chapter 5 studies the periodic forced response of a bladed disk system with wedge dampers. Chapter 6 summaries the conclusions of the research project and gives the recommendations for possible future work.
CHAPTER 2

PREDICTION OF FORCED RESPONSE OF BLADED DISK SYSTEMS
BY USING COMPLEX CONSTRAINED MODE SHAPES

2.1 Introduction

In predicting the forced response of a frictionally constrained bladed disk system, receptance based on free mode shapes is often used along with the Harmonic Balance Method, and the resulting nonlinear algebraic equations can be solved numerically to predict the forced response [Griffin, 1980; Menq and Griffin, 1985; Menq et al, 1986; Yang and Menq, 1997]. It is shown that when the contact interface is near fully slipping, the forced response can be predicted very well using a small number of free mode shapes. However, when the preload of the friction contact is relatively large and the interface is near fully stuck, a large number of free mode shapes are often necessary in order to correctly predict the forced response, particularly for a shrouded blade disk system. For shrouded blades, high preload at the shroud contact is usually desirable. The location of shroud contact along with high preload often results in significant mode shape change,
and consequently using free mode shapes for the prediction of the resonant response of a shrouded blade system becomes inefficient.

In this study, the concept of constrained mode shapes is employed to predict the forced response of a frictionally constrained bladed disk system. For a tuned bladed disk system, the constrained mode shapes can be calculated using a finite element model of a single blade along with the cyclic symmetric constraint that simulates a fully stuck friction contact. The concept of quasi-comparison functions proposed by Meirovitch and Kwak [1991] along with the classical Rayleigh-Ritz method can also be used to calculate the constrained mode shapes and can usually yield better rate of convergence. The resulting constrained mode shapes of a tuned bladed system are often complex and can be used to obtain the constrained receptance of the frictionally constrained blade. By using receptance to represent the linear structures and impedance to characterize the nonlinear friction interfaces, it was shown that in predicting the forced response of a frictionally constrained system all the linear degrees of freedom can be condensed to receptance and the modeling of friction interface can be separated from the complex structural models [Menq and Griffin, 1985]. Similar concept was used to obtain reduced models of structures with local non-linearities [Friswell et al., 1996]. However, the focus of this work is to develop an approach in which the constrained mode shapes, when available, can be used to predict the forced response of a frictionally constrained blade system even
when the friction interface is not fully stuck. Its application to the design of shroud
contact is of particular interest.

2.2 Prediction of Forced Response of Bladed Disk Systems by Using Complex Mode
Shapes

2.2.1 Cyclic symmetric structure and complex mode shapes

In turbine engine, turbine blades are positioned radially on the disk. The bladed
disk can be modeled as a cyclic symmetric structure because the turbine blades are
repetitive substructures on the periphery of the disk. In the calculation of the forced
response of a bladed disk system, which is a rotational periodic system [Thomas, 1979], a
great simplification can be obtained by assuming that the bladed disk system is tuned,
namely each blade of the system has exactly the same dynamic characteristics.

The tuned system assumption states that all the repeating substructures of the
structure are identical, and consequently have exactly the same vibrating frequency when
subjected to cyclic excitation force. It is also assumed that the excitation forces applied
to the adjacent substructures have the same amplitude. However, there exist a constant
phase difference between the adjacent excitation forces. In turbine engine design, the
constant phase angle is called the interblade phase angle, $\varphi$, which is defined as:
where $E$ is the engine order of the excitation force, and $N$ is the number of the blades in the bladed disk.

Figure 2.1 shows schematic of a section of a disk assembly. In this figure, $\mathbf{x}_k$ is the vector of nodal displacements of the $k^{th}$ blade, and $\mathbf{f}_k$ is the vector of excitation force of the $k^{th}$ blade. Based on the tuned system assumption, the relationships of the displacements and forces between the neighboring blades can be formulated as:

\begin{align}
\mathbf{x}_k &= \mathbf{x}_{k+1} e^{-j\varphi} = \mathbf{x}_{k-1} e^{j\varphi} \\
\mathbf{f}_k &= \mathbf{f}_{k+1} e^{-j\varphi} = \mathbf{f}_{k-1} e^{j\varphi}
\end{align}

By applying these relationships between neighboring blades, as long as the forced response of one blade is known, all the forced responses of the other blades can be readily obtained. Furthermore, in the applications of blade-to-blade, wedge dampers, and shroud friction dampers, the relative displacement between neighboring blades can be expressed in terms of the displacement of a single blade. This will be further discussed later.
Figure 2.1 Schematic of a section of a disk assembly.
The finite element model of a bladed disk with $N$ identical blades is as shown in equation (2.4):

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = 0 \quad (2.4)$$

where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices, and $\mathbf{X}$ is the nodal displacement vector. Note that the damping term is neglected. It is clear that if each blade has $J$ nodal displacements, the total degrees-of-freedom of the bladed disk are $NJ$. The nodal displacements are ordered so that the $J$ nodal displacements of the first blade are followed by $J$ nodal displacements of the second blade, and so on. A mode shape of the bladed disk, $\mathbf{X}$, can be depicted as:

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_N \end{bmatrix}^T \quad (2.5)$$

where $\mathbf{X}_k, k = 1, 2, \cdots, N$, is a vector of length $J$ containing the nodal displacements associated with the $k^{th}$ blade.

Since the bladed disk is cyclic symmetric, the normal modes of vibration may occur in degenerate orthogonal pairs. This is because that if a mode has a maximum displacement at some node on the blade, it is possible to rotate the mode shape through any angle and does not change the frequency of vibration. It can be shown that the
Degenerate normal modes vibrate at the same natural frequency. The normal modes of vibration can be categorized as single modes and double modes, depending on the relationship between the mode shapes of individual blades. Each normal mode of vibration corresponds to a mode number \( n \).

(i) Single modes:

Single modes correspond to mode number 0 and \( N/2 \), where mode number \( N/2 \) exists only when \( N \) is an even number. When mode number is 0, which is a rigid mode of vibration, each blade has the same mode shape as its adjacent blades, i.e.

\[
X_k = X_{k+1}, \quad k = 1, 2, \ldots, N
\]  

(2.6)

On the other hand, if each blade has the same mode shape as its adjacent blades, but is vibrating in antiphase with them, i.e.,

\[
X_k = -X_{k+1}, \quad k = 1, 2, \ldots, N
\]

(2.7)

the normal mode will correspond to mode number \( N/2 \), which occurs only in the cases when \( N \) is an even number. Figure 2.2 shows the mode shapes corresponding to mode number 0 and \( N/2 \), where the bladed disk has 16
blades. Figure 2.2 (a) shows the in-phase mode shape corresponding to mode number 0, and Figure 2.2 (b) shows the anti-phase mode shape corresponding to mode number 8.

(ii) Double modes:
Double modes correspond to mode number \( n, 0 < n < N/2 \), where each normal mode of vibration has two possible orthogonal mode shapes vibrating with same natural frequency. Furthermore, any linear combination of these two orthogonal mode shapes is also a possible mode shape that vibrates at the same natural frequency. Figure 2.3 shows two complex valued mode shapes corresponding to mode number 1, where the bladed disk has 16 blades. It is observed that these two mode shapes are indeed orthogonal.
Figure 2.2 Mode shapes of a bladed disk: (a) \( n = 0 \); (b) \( n = 8 \).
Figure 2.3 Two orthogonal mode shapes of a bladed disk: $n = 1$. 

---

Blade number (a) 

Blade number (b)
2.2.2 Cyclic symmetric constraints

In turbine jet engine industry, shrouds are often employed in turbine design to attenuate the blade vibration and at the same time to increase aeroelastatic stability of the turbine. During the engine operation, the shrouded fan blades rotate through the fluid flow with severe fluctuation, and as a result, the blades may bend and twist to cause the off-center shroud contact and to experience friction constraint. This friction constraint increases the stiffness of the shrouded blade, as a consequence of which the shrouded blade’s mode shapes change significantly from those of a blade without shroud constraint. Since the shrouded blade disk is cyclic symmetric, the friction constraint of the shroud contact is also cyclic symmetric. Furthermore, by assuming that the shrouded blade disk is a tuned system, the induced constraining forces on shroud contacts have the same amplitude, while differ from the adjacent shroud contact with an interblade phase angle, \( \varphi \), as defined in equation (2.1).

Figure 2.4 shows two shrouded blades contact each other through their protruding shrouds and the blade coordinate system \((xyz)\) is defined in accordance with the tangential \((x)\), axial \((y \text{ or } -y)\), and radial \((z)\) directions. The geometry of the 3D shroud contact is defined by two angles \( \psi \) (called shroud angle) and \( \phi \) (called inclination angle), whose detailed definitions are shown in Figure 2.5. These two angles define the orientation of the shroud contact plane.
Figure 2.4 Contact geometry of a shrouded blade system.
Figure 2.5 Definition of shroud angle $\psi$ and inclination angle $\phi$. 
A shroud coordinate system \((\nu\nu\nu)\) associates with this oriented plane can also be specified, in which \((\nu)\) axis is along the normal direction of the plane and the \((\mu)\) and \((\nu)\) axes are on the plane. First, a shroud contact plane with its normal pointing towards the \((x)\) axis is shown in Figure 2.5(a). By orienting the contact plane a \(\psi\) angle with respect to the \((z)\) axis, a new contact plane is obtained and shown in Figure 2.5(b). The \((x'y'z')\) coordinate system associates with this oriented plane, and its \((x')\) axis is along the normal direction of the plane and the \((y')\) and \((z')\) axes are on the plane. The final geometry of the 3D shroud contact can be obtained by rotating the oriented plane again a \(\phi\) angle with respect to the \((y')\) axis, and the configuration is shown in Figure 2.5(c). The \((x''y''z'')\) (or \(\nu\nu\nu\)) coordinate system associates with this oriented plane, and its \((x'')\) axis is along the normal direction of the plane and the \((y'')\) \((u)\) and \((z'')\) \((w)\) axes are on the plane. These two coordinate systems can be related to each other through a coordinate transformation matrix.

\[
T_0 = \begin{bmatrix}
\cos \psi \cos \phi & -\sin \psi & \cos \psi \sin \phi \\
\sin \psi \cos \phi & \cos \psi & \sin \psi \sin \phi \\
-\sin \phi & 0 & \cos \phi
\end{bmatrix}
\tag{2.8}
\]

Figure 2.6 shows a shrouded blade disk in which left and right shroud contacts are subjected to the induced constraining forces \(f_l\) and \(f_r\), respectively. The shrouded blade system is assumed to be tuned, therefore, the whole system can be represented by a single
shrouded blade having cyclic symmetry constraint applied at a pair of shroud contact points. In modeling the dynamics of a shrouded blade, the displacement vector of a finite element model can be divided into three segments in which $x_r$ is the displacement vector of the right shroud contact point, $x_l$ is that of the left shroud contact point, and $x_h$ represents the displacement vector of all the other nodes of the blade. The governing equation of motion can be formulated as:

$$
\begin{bmatrix}
    m_{bb} & m_{br} & m_{bl} \\
m_{rb} & m_{rr} & m_{rl} \\
m_{lb} & m_{lr} & m_{ll}
\end{bmatrix}
\begin{bmatrix}
    \dot{x}_b \\
    \dot{x}_r \\
    \dot{x}_l
\end{bmatrix}
+ \begin{bmatrix}
    c_{bb} & c_{br} & c_{bl} \\
c_{rb} & c_{rr} & c_{rl} \\
c_{lb} & c_{lr} & c_{ll}
\end{bmatrix}
\begin{bmatrix}
    \ddot{x}_b \\
    \ddot{x}_r \\
    \ddot{x}_l
\end{bmatrix}
+ \begin{bmatrix}
    k_{bb} & k_{br} & k_{bl} \\
k_{rb} & k_{rr} & k_{rl} \\
k_{lb} & k_{lr} & k_{ll}
\end{bmatrix}
\begin{bmatrix}
    x_b \\
    x_r \\
    x_l
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    f_r \\
    f_l
\end{bmatrix}
$$

(2.9)

where $m_i$, $c_i$, and $k_i$, $i,k = b,r,l$, are the associated mass, damping, and stiffness sub-matrices, and $\varphi$ is the interblade phase angle. Since the shrouded blades is cyclic symmetric and is assumed a tuned system, the displacement vector and the constraining force of the left and right shroud contacts can be related as:

$$
x_l = e^{-j\varphi} x_r
$$

(2.10)

$$
f_l = -e^{-j\varphi} f_r
$$

(2.11)
Figure 2.6 A tuned blade system having part-span shrouds.
By substituting equations (2.10) and (2.11) into equation (2.9), the governing equation of motion becomes

\[
\begin{bmatrix}
\mathbf{m}_{bb} & \mathbf{m}_{br} + e^{-j\alpha} \mathbf{m}_{bl} \\
\mathbf{m}_{rb} + e^{j\alpha} \mathbf{m}_{tb} & \mathbf{m}_{rr} + e^{-j\alpha} \mathbf{m}_{rt} + \mathbf{m}_{tt}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_b \\
\mathbf{x}_r
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{c}_{bb} & \mathbf{c}_{br} + e^{-j\alpha} \mathbf{c}_{bl} \\
\mathbf{c}_{rb} + e^{j\alpha} \mathbf{c}_{tb} & \mathbf{c}_{rr} + e^{-j\alpha} \mathbf{c}_{rt} + \mathbf{c}_{tt}
\end{bmatrix}
\begin{bmatrix}
\mathbf{\dot{x}}_b \\
\mathbf{\dot{x}}_r
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{k}_{bb} & \mathbf{k}_{br} + e^{-j\alpha} \mathbf{k}_{bt} \\
\mathbf{k}_{rb} + e^{j\alpha} \mathbf{k}_{tb} & \mathbf{k}_{rr} + e^{-j\alpha} \mathbf{k}_{rt} + \mathbf{k}_{tt}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_b \\
\mathbf{x}_r
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\] (2.12)

It is clear that the mass, damping, and stiffness matrices are in Hermitian form due to the cyclic symmetry constraint. By using the Modal Analysis Method, the finite element model can be solved for the natural frequencies and the associated mode shapes. Since the matrices are Hermitian, the mode shapes are complex numbered.

On the other hand, a stiffness matrix \( \mathbf{k}_s \) can be used to represent the shroud contact stiffness, as depicted in Figure 2.7. When examining the free vibration of the shrouded blade having the cyclic symmetry constraint, the equation of motion can be formulated as follows.

\[
\begin{bmatrix}
\mathbf{m}_{bb} & \mathbf{m}_{br} & \mathbf{m}_{bt} \\
\mathbf{m}_{rb} & \mathbf{m}_{rr} & \mathbf{m}_{rt} \\
\mathbf{m}_{tb} & \mathbf{m}_{rt} & \mathbf{m}_{tt}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_b \\
\mathbf{x}_r \\
\mathbf{x}_t
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{c}_{bb} & \mathbf{c}_{br} & \mathbf{c}_{bt} \\
\mathbf{c}_{rb} & \mathbf{c}_{rr} & \mathbf{c}_{rt} \\
\mathbf{c}_{tb} & \mathbf{c}_{rt} & \mathbf{c}_{tt}
\end{bmatrix}
\begin{bmatrix}
\mathbf{\dot{x}}_b \\
\mathbf{\dot{x}}_r \\
\mathbf{\dot{x}}_t
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{k}_{bb} & \mathbf{k}_{br} & \mathbf{k}_{bt} \\
\mathbf{k}_{rb} & \mathbf{k}_{rr} & \mathbf{k}_{rt} + \mathbf{k}_s \\
\mathbf{k}_{tb} & \mathbf{k}_{rt} - e^{-j\alpha} \mathbf{k}_s & \mathbf{k}_{tt} + \mathbf{k}_s
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_b \\
\mathbf{x}_r \\
\mathbf{x}_t
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\] (2.13)

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Figure 2.7 A fully locked tuned shrouded blade system.
It is worthy noting that the stiffness matrix is Hermitian due to the cyclic symmetric constraint. By using the Modal Analysis Method, the finite element model can be solved for the natural frequencies and the associated constrained mode shapes. Since the stiffness matrix is Hermitian, the mode shapes are complex numbered.

2.2.3 Resonant response of a shrouded blade

The governing equation of one shrouded blade can be formulated as:

\[ m\ddot{x} + c\dot{x} + kx = f_0 e^{j\omega f t} \]  \hspace{1cm} (2.14)

where \( x \) is the nodal displacement vector, \( f_0 \) is the force vector including the external excitation force and the constraining force due to shroud contact, \( m, c, \) and \( k \) are the mass, damping and stiffness matrices, and \( \omega_f \) is the excitation frequency. It is worthy noting that \( m, c, \) and \( k \) matrices are Hermitian due to the cyclic symmetric constraint. The excitation force can be represented as:

\[ f_0 = fe^{j\phi} \]  \hspace{1cm} (2.15)

where \( f \) is a complex vector representing the amplitude and phase of the force, and \( \phi \) is the interblade phase angle defined in equation (2.1).
By using the Modal Analysis Method, the governing equation (2.14) can be solved to obtain the natural frequencies and the associated mode shapes. Since \( m, c, \) and \( k \) matrices are Hermitian due to the cyclic symmetric constraint, the resulting mode shapes are complex numbered, denoted as \( \Phi \). The nodal displacement vector, \( x \), can be replaced by the modal vector, \( y \).

\[
x = \Phi y
\]  
(2.16)

By substituting equation (2.16) in equation (2.14), and pre-multiplying the complex conjugate transpose of the mode shapes, denoted as \( \Phi^H \), the resulting modal equation can be formulated as:

\[
\Phi^H m \Phi y + \Phi^H c \Phi y + \Phi^H k \Phi y = \Phi^H f_0 e^{jo t}
\]  
(2.17)

Consequently, the \( m, c, \) and \( k \) matrices are diagonalized simultaneously. If each blade has \( J \) nodal displacements, there will be \( J \) decoupled modal equations.

\[
\mathbf{m}, \ddot{y} + c, \dot{y} + k, y = \Phi^H f_0 e^{j\omega t}, \quad i = 1, 2, \ldots, J
\]  
(2.18)
where $m_i$, $c_i$, and $k_i$ are the $i^{th}$ diagonal elements of the corresponding diagonalized matrices. The modal response can be expressed as:

$$y_i = \Lambda_i \Phi_i^H f_0 e^{j\omega t}$$  \hspace{1cm} (2.19)

$$\Lambda_i = \left[ (k_i - \omega^2 m_i) + j(\omega c_i) \right]^{-1}$$  \hspace{1cm} (2.20)

Finally the resonant response of one blade can be formulated as:

$$x = \Phi \Lambda \Phi^H f_0 e^{j\omega t}$$  \hspace{1cm} (2.21)

where the total receptance matrix, $r$, is the summation of all the receptances calculated from each individual mode.

$$r = \Phi \Lambda \Phi^H = \sum_{i=1}^{J} \Phi_i \Lambda_i \Phi_i^H$$  \hspace{1cm} (2.22)

$$\Lambda = \text{diag}[\Lambda_1, \Lambda_2, \ldots, \Lambda_J]$$  \hspace{1cm} (2.23)
2.2.4 Resonant response of a shrouded blade system

The governing equation of a shrouded blade disk can be formulated as:

\[ M\ddot{X} + C\dot{X} + KX = F e^{i\omega_f t} \quad (2.24) \]

where \( X \) is the nodal displacement vector, \( F \) is the force vector including the external excitation force and the constraining force due to shroud contact, \( M, C, \) and \( K \) are the mass, damping and stiffness matrices, and \( \omega_f \) is the excitation frequency. It is worthy noting that \( M, C, \) and \( K \) matrices are Hermitian due to the cyclic symmetric constraint. It is clear that each blade has \( J \) nodal displacements, the total degrees-of-freedom of the bladed disk are \( NJ \). The nodal displacements are ordered so that the \( J \) nodal displacements of the first blade are followed by \( J \) nodal displacements of the second blade, and so on. A mode shape of the bladed disk, \( X \), can be depicted as:

\[ X = [X_1 \ X_2 \ \cdots \ X_N]^T \quad (2.25) \]

where \( X_k, k = 1,2,\cdots,N \), is a vector of length \( J \) containing the nodal displacements associated with the \( k^{th} \) blade. The excitation force \( F \) is represented as
\[ F = \begin{bmatrix} f_0 & f_1 & \cdots & f_{N-1} \end{bmatrix}^T \]  \hspace{1cm} (2.26)

\[ f_\ell = f e^{i\phi}, \quad \ell = 0, 1, 2, \ldots, (N-1) \]  \hspace{1cm} (2.27)

where \( f \) is the complex vector representing the amplitude and phase of the force, \( \phi \) is the interblade phase angle defined in equation (2.1), and \( f_\ell \) is the cyclic symmetric excitation force on the \( \ell \)th blade.

