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TOPICS IN MULTIFRACTAL FORMALISM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

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The local dimension of measure $\mu$ at a given point $x$ is defined by the limit of the ratio $\frac{\log \mu(B_r(x))}{\log r}$ as $r \to 0$, where $B_r(x)$ is the ball of radius $r$ centered at $x$. It is the basis of the multifractal formalism which connects local and global scaling behaviors of a multifractal measure. However, in the general case the classical multifractal formalism does not always work. In fact, the limit in the definition of local dimension may not exist. In the first part of this dissertation we show that in fact a typical probability measure on a separable metric space does not have a local dimension at a typical point. More precisely, we show that the lower local dimension of a typical measure is 0 almost everywhere and on a residual set, and the upper local dimension is infinite on a residual set and positive (in particular, equal to $n$ in case of $\mathbb{R}^n$) almost everywhere. Similar discrepancy can be observed in other characteristics of a measure, such as various dimensions of a measure. We also discuss multifractal spectra for a typical probability measure.

In Chapter 4, we introduce the more general multifractal formalism, changing the ratio in the definition of local dimension to $\frac{h(\mu(B_r(x)))}{g(r)}$ where $h, g$ are functions satisfying some monotonicity and scaling conditions. We apply a particular case - the "second-order" formalism - to the description of the hyperspace of compact subsets of a self-similar fractal set.
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## TABLE OF CONTENTS

Abstract ................................................................. ii
Acknowledgments .................................................. iii
Vita ................................................................. iv

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1 Introduction</strong></td>
<td>1</td>
</tr>
<tr>
<td><strong>2 Preliminaries</strong></td>
<td>4</td>
</tr>
<tr>
<td>2.1 Hausdorff and Packing Measure and Dimension</td>
<td>4</td>
</tr>
<tr>
<td>2.2 Multifractal Formalism</td>
<td>8</td>
</tr>
<tr>
<td>2.3 Self-similar Sets</td>
<td>13</td>
</tr>
<tr>
<td><strong>3 Typical Measures</strong></td>
<td>16</td>
</tr>
<tr>
<td>3.1 Introduction and Preliminaries</td>
<td>17</td>
</tr>
<tr>
<td>3.2 Main Theorems</td>
<td>22</td>
</tr>
<tr>
<td>3.3 Example</td>
<td>32</td>
</tr>
<tr>
<td>3.4 Multifractal Spectra of a Typical measure</td>
<td>34</td>
</tr>
<tr>
<td><strong>4 Generalized Multifractal Formalism</strong></td>
<td>41</td>
</tr>
<tr>
<td>4.1 Definitions and Theorems</td>
<td>41</td>
</tr>
<tr>
<td>4.2 Generalized Multifractal Description of a Hyperspace of Compact Subsets</td>
<td>52</td>
</tr>
</tbody>
</table>

Bibliography .................................................... 62

v
CHAPTER 1
INTRODUCTION

The term "fractal" was coined by Mandelbrot in [Man75] and refers to sets that have irregular details on arbitrarily small scales. The best known fractals, such as a middle-third Cantor set, have strictly self-similar structure, but fractals often exhibit some sort of approximate self-similarity. Fractals arise naturally in many areas of mathematics, for example dynamical systems. They can also be used to model a wide range of natural phenomena. The so-called "fractal dimensions" provide the method of description of the global geometric structure of fractal sets. The Hausdorff dimension is the oldest and most important of them. It is defined using Hausdorff measure $\mathcal{H}^d$, based on a measure theoretic construction by Caratheodory. Some classical papers on this topic can be found in [Edg93]. Packing and box-counting dimensions are also widely used. Usually the fractal dimension of such a set is strictly greater than its topological dimension.

The multifractal theory was introduced much later in order to provide a description of singularity structure of measures. It also allows us to investigate the fine local structure of fractal sets. Multifractal analysis can be used to model phenomena in which scaling occurs with a range of different power laws. In fact, it appeared first in works of physicists dealing with turbulence. The set (possibly fractal) on
which the given measure is supported can be decomposed into sets determined by particular values of local dimension. The local dimension of measure $\mu$ at a given point $x$ is defined by the limit of the ratio $\frac{\log \mu(B_r(x))}{\log r}$ as $r \to 0$, where $B_r(x)$ is the ball of radius $r$ centered at $x$. In [HJK+86] the function $f(\alpha)$ was introduced, which describes the dimension of a set on which the given measure $\mu$ has local dimension $\alpha$. Another approach [HP83] used generalized Rényi dimensions usually denoted $D_q$ or $\tau(q)$ to consider the global irregularities of $\mu(B_r(x))$ for fixed $r$ and then take the limit as $r \to 0$. Mostly heuristic arguments by various physicists connected these two descriptions via Legendre transform.

The multifractal spectrum was computed exactly for such important constructions as graph-directed fractals [EM92] and cookie-cutters [Ran89], and it was shown that in these cases $f(\alpha)$ is equal to the Legendre transform of $\tau(q)$. However, this is not always true in more general cases. Moreover, the limit in the definition of local dimension does not always exist. In the first part of this dissertation we show that in fact a typical probability measure on a separable metric space does not have a local dimension at a typical point. Thus the use of multifractal approach where it is not theoretically justified may potentially lead to an incorrect or incomplete analysis of some physical problems.

The second part of this dissertation is concerned with a generalization of multifractal formalism which may work in some cases where the classical theory fails to produce meaningful results. It uses the more general approach to fractal dimensions which associates the dimension of a set with some Hausdorff function $\phi(t)$. In particular, it is possible to use functions that approach zero faster than any power of $t$.
as $t \to 0$. This helps to describe the geometry of infinite-dimensional sets such as the hyperspace of compact subsets of some finite-dimensional set. The dimensions of such hyperspaces were investigated by McClure [McC97], [McC96]. However, all existing multifractal theories are based on functions $\phi(t) = t^s$. In the last part of this dissertation we attempt to develop a more general multifractal formalism appropriate in particular for infinite-dimensional sets.

The organization is as follows. In Chapter 2, we state definitions and some properties of several fractal dimensions, describe Olsen's multifractal formalism and the notion of self-similar sets.

In Chapter 3, we discuss the local dimension, various dimensions of a measure and multifractal spectra for a typical probability measure. In particular, we show that the lower local dimension of a typical measure is 0 almost everywhere and on a residual set, and the upper local dimension is infinite on a residual set and positive (in particular, equal to $n$ in case of $\mathbb{R}^n$) almost everywhere. Similar discrepancy can be observed in other characteristics of a measure.

In Chapter 4, we introduce the general multifractal formalism, changing the ratio in the definition of local dimension to $\frac{h(\mu(B_r(x)))}{g(r)}$ where $h, g$ are functions satisfying some monotonicity and scaling conditions. We apply a particular case - the "second-order" formalism - to the description of the hyperspace of compact subsets of a self-similar fractal set.
CHAPTER 2
PRELIMINARIES

In this chapter we will list necessary definitions and theorems concerning Hausdorff and other measures and dimensions (Section 2.1), Olsen's multifractal formalism (Section 2.2) and self-similar fractal sets (Section 2.3). Throughout this chapter $X$ will denote an arbitrary metric space with metric $d$, $\mathcal{P}(X)$ will denote the set of all Borel probability measures on $X$. Throughout the dissertation we will also denote an open ball with center $x$ and radius $r$ by $B_r(x)$ and a corresponding closed ball by $\overline{B}_r(x)$.

2.1 Hausdorff and Packing Measure and Dimension

Let $X$ be an arbitrary metric space with metric $d$. Let $\Phi$ denote the set of all Hausdorff functions, that is, nondecreasing right continuous functions $\phi : [0, a) \to [0, \infty)$ such that $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$. A $\delta$-cover of a set $E \subseteq X$ is a finite or countable collection of sets $E_i \in X$ such that $E \subseteq \cup_i E_i$ and $\text{diam}(E_i) \leq \delta$ for all $i$. For $\phi \in \Phi$, define a Hausdorff measure on $X$ corresponding to $\phi$ as follows:

$$\mathcal{H}_\delta^\phi(E) = \inf \left\{ \sum_i \phi(\text{diam}(E_i)) \mid \{E_i\}_i \text{ is a } \delta\text{-cover of } E \right\},$$

$$\mathcal{H}_0^\phi(E) = \lim_{\delta \to 0} \mathcal{H}_\delta^\phi(E).$$
It is shown in [Rog70], p.50 that $\mathcal{H}^\phi$ is a metric outer measure on $X$, so that the restriction of $\mathcal{H}^\phi$ on measurable subsets of $X$ is a Borel measure.

The partial ordering of the set $\Phi$ is introduced by writing

$$\phi \prec \psi \quad \text{if} \quad \lim_{t \to 0} \frac{\psi(t)}{\phi(t)} = 0$$

Here $\psi$ approaches zero more rapidly than $\phi$, but corresponds to a "larger dimension".

The following proposition is proved in [Rog70], p.79.

**Proposition 2.1.1.** Let $f, g, h$ be functions in $\Phi$ with $f < g < h$. If a set $E$ has $\sigma$-finite positive $\mathcal{H}^g$-measure, then $E$ has zero $\mathcal{H}^h$-measure and non-$\sigma$-finite $\mathcal{H}^f$-measure.

Thus the partial ordering of $\Phi$ induces the partial ordering of sets $E \subseteq X$ which is reflected in the notion of dimension. Namely, we write $\dim E \prec \phi$ if $\mathcal{H}^\phi(E) = 0$, $\dim E \succ \phi$ if $\mathcal{H}^\phi(E)$ is non-$\sigma$-finite, $\dim E \asymp \phi$ if $\mathcal{H}^\phi(E)$ is positive and $\sigma$-finite.

In many instances it is useful to define a totally ordered family of functions $\{\phi_s\}_{s>0} \subset \Phi$ such that $\phi_s \prec \phi_t$ if $s < t$. Then the proposition above shows that for any set $E$ there is a critical value

$$s_0 = \sup \{s \mid \mathcal{H}^{\phi_s}(E) = \infty\} = \inf \{s \mid \mathcal{H}^{\phi_s}(E) = 0\}$$

We call this critical value a *generalized Hausdorff dimension* in Chapter 3. For the family of functions $\phi_s(t) = t^s$ this is the commonly used *Hausdorff dimension*, denoted by $\dim E$. Other important families are $\phi_s(t) = 2^{-1/t^s}$ and $\phi_s(t) = 2^{-(\log 1/t)^s}$. They appear when we study certain sets of Hölder and analytic functions correspondingly (see [KT61]).
Packing measure, introduced by Taylor and Tricot [TT85], is defined as follows. A \( \delta \)-packing of a set \( E \subseteq X \) is a finite or countable collection of disjoint closed balls \( \{ B_r(x_i) \} \), with centers in \( E \) and such that \( 2r_i \leq \delta \) for every \( i \). For \( \phi \in \Phi \), let

\[
\bar{P}_\delta^\phi(E) = \sup \{ \sum_i \phi(2r_i) \mid \{ B_r(x_i) \} \text{ is a } \delta \text{-packing of } E \},
\]

\[
\bar{P}_\delta^\phi(E) = \lim_{\delta \to 0} \bar{P}_\delta^\phi(E).
\]

\( \bar{P}^\phi \) is not in general countably subadditive, so there is one more step necessary to obtain an outer measure, that is, defining

\[
\mathcal{P}^\phi(E) = \inf \{ \sum_i \mathcal{P}^\phi(E_i) \mid E \subseteq \cup_i E_i \}.
\]

Then usual and generalized packing dimensions are defined using \( \mathcal{P}^\phi \), exactly as in the case of Hausdorff dimensions. The usual packing dimension is denoted by \( \text{Dim } E \).

**Remark.** Originally, Taylor and Tricot defined packing measures using diameters of balls rather than their radii. This leads to the same result in \( \mathbb{R}^n \) but not in general metric spaces. Cutler [Cut95] has shown that the radius definition is more convenient in the sense that the regularity properties of packing measure are preserved in general metric space, which is not always the case with the diameter definition. This is also true about the important relationship

\[
\mathcal{P}^\phi(E) \geq \mathcal{H}^\phi(E) \text{ for any } E \subseteq X.
\]

Saint Raymond and Tricot [SRT88] introduced also a centered Hausdorff measure.
as a notion precisely dual to the packing measure. It is defined using centered \( \delta \)-covers of \( E \subseteq X \), that is, covers by balls centered in \( E \) with radii at most \( \delta \):

\[
\overline{C}_\delta^\phi(E) = \inf \{ \sum_i \phi(2r_i) \mid \{ B_{r_i}(x_i) \}_i \text{ is a centered } \delta \text{-cover of } E \}. 
\]

\[
\overline{C}_\delta^\phi(E) = \lim_{\delta \to 0} \overline{C}_\delta^\phi(E). 
\]

The set function \( \overline{C}_\delta^\phi \) is not necessarily monotone, so we need one more step to define a corresponding metric outer measure:

\[
C^\phi(E) = \sup_{F \subseteq E} \overline{C}_\delta^\phi(F). 
\]

Finally, we will use also a box-counting dimension which is easier to compute but does not possess many important properties of Hausdorff and packing dimensions. Let \( E \subseteq X \) and let \( N_\delta(E) \) be the minimum number of closed balls of diameter \( \delta \) required to cover \( E \). Then the upper box dimension is defined as

\[
\overline{\dim}_B(E) = \lim_{\delta \to 0} \frac{\log N_\delta(E)}{\log 1/\delta},
\]

and the lower box dimension as

\[
\underline{\dim}_B(E) = \lim_{\delta \to 0} \frac{\log N_\delta(E)}{\log 1/\delta}.
\]

Alternatively, instead of \( N_\delta(E) \) we can use the maximum number of disjoint closed balls of radius \( \delta \) which can be centered at points of \( E \). This will give the same values for the box dimensions (for the proof see e.g. [Fal90], p.39).
2.2 Multifractal Formalism

Olsen has defined multifractal generalizations of the centered Hausdorff measure and the packing measure. Most of the definitions and theorems in this section are taken from [Ols95]. The material in this section is used in Section 3.4 and Chapter 4.

Let $\mu$ be a Borel probability measure on $X$. Let $E \subseteq X$, $E \neq \emptyset$, $q, t \in \mathbb{R}$. Define

$$\overline{H}_{\mu, \delta}^{q, t}(E) = \inf \left\{ \sum_i (\mu(B_{r_i}(x_i)))^{q}(2r_i)^t \mid \{B_{r_i}(x_i)\}_i \text{ is a centered } \delta\text{-cover of } E \right\},$$

$$\overline{H}_{\mu, \delta}^{q, t}(\emptyset) = 0$$

$$\overline{H}_{\mu}^{q, t}(E) = \sup_{\delta > 0} \overline{H}_{\mu, \delta}^{q, t}(E)$$

$$\mathcal{H}_{\mu}^{q, t}(E) = \sup_{F \subseteq E} \overline{H}_{\mu}^{q, t}(F)$$

and similarly for the packing measure:

$$\overline{P}_{\mu, \delta}^{q, t}(E) = \sup \left\{ \sum_i (\mu(B_{r_i}(x_i)))^{q}(2r_i)^t \mid \{B_{r_i}(x_i)\}_i \text{ is a centered } \delta\text{-packing of } E \right\},$$

$$\overline{P}_{\mu, \delta}^{q, t}(\emptyset) = 0$$

$$\overline{P}_{\mu}^{q, t}(E) = \inf_{\delta > 0} \overline{P}_{\mu, \delta}^{q, t}(E)$$

$$\mathcal{P}_{\mu}^{q, t}(E) = \inf_{E \subseteq U, \varepsilon_i} \sum_i \overline{P}_{\mu}^{q, t}(E_i)$$
Proposition 2.2.1. $\mathcal{H}^q_t, \mathcal{P}^q_t$ are metric outer measures, hence measures on Borel subsets of $X$.

Proof. See [Ols95], Proposition 2.2 and 2.3.

