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A STRESS BASED THEORY DESCRIBING THE COUPLED THERMOELASTIC BEHAVIOR OF LAMINATED PLATES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Bryan Carl Foos, B.S., M.S.

* * * * *

The Ohio State University
1999

Dissertation Committee:

Dr. William E. Wolfe, Advisor
Dr. Henry R. Busby
Dr. Robert M. Sykes

Approved by

Advisor

Department of Civil Engineering
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1999
ABSTRACT

Under service conditions, advanced composite materials are often subjected to combined mechanical and thermal environments which may lead to unacceptable deformations or cause internal structural damage to the material. A fully coupled stress based discrete layer plate theory of a monoclinic material which accounts for combined mechanical and thermal loads is developed. Assuming small strain and isothermal material properties, the effects of temperature changes are confined to heat conduction. The theoretical model incorporates realistic thermomechanical stress distributions within and across each lamina, as well as the coupling contributions of the deformation/stress field and vice versa. The stress based plate theory is formulated on an assumed in-plane stress distribution though the lamina thickness. This results in the transverse shear and normal stress distributions being quadratic and cubic, respectively, over the lamina thickness. A variational formulation of the governing equations is developed. The governing equations coupled with thermal effects are restated in a self-adjoint form and generalized for a laminated plate including continuous displacements, temperatures, and transverse stress at the laminae interfaces. The governing function is specialized for certain problems of interest.
Dedicated to my Family and Friends
ACKNOWLEDGMENTS

I wish to express my sincere gratitude to my advisor Dr. William E. Wolfe for his encouragement and guidance throughout the development of this work. The support provided by Dr. Ranbir Singh Sandhu in my graduate studies and this work is greatly appreciated. Sincere thanks are extended to Dr. Henry R. Busby and Dr. Robert M. Sykes for reviewing the manuscript and providing their valuable suggestions and comments.

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## VITA

<table>
<thead>
<tr>
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<tr>
<td>November 2, 1965</td>
<td>Born - Fremont, Ohio, USA</td>
</tr>
<tr>
<td>1988</td>
<td>B.S Civil Engineering</td>
</tr>
<tr>
<td></td>
<td>The Ohio State University, Columbus, Ohio</td>
</tr>
<tr>
<td>1990</td>
<td>M.S. Civil Engineering</td>
</tr>
<tr>
<td></td>
<td>The Ohio State University, Columbus, Ohio</td>
</tr>
<tr>
<td>1990-1994</td>
<td>Aerospace Engineer</td>
</tr>
<tr>
<td></td>
<td>Wright Laboratory, Flight Dynamics Directorate</td>
</tr>
<tr>
<td></td>
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</tr>
<tr>
<td>1994-Present</td>
<td>Materials Engineer</td>
</tr>
<tr>
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</tbody>
</table>
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Major Field: Civil Engineering

Structural and Computational Mechanics, Mechanics of Composite Materials.

Structural Dynamics

Minor Field: Applied Mathematics
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iii</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>iv</td>
</tr>
<tr>
<td>Vita</td>
<td>vi</td>
</tr>
<tr>
<td>List of Figures</td>
<td>xii</td>
</tr>
<tr>
<td>Nomenclature</td>
<td>xiii</td>
</tr>
<tr>
<td>Chapters:</td>
<td></td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. Literature Review</td>
<td>6</td>
</tr>
<tr>
<td>2.1 Theory of Laminated Plates Including Thermal Effects</td>
<td>9</td>
</tr>
<tr>
<td>2.1.1 Displacement Based Models</td>
<td>10</td>
</tr>
<tr>
<td>2.1.1.1 Classical Plate Theory</td>
<td>11</td>
</tr>
<tr>
<td>2.1.1.2 First Order Shear Deformation</td>
<td>15</td>
</tr>
<tr>
<td>2.1.1.3 Higher Order Shear Deformation</td>
<td>19</td>
</tr>
<tr>
<td>2.1.1.4 Discrete Layer Theory</td>
<td>22</td>
</tr>
<tr>
<td>2.1.1.5 Other Displacement Based Approaches</td>
<td>26</td>
</tr>
<tr>
<td>2.1.1.6 Stress Based Theory</td>
<td>32</td>
</tr>
<tr>
<td>2.2 Heat Transfer Models</td>
<td>36</td>
</tr>
<tr>
<td>3. Derivation of Lamina Stresses</td>
<td>44</td>
</tr>
<tr>
<td>3.1 Equilibrium Equations of a Thermoelastic Body</td>
<td>44</td>
</tr>
<tr>
<td>3.2 Heat Transfer Models</td>
<td>44</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>------------------------------------------------------------</td>
</tr>
<tr>
<td>3.2</td>
<td>Thermoelastic Constitutive Law</td>
</tr>
<tr>
<td>3.3</td>
<td>Kinematics</td>
</tr>
<tr>
<td>3.4</td>
<td>Plate Equations of Equilibrium</td>
</tr>
<tr>
<td>3.5</td>
<td>Derivation of a Consistent Stress Field</td>
</tr>
<tr>
<td>4.</td>
<td>The Governing Field Equations</td>
</tr>
<tr>
<td>4.1</td>
<td>Variational Formulation</td>
</tr>
<tr>
<td>4.1.1</td>
<td>Restatement of the Governing Equations</td>
</tr>
<tr>
<td>4.2</td>
<td>Generalized Variational Formulation</td>
</tr>
<tr>
<td>4.3</td>
<td>Governing Field Equations</td>
</tr>
<tr>
<td>4.3.1</td>
<td>Constitutive Equations</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Equations of Equilibrium</td>
</tr>
<tr>
<td>4.3.3</td>
<td>Interface Equations</td>
</tr>
<tr>
<td>5.</td>
<td>Self-Adjoint Form of the Governing Equations</td>
</tr>
<tr>
<td>5.1</td>
<td>Basic Variational Principles</td>
</tr>
<tr>
<td>5.1.1</td>
<td>Initial Boundary Value Problem</td>
</tr>
<tr>
<td>5.1.2</td>
<td>Bilinear Mapping</td>
</tr>
<tr>
<td>5.1.3</td>
<td>Self-Adjoint Operator</td>
</tr>
<tr>
<td>5.1.4</td>
<td>Gateaux Differential of a Function</td>
</tr>
<tr>
<td>5.1.5</td>
<td>Basic Variational Problem</td>
</tr>
<tr>
<td>5.1.6</td>
<td>Coupled Problems</td>
</tr>
<tr>
<td>5.2</td>
<td>Self-Adjoint Form of Equations</td>
</tr>
<tr>
<td>5.2.1</td>
<td>Weighted Displacement Definitions</td>
</tr>
<tr>
<td>5.2.2</td>
<td>Constitutive Equations and Equations of Equilibrium</td>
</tr>
<tr>
<td>5.2.3</td>
<td>Interface Displacement Conditions</td>
</tr>
<tr>
<td>5.3</td>
<td>Integral Form of the Equations</td>
</tr>
<tr>
<td>5.3.1</td>
<td>Field Equations</td>
</tr>
<tr>
<td>5.3.2</td>
<td>Interfacial Continuity Conditions</td>
</tr>
<tr>
<td>6.</td>
<td>Operator Form of Governing Equations</td>
</tr>
<tr>
<td>6.1</td>
<td>Governing Equations</td>
</tr>
<tr>
<td>6.2</td>
<td>Adjointness of the Operator Matrices</td>
</tr>
<tr>
<td>6.2.1</td>
<td>Operator Matrix ([A]^k)</td>
</tr>
<tr>
<td>6.2.2</td>
<td>Operator Matrix ([\Xi]^k)</td>
</tr>
<tr>
<td>6.2.3</td>
<td>Operator Matrices ([B]^k) and ([\overline{B}]^k)</td>
</tr>
<tr>
<td>6.2.4</td>
<td>Operator Matrices ([C]^k) and ([\overline{C}]^k)</td>
</tr>
<tr>
<td>6.2.5</td>
<td>Operator Matrices ([A]^k) and ([\overline{A}]^k)</td>
</tr>
</tbody>
</table>
6.3 Consistent Boundary Operators ............................................ 117
6.4 Prescribed Boundary Conditions .......................................... 132

7. Generalized Variational Formulation ........................................ 135

7.1 Variational Formulation of the Self-Adjoint Problem ............. 135
7.2 Proof for Vanishing of the Gateaux Differential .................... 145
  7.2.1 Gateaux Differential with Respect to $\tilde{V}_\alpha^{(k)}$ ....... 145
  7.2.2 Gateaux Differential with Respect to $\hat{\phi}_\alpha^{(k)}$ ....... 146
  7.2.3 Gateaux Differential with Respect to $V_{(k)}$ ................. 148
  7.2.4 Gateaux Differential with Respect to $\hat{\phi}_3^{(k)}$ ......... 149
  7.2.5 Gateaux Differential with Respect to $\hat{\phi}_3^{(k)}$ ......... 150
  7.2.6 Gateaux Differential with Respect to $N_{\alpha\beta}^{(k)}$ ....... 152
  7.2.7 Gateaux Differential with Respect to $N_{33}^{(k)}$ ........... 153
  7.2.8 Gateaux Differential with Respect to $M_{\alpha\beta}^{(k)}$ ....... 154
  7.2.9 Gateaux Differential with Respect to $M_{33}^{(k)}$ ......... 155
  7.2.10 Gateaux Differential with Respect to $V_{\alpha}^{(k)}$ .......... 156
  7.2.11 Gateaux Differential with Respect to $\sigma_{\alpha3}^{-(k)}$ .... 158
  7.2.12 Gateaux Differential with Respect to $\sigma_{33}^{-(k)}$ ....... 161
  7.2.13 Gateaux Differential with Respect to $\theta^{-(k)}$ .......... 162
  7.2.14 Gateaux Differential with Respect to $\tilde{q}_{\alpha}^{(k)}$ ....... 166
  7.2.15 Gateaux Differential with Respect to $\tilde{q}_{\alpha}^{(k)}$ ....... 168
7.3 Order of Differentiability .................................................. 170

8. Specializations of the Variational Formulation ...................... 173

8.1 Reduction of Required Order of Differentiability ................. 174
8.2 Specializations of Governing Function ................................ 186

9. Summary and Conclusions .................................................... 195

Bibliography ................................................................. 199
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Lamina Geometry and Coordinate System</td>
<td>46</td>
</tr>
<tr>
<td>2.</td>
<td>Coordinate System for a Four-Layer Laminate</td>
<td>69</td>
</tr>
</tbody>
</table>
## NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>± a, ± b, ± ( \frac{h}{2} )</td>
<td>Plate dimensions</td>
</tr>
<tr>
<td>B_B</td>
<td>Bilinear mapping on ( V_R \times V_R )</td>
</tr>
<tr>
<td>( \delta_{ij} )</td>
<td>Kronecker delta</td>
</tr>
<tr>
<td>c</td>
<td>Specific heat at constant volume</td>
</tr>
<tr>
<td>E_{ijkl}</td>
<td>Isothermal elasticity tensor</td>
</tr>
<tr>
<td>( \varepsilon_{kl} )</td>
<td>Symmetric strain tensor</td>
</tr>
<tr>
<td>f_i</td>
<td>Body force vector</td>
</tr>
<tr>
<td>F_i</td>
<td>Generalized body force vector</td>
</tr>
<tr>
<td>g_i</td>
<td>Thermal gradient</td>
</tr>
<tr>
<td>g^{(k)}_\sigma</td>
<td>Traction boundary conditions</td>
</tr>
<tr>
<td>g^{(k)}_u</td>
<td>Displacement boundary conditions</td>
</tr>
<tr>
<td>K_{ij}</td>
<td>Coefficients of thermal conductivity</td>
</tr>
<tr>
<td>( \Gamma_{ij} )</td>
<td>Symmetric tensor of thermal coefficients</td>
</tr>
<tr>
<td>M_{\alpha\beta}</td>
<td>In-plane moment resultants</td>
</tr>
<tr>
<td>Symbol</td>
<td>Meaning</td>
</tr>
<tr>
<td>--------</td>
<td>---------</td>
</tr>
<tr>
<td>$M_{11}$</td>
<td>Transverse normal moment resultant</td>
</tr>
<tr>
<td>$N_{0\beta}$</td>
<td>In-plane stress resultants</td>
</tr>
<tr>
<td>$N_{33}$</td>
<td>Transverse normal stress resultant</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Temperature above the reference temperature</td>
</tr>
<tr>
<td>$\theta^z$</td>
<td>Boundary temperature term, e.g., at $\pm \frac{h}{2}$</td>
</tr>
<tr>
<td>$\sigma_{ij}$</td>
<td>Symmetric Cauchy stress tensor</td>
</tr>
<tr>
<td>$\sigma^z_{ij}$</td>
<td>$\sigma_{ij}$ at top and bottom, i.e., at $\pm \frac{h}{2}$</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Linear functional</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>Potential energy</td>
</tr>
<tr>
<td>$\pi_{ij}$</td>
<td>Coefficients of linear thermal expansion</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Mass density</td>
</tr>
<tr>
<td>$q_i$</td>
<td>Heat flux</td>
</tr>
<tr>
<td>$\bar{q}_\alpha$</td>
<td>Flux resultant</td>
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<tr>
<td>$\bar{q}_\alpha$</td>
<td>First moment of flux</td>
</tr>
<tr>
<td>$r$</td>
<td>Internal heat generation per unit time per unit volume</td>
</tr>
<tr>
<td>$R$</td>
<td>Open connected region in Euclidean space</td>
</tr>
<tr>
<td>$\overline{R}$</td>
<td>Closure of $R$</td>
</tr>
<tr>
<td>$S$</td>
<td>Boundary of $R$</td>
</tr>
<tr>
<td>$S_0$</td>
<td>Temperature specified boundary</td>
</tr>
<tr>
<td>$S_q$</td>
<td>Heat Flux specified boundary</td>
</tr>
</tbody>
</table>
$S_\sigma$  Traction specified boundary

$S_u$  Displacement specified boundary

$S_{ijkl}$  Elastic compliance tensor

$T_o$  Reference temperature

t  Time

$u_i$  Displacement

$\bar{u}_i$  Displacement resultant

$\bar{u}_i$  First moment of displacement

$\hat{u}_i$  Second moment of displacement

$\bar{V}^{(k)}_\alpha$  Generalized in-plane displacements

$\bar{V}^{(k)}_3$  Generalized transverse displacements

V  Total volume of body

$V_\alpha$  Transverse shear stress resultant

$\bar{\phi}^{(k)}_\alpha$  Generalized in-plane displacement

$\bar{\phi}^{(k)}_3$, $\hat{\phi}^{(k)}$  Generalized transverse displacement

$x_i$  Cartesian coordinates

$Z_0$  Initial conditions

*  Convolution integral
CHAPTER 1

INTRODUCTION

Advanced composite materials exhibit a number of unique properties, including high stiffness-to-weight and strength-to-weight ratios, high fatigue resistance, high damping, good dimensional stability, and negligible thermal expansion in the fiber direction. In addition, optimum design of laminated composite structures is possible by tailoring the variation of fiber orientation, stacking sequence, lamina thickness, and choice of fiber and matrix materials. Material response can be isotropic, orthotropic, or anisotropic depending upon the particular system configuration.

As the technology of composite materials becomes more developed, it will be practical to exploit their unique features for application in components subjected to not only mechanical loads but to thermal loads (e.g., high temperatures, high temperature gradients, and cyclic changes of temperature) as well. Examples are provided in high-speed aircraft, spacecraft, launch and reentry vehicles, advanced propulsion systems, missile technology, and high performance electronic circuitry packages. A review paper by Thornton (1992) describes the development of thermal structures.
from the early days of supersonic flight to more recent challenges presented by hypersonic flight. Under service conditions, these structures are often subjected to mechanical and/or thermal environments which present new challenges to structural engineers.

One critical technology common to all these structures is composite plate design, which requires very accurate evaluation of thermal deflections and stresses. Successful evaluation depends on the theory used to model the thermomechanical response of the given structure. The theory should accurately describe the kinematics, constitutive laws, boundary and initial conditions as well as interaction between the displacement and temperature fields. Models that can accurately reflect the fully coupled (mechanical and thermal) response of composite laminates are needed to properly use these materials. This dissertation deals with the development of a theoretical model to describe the thermomechanical response of composite laminates subjected to combined loadings.

An understanding of thermally induced stresses in anisotropic bodies is essential for a comprehensive study of material response to a temperature exposure. Because thermal stresses are associated with thermal strain incompatibility, they can exist without the action of external forces. In isotropic solid bodies, [thermal] stresses arise either from non-uniform temperature gradients or externally constrained boundaries. In laminated composites, thermal anisotropy can lead to stresses even under uniform temperature changes.
Whitney (1986) found that in composite laminates, relatively small amounts of mechanical energy can result in localized stress levels large enough to cause significant damage growth leading to progressive damage of the layers. Because of the heterogeneous layer construction and material anisotropy, the induced thermal stresses are further complicated by the internal mutual constraints and the thermal incompatibility between the layers. In addition to in-service exposures, composite systems are exposed to temperature changes during the manufacturing stage. For example, significant residual stresses often develop during the curing process due to a mismatch in the thermal expansion coefficients from layer to layer, mismatch in thermal expansion coefficients between the fiber and the matrix, cure shrinkage in the thermosetting matrices, and melting/solidification volumetric changes in thermoplastics.

In order to use composite materials efficiently, it is important to understand their response to changes in temperature and applied external loads. Without a model, such as the one developed in this dissertation, as new materials and uses of composite materials are developed, the increasing variety of designs will dictate an ever-expanding number of expensive and time-consuming laboratory tests for the determination of the material system response. The objective of the research presented in this dissertation is the development of a fully coupled theory capable of accurately predicting the localized thermomechanical stress fields in laminates subjected to combined loads. To be useful, the theoretical model must incorporate realistic thermomechanical stress distributions within and across each lamina, as well as the
coupling contributions of the deformation/stress fields to the ensuing temperature field and vice versa.

A review of the current literature on the determination of thermomechanical stress and displacement fields in composite plates is presented in Chapter 2. For coupled problems, the simultaneous solution of the temperature and stress field in the body is required. It was concluded from this survey that many of the current thermomechanical plate theories, which use an assumed through-the-thickness displacement distribution, are generally acceptable for predicting the global response of the system, such as deflections, but do not accurately predict localized thermal stress behavior. Because predictions of the local thermal stress are required to analyze regions of high thermal gradients, none of the existing thermomechanical laminated plate theories is satisfactory. However, an isothermal static laminate plate theory by Pagano (1978a,b) has been shown to accurately model regions of high stress gradients for specimens subjected to mechanical loads. It is the intent of the proposed research to extend Pagano's (1978a,b) isothermal stress-based discrete plate theory to include thermal effects. Unlike previous formulations, no assumptions about the displacement components are made: rather an assumed through-the-thickness in-plane stress distribution is used. The theory will be derived assuming isothermal material properties with temperature effects confined to heat conduction.

Chapter 2 contains a review of existing thermomechanical plate theories as well as heat conduction models. The lamina stresses for an assumed linear in-plane stress distribution are derived in Chapter 3. A coupled constitutive law is used to
incorporate the thermal effects. A variational formulation for the coupled theory of linear thermoelasticity is derived in Chapter 4. The governing equations obtained in Chapter 4 are put in self-adjoint form in Chapter 5. The operator form of the self-adjoint governing equations is presented in Chapter 6. The generalized variational formulation for the self-adjoint equations in operator form is carried out in Chapter 7, while specializations for selected problems are made in Chapter 8. Chapter 9 reports the conclusions as well as recommendations for future work.
CHAPTER 2

LITERATURE REVIEW

For over a century thermoelasticity has been the subject of well-known investigations [Duhamel (1837), Neumann (1838), and Voigt (1910)]. A tremendous interest in the analysis of thermoelastic behavior of laminated composites has emerged; see for example, Boley and Weiner (1960), Nowacki (1962), Parkus (1968), and Kovalenko (1969). This interest is due to the increased use of composite materials in aerospace and other various fields of modern technology.

It should be understood that the word thermoelasticity is used in this dissertation in its broadest meaning and embraces the classical theory of elasticity, the effects of temperature distribution on the stresses in (linear) elastic bodies, the theory of heat conduction, and the interaction between the deformation and temperature field. The present treatment brings together all these phenomena into the general framework of thermoelasticity and its variational treatment. The rate of change of temperature is assumed small in comparison to the speed of sound propagation in the material. Therefore thermal shock problems will not be considered, and the stress calculations are considered to be quasi-static. The stress in the body at each time instant consists of
a sequence of static analyses calculated from the temperature distribution at that time instant without the inertia effects. The importance of retaining the inertia terms in the equations of motion has been investigated by Boley and Barber (1957). They pointed out that among various beams and plates studied, the effects of inertia on the natural frequencies is most evident for thin plates under rapidly applied heating inputs. More references on the study of inertia effects for other structures can be found in articles by Stroud and Mayers (1971) and Jadeja and Loo (1974).

The successful performance of composite materials as heat shields and thermal protectants as well as load bearing structures in a thermal environment is contingent upon a thorough knowledge of the physical events which occur in the body while it is subjected to combined thermal and external forces. Also important is the development of an accurate analytical technique for predicting the material response. The phenomenon of coupling between thermodynamic and mechanical responses in solid media was first predicted by Duhamel (1837). The coupling contribution in the thermal analysis is a heat source in the heat balance equation while the coupling in the mechanical analysis is reflected in the constitutive equation. Most problems of thermoelasticity are solved by neglecting the coupling between the thermal and displacement fields, even though a thermal environment can cause critical components to distort beyond operational tolerances. Thus in order to predict the structural response accurately, a fully coupled thermal and mechanical analysis is required.

Thick plates are basically three-dimensional structures. The advantages gained by modeling them as two-dimensional problems has been the primary motivation for
the construction of plate theories. However, for layered composite plates, closed-form solutions of the initial value problem are not available for plates of arbitrary geometry, with layers of arbitrary orientation, and subjected to arbitrary loading or boundary conditions. Exact (closed-form) solutions for rectangular laminated plates have been presented only for simple geometrical and layer configurations, transverse load distributions, and boundary conditions.

Another solution technique, the perturbation procedure, has been used to solve certain problems. However, as with analytical solutions, the perturbation technique is only suitable for particular problems and thus the application is very limited. At the present time, a more feasible and practical approach is to use numerical methods to approach the complicated geometry, material constituency, and boundary and loading configurations encountered in the analysis of laminated plates. Various numerical methods have been developed to obtain approximate solutions such as the finite element method and the finite difference method.

The finite element method is the industry standard technique used to provide numerical approximations of initial-value problems, and many problems have been solved this way. Various theories, variational formulations, interpolation schemes, element geometries, and solution procedures have been introduced to construct numerical solutions to the initial-value problem. However, finite element analyses of thermal stresses in composites are based on theoretical formulations such as classical plate theory or shear deformation theories so that the numerical solution is no better than the limitations imposed by the underlying plate theory assumptions. The finite
element method is commonly used in conjunction with the Galerkin formulation or a variational method.

In the literature, considerable research effort has been devoted to the study of the thermomechanical response of laminated plates. In this chapter, several theories corresponding to two-dimensional plate problems are reviewed. A review of all the published work on the subject will not be attempted. The references cited have been selected to review the various models and are not necessarily the only significant contributions to the subject.

2.1 THEORY OF LAMINATED PLATES INCLUDING THERMAL EFFECTS

The isothermal theory of anisotropic thin elastic plates was developed principally by Boussinesq (1879), Voigt (1910), and Lechnitzkz (1936). Nadai (1925) developed but did not attempt to solve the governing equations for the deflection of an isotropic thin elastic plate subjected to a linear through-the-thickness temperature distribution. The problem of thermal bending of anisotropic plates was first studied by Pell (1946) who derived the equations governing the transverse deflection of a thin plate subjected to a linear through-the-thickness temperature distribution.

Since Pell's work, theories for the thermoelastic behavior of laminated plates have been extended, modified, and tailored for the analysis of composite material systems. In general, the models used to analyze laminated plates subjected to thermal loads are extensions of models used for isothermal static analyses. The following
sections give a brief review of many of the existing thermomechanical plate theories, including smeared and discrete models. These two approaches can be subdivided according to the nature of the assumed displacement or stress field through the plate thickness. Other notable solution techniques to model the thermoelastic behavior in the body are also reviewed.

Throughout this dissertation, standard index notation is used in which Greek indices take on values of 1 and 2 while Latin indices take on the values from 1 to 3. Superimposed dots and subscripts preceded by a comma denote, respectively, differentiation with respect to time and spatial coordinates indicated by the subscripts. Parentheses about a pair of free subscripts signifies the symmetric part of the tensor with respect to those subscripts.

2.1.1 Displacement Based Models

Displacement based plate theories are two-dimensional approximations of the three-dimensional elasticity theory obtained by prescribing a through-the-thickness distribution of the displacement components. In laminated plates, theories that assume a continuous displacement field over the whole laminate are referred to as smeared plate theories. Theories employing displacement assumptions over each individual lamina are called discrete layer theories.
2.1.1.1 Classical Plate Theory

Classical plate theory (CPT) for laminated composites follows the same philosophy as employed in the homogeneous isotropic plate theory [see Timoshenko and Woinowk-Krieger (1959)], and is based on the following fundamental assumptions:

1. In-plane displacements are small compared to the thickness of the plate.
2. Kirchhoff's Hypothesis, i.e., plane sections before bending remain plane
   and perpendicular to the midsurface after deformation.
3. The transverse normal strain, $\varepsilon_{33}$, is assumed zero.
4. The transverse normal stress, $\sigma_{33}$, is small compared to the in-plane normal stresses $\sigma_{11}$ and $\sigma_{22}$ and can be neglected, i.e., $\sigma_{33} = 0$.
5. Rotary inertia, the inertial resistance to rotational acceleration of the plate, is neglected.
6. The plate is constructed of linear elastic material with constitutive properties independent of temperature.

A mathematical representation for the assumed displacement field through the thickness of the laminate is

$$u_\alpha(x_i, t) = u'_\alpha(x_{\parallel}, t) - x_3 w'_\alpha$$

$$u_3(x_i, t) = w'_{\parallel}(x_{\parallel}, t)$$
where $u_{a}$, $u_{i}$ are the in-plane and out-of-plane displacement components respectively, $t$ is the time, and $u'_{a}$, $w'$ represent the midplane displacements along in-plane, $x_{B}$, and out-of-plane, $x_{1}$, directions, respectively. The number of independent unknowns is three.

Based on the assumptions of CPT. together with the thermoelastic version of Hooke's law to encompass thermal effects, Stavsky (1963a) extended Pell's (1946) work by developing a general thermoelastic theory for thin anisotropic plates subjected to a uniform temperature distribution. Stavsky (1963a) was the first to discuss not only the coupling phenomenon between transverse bending and in-plane stretching, as shown by Stavsky (1961) and Reissner and Stavsky (1961) for isothermal problems, but also the thermoelastic coupling present in the governing equations for the deflections of a plate.

Stavsky (1963b) applied the theory to obtain expressions for the deformations and stresses in a thin rectangular plate, simply supported along two infinitely long edges and subjected to uniform heating.

One limitation of classical plate theory is that it neglects the effects of transverse shear deformation, implying an infinite shear rigidity in the plate and leading to an over-estimate of the plate stiffness. As a consequence, the lateral thermal deflection is underpredicted, see Khdeir and Reddy (1991). and the natural frequencies are overpredicted, see Kao and Pao (1976).
Wu and Tauchert (1980a) investigated thermally induced deformations in symmetric specially orthotropic laminates. The method of Levy (1959) was used to determine the uncoupled flexural response of a rectangular plate having two parallel simply supported edges and subjected to a temperature distribution that does not vary in the direction parallel to the simple supports. Using a similar approach, Misra (1971) investigated the problem of a homogeneous orthotropic plate having two edges simply supported and two others clamped. A solution is given for the special case in which the temperature has a linear variation in the thickness direction only. Muskhuty (1976) considered a similar plate geometry with transient heating on one face and the other face thermally isolated.

Kolyano and Plyatsko (1969) considered the problem of a cylindrically anisotropic circular plate symmetrically heated over a portion of one face. Another study by Wu and Tauchert (1980b) considered the thermal deformation and stresses in rectangular antisymmetric cross-ply and angle-ply laminates. Following the procedure used by Whitney (1969) for the isothermal case, a double Fourier series solution was developed for special boundary condition problems to obtain exact closed-form solutions. They found that the effects of membrane-bending coupling is more significant for thin plates but becomes unimportant as the number of layers within a plate of given thickness increases.

As the thickness of the plate increases the transverse shear components become more pronounced, thus the accuracy of CPT decreases [see Whitney (1969) and Pagano (1969)]. In his study, Pagano (1969, 1970) compared solutions of several
specific boundary value problems subjected to a mechanical load corresponding to an exact (closed-form) solution from elasticity theory. Numerical results presented in conjunction with the closed-form solutions confirmed that CPT gave poor predictions of the deflections of the laminate at low span-to-depth ratios (less than 10) but converged to the closed-form solution as this ratio increased. In particular, observed plate deflections were shown to be considerably larger than those predicted by CPT. Similar trends have been found in the literature, see for example, Wang and Crossman (1977b), Ryabov (1967) and Tauchert (1986) for thermal problems.

Bert (1983) concluded that CPT and Bernoulli-Euler beam theory have similar limitations in that the transverse shear and normal strains are neglected and in-plane normal strain is distributed linearly rather than non-linearly through the plate thickness. CPT is also limited by neglecting rotary inertia. Boley and Barber (1957) pointed out that among the plates they studied, the effect of rotary inertia on the natural frequency was most evident for thin plates under rapidly applied heating input. Other investigations of thermally induced vibrations by Kraus (1966) and Lu and Sun (1965) have supported the findings of Boley and Barber (1957).

Another limitation in the formulation of CPT is that no provisions are included for any change of the material properties (elastic coefficients, thermal expansion coefficients, specific heat, etc.) with temperature. It is well known that for many materials some of these properties are highly dependent upon temperature. CPT can lead to significant errors when applied to even moderately thick plates. This is particularly true for composite materials in which the effective transverse shear
modulus is generally low compared to the effective elastic modulus along the fiber direction, see Murthy (1981).

2.1.1.2 First Order Shear Deformation Theory

Due to the inherent limitations of CPT it is not recommended for the thermal stress analysis of laminated composite plates and hence various thermoelastic formulations that include transverse shear effects have been proposed. The Reissner-Mindlin plate theory, commonly referred to as the First Order Shear Deformation Theory (FSDT), is an extension of Mindlin's (1951) isothermal homogeneous isotropic plate theory in which corrections for transverse shear and rotary inertia are included. Yang et al. (1966) and Whitney and Pagano (1970) extended Mindlin's (1951) theory to laminated plates consisting of an arbitrary number of bonded anisotropic layers. FSDT is based on the assumption that the transverse strain $\varepsilon_{33}$ is independent of the thickness coordinate $x_3$. This leads to a displacement field for which the in-plane displacement components vary linearly over the laminate thickness, and the transverse displacement is constant. This displacement field can be represented mathematically by the following expressions:

$$u_\alpha (x_i,t) = \dot{u}_\alpha (x_\beta,t) + x_3 \Phi_\alpha (x_\beta,t)$$

$$u_3 (x_i,t) = w' (x_\beta,t)$$

where $\Phi_\alpha$ are the rotations of the normals to the plate midsurface and the other terms
are defined as in CPT. The number of independent unknowns is five; namely, the in-plane midsurface displacements $u_\alpha$, the transverse displacement $w$, and the rotations $\varphi_\alpha$. Normals to the midsurface before deformation remain straight but not necessarily normal to the midsurface after deformation, but because FSDT is a smeared plate theory, it is assumed that the individual layer rotations are equal.

FSDT was initially developed for mechanical loading and did not take into account temperature effects. Studies by Das and Rath (1972), Bapa Rao (1979), Kolesnikov (1981), and Tolkachev and Shpeketrov (1980) extended FSDT for isotropic homogeneous plates to include thermoelastic behavior. A similar extension of the isothermal laminated plate theories of Yang et al. (1966) and Whitney and Pagano (1970) was carried out by Reddy and co-workers (1980a, b).

In their investigation Reddy and Hsu (1979, 1980b) presented a finite element formulation based on the assumptions of FSDT for anisotropic composite plates subjected to thermal and mechanical loads. A linear temperature variation through the thickness, consistent with the displacement distribution, was assumed to be of the form,

$$T(x_i) = T_o(x_\beta) + \frac{x_1}{h} T_1(x_\beta)$$ (2.5)

where $T$ is the temperature, $T_o$ and $T_1$ are variables which determine the in-plane temperature distribution, and $h$ is the plate thickness. Because exact solutions to the governing plate equations were not available for a plate of arbitrary geometry, constructed of arbitrarily oriented layers, of arbitrary boundary conditions, or subjected
to an arbitrary loading, the finite element solution was validated with a closed-form solution for a simply supported rectangular cross-ply laminated plate under sinusoidal and uniform loading (thermal, mechanical, and combined). A parametric study of the plate aspect ratio, side-to-thickness ratio, orientation of layers, and edge conditions showed the finite element results for the transverse plate deflections agreed closely with the closed-form solution. The temperature effects on material properties were neglected. Reddy et al. (1980a) extended this formulation to bi-modulus, (materials with properties different in tension and compression) cross-ply plates subjected to a sinusoidal distribution of mid-plane temperature.

Chen et al. (1985) investigated the thermal stress response of graphite/epoxy laminated plates subjected to combined mechanical and thermal loads. The temperature field was obtained using a one-dimensional finite difference code and included temperature dependent material properties, radiation and convective heat losses, and ablation. The structural response of the laminate was analyzed using the finite element method based on FSDT. Thermomechanical coupling was neglected in the analysis. Analytical predictions were compared with experimental data. Discrepancies between predicted and observed results increased with time due to the limitations of a one-dimensional heat model.

Murakami (1993) examined the accuracy of the thermoelastic version of FSDT by comparing results to CPT and a generalization of the closed-form solution developed by Pagano (1969), who considered cylindrical bending due to a sinusoidal transverse loading acting on the top surface of a plate. Although comparisons of the
predicted thermal displacements with the closed-form solution revealed some improvements of FSDT over those of CPT, both theories experienced limitations in accurately predicting the rapid variation of transverse displacement in each layer.

In the case of thermal deformations, the accuracy of the transverse normal strain is important. As in CPT, FSDT does not address the assumption that the transverse normal stress $\sigma_{33}$ is zero through the thickness of the laminate, thus FSDT is limited in its ability to accurately predict the thermal deformation behavior. Because FSDT assumes a constant rotation of the entire laminate, it does not allow for warping of the plate cross-section and can induce serious errors particularly in the case of laminates consisting of a large number of alternating layers of different material properties. Moreover, the effects of shear deformations are included in an ad hoc fashion requiring the use of arbitrary shear correction factors. The determination of the shear correction coefficients is a controversial topic [Mindlin (1951), Reissner (1945)] with different values given by various researchers.

The use of FSDT for a detailed thermomechanical stress analysis is limited due to the modeling assumptions for the transverse normal and shear stress/strain components. Many researchers have suggested that a higher order shear deformation plate theory is necessary for an accurate determination of the transverse thermal deformations and stresses. Theories of this type are discussed in the following section.
2.1.1.3 Higher Order Shear Deformation Theory

The FSDT proposed by Mindlin (1951) cannot properly account for the variation of transverse shear strain through the plate thickness. Consequently, many higher-order theories have been developed to eliminate this deficiency. It should be noted that the terminology "higher order" refers to the level of truncation of terms in a power series expansion for the assumed displacement field rather than the order of the final system of differential equations.

In general, higher-order shear deformation theories (HSDT) differ from one another by the assumed form of the in-plane and transverse displacement distribution across the plate thickness and on the boundary conditions on the transverse plate surfaces. With each additional term in the assumed displacement field, an additional dependent variable is introduced. Typically, HSDT assumes a quadratic or higher order distribution of the in-plane displacement components across the plate thickness and a constant or higher order distribution of the transverse displacement component; an alternative form is a linear distribution of the in-plane displacements and a linear or higher order transverse displacement. Other differences between HSDT and FSDT are stated below:

1. Normals to the midsurface of the plate before deformation are not required to be normal or straight after deformation.

2. The transverse normal strain $\varepsilon_{33}$ is, in general, not negligible.

3. The transverse normal stress $\sigma_{33}$ is not neglected.
The first isothermal higher order theory for an orthotropic material was proposed by Hildebrand et al. (1949). Since then many higher order theories have been developed, each with a different assumed displacement field but without including thermal effects; these works include Lo et al. (1977a, 1977b, 1978), Kristna Murthy (1986), Kristna Murthy and Velliachamy (1987), Reddy and Phan (1985), Whitney and Sun (1973), Hinrichsen and Palazotto (1986), Kant and Pandya (1988), Nelson and Lorch (1974), and Bhimaraddi and Stevens (1984). A historical account of isothermal shear-deformable plate theories can be found in articles by Lo et al. (1977a), Seide (1980), Bert (1984), Reddy (1984), Reissner (1985), and Noor and Burton (1989, 1990).

The inclusion of higher order terms in the series expansion of the assumed displacement field for isothermal cases has resulted in an increased accuracy (as compared to CPT and FSDT) in the prediction of the in-plane displacements, stresses, and extensional modes of vibration. [see Murakami (1986), Lo et al. (1977a), Nelson and Lorch (1974), Reddy and Phan (1985), and Whitney and Sun (1973)]. However, values for the transverse stresses and strains, especially across interlaminar boundaries, are still not accurately predicted with the use of the HSDT.

Using the higher order theory proposed by Reddy (1984), Khdeir et al. (1992) and Khdeir and Reddy (1991) investigated the thermal stresses and deflections of cross-ply laminated plates subjected to a sinusoidal temperature distribution, and the results were compared to those obtained using the thermoelastic versions of CPT and FSDT. The displacement field Khdeir and Reddy (1991) used was of the form
\[ u_t(x_t,t) = u^i(x_0,t) + x_1 \left[ \frac{x_1}{3} \left( \frac{x_1}{h} \right)^2 \left( \frac{x_1}{3} \right)^2 \right] \tag{2.6} \]

where \( \gamma \) is a variable that takes on a value of either one or zero and the other terms are as defined previously. The out-of-plane displacement is assumed constant in \( x_3 \). If \( \gamma = 0 \) the equations reduce to FSDT, and if \( \varphi_a = -w_{,t} \), the equations are reduced to CPT. Closed-form solutions were determined for the thermal deflections and stresses of rectangular laminates having two opposite edges simply supported and the remaining edges arbitrary combinations of free, clamped, and simply supported boundary conditions. The results showed that for the simply supported-clamped boundary conditions, the CPT underpredicts the plate thermal deformation response as compared to FSDT and HSDT. In addition, the results from CPT, FSDT, and HSDT converge for sufficiently thin plates, i.e., as the span-to-depth ratio, \( b/h \), increases, becoming insignificant for \( b/h > 20 \). Numerical results of the in-plane thermal stress through the thickness were presented only for the FSDT, and no stress predictions using the other theories were provided. The temperature dependence of the material properties was neglected in the analysis.

Murakami (1993) studied thermal deformations in layered elastic plates using a higher order theory in which the in-plane displacements varied as a cubic across the entire thickness of the plate with non-zero transverse normal strain. He compared these results to the thermoelastic version of the closed-form solution first proposed by Pagano (1969) and found that HSDT did not significantly improve the predicted value of the transverse displacement through the thickness. Similar results were found by
Jonnalagadd et al. (1993) who compared analytical results of central plate deflections to predictions from CPT, FSDT, and several HSDT's.

Because kinematic relations are, in general, satisfied for shear deformation theories, displacement predictions are acceptable; however, theories based on an assumed displacement field do not, in general, satisfy equilibrium equations. This results in discontinuous tractions at interlamina boundaries. While HSDT performs better than CPT and FSDT, HSDT is not suitable for a thermal stress analysis in regions of high thermal and stress gradients. Due to these deficiencies, smeared plate theories are not viable refined thermal stress analysis techniques for determining accurate, localized interlaminar thermal stresses in multilayered composite materials. Murakami (1993) pointed out that an improvement to these theories requires the use of assumed displacements in each layer (discrete theory) of the plate rather than over the entire plate thickness. The need to eliminate the deficiencies of the smeared plate theories has motivated the development of discrete laminate theories which are reviewed in the following section.

2.1.1.4 Discrete Layer Theory

The displacement based plate theories thus far reviewed assume a displacement distribution through the entire thickness of the multilayer plate. These models, commonly referred to as smeared plate theories because the properties of the individual plies are smeared into one equivalent thick lamina, employ an effective
stiffness for the entire plate. Hence, the theories are typically well-suited to the prediction of global plate response such as central deflections and low vibrational frequencies but cannot accurately predict localized phenomena. In addition, smeared plate theories violate the equilibrium conditions of the plate because stress continuity across lamina interfaces is not satisfied.

In contrast, displacement based discrete laminate theories (DLT) assume a displacement distribution for each lamina of the plate resulting in a piecewise continuous function through the plate thickness. Each layer of the system is modeled as a homogeneous, anisotropic plate and the governing equations of each individual layer are combined with the interlaminar continuity conditions to obtain global governing equations of the entire laminated plate.

Sun et al. (1968), Achenbach et al. (1968), Sun and Whitney (1973), Srinivas (1973), and Mau (1973) employed discrete displacement based theories for the isothermal analysis of laminate plates based on FSDT. Sun and Whitney (1973) studied the dynamic behavior of laminated plates by assuming a FSDT displacement field for each lamina while enforcing certain constraint conditions at the interfaces of the layers. They found that the effect of local shear deformation depends highly on the transverse shear rigidities of the constituent layers. However, the transverse normal stress was neglected and shear correction factors were required.

Srinivas (1973) also assumed the in-plane displacement components to be piecewise linear and the transverse displacement to be constant through the thickness. The governing differential equations and boundary conditions were derived using a
variational approach. The effects of transverse shear deformation and rotary inertia were included, but the interlaminar normal stress was neglected. While interface displacement continuity conditions were enforced, the interlaminar traction continuity conditions were violated. Shear correction factors were also required. Though accurate results for deflections, in-plane stresses, and natural frequencies were obtained, the approach is still not applicable for general stress analysis due to the shortcomings in the theory. Many other researchers including, Seide (1980), Di Sciuva (1986), and Owen and Li (1987a, b) have used this approach to investigate the response of a laminated composite, although thermal effects were not included.

Recently, investigators including Cho et al. (1990), Lee et al. (1990), Soldatos (1992), Wu and Kuo (1992) and Gaudenzi (1992), have used HSDT assumptions on a discrete layer basis to study the response of laminated plates. For example, Wu and Kuo (1992) presented a discrete laminate theory assuming cubic in-plane displacements and quadratic transverse displacements over each layer of the laminate. The displacement and traction continuity conditions at the ply interfaces were used as constraints in the formulation and introduced in the potential energy functional by the Lagrange multiplier method. Results for the cylindrical bending of a simply supported infinitely long symmetric cross-ply laminate compared well to Pagano's (1969) closed-form solution. However the theory is limited to static cylindrical bending, hence the extension to the thermodynamic analysis of composite plates is not available. Thus far, the reviewed discrete displacement based theories have been limited to mechanical loading and the inclusion of thermal effects has not been addressed.
Ryabov (1967) was among the first to use a discrete displacement based theory for a thermoelastic analysis of a multilayer plate. A displacement field, in accordance with CPT, was assumed over each layer of a multilayer slab which consisted of parallel isotropic layers of constant thickness subjected to a temperature variation over the thickness. Kao and Pao (1976) extended Ryabov's (1967) work by developing the governing equations for the thermally induced vibration of thin anisotropic plates. The temperature dependence of the material properties was assumed to be negligible. Numerical results, based on a Fourier series solution, for the natural frequencies and maximum transverse displacements were presented for symmetric cross-ply plates without comparison to other solution techniques.

In a recent study, Tanigaiwa et al. (1991) investigated the thermal bending of laminated rectangular plates due to partial heating by enforcing the CPT assumptions individually over each layer of the plate. They used the theory to evaluate the thermal deflection and in-plane thermal stresses in a simply supported and clamped metal matrix plate but did not compare their results to other solution techniques.

The discrete displacement based models in general yield better results than the smeared plate theories and have provided valuable insight into the understanding of the thermomechanical response of laminated composites. However the discrete displacement based plates theories predict accurate thermal/mechanical displacements and in-plane stresses but are limited in predicting the transverse stress distribution across a laminate thickness. Although much more published research work on isothermal shear-deformable theories exists, the extension to include thermal effects is
straightforward but of little benefit due to the limitations of the models as evidenced in isothermal cases. The localized thermomechanical stress analysis of laminates under combined thermal and mechanical loads is not forthcoming due to the inherent limitations of the displacement based approach.

2.1.1.5 Other Displacement Based Approaches

Other techniques such as three-dimensional elasticity models and the finite element method have improved predictions for the thermomechanical behavior of laminated composites. Three-dimensional elasticity solutions for the thermomechanical response of laminated plates have been presented in the literature for a variety of problems. Lee (1970) presented a three-dimensional uncoupled series solution based on CPT assumptions for an elastic plate subjected to a linear temperature distribution. It was found however that the model did not satisfy edge conditions.

Srinivas and Rao (1972) extended an isothermal three-dimensional series solution for uncoupled thermoelasticity to investigate the flexure of a class of simply supported rectangular laminates with piecewise variations of temperature across the thickness. There were no restrictions imposed on the thickness variation of stresses or displacements. Numerical solutions for the midsurface deflections of the plate for various thickness ratios were presented and compared to the CPT predictions. Results showed that for pure thermal loading, CPT predictions are very close to closed-form
solution even for thick laminates. However, it was noted that the convergence of the solution is poorer for stresses than for displacements.

Wang and Chou (1988) have analyzed the three-dimensional thermoelastic problem of a balanced, symmetric angle-ply laminate by separating the plate into two regions, the interior and boundary layer regions. The elasticity solution in the interior region is studied by the CPT assumptions while in the boundary layer region a perturbation method is applied. However, this type of approach leads to inconsistencies at the interface between the two regions.

Using CPT to derive the governing plate equations, Hussein et al. (1989a, b) developed an analytical solution using Airy stress functions to predict the thermal stresses in sandwich panels. Numerical results showed that the bonding stiffness has a strong effect on the panel's thermal stress response.

Dvorak and Chen (1992a, b) presented closed-form solutions for the evaluation of local thermomechanical stress fields by approximating the thermal load as an "effective" mechanical load, thereby reducing the problem to one of purely mechanical loads. This type of approach oversimplifies the problem and leads to errors in the predictions.

Webber and Morton (1993) developed an analytical solution for the calculation of the free-edge stress field due to thermal effects in composite laminated plates based on the principle of minimum complementary energy. Analytical results are compared to the FEM solutions of Herakovich (1976). The analytical solution has several
limitations because the interface displacement continuity conditions are not ensured, and the solution is limited to specific laminate configurations.

Noor et al. (1993) presented an analytical solution based on the strain energy of the plate system using sensitivity coefficients. The sensitivity coefficients measure the effect of the strain energy to the laminate configuration and material parameters. The strain energy and sensitivity coefficients obtained by the three-dimensional model are used as a basis by which the accuracy of the corresponding quantities obtained by the two-dimensional FSDT are assessed.

Reddy and Hsu (1979, 1980b), Tauchert (1989, 90), and Tanigawa et al. (1991) have all presented series type solutions for laminates of a particular geometry, support conditions, and loadings. Additional analytical techniques can be found in survey papers by Tauchert (1991) and Noor and Burton (1992a). Closed-form solutions for the response of laminated plates are not possible when the plate is of arbitrary geometry, constructed of arbitrarily-orientated layers, and subjected to arbitrary loading or boundary conditions. Thus, although closed-form solutions provide a useful insight into the behavior of laminated plates to thermal loading they are limited to a few specific problems.

In order to overcome the limitations and complexities of three-dimensional closed-form solutions, approximate techniques such as the finite element method (FEM) have been developed. The FEM is considered the most attractive numerical technique due to its versatility in handling thermomechanical problems with a variety of complexities.
Herakovich (1976) used the finite element method to investigate thermal edge effects for unidirectional boron/epoxy material bonded to sheets of aluminum and titanium. He showed the interlaminar thermal stress may be significantly greater than the stresses due to the mechanical loading.

Chen and Chen (1990) studied the thermal deformations and stresses induced by a nonuniform temperature field in composite laminated plates using the FEM based on CPT. The stiffness matrix and load vector were derived based on minimum potential energy. All three thermal displacements of the middle surface were expressed as products of one-dimensional first-order Hermite interpolation polynomials.

Wang and Wang (1993) developed a hybrid multi-layer element to analyze the thermomechanical stresses in composites subjected to a nonuniform temperature distribution. The resulting interfacial stresses in a two layer laminate beam compared well with the FEM result of Cho et al. (1989) but showed discrepancies near the free edges.

Much of the literature on the finite element analysis of laminated plates centers on investigations of new element types. Mau et al. (1972) proposed a stress hybrid element to model different laminae. Chang et al. (1990) used 12 node and 16 node volume elements in their analysis of thick laminated plates. Mukherjee and Sinha (1994) investigated the thermostructural response problem in thick laminated composites by employing a three-dimensional finite element technique with quadratic 20 node isoparametric brick elements. Advances in computational models for
composites subjected to thermal loads can be found in Noda (1991), Noor and Burton (1992b), Thornton (1993), and Argyris and Tenek (1997).

The finite element method is a very powerful and practical tool for the analysis of coupled thermomechanical plate problems. The diversity of the technique allows for computer implementation of complex composite systems composed of virtually any configuration and loading. In order to construct element solution procedures however, weighted residual methods (such as Galerkin’s) or variational methods are used. Limitations and/or disadvantages may be transferred to the finite element model by the theory used or the variational approximation employed in the analysis.

Many finite element formulations for the thermomechanical analysis of laminated plates use Galerkin’s method as the starting point. This method involves direct use of the weak form of the governing equations and does not require the existence of a functional. However, the method does not provide a consistent means for dealing with boundary conditions. A more systematic method is possible where a variational principle can be formulated for the problem.

Variational methods have been used to develop approximate theories for problems in coupled thermoelasticity [Biot (1955, 1956, 1959), Chambers (1956), Herrmann (1960, 1963), Boley and Tolins (1962), Ben-Amoz (1965), Nickell and Sackman (1966, 1968a, b), Inan (1972), and Bahar and Hetnarski (1977)]. The power of these approaches is evidenced by the automatic generation of the correct number of boundary conditions and their correct expressions for many complex problems.
Variational methods require the construction of a functional of the field state variables such that the variation of the functional vanishes if and only if the field equations together with boundary and initial conditions are satisfied. For linear operators, many researchers have studied the problem in an inner product space. Tonti (1967, 1972) noted that the self-adjointness of an operator is defined through the given bilinear mapping. Gurtin (1964a, b) used the convolution product as a bilinear mapping for initial boundary value problems. The convolution product was employed to reduce the initial boundary value problem into an equivalent boundary value problem containing the initial conditions explicitly.

Nickell and Sackman (1966, 1968a, b) derived several variational principles for the initial boundary value problem of fully coupled linear thermoelasticity for an inhomogeneous anisotropic continuum. A consistent set of field variables was employed and initial conditions were incorporated explicitly in the formulation using the convolution integral. Sandhu and co-workers (1970, 1971, 1972, 1975) extended this work by presenting a comprehensive treatment on the variational principles for finite element approximations applicable to linear, coupled field problems with nonhomogeneous boundary conditions and jump discontinuities. Introducing the concept of boundary operators consistent with the field operator, they developed a systematic procedure to obtain a consistent variational approach.
2.1.1.6 Stress Based Theory

The drawbacks of displacement based plate theories necessitate a search for a theory to more accurately describe the localized thermomechanical response of a laminated composite. One of the earliest stress based approaches to plate theory was due to Reissner (1947, 1950). He developed an isothermal stress based variational theory for the transverse bending of smeared homogeneous plates by assuming a linear variation of in-plane stress $\sigma_{\alpha\beta}$ across the entire plate thickness. The theory includes the effects of the transverse normal and shear stresses, important in the stress analysis of multilayered composite plates, and overcomes many of the deficiencies of the displacement based theories.

Voyiadjis and Baluch (1981) developed a smeared stress based theory for homogeneous plates, taking into account transverse normal stress and strain, transverse shear stress and strain, as well as rotatory inertia. The assumed distribution of the transverse shearing stresses and the transverse normal stress was similar to Reissner’s (1945) static theory, i.e., a quadratic and cubic distribution of transverse shear stresses and transverse normal stress, respectively, through the entire plate thickness. The assumed transverse stress field was written as

$$\sigma_{\alpha\gamma}(x_i,t) = \left[ \frac{3}{2} \frac{Q_{\alpha}(x_{\alpha},t)}{x_3} \right] \left[ 1 - \left( \frac{2x_3}{h} \right)^2 \right]$$

(2.7)

$$\sigma_{33}(x_i,t) = - \left[ \frac{p(x_{\alpha},t)}{4} \right] \left[ 2 - 3 \left( \frac{2x_3}{h} \right) + 2 \left( \frac{2x_3}{h} \right)^3 \right]$$

(2.8)
where \( h \) is the plate thickness, \( \sigma_{a3} \) and \( \sigma_{33} \) are the transverse shear and normal stress components respectively, \( Q_a \) is the shear stress resultant defined by

\[
Q_a = \int_{-h/2}^{h/2} \sigma_{a3} \, dx_3
\]

and \( p(x_{a3}, t) \) is the applied force. Voyiadjis and Baluch compared their predictions for wave speed to those of CPT, FSDT, and a closed-form solution. Their results showed that CPT was accurate only for long waves, whereas their results compared closely with the closed-form solution over a wide spectrum of wavelengths. Their model did suffer from limitations similar to other smeared plate theories [refer to Sections 2.1.1.1-2.1.1.3] and an extension to thermal effects was not presented.