By using the Modal Analysis Method, the governing equation (2.24) can be solved to obtain the natural frequencies and the associated mode shapes. Since \( M, C, \) and \( K \) matrices are Hermitian due to the cyclic symmetric constraint, the resulting mode shapes are complex numbered, denoted as \( \Psi \). The nodal displacement vector, \( X \), can be replaced by the modal vector, \( Y \).

\[ X = \Psi Y \]  \hspace{1cm} (2.28)

By substituting equation (2.28) in equation (2.24), and pre-multiplying the complex conjugate transpose of the mode shapes, denoted as \( \Psi^H \), the resulting modal equation can be formulated as:
\[ \Psi^H M \Psi \ddot{Y} + \Psi^H C \Psi \dot{Y} + \Psi^H K \Psi Y = \Psi^H F e^{j\omega t} \] (2.29)

Consequently, the \( M, C, \) and \( K \) matrices are diagonalized simultaneously. Since there are \( N \) blades in the bladed disk and each blade has \( J \) nodal displacements, there will be \( NJ \) decoupled modal equations.

\[ M_i \ddot{Y}_i + C_i \dot{Y}_i + K_i Y_i = \Psi_i^H F e^{j\omega_t}, \quad i = 1, 2, \ldots, NJ \] (2.30)

where \( M_i, C_i, \) and \( K_i \) are the \( i^{th} \) diagonal elements of the corresponding diagonalized matrices. The modal response can be expressed as:

\[ Y_i = A_i \Psi_i^H F e^{j\omega t} \] (2.31)

\[ A_i = \left[ \left( K_i - \omega^2 M_i \right) + j(\omega C_i) \right]^{-1} \] (2.32)

Finally the resonant response of one blade can be formulated as:

\[ X = \Psi A \Psi^H F e^{j\omega t} \] (2.33)
where the total receptance matrix, \( R \), is the summation of all the receptances calculated from each individual mode.

\[
R = \Psi A \Psi^H = \sum_{i=1}^{N} \Psi_i A_i \Psi_i^H
\]  

(2.34)

\[
A = \text{diag}[A_1, A_2, \ldots, A_N]
\]  

(2.35)

It is clear that the resonant response of a shrouded blade disk can be calculated by performing the modal analysis on the finite element model of the disk. However, in turbine engine design, each shrouded blade may consist of thousands of nodes. The resulting equation of motion will have the mass, damping, and stiffness matrices the order of tens of thousands square matrices. Consequently, it will require huge storage space and computer time in calculating the resonant response of a disk. One way to save the computation resources is to exploit the cyclic symmetric nature of a shrouded blade disk and assume the shrouded blades are tuned. By calculating the resonant response of one blade first, all the other blades' resonant response can be obtained accordingly.

Previously it is assumed that in a mode shape, \( X \), the nodal displacements are ordered so that the nodal displacements of the first blade are followed by nodal displacements of the second blade, and so on, as shown in equation (2.25). Since the
shrouded blades are cyclic symmetric and tuned, the mode shapes corresponding to mode number \( n \) have the same maximum displacements for each blades and an interblade phase angle, \( \varphi_n \), between the adjacent blades.

\[
\varphi_n = \frac{2\pi n}{N} \quad (2.36)
\]

Since the mode shapes of one shrouded blade, \( \Phi \), is known. The mode shapes of a shrouded blade disk corresponding to mode number \( n \) can be formulated as:

\[
\Psi_n = \Pi_n \Phi \quad (2.37)
\]

\[
\Pi_n = \begin{bmatrix}
I & e^{-j\varphi_n} & \cdots & e^{-j(N-1)\varphi_n}
\end{bmatrix}^T \quad (2.38)
\]

where \( I \) is an identical matrix with the same size as \( \Phi \). When the mode number \( n \) corresponds to double modes, there will be two orthogonal mode shapes. Since any linear combination of these two orthogonal mode shapes is also a mode shape, one way to formulate the mode shapes is to choose the real part and imaginary part of a complex mode shape as two orthogonal mode shapes.

\[
\begin{bmatrix}
\Psi_{nR} & \Psi_{ni}
\end{bmatrix} = \begin{bmatrix}
\text{real}(\Pi_n \Phi) & \text{imag}(\Pi_n \Phi)
\end{bmatrix} \quad (2.39)
\]
The mode shapes of a shrouded blade disk can be expressed as:

\[ \Psi = [\Psi_0, \Psi_{1R}, \Psi_{1L}, \ldots, \Psi_{N/2}] \]  \hspace{1cm} (2.40)

Note that the mode shape, \( \Psi_{N/2} \), exists only when there are even number of blades in the disk.

Thus the resonant response of the shrouded blade disk can be calculated based on that of one shrouded blade.

\[ X = \Psi \Lambda \Psi^H f e^{j\omega t} \]
\[ = \Pi_E \cdot \left( \Phi \Lambda \Phi^H f e^{j\omega t} \right) \]
\[ = \Pi_E \cdot x \]  \hspace{1cm} (2.41)

where

\[ \Pi_E = \begin{bmatrix} I & e^{-j\varphi} & \ldots & e^{-j(N-1)\varphi} \end{bmatrix}^T, \quad \varphi = \frac{2\pi E}{N} \]  \hspace{1cm} (2.42)
2.3 Prediction of Forced Response of Bladed Disk System by Using Free Mode Shapes

In the calculation of the forced response of a bladed disk system, a great simplification can be obtained by assuming that the bladed disk system is tuned, namely each blade of the system has exactly the same dynamic characteristics. In addition, the excitation of interest is that induced by the blades rotating through circumferential variations in the flow field. It can be shown that in effect each blade is exposed to a periodic excitation having the same amplitude but differing in phase by an amount which is proportional to the blade's angular location on the disk. It is assumed that the resonant response of the bladed system is also periodic and has the same fundamental period as the excitation. Thus the external excitation and the motions of the blades as well as the nonlinear constrained forces can be represented by infinite Fourier series. By truncating these series after the fundamental terms, an approximate solution assuming that the forced response is simply harmonic can be derived [Menq and Griffin, 1985; Menq et al, 1986; Yang and Menq, 1997].

In this approach, each blade vibrates in the same manner but with an interblade phase difference ($\phi$) from its adjacent blades. The interblade phase angle is defined as follows:
\[ \varphi = \frac{2\pi E}{N} \]  \hspace{1cm} (2.43)

where \( N \) is the total number of the blades in the system and \( E \) is the engine order of the excitation on the system. Furthermore, since each blade is constrained by friction contacts, the induced constraining force will have the same interblade phase difference from that on the adjacent blades. As a result, the response of the frictionally constrained blade system can be described by the response of a single blade. In this study, a blade-to-ground friction damper is employed first for illustration, and a shrouded blade system is studied later on. Figure 2.8 shows a blade constrained by a blade-to-ground friction damper. The equation of motion under a harmonic excitation can be expressed as:

\[ m\ddot{u}(t) + c\dot{u}(t) + ku(t) = f_x(t) - f_N(u, \dot{u}, i) \]  \hspace{1cm} (2.44)

where \( u \) is the nodal displacement vector, \( m \) is the mass matrix, \( c \) is the damping matrix, \( k \) is the stiffness matrix, \( f_x \) is the external harmonic excitation, and \( f_N \) is the nonlinear constrained force which is a function of the motion at the contact point. The finite element model can be three-dimensional. If the model contains \( n \) nodes, all the matrices will be \( 3n \times 3n \) matrices, and all the vectors will be \( 3n \)-element vectors.

The external harmonic excitation can be expressed as:

\[ f_x(t) = \cdots \]
Figure 2.8 A blade constrained by a blade-to-ground friction damper.
\( f_x(t) = f_x e^{j\omega t} \) \hspace{1cm} (2.45)

where \( \omega \) is the excitation frequency and \( f_x \) is a complex vector representing the magnitude and phase of the excitation. It is worthy noting that except for the element associated with the friction contact point the other elements of \( f_N \) are zeros. It is clear that the nonlinear aspect of the dynamic problem is embedded in the nonlinear friction force \( f_N \).

By using the Modal Analysis Method, the governing equation (2.44) can be solved to obtain the natural frequencies and the associated mode shapes. Since the frictional constraint is not included in the modal analysis, the resulting mode shapes are called free mode shapes in this study.

Let the free mode shapes be denoted as \( \Phi^f \). The receptance matrix of the blade can be calculated as:

\[
R^f = \begin{bmatrix} r_{kl}^f \end{bmatrix} = \sum_{i=1}^{3n} \Phi^f_i \Phi^f_i^T \left[ \left( k_i^f - \omega^2 m_i^f \right) + j(\omega c_i^f) \right]^{-1}
\]

(2.46)
where \( r'_{k} \) is defined as the steady state response of the \( k^{th} \) node due to unit harmonic excitation force at the \( i^{th} \) node, \( \Phi'_{i} \) is the \( i^{th} \) free mode shape, \( m'_{i} \) is the \( i^{th} \) modal mass, \( k'_{i} \) is the \( i^{th} \) modal stiffness, and \( c'_{i} \) is the \( i^{th} \) modal damping. It is evident that the total receptance is the summation of the receptance calculated from each individual mode.

When the blade is constrained by a friction damper, the resulting constrained force is characterized by the contact point's displacement, \( u_{c} \), and it can be considered as a feedback force that influences the response of the blade. This feedback effect is shown in Figure 2.9. By using the harmonic balance method, the constrained force can be approximated by a harmonic function having the same frequency as the external harmonic excitation, and its amplitude and phase are nonlinear functions of the contact point's displacement. These nonlinear functions are called impedances and can be found in references [Griffin, 1980, Menq and Griffin, 1985]. From the nonlinear feedback loop shown in Figure 2.9, it is evident that in the calculation of nonlinear forced response of a frictionally constrained blade, all the linear degrees of freedom can be condensed to the receptance and the modeling of friction contact can be separated from the complex structure model. This approach results in a set of nonlinear algebraic equations, which can be formulated as follows:

\[
\mathbf{u}_{c} = r'_{c} \mathbf{f}_{e} - r'_{c} \mathbf{f}_{N}(\mathbf{u}_{c})
\]  

(2.47)
Figure 2.9 Nonlinear feedback loop using free receptance.
where \( u_c \) is the harmonic response at the contact point, \( r_{ce}^{f} \) is the receptance at the contact point due to unit harmonic excitation force, and \( r_{ce}^{c} \) is the receptance at the contact point due to unit constrained force. It is worthy noting here that the receptance in equation (2.47) is obtained based on free mode shapes and the associated modal information. Equation (2.47) can then be solved iteratively to obtain the resonant response of the frictionally constrained blade.

In predicting the resonant response of a frictionally constrained blade, it is found that when the contact interface is near fully slipping, the resonant response can be predicted very well by using a small number of free mode shapes. However, when the preload of the friction contact is relatively large and the interface is near fully stuck, a large number of free mode shapes are required in order to correctly predict the resonant response. In general, it is difficult to know how many modes are necessary for accurate prediction. A typical approach is to observe if the predicted results converge as increasing the number of modes used in the necessary nonlinear analysis. It is evident that this approach is not the most efficient one. In this study, by examining the contribution of each individual mode to the total receptance at the contact point, it can be shown that their significance can be determined.

In this study, a blade-to-ground friction damper with stiffness of \( 0.6 \times 10^6 \, N/m \) is attached to a nodal point of the blade, which is modeled as a beam-like structure. The
excitation force is $1 \, \text{N}$, located at the same node as the friction damper. In Figure 2.10, the receptances of the first four modes of the frictionally constrained blade are compared. Those receptances are calculated using free mode shapes. Figure 2.10(a) shows the receptances ranging from 50Hz to 3500Hz, and Figure 1.10(b) enlarges the region of interest which is from 50Hz to 550Hz. It is clear that near the fully slipping region, the first mode's receptance dominates among all the modes, e.g., contributing 99.95% to the total receptance at 181Hz. On the other hand, near the fully stuck region, which is around 476Hz, the first three modes are necessary to accurately calculate the total receptance, e.g., contributing 99.98% to the total receptance. For comparison, the first mode contributes 87.94%, while the first two modes contribute 97.55%. Figure 2.11 shows the resonant response near the fully stuck region. It is observed that, if the receptance of the first mode only is employed for the analysis, the resonant frequency, which is 476Hz, is over-estimated to be 508Hz. However, if the first three modes are included in the analysis, the predicted resonant response is very close to that predicted by using all the available free mode shapes. In other words, for this example, the first three free mode shapes are required in order to represent the first frictionally constrained vibration mode. It is, however, clear that when the damper becomes stiffer, the number of required free mode shapes increases.
Figure 2.10 Receptance of the first four modes of the blade based on free mode shapes.
Figure 2.11 Prediction of the resonant response based on the free mode shapes.
2.4 Prediction of Forced Response of Bladed Disk System by Using Constrained Mode Shapes

In order to reduce the number of mode shapes that are necessary for accurate prediction of the resonant response near the fully stuck region, the concept of constrained mode shapes is introduced in this study. For the frictionally constrained bladed disk system shown in Figure 2.8, the constrained mode shapes are obtained from a finite element model of the blade along with the spring connecting to the contact point \( c \) to the ground.

With a constrained system in mind, the equation of motion of the frictionally constrained blade can be written in the following form:

\[
\mathbf{m}\ddot{\mathbf{u}}(\tau) + \mathbf{c}\dot{\mathbf{u}}(\tau) + \left( \mathbf{k} + \mathbf{k}_d \right) \mathbf{u}(\tau) = \mathbf{f}_x(\tau) - \left( \mathbf{f}_N(u, \dot{u}, \tau) - \mathbf{k}_d \mathbf{u}(\tau) \right)
\]  

(2.48)

where \( \mathbf{k}_d \) is the constraint stiffness matrix of the blade, in which all the elements are zero except that the one associated with the damper is \( k_d \). By using the Modal Analysis Method, the governing equation (2.48) can be solved to obtain the natural frequencies and the associated mode shapes. Because the constraint condition is included in the left hand side of the governing equation (2.48), the resulting mode shapes are called constrained.
mode shapes. Let the constrained mode shapes be denoted as $\Phi^c$. The receptance matrix of the constrained blade can be calculated as follows:

$$R^c = [r^c_{\alpha \beta}] = \sum_{i=1}^{3n} \Phi^c_i \Phi^c_i^T \left[ \left( k^c_i + \omega^2 m^c_i \right) + j \omega c^c_i \right]^{-1} \tag{2.49}$$

Superscript $s$ is used in equation (2.49) as a counterpart of the superscript $f$ in equation (2.46) for the current system is a constrained one.

Using the constrained receptance defined in equation (2.49), similar to the nonlinear feedback loop depicted in Figure 2.9, a nonlinear feedback loop, as shown in Figure 2.12, can be developed to signify the feedback effect of the constrained force given on the right hand side of equation (2.48). From this nonlinear feedback loop, it is again clear that equation (2.48) based on a finite element model of the blade can be condensed to a set of nonlinear algebraic equations.

$$\mathbf{u}_c = r^c_{cc} \mathbf{f}_e - r^c_{cc} \left( \mathbf{f}_n - k_d \mathbf{u}_c \right) \tag{2.50}$$
Figure 2.12 Nonlinear feedback loop using constrained receptance.
where $r^{s}_{ee}$ is the receptance at the contact point due to unit harmonic excitation force, $r^{s}_{ce}$ is the receptance at the contact point due to unit constrained force, and $k_d$ is the stiffness of the blade-to-ground damper. Equation (2.50) can be rewritten into a formula similar to that of equation (2.47) as follows.

$$u_e = \frac{r^{s}_{ee} f_e}{1 - r^{s}_{ee} k_d} - \frac{r^{s}_{ce} f_N(u_e)}{1 - r^{s}_{ce} k_d}$$  \hspace{1cm} (2.51)

For an external excitation $f_e$, equation (2.51) can be employed to solve iteratively for the resonant response $u_e$, which is the relative motion across the friction contact.

In predicting the resonant response when the contact interface is near fully stuck, it is found that one can use a small number of constrained mode shapes and still obtain correct prediction. This can be verified directly by comparing the receptances of each individual constrained mode shape. Figure 2.13(a) shows the receptances of the first four constrained mode shapes ranging from 50Hz to 3500Hz, and Fig. 2.13(b) enlarges the region of interest, which is from 50Hz to 550Hz. It is clear that near the fully stuck region the receptance of the first constrained mode dominates among all the modes, e.g., contributing 99.76% to the total receptance at 476Hz. This indicates that by simply using the first constrained mode, one can correctly predict the resonant response near the fully stuck region. However, when examining the receptance near the fully slipping region, it
can be seen that the receptance of the first constrained mode only contributes 84.58% at 181Hz. In other words, more constrained modes are required in order to correctly predict the resonant response near the fully slipping region. The predicted resonant responses of the frictionally constrained blade are shown in Figure 2.14. It is clear that the first constrained mode can be employed alone to correctly predict the resonant response near the fully stuck region. On the other hand, near the fully slipping region, it needs more constrained modes to predict the resonant response accurately. It should be pointed out that, when the contact interface is fully stuck, the required normal load can be estimated as $k_u r c f_e \mu$. This value can be used to determine if the interface is near fully stuck.
Figure 2.13 Receptance of the first four modes based on constrained mode shapes:

(a) 50 - 3500 Hz; (b) 50 - 550 Hz.
Figure 2.14 Prediction of the resonant response based on the constrained mode shapes.
2.5 Prediction of Forced Response of Bladed Disk System by Using Hybrid Mode Shapes

Based on the results from the previous two sections, it can be seen that it is possible to use hybrid mode shapes, namely the combined free and constrained mode shapes, to predict the resonant response of a frictionally constrained blade, and possibly very few hybrid vibration modes are needed to achieve the required accuracy. However, since the free and constrained mode shapes are not from the same set of eigenfunctions of a linear system, usual modal synthesis approach cannot be applied to combine the free and constrained mode shapes for the purpose of dynamic analysis. Moreover, the actual mode shapes depend on the degree of slipping that is present at the friction contact. This relationship is nonlinear and the degree of slipping is unknown before the response is predicted. Therefore, the usual variational approach cannot be applied to derive the eigenvalue problem.

Nevertheless, in this approach, the impedance $f_Y(u_c)$ is updated during each iteration, and therefore, it can be used to determine the degree of slipping and serve as a basis for interpolation that allows the free and constrained modes to be combined. The impedance $f_Y(u_c)$ has two components, namely the in-phase component $f_x(u_c)$ and the out of phase component $f_c(u_c)$. The in-phase component represents the resulting spring force at the friction contact and it has direct impact on the mode shape change of the
frictionally constrained blade. Along with the amplitude $A$ of the relative motion $u_c$, $f_s(u_c)$ can be used to estimate the equivalent spring constant of the friction contact.

$$k_e = \frac{|f_s(u_c)|}{A} \quad (2.52)$$

If $k_e = 0$, the interface is fully slipping, and the interface becomes fully stuck when $k_e$ approaches $k_d$. Therefore, an interpolation scheme based on the equivalent spring constant can be established as follows.

$$u_c = r^h_{ce} f_s - r^h_{cc} f_N(u_c) \quad (2.53)$$

where $r^h_{ce}$ and $r^h_{cc}$ are hybrid receptances defined as follows.

$$r^h_{ce} = \left(1 - \frac{k_e(u_c)}{k_d}\right) r^f_{ce} + \frac{k_e(u_c)}{k_d} \frac{r^f_{ce}}{1 - r^f_{ce} k_d} \quad (2.54)$$

$$r^h_{cc} = \left(1 - \frac{k_e(u_c)}{k_d}\right) r^f_{cc} + \frac{k_e(u_c)}{k_d} \frac{r^f_{cc}}{1 - r^f_{cc} k_d} \quad (2.55)$$
Using the hybrid receptance defined in equation (2.54) and (2.55), similar to the nonlinear feedback loop depicted in Figure 2.9 and 2.12, a nonlinear feedback loop, as shown in Figure 2.15, can be developed to signify the hybrid feedback effect of the friction force. Note here that in Figure 2.16, a constant $\alpha$ is defined as:

$$\alpha = 1 - \frac{k_c(u_c)}{k_d} \quad (2.56)$$

The constant $\alpha$ signifies the linear interpolation of the hybrid receptance based on the free receptance and the constrained receptance. For an external excitation $f_e$, equation (2.53) can be employed to solve iteratively for $u_e$.