It is easy to see from the definitions that there are critical numbers:

$$\dim^q_t(E) \text{ such that } \mathcal{H}^{q,t}_\mu(E) = \begin{cases} \infty & \text{for } t < \dim^q_\mu(E) \\ 0 & \text{for } t > \dim^q_\mu(E) \end{cases}$$

$$\text{Dim}^q_t(E) \text{ such that } \mathcal{P}^{q,t}_\mu(E) = \begin{cases} \infty & \text{for } t < \text{Dim}^q_\mu(E) \\ 0 & \text{for } t > \text{Dim}^q_\mu(E) \end{cases}$$

They are called, correspondingly, multifractal Hausdorff and multifractal packing dimensions of $E$.

For studying measures it is natural to consider multifractal dimensions of $E = \text{supp } \mu$. Denote $b_\mu(q) = \dim^q_\mu(\text{supp } \mu), B_\mu(q) = \text{Dim}^q_\mu(\text{supp } \mu)$. Clearly $b_\mu(q)$ and $B_\mu(q)$ are nonincreasing and also we have

$$b_\mu(1) = 0, \quad b_\mu(0) = \dim_\mu(\text{supp } \mu)$$

$$B_\mu(1) = 0, \quad B_\mu(0) = \text{Dim}_\mu(\text{supp } \mu)$$

$$0 \leq b_\mu(q) \leq B_\mu(q) \text{ for } q < 1,$$

$$b_\mu(q) \leq B_\mu(q) \leq 0 \text{ for } q > 1.$$
We will drop the subscript \( \mu \) when it is clear from the context.

Some properties of the multifractal measures and dimensions are true only if some additional requirements on measure \( \mu \) are satisfied. To define them, denote

\[
T_a(E) = \lim_{r \to 0} \sup_{x \in E} \frac{\mu(B_r(x))}{\mu(B_r(x))}
\]

Now let

\[
P_0(X) = \{ \mu \in \mathcal{P}(X) \mid \exists a > 1 : \forall x \in \text{supp} \mu : T_a(\{x\}) < \infty \}
\]

\[
P_1(X) = \{ \mu \in \mathcal{P}(X) \mid \exists a > 1 : T_a(\text{supp} \mu) < \infty \}
\]

The following proposition generalizes the relationship between Hausdorff and packing measure.

**Proposition 2.2.2.** Let \( \mu \in \mathcal{P}(\mathbb{R}^d) \) and \( q, t \in \mathbb{R} \). Then

(i) \( \mathcal{H}^q, (\mu, t) \leq \mathcal{P}^q, (\mu, t) \) for \( q \leq 0 \);

(ii) If \( \mu \in \mathcal{P}_0(\mathbb{R}^d) \), then \( \mathcal{H}^q, (\mu, t) \leq \mathcal{P}^q, (\mu, t) \) for \( q > 0 \);

(iii) There is an integer \( \zeta > 0 \) such that \( \mathcal{H}^q, (\mu, t) \leq \zeta \mathcal{P}^q, (\mu, t) \). In particular, \( \dim^q \mu \leq \dim^q \mu \).

**Proof.** See [Ols95], Proposition 2.4. \( \square \)

Another approach to the multifractal description of a measure involves a decomposition of its support into subsets of the same logarithmic measure density. The precise formulation of this is as follows. Let \( \mu \in \mathcal{P}(X) \) and \( x \in \text{supp} \mu \). Denote

\[
\alpha_\mu(x) = \lim_{\delta \to 0} \frac{\log \mu(B_\delta(x))}{\log \delta}, \quad \overline{\alpha}_\mu(x) = \lim_{\delta \to 0} \frac{\log \mu(B_\delta(x))}{\log \delta}
\] (2.1)
These limits are often called *upper and lower local dimensions* of measure $\mu$ at point $x$. If the upper and lower limits coincide, the common value is denoted $\alpha_\mu(x)$ and called the *local dimension*.

Now for any $\alpha > 0$ denote

$$X_\alpha = \{x \in \text{supp} \mu \mid \alpha_\mu(x) \geq \alpha\},$$

$$X^\alpha = \{x \in \text{supp} \mu \mid \alpha_\mu(x) \leq \alpha\},$$

$$\bar{X}_\alpha = \{x \in \text{supp} \mu \mid \bar{\alpha}_\mu(x) \geq \alpha\},$$

$$\bar{X}^\alpha = \{x \in \text{supp} \mu \mid \bar{\alpha}_\mu(x) \leq \alpha\}.$$

Finally let

$$f_\mu(\alpha) = \dim(X_\alpha \cap \bar{X}^\alpha), \quad F_\mu(\alpha) = \text{Dim}(X_\alpha \cap \bar{X}^\alpha).$$

The function $f_\mu(\alpha)$ is usually called "the multifractal spectrum" or "the singularity spectrum" of $\mu$. In all this notation we will drop the subscript $\mu$ whenever it is clear from context.

If $f : \mathbb{R} \to \mathbb{R}$ is any function, its *Legendre transform* is defined by $f^*(x) = \inf_y(xy + f(y))$. Notice that the result is a concave function. In some well-studied cases the multifractal spectra $f(\alpha)$ and $F(\alpha)$ are equal to Legendre transforms of $b(q)$ and $B(q)$ correspondingly, see, for example, [EM92],[Ran89],[Ols95]. In general, only the upper bounds for $f(\alpha)$ and $F(\alpha)$ can be obtained in such way. In particular, $f$ and $F$ need not always be concave.
Define

\[ a = \sup_{q>0} -\frac{b(q)}{q}, \quad \overline{a} = \inf_{q<0} -\frac{b(q)}{q} \]

\[ A = \sup_{q>0} -\frac{B(q)}{q}, \quad \overline{A} = \inf_{q<0} -\frac{B(q)}{q} \]

**Proposition 2.2.3.** For any \( \mu \in \mathcal{P}(X) \),

(i) \( a \leq \inf \overline{\alpha}_\mu(x) \leq \sup \overline{\alpha}_\mu(x) \leq \overline{A}, \quad A \leq \inf \underline{\alpha}_\mu(x) \leq \sup \underline{\alpha}_\mu(x) \leq \overline{a} \).

(ii) \( f_\mu(\alpha) \leq b^*(\alpha) \) if \( \alpha \in (a, \overline{a}) \), \( f_\mu(\alpha) = 0 \) otherwise.

(iii) \( F_\mu(\alpha) \leq B^*(\alpha) \) if \( \alpha \in (a, \overline{a}) \), \( F_\mu(\alpha) = 0 \) otherwise.

**Proof.** See [Ols95], Theorem 2.17. \( \square \)

Finally, we will use the multifractal analogs to the box dimensions which are defined as follows. For any \( E \subseteq X, q \in \mathbb{R} \) and \( \delta > 0 \) write

\[ T^q_{\mu,\delta}(E) = \inf \{ \sum_i (\mu(B_\delta(x_i)))^q \mid (B_\delta(x_i))_i \text{ is a centered covering of } E \}, \]

\[ \overline{T}^q_{\mu}(E) = \lim_{\delta \to 0} \frac{\log T^q_{\mu,\delta}(E)}{-\log \delta} \]

\[ L^q_{\mu}(E) = \lim_{\delta \to 0} \frac{\log T^q_{\mu,\delta}(E)}{-\log \delta} \]

There is also another way to define the multifractal box dimensions. Let

\[ S^q_{\mu,\delta}(E) = \sup \{ \sum_i (\mu(B_\delta(x_i)))^q \mid (B_\delta(x_i))_i \text{ is a centered packing of } E \}, \]
Note that these definitions do not always give the same dimension values as it happens in the non-multifractal case.

Note: If $X$ is compact, we may consider only finite covers by balls. Moreover, this box dimension doesn’t change if we consider open balls instead of closed. In Section 3.4 we will use open balls for the sake of convenience.

2.3 Self-similar Sets

Self-similar sets are examples of fractal sets with easily calculated dimensions. Let $f_1, \ldots, f_n, n \geq 2$ be similarities on a metric space $(X, d)$, that is,

$$d(f_i(x), f_i(y)) = r_i d(x, y)$$

for any $i = 1, \ldots, n, x, y \in X$

where $r_i$ are positive numbers called similarity ratios. Such a collection is called an iterated function system (IFS).

A set $F \subseteq X$ is an invariant set for this iterated function system if

$$F = \bigcup_{i=1}^{n} f_i(F)$$

Such a set is called self-similar. Hutchinson [Hut81] has shown that if $X$ is a complete metric space and all $f_i$’s are contractions, that is, $r_i < 1$ for every $i$, then there is
a unique nonempty compact invariant set for the given contracting iterated function system.

The iterated function system satisfies the open set condition if there exists a nonempty open set $V$ such that $\bigcup_{i=1}^{n} f_i(V) \subseteq V$ and $f_i(V) \cap f_j(V) = \emptyset$ for $i \neq j$. The similarity dimension of the invariant set of an IFS is a number $s$ such that $\sum_{i=1}^{n} r_i^s = 1$. If the open set condition is satisfied, the similarity dimension coincides with the Hausdorff dimension of the invariant set.

Now let us consider "string models" of such sets. Suppose we have $n$ contracting similarities $f_1, \ldots, f_n$ with ratios $r_1, \ldots, r_n$. Let $E = \{1, 2, \ldots, n\}$. We will consider the space of infinite strings on an alphabet $E$, denoted by $E^\omega$. It is also called a coding space for the given iterated function system. Denote also by $E^{(k)}$ the set of all $k$-letter strings from this alphabet, and let $E^{(*)} = \cup_{k=0}^{\infty} E^{(k)}$ be the set of all finite strings.

The metric on the space $E^{(*)}$ is defined in the following way. For any finite string $\alpha = (i_1 i_2 \ldots i_k) \in E^{(k)}$ let $d(\alpha) = r_{i_1} r_{i_2} \ldots r_{i_k}$ (and $d(\alpha) = 1$ if $\alpha$ is an empty string). Suppose two strings $\sigma, \tau \in E^{(*)}$ have longest common prefix $\alpha$. Then the distance between them is defined by $\rho(\sigma, \tau) = d(\alpha)$. Thus every finite string $\alpha$ defines a cylinder $[\alpha] = \{ \sigma \in E^{(*)} | \sigma = \alpha \tau \}$ with diameter $d(\alpha)$.

This space becomes a model for a self-similar set $F$ when we introduce a coding map $h : E^{(*)} \to X$. For $\alpha = (i_1 \ldots i_k) \in E^{(k)}$ let $F_\alpha = f_{i_1} \circ \cdots \circ f_{i_k}(F)$. Then we define

$$ h(\sigma) = \bigcap_{k=1}^{\infty} F_{\sigma|k} \quad \text{for } \sigma \in E^{(*)}, $$
where $\sigma|k$ denotes the finite string consisting of first $k$ letters of $\sigma$. The map $h$ is one-to-one if and only if $f_1(F), \ldots, f_n(F)$ are disjoint.
CHAPTER 3
TYPICAL MEASURES

Throughout this chapter $X$ will denote a complete separable metric space. Section 3.1 lists preliminary definitions and results, including various definitions of the dimension of a measure. In Section 3.2 we will show (Theorem 3.2.1) that for a typical probability measure $\alpha_\mu(x) = 0$ and $\overline{\alpha}_\mu(x) = \infty$ for all $x$ except a set of first category. Also $\alpha_\mu(x) = 0$ almost everywhere and with some additional conditions on $X$ there is a corresponding result for upper local dimension: in particular, we show that a typical measure $[0,1]^d$ has $\overline{\alpha}_\mu(x) = d$ almost everywhere (Theorem 3.2.4).

There are similar results concerning “global” dimensions of probability measures. Theorems 3.2.2 and 3.2.3 show in particular that the Hausdorff dimension of a typical measure on any compact separable space equals 0 and the packing dimension of a typical measure on $[0,1]^d$ equals $d$. Section 3.3 gives an example of a measure for which upper and lower local dimensions are different almost everywhere. Results similar to Theorem 3.2.1 are obtained also for invariant subspaces of $\mathcal{P}(X)$, and in section 3.4 there are some theorems concerning the multifractal spectra of a typical measure.
3.1 Introduction and Preliminaries

Upper and lower local dimensions $\alpha^\mu(x), \alpha^\mu(x)$ of a measure $\mu \in \mathcal{P}(X)$: have been extensively studied (see e.g. [Cut92], [Ols95], [Tam95], [You82] and many others). Note that the limits in their definitions (2.1) are the same for closed and open balls. $\alpha^\mu(x)$ and $\alpha^\mu(x)$ have been shown to coincide almost everywhere in some important particular cases such as the ergodic invariant measures of smooth diffeomorphisms with nonzero Lyapunov exponents [You82]. If this happens, a measure is called regular. The question arises whether $\alpha^\mu(x)$ and $f(\alpha)$ can be used to describe more general situations. We will show that this is in fact not the case for a typical probability measure (in the sense of category).

Some relations between different notions of regularity are discussed in [Tay]. There are some results as well describing situations where $\alpha^\mu(x) \neq \alpha^\mu(x)$. For example, Taylor in [Tay95] shows that this happens for super Brownian motion. Shereshevsky in [She91] shows that under some conditions, if $\mu$ is an invariant measure of a smooth diffeomorphism, the set where $\alpha^\mu(x) \neq \alpha^\mu(x)$ is dense and has positive Hausdorff dimension. Finally, Haase in [Haa92] shows that if $x \in X$ then for a typical measure $\mu \in \mathcal{P}(X)$ (that is, all measures up to a set of first category) $\alpha^\mu(x) = 0$, and if $x$ is a non-isolated point of $X$, then for a typical measure $\alpha^\mu(x) = \infty$. Theorem 3.2.1 is basically a generalization of this result - we prove that for a typical measure these equalities hold not only at a given point, but at a typical point in the sense of category.
An interesting question, connected with this, is about the dimension of a typical measure, especially Hausdorff and packing dimension. The dimensions on other spaces have been explored before, such as the dimension of typical compact set or a graph of typical continuous function. For example, Hausdorff dimension of a typical compact subset of $\mathbb{R}^d$ is 0 and the upper entropy dimension is $d$ [Gru89]. We show later in this paper that the probability measures behave similarly in this matter.

We will use the following well-known relations:

\[
\dim\{x \in X \mid \alpha(x) \leq \alpha\} \leq \alpha \quad (3.1)
\]

\[
\dim\{x \in X \mid \bar{\alpha}(x) \leq \alpha\} \leq \alpha \quad (3.2)
\]

\[
\text{if } \mu(A) > 0, A \subseteq \{x \in X \mid \alpha(x) \geq \alpha\} \text{ then } \dim A \geq \alpha \quad (3.3)
\]

\[
\text{if } \mu(A) > 0, A \subseteq \{x \in X \mid \bar{\alpha}(x) \geq \alpha\} \text{ then } \dim A \geq \alpha \quad (3.4)
\]

For the proof see e.g. [Cut95],[Ols95]. Note that (3.4) would not be true for all metric spaces if packing measure were defined using diameters, but true under some fairly general conditions.

Recall that $\mathcal{P}(X)$ denotes the set of all Borel probability measures on $X$. The weak* topology on $\mathcal{P}(X)$ is characterized by the following proposition:

**Proposition 3.1.1.** Let $\mu, \{\mu_n\}_{n=1}^{\infty}$ be measures in $\mathcal{P}(X)$. Then the following statements are equivalent:

(a) $\mu_n \rightarrow \mu$ in weak* topology.
(b) \( \lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu \) for any bounded continuous function \( f \).

(c) \( \lim_{n \to \infty} \mu_n(F) \leq \mu(F) \) for every closed set \( F \).

(d) \( \lim_{n \to \infty} \mu_n(G) \geq \mu(G) \) for every open set \( G \).

(e) \( \lim_{n \to \infty} \mu_n(A) = \mu(A) \) for every Borel set \( A \) with boundary of \( \mu \)-measure 0.

Proof. This is a version of slightly more general Theorem 6.1 in [Par67].

If \( X \) is complete and separable, \( \mathcal{P}(X) \) is also complete, so we can use an expression "a typical measure" to signify that all measures except a set of first category in \( \mathcal{P}(X) \) have the desired properties.

**Proposition 3.1.2.** The probability measures with finite support are dense in \( \mathcal{P}(X) \).

Proof. See Theorem 6.3 in [Par67].

It follows that \( \mathcal{P}(X) \) is separable since \( X \) is separable.