Reissner's work (1947, 1950) was extended to model laminated plates by Pagano (1978a, b), who assumed a linear variation of in-plane stresses over each lamina with the transverse stress components being obtained from three-dimensional equilibrium equations. In his method all six stress components are, in general, non-zero, and traction and displacement continuity conditions at interfaces between adjacent layers are satisfied. This theory was applied to a free edge delamination coupon problem in which laminates of finite width were subjected to a uniform axial strain. Results were compared with an existing finite element solution of the same system [Wang and Crossman (1977a,b)], and it closely approximated the free edge stress field which is generally a site of high stress gradients. Pagano's (1978a) stress based model overcame many of the limitations associated with a displacement based
model but was formulated for isothermal conditions (expansional strains were included and assumed constant) and the extension to thermal conditions was not presented.

As the number of layers in the composite was increased, the complexity of Pagano’s model became computationally prohibitive. To overcome this difficulty Pagano and Soni (1983) developed a global-local laminate variational model. This model divided the laminate into two sections for analysis, one for local and the other for global response. For critical areas that require localized stress analysis, Pagano’s (1978a) model was used, while for other areas the numerically simpler displacement based approach after Whitney and Sun’s (1973) higher order theory was used. This model was intended to solve complicated problems of multilayered laminated plates; however, technical difficulties arose when dividing regions into local or global domains because locations of critical points are seldom known in advance. This approach resulted in a variationally inconsistent plate model especially in the vicinity of the global/local interface.

Chyou (1989) reduced the governing equations of Pagano (1978a) by eliminating the terms $N_{33}$ and $M_{33}$ [terms introduced by Pagano (1978a) for convenience and defined as the transverse force and moment resultants, respectively] as well as two weighted displacement terms. He then put the reduced equations in self-adjoint form to ensure convergence and incorporated them in a finite element formulation.
Schoeppner (1991) further extended Pagano’s (1978a) and Chyou’s (1989) work to include inertia terms so that the dynamic analysis of laminated plates could be investigated. The theory is general in its formulation and allows for interlaminar discontinuities such as delaminations or manufacturing defects. Hamilton’s variational principle was used to derive the plate equations of motion, the plate constitutive relationships, and the interface continuity conditions. The governing field equations were written in a self-adjoint form to ensure convergence. A governing functional of the field variables was then constructed such that the variation of the functional vanishes if and only if the governing field equations are identically satisfied. Specializations of the governing functional were carried out for implementation into a finite element program. Schoeppner (1991) compared the stress (in-plane and transverse) and displacement fields for a free edge graphite/epoxy delamination coupon subjected to a uniform strain in the longitudinal direction to those of Pagano (1978a) and found essentially the same results with small differences attributed to the different numerical calculation procedures used by the two authors.

Schoeppner et al. (1993) compared results of the free vibration analysis of isotropic and laminated plates using his theory to the closed-form solution of Srinivas et al. (1970). Schoeppner’s (1991) stress based theory predicted accurate global behavior such as displacements and frequencies as well as local behavior such as interlaminar transverse and normal stresses. Butalia (1996) further extended Schoeppner’s (1991) work to include time-dependent mass density for a material subjected to dynamic loading.
In the author's opinion, the ability to accurately model both the global and local response of laminated composite plates renders this theory as the most suitable choice for extension to the prediction of localized thermal stresses in laminated plates. In order to fully describe the behavior and failure of a laminate under thermomechanical loading, an accurate description for both the temperature field in the body and also through-the-thickness stress distributions must be identified.

Review of elastic and thermoelastic theories of laminated plates indicates the suitability of extending a discrete stress based theory to include thermal effects to accurately predict thermal stresses even in regions of high stress. The ability to accurately predict transverse normal and shear stresses in composite plates will provide a valuable tool in the proper design and testing of such systems because interlaminar stresses can cause severe damage in laminated composites.

2.2 HEAT TRANSFER MODELS

A heat transfer model is required to describe the temperature field in a composite body. In this dissertation, thermal effects will be coupled to the plate theory with the temperature as a field variable. To accurately predict the thermal stresses in a composite, the thermal response of the material must first be determined. It is customary to categorize the various heat transfer processes into three basic types or modes: namely, conduction, convection, and radiation. Numerous books have been
written on the study of heat transfer in materials, examples are the texts by Ozisik (1980), Chapman (1984), and Kakac and Yener (1993).

Although most practical heat transfer problems involve at least two and sometimes all three modes of heat transfer, one mode is predominate and many analytical studies can reasonably limit the analysis by assuming the contributions of some modes to be negligible. In this dissertation the formulation will be limited to heat conduction only.

In general, a heat conduction problem consists of determining the temperature at any time and at any point within a specified solid which had been heated to a known initial temperature distribution and whose surface has been subjected to a known set of boundary conditions. The general three-dimensional differential equation of heat conduction for an anisotropic solid including thermoelastic coupling with heat generation within the body is given by the following [see, for example, Boley and Weiner (1960) for a comprehensive derivation]

\[ -q_{i\alpha} + r = \rho c \dot{\theta} + T_0 \Gamma_{ij} \dot{\varepsilon}_{ij} \]  

(2.10)

where \( q_{i\alpha} \), \( r \), \( \rho \), \( c \), \( \theta \), \( T_0 \), \( \Gamma_{ij} \), \( \varepsilon_{ij} \) are respectively, the heat flux, internal heat generation per unit time per unit volume, mass density, specific heat at constant volume, temperature, reference temperature at which zero strain yields zero stress, tensor of thermal coefficients, and strain tensor. Heat generation in the body may be due to nuclear, electrical, chemical, gamma-ray, or other sources that may be a function of time and/or position. The principle of conservation of energy states that there is a
relationship between the stress, strain, and temperature field. In one interpretation, conservation of energy indicates that variations of stresses and strains within the solid alter the heat flow and thermal energy. The equation expresses a conversion of mechanical to thermal energy. The conversion of mechanical energy to thermal energy is a well known phenomenon which has been studied extensively [see for example, Nowinski (1978)].

The heat flux components are related to the temperature gradient by Fourier’s Law (1822). For an anisotropic material, Fourier’s Law states

\[ q_i = -K_{ij} \theta_{ij} \]  

where \( K_{ij} \) are the coefficients of thermal conductivity. Equation (2.10) can therefore be restated as

\[ (K_{ij} \theta_{ij})_t + r = \rho c \dot{\theta} + T_v \Gamma_{ij} \dot{\varepsilon}_{ij} \]  

This expression describes, in differential form, the dependence of the temperature in the solid on the spatial coordinates and time. Its solution determines the temperature as a function of position and time in the body.

Considerable effort has been devoted to the development of closed-form, approximate, and numerical methods of solution for heat conduction problems. Closed-form solutions for specific heat conduction problems have been obtained using the method of separation of variables, integral transformation techniques, Laplace transformation techniques [Padovan (1972, 1974a, b, c, d), Cobble (1974), Mehta (1977)], and approximate techniques such as the Rayleigh-Ritz and Galerkin methods.
For example, Chang et al. (1973) and Chang and Hsou (1977) used Green's function for the solution of problems of steady state and unsteady state heat conduction by transforming differential equations into integral equations. Once the Green's function was known, the temperature could be determined using the Green's formula and numerical integration techniques. However this type of analysis is limited in its applicability to problems in which the Green's function can be determined.

Although a great number of problems have been solved analytically, only a limited number of relatively simple geometrical shapes (e.g., spheres, infinite slabs, cylinders, etc.) can be handled. Also, usually only relatively simple boundary conditions may be applied. Due to these limitations most heat conduction problems of practical value do not have analytical solutions. Numerical techniques have been developed to handle these problems of greater complexity.

Numerical methods yield values for temperature at discrete points within the body and only at discrete time instances. The detail and accuracy of the answer obtained depend mainly on the amount of effort one wishes to expend and on the simplifying assumptions used.

The finite difference method is widely used in the solution of both transient and steady-state heat conduction problems. The basic concept of this method is the approximation of partial derivatives at a given point by finite differences, thus converting the solution of the partial differential equations to the solution of coupled algebraic equations for temperature at a select number of nodal points within the region.
In most engineering formulations it is possible to introduce certain simplifying assumptions. A commonly used simplification is the omission of the mechanical coupling term in the heat balance equation, equation (2.12). In this dissertation a formulation will be derived for the class of coupled quasi-static problems in which the changes in temperature proceed slowly and mechanical inertia effects are neglected. Despite the omission of the inertia terms, the displacement and temperature fields remain functions of time. This feature distinguishes the quasi-static problem from those of the steady-state problem.

The replacement of the actual thermoelastic problem by an equivalent quasi-static problem is proposed for situations in which changes in the temperature field proceed slowly. As a consequence, the thermoelastic process may be perceived as a sequence of equilibrium states. This assertion is, of course, not valid for problems in which the temperature changes rapidly, such as during the propagation of thermoelastic waves in the aftermath of a thermal shock. Stationary or steady-state thermoelastic problems will also not be considered in the present formulation since this would result in the time-independence of the functions involved, thus suppressing the time derivative functions in the coupling terms.

Many investigations of the thermal response of composites have been limited to one-dimensional finite difference analysis [Pering et al. (1980), Griffis et al. (1981a, b), Henderson and Wiecek (1987)]. However due to the anisotropic nature of composite materials the errors in the temperature predictions increase with increasing time and these methods are not recommended for a detailed analysis.
Due to the inherent limitations of a one-dimensional heat conduction model, two-dimensional finite difference models have been developed. Griffis et al. (1986), Fanucci (1987), and Chang (1987) extended an earlier one-dimensional heat transfer model to a two-dimensional analysis to predict the thermal/structural response of composite materials subjected simultaneously to mechanical and thermal loads. The authors attributed several possible sources of error between the predicted and experimental results. They pointed out that the two-dimensional finite difference method applied to a complex thermal system such as a composite body leads to errors in the temperature predictions because it oversimplifies the physical events occurring in the composite body.

Gularte et al. (1988) presented a procedure to predict failure in laminated composite plates subjected to simultaneous localized heating and mechanical loads. The results from the two-dimensional finite difference thermal analysis were used as inputs for the stress and failure calculations. To analyze the composite structure a finite element approach based on a thermoelastic version of the FSDT was used. A comparison of experimental data and analytical predictions indicates disparities which the authors attribute to inaccuracies in the thermal calculations and the use of FSDT which may be overly simplistic to model the complex stress state.

Due to the limitations of one- and two-dimensional analyses, a three-dimensional heat conduction model is necessary to accurately determine both in-plane and through-the-thickness temperature profiles. Milke and Vizzini (1991) formulated a three-dimensional heat conduction model while omitting the thermomechanical
coupling term. This model allowed for temperature-dependent material properties and arbitrary locations of heat sources and sinks in addition to realistic boundary conditions. Based upon Fourier's Law, the equation for three-dimensional heat conduction in three mutually perpendicular directions in an anisotropic material was simplified by assuming that, over any small region of the laminate, the temperature can be assumed to be uniform in this region, leading to the following expression:

\[
K_{11} \frac{\partial^2 \theta}{\partial x^2} + K_{22} \frac{\partial^2 \theta}{\partial y^2} + K_{33} \frac{\partial^2 \theta}{\partial z^2} + (K_{12} + K_{21}) \frac{\partial^2 \theta}{\partial x \partial y} = \rho c \frac{\partial \theta}{\partial t} \ . \tag{2.13}
\]

This expression was applied successively over the numerous small isothermal regions and over time to determine the temperature distribution through the laminate. The three-dimensional thermal response using the finite difference provided good correlation between experimental data and predicted results.

It should be noted that several heat transfer models, including finite element models, have been presented in the literature for the thermal analysis of composite materials [see for example the extensive survey paper on the subject by Noor and Burton (1992b)]. However, due to the complex nature of anisotropic inhomogeneous multilayered composites, a three-dimensional analysis appears necessary due to the limitations of one- and two-dimensional approaches.

In order to fully and accurately describe the coupled thermoelastic behavior of a laminate to combined thermal and mechanical loading, a description of the stress and temperature field through the thickness must be identified. This would seem to the author to require a coupled ply-by-ply structural analysis rather than a smeared plate
theory and a coupled three-dimensional thermal response model. After reviewing the present state of the art on the coupled thermomechanical response of laminated plates, it is apparent that there is no completely acceptable solution for modeling a composite structure under combined loading. However, the shortcomings of current techniques are not necessarily in the laminate theory or the thermal response model, but rather in the application and extension of existing theories to incorporate the coupled response in a thermomechanical analysis. The following chapters will present a methodology for combining mechanical and thermal methods into a theory to describe the response of laminated composites subjected to combined loads.
CHAPTER 3

DERIVATION OF LAMINA STRESSES

Starting with the equilibrium of an elastic solid combined with the thermoelastic constitutive law and kinematic relations, the equations of generalized equilibrium in two dimensions are derived for a lamina subjected to combined mechanical and thermal loads. The derivation presented here is an extension of the model developed by Chyou (1989). For an assumed linear through-the-thickness distribution of in-plane stress components (mechanical and thermal), the transverse stresses are derived using a linear thermoelastic constitutive relationship. This gives rise to quadratic and cubic distributions through the lamina thickness of the transverse shear and normal stresses, respectively.

3.1 EQUILIBRIUM EQUATIONS OF A THERMOELASTIC BODY

Consider a rectangular plate of uniform thickness h which is homogeneous and linearly thermoelastic. The origin of the rectangular Cartesian coordinate system is
taken to be located at the mid-plane of the plate so that plate boundaries are given by
\[ x_1 = \pm a, \quad x_2 = \pm b, \quad \text{and} \quad x_3 = \pm \frac{h}{2} \] as shown in Figure 1.

The formulation begins with the differential equations of equilibrium for three-dimensional elasticity which can be written as:

\[ \sigma_{ij,j} + f_i = 0. \]  

Separating the equations into in-plane and out-of-plane expressions gives:

\[ \sigma_{\alpha \beta, \beta} + \sigma_{\alpha 3, 3} + f_\alpha = 0 \]  \hspace{1cm} (3.2)
\[ \sigma_{\alpha 3, \alpha} + \sigma_{33, 3} + f_3 = 0 \]  \hspace{1cm} (3.3)

where \( \sigma_{ij} \) are the components of the symmetric Cauchy stress tensor and \( f_i \) are the components of the body force vector per unit volume.

### 3.2 THERMOELASTIC CONSTITUTIVE LAW

For an isothermal, anisotropic, linear elastic material, the generalized Hooke's law can be written as

\[ \sigma_{ij} = E_{ijkl} \varepsilon_{kl} \]  \hspace{1cm} (3.4)

where \( \varepsilon_{kl} \) are the components of the symmetric strain tensor, and \( E_{ijkl} \) are the components of the rate independent isothermal elasticity tensor. Assuming the absence of body couples, the existence of a strain energy function and the symmetry of \( \sigma_{ij} \) and \( \varepsilon_{kl} \),

\[ E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij} \]
Figure 1. Lamina Geometry and Coordinate System
which reduces the number of independent constants from 81 to 21. If the material is monoclinic, i.e. one which exhibits symmetry about \( x_3 = 0 \), the number of independent constants is further reduced to 13. The constitutive equations are then of the form:

\[
\begin{align*}
\sigma_{\alpha\beta} &= E_{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} + E_{\alpha\beta33} \varepsilon_{33} \\
\sigma_{13} &= \sigma_{31} = 2 E_{\alpha3\beta3} \varepsilon_{\beta3} \\
\sigma_{33} &= E_{33\gamma\delta} \varepsilon_{\gamma\delta} + E_{3333} \varepsilon_{33}
\end{align*}
\]

(3.5)

or equivalently

\[
\begin{align*}
\varepsilon_{\alpha\beta} &= S_{\alpha\beta\gamma\delta} \sigma_{\gamma\delta} + S_{\alpha\beta33} \sigma_{33} \\
\varepsilon_{13} &= \varepsilon_{31} = 2 S_{\alpha3\beta3} \sigma_{\beta3} \\
\varepsilon_{33} &= S_{33\gamma\delta} \sigma_{\gamma\delta} + S_{3333} \sigma_{33}
\end{align*}
\]

(3.6)

where \( S_{ijkl} \) are the components of the rate independent isothermal compliance tensor with the symmetry properties

\[
S_{ijkl} = S_{jikl} = S_{ijlk} = S_{klij}.
\]

For non-isothermal situations, the Duhamel-Neuman postulate states that the total strain in a material is the sum of the contributions from the mechanical and thermal loads. This can be represented mathematically by

\[
\varepsilon_{ij} = \varepsilon_{ij}^{\text{M}} + \varepsilon_{ij}^{\text{T}}
\]

where \( \varepsilon_{ij}^{\text{M}} \) and \( \varepsilon_{ij}^{\text{T}} \) are the components of strain associated with the mechanical and thermal loads, respectively. \( \varepsilon_{ij}^{\text{T}} \) can be written in terms of \( \pi_{ij} \) and \( \Delta \theta \) (\( \Delta \theta = \theta - T_0 \)) which are the coefficients of the linear thermal expansion tensor and change in temperature.
Δθ, from the initial stress-free temperature, T₀, respectively. Without loss of
generality assume
\[ ε_{ij}^T = π_{ij} θ. \] (3.7)

The thermoelastic constitutive relationships are then
\[ ε_{αβ} = S_{αβγδ} σ_{γδ} + S_{αβ33} σ_{33} + π_{αβ} θ \]
\[ ε_{33} = ε_{33} = 2 S_{α3β3} σ_{β3} \]
\[ ε_{33} = S_{33γ6} σ_{γ6} + S_{3333} σ_{33} + π_{33} θ. \] (3.8)

Note that for the case of a monoclinic material, π_{α3} is identically zero. For a linear
system, the total stress in the body can be written as the superposition of the
mechanical and thermal contributions:
\[ σ_{αβ} = E_{αβγ6} ε_{γ6} + E_{αβ33} ε_{33} + Γ_{αβ} θ \]
\[ σ_{33} = E_{33γ6} ε_{γ6} + E_{3333} ε_{33} + Γ_{33} θ \] (3.9)

where Γ_{ij} is a second order tensor of thermal coefficients given by
\[ Γ_{ij} = -E_{ijkl} π_{kl}. \] (3.10)

In this study it is assumed that the fiber and matrix are both thermally
anisotropic (i.e., the coefficients of thermal expansion vary with direction). In
addition, it is assumed that the thermal stress-strain relationships, and hence the
elasticity, compliance, and thermal expansion tensors are independent of the time
variable (i.e., rate independent material properties).
3.3 KINEMATICS

For small deformation theory, the kinematic relations for linear thermoelasticity can be written as:

\[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) = u_{(i,j)} \]  

(3.11)

where \( u_i \) are the components of the displacement vector.

3.4 PLATE EQUATIONS OF EQUILIBRIUM

In this section, the three-dimensional problem is reduced to two dimensions by integrating the equilibrium equations over the transverse direction, \( x_3 \), of the lamina. Integrating equations (3.2) and (3.3) over the thickness of the lamina yields:

\[ \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} \, dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{33,3} \, dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_\alpha \, dx_3 = 0 \]  

(3.12)

\[ \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{33,\alpha} \, dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{33,3} \, dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_3 \, dx_3 = 0 \]  

(3.13)

Integrating the first moment of equation (3.2) over the thickness of the lamina gives:

\[ \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta,\beta} \, x_3 \, dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{33,3} \, x_3 \, dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_\alpha \, x_3 \, dx_3 = 0 \]  

(3.14)
Define the in-plane force resultants and boundary terms as:

\[
N_{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} \, dx_3 \quad (3.15)
\]

\[
M_{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} x_3 \, dx_3 \quad (3.16)
\]

\[
V_\alpha = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha 3} \, dx_3 \quad (3.17)
\]

\[
\sigma_{\alpha 3}^+ = \sigma_{33} (x_1, x_2, \frac{h}{2}) \quad (3.18)
\]

\[
\sigma_{\alpha 3}^- = \sigma_{33} (x_1, x_2, -\frac{h}{2}) \quad (3.19)
\]

where the components of \( \sigma_{ij} \) are as defined in equation (3.9). Assuming that the body force per unit volume is constant over the thickness of the lamina, define:

\[
F_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i \, dx_3 = h f_i \quad (3.20)
\]

Then equations (3.12), (3.13), and (3.14) can be written as:

\[
N_{\alpha\beta,\beta} + (\sigma_{\alpha 3}^+ - \sigma_{\alpha 3}^-) + F_\alpha = 0 \quad (3.21)
\]

\[
V_{\alpha,\alpha} + (\sigma_{33}^+ - \sigma_{33}^-) + F_3 = 0 \quad (3.22)
\]

\[
M_{\alpha\beta,\beta} + \frac{h}{2} (\sigma_{\alpha 3}^+ + \sigma_{\alpha 3}^-) - V_\alpha = 0 \quad (3.23)
\]
Equations (3.21), (3.22), and (3.23) represent the generalized equations of equilibrium for a lamina. The formulation thus far is exact, i.e., no assumptions regarding the distribution of stresses, displacements, or temperature have been made.

3.5 DERIVATION OF A CONSISTENT STRESS FIELD

Definitions (3.15), (3.16), and (3.17) express $N_{a\beta}$, $M_{a\beta}$, and $V_\alpha$ as generalized stress and moment resultants of the in-plane components of the stress tensor. The inverse relationship, i.e., the stress distribution for a given $N_{a\beta}$, $M_{a\beta}$, and $V_\alpha$ is not uniquely defined. However if an assumption is made regarding the distribution of some of the components of $\sigma_{ij}$, the distribution of the others may be determined. For a homogeneous plate Reissner (1947, 50) assumed a linear in-plane stress distribution across the thickness of the plate.

In the current formulation, the in-plane stress distribution is assumed to be linear across thickness of the lamina as $\sigma_{a\beta} = A_{a\beta} + B_{a\beta}x$. Substituting this expression into equations (3.15) and (3.16) and solving for the variables $A_{a\beta}$ and $B_{a\beta}$ gives the in-plane stresses in terms of in-plane stress resultants as:

$$\sigma_{a\beta} = \frac{N_{a\beta}}{h} + \frac{12 M_{a\beta}}{h^3} x.$$  \hspace{1cm} (3.24)

Using the superposition principle, the contribution of stresses from the thermal and mechanical loads can be solved independently and then added together. Since the total stress distribution is assumed to be linear across the thickness of the lamina, the
thermal stress distribution is assumed linear and hence the temperature distribution is also assumed to be linear as:

\[
\theta(x_1, x_2, x_3, t) = \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-)
\]  

(3.25)

where

\[
\theta^+ = \theta(x_1, x_2, \frac{h}{2}, t)
\]

\[
\theta^- = \theta(x_1, x_2, \frac{-h}{2}, t)
\]

are the boundary temperature terms. The distribution of the temperature is now in terms of boundary temperature terms, \( \theta^+ \) and \( \theta^- \), \( h \) and \( x_3 \).

Integrating equation (3.2) through the thickness of the lamina and rearranging yields:

\[
\sigma_{a_3} = \sigma_{a_3}^+ - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{a\beta, \beta} d\eta_3 - \int_{-\frac{h}{2}}^{\frac{h}{2}} f_\alpha d\eta_3.
\]  

(3.26)

Substituting equation (3.24) into (3.26) gives:

\[
\sigma_{a_3} = \sigma_{a_3}^+ - \int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{ \frac{N_{a\beta, \beta}}{h} + \frac{12 M_{a\beta, \beta}}{h^3} \eta_3 \right\} d\eta_3 - \int_{-\frac{h}{2}}^{\frac{h}{2}} f_\alpha d\eta_3.
\]  

(3.27)
Combining equations (3.21) and (3.23) with equations (3.27) and simplifying results in the following expression for the transverse shear stress:

\[
\sigma_{a3} = (\sigma_{a3}^+ - \sigma_{a3}^-) \cdot \frac{x_3}{h} + \frac{1}{4} \left(\frac{\sigma_{a}^+ + \sigma_{a}^-}{2} \right) \left(\frac{12}{h^3} x_3^2 - 1\right) + \frac{3}{2h} V_\alpha \left(1 - \frac{4}{h^2} x_3^3\right)
\]

(3.28)

To derive an expression for the transverse normal stress \(\sigma_{33}\), equation (3.3) is integrated in the through-the-thickness direction.

\[
\sigma_{33} = \sigma_{33}^- - \int_{h/2}^{h/2} \sigma_{a3,3} \, d\eta_3 - \int_{h/2}^{h/2} f_3 \, d\eta_3
\]

(3.29)

Differentiating equation (3.28) with respect to \(x_3\), substituting into equation (3.29) and using equation (3.22) yields:

\[
\sigma_{33} = \frac{1}{2} \left(\sigma_{33}^+ + \sigma_{33}^-\right) - \left\{ \left(\sigma_{a3,3}^+ - \sigma_{a3,3}^-\right) \left(\frac{x_3^2}{2h} - \frac{h}{8}\right) + \left(\sigma_{a3,3}^+ + \sigma_{a3,3}^-\right) \right\} - \frac{3}{2} \left(\sigma_{33}^+ - \sigma_{33}^-\right) \left(\frac{x_3}{h} - \frac{4}{3} \frac{x_3^3}{h^3}\right) - f_3 \left(\frac{2}{h^2} \frac{x_3^2}{2} - \frac{x_3}{2}\right).
\]

(3.30)

Following Pagano’s (1978a, b) formulation, two new terms, the transverse generalized force and moment resultants, are introduced and defined as

\[
N_{33} = \int_{h/2}^{h/2} \sigma_{33} \, dx_3
\]

(3.31)

\[
M_{33} = \int_{h/2}^{h/2} \sigma_{33} \, x_3 \, dx_3.
\]

(3.32)
Substituting equation (3.30) into equations (3.31) and (3.32) yields:

\[ N_{33} = \frac{h}{2} (\sigma_{33}^+ + \sigma_{33}^-) + \frac{h^2}{12} (\sigma_{a3,3}^+ - \sigma_{a3,3}^-) \]  
\[ M_{33} = \frac{h^3}{120} (\sigma_{a3,3}^+ + \sigma_{a3,3}^-) + \frac{h^2}{10} (\sigma_{33}^+ - \sigma_{33}^-) + \frac{h^3}{60} f_3. \]

Rearranging, equation (3.33) is written as

\[ (\sigma_{a3,3}^+ - \sigma_{a3,3}^-) = \frac{12}{h^2} \left\{ N_{33} - \frac{h}{2} (\sigma_{33}^+ + \sigma_{33}^-) \right\}. \]

Similarly, equation (3.34) is rewritten as

\[ (\sigma_{a3,3}^+ + \sigma_{a3,3}^-) = \frac{120}{h^3} \left\{ M_{33} - \frac{h^2}{10} (\sigma_{33}^+ - \sigma_{33}^-) \right\} - 2 f_3. \]

Substituting equation (3.35) and (3.36) into (3.30) yields the final form for the transverse normal stress distribution across the thickness of a lamina.

\[ \sigma_{33} = \frac{(\sigma_{33}^+ + \sigma_{33}^-)}{4} \left( \frac{12 x_3^2}{h^3} - 1 \right) + \frac{(\sigma_{33}^+ - \sigma_{33}^-)}{4} \left( \frac{40 x_3^3}{h^4} - \frac{6 x_3^4}{h} \right) \]

\[ + \frac{3 N_{33}}{2h} \left( 1 - \frac{4 x_3^2}{h^2} \right) + \frac{15 M_{33}}{h^2} \left( \frac{2 x_3^3}{h} - \frac{8 x_3^4}{h^3} \right). \]  

In this chapter, an approximation based on an assumed distribution of in-plane stresses [refer to equation (3.24)] has been assumed. Using the three dimensional equations of equilibrium for a lamina with small thickness, the out-of-plane transverse stresses have been derived [refer to equations (3.28) and (3.37) respectively].
CHAPTER 4

THE GOVERNING FIELD EQUATIONS

The initial-boundary value problem of interest consists of the equations of thermoelasticity including heat conduction. In this chapter, the field equations are derived for a laminate. The generalized variational technique presented in this section is based on that given by Sandhu and Salaam (1975) and is used to derive the field equations.

4.1 VARIATIONAL FORMULATION

The fundamental system of field equations for a linear coupled thermoelastic solid characterizing anisotropic materials is stated for reference. The thermal gradient-temperature relations, heat conduction equations, and energy balance equations can be expressed respectively as:

\[ g_i = \theta_i \]  

\[ q_i = -K_{ij} g_j \]  

\[ -q_{i,1} + r = \rho c \dot{\theta} + T_0 \Gamma_{ij} \dot{\varepsilon}_{ij} \]
where $g_i$ is the thermal gradient vector. $K_{ij}$ are the components of the symmetric thermal conductivity tensor, and the terms, as defined in equation (2.10), are as follows:

- $q$ is the heat flux
- $r$ is the internal heat generation per unit time per unit volume
- $\rho$ is the mass density
- $c$ is the specific heat at constant volume
- $\theta$ is the temperature above the reference temperature
- $T_0$ is the reference temperature at which zero strain yields zero stress
- $\Gamma$ is the symmetric tensor of thermal coefficients
- $\varepsilon$ is the symmetric strain tensor.

The kinematic equations, (3.11), thermoelastic constitutive relations, (3.9), and equilibrium equations, (3.1) respectively, are restated for reference as:

\begin{align*}
\varepsilon_{ij} &= \frac{1}{2} (u_{i,j} + u_{j,i}) = u_{(i,j)} \quad (4.4) \\
\sigma_{ij} &= E_{ijkl} \varepsilon_{kl} + \Gamma_{ij} \theta \quad (4.5) \\
\sigma_{ij,i} + f_i &= 0. \quad (4.6)
\end{align*}

Equations (4.1) - (4.6) represent the governing equations of coupled thermoelasticity.
4.1.1 Restatement of the Governing Equations

A major motivation for recasting of the initial-boundary value problem of coupled thermoelasticity is to eliminate the time derivatives from the equations and incorporate the initial conditions explicitly into the field equations and into the functionals which arise in the variational formulation. It is convenient to use Gurtin’s (1963, 64) bilinear mapping. The non-degenerative bilinear mapping of Gurtin is given by

$$ B_R \times_1 (u, v) = \int_R u \ast v \, dR = \int_R \int_0^t u(x, t-\tau) v(x, \tau) \, d\tau $$

(4.7)

where the " \ast " denotes the convolution integral and satisfies distributive, associative, and commutative laws. u and v are functions of position and time, R is an open connected bounded region in Euclidean space with the interior volume V and the surface boundary S. Dividing equation (4.3) first by $T_0$, the convolution integral can be applied to the terms on the right hand side as

$$ g^{\ast} \frac{\rho c}{T_0} \dot{\theta} = \int_0^t \frac{\rho c}{T_0} \frac{\partial \theta(t-\tau)}{\partial(t-\tau)} \, d\tau $$

$$ = -\frac{\rho c}{T_0} \left[ \frac{\partial \theta(t-\tau)}{\partial(t-\tau)} \right]_{\tau=0}^{t=0} $$

$$ = \frac{\rho c}{T_0} \theta(x, t) - \frac{\rho c}{T_0} \theta(x, 0) $$

57
\[ \frac{\partial c}{\partial T} \theta - \frac{\partial c}{\partial T} \theta_0 \]  

(4.8)

where \( \theta (x, 0) = \theta_0 \), the temperature at time \( t = 0 \).

\[ g' \Gamma_{ij} \dot{\epsilon}_{ij} = \int_0^t \Gamma_{ij} \frac{\partial \epsilon_{ij}(t-\tau)}{\partial(t-\tau)} \, d\tau \]

\[ = -\Gamma_{ij} \left[ \epsilon_{ij}(t-\tau) \right]_{\tau=0} \]

\[ = \Gamma_{ij} \epsilon_{ij}(x, t) - \Gamma_{ij} \epsilon_{ij}(x, 0) \]

\[ = \Gamma_{ij} \epsilon_{ij} - \Gamma_{ij} d_{i,j} \]  

(4.9)

where \( \epsilon_{ij}(x, 0) = u_{(i,j)}(x, 0) = d_{i,j} \) and the function \( g' = g'(t) = 1 \) for \( 0 \leq t < \infty \).

Equation (4.3) can be restated as:

\[ -\Gamma_{ii} q_{i} + \frac{g'}{T_0} r = \frac{\partial c}{\partial T} \theta - \frac{\partial c}{\partial T} \theta_0 + \Gamma_{ij} \epsilon_{ij} - \Gamma_{ij} d_{i,j} \]  

(4.10)

Define the following term as:

\[ b(x, t) = \left[ \frac{g'}{T_0} \cdot r \right](x, t) + \frac{\partial c}{\partial T} \theta_0 + \Gamma_{ij} d_{i,j} \]  

(4.11)

Therefore equations (4.1) - (4.3) can be restated as:

\[ \frac{g'}{T_0} g_i = \frac{g'}{T_0} \theta_i \]  

(4.12)

\[ \frac{g'}{T_0} q_i = -\frac{g'}{T_0} K_{ij} g_j \]  

(4.13)

\[ -\frac{g'}{T_0} q_{i,i} + b = \frac{\partial c}{\partial T} \theta + \Gamma_{ij} \epsilon_{ij} \]  

(4.14)
The restated coupled field equations can be written in matrix form as \( Aw = \nu \)

on \( R \) where \( A \) is the matrix of linear operators:

\[
A = \begin{bmatrix}
0 & 0 & -\frac{1}{2}(\delta_{ik} \frac{\partial}{\partial j} + \delta_{jk} \frac{\partial}{\partial i}) & 0 & 0 & 0 \\
0 & E_{ijkl} & -1 & \Gamma_{ij} & 0 & 0 \\
\frac{1}{2}(\delta_{ik} \frac{\partial}{\partial l} + \delta_{jk} \frac{\partial}{\partial k}) & -1 & 0 & 0 & 0 & 0 \\
0 & \Gamma_{kl} & 0 & \rho c \frac{\partial}{\partial T} & 0 & \frac{g'}{T} \frac{\partial}{\partial T} \\
0 & 0 & 0 & 0 & \frac{K_{kl}}{T} & \frac{g'}{T} \frac{\partial}{\partial T} \\
0 & 0 & 0 & 0 & -\frac{g'}{T} \frac{\partial}{\partial T} & \frac{g'}{T} \frac{\partial}{\partial T} & 0
\end{bmatrix}
\]

\( w = \begin{bmatrix} u_i \\ \varepsilon_{ij} \\ \sigma_{ij} \\ \theta \\ g_k \\ q_i \end{bmatrix} \quad \text{and} \quad \nu = \begin{bmatrix} f_k \\ 0 \\ 0 \\ b \\ 0 \end{bmatrix}. \quad (4.15)

where \( \delta_{ik} \) is the Kronecker delta.

Equation (4.15) includes the equilibrium, thermoelastic constitutive relationships, kinematic equations, heat balance equation, Fourier's Law, and thermal gradient relations. Associated with this coupled system of field equations are the displacement, traction, temperature, and heat flux consistent boundary conditions which can be expressed respectively as:

\(- u_i = - S_i \) on \( S_u \) \quad (4.16)

\( n_j \sigma_{ij} = t_i \) on \( S_\sigma \) \quad (4.17)
\[
\frac{g'}{T_c} \cdot n_j \hat{\theta} = \frac{g'}{T_c} \cdot n_j \hat{\theta} \text{ on } S_0 \quad (4.18)
\]

\[
- \frac{g'}{T_c} \cdot n_j q_j = - \frac{g'}{T_c} \cdot \hat{Q} \text{ on } S_q \quad (4.19)
\]

as well as the internal jump conditions, expressed as:

\[- u'_i = - g'_i \text{ on } S'_u \quad (4.20)\]

\[n_j \sigma'_{ij} = h'_i \text{ on } S'_\sigma \quad (4.21)\]

\[(\frac{g'}{T_c} \cdot n_j \hat{\theta})' = c_i' \text{ on } S'_0 \quad (4.22)\]

\[-(\frac{g'}{T_c} \cdot n_j q_j)' = - d_i' \text{ on } S'_q \quad (4.23)\]

where \(S_i\) are the prescribed surface displacements, \(t_i\) are the prescribed surface tractions, \(\hat{\theta}\) are the prescribed surface temperatures, and \(\hat{Q}\) are the prescribed surface heat flows. \(S_u, S_\sigma, S_\theta,\) and \(S_q\) represent the portion of the boundary on which displacement, traction, temperature, and heat flux respectively are specified. while \(S'_u, S'_\sigma, S'_\theta,\) and \(S'_q\) represent the portion of the internal region \(R\) on which displacement jump, traction jump, temperature jump, and heat flux jump conditions, respectively are specified. Components of the unit normal vector are denoted by \(n_i\). The surfaces are such that

\[S_u + S_\sigma = S_0 + S_q = S\]

\[S_u \cap S_\sigma = 0 \text{ and } S_0 \cap S_q = 0\]

and \(S'_u, S'_\sigma, S'_\theta, S'_q \subset R\)
The governing function can be written using equations (4.15) - (4.23) as:

\[ 2\Omega = < u_i, (\sigma_{ij} - 2f_i) > R + < e_{ij}, (E_{ijkl} e_{kl} - \sigma_{ij} + \Gamma_{ij} \theta) > R + < \sigma_{ij}, (u_{ij} - e_{ij}) > R \\
+ < \theta, (\Gamma_{ij} e_{ij} + \frac{\rho c}{T_s} \theta + \frac{g'}{T_s} q_{ij} - 2b) > R + < g_i, (\frac{g'}{T_o} * K_{ij} g_i + \frac{g'}{T_s} * q_{ij} ) > R \\
+ < q_{ij}, (\frac{g'}{T_s} * \theta_{ij} + \frac{g'}{T_s} * g_i ) > R + < n_i, \sigma_{ij}, (-u_{ij} + 2S_i ) > S_s \\
+ < u_i, (n_j \sigma_{ij} - 2t_j ) > S_v + < n_j q_i, (\frac{g'}{T_s} * n_j \theta - 2 \frac{g'}{T_s} * n_j \dot{\theta} ) > S_q \\
+ < n_j \theta, (\frac{g'}{T_s} * n_j q_j + 2 \frac{g'}{T_s} \dot{Q}) > S_q + < n_j \sigma_{ij}, (-u_{ij}' + 2g_i' ) > S_q \\
+ < u_i, (n_j \sigma_{ij}' - 2h_j') > S_v + < n_j q_i, (\frac{g'}{T_s} * n_j \theta' - 2c_i' ) > S_q' \\
+ < n_j \theta, (\frac{g'}{T_s} * n_j q_j' + 2 d_i' ) > S_q' \] (4.24)

Defining an inner product space as

\[ < u, v > R = \int_R u v \, dR \] (4.25)

Using equation (4.25), equation (4.24) takes the form:

\[ 2\Omega = -\int_V \left[ u_i (\sigma_{ij} + 2f_i) + e_{ij} (-E_{ijkl} e_{kl} + \sigma_{ij} - \Gamma_{ij} \theta) + \sigma_{ij} (-u_{ij} + e_{ij}) + \theta (-\Gamma_{ij} e_{ij} - \frac{\rho c}{T_s} \theta - \frac{g'}{T_s} * q_{ij} + 2b) + g_i (-\frac{g'}{T_o} * K_{ij} g_i - \frac{g'}{T_s} * q_{ij} ) + q_i (\frac{g'}{T_s} * \theta_{ij} - \frac{g'}{T_s} * g_i ) \right] \, dV \]
The complementary form of the governing function is obtained by identically satisfying the thermoelastic constitutive law \( (\varepsilon_{ij} = S_{ijkl}\sigma_{kl} + \pi_{ij}\theta) \) and by identically satisfying Fourier’s Law \( (g_i = -\lambda_{ij}q_j) \) thereby eliminating both \( \varepsilon_{ij} \) and \( g_i \) from the governing function. This gives:

\[
2\Pi_{ij} = -\int_V \left[ u_i (\sigma_{ij,j} + 2f_{ij}) + \sigma_{ij} (-u_{j,i} + S_{ijkl}\sigma_{kl} + \pi_{ij}\theta) + \theta (-\Gamma_{ij} S_{ijkl}\sigma_{kl} - \Gamma_{ij}\pi_{ij}\theta - \frac{\rho c}{T_o} \theta - \frac{g'}{T_o} \ast q_{i,i} + 2 \{ \frac{g'}{T_o} \ast r + \frac{\rho c}{T_o} \theta(x, 0) + \Gamma_{ij} S_{ijkl}\sigma_{kl}(x, 0) + \Gamma_{ij} \pi_{ij}\theta(x, 0) \}) + q_i \left( \frac{g'}{T_o} \ast \theta_{i,i} + \frac{g'}{T_o} \ast \lambda_{ij}q_j \right) \right] dV
\]

\[
-\int_{s_a} n_j \sigma_{ij} (u_i - 2S_{ij}) dS + \int_{s_a} u_i (n_j \sigma_{ij} - 2t_{ij}) dS
\]

\[
+ \int_{s_a} n_j q_i \left( \frac{g'}{T_o} \ast n_j \theta - 2 \frac{g'}{T_o} \ast n_j \hat{\theta} \right) dS - \int_{s_a} n_j \theta \left( \frac{g'}{T_o} \ast n_j q_i - 2 \frac{g'}{T_o} \ast \hat{Q} \right) dS
\]
\[- \int_{s_4} n_i \sigma_{ij} (u_i' - 2g_i') \, dS + \int_{s_4} u_i (n_i \sigma_{ij}' - 2h_i') \, dS \]

\[+ \int_{s_4} n_i q_i' (\frac{g_i'}{T_o} \cdot n_i \theta' - 2c_i') \, dS - \int_{s_4} n_i \theta (\frac{g_i'}{T_o} \cdot n_i q_i - 2d_i') \, dS \quad (4.27)\]

Using the divergence theorem,

\[
\int_{V} u_i \sigma_{ij} \, dV = - \int_{V} u_{ij} \sigma_{ij} \, dV + \int_{s_4} u_i \sigma_{ij} n_i \, dS + \int_{s_4} u_i \sigma_{ij}' n_i \, dS \\
+ \int_{s_4} u'_i \sigma_{ij} n_i \, dS + \int_{s_4} u_i \sigma_{ij}' n_i \, dS \quad (4.28)
\]

Substituting equation (4.28) into equation (4.27), the spatial derivatives on \( \sigma_{ij} \) are eliminated. This gives:

\[2\Pi_{(III)} = - \int_{V} \left[ -u_{ij} \sigma_{ij} + 2u_if_i - u_{ij} \sigma_{ij} + \sigma_{ij} S_{ijkl} \sigma_{kl} + \sigma_{ij} \pi_{ij} \theta \right. \]

\[+ \theta (-\Gamma_{ij} S_{ijkl} \sigma_{kl} - \Gamma_{ij} \pi_{ij} \theta - \frac{\rho c}{T_o} \theta - \frac{g_i'}{T_o} \cdot q_{i,j} + 2\{ \frac{g_i'}{T_o} \cdot \tau + \frac{\rho c}{T_o} \theta(x, 0) 
\]

\[+ \Gamma_{ij} S_{ijkl} \sigma_{kl}(x, 0) + \Gamma_{ij} \pi_{ij} \theta(x, 0) \}) + q_i (\frac{g_i'}{T_o} \cdot \theta_i + \frac{g_i'}{T_o} \cdot \lambda q_i ) \] \]dV

\[- 2 \int_{s_4} n_i \sigma_{ij} (u_i - S_i) \, dS - 2 \int_{s_4} u_i t_i \, dS \\
2 \int_{s_4} n_i \sigma_{ij} (u_i' - g_i') \, dS - 2 \int_{s_4} u_i n_i \, dS \]
\[ + \int_{s_u} n_iq_i \left( \frac{g'}{T_c} \star n_j \theta - 2 \frac{g'}{T_c} \star n_j \theta \right) dS - \int_{s_u} n_i \theta \left( \frac{g'}{T_c} \star n_j q_j - 2 \frac{g'}{T_c} \star \hat{Q} \right) dS \\
+ \int_{s_u} n_iq_i \left( \frac{g'}{T_c} \star n_j \theta' - 2c \right) dS - \int_{s_u} n_i \theta \left( \frac{g'}{T_c} \star n_j q_j' - 2d \right) dS \quad (4.29) \]

Rewriting equation (4.29) and expanding \( \theta \):

\[ \Gamma_{III} = -\int \left[ u_i f_i \left\{ \frac{1}{2} \left( u_{ij} + u_{ji} \right) \sigma_{ij} + \frac{1}{2} \sigma_{ii} \sigma_{kl} \sigma_{kl} + \sigma_{ij} \pi_{ij} \left( \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right) \right\} + \frac{1}{2} \left( \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right) \right] \left\{ \frac{g'}{T_c} \star q_{\alpha \alpha} \right\} \]

\[ - \frac{1}{2} \left( \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right) \left\{ \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right\} \left\{ \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right\} \]

\[ - \frac{1}{2} \left( \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right) \left\{ \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right\} \]

\[ + \left\{ \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right\} \left\{ \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right\} \left\{ \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right\} \]

\[ + \Gamma_{ij} \pi_{ij} \theta(x,0) \} + \frac{1}{2} q_o \frac{g'}{T_c} \star \left\{ \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right\} \]

\[ + \frac{1}{2} q_o \frac{g'}{T_c} \star \lambda_{\alpha \beta} q_{\beta} \right\} dV - \int_{s_u} n_i \sigma_{ij} \left( u_i - S_i \right) dS - \int_{s_u} u_i t_i dS \\
- \int_{s_u} n_i \sigma_{ij} \left( u_i' - g_i' \right) dS - \int_{s_u} u_i h'_i dS \]
\[\begin{align*}
\frac{1}{2} \int_{S} n_{ij} \left( g' \cdot n_{i} \theta - 2 g' \cdot n_{i} \tilde{Q} \right) dS - \frac{1}{2} \int_{S} n_{i} \theta \left( \frac{g'}{T} \cdot n_{i} q_{j} - 2 \frac{g'}{T} \cdot \tilde{Q} \right) dS \\
+ \frac{1}{2} \int_{S} n_{ij} \left( g' \cdot n_{i} \theta' - 2 c' \right) dS - \frac{1}{2} \int_{S} n_{i} \theta \left( \frac{g'}{T} \cdot n_{i} q_{j}' - 2 d' \right) dS
\end{align*}\]

Hence, the governing function can be expressed as:

\[J = \Pi_{(iii)} (u_{i}, u_{i,j}, \sigma_{ij}, \theta^{+}, \theta^{-}, \theta_{\alpha}^{+}, \theta_{\alpha}^{-}, q_{\alpha}, q_{\alpha,\alpha})\]

Taking the first variation of equation (4.30) yields:

\[\delta J = \frac{\partial \Pi_{(iii)}}{\partial u_{i}} \delta u_{i} + \frac{\partial \Pi_{(iii)}}{\partial u_{i,j}} \delta u_{i,j} + \frac{\partial \Pi_{(iii)}}{\partial \sigma_{ij}} \delta \sigma_{ij} + \frac{\partial \Pi_{(iii)}}{\partial \theta^{+}} \delta \theta^{+} + \frac{\partial \Pi_{(iii)}}{\partial \theta^{-}} \delta \theta^{-}
\]

Using equation (4.32) gives:

\[\delta J = \int_{V} \left\{ f_{i} \delta u_{i} - \sigma_{ij} \delta u_{i,j} + \left[ \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) + S_{ijkl} \sigma_{kl} \right] \right.\]

\[+ \pi_{ij} \left\{ \frac{\theta^{+} + \theta^{-}}{2} + \frac{x_{i}}{h} (\theta^{+} - \theta^{-}) \right\} \delta \sigma_{ij} + \left[ \sigma_{ij} \pi_{ij} - (\Gamma_{ij} \pi_{ij} + \frac{pc}{T}) \right] \left\{ \frac{\theta^{+} + \theta^{-}}{2} ight.\]

\[+ \frac{x_{i}}{h} (\theta^{+} - \theta^{-}) - \frac{1}{2} \frac{g'}{T_{i}} \cdot q_{\alpha,\alpha} + \frac{g'}{T_{i}} \cdot \tilde{r} + \left( \frac{pc}{T_{i}} + \Gamma_{ij} \pi_{ij} \right) \theta(x,0) - \pi_{ij} \sigma_{ij}(x,0)\]

\[\left. + \frac{x_{i}}{h} (\theta^{+} - \theta^{-}) \right\} \delta \theta^{+} + \left[ \sigma_{ij} \pi_{ij} - (\Gamma_{ij} \pi_{ij} + \frac{pc}{T}) \right] \left\{ \frac{\theta^{+} + \theta^{-}}{2} + \frac{x_{i}}{h} (\theta^{+} - \theta^{-}) \right\}\]

65
\[-\frac{1}{2} \frac{g'}{T_s} \ast q_{\alpha\alpha} + \frac{g'}{T_s} \ast r + \left( \frac{\partial c}{T} + T \frac{\varphi}{\partial x} \right) \theta(x,0) - \varpi_{ij} \sigma_{ij}(x,0) \left( \frac{1}{2} - \frac{x}{h} \right) \delta \theta \]

\[+ \frac{1}{2} \left( \frac{1}{2} + \frac{x}{h} \right) \frac{g'}{T_s} \ast q_{\alpha} \delta \theta^*_{\alpha} + \frac{1}{2} \left( \frac{1}{2} - \frac{x}{h} \right) \frac{g'}{T_s} \ast q_{\alpha} \delta \theta^-_{\alpha} \]

\[+ \frac{1}{2} \left( \frac{g'}{T_s} \ast \left\{ \frac{\theta^*_{\alpha} + \theta^-_{\alpha}}{2} + \frac{x}{h} (\theta^*_{\alpha} - \theta^-_{\alpha}) \right\} + \frac{2}{T_s} \lambda_{\alpha \beta} q_{\beta} \right) \delta q_{\alpha} \]

\[+ \frac{1}{2} \frac{g'}{T_s} \ast \left\{ \frac{\theta^* + \theta^-}{2} + \frac{x}{h} (\theta^* - \theta^-) \right\} \delta q_{\alpha \alpha} \right\} dV \]

\[+ \int_{s_{u}} t_{i} \delta u_{i} dS + \int_{s_{t}} h'_{i} \delta u_{i} dS + \int_{s_{u}} (u_{i} - S_{i}) \delta (n_{i} \sigma_{ij}) dS \]

\[+ \int_{s_{u}} (u_{i}' - g_{i}') \delta (n_{i} \sigma_{ij}) dS - \frac{1}{2} \int_{s_{u}} \left( \frac{g'}{T_s} \ast n_{i} \theta - 2 \frac{g'}{T_s} \ast n_{i} \dot{\theta} \right) \delta q_{i} dS \]

\[+ \frac{1}{2} \int_{s_{u}} \left( \frac{g'}{T_s} \ast n_{i} q_{i} - 2 \frac{g'}{T_s} \ast \dot{Q} \right) \delta \theta dS - \frac{1}{2} \int_{s_{u}} \left( \frac{g'}{T_s} \ast n_{i} \theta' - 2 c_{i} \right) \delta q_{i} dS \]

\[+ \frac{1}{2} \int_{s_{u}} \left( \frac{g'}{T_s} \ast n_{i} q_{i}' - 2 d_{i}' \right) \delta \theta dS = 0 \]  

(4.33)

Applying the divergence theorem to \[\int_{v} \sigma_{ij} \delta u_{i,j} dV, \int_{v} \frac{1}{2} \frac{g'}{T_s} \ast q_{\alpha} \delta \theta^*_{\alpha} dV. \]

\[\int_{v} \frac{1}{2} \frac{g'}{T_s} \ast q_{\alpha} \delta \theta^-_{\alpha} dV \text{ and } \int_{v} \frac{1}{2} \frac{g'}{T_s} \ast \theta \delta q_{\alpha \alpha} dV \text{ terms, the } \delta u_{i,j}, \delta \theta^*_{\alpha}, \delta \theta^-_{\alpha} \text{ and } \delta q_{\alpha \alpha} \]

terms respectively can be eliminated from equation (4.33) as:

66
\[ \delta J = \int_V \left\{ \delta u_i [\sigma_{ij} + f_i] + \delta \sigma_{ij} \left[ - \frac{1}{2} (u_{ij} + u_{ji}) + S_{ijkl} \sigma_{kl} + \pi_{ij} \left( \frac{\theta^+ + \theta^-}{2} \right) \right] \right. \\
\left. + \frac{x_3}{h} (\theta^+ - \theta^-) \right\} + \delta \theta^* [\sigma_{ij} \pi_{ij} - (\Gamma \pi_{ij} + \frac{\rho c}{T_0}) \left\{ \frac{\theta^+ + \theta^-}{2} + \frac{x_3}{h} (\theta^+ - \theta^-) \right\} \\
- \frac{g^*}{T_0} q_{\alpha \alpha} + \frac{g^*}{T_0} \ast r + \left( \frac{\rho c}{T_0} + \Gamma \pi_{ij} \right) \theta (x, 0) - \pi_{ij} \sigma_{ij} (x, 0) \left[ \frac{1}{2} + \frac{x_3}{h} \right] \right. \\
+ \delta \theta^* [\sigma_{ij} \pi_{ij} - (\Gamma \pi_{ij} + \frac{\rho c}{T_0}) \left\{ \frac{\theta^+ + \theta^-}{2} + \frac{x_3}{h} (\theta^+ - \theta^-) \right\} - \frac{g^*}{T_0} \ast q_{\alpha \alpha} + \frac{g^*}{T_0} \ast r \\
+ \left( \frac{\rho c}{T_0} + \Gamma \pi_{ij} \right) \theta (x, 0) - \pi_{ij} \sigma_{ij} (x, 0) \left[ \frac{1}{2} - \frac{x_3}{h} \right] \right. \\
+ \delta q_{4d} \left\{ \frac{g^*}{T_0} \left\{ \frac{\theta^+ + \theta^-}{2} + \frac{x_3}{h} (\theta^+ - \theta^-) \right\} + \frac{g^*}{T_0} \ast \lambda_{\alpha \beta} q_{\beta} \right\} \right\} dV \\
+ \int_{s_a} (u_i - S_i) \delta (n_j \sigma_{ij}) dS + \int_{s_a} (u_i' - g_i') \delta (n_j \sigma_{ij}) dS \\
- \int_{s_a} (n_j \sigma_{ij} - 1_i) \delta u_i dS - \int_{s_a} (n_j \sigma_{ij} - h_i') \delta u_i dS \\
- \int_{s_4} \left( \frac{g^*}{T_0} n_i \theta - \frac{g^*}{T_0} n_i \hat{\theta} \right) \delta q_{4d} dS + \int_{s_4} \left( \frac{g^*}{T_0} n_i q_j - \frac{g^*}{T_0} \hat{Q} \right) \delta \theta dS \\
- \int_{s_4} \left( \frac{g^*}{T_0} n_i \theta' - c_i' \right) \delta q_{4d} dS + \int_{s_4} \left( \frac{g^*}{T_0} n_i q_j' - d_i' \right) \delta \theta dS = 0 \quad (4.34) \]

This is the function that will be used to derive the governing field equations for the coupled linear thermoelastic medium under consideration.
The derivation thus far has been limited to a single lamina with applied tractions/displacements (temperature/heat flux) on its boundaries. Consider a laminated body composed of N layers with the volume of each layer represented by $V_k$ ($k=1,2,3,...,N$), and a stack of N laminae bonded such that layer $k=1$ is the top lamina and layer $k=N$ is the bottom lamina of the laminate. A representative four-layer laminate system is depicted in Figure 2. Observe that the surface $S$ represents the boundary, i.e. the edges of the laminae, as well as the top of the first lamina and bottom of the N-th lamina. The surface $S'$ represents the interface surfaces between individual lamina. Then, equation (4.34) takes the form:

$$
\delta J = \sum_{k=1}^{N} \int_{V_k} \left\{ \delta u_i [\sigma_{ij} + f_i] + \delta \sigma_{ij} \left[ -\frac{1}{2} (u_{ij} + u_{ji}) + S_{ijkl} \sigma_{kl} + \pi_{ij} \left( \frac{\theta^+ + \theta^-}{2} \right) \right.ight.