Figure 2.16 shows the predicted resonant responses of the frictionally constrained blade using hybrid receptances. The receptances are calculated using the first free mode and the first constrained mode. In the same figure, the results are compared to those predicted by using six constrained modes. It is clear that the hybrid receptances can be employed to correctly predict the resonant response over the range from fully slipping to fully stuck.
Figure 2.15 Nonlinear feedback loop using hybrid receptance.
Figure 2.16 Prediction of the resonant response based on hybrid mode shapes.
2.6 Forced Response of a Shrouded Blade System

In turbine jet engine industry, shrouds are often employed in turbine design to attenuate the blade vibration and at the same time to increase aeroelastatic stability of the turbine. During the engine operation, the shrouded fan blades rotate through the fluid flow with severe fluctuation, and as a result, the blades may bend and twist to cause the off-center shroud contact and to experience friction constraint. This friction constraint increases the stiffness of the shrouded blade, as a consequence of which the shrouded blade's mode shapes change significantly from those of a blade without shroud constraint.

2.6.1 Shroud contact kinematics

Figure 2.17 shows two shrouded blades contact each other through their protruding shrouds and the global coordinate system \((xyz)\) is defined in accordance with the tangential \((x)\), axial \((y\) or \(-y)\), and radial \((z)\) directions. The geometry of the 3D shroud contact is defined by two angles \(\psi\) (called shroud angle) and \(\phi\) (called inclination angle), whose detailed definitions are shown in Figure 2.5 and explained in Section 2.2.2. These two angles define the orientation of the shroud contact plane. A local coordinate system \((uvw)\) associates with this oriented plane can also be specified, in which \((v)\) axis is along the normal direction of the plane and the \((u)\) and \((w)\) axes are on the plane.
These two coordinate systems can be related to each other through a coordinate transformation matrix.

\[
T_0 = \begin{bmatrix}
\cos \psi \cos \phi & -\sin \psi & \cos \psi \sin \phi \\
\sin \psi \cos \phi & \cos \psi & \sin \psi \sin \phi \\
-\sin \phi & 0 & \cos \phi
\end{bmatrix}
\]  

(2.57)

Figure 2.18 shows a shrouded blade system in which a stiffness matrix \(k_s\) is used to represent the contact stiffness. The shrouded blade system is assumed to be tuned, therefore, the whole system can be represented by a single shrouded blade having cyclic symmetry constraint applied at a pair of shroud contact points. In modeling the dynamics of a shrouded blade, the displacement vector of a finite element model can be divided into three segments in which \(x_r\) is the displacement vector of the right shroud contact point, \(x_l\) is that of the left shroud contact point, and \(x_b\) represents the displacement vector of all the other nodes of the blade.

Since the two neighboring blades vibrate in the same manner but with the interblade phase angle (\(\varphi\)) difference, the relative motion of the two neighboring shrouds at the right shroud contact point, \(w_r\), can be derived as follows.

\[
w_r = T_1 \begin{bmatrix} x_r \\ x_l \end{bmatrix}
\]  

(2.58)
Figure 2.17 Contact geometry of a shrouded blade system.
Figure 2.18 A tuned blade system having part-span shroud.
where $T_i$ is the interblade relative displacement transformation matrix.

$$T_i = \begin{bmatrix} I_{3 \times 3} & -e^{-\omega s I_{3 \times 3}} \end{bmatrix} \quad (2.59)$$

It is worthy noting that only the relative motion at the right shroud contact point is derived. Since the shrouded blade system is assumed to be tuned, the relative motion and the resulting constrained force of the left shroud contact point can be related to those of the right shroud contact point by using the condition of cyclic symmetry. In the calculation of resonant response, both $x_r$ and $x_s$ are assumed to have elliptical motions, therefore, the resulting relative motion $w_r$ also has elliptical trajectory in the 3D space.

Using the available shroud contact model [Yang, 1996], the constrained force at the right shroud contact point can be determined. In the shroud contact model, the nonlinear relationship between the constrained force and the relative motion is defined in the local coordinate system. By performing a coordinate transformation, the relative motion can be transferred to the local coordinate system as follows.

$$u_r = T_0^T w_r \quad (2.60)$$
In the local coordinate system, the resulting constrained force $p_r$ is a nonlinear function of the relative displacement $u_r$.

$$p_r = p_r(u_r) \quad (2.61)$$

Note that in the case when the shroud contact point is fully stuck, the resulting constrained force is nothing more than the spring force due to the contact stiffness.

$$p_r = k'_s u_r \quad (2.62)$$

where $k'_s$ is the contact stiffness in the local coordinate system. In other words,

$$k'_s = T_0^T k_s T_0 \quad (2.63)$$

After the constrained force is calculated, it can be transferred back to the global coordinate system using the coordinate transformation matrix.

$$f_r = T_0 p_r \quad (2.64)$$

Finally, the constrained forces at the two shroud contact points can be derived as follows.
2.6.2 Constrained receptance of a shrouded blade

When examining the free vibration of a shrouded blade having the cyclic symmetry constraint, the equation of motion can be formulated as follows.

\[
\begin{bmatrix}
\mathbf{m}_{b} & \mathbf{m}_{r} & \mathbf{m}_{\ell}
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_{b} \\
\ddot{x}_{r} \\
\ddot{x}_{\ell}
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{c}_{b} & \mathbf{c}_{r} & \mathbf{c}_{\ell}
\end{bmatrix}
\begin{bmatrix}
\dot{x}_{b} \\
\dot{x}_{r} \\
\dot{x}_{\ell}
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{k}_{bb} & \mathbf{k}_{br} & \mathbf{k}_{b\ell} \\
\mathbf{k}_{rb} & \mathbf{k}_{rr} + \mathbf{k}_{r\ell} - e^{i\phi} \mathbf{k}_{r}\ell & \mathbf{k}_{r\ell} \\
\mathbf{k}_{\ell b} & \mathbf{k}_{\ell r} - e^{-i\phi} \mathbf{k}_{r}\ell & \mathbf{k}_{\ell\ell} + \mathbf{k}_{\ell}\ell
\end{bmatrix}
\begin{bmatrix}
x_{b} \\
x_{r} \\
x_{\ell}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]  

(2.66)

where \( \mathbf{m}_{j} \), \( \mathbf{c}_{j} \), and \( \mathbf{k}_{j} \), \( i, j = b, r, \ell \), are the associated mass, damping, and stiffness sub-matrices, and \( \phi \) is the interblade phase angle. It is worthy noting that the stiffness matrix is Hermitian due to the cyclic symmetry constraint. By using the Modal Analysis Method, the finite element model can be solved for the natural frequencies and the associated constrained mode shapes. Since the stiffness matrix is Hermitian, the constrained mode shapes are complex. The complex constrained mode shape matrix is denoted as \( \Phi^{*} \). If the model contains \( n \) nodes, the dimension of this matrix is \( 3n \times 3n \).

The total receptance is the summation of the receptance of each individual mode.
\[ r = \left[ r_{kl} \right] = \sum_{i=1}^{3a} \left( \Phi_i^* \Lambda_i \Phi_i^{H*} \right) \]  

(2.67)

and

\[ \Lambda_i = \left( (k_i - \omega^2 m_i) + j(\omega c_i) \right)^{-1} \]  

(2.68)

where \( r_{kl} \) is defined as the steady state response of the \( k \)th node due to unit harmonic excitation force at the \( \ell \)th node, \( \Phi_i^* \) is the \( i \)th constrained mode shape, \( \Phi_i^{H*} \) is the complex conjugate transpose of \( \Phi_i^* \), \( m_i \) is the \( i \)th modal mass, \( k_i \) is the \( i \)th modal stiffness, and \( c_i \) is the \( i \)th modal damping.

### 2.6.3 Prediction of resonant response

Using the constrained receptance defined in equations (2.67) and (2.68), similar to the nonlinear feedback loop depicted in Figure 2.12, a nonlinear feedback loop, as shown in Figure 2.19, can be developed to signify the feedback effect of the constrained force on the shrouded blade. From this nonlinear feedback loop, it is again clear that the equation of motion based on a finite element model of the shrouded blade can be condensed to a set of nonlinear algebraic equations.

\[
\begin{bmatrix}
x_r \\
x_{r*}
\end{bmatrix} = r_{r*}^* f - r_{r*}^* T_1 T_0 \left\{ p_r(u_r) - k_r' u_r \right\}
\]  

(2.69)
Figure 2.19 Nonlinear feedback loop of a shrouded blade.
where \( r_{ce}^f \) is the receptance at the shroud contact point due to unit harmonic excitation force, and \( r_{ce}^c \) is the receptance at the shroud contact points due to unit constrained force. In equation (2.69), the relative motion can be related to the unknown displacement as follow.

\[
\mathbf{u}_r = \mathbf{T}_0^T \begin{bmatrix} x_r \\ x_e \end{bmatrix}
\]

(2.70)

In this study, a shrouded blade system consisting of 40 blades is used as an example to illustrate the proposed approach. Each blade is modeled as a beam-like structure. A stiffness matrix is used to represent the contact stiffness between two adjacent blades. The natural frequencies and the associated constrained mode shapes are calculated using equation (2.66). Figure 2.20 shows the receptances of the first four modes based on the constrained mode shapes. The frequency range in Figure 2.20(a) is from 50Hz to 3500Hz, and Figure 2.20(b) enlarges the region of interest, which is from 50Hz to 550Hz. It can be seen that the receptance of the first mode dominates in the fully stuck region, e.g., contributing 99.61% to the total receptance at 406Hz. This implies that the first constrained mode shape can be employed to predict the resonant response in the fully stuck region very well. However, the first mode contributes only 80.37% in the fully slipping region, which is not good enough for accurate prediction. Figure 2.21 shows the comparison of the resonant responses predicted by using two different models.
The first model uses the first mode only and the second one uses all available modes. For an external nodal excitation force of $1 N$ located at the shroud contact nodal point, when the preload is high, higher than $10 N$, the shroud contact is very near fully stuck. Therefore, the prediction based on the first constrained mode should be very accurate. It can be observed from Figure 2.21 that when the preload is $250 N$ the shroud contact is fully stuck and the two models have exactly the same prediction. Only when the preload decreases to $10 N$, the discrepancy becomes noticeable. On the other hand, the model using the first constrained mode overestimates the resonant frequency of the shrouded blade when the interface is fully slipping. It is evident that the proposed hybrid receptances can also be applied to the shrouded blade system.
Figure 2.20 Receptance of the first four modes of the blade based on the constrained mode shapes: (a) 50 - 3500 Hz; (b) 50 - 550 Hz.
Figure 2.21 Prediction of the resonant response of the shrouded blade based on the constrained mode shapes.
2.7 Conclusions

In this study, a blade constrained by a blade-to-ground friction damper is first used as an example to illustrate the concept of constrained mode shape and how it can be used to predict the resonant response of a frictionally constrained blade. It is shown that by examining each mode's contribution to the receptance at the friction contact point, the importance of each individual mode to the prediction of the resonant response of a frictionally constrained blade can be determined. Furthermore, by comparing the receptances calculated from free mode shapes and those from constrained mode shapes, it is found that in the neighborhood of the fully slipping region, the prediction of resonant response requires fewer number of modes when using free mode shapes compared to using constrained mode shapes. On the other hand, in the neighborhood of the fully stuck region, it requires fewer number of modes if constrained mode shapes are used.

The approach is then extended to predict the resonant response of a shrouded blade system. For a tuned blade system, the constrained mode shapes can be calculated using a finite element model of a single shrouded blade along with the cyclic symmetry constraint that simulates a fully stuck shroud constraint. The resulting constrained mode shapes are often complex and can be used to obtain the constrained receptance of the shrouded blade. Using the constrained receptance and the available shroud contact model, a nonlinear feedback loop can be developed to signify the feedback effect of the...
constrained force on the shrouded blade. From this nonlinear feedback loop, it is shown that under a harmonic excitation the equation of motion based on a finite element model of the shrouded blade can be condensed to a set of nonlinear algebraic equations. In this work, a shrouded blade system is employed to demonstrate the effectiveness of the developed approach. Similar to the results obtained for the blade-to-ground damper, it is found that in the neighborhood of the fully stuck region, it requires fewer number of modes for accurate prediction of the resonant response of a shrouded blade system if constrained mode shapes are used. In shroud design, high preload is often desirable. Therefore, using the constrained mode shapes for the prediction of resonant response is preferred.

Moreover, the concept of hybrid receptance is introduced so as to yield very accurate prediction of the resonant response based on only very few vibration modes. It is shown that the hybrid receptances can be employed to correctly predict the resonant response over the range from fully slip to fully stuck.
CHAPTER 3

PERIODIC FORCED RESPONSE OF STRUCTURES HAVING THREE-DIMENSIONAL FRICTIONAL CONSTRAINTS

3.1 Introduction

Many mechanical systems have moving components that are mutually constrained through frictional contacts. When subjected to cyclic excitations, a contact interface may undergo constant changes among sticks, slips, and separations, which lead to very complex contact kinematics. In previous studies of dry friction damper systems, according to Coulomb friction law, the friction coefficient at the contact interface is usually assumed to be constant, the relative motion across the friction contact point is often one-dimensional, and the system is subjected to constant normal load [Griffin, 1980]. It usually results in very simple contact kinematics and can be used to obtain analytical solutions for single-degree-of-freedom systems [Den Hartog, 1931; Wang, 1996]. It can also integrate with Multi-Harmonic Balance Method to yield approximate solutions for single-degree-of-freedom systems [Cameron and Griffin, 1989; Ren and Beards, 1994; Wang and Chen, 1993] and multiple-degree-of-freedom systems [Ferri and
Dowell, 1989; Guillen and Pierre, 1996; Pierre et al, 1985]. It was shown by Menq et al [1986] that the normal load across the friction contact point may vary dynamically with the vibratory motion of a single-degree-of-freedom system, and causes the contact point to stick, slip and possibly separate. A two-dimensional friction contact model that characterizes the transitions among sticks, slips, and separations of the contact interface was recently proposed by Yang and Menq [1997]. In this study, analytical criteria were developed to predict the transitions among sticks, slips, and separations of the friction contact interface.

In order to characterize the three-dimensional contact kinematics of a friction contact, Yang and Menq [1998a] proposed a three-dimensional friction contact model and developed analytical criteria to predict the transitions among sticks, slips, and separations of the friction contact when the resulting relative motion is three-dimensional. These analytical criteria were employed to simulate the hysteresis loop for a given relative motion at the contact interface, so as to characterize the equivalent friction damping and nonlinear stiffness induced by the friction constraint. They were integrated with the single-term Harmonic Balance Method to predict the resonant response of a frictionally constrained three-degrees-of-freedom oscillator. Since the single-term Harmonic Balance Method was used, the effect of the super-harmonic components of the constrained force on the structure’s vibration was ignored. In addition, the resulting relative motion at the friction contact is assumed to be harmonic. Since the actual relative motion often has
super-harmonic components, this assumption hinders accurate predictions of the transitions among sticks, slips, and separations of the friction contact.

In this work, a 3D friction contact model [Yang and Menq, 1998a] is employed to predict the periodic forced response of structures having 3D frictional constraints. In the friction contact model, a contact plane is defined and its orientation is assumed invariant. The resulting relative motion across the two contacting surfaces can be decomposed into two components: in-plane tangential motion on the contact plane and normal component perpendicular to the plane. The in-plane tangential relative motion is often two-dimensional, and it induces stick-slip friction. The normal relative motion can cause variation of the contact normal load and possible separation of the two contacting surfaces. Analytical criteria based on this friction contact model are used to determine the transitions among sticks, slips, and separations of the friction contact, and subsequently the constrained force which consists of the induced stick-slip friction force on the contact plane and the contact normal load. The resulting constrained force is often a periodic function and can be considered as a feedback force that influences the response of the constrained structures. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be integrated with the receptance of the structures so as to calculate the forced response of the constrained structures. It results in a set of nonlinear algebraic equations that can be solved iteratively to yield the relative motion as well as the constrained force at the friction contact. This method is used to
predict the periodic forced response of a frictionally constrained 3-DOF oscillator. The predicted results are compared with those of the direct time integration method so as to validate the proposed method. In addition, the effect of super-harmonic components on the resonant response and jump phenomenon is examined.

3.2 3D Contact Kinematics

When two vibrating bodies are mutually constrained by a friction contact, as shown in Figure 3.1, the periodic relative motion across the two contacting surfaces is usually three-dimensional, and is often not parallel to the contact plane. In order to analyze the induced stick-slip friction, the periodic relative motion in the 3D space can be decomposed into an in-plane periodic motion on the contact plane and a periodically varying component normal to the contact plane. The in-plane periodic motion can induce stick-slip friction, and thus can attenuate the resonant response of the constrained mechanical systems. On the other hand, the normal component tends to alter the normal load across the interface; and this effect, in extreme circumstances, may lead to a separation of the interface. It should be noted that the variable normal load is taken as the sum of the initial contact force at equilibrium plus a term that is proportional to the periodically varying normal component of the relative motion.
Figure 3.1 3D contact kinematics.
In the 3D contact model proposed by Yang and Menq [1998a], the contact interface between two vibrating bodies can be modeled as a substructure that contains a massless elastic element and a friction contact point, as depicted in Figure 3.2. In this model, the elastic element accounts for the shear and normal stiffness of the interface. It is characterized by a $2 \times 2$ stiffness matrix, $K_u$, for the shear stiffness and a spring constant, $k_n$, for the normal stiffness. The friction contact point, that is assumed to obey the Coulomb friction law with the friction coefficient $\mu$ when in contact with Body 2, can undergo tangential stick-slip motion, and may experience intermittent separation from Body 2 when the normal relative motion ($v$) becomes large. The contact interface is assumed to have either a preload or an initial gap (as designated by $n_0$). This model allows a negative preload to represent the situation when the interface has an initial gap; the equivalent preload across the interface with a gap $e$ is calculated as $-k_e$. In this model, $u$ and $v$ are the input tangential relative motion and normal relative motion of the contact interface respectively, and they can be evaluated as the motion of Body 1 with respect to Body 2. In this model, the $2 \times 2$ shear stiffness matrix $K_u$ is used because the tangential relative motion is two-dimensional. If the shear stiffness property is isotropic, a spring constant $k_u$ can be used, and the $2 \times 2$ shear stiffness matrix becomes $k_u I$, in which $I$ is the $2 \times 2$ identity matrix.
Figure 3.2 A model of the friction interface experiencing 3D contact kinematics.
3.2.1 Constrained force

The constrained force consists of two components: the induced stick-slip friction on the contact plane and the variable normal force. Since the friction force is completely characterized by the relative motion, it will not lose generality to assume one of the contacting surfaces is the ground. With this assumption, the input tangential relative motion $u$, the slip motion of the contact point $w$, and the induced stick-slip friction $f$ are vectors parallel to the ground; the normal relative motion $v$ and the normal load $n$ are scalars. The friction force, acting on the ground, can be expressed as:

$$ f = K_u (u - w) $$  \hspace{1cm} (3.1)

The normal load is taken as the sum of the preload $n_0$ plus the variation caused by $v$, and it can be expressed as:

$$ n = \begin{cases} n_0 + k_v v & \text{when } v \geq -n_0/k_v \\ 0 & \text{when } v < -n_0/k_v \end{cases} $$  \hspace{1cm} (3.2)

This friction contact model can be applied to the most general 3D friction contact problem, where the orientation of the contact plane may oscillate when the structure vibrates. However, in this work, we limit its applications to the case, in which the
orientation of the contact plane can be assumed to be invariant. In many mechanical systems, this assumption is reasonable if the amplitude of vibration is relatively small when compared to the overall dimension of the structure.

3.2.2 Stick, slip, and separation

Depending on the amplitude and phase of the various components of the vibratory motion, the friction contact may either stick, slip, or separate during a cycle of oscillation. When the vibratory motion is really small, the contact point sticks and the friction force is proportional to the displacement $u$ with reference to $w = 0$, as implied in equation (3.1). According to the Coulomb friction law, the induced stick-slip friction is always limited to the varying slip load $\mu n$. During the course of the vibration, the interface may reach a point where the friction force tends to exceed the slip load and begin to slip. Subsequently, the friction force remains equal to the slip load, and slip takes place along the direction of the friction force until the contact point sticks again. In other words, the stick and slip conditions can be expressed as follows:

Stick-state:  $|f = K_u (u - u_0) + f_0| < \mu n \quad w = 0$  (3.3)

Slip-state:  $f = \mu n \frac{\dot{w}}{|\dot{w}|} \quad \dot{w} \neq 0$  (3.4)
where \( u_0 \) and \( f_0 \) are the initial values of \( u \) and \( f \) at the beginning of the stick state.