\( \mathcal{P}(X) \) with the weak* topology can be metrized in several ways. In particular for \( X \) separable a *Prokhorov metric* \( p \) can be used:

\[
p(\mu, \nu) = \inf\{\epsilon > 0 \mid \mu(A) \leq \nu(A_\epsilon) + \epsilon \text{ and } \nu(A) \leq \mu(A_\epsilon) + \epsilon \text{ for any Borel set } A\},
\]

where \( A_\epsilon \) is the \( \epsilon \)-neighborhood of \( A \) in \( X \).

The inequalities (3.1)-(3.4) can also be used to establish a connection between the local and "global" dimensions of a measure. The latter can be computed in a number of ways. We will need the following definitions:

\[
\dim^* \mu = \inf\{\dim Y \mid Y \subseteq X, \mu(Y) = 1\},
\]
\[ \dim_* \mu = \inf \{ \dim Y \mid Y \subseteq X, \mu(Y) > 0 \} , \]
\[ \text{Dim}^* \mu = \inf \{ \text{Dim} Y \mid Y \subseteq X, \mu(Y) = 1 \} , \]
\[ \text{Dim}_* \mu = \inf \{ \text{Dim} Y \mid Y \subseteq X, \mu(Y) > 0 \} , \]
\[ \overline{C}(\mu) = \lim_{\delta \to 0} \inf \{ \dim_B(Y) \mid Y \subseteq X, \mu(Y) \geq 1 - \delta \} , \]
\[ C(\mu) = \lim_{\delta \to 0} \inf \{ \dim_B(Y) \mid Y \subseteq X, \mu(Y) \geq 1 - \delta \} , \]

Ledrappier dimensions are defined as follows. Suppose \( \mu \) is supported on a totally bounded set. Let \( N_\mu(\epsilon, \delta) \) be the minimal number of balls of radius \( \epsilon \) which cover a set of measure greater than \( 1 - \delta \). Then
\[
\overline{C}_L(\mu) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\log(N_\mu(\epsilon, \delta))}{\log(1/\epsilon)} , \tag{3.5}
\]
\[
C_L(\mu) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{\log(N_\mu(\epsilon, \delta))}{\log(1/\epsilon)} . \tag{3.6}
\]
(here \( \lim \) is the same as \( \sup \) due to monotonicity).

**Proposition 3.1.3.**
\[ \dim^* \mu \leq C_L(\mu) \leq C(\mu) , \]
\[ \overline{C}_L(\mu) \leq \overline{C}(\mu) = \text{Dim}^* \mu \]

**Proof.** The last equality is proved in [Tam95]; everything else is proved in [You82]. □
Let us introduce also a few more global characteristics of a measure:

\[
\overline{C}_{L^*}(\mu) = \lim_{\delta \to 1} \lim_{\epsilon \to 0} \frac{\log(N_{\mu}(\epsilon, \delta))}{\log(1/\epsilon)},
\]

\[
\underline{C}_{L^*}(\mu) = \lim_{\delta \to 1} \lim_{\epsilon \to 0} \frac{\log(N_{\mu}(\epsilon, \delta))}{\log(1/\epsilon)},
\]

(here \(\lim\) is the same as \(\inf_{0 < \delta < 1}\);)

\[
\overline{C}_*(\mu) = \inf \{ \overline{\dim} B | Y \subseteq X, \mu(Y) > 0 \},
\]

\[
\underline{C}_*(\mu) = \inf \{ \underline{\dim} B | Y \subseteq X, \mu(Y) > 0 \},
\]

**Proposition 3.1.4.**

\[
\underline{C}_{L^*}(\mu) \leq \overline{C}_{L^*}(\mu) \leq \overline{C}_*(\mu) = \text{Dim}_* \mu.
\]

**Proof.** The first two inequalities are obvious, as is the inequality \(\text{Dim}_* \mu \leq \overline{C}_*(\mu)\).

For any \(D > \text{Dim}_* \mu\) there is a set \(Y \subseteq X\) such that \(\mu(Y) > 0\) and \(\text{Dim} Y < D\). Note that \(\text{Dim} Y = \inf \{ \sup_i \overline{\dim} B A_i | Y \subseteq \cup A_i \} \) (for proof see e.g. [Cut95]). Hence there is at most countable collection of sets \(\{A_i\}\) such that \(\sup_i \overline{\dim} B A_i < D\) and \(\mu(\cup A_i) \geq \mu(Y) > 0\). It follows that there is an \(A_i\) with \(\mu(A_i) > 0, \overline{\dim} B A_i < D\), hence \(\text{Dim}_* \mu \geq \overline{C}_*(\mu)\).

**Proposition 3.1.5.** Let \(\mu\) be a probability measure on a compact space \(X\). If for some \(d\) there are numbers \(c > 0\) and \(R > 0\) such that

\[
\mu(B_r(x)) \leq cr^d
\]

for all \(x \in X, 0 < r < R\), then \(C_{L^*}(\mu) \geq d\).
Proof. Suppose that for some $\delta$

$$\lim_{\epsilon \to 0} \frac{\log N_\mu(\epsilon, \delta)}{\log(1/\epsilon)} < d.$$ 

Then there are $a < d$ and $\epsilon_i \downarrow 0, \epsilon_i < R$ such that $N_\mu(\epsilon_i, \delta) < \epsilon_i^{-a}$ for all $l$. Hence for any $l$ there are $N_\mu(\epsilon_i, \delta)$ balls $\{B_{\epsilon_i}^l\}$ with $\mu(\bigcup_i B_{\epsilon_i}^l) > 1 - \delta$. But then

$$1 - \delta < \mu(\bigcup_i B_{\epsilon_i}^l) \leq cN_\mu(\epsilon_i, \delta)\epsilon_i^d < c\epsilon_i^{d-a}$$

for all $l$, which contradicts the fact that $\epsilon_i^{d-a} \to 0$. \qed

3.2 Main Theorems

**Theorem 3.2.1.** Let $X$ be a complete separable metric space. Then for a typical measure $\mu$ in $\mathcal{P}(X)$ there is a residual Borel set $A_\mu$ in $X$ such that for any $x \in A_\mu$ we have $\alpha_\mu(x) = 0$. If $X$ has no isolated points, then in addition we can have $\alpha_\mu(x) = \infty$ for $x \in A_\mu$.

Proof. If $\mu_n \to \mu$ in the weak* topology, then $\lim_{n \to \infty} \mu_n(G) \geq \mu(G)$ for all open $G$ and $\lim_{n \to \infty} \mu_n(F) \leq \mu(F)$ for all closed $F$ (Proposition 3.1.1). It follows that for fixed $x$ and $r$ the ratio $\frac{\log \mu(B_r(x))}{\log r}$ is lower semicontinuous and $\frac{\log \mu(B_r(x))}{\log r}$ is upper semicontinuous with respect to $\mu$.

Consider closed balls first. Then

$$\Omega_{a,x,R} = \left\{ \mu \in \mathcal{P}(X) \mid \sup_{r < R} \frac{\log \mu(B_r(x))}{\log r} > a \right\} = \bigcup_{r < R} \left\{ \mu \in \mathcal{P}(X) \mid \frac{\log \mu(B_r(x))}{\log r} > a \right\}$$

is open for any $0 < R < 1, a > 0, x \in X$. 

22
Now we want to show that $\Omega_{a,x,R}$ is dense in $\mathcal{P}(X)$. Let $\mu \in \mathcal{P}(X)$ and suppose that $\mu \notin \Omega_{a,x,R}$, that is, $\frac{\log \mu(B_r(x))}{\log r} \leq a$ for all $r < R$.

Fix any $\epsilon > 0$. If $\mu(\{x\}) \neq 0$, in case $X$ has no isolated points we can find a measure $\nu \in \mathcal{P}(X)$ such that $p(\mu, \nu) < \epsilon/2$, and $\nu(\{x\}) = 0$. Otherwise let $\nu = \mu$.

We construct $\mu_\epsilon \in \mathcal{P}(X)$ as follows. Pick some $s > a$ and some $r < R$ such that $\nu(B_r(x)) < \epsilon/4$. If $\nu(B_r(x)) \leq r^s$, we are done. Otherwise let $\mu_\epsilon(B_r(x)) = r^s$ and $\mu_\epsilon(A) = C \nu(A)$ for $A \subseteq X \setminus B_r(x)$, where $C = (1 - r^s)/(1 - \nu(B_r(x)))$ so that $\mu_\epsilon(X) = 1$. Then $\frac{\log \mu_\epsilon(B_r(x))}{\log r} > a$ and $p(\mu, \mu_\epsilon) \leq \epsilon/2 + \epsilon/4 + (C - 1)(1 - \nu(B_r(x))) = 3\epsilon/4 + (1 - r^s) - 1 + \nu(B_r(x)) < \epsilon$. Hence $\Omega_{a,x,R}$ is dense in $\mathcal{P}(X)$.

Let $\{x_i\}_{i=1}^\infty$ be a countable dense subset of $X$. Then

$$\Omega_{a,R} = \left\{ \mu \in \mathcal{P}(X) \mid \sup_{r < R} \frac{\log \mu(B_r(x_i))}{\log r} > a \text{ for all } i \right\}$$

is a countable intersection of open dense sets, i.e. residual for any $a > 0, 0 < R < 1$.

The same is true for $\Omega_a = \bigcap_{n} \Omega_{a,1/n}$.

Now $\frac{\log \mu(B_r(x))}{\log r}$ is also lower semicontinuous with respect to $x$. so

$$A_{a,\mu,R} = \left\{ x \in X \mid \sup_{r < R} \frac{\log \mu(B_r(x))}{\log r} > a \right\}$$

is open for any $a > 0, 0 < R < 1, \mu \in \mathcal{P}(X)$.

For any fixed $\mu \in \Omega_a$, $A_{a,\mu,1/n}$ is open and dense (since $\{x_i\}_{i=1}^\infty \subseteq A_{a,\mu,1/n}$), hence

$$A_{a,\mu} = \bigcap_n A_{a,\mu,1/n} = \left\{ x \in X \mid \sup_{r < 1/n} \frac{\log \mu(B_r(x))}{\log r} > a \text{ for all } n \geq 1 \right\}$$

is residual in $X$. But

$$A_{a,\mu} \subseteq \left\{ x \in X \mid \lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r} \geq a \right\} ,$$
so
\[ \Omega_a \subseteq \left\{ \mu \in \mathcal{P}(X) \mid \lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r} \geq a \text{ on a residual subset of } X \right\} . \]

Now take the intersection \( \Omega = \bigcap_{n=1}^{\infty} \Omega_{1/n} \) and let the corresponding subset of \( X \) for a fixed \( \mu \in \Omega \) be \( A_{\mu} = \bigcap_{n=1}^{\infty} A_{1/n, \mu} \). This concludes the proof for the upper local dimension.

The proof is similar for the lower local dimension. Using open balls, we can show that
\[ \Omega_b = \left\{ \mu \in \mathcal{P}(X) \mid \lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r} \leq b \text{ on a residual subset of } X \right\} \]
is residual for any \( b > 0 \). We need only to change sup to inf and reverse inequality signs in the proof above. (Also we will not need to consider the case \( \mu(\{x\}) \neq 0 \) separately, just take \( \mu_\epsilon = r^s \) for some \( s < b \) and some small enough \( r < R \), so the condition that \( X \) has no isolated points is not necessary here).

Take the intersection \( \Omega = \bigcap_{n=1}^{\infty} \Omega_{1/n} \) to conclude the proof for \( \alpha_{\mu}(x) \). Then \( \Omega \cap \Omega \) gives the residual set of measures for which \( \alpha_{\mu}(x) = 0, \overline{\alpha}_{\mu}(x) = \infty \) for most \( x \in X \).

The natural question arising next is whether we can take \( A \) to be a set of positive measure. By (3.4), of course, we cannot have \( \overline{\alpha}_{\mu}(x) > \text{Dim } X \) on a set of positive measure, so we can hope only to get \( \overline{\alpha}_{\mu}(x) = \text{Dim } X \). We will show that we can in fact have \( \alpha_{\mu}(x) = 0 \) and \( \overline{\alpha}_{\mu}(x) = \text{Dim } X \) almost everywhere with some additional conditions on \( X \) for the latter. To this end we need first to consider the "global" dimension of a typical measure.

24
Theorem 3.2.2. If $X$ is a compact separable metric space, a typical measure $\mu \in \mathcal{P}(X)$ has $\dim^* \mu = C_L(\mu) = 0$.

Theorem 3.2.3. Let $X$ be a compact separable metric space. Suppose there exists a probability measure $\lambda \in \mathcal{P}(X)$ which is positive on all open sets and has $C_L(\lambda) \geq d$. Then a typical measure $\mu \in \mathcal{P}(X)$ has $\dim_\mu \mu \geq C_L(\mu) \geq d$.

Note. This is true, in particular, with $\lambda$ being the Lebesgue measure on $[0,1]^d$. We can also let $X$ be a self-similar fractal set of Hausdorff dimension $d$, with $\lambda$ being the Hausdorff measure $\mathcal{H}^d$. See Proposition 3.1.5.

We will need several lemmas to prove these theorems. In what follows, we will use the open balls in definition of $N_\mu(\epsilon, \delta)$, which does not change the limits.

Lemma 3.2.1. For $X$ a compact separable metric space, $N_\mu(\epsilon, \delta)$ is upper semicontinuous with respect to $\mu$.

Proof. Let $\mu_n \to \mu$ and $N_0(\epsilon, \delta) = \lim_{n \to \infty} N_{\mu_n}(\epsilon, \delta)$. Since $N_{\mu_n}(\epsilon, \delta)$ is integer, taking subsequences if needed, we may assume that $N_{\mu_n}(\epsilon, \delta) = N_0(\epsilon, \delta)$. For any $N < N_0(\epsilon, \delta)$ the inequality $N_{\mu_n}(\epsilon, \delta) > N$ means that for any $N$ open balls $\{B_i(x_i)\}_{i=1}^N$ we have $\mu_n(\bigcup_{i=1}^N B_i(x_i)) \leq 1 - \delta$. Hence

$$\mu \left( \bigcup_{i=1}^N B_i(x_i) \right) \leq \lim_{n \to \infty} \mu_n \left( \bigcup_{i=1}^N B_i(x_i) \right) \leq 1 - \delta \quad \text{from Proposition 3.1.1.}$$

so $N_\mu(\epsilon, \delta) > N$. It follows that $N_\mu(\epsilon, \delta) \geq N_0(\epsilon, \delta)$. \qed

Proof of Theorem 3.2.2. Since $N_\mu(\epsilon, \delta)$ is upper semicontinuous with respect to $\mu$ by Lemma 3.2.1, so is $\frac{\log(N_\mu(\epsilon, \delta))}{\log(1/\epsilon)}$. Hence

$$\Omega_{a,\epsilon_0,\delta} = \left\{ \mu \in \mathcal{P}(X) \mid \inf_{\epsilon < \epsilon_0} \frac{\log(N_\mu(\epsilon, \delta))}{\log(1/\epsilon)} < a \right\}$$

25
is open for any \( a, \epsilon_0 > 0, 0 < \delta < 1 \).

To show that \( \Omega_{a, \epsilon_0, \delta} \) is dense in \( \mathcal{P}(X) \), let \( \mu \in \mathcal{P}(X) \). For any \( \rho > 0 \) by Proposition 3.1.2 there is a measure \( \mu_\rho \) with finite support such that \( p(\mu, \mu_\rho) < \rho \). It means that for any \( \delta \), \( N_{\mu_\rho}(\epsilon, \delta) \) stays bounded as \( \epsilon \to 0 \), so for any \( a > 0, \epsilon_0 > 0 \) there is an \( \epsilon < \epsilon_0 \) such that \( \frac{\log N_{\mu_\rho}(\epsilon, \delta)}{\log(1/\epsilon)} < a \).

Now we have

\[
\Omega_{a, \delta} = \left\{ \mu \in \mathcal{P}(X) \mid \lim_{\epsilon \to 0} \frac{\log(N_{\mu}(\epsilon, \delta))}{\log(1/\epsilon)} \leq a \right\}
\]

and this intersection is a dense \( G_\delta \) set for any \( a > 0, 0 < \delta < 1 \). Taking intersections \( \cap_{n=1}^{\infty} \cap_{m=1}^{\infty} \Omega_{1/n, 1/m} \), we get the result. \( \square \)

**Note.** The result concerning Hausdorff dimension can also be shown as follows. By Theorem 3.2.1 and (3.1) there is a residual Borel set \( A \subseteq X \) with \( \dim A = 0 \). It can be shown (see proof of Lemma 2 in [Bro77]) that for any residual Borel set in \( X \) there is a residual set of measures in \( \mathcal{P}(X) \) concentrated on this set. But then for a typical measure \( \mu \in \mathcal{P}(X) \) we have \( \mu(A) = 1 \), so \( \dim^* \mu = 0 \).