+ \frac{x_1}{h} (\theta^+ - \theta^-) \right\} + \delta \theta^+ \left[ \sigma_{ij} \pi_{ij} - (\Gamma_{ij} \pi_{ij} + \frac{\rho c}{T_e}) \right] \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right\}

- \frac{g'}{T_e} * q_{a,a} + \frac{g'}{T_o} * r + \left( \frac{\rho c}{T_e} + \Gamma_{ij} \pi_{ij} \right) \theta(x,0) - \pi_{ij} \sigma_{ij}(x,0) \left( \frac{1}{2} + \frac{x_1}{h} \right)

+ \delta \pi_{ij} \left[ \sigma_{ij} \pi_{ij} - (\Gamma_{ij} \pi_{ij} + \frac{\rho c}{T_e}) \right] \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right\}

- \frac{g'}{T_e} * q_{a,a} + \frac{g'}{T_o} * r + \left( \frac{\rho c}{T_e} + \Gamma_{ij} \pi_{ij} \right) \theta(x,0) - \pi_{ij} \sigma_{ij}(x,0) \left( \frac{1}{2} - \frac{x_1}{h} \right)

+ \delta q_{a} \left[ \frac{g'}{T_e} \left( \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right) \right] + \frac{g'}{T_e} * \lambda_{ab} q_{b} \right) \right\}^{k} dV_{(k)}

68
Figure 2. Coordinate System for a Four-Layer Laminate
Vanishing of the volume integral terms in equation (4.35) requires satisfaction of the kinematic relations, constitutive equations, equations of equilibrium, heat balance, thermal gradient equation, and Fourier's Law for each layer. Vanishing of the area integrals requires that one term in each of the products in the integral be prescribed at each point on the boundary.

4.2 GENERALIZED VARIATIONAL FORMULATION

The governing variational equations are obtained by making appropriate substitutions into equation (4.35) and making suitable specializations. Substituting
equations (3.8) and (3.11) in (4.35) and separating the in-plane and transverse terms gives:

\[
\delta J = \sum_{k=1}^{N} \int_{V_k} \left\{ \left[ -u_{(\alpha,\beta)} + S_{\alpha\beta\gamma\delta} \sigma_{\gamma\delta} + S_{\alpha\beta33} \sigma_{33} + \pi_{\alpha\beta}(\theta^+ + \theta^-) + \frac{x_1}{h} (\theta^+ - \theta^-) \right] \delta \sigma_{\alpha\beta} 
\right. \\
+ [-u_{(\alpha,3)} + 2S_{\alpha333} \sigma_{33}] \delta \sigma_{\alpha3} + [-u_{(3,3)} + S_{33\gamma\delta} \sigma_{\gamma\delta} + S_{3333} \sigma_{33}] \\
+ \pi_{33} \left[ \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right] \delta \sigma_{33} \left[ \frac{\sigma_{\alpha\beta\beta} + \sigma_{\alpha33} + f_{\alpha}}{h} \right] \delta u_{\alpha} \\
+ [\sigma_{\alpha33} + \sigma_{333} + f_{3}] \delta u_{3} \left[ \delta \theta \right] \left[ \sigma_{\alpha\beta\pi_{\alpha\beta}} + \sigma_{33}\pi_{33} - (\Gamma_{\alpha\beta\pi_{\alpha\beta}} + \Gamma_{33}\pi_{33}) \right] \\
+ \frac{\rho c}{T_0} \left[ \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right] - \frac{\delta \theta}{T_0} \left[ \sigma_{\alpha\beta\pi_{\alpha\beta}} + \sigma_{33}\pi_{33} - (\Gamma_{\alpha\beta\pi_{\alpha\beta}} + \Gamma_{33}\pi_{33}) \right] \\
+ \frac{\rho c}{T_0} \left[ \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right] \left( \delta \pi_{33}\sigma_{33}(x,0) \right) \\
+ \delta \theta \left[ \sigma_{\alpha\beta\pi_{\alpha\beta}} + \sigma_{33}\pi_{33} - (\Gamma_{\alpha\beta\pi_{\alpha\beta}} + \Gamma_{33}\pi_{33}) \right] \\
+ \frac{x_1}{h} \left( \theta^+ - \theta^- \right) [\sigma_{\alpha\beta}\sigma_{\alpha\beta}(x,0) - \pi_{33}\sigma_{33}(x,0)] \left( \frac{1}{2} - \frac{x_1}{h} \right) \\
+ \delta \sigma_{\alpha}[ \frac{\delta \sigma_{\alpha}}{T_0} \left[ \frac{\theta^+ + \theta^-}{2} + \frac{x_1}{h} (\theta^+ - \theta^-) \right] \left[ \sigma_{\alpha\beta}\sigma_{\alpha\beta}(x,0) - \pi_{33}\sigma_{33}(x,0) \right] \left( \frac{1}{2} - \frac{x_1}{h} \right) \right\} \text{d}V(k)
\]
+ \int_{S_a} \left( u_i - S_i \right) \delta(n_j \sigma_{ij}) \, dS - \int_{S_a} \left( n_i \sigma_{ij} - t_i \right) \delta u_i \, dS

+ \sum_{k=1}^{N} \int_{S_a} \left\{ \left( u_i' - g_i' \right) \delta(n_j \sigma_{ij}) \right\}^{(k)} \, dS

- \sum_{k=1}^{N} \int_{S_a} \left\{ \left( n_j \sigma_{ij}' - h_i' \right) \delta u_i \right\}^{(k)} \, dS - \int_{S_a} \left( \frac{g_i'}{T_s} * n_j, \theta - \frac{g_i'}{T_s} * n_j, \theta \right) \delta q_j \, dS

+ \int_{S_a} \left( \frac{g_i'}{T_s} * n_j q_j' - \frac{g_i'}{T_s} * Q \right) \delta \theta \, dS

- \sum_{k=1}^{N} \int_{S_a} \left\{ \left[ \left( \frac{g_i'}{T_s} * n_j, \theta \right)' - c_i \right] \delta q_j \right\}^{(k)} \, dS

+ \sum_{k=1}^{N} \int_{S_a} \left\{ \left( \frac{g_i'}{T_s} * n_j q_j' - d_i \right) \delta \theta \right\}^{(k)} \, dS = 0 \quad (4.36)

For notational convenience the following generalized definitions are introduced:

\[
(\vec{g}, \vec{g}, \vec{g}) \equiv \int_{-h_k}^{h_k} g \left( 1, \frac{2x_1}{h}, \frac{4x_1^3}{h^2} \right) \frac{2}{h} \, dx_1
\quad (4.37)

The governing equations are derived by substituting the stress components, equations (3.24), (3.28) and (3.37) into equation (4.36). Performing the integration over the lamina thickness and simplifying, the governing variational equation is:

\[
\delta J = \sum_{k=1}^{N} \int_{a}^{b} \int_{-a}^{+b} \left\{ \frac{\delta N_{\alpha\beta}}{h} \left[ -\frac{h}{2} \bar{u}_{(\alpha,\beta)} + S_{\alpha\beta \gamma \delta} N_{\gamma \delta} + S_{\alpha\beta 33} N_{33} + \pi_{\alpha \beta} \frac{h}{2} (\theta' + \theta') \right] \right\} \, \, dx_1 \, \, dx_2
\]

72
\[ + \frac{12\delta M_{\alpha \beta}}{h^3} \left[ -\frac{h^2}{4} \bar{u}_{(\alpha, \beta)} + S_{\alpha \beta \gamma \delta} M_{\gamma \delta} + S_{\alpha \beta 33} M_{33} + \pi_{\alpha \beta} \frac{h^2}{12} (\theta^- - \theta^+) \right] \]

\[ + \delta \sigma_{\alpha 3} \left[ -\frac{1}{2} u^-_\alpha + \frac{3}{4} \bar{u}_\alpha + \frac{1}{4} \bar{u}_\alpha - \frac{3 h}{16} \hat{u}_{3, \alpha} - \frac{h}{8} \bar{u}_{3, \alpha} + \frac{h}{16} \bar{u}_{3, \alpha} \right. \]

\[ + 2 S_{\alpha 3 \beta 3} \left( \frac{(4\sigma^-_{\beta 3} - \sigma^+_{\beta 3}) h}{30} - \frac{V_\beta}{10} \right) \]

\[ + \delta \sigma_{\alpha 3} \left[ -\frac{1}{2} u^-_\alpha + \frac{3}{4} \bar{u}_\alpha - \frac{1}{4} \bar{u}_\alpha - \frac{3 h}{16} \hat{u}_{3, \alpha} + \frac{h}{8} \bar{u}_{3, \alpha} + \frac{h}{16} \bar{u}_{3, \alpha} \right. \]

\[ + 2 S_{\alpha 3 \beta 3} \left( \frac{(4\sigma^-_{\beta 3} - \sigma^+_{\beta 3}) h}{30} - \frac{V_\beta}{10} \right) \]

\[ + \delta V_\alpha \left[ -\frac{3}{2h} \bar{u}_\alpha - \frac{3}{8} (\bar{u}_{3, \alpha} - \hat{u}_{3, \alpha}) + \frac{12}{5h} S_{\alpha 3 \beta 3} V_\beta - \frac{1}{5} S_{\alpha 3 \beta 3} (\sigma^+_{\beta 3} + \sigma^-_{\beta 3}) \right] \]

\[ + \delta N_{33} \left[ -\frac{3}{h} \bar{u}_3 + \frac{1}{h} S_{33 \alpha \beta} N_{\alpha \beta} + \frac{6}{5h} S_{3333} N_{33} - \frac{1}{10} S_{3333} (\sigma^-_{33} + \sigma^+_{33}) \right. \]

\[ + \frac{\pi_{33}}{2} (\theta^- + \theta^+) \]

\[ + \delta M_{33} \left[ -\frac{45}{h^2} \hat{u}_3 + \frac{15}{h^2} \bar{u}_3 + \frac{12}{h^3} S_{33 \alpha \beta} M_{\alpha \beta} + \frac{120}{7h^3} S_{3333} M_{33} \right. \]

\[ - \frac{3}{7h} S_{3333} (\sigma^-_{33} + \sigma^+_{33}) + \frac{\pi_{33}}{h} (\theta^- - \theta^+) \]

\[ + \delta \sigma^+_{33} \left[ -u^+_3 + \frac{15}{4} \hat{u}_3 + \frac{3}{4} \bar{u}_3 - \frac{3}{4} \bar{u}_3 - \frac{1}{10} S_{3333} N_{33} \right. \]

73
\[
\begin{align*}
- \frac{3}{7 \hbar} S_{3333} M_{33} &+ \frac{h}{70} S_{3333}(6 \sigma_{33}^+ + \sigma_{33}^-) \\
+ \delta \sigma_{33}^+ &\left[ \frac{15}{4} \dot{u}_3 + \frac{3}{2} \ddot{u}_3 + \frac{3}{4} \dot{u}_3 - \frac{1}{10} S_{3333} N_{33} \\
+ \frac{3}{7 \hbar} S_{3333} M_{33} &+ \frac{h}{70} S_{3333}(6 \sigma_{33}^+ + \sigma_{33}^-) \right] \\
+ \frac{h}{2} \delta \bar{u}_a &\left[ \frac{1}{h} N_{a3,3} + \frac{1}{h} (\sigma_{a3}^+ - \sigma_{a3}^-) + \frac{1}{h} F_a \right] \\
+ \frac{h^2}{4} \delta \bar{u}_a &\left[ \frac{12}{h^3} M_{a3,3} + \frac{6}{h^2} (\sigma_{a3}^+ + \sigma_{a3}^-) - \frac{12 V_a}{h} \right] \\
+ \frac{h}{2} \delta \bar{u}_3 &\left[ \frac{3}{2h} V_{3,3} - \frac{1}{4} (\sigma_{a3}^+ + \sigma_{a3}^-) - \frac{3}{2h} (\sigma_{33}^+ - \sigma_{33}^-) + \frac{30}{h^3} M_{33} + \frac{1}{h} F_3 \right] \\
+ \frac{h^2}{4} \delta \bar{u}_3 &\left[ \frac{1}{h} (\sigma_{a3}^+ - \sigma_{a3}^-) + \frac{6}{h^2} (\sigma_{33}^+ + \sigma_{33}^-) - \frac{12}{h} N_{33} \right] \\
+ \frac{h^3}{8} \delta \bar{u}_3 &\left[ \frac{3}{h^2} (\sigma_{a3}^+ + \sigma_{a3}^-) - \frac{6}{h^3} V_{3,3} + \frac{30}{h^3} (\sigma_{33}^+ - \sigma_{33}^-) - \frac{360}{h^3} M_{33} \right] \\
+ \delta \theta^+ &\left[ \frac{\pi_{a3}}{2} N_{a3} + \frac{\pi_{a3}}{h} M_{a3} + \frac{\pi_{33}}{2} N_{33} + \frac{\pi_{33}}{h} M_{33} \right] \\
- (\Gamma_{a3} \pi_{a3} + \Gamma_{33} \pi_{33} + \frac{p c}{T_o} \left( \frac{\theta^+ + \theta^-}{4} \right) + \frac{\theta^+ - \theta^-}{12} \right) \\
- \frac{g'}{T_o} \left[ \frac{h}{4} (\bar{q}_{a3} + \bar{q}_{a3}) + \frac{h}{2} \frac{g'}{T_o} \right] - \frac{\pi_{a3}}{2} N_{a3}(x, 0) \\
\end{align*}
\]
\begin{align*}
&\quad - \frac{\pi_{ab}}{h} M_{ab}(x, 0) - \frac{\pi_{13}}{2} N_{33}(x, 0) - \frac{\pi_{33}}{h} M_{33}(x, 0) \\
&\quad + (\Gamma_{ab}\pi_{ab} + \Gamma_{33}\pi_{33} + \frac{\rho c}{T_0}) \left( \frac{\theta^+ + \theta^-}{4} h + \frac{\theta^+ - \theta^-}{12} h \right) \\
&\quad + \delta \theta^- \left[ \frac{\pi_{ab}}{2} N_{ab} - \frac{\pi_{ab}}{h} M_{ab} + \frac{\pi_{33}}{2} N_{33} - \frac{\pi_{33}}{h} M_{33} \\
&\quad - (\Gamma_{ab}\pi_{ab} + \Gamma_{33}\pi_{33} + \frac{\rho c}{T_0}) \left( \frac{\theta^+ + \theta^-}{4} h - \frac{\theta^+ - \theta^-}{12} h \right) \right] \\
&\quad - \frac{g'}{T_i} \left( \frac{h}{4} (\bar{q}_{aa} - \bar{q}_{aa}) \right) + \frac{h}{2} \frac{g'}{T_i} \theta^+ - \frac{\pi_{ab}}{2} N_{ab}(x, 0) \\
&\quad + \frac{\pi_{ab}}{h} M_{ab}(x, 0) - \frac{\pi_{13}}{2} N_{33}(x, 0) + \frac{\pi_{33}}{h} M_{33}(x, 0) \\
&\quad + (\Gamma_{ab}\pi_{ab} + \Gamma_{33}\pi_{33} + \frac{\rho c}{T_0}) \left( \frac{\theta^+ + \theta^-}{4} h - \frac{\theta^+ - \theta^-}{12} h \right) \right] \\
&\quad + \frac{\delta \bar{q}_{aa}}{2} \left[ \frac{g'}{T_i} \left( \frac{h}{2} (\theta_{a\a} + \theta_{a\a}) + \frac{h}{2} \lambda_{ab} \bar{q}_{bb} \right) \right] \\
&\quad + \left( \frac{g'}{T_i} \left( \frac{h}{4} (\theta_{a\a} - \theta_{a\a}) + \frac{3h}{4} \lambda_{ab} \bar{q}_{bb} \right) \right) {^{(k)}} d x_1 d x_2 \\
&\quad + \int_{S_a} (u_i - S_i) \delta(n_j \sigma_{ij}) d S - \int_{S_a} (n_j \sigma_{ij} - t_i) \delta u_i d S \\
&\quad + \sum_{k=1}^{N} \sum_{i=0}^{i} \int_{S_{ak}} \left\{ (u'_{i} - g'_{i}) \delta(n_j \sigma_{ij}) \right\} {^{(k)}} d S
\end{align*}
\[
- \sum_{k=1}^{N} \int_{s_{nk}} \int_{0}^{1} \{ (n_j \sigma_{ij}^{\prime} - h_{ij}^{\prime}) \delta u_{ij} \}^{(k)} dS \\
- \int_{s_a} \left( \frac{g \cdot n_j \theta}{T_a} - \frac{g \cdot n_j \hat{\theta}}{T_a} \right) \delta q_j dS \\
+ \int_{s_a} \left( \frac{g \cdot n_j q_j}{T_a} - \frac{g \cdot \hat{Q}}{T_a} \right) \delta \theta dS \\
- \sum_{k=1}^{N} \int_{s_{nk}} \{ (\frac{g \cdot n_j \theta}{T_a} - c_j) \delta q_j \}^{(k)} dS \\
+ \sum_{k=1}^{N} \int_{s_{nk}} \{ (\frac{g \cdot n_j q_j}{T_a} - d_j) \delta \theta \}^{(k)} dS = 0
\]

(4.38)

4.3 GOVERNING FIELD EQUATIONS

The appropriate field equations are found by setting to zero the expression in the brackets corresponding to each of the arbitrary admissible variations of the field variables. The field equations must be satisfied in each layer of the laminate. The field variables are \( N_{ij}, M_{ij}, V_{ij}, N_{33}, M_{33}, \overline{q}_a, \overline{q}_a, \sigma_{a3}, \sigma_{a3}, \sigma_{33}, \sigma_{33}, \overline{u}_a, \overline{u}_a, \overline{u}_3, \overline{u}_3, \theta^+, \) and \( \theta^- \) as indicated in equation (4.38).
Constitutive Equations

For an arbitrary admissible variation of the generalized resultants \( N_{\alpha\beta}, M_{\alpha\beta}. \)

\( V_{\alpha}. N_{33}, M_{33}. \tilde{q}_{\alpha}, \) and \( \tilde{q}_{\alpha} \) the following constitutive equations are obtained:

\[
\ddot{u}_{(\alpha,\beta)} = \frac{2}{h} [S_{\alpha\beta\gamma\delta} N_{\gamma\delta} + S_{\alpha\beta 33} N_{33} + \pi_{\alpha\beta} \frac{h}{2} (\theta^+ + \theta^-)] \tag{4.39}
\]

\[
\ddot{u}_{(\alpha,\beta)} = \frac{4}{h^2} [S_{\alpha\beta\gamma\delta} M_{\gamma\delta} + S_{\alpha\beta 33} M_{33} + \pi_{\alpha\beta} \frac{h^2}{12} (\theta^+ - \theta^-)] \tag{4.40}
\]

\[
\ddot{u}_{3\alpha} - \ddot{u}_{3\alpha} - \frac{4}{h} \ddot{u}_{\alpha} = \frac{8}{15} S_{33\beta 3} (\sigma_{33}^+ + \sigma_{33}^-) - \frac{32}{5h} S_{33\beta 3} V_{\beta} \tag{4.41}
\]

\[
6 \ddot{u}_3 = 2 S_{33\alpha\beta} N_{\alpha\beta} + \frac{12}{5} S_{3333} N_{33} - \frac{h}{5} S_{3333} (\sigma_{33}^+ + \sigma_{33}^-) + \pi_{33} h (\theta^+ + \theta^-) \tag{4.42}
\]

\[
3 \ddot{u}_3 - \ddot{u}_3 = \frac{4}{5h} S_{33\alpha\beta} M_{\alpha\beta} + \frac{8}{7h} S_{3333} M_{33} - \frac{h}{35} S_{3333} (\sigma_{33}^+ - \sigma_{33}^-) + \frac{h}{15} (\theta^+ - \theta^-) \tag{4.43}
\]

\[
\frac{g'}{T_c} (\theta_{\alpha}^+ + \theta_{\alpha}^-) = - \frac{g'}{T_c} \lambda_{\alpha\beta} \tilde{q}_\beta \tag{4.44}
\]

\[
\frac{g'}{T_c} (\theta_{\alpha}^- - \theta_{\alpha}^-) = - 3 \frac{g'}{T_c} \lambda_{\alpha\beta} \tilde{q}_\beta \tag{4.45}
\]
4.3.2 Equations of Equilibrium

For an arbitrary admissible variation of the weighted displacement terms $\bar{u}_a$, $\bar{u}_i$, $\bar{u}_j$, $\bar{u}_1$, the following generalized equations of equilibrium are obtained:

\begin{align*}
N_{a\beta,\beta} + (\sigma_{a3}^+ - \sigma_{a3}^-) + F_a &= 0 \\
M_{a\beta,\beta} + \frac{h}{2} (\sigma_{a3}^+ + \sigma_{a3}^-) - V_a &= 0 \\
V_{a,\alpha} - \frac{h}{6} (\sigma_{a3,\alpha}^+ + \sigma_{a3,\alpha}^-) - (\sigma_{33}^- - \sigma_{33}^+) + \frac{20}{h^3} M_{33} + \frac{2}{3} F_3 &= 0 \\
N_{33} - \frac{h^2}{12} (\sigma_{a3,\alpha}^+ - \sigma_{a3,\alpha}^-) - \frac{h}{2} (\sigma_{33}^+ + \sigma_{33}^-) &= 0 \\
60 \frac{h}{h^3} M_{33} - \frac{h}{2} (\sigma_{a3,\alpha}^+ + \sigma_{a3,\alpha}^-) + V_{a,\alpha} - 5 (\sigma_{33}^+ - \sigma_{33}^-) &= 0
\end{align*}

(4.46) (4.47) (4.48) (4.49) (4.50)

4.3.3 Interface Equations

For an arbitrary admissible variation of the transverse stress components on the top and bottom boundaries of the plate, $\sigma_{a3}^+$, $\sigma_{a3}^-$, $\sigma_{33}^+$, and $\sigma_{33}^-$, the following interface displacement equations are obtained:

\begin{equation}
\dot{u}_a^* = -h \left( \frac{3}{8} \ddot{u}_{3,a} - \frac{1}{8} \ddot{u}_{3,\alpha} - \frac{3}{2h} \ddot{u}_a \right) - \left( \frac{h}{4} \ddot{u}_{3,\alpha} - \frac{1}{2} \ddot{u}_a \right) \\
+ 4 S_{a3,\beta} \left[ \frac{(4\sigma_{33}^- - \sigma_{33}^+)h}{30} - \frac{V_\beta}{10} \right]
\end{equation}

(4.51)
\[ u_\alpha^* = h \left( \frac{3}{8} \bar{u}_{1,\alpha} - \frac{1}{8} \bar{u}_{3,\alpha} - \frac{3}{2h} \bar{u}_{\alpha} \right) - \left( \frac{h}{4} \bar{u}_{3,\alpha} - \frac{1}{2} \bar{u}_{\alpha} \right) \]

\[ -4 S_{\alpha \beta 3} \left[ \frac{(4\sigma_{\beta 3}^- - \sigma_{\beta 3}^+)}{30} - \frac{V_\beta}{10} \right] \]  

(4.52)

\[ u^*_3 = \frac{3}{4} (5 \bar{u}_3 - \bar{u}_1) + \frac{3}{2} \bar{u}_3 \]

\[ + \frac{1}{70h} S_{3333} \left[ (6\sigma_{33}^- + \sigma_{33}^+) h^2 - 7h N_{33} - 30 M_{33} \right] \]  

(4.53)

\[ u^*_3 = \frac{3}{4} (5 \bar{u}_3 - \bar{u}_1) - \frac{3}{2} \bar{u}_3 \]

\[ - \frac{1}{70h} S_{3333} \left[ (6\sigma_{33}^- + \sigma_{33}^+) h^2 - 7h N_{33} + 30 M_{33} \right] \]  

(4.54)

For an arbitrary admissible variation of boundary temperature terms, \( \theta^+ \) and \( \theta^- \), the following equations are obtained:

\[ \frac{\pi_{\alpha \beta}}{2} N_{\alpha \beta} + \frac{\pi_{\alpha \beta}}{h} M_{\alpha \beta} + \frac{\pi_{33}}{2} N_{33} + \frac{\pi_{33}}{h} M_{33} \]

\[-(\Gamma_{\alpha \beta} \pi_{\alpha \beta} + \Gamma_{33} \pi_{33} + \frac{\rho c}{T_o} \left( \frac{\theta^+ + \theta^-}{4} h + \frac{\theta^+ - \theta^-}{12} h \right) \]

\[-\frac{g'}{T_o} \left( \bar{q}_{a,\alpha} + \bar{q}_{a,\alpha} \right) + \frac{h}{2} \Gamma_{\alpha \beta} \frac{\pi_{\alpha \beta}}{2} N_{\alpha \beta}(x, 0) \]

\[ - \frac{\pi_{\alpha \beta}}{h} M_{\alpha \beta}(x, 0) - \frac{\pi_{33}}{2} N_{33}(x, 0) - \frac{\pi_{33}}{h} M_{33}(x, 0) \]

\[ + (\Gamma_{\alpha \beta} \pi_{\alpha \beta} + \Gamma_{33} \pi_{33} + \frac{\rho c}{T_o} \left( \frac{\theta^+ + \theta^-}{4} h + \frac{\theta^+ - \theta^-}{12} h \right) = 0 \]  

(4.55)
The initial boundary value problem has been expressed in terms of fourteen constitutive equations, seven equations of equilibrium, six interface displacement equations, and two thermal energy balance equations with 29 field variables. These field variables are \( N_{\alpha\beta}, M_{\alpha\beta}, V_{\alpha}, N_{33}, M_{33}, \bar{u}_\alpha, \bar{u}_\alpha, \bar{u}_3, \bar{u}_3, \bar{\sigma}_{\alpha\beta}, \sigma_{33}, \sigma_{13}, \sigma_{\alpha\beta}, \theta^+, \theta^- \). Temperature terms arise from the governing thermoelastic relationships. Temperature terms also appear as coupling in four constitutive equations and in the two thermal balance equations.

The governing field equations which were derived in this chapter, can be represented in an operator matrix form as \( [A^* \{ u \} \). The operator matrix \([A^* \) for the constitutive equations and the equations of equilibrium is given by:

\[
\frac{\pi_{\alpha\beta}}{2} N_{\alpha\beta} - \frac{\pi_{\alpha\beta}}{h} M_{\alpha\beta} + \frac{\pi_{33}}{2} N_{33} - \frac{\pi_{13}}{h} M_{33}
\]

\[-(\Gamma_{\alpha\beta} \pi_{\alpha\beta}) + \Gamma_{33} \pi_{33} + \frac{pc}{T_0} \left( \frac{\theta^+ + \theta^-}{4} h - \frac{\theta^+ - \theta^-}{12} h \right)
\]

\[-\frac{g'}{T_0} \frac{h}{4} (\bar{q}_{\alpha\alpha} - \bar{q}_{\alpha\alpha}) + \frac{h}{2} \frac{g'}{T_0} - r - \frac{\pi_{\alpha\beta}}{2} N_{\alpha\beta}(x, 0)
\]

\[+ \frac{\pi_{\alpha\beta}}{h} M_{\alpha\beta}(x, 0) - \frac{\pi_{33}}{2} N_{33}(x, 0) + \frac{\pi_{33}}{h} M_{33}(x, 0)
\]

\[+ (\Gamma_{\alpha\beta} \pi_{\alpha\beta}) + \Gamma_{13} \pi_{13} + \frac{pc}{T_0} \left( \frac{\theta^+ + \theta^-}{4} h - \frac{\theta^+ - \theta^-}{12} h \right) = 0 \quad (4.56)
\]
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & \frac{1}{2} \Gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \Gamma_1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{h} \frac{\partial}{\partial \gamma} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} \Gamma_2 & 0 & 0 & 0 & 0 & \frac{2}{h} S_{\phi \theta \phi} & \frac{2}{h} S_{\phi \theta \phi 33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} \Gamma_2 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{\hbar^2} S_{\phi \theta \phi} & \frac{4}{\hbar^2} S_{\phi \theta \phi 33} & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -\frac{4}{5 \hbar} S_{\phi \theta \phi} & -\frac{8}{7 \hbar} S_{3333} & 0 & 0 & 0 \\
0 & -\frac{4}{5 \hbar} \delta_{\phi \theta} & -\frac{\partial}{\partial \rho} & 0 & -\frac{\partial}{\partial \rho} & 0 & 0 & 0 & 0 & \frac{32}{5 \hbar} S_{\phi \phi 33} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{e^*}{T_r} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{e^*}{T_r} \\
\end{bmatrix}
\]

and
\[
\{u^*\}^T = \left( \bar{u}_\alpha \quad \bar{\bar{u}}_\alpha \quad \bar{u}_i \quad \bar{\bar{u}}_i \quad \bar{N}_{\alpha \beta} \quad \bar{N}_{33} \quad \bar{M}_{\alpha \beta} \quad \bar{M}_{33} \quad \bar{V}_\alpha \quad \bar{q}_\alpha \quad \bar{q}_\alpha \right) (4.57)
\]

It is noted that this operator matrix is not complete, additional terms which are included in the governing equations are not shown. This operator matrix is presented to illustrate that the governing equations are not in self-adjoint form. Complete definitions of all the symbols given in [A\textsuperscript{o}] will be given in the following chapter. The following chapter presents a method of obtaining a self-adjoint form for these equations.
CHAPTER 5

SELF-ADJOINT FORM OF GOVERNING EQUATIONS

To implement the theory developed in the previous Chapters into a finite element analysis, a self-adjoint form of the governing equations is desirable so that a Ritz-type variational formulation can be used. The governing equations are re-arranged by defining generalized displacements to achieve a self-adjoint form.

5.1 BASIC VARIATIONAL PRINCIPLES

In this section a review of the basic principles of variational theory are presented along with definitions of some of the mathematical terms involved in the formulation. The variational principles presented here were developed by Sandhu and Salaam (1975), Sandhu (1976) and Al-Ghotani (1986).
5.1.1 Initial Boundary Value Problem

An initial-boundary value problem can be represented by the following sets of equations:

\[ A(u) = f \text{ on } R \times [0, \infty) \]  
(5.1)

\[ C(u) = g \text{ on } S \times [0, \infty) \]  
(5.2)

\[ u = d \text{ on } \overline{R} \text{ at } t = 0 \]  
(5.3)

where \( R \) is an open connected region of interest in an Euclidean space, \( S \) is the boundary of \( R \), and \( \overline{R} \) is the closure of \( R \). The term \( u \) represents the field variables.

Linear field operators, \( A \) and \( C \), are defined such that:

\[ A: W_R \rightarrow V_R \]

\[ C: W_S \rightarrow V_S \]

where \( W_R \) and \( W_S \) are the domains of \( A \) and \( C \), respectively, while \( V_R \) and \( V_S \) are the ranges of \( A \) and \( C \), respectively. The field operators \( A \) and \( C \) are said to be linear if:

\[ A(au + bw) = aA(u) + bA(w) \text{ for all } u, w \in W_R \]

\[ C(au + bw) = aC(u) + bC(w) \text{ for all } u, w \in W_S \]  
(5.4)

Solution of the initial boundary value problem implies determination of \( u \in W_R \) for a given \( f \in V_R \), \( g \in V_S \) and \( d \in V_S \) such that equations (5.1) - (5.3) are satisfied.
5.1.2 Bilinear Mapping

Given $V$ and $T$ as linear vector spaces, a mapping $B: V \times V \rightarrow T$ is said to be a bilinear mapping if:

\[ B(u, w) = B(w, u) \]
\[ B(au_1 + bu_2, w) = aB(u_1, w) + bB(u_2, w) \]
\[ B(u, aw_1 + bw_2) = aB(u, w_1) + bB(u, w_2) \]

where $a$ and $b$ are scalars, $u$ and $w \in V$ and $B(u, w) \in T$. Further, the bilinear mapping is said to be non-degenerate if:

\[ B(u, w) = 0 \text{ for all } u \in V \text{ if and only if } w = 0 \]

We shall adopt the notation:

\[ B_R(u, w) \equiv \langle u, w \rangle_R \] (5.5)

where $R$ is the domain of the linear vector space.

5.1.3 Self-Adjoint Operator

Let $A$ be a linear operator on a vector space $V$. Then operator $A^*: V \rightarrow V$ is said to be the adjoint of $A$ with respect to the bilinear mapping $\langle \ , \ \rangle_R: V \times V \rightarrow T$ if:

\[ \langle u, Aw \rangle_R = \langle w, A^*u \rangle_R + D_5(w, u) \text{ for all } u, w \in V \] (5.6)
where \( D_s(w, u) \) represents the boundary terms. If \( A = A^* \), then \( A \) is said to be self-adjoint. If \( A \) is self-adjoint, then \( D_s(w, u) \) is anti-symmetric, i.e.,

\[
D_s(w, u) = -D_s(u, w) \quad \text{for all } u, w \in V
\]

Alternatively, operator \( A \) is said to be symmetric with respect to the bilinear mapping \(< \cdot, \cdot >_R: V \times V \rightarrow T \) if

\[
<u, Aw>_R = <w, Au>_R
\]

5.1.4 Gateaux Differential of a Function

Consider a continuous function \( \Omega: V \rightarrow T \). The Gateaux differential of \( \Omega \) can then be defined as:

\[
\Delta_w \Omega(u) = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \left[ \Omega(u + \lambda w) - \Omega(u) \right] \quad (5.7)
\]

provided the limit exists. Note that if \( u, w \in V \) then \( u + \lambda w \in V \). If the Gateaux differential exits at every point in the neighborhood of \( u \), then we can write

\[
\Delta_w \Omega(u) = \frac{d}{d\lambda} \Omega(u + \lambda w) \bigg|_{\lambda=0} \quad (5.8)
\]

where the path is \( w \in V \) and \( \lambda \) is a scalar.
5.1.5 Basic Variational Problem

Consider the boundary value problem as defined by equation (5.1) with homogenous boundary conditions. Using the inner product as the bilinear mapping on the region R, Mikhlin (1965) showed that the functional

\[ Q(u) = \langle u, Au - 2f \rangle_R = \langle u, Au \rangle_R - 2\langle u, f \rangle_R \]  \hspace{1cm} (5.9)

assumes its minimum value for the unique solution \( u_0 \) where operator \( A \) is self-adjoint and positive definite. Conversely, the \( u \) that minimizes the function \( \Omega(u) \) is the solution to the initial-value problem. Taking the Gateaux differential along the path \( w \) for an arbitrary \( w \in V \) and assuming operator \( A \) as self-adjoint yields:

\[ \Delta_w \Omega(u) = \langle u, Aw \rangle_R + \langle w, Au \rangle_R - 2\langle w, f \rangle_R \]

\[ = 2\langle w, (Au - f) \rangle \]  \hspace{1cm} (5.10)

if and only if (5.1) is satisfied. Use of the Gateaux differential eliminates the need for \( A \) to be positive. Non-homogenous boundary conditions can be incorporated in the problem as [Sandhu and Salaam (1975)]:

\[ \Omega(u) = \langle u, Au - 2f \rangle_R + \langle u, Cu - 2g \rangle_S \]  \hspace{1cm} (5.11)

where \( A \) is a linear self-adjoint field operator and \( C \) is consistent with \( A \). Consistency of boundary operators with respect to field operators is discussed in the following subsection.
5.1.6 Coupled Problems

Consider a coupled initial-boundary value problem with "n" independent field variables. Equations (5.1) and (5.2) become:

\[ \sum_{j=1}^{n} A_{ij} u_j = f_i \text{ on } \Omega \]  \hspace{1cm} (5.12)

\[ \sum_{j=1}^{n} C_{ij} u_j = g_i \text{ on } S_i, \quad i = 1, 2, 3, \ldots, n \]  \hspace{1cm} (5.13)

where \( S = \bigcup_{i=1}^{n} S_i \) and \( n \) is the number of independent field variables. The matrix of field operators \( A_{ij} \) and matrix of boundary operators \( C_{ij} \) are such that

\[ A_{ij}: W_{R_i} \to V_{R_i}, \quad i, j = 1, 2, 3, \ldots, n \]

\[ C_{ij}: W_{S_i} \to V_{S_i}, \quad i, j = 1, 2, 3, \ldots, n \]

where \( W_{R_i} \) and \( W_{S_i} \) are subspaces of \( V_{R_i} \) and \( V_{S_i} \), respectively. Therefore, a bilinear mapping \( < \cdot, \cdot >_R \) on \( V_R \) is defined as:

\[ <u, v>_R = <u_1, v_1>_R + <u_2, v_2>_R + \ldots + <u_n, v_n>_R \]

The matrix of operators \( A_{ij} \) is said to be self-adjoint with respect to the bilinear mapping \( < \cdot, \cdot >_R \) if:

\[ \sum_{j=1}^{n} <u_j, A_{ji} v_i>_R = <v_i, \sum_{j=1}^{n} A_{ij} u_j>_R + D_S(u_i, v_j) \quad i = 1, 2, 3, \ldots, n \]  \hspace{1cm} (5.14)
The matrix of boundary operators $C_{ij}$ is said to be consistent with the self-adjoint matrix of operators $A_{ij}$ if:

$$<v_i, \sum_{j=1}^{n} C_{ij}u_j>_S = \sum_{j=1}^{n} <u_i, C_{ji}v_j>_S + D_S(u_i, v_j)$$

i.e.

$$D_S(u_i, v_j) = <v_i, \sum_{j=1}^{n} C_{ij}u_j>_S - \sum_{j=1}^{n} <u_j, C_{ji}v_i>_S$$  \hspace{1cm} (5.15)

Substituting equation (5.15) in (5.14) yields:

$$\sum_{j=1}^{n} <u_i, A_{ij}v_j>_R = <v_i, \sum_{j=1}^{n} A_{ij}u_j>_R + <v_i, \sum_{j=1}^{n} C_{ij}u_j>_S - \sum_{j=1}^{n} <u_j, C_{ji}v_i>_S$$  \hspace{1cm} (5.16)

Internal discontinuities can be added to equations (5.12) and (5.13) directly as shown by Sandhu and Salaam (1975) by defining them as:

$$\sum_{j=1}^{n} (C_{ij}u_i)^' = g_i \text{ on } S_i \hspace{1cm} i = 1, 2, 3, \ldots n$$  \hspace{1cm} (5.17)

where the primed superscript denotes the jump discontinuity along the element boundary $S_i$ embedded in the domain $R$, and $g_i$ are the prescribed jump discontinuity values. The internal discontinuities can included in the above formulation by adding a term and defining the bilinear mapping over $R$ as the sum of maps over individual elements.
5.2 SELF-ADJOINT FORM OF EQUATIONS

The governing field equations derived in the previous chapter consist of fourteen constitutive equations (4.39) - (4.45) and seven equations of equilibrium (4.46) - (4.50). For a laminate, there are six interface displacement quantities given by equations (4.51) - (4.54). In addition there are two thermal energy balance equations for a total of 29 field variables per lamina. In their present form, the field equations derived in Chapter 4 are not self-adjoint. In the following, the field equations will be restated in a self-adjoint form.

5.2.1 Weighted Displacement Definitions

Rearranging equation (4.41) yields:

\[ -\frac{\partial}{\partial \rho} \left( \frac{3}{4} (\bar{u}_3 - \hat{u}_3) \right) - \delta_{\rho} \left( \frac{3}{\rho} \bar{u}_3 \right) - \frac{2}{5} S_{p3y3} \left( \sigma_{y1}^* + \sigma_{y1}^- \right) + \frac{24}{5h} S_{p3y3} V_7 = 0 \]  

(5.18)

Rearranging equation (4.42) yields:

\[ -\frac{3\hat{u}_3}{h} + \frac{1}{h} S_{333} N_{y3} + \frac{6}{5h} S_{3333} N_{33} - \frac{1}{10} S_{3333}(\sigma_{y1}^* + \sigma_{y1}^-) + \frac{1}{2} \pi_{33} (\theta^* + \theta) = 0 \]

(5.19)
Rearranging equation (4.43) yields:

\[
\frac{15}{h^2} (3\ddot{u}_3 - \ddot{u}_3) - \frac{12}{h^3} S_{333\delta} M_{\gamma\delta} - \frac{120}{7h^3} S_{3333} M_{333} + \frac{3}{7h} S_{3333}(\sigma_{33}^* - \sigma_{33}^-) \\
- \frac{1}{h} \pi_{33} (\theta^* - \theta^-) = 0
\]  

(5.20)

5.2.2 Constitutive Equations and Equations of Equilibrium

Defining operators \( \Gamma_1 \) and \( \Gamma_2 \) as:

\[
\Gamma_1 = \delta_{\alpha\gamma} \frac{\partial}{\partial \beta} + \delta_{\beta\gamma} \frac{\partial}{\partial \alpha} 
\]

(5.21)

\[
\Gamma_2 = \delta_{\mu\nu} \frac{\partial}{\partial \rho} + \delta_{\nu\mu} \frac{\partial}{\partial \mu}
\]

(5.22)

Equations (4.46) - (4.48) can be written in the form:

\[
\frac{1}{2} \Gamma_1 N_{\alpha\beta} + (\sigma_{\gamma3}^* - \sigma_{\gamma3}^-) + F_\gamma = 0
\]

(5.23)

\[
\frac{1}{2} \Gamma_1 M_{\alpha\beta} + \frac{h}{2} (\sigma_{\gamma3}^* + \sigma_{\gamma3}^-) - V_\gamma = 0
\]

(5.24)

\[
\frac{\partial}{\partial \gamma} V_\gamma + (\sigma_{33}^* - \sigma_{33}^-) + F_3 = 0
\]

(5.25)
Equations (4.49) and (4.50) can be rewritten as:

\[-N_{33} + \frac{h}{2} \left( \sigma_{13}^{\ast} + \sigma_{33}^{\ast} \right) + \frac{h^2}{12} \left( \sigma_{\gamma \gamma}^{\ast} - \sigma_{\gamma \gamma}^{-} \right) = 0 \tag{5.26}\]

and

\[-M_{33} + \frac{h^3}{120} \left( \sigma_{\gamma \gamma}^{\ast} + \sigma_{\gamma \gamma}^{-} \right) + \frac{h^2}{10} \left( \sigma_{33}^{\ast} - \sigma_{33}^{-} \right) + \frac{h^2}{60} F_3 = 0 \tag{5.27}\]

Equation (4.39) can be written in the form

\[-\frac{1}{2} \Gamma_2 \left( \frac{\ddot{u}_x}{2} \right) + \frac{1}{h} S_{\mu p q} N_{\alpha \beta} + \frac{1}{h} S_{\mu p q}^3 N_{33} + \frac{l}{2} \pi_{\alpha \beta}(\theta^* + \theta^-) = 0 \tag{5.28}\]

Equation (4.40) can be written as:

\[-\frac{1}{2} \Gamma_2 \left( \frac{3 \ddot{u}_x}{h} \right) + \frac{12}{h^3} S_{\mu p q} M_{\alpha \beta} + \frac{12}{h^3} S_{\mu p q}^3 M_{33} + \frac{l}{h} \pi_{\alpha \beta}(\theta^* - \theta^-) = 0 \tag{5.29}\]

Equations (4.44) and (4.45) can be written as:

\[\frac{g'}{T} \ast (\Theta_{13}^{\ast} + \Theta_{13}^{-}) = - \frac{g'}{T} \ast \lambda_{\alpha \beta} \ddot{q}_\beta \tag{5.30}\]

\[\frac{g'}{T} \ast (\Theta_{33}^{\ast} - \Theta_{33}^{-}) = - 3 \frac{g'}{T} \ast \lambda_{\alpha \beta} \ddot{q}_\beta \tag{5.31}\]

The equations of equilibrium and constitutive laws have been restated as given by equations (5.18) - (5.31).
5.2.3 Interface Displacement Conditions

Equation (4.51) may be rewritten as:

\[ u_\rho^* = \left( \frac{\bar{u}}{2} \right) + \frac{h}{2} \left( \frac{3 \bar{u}}{h} \right) - \frac{h^4}{120} \left\{ \frac{15}{h^2} (3 \bar{u}_3 - \bar{u}_1) \right\} \rho - \frac{h^2}{12} \left( \frac{3 \bar{u}}{h} \right)_\rho - \frac{2}{5} \sum_{ij} V_{ij} + \frac{2h}{15} \sum_{ij} \left( 4 \sigma_{ij}^* - \sigma_{ij}^* \right) \]

Similarly, equation (4.52) may be written as:

\[ u_\rho^- = \left( \frac{\bar{u}}{2} \right) - \frac{h}{2} \left( \frac{3 \bar{u}}{h} \right) + \frac{h^4}{120} \left\{ \frac{15}{h^2} (3 \bar{u}_3 - \bar{u}_1) \right\} \rho - \frac{h^2}{12} \left( \frac{3 \bar{u}}{h} \right)_\rho + \frac{2}{5} \sum_{ij} V_{ij} - \frac{2h}{15} \sum_{ij} \left( 4 \sigma_{ij}^* - \sigma_{ij}^* \right) \]

and the expressions for \( u_\rho^* \) and \( u_\rho^- \), equations (4.53) and (4.54), can be written as:

\[ u_\rho^* = \frac{h^2}{10} \left\{ \frac{15}{h^2} (3 \bar{u}_3 - \bar{u}_1) \right\} + \frac{3}{4} \left( \bar{u}_3 - \bar{u}_1 + \frac{1}{2} \left( \frac{3 \bar{u}}{h} \right) + \frac{1}{10} \sum_{33} \right \}

\[ u_\rho^- = \frac{h^2}{10} \left\{ \frac{15}{h^2} (3 \bar{u}_3 - \bar{u}_1) \right\} + \frac{3}{4} \left( \bar{u}_3 - \bar{u}_1 \right) \left( \frac{3 \bar{u}}{h} \right) + \frac{1}{10} \sum_{33} \]

and

\[ u_3^* = \frac{h^2}{10} \left\{ \frac{15}{h^2} (3 \bar{u}_3 - \bar{u}_1) \right\} + \frac{3}{4} \left( \bar{u}_3 - \bar{u}_1 \right) - \frac{1}{10} \sum_{33} \]

\[ u_3^- = \frac{h^2}{10} \left\{ \frac{15}{h^2} (3 \bar{u}_3 - \bar{u}_1) \right\} + \frac{3}{4} \left( \bar{u}_3 - \bar{u}_1 \right) \left( \frac{3 \bar{u}}{h} \right) - \frac{1}{10} \sum_{33} \]

92
Equations (5.32) - (5.35) represent the restated form of the inplane and transverse interface displacement components of a lamina, respectively.

5.3 INTEGRAL FORM OF THE EQUATIONS

Consider a $N$ layered laminate with $k=1$ and $k = N$ representing the top and bottom laminae, respectively (see Figure 2). The self-adjoint form of the governing equations will now be written for the laminate.