During the cycle of motion, the applied variable normal load may vanish to cause the interface to separate; consequently, the friction force is not present.

### 3.2.3 Stick/slip/separation transition criteria

In order to evaluate the resulting periodic constrained force at the friction contact, analytical criteria are developed to determine the transitions among sticks, slips, and separations, when experiencing variable normal load [Yang and Menq, 1998a]. The analytical criteria can be summarized as follows.

(i) **Stick-to-slip transition**

This transition occurs when the friction force on the tangential plane reaches the varying slip load \( \mu n \). That is,

\[
|f| = K_u (u - u_0) + f_0 \geq \mu n \tag{3.5}
\]

To ensure that the magnitude of the friction force has a tendency to exceed the slip load, the following constraint is imposed:

\[
|f| > \mu \dot{n} \tag{3.6}
\]

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(ii) Slip-to-stick transition

During the slip state, according to the Coulomb friction law, it was shown in reference [Yang and Menq, 1998a] that the friction force can be solved from an initial value problem:

\[ \dot{f} = K_u \left( \dot{u} - \frac{f^T K_u \dot{u} - \mu^2 \dot{n} \ddot{n}}{f^T K_u f} \right) \]  \hspace{1cm} (3.7)

The slip-to-stick transition occurs when the velocity of the relative motion \( \dot{w} \) equals 0, which implies:

\[ f^T K_u \dot{u} - \mu^2 n \ddot{n} = 0 \]  \hspace{1cm} (3.8)

Since the initial friction force at the beginning of the slip condition is known, the initial value problem of equation (3.7) can be solved by using a numerical integration scheme such as the 4th order Runge-Kutta method to obtain the friction force \( f \). Once the friction force is obtained, the criterion of equation (3.8) can be used to predict the occurrence of the slip-to-stick transition.
(iii) Stick/slip-to-separation transition

The transition from stick/slip to separation occurs when the normal load vanishes. In addition, the normal load should be decreasing at this moment to ensure the occurrence of the separation. Hence the transition criteria can be formulated as:

\[ n = 0, \quad \dot{n} < 0 \]  \hspace{1cm} (3.9)

(iv) Separation-to-stick/slip transition

Similarly, the separation ends when the normal load is about to develop on the contact plane. Therefore, the moment of this transition can be determined by the criterion:

\[ n = 0, \quad \dot{n} \geq 0 \]  \hspace{1cm} (3.10)

When the normal load and the friction force begin to develop on the contact plane at the end of the separation, their rate of change at the moment determine whether the following state is either stick or slip.

\[ \dot{u}^T K_a^T K_a u < \mu^2 \dot{n}^2 \Rightarrow \text{Stick begins} \] \hspace{1cm} (3.11)
\[ \mathbf{u}^T \mathbf{K}_u^T \mathbf{K}_u \mathbf{u} \geq \mu^2 \mathbf{n}^2 \Rightarrow \text{Slip begins} \quad (3.12) \]

It should be pointed out that the incipient slip condition is regarded as an one-dimensional case, because the friction force is not present at this moment and the slip action will be developed along \( \mathbf{u} \). Thus, according to the Coulomb friction law, the rate of change of the developing friction force can be expressed as:

\[ \dot{\mathbf{f}} = \mu \dot{n} \frac{\mathbf{u}}{||\mathbf{u}||} \quad (3.13) \]

Once the friction force develops, it can be further determined by solving the initial value problem of equation (3.7).

In this study, these criteria are used to simulate hysteresis loops of the friction contact, when experiencing periodic relative motions. With these hysteresis loops, the constrained force can be linked to the relative motion between two contacting bodies. By using Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions and employed in Multi-Harmonic Balance Method to predict the periodic response.
3.3 Multi-Harmonic Balance Method

The equation of motion for a structure experiencing three-dimensional frictional constraints under external periodic excitations can be expressed as:

\[
M \ddot{U}(t) + C \dot{U}(t) + KU(t) = f_e(t) - f_N(U, \dot{U}, t) \tag{3.14}
\]

where \( U \) is the nodal displacement vector, \( M \) the mass matrix, \( C \) the damping matrix, \( K \) the stiffness matrix, \( f_e \) the external periodic excitation, and \( f_N \) the nonlinear constrained force, which is a function of the motion at the contact point. The model is three-dimensional. If the model contains \( n \) nodes, all the matrices are \( 3n \times 3n \) matrices, and all the vectors have \( 3n \) elements.

The external periodic excitation can be resolved into Fourier series:

\[
f_e = \sum_{k=1}^{\infty} f_k^e e^{i k \omega t} \tag{3.15}
\]

in which \( \omega \) is the fundamental frequency of the excitation force, and \( f_k^e \) represents the magnitude and phase of its \( k^{th} \) harmonic component. In the following analysis, the external periodic excitation is approximated as:
where $m$ is the number of harmonic components considered in the analysis. From linear vibration theory, without nonlinear constrained force, the forced response of the structure due to external periodic excitation can be obtained by standard harmonic analysis. The complex receptance matrix corresponding to the $k^{th}$ harmonic frequency can be expressed as:

$$
R_k = \left[ r_{il,k} \right] = \left[ K - (k\omega)^2 M + j(k\omega)C \right]^{-1}
$$  \hspace{1cm} (3.17)

where $r_{il,k}$ is defined as the harmonic steady state response of the $i^{th}$ node due to the $k^{th}$ unit harmonic excitation force at the $e^{th}$ node.

Assume that there are $p$ friction contact points. For simplicity, these contact points are assigned to be the first $p$ nodes of the model. Therefore, the displacement vector of these friction contact points is expressed as:

$$
U_c = \left[ u_i \right] \quad i = 1, 2, \cdots, p
$$  \hspace{1cm} (3.18)
Since the external excitation is periodic, the steady state response and the induced constrained force are assumed to be periodic. The steady state periodic response at the $i^{th}$ node can be expressed as:

$$u_i = \sum_{k=0}^{m} u_{i,k} e^{ij\omega t}$$  \hspace{1cm} (3.19)

The steady response at each node consists of two components. The first one is due to the external excitation force while the second component caused by the constrained forces at the friction contact points.

With the complex receptance matrix, the $k^{th}$ harmonic component of the steady state response at the $i^{th}$ node due to the external excitation force can be obtained as:

$$u_{i,k}^e = \sum_{\ell=1}^{n} r_{i,\ell,k} f_{\ell,k}^e$$  \hspace{1cm} (3.20)

where $u_{i,k}^e$ is the $k^{th}$ harmonic response of the $i^{th}$ node due to the external excitation and $f_{\ell,k}^e$ is the $k^{th}$ harmonic component of the external excitation at the $\ell^{th}$ node.

---

1 In order to determine the constrained force at the $i^{th}$ contact node, the relative displacement across the contact interface is needed. In this manuscript, the second body of each contact interface is assumed to be the rigid ground. Therefore, instead of using the relative displacement, the displacement of the contact node is used.
The constrained force at these friction contact points can be obtained by employing the 3D friction contact model. For any given displacement at a contact node, discrete simulation for the induced stick-slip friction is performed. Since analytical transition criteria are used in the simulation, it takes at most few cycles to obtain the steady state constrained force. By appropriately selecting the number of points in the discrete simulation, the steady state constrained force can be resolved into Fourier series by using FFT algorithm:

\[ f_i^N = f_i^N(u_i) = \sum_{k=0}^{N} f_{i,k}^N(u_i)e^{jk2\pi} \]  

(3.21)

where \( f_{i,k}^N \) represents the magnitude and phase of the \( k^{th} \) harmonic component of the constrained force at the \( i^{th} \) friction contact point\(^2\). In the previous studies [Ferri and Dowell, 1988; Ren and Beards, 1994; Wang, 1996; Wang and Chen, 1993], the even harmonic components of the response were not needed due to the assumption of constant normal load. However, in this study, the constrained force is a periodic function having both odd and even harmonic components, due to the periodically varying normal load.

\(^2\) Sub-harmonic components can also be included in the approach presented in this work. However, they are ignored in the current manuscript.
These constrained forces can be considered as feedback forces that come to influence the response of the entire structure. Their effects to the motion at these $p$ contact points can be expressed as follows:

$$
\sum_{k=0}^{m} \mathbf{u}_{i,k} e^{j \omega t} = \sum_{k=1}^{m} \mathbf{u}_{i,k}^* e^{j \omega t} - \sum_{k=0}^{p} \sum_{\ell=1}^{r} \mathbf{r}_{i,k} \mathbf{f}_{i,k}^N (\mathbf{u}_\ell) e^{j \omega t} \quad i = 1, 2, \cdots, p
$$

According to Multi-Harmonic Balance Method, by equating the coefficients for each harmonic frequency, a set of nonlinear algebraic equations can be obtained.

$$
\mathbf{u}_{i,k} = \mathbf{u}_{i,k}^* - \sum_{\ell=1}^{r} \mathbf{r}_{i,k} \mathbf{f}_{i,k}^N (\mathbf{u}_\ell) \quad i = 1, 2, \cdots, p; \ k = 0, 1, \cdots, m
$$

This set of nonlinear algebraic equations has the unknown $\{\mathbf{u}_{i,k}\}$, which can be solved iteratively by using Newton-Raphson algorithm. After knowing $\{\mathbf{u}_{i,k}\}$, the constrained force can be calculated by using equation (3.21), and the constrained force along with the receptance can be used to calculate the periodic response of the entire system.
3.4 Periodic Forced Response of A Three-Degrees-of-Freedom Oscillator

A three-degrees-of-freedom oscillator is considered in this study to illustrate the ability of the proposed method in predicting the periodic response of structures experiencing 3D frictional constraints. The oscillator, as depicted in Figure 3.3, can move in the $xyz$ space, and either is brought into contact with the ground by a preload $n_0$ or has an initial gap in between. The interface between the oscillator and the ground is modeled as a flexible Coulomb friction contact, which is shown in Figure 3.2. When subjected to external excitation, the $xy$ motion of the oscillator is restricted by friction, while the $z$ motion may cause the normal load across the interface to vary. Instead of using the conventional mass-spring-dashpot notation, this 3DOF system can be described alternatively by its modal information involving three vibration modes. It can be shown that at least two vibration modes involved in the frequency range of interest are needed to result in a response having three-dimensional periodic motion. To simplify the analysis, only two modes are considered, and the third mode is neglected from the analysis by letting its natural frequency out of the frequency range of interest. The system parameters of the 3DOF oscillator under investigation along with the harmonic modal excitation are shown in Table 3.1. The parameters of the friction interface used in this investigation are: $\mu = 0.5$, $K_u = diagonal[20 \ 20]$, and $k_v = 20$. 

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Figure 3.3 A three-degrees-of-freedom oscillator having a 3D frictional constraint.
Table 3.1 Modal information of the 3DOF oscillator and the excitation.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mass</th>
<th>Frequency (Hz)</th>
<th>Damping ratio</th>
<th>Mode shape</th>
<th>Excitation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.5</td>
<td>0.01</td>
<td>$(1 \ 1 \ 0.8)^T$</td>
<td>$1.0 \angle 0^\circ$</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>3.0</td>
<td>0.01</td>
<td>$(-1 \ 1 \ 0.6)^T$</td>
<td>$1.0 \angle 0^\circ$</td>
</tr>
</tbody>
</table>
3.4.1 Shift of resonant frequency

In this study, the first three harmonic components of the periodic response are included in the analysis. Under various levels of preload, the periodic responses of the 3DOF oscillator are calculated and shown in Figure 3.4. Since the resulting responses along the three axes are similar, only the amplitude of the response along the $x$ direction is presented in the figure. It can be observed that there exist two limit cases, which are referred to as the fully separate case and fully stuck case. Both cases are linear problems because the nonlinear contact force does not appear in the analysis. The fully separate case occurs when the interface has such a large initial gap that the vibrating oscillator cannot make contact with the ground. Since the contact force is not present, two resonant responses corresponding to the first two natural frequencies, 0.5 and 3.0 Hz, of the system can be clearly seen. In the opposite way, when the preload of the interface exceeds a level depending on the external excitation, the interface remains fully stuck. In this case, the friction contact does not dissipate energy but provides additional stiffness, which arises from the compliance of the interface, to the system to cause higher resonant frequencies at 3.59 and 4.66 Hz.
Figure 3.4 Prediction of periodic forced response of the 3DOF oscillator using 3-Terms Harmonic Balance Method.
3.4.2 Reduction of resonant peak

In between the two linear cases, the nonlinear constrained force, including the stick-slip friction and the variable normal load, appears to affect the response of the system. The attenuation effect of the stick-slip friction can be clearly seen from the results. As the preload increases, the resonant response decreases until the minimum response is reached. Beyond this preload, the damping effect tends to reduce gradually towards the fully stuck case. The preload that gives the minimum response is known as the optimal preload.

Since the stick-slip friction has higher harmonic components, it is possible that the higher harmonics of the oscillator can be excited and result in internal resonance. As can be seen in Figure 3.4, for example, when the preload is -12, the internal resonance can be observed at two resonant frequencies 0.33 and 0.21 Hz, where the second and third harmonics are significant. This also indicates that the even harmonic components can no longer be ignored because they may affect the accuracy in predicting the periodic response.

When only the fundamental harmonic component is considered, the forced responses of the 3DOF oscillator under various levels of preload are calculated and shown in Figure 3.5. By comparing the predicted responses with those of 3-terms
Harmonic Balance Method, it can be observed that single-term Harmonic Balance Method often over-estimates the forced response. Furthermore, single-term Harmonic Balance Method can not predict the internal resonance in the forced response.
Figure 3.5 Prediction of forced response of the 3DOF oscillator using Single-Term Harmonic Balance Method.
3.4.3 Comparison with time integration method

The comparison of the predicted results with those of the time integration method is also shown in Figures 3.4 and 3.5, in which the discrete data points denote the time integration solutions. All the comparisons are made in the frequency near resonance. From the results, it is apparent that the 3-terms Harmonic Balance Method can provide more accurate solutions than the single-term Harmonic Balance Method. In the work of Yang and Menq [1998a], by using the single-term Harmonic Balance Method, the prediction of the resonant response results in discrepancies in the cases that have either small preload or small gap.

In this work, by including the super-harmonic components in the analysis, Multi-Harmonic Balance Method can provide accurate prediction of the periodic response. Figure 3.6 shows the comparison of the predicted forced response by using 1, 3 and 7-terms Harmonic Balance Method and the time integration method when the preload is 0. In this case, the super-harmonic components effect the forced response significantly within the frequency range between 0 and 1 Hz. It can be seen that the single-term Harmonic Balance Method over-estimates the peak resonance by 130%, and the 3-terms Harmonic Balance Method over-estimates the peak resonance by 17%. However, when the first seven harmonic components are analyzed, the predicted forced response agrees with the time integration results very well.
Figure 3.6 Comparison of periodic response of the 3DOF oscillator using Multi-Harmonic Balance Method and Time Integration Method ($\alpha = 0$).
For further illustration, the steady state oscillation trajectories of the displacement in the $x$ direction are compared in Figure 3.7 when the external excitation frequency is 0.77 Hz. It is clear that the 7-terms Harmonic Balance Method can provide very accurate prediction of the fundamental harmonic component as well as super-harmonic components. It is worth noting that the super-harmonic response of the structure may not be the major reason of the very large discrepancies in Figure 3.7. It is possible that small super-harmonic components can induce significant changes on stick-slip transition that lead to large discrepancies in the prediction of the constrained forces.

The time consumption of the Time Integration Method and the Multi-Harmonic Balance Method for the prediction of periodic forced response is compared as shown in Table 3.2. It is evident that the Time Integration Method requires much longer time because it needs to go through the transient response before reaching steady state response. On the other hand, it can be observed that the Multi-Harmonic Balance Method will consume more time when more super-harmonic components are included in the analysis.
Figure 3.7 Comparison of periodic trajectories of the 3DOF oscillator using Multi-Harmonic Balance Method and Time Integration Method ($\omega = 0.77\text{Hz}, n_0 = 0$).
<table>
<thead>
<tr>
<th>Method</th>
<th>Time Consumption (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single-term HBM</td>
<td>93</td>
</tr>
<tr>
<td>3-terms HBM</td>
<td>427</td>
</tr>
<tr>
<td>7-terms HBM</td>
<td>2343</td>
</tr>
<tr>
<td>Time Integration Method</td>
<td>3438</td>
</tr>
</tbody>
</table>

Table 3.2 Comparison of time consumption of Time Integration Method and Multi-Harmonic Balance Method.
3.4.4 Jump phenomenon

In addition to its influence on the friction characteristic, the variable normal load can directly impose nonlinear stiffness on the system. This nonlinear stiffness arises from the intermittent separation of the contact surface during the course of vibration. It has been known that this nonlinearity can result in a multi-valued response that can lead to so called “jump phenomenon” [Thomson, 1988] and the standard continuation technique [Allgower and Georg, 1990] can be used to obtain the multi-valued resonant response. In Figure 3.4, two different types of jump phenomenon can be clearly seen although the effect of the variable normal load is mixed with that of friction. The first one occurs when the interface has a moderate initial gap ($n_0 = -12$); as the amplitude of the vibratory motion increases, the interface will stay in contact for some period to impose a “hardening spring” effect on the response causing the resonant peak to bend towards higher frequencies. The other jump phenomenon, however, occurs when a moderate preload is applied ($n_0 = 2$). The increase in the amplitude of the motion causes the preloaded interface to separate, and as a result, the interface can not provide stiffness to the system temporarily. The overall effect of the temporary separation is similar to the effect of a “softening spring” that gives rise to the response with a resonance peak bending towards lower frequencies.
Figure 3.8 enlarges the “spring hardening” phenomenon which occurs between 0.759 and 0.782 Hz, when the preload is -12, and compares the resonant response predicted by the 3-terms Harmonic Balance Method and the single-term Harmonic Balance Method. It can be seen that the single-term Harmonic Balance Method overestimates the peak response by eighty percent. It should also be noted that one of the multiple solutions from the harmonic balance method shown as the dotted curve is unstable [Thomson, 1988]; while separated by the unstable response, the stable response consists of two curves, which are referred to as the upper and lower branches. Figure 3.9 enlarges two “spring softening” phenomena when the preload is 2. The one occurs between 2.35 and 2.38 Hz is shown in Figure 3.9 (a) and the other between 3.21 and 3.39 Hz in Figure 3.9 (b). In this figure, the resonant response predicted by the single-term Harmonic Balance Method is compared to that by the 3-terms Harmonic Balance Method. It is found that the 3-terms Harmonic Balance Method predicts the jump phenomenon more accurately than the single-term Harmonic Balance Method, which often overestimates the resonant response at the region near jump. In addition, it can be observed that the second harmonic component can no longer be ignored because of the periodically varying normal load.
Figure 3.8 “Spring Hardening” jump phenomenon: $n_0 = -12$. 
Figure 3.9 Two “spring softening” jump phenomena at $n_0 = 2$; (a) jump between 2.35 and 2.38; (b) jump between 3.21 and 3.39.
3.5 Conclusions

In this work, a 3D friction contact model is employed to predict the periodic response of structures having 3D frictional constraints. When subjected to periodic excitation, the resulting relative motion at the friction contact interface is assumed to be periodic in the three-dimensional space. Based on the 3D friction contact model, analytical criteria are used to determine the transitions among sticks, slips, and separations of the contact interface and are used to simulate hysteresis loops of the induced constrained force, when experiencing periodic relative motion. The constrained force can be considered as a feedback force that influences the response of the frictionally constrained structures. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions so as to predict the periodic response of frictionally constrained structures. Due to the periodically changing normal load, both even and odd harmonic components need be included in the analysis.