**Lemma 3.2.2.** For \( X \) a compact separable metric space, \( N_{\mu}(\epsilon, \delta) \) is left continuous with respect to \( \epsilon \).

**Proof.** Let \( \epsilon \) be a discontinuity point of \( N_{\mu}(\epsilon, \delta) \) and let \( \epsilon_n \uparrow \epsilon \). Let \( N_0 = \lim_{\epsilon_n \to \epsilon} N_{\mu}(\epsilon_n, \delta) \) (which exists since \( N_{\mu}(\epsilon, \delta) \) is a decreasing function of \( \epsilon \)). Since \( N_{\mu}(\epsilon, \delta) \) is integer, for large \( n \) we have \( N_{\mu}(\epsilon_n, \delta) = N_0 \). Pick any \( N < N_0 \) and any \( N \) open balls \( \{B_\epsilon(x_i)\}_{i=1}^{N} \). Then for large \( n \) we have \( \mu(\bigcup_{i=1}^{N} B_\epsilon(x_i)) \leq 1 - \delta \). Hence \( \mu(\bigcup_{i=1}^{N} B_\epsilon(x_i)) = \lim_{n \to \infty} \mu(\bigcup_{i=1}^{N} B_\epsilon(x_i)) \leq 1 - \delta, \) so \( N_{\mu}(\epsilon, \delta) > N \). By monotonicity \( N_{\mu}(\epsilon, \delta) = N_0 \). \( \square \)
Lemma 3.2.3. If \( X \) is a compact separable metric space, for any \( \mu, \{\mu_n\}_{n=1}^{\infty} \) with \( \mu_n \to \mu \), any \( \epsilon > 0, 0 < \delta < 1, 0 < \nu < \min(\delta, 1 - \delta) \) we have

\[
N_\mu(\epsilon + \nu, \delta + \nu) \leq \lim_{n \to \infty} N_\mu_n(\epsilon, \delta)
\]

Proof. Let \( N_0(\epsilon, \delta) = \lim_{n \to \infty} N_{\mu_n}(\epsilon, \delta) \). Since \( N_{\mu_n}(\epsilon, \delta) \) is integer, taking subsequences we can assume that \( N_{\mu_n}(\epsilon, \delta) = N_0(\epsilon, \delta) \). This means that for each \( n \) there are \( N_0(\epsilon, \delta) \) balls \( B_\epsilon(x^n_i) \) such that \( \mu_n(\bigcup_{i=1}^{N_0} B_\epsilon(x^n_i)) > 1 - \delta \).

Pick any \( \nu \) as in the statement of the lemma. For large enough \( n \) we have

\[
p(\mu_n, \mu) < \nu,
\]

where \( p \) is a Prokhorov metric. For any Borel set \( A \) we have then \( \mu_n(A) \leq \mu(A_\nu) + \nu \). Let \( A = \bigcup_{i=1}^{N_0} B_\epsilon(x_i^n) \). Then \( A_\nu = \bigcup_{i=1}^{N_0} B_{\epsilon+\nu}(x_i^n) \), so we have

\[
\mu(\bigcup_{i=1}^{N_0} B_{\epsilon+\nu}(x_i^n)) \geq \mu_n(\bigcup_{i=1}^{N_0} B_\epsilon(x_i^n)) - \nu > 1 - \delta - \nu,
\]

hence \( N_\mu(\epsilon + \nu, \delta + \nu) \leq N_0(\epsilon, \delta) \). \( \square \)

Lemma 3.2.4. Let \( X \) be a compact metric space. Suppose there exists a probability measure \( \lambda \in \mathcal{P}(X) \) which is positive on all open sets and has \( C_{L^*}(\lambda) \geq d \). Then the set of measures with \( C_{L^*}(\mu) \geq d \) is dense in \( \mathcal{P}(X) \).

Proof. Pick any \( \mu \in \mathcal{P}(X) \) and \( \eta > 0 \). Since \( \lambda \) is positive on balls, using finite cover of \( X \) by balls of radius \( \eta/2 \) we can construct an \( \eta \)-partition \( \{I^\eta_k\}_{k=1}^{K(\eta)} \) of \( X \) with \( |I^\eta_k| < \eta \) and \( \lambda(I^\eta_k) > 0 \) for all \( k \). Construct a new measure \( \mu_\eta = \sum_k c^\eta_k \lambda^\eta_k \), where \( \lambda^\eta_k \) is \( \lambda \) restricted to \( I^\eta_k \), and \( c^\eta_k = \mu(I^\eta_k)/\lambda(I^\eta_k) \). Then \( p(\mu, \mu_\eta) \leq \eta \). Suppose for some \( \delta \)

\[
\lim_{\epsilon \to 0} \frac{\log N_{\mu_\eta}(\epsilon, \delta)}{\log(1/\epsilon)} < d.
\]
Then there are \( a < d \) and \( \epsilon_0 > 0 \) such that \( N_{\mu_\epsilon}(\epsilon, \delta) < \epsilon^{-a} \) for all \( \epsilon < \epsilon_0 \). Hence for any such \( \epsilon \) there are \( N_{\mu_\epsilon}(\epsilon, \delta) \) balls \( \{B_i^\epsilon\} \) with \( \mu_\eta(\cup_i B_i^\epsilon) > 1 - \delta \). But then

\[
1 - \delta < \mu_\eta(\bigcup_i B_i^\epsilon) = \sum_k c_k\lambda_k^\eta(\bigcup_i B_i^\epsilon) \leq C(\eta)\lambda(\bigcup_i B_i^\epsilon),
\]

where \( C(\eta) = \max c_k^\eta \). It means that \( N_\lambda(\epsilon, \delta_0) \leq \epsilon^{-a} \) for all \( \epsilon < \epsilon_0 \), where \( \delta_0 = 1 - (1 - \delta)/C(\eta) \). Hence we have

\[
\lim_{\epsilon \to 0} \frac{\log N_\lambda(\epsilon, \delta_0)}{\log(1/\epsilon)} \leq a < d,
\]

which contradicts the assumption that \( \overline{C}_L(\lambda) \geq d \).

\( \square \)

**Proof of Theorem 3.2.3.** By Lemmas 3.2.1, 3.2.3 for any \( \epsilon, \delta, \nu, \mu, \mu_n \to \mu \) as in the statements of the lemmas we have

\[
N_\mu(\epsilon + \nu, \delta + \nu) \leq \lim_{n \to \infty} N_{\mu_n}(\epsilon, \delta) \leq \lim_{n \to \infty} N_{\mu_n}(\epsilon, \delta) \leq N_\mu(\epsilon, \delta)
\]

As \( \epsilon \) or \( \delta \) increase, \( N_\mu(\epsilon, \delta) \) decreases. Fix \( \mu \in \mathcal{P}(X) \). Consider all lines \( \delta = \epsilon + \alpha \) on the plane with \( \alpha \) rational. On each of these lines \( N_\mu(\epsilon, \delta) \) is monotone, so it has countably many discontinuities there. All but countably many lines \( \delta = \delta_0 \) do not pass through any of these discontinuities. Denote the set of such \( \delta_0 \)'s by \( D_\mu \). For each \( \delta_0 \in D_\mu \) there is a dense set \( E_\mu(\delta) = \{\delta_0 - \alpha \mid \alpha \in \mathbb{Q}\} \) such that for \( \delta \in D_\mu, \epsilon \in E_\mu(\delta) \) we have \( N_\mu(\epsilon + \nu, \delta + \nu) \to N_\mu(\epsilon, \delta) \) as \( \nu \to 0 \). It follows that for \( \delta \in D_\mu, \epsilon \in E_\mu(\delta) \) all the inequalities above become equalities, which means \( N_\mu(\epsilon, \delta) = \lim_{n \to \infty} N_{\mu_n}(\epsilon, \delta) \) for any \( \mu_n \to \mu \), making \( N_\mu(\epsilon, \delta) \) continuous at \( \mu \).

By Lemma 3.2.4 we can choose a countable dense set of measures

\[
\mathcal{M} \subseteq \left\{ \mu \in \mathcal{P}(X) \mid \inf_{\delta} \lim_{\epsilon \to 0} \frac{\log(N_\mu(\epsilon, \delta))}{\log(1/\epsilon)} \geq d \right\}
\]

28
Let $D = \cap \{D_\mu \mid \mu \in \mathcal{M}\}$. $D$ contains all but countably many points of $(0,1)$. Pick any $\delta \in D$. Fix $\eta > 0, \epsilon_0 > 0$. For any $\mu \in \mathcal{M}$ there is $\epsilon_1 < \epsilon_0$ such that
\[
\frac{\log N_\mu(\epsilon_1, \delta)}{\log(1/\epsilon_1)} > d - \frac{\eta}{2}
\] (3.7)

By Lemma 3.2.2 $N_\mu(\epsilon, \delta)$ is left continuous, hence there is an interval $(\epsilon_2, \epsilon_1]$ on which $\frac{N_\mu(\epsilon, \delta)}{\log(1/\epsilon)} > d - \eta/2$. But $E_\mu(\delta)$ is dense in $(0,1)$, so the set $E_\mu(\delta) \cap (0, \epsilon_1)$ is not empty. For any $\epsilon$ in this set, $\frac{\log N_\mu(\epsilon, \delta)}{\log(1/\epsilon)}$ is continuous at $\mu$, so there is an open neighborhood of $\mu$ such that for any measure $\nu$ in this neighborhood we have
\[
\left| \frac{\log N_\nu(\epsilon, \delta)}{\log(1/\epsilon)} - \frac{\log(N_\mu(\epsilon, \delta))}{\log(1/\epsilon)} \right| < \frac{\eta}{2}
\] (3.8)

Hence by (3.7) and (3.8)
\[
\frac{\log N_\nu(\epsilon, \delta)}{\log(1/\epsilon)} > d - \eta,
\]
so
\[
\sup_{\epsilon < \epsilon_0} \frac{\log N_\nu(\epsilon, \delta)}{\log(1/\epsilon)} > d - \eta.
\]

It follows that for any $\delta \in D, \eta, \epsilon_0 > 0$
\[
\Omega_{\epsilon_0, \delta, \eta} = \left\{ \mu \in \mathcal{P}(X) \mid \sup_{\epsilon < \epsilon_0} \frac{\log(N_\mu(\epsilon, \delta))}{\log(1/\epsilon)} > d - \eta \right\}
\]
is an open dense set in $\mathcal{P}(X)$. Then
\[
\Omega_{\delta, \eta} = \left\{ \mu \in \mathcal{P}(X) \mid \lim_{\epsilon \to 0} \frac{\log(N_\mu(\epsilon, \delta))}{\log(1/\epsilon)} \geq d - \eta \right\} \supseteq \bigcap_{n=1}^{\infty} \Omega_{1/n, \delta, \eta}
\]
is residual in $\mathcal{P}(X)$. Taking intersections $\bigcap_{n=1}^{\infty} \Omega_{1/n, \delta, \eta}$ concludes the proof. \qed
Theorem 3.2.4. If $X$ is a compact separable metric space, then a typical measure $\mu \in \mathcal{P}(X)$ has $\alpha_\mu(x) = 0$ a.e. with respect to $\mu$. If there is a probability measure $\lambda \in \mathcal{P}(X)$ which is positive on all open sets and has $\overline{C}_{L^*}(\lambda) \geq d$ then for a typical measure $\mu \in \mathcal{P}(X)$ also $\overline{c}_\mu(x) \geq d$ a.e.

Proof. If $\dim^* \mu = 0$, by (3.3) we have $\alpha_\mu(x) = 0$ a.e.

If $\dim_* \mu \geq d$, by (3.2) we have $\overline{c}_\mu(x) \geq d$ a.e. Theorems 3.2.3 and 3.2.2 now give the desired result. \qed

So far we have considered the space of all probability measures on $X$, but it is possible to look also at some subspaces of it, for example, subspaces of measures invariant with respect to some transformation. We will consider here one important class of such subspaces.

By a dynamical system we will understand here a compact metric space $X$ together with a homeomorphism $T : X \to X$. Denote by $\mathcal{P}_T(X)$ the space of all Borel probability measures on $X$ which are invariant with respect to $T$, that is, $\mu(T^{-1}(A)) = \mu(A)$ for any Borel subset $A$ of $X$. $\mathcal{P}_T(X)$ is a closed nowhere dense (if $T$ is not trivial) subspace of $\mathcal{P}(X)$ in the weak* topology ([DGS76], Proposition (3.5)). We will now consider how a typical invariant measure behaves with respect to local dimension.

The periodic measures in the statement of the next theorem are measures with mass $1/p$ at each of the points $x, Tx, ..., T^{p-1}x$, where $x$ is a periodic point of minimal period $p$.

Theorem 3.2.5. Let $(X, T)$ be a dynamical system on a complete separable metric space, with the property that periodic measures are dense in $\mathcal{P}_T(X)$. Then for a
typical measure \( \mu \) in \( \mathcal{P}_T(X) \) there is a residual Borel set \( A_\mu \) in \( X \) such that for any \( x \in A_\mu \) we have \( \alpha_\mu(x) = 0 \). If \( X \) has no isolated points, then in addition we can have \( \overline{\alpha}_\mu(x) = \infty \) for \( x \in A_\mu \).

**Proof.** The proof of Theorem 3.2.1 can work here if we show that

\[
\Omega_{a,x,R} = \left\{ \mu \in \mathcal{P}_T(X) \mid \sup_{r < R} \frac{\log \mu(B_r(x))}{\log r} > a \right\}
\]

is dense in \( \mathcal{P}_T(X) \) for any \( R > 0, a > 0 \) and \( x \) in a dense subset of \( X \), and similarly for lower dimension.

Pick any point \( x \in X \), any \( R > 0 \) and any invariant measure \( \mu \in \mathcal{P}_T(X) \) and suppose \( \mu \notin \Omega_{a,x,R} \). There is a sequence of periodic measures \( \mu_n \) which converges to \( \mu \) and such that \( x \notin \text{supp} \mu_n \) for any \( n \). Pick any \( b > a \) and for any \( n \) choose \( r_n < R \) so that \( B_{r_n}(x) \cap \text{supp} \mu_n = \emptyset \). Let \( C = r^b/b\mu(B_{r_n}(x)) \) \( (C < 1 \text{ since } \mu \notin \Omega_{a,x,R}) \) and let \( \nu_n = C\mu + (1 - C)\mu_n \). Then \( \nu_n \) is invariant, \( \nu_n \in \Omega_{a,x,R} \) for any \( n \) and \( \nu_n \to \mu \). That shows that \( \Omega_{a,x,R} \) is dense in \( \mathcal{P}_T(X) \). For the lower bound do the same for any \( b < a \) and let \( \nu_n = C\mu - (C - 1)\mu_n \). \( \square \)

It is proven in [DGS76], Proposition (21.8) that for systems on compact metric spaces which satisfy the following specification property, first introduced by Bowen [Bow71], periodic measures are dense in \( \mathcal{P}_T(X) \).

**Definition 3.2.1.** \((X, T)\) is said to satisfy the specification property if for any \( \epsilon > 0 \) there exists an integer \( M(\epsilon) \) such that for any \( k \geq 2 \), for any \( k \) points \( x_1, \ldots, x_k \in X \), for any integers \( a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k \) with \( a_i - b_{i-1} \geq M(\epsilon) \) for
$2 \leq i \leq k$ and for any integer $p$ with $p \geq M(\epsilon) + b_k - a_1$, there exists a point $x \in X$ with $T^p x = x$ such that

$$d(T^n x, T^m x_i) \leq \epsilon$$

for $a_i \leq n \leq b_i, 1 \leq i \leq k$.

Some examples of such systems are shifts on any compact metric state space and Axiom A diffeomorphisms.