5.3.1 Field Equations

The following definitions are introduced for the $k$th lamina:

\[ \\bar{V}_p^{(k)} = \frac{\bar{u}_p^{(k)}}{2} \quad (5.37) \]

\[ \\bar{\phi}_p^{(k)} = \frac{3 \bar{u}_p^{(k)}}{h_k} \quad (5.38) \]

\[ \\bar{V}_3^{(k)} = \frac{3}{4} (\bar{u}_3 - \bar{u}_3)^{(k)} \quad (5.39) \]

\[ \bar{\phi}_3^{(k)} = \frac{15}{h_k^2} (3 \bar{u}_3 - \bar{u}_3)^{(k)} \quad (5.40) \]

\[ \bar{\phi}_3^{(k)} = \frac{3 \bar{u}_3^{(k)}}{h_k} \quad (5.41) \]
Introducing these definitions into the governing equations gives the final form of the equations as:

\[
\frac{1}{2} \Gamma_1 N_{\alpha\beta}^{(k)} + (\sigma_{\gamma\gamma}^{+(k)} - \sigma_{\gamma\gamma}^{-(k)}) + F^{(k)} = 0
\] (5.42)

\[
\frac{1}{2} \Gamma_1 M_{\alpha\beta}^{(k)} + \frac{h_k}{2} (\sigma_{\gamma\gamma}^{+(k)} + \sigma_{\gamma\gamma}^{-(k)}) - V^{(k)} = 0
\] (5.43)

\[
\frac{\partial}{\partial \gamma} V^{(k)} + (\sigma_{33}^{+(k)} - \sigma_{33}^{-(k)}) + F_3^{(k)} = 0
\] (5.44)

\[
-M^{(k)}_{33} + \frac{h_k^3}{120} (\sigma_{\gamma\gamma}^{+(k)} + \sigma_{\gamma\gamma}^{-(k)}) + \frac{h_k^2}{10} (\sigma_{33}^{+(k)} - \sigma_{33}^{-(k)}) + \frac{h_k^2}{60} F_3^{(k)} = 0
\] (5.45)

\[
-N_{33}^{(k)} + \frac{h_k}{2} (\sigma_{33}^{+(k)} + \sigma_{33}^{-(k)}) + \frac{h_k^2}{12} (\sigma_{\gamma\gamma}^{+(k)} - \sigma_{\gamma\gamma}^{-(k)}) = 0
\] (5.46)

Using definitions (5.37) - (5.41), equation (5.28) can be rewritten in the following form:

\[
-\frac{1}{2} \Gamma_2 V^{(k)}_{\rho} + \frac{1}{h_k} S_{\alpha\beta\rho}^{(k)} N_{\alpha\beta}^{(k)} + \frac{1}{h_k} S_{\alpha\beta\gamma}^{(k)} N_{\gamma\beta}^{(k)} + \frac{1}{2} \pi_{\alpha\beta}^{(k)} (\theta^{+(k)} + \theta^{-(k)}) = 0
\] (5.47)

Equations (5.19) can be restated as:

\[
-\phi_{\gamma}^{(k)} + \frac{1}{h_k} S_{33\gamma}^{(k)} N_{\gamma}^{(k)} + \frac{6}{5h_k} S_{3333}^{(k)} N_{33}^{(k)} - \frac{1}{10} S_{3333}^{(k)} (\sigma_{33}^{+(k)} + \sigma_{33}^{-(k)})
\]

\[
+ \frac{1}{2} \pi_{33}^{(k)} (\theta^{+(k)} + \theta^{-(k)}) = 0
\] (5.48)

94
Restating equation (5.20) yields:

\[- \frac{1}{2} \Gamma \varphi_p^{(k)} + \frac{12}{h_k} S_{\mu \nu \rho}^{(k)} M_{\rho \beta}^{(k)} + \frac{12}{h_k} S_{\mu \nu \rho 33}^{(k)} M_{33}^{(k)} + \frac{1}{h_k} \pi_{\alpha \beta}^{(k)} (\theta^{+^{(k)}} - \theta^{-^{(k)}}) = 0\]

(5.49)

\[- \varphi_3^{(k)} + \frac{12}{h_k} S_{3336}^{(k)} M_{66}^{(k)} + \frac{120}{7h_k} S_{3333}^{(k)} M_{33}^{(k)} - \frac{3}{7h_k} S_{3333}^{(k)} (\sigma_3^{+^{(k)}} - \sigma_3^{-^{(k)}})\]

\[+ \frac{1}{h_k} \pi_{33}^{(k)} (\theta^{+^{(k)}} - \theta^{-^{(k)}}) = 0\]

(5.50)

\[- \frac{\partial}{\partial \rho} \tilde{V}_i^{(k)} - \delta_{\mu \rho} \tilde{\varphi}_p^{(k)} - \frac{2}{5} S_{\rho 3}^{(k)} (\sigma_{\gamma 3}^{+^{(k)}} + \sigma_{\gamma 3}^{-^{(k)}}) + \frac{24}{5h_k} S_{\rho 3}^{(k)} V_{(k)}^{(k)} = 0\]

(5.51)

and we have the following:

\[\frac{h}{4} \frac{g'}{T_c} (\theta_{\alpha}^{+^{(k)}} + \theta_{\alpha}^{-^{(k)}}) = - \frac{h}{4} \frac{g'}{T_c} \lambda_{\alpha \beta}^{(k)} \tilde{q}_\beta^{(k)}\]

(5.52)

\[\frac{h}{4} \frac{g'}{T_c} (\theta_{\alpha}^{+^{(k)}} - \theta_{\alpha}^{-^{(k)}}) = - \frac{3h}{4} \frac{g'}{T_c} \lambda_{\alpha \beta}^{(k)} \tilde{q}_\beta^{(k)}\]

(5.53)

### 5.3.2 Interfacial Continuity Conditions

The interface displacement, traction and temperature continuity conditions can be written as:

\[u_i^{-(k)} = u_i^{+(k+1)}\]

(5.54)

\[\sigma_{i3}^{-(k)} = \sigma_{i3}^{+(k+1)}\]

(5.55)

\[\theta^{-(k)} = \theta^{+(k+1)}\]

(5.56)
Substituting equations (5.32) and (5.33) in equation (5.54) for \( i = 1 \) and 2, and using definitions (5.37) - (5.41) the inplane displacement continuity equations are obtained as:

\[
- \vec{V}_p^{(k)} + \frac{h^2_k}{2} \tilde{\phi}_p^{(k)} + \frac{h_k^4}{12} \tilde{\phi}_3^{(k)} - \frac{h_k^2}{120} \tilde{\phi}_3^{(k)} - \frac{2}{5} S_{\rho \gamma}^{(k)} V_r^{(k)} \\
+ \frac{2h^2_k}{15} S_{\rho \gamma}^{(k)} (4 \sigma_{\gamma}^{(-k)} - \sigma_{\gamma}^{=(-k+1)}) \\
+ \vec{V}_p^{(k+1)} + \frac{h_{k+1}^2}{2} \tilde{\phi}_p^{(k+1)} - \frac{h_{k+1}^2}{12} \tilde{\phi}_3^{(k+1)} - \frac{h^2_{k+1}}{120} \tilde{\phi}_3^{(k+1)} - \frac{2}{5} S_{\rho \gamma}^{(k+1)} V_r^{(k+1)} \\
+ \frac{2h_{k+1}^2}{15} S_{\rho \gamma}^{(k+1)} (4 \sigma_{\gamma}^{(-k)} - \sigma_{\gamma}^{=(-k+1)}) = 0
\]

Equations (5.57) and (5.58) are the revised form of the displacement continuity conditions at lamina interfaces. Equation (5.55) gives the traction continuity condition.
at lamina interfaces. Substituting equations (4.55) and (4.56) in equation (5.56) yields the thermal balance equation as:

\[
\frac{1}{2} \pi_{\alpha \beta}^{(k)} N_{\alpha \beta}^{(k)} - \frac{1}{h_k} \pi_{\alpha \beta}^{(k)} M_{\alpha \beta}^{(k)} + \frac{1}{2} \pi_{33}^{(k)} N_{33}^{(k)} - \frac{1}{h_k} \pi_{33}^{(k)} M_{33}^{(k)}
\]

\[- (\Gamma_{\alpha \beta}^{(k)} \pi_{\alpha \beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)}) + \frac{\rho^{(k)} c^{(k)}}{T_o} \cdot \left( \frac{2h_k}{12} \theta^{-(k-1)} + \frac{4h_k}{12} \theta^{-(k)} \right) \]

\[- \frac{g'}{T_c} \cdot \left( \frac{\bar{q}_{\alpha, \alpha}^{(k)} - \bar{q}_{\alpha, \alpha}^{(k)}}{4} \right) + \frac{h_k}{T_o} \cdot \frac{g'}{2} \pi_{\alpha \beta}^{(k)} N_{\alpha \beta}^{(k)}(x, 0) \]

\[- \frac{1}{h_k} \pi_{\alpha \beta}^{(k)} M_{\alpha \beta}^{(k)}(x, 0) - \frac{1}{2} \pi_{33}^{(k)} N_{33}^{(k)}(x, 0) + \frac{1}{h_k} \pi_{33}^{(k)} M_{33}^{(k)}(x, 0) \]

\[- (\Gamma_{\alpha \beta}^{(k+1)} \pi_{\alpha \beta}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)}) + \frac{\rho^{(k+1)} c^{(k+1)}}{T_o} \cdot \left( \frac{2h_k}{12} \theta^{-(k-1)} + \frac{4h_k}{12} \theta^{-(k)} \right) \]

\[- \frac{1}{2} \pi_{\alpha \beta}^{(k+1)} N_{\alpha \beta}^{(k+1)} + \frac{1}{h_{k+1}} \pi_{\alpha \beta}^{(k+1)} M_{\alpha \beta}^{(k+1)} + \frac{1}{2} \pi_{33}^{(k+1)} N_{33}^{(k+1)} + \frac{1}{h_{k+1}} \pi_{33}^{(k+1)} M_{33}^{(k+1)} \]

\[- (\Gamma_{\alpha \beta}^{(k+1)} \pi_{\alpha \beta}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)}) + \frac{\rho^{(k+1)} c^{(k+1)}}{T_o} \cdot \left( \frac{4h_{k+1}}{12} \theta^{-(k)} \right) \]

\[- \frac{2h_{k+1}}{12} \theta^{-(k+1)} \] - \[\frac{g'}{T_c} \cdot \left( \frac{\bar{q}_{\alpha, \alpha}^{(k+1)} - \bar{q}_{\alpha, \alpha}^{(k+1)}}{4} \right) + \frac{h_{k+1}}{T_o} \cdot \frac{g'}{2} \pi_{\alpha \beta}^{(k+1)} N_{\alpha \beta}^{(k+1)}(x, 0) \]

\[- \frac{1}{2} \pi_{\alpha \beta}^{(k+1)} N_{\alpha \beta}^{(k+1)}(x, 0) - \frac{1}{h_{k+1}} \pi_{\alpha \beta}^{(k+1)} M_{\alpha \beta}^{(k+1)}(x, 0) - \frac{1}{2} \pi_{33}^{(k+1)} N_{33}^{(k+1)}(x, 0)\]
\[-\frac{1}{h_{k+1}} \pi_{33}^{(k+1)} M_{33}^{(k+1)}(x, 0) + (\Gamma_{\alpha}\pi_{\alpha\beta}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)})
\]

\[+ \frac{\rho^{(k+1)} c^{(k+1)}}{T_o} \left( \frac{4h_{k+1}}{12} \theta^{(k)} - \frac{2h_{k+1}}{12} \theta^{(k+1)} \right) = 0 \] (5.59)

This can be simplified by writing:

\[
\frac{1}{2} \pi_{\alpha\beta}^{(k)} N_{\alpha\beta}^{(k)} - \frac{1}{h_{k+1}} \pi_{\alpha\beta}^{(k)} M_{\alpha\beta}^{(k)} + \frac{1}{2} \pi_{33}^{(k)} N_{33}^{(k)} - \frac{1}{h_{k+1}} \pi_{33}^{(k)} M_{33}^{(k)}
\]

\[- (\Gamma_{\alpha}\pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)}) + \frac{\rho^{(k)} c^{(k)}}{T_o} \left( \frac{2h_{k}}{12} \theta^{(k)} + \frac{4h_{k}+1}{12} \theta^{(k+1)} \right) \]

\[- \frac{g'}{T_o} \frac{h_{k+1}}{4} (\tilde{q}_{\alpha\alpha}^{(k)} - \tilde{q}_{\beta\beta}^{(k)}) + \frac{h_{k+1}}{2} \frac{g'}{T_o} \tilde{r}_{(k+1)}^{(k)} \]

\[+ \frac{1}{2} \pi_{\alpha\beta}^{(k+1)} N_{\alpha\beta}^{(k+1)} + \frac{1}{h_{k+1}} \pi_{\alpha\beta}^{(k+1)} M_{\alpha\beta}^{(k+1)} + \frac{1}{2} \pi_{33}^{(k+1)} N_{33}^{(k+1)} + \frac{1}{h_{k+1}} \pi_{33}^{(k+1)} M_{33}^{(k+1)} \]

\[- (\Gamma_{\alpha}\pi_{\alpha\beta}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)}) + \frac{\rho^{(k+1)} c^{(k+1)}}{T_o} \left( \frac{4h_{k+1}}{12} \theta^{(k)} \right) \]

\[+ \frac{2h_{k+1}}{12} \theta^{(k+1)} \right) + \frac{g'}{T_o} \frac{h_{k+1}}{4} (\tilde{q}_{\alpha\alpha}^{(k+1)} + \tilde{q}_{\beta\beta}^{(k+1)}) \]

\[+ \frac{h_{k+1}}{2} \frac{g'}{T_o} \tilde{r}_{(k+1)}^{(k+1)} + Z_b^{(k)} = 0 \] (5.60)

where

\[Z_b^{(k)} = - \frac{1}{2} \pi_{\alpha\beta}^{(k+1)} N_{\alpha\beta}^{(k+1)}(x, 0) - \frac{1}{h_{k+1}} \pi_{\alpha\beta}^{(k+1)} M_{\alpha\beta}^{(k+1)}(x, 0) \]

\[- \frac{1}{2} \pi_{33}^{(k+1)} N_{33}^{(k+1)}(x, 0) - \frac{1}{h_{k+1}} \pi_{33}^{(k+1)} M_{33}^{(k+1)}(x, 0) + (\Gamma_{\alpha\beta} \pi_{\alpha\beta}^{(k+1)}) \]

98
\[ + \Gamma_{33}^{(k+l)} \pi_{33}^{(k+l)} + \frac{\rho^{(k+l)} c^{(k+l)}}{T_o} \left( \frac{4h_{k+l}}{12} \theta_o^{(k)} + \frac{2h_{k+l}}{12} \theta_o^{(k+l)} \right) \]

\[- \frac{1}{2} \pi^{(k)}_{\alpha\beta} N^{(k)}_{\alpha\beta} (x, 0) + \frac{1}{h_k} \pi^{(k)}_{\alpha\beta} M^{(k)}_{\alpha\beta} (x, 0) - \frac{1}{2} \pi^{(k)}_{33} N^{(k)}_{33} (x, 0) \]

\[+ \frac{1}{h_k} \pi^{(k)}_{33} M^{(k)}_{33} (x, 0) + (\Gamma^{(k)}_{\alpha\beta} \pi^{(k)}_{\alpha\beta} + \Gamma^{(k)}_{33} \pi^{(k)}_{33} + \frac{\rho^{(k)} c^{(k)}}{T_o}) \]

\[\left\{ \frac{2h_k}{12} \theta^{-,(k-l)} + \frac{4h_k}{12} \theta^{-,(k)} \right\} \quad (5.61)\]

The interface displacement continuity conditions associated with the self-adjoint form of the field equations are given by equations (5.57) and (5.58). These equations, along with the thermal balance equation, complete the set of equations required to solve for the unknown field variables and interface stresses.

In this chapter, the governing equations derived in Chapter 4 were rewritten in a self-adjoint form in the sense of Gurtin's convolution bilinear mapping. Expressions for the interface displacements have been written to assure displacement continuity between layers in a laminate system. The coupling between the layers will be incorporated into the laminate system of equations discussed in the next chapter.
CHAPTER 6

OPERATOR FORM OF GOVERNING EQUATIONS

In this chapter, the field equations and interface continuity conditions for the laminate will be expressed in operator form. It will be proved that the field operator is self-adjoint and that the boundary operator is consistent with the field operator.

6.1 GOVERNING EQUATIONS

The set of self-adjoint field equations (5.42) - (5.53) for the k-th lamina, can be expressed in operator form as:

\[
\begin{bmatrix}
  A^{(k)} & B^{(k)} \\
  C^{(k)} & D_{a}^{(k)}
\end{bmatrix}
\begin{bmatrix}
  \{u\}^{(k)} \\
  \{\sigma\}^{(k)}
\end{bmatrix}
+ \begin{bmatrix}
  \{\sigma\}^{(k)} \\
  \{F\}^{(k)}
\end{bmatrix} = 0
\] (6.1)

where \(A^{(k)}\), \(B^{(k)}\), \(C^{(k)}\) and \(D_{a}^{(k)}\) are the field operator matrices for the k-th layer (k = 1, 2, . . . N). \(\{u\}^{(k)}\) and \(\{\sigma\}^{(k)}\) represent the vectors of field variables and \(\{F\}^{(k)}\) is the vector of generalized body. Explicitly, the self-adjoint linear operator \(A^{(k)}\) is given by:
The vector of field variables for the problem are defined as:

\[
\{u\}^{(k)}_T = [ \overline{v}_{(k)} \overline{\phi}_{(k)} \overline{v}_3 \overline{\phi}_3 \overline{\phi}_{3}^{(k)} N_{(k)} N_{33} N_{(k)}^{(k)} M_{(k)} M_{33} M_{(k)}^{(k)} V_{(k)} q_{(k)} \alpha \bar{q}_{(k)} ]
\]  

(6.3)

\[
\{\sigma\}^{z(k)} = [ \sigma_{(33)}^{z(k)} \sigma_{(33)}^{z(k)} \theta_{(33)}^{z(k)} ]
\]  

(6.4)

The off-diagonal terms of \([A]^{(k)}\) constitute adjoint pairs with respect to Gurtin's bilinear mapping. The operators \([B]^{(k)}\) and \([C]^{(k)}\) for the transverse stress and thermal components at the bottom and top of a lamina are given by:
The operator and vector of body force terms are given by:

\[
[D_u]^{(k)} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \frac{h_k^2}{60} & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]  

(6.6)
The generalized body force vector is defined as:

\[
\{F\}^{(k)} = \begin{bmatrix} F_1^{(k)} \\ F_2^{(k)} \\ F_3^{(k)} \end{bmatrix}
\]  

(6.7)

Further, the interface continuity equations (5.57) - (5.59), for the k-th layer, can be written as:

\[
\begin{aligned}
&\left[\Lambda\right]^{(k)} \{\sigma\}^{-(k-1)} + \left[B\right]^{(k)} \{\mathbf{u}\}^{(k)} + \left[\Xi\right]^{(k)} \{\sigma\}^{-(k)} + \left[C\right]^{(k+1)} \{\mathbf{u}\}^{(k+1)} \\
&+ \left[\overline{\Lambda}\right]^{(k+1)} \{\sigma\}^{-(k-1)} + \left[D,\right]^{(k)} \{\mathbf{r}\}^{(k)} + \left[\overline{D,}\right]^{(k+1)} \{\mathbf{r}\}^{(k+1)} + \{Z,\}^{(k)} = 0
\end{aligned}
\]

(6.8)
The boundary operators are defined as:

\[
[B]^{k \gamma} = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
\frac{h_k}{2} & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
-\frac{h_k^'}{120 \partial \gamma} & -\frac{h_k^'}{10} & 0 & 0 \\
\frac{h_k^'}{12 \partial \gamma} & \frac{h_k}{2} & 0 & 0 \\
0 & 0 & \frac{\pi_{\phi \phi}^{(k)}}{2} & 0 \\
0 & -\frac{1}{10} S_{xxx}^{(k)} & \frac{\pi_{\phi \phi}^{(k)}}{2} & \frac{\pi_{\phi \phi}^{(k)}}{h_k} \\
0 & 0 & \frac{3}{h_k} S_{xxx}^{(k)} & \frac{\pi_{\phi \phi}^{(k)}}{h_k} \\
-\frac{2}{5} S_{\phi \phi}^{(k)} & 0 & 0 & 0 \\
0 & 0 & \frac{g' h_k}{T_c} \frac{\partial}{4 \partial \gamma} & 0 \\
0 & 0 & \frac{g' h_k}{T_c} \frac{\partial}{4 \partial \gamma} & 0 \\
0 & 0 & \frac{g' h_k}{T_c} \frac{\partial}{4 \partial \gamma} & 0 \\
0 & 0 & \frac{g' h_k}{T_c} \frac{\partial}{4 \partial \gamma} & 0 \\
\end{bmatrix}
\]

\[
[C]^{k \gamma} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\frac{h_k}{2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\frac{h_k^'}{120 \partial \gamma} & -\frac{h_k^'}{10} & 0 & 0 \\
-\frac{h_k^'}{12 \partial \gamma} & \frac{h_k}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{\pi_{\phi \phi}^{(k)}}{2} \\
0 & -\frac{1}{10} S_{xxx}^{(k)} & 0 & \frac{\pi_{\phi \phi}^{(k)}}{2} \\
0 & 0 & 0 & \frac{\pi_{\phi \phi}^{(k)}}{h_k} \\
0 & 0 & 0 & \frac{3}{h_k} S_{xxx}^{(k)} \\
-\frac{2}{5} S_{\phi \phi}^{(k)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{g' h_k}{T_c} \frac{\partial}{4 \partial \gamma} \\
0 & 0 & 0 & \frac{g' h_k}{T_c} \frac{\partial}{4 \partial \gamma} \\
\end{bmatrix}
\]

(6.9)
\[
[\Xi]^{(k)} = \begin{bmatrix}
\Xi_{11}^{(k)} & 0 & 0 \\
0 & \Xi_{22}^{(k)} & 0 \\
0 & 0 & \Xi_{33}^{(k)} \\
\end{bmatrix}
\]

(6.10)

where

\[
\Xi_{11}^{(k)} = \frac{8}{15} [ h_k S_{p\gamma 3}^{(k)} + h_{k-1} S_{p\gamma 3}^{(k-1)} ]
\]

\[
\Xi_{22}^{(k)} = \frac{3}{35} [ h_k S_{333}^{(k)} + h_{k-1} S_{333}^{(k-1)} ]
\]

\[
\Xi_{33}^{(k)} = -\frac{1}{3} \left[ h_k \left\{ \Gamma_{\alpha\beta}^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho_{(k)}^{(k)} c_{(k)}}{T_o} \right\} + 
\right.
\]

\[
\left. + h_{k-1} \left\{ \Gamma_{\alpha\beta}^{(k-1)} \pi_{\alpha\beta}^{(k-1)} + \Gamma_{33}^{(k-1)} \pi_{33}^{(k-1)} + \frac{\rho_{(k-1)}^{(k-1)} c_{(k-1)}}{T_o} \right\} \right]
\]

and

\[
[A]^{(k)} = \begin{bmatrix}
A_{11}^{(k)} & 0 & 0 \\
0 & A_{22}^{(k)} & 0 \\
0 & 0 & A_{33}^{(k)} \\
\end{bmatrix}
\]

(6.11)

where

\[
A_{11}^{(k)} = -\frac{2}{15} h_k S_{p\gamma 3}^{(k)}
\]

\[
A_{22}^{(k)} = \frac{h_k}{70} S_{333}^{(k)}
\]

\[
A_{33}^{(k)} = -\frac{1}{6} h_k \left\{ \Gamma_{\alpha\beta}^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho_{(k)}^{(k)} c_{(k)}}{T_o} \right\}
\]
and

\[
\begin{bmatrix}
\Lambda^{(k)}
\end{bmatrix}^{(k)} = \begin{bmatrix}
\Lambda_{11}^{(k)} & 0 & 0 \\
0 & \Lambda_{22}^{(k)} & 0 \\
0 & 0 & \Lambda_{33}^{(k)}
\end{bmatrix}
\]

(6.12)

where

\[
\Lambda_{11}^{(k)} = \frac{-2}{15} h_k S^{(k)}_{p3y} \\
\Lambda_{22}^{(k)} = \frac{h_k}{70} S^{(k)}_{3333} \\
\Lambda_{33}^{(k)} = \frac{1}{6} h_k \left\{ \Gamma_{ab}^{(k)} \pi_{ab}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho^{(k)} c^{(k)}}{T_o} \right\}
\]

Additionally,

\[
\begin{bmatrix}
D, \end{bmatrix}^{(k)} = \begin{bmatrix}
0 \\
0 \\
\frac{h_k g^{(k)}}{2 T_o}
\end{bmatrix} \\
\begin{bmatrix}
\overline{D}, \end{bmatrix}^{(k)} = \begin{bmatrix}
0 \\
0 \\
\frac{h_k g^{(k)}}{2 T_o}
\end{bmatrix} \\
\{Z_a\}^{(k)} = \begin{bmatrix}
0 \\
0 \\
Z^{(k)}_b
\end{bmatrix}
\]

(6.13)

The components \{\sigma\}^{(1)} and \{\sigma\}^{-(N)} are given for a problem and appear as forcing functions defined as:

\[
\{\sigma\}^{(1)} = \begin{bmatrix}
\hat{\sigma}^{(0)}_{y3} \\
\hat{\sigma}^{(0)}_{33} \\
\hat{\sigma}^{(0)}_{33}
\end{bmatrix} \quad \text{and} \quad \{\sigma\}^{-(N)} = \begin{bmatrix}
\hat{\sigma}^{(N)}_{y3} \\
\hat{\sigma}^{(N)}_{33} \\
\hat{\sigma}^{(N)}_{33}
\end{bmatrix}
\]

where the superscript (0) denotes the prescribed values at the top of the 1st layer and the superscript (N) denotes the bottom of the Nth layer. Further, the following definitions are introduced:
\[\{Q\}^{(i)} = [\Lambda]^{(i)} \{\sigma\}^{*(i)} \quad (6.14)\]

\[
\begin{align*}
&= \left\{ \begin{array}{c}
- \frac{2h_1}{15} S_{p33}^z \dot{\sigma}_y^z \\
\frac{h_1}{70} S_{3333} \dot{\sigma}_y^0 \\
- \frac{1}{6} \left( \Gamma_{\alpha\beta}^{(i)} \pi_{\alpha\beta}^{(i)} + \Gamma_3^{(i)} \pi_3^{(i)} + \frac{\rho^{(i)} c^{(i)}}{T_0} \right) \dot{\theta}^{(0)}
\end{array} \right. 
\end{align*}
\]

and

\[\{Q\}^{(N-1)} = [\Lambda]^{(N)} \{\sigma\}^{-(N)} \quad (6.15)\]

\[
\begin{align*}
&= \left\{ \begin{array}{c}
- \frac{2h_N}{15} S_{p33}^z \dot{\sigma}_y^z \\
\frac{h_N}{70} S_{3333} \dot{\sigma}_y^0 \\
- \frac{1}{6} \left( \Gamma_{\alpha\beta}^{(N)} \pi_{\alpha\beta}^{(N)} + \Gamma_3^{(N)} \pi_3^{(N)} + \frac{\rho^{(N)} c^{(N)}}{T_0} \right) \dot{\theta}^{(N)}
\end{array} \right. 
\end{align*}
\]
Define:

\[
\begin{align*}
\{P\}^{(1)} & \equiv [C]^{(1)} \{\sigma\}^{**(1)} = \\
& = \begin{bmatrix}
\hat{\sigma}_{\gamma^3}^{(0)} \\
\frac{h_1}{2} \hat{\sigma}_{\gamma^3}^{(0)} \\
\hat{\sigma}_{33}^{(0)} \\
\frac{h_1^3}{120} \hat{\sigma}_{\gamma^3,\gamma}^{(0)} + \frac{h_1^2}{10} \hat{\sigma}_{33}^{(0)} \\
\frac{h_1^2}{12} \hat{\sigma}_{\gamma^3,\gamma}^{(0)} + \frac{h_1}{2} \hat{\sigma}_{33}^{(0)} \\
\frac{\pi_{\alpha\beta}^{(1)}}{2} \hat{\theta}^{(0)} \\
-\frac{1}{10} S_{3333}^{(1)} \hat{\sigma}_{33}^{(0)} + \frac{\pi_{33}^{(1)}}{2} \hat{\theta}^{(0)} \\
\frac{\pi_{\alpha\beta}^{(1)}}{h_1} \hat{\theta}^{(0)} \\
-\frac{3}{7h_1} S_{3333}^{(1)} \hat{\sigma}_{33}^{(0)} + \frac{\pi_{33}^{(1)}}{h_1} \hat{\theta}^{(0)} \\
-\frac{2}{5} S_{\rho\gamma\gamma}^{(1)} \hat{\sigma}_{\gamma^3}^{(0)} \\
\frac{g'}{T_o} \frac{h_1}{4} \hat{\sigma}_{\gamma^3}^{(0)} \\
\frac{g'}{T_o} \frac{h_1}{4} \hat{\theta}^{(0)} \gamma 
\end{bmatrix}
\end{align*}
\] (6.16)
Combining equations (6.1), (6.8), (6.14) - (6.17), the set of governing equations may be expressed as:

\[
[W] \{X\} = \{\psi\}
\]

(6.18)

where

\[
\{\psi\} = -(\{I\} + \{Y\} + \{Z\})
\]

(6.19)
The operator matrix is defined as:

\[
[W] = \begin{bmatrix}
[A]^n & [B]^n & 0 & 0 & 0 & 0 \\
0 & [C]^b & [A]^b & [B]^b & 0 & 0 \\
0 & 0 & 0 & [C]^n & [A]^n & [B]^n \\
0 & 0 & 0 & 0 & [A]^n & [B]^n \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Further,

\[
{X} = \begin{bmatrix}
{u}^{(1)} \\
{\sigma}^{(1)} \\
{u}^{(2)} \\
{\sigma}^{(2)} \\
{u}^{(3)} \\
{\sigma}^{(3)} \\
{u}^{(4)} \\
{\sigma}^{(4)} \\
{\cdots} \\
{\sigma}^{(N-2)} \\
{u}^{(N-1)} \\
{\sigma}^{(N-1)} \\
{u}^{(N)} \\
\end{bmatrix}
\]

\[
{I} = \begin{bmatrix}
[D_u]^{(1)} \{F\}^{(1)} \\
[D_s]^{(1)} \{r\}^{(1)} + [\overline{D_s}]^{(2)} \{r\}^{(2)} \\
[D_u]^{(2)} \{F\}^{(2)} \\
[D_s]^{(2)} \{r\}^{(2)} + [\overline{D_s}]^{(3)} \{r\}^{(3)} \\
[D_u]^{(3)} \{F\}^{(3)} \\
[D_s]^{(3)} \{r\}^{(3)} + [\overline{D_s}]^{(4)} \{r\}^{(4)} \\
[D_u]^{(4)} \{F\}^{(4)} \\
[D_s]^{(4)} \{r\}^{(4)} + [\overline{D_s}]^{(5)} \{r\}^{(5)} \\
\cdots \\
[D_u]^{(N-2)} \{F\}^{(N-2)} \\
[D_s]^{(N-2)} \{r\}^{(N-2)} + [\overline{D_s}]^{(N-1)} \{r\}^{(N-1)} \\
[D_u]^{(N-1)} \{F\}^{(N-1)} \\
[D_s]^{(N-1)} \{r\}^{(N-1)} + [\overline{D_s}]^{(N)} \{r\}^{(N)} \\
[D_u]^{(N)} \{F\}^{(N)} \\
\end{bmatrix}
\]
6.2 ADJOINTNESS OF THE OPERATOR MATRICES

In order for the field operator matrix \([W]\) to be self-adjoint in the sense of equation (5.16), it is sufficient that:

i) \([A]^{(k)}\) and \([\Xi]^{(k)}\) be self-adjoint,

ii) \([B]^{(k)}\) and \([\overline{B}]^{(k)}\) be adjoint pairs,

iii) \([C]^{(k)}\) and \([\overline{C}]^{(k)}\) be adjoint pairs.

iv) \([\Lambda]^{(k)}\) and \([\overline{\Lambda}]^{(k)}\) be adjoint pairs.

In order to prove the self-adjointness of the field operator, Green's theorem is used which can be expressed as:
\[ \int_{R} g \ast (u \nu) \alpha dR = \int_{R} g \ast (\nu u_{,\alpha} + u \nu_{,\alpha}) dR = \int_{S} g \ast (u \nu) dS \quad (6.20) \]

The symbol \(< . , >_{R}\) represents Gurtin's bilinear convolution mapping defined by equation (5.67) as:

\[ < u , v >_{R} \equiv \int_{R} (u \ast v) dR \quad (6.21) \]

where \( u , v \in R \times [0, \infty) \). Hence, Green's theorem takes the form:

\[ < v , g \ast u_{,\alpha} >_{R} + < u , g \ast v_{,\alpha} >_{R} = < u , g \ast v >_{S} \]

or rearranged

\[ < v , g \ast u_{,\alpha} >_{R} = -< u , g \ast v_{,\alpha} >_{R} + < u , g \ast v >_{S} \quad (6.22) \]

6.2.1 Operator Matrix \( [A]^{(k)} \)

Let us consider the components \( A_{16}^{(k)} \) and \( A_{61}^{(k)} \) of the operator matrix \( A \), equation (6.3), defined by the respective subscript notation. Taking the Gurtin product of \( \tilde{v}_{\gamma}^{(k)} \) for an arbitrary function in domain of \( \Gamma_{2} \), with \( \frac{1}{2} \Gamma_{1} N_{\alpha\beta}^{(k)} \) and applying Green's theorem yields:

\[ < \tilde{v}_{\gamma}^{(k)} , A_{16}^{(k)} N_{\alpha\beta}^{(k)} >_{R^{1} \Gamma} = < \tilde{v}_{\gamma}^{(k)} , \frac{1}{2} \Gamma_{1} N_{\alpha\beta}^{(k)} >_{R^{1} \Gamma} , \]

\[ = -< N_{\mu\alpha}^{(k)} , \frac{1}{2} \Gamma_{2} \tilde{v}_{\gamma}^{(k)} >_{R^{1} \Gamma} + < N_{\alpha\beta}^{(k)} \eta_{\beta} \tilde{v}_{\alpha}^{(k)} >_{S^{1} \Gamma} , \]

\[ = < N_{\mu\alpha}^{(k)} , A_{61}^{(k)} \tilde{v}_{\gamma}^{(k)} >_{R^{1} \Gamma} + < N_{\alpha\beta}^{(k)} \eta_{\beta} \tilde{v}_{\alpha}^{(k)} >_{S^{1} \Gamma} \quad (6.23) \]
where \( R_{1}^{(k)} \) is and region of the \( k \)-th layer and \( S_{1}^{(k)} \) is its boundary. For operators \( A_{28}^{(k)} \) and \( A_{82}^{(k)} \), taking the Gurtin product of \( \bar{\phi}_{\gamma}^{(k)} \) for an arbitrary function in domain of \( \Gamma_{2} \).

with \( \frac{1}{2} \Gamma_{1} M_{\alpha\beta}^{(k)} \) yields:

\[
< \bar{\phi}_{\gamma}^{(k)} , A_{28}^{(k)} M_{\alpha\beta}^{(k)} >_{R^{k}} = < \bar{\phi}_{\gamma}^{(k)} , \frac{1}{2} \Gamma_{1} M_{\alpha\beta}^{(k)} >_{R^{k}},
\]

\[
= -< M_{\mu \alpha}^{(k)} \cdot \frac{1}{2} \Gamma_{1} \Phi_{\gamma}^{(k)} >_{R^{k}}, + < M_{\alpha \beta}^{(k)} \eta_{\beta} , \bar{\phi}_{\gamma}^{(k)} >_{S^{k}},
\]

\[
= < M_{\mu \alpha}^{(k)} A_{82}^{(k)} \Phi_{\gamma}^{(k)} >_{R^{k}}, + < M_{\alpha \beta}^{(k)} \eta_{\beta} , \bar{\phi}_{\gamma}^{(k)} >_{S^{k}} ,
\]  \[(6.24)\]

For operators \( A_{10}^{(k)} \) and \( A_{110}^{(k)} \), taking the Gurtin product of an arbitrary function \( \bar{v}_{3}^{(k)} \) with respect to \( \frac{\partial}{\partial \gamma} V_{\gamma}^{(k)} \):

\[
< \bar{v}_{3}^{(k)} , A_{110}^{(k)} V_{\gamma}^{(k)} >_{R^{k}} = < \bar{v}_{3}^{(k)} , \frac{\partial}{\partial \gamma} V_{\gamma}^{(k)} >_{R^{k}},
\]

\[
= -< V_{\gamma}^{(k)} \cdot \frac{\partial}{\partial \gamma} \bar{v}_{3}^{(k)} >_{R^{k}}, + < V_{\gamma}^{(k)} \eta_{\gamma} , \bar{v}_{3}^{(k)} >_{S^{k}},
\]

\[
= < V_{\gamma}^{(k)} A_{110}^{(k)} \bar{v}_{3}^{(k)} >_{R^{k}}, + < V_{\gamma}^{(k)} \eta_{\gamma} , \bar{v}_{3}^{(k)} >_{S^{k}},
\]  \[(6.25)\]

Further, operator pairs, \( A_{210}^{(k)} & A_{102}^{(k)} \), \( A_{67}^{(k)} & A_{76}^{(k)} \), \( A_{69}^{(k)} & A_{96}^{(k)} \), \( A_{79}^{(k)} & A_{97}^{(k)} \), and \( A_{57}^{(k)} & A_{75}^{(k)} \) are linear algebraic operators which are transpose of each other and hence constitute self-adjoint pairs. Additionally, diagonal operators \( A_{i}^{(k)} \), \( i=1,\ldots,12 \)
are symmetric tensors in the sense of equation (5.6) with $D_s(w,u) = 0$. Hence the operator matrix $[A]^{(k)}$ is self-adjoint with respect to Gurtin's bilinear mapping [refer equation (5.67)].

6.2.2 Operator Matrix $[\Xi]^{(k)}$

Consider the operator $\Xi_{11}^{(k)}$. Taking the Gurtin product of an arbitrary function $\sigma_{p3}^{-(k)}$ in the domain of $\Xi_{11}^{(k)}$, with $\Xi_{11}^{(k)} \sigma_{\gamma 3}^{-(k)}$ yields:

$$<\sigma_{p3}^{-(k)} \cdot \Xi_{11}^{(k)} \sigma_{\gamma 3}^{-(k)} >_{R^{4x4}} = <\sigma_{p3}^{-(k)} \cdot \frac{8}{15} [h_k S_{p3\gamma 3}^{(k)} + h_{k+1} S_{p3\gamma 3}^{(k+1)}] \sigma_{\gamma 3}^{-(k)} >_{R^{4x4}},$$

$$= <\sigma_{\gamma 3}^{-(k)} \cdot \frac{8}{15} [h_k S_{p3\gamma 3}^{(k)} + h_{k+1} S_{p3\gamma 3}^{(k+1)}] \sigma_{p3}^{-(k)} >_{R^{4x4}},$$

$$= <\sigma_{\gamma 3}^{-(k)} \cdot \Xi_{11}^{(k)} \sigma_{p3}^{-(k)} >_{R^{4x4}}. \tag{6.26}$$

Similarly operators $\Xi_{22}^{(k)}$ and $\Xi_{33}^{(k)}$ are symmetric in sense of equation (5.6) with $D_s(w,u) = 0$. Hence the operator matrix $[\Xi]^{(k)}$ is self-adjoint with respect to Gurtin's bilinear mapping [refer equation (5.67)].
6.2.3 Operator Matrices $[B]^{(k)}$ and $[\overline{B}]^{(k)}$

Let us consider the operators $B_{41}^{(k)}$ and $\overline{B}_{14}^{(k)}$ that belong respectively to the operator matrices $[B]^{(k)}$ and $[\overline{B}]^{(k)}$. Taking Gurtin product of an arbitrary function $\tilde{\phi}_3^{(k)}$ in domain of $\overline{B}_{14}^{(k)}$, with $B_{41}^{(k)} \sigma_{\gamma 3}^{(-k)}$ and applying Green's theorem yields:

$$<\tilde{\phi}_3^{(k)}, B_{41}^{(k)} \sigma_{\gamma 3}^{(-k)} >_{R^{*k}} = <\tilde{\phi}_3^{(k)}, \frac{h_k}{120} \sigma_{\gamma 3}^{(-k)} >_{R^{*k}},$$

$$= - <\sigma_{\gamma 3}^{(-k)} \frac{h_k}{120} \tilde{\phi}_3^{(k)} >_{R^{*k}} + <\sigma_{\gamma 3}^{(-k)} \eta_{\gamma}, \frac{h_k}{120} \tilde{\phi}_3^{(k)} >_{S^{*k}},$$

$$= <\sigma_{\gamma 3}^{(-k)} \overline{B}_{14}^{(k)} \tilde{\phi}_3^{(k)} >_{R^{*k}} + <\sigma_{\gamma 3}^{(-k)} \eta_{\gamma}, \frac{h_k}{120} \tilde{\phi}_3^{(k)} >_{S^{*k}}, \quad (6.27)$$

Hence operators $B_{41}^{(k)}$ and $\overline{B}_{14}^{(k)}$ constitute adjoint pairs. Similarly it can be proved that operator pairs, $B_{51}^{(k)}$ & $\overline{B}_{15}^{(k)}$, $B_{11}^{(k)}$ & $\overline{B}_{31}^{(k)}$, and $B_{12}^{(k)}$ & $\overline{B}_{32}^{(k)}$ form adjoint pairs. Further the operator pairs, $B_{11}^{(k)}$ & $\overline{B}_{11}^{(k)}$, $B_{21}^{(k)}$ & $\overline{B}_{12}^{(k)}$, $B_{32}^{(k)}$ & $\overline{B}_{23}^{(k)}$, $B_{42}^{(k)}$ & $\overline{B}_{24}^{(k)}$, $B_{52}^{(k)}$ & $\overline{B}_{25}^{(k)}$, $B_{72}^{(k)}$ & $\overline{B}_{27}^{(k)}$, $B_{92}^{(k)}$ & $\overline{B}_{29}^{(k)}$, $B_{63}^{(k)}$ & $\overline{B}_{36}^{(k)}$, $B_{73}^{(k)}$ & $\overline{B}_{37}^{(k)}$, $B_{83}^{(k)}$ & $\overline{B}_{38}^{(k)}$, $B_{93}^{(k)}$ & $\overline{B}_{93}^{(k)}$, and $B_{101}^{(k)}$ & $\overline{B}_{101}^{(k)}$ are algebraic expressions which are transpose of each other. Hence operators $[B]^{(k)}$ and $[\overline{B}]^{(k)}$ constitute adjoint pairs.
6.2.4 Operator Matrices $[C]^{(k)}$ and $[\overline{C}]^{(k)}$

Following the method used to show the adjointness of $[B]^{(k)}$ and $[\overline{B}]^{(k)}$, it can be shown that $[C]^{(k)}$ and $[\overline{C}]^{(k)}$ constitute adjoint pairs.

6.2.5 Operator Matrices $[\Lambda]^{(k)}$ and $[\overline{\Lambda}]^{(k)}$

Let us consider the operators $\Lambda_{11}^{(k)}$ and $\overline{\Lambda}_{11}^{(k)}$ of the operator matrices $[\Lambda]^{(k)}$ and $[\overline{\Lambda}]^{(k)}$. Taking the Gurtin product of an arbitrary function $\tilde{\sigma}_{\gamma 3}^{+(k)}$ in domain of $\Lambda_{11}^{(k)}$, with $\overline{\Lambda}_{11}^{(k)} \sigma_{\gamma 3}^{+(k)}$ and applying Green's theorem gives:

$$<\tilde{\sigma}_{\rho 3}^{+(k)} \cdot \overline{\Lambda}_{11}^{(k)} \sigma_{\gamma 3}^{+(k)} > _{R^{**}} = <\tilde{\sigma}_{\rho 3}^{+(k)} \cdot \frac{2}{15} h_k S_{\rho 3}^{(k)} \sigma_{\gamma 3}^{+(k)} > _{R^{**}}$$

$$= <\sigma_{\gamma 3}^{-(k)} \cdot \frac{2}{15} h_k S_{\rho 3}^{(k)} \overline{\sigma}_{\rho 3}^{+(k)} > _{R^{**}}$$

$$= <\sigma_{\gamma 3}^{-(k)} \cdot \Lambda_{11}^{(k)} \overline{\sigma}_{\gamma 3}^{+(k)} > _{R^{**}} \quad (6.28)$$

Similarly operators $\Lambda_{22}^{(k)}$ & $\overline{\Lambda}_{22}^{(k)}$ and $\Lambda_{33}^{(k)}$ & $\overline{\Lambda}_{33}^{(k)}$ are symmetric in sense of equation (5.6) with $D_{\rho}(w,u) = 0$. Hence, operator matrices $[\Lambda]^{(k)}$ and $[\overline{\Lambda}]^{(k)}$ constitute adjoint pairs.
In sections 6.2.1 - 6.2.5, it has been proved that $[A]^k$ and $[\Xi]^k$ are self-adjoint. $[B]^k$ and $[\overline{B}]^k$, $[C]^k$ and $[\overline{C}]^k$, and $[\Lambda]^k$ and $[\overline{\Lambda}]^k$ are adjoint pairs. Hence the operator matrix $[W]$ for the global system in equation (6.18) is self-adjoint with respect to Gurtin's bilinear mapping [refer equation (5.67)].

6.3 CONSISTENT BOUNDARY OPERATORS

The consistent boundary operators are derived by using equation (5.16). These operators are found by considering the non-zero operators of the field operator matrix $[W]$ and taking the Gurtin product of a typical arbitrary set of $\{\overline{u}\}^k$ with the corresponding set of equations. The Gurtin product of a typical set of equations (6.1) can be expressed as:

$$
<\{\overline{u}\}^k \cdot [C]^k \{\sigma\}^{-(k-1)} + [A]^k \{u\}^k + [B]^k \{\sigma\}^{-(k)} + [D_u]^k \{F\}^k >_{\overline{r}^k},
$$

$$
= <\{\sigma\}^{-(k-1)} . [C]^k \{\overline{u}\}^k >_{\overline{r}^k}, + <\{u\}^k \cdot [A]^k \{\overline{u}\}^k >_{\overline{r}^k},
$$

$$
+ <\{\sigma\}^{-(k)} . [B]^k \{\overline{u}\}^k >_{\overline{r}^k}, + <\{\sigma\}^{-(k)} . [L]^k \{\overline{u}\}^k >_{\overline{s}^k},
$$

$$
+ <\{\sigma\}^{-(k)} . [G]^k \{\overline{u}\}^k >_{\overline{s}^k}, + <\{u\}^k \cdot [J]^k \{\overline{u}\}^k >_{\overline{s}^k},
$$

$$
- <\{\overline{u}\}^k \cdot [G]^k \{\sigma\}^{-(k-1)} + [J]^k \{u\}^k + [L]^k \{\sigma\}^{-(k)} >_{\overline{s}^k},
$$

$$
+ <[D_u]^k \{F\}^k \cdot \{\overline{u}\}^k >_{\overline{r}^k}, \tag{6.29}
$$

where the matrices $[G]^k$, $[\overline{G}]^k$, $[L]^k$, $[\overline{L}]^k$, and $[J]^k$ are yet to be determined.
The elements of these matrices are determined based on the selection of an arbitrary \( \{ \tilde{u} \}^{(k)} \).

Selecting \( \{ \tilde{u} \}^{(k)} \) = \( \begin{bmatrix} \tilde{v}_{\alpha}^{(k)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \) and substituting this into the left hand side of equation (6.29) and using equation (6.23) gives:

\[
< \tilde{v}_{\alpha}^{(k)} \cdot \sigma_{\alpha \beta}^{- (k-1)} + N_{\alpha \beta}^{(k)} \cdot \sigma_{\alpha \beta}^{- (k)} >_{R^{(k)}} = < \sigma_{\alpha \beta}^{- (k-1)} \cdot \tilde{v}_{\alpha}^{(k)} >_{R^{(k)}} + < N_{\alpha \beta}^{(k)} \cdot \tilde{v}_{\alpha}^{(k)} >_{R^{(k)}}
\]

\[
+ < \sigma_{\alpha \beta}^{- (k)} \cdot \tilde{v}_{\alpha}^{(k)} >_{R^{(k)}} + < N_{\alpha \beta}^{(k)} \eta_{\beta} \cdot \tilde{v}_{\alpha}^{(k)} >_{S^{(k)}},
\]

(6.30)

To obtain the consistent form of the boundary terms given by (6.29) we write

\[
< N_{\alpha \beta}^{(k)} \eta_{\beta} \cdot \tilde{v}_{\alpha}^{(k)} >_{S^{(k)}} = < N_{\alpha \beta}^{(k)} \eta_{\beta} \cdot \tilde{v}_{\alpha}^{(k)} >_{S^{(k)}} - < \tilde{v}_{\alpha}^{(k)} \cdot \eta_{\beta} >_{S^{(k)}},
\]

(6.31)

where \( S_{1}^{(k)} \cup S_{2}^{(k)} = S_{u}^{(k)} \) and \( S_{1}^{(k)} \cap S_{2}^{(k)} = \emptyset \).

To find the next set of elements in the consistent boundary operator matrix, select \( \{ \tilde{u} \}^{(k)} = \begin{bmatrix} \tilde{v}_{\alpha}^{(k)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \). Substituting into the left hand side of equation (6.31) and using equation (6.24) gives:

\[
< \tilde{v}_{\alpha}^{(k)} \cdot \frac{h_{k}}{2} \sigma_{\alpha \beta}^{- (k-1)} + M_{\alpha \beta}^{(k)} \cdot \tilde{v}_{\alpha}^{(k)} + \frac{h_{k}}{2} \sigma_{\alpha \beta}^{- (k)} >_{R^{(k)}} = < \sigma_{\alpha \beta}^{- (k-1)} \cdot \frac{h_{k}}{2} \tilde{v}_{\alpha}^{(k)} >_{R^{(k)}} + < M_{\alpha \beta}^{(k)} \cdot \tilde{v}_{\alpha}^{(k)} >_{R^{(k)}} + < \frac{h_{k}}{2} \sigma_{\alpha \beta}^{- (k)} \tilde{v}_{\alpha}^{(k)} >_{R^{(k)}}
\]

\[
+ < \sigma_{\alpha \beta}^{- (k)} \cdot \frac{h_{k}}{2} \tilde{v}_{\alpha}^{(k)} >_{R^{(k)}} + < M_{\alpha \beta}^{(k)} \eta_{\beta} \cdot \tilde{v}_{\alpha}^{(k)} >_{S^{(k)}},
\]

(6.32)
Rewriting the boundary term yields:

\[ \langle M_{ab}^{(k)} \eta_{ib}, \tilde{\Phi}_{i}^{(k)} \rangle_{S_1^{(k)}} = \langle M_{ab}^{(k)} \eta_{ib}, \tilde{\Phi}_{i}^{(k)} \rangle_{S_1^{(k)}} - \langle \tilde{\Phi}_{i}^{(k)} - M_{ab}^{(k)} \eta_{ib} \rangle_{S_1^{(k)}} \]  

(6.33)

where \( \overline{S}_3^{(k)} \cup \overline{S}_4^{(k)} = \overline{S}_a^{(k)} \) and \( \overline{S}_3^{(k)} \cap \overline{S}_4^{(k)} = 0 \).

Selecting \( \{u\}'^{(k)} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \) gives, after substituting into (6.29) and using (6.25):

\[ \langle \tilde{V}_3^{(k)}, \sigma_{33}^{-(k-l)} + V_{a,a}^{(k)} - \sigma_{33}^{-(k)} \rangle_{R^{(k)}} = \langle \sigma_{33}^{-(k-l)}, \tilde{V}_3^{(k)} \rangle_{R^{(k)}} + \langle V_{a}^{(k)}, -\tilde{V}_3^{(k)} \rangle_{R^{(k)}} + \langle \sigma_{33}^{-(k)}, -\tilde{V}_3^{(k)} \rangle_{R^{(k)}} \]

\[ + \langle V_{a}^{(k)}, \eta_{a} \cdot \tilde{V}_3^{(k)} \rangle_{S_{1}^{(k)}}, \]  

(6.34)

The form of equation (5.16) is realized by writing the boundary terms as:

\[ \langle V_{a}^{(k)}, \eta_{a} \cdot \tilde{V}_3^{(k)} \rangle_{S_{1}^{(k)}} = \langle V_{a}^{(k)}, \eta_{a} \cdot \tilde{V}_3^{(k)} \rangle_{S_{1}^{(k)}} - \langle \tilde{V}_3^{(k)} \cdot -V_{a}^{(k)} \eta_{a} \rangle_{S_{1}^{(k)}}, \]  

(6.35)

where \( \overline{S}_5^{(k)} \cup \overline{S}_6^{(k)} = \overline{S}_a^{(k)} \) and \( \overline{S}_5^{(k)} \cap \overline{S}_6^{(k)} = 0 \).