The developed method is used to predict the periodic response of a frictionally constrained 3-DOF oscillator. The predicted nonlinear response shows three distinct features: (1) shifted resonant frequency due to the additional spring constant introduced by the friction constraint, (2) damped resonant response due to the additional friction damping introduced by frictional slip, (3) multi-valued response leading to a jump
phenomenon due to intermittent interface separation. The predicted results are also compared with those of the direct time integration method so as to validate the proposed method. In addition, the effect of super-harmonic components on the resonant response and jump phenomenon is examined. It was found that single-term Harmonic Balance Method often over-estimates the resonant response of a frictionally constrained structure and can not predict the internal resonance in the forced response. It is also found that small super-harmonic components can induce significant changes on stick-slip transition and lead to large discrepancies in the prediction of the constrained forces. It is shown that the Multi-Harmonic Balance Method is more efficient than the Time Integration method in the prediction of the periodic forced response. However, it will take more time when more super-harmonic components are included in the analysis.
CHAPTER 4

PREDICTION OF PERIODIC FORCED RESPONSE OF BLADES HAVING 3D NONLINEAR SHROUD CONSTRAINTS

4.1 Introduction

In turbine engine design, shroud constraints are often employed to attenuate the blade vibration and at the same time to increase aeroelastic stability of the turbine. During the engine operation, the shrouded blades rotate through the fluid flow with severe fluctuation, and as a result, the blades may bend and twist to cause the off-center shroud contact to experience friction constraint. Therefore, the relative motion across the shroud contact is often three-dimensional. The relative motion can cause variations of the contact normal load that result in very complicated contact kinematics including stick, slip, and separation. The resulting constraint force can add friction damping as well as nonlinear spring force to the bladed disk system [Cameron et al, 1990; Ferri, 1996; Griffin, 1980]. In order to analytically evaluate the performance of the shroud and to help the design, it is necessary to consider the 3D shroud constraint so that the forced response of the shrouded blades can be accurately predicted.
In the shrouded blade system, the protruding shrouds constrain the blade motions through a contact plane. The relative motion across the contact plane can be decomposed into two components: in-plane tangential motion on the contact plane and normal component perpendicular to the contact plane. The in-plane tangential relative motion is often two-dimensional, and it can induce stick-slip friction [Menq and Yang, 1998a; Yang and Menq, 1997; Griffin and Menq, 1991; Menq et al., 1991]. On the other hand, the normal relative motion can cause variation of the contact normal load and, in extreme circumstances, separation of contact interface [Menq et al., 1986; Yang and Menq, 1997].

In previous studies of dry friction damper systems, according to Coulomb friction law, the friction coefficient at the contact interface is usually assumed to be constant, the relative motion across the friction contact point is often one-dimensional, and the system is subjected to constant normal load [Griffin, 1980]. It usually results in very simple contact kinematics and can be used to obtained analytical solutions for single-degree-of-freedom systems [Den Hartog, 1931; Wang, 1996], or to integrate with Multi-Harmonic Balance Method to yield approximate solutions for single-degree-of-freedom systems [Cameron and Griffin, 1989; Ren and Beards, 1994; Wang and Chen, 1993] and multiple-degree-of-freedom systems [Ferri and Dowell, 1988; Guillen and Pierre, 1996; Pierre et al., 1985]. Yang and Menq [1997] proposed a 2D version of shroud contact kinematics, in which the contact interface retains the normal component of the relative motion that causes normal load variation, while the in-plane tangential component of the relative
motion degenerates into linear motion. In other words, the assumed blade motion has only two components, namely axial and tangential components. To take the radial component into account, Yang et al. [1998a] proposed a simplified three-dimensional shroud contact kinematics, where the two-dimensional in-plane tangential relative motion is assumed to be elliptical and is decomposed into two linear motions along the principal major and minor axes of the ellipse. A variable normal load friction force model [Yang and Menq, 1997] is then applied separately to each individual linear motion to estimate the equivalent stiffness and damping of the shroud contact. Yang and Menq [1998a] proposed a 3D shroud contact model, in which the joined effect of the 2D tangential relative motion and the normal relative motion is examined. They developed a set of analytical criteria to determine the transitions among stick, slip, and separation, when experiencing variable normal load. With these transition criteria, the constrained force can be predicted for any given 3D periodic relative motion across the contact interface.

In this work, the 3D shroud contact model [Yang and Menq, 1998a] is employed to obtain the constrained force at the shroud contact of a shrouded blade system. The bladed system is assumed to be tuned and the assumed blade motion has three components, namely axial, tangential, and radial components. In the shroud contact model, a contact plane is defined and its orientation is assumed invariant. When subjected to periodic excitation, the resulting three-dimensional relative motion at the shroud contact as well as the constrained force are assumed to be periodic. The
constrained force can be considered as a feedback force that influences the response of the shrouded blade. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions. In the calculation of the nonlinear forced response of a shrouded blade, all the linear degrees of freedom can be condensed to receptance and the modeling of shroud contact can be separated from the complex structure model. This approach results in a set of nonlinear algebraic equations which can be solved iteratively to yield the periodic forced response of blades having 3D nonlinear shroud constraint.

4.2 3D Shroud Contact

Figure 4.1 shows a shrouded blade system with two neighboring blades contacting each other through the protruding shrouds. When subjected to cyclic excitation, the vibratory motion of the shrouded blade can be assumed to be periodic, and the resulting relative motion across the shroud contact is also periodic in the 3D space. In modeling the shroud contact, a “substructure” can be used to represent the friction interface that contains the contact plane and small portions of the two neighboring shrouds, and the substructure can be modeled as two massless elastic elements that are held together by a preload $n_0$. The points $A$ and $B$ are the outermost points of these two elastic elements; and the difference of their respective motions can describe the 3D periodic relative motion of the two neighboring shrouds. The periodic relative motion is often not parallel
to the contact plane. In order to analyze the induced friction, the periodic relative motion in the 3D space can be decomposed into an in-plane periodic motion on the contact plane and a periodically varying component normal to the contact plane.
Figure 4.1 Shroud contact of two neighboring shrouded blades.
4.2.1 Shroud contact geometry

Two shrouded blades contact each other through their protruding shrouds as shown in Figure 4.2, and the \( xyz \) coordinate system (called blade coordinate system) is defined in accordance with the tangential (\( x \)), axial (\( y \)), and radial (\( z \)) directions. The contact plane of the 3D shroud contact is defined by two angles \( \psi \) (called shroud angle) and \( \phi \) (called inclination angle). A \( uvw \) coordinate system (called shroud coordinate system) can be defined, where \( v \) axis is along the normal direction of the contact plane, and \( u \) and \( w \) axes are on the contact plane. In this work, the blade coordinate system is specified by three basis unit vectors, namely \( [\hat{x} \ \hat{y} \ \hat{z}] \), and the shroud coordinate system is defined as \( [\hat{v} \ \hat{u} \ \hat{w}] \). These two coordinate systems can be related to each other as follows:

\[
[\hat{v} \ \hat{u} \ \hat{w}] = [\hat{x} \ \hat{y} \ \hat{z}] T_0 \tag{4.1}
\]

where \( T_0 \) is the coordinate transformation matrix [Yang et al, 1998a].

\[
T_0 = \begin{bmatrix}
\cos \psi \cos \phi & -\sin \psi & \cos \psi \sin \phi \\
\sin \psi \cos \phi & \cos \psi & \sin \psi \sin \phi \\
-\sin \phi & 0 & \cos \phi
\end{bmatrix} \tag{4.2}
\]
Figure 4.2 Contact geometry of a shrouded blade system.
In current design practice, the inclination angle $\phi$ is often set to be zero.

The decomposition of the 3D relative motion is shown in Figure 4.3 schematically. The in-plane periodic motion can induce stick-slip friction, and thus can attenuate the periodic response of the shrouded blades. On the other hand, the normal component tends to alter the normal load across the interface; and this effect, in extreme circumstances, may lead to separation of the interface. It should be noted that the variable normal load is taken as the sum of the initial contact pressure at equilibrium plus a term that is proportional to the periodically varying normal component of the relative motion. Since this decomposition is to transfer the relative motion from the blade coordinate system to the shroud coordinate system, it can be carried out by performing a coordinate transformation on the 3D relative motion by using equation (4.1).

4.2.2 3D shroud contact model

In the 3D shroud contact model proposed by Yang and Menq [1998a], the contact interface between two vibrating shrouds can be modeled as a substructure that contains a massless elastic element and a friction contact point, as depicted in Figure 4.4. In this model, the elastic element accounts for the shear and normal stiffness of the substructure, and it is characterized by a $2 \times 2$ stiffness matrix $K_u$ for the shear stiffness and a spring constant $k_r$ for the normal stiffness.
Periodic relative motion in 3D space

In-plane (2D) periodic motion

Periodically varying normal component causing normal load variation

Figure 4.3 Decomposition of 3D periodic relative motion.
Figure 4.4 A 3D shroud contact model.
The friction contact point, that is assumed to obey the Coulomb friction law with the friction coefficient $\mu$ when in contact with Body 2, can undergo tangential stick-slip motion, and may experience intermittent separation from Body 2 when the normal relative motion ($v$) becomes large. The contact interface is assumed to have either a preload or an initial gap (as designated by $n_y$). This model allows a negative preload to represent the situation when the interface has an initial gap; the equivalent preload across the interface with a gap $e$ is calculated as $-k_e e$. In this model, $u$ and $v$ are the input tangential relative motion and normal relative motion of the contact interface respectively, and they can be evaluated as the motion of Body 1 with respect to Body 2.

Referring to Section 3.2, analytical criteria between stick, slip and separation are used to simulate hysteresis loops of the friction contact, when experiencing periodic relative motion. With these hysteresis loops, the resulting constrained force can be characterized by the relative motion between two neighboring shrouds. By using Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions and employed in Multi-Harmonic Balance Method to solve for the periodic response.
4.3 Prediction of Periodic Forced Response of A Shrouded Blade System

In the analysis of a shrouded blade system, a great simplification can be obtained by assuming that the bladed system is tuned, namely each blade of the system has exactly the same dynamic characteristics. In addition, the excitation of interest is that induced by the blades rotating through circumferential variations in the flow field. It can be shown that in effect each blade is exposed to a periodic excitation having the same amplitude but differing in phase by an amount which is proportional to the blade's angular location on the disk. It is assumed that the forced response of the bladed system is also periodic and has the same fundamental period as the excitation. Thus the external excitation and the motions of the blades as well as the nonlinear constrained forces can be represented by infinite Fourier series. By truncating these series after the $m$th terms, an approximate solution assuming that the forced response is periodic can be derived. In this approach, each blade vibrates in the same manner but with a proportional interblade phase difference ($k\phi$) for $k$th harmonic component from its adjacent blades. The interblade phase angle is defined as follows:

$$\phi = \frac{2\pi E}{N}$$  \hspace{2cm} (4.3)

\footnote{Sub-harmonic components can also be included in the approach presented in this work. However, they are ignored in the current manuscript.}
where \( N \) is the total number of the blades in the system and \( E \) is the engine order of the excitation on the system.

The equation of motion of a shrouded blade subjected to a periodic excitation can be expressed as follows:

\[
\mathbf{m}\ddot{\mathbf{x}} + \mathbf{c}\dot{\mathbf{x}} + \mathbf{k}\mathbf{x} = \mathbf{f}_e - \mathbf{f}_N
\]  

(4.4)

where \( \mathbf{x} \) is the nodal displacement vector, \( \mathbf{m} \) is the mass matrix, \( \mathbf{c} \) is the damping matrix, \( \mathbf{k} \) is the stiffness matrix, \( \mathbf{f}_e \) is the external periodic excitation, and \( \mathbf{f}_N \) is the nonlinear constrained force which is a nonlinear function of the relative motion at the shroud contact. The finite element model is three-dimensional and if the model contains \( n \) nodes, all the matrices will be \( 3n \times 3n \) matrices, and all the vectors will be \( 3n \)-element vectors.

The external periodic excitation can be expressed as follows.

\[
f_e(t) = \sum_{k=0}^{m} f_{x,k} e^{j\omega t}
\]  

(4.5)
where \( \omega \) is the fundamental excitation frequency and \( f_{e,k} \) is a complex vector representing the amplitude and phase for the \( k^{th} \) harmonic component of the excitation. Here the external periodic excitation is assumed to include up to \( m^{th} \) harmonic component. It should be pointed out that except for the elements associated with those shroud contact points the other elements of \( f_{e} \) are zeros. It is clear that the nonlinear aspect of the dynamic problem is embedded in the nonlinear friction force \( f_{v} \). By using the Modal Analysis Method, the mode shape matrix can be obtained and is denoted as \( \Phi \). Using the mode shape matrix and the associated modal information, the receptance of the blade can be derived as:

\[
\mathbf{r} = [r_{pt,k}] = \sum_{i=1}^{3n} \left( \Phi_i \Lambda_{i,k} \Phi_i^T \right) \tag{4.6}
\]

and

\[
\Lambda_{i,k} = \left[ (\mathbf{k}_i - k^2 \omega^2 \mathbf{m}_i) + j(\kappa \omega \mathbf{c}_i) \right]^{-1} \tag{4.7}
\]

where \( r_{pt,k} \) is defined as the steady state response of the \( p^{th} \) node due to unit \( k^{th} \) harmonic excitation force at the \( \ell^{th} \) node, \( \Phi_i \) is the \( i^{th} \) mode shape, \( \mathbf{m}_i \) is the \( i^{th} \) modal mass, \( \mathbf{k}_i \) is the \( i^{th} \) modal stiffness, and \( \mathbf{c}_i \) is the \( i^{th} \) modal damping.
In this work, the periodic three-dimensional motion in the blade coordinate system can be represented as follows:

\[
x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \approx \left[ \sum_{k=0}^{m} A_{x,k} e^{j\omega k} \right] \text{j} + \sum_{k=0}^{m} A_{y,k} e^{j\omega k} \text{j} + \sum_{k=0}^{m} A_{z,k} e^{j\omega k} \text{j}
\]

where \( \omega \) is the fundamental oscillating frequency, and \( A_{i,k} \), \( i = x, y, z \), are the complex vectors representing the amplitude and phase angle of the \( k^{th} \) harmonic component along tangential, axial and radial axes. For a shrouded blade, several pairs of shroud contact points can be defined. For each pair of shroud contact points, one is on the right and the other left and their motions are denoted as \([x_r \quad x_t]^T\). For convenience, this vector can be arranged as:

\[
[x_r \quad x_t]^T = [(x_r \quad x_t)_k]^T = [x_{r,0} \quad x_{t,0} \quad x_{r,1} \quad x_{t,1} \cdots x_{r,m} \quad x_{t,m}]^T
\]

4.3.1 Relative motion at shroud contact

Since the shrouded blade system is assumed to be tuned, the condition of cyclic symmetry can be applied when deriving the relative motion of a shroud contact. Take the
relative motion between the point $B$ of the right shroud and the point $A$ of the left shroud in Figure 4.1 as an example. First, the motions of the two contact points of the first shrouded blade (left one) are defined and they are $[x_r \ x_t]^T$. Therefore, the motion of point $B$ is now $x_r$ and the motion of point $A$ differs from $x_r$ with the proportional interblade phase angle ($k\varphi$) for the $k^{th}$ harmonic component. As a result, the relative motion of the two neighboring shrouds, $w_r$, can be derived as

$$w_r = T_i [(x_r \ x_t)_k]^T, \ k = 0,1,\ldots,m$$  \hspace{1cm} (4.10)$$

where $T_i$ is the interblade relative displacement transformation matrix.

$$T_i = \begin{bmatrix} diag(T_{i,0}, T_{i,1}, \ldots, T_{i,m}) \end{bmatrix} \hspace{1cm} (4.11)$$

$$T_{i,k} = [I_{3x3} \ -e^{-jk\varphi} I_{3x3}], \ k = 0,1,\ldots,m \hspace{1cm} (4.12)$$

It is worthy noting that only the relative motion at the right shroud contact point is derived. Since the shrouded blade system is assumed to be tuned, the relative motion and the resulting constrained force of the left shroud contact point can be related to those of the right shroud contact point by using the condition of cyclic symmetry. Since both $x_r$,
and $x_r$ are periodic motions, the resulting relative motion $w_r$ also has periodic trajectory in the 3D space.

Since this relative motion $w_r$ is in the blade coordinate system, it can be transformed to the shroud coordinate system by using the transformation defined in equation (4.1):

$$u_r = T_{BS}^T w_r$$  

$$T_{BS} = \begin{bmatrix} diag(T_{0,0}, T_{0,1}, \ldots, T_{0,m}) \end{bmatrix}, T_{0,k} = T_0, \quad k = 0, 1, \ldots, m$$  

where $T_{BS}$ is the coordinate transformation matrix for 3D periodic relative motion. The 3D relative motion in the shroud coordinate system can be expressed as follows:

$$u_r = \begin{bmatrix} v \\ u \\ w \end{bmatrix} \approx \begin{bmatrix} \sum_{k=0}^{m} A_{v,k} e^{j\omega k} \\ \sum_{k=0}^{m} A_{u,k} e^{j\omega k} \\ \sum_{k=0}^{m} A_{w,k} e^{j\omega k} \end{bmatrix}$$  

where $A_{i,k}$, $i = v, u, w$, are the complex vectors representing the amplitude and phase angle of the $k^{th}$ harmonic component along $v$, $u$ and $w$ axes.
4.3.2 Constrained force at shroud contact

After the decomposition, the \( u \) and \( w \) components of the relative motion follow a periodic trajectory on the contact plane and can induce the stick-slip friction, while the \( v \) component can cause the normal load across the interface to vary dynamically. It is apparent that the \( u \) and \( w \) motions are coupled together when inducing stick-slip friction. Using the 3D contact model proposed by Yang and Menq [1998a], the constrained force at the right shroud contact point can be determined. By using Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions and can be expressed as follows:

\[
\mathbf{p}_r = \begin{bmatrix}
\mathbf{p}_r(v) \\
\mathbf{p}_r(v,u,w) \\
\mathbf{p}_r(v,u,w)
\end{bmatrix} \approx \begin{bmatrix}
\sum_{k=0}^{m} \mathbf{p}_{rv,k} e^{jk\omega t} \\
\sum_{k=0}^{m} \mathbf{p}_{ru,k} e^{jk\omega t} \\
\sum_{k=0}^{m} \mathbf{p}_{rw,k} e^{jk\omega t}
\end{bmatrix}
\]

(4.16)

where \( \mathbf{p}_{ri,k} \), \( i = v, u, w \), are the complex Fourier coefficients of the \( k^{th} \) harmonic component along \( v \), \( u \) and \( w \) axes. Then, the constrained force can be transformed back to the blade coordinate system:

\[
f_r = T_{B0} \mathbf{p}_r
\]

(4.17)
Furthermore, the constrained forces at a pair of shroud contact points can be related to the force at the right shroud contact point using the interblade relative displacement transformation matrix.

\[
\begin{bmatrix}
  f_r \\
  f_r
\end{bmatrix} = T_i^H f_r
\]  

(4.18)

where \( T_i^H \) is the complex conjugate transpose of \( T_i \).

4.3.3 Nonlinear algebraic equations

When the blade is constrained by its neighboring blades through shroud contacts, the resulting constrained forces are characterized by the displacements of a pair of contact points, \([x_r \ x_r]^T\), and they can be considered as feedback forces that influence the response of the blade. This feedback effect along with the contact kinematics is shown in Figure 4.5. From the nonlinear feedback loop shown in Figure 4.5, it is evident that in the calculation of nonlinear forced response of a shrouded blade, all the linear degrees of freedom can be condensed to receptance and the modeling of friction contact can be separated from the complex structure model. This approach results in a set of nonlinear algebraic equations, which can be formulated as follows:
Figure 4.5 Nonlinear feedback loop of a shrouded blade.
\[ u_r = T_{BS}^T T_1 \{ r_{ce} f_r - r_{cc} T_{BS}^H p_r(u_r) \} \]  \hspace{1cm} (4.19)

where \( r_{ce} \) is the receptance at the shroud contact points due to unit harmonic excitation force, and \( r_{cc} \) is the receptance at the shroud contact points due to unit constrained force.

For simplicity of demonstration, each shrouded blade is assumed to contain a pair of shroud contact points. By using the Fast Fourier Transform, the constrained force can be approximated by harmonic functions having the same fundamental frequency as the external periodic excitation, and its amplitude and phase are nonlinear functions of the relative motions of the pair of contact points. By using the Multi-Harmonic Balance Method, the nonlinear algebraic equations become

\[ u_{r,k} = \tau r_{ce,k} f_{r,k} - \tau r_{cc,k} p_{r,k}(u_r), \quad k = 0,1,2,\ldots,m \]  \hspace{1cm} (4.20)

where

\[ \tau r_{ce,k} = T_{0,k}^T T_{1,k} r_{ce,k} \]  \hspace{1cm} (4.21)

\[ \tau r_{cc,k} = T_{0,k}^T T_{1,k} r_{cc,k} T_{1,k}^H T_{0,k} \]  \hspace{1cm} (4.22)

This set of nonlinear algebraic equations have the unknown \( \{ u_{r,k} \} \), and can be solved iteratively by using Newton-Raphson algorithm. With the solution \( \{ u_{r,k} \} \), the
constrained force can be obtained by using equation (4.16). The periodic response of the shrouded blade system can be calculated by using the resulting constrained force together with the receptance.