### 3.3 Example

Let us now construct a simple example of a measure $\mu$ with $\underline{a}_\mu(x) < \overline{a}_\mu(x)$ a.e. The technique used here is common with this type of problem; see, for instance, Example 5.1 in [TT86] or a very detailed account of a similar example [Cut95].

Consider a Cantor-type set $F \subset (0, 1)$ constructed as follows. Let $\{a_i\}_{i=1}^\infty$ be a sequence of positive integers. Let $p_1, p_2$ be positive integers and $r_1, r_2 > 0$ such that $p_1 r_1 < 1, p_2 r_2 < 1$. Replace $I_0 = [0, 1]$ by $p_1$ disjoint intervals of length $r_1$. Repeat this with each of the resulting intervals and so on. Do this $a_1$ times; then do the same with parameters $p_2, r_2$ a$_2$ times; then again with $p_1, r_1$ a$_3$ times and so on. Let $\mu$ be a probability measure which is equally divided between all intervals on each step. Let $c_k = \sum_{i=1}^k a_i, b_k = \sum_{i=1}^k a_{2i}, d_k = \sum_{i=1}^k a_{2i-1}$. We will denote by $I_n$ any interval of construction after $c_n$ steps.

Then for the length of intervals we have:

$$|I_{2k}| = r_1^{d_k} r_2^{b_k}, \quad |I_{2k+1}| = r_1^{d_k+1} r_2^{b_k},$$
and for the measure
\[ \mu(I_{2k}) = p_1^{-d_k} p_2^{-b_k}, \quad \mu(I_{2k+1}) = p_1^{-d_{k+1}} p_2^{-b_k}, \]
hence
\[ \alpha_{2k} := \frac{\log \mu(I_{2k})}{\log |I_{2k}|} = \frac{-d_k \log p_1 - b_k \log p_2}{d_k \log r_1 + b_k \log r_2} = -\frac{\log p_2}{\log r_2} \frac{d_k \log r_1 + 1}{d_k \log r_2 + 1}, \]
\[ \alpha_{2k+1} := \frac{\log \mu(I_{2k+1})}{\log |I_{2k+1}|} = -\frac{\log p_2}{\log r_2} \frac{d_{k+1} \log r_1 + 1}{d_{k+1} \log r_2 + 1}. \]

Suppose we choose \( \{a_i\} \) so that \( \lim_{k \to \infty} (d_{k+1}/b_k) > \lim_{k \to \infty} (d_k/b_k) \) (that is, \( \lim_{k \to \infty} (a_{2k+1}/ \sum_{i=1}^{k} a_{2i}) > 0 \)). Then we have
\[ \alpha = \lim_{k \to \infty} \frac{\log \mu(I_{2k})}{\log |I_{2k}|} < \lim_{k \to \infty} \frac{\log \mu(I_{2k+1})}{\log |I_{2k+1}|} = \bar{\alpha} \]

For \( x \in F \) let now \( \{I_n\}_{n=0}^\infty \) be a sequence of intervals converging to \( x \). Let \( \epsilon_k = |I_{2k}|. \) We have \( B_{\epsilon_k}(x) \supseteq I_{2k} \), so
\[ \frac{\log \mu(B_{\epsilon_k}(x))}{\log \epsilon_k} \leq \frac{\log \mu(I_{2k})}{\log |I_{2k}|}, \]
hence
\[ \lim_{\epsilon \to 0} \frac{\log \mu(B_{\epsilon}(x))}{\log \epsilon} \leq \alpha \]

After \( c_{2k+1} \) steps the distance between two intervals of construction is at least \( c|I_{2k+1}| \), where \( c \) is the minimal distance between intervals after the first step. Let \( \delta_k = c|I_{2k+1}| \). Then \( F \cap B_{\delta_k}(x) \subseteq I_{2k+1}, \) so
\[ \frac{\log \mu(B_{\delta_k}(x))}{\log \delta_k} \geq \frac{\log \mu(I_{2k+1})}{\log c|I_{2k+1}|}, \]
hence

\[
\lim_{\epsilon \to 0} \frac{\log \mu(B_\epsilon(x))}{\log \epsilon} \geq \alpha
\]

3.4 Multifractal Spectra of a Typical measure

First we will show that a typical measure is not "nice" in the sense that it does not belong to \( \mathcal{P}_0 \). This shows that most measures fail some of the properties listed in chapter 2 which are often taken for granted.

**Theorem 3.4.1.** If \( X \) is a complete separable metric space, a typical measure \( \mu \in \mathcal{P}(X) \) does not belong to \( \mathcal{P}_0(X) \).

**Proof.** First, note that for a typical measure we have \( \text{supp} \mu = X \). Indeed, if \( \text{supp} \mu \neq X \), there is an open ball \( B \in X \setminus \text{supp} \mu \). For any \( x \in B \) we have \( \alpha_\mu(x) = \infty \). while for a typical measure \( \alpha_\mu(x) = 0 \) on a dense subset of \( X \).

Now fix some \( a > 1 \) and consider the function

\[
T_\mu(r, x) = \begin{cases} 
\frac{\mu(B_{ar}(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0, \\
\infty & \text{otherwise}.
\end{cases}
\]

Fix \( x \in X, r > 0 \) and let \( \mu_n \to \mu \). Then we have \( \lim_{n \to \infty} \mu_n(B_{ar}(x)) \geq \mu(B_{ar}(x)) \) and \( \lim_{n \to \infty} \mu_n(B_r(x)) \leq \mu(B_r(x)) \). From the second inequality it follows also that \( T_\mu(r, x) = \infty \) implies \( T_{\mu_n}(r, x) = \infty \) eventually. Hence we have

\[
\lim_{n \to \infty} \frac{\mu_n(B_{ar}(x))}{\mu_n(B_r(x))} \geq \frac{\mu(B_{ar}(x))}{\mu(B_r(x))},
\]

that is, \( T_\mu(r, x) \) is lower semicontinuous with respect to \( \mu \).
It follows that \( \Omega_{x,b,R} = \{ \mu \in \mathcal{P}(X) \mid \sup_{r<R} T_\mu(r,x) > b \} \) is open in \( \mathcal{P}(X) \) for any \( R > 0, b > 1 \). It is enough to show that this set is dense: then we will have that \( \{ \mu \in \mathcal{P}(X) \mid \lim_{r \to 0} T_\mu(r,x) = \infty \} \) is residual for any \( x \in X \) (in fact, \( T_\mu(r,x) = \infty \) on a residual subset of \( X \) for a typical measure), as in Theorem 3.2.1. Since \( T_\mu(r,x) \leq \frac{\mu(B_{ar}(x))}{\mu(B_r(x))} \) (which is the ratio that appears in Olsen's definition of \( \mathcal{P}_0 \)) and for a typical measure \( x \in \text{supp} \mu \), it follows that a typical measure is not in \( \mathcal{P}_0(X) \).

To show that \( \Omega_{x,b,R} \) is dense for any \( x \in X, b > 1, R > 0 \), pick any \( \nu \in \mathcal{P}(X) \) and any \( \epsilon > 0 \). If \( \nu(\{x\}) \neq 0 \), choose \( \mu \in \mathcal{P}(X) \) such that \( \mu(\{x\}) = 0 \) and \( p(\mu, \nu) < \epsilon/2 \); otherwise let \( \mu = \nu \). Suppose \( \sup_{r<R} T_\mu(r,x) < b \). Then, in particular, \( x \in \text{supp} \mu \).

Pick some \( d > b \) and \( r < R \) so that \( \mu(B_{ar}(x)) < \epsilon/(2d) \). Construct a measure \( \mu_\epsilon \) in a following way: let \( \mu_\epsilon(B_{ar}(x)) = d\mu(B_r(x)) \), let \( \mu_\epsilon(B_r(x)) = \mu(B_r(x)) \) and \( \mu_\epsilon(A) = C\mu(A) \) for \( A \in X \setminus B_{ar}(x) \), where \( C = (1 - d\mu(B_r(x)))/(1 - \mu(B_{ar}(x))) \).

Then \( p(\nu, \mu_\epsilon) \leq \epsilon \) and \( \sup_{r<R} T_{\mu_\epsilon}(x,r) > b \). \( \square \)

Now let us explore the typical behavior of multifractal spectra in a complete separable metric space.

**Proposition 3.4.1.** If \( X \) is a complete separable metric space, a typical measure has \( B(q) = 0 \) for \( q \geq 1 \).

**Proof.** By [Ols95], Theorem 2.17, \( A_\mu \leq \inf \alpha_\mu(x) \). A typical measure has \( \alpha_\mu(x) = 0 \) a.e. hence \( A_\mu \leq 0 \). By definition then, \( -B(q)/q \leq 0 \) for \( q > 0 \), that is, \( B(q) \geq 0 \) for \( q > 0 \). But \( B(q) \leq 0 \) for \( q \geq 1 \), so for a typical measure \( B(q) = 0 \) for \( q \geq 1 \). \( \square \)
Proposition 3.4.2. If $X = [0, 1]^d$, a typical measure has $b_\mu(q) = L^q_\mu(X) = C^q_\mu(X) = d - dq$ for $q < 0$.

Proof. 1. $L^q_\mu(X) \geq b(q) \geq d - dq$.

By [Ols95], Theorem 2.17, $\overline{\alpha}_\mu \geq \sup \alpha_\mu(x)$. For any $a < d$ we have

$$\dim \{x \in X | \alpha_\mu(x) \leq a\} \leq a,$$

but $\dim X = d$, so $\sup \alpha_\mu(x) \geq a$; moreover,

$$\dim(X_a) = \dim \{x \in \text{supp } \mu | \alpha_\mu(x) \geq d\} = d.$$

We have then $\overline{\alpha}_\mu \geq d$. By definition, $b(q)/q \geq d$ for $q < 0$, that is, $b(q) + dq \geq 0$ for $q < 0$. By [Ols95], Proposition 2.5 (iii), we have $\dim X_a \leq dq + b(q)$ for $q < 0$. so $b(q) \geq d - dq$.

2. $b(q) \leq L^q_\mu(X) \leq d - dq$.

Here we will use open balls in the definition of $L^q_\mu$. First, let us prove that $T^q_{\mu,\delta}$ is upper semicontinuous with respect to $\mu$ for any $q < 0, \delta > 0$. Let $\mu_n \to \mu$ and pick any $T < \lim_{n \to \infty} T^q_{\mu_n,\delta}$. Taking subsequences, we may assume that $T_{\mu_n,\delta} > T$ for any $n$.

Now for any finite cover $(B_i)$ by balls of radius $\delta$ we have that $\sum_i \mu_n(B_i)^q > T$ for any $n$. Since $B_i$ is open, $\lim \mu_n(B_i) \geq \mu(B_i)$, so for $q < 0$ we have $\lim \mu_n^q(B_i) \leq \mu^q(B_i)$.

Hence

$$\sum_i \mu_q(B_i) \geq \sum_i \lim_{n \to \infty} \mu_n^q(B_i) \geq \lim_{n \to \infty} \sum_i \mu_n^q(B_i) > T,$$

which shows that $T^q_{\mu,\delta} \geq T$. 

36
It follows that

$$\{ \mu : \inf_{\delta < \delta_0} \frac{\log T_{\mu, \delta}^q}{\log \delta} < a \}$$

is open for any $a, q, \delta_0$. Now let us prove that it is dense for any $a > d - dq$.

Let $\mu \in \mathcal{P}(X)$ and let $\lambda$ denote the Lebesgue measure. For any $\eta > 0$ we can construct an $\eta$-partition $\{ I_k \}_{k=1}^{K(\eta)}$ of $X$ with $|I_k| < \eta$ and $\lambda(I_k) > 0$ for all $k$. Construct a new measure $\mu_\eta = \sum_k c_k \lambda_k$, where $\lambda_k$ is $\lambda$ restricted to $I_k$, and $c_k = \mu(I_k)/\lambda(I_k)$. Then $p(\mu, \mu_\eta) \leq \eta$. For $\delta < \delta_0$ we can choose a cover $(B_i)$ of $X$ by balls of radius $\delta$ so that the number of balls is less than $C\delta^{-d}$ for some $C$. We have

$$\mu_\eta(B_i) \geq \min(c_k)\lambda(B_i) = K \min(c_k)\delta^d,$$

hence for $q < 0$

$$T_{\mu_\eta, \delta}^q \leq \sum_i \mu_\eta(B_i)^q \leq \sum_i (K \min(c_k)\delta^d)^q \leq C_0\delta^{-d}\delta^{dq} < \delta^{-a}$$

for small enough $\delta$, so

$$\inf_{\delta < \delta_0} \frac{\log T_{\mu_\eta, \delta}^q}{\log \delta} < a$$

It follows that $\{ \mu \in \mathcal{P}(X) : L^q_\mu(X) \leq d - dq \}$ is residual in $\mathcal{P}(X)$. □

Note. Similarly we can show that $L^q_\mu(X) \leq d - dq$ for a typical measure if $q \geq 0$. Also, using finitely supported measures instead of Lebesgue, we can show that typically $\overline{C}^q_\mu(X) = 0$ for $q \geq 1$ and $\overline{C}^q_\mu(X) = \overline{C}^q_\mu(X) = \infty$ for $q < 0$.

We will also make use here of the “coarse” multifractal formalism (the approach described in Section 2.2 is called the “fine” theory). Following [Rie95], let

$$M_{\mu, \delta}(q) = \sum (\mu(B_i'))^q,$$
\[
\overline{\beta}_\mu(q) = \lim_{\delta \to 0} \frac{\log M_{\mu,\delta}(q)}{-\log \delta}, \quad \beta_\mu(q) = \lim_{\delta \to 0} \frac{\log M_{\mu,\delta}(q)}{-\log \delta}
\]

where the sum is over all \( \delta \)-boxes \( B = \prod_{k=1}^d [l_k\delta, (l_k + 1)\delta), l_k \in \mathbb{Z} \) with nonzero \( \mu \)-measure, and \( B' \) denotes the box \( B \) together with its neighbors. For more details on this approach see for example [Fal97].

We can then consider how this spectrum behaves for a typical probability measure.

**Lemma 3.4.1.** For all \( \mu \in \mathcal{P}(X) \) and \( q < 0 \) we have \( \overline{\beta}_\mu(q) = \overline{\mathcal{L}}^q_{\mu}(\text{supp } \mu) = \overline{\mathcal{C}}^q_{\mu}(\text{supp } \mu), \beta_\mu(q) = \mathcal{L}^q_{\mu}(\text{supp } \mu) = \mathcal{C}^q_{\mu}(\text{supp } \mu) \).

**Proof.** The equalities for \( L \) and \( C \) are proved in [Ols95], Proposition 2.19. For any \( \delta > 0 \) and any \( \delta \)-box \( B_i \) with \( \mu(B_i) > 0 \) we can find \( x_i \in B_i \cup \text{supp } \mu \). Then \( B_i' \subseteq B_{2\sqrt{d}\delta}(x_i) \). Thus we have a centered covering of \( \text{supp } \mu \) by balls \( \{ B_{2\sqrt{d}\delta}(x_i) \} \) with \( \sum(\mu(B_{2\sqrt{d}\delta}(x_i)))^q \leq \sum(\mu(B'_i))^q \), so \( M_{\mu,\delta}(q) \geq T^q_{\mu,2\sqrt{d}\delta}(\text{supp } \mu) \). Taking logarithms and limits, we have \( \overline{\beta}_\mu(q) \geq \overline{\mathcal{L}}^q_{\mu} \) and similarly for lower limits.

To prove another direction, for any \( \delta \)-box \( B_i \) with nonzero measure consider a ball \( B_\delta(x_i) \), where \( x_i \) is some point in \( B_i \cup \text{supp } \mu \). Then \( B_\delta(x_i) \subseteq B' \) and if the neighborhoods \( B'_i_1 \) and \( B'_i_2 \) of two boxes do not intersect, then the corresponding balls also do not intersect. It follows that we have at most \( 3^d \) centered packings of \( \text{supp } \mu \) by these balls. Hence

\[
\sum(\mu(B'_i))^q \leq \sum(\mu(B_\delta(x_i)))^q \leq 3^d S^q_{\mu,\delta}(\text{supp } \mu),
\]

so \( \overline{\beta}_\mu(q) \leq \overline{\mathcal{C}}^q_{\mu} \) and \( \beta_\mu(q) \leq \mathcal{C}^q_{\mu} \).