For \( \{u\}'^{(k)} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \) we have:

\[ \langle \Phi_{3}^{(k)}, \frac{h_k^2}{10} \sigma_{33}^{-(k-l)} + \frac{h_k^2}{120} \sigma_{a3,a}^{-(k-l)} - M_{33}^{(k)} + \frac{h_k^2}{120} \sigma_{33,a}^{-(k)} - \frac{h_k^2}{10} \sigma_{33}^{-(k)} \rangle_{R^{(k)}} \]

\[ = \langle \sigma_{a3}^{-(k-l)}, -\frac{h_k^2}{120} \Phi_{3}^{(k)} \rangle_{R^{(k)}} + \langle \sigma_{33}^{-(k-l)}, -\frac{h_k^2}{10} \Phi_{3}^{(k)} \rangle_{R^{(k)}} + \langle M_{33}^{(k)}, -\Phi_{3}^{(k)} \rangle_{R^{(k)}} \]

\[ + \langle \sigma_{a3}^{-(k-l)}, -\frac{h_k^2}{120} \Phi_{3}^{(k)} \eta_{a} \rangle_{S_{1}^{(k)}} + \langle \sigma_{33}^{-(k-l)}, -\frac{h_k^2}{120} \Phi_{3}^{(k)} \eta_{a} \rangle_{S_{1}^{(k)}}, \]  

(6.36)
Similarly if \( \{\vec{u}\}^{(k)} = [0 0 0 0 0 0 0 0 0 0 0] \) we get:

\[
\begin{align*}
\langle \phi_3^{(k)} \rangle &= \frac{h_k}{2} \sigma_{33}^{(k-1)} + \frac{h_k^2}{12} \sigma_{a3a}^{(k-1)} + \frac{h_k}{2} \sigma_{33a}^{(k-1)} - N_{33}^{(k)} - \frac{h_k^2}{12} \sigma_{a3a}^{(k)} > R^{(k)}, \\
\langle \phi_3^{(k)} \rangle &= \langle \sigma_{a3}^{(k-1)} \cdot \frac{h_k}{2} \phi_3^{(k)} \rangle > R^{(k)}, + \langle \sigma_{a3}^{(k-1)} \cdot \frac{h_k}{2} \phi_3^{(k)} \rangle > R^{(k)}, \\
\langle \sigma_{a3}^{(k-1)} \cdot \frac{h_k}{2} \phi_3^{(k)} \rangle > S^{(k)}, + \langle \sigma_{a3}^{(k-1)} \cdot \frac{h_k}{2} \phi_3^{(k)} \rangle > S^{(k)}.
\end{align*}
\]

(6.37)

The next set of boundary operator matrix elements are determined by selecting

\( \{\vec{u}\}^{(k)} = [0 0 0 0 0 0 0 0 0 0 0] \) where upon substitution into (6.23) and using Green’s theorem, results in the following:

\[
\begin{align*}
\langle \bar{N}_{ab}^{(k)} \rangle &= \frac{\pi_{ab}^{(k)}}{2} \theta^{(k-1)} + \frac{\pi_{ab}^{(k)}}{2} \theta^{(k)} + \frac{1}{h_k} S_{a33}^{(k)} N_{33}^{(k)} - \bar{V}_{(a \beta)}^{(k)} + \frac{1}{h_k} S_{a3 \beta \gamma}^{(k)} N_{(a \beta \gamma)}^{(k)} > R^{(k)}, \\
\langle \bar{N}_{ab}^{(k)} \rangle &= < \theta^{(k-1)}, \frac{\pi_{ab}^{(k)}}{2} \bar{N}_{ab}^{(k)} > R^{(k)} + < \theta^{(k)}, \frac{\pi_{ab}^{(k)}}{2} \bar{N}_{ab}^{(k)} > R^{(k)} + < \bar{V}_{(a \beta)}^{(k)} \cdot \bar{N}_{ab}^{(k)} > R^{(k)}, \\
\langle \bar{N}_{ab}^{(k)} \rangle &= < N_{ab}^{(k)} \cdot \frac{1}{h_k} S_{a3 \mu \rho}^{(k)} N_{ab}^{(k)} > R^{(k)} + < \bar{V}_{(a \beta)}^{(k)} \cdot \bar{N}_{ab}^{(k)} > S^{(k)}, \\
\langle \bar{N}_{ab}^{(k)} \rangle &= < N_{33}^{(k)} \cdot \frac{1}{h_k} S_{a3 \mu \rho}^{(k)} \bar{N}_{ab}^{(k)} > R^{(k)}.
\end{align*}
\]

(6.38)

To take the form of (5.16) we write the form of the boundary term as:

\[
\begin{align*}
\langle \bar{N}_{ab}^{(k)} \cdot \bar{V}_{(a \beta)}^{(k)} \eta_{(a \beta)} > S^{(k)} &= < \bar{V}_{(a \beta)}^{(k)} \cdot \bar{N}_{ab}^{(k)} \eta_{(a \beta)} > S^{(k)} - < \bar{N}_{ab}^{(k)} \cdot \bar{V}_{(a \beta)}^{(k)} \eta_{(a \beta)} > S^{(k)}.
\end{align*}
\]

(6.39)
Proceeding similarly by substituting \( \{ \tilde{u} \}^T_{i(k)} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \)
yields:

\[
<\tilde{N}_{33}^{(k)} > \frac{\pi_{33}^{(k)}}{2} \theta^{-(k-1)} + \frac{\pi_{33}^{(k)}}{2} \theta^{-(k)} - \frac{1}{10} S_{3333}^{(k)} \sigma_{33}^{-(k-1)} - \frac{1}{10} S_{3333}^{(k)} \sigma_{33}^{-(k)} + \frac{1}{h_k} S_{3333}^{(k)} N_{a\beta}^{(k)}
+ \frac{6}{5h_k} S_{3333}^{(k)} N_{33}^{(k)} - \phi_{3}^{(k)} > R^{*},
\]

\[
=<\theta^{-(k-1)} \frac{\pi_{33}^{(k)}}{2} \tilde{N}_{33}^{(k)} > R^{*} + <\theta^{-(k)} \frac{\pi_{33}^{(k)}}{2} \tilde{N}_{33}^{(k)} > R^{*},
\]

\[
+ <\sigma_{33}^{-(k-1)} \frac{1}{10} S_{3333}^{(k)} \tilde{N}_{33}^{(k)} > R^{*} + <\sigma_{33}^{-(k)} \frac{1}{10} S_{3333}^{(k)} \tilde{N}_{33}^{(k)} > R^{*},
\]

\[
+ <\phi_{3}^{(k)} \tilde{N}_{33}^{(k)} > R^{*} + <N_{33}^{(k)} \frac{6}{5h_k} S_{3333}^{(k)} \tilde{N}_{33}^{(k)} > R^{*},
\]

\[
+ <N_{a\beta}^{(k)} \frac{1}{h_k} S_{3333}^{(k)} \tilde{N}_{33}^{(k)} > R^{*}, \tag{6.40}
\]

For \( \{ \tilde{u} \}^T_{i(k)} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \) equation (6.31) can be written as:

\[
<\tilde{M}_{a\beta}^{(k)} \frac{\pi_{a\beta}^{(k)}}{h_k} \theta^{-(k-1)} - \frac{\pi_{a\beta}^{(k)}}{h_k} \theta^{-(k)} - \phi_{(a,\beta)}^{(k)} > M_{a\beta}^{(k)} + \frac{12}{h_k} S_{a\beta33}^{(k)} M_{a\beta}^{(k)} + \frac{12}{h_k} S_{a\beta33}^{(k)} M_{a\beta}^{(k)} > R^{*},
\]

\[
=<\theta^{-(k-1)} \frac{\pi_{a\beta}^{(k)}}{h_k} \tilde{M}_{a\beta}^{(k)} > R^{*} + <\theta^{-(k)} \frac{\pi_{a\beta}^{(k)}}{h_k} \tilde{M}_{a\beta}^{(k)} > R^{*} + <\phi_{(a,\beta)}^{(k)} \tilde{M}_{a\beta}^{(k)} > R^{*},
\]

\[
+ <M_{a\beta}^{(k)} \frac{12}{h_k} S_{a\beta33}^{(k)} \tilde{M}_{a\beta}^{(k)} > R^{*} + <M_{a\beta}^{(k)} \frac{12}{h_k} S_{a\beta33}^{(k)} \tilde{M}_{a\beta}^{(k)} > R^{*},
\]

\[
+ <\tilde{M}_{a\beta}^{(k)} \phi_{(a)} \eta_{\beta} > R^{*}. \tag{6.41}
\]

121
where the following boundary terms can be written as:

\[
<\tilde{M}_{\alpha\beta}^{(k)} \cdot \tilde{\phi}_\alpha^{(k)} \eta_\beta>_S^{(a)} = <\tilde{\phi}_\alpha^{(k)} \cdot \tilde{M}_{\alpha\beta}^{(k)} \eta_\beta>_S^{(a)} = <\tilde{\phi}_\alpha^{(k)} \cdot \tilde{M}_{\alpha\beta}^{(k)} \eta_\beta>_S^{(a)}.
\] (6.42)

Selecting \(\{\tilde{u}\}^{(k)}_T = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ M_{33}^{(k)} \ 0 \ 0 \ 0]\) and substituting into equation (6.37) gives:

\[
<\tilde{M}_{33}^{(k)} \cdot \frac{\pi_{33}^{(k)}}{h_k} \theta^{-(k-1)} - \frac{\pi_{33}^{(k)}}{h_k} \theta^{-(k)} - \tilde{\phi}_3^{(k)} - \frac{3}{7h_k} S_{3333}^{(k)} \sigma_{33}^{-(k-1)} + \frac{3}{7h_k} S_{3333}^{(k)} \sigma_{33}^{(k)}
\]

\[+ \frac{12}{h_k^3} S_{3333}^{(k)} M_{33}^{(k)} + \frac{120}{7h_k^3} S_{3333}^{(k)} M_{33}^{(k)} > R^{(k)}.
\]

\[= <\theta^{-(k-1)} \cdot \frac{\pi_{33}^{(k)}}{h_k} \tilde{M}_{33}^{(k)} > R^{(k)} + <\theta^{-(k)} \cdot \frac{\pi_{33}^{(k)}}{h_k} \tilde{M}_{33}^{(k)} > R^{(k)},
\]

\[+ <\sigma_{33}^{(k)} \cdot \frac{3}{7h_k} S_{3333}^{(k)} \tilde{M}_{33}^{(k)} > R^{(k)} + <\tilde{\phi}_3^{(k)} \cdot \tilde{M}_{33}^{(k)} > R^{(k)},
\]

\[+ <M_{33}^{(k)} \cdot \frac{120}{7h_k^3} S_{3333}^{(k)} \tilde{M}_{33}^{(k)} > R^{(k)} + <M_{33}^{(k)} \cdot \frac{12}{h_k^3} S_{3333}^{(k)} \tilde{M}_{33}^{(k)} > R^{(k)},
\]

\[+ <\sigma_{33}^{(k)} \cdot \frac{3}{7h_k} S_{3333}^{(k)} \tilde{M}_{33}^{(k)} > R^{(k)}.
\] (6.43)

To find the next set of elements in the consistent boundary operator matrix, selecting \(\{\tilde{u}\}^{(k)}_T = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \tilde{V}_\gamma^{(k)} \ 0 \ 0].\) Then we have:

\[
<\tilde{V}_\gamma^{(k)} \cdot -\frac{2}{5} S_{\rho\gamma3}^{(k)} \sigma_{\gamma3}^{-(k-1)} - \tilde{\phi}_\rho^{(k)} - \tilde{V}_\gamma^{(k)} + \frac{24}{5h_k} S_{\rho\gamma3}^{(k)} \gamma_\gamma^{(k)} - \frac{2}{5} S_{\rho\gamma3}^{(k)} \sigma_{\gamma3}^{-(k)} > R^{(k)}.
\]

\[= <\sigma_{\gamma3}^{-(k-1)} \cdot -\frac{2}{5} S_{\rho\gamma3}^{(k)} \tilde{V}_\gamma^{(k)} > R^{(k)} + <\tilde{\phi}_\rho^{(k)} \cdot \tilde{V}_\gamma^{(k)} > R^{(k)} + <\tilde{V}_\gamma^{(k)} \cdot \tilde{V}_\alpha \cdot \gamma_\alpha > R^{(k)}.
\]
where:

\[
< \tilde{V}_\alpha \eta_{\alpha}, - \tilde{V}_3 >_s, = < \tilde{V}_3, - \tilde{V}_\alpha \eta_\alpha >_{s, k}, - < \tilde{V}_\alpha \eta_\alpha, \tilde{V}_3 >_{s, k}.
\]  

(6.45)

Selecting \( \{ \bar{u} \}^{(k)} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \) then we have:

\[
< \tilde{q}_\alpha < \frac{g'}{T_0} \theta^{-k-1}_\alpha + \frac{g'}{T_0} \theta^{-k}_\alpha + \frac{g'}{T_0} \lambda^{(k)} \eta_{\alpha} >_{R^{k1}},
\]  

(6.46)

Selecting \( \{ \bar{u} \}^{(k)} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \) then we have:

\[
< \tilde{q}_\alpha < \frac{g'}{T_0} \theta^{-k}_\alpha + \frac{g'}{T_0} \theta^{-k-1}_\alpha + \frac{3h_k}{T_0} \lambda^{(k)} \eta_{\alpha} >_{R^{k1}},
\]  

(6.47)
Taking the convolution of \( \{ \bar{\sigma} \}^{-k} \) with equation (6.8) gives:

\[
<\{\bar{\sigma}\}^{-k} \cdot [\Lambda]^{(k)} \{\sigma\}^{-k-1} + [B]^{(k)} \{u\}^{(k)} + [\Xi]^{(k)} \{\sigma\}^{-k} + [C]^{(k+1)} \{u\}^{(k+1)} \\
+ [\bar{\Lambda}]^{(k+1)} \{\sigma\}^{-k+1} + [D_y]^{(k)} \{r\}^{(k)} + [\bar{D}_y]^{(k+1)} \{r\}^{(k+1)} + \{Z_i\}^{(k)} >_{R^k},
\]

\[
= <\{\sigma\}^{-k-1} \cdot [\Lambda]^{(k)} \{\bar{\sigma}\}^{(k)} >_{R^k} + <\{u\}^{(k)} \cdot B]^{(k)} \{\bar{\sigma}\}^{-k} >_{R^k},
\]

\[
+ <\{\sigma\}^{-k+1} \cdot [\Xi]^{(k)} \{\bar{\sigma}\}^{(k)} >_{R^k} + <\{u\}^{(k+1)} \cdot C]^{(k+1)} \{\bar{\sigma}\}^{-k} >_{R^k},
\]

\[
+ <\{\sigma\}^{-k+1} \cdot [\Psi]^{(k)} \{\bar{\sigma}\}^{(k)} >_{S^k} + <\{\sigma\}^{-k+1} \cdot [T]^{(k+1)} \{\bar{\sigma}\}^{(k)} >_{S^k},
\]

\[
- <\{\bar{\sigma}\}^{-k} \cdot [T]^{(k)} \{\sigma\}^{-k-1} + [L]^{(k)} \{u\}^{(k)} + [\Psi]^{(k)} \{\sigma\}^{-k} \\
+ [\bar{\Gamma}]^{(k+1)} \{u\}^{(k+1)} + [\bar{T}]^{(k+1)} \{\sigma\}^{-k+1} >_{S^k},
\]

\[
+ <[D_y]^{(k)} \{r\}^{(k)} \cdot \{\bar{\sigma}\}^{(k)} >_{R^k} + <[\bar{D}_y]^{(k+1)} \{r\}^{(k+1)} \cdot \{\bar{\sigma}\}^{(k)} >_{R^k},
\]

\[
+ <\{Z_i\}^{(k)} \cdot \{\bar{\sigma}\}^{-k} >_{R^k}. \tag{6.48}
\]

The matrices \([T]^{(k)}\), \([\bar{T}]^{(k)}\), \([\Psi]^{(k)}\), \([L]^{(k)}\) and \([\bar{\Gamma}]^{(k)}\) need to be determined by selecting an arbitrary \( \{ \bar{\sigma} \}^{-k} \).

Selecting \( \{ \bar{\sigma} \}^{-k} = [\bar{\sigma}^{-k}_{\gamma \beta} \ 0 \ 0] \) and substituting into the left hand side of equation (6.47) gives:

\[
<\bar{\sigma}^{-k}_{\gamma \beta} \cdot \frac{2}{15} h_k S^{(k)}_{\rho \lambda 3} \bar{\sigma}^{-k-1}_{\beta \gamma} - \bar{V}^{(k)}_{\gamma} + \frac{h_k}{2} \bar{\phi}^{(k)}_{\gamma} - \frac{h^2_k}{120} \bar{\phi}_{\gamma 3}^{(k)} + \frac{h^2_k}{12} \bar{\phi}_{\gamma 3}^{(k)} + \frac{2}{5} S^{(k)}_{\rho \lambda 3} \bar{V}^{(k)}_{\rho}
\]

124
\[
\begin{align*}
&+ \frac{8}{15} \left[ h_k \sigma_{y3}^{(k)} + h_{k+1} \sigma_{y3}^{(k+1)} \right] \gamma_3 + \nabla_y^{(k+1)} + \frac{h_{k+1}}{2} \phi_{\gamma}^{(k+1)} - \frac{h_{k+1}^2}{120} \phi_{\gamma}^{(k+1)} \\
&- \frac{h_{k+1}^2}{12} \phi_{\gamma}^{(k+1)} - \frac{2}{5} h_k \sigma_{y3}^{(k)} \gamma_3 + \frac{2h_{k+1}}{15} \sigma_{y3}^{(k+1)} \gamma_3 > R^y, \\
&= <\sigma_{y3}^{-(k-1)} - \frac{2}{15} h_k \sigma_{y3}^{(k)} \gamma_3 > R^y + <\nabla_y^{(k)} - \sigma_{y3}^{-(k)} > R^y + <\phi_{\gamma}^{(k)} - \frac{h_k}{2} \sigma_{y3}^{-(k)} > R^y, \\
&+ <\phi_{\gamma}^{(k)} - \frac{h_k}{120} \sigma_{y3,\gamma}^{-(k)} > R^y, \\
&+ <\phi_{\gamma}^{(k+1)} - \frac{h_k^2}{12} \sigma_{y3,\gamma}^{-(k)} > R^y, \\
&+ <\nabla_y^{(k+1)} - \sigma_{y3}^{-(k)} > R^y + <\phi_{\gamma}^{(k+1)} - \frac{h_{k+1}}{2} \sigma_{y3}^{-(k)} > R^y, \\
&+ <\phi_{\gamma}^{(k+1)} - \frac{h_{k+1}^2}{120} \sigma_{y3,\gamma}^{-(k)} > R^y, \\
&+ <\phi_{\gamma}^{(k+1)} - \frac{h_{k+1}^2}{12} \sigma_{y3,\gamma}^{-(k)} > R^y, \\
&+ <\nabla_y^{(k+1)} - \sigma_{y3}^{-(k)} > R^y + <\phi_{\gamma}^{(k+1)} - \frac{h_{k+1}}{2} \sigma_{y3}^{-(k)} > R^y, \\
&+ <\phi_{\gamma}^{(k+1)} - \frac{h_{k+1}^2}{120} \sigma_{y3,\gamma}^{-(k)} > R^y, \\
&+ <\phi_{\gamma}^{(k+1)} - \frac{h_{k+1}^2}{12} \sigma_{y3,\gamma}^{-(k)} > R^y, \\
&+ <\phi_{\gamma}^{(k+1)} - \frac{h_{k+1}^2}{120} \sigma_{y3,\gamma}^{-(k)} > R^y, \\
&+ <\phi_{\gamma}^{(k+1)} - \frac{h_{k+1}^2}{12} \sigma_{y3,\gamma}^{-(k)} > R^y, \\
&+ <\phi_{\gamma}^{(k+1)} - \frac{h_{k+1}^2}{120} \sigma_{y3,\gamma}^{-(k)} > R^y, \\
&+ <\phi_{\gamma}^{(k+1)} - \frac{h_{k+1}^2}{12} \sigma_{y3,\gamma}^{-(k)} > R^y. \\
\end{align*}
\]

(6.49)

For \( \tilde{\sigma}_{y} = [0 \quad \tilde{\sigma}_{y3}^{-(k)} \quad 0] \) we have:

\[
<\tilde{\sigma}_{y3}^{-(k)} - \frac{h_k}{70} S_{33}^{(k)} \sigma_{y3}^{-(k-1)} - \nabla_y^{(k)} - \frac{h_k^2}{10} \tilde{\phi}_{\gamma}^{(k)} + \frac{h_k}{2} \phi_{\gamma}^{(k)} - \frac{1}{10} S_{33}^{(k)} N_{33}^{(k)} + \frac{3}{7h_k} S_{33}^{(k)} M_{33}^{(k)}
\]

\[
+ \frac{3}{35} \left[ h_k S_{33}^{(k)} + h_{k+1} S_{33}^{(k+1)} \right] \sigma_{y3}^{-(k)} + \frac{h_{k+1}}{10} \phi_{\gamma}^{(k+1)} + \frac{h_{k+1}^2}{2} \phi_{\gamma}^{(k+1)}
\]

125
For \( \{\sigma\}^{-(k)} \) we have:

\[
<\tilde{\sigma}^{-(k)}_k> = \frac{h_k}{6} \left[ \Gamma_{ab}^{(k)} \pi_{ab}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho_{ab}^{(k) C^{(k)}}}{T_o} \right] \theta^{-(k-1)} + \frac{\pi_{ab}^{(k)}}{2} N_{ab}^{(k)} + \frac{\pi_{33}^{(k)}}{2} N_{33}^{(k)}
\]

\[
- \frac{\pi_{ab}^{(k)}}{h_k} M_{ab}^{(k)} - \frac{\pi_{33}^{(k)}}{h_k} M_{33}^{(k)} - \frac{g'}{4} h_k \tilde{q}_{\alpha,\alpha} + \frac{g'}{4} h_k \tilde{q}_{\alpha,\alpha}
\]

\[
- \frac{1}{3} \left[ h_k \left\{ \Gamma_{ab}^{(k)} \pi_{ab}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho_{ab}^{(k) C^{(k)}}}{T_o} \right\} \right]
\]

\[
+ h_{k+1} \left\{ \Gamma_{ab}^{(k+1)} \pi_{ab}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)} + \frac{\rho_{ab}^{(k+1) C^{(k+1)}}}{T_o} \right\}
\]

126
\[
\frac{\pi_{\alpha\beta}^{(k+1)}}{2} N_{\alpha\beta}^{(k+1)} + \frac{\pi_{33}^{(k+1)}}{2} N_{33}^{(k+1)} + \frac{\pi_{\alpha\beta}^{(k+1)}}{h_{k+1}} M_{\alpha\beta}^{(k+1)} + \frac{\pi_{33}^{(k+1)}}{h_{k+1}} M_{33}^{(k+1)}
\]

\[- \frac{g'}{T_o} \frac{h_{k+1}}{4} \tilde{q}_{\alpha\alpha}^{(k+1)} - \frac{g'}{T_o} \frac{h_{k+1}}{4} \tilde{q}_{\alpha\alpha}^{(k+1)}
\]

\[- \frac{h_{k+1}}{6} \left[ \Gamma_{\alpha\beta}^{(k+1)} \pi_{\alpha\beta}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)} + \frac{\rho^{(k+1)} c^{(k+1)}}{T_o} \right] \theta^{-(k+1)} > R_{k+1}
\]

\[= \frac{h_{k+1}}{6} \left[ \Gamma_{\alpha\beta}^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho^{(k)} c^{(k)}}{T_o} \right] \tilde{\theta}^{-(k)} > R_{k+1}
\]

\[+ \frac{N_{\alpha\beta}^{(k)}}{2} \frac{\pi_{\alpha\beta}^{(k)}}{h_{k}} \tilde{\theta}^{-(k)} > R_{k+1} + \frac{N_{33}^{(k)}}{2} \frac{\pi_{33}^{(k)}}{h_{k}} \tilde{\theta}^{-(k)} > R_{k+1}
\]

\[+ \frac{M_{\alpha\beta}^{(k)}}{h_{k+1}} \pi_{\alpha\beta}^{(k)} > R_{k+1} + \frac{M_{33}^{(k)}}{h_{k+1}} \pi_{33}^{(k)} > R_{k+1}
\]

\[+ \frac{\tilde{q}_{\alpha\alpha}^{(k)}}{\Gamma_{\alpha\beta}^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho^{(k)} c^{(k)}}{T_o}} + \frac{h_{k+1}}{4} \frac{\tilde{q}_{\alpha\alpha}^{(k)}}{\Gamma_{\alpha\beta}^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho^{(k)} c^{(k)}}{T_o}}
\]

\[= \frac{h_{k+1}}{3} \left[ \Gamma_{\alpha\beta}^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho^{(k)} c^{(k)}}{T_o} \right] \tilde{\theta}^{-(k)} > R_{k+1}
\]

\[+ \frac{N_{\alpha\beta}^{(k+1)}}{2} \frac{\pi_{\alpha\beta}^{(k+1)}}{h_{k+1}} \tilde{\theta}^{-(k)} > R_{k+1} + \frac{N_{33}^{(k+1)}}{2} \frac{\pi_{33}^{(k+1)}}{h_{k+1}} \tilde{\theta}^{-(k)} > R_{k+1}
\]

\[+ \frac{M_{\alpha\beta}^{(k+1)}}{h_{k+1}} \frac{\pi_{\alpha\beta}^{(k+1)}}{h_{k+1}} \tilde{\theta}^{-(k)} > R_{k+1} + \frac{M_{33}^{(k+1)}}{h_{k+1}} \frac{\pi_{33}^{(k+1)}}{h_{k+1}} \tilde{\theta}^{-(k)} > R_{k+1}
\]

\[+ \frac{\tilde{q}_{\alpha\alpha}^{(k+1)}}{\Gamma_{\alpha\beta}^{(k+1)} \pi_{\alpha\beta}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)} + \frac{\rho^{(k+1)} c^{(k+1)}}{T_o}} + \frac{h_{k+1}}{4} \frac{\tilde{q}_{\alpha\alpha}^{(k+1)}}{\Gamma_{\alpha\beta}^{(k+1)} \pi_{\alpha\beta}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)} + \frac{\rho^{(k+1)} c^{(k+1)}}{T_o}}
\]

\[= \frac{h_{k+1}}{3} \left[ \Gamma_{\alpha\beta}^{(k+1)} \pi_{\alpha\beta}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)} + \frac{\rho^{(k+1)} c^{(k+1)}}{T_o} \right] \tilde{\theta}^{-(k)} > R_{k+1}
\]

127
The consistent boundary operators for \( N \) layers can be written collectively as follows:

\[
[R] \{ X \} = \{ H \}
\]

(6.52)
The vector of field variables and vector of prescribed boundary conditions are given as:

\[
{X} = \begin{bmatrix}
{u}^{(1)} \\
{\sigma}^{(1)} \\
{u}^{(2)} \\
{\sigma}^{(2)} \\
{u}^{(3)} \\
{\sigma}^{(3)} \\
{u}^{(4)} \\
{\sigma}^{(4)} \\
\vdots \\
{\sigma}^{(N-2)} \\
{u}^{(N-1)} \\
{\sigma}^{(N-1)} \\
{u}^{(N)}
\end{bmatrix}
\]

Also,

\[
{H} = \begin{bmatrix}
{g_u}^{(1)} \\
{g_{\sigma}}^{(1)} \\
{g_u}^{(2)} \\
{g_{\sigma}}^{(2)} \\
{g_u}^{(3)} \\
{g_{\sigma}}^{(3)} \\
{g_u}^{(4)} \\
{g_{\sigma}}^{(4)} \\
\vdots \\
{g_{\sigma}}^{(N-2)} \\
{g_u}^{(N-1)} \\
{g_{\sigma}}^{(N-1)} \\
{g_u}^{(N)}
\end{bmatrix}
\]

Also,

\[
[J]^u = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -\eta_p & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\eta_p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\eta_p & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\eta_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \eta_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \eta_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \eta_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \eta_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \eta_p & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \eta_p & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_p & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_p & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_p & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_p
\end{bmatrix}
\]
\[
[\mathbf{G}]^{kr} =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{h_k}{120} \eta_y & 0 & 0 & 0 \\
\frac{h_k^2}{12} \eta_y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{g'}{T_o} \frac{h_k}{4} \eta_y & 0 \\
0 & 0 & \frac{g'}{T_o} \frac{h_k}{4} \eta_y & 0
\end{bmatrix}
\]
The vector $\{H\}$ denotes the prescribed boundary conditions. Equation (6.52) can be rewritten in two parts as:

$$\begin{bmatrix} 
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{h_k}{120} \eta_y & 0 & 0 \\
-\frac{h_k}{12} \eta_y & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{g'}{T_o} h_k / 4 \eta_y & 0 & 0 \\
0 & 0 & -\frac{g'}{T_o} h_k / 4 \eta_y 
\end{bmatrix}$$

$$[\psi]^{(k)} = [T]^{(k)} = [\overline{T}]^{(k)} = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}$$

Equation (6.53) can be rewritten in two parts as:

$$[J]^{(k)} \{u\}^{(k)} = \{g_u\}^{(k)} \quad \text{on } S_u^{(k)} \tag{6.53}$$

$$[\overline{L}]^{(k)} \{u\}^{(k)} + [\overline{G}]^{(k+1)} \{u\}^{(k+1)} = \{g_\sigma\}^{(k)} \quad \text{on } S_\sigma^{(k)} \tag{6.54}$$

131
6.4 PRESCRIBED BOUNDARY CONDITIONS

In order to describe the problem in the form proposed by Sandhu and Salaam (1975), the boundary conditions [equations (6.53) and (6.54)] need to be specified in a consistent form. Using equation (5.11), the consistent form of the boundary conditions can be written as:

\[
<\{u\}^{(k)}\cdot [J]^{(k)}\{u\}^{(k)} - 2\{g_u\}^{(k)} >_s + <\{\sigma\}^{(k)}\cdot [\Sigma]^{(k)}\{u\}^{(k)} >_s \\
+ [\Sigma]^{(k-1)}\{u\}^{(k-1)} - 2\{g_\sigma\}^{(k)} >_s = 0 \tag{6.55}
\]

Substituting the definitions presented in Section 6.3, equation (6.55) takes the form:

\[
<\tilde{\nabla}^{(k)}_\alpha \cdot N^{(k)}_{\alpha \beta} \eta_\beta - 2g_1^{(k)} >_s + <\tilde{\phi}^{(k)}_\alpha \cdot -M^{(k)}_{\alpha \beta} \eta_\beta - 2g_1^{(k)} >_s \\
+ <\nabla^{(k)}_3 \cdot \tilde{V}^{(k)}_\alpha \eta_\alpha - 2g_5^{(k)} >_s + <\tilde{\psi}^{(k)}_3 \cdot -2g_u^{(k)} >_s + <\tilde{\phi}^{(k)}_3 \cdot -2g_u^{(k)} >_s \\
+ <\nabla^{(k)}_{\alpha \beta} \cdot \tilde{V}^{(k)}_\alpha \eta_\alpha - 2g_2^{(k)} >_s + <\nabla^{(k)}_{33} \cdot -2g_7^{(k)} >_s + <M^{(k)}_{\alpha \beta} \cdot \tilde{\phi}^{(k)}_\alpha \eta_\beta - 2g_4^{(k)} >_s \\
+ <M^{(k)}_{33} \cdot -2g_8^{(k)} >_s + <V^{(k)}_\alpha \cdot \nabla^{(k)}_\gamma \eta_\alpha - 2g_\sigma^{(k)} >_s + <q^{(k)}_\alpha \cdot -2g^{(k)} >_s \\
+ <q^{(k)}_\alpha \cdot -2g^{(k)} >_s + <\Theta^{(k)}_\gamma \cdot \frac{h_k^2}{12} \hat{\phi}^{(k)}_3 \eta_\gamma - \frac{h_k^2}{12} \hat{\phi}^{(k)}_3 \eta_\gamma + \frac{h_k^{k+1}}{120} \hat{\phi}^{(k-1)}_3 \eta_\gamma \\
+ \frac{h_k^{k+1}}{12} \hat{\phi}^{(k-1)}_3 \eta_\gamma - 2g^{(k)} >_s + <\Theta^{(k)}_\gamma \cdot \frac{g^' h_k}{T_0} \hat{q}^{(k)}_\alpha \eta_\alpha - \frac{g^'}{T_0} \frac{h_k}{4} \hat{q}^{(k)}_\alpha \eta_\alpha \\
+ \frac{g^'}{T_0} \frac{h_k^{k+1}}{4} \hat{q}^{(k+1)}_\alpha \eta_\alpha - 2g^{(k)} >_s = 0 \tag{6.56}
\]
The specified boundary conditions can be expressed as:

\[- N^{(k)}_{\alpha \beta} \eta_\beta = g^{(k)}_1 \quad \text{on } S^{(k)}_1 \]  \hspace{1cm} (6.57)

\[- M^{(k)}_{\alpha \beta} \eta_\beta = g^{(k)}_3 \quad \text{on } S^{(k)}_3 \]  \hspace{1cm} (6.58)

\[- V^{(k)}_{\alpha} \eta_\alpha = g^{(k)}_5 \quad \text{on } S^{(k)}_5 \]  \hspace{1cm} (6.59)

\[\overline{V}^{(k)}_{\alpha} \eta_\alpha = g^{(k)}_2 \quad \text{on } S^{(k)}_2 \]  \hspace{1cm} (6.60)

\[\bar{\phi}^{(k)}_{\alpha} \eta_\beta = g^{(k)}_4 \quad \text{on } S^{(k)}_4 \]  \hspace{1cm} (6.61)

\[\overline{V}^{(k)}_3 \eta_\alpha = g^{(k)}_6 \quad \text{on } S^{(k)}_6 \]  \hspace{1cm} (6.62)

and the continuity conditions are expressed as:

\[\frac{h^2_k}{120} \overline{\phi}^{(k)}_3 \eta_\gamma = \frac{h^2_k}{120} \overline{\phi}^{(k)}_3 \eta_\gamma + \frac{h^{2}_{k+1}}{120} \overline{\phi}^{(k+1)}_3 \eta_\gamma + \frac{h^{2}_{k+1}}{120} \overline{\phi}^{(k+1)}_3 \eta_\gamma = g^{(k)}_\sigma \quad \text{on } S^{(k)}_\sigma \]  \hspace{1cm} (6.63)

and

\[\frac{g'}{T_o} \frac{h_k}{4} \bar{q}^{(k)}_\alpha \eta_\alpha - \frac{g'}{T_o} \frac{h_k}{4} \bar{q}^{(k)}_\alpha \eta_\alpha + \frac{g'}{T_o} \frac{h_{k+1}}{4} \bar{q}^{(k+1)}_\alpha \eta_\alpha + \frac{g'}{T_o} \frac{h_{k+1}}{4} \bar{q}^{(k+1)}_\alpha \eta_\alpha \]

\[+ \frac{g'}{T_o} \frac{h_{k+1}}{4} \overline{\bar{q}^{(k+1)}_\alpha} \eta_\alpha = g^{(k)}_o \quad \text{on } S^{(k)}_o \]  \hspace{1cm} (6.64)

The self-adjoint form of the field equations is given by equations (5.42) - (5.53). In addition, a consistent set of boundary operators have been derived to represent the boundary conditions in the governing function. It is noted that a set of jump discontinuities at interlaminar boundaries can be easily included in the theory to account for discontinuities at interelement boundaries. The internal discontinuity
The conditions are of the same form as the continuity conditions (6.57) - (6.62) except that the equations are valid on $S'$, the internal boundaries of domain $R$ rather than $S$ the external boundaries of $R$. The jump discontinuity conditions will be incorporated into the governing function in the following chapter.
CHAPTER 7

GENERALIZED VARIATIONAL FORMULATION

The governing function for the self-adjoint form of the field equations and boundary conditions, derived in Chapter 6, is formulated. Vanishing of the Gateaux differential along arbitrary paths in space of admissible functions is carried out. The order of differentiability of the field variables is investigated.

7.1 VARIATIONAL FORMULATION OF THE SELF-ADJOINT PROBLEM

An initial boundary value problem with N independent field variables, defined by (5.12), (5.13) and (5.17), has for a governing functional [Sandhu and Salaam (1975)]:

\[
\Omega = \sum_{i=1}^{N} \left[ <X_i \cdot \sum_{j=1}^{n} \psi_{ij} X_j - 2\psi_i>_{\mathcal{R}_i} + <X_i, \sum_{j=1}^{n} \psi_{ij} X_j - 2\psi_i>_{\mathcal{S}_i} \right] \\
+ <X_i, \sum_{j=1}^{n} (\psi_{ij} X_j)' - 2\psi_i>_{\mathcal{S}_i} 
\] (7.1)
For the problem at hand, substituting the field equations (6.18), the consistent boundary condition (6.57) - (6.62) along with the jump discontinuity conditions into equation (7.1), the governing function is:

\[
\Omega(u, \sigma) = \sum_{k=1}^{N} <\{u\}^{(k)}, [A]^{(k)} \{u\}^{(k)}>_{R^{k+1}} + \sum_{k=1}^{N-1} <\{u\}^{(k)}, [B]^{(k)} \{\sigma\}^{-(k)}>_{R^{k+1}} + \sum_{k=1}^{N} <\{u\}^{(k)}, [C]^{(k)} \{\sigma\}^{-(k)}>_{R^{k+1}} + \sum_{k=1}^{N} <\{u\}^{(k)}, [D]^{(k)} \{F\}^{(k)}>_{R^{k+1}} + 2 <\{u\}^{(1)}, [P]^{(1)}>_{R^{1}}, 2 <\{u\}^{(N)}, [P]^{(N)}>_{R^{N}},
\]

\[
+ \sum_{k=1}^{N-1} <\{\sigma\}^{-(k)}, [B]^{(k)} \{u\}^{(k)}>_{R^{k+1}} + \sum_{k=1}^{N-1} <\{\sigma\}^{-(k)}, [C]^{(k)} \{\sigma\}^{-(k)}>_{R^{k+1}} + \sum_{k=1}^{N-1} <\{\sigma\}^{-(k)}, [D]^{(k)} \{F\}^{(k)}>_{R^{k+1}} + 2 <\{\sigma\}^{-(1)}, [Q]^{(1)}>_{R^{1}}, 2 <\{\sigma\}^{-(N-1)}, [Q]^{(N-1)}>_{R^{N}},
\]

\[
+ \sum_{k=1}^{N} <\{u\}^{(k)}, [J]^{(k)} \{u\}^{(k)} - 2 \{g_u\}^{(k)}>_{S^{1}},
\]

\[
+ <\{\sigma\}^{-(1)}, [L]^{(1)} \{u\}^{(1)} + [\overline{G}]^{(2)} \{u\}^{(2)} - 2 \{g_\sigma\}^{(1)}>_{S^{1}},
\]

136
\[
+ \sum_{k=2}^{N-2} \langle \{\sigma\}^{-k}, [L]^{k-1} \{u\}^{(k-1)} \{u\}^{(k)} - 2\{g\}^{(k)} \rangle_{s^{n}} \\
+ \langle \{\sigma\}^{-(N-1)}, [L]^{(N-1)} \{u\}^{(N-1)} + [G]^{(N)} \{u\}^{(N)} - 2\{g\}^{(N-1)} \rangle_{s^{n}} \\
+ \sum_{k=1}^{N} \langle \{u\}^{(k)} \{J\}^{(k)} \{u\}^{(k)} - 2\{g\}^{(k)} \rangle_{s^{n}} \\
+ \langle \{\sigma\}^{-(1)}, [L]^{(1)} \{u\}^{(1)} + [G]^{(1)} \{u\}^{(1)} - 2\{g\}^{(1)} \rangle_{s^{n}} \\
+ \sum_{k=2}^{N-2} \langle \{\sigma\}^{-k}, [L]^{(k)} \{u\}^{(k)} + [G]^{(k)} \{u\}^{(k)} - 2\{g\}^{(k)} \rangle_{s^{n}} \\
+ \langle \{\sigma\}^{-(N-1)}, [L]^{(N-1)} \{u\}^{(N-1)} + [G]^{(N)} \{u\}^{(N)} - 2\{g\}^{(N-1)} \rangle_{s^{n}} \\
\tag{7.2}
\]

where \(R^{(k)}\) is the two-dimensional region of the \(k^{th}\) lamina and \(S^{(k)}_{u}\), \(S^{(k)}_{\sigma}\) symbolically represent appropriate portions of the boundary \(R^{(k)}\). \(S^{(k)}_{u}\) and \(S^{(k)}_{\sigma}\) represent appropriate subsets of internal boundaries in the region. Substituting (6.2) - (6.17) into equation (7.2) gives the explicit form of the function, including jump discontinuities:

\[
\Omega(u,\sigma) = 2\langle \bar{\n}^{(1)}, \hat{\sigma}^{(0)} \rangle_{R^{(1)}} + 2\langle \bar{\phi}^{(1)}_{3}, \frac{h_{1}}{2} \hat{\sigma}^{(0)}_{33} \rangle_{R^{(1)}} + 2\langle \bar{\n}^{(1)}, \hat{\sigma}^{(0)} \rangle_{R^{(1)}} \\
+ 2\langle \bar{\n}^{(1)}, \frac{h_{1}^{3}}{120} \hat{\sigma}^{(0)}_{\gamma \gamma \gamma} + \frac{h_{1}^{2}}{10} \hat{\sigma}^{(0)}_{33} \rangle_{R^{(1)}} + 2\langle \bar{\phi}^{(1)}_{3}, \frac{h_{1}^{2}}{12} \hat{\sigma}^{(0)}_{\gamma \gamma \gamma} + \frac{h_{1}}{2} \hat{\sigma}^{(0)}_{33} \rangle_{R^{(1)}} \\
+ 2\langle N^{(1)}_{\sigma \phi}, \frac{\pi_{\sigma \phi}^{(1)}}{2} \hat{\theta}^{(0)} \rangle_{R^{(1)}} + 2\langle N^{(1)}_{33}, -\frac{1}{10} S_{3333}^{(1)} \hat{\sigma}^{(0)}_{33} + \frac{\pi_{33}^{(1)}}{2} \hat{\theta}^{(0)} \rangle_{R^{(1)}} \\
+ 2\langle M^{(1)}_{\sigma \phi}, \frac{\pi_{\sigma \phi}^{(1)}}{h_{1}} \hat{\theta}^{(0)} \rangle_{R^{(1)}} + 2\langle M^{(1)}_{33}, -\frac{3}{7h_{1}} S_{3333}^{(1)} \hat{\sigma}^{(0)}_{33} + \frac{\pi_{33}^{(1)}}{h_{1}} \hat{\theta}^{(0)} \rangle_{R^{(1)}}
\]

137
\[ + 2<\mathcal{V}_{\alpha}^{(1)} \cdot \left( -\frac{2}{5} S_{\rho \gamma}^{(1)} \hat{\sigma}_{\gamma}^{(0)} \right) >_{R^{(1)}} + 2<\hat{q}_{\alpha}^{(1)} \cdot \frac{g'}{T_o} \cdot \frac{h_1}{4} \hat{\theta}_{\gamma}^{(0)} >_{R^{(1)}} \]

\[ + 2<\tilde{q}_{\alpha}^{(1)} \cdot \frac{g'}{T_o} \cdot \frac{h_1}{4} \hat{\theta}_{\gamma}^{(0)} >_{R^{(1)}} + 2<\mathcal{V}_{\alpha}^{(N)} \cdot \hat{\sigma}_{\alpha 3}^{(N)} >_{R^{(N)}} + 2<\tilde{q}_{\alpha}^{(N)} \cdot \frac{h_N}{2} \hat{\sigma}_{\alpha 3}^{(N)} >_{R^{(N)}} \]

\[ + 2<\mathcal{V}_{\gamma}^{(N)} \cdot \hat{\sigma}_{33}^{(N)} >_{R^{(N)}} + 2<\tilde{q}_{\gamma}^{(N)} \cdot \frac{h_N}{120} \hat{\sigma}_{\gamma 3}^{(N)} >_{R^{(N)}} + 2<\phi_{\gamma}^{(N)} \cdot \frac{h_N}{10} \hat{\sigma}_{33}^{(N)} >_{R^{(N)}} \]

\[ + 2<\phi_{\gamma}^{(N)} \cdot -\frac{h_N}{12} \hat{\sigma}_{\gamma 3}^{(N)} + \frac{h_N}{2} \hat{\sigma}_{33}^{(N)} >_{R^{(N)}} + 2<N_{\alpha \beta}^{(N)} \cdot \frac{\pi_{\alpha \beta}^{(N)}}{2} \hat{\theta}_{\gamma}^{(N)} >_{R^{(N)}} \]

\[ + 2<N_{33}^{(N)} \cdot \frac{1}{10} S_{3333}^{(N)} \hat{\sigma}_{33}^{(N)} + \frac{\pi_{33}^{(N)}}{2} \hat{\theta}_{\gamma}^{(N)} >_{R^{(N)}} + 2<M_{\alpha \beta}^{(N)} \cdot \frac{\pi_{\alpha \beta}^{(N)}}{h_N} \hat{\theta}_{\gamma}^{(N)} >_{R^{(N)}} \]

\[ + 2<M_{33}^{(N)} \cdot \frac{3}{7h_N} S_{3333}^{(N)} \hat{\sigma}_{33}^{(N)} - \frac{\pi_{33}^{(N)}}{h_N} \hat{\theta}_{\gamma}^{(N)} >_{R^{(N)}} + 2<\mathcal{V}_{\alpha}^{(N)} \cdot -\frac{2}{5} S_{\rho \gamma}^{(N)} \hat{\sigma}_{\gamma 3}^{(N)} >_{R^{(N)}} \]

\[ + 2<\tilde{q}_{\alpha}^{(N)} \cdot \frac{g'}{T_o} \cdot \frac{h_N}{4} \hat{\theta}_{\gamma}^{(N)} >_{R^{(N)}} + 2<\tilde{q}_{\alpha}^{(N)} \cdot \frac{g'}{T_o} \cdot \frac{h_N}{4} \hat{\theta}_{\gamma}^{(N)} >_{R^{(N)}} \]

\[ + \sum_{k=2}^{N} \left\{ <\mathcal{V}_{\alpha}^{(k)} \cdot \sigma_{\alpha 3}^{-(k-1)} >_{R^{(k)}} + <\phi_{\alpha}^{(k)} \cdot \frac{h_k}{2} \sigma_{\alpha 3}^{-(k-1)} >_{R^{(k)}} + <\mathcal{V}_{\gamma}^{(k)} \cdot \sigma_{33}^{-(k-1)} >_{R^{(k)}} \right\} \]

\[ + <\hat{\phi}_{\gamma}^{(k)} \cdot \frac{h_k}{120} \sigma_{\gamma 3}^{-(k-1)} + \frac{h_k}{10} \sigma_{33}^{-(k-1)} >_{R^{(k)}} + <\hat{\phi}_{3}^{(k)} \cdot \frac{h_k}{12} \sigma_{33}^{-(k-1)} >_{R^{(k)}} \]

\[ + \frac{h_k}{2} \sigma_{33}^{-(k-1)} >_{R^{(k)}} + <N_{\alpha \beta}^{(k)} \cdot \frac{\pi_{\alpha \beta}^{(k)}}{2} \theta^{-(k-1)} >_{R^{(k)}} \]

\[ + <N_{33}^{(k)} \cdot -\frac{1}{10} S_{3333}^{(k)} \sigma_{33}^{-(k-1)} + \frac{\pi_{33}^{(k)}}{2} \theta^{-(k-1)} >_{R^{(k)}} \]

\[ + <M_{\alpha \beta}^{(k)} \cdot \frac{\pi_{\alpha \beta}^{(k)}}{h_k} \theta^{-(k-1)} >_{R^{(k)}} + <M_{33}^{(k)} \cdot -\frac{3}{7h_k} S_{3333}^{(k)} \sigma_{33}^{-(k-1)} >_{R^{(k)}} \]

138
\[ + \frac{\pi_{33}^{(k)}}{h_k} \theta^{-(k-1)} > R^{\alpha \beta} + < \bar{V}_{\alpha}^{(k)} > - \frac{2}{5} S_{\rho \gamma \lambda}^{(k)} \sigma_{\gamma \lambda}^{-(k-1)} > R^{\alpha \beta}, \]

\[ + < q_{\alpha}^{(k)} \cdot g' \cdot \frac{h_k}{T_0} \theta^{-(k-1)} > R^{\alpha \beta}, + < \bar{q}_{\alpha}^{(k)} \cdot g' \cdot \frac{h_k}{T_0} \theta^{-(k-1)} > R^{\alpha \beta}. \]

\[ + \sum_{k=1}^{N} \{ < \bar{V}_{\alpha}^{(k)} > - \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, + < \bar{\phi}_{\alpha}^{(k)} > \frac{h_k}{2} \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, + < \bar{V}_{\alpha}^{(k)} > - \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, \]

\[ + < \bar{V}_{\alpha}^{(k)} > - \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, + < \bar{\phi}_{\alpha}^{(k)} > \frac{h_k}{2} \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, + < \bar{V}_{\alpha}^{(k)} > - \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, \]

\[ + < \frac{\phi_{\alpha}^{(k)}}{120} \sigma_{\gamma \lambda}^{-(k)} > R^{\alpha \beta}, + < \phi_{\alpha}^{(k)} > \frac{h_k}{10} \sigma_{\gamma \lambda}^{-(k)} > R^{\alpha \beta}, + < \phi_{\alpha}^{(k)} > \frac{h_k}{12} \sigma_{\gamma \lambda}^{-(k)} > R^{\alpha \beta}, \]

\[ + \frac{h_k}{2} \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, + < \frac{\phi_{\alpha}^{(k)}}{h_k} > \frac{h_k}{2} \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, \]

\[ + < \frac{\pi_{33}^{(k)}}{h_k} > \frac{h_k}{720} S_{\gamma \lambda}^{(k)} \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, + < \frac{\pi_{33}^{(k)}}{h_k} > \frac{h_k}{720} \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, \]

\[ + < \frac{\pi_{33}^{(k)}}{h_k} > \frac{h_k}{720} S_{\gamma \lambda}^{(k)} \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, + < \frac{\pi_{33}^{(k)}}{h_k} > \frac{h_k}{720} \sigma_{\alpha \beta}^{-(k)} > R^{\alpha \beta}, \]

\[ + < \bar{V}_{\alpha}^{(k)} > - \frac{2}{5} S_{\rho \gamma \lambda}^{(k)} \sigma_{\gamma \lambda}^{-(k)} > R^{\alpha \beta}, + < \bar{q}_{\alpha}^{(k)} \cdot g' \cdot \frac{h_k}{T_0} \theta^{-(k-1)} > R^{\alpha \beta}, \]

\[ + < q_{\alpha}^{(k)} \cdot g' \cdot \frac{h_k}{T_0} \theta^{-(k-1)} > R^{\alpha \beta}. \]

\[ + \sum_{k=1}^{N} \{ < \bar{V}_{\alpha}^{(k)} >, N_{\alpha \beta}^{(k)} > R^{\alpha \beta}, + < \bar{\phi}_{\alpha}^{(k)} >, M_{\alpha \beta}^{(k)} > R^{\alpha \beta}, + < \bar{V}_{\alpha}^{(k)} >, V_{\alpha \alpha}^{(k)} > R^{\alpha \beta}, \]

\[ + < \phi_{\alpha}^{(k)} >, - M_{33}^{(k)} > R^{\alpha \beta}, + < \bar{\phi}_{\alpha}^{(k)} >, - N_{33}^{(k)} > R^{\alpha \beta}, \]

\[ + < N_{\alpha \beta}^{(k)} >, - \bar{V}_{(\alpha \beta)}^{(k)} > R^{\alpha \beta}, + < \frac{1}{h_k} S_{\alpha \beta \mu \rho}^{(k)} N_{\mu \rho}^{(k)} + < \frac{1}{h_k} S_{\alpha \beta 33}^{(k)} N_{33}^{(k)} > R^{\alpha \beta} \]

139
\[
+ \langle N_{33}^{(k)} - \phi_j^{(k)} \rangle + \frac{1}{h_k} S_{33\alpha\beta}^{(k)} \Delta N_{\alpha\beta}^{(k)} + \frac{6}{5h_k} S_{3333}^{(k)} \Delta N_{33}^{(k)} >_{R^{1\alpha}} \\
+ \langle M_{\alpha\beta}^{(k)} - \phi_j^{(k)} \rangle + \frac{12}{h_k^3} S_{\mu\alpha\beta}^{(k)} \Delta M_{\alpha\beta}^{(k)} + \frac{12}{h_k^3} S_{\alpha\beta33}^{(k)} \Delta M_{33}^{(k)} >_{R^{1\alpha}} \\
+ \langle M_{33}^{(k)} - \phi_j^{(k)} \rangle + \frac{12}{h_k^3} S_{33\alpha\beta}^{(k)} \Delta M_{\alpha\beta}^{(k)} + \frac{120}{7h_k^3} S_{\alpha\beta33}^{(k)} \Delta M_{33}^{(k)} >_{R^{1\alpha}} \\
+ \langle V_{\alpha}^{(k)} - \phi_j^{(k)} \rangle - \nabla_{\beta} V_{\alpha}^{(k)} + \frac{24}{5h_k} S_{\alpha\beta33}^{(k)} \nabla_{\rho} V_{\alpha}^{(k)} >_{R^{1\alpha}} \\
+ \langle \overline{a}_{\alpha}^{(k)} \rangle \frac{h_k}{4} \frac{g'}{T_o} \lambda_{\alpha\beta} \overline{a}_{\beta}^{(k)} >_{R^{1\alpha}}, \quad + \langle \overline{a}_{\alpha}^{(k)} \rangle \frac{3h_k}{4} \frac{g'}{T_o} \lambda_{\alpha\beta} \overline{a}_{\beta}^{(k)} >_{R^{1\alpha}}, \quad \}
\]

\[
+ 2 \sum_{k=1}^{N} \left\{ \left\langle \overline{V}_{\alpha}^{(k)} \right\rangle, F_{\alpha}^{(k)} >_{R^{1\alpha}}, \left\langle \overline{V}_{\alpha}^{(k)} \right\rangle, F_{\alpha}^{(k)} >_{R^{1\alpha}}, \left\langle \overline{\phi}_3^{(k)} \right\rangle, \frac{h_k^2}{60} F_{3}^{(k)} >_{R^{1\alpha}} \right\}
\]

\[
+ 2 \langle \sigma_{\alpha 3}^{-(1)} \rangle, -\frac{2}{15} h_k S_{\alpha 3p 3}^{(1)} \hat{\sigma}_{\rho 3}^{(0)} >_{R^{1\nu}} + 2 \langle \sigma_{33}^{-(1)} \rangle, \frac{h_k}{70} S_{3333}^{(1)} \hat{\sigma}_{33}^{(0)} >_{R^{1\nu}} \\
+ 2 \langle \theta^{-(1)} \rangle, -\frac{1}{6} h_k \left\{ \Gamma_{\alpha\beta}^{(1)} \pi_{\alpha\beta}^{(1)} + \Gamma_{33}^{(1)} \pi_{33}^{(1)} + \frac{\rho_{\nu}^{(1)}}{T_o} \right\} \hat{\theta}^{(0)} >_{R^{1\nu}} \\
+ 2 \langle \sigma_{\alpha 3}^{-(N-1)} \rangle, -\frac{2}{15} h_N S_{\alpha 3p 3}^{(N)} \hat{\sigma}_{\rho 3}^{(N)} >_{R^{1\nu}}, + 2 \langle \sigma_{33}^{-(N-1)} \rangle, \frac{h_N}{70} S_{3333}^{(N)} \hat{\sigma}_{33}^{(N)} >_{R^{1\nu}}, \\
+ 2 \langle \theta^{-(N-1)} \rangle, -\frac{1}{6} h_N \left\{ \Gamma_{\alpha\beta}^{(N)} \pi_{\alpha\beta}^{(N)} + \Gamma_{33}^{(N)} \pi_{33}^{(N)} + \frac{\rho_{\nu}^{(N)}}{T_o} \right\} \hat{\theta}^{(N)} >_{R^{1\nu}}, \\
+ \sum_{k=2}^{N-1} \left\{ \left\langle \sigma_{\alpha 3}^{-(k)} \rangle, -\frac{2}{15} h_k S_{\alpha 3p 3}^{(k)} \sigma_{\rho 3}^{-(k)} >_{R^{1\nu}}, \left\langle \sigma_{33}^{-(k)} \rangle, \frac{h_k}{70} S_{3333}^{(k)} \sigma_{33}^{-(k)} >_{R^{1\nu}} \right\}
\]

\[
+ \langle \theta^{-(k)} \rangle, -\frac{1}{6} h_k \left\{ \Gamma_{\alpha\beta}^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho_{\nu}^{(k)}}{T_o} \right\} \theta^{-(k-1)} >_{R^{1\nu}}, \right\}
\]