4.4 Comparison with Direct Time Integration Method

In order to verify the solution procedure presented in this work, the predicted results are compared to those of direct time integration method. Since it is very time consuming when using direct time integration method for a bladed system, a simplified system is considered. A blade with shroud-to-ground friction damper, as shown in Figure 4.6, is tested to validate the proposed solution procedure. In this test, the 3D shroud contact model is employed to predict the periodic response of a shrouded blade, which is constrained by the ground. In this study, the first five vibration modes of the shrouded blade are employed to calculate the receptances and three harmonic terms are included in the calculation of nonlinear forced response. Various levels of preload, ranging from fully separate to fully stuck case, are applied and the predicted results are shown as solid curves in Figure 4.7, along with discrete data points, which are the results of the direct time integration method. In the figure, the frequency is normalized with respect to the first mode natural frequency, and the amplitude is normalized with respect to the peak value of the fully separation case. It is seen that the Multi-Harmonic Balance Method can accurately predict the periodic response of the shrouded blade when the shroud contact is
either preloaded or having an initial gap. In the same figure, the predicted resonance response for the case $n_0 = -5000$ using single-term Harmonic Balance Method is also compared. It is apparent that in this case the nonlinear spring force of the shroud constraint exhibits a hardening effect that causes a jump phenomenon and the single-term Harmonic Balance Method can not accurately predict the resonance response.
Figure 4.6 A blade with shroud-to-ground friction damper.
Figure 4.7 Periodic response of a blade with shroud-to-ground friction damper.
4.5 Periodic Forced Response of A Shrouded Blade System

The proposed method is applied to predict the periodic response of a shrouded blade system. In this shrouded blade system, the first twenty vibration modes are employed in the analysis.

4.5.1 Shift of resonant frequency

It is known that when changing the preload there exist two limit cases, namely the fully separate case and fully stuck case [Yang and Menq, 1998a]. Since the nonlinear contact force does not appear in the analysis, both cases are linear problems. The fully separate case occurs when the interface has such a large initial gap that the vibrating neighboring shrouds can not make contact with each other. Since the contact force is not present, the resonant response corresponding to the natural frequencies of the system can be clearly seen. On the other hand, when the preload of the interface exceeds a level depending on the external excitation, the shroud contact interface remains fully stuck. In this case, the shroud contact does not dissipate energy. However, it provides additional stiffness, which arises from the shroud constraint, to the system to cause higher resonant frequencies.
The frequency shifts of the first three vibration modes of the shrouded blade system subjected to nodal force external excitation are shown in Figure 4.8. In the figure the frequency is normalized with respect to the first mode natural frequency, and the amplitude is normalized with respect to the peak of the first mode for the fully separation case. Since the resulting responses along the three axes are similar, only the normalized amplitude of the response along the axial direction is presented in the figure. In the first vibration mode, the resonant peak shifts from fully separation at normalized frequency 1.0 to fully stuck at normalized frequency 2.08, and the amplitude of the peak resonance is reduced by 54.5%. For the second vibration mode, the resonant peak shifts from fully separation at normalized frequency 2.14 to fully stuck at normalized frequency 4.79, and the amplitude of the peak resonance is reduced by 93.3%. For the third vibration mode, the resonant peak shifts from fully separation at normalized frequency 2.33 to fully stuck at normalized frequency 6.85, and the amplitude of the peak resonance is reduced by 89.1%.
Figure 4.8 Periodic forced response of a shrouded blade system: resonant shift of resonant frequency.
4.5.2 Reduction of resonant peak

In between the two linear cases, the constraint force consists of nonlinear friction force and the variable normal load. The significance of the variation of the contact normal load depends on the direction of the resulting relative motion at the shroud contact and the orientation of the contact plane. If the variation of the contact normal load is not significant, the effect of a shroud constraint is not very different from that of a platform damper. This is demonstrated by the attenuation effect of the induced friction on the resonant response of the first vibration mode. Figure 4.9 shows the tracking curves of the first vibration mode of the shrouded blade system when changing the contact preload. Two sets of curves are shown. The solid curves are the predicted results using the 3-terms Harmonic Balance Method and the dashed curves the single-term Harmonic Balance Method. It is seen that as the preload increases, the resonant peak decreases until the minimum response is reached at \( n_o = 2 \). Beyond this preload, the damping effect tends to reduce gradually towards the fully stuck case. The preload that gives the minimum response is known as the optimal preload.

Since the induced friction force has super-harmonic components, it is possible that the super-harmonics of the shrouded blade system can be excited and internal resonance can occur. As can be seen in Figure 4.9, when the preload is 0, the internal resonance can be observed at normalized frequencies 0.75 and 1.45 based on the results
using 3-terms Harmonic Balance Method. It appears that the single-term Harmonic Balance Method tends to over-estimate the resonant peak although for the first vibration mode the problem is not obvious.
Figure 4.9 Periodic forced response of a shrouded blade system: attenuation of resonant amplitude.
4.5.3 Jump phenomenon

In addition to its influence on the friction characteristic, the variable normal load can directly impose nonlinear stiffness on the system. This nonlinear stiffness arises from the intermittent separation of the contact surface during the course of vibration. It has been known that this nonlinearity can result in a multi-valued response that can lead to a jump phenomenon [Thomson, 1988]. A multi-valued response can be solved by using the standard continuation technique [Allgower and Georg, 1990]. In this study, a jump phenomenon can be observed at the third vibration mode, as shown in Figure 4.10, when a moderate preload ($n_0 = 2$) is applied. The increase in the resonant amplitude causes the preloaded interface to separate, and as a result, the interface can not provide stiffness to the system temporarily. The overall effect of the temporary separation is similar to the effect of a “softening spring” that gives rise to the response with a resonance peak bending towards lower frequencies. It should also be noted that one of the multiple solutions from the harmonic balance method shown as the dotted curve is unstable [Thomson, 1988]; while separated by the unstable response, the stable response consists of two curves, which are referred to as the upper and lower branches. In this figure, the resonant response predicted by the single-term Harmonic Balance Method is compared to that by the 3-terms Harmonic Balance Method. It is seen that the single-term Harmonic Balance Method well over-estimates the resonant response at the region
near jump. Furthermore, the internal resonance can be clearly observed from the predicted results using 3-terms Harmonic Balance Method.
Figure 4.10 Periodic forced response of a shrouded blade system: jump phenomenon at third vibration mode.
4.6 Conclusions

In this work, a 3D shroud contact model is employed to predict the periodic response of blades having 3D nonlinear shroud constraint. When subjected to periodic excitation, the resulting relative motion at the shroud contact is assumed to be periodic in three-dimensional space. Based on the 3D shroud contact model, analytical criteria are used to determine the transitions between stick, slip, and separation of the contact interface and are used to simulate hysteresis loops of the induced constrained force, when experiencing periodic relative motion. The constrained force can be considered as a feedback force that influences the response of the shrouded blade. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions so as to predict the periodic response of a shrouded blade. This approach results in a set of nonlinear algebraic equations, which can be solved iteratively to yield the periodic response of blades having 3D nonlinear shroud constraint.

The predicted nonlinear response shows three distinct features: (1) shifted resonant frequency due to the additional spring constant introduced by the shroud constraint, (2) damped resonant response due to the additional friction damping introduced by frictional slip, (3) multi-valued response leading to a jump phenomenon due to intermittent interface separation. The predictive ability of the proposed approach
has important implications to the design of the shroud contact. In the design of the
shroud contact, the preload is one of the important parameters to control the effectiveness
of the shroud contact. Since the attenuation effect of the shroud contact on resonant
vibration can be accurately predicted over a wide range of preload using the proposed
approach, the designer can achieve the optimal preload to maximize the performance of
the shroud contact in dissipating vibratory energy. Moreover, the proposed approach can
also facilitate the design of shroud angle, which is another important parameter to be
considered in the design of the shroud contact.
CHAPTER 5

PREDICTION OF PERIODIC FORCED RESPONSE OF BLADED DISK SYSTEMS WITH WEDGE DAMPERS

5.1 Introduction

For turbomachinery, one of the major failures is the high cycle fatigue of turbine blade, which is mostly caused by blade vibration. In turbine engine design, friction dampers are often employed in attenuating the blade vibration as well as increasing aeroelastic stability of the turbine. Examples of the application include platform dampers such as blade-to-ground and blade-to-blade damper, and shroud contacts [Griffin, 1990]. However, the platform dampers and shroud contacts are characterized by friction force models involving only one friction interface. In these friction contacts, the evaluation of friction force can be assumed as decoupled from the influence of other interfaces. On the other hand, a wedge damper is an example that involves two coupled interfaces, whose configuration is shown in Figure 5.1. The wedge damper is fitted in a V-shape slot between the inclined platforms of the adjacent blades. During engine operation, the rotation of the bladed disk introduces the centrifugal force that pressed the wedge damper
against the platforms. The vibration of the blades will introduce relative motions in those two coupled interfaces that result in very complicated contact kinematics including stick and slip. The resulting constraint force can add friction damping as well as nonlinear spring force to the bladed disk system [Cameron et al, 1990; Ferri, 1996; Griffin, 1980].

To analytically evaluate the performance of wedge damper and to help the design, it is necessary to consider the modeling of wedge damper so that the forced response of the bladed disk system can be accurately predicted. In previous studies of dry friction damper systems, according to Coulomb friction law, the friction coefficient at the contact interface is usually assumed to be constant, the relative motion across the friction contact point is often one-dimensional, and the system is subjected to constant normal load [Griffin, 1980]. It usually results in very simple contact kinematics and can be used to obtained analytical solutions for single-degree-of-freedom systems [Den Hartog, 1931; Wang, 1996], or to integrate with Multi-Harmonic Balance Method to yield approximate solutions for single-degree-of-freedom systems [Cameron and Griffin, 1989; Ren and Beards, 1994; Wang and Chen, 1993] and multiple-degree-of-freedom systems [Ferri and Dowell, 1988; Guillen and Pierre, 1996; Pierre et al, 1985].

In practice, the wedge damper contact interface can be curved shaped or flat. In a curved wedge damper, the contact kinematics allow rolling motion of the damper [Pfeiffer and Hajek, 1992; Sexto et al., 1997; Csaba and Andersson, 1997]. On the other
hand, when the wedge damper has flat contact surfaces, the friction contacts may stick and slip alternately [Yang and Menq, 1998b, 1998c].

In this work, a wedge damper contact model proposed by Yang and Menq [1998b, 1998c] is employed to obtain the constrained force at the wedge damper contact of a bladed disk system. The bladed disk system is assumed to be tuned and the assumed blade motion has two components, namely tangential and radial components. When subjected to periodic excitation, the resulting relative motion at the wedge damper as well as the constrained force are assumed to be periodic. The constrained force can be considered as a feedback force that influences the response of the blade. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions. In the calculation of the nonlinear forced response of a blade, all the linear degrees of freedom can be condensed to receptance and the modeling of wedge damper can be separated from the complex structure model. This approach results in a set of nonlinear algebraic equations which can be solved iteratively to yield the periodic forced response of blades having wedge damper constraint.
Figure 5.1 A bladed disk system with wedge dampers.
5.2 Modeling of Wedge Dampers

The detailed configuration of a wedge damper is as shown in Figure 5.2. The geometry of the wedge damper can be characterized by the damper angles $\alpha$ and $\beta$ between the inclined platforms and the wedge damper. The centrifugal force $N$ will press the wedge damper against the inclined platforms of the adjacent blades. The following discussion is summarized from the work of Yang and Menq [1998b, 1998c].

The $xy$ coordinate system (called blade coordinate system) is defined in accordance with tangential ($x$) and radial ($y$) directions. A $uv$ coordinate system (called wedge damper coordinate system) is defined along the inclined surfaces. In this study, the blade coordinate system is specified by two basis unit vectors $[\hat{x} \; \hat{y}]$, and the wedge damper coordinate system is defined as $[\hat{u} \; \hat{v}]$. These two coordinate systems can be related to each other as follows

$$
[\hat{u} \; \hat{v}] = [\hat{x} \; \hat{y}] T_0 \tag{5.1}
$$

where $T_0$ is the coordinate transformation matrix.

$$
T_0 = \frac{1}{\sin(\alpha + \beta)} \begin{bmatrix} -\cos\beta & -\cos\alpha \\ -\sin\beta & \sin\alpha \end{bmatrix} \tag{5.2}
$$
Figure 5.2 Configuration of a wedge damper.
5.2.1 Wedge damper contact kinematics

The motions of the blades, $d_1$ and $d_2$, can be expressed in the wedge damper coordinate system

\[ \mathbf{d}_1 = d_{1,w} \mathbf{\hat{u}} + d_{1,v} \mathbf{\hat{v}} \]  
\[ (5.3) \]

\[ \mathbf{d}_2 = d_{2,w} \mathbf{\hat{u}} + d_{2,v} \mathbf{\hat{v}} \]  
\[ (5.4) \]

Consequently, the damper motion can be characterized by the motions of the blades as follows

\[ \mathbf{d}_{\text{damper}} = d_{2,w} \mathbf{\hat{u}} + d_{1,v} \mathbf{\hat{v}} \]  
\[ (5.5) \]

Thus, the relative motions at both contacts can be expressed as

\[ d_r = d_{1,w} - d_{2,w} \]  
\[ (5.6) \]

\[ d_v = d_{2,v} - d_{1,v} \]  
\[ (5.7) \]
5.2.2 Wedge damper contact model

Figure 5.3 shows a model of a wedge damper [Yang and Menq, 1998b, 1998c], in which two inclined friction interfaces are on both sides of a wedge object that is assumed to be rigid and subjected to a centrifugal force $N$. The inclined angles, $\alpha$ and $\beta$, are the angles from the direction of the centrifugal force, and the angle $\alpha + \beta$ is the damper angle of the wedge damper. In this model, each interface consists of one flexible element of stiffness $k_1 \ (k_2)$ and one friction contact point of friction coefficient $\mu_1 \ (\mu_2)$.

The relative motions of the interfaces, $d_u$ and $d_v$, can be calculated from the blade motions using equations (5.6) and (5.7). They are termed the input motions in the following discussion. The slip motions of the contact points, $w_1$ and $w_2$, characterize the stick/slip condition of the interfaces. When the contact point reverses the direction ($\dot{w}_1 = 0 \ \text{or} \ \dot{w}_2 = 0$), the slip-to-stick transition takes place. However, $\dot{w}_1$ and $\dot{w}_2$ are not explicitly known, and how to relate these two slip motions to the input motions becomes important when analyzing the stick-slip phenomenon. It should be pointed out that this complexity does not appear when the single-interface model with constant normal load is considered. It can be shown that, in the case of the simple contact kinematics, the contact point reverses the direction when the input motion reverses the direction.
Figure 5.3 Wedge damper contact model.
To simplify the formulations, the following nondimensionalized variables are defined:

\[ \bar{f}_{N,1} = \frac{f_{N,1}}{N}, \quad \bar{N}_1 = \frac{N_1}{N}, \quad \bar{u} = \frac{k_1 d_x}{N}, \quad \bar{w}_1 = \frac{k_1 w_1}{N} \]

\[ \bar{f}_{N,2} = \frac{f_{N,2}}{N}, \quad \bar{N}_2 = \frac{N_2}{N}, \quad \bar{v} = \frac{k_2 d_x}{N}, \quad \bar{w}_2 = \frac{k_2 w_2}{N} \]

In this model, the friction forces are equal to the spring forces at any instant, and can be expressed as:

\[ \bar{f}_{N,1} = \bar{u} - \bar{w}_1 \] \hspace{1cm} (5.9)

\[ \bar{f}_{N,2} = \bar{v} - \bar{w}_2 \] \hspace{1cm} (5.10)

Furthermore, each friction contact obeys the Coulomb friction law:

(i) Stick condition (E):

\[-\bar{N}_1 \leq \bar{f}_{N,1} \leq \bar{N}_1 \quad \text{and} \quad \ddot{\bar{w}}_1 = 0 \quad \text{for interface 1} \] \hspace{1cm} (5.11)

\[-\bar{N}_2 \leq \bar{f}_{N,2} \leq \bar{N}_2 \quad \text{and} \quad \ddot{\bar{w}}_2 = 0 \quad \text{for interface 2} \] \hspace{1cm} (5.12)

(ii) Positive slip condition (P):

\[ \bar{f}_{N,1} = \bar{N}_1 \quad \text{and} \quad \dot{\bar{w}}_1 \geq 0 \quad \text{for interface 1} \] \hspace{1cm} (5.13)
(iii) Negative slip condition ($N$):

\[
\tilde{f}_{N,1} = -\tilde{N}_1 \quad \text{and} \quad \hat{w}_1 \leq 0 \quad \text{for interface 1} \tag{5.15}
\]
\[
\tilde{f}_{N,2} = -\tilde{N}_2 \quad \text{and} \quad \hat{w}_2 \leq 0 \quad \text{for interface 2} \tag{5.16}
\]

Under the influence of the normal load $\tilde{N}_1$, the input motion $\tilde{u}$ can induce the friction force $\tilde{f}_{N,1}$, which affects the normal load of $\tilde{N}_2$ through the force balance of the damper, and subsequently alters the friction force $\tilde{f}_{N,2}$. As a result, the friction force depends on not only the input motion of its associated interface but also the input motion of the other interface. The force balance of the damper, the center role of the coupling, can be formulated as follows by neglecting the inertia of the wedge damper:

\[
\tilde{N}_1 = -\frac{\mu_1 \cos(\alpha + \beta)}{\sin(\alpha + \beta)} \tilde{f}_{N,1} + \frac{\mu_1}{\sin(\alpha + \beta)} \tilde{f}_{N,2} + \frac{\mu_1 \cos \beta}{\sin(\alpha + \beta)} \tag{5.17}
\]
\[
\tilde{N}_2 = -\frac{\mu_2}{\sin(\alpha + \beta)} \tilde{f}_{N,1} + \frac{\mu_2 \cos(\alpha + \beta)}{\sin(\alpha + \beta)} \tilde{f}_{N,2} + \frac{\mu_2 \cos \alpha}{\sin(\alpha + \beta)} \tag{5.18}
\]

From the above equations, it is clear that the normal loads $\tilde{N}_1$ and $\tilde{N}_2$ are affected by both friction forces $\tilde{f}_{N,1}$ and $\tilde{f}_{N,2}$. 

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5.2.3 Stick-slip characteristics

According to the Coulomb friction law, each interface of the damper may stick, slip towards the positive direction, or slip towards the negative direction. Thus, the state of the wedge damper can be defined as the combination of the stick-slip motions of the two friction interfaces. The state of the damper is represented by a two-letter symbol, the first letter of which is referred to as the condition of interface 1 and the second letter as the condition of interface 2, in which the letter E designates the stick condition, the letter P the positive-slip condition, and the letter N the negative-slip condition. There are nine possible states, which can be classified into three types: one stick state, four single-slip states, and four double-slip states. The stick state denotes the condition of both interfaces sticking (EE). The single-slip state denotes the condition of one interface sticking and the other slipping (EP, EN, PE, NE). The double-slip state denotes the condition of both interfaces slipping (PP, PN, NP, NN).

By using the Coulomb friction law and the force balance of the wedge damper (Equations (5.11 - 5.18)), the stick-slip configuration of the wedge damper can be visualized by the diagram of Figure 5.4. In this configuration, the friction forces limited by the slip load lie within a range that is shown as the shaded area and consists of the nine states that are mentioned above. For example, considering interface 1, the two projective lines ($\vec{f}_{N,1} = \vec{N}_1$ and $\vec{f}_{N,1} = -\vec{N}_1$) represent the positive and negative slip conditions, and
the region between these two lines is the stick condition for interface 1. Interface 2 also has a similar situation of its stick-slip conditions being bounded within the two projective lines \( \tilde{f}_{N,2} = \tilde{N}_2 \) and \( \tilde{f}_{N,2} = -\tilde{N}_2 \). Considering the two interfaces simultaneously, the double-slip state of the wedge damper is shown as the intersection of the projective lines of different interfaces, the single-slip state is shown as the line segment between two double-slip states, and the stick state is the region bounded by the four line segments of the single-slip state.