**Lemma 3.4.2.** For all \( \mu \in \mathcal{P}(X) \) and \( q \geq 0 \), \( \overline{\beta}_\mu(q) \geq \overline{\mathcal{C}}^q_{\mu}(\text{supp } \mu), \beta_\mu(q) \geq \mathcal{C}^q_{\mu}(\text{supp } \mu) \).
Proof. For $\delta > 0$ consider any centered packing $\{B_\delta(x_i)\}$ of $\text{supp}\mu$. For each $i$ find a $\delta$-box $B_i$ such that $x_i \in B_i$. Then $B_\delta(x_i) \subseteq B_i'$ and the number of balls which correspond to the same box is bounded by some $K = K(d)$. Hence

$$\sum (\mu(B_\delta(x_i)))^q \leq \sum (\mu(B_i'))^q \leq KM_{\mu,\delta}(q)$$

It follows that $S_{\mu,\delta}(\text{supp}\mu) \leq KM_{\mu,\delta}(q)$, which gives the desired inequalities. □

Since obviously $\beta_{\mu}(q) \leq 0$ for $q \geq 1$, we have

**Lemma 3.4.3.** If $X = [0, 1]^d$, for a typical probability measure $\mu \in \mathcal{P}(X)$ and $q > 0$ we have $\beta_{\mu}(q) \leq d - dq$.

**Proof.** Consider $\overline{M}_{\mu,\delta}(q)$ which is the same sum as $M_{\mu,\delta}(q)$ but using closed boxes.

Let $\mu_n \to \mu$. For any closed box $B$ we have then $\lim_{\delta \to 0} (\mu_n(B))^q \leq (\mu(B))^q$, so $\overline{M}_{\mu,\delta}(q)$ is upper semicontinuous with respect to $\mu$ (using also the fact that $\mu(B) = 0$ implies $\mu_n(B) = 0$ for all $n$). Hence the set

$$\{\mu \mid \inf_{\delta < \delta_0} \frac{\log \overline{M}_{\mu,\delta}(q)}{-\log \delta} < a\}$$

is open for any $a, q, \delta_0$. We can show that it is dense for any $a > d - dq$ similarly to the proof for $T_{\mu,\delta}^q$ in Proposition 3.4.2. Since $\overline{M}_{\mu,\delta}(q) \geq M_{\mu,\delta}(q)$ for $q \geq 0$, the result follows. □

**Lemma 3.4.4.** $\beta_{\mu}(q) \leq \beta_{\mu}(q)$ for all $\mu \in \mathcal{P}(X)$ and $q \geq 0$.

**Proof.** Pick any $t > \beta_{\mu}(q)$. Then there is a sequence $\delta_n \downarrow 0$ such that

$$\frac{\log M_{\mu,\delta_n}(q)}{-\log \delta_n} < t$$

for any $n$. 

39
that is, $\sum (\mu(B_i^n))^q < \delta_n^{-t}$, where the sum is over all $\delta_n$-boxes with nonzero measure.

Let $E \subseteq \text{supp} \mu$ and let $I_n = \{ i \mid B_i^n \cap E \neq \emptyset \}$. For any $i \in I_n$ we can find balls $B_{\delta_n}(x_1), \ldots, B_{\delta_n}(x_{k(i)})$ so that their centers lie in $B_i^n \cap E$, the union of these balls covers $B_i^n \cup E$ and their number is at most $K = K(d)$. Then we have $B_{\delta_n}(x_i) \subseteq (B_i^n)'$ for $l = 1, \ldots, k(i)$, hence

$$
\overline{H}_{\mu, \delta_n}^{q,t}(E) \leq \sum_{i \in I_n} \sum_{l=1}^{k(i)} (\mu(B_{\delta_n}(x_i)))^q (2\delta_n)^t
\leq 2^t K \sum_{i \in I_n} (\mu(B_i^n))^q \delta_n^t \leq 2^t K.
$$

Letting $n \to \infty$ we get $\overline{H}_{\mu}^{q,t}(E) \leq 2^t K$ for all $E \subseteq \text{supp} \mu$, so $H_{\mu}^{q,t}(\text{supp} \mu) \leq 2^t K$ and we have $b_{\mu}(q) \leq t$. \hfill \Box

Combining all these results and noting that typically $\text{supp} \mu = X$ (see the proof of Theorem 3.4.1), we have

**Theorem 3.4.2.** If $X = [0, 1]^d$, then for a typical probability measure $\mu \in \mathcal{P}(X)$:

1. $b_{\mu}(q) = L_{\mu}^q(X) = C_{\mu}^q(X) = \beta_{\mu}(q) = d - dq$ for $q < 0$;

2. $\overline{\beta}_{\mu}(q) = L_{\mu}^q(X) = C_{\mu}^q(X) = \infty$ for $q < 0$;

3. $C_{\mu}^q(X) \leq \beta_{\mu}(q) \leq d - dq$, $b_{\mu}(q) \leq \beta_{\mu}(q) \leq d - dq$, $L_{\mu}^q(X) \leq d - dq$ for $q \geq 0$;

4. $\overline{\beta}_{\mu}(q) = C_{\mu}^q(X) = B_{\mu}(q) = 0$ for $q \geq 1$. 

40
CHAPTER 4
GENERALIZED MULTIFRACTAL FORMALISM

In this chapter we will describe a generalized multifractal formalism and prove the corresponding versions of theorems from Section 2.2. A particular case of this is the "second order" multifractal formalism applied in Section 4.2 to describe a hyperspace of compact subsets of a self-similar fractal set, which is a common example of infinite-dimensional metric space.

4.1 Definitions and Theorems

Let $h(x)$ and $g(x)$ be functions on $(0, a), 0 < a \leq \infty$ with the following properties:

(i) $h(x), g(x)$ are continuous, positive and strictly decreasing on $(0, a)$;

(ii) $\lim_{x \to 0} h(x) = \lim_{x \to 0} g(x) = \infty$, $\lim_{x \to a} h(x) = 0$;

(iii) $\lim_{x \to \infty} \frac{h^{-1}(kx)}{h^{-1}(x)} = 0$ for any $k > 1$;

(iv) $\lim_{x \to 0} (g(x) - g(kx)) < \infty$ for $k > 1$.

Conditions (i) and (ii) guarantee that $h^{-1}(x)$ is defined on $(0, \infty)$. Consider a family of functions $\phi_s(x)$ defined by:

$$\phi_s(x) = \begin{cases} h^{-1}(sg(x)) & \text{for } x > 0, \\ 0 & \text{for } x = 0. \end{cases}$$
Clearly \( \phi_s(x) \) are continuous and increasing on \([0, \infty)\), thus a family of Hausdorff functions. Also let us formally set \( h^{-1}(0) = a \), including the case when \( a = \infty \).

It is easy to see that (iii) and (iv) imply the following:

\[
\lim_{x \to 0} \frac{\phi_{s_1}(kx)}{\phi_{s_2}(x)} = 0 \quad \text{for} \quad s_1 > s_2 \quad \text{and any} \quad k \geq 1
\]

(4.1)

Thus we can define generalized Hausdorff and packing dimensions based on the family of functions \( \phi_s \):

\[
\dim_{h,g}(E) = \sup \{ s > 0 \mid H^{\phi_s}(E) = \infty \} = \inf \{ s > 0 \mid H^{\phi_s}(E) = 0 \},
\]

\[
\text{Dim}_{h,g}(E) = \sup \{ s > 0 \mid P^{\phi_s}(E) = \infty \} = \inf \{ s > 0 \mid P^{\phi_s}(E) = 0 \}
\]

Some examples of commonly encountered families of functions include:

1. \( \phi_s(x) = x^s \) corresponding to \( h(x) = g(x) = -\log x \) gives the usual Hausdorff and packing dimensions \( H^s, P^s \);

2. \( \phi_s(x) = 2^{-x^s} \) corresponding to \( h(x) = \log_2 \log x, g(x) = -\log_2(x) \);

3. \( \phi_s(x) = 2^{-(\log_21/x)^s} \) corresponding to \( h(x) = g(x) = \log_2 \log_2 1/x \).

The last two families appear when we study certain sets of Hölder and analytic functions correspondingly (see [KT61]).

We will also need Hausdorff and packing measures based on a family of functions \( \bar{\phi}_s(x) = \phi_s(x/2) \). Property (4.1) ensures that this family gives the same values of generalized Hausdorff and packing dimensions.

For \( \mu \in \mathcal{P}(X) \) consider also generalized local dimension, defined by:

\[
\alpha_{\mu}^{h,g}(x) = \lim_{\delta \to 0} \frac{h(\mu(B_\delta(x)))}{g(\delta)}, \quad \underline{\alpha}_{\mu}^{h,g}(x) = \lim_{\delta \to 0} \frac{h(\mu(B_\delta(x)))}{g(\delta)}
\]
We will drop superscripts \( h, g \) from this notation whenever it is clear from the context.

Denote \( X_\alpha = \{ x \in \text{supp} \mu \mid \alpha_\mu(x) \geq \alpha \} \) and similarly for \( X^\alpha, \overline{X}_\alpha, \overline{X}^\alpha \) and denote \( f(\alpha) = \dim_{h,g}(X_\alpha \cap \overline{X}^\alpha), F(\alpha) = \dim_{h,g}(X_\alpha \cap \overline{X}^\alpha) \).

Let \( E \subseteq X, q, t \in \mathbb{R} \). Define

\[
\overline{\mathcal{H}}_{\mu,\delta}(E) = \inf \{ \sum_i h^{-1}(q h(\mu(B_r(x_i)))) + t g(r_i) \mid (B_r(x_i))_i \text{ is a centered } \delta\text{-cover of } E \}, E \neq \emptyset
\]

\[
\overline{\mathcal{H}}_{\mu,\delta}(\emptyset) = 0
\]

\[
\overline{\mathcal{H}}_{\mu,q}(E) = \sup_{\delta > 0} \overline{\mathcal{H}}_{\mu,\delta}(E)
\]

\[
\mathcal{H}_{\mu,q}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}_{\mu,q}(F)
\]

and similarly for the packing measure:

\[
\overline{\mathcal{P}}_{\mu,\delta}(E) = \sup \{ \sum_i h^{-1}(q h(\mu(B_r(x_i)))) + t g(r_i) \mid (B_r(x_i))_i \text{ is a centered } \delta\text{-packing of } E \}, E \neq \emptyset
\]

\[
\overline{\mathcal{P}}_{\mu,\delta}(\emptyset) = 0
\]

\[
\overline{\mathcal{P}}_{\mu,q}(E) = \inf_{\delta > 0} \overline{\mathcal{P}}_{\mu,\delta}(E)
\]

\[
\mathcal{P}_{\mu,q}(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \overline{\mathcal{P}}_{\mu,q}(E_i)
\]
Almost the same proof as in the usual case shows that these are metric outer measures. There are also numbers

$$\dim_{h,g}^q(E) = \sup \{ t \mid \mathcal{H}^q(E) = \infty \} = \inf \{ t \mid \mathcal{H}^q(E) = 0 \},$$

$$\dim_{h,g}^q(E) = \sup \{ t \mid \mathcal{P}^q(E) = \infty \} = \inf \{ t \mid \mathcal{P}^q(E) = 0 \}.$$

Let $b(q) = \dim_{h,g}^q(\text{supp } \mu)$, $B(q) = \dim_{h,g}^q(\text{supp } \mu)$. Clearly $b(q)$ and $B(q)$ are non-increasing and also we have

$$b(1) = 0, \quad b(0) = \dim_{h,g}(\text{supp } \mu)$$

$$B(1) = 0, \quad B(0) = \dim_{h,g}(\text{supp } \mu)$$

Define also

$$\underline{a} = \sup_{q>0} - \frac{b(q)}{q}, \quad \overline{a} = \inf_{q<0} - \frac{b(q)}{q}$$

$$\underline{A} = \sup_{q>0} - \frac{B(q)}{q}, \quad \overline{A} = \inf_{q<0} - \frac{B(q)}{q}$$

Proposition 4.1.1.

(i) $X^\alpha = \emptyset$ for $\alpha < \overline{A}$, (ii) $X^\alpha = \emptyset$ for $\alpha > \overline{a}$

(iii) $\overline{X}^\alpha = \emptyset$ for $\alpha > \overline{A}$, (iv) $\overline{X}^\alpha = \emptyset$ for $\alpha < \underline{a}$

44
Proof. (i) Let $\alpha < A$ and $x \in X^\alpha$. By the definition of $A$ there are numbers $\epsilon, q > 0$ such that $\alpha + \epsilon < -\frac{B(q)}{q}$. Let $t = -q(\alpha + \epsilon)$. Since $x \in X^\alpha$,

$$\lim_{r \to 0} \frac{h(\mu(B_r(x)))}{g(r)} \leq \alpha < \alpha + \epsilon$$

Hence there is a sequence $r_n \downarrow 0$ so that $0 < r_n < 1/n$ and

$$\frac{h(\mu(B_{r_n}(x)))}{g(r_n)} < \alpha + \epsilon$$

Thus

$$\overline{P}^{q,t}_{\mu,1/n}([x]) \geq h^{-1}(qh(\mu(B_{r_n}(x))) + tg(r_n))$$

$$\geq h^{-1}((q(\alpha + \epsilon) + t)g(r_n)) = h^{-1}(0) \quad (4.2)$$

It follows that $\overline{P}^{q,t}_{\mu}([x]) \geq h^{-1}(0)$, hence $\overline{P}^{q,t}_{\mu}([x]) \geq h^{-1}(0) > 0$ and so $-q(\alpha + \epsilon) = t \leq \text{Dim}_{h,q}([x]) \leq B(q)$ which contradicts the choice of $q$ and $\epsilon$.

(ii) Let $\alpha > \bar{\alpha}$ and $x \in X^\alpha$. By the definition of $\bar{\alpha}$ there are numbers $\epsilon > 0, q < 0$ such that $\alpha - \epsilon < \frac{-b(q)}{q}$. Let $t = -q(\alpha - \epsilon)$. Since $x \in X^\alpha$,

$$\lim_{r \to 0} \frac{h(\mu(B_r(x)))}{g(r)} \geq \alpha > \alpha - \epsilon$$

Hence there is $r_0 > 0$ such that for all $0 < r < r_0$ we have

$$\frac{h(\mu(B_{r_n}(x)))}{g(r_n)} \geq \alpha - \epsilon$$

Thus

$$h^{-1}(qh(\mu(B_{r_n}(x))) + tg(r_n)) \geq h^{-1}((q(\alpha - \epsilon) + t)g(r_n)) = h^{-1}(0)$$
It follows that
\[ \mathcal{H}_\mu^{q,t}(\{x\}) \geq \overline{\mathcal{H}}_\mu^{q,t}(\{x\}) \geq \overline{\mathcal{H}}_{\mu,\tau_0}^{q,t}(\{x\}) \geq h^{-1}(0), \]
and so \(-q(\alpha - \epsilon) = t \leq \dim_{h,q}^q(\{x\}) \leq b(q)\) which contradicts the choice of \(q\) and \(\epsilon\).

\[ \square \]

**Theorem 4.1.1.** \(f(\alpha) \leq \inf_q(\alpha q + b(q))\) for \(a \leq \alpha \leq \overline{a}\).

**Proof.** It is enough to show that if \(\alpha q + b(q) \geq 0\), then \(\dim_{h,q}(X^\alpha) \leq \alpha q + b(q)\) for \(q \geq 0\) and \(\dim_{h,q}(X_\alpha) \leq \alpha q + b(q)\) for \(q \leq 0\). For \(q = 0\) this is true, so let \(q > 0\) first.