140
+ \sum_{k=1}^{N-2} \left\{ \left< \sigma_{\alpha}^{-(k)} \right> - \frac{2}{15} h_{k+1} \sigma_{\alpha}(k) \sigma_{\rho}^{(k+1)} >_{R_{k+1}} \right. \\
+ \frac{2}{70} h_{k+1} S_{3333}(k) \sigma_{\rho}^{(k+1)} >_{R_{k+1}} \\
+ \left< \sigma_{33}^{-(k)} \right> \frac{1}{6} h_{k+1} \left\{ \left< \pi^{(k+1)} \right> + \frac{\rho^{(k+1)} c^{(k+1)}}{T_0} \right\} \theta^{-(k)} >_{R_{k+1}} \right\} \\
+ \sum_{k=1}^{N-1} \left\{ \left< \sigma_{\alpha}^{-(k)} \right> + \frac{8}{15} \left[ h_{k} S_{33}^{(k)} + h_{k+1} S_{3333}^{(k+1)} \right] \sigma_{\rho}^{(k)} >_{R_{k+1}} \right. \\
+ \frac{3}{35} \left[ h_{k} S_{3333}^{(k)} + h_{k+1} S_{3333}^{(k+1)} \right] \sigma_{33}^{(k)} >_{R_{k+1}} \\
+ \left< \theta^{-(k)} \right> \frac{1}{3} \left[ h_{k} \left\{ \left< \rho_{\alpha}^{(k)} \right> + \frac{\rho^{(k+1)} c^{(k+1)}}{T_0} \right\} \right. \\
+ h_{k+1} \left\{ \left< \rho_{\alpha}^{(k+1)} \right> + \frac{\rho^{(k+1)} c^{(k+1)}}{T_0} \right\} \theta^{-(k)} >_{R_{k+1}} \right\} \\
+ \sum_{k=1}^{N-1} \left\{ \left< \sigma_{\alpha}^{-(k)} \right> - \frac{h_{k}}{2} \phi^{(k)} + \frac{h_{k}}{12} \phi_{3}^{(k)} - \frac{h_{k}}{12} \phi_{3,\alpha}^{(k)} - \frac{2}{5} S_{3333}^{(k)} \phi_{\rho}^{(k)} >_{R_{k+1}} \right. \\
+ \frac{h_{k}^{2}}{10} \phi_{3}^{(k)} + \frac{h_{k}^{2}}{10} \phi_{3}^{(k)} - \frac{1}{10} S_{3333}^{(k)} N_{33}^{(k)} \\
+ \frac{3}{7 h_{k}^{2}} S_{3333}^{(k)} M_{33}^{(k)} >_{R_{k+1}} + \left< \theta^{-(k)} \right> \frac{\pi_{\alpha}^{(k)}}{2} N_{\alpha}^{(k)} + \frac{\pi_{3}^{(k)}}{2} N_{33}^{(k)} \\
- \frac{\pi_{\alpha}^{(k)}}{h_{k}^{2}} \frac{\pi_{\alpha}^{(k)}}{h_{k}^{2}} M_{\alpha}^{(k)} - \frac{\pi_{3}^{(k)}}{h_{k}^{2}} M_{33}^{(k)} - \frac{g^{(k)}}{T_0} \frac{h_{k}^{2}}{4} - \frac{h_{k}^{2}}{T_0} \frac{g^{(k)}}{4} + \frac{\phi_{3}^{(k+1)}}{h_{k+1}^{2}} \phi_{3}^{(k+1)} >_{R_{k+1}} \right\} \\
+ \sum_{k=1}^{N-1} \left\{ \left< \sigma_{\alpha}^{-(k)} \right> - \frac{h_{k+1}^{2}}{10} \phi_{\alpha}^{(k+1)} - \frac{h_{k+1}^{2}}{12} \phi_{3,\alpha}^{(k+1)} - \frac{h_{k+1}^{2}}{12} \phi_{3,\alpha}^{(k+1)} >_{R_{k+1}} \right. \\
+ \frac{h_{k+1}^{2}}{12} \phi_{3,\alpha}^{(k+1)} \right\} \\
141
\[-\frac{2}{5} \sigma_{\alpha \beta_3}^{(k+1)} \mathbf{V}_p^{(k+1)} > R^{(k+1)} + \langle \sigma_{33}^{(k)} \cdot \mathbf{V}_3^{(k+1)} \rangle + \frac{h_3^2}{10} \phi_3^{(k+1)} + \frac{h_{k+1}}{2} \phi_3^{(k+1)}\]

\[-\frac{1}{10} S_{333}^{(k+1)} N_{33}^{(k+1)} - \frac{3}{7 h_{k+1}} S_{333}^{(k+1)} M_{33}^{(k+1)} > R^{(k+1)} + \langle \theta^{(k)} \rangle \frac{\pi_{\alpha \beta}^{(k+1)}}{2} N_{\alpha \beta}^{(k+1)}\]

\[+ \frac{\pi_{33}^{(k+1)}}{2} N_3^{(k+1)} + \frac{\pi_{\alpha \beta}^{(k+1)}}{h_{k+1} M_{\alpha \beta}^{(k+1)}} + \frac{\pi_{33}^{(k+1)}}{h_{k+1} M_3^{(k+1)}} - \frac{g'}{T_o} \hbar_{k+1} \bar{q}_{\alpha \alpha}^{(k+1)}\]

\[-\frac{g'}{T_o} \frac{h_{k+1}}{4} \bar{q}_{\alpha \alpha}^{(k+1)} > R^{(k+1)} \]}

\[+ 2 \sum_{k=1}^{N-1} \{ \langle \theta^{(k)} \rangle \frac{h_k g'}{2 T_o} \bar{r}^{(k)} > R^{(k)} \} \]

\[+ 2 \sum_{k=1}^{N-1} \{ \langle \phi^{(k)} \rangle \frac{g'}{2 T_o} \bar{r}^{(k+1)} > R^{(k+1)} \} + 2 \sum_{k=1}^{N-1} \{ \langle \theta^{(k)} \rangle \cdot Z_o^{(k)} > R^{(k)} \} \]

\[+ \langle \bar{\mathbf{V}}_\alpha \rangle , - N^{(1)}_{\alpha \beta} \eta_\beta - 2 \phi^{(1)}_\alpha > S^{(1)}_{\alpha \alpha} \]

\[+ \sum_{k=1}^{N} \{ \langle \bar{\mathbf{V}}_\alpha \rangle , - N^{(k)}_{\alpha \beta} \eta_\beta - 2 \phi^{(k)}_\alpha > S^{(k)}_{\alpha \alpha} \} + \langle \bar{\mathbf{M}}^{(k)}_{\alpha \beta} \eta_\beta - 2 \phi^{(k)}_\alpha > S^{(k)}_{\alpha \alpha} \}

\[+ \langle \bar{\mathbf{V}}_3 \rangle \cdot \mathbf{V}_3^{(k)} \eta_\alpha - 2 \phi^{(k)}_3 \eta_\alpha > S^{(k)}_{\alpha \alpha} + \langle \bar{\mathbf{V}}_\alpha \rangle \cdot \mathbf{V}_3^{(k)} \eta_\alpha - 2 \phi^{(k)}_3 \eta_\alpha > S^{(k)}_{\alpha \alpha} \]

\[+ < \mathbf{M}_{\alpha \beta}^{(k)} \cdot \mathbf{V}_3^{(k)} \eta_\alpha - 2 \phi^{(k)}_3 \eta_\alpha > S^{(k)}_{\alpha \alpha} + < \mathbf{V}_3^{(k)} \cdot \mathbf{V}_3^{(k)} \eta_\alpha - 2 \phi^{(k)}_3 \eta_\alpha > S^{(k)}_{\alpha \alpha} \}

\[+ < \sigma_{\alpha j}^{(1)} \frac{h^3}{120} \phi_3^{(1)} \eta_\alpha - \frac{h^2}{12} \phi_3^{(1)} \eta_\alpha + \frac{h^2}{12} \phi_3^{(1)} \eta_\alpha + \frac{h^2}{12} \phi_3^{(1)} \eta_\alpha + \frac{h^2}{12} \phi_3^{(1)} \eta_\alpha + \frac{h^2}{12} \phi_3^{(1)} \eta_\alpha > S^{(1)}_{\alpha j} \]

\[+ < \theta^{(1)} \cdot \frac{g'}{T_o} \frac{h_2}{4} \bar{q}_{\alpha}^{(1)} \eta_\alpha - \frac{g'}{T_o} \frac{h_2}{4} \bar{q}_{\alpha}^{(1)} \eta_\alpha + \frac{g'}{T_o} \frac{h_2}{4} \bar{q}_{\alpha}^{(1)} \eta_\alpha + \frac{g'}{T_o} \frac{h_2}{4} \bar{q}_{\alpha}^{(1)} \eta_\alpha > S^{(1)}_{\alpha j} \]

\[+ \frac{g'}{T_o} \frac{h_2}{4} \bar{q}_{\alpha}^{(2)} \eta_\alpha - 2 \phi^{(1)}_\alpha > S^{(1)}_{\alpha j} \]

142
The field equations as well as the boundary conditions of the problem are satisfied.

\[ + \frac{g'}{T_0} \cdot \frac{h^i}{4} (\bar{q}^{(1)}_a) \eta_\alpha - 2 g_\alpha^{(1)} \eta_\alpha > s^{(1)}_\alpha, \]

\[ + \sum_{k=2}^{N-2} \left\{ < \sigma^{-(k)}_{\alpha_3} \cdot \frac{h^i}{120} (\bar{\phi}^{(k)}_3) \eta_\alpha - \frac{h^i}{12} (\bar{\phi}^{(k)}_3) \eta_\alpha + \frac{h^k}{120} (\bar{\phi}^{(k+1)}_3) \eta_\alpha \right\} \]

\[ + \frac{h^k}{12} (\bar{\phi}^{(k+1)}_3) \eta_\alpha - 2 g_\alpha^{(k)} \eta_\alpha > s^{(k)}_\alpha, \]

\[ + < \theta^{-1(k)} \cdot \frac{g'}{T_0} \cdot \frac{h^k}{4} (\bar{q}^{(k)}_a) \eta_\alpha - \frac{g'}{T_0} \cdot \frac{h^k}{4} (\bar{q}^{(k)}_a) \eta_\alpha \]

\[ + \frac{g'}{T_0} \cdot \frac{h^k}{4} (\bar{q}^{(k+1)}_a) \eta_\alpha + \frac{g'}{T_0} \cdot \frac{h^k}{4} (\bar{q}^{(k+1)}_a) \eta_\alpha - 2 g_\alpha^{(k)} \eta_\alpha > s^{(k)}_\alpha. \]

\[ + < \sigma^{-(N-1)}_{\alpha_3} \cdot \frac{h^i}{120} (\bar{\phi}^{(N-1)}_3) \eta_\alpha - \frac{h^i}{12} (\bar{\phi}^{(N-1)}_3) \eta_\alpha + \frac{h^i}{120} (\bar{\phi}^{(N)}_3) \eta_\alpha \]

\[ + \frac{h^i}{12} (\bar{\phi}^{(N)}_3) \eta_\alpha - 2 g_\alpha^{(N-1)} \eta_\alpha > s^{(N-1)}_\alpha, \]

\[ + < \theta^{-1(N-1)} \cdot \frac{g'}{T_0} \cdot \frac{h^{N-1}}{4} (\bar{q}^{(N-1)}_a) \eta_\alpha - \frac{g'}{T_0} \cdot \frac{h^{N-1}}{4} (\bar{q}^{(N-1)}_a) \eta_\alpha + \frac{g'}{T_0} \cdot \frac{h^{N-1}}{4} (\bar{q}^{(N)}_a) \eta_\alpha \]

\[ + \frac{g'}{T_0} \cdot \frac{h^{N-1}}{4} (\bar{q}^{(N)}_a) \eta_\alpha - 2 g_\alpha^{(N-1)} \eta_\alpha > s^{(N-1)}_\alpha. \]

We shall show that the Gateaux differential of this function vanishes if and only if the field equations as well as the boundary conditions of the problem are satisfied.
7.2 PROOF FOR VANISHING OF THE GATEAUX DIFFERENTIAL

In this section, the Gateaux differential [refer equation (5.8)] of the governing function [equation (7.3)] will be evaluated along arbitrary paths in the space of admissible functions. It is necessary and sufficient to consider paths in the space of admissible states of one variable at a time while the remaining variables have zero components.

7.2.1 Gateaux Differential with Respect to $\overline{V}_a^{(k)}$

The Gateaux differential of the governing function $\Omega$, with respect to $\overline{V}_a^{(k)}$, along an arbitrary path $\overline{x}_a^{(k)}$, can be expressed as:

$$
\Delta_{\overline{V}_a^{(k)}} \Omega = <\overline{x}_a^{(k)}, \sigma_{\alpha \delta}^{-(k-1)} - \sigma_{\alpha \delta}^{-(k)} + N_{\alpha \beta}^{(k)} + 2 F_{\alpha}^{(k)} - \sigma_{\alpha \delta}^{-(k)} + \sigma_{\alpha \delta}^{-(k-1)}>_{R^{(k)}},
$$

$$
+ <N_{\alpha \beta}^{(k)} \cdot \overline{x}_a^{(k)}>_{R^{(k)}}, + <\overline{x}_a^{(k)}, - N_{\alpha \beta}^{(k)} \eta_{\beta} - 2 E_{i}^{(k)} >_{S_{i}^{(k)}},
$$

$$
+ <N_{\alpha \beta}^{(k)} \cdot \overline{x}_a^{(k)} \eta_{\beta}>_{S_{i}^{(k)}}, + <\overline{x}_a^{(k)}, - (N_{\alpha \beta}^{(k)}) \eta_{\beta} - 2 g_{i}^{(k)} >_{S_{i}^{(k)}},
$$

$$
+ <N_{\alpha \beta}^{(k)} \cdot (\overline{x}_a^{(k)}) \eta_{\beta}>_{S_{i}^{(k)}}, \quad (7.4)
$$

Using Green's theorem as given by equation (6.22):

$$
<N_{\alpha \beta}^{(k)} \cdot \overline{x}_a^{(k)}>_{R^{(k)}}, = <\overline{x}_a^{(k)}, N_{\alpha \beta}^{(k)}>_R^{(k)}, - <N_{\alpha \beta}^{(k)} \cdot \overline{x}_a^{(k)} \eta_{\beta}>_{S_{i}^{(k)}},
$$

$$
- <N_{\alpha \beta}^{(k)} \cdot (\overline{x}_a^{(k)}) \eta_{\beta}>_{S_{i}^{(k)}}.
$$
Recalling equation (6.31), the above expression can be written as:

\[
< N_{\alpha \beta}^{(k)} - \bar{N}_{(\alpha, \beta)} >_{R^k} = < \bar{x}_\alpha^{(k)} \cdot N_{\alpha \beta}^{(k)} >_{R^k} + < \bar{x}_\alpha^{(k)} \cdot N_{\alpha \beta}^{(k)} \eta_\beta >_{S^k_i},
\]

\[
- < N_{\alpha \beta}^{(k)} \cdot \bar{x}_\alpha^{(k)} \eta_\beta >_{S^k_i} + < \bar{x}_\alpha^{(k)} \cdot (N_{\alpha \beta}^{(k)})' \eta_\beta >_{S^k_i}, - < N_{\alpha \beta}^{(k)} \cdot (\bar{x}_\alpha^{(k)})' \eta_\beta >_{S^k_i}.
\]

(7.5)

Using equation (7.5), equation (7.4) can be expressed as:

\[
\Delta_{\alpha(k)} \Omega = 2 < \bar{x}_\alpha^{(k)} \cdot N_{\alpha \beta}^{(k)} + (\sigma_{\alpha 3}^{-(k-1)} - \sigma_{\alpha 3}^{-(k)}) + F^{(k)} >_{R^k},
\]

\[
+ 2 < \bar{x}_\alpha^{(k)} \cdot N_{\alpha \beta} \eta_\beta - g^{(k)}_1 >_{S^k_i} + 2 < (N_{\alpha \beta}^{(k)})' \eta_\beta - g^{(k)}_1 >_{S^k_i}.
\]

(7.6)

The vanishing of \( \Delta_{\alpha(k)} \Omega \) for all \( \bar{x}_\alpha^{(k)} \), for Gurtin’s non-degenerate bilinear mapping, equation (4.15), gives:

\[
N_{\alpha \beta}^{(k)} + (\sigma_{\alpha 3}^{-(k-1)} - \sigma_{\alpha 3}^{-(k)}) + F^{(k)} = 0 \quad \text{on } R^k
\]

\[
N_{\alpha \beta} \eta_\beta = g^{(k)}_1 \quad \text{on } S^k_i
\]

\[
(\sigma_{\alpha 3}^{-(k)})' \eta_\beta = g^{(k)}_1 \quad \text{on } S^k_i
\]

7.2.2 Gateaux Differential with Respect to \( \bar{\varphi}_\alpha^{(k)} \)

The Gateaux differential of the governing function, \( \Omega \), with respect to \( \bar{\varphi}_\alpha^{(k)} \), along an arbitrary path \( \bar{y}_\alpha^{(k)} \), can be expressed as:

\[
\Delta_{\bar{y}_\alpha^{(k)}} \Omega = \frac{h_k}{2} \sigma_{\alpha 3}^{-(k-1)} + \frac{h_k}{2} \sigma_{\alpha 3}^{-(k)} + M_{\alpha \beta}^{(k)} - V^{(k)} \bar{\varphi}_\alpha^{(k)} + \frac{h_k}{2} \sigma_{\alpha 3}^{-(k)}
\]

146
Using Green's theorem and using (6.33) gives:

\[
< M_{\alpha \beta}^{(k)} \cdot \bar{y}_{(\alpha \beta)} >_{R^{(k)}} = < \bar{y}_{\alpha}^{(k)}, M_{\alpha \beta}^{(k)} >_{R^{(k)}} + < M_{\alpha \beta}^{(k)} \cdot \bar{y}_{\alpha}^{(k)} \eta_{\beta} >_{S_{l}^{(k)}}
\]

\[- < M_{\alpha \beta}^{(k)} \cdot \bar{y}_{\alpha}^{(k)} \eta_{\beta} >_{S_{r}^{(k)}} + < \bar{y}_{\alpha}^{(k)}, ( M_{\alpha \beta}^{(k)} \cdot \eta_{\beta} >_{S_{l}^{(k)}} - < M_{\alpha \beta}^{(k)} \cdot ( \bar{y}_{\alpha}^{(k)} ) \eta_{\beta} >_{S_{r}^{(k)}}.
\]

(7.8)

Using equation (7.8), equation (7.7) can be expressed as:

\[
\Delta_{\bar{y}_{\alpha}^{(k)}}, \Omega = 2 < \bar{y}_{\alpha}^{(k)}, M_{\alpha \beta}^{(k)} >_{R^{(k)}} + \frac{\hbar}{2} ( \sigma_{\alpha \beta}^{-(k-1)} - \sigma_{\alpha 3}^{-(k-1)} ) \cdot V_{\alpha}^{(k)} >_{R^{(k)}}
\]

\[+ 2 < \bar{y}_{\alpha}^{(k)}, - M_{\alpha \beta}^{(k)} \eta_{\beta} - g_{3}^{(k)} >_{S_{l}^{(k)}} + 2 < \bar{y}_{\alpha}^{(k)}, ( M_{\alpha \beta}^{(k)} ) \eta_{\beta} - g_{3}^{(k)} >_{S_{r}^{(k)}}.
\]

(7.9)

The vanishing of \(\Delta_{\bar{y}_{\alpha}^{(k)}}, \Omega\) for all \(\bar{y}_{\alpha}^{(k)}\) gives:

\[
M_{\alpha \beta}^{(k)} + \frac{\hbar}{2} ( \sigma_{\alpha \beta}^{-(k-1)} + \sigma_{\alpha 3}^{-(k-1)} ) \cdot V_{\alpha}^{(k)} = 0 \quad \text{on } R^{(k)}
\]

\[- M_{\alpha \beta}^{(k)} \eta_{\beta} = g_{3}^{(k)} \quad \text{on } S_{l}^{(k)}
\]

\[- ( M_{\alpha \beta}^{(k)} ) \eta_{\beta} = g_{3}^{(k)} \quad \text{on } S_{r}^{(k)}
\]
7.2.3 Gateaux Differential with Respect to $\vec{v}_3^{(k)}$

The Gateaux differential of the governing function $Q$ with respect to $\vec{v}_3^{(k)}$ along an arbitrary path $\vec{x}_3^{(k)}$ can be expressed as:

\begin{equation}
\Delta_{\vec{x}_3^{(k)}} Q = <\vec{x}_3^{(k)}, \sigma_{33}^{(k-1)} - \sigma_{33}^{(k)} + V_{\alpha \alpha}^{(k)} + 2F_3^{(k)} - \sigma_{33}^{(k)} + \sigma_{33}^{(k-1)}>_{R^{(k)}},
\end{equation}

\begin{equation}
+ <V_{\alpha}^{(k)} \cdot - \vec{x}_3^{(k)}>_{R^{(k)}}, + <\vec{x}_3^{(k)} \cdot - V_{\alpha}^{(k)} \eta_{\alpha} - 2g_5^{(k)}>_{s_3^{(k)}}, + <\vec{v}_{\alpha}^{(k)}>,
\end{equation}

\begin{equation}
\vec{x}_3^{(k)} \eta_{\alpha}>_{s_3^{(k)}},
\end{equation}

\begin{equation}
+ <\vec{x}_3^{(k)} \cdot -( V_{\alpha}^{(k)})^{\prime} \eta_{\alpha} - 2g_5^{(k)}>_{s_3^{(k)}}, + <\vec{v}_{\alpha}^{(k)} \cdot ( \vec{x}_3^{(k)})^{\prime} \eta_{\alpha}>_{s_3^{(k)}},
\end{equation}

(7.10)

Using Green's theorem and using (6.35) gives:

\begin{equation}
<V_{\alpha}^{(k)} \cdot - \vec{x}_3^{(k)}>_{R^{(k)}}, = <\vec{x}_3^{(k)} \cdot V_{\alpha \alpha}^{(k)}>_{R^{(k)}}, + <\vec{x}_3^{(k)} \cdot - V_{\alpha}^{(k)} \eta_{\alpha}>_{s_3^{(k)}},
\end{equation}

\begin{equation}
- <V_{\alpha}^{(k)} \cdot \vec{x}_3^{(k)} \eta_{\alpha}>_{s_3^{(k)}}, + <\vec{x}_3^{(k)} \cdot -( V_{\alpha}^{(k)})^{\prime} \eta_{\alpha}>_{s_3^{(k)}}, - <V_{\alpha}^{(k)} \cdot ( \vec{x}_3^{(k)})^{\prime} \eta_{\alpha}>_{s_3^{(k)}},
\end{equation}

(7.11)

Using equation (7.11), equation (7.10) can be expressed as:

\begin{equation}
\Delta_{\vec{x}_3^{(k)}} Q = 2 <\vec{x}_3^{(k)} \cdot V_{\alpha \alpha}^{(k)} + ( \sigma_{33}^{(k-1)} - \sigma_{33}^{(k)}) + F_3^{(k)}>_{R^{(k)}},
\end{equation}

\begin{equation}
+ 2 <\vec{x}_3^{(k)} \cdot - V_{\alpha}^{(k)} \eta_{\alpha} - g_5^{(k)}>_{s_3^{(k)}}, + 2 <\vec{x}_3^{(k)} \cdot -( V_{\alpha}^{(k)})^{\prime} \eta_{\alpha} - g_5^{(k)}>_{s_3^{(k)}},
\end{equation}

(7.12)

The vanishing of $\Delta_{\vec{x}_3^{(k)}} Q$ for all $\vec{x}_3^{(k)}$ gives:

\begin{equation}
V_{\alpha \alpha}^{(k)} + ( \sigma_{33}^{(k-1)} - \sigma_{33}^{(k)}) + F_3^{(k)} = 0 \quad \text{on } R^{(k)}
\end{equation}

148
7.2.4 Gateaux Differential with Respect to $\phi^{(k)}_3$

The Gateaux differential of the governing function $\Omega$, with respect to $\phi^{(k)}_3$, along an arbitrary path $\bar{\gamma}^{(k)}_3$, can be expressed as:

$$\Delta_{\bar{\gamma}^{(k)}_3} \Omega = \langle \bar{y}^{(k)}_3, \frac{h^3}{120} \sigma_{\gamma,\gamma}^{(k-l)} \rangle + \frac{h^3}{120} \sigma_{\gamma,\gamma}^{(k)} + \frac{h^3}{10} \sigma_{\gamma,\gamma}^{(k-l)} - \frac{h^3}{10} \sigma_{\gamma,\gamma}^{(k)} - M_{33}^{(k)}$$

$$+ \frac{2}{60} F_3^{(k)} + \frac{h^3}{10} \sigma_{\gamma,\gamma}^{(k-l)} - \frac{h^3}{10} \sigma_{\gamma,\gamma}^{(k)} - M_{33}^{(k)} > R^{(k)}$$

$$+ <\sigma_{\alpha^3}, - \frac{h^3}{120} \bar{y}^{(k)}_3 >_{R^{(k)}}, + <\sigma_{\alpha^3}, - \frac{h^3}{120} \bar{y}^{(k)}_3 >_{R^{(k)}}$$

$$+ <\sigma_{\alpha^3}, \frac{h^3}{120} \bar{y}^{(k)}_3 \eta_{\alpha} >_{S^{(k)}}, + <\sigma_{\alpha^3}, \frac{h^3}{120} \bar{y}^{(k)}_3 \eta_{\alpha} >_{S^{(k)}},$$

$$+ <\sigma_{\alpha^3}, \frac{h^3}{120} \bar{y}^{(k)}_3 \eta_{\alpha} >_{S^{(k)}}, + <\sigma_{\alpha^3}, \frac{h^3}{120} \bar{y}^{(k)}_3 \eta_{\alpha} >_{S^{(k)}},$$

$$+ <\sigma_{\alpha^3}, \frac{h^3}{120} \bar{y}^{(k)}_3 \eta_{\alpha} >_{S^{(k)}}, + <\sigma_{\alpha^3}, \frac{h^3}{120} \bar{y}^{(k)}_3 \eta_{\alpha} >_{S^{(k)}}, \quad (7.13)$$

Again using Green’s theorem:

$$<\sigma_{\alpha^3}, \frac{h^3}{120} \bar{y}^{(k)}_3 >_{R^{(k)}}, = - <\bar{y}^{(k)}_3, \frac{h^3}{120} \sigma_{\gamma,\gamma}^{(k-l)} >_{R^{(k)}}$$

$$+ <\bar{y}^{(k)}_3, \frac{h^3}{120} \sigma_{\alpha^3,\gamma}^{(k-l)} >_{S^{(k)}}, + <\bar{y}^{(k)}_3, \frac{h^3}{120} \sigma_{\alpha^3,\gamma}^{(k-l)} >_{S^{(k)}},$$

$$+ <\bar{y}^{(k)}_3, \frac{h^3}{120} \sigma_{\alpha^3,\gamma}^{(k-l)} >_{S^{(k)}}, + <\bar{y}^{(k)}_3, \frac{h^3}{120} \sigma_{\alpha^3,\gamma}^{(k-l)} >_{S^{(k)}}, \quad (7.14)$$

149
Equation (7.13) can be expressed using equations (7.14) and (7.15) as:

\[ A_{y,\alpha} = 2 \sum_{i=1}^{n} \left( \frac{h^2}{120} \sigma_{y,\alpha}^{(k)} + \frac{h^2}{10} \sigma_{y,\alpha}^{(k-1)} - \frac{h^2}{33} \sigma_{y,\alpha}^{(k)} - M_{y,\alpha}^{(k)} \right) \]

Equation (7.13) can be expressed using equations (7.14) and (7.15) as:

\[ \Delta_{y,\alpha}, \Omega = 2 \sum_{i=1}^{n} \left( \frac{h^2}{120} \sigma_{y,\alpha}^{(k-1)} + \frac{h^2}{10} \sigma_{y,\alpha}^{(k-1)} - \frac{h^2}{33} \sigma_{y,\alpha}^{(k)} - M_{y,\alpha}^{(k)} \right) \]

The vanishing of \( \Delta_{y,\alpha}, \Omega \) for all \( y_{ij}^{(k)} \) gives:

\[ - M_{y,\alpha}^{(k)} + \frac{h^2}{120} \sigma_{y,\alpha}^{(k-1)} + \frac{h^2}{10} \sigma_{y,\alpha}^{(k-1)} - \frac{h^2}{33} \sigma_{y,\alpha}^{(k)} + \frac{h^2}{60} F_{3}^{(k)} = 0 \text{ on } R^{(k)} \]

7.2.5 Gateaux Differential with Respect to \( \phi_{i}^{(k)} \)

The Gateaux differential of the governing function \( \Omega \), with respect to \( \phi_{i}^{(k)} \), along an arbitrary path \( z_{i}^{(k)} \), can be expressed as:

\[ \Delta_{z_{i}^{(k)}, \Omega} = \left( \sum_{i=1}^{n} \left( \frac{h^2}{12} \sigma_{i,\alpha}^{(k-1)} - \frac{h^2}{12} \sigma_{i,\alpha}^{(k-1)} - N_{3,\alpha}^{(k)} + \frac{h^2}{2} \sigma_{3,\alpha}^{(k-1)} + \frac{h^2}{2} \sigma_{3,\alpha}^{(k)} \right) \right) \]

\[ + \sum_{i=1}^{n} \left( \frac{h^2}{12} \sigma_{i,\alpha}^{(k-1)} - \frac{h^2}{12} \sigma_{i,\alpha}^{(k-1)} - N_{3,\alpha}^{(k)} + \frac{h^2}{2} \sigma_{3,\alpha}^{(k-1)} + \frac{h^2}{2} \sigma_{3,\alpha}^{(k)} - N_{3,\alpha}^{(k)} \right) \]
Using Green's theorem gives:

\[
<\sigma_{a_3}^{-k(k)} + \frac{h_k^2}{12} \overline{z} \eta_\alpha \sigma_{a_3}^{-k(k)} + \frac{6_h}{12} \overline{z} \eta_\alpha \sigma_{a_3}^{-k(k)} + \frac{h_k^2}{12} (\overline{z} \eta_\alpha \sigma_{a_3}^{-k(k)}) > S_a \tag{7.16}
\]

Equation (7.16) can be expressed using equations (7.17) and (7.18) as:

\[
\Delta_{z_i^k} \Omega = 2 <z_3^{(k)} - N_{33}^{(k)} + \frac{h_k^2}{12} (\sigma_{a_3}^{-k(k)} - \sigma_{a_3}^{-k(k)}) + \frac{h_k}{2} (\sigma_{33}^{(k)} - \sigma_{33}^{(k)}) > R^{(k)}
\]

The vanishing of \( \Delta_{z_i^k} \Omega \) for all \( z_3^{(k)} \) gives:

\[
-N_{33}^{(k)} + \frac{h_k^2}{12} (\sigma_{a_3}^{-k(k)} - \sigma_{a_3}^{-k(k)}) + \frac{h_k}{2} (\sigma_{33}^{(k)} - \sigma_{33}^{(k)}) = 0 \quad \text{on} \quad R^{(k)}
\]
7.2.6 Gateaux Differential with Respect to $N^{(k)}_{\alpha\beta}$

The Gateaux differential of the governing function, $\Omega$, with respect to $N^{(k)}_{\alpha\beta}$, along an arbitrary path $\eta^{(k)}_{\alpha\beta}$, can be expressed as:

$$
\Delta_{\eta^{(k)}_{\alpha\beta}} \Omega = <\eta^{(k)}_{\alpha\beta} \cdot \frac{\pi^{(k)}_{\alpha\beta}}{2} \theta^{-(k-1)} + \frac{\pi^{(k)}_{\alpha\beta}}{2} \theta^{-(k)} - \nabla^{(k)}_{(\alpha\beta)} + 2 \frac{1}{h_k} S^{(k)}_{\alpha\beta\mu\nu} N^{(k)}_{\mu\nu} >_R^{(k)} + 2 <\nabla^{(k)}_{\alpha} \cdot \eta^{(k)}_{\alpha\beta} \eta_{\beta}>_R^{(k)} + <\eta^{(k)}_{\alpha\beta} \cdot \nabla^{(k)}_{\alpha} \eta_{\beta} - g^{(k)}_{2} >_s^{(k)} + <\eta^{(k)}_{\alpha\beta} \cdot (\nabla^{(k)}_{\alpha} \eta_{\beta} - g^{(k)}_{2}) >_s^{(k)},
$$

Using Green's theorem gives:

$$
<\nabla^{(k)}_{\alpha} \cdot \eta^{(k)}_{\alpha\beta} >_R^{(k)} = -<\eta^{(k)}_{\alpha\beta} \cdot \nabla^{(k)}_{(\alpha\beta)} >_R^{(k)} + <\eta^{(k)}_{\alpha\beta} \cdot \nabla^{(k)}_{\alpha} \eta_{\beta} >_s^{(k)} + <\eta^{(k)}_{\alpha\beta} \cdot (\nabla^{(k)}_{\alpha} \eta_{\beta} - g^{(k)}_{2}) >_s^{(k)},
$$

Equation (7.19) can be expressed using equation (7.20) as:

$$
\Delta_{\eta^{(k)}_{\alpha\beta}} \Omega = 2<\eta^{(k)}_{\alpha\beta} \cdot \nabla^{(k)}_{(\alpha\beta)} >_R^{(k)} + <\eta^{(k)}_{\alpha\beta} \cdot \nabla^{(k)}_{\alpha} \eta_{\beta} >_s^{(k)} + 2<\eta^{(k)}_{\alpha\beta} \cdot \nabla^{(k)}_{\alpha} \eta_{\beta} - g^{(k)}_{2} >_s^{(k)} + 2<\eta^{(k)}_{\alpha\beta} \cdot (\nabla^{(k)}_{\alpha} \eta_{\beta} - g^{(k)}_{2}) >_s^{(k)},
$$
The vanishing of $\Delta_{\eta_{\alpha\beta}}^k \Omega$ for all $\eta_{\alpha\beta}^{(k)}$ gives:

$$- \nabla^{(k)}_{(\alpha \beta)} + \frac{1}{h_k} S_{\alpha\beta \mu \rho}^{(k)} N^{(k)}_{\mu \rho} + \frac{1}{h_k} S_{\mu \rho \nu \sigma}^{(k)} N^{(k)}_{\nu \sigma} + \frac{\pi_{\mu \nu}^{(k)}}{2} (\theta^{-(k-1)} + \theta^{-(k)}) = 0 \text{ on } R^{(k)}$$

$$\nabla_{(\alpha)}^{(k)} \eta_{\beta} = g_{(k)}^{(2)} \text{ on } S_{(k)}^{(2)}$$

$$\nabla^{(k)}_{(\alpha \beta)} \eta_{\beta} = g_{(k)}^{(2)} \text{ on } S_{(k)}^{(2)}$$

7.2.7 Gateaux Differential with Respect to $N_{33}^{(k)}$

The Gateaux differential of the governing function, $\Omega$, with respect to $N_{33}^{(k)}$, along an arbitrary path $\eta_{33}^{(k)}$, can be expressed as:

$$\Delta_{\eta_{33}^{(k)}} \Omega = \langle \eta_{33}^{(k)} \rangle - \frac{1}{10} S_{3333}^{(k)} \sigma_{33}^{-(k-1)} + \frac{1}{2} \pi^{(k)}_{33} \theta^{-(k-1)} - \frac{1}{10} S_{3333}^{(k)} \sigma_{33}^{-(k)}$$

$$+ \frac{1}{2} \pi^{(k)}_{33} \theta^{-(k)} \cdot \phi_{3}^{(k)} + \frac{1}{h_k} S_{33 \gamma 6}^{(k)} N_{\gamma 6}^{(k)} - \frac{1}{h_k} S_{33 \gamma 6}^{(k)} N_{\gamma 6}^{(k)}$$

$$+ 2 \frac{6}{5 h_k} S_{3333}^{(k)} N_{33}^{(k)} - \frac{1}{10} S_{3333}^{(k)} \sigma_{33}^{-(k-1)} + \frac{1}{2} \pi^{(k)}_{33} \theta^{-(k-1)} - \frac{1}{10} S_{3333}^{(k)} \sigma_{33}^{-(k)}$$

$$+ \frac{1}{2} \pi_{33}^{(k)} \theta^{-(k)} \rangle_{R^{(k)}}, \quad (7.22)$$

Equation (7.22) can be restated as:

$$\Delta_{\eta_{33}^{(k)}} \Omega = 2 \langle \eta_{33}^{(k)} \rangle - \frac{1}{10} S_{3333}^{(k)} \sigma_{33}^{-(k-1)} + \sigma_{33}^{-(k)} + \frac{1}{2} \pi_{33}^{(k)} (\theta^{-(k-1)} + \theta^{-(k)})$$

$$- \phi_{3}^{(k)} + \frac{1}{h_k} S_{33 \gamma 6}^{(k)} N_{\gamma 6}^{(k)} + \frac{6}{5 h_k} S_{3333}^{(k)} N_{33}^{(k)} \rangle_{R^{(k)}},$$

153
The vanishing of $\Delta_{\eta^{(i)}} \Omega$ for all $\eta^{(3)}_{33}$ gives:

$$- \phi^{(k)}_3 + \frac{1}{h_k} S^{(k)}_{33\rho} N^{(k)}_{\rho} + \frac{6}{5h_k} S^{(k)}_{3333} N^{(k)}_{33} - \frac{1}{10} S^{(k)}_{3333} (\sigma^{(k-1)}_{33} + \sigma^{(k)}_{33})$$

$$+ \frac{1}{2} \pi^{(k)}_{33} (\theta^{-(k-1)} + \theta^{-(k)}) = 0 \text{ on } R^{(k)}$$

7.2.8 Gateaux Differential with Respect to $M^{(k)}_{\alpha\beta}$

The Gateaux differential of the governing function $\Omega$ with respect to $M^{(k)}_{\alpha\beta}$, along an arbitrary path $m^{(k)}_{\alpha\beta}$, can be expressed as:

$$\Delta_{m^{(k)}_{\alpha\beta}} \Omega = <m^{(k)}_{\alpha\beta}, \frac{1}{h_k} \pi^{(k)}_{\alpha\beta} \theta^{-(k-1)} - \frac{1}{h_k} \pi^{(k)}_{(\alpha\beta)} \theta^{-(k)} - \phi^{(k)}_{(\alpha\beta)} + 2 \frac{12}{h_k} S^{(k)}_{\mu\rho\alpha\beta} M^{(k)}_{\mu\rho\beta}$$

$$+ 2 \frac{12}{h_k} S^{(k)}_{\mu\rho33} M^{(k)}_{333} + \frac{1}{h_k} \pi^{(k)}_{\alpha\beta} \theta^{-(k-1)} - \frac{1}{h_k} \pi^{(k)}_{(\alpha\beta)} \theta^{-(k)}>_{R^{(k)}}$$

$$+ <\Phi^{(k)}_{(\alpha)} \cdot m^{(k)}_{(\alpha\beta)} >_{R^{(k)}}$$

$$+ <\Phi^{(k)}_{(\alpha)} \cdot m^{(k)}_{(\alpha\beta)} \eta_{(\beta)}>_{R^{(k)}} + <m^{(k)}_{(\alpha\beta)} \eta_{(\beta)} \cdot 2 g^{(k)}_{44}, g^{(k)}_{44}>_{R^{(k)}}$$

$$+ <\Phi^{(k)}_{(\alpha)} \cdot (m^{(k)}_{(\alpha\beta)}) \eta_{(\beta)}>_{R^{(k)}} + <m^{(k)}_{(\alpha\beta)} \cdot (\Phi^{(k)}_{(\alpha)} \eta_{(\beta)} \cdot 2 g^{(k)}_{44})>_{R^{(k)}}$$

$$+ <\Phi^{(k)}_{(\alpha)} \cdot (m^{(k)}_{(\alpha\beta)}) \eta_{(\beta)}>_{R^{(k)}} + <m^{(k)}_{(\alpha\beta)} \cdot (\Phi^{(k)}_{(\alpha)} \eta_{(\beta)} \cdot 2 g^{(k)}_{44})>_{R^{(k)}}$$

Using Green's theorem:

$$<\Phi^{(k)}_{(\alpha)} \cdot m^{(k)}_{(\alpha\beta)} >_{R^{(k)}} = - <m^{(k)}_{(\alpha\beta)} \cdot \Phi^{(k)}_{(\alpha\beta)}>_{R^{(k)}} + <\Phi^{(k)}_{(\alpha)} \cdot m^{(k)}_{(\alpha\beta)} \eta_{(\beta)}>_{R^{(k)}}$$

$$+ <\Phi^{(k)}_{(\alpha)} \cdot (m^{(k)}_{(\alpha\beta)}) \eta_{(\beta)}>_{R^{(k)}}$$

$$+ <\Phi^{(k)}_{(\alpha)} \cdot (m^{(k)}_{(\alpha\beta)}) \eta_{(\beta)}>_{S^{(k)}}$$

(7.24)
Equation (7.23) can be expressed using equation (7.24) as:

\[
\Delta_{m_{g}^{(k)}} \Omega = 2 < m_{g}^{(k)} \cdot \frac{1}{h_{k}} \pi_{\alpha \beta}^{k} (\theta^{-(k-1)} - \theta^{-(k)}) - \Phi_{(\alpha \beta)}^{(k)} + \frac{12}{h_{k}^{3}} S^{(k)}_{\mu \rho \sigma \beta} M^{(k)}_{\sigma \beta} \\
+ \frac{12}{h_{k}^{3}} S^{(k)}_{\mu \rho 33} M^{(k)}_{33} > \mathcal{R}^{(k)} \quad + 2 < m_{g}^{(k)} \cdot \Phi_{(\alpha \beta)}^{(k)} \eta_{\beta} - g_{4}^{(k)} > \mathcal{S}_{4}^{(k)} \\
+ 2 < m_{g}^{(k)} \cdot (\Phi_{(\alpha \beta)}^{(k)} \cdot \eta_{\beta} - g_{4}^{(k)}) > \mathcal{S}_{4i}^{(k)}.
\]

The vanishing of \( \Delta_{\eta_{\beta}} \Omega \) for all \( m_{g}^{(k)} \) implies:

\[
- \Phi_{(\alpha \beta)}^{(k)} + \frac{12}{h_{k}^{3}} S^{(k)}_{\mu \rho \sigma \beta} M^{(k)}_{\sigma \beta} + \frac{12}{h_{k}^{3}} S^{(k)}_{\mu \rho 33} M^{(k)}_{33} \\
+ \frac{1}{h_{k}^{3}} \pi_{\alpha \beta}^{k} (\theta^{-(k-1)} - \theta^{-(k)}) = 0 \quad \text{on} \quad \mathcal{R}^{(k)}
\]

\[
\Phi_{(\alpha \beta)}^{(k)} \eta_{\beta} = g_{4}^{(k)} \quad \text{on} \quad \mathcal{S}_{4}^{(k)}
\]

\[
(\Phi_{(\alpha \beta)}^{(k)} \cdot \eta_{\beta} = g_{4}^{(k)}) \quad \text{on} \quad \mathcal{S}_{4i}^{(k)}
\]

7.2.9 Gateaux Differential with Respect to \( M_{33}^{(k)} \)

The Gateaux differential of the governing function, \( \Omega \), with respect to \( M_{33}^{(k)} \), along an arbitrary path \( m_{33}^{(k)} \), can be expressed as:

\[
\Delta_{m_{33}^{(k)}} \Omega = < m_{33}^{(k)} \cdot - \frac{3}{7h_{k}} S_{3333}^{(k)} \sigma_{33}^{(k-1)} - \frac{1}{h_{k}} \pi_{33}^{k} \theta^{-(k-1)} - \Phi_{3}^{(k)} + \frac{3}{7h_{k}} S_{3333}^{(k)} \sigma_{33}^{(k)} \\
- \frac{1}{h_{k}} \pi_{33}^{k} \theta^{-(k)} + \frac{12}{h_{k}^{3}} S_{3333}^{(k)} M_{33}^{(k)} - \Phi_{3}^{(k)} + 2 \frac{120}{7h_{k}^{3}} S_{3333}^{(k)} M_{33}^{(k)} > \mathcal{S}_{33}^{(k)}
\]
Equation (7.25) can be expressed as:

\[
\Delta_{m_{ij}} \Omega = 2 < m_{ij}^{(k)} \cdot - \phi_{ij}^{(k)} + \frac{12}{h_k^2} S_{ij}^{(k)} M_{ij}^{(k)} + \frac{120}{7h_k^3} S_{ij}^{(k)} M_{ij}^{(k)} - \frac{3}{7h_k} S_{ij}^{(k)} (\sigma_{ij}^{(k)} - \sigma_{ij}^{(k)}) + \frac{1}{h_k} \pi_{ij}^{(k)} (\theta^{(k)} - \theta^{(k)}) >_{R_k}.
\]  

Equation (7.25) can be expressed as:

\[
\Delta_{m_{ij}} \Omega = 2 < m_{ij}^{(k)} \cdot - \phi_{ij}^{(k)} + \frac{12}{h_k^2} S_{ij}^{(k)} M_{ij}^{(k)} + \frac{120}{7h_k^3} S_{ij}^{(k)} M_{ij}^{(k)} - \frac{3}{7h_k} S_{ij}^{(k)} (\sigma_{ij}^{(k)} - \sigma_{ij}^{(k)}) + \frac{1}{h_k} \pi_{ij}^{(k)} (\theta^{(k)} - \theta^{(k)}) >_{R_k}.
\]  

The vanishing of \( \Delta_{m_{ij}} \Omega \) for all \( m_{ij}^{(k)} \) gives:

\[
- \phi_{ij}^{(k)} + \frac{12}{h_k^2} S_{ij}^{(k)} M_{ij}^{(k)} + \frac{120}{7h_k^3} S_{ij}^{(k)} M_{ij}^{(k)} - \frac{3}{7h_k} S_{ij}^{(k)} (\sigma_{ij}^{(k)} - \sigma_{ij}^{(k)}) + \frac{1}{h_k} \pi_{ij}^{(k)} (\theta^{(k)} - \theta^{(k)}) = 0 \quad \text{on } R_k
\]

7.2.10 Gateaux Differential with Respect to \( V_{\alpha}^{(k)} \)

The Gateaux differential of the governing function, \( \Omega \), with respect to \( V_{\alpha}^{(k)} \), along an arbitrary path  \( q_{\alpha}^{(k)} \), can be expressed as:

\[
\Delta_{q_{\alpha}} \Omega = < q_{\alpha}^{(k)} \cdot - \frac{2}{5} S_{\alpha y 3}^{(k)} \sigma_{y 3}^{(k)} - \frac{2}{5} S_{\alpha y 3}^{(k)} \sigma_{y 3}^{(k)} - \phi_{\alpha}^{(k)} - \phi_{\alpha}^{(k)} >_{V_{3,\alpha}}
\]

\[+\frac{24}{5h_k} S_{\alpha y 3}^{(k)} V_{y 3}^{(k)} - \frac{2}{5} S_{\alpha y 3}^{(k)} \sigma_{y 3}^{(k)} - \frac{2}{5} S_{\alpha y 3}^{(k)} \sigma_{y 3}^{(k)} >_{R_k}.
\]
Green's theorem gives:

\[ \langle \nabla_3^{(k)} \cdot q_{,\alpha}^{(k)} \rangle_{R^{(k)}} = \langle q_{,\alpha}^{(k)} \cdot \nabla_3^{(k)} \rangle_{R^{(k)}} + \langle q_{,\alpha}^{(k)} \cdot \nabla_3^{(k)} \rangle_{S_{\alpha}^{(k)}} \]

Equation (7.27) can be expressed using equation (7.28) as:

\[ \Delta_{q_{,\alpha}} \Omega = 2 \langle q_{,\alpha}^{(k)} \cdot \nabla_3^{(k)} \rangle_{R^{(k)}} - \frac{2}{5} S_{p,3y}^{(k)} (\sigma_{y}^{(k-1)} + \sigma_{y}^{(k)}) + 2 < q_{,\alpha}^{(k)} \cdot \nabla_3^{(k)} \rangle_{S_{\alpha}^{(k)}} + \frac{24}{5h} S_{\alpha,3y}^{(k)} V_{y}^{(k)} >_{R^{(k)}} + 2 < q_{,\alpha}^{(k)} \cdot \nabla_3^{(k)} \rangle_{S_{\alpha}^{(k)}} \]

The vanishing of \( \Delta_{q_{,\alpha}} \Omega \) for all \( q_{,\alpha}^{(k)} \) gives:

\[ - \nabla_3^{(k)} \cdot q_{,\alpha}^{(k)} - \frac{2}{5} S_{p,3y}^{(k)} (\sigma_{y}^{(k-1)} + \sigma_{y}^{(k)}) + \frac{24}{5h} S_{\alpha,3y}^{(k)} V_{y}^{(k)} = 0 \text{ on } R^{(k)} \]

\[ \nabla_3^{(k)} \eta_{\alpha} = g_{6}^{(k)} \text{ on } S_{6}^{(k)} \]

\[ (\nabla_3^{(k)}) \cdot \eta_{\alpha} = g_{6}^{(k)} \text{ on } S_{6i}^{(k)} \]
7.2.11 Gateaux Differential with Respect to $\sigma_{a3}^{-(k)}$

The Gateaux differential of the governing function, $\Omega$, with respect to $\sigma_{a3}^{-(k)}$, along an arbitrary path $\tau_{a3}^{-(k)}$, can be expressed as:

$$
\Delta_{\tau_{a3}^{-(k)}} \Omega = \langle \tau_{a3}^{-(k)}, \nabla \sigma_{a3}^{-(k)} \rangle + \frac{h_{k+1}}{2} \bar{\phi}_{a}^{(k+1)} - \frac{2}{5} \nabla_{\alpha}^{(k+1)} V_{\gamma}^{(k+1)} - \nabla_{\gamma}^{(k+1)} V_{\alpha}^{(k+1)} + \frac{h_{k}}{2} \bar{\phi}_{a}^{(k)}
$$

$$
- \frac{2}{5} S_{\rho\gamma\lambda}^{(k)} V_{\gamma}^{(k)} - \frac{2h_{k}}{15} h_{k} S_{\alpha\beta\gamma}^{(k)} \sigma_{\gamma}^{(k+1)} - \frac{2}{5} S_{\rho\gamma\lambda}^{(k)} V_{\gamma}^{(k+1)} - \frac{2h_{k+1}}{15} h_{k} S_{\alpha\beta\gamma}^{(k+1)} \sigma_{\gamma}^{(k+1)}
$$

$$
+ \frac{2}{15} [h_{k} S_{\alpha\beta\gamma}^{(k)} + h_{k+1} S_{\beta\alpha\gamma}^{(k+1)}] \sigma_{\gamma}^{(k-1)} + \nabla^{(k+1)} V_{\alpha}^{(k+1)} + \frac{h_{k+1}}{2} \bar{\phi}_{a}^{(k+1)}
$$

$$
- \frac{h_{k+1}}{12} \bar{\phi}_{a}^{(k+1)} - \frac{h_{k+1}}{120} \phi_{a}^{(k+1)} > R^{k+1} + \langle \bar{\phi}_{a}^{(k)} \cdot \frac{h_{k}}{120} \tau_{a3}^{-(k)} > R^{k+1}
$$

$$
+ \langle \bar{\phi}_{a}^{(k)} \cdot \frac{h_{k}}{120} \tau_{a3}^{-(k-1)} > R^{k+1}
$$

$$
+ \langle \bar{\phi}_{a}^{(k)} \cdot \frac{h_{k}}{120} \tau_{a3}^{-(k+1)} > R^{k+1}
$$

$$
+ \langle \tau_{a3}^{-(k)} \cdot \frac{h_{k}^{3}}{120} \bar{\phi}_{a}^{(k)} \eta_{a} - \frac{h_{k}^{2}}{12} \bar{\phi}_{a}^{(k)} \eta_{a} + \frac{h_{k+1}}{120} \phi_{a}^{(k+1)} \eta_{a} + \frac{h_{k+1}}{12} \phi_{a}^{(k+1)} \eta_{a} - 2 g_{a}^{(k)} > s_{a}^{(k)}
$$

$$
+ \langle \tau_{a3}^{-(k)} \cdot \frac{h_{k}}{120} (\bar{\phi}_{a}^{(k)} \eta_{a} - \frac{h_{k}}{12} (\bar{\phi}_{a}^{(k)}) \eta_{a} + \frac{h_{k+1}}{120} (\phi_{a}^{(k+1)} \eta_{a} - 2 g_{a}^{(k)}) > s_{a}^{(k)}
$$

(7.30)

158
Using Green’s theorem:

\[
\frac{\hat{\phi}_3^{(k)}}{120} \tau_{3,3}^{(k)} - \frac{\hat{\phi}_3^{(k)}}{120} \tau_{3,3}^{(k)} = -\frac{\hat{\phi}_3^{(k)}}{120} \eta^{(k)} + \frac{h_k^{(k)}}{120} \phi_3^{(k)} \eta^{(k)} + \frac{h_k^{(k)}}{120} \phi_3^{(k)} \eta^{(k)}
\]

(7.31)

\[
\frac{\hat{\phi}_3^{(k)}}{12} \tau_{3,3}^{(k)} = -\frac{\hat{\phi}_3^{(k)}}{12} \tau_{3,3}^{(k)} + \frac{h_k^{(k)}}{12} \phi_3^{(k)} \eta^{(k)} + \frac{h_k^{(k)}}{12} \phi_3^{(k)} \eta^{(k)}
\]

(7.32)

\[
\frac{\hat{\phi}_3^{(k+1)}}{120} \tau_{3,3}^{(k+1)} = -\frac{\hat{\phi}_3^{(k+1)}}{120} \tau_{3,3}^{(k+1)} + \frac{h_k^{(k+1)}}{120} \phi_3^{(k+1)} \eta^{(k+1)} + \frac{h_k^{(k+1)}}{120} \phi_3^{(k+1)} \eta^{(k+1)}
\]

(7.33)

\[
\frac{\hat{\phi}_3^{(k+1)}}{12} \tau_{3,3}^{(k+1)} = -\frac{\hat{\phi}_3^{(k+1)}}{12} \tau_{3,3}^{(k+1)} + \frac{h_k^{(k+1)}}{12} \phi_3^{(k+1)} \eta^{(k+1)} + \frac{h_k^{(k+1)}}{12} \phi_3^{(k+1)} \eta^{(k+1)}
\]

(7.34)

Equation (7.30) can be expressed using equations (7.31) - (7.34) as:

\[
\Delta_{\rho^{(k)}} = 2\tau_{\rho^{(k)}} - \frac{2h_k}{5} S_{\rho^{(k)}}^{(k)} + \frac{2h_k}{5} S_{\rho^{(k)}}^{(k)} (4\sigma_{\gamma^{(k)}} - \sigma_{\gamma^{(k-1)}})
\]

(7.35)

\[
+ \frac{\sqrt{h_k}}{5} S_{\rho^{(k)}}^{(k)} \phi_3^{(k)} \eta^{(k)} + \frac{\sqrt{h_k}}{5} S_{\rho^{(k)}}^{(k)} \phi_3^{(k)} \eta^{(k)}
\]

(7.36)
\[
+ \frac{2h_{k+1}}{15} S_{\rho\gamma\delta}^{(k+1)}(4\sigma_{\gamma\delta}^{(k)} - \sigma_{\gamma\delta}^{-(k+1)}) \rangle_{\mathcal{R}^k}, \\
+ 2<\tau_{\rho_3}^{(k)} \cdot \frac{h_k}{120} \hat{\phi}_3^{(k)} \eta_\alpha - \frac{h_k}{12} \hat{\phi}_3^{(k)} \eta_\alpha + \frac{h_{k+1}}{120} \hat{\phi}_3^{(k+1)} \eta_\alpha + \frac{h_{k+1}}{12} \hat{\phi}_3^{(k+1)} \eta_\alpha - g_\sigma^{(k)} \rangle_{S^k}, \\
+ 2<\tau_{\rho_3}^{(k)} \cdot \frac{h_k}{120} (\hat{\phi}_3^{(k)})' \eta_\alpha - \frac{h_k}{12} (\hat{\phi}_3^{(k)})' \eta_\alpha + \frac{h_{k+1}}{120} (\hat{\phi}_3^{(k+1)})' \eta_\alpha + \frac{h_{k+1}}{12} (\hat{\phi}_3^{(k+1)})' \eta_\alpha \\
+ \frac{h_{k+1}}{12} (\hat{\phi}_3^{(k+1)})' \eta_\alpha - g_\sigma^{(k)} \rangle_{S^k_{\sigma}}, \\
\]