Furthermore, the stick-slip characteristics can be derived by considering the force balance of the wedge damper (Equations (5.17, 5.18)) and the Coulomb friction law (Equations (5.9, 5.10)). The stick-slip characteristics are summarized in Table 5.1.
Figure 5.4 Stick-slip configuration of a wedge damper.
<table>
<thead>
<tr>
<th>State</th>
<th>$\tilde{t}_{N,1}$</th>
<th>$\tilde{w}_1$</th>
<th>$\tilde{t}_{N,2}$</th>
<th>$\tilde{w}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$EE$</td>
<td>$(\ddot{u} - \ddot{u}<em>0) + \tilde{F}</em>{N,1}^0$</td>
<td>0</td>
<td>$(\ddot{v} - \ddot{v}<em>0) + \tilde{F}</em>{N,2}^0$</td>
<td>0</td>
</tr>
<tr>
<td>$PE$</td>
<td>$N_1$</td>
<td>$\ddot{u} - \frac{\mu_1}{s + \mu_1 c} \ddot{v}$</td>
<td>$(\ddot{v} - \ddot{v}<em>0) + \tilde{F}</em>{N,2}^0$</td>
<td>0</td>
</tr>
<tr>
<td>$NE$</td>
<td>$-N_1$</td>
<td>$\ddot{u} + \frac{\mu_1}{s - \mu_1 c} \ddot{v}$</td>
<td>$(\ddot{v} - \ddot{v}<em>0) + \tilde{F}</em>{N,2}^0$</td>
<td>0</td>
</tr>
<tr>
<td>$EP$</td>
<td>$(\ddot{u} - \ddot{u}<em>0) + \tilde{F}</em>{N,1}^0$</td>
<td>0</td>
<td>$N_2$</td>
<td>$\frac{\mu_2}{s - \mu_2 c} \ddot{u} + \ddot{v}$</td>
</tr>
<tr>
<td>$EN$</td>
<td>$(\ddot{u} - \ddot{u}<em>0) + \tilde{F}</em>{N,1}^0$</td>
<td>0</td>
<td>$-N_2$</td>
<td>$\frac{-\mu_2}{s + \mu_2 c} \ddot{u} + \ddot{v}$</td>
</tr>
<tr>
<td>$PP$</td>
<td>$N_1$</td>
<td>$\ddot{u}$</td>
<td>$N_2$</td>
<td>$\ddot{v}$</td>
</tr>
<tr>
<td>$PN$</td>
<td>$N_1$</td>
<td>$\ddot{u}$</td>
<td>$-N_2$</td>
<td>$\ddot{v}$</td>
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<td>$\ddot{u}$</td>
<td>$N_2$</td>
<td>$\ddot{v}$</td>
</tr>
<tr>
<td>$NN$</td>
<td>$-N_1$</td>
<td>$\ddot{u}$</td>
<td>$-N_2$</td>
<td>$\ddot{v}$</td>
</tr>
</tbody>
</table>

$s = \sin(\alpha + \beta), \quad c = \cos(\alpha + \beta)$

Table 5.1 Stick-slip characteristics
5.2.4 Analytical transition criteria

Analytical transition criteria are developed to predict when the transitions take place between states, so as to evaluate the induced friction force. For the dual-interface friction force model, there are 28 possible transitions. This variety indeed reflects the complexity of this problem. Each criterion requires one equation to prescribe when the transition takes place and a number of constraints to guarantee the transition to be satisfied. In general, the criteria can be set up by considering when the friction force reaches the slip load for the stick-to-slip transition or when the contact point reverses its direction for the slip-to-stick transition, and if the stick-slip conditions after the transition are satisfied. Take the transition from state $EP$ to state $EE$ for example. Before the transition, interface 1 sticks and interface 2 slips towards the positive direction; after the transition, interface 1 remains stuck but interface 2 becomes stuck. Thus, the transition occurs when the contact point of interface 2 reverses the direction, i.e. $\dot{w}_2 = 0$ and $\ddot{w}_2 \leq 0$; the latter one is required to ensure the contact point of interface 2 has a tendency towards the negative direction when it reaches the extreme. Using the stick-slip characteristic of state $EP$, the transition criterion becomes:

\[
\begin{align*}
\left[\frac{\mu_2}{\sin(\alpha + \beta) - \mu_2 \cos(\alpha + \beta)}\right] \ddot{u} + \dot{v} &= 0 \\
\left[\frac{\mu_2}{\sin(\alpha + \beta) - \mu_2 \cos(\alpha + \beta)}\right] \dddot{u} + \ddot{v} &\leq 0
\end{align*}
\]

(5.19)

The analytical transition criteria are summarized in Table 5.2.
<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Transition criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>EE</em></td>
<td><em>EP</em></td>
<td>$\dot{f}_{N,2} - \dot{N}<em>2 = 0$ $\dot{f}</em>{N,2} - \dot{N}_2 \geq 0$</td>
</tr>
<tr>
<td><em>EN</em></td>
<td><em>EP</em></td>
<td>$\dot{f}_{N,2} + \dot{N}<em>2 = 0$ $\dot{f}</em>{N,2} + \dot{N}_2 \leq 0$</td>
</tr>
<tr>
<td><em>PE</em></td>
<td><em>EP</em></td>
<td>$\dot{f}_{N,1} - \dot{N}<em>1 = 0$ $\dot{f}</em>{N,1} - \dot{N}_1 \geq 0$</td>
</tr>
<tr>
<td><em>NE</em></td>
<td><em>EP</em></td>
<td>$\dot{f}_{N,1} + \dot{N}<em>1 = 0$ $\dot{f}</em>{N,1} + \dot{N}_1 \leq 0$</td>
</tr>
<tr>
<td><em>EE</em></td>
<td><em>NP</em></td>
<td>$\left[\mu_2/(s - \mu_2 c)\right] \ddot{u} + \ddot{v} = 0$ $\left[\mu_2/(s - \mu_2 c)\right] \ddot{u} + \ddot{v} \leq 0$</td>
</tr>
<tr>
<td><em>EN</em></td>
<td><em>NP</em></td>
<td>$\ddot{u} \leq 0$ $\ddot{v} \geq 0$</td>
</tr>
<tr>
<td><em>PE</em></td>
<td><em>NP</em></td>
<td>$\dot{f}_{N,1} - \dot{N}_1 = 0$ $\ddot{u} \geq 0$ $\ddot{v} \leq 0$</td>
</tr>
<tr>
<td><em>PP</em></td>
<td><em>NP</em></td>
<td>$\dot{f}_{N,1} - \dot{N}_1 = 0$ $\ddot{u} \geq 0$ $\ddot{v} \leq 0$</td>
</tr>
<tr>
<td><em>EE</em></td>
<td><em>EN</em></td>
<td>$\ddot{u} - \left[\mu_1/(s + \mu_1 c)\right] \ddot{v} = 0$ $\ddot{u} - \left[\mu_1/(s + \mu_1 c)\right] \ddot{v} \leq 0$</td>
</tr>
<tr>
<td><em>EN</em></td>
<td><em>EN</em></td>
<td>$\ddot{u} \leq 0$ $\ddot{v} \leq 0$ $\ddot{u} - \left[\mu_1/(s + \mu_1 c)\right] \ddot{v} \geq 0$</td>
</tr>
<tr>
<td><em>PN</em></td>
<td><em>EN</em></td>
<td>$\dot{f}_{N,1} - \dot{N}_1 = 0$ $\ddot{u} \geq 0$ $\ddot{v} \leq 0$</td>
</tr>
<tr>
<td><em>PP</em></td>
<td><em>EN</em></td>
<td>$\dot{f}_{N,1} - \dot{N}_1 = 0$ $\ddot{u} \geq 0$ $\ddot{v} \leq 0$</td>
</tr>
<tr>
<td><em>EE</em></td>
<td><em>NE</em></td>
<td>$\ddot{u} + \left[\mu_1/(s + \mu_1 c)\right] \ddot{v} = 0$ $\ddot{u} + \left[\mu_1/(s + \mu_1 c)\right] \ddot{v} \geq 0$</td>
</tr>
<tr>
<td><em>NP</em></td>
<td><em>NE</em></td>
<td>$\dot{f}_{N,2} + \dot{N}_2 = 0$ $\ddot{u} \leq 0$ $\ddot{v} \geq 0$</td>
</tr>
<tr>
<td><em>EN</em></td>
<td><em>NE</em></td>
<td>$\ddot{u} \geq 0$ $\ddot{v} \leq 0$</td>
</tr>
<tr>
<td><em>NN</em></td>
<td><em>NE</em></td>
<td>$\dot{f}_{N,2} + \dot{N}_2 = 0$ $\ddot{u} \leq 0$ $\ddot{v} \leq 0$</td>
</tr>
<tr>
<td><em>PP</em></td>
<td><em>EP</em></td>
<td>$\ddot{u} = 0$ $\ddot{v} \geq 0$</td>
</tr>
<tr>
<td><em>PE</em></td>
<td><em>EP</em></td>
<td>$\ddot{v} = 0$ $\ddot{u} \geq 0$ $\ddot{v} \leq 0$</td>
</tr>
<tr>
<td><em>EN</em></td>
<td><em>EP</em></td>
<td>$\ddot{u} = 0$ $\ddot{v} \leq 0$</td>
</tr>
<tr>
<td><em>PE</em></td>
<td><em>EN</em></td>
<td>$\ddot{v} = 0$ $\ddot{u} \geq 0$ $\ddot{v} \geq 0$</td>
</tr>
<tr>
<td><em>EN</em></td>
<td><em>NN</em></td>
<td>$\ddot{u} = 0$ $\ddot{v} \geq 0$</td>
</tr>
<tr>
<td><em>NE</em></td>
<td><em>NN</em></td>
<td>$\ddot{v} = 0$ $\ddot{u} \leq 0$ $\ddot{v} \geq 0$</td>
</tr>
<tr>
<td><em>EP</em></td>
<td><em>NN</em></td>
<td>$\ddot{v} = 0$ $\ddot{u} \leq 0$ $\ddot{v} \geq 0$</td>
</tr>
<tr>
<td><em>NP</em></td>
<td><em>NE</em></td>
<td>$\ddot{v} = 0$ $\ddot{u} \leq 0$ $\ddot{v} \leq 0$</td>
</tr>
</tbody>
</table>

$s = \sin(\alpha + \beta), \quad c = \cos(\alpha + \beta)$

Table 5.2 Analytical transition criteria

179
5.2.5 Estimation of Fourier coefficients

The analytical transition criteria are employed to simulate hysteresis loops of the friction contacts, when experiencing periodic relative motion. With these hysteresis loops, the resulting constrained force can be characterized by the relative motion between two neighboring blades and wedge damper. By using Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions and employed in Multi-Harmonic Balance Method to solve for the periodic response.

In this wedge damper friction force model, the coupling through the force balance of the damper leads to a complex stick-slip mechanism in which the stick-slip motions of the interfaces interact with each other. The estimation of the Fourier coefficients is as shown in Figure 5.5. For periodic input motions, the stick-slip transition conditions are first evaluated so as to predict the induced periodic friction forces. Then the Fourier coefficients can be obtained by using Fast Fourier Transform. To be specific for the case of the wedge damper model, when the periodic input motions, say

\[ \bar{u} = \sum_{k=1}^{m} A_k \sin(k\omega t + \phi_k) \]  
\[ \bar{v} = \sum_{k=1}^{m} B_k \cos(k\omega t + \phi_k) \]

are considered, the friction forces are approximated by the first \( m \) harmonic components of their Fourier series as
\[ u = \sum_{k=1}^{m} A_k \sin(\omega t + \phi_k) \]

\[ \bar{u} = \bar{u} \]

\[ v = \sum_{k=1}^{m} B_k \sin(\omega t + \phi_k) \]

\[ \bar{v} = \bar{v} \]

Force balance of the damper

\[ f_{N1} \approx \sum_{k=1}^{m} f_{k1} \sin(\omega t + \phi_k) + f_{k2} \cos(\omega t + \phi_k) \]

\[ f_{N2} \approx \sum_{k=1}^{m} f_{k3} \sin(\omega t + \phi_k) + f_{k4} \cos(\omega t + \phi_k) \]

Figure 5.5 Estimation of Fourier coefficients.
\[ \tilde{f}_{N1} \approx \sum_{k=1}^{m} \tilde{f}_{s,k} \sin(k\omega t + \phi_k) + \tilde{f}_{c,k} \cos(k\omega t + \phi_k) \]  
\[ \tilde{f}_{N2} \approx \sum_{k=1}^{m} \tilde{f}_{s,k}^2 \sin(k\omega t + \varphi_k) + \tilde{f}_{c,k}^2 \cos(k\omega t + \varphi_k) \]

where \( \tilde{f}_{s,k} \) and \( \tilde{f}_{c,k} \) are the Fourier coefficients of the \( k \)th harmonic component.

One example of the effects of the super-harmonic components is as shown in Figure 5.6. In this plot, the fundamental harmonic components, \( A_t \) and \( B_t \), of the input motions, \( \tilde{u} \) and \( \tilde{v} \), vary from 0 to 5 respectively and the resulting friction forces are simulated. The Fourier coefficients of the first three harmonic components are compared here. It is clear that the super-harmonic components are significant and can not be neglected.
Figure 5.6 Fourier coefficients of a friction force trajectory: (a) first harmonic; (b) second harmonic; (c) third harmonic.
5.3 Prediction of Periodic Response of Bladed Disk System with Wedge Dampers

In the analysis of a bladed disk system, a great simplification can be obtained by assuming that the bladed disk system is tuned, namely each blade of the system has exactly the same dynamic characteristics. In addition, the excitation of interest is that induced by the blades rotating through circumferential variations in the flow field. It can be shown that in effect each blade is exposed to a periodic excitation having the same amplitude but differing in phase by an amount which is proportional to the blade's angular location on the disk. It is assumed that the forced response of the bladed disk system is also periodic and has the same fundamental period as the excitation. Thus the external excitation and the motions of the blades as well as the nonlinear constrained forces can be represented by infinite Fourier series. By truncating these series after the $m^{th}$ terms, an approximate solution assuming that the forced response is periodic can be derived. In this approach, each blade vibrates in the same manner but with a proportional interblade phase difference ($k\varphi$) for $k^{th}$ harmonic component from its adjacent blades. The interblade phase angle is defined as follows:

$$\varphi = \frac{2\pi E}{N}$$  \hspace{1cm} (5.22)

---

1 Sub-harmonic components can also be included in the approach presented in this paper. However, they are ignored in the current manuscript.
where \( N \) is the total number of the blades in the system and \( E \) is the engine order of the excitation on the system.

A tuned bladed disk system with wedge dampers is shown in Figure 5.7. The equation of motion of a blade constrained by wedge dampers and subjected to a periodic excitation can be expressed as follows:

\[
mx + cx + kx = f_e - f_N
\]  

(5.23)

where \( x \) is the nodal displacement vector, \( m \) is the mass matrix, \( c \) is the damping matrix, \( k \) is the stiffness matrix, \( f_e \) is the external periodic excitation, and \( f_N \) is the nonlinear constrained force which is a nonlinear function of the relative motion at the wedge damper. The finite element model is two-dimensional and if the model contains \( n \) nodes, all the matrices will be \( 2n \times 2n \) matrices, and all the vectors will be \( 2n \)-element vectors.

The external periodic excitation can be expressed as follows.

\[
f_e(t) = \sum_{k=0}^{m} f_{e,k} e^{jk\omega t}
\]  

(5.24)
Figure 5.7 A tuned bladed disk system with wedge dampers.
where $\omega$ is the fundamental excitation frequency and $\mathbf{f}_{r,k}$ is a complex vector representing the amplitude and phase for the $k^{th}$ harmonic component of the excitation. Here the external periodic excitation is assumed to include up to $m^{th}$ harmonic component. It should be pointed out that except for the elements associated with those friction contact points the other elements of $\mathbf{f}_N$ are zeros. It is clear that the nonlinear aspect of the dynamic problem is embedded in the nonlinear friction force $\mathbf{f}_N$. By using the Modal Analysis Method, the mode shape matrix can be obtained and is denoted as $\Phi$. Using the mode shape matrix and the associated modal information, the receptance of the blade can be derived as:

$$r = \left[r_{pt,k}\right] = \sum_{i=1}^{\Delta} \left(\Phi_i \Lambda_{i,k} \Phi_i^T\right)$$  \hspace{1cm} (5.25)

and

$$\Lambda_{i,k} = \left[\left(k_i - k^2 \omega^2 m_i\right) + j(k\omega c_i)\right]^{-1}$$ \hspace{1cm} (5.26)

where $r_{pt,k}$ is defined as the steady state response of the $p^{th}$ node due to unit $k^{th}$ harmonic excitation force at the $i^{th}$ node, $\Phi_i$ is the $i^{th}$ mode shape, $m_i$ is the $i^{th}$ modal mass, $k_i$ is the $i^{th}$ modal stiffness, and $c_i$ is the $i^{th}$ modal damping.
In this work, the periodic two-dimensional motion in the blade coordinate system can be represented as follows:

\[
\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \approx \left[ \sum_{k=1}^{m} A_{x,k} e^{jk\omega t} \right] \left[ \sum_{k=1}^{m} A_{y,k} e^{jk\omega t} \right]
\]

(5.27)

where \( \omega \) is the fundamental oscillating frequency, and \( A_{i,k}, i = x, y, \) are the complex vectors representing the amplitude and phase angle of the \( k \)th harmonic component along tangential and radial axes. For a blade constrained by wedge dampers, several pairs of friction contact points can be defined. For each pair of friction contact points, one is on the right and the other left and their motions are denoted as \( \begin{bmatrix} x_p & x_q \end{bmatrix}^T \). For convenience, this vector can be arranged as:

\[
\begin{bmatrix} x_p & x_q \end{bmatrix}^T = \begin{bmatrix} x_{p,1} & x_{q,1} & x_{p,2} & x_{q,2} & \cdots & x_{p,m} & x_{q,m} \end{bmatrix}^T
\]

(5.28)

5.3.1 Relative motion at wedge damper contact points

Since the bladed disk system is assumed to be tuned, the condition of cyclic symmetry can be applied when deriving the relative motion of a wedge damper. Take the
relative motion between the point $p$ of the right platform and the point $q$ of the left platform in Figure 5.7 as an example. First, the motions of the two contact points of the blade are defined and they are $[x_p \ x_q]^T$. Therefore, the motion of point $p$ is now $x_p$ and the motion of point $q$ differs from $x_q$ with the proportional interblade phase angle $(k\phi)$ for the $k$th harmonic component. As a result, the relative motion of the wedge damper, $w_d$, can be derived as

$$w_d = T_1\begin{bmatrix} x_p \\ x_q \end{bmatrix}^T, \quad k = 1,2,\cdots,m$$  \hspace{1cm} (5.29)

where $T_1$ is the relative displacement transformation matrix.

$$T_1 = \begin{bmatrix} \text{diag}(T_{1,1},T_{1,2},\cdots,T_{1,m}) \end{bmatrix}$$  \hspace{1cm} (5.30)

$$T_{1,k} = \begin{bmatrix} I_{2\times2} \\ -e^{-jk\phi}I_{2\times2} \end{bmatrix}, \quad k = 1,2,\cdots,m$$  \hspace{1cm} (5.31)

It is worthy noting that only the relative motion at the right wedge damper is derived. Since the bladed disk system is assumed to be tuned, the relative motion and the resulting constrained force of the left wedge damper can be related to those of the right wedge damper by using the condition of cyclic symmetry. Since both $x_p$ and $x_q$ are periodic motions, the resulting relative motion $w_d$ also has periodic trajectory in the 2D space.
Since this relative motion $w_d$ is in the blade coordinate system, it can be transformed to the wedge damper coordinate system by using the transformation defined in equation (5.2):

$$u_d = T_{BW}^T w_d \quad (5.32)$$

$$T_{BW} = \left[ \text{diag} \left( T_{0,1}, T_{0,2}, \ldots, T_{0,m} \right) \right], T_{0,k} = T_0, \quad k = 1, 2, \ldots, m \quad (5.33)$$

where $T_{BW}$ is the coordinate transformation matrix for 2D periodic relative motion. The 2D relative motion in the wedge damper coordinate system can be expressed as follows:

$$u_d \approx \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{m} A_{u,k} e^{jk\omega t} \\ \sum_{k=1}^{m} A_{v,k} e^{jk\omega t} \end{bmatrix} \quad (5.34)$$

where $A_{i,k}, i = u, v$, are the complex vectors representing the amplitude and phase angle of the $k^{th}$ harmonic component along $u$ and $v$ axes.
5.3.2 Constrained force at wedge damper contact points

The $u$ and $v$ components of the relative motion follow a periodic trajectory on the contact planes and can induce the stick-slip friction. It is apparent that the $u$ and $v$ motions are coupled together when inducing stick-slip friction. Using the wedge damper friction contact model proposed by Yang and Menq [1998b, 1998c], the constrained force at the right wedge damper can be determined. By using Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions and can be expressed as follows:

\[
P_d = \begin{bmatrix} p_{du}(u, v) \\ p_{dv}(u, v) \end{bmatrix} \approx \sum_{k=1}^{m} \frac{m}{\sum_{k=1}^{m}} p_{du,k} e^{jk\alpha_f} \]  

(5.35)

where $p_{di,k}$, $i = u, v$, are the complex Fourier coefficients of the $k^{th}$ harmonic component along $u$ and $v$ axes. Then, the constrained force can be transformed back to the blade coordinate system:

\[
f_d = T_{BW} p_d \]  

(5.36)
Furthermore, the constrained forces at a pair of wedge damper contact points can be related to the force at the right wedge damper using the interblade relative displacement transformation matrix.

\[
\begin{bmatrix}
  f_{np} \\
  f_{nq}
\end{bmatrix} = T_i^H f_d
\]  

(5.37)

where $T_i^H$ is the complex conjugate transpose of $T_i$.