Let
\[ A_n = \{ x \in X^\alpha \mid \frac{h(\mu(B_r(x)))}{g(r)} \leq \alpha + \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{n} \}. \]

Then \(A_n \uparrow X^\alpha\). For \(n \in \mathbb{N}\) and \(0 < \epsilon < 1/n\) let \((B_n(x_i))_i\) be a centered \(\epsilon\)-cover of \(A_n\). Then \(h(\mu(B_n(x_i))) \leq (\alpha + \delta/q)g(r)\), so
\[ \mathcal{H}_{\epsilon}^{\phi_{\alpha q + t + \delta}}(A_n) \leq \sum_i \phi_{\alpha q + t + \delta}(1/2 \text{ diam } B_r(x_i)) \leq \sum_i h^{-1}(-(\alpha q + t + \delta)g(r_i)) \]
\[ \leq \sum_i h^{-1}(qh(\mu(B_r(x_i))) + tg(r_i)) \]

Hence we have
\[ \mathcal{H}_{\epsilon}^{\phi_{\alpha q + t + \delta}}(A_n) \leq \overline{\mathcal{H}}_{\mu,q}^{q,t}(A_n) \text{ for } \epsilon < \frac{1}{n} \]

Letting \(\epsilon \to 0\), we have \(\mathcal{H}_{\epsilon}^{\phi_{\alpha q + t + \delta}}(A_n) \leq \overline{\mathcal{H}}_{\mu,q}^{q,t}(A_n) \leq \mathcal{H}_{\mu,q}^{q,t}(A_n)\) for all \(n\), hence
\[ \mathcal{H}_{\epsilon}^{\phi_{\alpha q + t + \delta}}(X^\alpha) \leq \overline{\mathcal{H}}_{\mu,q}^{q,t}(X^\alpha). \]

It follows that for any \(t > b(q)\) and any \(\delta > 0\)
\[ \alpha q + t + \delta \geq \dim_{h,q}(X^\alpha). \]
The second inequality is proved similarly, taking

\[ A_n = \{ x \in X_\alpha \mid \frac{h(\mu(B_r(x)))}{g(r)} \geq \alpha - \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{n} \}. \]

\[ \square \]

**Theorem 4.1.2.** \( F(\alpha) \leq \inf_q (q\alpha + B(q)) \) for \( \underline{a} \leq \alpha \leq \overline{a} \).

**Proof.** It is enough to show that if \( \alpha q + B(q) \geq 0 \), then \( \text{Dim}_h g(X_\alpha) \leq \alpha q + B(q) \) for \( q \geq 0 \) and \( \text{Dim}_h g(X_\alpha) \leq \alpha q + B(q) \) for \( q \leq 0 \). For \( q = 0 \) this is true, so let \( q > 0 \) first. Let

\[ A_n = \{ x \in X_\alpha \mid \frac{h(\mu(B_r(x)))}{g(r)} \leq \alpha + \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{n} \}. \]

For \( n \in \mathbb{N}, E \subseteq A_n \) and \( 0 < \varepsilon < 1/n \) let \( (B_r(x_i))_i \) be a centered \( \varepsilon \)-packing of \( A_n \). Then \( h(\mu(B_r(x_i))) \leq (\alpha + \delta/q)g(r) \), so

\[ \sum_i h^{-1}((\alpha q + t + \delta)g(r_i)) \leq \sum_i h^{-1}(qh(\mu(B_r(x_i)))) + tg(r_i)) \]

Then we have

\[ \overline{P}_{\varepsilon}^{\alpha q + t + \delta} (E) \leq \overline{P}_{\mu}^{\alpha q + t} (E) \text{ for } \varepsilon < \frac{1}{n} \]

Letting \( \varepsilon \to 0 \), we have \( \overline{P}_{\alpha q + t + \delta} (E) \leq \overline{P}_{\mu}^{\alpha q + t} (E) \) for all \( E \subseteq A_n \)

Now let \( A_n \subseteq \cup_i E_i \). Then

\[ \overline{P}_{\alpha q + t + \delta} (A_n) \leq \sum_i \overline{P}_{\alpha q + t + \delta} (A_n \cap E_i) \leq \sum_i \overline{P}_{\mu}^{\alpha q + t} (A_n) \]

\[ \leq \sum_i \overline{P}_{\mu}^{\alpha q + t} (A_n \cap E_i) \leq \sum_i \overline{P}_{\mu}^{\alpha q + t} (E_i) \]
Hence $\mathcal{P}^{q,t}(A_n) \leq \mathcal{P}_\mu^{q,t}(A_n)$ for all $n$. Since $\overline{X}^\alpha = \cup_n A_n$, this is also true for $\overline{X}^\alpha$. It follows that for any $t > B(q)$ and any $\delta > 0$

$$aq + t + \delta \geq \dim_{h\mu}(\overline{X}^\alpha).$$

The second inequality is proved similarly, taking

$$A_n = \{ x \in X^\alpha \mid \frac{h(\mu(B_r(x)))}{g(r)} \geq \alpha - \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{n} \}.$$

It is useful to define a "nice" class of measures analogous to $\mathcal{P}_0(X)$ for the usual multifractal case. Let

$$\mathcal{P}_h(X) = \{ \mu \in \mathcal{P}(X) \mid \exists a > 1 : \forall x \in \text{supp } \mu : \lim_{r \to 0} (h(\mu(B_r(x))) - h(\mu(B_{ar}(x)))) < \infty \}$$

Clearly if the limit here is finite for some $a > 1$, it is also finite for all $a > 1$.

To prove the next result, we will need the following variant of the covering theorem for general metric spaces:

**Theorem 4.1.3.** Let $X$ be a separable metric space and $B$ a family of closed balls in $X$ such that

$$\sup\{ \text{diam}(B) \mid B \in B \} < \infty.$$

Then there is a finite or countable sequence $B_{r_i}(x_i) \in B$ of disjoint balls such that

$$\bigcup_{B \in B} B \subseteq \bigcup_i B_{5r_i}(x_i)$$
Proof. See [Mat95], Thm. 2.1 and discussion. □

**Theorem 4.1.4.** If $X$ is a separable metric space, then for any $\mu \in \mathcal{P}(X)$, $E \subseteq X$ we have $\dim_{h, g}^q(E) \leq \dim_{h, g}^q(\mathcal{E})$ for $q \leq 0$. If $\mu \in \mathcal{P}_h(X)$, then this inequality also holds for $q > 0$.

**Proof.** First let $q > 0$ and $\mu \in \mathcal{P}_h(X)$. Let $E \in X$. For $n \in \mathbb{N}$ define

$$E_n = \{x \in E \mid h(\mu(B_r(x))) - h(\mu(B_{5r}(x))) < n \text{ for } 0 < r < \frac{1}{n}\}$$

Fix some $n \in \mathbb{N}$ and let $F \subseteq E_n$. We will show first that $\overline{\mathcal{P}}_{\mu, t}^q(F) \leq \overline{\mathcal{P}}_{\mu, t}^q(F)$.

We may assume that $\overline{\mathcal{P}}_{\mu, t}^q(F) < \infty$. Let $\delta > 0$. Let

$$B = \{B_r(x) \mid x \in F, r < \min(\delta/5, 1/n)\}.$$ 

Then by Theorem 4.1.3 we can find a finite or countable subfamily of disjoint balls $(B_{r_i}(x_i)) \subseteq B$ such that

$$\bigcup_{B \in B} B \subseteq \bigcup_{i} B_{5r_i}(x_i)$$

First let $t \leq 0$. Then $tg(r_i) \leq tg(5r_i)$ and so

$$\overline{\mathcal{P}}_{\mu, \delta}^q(F) \geq \sum_i h^{-1}(qh(\mu(B_{r_i}(x_i))) - t \log r_i)$$

$$\geq \sum_i h^{-1}(qh(\mu(B_{5r_i}(x_i))) + qn + tg(5r_i))$$

Let $t > 0$. Then by property (iv) and continuity of $g(x)$ there is $M > 0$ such that $g(r_i) - g(5r_i) < M$ for all $r_i$. Hence we have
\[
\overline{P}_{\mu,\delta}^{q,t}(F) \geq \sum_i h^{-1}(qh(\mu(B_{r_i}(x_i))) + qn + tg(r_i) + tM)
\]

It is clear from (iii) that \(\lim_{x \to \infty} \frac{h^{-1}(kx + a)}{h^{-1}(x)} = 0\) for any \(a \in \mathbb{R}\) and \(k > 1\). Hence in the definitions of generalized Hausdorff and packing measures we can use functions \(h^{-1}(qh(\mu(B_{r_i}(x_i))) + tg(r_i) + a)\) without changing \(\dim_h^q\) and \(\dim_h^q\). We will use notation \(\mathcal{H}'\) and \(\mathcal{P}'\) for such modified measures.

Thus both in case \(t < 0\) and \(t > 0\) we have

\[
\overline{P}_{\mu,\delta}^{q,t}(F) \geq \overline{H}_{\mu,\delta}'(F),
\]

where \(\mathcal{H}'\) is of course denoting different measures in different cases. Letting \(\delta \to 0\), we have

\[
\overline{P}_{\mu}^{q,t}(F) \geq \overline{H}_{\mu}'(F) \text{ for all } F \subseteq E_n.
\]

Let \(E_n \subseteq \cup_i F_i\). Then

\[
\mathcal{H}_{\mu}'(E_n) = \mathcal{H}_{\mu}'(\cup_i (F_i \cap E_n)) \leq \sum_i \mathcal{H}_{\mu}'(F_i \cap E_n)
\]

\[
\leq \sum_{i} \sup_{F \subseteq F_i \cap E_n} \overline{H}_{\mu}^{q,t}(F) \leq \sum_{i} \sup_{F \subseteq F_i \cap E_n} \overline{P}_{\mu}^{q,t}(F) \leq \sum_{i} \overline{P}_{\mu}^{q,t}(F_i)
\]

Hence \(\mathcal{H}_{\mu}'(E_n) \leq \mathcal{P}_{\mu}'(E_n)\) for all \(n\), and since \(E_n \uparrow E\), we have

\[
\mathcal{H}_{\mu}'(E) \leq \mathcal{P}_{\mu}'(E)
\]

It follows that \(\dim_h^q(E) \leq \dim_h^q(E)\).
For \( q < 0 \) the argument is similar, except that we do not need \( \mu \in \mathcal{P}_h(X) \) because
\[ qh(\mu(B_{r_i}(x_i))) \leq qh(\mu(B_{sr_i}(x_i))) \quad \text{for all } x \in E, \] and also \( \text{Dim}_{h,q}^t(E) > 0 \) if \( q < 0 \), so there is no need to consider case \( t < 0 \).

The usual multifractal formalism corresponds to \( h(x) = g(x) = -\log x \). We will consider also one more example - a “second order” multifractal formalism which corresponds to \( h(x) = \log \log(1/x), g(x) = -\log x \). For the sake of convenience we will use powers of 2 rather than powers of \( e \), resulting in the following definitions:

\[
\phi_s(x) = 2^{-s-x}
\]

\[
\alpha_\mu(x) = \lim_{\delta \to 0} \frac{\log \log(1/\mu(B_\delta(x)))}{\log(1/\delta)}, \quad \overline{\alpha}_\mu(x) = \lim_{\delta \to 0} \frac{\log \log(1/\mu(B_\delta(x)))}{\log(1/\delta)}
\]

We will denote “second order” Hausdorff and packing dimension by

\[
\text{dim}_2(E) = \sup\{s > 0 \mid \mathcal{H}^{\phi_s}(E) = \infty\} = \text{inf}\{s > 0 \mid \mathcal{H}^{\phi_s}(E) = 0\},
\]

\[
\text{Dim}_2(E) = \sup\{s > 0 \mid \mathcal{P}^{\phi_s}(E) = \infty\} = \text{inf}\{s > 0 \mid \mathcal{P}^{\phi_s}(E) = 0\}
\]

and the multifractal measures are computed by

\[
\overline{\mathcal{H}}_{h,\delta}^{q,t}(E) = \inf\left\{ \sum_i 2^{-\left(\log(1/\mu(B_{r_i}(x_i)))\right)\gamma_r^{-t}} \mid (B_{r_i}(x_i))_i \text{ is a centered} \delta - \text{cover of } E, \ E \neq \emptyset \right\}
\]

\[
\underline{\mathcal{H}}_{h,\delta}^{q,t}(E) = \sup\left\{ \sum_i 2^{-\left(\log(1/\mu(B_{r_i}(x_i)))\right)\gamma_r^{-t}} \mid (B_{r_i}(x_i))_i \text{ is a centered} \delta - \text{packing of } E, \ E \neq \emptyset \right\}
\]
Remark. The results of Chapter 3, namely, Theorems 3.2.1, 3.2.2, 3.2.3 and 3.2.4 are still true in case of a generalized local dimension. It is easy to see that their proofs work for the generalized local dimensions $\alpha_{h, g}^\mu(x)$ and $\overline{\alpha}_{h, g}^\mu(x)$ if we use properties (i)-(iv) of functions $h, g$ and change the definitions in section 3.1 correspondingly; for example, upper box dimension is defined as

$$\overline{\dim}_B^{h, g}(E) = \lim_{\delta \to 0} \frac{h(1/N_\delta(E))}{g(\delta)},$$

definition 3.5 becomes

$$\overline{C}_L^{h, g}(\mu) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{h(1/N_{\mu}(\epsilon, \delta))}{g(\epsilon)},$$

and so on.

4.2 Generalized Multifractal Description of a Hyperspace of Compact Subsets

Given a metric space $X$ with metric $d$, the hyperspace $\mathcal{K}(X)$ is defined to be the set of nonempty compact subsets of $X$. The Hausdorff metric on this space is defined as follows. For $A \in X$ let the $\delta$-neighborhood of $A$ be

$$A_\delta = \{ y \mid d(x, y) < \delta \text{ for some } x \in A \}.$$  

Then for $A, B \in \mathcal{K}(X)$ let

$$\bar{d}(A, B) = \inf\{ \delta \mid A \subseteq B_\delta \text{ and } B \subseteq A_\delta \}.$$  

It is known that if $X$ is compact then $\mathcal{K}(X)$ is also compact (see e.g. [Edg90], p.67).
Let $E \in \mathbb{R}^d$ be a self-similar fractal which is an invariant set for some iterated function system $f_1, \ldots, f_m$ with similarity ratios $r_1 = \cdots = r_m = r$ satisfying the open set condition (see Section 2.3). We will investigate the second-order multifractal formalism for the hyperspace $\mathcal{K}(E)$. Let $\Omega = \{1, \ldots, m\}^N$ with the metric defined by $r_1 = \cdots = r_m = r$ be a coding space for $E$ and let $\mathcal{K}(\Omega)$ be the hyperspace of compact subsets of $\Omega$. Denote by $\hat{h} : \Omega \to E$ the natural coding map and by $h : \mathcal{K}(\Omega) \to \mathcal{K}(E)$ the corresponding map for hyperspaces defined by $h(X) = \{\hat{h}(x) | x \in X\}$.

Let $\Omega_k$ be the set of all finite strings of length $k$, and for any $A \subseteq \Omega_k$ define a $k$-set $\tilde{A} = \{C \in \mathcal{K}(\Omega) | \{\alpha \in \Omega_k | [\alpha] \cap C \neq \emptyset\} = A\}$. Denote also $\tilde{A} = h(\tilde{A})$. Then clearly $\text{diam} \tilde{A} = r^k$. For any $X \in \mathcal{K}(E)$ and any $k$ we can find a $k$-set $\tilde{A}_k(X)$ corresponding to $X$ by choosing $A = \{\alpha \in \Omega_k | h([\alpha]) \cap X \neq \emptyset\}$.

Construct a probability measure $\mu$ on $\mathcal{K}(\Omega)$ in the following way. Let $\mu(\mathcal{K}(\Omega)) = 1$ and for any $k$-set $\tilde{A}$ distribute its measure evenly among all $(k+1)$-sets $\tilde{B} \subset \tilde{A}$. Suppose $A = \{\alpha_1, \ldots, \alpha_n\}$; then $B$ contains at least 1 and at most $m$ descendants of each $\alpha_i$, so there are $(2^m - 1)^n$ such sets $\tilde{B}_i$ and $\mu(\tilde{B}_i) = (2^m - 1)^{-n}\mu(\tilde{A})$ for each of them. It follows that for any $(k+1)$-set $\tilde{B}$ we have $\mu(\tilde{B}) = \gamma^{n_1 + \cdots + n_k}$. where $\gamma = (2^m - 1)^{-1}$ and $n_i$ is a number of different strings that we get considering the first $i$ letters in strings included in the underlying set $B$. This measure corresponds to a measure $\mu$ on $\mathcal{K}(E)$, defined by $\mu(F) = \mu(h^{-1}(F))$.