(7.35)

The vanishing of \(\Delta_{\tau_{\rho_3}^{(k)}} \Omega\) for all \(\tau_{\rho_3}^{(k)}\) gives:

\[
-\overline{\nabla}_\rho^{(k)} + \frac{h_k}{2} \hat{\phi}_\rho^{(k)} + \frac{h_k}{12} \hat{\phi}_3^{(k)} - \frac{h_k}{120} \hat{\phi}_3^{(k)} - \frac{2}{5} S_{\rho\gamma\delta}^{(k)} V_\gamma^{(k)} \\
+ \frac{2h_k}{15} S_{\rho\gamma\delta}^{(k)}(4\sigma_{\gamma\delta}^{(k)} - \sigma_{\gamma\delta}^{-(k+1)}) \\
+ \overline{\nabla}_\rho^{(k+1)} + \frac{h_{k+1}}{2} \hat{\phi}_\rho^{(k+1)} + \frac{h_{k+1}}{12} \hat{\phi}_3^{(k+1)} - \frac{h_{k+1}}{120} \hat{\phi}_3^{(k+1)} - \frac{2}{5} S_{\rho\gamma\delta}^{(k+1)} V_\gamma^{(k+1)} \\
+ \frac{2h_{k+1}}{15} S_{\rho\gamma\delta}^{(k+1)}(4\sigma_{\gamma\delta}^{(k)} - \sigma_{\gamma\delta}^{-(k+1)}) = 0 \\
\text{on } \mathcal{R}^{(k)} \\
\frac{h_k}{120} \hat{\phi}_3^{(k)} \eta_\alpha - \frac{h_k}{12} \hat{\phi}_3^{(k)} \eta_\alpha + \frac{h_{k+1}}{120} \hat{\phi}_3^{(k+1)} \eta_\alpha + \frac{h_{k+1}}{12} \hat{\phi}_3^{(k+1)} \eta_\alpha = g_\sigma^{(k)} \\
\text{on } S_{\sigma}^{(k)} \\
\frac{h_k}{120} (\hat{\phi}_3^{(k)})' \eta_\alpha - \frac{h_k}{12} (\hat{\phi}_3^{(k)})' \eta_\alpha + \frac{h_{k+1}}{120} (\hat{\phi}_3^{(k+1)})' \eta_\alpha + \frac{h_{k+1}}{12} (\hat{\phi}_3^{(k+1)})' \eta_\alpha = g_\sigma^{(k)} \\
\text{on } S_{\sigma}^{(k)}
\]

160
7.2.12 Gateaux Differential with Respect to $\sigma_{33}^{-(k)}$

The Gateaux differential of the governing function, $\Omega$, with respect to $\sigma_{33}^{-(k)}$, along an arbitrary path $\tau_{33}^{(k)}$, can be expressed as:

$$
\Delta_{\tau_{33}^{(k)}, \Omega} = \epsilon \tau_{33}^{(k)} \cdot -\nabla_{33}^{(k)} - \frac{h_{k}^{2}}{10} \phi_{3}^{(k)} + \frac{h_{k}}{2} \phi_{j}^{(k)} - \frac{1}{10} S_{3333}^{(k)} N_{33}^{(k)} + \frac{7}{3h_{k}} S_{3333}^{(k)} M_{33}^{(k)} \\
+ \frac{7}{3h_{k+1}} S_{3333}^{(k+1)} M_{33}^{(k+1)} + 2 \frac{h_{k}}{70} S_{3333}^{(k)} N_{33}^{(k)} + 2 \frac{h_{k+1}}{70} S_{3333}^{(k+1)} \sigma_{33}^{-(k-1)} \\
+ 2 \frac{3}{35} \left[ h_{k} S_{3333}^{(k)} + h_{k+1} S_{3333}^{(k+1)} \right] - \nabla_{33}^{(k)} - \frac{h_{k}^{2}}{10} \phi_{3}^{(k)} + \frac{h_{k}}{2} \phi_{3}^{(k)} - \frac{1}{10} S_{3333}^{(k)} N_{33}^{(k)} \\
+ \frac{7}{3h_{k}} S_{3333}^{(k)} M_{33}^{(k)} + \nabla_{33}^{(k+1)} - \frac{h_{k+1}^{2}}{10} \phi_{3}^{(k+1)} + \frac{h_{k+1}}{2} \phi_{3}^{(k+1)} - \frac{1}{10} S_{3333}^{(k+1)} N_{33}^{(k+1)} \\
- \frac{7}{3h_{k+1}} S_{3333}^{(k+1)} M_{33}^{(k+1)} > r^{(k)}.
$$

(7.36)

The vanishing of $\Delta_{\tau_{33}^{(k)}, \Omega}$, for all $\tau_{33}^{(k)}$, gives:

$$
- \nabla_{33}^{(k)} - \frac{h_{k}^{2}}{10} \phi_{3}^{(k)} + \frac{h_{k}}{2} \phi_{3}^{(k)} - \frac{1}{10} S_{3333}^{(k)} N_{33}^{(k)} + \frac{7}{3h_{k}} S_{3333}^{(k)} M_{33}^{(k)} \\
+ \frac{h_{k}}{70} S_{3333}^{(k)} \left( 6 \sigma_{33}^{-(k)} + \sigma_{33}^{-(k-1)} \right) 
$$

161
The Gateaux differential of the governing function, \( \Omega \), with respect to \( \theta^{-(k)} \), along an arbitrary path \( T^{-(k)} \), can be expressed as:

\[
\Delta_{T^{-(k)}} \Omega = <T^{-(k)} \cdot \frac{1}{2} \pi_{\alpha\beta}^{(k+1)} N_{\alpha\beta}^{(k+1)} + \frac{1}{h_{k+1}} \pi_{\alpha\beta}^{(k+1)} M_{\alpha\beta}^{(k+1)} + \frac{1}{2} \pi_{33}^{(k+1)} N_{33}^{(k+1)}

+ \frac{1}{h_{k+1}} \pi_{33}^{(k+1)} M_{33}^{(k+1)} - \frac{g'}{T} \cdot \frac{h_k}{4} (\tilde{q}_{\alpha,\alpha}^{(k)} - \tilde{\tilde{q}}_{\alpha,\alpha}^{(k)}) + \frac{1}{2} \pi_{\alpha\beta}^{(k)} N_{\alpha\beta}^{(k)}

- \frac{1}{h_k} \pi_{\alpha\beta}^{(k)} M_{\alpha\beta}^{(k)} + \frac{1}{2} \pi_{33}^{(k)} N_{33}^{(k)} - \frac{1}{h_k} \pi_{33}^{(k)} M_{33}^{(k)} + \frac{1}{2} \pi_{\alpha\beta}^{(k)} N_{\alpha\beta}^{(k)}

- \frac{1}{h_k} \pi_{\alpha\beta}^{(k)} M_{\alpha\beta}^{(k)} + \frac{1}{2} \pi_{33}^{(k)} N_{33}^{(k)} - \frac{1}{h_k} \pi_{33}^{(k)} M_{33}^{(k)} + \frac{1}{2} \pi_{\alpha\beta}^{(k)} N_{\alpha\beta}^{(k)}

+ \frac{1}{h_{k+1}} \pi_{\alpha\beta}^{(k+1)} M_{\alpha\beta}^{(k+1)} + \frac{1}{2} \pi_{33}^{(k+1)} N_{33}^{(k+1)} + \frac{1}{h_{k+1}} \pi_{33}^{(k+1)} M_{33}^{(k+1)}

- \frac{g'}{T} \cdot \frac{h_{k+1}}{4} (\tilde{q}_{\alpha,\alpha}^{(k+1)} + \tilde{\tilde{q}}_{\alpha,\alpha}^{(k+1)}) - \frac{h_k}{6} (\Gamma^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)}

+ \rho^{(k)} c^{(k)} \theta^{-(k)} - \frac{h_{k+1}}{6} (\Gamma^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho^{(k)} c^{(k)}}{T_o} \theta^{-(k+1)})
\]
\[-2\frac{1}{3} \left\{ \Gamma_{\alpha\beta}^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho^{(k)} c^{(k)}}{T_o} \right\} + h_{k-1} \left\{ \Gamma_{\alpha\beta}^{(k-1)} \pi_{\alpha\beta}^{(k-1)} + \Gamma_{33}^{(k-1)} \pi_{33}^{(k-1)} + \frac{\rho^{(k-1)} c^{(k-1)}}{T_o} \right\} \]

\[+ \Gamma_{33}^{(k)} \theta^{(k)} + 2 Z_{b}^{(k)} + \frac{h_{k+1}}{2} \frac{g'}{T_o} \sigma_{f}^{(k)} \]

\[+ h_{k} \frac{g'}{T_o} \Gamma^{(k)} > R^{(k)}, + \frac{h_{k}}{4} \frac{g'}{T_o} \Gamma_{\alpha}^{(k-1)} > R^{(k)}, \]

\[+ \frac{\pmb{q}_{\alpha}^{(k)} \cdot \frac{g'}{T_o} h_{k} \Gamma_{\alpha}^{(k-1)} > R^{(k)}, + \frac{\pmb{q}_{\alpha}^{(k)} \cdot \frac{g'}{T_o} h_{k} \Gamma_{\alpha}^{(k-1)} > R^{(k)}, \]

\[+ \frac{\pmb{q}_{\alpha}^{(k)} \cdot \frac{g'}{T_o} h_{k} \Gamma_{\alpha}^{(k-1)} > R^{(k)}, \]

\[+ \Gamma_{\alpha}^{(k-1)} \eta_{\alpha} - \frac{g'}{T_o} \pmb{q}_{\alpha}^{(k)} \eta_{\alpha} + \frac{g'}{T_o} \pmb{q}_{\alpha}^{(k)} \eta_{\alpha} + \frac{h_{k}}{4} \frac{g'}{T_o} \pmb{q}_{\alpha}^{(k)} \eta_{\alpha} + \frac{h_{k+1}}{4} \frac{g'}{T_o} \pmb{q}_{\alpha}^{(k+1)} \eta_{\alpha} \]

\[+ \frac{g'}{T_o} \frac{h_{k}}{4} (\pmb{q}_{\alpha}^{(k)}) \eta_{\alpha} - \frac{g'}{T_o} \frac{h_{k}}{4} (\pmb{q}_{\alpha}^{(k)}) \eta_{\alpha} + \frac{g'}{T_o} \frac{h_{k+1}}{4} (\pmb{q}_{\alpha}^{(k+1)}) \eta_{\alpha} \]

\[+ \frac{g'}{T_o} \frac{h_{k+1}}{4} (\pmb{q}_{\alpha}^{(k)}) \eta_{\alpha} - 2 \frac{g'}{T_o} \pmb{q}_{\alpha}^{(k)} < s^{(k)}, \]

\[+ \frac{g'}{T_o} \frac{h_{k+1}}{4} (\pmb{q}_{\alpha}^{(k+1)}) \eta_{\alpha} - 2 \frac{g'}{T_o} \pmb{q}_{\alpha}^{(k)} < s^{(k)}. \]

(7.37)

Using the divergence theorem:

\[< \pmb{q}_{\alpha}^{(k)} \cdot \frac{g'}{T_o} h_{k} \Gamma_{\alpha}^{(k-1)} > R^{(k)} = -< \Gamma_{\alpha}^{(k-1)} \cdot \frac{g'}{T_o} \pmb{q}_{\alpha}^{(k)} > R^{(k)}, \]

\[+ < \Gamma_{\alpha}^{(k)} \pmb{q}_{\alpha}^{(k)} > s^{(k)}, + < \Gamma_{\alpha}^{(k)} \pmb{q}_{\alpha}^{(k)} > s^{(k)}, \]

(7.38)
\begin{align*}
\langle q'^{(k)}_a, - \frac{g'}{T_i} \frac{h_i}{4} T_{\alpha}^{-(k)} \rangle >_{R_{\gamma}} = - \langle T_{-^{(k)}} - \frac{g'}{T_i} \frac{h_i}{4} \overline{q'^{(k)}_a} \rangle >_{R_{\gamma}} \\
+ \langle T_{-^{(k)}} - \frac{g'}{T_i} \frac{h_i}{4} \overline{q'^{(k)}_a} \eta^{(k)} >_{\gamma} \rangle + \langle T_{-^{(k)}} - \frac{g'}{T_i} \frac{h_i}{4} (\overline{q'^{(k)}_a})^{(k)} \eta^{(k)} >_{\gamma} \rangle.
\end{align*}
(7.39)

\begin{align*}
\langle \overline{q^{(k+1)}_a}, \frac{g'}{T_i} \frac{h_{k+1}}{4} T_{\alpha}^{-(k)} \rangle >_{R_{\gamma}} = - \langle T_{-^{(k)}} - \frac{g'}{T_i} \frac{h_{k+1}}{4} \overline{q^{(k+1)}_a} \rangle >_{R_{\gamma}} \\
+ \langle T_{-^{(k)}} - \frac{g'}{T_i} \frac{h_{k+1}}{4} \overline{q^{(k+1)}_a} \eta^{(k+1)} >_{\gamma} \rangle + \langle T_{-^{(k)}} - \frac{g'}{T_i} \frac{h_{k+1}}{4} (\overline{q^{(k+1)}_a})^{(k+1)} \eta^{(k+1)} >_{\gamma} \rangle.
\end{align*}
(7.40)

\begin{align*}
\langle \overline{q^{(k+1)}_a}, \frac{g'}{T_i} \frac{h_{k+1}}{4} T_{\alpha}^{-(k)} \rangle >_{R_{\gamma}} = - \langle T_{-^{(k)}} - \frac{g'}{T_i} \frac{h_{k+1}}{4} \overline{q^{(k+1)}_a} \rangle >_{R_{\gamma}} \\
+ \langle T_{-^{(k)}} - \frac{g'}{T_i} \frac{h_{k+1}}{4} \overline{q^{(k+1)}_a} \eta^{(k+1)} >_{\gamma} \rangle + \langle T_{-^{(k)}} - \frac{g'}{T_i} \frac{h_{k+1}}{4} (\overline{q^{(k+1)}_a})^{(k+1)} \eta^{(k+1)} >_{\gamma} \rangle.
\end{align*}
(7.41)

Therefore using (7.38) - (7.41), (7.37) can be expressed as:

\[
\Delta_{T^{-k}, \Omega} = 2 \langle T_{-^{(k)}} \frac{1}{2} \pi^{(k)}_{\alpha \beta} \ N^{(k)}_{\alpha \beta} - \frac{1}{h_k} \pi^{(k)}_{\alpha \beta} M^{(k)}_{\alpha \beta} + \frac{1}{2} \pi^{(k)}_{33} N^{(k)}_{33} - \frac{1}{h_k} \pi^{(k)}_{33} M^{(k)}_{33} \\
- (\Gamma^{(k)}_{\alpha \beta} \pi^{(k)}_{\alpha \beta} + \Gamma^{(k)}_{33} \pi^{(k)}_{33} + \Theta^{(k)}_{\alpha \beta} \pi^{(k)}_{\alpha \beta}) \left( \frac{2h_k}{12} \theta^{-^{(k-1)}} - \frac{4h_k}{12} \theta^{-^{(k)}} \right) \right \}
\]

\[
- \frac{g'}{T_i} \frac{h_i}{4} (\overline{q^{(k)}_a} - \overline{q^{(k)}_a}) + \frac{h_k}{2} \frac{g'}{T_o} f^{(k)}
\]

\[
+ \frac{1}{2} \pi^{(k+1)}_{\alpha \beta} N^{(k+1)}_{\alpha \beta} + \frac{1}{h_{k+1}} \pi^{(k+1)}_{\alpha \beta} M^{(k+1)}_{\alpha \beta} + \frac{1}{2} \pi^{(k+1)}_{33} N^{(k+1)}_{33} + \frac{1}{h_{k+1}} \pi^{(k+1)}_{33} M^{(k+1)}_{33}
\]

164
\[-(\Gamma^{(k+1)}_{\alpha\beta}\pi_{\alpha\beta}^{(k+1)} + \Gamma^{(k+1)}_{33}\pi_{33}^{(k+1)} + \frac{\rho^{(k+1)}c^{(k+1)}T_0}{T_o})\{\frac{4h_{k+1}}{12}\theta^{-(k)}\} + \frac{2h_{k+1}}{12}\theta^{-(k+1)}\} \cdot \frac{g^*}{T_o} \cdot \frac{h_{k+1}}{4} (\tilde{q}_{\alpha,\alpha}^{(k+1)} + \tilde{q}_{\alpha,\alpha}^{(k+1)}) \]

\[-+ \frac{h_{k+1}}{2\frac{g^*}{T_o}} \Gamma^{(k+1)} + Z_b^{(k)} \right]

\[-+ 2\frac{T^{-(k)}}{T_o} \cdot \frac{g^*}{T_o} \cdot \frac{h_{k+1}}{4} (\tilde{q}_{\alpha}^{(k+1)})^\eta_{\alpha} + \frac{g^*}{T_o} \cdot \frac{h_{k+1}}{4} (\tilde{q}_{\alpha}^{(k+1)})^\eta_{\alpha} + \frac{g^*}{T_o} \cdot \frac{h_{k+1}}{4} (\tilde{q}_{\alpha}^{(k+1)})^\eta_{\alpha} \]

\[-+ \frac{g^*}{T_o} \cdot \frac{h_{k+1}}{4} (\tilde{q}_{\alpha}^{(k+1)})^\eta_{\alpha} - g_{\sigma}^{(k)} > s_{\sigma}^{(k)} \]

The vanishing of \(\Delta_{\epsilon_{\nu},\Omega}\), for all \(T^{-(k)}\) gives:

\[-\frac{1}{2} \pi_{\alpha\beta}^{(k)} N_{\alpha\beta}^{(k)} - \frac{1}{h_k} \pi_{\alpha\beta}^{(k)} M_{\alpha\beta}^{(k)} + \frac{1}{2} \pi_{33}^{(k)} N_{33}^{(k)} - \frac{1}{h_k} \pi_{33}^{(k)} M_{33}^{(k)} \]

\[--(\Gamma^{(k)}_{\alpha\beta}\pi_{\alpha\beta}^{(k)} + \Gamma^{(k)}_{33}\pi_{33}^{(k)} + \frac{\rho^{(k)}c^{(k)}T_0}{T_o})\{\frac{2h_k}{12}\theta^{-(k-1)} + \frac{4h_k}{12}\theta^{-k}\} \]

\[-- \frac{g^*}{T_o} \cdot \frac{h_k}{4} (\tilde{q}_{\alpha,\alpha}^{(k)} - \tilde{q}_{\alpha,\alpha}^{(k)}) + \frac{h_k}{2\frac{g^*}{T_o}} \Gamma^{(k)} \]

\[+ \frac{1}{2} \pi_{\alpha\beta}^{(k+1)} N_{\alpha\beta}^{(k+1)} + \frac{1}{h_{k+1}} \pi_{\alpha\beta}^{(k+1)} M_{\alpha\beta}^{(k+1)} + \frac{1}{2} \pi_{33}^{(k+1)} N_{33}^{(k+1)} + \frac{1}{h_{k+1}} \pi_{33}^{(k+1)} M_{33}^{(k+1)} \]

\[--(\Gamma^{(k+1)}_{\alpha\beta}\pi_{\alpha\beta}^{(k+1)} + \Gamma^{(k+1)}_{33}\pi_{33}^{(k+1)} + \frac{\rho^{(k+1)}c^{(k+1)}T_0}{T_o})\{\frac{4h_{k+1}}{12}\theta^{-k}\} \]

165
The Gateaux differential of the governing function, $\Omega$, with respect to $\tilde{q}^{(k)}_\alpha$, along an arbitrary path $\tilde{\beta}^{(k)}_\alpha$, can be expressed as:

$$
\Delta_{\tilde{\beta}^{(k)}_\alpha, \Omega} = \langle \tilde{\beta}^{(k)}_\alpha, \cdot \rangle \cdot \frac{g'}{T_c} + \frac{h_k}{4} \frac{g'}{T_c} \theta^{-(k-1)} + \frac{g'}{T_c} \theta^{-(k)} + \frac{h_k}{4} \frac{g'}{T_c} \lambda^{(k)} \tilde{q}^{(k)}_\alpha + \frac{h_k}{4} \frac{g'}{T_c} \lambda^{(k)} \tilde{q}^{(k)}_\alpha + \frac{h_k}{4} \frac{g'}{T_c} \lambda^{(k)} \tilde{q}^{(k)}_\alpha >_{R^k},
$$

$$
+ \langle \theta^{-(k)} \cdot \frac{g'}{T_c} \tilde{\beta}^{(k)}_\alpha, \cdot \rangle >_{R^k}, + \langle \theta^{-(k)} \cdot \frac{g'}{T_c} \tilde{\beta}^{(k)}_\alpha, \cdot \rangle >_{R^k},
$$

166
Using the divergence theorem:

\[ \langle \theta^{(-k)} \rangle \cdot \frac{g'}{T_0} \cdot \frac{h_k}{4} \bar{\beta}_{\alpha}^{(k)} \eta_{\alpha} >_{s_\alpha^k} + \langle \theta^{(-k)} \rangle \cdot \frac{g'}{T_0} \cdot \frac{h_{k+1}}{4} \bar{\beta}_{\alpha}^{(k+1)} \eta_{\alpha} >_{s_\alpha^k} \]

\[ + \langle \theta^{(-k)} \rangle \cdot \frac{g'}{T_0} \cdot \frac{h_k}{4} (\bar{\beta}_{\alpha}^{(k)})' \eta_{\alpha} >_{s_\alpha^k} + \langle \theta^{(-k)} \rangle \cdot \frac{g'}{T_0} \cdot \frac{h_{k+1}}{4} (\bar{\beta}_{\alpha}^{(k+1)})' \eta_{\alpha} >_{s_\alpha^k}. \]

(7.42)

Substituting (7.43) and (7.44) into (7.42) results in:

\[ \Delta \bar{\beta}_{\alpha}^{(k)} \Omega = 2 \langle \bar{\beta}_{\alpha}^{(k)} \rangle \cdot \frac{g'}{T_0} \cdot \frac{h_k}{4} \theta_{\alpha}^{(-k)} + \langle \theta_{\alpha}^{(-k)} \rangle \cdot \frac{h_k}{4} \theta_{\alpha}^{(-k)} + \frac{g'}{T_0} \cdot \frac{h_k}{4} \lambda_{\alpha\beta}^{(k)} \bar{q}_{\beta}^{(k)} >_{R^{(k)}}. \]

(7.44)

Substituting (7.43) and (7.44) into (7.42) results in:

\[ \frac{h}{4T_0} \cdot \left( \theta_{\alpha}^{(-k-1)} + \theta_{\alpha}^{(-k)} \right) + \frac{h}{4T_0} \cdot \lambda_{\alpha\beta}^{(k)} \bar{q}_{\beta}^{(k)} = 0 \quad \text{on } R^{(k)} \]

167
7.2.15 Gateaux Differential with Respect to $\bar{q}_{\alpha}^{(k)}$

The Gateaux differential of the governing function, $\Omega$, with respect to $\bar{q}_{\alpha}^{(k)}$, along an arbitrary path $\bar{e}_{\alpha}^{(k)}$, can be expressed as:

$$\Delta_{\bar{e}_{\alpha}^{(k)}} \Omega = \langle \bar{e}_{\alpha}^{(k)}, \frac{g'}{T_c} \cdot \frac{h_k}{4} \theta_{\alpha}^{(k-1)} - \frac{g'}{T_c} \cdot \frac{h_k}{4} \theta_{\alpha}^{(k)} + 2 \frac{3h_k}{T_c} \frac{g'}{T_c} \lambda_{\alpha \beta}^{(k)} \bar{q}_{\beta}^{(k)} \rangle_{R^k},$$

$$+ \langle \theta^{(k)}, \frac{g'}{T_c} \cdot \frac{h_k}{4} \bar{e}_{\alpha \alpha}^{(k)} \rangle_{R^k},$$

$$+ \langle \theta^{(k)}, \frac{g'}{T_c} \cdot \frac{h_k}{4} \bar{e}_{\alpha}^{(k)} \eta_\alpha \rangle_{S_{\alpha}^{(k)}},$$

$$+ \langle \theta^{(k)}, \frac{g'}{T_c} \cdot \frac{h_{k+1}}{4} \bar{e}_{\alpha}^{(k+1)} \eta_\alpha \rangle_{S_{\alpha}^{(k+1)}},$$

$$+ \langle \theta^{(k)}, \frac{g'}{T_c} \cdot \frac{h_k}{4} \frac{\bar{e}_{\alpha}^{(k)}}{T_c} \eta_\alpha \rangle_{S_{\alpha}^{(k)}},$$

$$+ \langle \theta^{(k)}, \frac{g'}{T_c} \cdot \frac{h_{k+1}}{4} \frac{\bar{e}_{\alpha}^{(k+1)}}{T_c} \eta_\alpha \rangle_{S_{\alpha}^{(k+1)}},$$

(7.45)

Using the divergence theorem:

$$\langle \theta^{(k)}, \frac{g'}{T_c} \cdot \frac{h_k}{4} \bar{e}_{\alpha}^{(k)} \rangle_{R^k} = - \langle \bar{e}_{\alpha}^{(k)}, \frac{g'}{T_c} \cdot \frac{h_k}{4} \theta_{\alpha}^{(k)} \rangle_{R^k},$$

$$+ \langle \theta^{(k)}, \frac{g'}{T_c} \cdot \frac{h_k}{4} \bar{e}_{\alpha}^{(k)} \eta_\alpha \rangle_{S_{\alpha}^{(k)}},$$

$$+ \langle \theta^{(k)}, \frac{g'}{T_c} \cdot \frac{h_{k+1}}{4} \bar{e}_{\alpha}^{(k+1)} \eta_\alpha \rangle_{S_{\alpha}^{(k+1)}},$$

(7.46)
Equation (7.45), using (7.46) and (7.47), yield:

\[
\begin{align*}
\Delta_{\alpha \xi} \eta = 2 & \varepsilon_{\alpha}^{(k)} \frac{g^'}{T} \left( \theta_{\alpha}^{(k)} + \frac{h}{T} \theta_{\alpha}^{(k-1)} \right) + \frac{g^'}{T} \left( \theta_{\alpha}^{(k)} + \frac{h}{T} \theta_{\alpha}^{(k-1)} \right) + \frac{3h}{T} \frac{g^'}{T} \lambda_{\alpha \beta}^{(k)} \bar{q}_{\beta}^{(k)} \right) > R^{(k)} \ ,
\end{align*}
\]

The vanishing of \( \Delta_{\alpha \xi} \eta \), for all \( \varepsilon_{\alpha}^{(k)} \) gives:

\[
\frac{h}{T} \frac{g^'}{T} \left( \theta_{\alpha}^{(k)} + \frac{h}{T} \theta_{\alpha}^{(k-1)} \right) + \frac{3h}{T} \frac{g^'}{T} \lambda_{\alpha \beta}^{(k)} \bar{q}_{\beta}^{(k)} = 0 \quad \text{on} \ R^{(k)}
\]

It has been shown that the vanishing of the Gateaux differential for all \{u\}^{(k)} and \{\sigma\}^{(k)} implies satisfaction of the field equations, the continuity conditions, jump discontinuity conditions, the initial conditions and the boundary conditions. Therefore, the basic function governing the laminated composite plate is given by equation (7.3). The function is general in the sense that it admits \( V_{\alpha}^{(k)} \), \( \Phi_{\alpha}^{(k)} \), \( V_{3}^{(k)} \), \( \Phi_{3}^{(k)} \), \( \Phi_{3}^{(k)} \), \( N_{\alpha \beta}^{(k)} \), \( N_{33}^{(k)} \), \( M_{\alpha \beta}^{(k)} \), \( M_{33}^{(k)} \), \( V_{\alpha}^{(k)} \), \( \sigma_{\gamma \gamma}^{(k)} \), \( \sigma_{33}^{(k)} \), \( \theta^{(k)} \), \( q_{\alpha}^{(k)} \) and \( \bar{q}_{\alpha}^{(k)} \) as field variables. Up to this point, it is not required that the admissible field variables identically satisfy any of the field equations, boundary conditions or initial conditions.
7.3 ORDER OF DIFFERENTIABILITY

The field variables \( \bar{v}_\alpha^{(k)} \), \( \bar{\phi}_\alpha^{(k)} \), \( \bar{v}_3^{(k)} \), \( \bar{\phi}_3^{(k)} \), \( \bar{M}_{\alpha\beta}^{(k)} \), \( \bar{M}_{33}^{(k)} \), \( \bar{V}_\alpha^{(k)} \), \( \bar{q}_\alpha^{(k)} \), \( \bar{q}_3^{(k)} \), \( \sigma_{33}^{(k)} \), \( \sigma_{y3}^{(k)} \) and \( \theta_{-1}^{(k)} \) belong to the intersection of the domains of the set of operators which act on them. Defining \( C^d \) as the space of functions whose derivatives are continuous up to order "q," the admissible state is ensured if the following are satisfied:

(a) \( v_f^{(k)} \), \( \phi_f^{(k)} \), \( v_3^{(k)} \), \( \phi_3^{(k)} \), \( N_{\alpha\beta}^{(k)} \), \( N_{33}^{(k)} \), \( M_{\alpha\beta}^{(k)} \), \( M_{33}^{(k)} \), \( V_\alpha^{(k)} \), \( \sigma_{33}^{(k)} \), \( \bar{q}_\alpha^{(k)} \), \( \bar{q}_3^{(k)} \).

(b) \( \sigma_{y3}^{-(k)} \in C^2 \).

To ensure that all the differential equations can be satisfied by an element in the set of admissible states, it is necessary that various field variables have appropriate smoothness. The field equations (5.42) - (5.44) necessitate:

(a) \( \sigma_{33}^{-(k)} \), \( \sigma_{y3}^{-(k)} \in C^0 \).

(b) \( V_\alpha^{(k)} \in C^1 \), at least one order of continuous differentiability higher than \( \sigma_{33}^{-(k)} \).

(c) \( N_{\alpha\beta}^{(k)} \in C^1 \), at least one order of continuous differentiability higher than \( \sigma_{y3}^{-(k)} \).

170
(d) $M_{ab}^{(k)} \in C^2$, at least one order of continuous differentiability higher than $V^{(k)}_\alpha$.

For $\sigma_{ij}^{(k)} \in C^0$, equations (5.45) - (5.47) and (5.49) require that:

(a) $\sigma_{ij}^{(k)} \in C^1$, at least one order of continuous differentiability higher than $\sigma_{33}^{(k)}$, therefore from (d) above $N_{ab}^{(k)} \in C^2$

(b) $\overline{V}^{(k)}_\alpha \in C^{3}$, at least one order of continuous differentiability higher than $N_{ab}^{(k)}$.

(c) $N_{33}^{(k)}, \theta^{-(k)} \in C^0$.

(d) $\phi^{(k)}_\alpha \in C^3$, at least one order of continuous differentiability higher than $M_{ab}^{(k)}$.

(e) $M_{33}^{(k)} \in C^0$.

Field equations (5.52) and (5.53) require that:

(a) $\theta^{-(k)} \in C^1$, at least one order of continuous differentiability higher than $\tilde{q}^{(k)}_\alpha$ and $\tilde{q}^{(k)}_{\alpha}$.

Field equations (5.48), (5.50) and (5.51), and require that:

(a) $\overline{V}^{(k)}_3 \in C^4$, at least one order of continuous differentiability higher than $\overline{\phi}^{(k)}_\alpha$.

(b) $\overline{\varphi}^{(k)}_3, \hat{\phi}^{(k)}_3 \in C^0$

Equations (5.57) and (5.58) require that for $\sigma_{ij}^{(k)} \in C^1$
(a) \( \overline{V}_a^{(k)} \cdot \overline{\phi}_a^{(k)} \cdot \overline{V}_a^{(k)} \) are at least the same order as \( \sigma_{\gamma3}^{(k)} \).

(b) \( \overline{\phi}_3^{(k)} \cdot \overline{\phi}_3^{(k)} \in C^2 \), at least one order of continuous differentiability higher than \( \sigma_{\gamma3}^{(k)} \).

Equations (5.59) requires that:

(a) \( \overline{q}_a^{(k)} \cdot \overline{q}_a^{(k)} \in C^1 \).

Combining the above requirements of the field equations, we have the following restrictions upon the field variables so that the differential equations can be simultaneously meaningful:

\[
\begin{align*}
\overline{V}_a^{(k)} & \in C^4 & \overline{\phi}_a^{(k)} & \in C^4 & \overline{V}_3^{(k)} & \in C^5 & \overline{\phi}_3^{(k)} & \in C^3 & \overline{\phi}_3^{(k)} & \in C^1 \\
N_{\alpha\beta}^{(k)} & \in C^3 & N_{33}^{(k)} & \in C^1 & M_{\alpha\beta}^{(k)} & \in C^3 & M_{33}^{(k)} & \in C^1 & V_\alpha^{(k)} & \in C^2 \\
\sigma_{\gamma3}^{(k)} & \in C^2 & \sigma_{33}^{(k)} & \in C^1 & \overline{q}_a^{(k)} & \in C^1 & \overline{q}_a^{(k)} & \in C^1 & \theta^{(k)} & \in C^2
\end{align*}
\]

(7.48)
CHAPTER 8

SPECIALIZATIONS OF THE VARIATIONAL FORMULATION

As discussed in Section 7.3, the highest order of continuous differentiability required for the present theory is order five. In order to find a solution for the governing function, it is advantageous to decrease the required order of differentiability. Various forms of the governing function can be obtained depending on the selection of the field variable on which the order of differentiation is to be reduced.

The complexity of the governing function depends on the requirements of the problem to be analyzed using the theory. Therefore, if certain assumptions about the behavior of the laminate are made, specialized forms of the formulation are obtained. For example, if delamination of the laminate is not considered, jump discontinuities conditions can be eliminated. This chapter presents various specializations of the formulation.
Elimination of the derivatives of the generalized force and flux variables can be achieved using existing relationships. In order to reduce the order of differentiability, the following operations are carried out:

(a) Eliminate the derivatives on $N^{(k)}_{\alpha\beta}$ using equations (7.20)

(b) Eliminate the derivatives on $M^{(k)}_{\alpha\beta}$ using equations (7.24)

(c) Eliminate the derivatives on $V^{(k)}_{\alpha}$ using equation (7.28)

(d) Eliminate the derivatives on $\bar{q}^{(k)}_{\alpha}$ using equations (7.43) and (7.44)

(e) Eliminate the derivatives on $\bar{q}^{(k)}_{\alpha}$ using equations (7.46) and (7.47).

Hence equation (7.3) can be written using the above substitutions and equations (7.14), (7.15), (7.17) and (7.18) as:

\[
\Omega_1(u, \sigma) = 2<\nabla^{(1)}_{\alpha} \cdot \bar{\sigma}^{(0)}_{\alpha\beta} >_{R^{1\prime}} + 2<\nabla^{(1)}_{\alpha} \cdot \frac{h}{2} \bar{\sigma}^{(0)}_{\alpha\beta} >_{R^{1\prime\prime}} + 2<\nabla^{(1)}_{\alpha} \cdot \bar{\sigma}^{(0)}_{\alpha\beta} >_{R^{1\prime\prime}}
\]

\[
+ 2<\nabla^{(1)}_{\alpha} \cdot \frac{h_1}{120} \bar{\sigma}^{(0)}_{\gamma\gamma\gamma} >_{R^{1\prime\prime}} + 2<\nabla^{(1)}_{\alpha} \cdot \frac{h_1^2}{10} \bar{\sigma}^{(0)}_{\gamma\gamma\gamma} >_{R^{1\prime\prime}} + 2<\nabla^{(1)}_{\alpha} \cdot \frac{h_1}{2} \bar{\sigma}^{(0)}_{\gamma\gamma\gamma} >_{R^{1\prime\prime}}
\]

\[
+ 2<N^{(1)}_{\alpha\beta} \cdot \frac{\pi_{\alpha\beta}}{2} \bar{\theta}^{(0)} >_{R^{1\prime\prime}} + 2<N^{(1)}_{\alpha\beta} \cdot \frac{1}{10} S^{(1)}_{3333} \bar{\sigma}^{(0)}_{33} + \frac{\pi^{(1)}_{33}}{2} \bar{\theta}^{(0)} >_{R^{1\prime\prime}}
\]

\[
+ 2<M^{(1)}_{\alpha\beta} \cdot \frac{\pi_{\alpha\beta}}{h_1} \bar{\theta}^{(0)} >_{R^{1\prime\prime}} + 2<M^{(1)}_{\alpha\beta} \cdot \frac{3}{7h_1} S^{(1)}_{3333} \bar{\sigma}^{(0)}_{33} + \frac{\pi^{(1)}_{33}}{h_1} \bar{\theta}^{(0)} >_{R^{1\prime\prime}}
\]

\[
+ 2<V^{(1)}_{\alpha} \cdot \frac{2}{5} S^{(1)}_{p33} \bar{\sigma}^{(0)}_{\gamma\gamma} >_{R^{1\prime\prime}} + 2<q^{(1)}_{\alpha} \cdot \frac{G}{T_e} * \frac{h_1}{4} \bar{\theta}^{(0)} >_{R^{1\prime\prime}}
\]
\[ + 2 < q^{(1)}_a \cdot \frac{g'}{T_o} - \frac{h}{4} \theta^{(0)}_\gamma > R'^k + 2 < \overline{v}_a \cdot -\hat{\sigma}^{(N)}_{\alpha_3} > R'^N + 2 < \overline{\phi}_a \cdot \frac{h_N}{2} \hat{\sigma}^{(N)}_{\alpha_3} > R'^N, \]

\[ + 2 < \overline{\nu}_3^{(N)} \cdot -\hat{\sigma}^{(N)}_{33} > R'^N + 2 < \overline{\phi}_3^{(N)} \cdot \frac{h_N}{120} \hat{\sigma}^{(N)}_{33,\gamma} - \frac{h_N^2}{10} \hat{\sigma}^{(N)}_{33} > R'^N, \]

\[ + 2 < \overline{\phi}_3^{(N)} \cdot -\frac{h_N^2}{12} \hat{\sigma}^{(N)}_{33} > R'^N + 2 < \overline{\phi}_3^{(N)} \cdot -\frac{h_N^2}{2} \hat{\sigma}^{(N)}_{33} > R'^N, \]

\[ + 2 < N^{(N)}_{33} \cdot -\frac{1}{10} S^{(N)}_{3333} \hat{\sigma}^{(N)}_{33} + \frac{\pi^{(N)}_{33}}{2} - \frac{\hat{\theta}^{(N)}}{h_N} > R'^N, \]

\[ + 2 < \hat{M}_{33}^{(N)} \cdot -\frac{3}{7h_N} S^{(N)}_{3333} \hat{\sigma}^{(N)}_{33} - \frac{\pi^{(N)}_{33}}{2} - \frac{\hat{\theta}^{(N)}}{h_N} > R'^N, \]

\[ + 2 < \overline{\theta}^{(N)}_3 > R'^N, \]

\[ + 2 < \overline{\theta}^{(N)}_3 > R'^N, \]

\[ + 2 \sum_{k=2}^N \{ < \overline{v}_a^{(k)} \cdot \sigma_{\alpha_3}^{(-k-1)} > R'^k + < \overline{\phi}_a^{(k)} \cdot \frac{h_k}{2} \sigma_{\alpha_3}^{(-k-1)} > R'^k + < \overline{\nu}_3^{(k)} \cdot \sigma_{33}^{(-k-1)} > R'^k, \]

\[ + < \overline{\phi}_3^{(k)} \cdot \frac{h_k^3}{120} \sigma_{33,\gamma}^{(-k-1)} + \frac{h_k^3}{10} \sigma_{33}^{(-k-1)} > R'^k, \]

\[ + \frac{h_k}{2} \sigma_{33}^{(-k-1)} > R'^k, + < N^{(k)}_{33} \cdot -\frac{1}{10} S^{(k)}_{3333} \sigma_{33}^{(-k-1)} + \frac{\pi^{(k)}_{33}}{2} - \frac{\theta^{(-k-1)}}{h_k} > R'^k, \]

\[ + < M^{(k)}_{33} \cdot \frac{\pi^{(k)}_{33}}{h_k} \theta^{(-k-1)} > R'^k, + < \overline{\nu}_3^{(k)} \cdot -\frac{3}{7h_k} S^{(k)}_{3333} \sigma_{33}^{(-k-1)} > R'^k, \]

\[ + \frac{\pi^{(k)}_{33}}{h_k} \theta^{(-k-1)} > R'^k, + < \overline{\nu}_3^{(k)} \cdot -\frac{2}{5} S^{(k)}_{3333} \sigma_{33}^{(-k-1)} > R'^k, \]

175
\[ + \left\{ \begin{array}{l}
+ \langle \tilde{q}_\alpha^{(k)} \cdot \frac{g'}{T_o} \frac{\hbar_k}{4} \theta_{\alpha}^{-(k-1)} \rangle_{\mathbb{R}^{(k)}} + \langle \tilde{q}_\alpha^{(k)} \cdot \frac{g'}{T_o} \frac{\hbar_k}{4} \theta_{\alpha}^{-(k-1)} \rangle_{\mathbb{R}^{(k)}} \end{array} \right\} \\
+ 2 \sum_{k=1}^{N-1} \left\{ \begin{array}{l}
+ \langle \tilde{V}_\alpha^{(k)} \cdot -\sigma_{\alpha 3}^{(k)} \rangle_{\mathbb{R}^{(k)}} + \langle \tilde{\phi}_3^{(k)} \cdot \frac{\hbar_k}{2} \sigma_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} + \langle \tilde{V}_3^{(k)} \cdot -\sigma_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} \\
+ \langle \tilde{\phi}_3^{(k)} \cdot \frac{\hbar_k}{120} \sigma_{33}^{(k)} - \frac{h_k^2}{10} \sigma_{33}^{(k)} - \frac{h_k^2}{12} \sigma_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} + \langle \tilde{\phi}_3^{(k)} \cdot -\frac{\hbar_k}{12} \sigma_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} \\
+ \frac{h_k}{2} \sigma_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} + \langle N_{ab}^{(k)} \cdot \frac{\pi_{ab}^{(k)}}{2} \theta^{-(k)} \rangle_{\mathbb{R}^{(k)}} + \langle N_{33}^{(k)} \cdot -\frac{1}{10} \langle \mathbb{S}_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} \sigma_{33}^{(k)} \\
+ \frac{h_k}{2} \sigma_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} + \langle M_{33}^{(k)} \cdot \frac{3}{7h_k} \mathbb{S}_{333}^{(k)} \sigma_{33}^{(k)} - \frac{\pi_{33}^{(k)}}{h_k} \theta^{-(k)} \rangle_{\mathbb{R}^{(k)}} \\
+ \langle \bar{V}_\alpha^{(k)} \cdot -\frac{2}{5} \mathbb{S}_{333}^{(k)} \sigma_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} + \langle \tilde{q}_\alpha^{(k)} \cdot \frac{g'}{T_o} \frac{\hbar_k}{4} \theta_{\alpha}^{-(k)} \rangle_{\mathbb{R}^{(k)}} \\
+ \langle \tilde{q}_\alpha^{(k)} \cdot \frac{g'}{T_o} \frac{\hbar_k}{4} \theta_{\alpha}^{-(k)} \rangle_{\mathbb{R}^{(k)}} \end{array} \right\} \\
+ \sum_{k=1}^{N} \left\{ \begin{array}{l}
+ \langle N_{ab}^{(k)} \cdot -\bar{V}_{ab}^{(k)} \rangle_{\mathbb{R}^{(k)}} + \frac{1}{h_k} \mathbb{S}_{ab\mu\rho}^{(k)} N_{\mu\rho}^{(k)} + \frac{1}{h_k} \mathbb{S}_{ab33}^{(k)} N_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} \\
+ \langle N_{33}^{(k)} \cdot -2 \bar{\phi}_3^{(k)} + \frac{1}{h_k} \mathbb{S}_{333}^{(k)} N_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} + \frac{6}{5h_k} \mathbb{S}_{333}^{(k)} N_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} \\
+ \langle M_{ab}^{(k)} \cdot -2 \bar{\phi}_{ab}^{(k)} + \frac{12}{h_k^3} \mathbb{S}_{ab\mu\rho}^{(k)} M_{\mu\rho}^{(k)} + \frac{12}{h_k^3} \mathbb{S}_{ab33}^{(k)} M_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} \\
+ \langle M_{33}^{(k)} \cdot -2 \bar{\phi}_3^{(k)} + \frac{12}{h_k^3} \mathbb{S}_{333}^{(k)} M_{33}^{(k)} + \frac{120}{7h_k^3} \mathbb{S}_{3333}^{(k)} M_{33}^{(k)} \rangle_{\mathbb{R}^{(k)}} \\
\end{array} \right\} \\
\right. \\
176
\[ + \langle V^{(k)}_{\alpha} \rangle - 2 \langle \bar{q}^{(k)}_{\alpha} \rangle - 2 \langle \bar{V}_{3,\alpha} \rangle + \frac{24}{5h_{k}} S^{(k)}_{\alpha3p3} V^{(k)}_{\rho} > \]
\[ + \langle \bar{q}^{(k)}_{\alpha} \rangle \cdot \frac{h_{k}}{4} \frac{g'_{\lambda \alpha \beta}}{T_{o}} \sigma^{(k)}_{\alpha \beta} > R^{(k)}_{\alpha} + \langle \bar{q}^{(k)}_{\alpha} \rangle \cdot \frac{3h_{k}}{4} \frac{g'_{\lambda \alpha \beta}}{T_{o}} \lambda \alpha \beta \bar{q}^{(k)}_{\beta} > R^{(k)}_{\alpha} \]
\[ + \sum_{k=1}^{N-1} \{ <\sigma_{\alpha^3}^{(k)} \cdot \frac{8}{15} [ h_k S_{\alpha^3 \beta^3}^{(k)} + h_{k+1} S_{\alpha^3 \beta^3}^{(k+1)} ] \sigma_{\beta^3}^{-(k)} >_{R^k} \}
\]
\[ + <\sigma_{33}^{-(k)} \cdot \frac{3}{5} [ h_k S_{3333}^{(k)} + h_{k+1} S_{3333}^{(k+1)} ] \sigma_{33}^{-(k)} >_{R^k} \]
\[ + <\theta^{-k} \cdot \frac{1}{3} [ h_k \{ \Gamma_{\alpha \beta}^{(k)} \pi_{\alpha \beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho^{(k)} \epsilon^{(k)}}{T_o} \} + h_{k+1} \{ \Gamma_{\alpha \beta}^{(k+1)} \pi_{\alpha \beta}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)} + \frac{\rho^{(k+1)} \epsilon^{(k+1)}}{T_o} \} ] \theta^{-k} >_{R^k} \}
\]
\[ + 2 \sum_{k=1}^{N-1} \{ <\theta^{-k} \cdot \frac{h_k \epsilon^{(k)}}{2 T_o} r^{(k)} >_{R^k} \} + 2 \sum_{k=1}^{N-1} \{ <\theta^{-k} \cdot \frac{h_{k+1} \epsilon^{(k+1)}}{2 T_o} r^{(k+1)} >_{R^{k+1}} \} + 2 \sum_{k=1}^{N-1} \{ <\theta^{-k} \cdot \eta^{-b} >_{R^k} \}
\]
\[ + \sum_{k=1}^{N} \{ <\overline{V}^{(k)}_{\alpha} \cdot -2 g_{\alpha}^{(k)} >_{s_1^k} + <\overline{\phi}_{\alpha}^{(k)} \cdot -2 g_{\alpha}^{(k)} >_{s_1^k} + <\overline{\nu}_{\alpha}^{(k)} \cdot 2 g_{\alpha}^{(k)} >_{s_1^k} + <\overline{\phi}_{\alpha}^{(k)} \cdot -2 g_{\alpha}^{(k)} >_{s_1^k} + <\overline{\nu}_{\alpha}^{(k)} \cdot 2 g_{\alpha}^{(k)} >_{s_1^k} + 2 <\overline{V}_{\alpha}^{(k)} \cdot \overline{\nu}_{\alpha}^{(k)} \eta_{\alpha}^{(k)} - g_{\alpha}^{(k)} >_{s_1^k} + 2 <\overline{V}_{\alpha}^{(k)} \cdot \overline{\nu}_{\alpha}^{(k)} \eta_{\alpha}^{(k)} - g_{\alpha}^{(k)} >_{s_1^k} \}
\]
\[ + <\sigma_{\alpha^3}^{(-1)} \cdot -2 g_{\alpha}^{(-1)} >_{s_{-1}^\alpha} + <\theta^{-1} \cdot -2 g_{\alpha}^{(-1)} >_{s_{-1}^\alpha} + <\sigma_{\alpha^3}^{(-N)} \cdot -2 g_{\alpha}^{(-N)} >_{s_{-N}^\alpha} + <\theta^{-N} \cdot -2 g_{\alpha}^{(-N)} >_{s_{-N}^\alpha} + \sum_{k=2}^{N-2} \{ <\sigma_{\alpha^3}^{-(k)} \cdot -2 g_{\alpha}^{(k)} >_{s_{-1}^\alpha} + <\theta^{-k} \cdot -2 g_{\alpha}^{(k)} >_{s_{-1}^\alpha}\}
\]
\[ + <\sigma_{\alpha^3}^{-(N-2)} \cdot -2 g_{\alpha}^{(N-2)} >_{s_{-N+1}^\alpha} + <\theta^{-(N-1)} \cdot -2 g_{\alpha}^{(N-1)} >_{s_{-N+1}^\alpha} + \sum_{k=1}^{N} \{ <\overline{\nu}_{\alpha}^{(k)} \cdot -2 g_{1}^{(k)} >_{s_{-1}^1} + <\overline{\phi}_{\alpha}^{(k)} \cdot -2 g_{3}^{(k)} >_{s_{-1}^3} \}
\]
Alternatively, formulations that do not contain derivatives of the kinematic variables can be obtained by carrying out the following operations:

(a) Eliminate the derivatives on $\overline{v}_a^{(k)}$ using of equation (7.5)

(b) Eliminate the derivatives on $\overline{\phi}_a^{(k)}$ using of equation (7.8)

(c) Eliminate the derivatives on $\overline{v}_3^{(k)}$ using of equation (7.11)

(d) Eliminate the derivatives on $\hat{\phi}_3^{(k)}$ using of equations (7.14) and (7.15)

(e) Eliminate the derivatives on $\hat{\phi}_3^{(k)}$ using of equations (7.17) and (7.18)

Performing these substitutions we get:

$$J_1(u, \sigma) = 2<\overline{v}_1^{(1)} \cdot \hat{\sigma}_a^{(0)} > R^{(i)} + 2<\overline{\phi}_a^{(1)} \cdot \frac{h_1}{2} \hat{\sigma}_a^{(0)} > R^{(i)} + 2<\overline{v}_3^{(1)} \cdot \hat{\sigma}_3^{(0)} > R^{(i)}$$

$$+ 2<\hat{\phi}_3^{(1)} \cdot \frac{h_1^2}{120} \hat{\sigma}_{33}^{(0)} > R^{(i)} + 2<\hat{\phi}_3^{(1)} \cdot \frac{h_1}{12} \hat{\sigma}_{33}^{(0)} > R^{(i)} + 2<\frac{\pi_{ab}^{(1)}}{2} \hat{\theta}_{ab}^{(0)} > R^{(i)} + 2<\frac{\pi_{ab}^{(1)}}{2} \hat{\theta}_{ab}^{(0)} > R^{(i)}.$$
\[ + 2 \langle M_\alpha^{(1)} \rangle \frac{\pi_\alpha^{(1)}}{h_1} \hat{\theta}^{(0)} > R^{(s)} + 2 \langle M_\alpha^{(1)} \rangle \frac{3}{7h_1} S_{3333}^{(1)} \hat{\sigma}_{33}^{(0)} + 2 \langle \pi_\alpha^{(1)} \rangle \frac{\sigma_\alpha^{(1)}}{h_1} \hat{\theta}^{(0)} > R^{(s)} \]

\[ + 2 \langle V_\alpha^{(1)} \rangle \frac{\hat{\theta}^{(0)}_T}{S^{(1)}_{\alpha\gamma}} > R^{(s)} + 2 \langle \pi_\alpha^{(1)} \rangle \frac{\hat{\theta}^{(0)}_T}{4} > R^{(s)} \]

\[ + 2 \langle \overline{q}_\alpha^{(1)} \rangle \frac{\hat{\theta}^{(0)}_T}{4} > R^{(s)} + 2 \langle \overline{v}_\alpha^{(1)} \rangle \cdot \hat{\hat{\sigma}}_{\alpha3}^{(N)} > R^{(s)} + 2 \langle \phi_\alpha^{(N)} \rangle \frac{h_N}{2} \cdot \hat{\sigma}_{\alpha3}^{(N)} > R^{(s)} \]

\[ + 2 \langle \overline{v}_3^{(N)} \rangle \cdot \hat{\hat{\sigma}}_{33}^{(N)} > R^{(s)} + 2 \langle \phi_3^{(N)} \rangle \frac{h_N}{12} \cdot \hat{\sigma}_{33}^{(N)} > R^{(s)} \]

\[ + 2 \langle \phi_3^{(N)} \rangle \frac{h_N}{12} \cdot \hat{\sigma}_{33}^{(N)} > R^{(s)} + 2 \langle \phi_3^{(N)} \rangle \frac{h_N}{10} \cdot \hat{\sigma}_{33}^{(N)} > R^{(s)} \]

\[ + 2 \langle N_3^{(N)} \rangle \cdot \hat{\sigma}_{33}^{(N)} > R^{(s)} + 2 \langle N_3^{(N)} \rangle \cdot \hat{\sigma}_{33}^{(N)} > R^{(s)} \]

\[ + 2 \langle N_1^{(N)} \rangle \cdot \hat{\sigma}_{33}^{(N)} > R^{(s)} + 2 \langle N_1^{(N)} \rangle \cdot \hat{\sigma}_{33}^{(N)} > R^{(s)} \]

\[ + 2 \langle M_3^{(N)} \rangle \cdot \hat{\sigma}_{33}^{(N)} > R^{(s)} + 2 \langle M_3^{(N)} \rangle \cdot \hat{\sigma}_{33}^{(N)} > R^{(s)} \]

\[ + 2 \langle q_\alpha^{(N)} \rangle \frac{h_N}{4} \cdot \hat{\theta}^{(N)} > R^{(s)} + 2 \langle q_\alpha^{(N)} \rangle \frac{h_N}{4} \cdot \hat{\theta}^{(N)} > R^{(s)} \]

\[ + 2 \langle q_\alpha^{(N)} \rangle \frac{h_N}{4} \cdot \hat{\theta}^{(N)} > R^{(s)} + 2 \langle q_\alpha^{(N)} \rangle \frac{h_N}{4} \cdot \hat{\theta}^{(N)} > R^{(s)} \]

\[ + \sum_{k=2}^{N} \left\{ 2 \langle v_\alpha^{(k)} \rangle \cdot \sigma_{\alpha3}^{(k-1)} > R^{(s)} + 2 \langle \phi_\alpha^{(k)} \rangle \frac{h_k}{2} \cdot \sigma_{\alpha3}^{(k-1)} > R^{(s)} + 2 \langle v_\alpha^{(k)} \rangle \cdot \sigma_{33}^{(k-1)} > R^{(s)} \right\} \]

\[ + 2 \langle \phi_3^{(k)} \rangle \frac{h_k^4}{120} \cdot \sigma_{33}^{(k-1)} + 2 \langle \phi_3^{(k)} \rangle \frac{h_k^4}{10} \cdot \sigma_{33}^{(k-1)} > R^{(s)} + 2 \langle \phi_3^{(k)} \rangle \frac{h_k^4}{12} \cdot \sigma_{33}^{(k-1)} > R^{(s)} \]

\[ + \frac{h_k}{2} \cdot \sigma_{33}^{(k-1)} > R^{(s)} + \langle N_3^{(k)} \rangle \cdot \frac{\pi_{ab}^{(k)}}{2} \cdot \theta^{(k-1)} > R^{(s)} \]

\[ + \langle N_3^{(k)} \rangle \cdot \frac{1}{10} \cdot \sigma_{33}^{(k-1)} + \langle N_3^{(k)} \rangle \cdot \frac{\pi_{33}^{(k)}}{2} \cdot \theta^{(k-1)} > R^{(s)} \]