5.3.3 Nonlinear algebraic equations

When the blade is constrained by the wedge dampers, the resulting constrained forces are characterized by the displacements of a pair of contact points, $[x_p \quad x_q]^T$, and they can be considered as feedback forces that influence the response of the blade. This feedback effect along with the contact kinematics is shown in Figure 5.8. From the nonlinear feedback loop shown in Figure 5.8, it is evident that in the calculation of nonlinear forced response of a bladed disk system, all the linear degrees of freedom can be condensed to receptance and the modeling of friction contact can be separated from the complex structure model. This approach results in a set of nonlinear algebraic equations, which can be formulated as follows:
Figure 5.8 Nonlinear feedback loop.
\[
\mathbf{u}_d = \mathbf{T}_{BI}^T \mathbf{T}_1 \left\{ r_{ce} \mathbf{f}_e - r_{cc} \mathbf{T}_1^H \mathbf{T}_{BI} \mathbf{p}_d (\mathbf{u}_d) \right\}
\] (5.38)

where \( r_{ce} \) is the receptance at the wedge damper contact points due to unit harmonic excitation force, and \( r_{cc} \) is the receptance at the wedge damper contact points due to unit constrained force. For simplicity of demonstration, each blade is assumed to contain a pair of wedge damper contact points. By using the Fast Fourier Transform, the constrained force can be approximated by harmonic functions having the same fundamental frequency as the external periodic excitation, and its amplitude and phase are nonlinear functions of the relative motions of the pair of contact points. By using the Multi-Harmonic Balance Method, the nonlinear algebraic equations become

\[
\mathbf{u}_{d,k} = \mathbf{T}_{ce,k}^T \mathbf{f}_{e,k} - \mathbf{T}_{cc,k}^T \mathbf{p}_{d,k} (\mathbf{u}_d), \quad k = 1, 2, \ldots, m
\] (5.39)

where

\[
\mathbf{T}_{ce,k} = \mathbf{T}_{0,k}^T \mathbf{T}_{1,k} \mathbf{r}_{ce,k}
\] (5.40)

\[
\mathbf{T}_{cc,k} = \mathbf{T}_{0,k}^T \mathbf{T}_{1,k} \mathbf{r}_{cc,k} \mathbf{T}_{1,k}^H \mathbf{T}_{0,k}
\] (5.41)

This set of nonlinear algebraic equations have the unknown \( \{ \mathbf{u}_{d,k} \} \), and can be solved iteratively by using Newton-Raphson algorithm. With the solution \( \{ \mathbf{u}_{d,k} \} \), the constrained force can be obtained by using equation (5.35). The periodic response of the
tuned bladed disk system with wedge dampers can be calculated by using the resulting constrained force together with the receptance.

5.4 Periodic Forced Response of Bladed Disk System with Wedge Dampers

The proposed method is applied to predict the periodic response of a tuned bladed disk system with wedge dampers. In this bladed disk system, the first four vibration modes are employed in the analysis.

5.4.1 Shift of resonant frequency

It is known that when changing the damper load there exist two limit cases, namely the fully slip case and fully stuck case [Yang and Menq, 1997]. Since the nonlinear contact force does not appear in the analysis, both cases are linear problems. The fully slip case occurs when the damper load is zero and the wedge damper can not provide any friction at all. Since the contact force is not present, the resonant response corresponding to the natural frequencies of the system can be clearly seen. On the other hand, when the damper load of the interface exceeds a level depending on the external excitation, the friction interfaces remain fully stuck. In this case, the friction contact does not dissipate energy. However, it provides additional stiffness, which arises from the wedge damper contact, to the system to cause higher resonant frequencies. The
frequency shift of the first vibration mode of the tuned bladed disk system subjected to external excitation are shown in Figure 5.9. Since the resulting responses along the two axes are similar, only the amplitude of the response along the tangential direction is presented in the figure. In the first vibration mode, the resonant peak shifts from fully slip at frequency 269.8 Hz to fully stuck at frequency 294.4 Hz, and the amplitude of the peak resonance is reduced by 62.6%.
Figure 5.9 Periodic forced response of a bladed disk system with wedge dampers: resonant shift of resonant frequency.
5.4.2 Reduction of resonant response

In between the two linear cases, the constraint force consists of nonlinear friction force. The significance of the variation of the damper load is demonstrated by the attenuation effect of the induced friction on the forced response of the first vibration mode. Figure 5.9 shows the tracking curves of the first vibration mode of the tuned bladed disk system when changing the damper load. Two sets of curves are shown. The solid curves are the predicted results using the 3-terms Harmonic Balance Method and the dashed curves the single-term Harmonic Balance Method. It is seen that as the damper load increases, the resonant peak decreases until the minimum response is reached at $N = 1$. Beyond this damper load, the damping effect tends to reduce gradually towards the fully stuck case. The damper load that gives the minimum response is known as the optimal damper load. It should be pointed out that the single-term Harmonic Balance Method under-estimates the resonant response when the damper load is less than the optimal damper load, and over-estimates the resonant response when the damper load is greater than the optimal damper load.
5.4.3 Comparison with time integration method

The time integration method is employed to validate the proposed approach. However, since it is time-consuming and ineffective to obtain the forced response of a tuned bladed disk by using the time integration method, a test beam is employed to validate the proposed solution procedure. The comparison of the predicted results using single-term Harmonic Balance Method with those of the time integration method is shown in Figures 5.10, in which the discrete data points denote the time integration solutions. The results have been reported by Yang and Menq [1998c]. All the comparisons are made in the frequency near resonance. In this figure, the prediction of the resonant response shows discrepancies in the cases that the damper load is either less or greater than the optimal damper load. By comparing with Figure 5.9, it is found that Multi-Harmonic Balance Method can provide accurate prediction of the periodic response by including the super-harmonic components in the analysis.
1.0 Fully Stuck

Fully Slip

N=35 N'

Discrete data: time integration method

^ 0.6 N=0.15 N'

N=0.25 N'

N=N'

0.0 1.00 1.15 1.05 1.10

Normalized frequency

Figure 5.10 Periodic forced response of a bladed disk system with wedge dampers: comparison of single-term Harmonic Balance Method and time integration method [courtesy Yang and Menq (1998c)].
5.4.4 Periodic response

The periodic response of the tuned bladed disk system with wedge damper can be clearly observed at the second vibration mode as shown in Figure 5.11. Here the periodic response predicted by the single-term Harmonic Balance Method is compared to that by the 3-terms Harmonic Balance Method. The first three harmonic components of the periodic response predicted by the 3-terms Harmonic Balance Method are also compared in this figure. It is found that the second harmonic component is as much as 16.5% of the first harmonic component. Furthermore, a peak resonance at 1428 Hz can be clearly observed, where the second mode natural frequency is excited. In this case, the single-term Harmonic Balance Method fails to predict the peak.

It should be pointed out that the super-harmonic components not only affects the resonant amplitude but also the resonant frequency shift. This effect is shown in Figure 5.12 where the tuned bladed disk is the same as in Section 5.4.1 except that the wedge damper stiffness is intentionally raised by an order. In this plot, the predicted forced response at the optimal damper load using 3-terms Harmonic Balance Method is compared with that of single-term Harmonic Balance Method. In this case, the resonant frequency shifts from 270 Hz to 293 Hz when super-harmonic components are included in analysis, while the resonant frequency shifts from 270 Hz to 300 Hz when only the
fundamental harmonic component is analyzed. This suggests that the super-harmonic components shall be considered in the design of the optimal damper load.
Figure 5.11 Periodic forced response of a bladed disk system with wedge dampers: super-harmonic components.
Figure 5.12  Periodic forced response of a bladed disk system with wedge dampers: optimal damper load.
5.5 Conclusions

In this chapter, a wedge damper contact model is employed to predict the periodic response of blades having wedge damper constraint. When subjected to periodic excitation, the resulting relative motion at the wedge damper is assumed to be periodic. Based on the wedge damper contact model, analytical criteria are used to determine the transitions between stick and slip of the contact interface and are used to simulate hysteresis loops of the induced constrained force, when experiencing periodic relative motion. The constrained force can be considered as a feedback force that influences the response of the blade. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions so as to predict the periodic response of the blade. This approach results in a set of nonlinear algebraic equations, which can be solved iteratively to yield the periodic response of a tuned bladed disk system with wedge dampers.

In this work, it is shown that the resonant frequency will shift to higher frequency due to the additional spring constant introduced by the wedge damper. It is also observed that the forced response is damped due to the additional friction damping introduced by frictional slip. In addition, it is found that the super-harmonic components can affect the prediction of the forced response, especially when the damper load is near the optimal damper load. It is shown that the super-harmonic components not only affects the
resonant amplitude but also the resonant frequency shift. The predictive ability of the proposed approach has important implications to the design of the wedge damper. In the design of the wedge damper, the damper load is one of the important parameters to control the effectiveness of the wedge damper. Since the attenuation effect of the wedge damper on resonant vibration can be accurately predicted over a wide range of damper load using the proposed approach, the designer can achieve the optimal damper load to maximize the performance of the wedge damper in dissipating vibratory energy. Moreover, the proposed approach can also facilitate the design of damper angle, which is another important parameter to be considered in the design of the wedge damper.
6.1 Conclusions

In turbine jet engine, one of the major failures is the high cycle fatigue of turbine blade, due to blade’s vibration resonance within the engine operating range. One way to reduce the vibration resonance is to provide friction damper. Friction damper not only introduces damping effect that attenuates resonant vibration, but also provides stiffness that increases aeroelastic stability of the turbine blade. With appropriate friction contact model, one can evaluate the induced friction force and incorporate the Harmonic Balance Method along with receptance formulation to predict the forced response of a bladed disk system. This approach can separate all the linear degrees of freedom of the system from the nonlinear part so that the nonlinear problem can be solved by considering the frictionally constrained part of the system. In the literature, the receptance is usually based on the finite element model that excludes the frictionally constraints. When the friction contact is near fully stuck, this approach requires more vibration mode information than a finite element model that includes the modeling of friction contact.
Moreover, when applying the Harmonic Balance Method, only the fundamental harmonic component is taken into account. Since the rotating turbine blades are subjected to periodic excitation in the air flow field, the forced response is also periodic. Consequently, the dynamic response of the frictionally constrained systems may behave differently when the super-harmonic components are included in the analysis.

In this study, the concept of constrained mode shape is used to predict the resonant response of a frictionally constrained blade. For a tuned blade system, the constrained mode shapes can be calculated using a finite element model of a single blade along with the cyclic symmetry constraint that simulates a fully stuck friction constraint. The resulting constrained mode shapes are often complex and can be used to obtain the constrained receptance of the blade. Using the constrained receptance and the available friction contact model, a nonlinear feedback loop can be developed to signify the feedback effect of the constrained force on the blade. From this nonlinear feedback loop, it is shown that under a harmonic excitation the equation of motion based on a finite element model of the blade can be condensed to a set of nonlinear algebraic equations. It is shown that by examining each mode’s contribution to the receptance at the friction contact point, the importance of each individual mode to the prediction of the resonant response of a frictionally constrained blade can be determined. Furthermore, by comparing the receptances calculated from free mode shapes and those from constrained mode shapes, it is found that in the neighborhood of the fully slipping region, the
prediction of resonant response requires fewer number of modes when using free mode shapes compared to using constrained mode shapes. On the other hand, in the neighborhood of the fully stuck region, it requires fewer number of modes if constrained mode shapes are used.

In this work, a shrouded blade system is employed to demonstrate the effectiveness of the developed approach. It is found that in the neighborhood of the fully stuck region, it requires fewer number of modes for accurate prediction of the resonant response of a shrouded blade system if constrained mode shapes are used. In shroud design, high preload is often desirable. Therefore, using the constrained mode shapes for the prediction of resonant response is preferred.

Moreover, the concept of hybrid receptance is introduced so as to yield very accurate prediction of the resonant response based on only very few vibration modes. It is shown that the hybrid receptances can be employed to correctly predict the resonant response over the range from fully slip to fully stuck.

In this work, a 3D friction contact model is employed to predict the periodic response of structures having 3D frictional constraints. When subjected to periodic excitation, the resulting relative motion at the friction contact interface is assumed to be periodic in the three-dimensional space. Based on the 3D friction contact model,
analytical criteria are used to determine the transitions among sticks, slips, and separations of the contact interface and are used to simulate hysteresis loops of the induced constrained force, when experiencing periodic relative motion. The constrained force can be considered as a feedback force that influences the response of the frictionally constrained structure. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions so as to predict the periodic response of frictionally constrained structures. Due to the periodically changing normal load, both even and odd harmonic components need be included in the analysis.

The developed method is used to predict the periodic response of a frictionally constrained 3-DOF oscillator. The predicted nonlinear response shows three distinct features: (1) shifted resonant frequency due to the additional spring constant introduced by the friction constraint, (2) damped resonant response due to the additional friction damping introduced by frictional slip, (3) multi-valued response leading to a jump phenomenon due to intermittent interface separation. The predicted results are also compared with those of the direct time integration method so as to validate the proposed method. In addition, the effect of super-harmonic components on the resonant response and jump phenomenon is examined. It was found that single-term Harmonic Balance Method often over-estimates the resonant response of a frictionally constrained structure and can not predict the internal resonance in the forced response. It is also found that
small super-harmonic components can induce significant changes on stick-slip transition and lead to large discrepancies in the prediction of the constrained forces.

Furthermore, the 3D shroud contact model is employed to predict the periodic response of blades having 3D nonlinear shroud constraint. When subjected to periodic excitation, the resulting relative motion at the shroud contact is assumed to be periodic in three-dimensional space. Based on the 3D shroud contact model, analytical criteria are used to determine the transitions between stick, slip, and separation of the contact interface and are used to simulate hysteresis loops of the induced constrained force, when experiencing periodic relative motion. The constrained force can be considered as a feedback force that influences the response of the shrouded blade. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions so as to predict the periodic response of a shrouded blade. This approach results in a set of nonlinear algebraic equations, which can be solved iteratively to yield the periodic response of blades having 3D nonlinear shroud constraint.

The predicted nonlinear response shows three distinct features: (1) shifted resonant frequency due to the additional spring constant introduced by the shroud constraint, (2) damped resonant response due to the additional friction damping introduced by frictional slip, (3) multi-valued response leading to a jump phenomenon.
due to intermittent interface separation. The predictive ability of the proposed approach has important implications to the design of the shroud contact. In the design of the shroud contact, the preload is one of the important parameters to control the effectiveness of the shroud contact. Since the attenuation effect of the shroud contact on resonant vibration can be accurately predicted over a wide range of preload using the proposed approach, the designer can achieve the optimal preload to maximize the performance of the shroud contact in dissipating vibratory energy. Moreover, the proposed approach can also facilitate the design of shroud angle, which is another important parameter to be considered in the design of the shroud contact.

In this work, a wedge damper contact model is employed to predict the periodic response of blades having wedge damper constraints. Unlike the shrouded blade systems, where shroud contacts are characterized by friction force models involving only one friction interface, a wedge damper involves two coupled interfaces that result in very complicated contact kinematics. The constrained force can be simulated by using the wedge damper contact model, when experiencing periodic relative motion, and it can be considered as a feedback force that influences the response of the blade. By using the Multi-Harmonic Balance Method along with Fast Fourier Transform, the constrained force can be approximated by a series of harmonic functions so as to predict the periodic response of the blade. This approach results in a set of nonlinear algebraic equations,
which can be solved iteratively to yield the periodic response of a tuned bladed disk system with wedge dampers.

In this research, it is shown that the resonant frequency will shift to higher frequency due to the additional spring constant introduced by the wedge damper. It is also observed that the forced response is damped due to the additional friction damping introduced by frictional slip. In addition, it is found that the super-harmonic components can affect the prediction of the forced response, especially when the damper load is near the optimal damper load. It is shown that the super-harmonic components not only affects the resonant amplitude but also the resonant frequency shift. The predictive ability of the proposed approach has important implications to the design of the wedge damper. In the design of the wedge damper, the damper load is one of the important parameters to control the effectiveness of the wedge damper. Since the attenuation effect of the wedge damper on resonant vibration can be accurately predicted over a wide range of damper load using the proposed approach, the designer can achieve the optimal damper load to maximize the performance of the wedge damper in dissipating vibratory energy. Furthermore, the proposed approach can also contribute to the design of damper angle, which is another important parameter to be considered in the design of the wedge damper.
The proposed solution approach in the prediction of periodic forced response is shown to be more efficient when compared to the Time Integration Method, which needs to get through transient response before reaching steady state solutions. However, it is also shown that the Multi-Harmonic Balance Method will consume more time when more super-harmonic components are included in the solutions.

The major contributions of this research are summarized below:

(1) Constrained mode shapes are incorporated in the formulation of the receptance. By using the receptance based on the constrained mode shapes, the Harmonic Balance Method can accurately predict dynamics behaviors of frictionally constrained blades near the fully stuck region with just a few modes of constrained mode shapes. Furthermore, the hybrid receptance, in which free and constrained mode shapes are combined together, is shown to yield very accurate prediction of the resonant response over the range from fully slip to fully stuck with only very few vibration modes.

(2) The Multi-Harmonic Balance Method and Fast Fourier Transform are employed in the prediction of periodic forced response of structures having three-dimensional friction constraints. The proposed approach is applied to not only a three degrees-of-freedom oscillator, but also a tuned bladed disk system with three
dimensional nonlinear shroud constraints. The predictive ability of the proposed approach can provide guidelines for the design of damping devices in mechanical systems having three-dimensional friction constraints.

(3) Multi-valued periodic response leading to a jump phenomenon due to intermittent interface separation at the friction contact are studied. It is shown that the nonlinear spring force, due to the friction contact, can cause a jump phenomenon. Moreover, the characteristics of the jump phenomenon differ significantly when the super-harmonic components are included in analysis.

(4) A wedge damper contact model is employed to predict the periodic response of blades having wedge damper constraints, in which a wedge damper involves two coupled interfaces that result in very complicated contact kinematics. The proposed approach can provide guidelines for the design of the wedge damper, such as the damper weight and wedge damper angle.

(5) The proposed solution approach in the prediction of periodic forced response is shown to be efficient and effective when compared with the Time Integration Method. This will provide design engineers an effective approach in the design of friction dampers in turbine blades.
6.2 Future Work

For the improvement and extension of this research work, the recommended future work is identified as follows.

(1) The free and constrained mode shapes are employed respectively in the representation of receptance in this study. Moreover, the free and constrained mode shapes can be combined together to result in hybrid mode shapes. It is applied to a bladed disk system with one-dimensional blade-to-ground friction dampers, and yields very accurate prediction of the resonant response based on only very few vibration modes. However, when extending to two-dimensional case, the interpolation of the receptance between free and constrained modes is still not clear. Further investigation of physical insight for the interpolation of the receptance in two-dimensional and three-dimensional friction dampers are recommended.

(2) The super-harmonic components are considered in the prediction of the periodic forced response of frictional constrained turbine blade systems in this research. However, the sub-harmonic components may come to affect the periodic forced response significantly. The inclusion of sub-harmonic components in the prediction of the periodic forced response is recommended.
(3) A two-dimensional wedge damper model is employed to predict the periodic forced response of a tuned blade system with wedge dampers. In this model, the wedge damper is assumed as rigid body, and the constraining forces are assumed along the tangential and radial axes. During engine operation, the blades may bend and twist to experience friction constraint, and the constraining forces are often three-dimensional. In this case, in addition to the force balance along the tangential, radial, and axial axes, the balance of the moment along the three axes shall be taken into account. The development of six degrees-of-freedom wedge damper model is recommended.

(4) The Time Integration Method is employed to validate the effectiveness of the proposed solution method for the periodic forced response in this research. However, for a tuned bladed disk system with frictional constraints such as shroud contacts or wedge dampers, it is very difficult to simulate all the friction contacts by using the Time Integration Method. To verify the prediction results of the periodic forced response, it is recommended that experimental turbine blades with frictional constraints shall be tested in the future works.
LIST OF REFERENCES


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