Consider now the case of pairwise disjoint construction, that is, where all sets $f_1(E), \ldots, f_m(E)$ are disjoint. Let $X \in \mathcal{K}(E)$ and let us explore the behavior of $\mu(B_\epsilon(X))$ as $\epsilon \to 0$. Choose $k_1$ so that $r^{k_1} \leq \epsilon < r^{k_1-1}$. Consider a $k_1$-set
\( \tilde{A}_{k_1}(X) \) corresponding to \( X \). Then any set \( Y \in \tilde{A}_{k_1}(X) \) lies at a distance of at most \( \text{diam} \tilde{A}_{k_1}(X) = r^{k_1} \) from \( X \), so \( \tilde{A}_{k_1}(X) \subseteq B_\varepsilon(X) \).

Now let \( c \) be the smallest distance between sets obtained on the first step of constructing \( E \) and let \( k_2 \) be such that \( c^{k_2-1} \leq \varepsilon < c^{k_2} \). Since the distance between basic sets on the \( k \)th step is at least \( c^k \), we have \( B_\varepsilon(X) \subseteq \tilde{A}_{k_2}(X) \).

Hence we have

\[
\frac{\log \log(1/\mu(\tilde{A}_{k_2}(X)))}{-\log(c/r)r^{k_2}} \leq \frac{\log \log(1/\mu(B_\varepsilon(X)))}{\log(1/\varepsilon)} \leq \frac{\log \log(1/\mu(\tilde{A}_{k_1}(X)))}{-\log r^{k_1}}
\]

and so

\[
\lim_{\varepsilon \to 0} \frac{\log \log(1/\mu(B_\varepsilon(X)))}{\log(1/\varepsilon)} = \lim_{k \to \infty} \frac{\log \log(1/\mu(\tilde{A}_k(X)))}{-\log r^k}
\]

\[
= \lim_{k \to \infty} \frac{\log \log(1/\mu(\tilde{A}_k(X)))}{-\log r^k}
\]

\[
= \lim_{k \to \infty} \frac{\log(n_1(X) + \cdots + n_k(X))}{-k \log r}
\]

whenever one of these limits exists.

Denote

\[
\overline{\beta}(X) = \lim_{k \to \infty} \frac{\log(n_1(X) + \cdots + n_k(X))}{-\log r^k}
\]

\[
\underline{\beta}(X) = \lim_{k \to \infty} \frac{\log(n_1(X) + \cdots + n_k(X))}{-\log r^k}
\]

**Proposition 4.2.1.** For any \( X \in E \) we have

\[
\overline{\beta}(X) = \overline{\dim}_B X, \quad \underline{\beta}(X) = \underline{\dim}_B X
\]
Proof. Consider basic sets \( h(\sigma), \sigma \in \Omega^* \). It is clear that we can compute the box dimensions using covers of these sets only. The minimal number of such sets of radius \( r^k \) necessary to cover \( X \), is exactly \( n_k(X) \), so

\[
\dim_B X = \lim_{k \to \infty} \frac{\log n_k(X)}{-k \log r^k}, \quad \dim_B X = \lim_{k \to \infty} \frac{\log n_k(X)}{-k \log r^k}
\]

The result follows immediately from the fact that \( n_k(X) < n_1(X) + \cdots + n_k(X) \leq kn_k(X) \). \( \Box \)

In particular, if \( \beta(X) < d \), then \( X \) is nowhere dense. Note that by the result of Gruber [Gru89], for most sets in the sense of category we have \( \dim_B X = d, \dim_B X = 0 \), but clearly it is not so in the sense of measure.

Now for \( \alpha \geq 0 \) let

\[
E_\alpha = \{ X \in \mathcal{K}(E) \mid \lim_{k \to \infty} \frac{\log(n_k(X))}{-k \log r} = \alpha \}, \quad (4.3)
\]

where \( n_i(X) \) is the number of different strings of length \( i \) that can begin strings included in \( h^{-1}(X) \). In the disjoint case, we have

\[
E_\alpha = \{ X \in \mathcal{K}(E) \mid \lim_{k \to \infty} \frac{\log \log(1/\mu(B_r(X)))}{\log(1/\epsilon)} = \alpha \}
\]

Since \( n_k \leq m^k \), \( E_\alpha = \emptyset \) for \( \alpha > s = -\log m/\log r \).

Now let us compute the multifractal spectra of \( \mathcal{K}(E) \). We will use the following lemma which is proven in [McC97].

Lemma 4.2.1. For a separable metric space \( X \) with a positive Borel measure \( \mu, x \in X \) and \( \delta > 0 \) let

\[
\mu_\delta(x) = \sup \{ \mu(U) : x \in U \text{ and } \text{diam}(U) \leq \delta \}.
\]
Let $\delta_k \downarrow 0$. Suppose that $\phi, \psi \in \Phi$ satisfy $\phi(\delta_k) \leq A\psi(\delta_{k+1})$ for all $k \in \mathbb{N}$. Let $E \subseteq X$ be a Borel set which satisfies $\mu(E) > 0$ and

$$\overline{D}_\mu(x, (\delta_k)_k) \equiv \lim_{k \to 0} \frac{\mu_\delta(x)}{\phi(\delta_k)} < M < \infty \text{ for every } x \in E.$$ 

Then $\mathcal{H}^\psi(E) \geq \frac{\mu(E)}{M} > 0$.

**Theorem 4.2.1.** Let $E$ be a self-similar fractal described above and let $E_\alpha$ be as in (4.3). Then

$$\dim_2(E_\alpha) = \text{Dim}_2(E_\alpha) = \alpha \quad \text{for } 0 < \alpha < s.$$ 

If $f_1(E), \ldots, f_m(E)$ are disjoint, this means $f(\alpha) = F(\alpha) = \alpha$ for $0 < \alpha < s$.

**Proof.** The proof consists of two parts.

(i) For any $\alpha_0 > \alpha > 0$ we have $\dim_2 E_\alpha \leq \alpha_0$.

Let $\alpha_0 > \alpha$ and pick some $t, \alpha < t < \alpha_0$. Let

$$E^K_\alpha = \{ X \in E_\alpha \mid n_k(X) < r^{-kt} \text{ for all } k > K \}.$$ 

Then $E^K_\alpha \uparrow E_\alpha$.

Take any $K$ and let $0 < \delta < r^K$. Let $\mathcal{B} = (B_{r_i}(x_i))_i$ be a centered $\delta$-packing of $E^K(\alpha)$. Consider subpackings

$$\mathcal{B}^k = \{ B_{r_i}(x_i) \in \mathcal{B} \mid r^{k+1} < r_i \leq r^k \} \text{ for } k \in \mathbb{N}.$$ 

Then if $B_{r_i}(x_i) \in \mathcal{B}^k$, we have $B_{r_i}(x_i) \supseteq \tilde{A}_{k+1}(x_i)$ and hence the number of balls in $\mathcal{B}^k$ is at most the number of different $(k + 1)$-sets with $n_k < r^{-kt}$ for any $k > K$. Denote
this number by $C(k)$. The number of possible descendants of a particular $k$-set $\tilde{A}$ is $(2^{m} - 1)^{n_{k}(\tilde{A})}$, so for the number of balls in $B^{k}$ we have the estimate

$$C(k) < C(k - 1)2^{m_{r}^{-1}}2^{kt}$$

$$< \cdots < C(K)2^{m_{r}^{-1}}2^{kt} \text{ where } M = m_{r}^{-1}$$

$$< 2^{m_{K}r^{-Kt}2^{M_{r}^{-1}}} \text{ since } n_{i} \leq n_{K} \text{ for } i < K.$$  

It follows that

$$\sum_{Br, (z_{i}) \in B} \phi_{\alpha_{0}}(r_{i}) \leq \sum_{K}^{\infty} \sum_{Br, (z_{i}) \in B^{k}} 2^{-r_{i}^{-\kappa\alpha_{0}}}$$

$$\leq \sum_{K}^{\infty} 2^{m_{K}r^{-Kt}2^{M_{r}^{-1}}}2^{-r_{i}^{-\kappa\alpha_{0}}}$$

$$\leq \sum_{K}^{\infty} 2^{(K+M)t^{-1}}2^{-r_{i}^{-\kappa\alpha_{0}}} \rightarrow 0 \text{ as } K \rightarrow \infty \text{ since } t < \alpha_{0}.$$  

Hence

$$P^{\phi_{\alpha_{0}}}(E_{a}^{K}) \leq P^{\phi_{\alpha_{0}}}(E_{a}^{K}) \leq \overline{P}^{\phi_{\alpha_{0}}}(E_{a}^{K}) < S$$  

for some constant $S$ which does not depend on $K$, and so $P^{\phi_{\alpha_{0}}}(E_{a}) < S$ which proves the first part.

(ii) For any $0 < \alpha_{0} < \alpha < s$, we have $\dim_{2} E_{a} \geq \alpha_{0}.$

Consider an auxiliary measure $\nu$, constructed in the same manner as $\mu$ but with distributing measure only among those $(k + 1)\text{-subsets of a } k\text{-set } \tilde{A} \text{ for which } n_{k+1} = [r^{-1}(k+1)^{\alpha}], \text{ and define } \nu(F) = \nu(h^{-1}(F)). \text{ Then supp } \nu \subset E_{a}. \text{ For any } (k+1)\text{-set } \tilde{A} \text{ which is a descendant of a } k\text{-set } \tilde{B} \text{ we have } \nu(\tilde{A}) = 1/C(n_{k}, n_{k+1})\nu(\tilde{B}), \text{ where } C(n_{k}, n_{k+1}) \text{ is a number of possible ways to choose } n_{k+1} \text{ descendants of } n_{k}.
strings so that there is at least one descendant of each of them. Hence \( \check{\nu}(A) = 1/(C(n_0, n_1)C(n_1, n_2), \ldots, C(n_k, n_{k+1})). \) To estimate this, notice that

\[
C(n_k, n_{k+1}) \geq \frac{(m-1)n_k}{n_{k+1} - n_k} = \frac{(m-1)n_k}{(n_{k+1} - n_k)!(m n_k - n_{k+1})!} \times \frac{(m-1)n_{k+1/2}}{(m-1)n_k^{m-1} n_{k+1/2}}
\]

(by Stirling’s formula)

\[
= \frac{(m-1)^{m-1} n_{k+1/2}}{(2\pi n_k)^{1/2} (n_{k+1}/n_k - 1)^{m-1} n_{k+1/2} (m - n_{k+1}/n_k) m n_k - n_{k+1} + 1/2}
\]

Hence, letting \( M \) denote the expression in brackets, we have

\[
\check{\nu}(A) \leq C^{-k} K_1 M^{-K_2 r^{-\alpha} r^{-k(k-1)/4}},
\]

and so

\[
\frac{\nu(A)}{2^{-r^{-\alpha} - 0}} \to 0 \text{ for } \alpha_0 < \alpha
\]

and the result follows by Lemma 4.2.1.

\( \square \)

**Theorem 4.2.2.** Let \( E \) be a self-similar fractal described above. Then for \( K(E) \) we have

\[
b(\mu) = B(\mu) = \begin{cases} s(1-q) & \text{if } q \leq 0, \\ 0 & \text{if } q \geq 1 \end{cases},
\]

\[
B(\mu) \geq b(\mu) \geq s(1-q) \text{ if } 0 < q < 1
\]
Proof. We know that \( b(1) = B(1) = 0 \). Let \( q > 1 \). By monotonicity \( b(q) \leq 0 \), \( B(q) \leq 0 \) for \( q > 1 \), so it is enough to show that for any \( t < 0 \) we have \( \mathcal{H}_{\mu, \delta}^{q,t}(F) > 0 \) for some \( F \subseteq \text{supp} \mu, \delta > 0 \).

Let \( F = \{X\} \) for some \( X = \{x\}, x \in \Omega \). For any ball \( B_\delta(X) \) pick \( k \) so that \( r^k \leq \delta < r^{k-1} \). Then this ball contains a \( k \)-set \( \tilde{A}_k(X) \), and \( \tilde{A}_k(X) \) is defined by one cylinder \( [x|k] \) so that \( n_1(X) + \cdots + n_k(X) = k \). Hence

\[
\log(1/\mu(B_\delta(X))) \leq \log(1/\mu(\tilde{A}_k(X))) = k \log(1/\gamma),
\]

and the last expression approaches 1 as \( k \to \infty \), so for a small enough \( \delta \) we have \( \mathcal{H}_{\mu, \delta}^{q,t}(\{X\}) > 0 \).

Since the space \( \mathcal{K}(\Omega) \) is separable, by Theorem 4.1.4 we have \( B(q) = 0 \) for \( q \geq 1 \).

Now it follows from Theorem 4.1.1 that

\[
b(q) \geq \sup_\alpha (f(\alpha) - q\alpha)
\]

Since \( f(s) = s \), we have \( s(1 - q) \leq b(q) \leq B(q) \) for all \( q \). To finish the proof, it is enough to show that \( B(q) \leq s(1 - q) \) for \( q < 0 \).

Let \( q < 0 \). It is enough to prove that \( \mathcal{P}_{\mu}^{q,t}(\text{supp} \mu) < \infty \) for \( t > s(1 - q) \). Let \( t = s(1 - q) + \epsilon \) for some \( \epsilon > 0 \). Take any \( \delta > 0 \) and any centered \( \delta \)-packing of \( \mathcal{K}(\Omega) \) by balls \( B = (B_r(x_i))_i \). Consider subpackings

\[
B^k = \{B_r(x_i) \in B \mid r^{k+1} < r_i \leq r^k\} \text{ for } k \in \mathbb{N}.
\]
Then if $B_{r_i}(x_i) \in B^k$, we have $B_{r_i}(x_i) \supseteq \tilde{A}_{k+1}(x_i)$ and hence

$$\log(1/\mu(B_{r_i}(x_i))) \leq \log(1/\mu(\tilde{A}_{k+1}(x_i))) = (n_1(x_i) + \cdots + n_k(x_i)) \log(1/\gamma)$$

$$< 2m^k \log(1/\gamma) \quad \text{since } n_j \leq m^j. \quad (4.4)$$

Hence for $q < 0$, using the fact that $m^k = r^{-ks}$, we have

$$\sum_{B_{r_i}(x_i) \in B^k} 2^{-(\log(1/\mu(B_{r_i}(x_i)))q + t)} \leq \sum_{B_{r_i}(x_i) \in B^k} 2^{-(\log(1/\mu(\tilde{A}_{k+1}(x_i)))q + t)}$$

$$\leq \sum_{B_{r_i}(x_i) \in B^k} 2^{-(2\log(1/\gamma))qm^k + t}$$

$$\leq \sum_{B_{r_i}(x_i) \in B^k} 2^{-Mr^{-ks}q + t} \quad \text{where } M = (2\log(1/\gamma))^q$$

$$= \sum_{B_{r_i}(x_i) \in B^k} 2^{-Mr^{-k(s+e)}}$$

$$\leq 2^{-Mr^{-k(s+e)-1}} 2m^k \quad \text{since there are at most } 2m^k - 1 \text{ different } k \text{- sets}$$

$$= 2^{-Mr^{-k(s+e)}} 2m^k$$

It follows that

$$\sum_{B_{r_i}(x_i) \in B} 2^{-(\log(1/\mu(B_{r_i}(x_i)))q + t)} \leq \sum_{k=0}^{\infty} 2^{-Mr^{-k(s+e)} + k\gamma s} \leq \sum_{k=1}^{\infty} 2^{-r^{-ks}} < \infty,$$

and the last expression does not depend on the packing, so $\overline{p}_{q,t} < \infty$. \hfill \Box

So in this case, since $f(\alpha) = \alpha$ for $0 \leq \alpha \leq s$, $f(\alpha)$ is exactly the Legendre transform of $b(q)$. The fact that $b(q) = 0$ for $q > 1$ indicates the presence of subsets of points where $\mu(B_r(X))$ decreases slower than $2^{-1/r^e}$. In fact, at all points $X$ of
$\mathcal{K}(\Omega)$ which are finite sets of $\Omega$, $\mu$ has a finite usual local dimension. Indeed, for $X = \{x_1, \ldots, x_l\}$ we have $n_k(X) = l$ for large enough $k$, so

$$\lim_{k \to \infty} \frac{\log(\mu(A_k(X)))}{\log r^k} = l \log \gamma / \log r.$$