180
\[ + \langle M_{\alpha\beta}^{(k)} \cdot \frac{\pi_{\alpha\beta}^{(k)}}{h_k} \theta^{-(k-1)} \rangle_{R^{(k)}} + \langle M_{33}^{(k)} \cdot -\frac{3}{7h_k} S_{333}^{(k)} \sigma_{33}^{(k-1)} \rangle_{R^{(k)}} \]

\[ + \frac{\pi_{33}^{(k)}}{h_k} \theta^{-(k-1)} \rangle_{R^{(k)}} + \langle V_{\alpha}^{(k)} \cdot -\frac{2}{5} S_{\rho 3}^{(k)} \sigma_{\gamma 3}^{(k-1)} \rangle_{R^{(k)}} \]

\[ + \langle \bar{q}_{\alpha}^{(k)} \cdot \frac{g'}{T_o} \frac{h_k}{4} \theta_{\gamma}^{-(k-1)} \rangle_{R^{(k)}} + \langle \bar{q}_{\alpha}^{(k)} \cdot \frac{g'}{T_o} \frac{h_k}{4} \theta_{\gamma}^{-(k-1)} \rangle_{R^{(k)}} \}

\[ + \sum_{k=1}^{N-1} \left\{ 2\langle \bar{V}_{\alpha}^{(k)} \cdot -\sigma_{\alpha 3}^{(k)} \rangle_{R^{(k)}} + 2\langle \bar{V}_{\alpha}^{(k)} \cdot \frac{h_k}{2} \sigma_{\alpha 3}^{(k)} \rangle_{R^{(k)}} + 2\langle \bar{V}_{3}^{(k)} \cdot -\sigma_{33}^{(k)} \rangle_{R^{(k)}} \right\}

\[ + 2\langle \bar{\phi}_{3}^{(k)} \cdot \frac{h_k}{120} \sigma_{33}^{(k)} \rangle_{R^{(k)}} + \langle N_{\alpha}^{(k)} \cdot \frac{\pi_{\alpha\beta}^{(k)}}{2} \theta^{-(k)} \rangle_{R^{(k)}} + \langle N_{33}^{(k)} \cdot -\frac{1}{10} S_{333}^{(k)} \sigma_{33}^{(k)} \rangle_{R^{(k)}} \]

\[ + \frac{\pi_{33}^{(k)}}{2} \theta^{-(k)} \rangle_{R^{(k)}} + \langle M_{\alpha\beta}^{(k)} \cdot \frac{\pi_{\alpha\beta}^{(k)}}{h_k} \theta^{-(k)} \rangle_{R^{(k)}} \]

\[ + \langle M_{33}^{(k)} \cdot \frac{3}{7h_k} S_{333}^{(k)} \sigma_{33}^{(k)} \cdot -\frac{\pi_{33}^{(k)}}{h_k} \theta^{-(k)} \rangle_{R^{(k)}} \]

\[ + \langle V_{\alpha}^{(k)} \cdot -\frac{2}{5} S_{\rho 3}^{(k)} \sigma_{\gamma 3}^{(k)} \rangle_{R^{(k)}} + \langle \bar{q}_{\alpha}^{(k)} \cdot \frac{g'}{T_o} \frac{h_k}{4} \theta_{\gamma}^{-(k)} \rangle_{R^{(k)}} \]

\[ + \langle \bar{q}_{\alpha}^{(k)} \cdot \frac{g'}{T_o} \frac{h_k}{4} \theta_{\gamma}^{-(k)} \rangle_{R^{(k)}} \} \]

\[ + \sum_{k=1}^{N} \left\{ 2\langle \bar{V}_{\alpha}^{(k)} \cdot N_{\alpha\beta}^{(k)} \rangle_{R^{(k)}} + 2\langle \bar{\phi}_{\alpha}^{(k)} \cdot M_{\alpha\beta}^{(k)} \rangle_{R^{(k)}} + 2\langle \bar{V}_{3}^{(k)} \cdot V_{\alpha\alpha}^{(k)} \rangle_{R^{(k)}} \right\}

\[ + \langle N_{\alpha\beta}^{(k)} \cdot \frac{1}{h_k} S_{\alpha\beta}^{(k)} N_{\gamma\gamma}^{(k)} \rangle_{R^{(k)}} + \langle N_{\alpha\beta}^{(k)} \cdot \frac{1}{h_k} S_{\alpha\beta 33}^{(k)} N_{33}^{(k)} \rangle_{R^{(k)}} \]
\[ + < N_{33}^{(k)} > - 2 \phi_{3}^{(k)} + \frac{1}{h_k} S_{330\beta}^{(k)} N_{\alpha\beta}^{(k)} + \frac{6}{5h_k} S_{3333}^{(k)} N_{33}^{(k)} >_{R^{(k)}} , \]

\[ + < M_{\alpha\beta}^{(k)} > \frac{12}{h_k} S_{\mu\nu\alpha\beta}^{(k)} M_{\mu\nu}^{(k)} + \frac{12}{h_k} S_{\alpha\beta33}^{(k)} M_{33}^{(k)} >_{R^{(k)}} , \]

\[ + < M_{33}^{(k)} > - 2 \phi_{3}^{(k)} + \frac{12}{h_k} S_{330\beta}^{(k)} M_{\alpha\beta}^{(k)} + \frac{120}{7h_k} S_{3333}^{(k)} M_{33}^{(k)} >_{R^{(k)}} , \]

\[ + < V_{\alpha}^{(k)} > - 2 \phi_{\alpha}^{(k)} + \frac{24}{5h_k} S_{\alpha\beta\rho\delta}^{(k)} V_{\rho}^{(k)} >_{R^{(k)}} , \]

\[ + < \rho_{\alpha}^{(k)} > \frac{h_k g^{'}}{4 T_0} \lambda_{\alpha\beta} \rho_{\beta}^{(k)} >_{R^{(k)}} , \]

\[ + < \rho_{\alpha}^{(k)} > \frac{3h_k g^{'}}{4 T_0} \lambda_{\alpha\beta} \rho_{\beta}^{(k)} >_{R^{(k)}} , \]

\[ + 2 \sum_{k=1}^{N} \{ < \bar{V}_{\alpha}^{(k)} > \bar{F}_{\alpha}^{(k)} >_{R^{(k)}} , + < \bar{V}_{3}^{(k)} > \bar{F}_{3}^{(k)} >_{R^{(k)}} , + < \bar{\phi}_{3}^{(k)} > \frac{h_k^2}{60} \bar{F}_{3}^{(k)} >_{R^{(k)}} . \}

\[ + 2 < \sigma_{3}^{(l)} > \frac{2}{15} h_1 S_{3303}^{(l)} \phi_{3}^{(l)} >_{R^{(l)}} , + 2 < \sigma_{33}^{(l)} > \frac{h_1}{70} S_{3333}^{(l)} \phi_{33}^{(l)} >_{R^{(l)}} , \]

\[ + 2 < \theta_{3}^{(l)} > \frac{1}{6} h_1 \{ \Gamma_{\alpha\beta}^{(l)} \pi_{\alpha\beta}^{(l)} + \Gamma_{33}^{(l)} \pi_{33}^{(l)} + \frac{\rho_{3}^{(l)} c_{3}^{(l)}}{T_0} \} \theta_{3}^{(l)} >_{R^{(l)}} . \]

\[ + 2 < \sigma_{3}^{(N-l)} > \frac{2}{15} h_N S_{3303}^{(N-l)} \phi_{3}^{(N-l)} >_{R^{(N-l)}} , + 2 < \sigma_{33}^{(N-l)} > \frac{h_N}{70} S_{3333}^{(N-l)} \phi_{33}^{(N-l)} >_{R^{(N-l)}} , \]

\[ + 2 < \theta_{3}^{(N-l)} > \frac{1}{6} h_N \{ \Gamma_{\alpha\beta}^{(N-l)} \pi_{\alpha\beta}^{(N-l)} + \Gamma_{33}^{(N-l)} \pi_{33}^{(N-l)} + \frac{\rho_{3}^{(N-l)} c_{3}^{(N-l)}}{T_0} \} \theta_{3}^{(N-l)} >_{R^{(N-l)}} . \]

\[ + \sum_{k=2}^{N-1} \{ < \sigma_{3}^{(k)} > \frac{2}{15} h_k S_{3303}^{(k)} \phi_{3}^{(k)} >_{R^{(k)}} , + < \sigma_{33}^{(k)} > \frac{h_k}{70} S_{3333}^{(k)} \phi_{33}^{(k)} >_{R^{(k)}} , \]

\[ + < \theta_{3}^{(k)} > \frac{1}{6} h_k \{ \Gamma_{\alpha\beta}^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho_{3}^{(k)} c_{3}^{(k)}}{T_0} \} \theta_{3}^{(k)} >_{R^{(k)}} . \}

182
\[ + \sum_{k=1}^{N-1} \left\{ \frac{2}{15} \sigma_{\alpha \beta}^{(k+1)} \rho_{\alpha \beta}^{(k+1)} \right\}_{R^{k+1}} \]

\[ + \frac{\sigma_{\alpha \beta}^{(k+1)}}{70} S_{\alpha \beta}^{(k+1)} \sigma_{\alpha \beta}^{(k+1)} \rightarrow R^{k+1} \]

\[ + \frac{\theta^{(k+1)}}{6} h_{k+1} \{ \gamma_{\alpha a}^{(k+1)} \pi_{\alpha a}^{(k+1)} + \gamma_{\alpha b}^{(k+1)} \pi_{\alpha b}^{(k+1)} + \frac{\rho_{\alpha a}^{(k+1)} c_{\alpha a}^{(k+1)}}{T_o} \} \theta^{(k+1)} \rightarrow R^{k+1} \} \]

\[ + \sum_{k=1}^{N-1} \left\{ \frac{8}{15} \left[ h_k S_{\alpha \beta}^{(k)} + h_{k+1} S_{\alpha \beta}^{(k+1)} \right] \sigma_{\alpha \beta}^{(k)} \rightarrow R^{k+1} \}

+ \frac{\sigma_{\alpha \beta}^{(k)}}{35} \left[ h_k S_{\alpha \beta}^{(k+1)} + h_{k+1} S_{\alpha \beta}^{(k+1)} \right] \sigma_{\alpha \beta}^{(k)} \rightarrow R^{k+1} \}

\[ + \frac{\theta^{(k+1)}}{3} h_k \{ \gamma_{\alpha a}^{(k)} \pi_{\alpha a}^{(k)} + \gamma_{\alpha b}^{(k)} \pi_{\alpha b}^{(k)} + \frac{\rho_{\alpha a}^{(k)} c_{\alpha a}^{(k)}}{T_o} \} \theta^{(k+1)} \rightarrow R^{k+1} \} \]

\[ + \sum_{k=1}^{N-1} \left\{ \frac{\pi_{\alpha a}^{(k)}}{10} S_{\alpha \beta}^{(k)} N_{\beta}^{(k)} + \frac{3}{7h_k} S_{\alpha \beta}^{(k+1)} M_{\beta}^{(k)} \rightarrow R^{k+1} \right\} \}

\[ + \frac{\pi_{\alpha a}^{(k)}}{2} N_{\beta}^{(k)} - \frac{\pi_{\alpha a}^{(k)}}{h_k} M_{\beta}^{(k)} - \frac{\pi_{\alpha a}^{(k)}}{h_k} M_{\beta}^{(k)} - \frac{\gamma^{(k)}}{T_o} \frac{h_k}{4} q_{\alpha \alpha}^{(k)} + \]

\[ \frac{\gamma^{(k)}}{T_o} \frac{h_k}{4} q_{\alpha \alpha}^{(k)} \rightarrow R^{k+1} \} \]

\[ + \sum_{k=1}^{N-1} \left\{ \frac{\sigma_{\alpha \beta}^{(k)}}{2} S_{\alpha \beta}^{(k+1)} V_{\beta}^{(k+1)} \rightarrow R^{k+1} \right\} \}

\[ + \frac{\sigma_{\alpha \beta}^{(k)}}{10} S_{\alpha \beta}^{(k+1)} N_{\beta}^{(k+1)} \rightarrow R^{k+1} \}

183
\[- \frac{3}{7} \frac{S_{333}^{(k+1)} M_{33}^{(k+1)}}{J_{333}} >_{R^{*} \alpha} \phi_{\alpha}^{(k+1)} \frac{2}{g_{\alpha}} N_{\alpha}^{(k+1)} \]

\[+ \frac{\pi_{33}^{(k+1)}}{2} N_{33}^{(k+1)} + \frac{\pi_{33}^{(k+1)}}{h_{k+1}} M_{33}^{(k+1)} + \pi_{33}^{(k+1)} M_{33}^{(k+1)} - \frac{g'}{T_{o}} \frac{h_{k+1} q_{\alpha}^{(k+1)}}{4} \]

\[\frac{2}{g_{\alpha}} N_{\alpha}^{(k+1)} >_{R^{*} \alpha} \}

\[+ 2 \sum_{k=1}^{N-1} \{ <\theta^{-1}, \frac{h_{k} g'}{2 T_{o}} f^{(k)}>_{R^{*}} \} + 2 \sum_{k=1}^{N-1} \{ <\theta^{-1}, \frac{h_{k+1} g'}{2 T_{o}} f^{(k)}>_{R^{*}} \}

\[+ 2 \sum_{k=1}^{N} \{ <\theta^{-1}, Z_{\alpha}^{(k)}>_{R^{*}} \}

\[+ \sum_{k=1}^{N} \{ 2<\bar{\nu}_{\alpha}^{(k)} - N_{\alpha}^{(k)} \eta_{\beta} - g_{1}^{(k)}>_{s_{1}^{*}} + 2<\phi_{\alpha}^{(k)} - M_{\alpha}^{(k)} \eta_{\beta} - g_{1}^{(k)}>_{s_{1}^{*}} \]

\[+ 2<\bar{\nu}_{3}^{(k)} - V_{\alpha}^{(k)} \eta_{\alpha} - g_{3}^{(k)}>_{s_{3}^{*}} + <N_{\alpha}^{(k)} - 2 g_{2}^{(k)}>_{s_{2}^{*}} \]

\[+ <M_{\alpha}^{(k)} - 2 g_{4}^{(k)}>_{s_{3}^{*}} + <V_{\alpha}^{(k)} - 2 g_{6}^{(k)}>_{s_{2}^{*}} \}

\[+ <\sigma_{\alpha}^{(-1)} - 2 g_{\alpha}^{(-1)}>_{s_{0}^{*}} + <\theta^{-1}, \frac{g'}{T_{o}} \frac{h_{1}}{q_{\alpha}^{(-1)}} \eta_{\alpha} - \frac{g'}{T_{o}} \frac{h_{1}}{q_{\alpha}^{(-1)}} \eta_{\alpha} + \frac{g'}{T_{o}} \frac{h_{2}}{q_{\alpha}^{(2)}} \eta_{\alpha} - 2 g_{\alpha}^{(-1)}>_{s_{0}^{*}} \]

\[+ \sum_{k=2}^{N-1} \{ <\sigma_{\alpha}^{(-1)} - 2 g_{\alpha}^{(-1)}>_{s_{0}^{*}} + <\theta^{-1}, \frac{h_{k}}{T_{o}} q_{\alpha}^{(-1)} \eta_{\alpha} - \frac{h_{k}}{T_{o}} q_{\alpha}^{(-1)} \eta_{\alpha} + \frac{h_{k}}{T_{o}} q_{\alpha}^{(k+1)} \eta_{\alpha} - 2 g_{\alpha}^{(k)}>_{s_{0}^{*}} \}

+ <\sigma_{\alpha}^{(-N-1)} - 2 g_{\alpha}^{(-N-1)}>_{s_{0}^{(-N-1)}}

184
\[ + <\theta^{-(N-1)}, \frac{g'}{T_o} * \frac{h_{N-1}}{4} \bar{\eta}_{\alpha}^{(N-1)} \eta_\alpha - \frac{g'}{T_o} * \frac{h_{N-1}}{4} \bar{\eta}_{\alpha}^{(N-1)} \eta_\alpha + \frac{g'}{T_o} * \frac{\bar{h}_{\alpha}}{4} \bar{\eta}_{\alpha}^{(N)} \eta_\alpha \]

\[ + \frac{g'}{T_o} * \frac{h_{\alpha}}{4} \bar{\eta}_{\alpha}^{(N)} \eta_\alpha - 2g_{\alpha}^{(N-1)} > s_{\alpha}^{N-1} \]

\[ + \sum_{k=1}^{N} \{ 2<\overline{v}_{\alpha}^{(k)}, -(N_{\alpha})^\dagger \eta_{\beta} - g_{1}^{(k)} >_{s_{\alpha}^{k}} + 2<\phi_{\alpha}^{(k)}, -(M_{\alpha})^\dagger \eta_{\beta} - g_{3}^{(k)} >_{s_{\alpha}^{k}} \]

\[ + 2<\overline{v}_{\alpha}^{(k)}, -(N_{\alpha})^\dagger \eta_{\alpha} - g_{5}^{(k)} >_{s_{\alpha}^{k}} + <N_{\alpha}^{(k)}, -2g_{2}^{(k)} >_{s_{\alpha}^{k}} \]

\[ + <M_{\alpha}^{(k)}, -2g_{4}^{(k)} >_{s_{\alpha}^{k}} + <\nu_{\alpha}^{(k)}, -2g_{6}^{(k)} >_{s_{\alpha}^{k}} \} \]

\[ + <\sigma_{\alpha}^{-(1)}, -2g_{\alpha}^{(1)} >_{s_{\alpha}^{1}} \]

\[ + <\theta^{-(1)}, \frac{g'}{T_o} * \frac{h_{1}}{4} (\bar{q}_{\alpha}^{(1)})^\dagger \eta_{\alpha} - \frac{g'}{T_o} * \frac{h_{1}}{4} (\bar{q}_{\alpha}^{(1)})^\dagger \eta_{\alpha} + \frac{g'}{T_o} * \frac{h_{2}}{4} (\bar{q}_{\alpha}^{(2)})^\dagger \eta_{\alpha} \]

\[ + \frac{g'}{T_o} * \frac{h_{2}}{4} (\bar{q}_{\alpha}^{(2)})^\dagger \eta_{\alpha} - 2g_{\alpha}^{(1)} >_{s_{\alpha}^{1}} \]

\[ + \sum_{k=2}^{N-2} \{ <\sigma_{\alpha}^{-(k)}, -2g_{\alpha}^{(k)} >_{s_{\alpha}^{k}} + <\theta^{-(k)}, \frac{g'}{T_o} * \frac{h_{k}}{4} (\bar{q}_{\alpha}^{(k)})^\dagger \eta_{\alpha} - \frac{g'}{T_o} * \frac{h_{k}}{4} (\bar{q}_{\alpha}^{(k)})^\dagger \eta_{\alpha} \]

\[ + \frac{g'}{T_o} * \frac{h_{k+1}}{4} (\bar{q}_{\alpha}^{(k+1)})^\dagger \eta_{\alpha} + \frac{g'}{T_o} * \frac{h_{k+1}}{4} (\bar{q}_{\alpha}^{(k+1)})^\dagger \eta_{\alpha} - 2g_{\alpha}^{(k)} >_{s_{\alpha}^{k}} \} \]

\[ + <\sigma_{\alpha}^{-(N-1)}, -2g_{\alpha}^{(N-1)} >_{s_{\alpha}^{N-1}} \]

\[ + <\theta^{-(N-1)}, \frac{g'}{T_o} * \frac{h_{N-1}}{4} (\bar{q}_{\alpha}^{(N-1)})^\dagger \eta_{\alpha} - \frac{g'}{T_o} * \frac{h_{N-1}}{4} (\bar{q}_{\alpha}^{(N-1)})^\dagger \eta_{\alpha} + \frac{g'}{T_o} * \frac{h_{N}}{4} (\bar{q}_{\alpha}^{(N)})^\dagger \eta_{\alpha} \]

\[ + \frac{g'}{T_o} * \frac{h_{N}}{4} (\bar{q}_{\alpha}^{(N)})^\dagger \eta_{\alpha} - 2g_{\alpha}^{(N-1)} >_{s_{\alpha}^{N-1}} \]

\[ \text{(8.2)} \]

185
8.2 FORMULATION FOR SPECIALIZED PROBLEMS

By forcing some of the field equations and/or boundary conditions to be satisfied identically, the number of field variables is reduced and some specializations of the variational formulation are realized. For the boundary value problem considered, if the set of admissible states is restricted to one that identically satisfies the constitutive equations (5.47) - (5.53) and assuming that body force and internal heat generation terms are negligible, the function $\Omega_1$ is specialized to:

$$
\Omega_2(u, \sigma) = 2<\nabla^{(1)}_\alpha \cdot \hat{\sigma}^{(0)}_{\alpha 3}>_{R^{(1)}} + 2<\phi^{(1)}_3 \cdot \frac{h_1}{2} \hat{\sigma}^{(0)}_{\alpha 3}>_{R^{(1)}} + 2<\nabla^{(1)}_3 \cdot \hat{\sigma}^{(0)}_{33}>_{R^{(1)}}
$$

$$
+ 2<\phi^{(1)}_3 \cdot \frac{h_1^2}{120} \hat{\sigma}^{(0)}_{33,3} + \frac{h_1^2}{10} \hat{\sigma}^{(0)}_{33}>_{R^{(1)}} + 2<\phi^{(1)}_3 \cdot \frac{h_1^2}{12} \hat{\sigma}^{(0)}_{33,3} + \frac{h_1}{2} \hat{\sigma}^{(0)}_{33}>_{R^{(1)}}
$$

$$
+ 2<\nabla^{(N)}_\alpha \cdot \hat{\sigma}^{(N)}_{\alpha 3}>_{R^{(N)}} + 2<\phi^{(N)}_3 \cdot \frac{h_N}{2} \hat{\sigma}^{(N)}_{\alpha 3}>_{R^{(N)}}
$$

$$
+ 2<\nabla^{(N)}_3 \cdot \hat{\sigma}^{(N)}_{33}>_{R^{(N)}} + 2<\phi^{(N)}_3 \cdot \frac{h_N^2}{120} \hat{\sigma}^{(N)}_{33,3} - \frac{h_N^2}{10} \hat{\sigma}^{(N)}_{33}>_{R^{(N)}}
$$

$$
+ 2<\phi^{(N)}_3 \cdot \frac{h_N^2}{12} \hat{\sigma}^{(N)}_{33,3} + \frac{h_N}{2} \hat{\sigma}^{(N)}_{33}>_{R^{(N)}}
$$

$$
+ 2 \sum_{k=2}^N \left\{ <\nabla^{(k)}_\alpha \cdot \sigma^{-(k-1)}_{\alpha 3}>_{R^{(k)}} + <\phi^{(k)}_3 \cdot \frac{h_k}{2} \sigma^{-(k-1)}_{\alpha 3}>_{R^{(k)}} + <\nabla^{(k)}_3 \cdot \sigma^{-(k-1)}_{33}>_{R^{(k)}}
$$

$$
+ <\phi^{(k)}_3 \cdot \frac{h_k^2}{120} \sigma^{-(k-1)}_{33,3} + \frac{h_k^2}{10} \sigma^{-(k-1)}_{33}>_{R^{(k)}} + <\phi^{(k)}_3 \cdot \frac{h_k^2}{12} \sigma^{-(k-1)}_{33,3} + \frac{h_k}{2} \sigma^{-(k-1)}_{33}>_{R^{(k)}} \right\}
$$

186
\[ + 2 \sum_{k=1}^{N-1} \left\{ \phi^{(k)}_3 \cdot \frac{h_k^3}{120} \sigma^{-1}_{\gamma \gamma} - \frac{h_k^2}{10} \sigma^{-1}_{33} > R^{(k)} \right\} \]

\[ + \phi^{(k)}_3 \cdot \frac{h_k^3}{12} \sigma^{-1}_{\gamma \gamma} \]

\[ + \frac{h_k^2}{2} \sigma^{-1}_{33} > R^{(k)} \}

\[ + \sum_{k=1}^{N} \left\{ N^{(k)}_{\alpha \beta} \cdot \frac{1}{h_k} S^{(k)}_{\alpha \mu \rho} N^{(k)}_{\mu \rho} > R^{(k)} \right\} \]

\[ + \left\{ \frac{6}{5h_k} S^{(k)}_{3333} N^{(k)}_{33} > R^{(k)} \right\} \]

\[ + \left\{ \frac{12}{h_k} S^{(k)}_{\mu \rho \alpha \beta} M^{(k)}_{\alpha \beta} > R^{(k)} \right\} \]

\[ + \left\{ \frac{120}{7h_k} S^{(k)}_{3333} M^{(k)}_{33} > R^{(k)} \right\} \]

\[ + \left\{ \frac{24}{5h_k} S^{(k)}_{\alpha 3 \rho} V^{(k)}_\rho > R^{(k)} \right\} \]

\[ + \left\{ \frac{3h_k g'}{4T_o} \lambda_{\alpha \beta} \tilde{q}^{(k)}_\beta > R^{(k)} \right\} \]

\[ + \left\{ \frac{3h_k g'}{4T_o} \lambda_{\alpha \beta} \tilde{q}^{(k)}_\beta > R^{(k)} \right\} \]

\[ + 2<\sigma^{(1)}_{33} \cdot \frac{2}{15} h_1 S^{(1)}_{\alpha 3 \rho} \hat{\sigma}^{(0)}_\rho > R^{(i)} \right\} + 2<\sigma^{(1)}_{33} \cdot \frac{h_1}{70} S^{(1)}_{3333} \hat{\sigma}^{(0)}_3 > R^{(i)} \right\} \]

\[ + 2<\theta^{(-1)} \cdot \frac{1}{6} h_1 (\Gamma^{(1)}_{\alpha \beta} \pi^{(1)}_{\alpha \beta} + \Gamma^{(1)}_{33} \pi^{(1)}_{33} + \rho^{(1)}_{\alpha \beta} C^{(1)}_{\alpha \beta}) \hat{\theta}^{(0)} > R^{(i)} \right\} \]

\[ + 2<\theta^{(-N+1)} \cdot \frac{2}{15} h_N S^{(N)}_{\alpha 3 \rho} \hat{\sigma}^{(N)}_\rho > R^{(N+1)} \right\} + 2<\sigma^{(N-1)}_{33} \cdot \frac{h_N}{70} S^{(N)}_{3333} \hat{\sigma}^{(N)}_3 > R^{(N)} \right\} \]

\[ + 2<\theta^{(-N+1)} \cdot \frac{1}{6} h_N (\Gamma^{(N)}_{\alpha \beta} \pi^{(N)}_{\alpha \beta} + \Gamma^{(N)}_{33} \pi^{(N)}_{33} + \rho^{(N)}_{\alpha \beta} C^{(N)}_{\alpha \beta}) \hat{\theta}^{(N)} > R^{(N)} \right\} \]

\[ + \sum_{k=2}^{N-1} \left\{ <\sigma^{(k)}_{\alpha \beta} \cdot \frac{2}{15} h_k S^{(k)}_{\alpha 3 \rho} \sigma^{(k-1)}_{\rho 3} > R^{(k)} \right\} + <\sigma^{(k)}_{33} \cdot \frac{h_k}{70} S^{(k)}_{3333} \sigma^{(k-1)}_{33} > R^{(k)} \right\} \]
\[
+ \langle \theta^{-t(k)} \rangle, -\frac{1}{6} h_k \left( \Gamma_{\alpha\beta}^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho^{(k)} c^{(k)}}{T_0} \right) \theta^{-t(k-1)} > R^{k+1} \}
\]
\[
+ \sum_{k=1}^{N-2} \left\{ \langle \sigma_{\alpha3}^{(k)} \cdot -\frac{2}{15} h_{k+1} S_{\alpha3\beta3}^{(k+1)} \sigma_{\beta3}^{(k+1)} > R^{k+1} \rangle, + \langle \sigma_{33}^{(k)} \cdot \frac{h_{k+1}}{70} S_{3333}^{(k+1)} \sigma_{33}^{(k+1)} > R^{k+1} \rangle \right\}
\]
\[
+ \theta^{-t(k)} \cdot -\frac{1}{6} h_{k+1} \left( \Gamma_{\alpha\beta}^{(k+1)} \pi_{\alpha\beta}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)} + \frac{\rho^{(k+1)} c^{(k+1)}}{T_0} \right) \theta^{-t(k)} > R^{k+1} \}
\]
\[
+ \sum_{k=1}^{N-1} \left\{ \langle \sigma_{\alpha3}^{(k)} \cdot \frac{8}{15} \left[ h_k S_{\alpha3\beta3}^{(k)} + h_{k+1} S_{\alpha3\beta3}^{(k+1)} \right] \sigma_{\beta3}^{(k)} > R^{k+1} \rangle, + \langle \sigma_{33}^{(k)} \cdot \frac{3}{35} \left[ h_k S_{33\beta3}^{(k)} + h_{k+1} S_{33\beta3}^{(k+1)} \right] \sigma_{33}^{(k)} > R^{k+1} \rangle \right\}
\]
\[
+ \theta^{-t(k)} \cdot -\frac{1}{3} \left[ h_k \left\{ \Gamma_{\alpha\beta}^{(k)} \pi_{\alpha\beta}^{(k)} + \Gamma_{33}^{(k)} \pi_{33}^{(k)} + \frac{\rho^{(k)} c^{(k)}}{T_0} \right\} \right.
+ h_{k+1} \left\{ \Gamma_{\alpha\beta}^{(k+1)} \pi_{\alpha\beta}^{(k+1)} + \Gamma_{33}^{(k+1)} \pi_{33}^{(k+1)} + \frac{\rho^{(k+1)} c^{(k+1)}}{T_0} \right\} \left| \theta^{-t(k)} > R^{k+1} \right\}
\]
\[
+ 2 \sum_{k=1}^{N-1} \left\{ \langle \theta^{-t(k)} \rangle, Z_b^{(k)} > R^{k+1} \right\}
\]
\[
+ \sum_{k=1}^{N} \left\{ \langle \overline{\nabla}_{\alpha}^{(k)} \cdot -2 g_{\alpha}^{(k)} >_{s_{18}}, + \langle \overline{\phi}_{\alpha}^{(k)} \cdot -2 g_{\phi}^{(k)} >_{s_{18}}, + \langle \overline{\phi}_{3}^{(k)} \cdot -2 g_{3}^{(k)} >_{s_{18}}, + 2 < N_{\alpha\beta}^{(k)} \cdot \overline{\nabla}_{\alpha}^{(k)} \eta_{\beta} - g_{2}^{(k)} >_{s_{18}}, + 2 < M_{\alpha\beta}^{(k)} \cdot \overline{\phi}_{\alpha}^{(k)} \eta_{\beta} - g_{2}^{(k)} >_{s_{18}}, + 2 < N_{3}^{(k)} \cdot \overline{\nabla}_{3}^{(k)} \eta_{3} - g_{2}^{(k)} >_{s_{18}}, + 2 < V_{\alpha}^{(k)} \cdot \overline{\nabla}_{3}^{(k)} \eta_{\alpha} - g_{2}^{(k)} >_{s_{18}}, + 2 < \sigma_{\alpha3}^{(l)} \cdot -2 g_{\sigma}^{(l)} >_{s_{18}}, + \theta^{-t(l)} \cdot -2 g_{\sigma}^{(l)} >_{s_{18}} \right\}
\]

188
\begin{align*}
+ \sum_{k=2}^{N} \left\{ <\sigma_{\alpha 3}^{(k)}, -2g_{\sigma}^{(k)}>_{\gamma_{1}}, + <\theta^{(k)}, -2g_{\sigma}^{(k)}>_{\gamma_{1}} \right\} \\
+ <\sigma_{\alpha 3}^{(N-1)}, -2g_{\sigma}^{(N-1)}>_{\gamma_{N-1}}, + <\theta^{(N-1)}, -2g_{\sigma}^{(N-1)}>_{\gamma_{N-1}} \\
+ \sum_{k=1}^{N} \left\{ <\nu_{\alpha}^{(k)}, -2g_{1}^{(k)}>_{\gamma_{1}}, + <\phi_{\alpha}^{(k)}, -2g_{3}^{(k)}>_{\gamma_{1}} \right\} \\
+ <\nu_{3}^{(k)}, -2g_{5}^{(k)}>_{\gamma_{1}}, + 2<N_{\alpha \beta}^{(k)}(\nu_{\alpha}^{(k)})\eta_{\beta} - g_{2}^{(k)}>_{\gamma_{1}}, \\
+ 2<M_{\alpha \beta}^{(k)}(\phi_{\alpha}^{(k)})\eta_{\beta} - g_{4}^{(k)}>_{\gamma_{1}}, + 2<V_{\alpha}^{(k)}(\nu_{3}^{(k)})\eta_{\alpha} - g_{6}^{(k)}>_{\gamma_{1}} \right\} \\
+ <\sigma_{\alpha 3}^{(-1)}, -2g_{\sigma}^{(-1)}>_{\gamma_{0}}, + <\theta^{(-1)}, -2g_{\sigma}^{(-1)}>_{\gamma_{0}} \\
+ \sum_{k=2}^{N-2} \left\{ <\sigma_{\alpha 3}^{(-1)}, -2g_{\sigma}^{(-1)}>_{\gamma_{1}}, + <\theta^{(-1)}, -2g_{\sigma}^{(-1)}>_{\gamma_{1}} \right\} \\
+ <\sigma_{\alpha 3}^{(N-1)}, -2g_{\sigma}^{(N-1)}>_{\gamma_{N-1}}, + <\theta^{(N-1)}, -2g_{\sigma}^{(N-1)}>_{\gamma_{N-1}}. \tag{8.3}
\end{align*}

Here $N_{\alpha \beta}^{(k)}$, $N_{33}^{(k)}$, $M_{\alpha \beta}^{(k)}$, $M_{33}^{(k)}$, $V_{\alpha}^{(k)}$, $\phi_{\alpha}^{(k)}$, and $\overline{\phi}_{\alpha}^{(k)}$ are not independent variables but are defined by $\nu_{\alpha}^{(k)}$, $\phi_{\alpha}^{(k)}$, $\overline{\nu}_{3}^{(k)}$, $\overline{\phi}_{3}^{(k)}$, $\overline{\phi}_{\alpha}^{(k)}$, $\sigma_{13}^{(k)}$, and $\theta^{(k)}$ through the constitutive relations (5.47) - (5.53). If $\nu_{\alpha}^{(k)}$, $\phi_{\alpha}^{(k)}$, $\overline{\nu}_{3}^{(k)}$, $\overline{\phi}_{3}^{(k)}$, $\overline{\phi}_{\alpha}^{(k)}$, $\sigma_{13}^{(k)}$, and $\theta^{(k)}$ are restricted to being continuous across all internal boundaries, the physical problem has no discontinuities, i.e., $g_{i}^{(k)}$ vanish. Satisfying identically the displacement boundary conditions, (6.60) - (6.62) for $S_{u}^{(k)}$ and for $S_{w}^{(k)}$, the traction-free boundary conditions, and substituting the appropriate expressions from the constitutive equations, $\Omega_{2}$ leads to:

189
\[
\Omega_\tau(u, \sigma) = 2\langle \tilde{v}_\alpha^{(1)} , \hat{\sigma}_{a3}^{(0)} \rangle_{R^1} + 2\langle \tilde{\phi}_\alpha^{(1)} , \frac{h_1}{2} \hat{\sigma}_{a3}^{(0)} \rangle_{R^1} + 2\langle \tilde{v}_3^{(1)} , \hat{\sigma}_{33}^{(0)} \rangle_{R^1} \\
+ 2\langle \tilde{\phi}_3^{(1)} , \frac{h_1}{120} \hat{\sigma}_{\gamma33}^{(0)} + \frac{h_1^2}{10} \hat{\sigma}_{33}^{(0)} \rangle_{R^1} + 2\langle \tilde{\phi}_3^{(1)} , \frac{h_1}{12} \hat{\sigma}_{\gamma33}^{(0)} + \frac{h_1}{2} \hat{\sigma}_{33}^{(0)} \rangle_{R^1} \\
+ 2\langle \tilde{v}_3^{(N)} , \hat{\sigma}_{33}^{(N)} \rangle_{R^N} + 2\langle \tilde{\phi}_3^{(N)} , \frac{h_N}{2} \hat{\sigma}_{a3}^{(N)} \rangle_{R^N} \\
+ 2\langle \tilde{v}_3^{(N)} , \hat{\sigma}_{33}^{(N)} \rangle_{R^N} + 2\langle \tilde{\phi}_3^{(N)} , \frac{h_N}{120} \hat{\sigma}_{\gamma33}^{(N)} - \frac{h_N^2}{10} \hat{\sigma}_{33}^{(N)} \rangle_{R^N} \\
+ 2\langle \tilde{\phi}_3^{(N)} , -\frac{h_N}{12} \hat{\sigma}_{\gamma33}^{(N)} + \frac{h_N}{2} \hat{\sigma}_{33}^{(N)} \rangle_{R^N}, \\
+ \sum_{k=2}^{N} \left\{ \langle \tilde{v}_\alpha^{(k)} , \sigma_{a3}^{-(k-1)} \rangle_{R^k}, + \langle \tilde{\phi}_\alpha^{(k)} , \frac{h_k}{2} \sigma_{a3}^{-(k-1)} \rangle_{R^k}, + \langle \tilde{v}_3^{(k)} , \sigma_{33}^{-(k-1)} \rangle_{R^k}, \\
+ \langle \tilde{\phi}_3^{(k)} , \frac{h_k^2}{12} \sigma_{\gamma33}^{-(k-1)} + \frac{h_k}{10} \sigma_{33}^{-(k-1)} \rangle_{R^k}, + \langle \tilde{\phi}_3^{(k)} , \frac{h_k}{12} \sigma_{\gamma33}^{-(k-1)} \rangle_{R^k} \right\} \\
+ \sum_{k=1}^{N-1} \left\{ \langle \tilde{v}_\alpha^{(k)} , \sigma_{a3}^{-(k)} \rangle_{R^k}, + \langle \tilde{\phi}_\alpha^{(k)} , \frac{h_k}{2} \sigma_{a3}^{-(k)} \rangle_{R^k}, + \langle \tilde{v}_3^{(k)} , \sigma_{33}^{-(k)} \rangle_{R^k}, \\
+ \langle \tilde{\phi}_3^{(k)} , \frac{h_k^2}{12} \sigma_{\gamma33}^{-(k)} - \frac{h_k}{10} \sigma_{33}^{-(k)} \rangle_{R^k}, + \langle \tilde{\phi}_3^{(k)} , \frac{h_k}{12} \sigma_{\gamma33}^{-(k)} \rangle_{R^k} \right\} \\
+ \sum_{k=1}^{N} \langle N_{ab}^{(k)} , \frac{\pi_{ab}^{(k)}}{2} \theta^{-(k-1)} \rangle_{R^k}, + \langle N_{33}^{(k)} , -\frac{1}{10} S_{3333}^{(k)} \sigma_{33}^{-(k-1)} \rangle_{R^k} + \langle \frac{\pi_{ab}^{(k)}}{2} \theta^{-(k-1)} \rangle_{R^k}, \\
+ \langle M_{ab}^{(k)} , \frac{\pi_{ab}^{(k)}}{h_k} \theta^{-(k-1)} \rangle_{R^k}, + \langle M_{33}^{(k)} , -\frac{3}{7h_k} S_{3333}^{(k)} \sigma_{33}^{-(k-1)} \rangle_{R^k}. \right\} 
\]
\[ + \frac{\pi^{(k)}}{h_k} \theta^{-(k-1)} >_{R^k}, + < V^{(k)}_\alpha > \frac{2}{5} S^{(k)}_{\rho y 3} \sigma^{-(k-l)}_{\gamma 3} >_{R^k}, \]

\[ + < q^{(k)}_\alpha \cdot \frac{g'}{T} \cdot \frac{h_k}{4} \theta^{-(k-l)} >_{R^k}, + < q^{(k)}_\alpha \cdot \frac{g'}{T} \cdot \frac{h_k}{4} \theta^{-(k-l)} >_{R^k}, \}

\[ + \sum_{k=1}^N \left\{ < N^{(k)}_{\alpha \beta} \cdot \frac{\pi^{(k)}_{\alpha \beta}}{2} \theta^{-(k)}>_{R^k}, + < N^{(k)}_{33} \cdot \frac{1}{10} S^{(k)}_{3333} \sigma^{-(k)}_{33} >_{R^k}, + \frac{\pi^{(k)}_{33}}{2} \theta^{-(k)}>_{R^k}, \right. \]

\[ + < M^{(k)}_{\alpha \beta} \cdot \frac{\pi^{(k)}_{\alpha \beta}}{h_k} \theta^{-(k)}>_{R^k}, + < M^{(k)}_{33} \cdot \frac{3}{7h_k} S^{(k)}_{3333} \sigma^{-(k)}_{33} >_{R^k}, - \frac{\pi^{(k)}_{33}}{h_k} \theta^{-(k)}>_{R^k}, \]

\[ + < V^{(k)}_\alpha > \frac{2}{5} S^{(k)}_{\rho y 3} \sigma^{-(k)}_{\gamma 3} >_{R^k}, + < q^{(k)}_\alpha \cdot \frac{g'}{T} \cdot \frac{h_k}{4} \theta^{(k)}>_{R^k}, \]

\[ + < q^{(k)}_\alpha \cdot \frac{g'}{T} \cdot \frac{h_k}{4} \theta^{-(k)}>_{R^k}, \} \]

\[ + \sum_{k=1}^N \left\{ < N^{(k)}_{\alpha \beta} \cdot < V^{(k)}_{(\alpha \beta)}>_{R^k}, + < N^{(k)}_{33} \cdot \frac{1}{2} \phi^{(k)}_{\rho 3} >_{R^k}, + < M^{(k)}_{\alpha \beta} \cdot \frac{1}{2} \phi^{(k)}_{\alpha \beta} >_{R^k}, \right. \]

\[ + < M^{(k)}_{33} \cdot \frac{3}{7h_k} \phi^{(k)}_{33} >_{R^k}, + < V^{(k)}_\alpha \cdot \frac{3h_k}{4} \frac{g'}{T} \phi^{(k)}_\alpha >_{R^k}, \]

\[ + < q^{(k)}_\alpha \cdot \frac{h_k}{4} \frac{g'}{T} \lambda^{(k)}_{\alpha \beta} >_{R^k}, + < q^{(k)}_\alpha \cdot \frac{3h_k}{4} \frac{g'}{T} \lambda^{(k)}_{\alpha \beta} >_{R^k}, \} \]

\[ + 2 < \sigma^{(i)}_{a3} \cdot \frac{2}{15} h_l S^{(i)}_{a3p3} \Phi^{(0)}_{\rho 3} >_{R^{ii}}, + 2 < \sigma^{(i)}_{33} \cdot \frac{h_l}{70} S^{(i)}_{3333} \Phi^{(0)}_{33} >_{R^{ii}}, \]

\[ + 2 < \theta^{-(i)} \cdot \frac{1}{6} h_l (\Gamma^{(i)}_{\alpha \beta} \pi^{(i)}_{\alpha \beta} + \Gamma^{(i)}_{33} \pi^{(i)}_{33} + \rho^{(i)}_{C}^{(i)} \frac{C^{(i)}}{T}) \Phi^{(0)} >_{R^{ii}} \]
\[ \sum_{i=1}^{\infty} \left\{ \begin{array}{l} i \cdot \theta \cdot \text{d} + \frac{1}{(1+\eta)^3} \cdot \eta - 1 \\ \end{array} \right. \]
Substituting the constitutive equations (5.47) - (5.53) for the quantities \( N^{(k)}_{\alpha\beta} \), \( N_{33}^{(k)} \), \( M_{\alpha\beta}^{(k)} \), \( M_{33}^{(k)} \), \( {\bar{V}}_{\alpha}^{(k)} \), \( \bar{q}_{\alpha}^{(k)} \), and \( \bar{q}_{\alpha}^{(k)} \) in \( \Omega_2 \) results in reducing the independent field variables to \( \bar{V}_{\alpha}^{(k)} \), \( \bar{\phi}_{3}^{(k)} \), \( \bar{\phi}_{13}^{(k)} \), \( \sigma_{i3}^{(k)} \), and \( \theta^{(k)} \). The number of independent field variables for \( \Omega_3 \) is \( 11N+4 \) (\( N \) is the number of layers) compared with \( 13N \) for Pagano (1978a, b) for obtaining numerical solutions for an isothermal static problem.

The dependent field variables \( N^{(k)}_{\alpha\beta} \), \( N_{33}^{(k)} \), \( M_{\alpha\beta}^{(k)} \), \( M_{33}^{(k)} \), \( {\bar{V}}_{\alpha}^{(k)} \), \( \bar{q}_{\alpha}^{(k)} \), and \( \bar{q}_{\alpha}^{(k)} \) are defined in terms of the kinematic and transverse stress variables by using the constitutive relations (5.47) - (5.53) and upon appropriate rearrangement of these relations, we get:

\[
N_{\mu\nu}^{(k)} = E_{\alpha\beta\mu\nu}^{(k)} \left[ \frac{h_k}{2} \pi^{(k)}_{\alpha\beta} \left( \theta^{-(k-1)} + \theta^{-(k)} \right) \right] \]

\[
N_{33}^{(k)} = E_{3333}^{(k)} \left[ -\frac{5h_k}{6} \phi_{3}^{(k)} - \frac{5}{6} \pi_{3333}^{(k)} N_{\nu\rho}^{(k)} + \frac{h_k}{12} \pi_{3333}^{(k)} \left( \sigma_{33}^{-(k-1)} + \sigma_{33}^{-(k)} \right) \right] \]

\[
M_{\mu\nu}^{(k)} = E_{\alpha\beta\mu\nu}^{(k)} \left[ \frac{h_k}{12} \phi_{\alpha\beta}^{(k)} - \frac{h_k}{12} \pi_{3333}^{(k)} M_{33}^{(k)} + \frac{h_k^2}{30} \pi_{3333}^{(k)} \left( \sigma_{33}^{-(k-1)} + \sigma_{33}^{-(k)} \right) \right] \]

\[
M_{33}^{(k)} = E_{3333}^{(k)} \left[ -\frac{7h_k^2}{120} \bar{q}_{3}^{(k)} - \frac{7}{10} \pi_{3333}^{(k)} \pi_{3333}^{(k)} + \frac{h_k^2}{40} \pi_{3333}^{(k)} \left( \sigma_{33}^{-(k-1)} + \sigma_{33}^{-(k)} \right) \right] \]

193
\[ V_{\gamma}^{(k)} = E_{\alpha_3\gamma_3}^{(k)} \left[ \frac{5h_k}{24} \overline{V}_{\gamma,a}^{(k)} + \frac{5h_k}{24} \overline{\phi}_{\alpha}^{(k)} + \frac{h_k}{12} S_{\beta_\gamma 3}^{(k)} \left[ \sigma_{\gamma_3}^{(k-1)} + \sigma_{\gamma_3}^{(k)} \right] \right] \quad (8.9) \]

\[ \overline{q}_{\alpha}^{(k)} = - \frac{g'}{T} \left[ K_{\alpha\beta}^{(k)} (\theta_{\alpha}^{(k-1)} + \theta_{\beta}^{(k)}) \right] \quad (8.10) \]

\[ \overline{q}_{\alpha}^{(k)} = - \frac{g'}{T} \left[ \frac{1}{3} K_{\alpha\beta}^{(k)} (\theta_{\alpha}^{(k-1)} - \theta_{\beta}^{(k)}) \right] \quad (8.11) \]

where \([ E_{\alpha\beta\gamma}^{(k)} ] invitations \) and \([ S_{\alpha\beta\gamma}^{(k)} ] invitations \)

The governing function given by equation (8.4) may be used to analyze laminated plates of monoclinic material under combined mechanical and thermal loads if the conditions set forth in the specializations are valid. By reducing the required order of differentiability, the admissible function space is extended to include lower order functions. The specializations do limit the problem to a laminated plate in which there are no pre-existing delaminations.

In order to analyze laminates with delaminations, it would be necessary to include the jump discontinuity terms in the formulation. The specializations of the governing function have led to a reduction in the number of field variables from 25N+4 to 11N+4. However, equation 7.3 is the most general form of the governing function. It provides the explicit form of the governing function for the initial boundary value problem including the governing coupled equations for the body, jump discontinuity conditions, initial conditions and boundary conditions.
SUMMARY AND CONCLUSIONS

In the research presented herein, a fully coupled theory capable of accurately predicting the localized thermomechanical stress fields in laminates subjected to combined loads has been developed. The starting point for the theoretical model was an isothermal stress based formulation [Pagano (1978a,b) and Chyou (1989)] that has been shown to accurately predict interlaminar stress fields in regions of high stress gradients. To be useful, the theoretical model developed incorporates realistic thermomechanical stress distributions within and across each lamina, as well as the coupling contributions of the deformation/stress fields and vice versa. The stress based model is formulated on an assumed linear in-plane stress distribution through the lamina thickness. The theory was derived assuming isothermal material properties with temperature effects confined to heat conduction.

The general variational principle based on the self-adjoint operator matrices and the consistent boundary operator matrices was specialized to reduce the number of
independent field variables. In the development of the coupled thermomechanical laminated plate theory described in this dissertation the following were accomplished:

1. For an assumed linear in-plane stress distribution, expressions for the remaining stress components to satisfy the equations of equilibrium were derived.

2. The governing field equations for the coupled thermomechanical model were derived using Hamilton's variational principle.

3. The governing plate equations coupled with thermal effects were restated in a self-adjoint form and generalized for a laminated plate including continuous displacements, temperatures, and transverse stresses at the laminae interfaces.

4. A governing function was derived and specialized based on specific assumptions.

The thermomechanical plate theory developed in this dissertation is general and can account for interlaminar discontinuities (e.g., manufacturing defects and delaminations). The model incorporates thermal coupling terms in the mechanical constitutive equations as well as the coupling term in the thermal balance expression. The ability to accurately model the coupled thermomechanical stress and displacement behavior of laminated composite plates and the ability to account for the delaminations suggest future application in the area of damage detection and prediction.

Many applications and extensions to the present model are evident. The present model accounts for composite laminates under a static mechanical load coupled with transient heat conduction effects. The development of a coupled
thermomechanical model to include inertia effects is required to model low velocity impact events. To fully characterize a combined thermal load and dynamic mechanical load event, the inclusion of a method to predict impactor-induced surface pressure distribution and a suitable stress based failure criterion should be included in the analysis to model damage progression.

In some circumstances nonlinear effects may become important. Under these conditions, it may be necessary to model large deformations as well as nonlinear material behavior. Depending on the required degree of accuracy and, in general, the larger the temperature deviation from room temperature, it may also become important to incorporate temperature dependent material properties into the model.

To accurately predict the thermal stresses in a laminated composite, the thermal response of the material must first be determined. In this dissertation, the formulation was limited to heat conduction only. Most practical heat transfer problems involve at least two and sometimes all three modes of heat transfer (conduction, convection, and radiation) and future advances of the model should incorporate these additional processes.

The model and its extensions need to be implemented into a finite element program for a rigorous analysis. The computational effort of any such effort also requires investigations to achieve a computationally efficient numerical model which can predict the coupled stress distribution and ensuing damage progression in a laminated composite subjected to combined heat transfer processes and as well as
dynamic mechanical loading. Further, the model needs to be experimentally verified for transverse stresses and damage prediction under various loading conditions.
BIBLIOGRAPHY


Levy, pg. 113 of Timoshenko and Woinowsky-Krieger (1959) referenced herein.


209


