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FINITE GROUPS
OF CHAIN DIFFERENCE ONE

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
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* * * * *

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ABSTRACT

In a finite group, connections between the lattice of subgroups and other group theoretic properties of the group have been studied by numerous researchers. This dissertation investigates groups in which the lengths of any two unrefinable subgroup chains differ by at most one. In 1991, Brewster, Ward, and Zimmermann published a paper entitled “Finite groups of chain difference one”. Their approach was to analyze simple groups using the classification of finite simple groups.

In this dissertation we take a different approach to the classification of groups of chain difference one. In the case $G$ is a nonsolvable group, an inductive proof relying primarily on Alperin's Fusion Theorem establishes that such a group has either a dihedral or semidihedral Sylow 2-subgroup. At this point deep results of Gorenstein-Walter and Alperin-Bender-Glauberman allow us to make the final identification. In the solvable case, elementary group theory techniques are used.
Dedicated to Mike, Anna, and Ben
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Connections between the lattice of subgroups of a finite group and other group theoretic properties of the group have been studied by numerous researchers, notably M. Suzuki [Su3]. One such property is the difference between the lengths of a longest and shortest unrefinable chain in \( G \). If \( G \) is a finite group we denote by \( l(G) \) and \( \lambda(G) \) respectively the lengths of the longest and shortest unrefinable chains in the lattice of subgroups of \( G \) and we set

\[
\text{cd}(G) = l(G) - \lambda(G).
\]

We call this the chain difference of the finite group \( G \).

**Main Theorem.** Let \( G \) be a finite group with \( \text{cd}(G) = 1 \). Then one of the following holds:

1. \( G/O_r(G) \) is supersolvable for some prime \( r \) and \( G \) is \( p \)-nilpotent for \( p \) the smallest prime divisor of \( |G| \); or
2. \( O^2(G) = \langle O_2(G) \times O_2'(G), x \rangle \) with \( O_2(G) \cong Z_2 \times Z_2 \) or \( Q_8 \), and \( x^3 \in O_2'(G) \) and \( [O_2(G), x] = O_2(G) \); or
3. \( E(G) \) is quasisimple, \( G/E(G) \) is supersolvable and \( E(G)/Z(E(G)) \) is isomorphic to \( A_5, A_6 \) or to \( PSL_2(p) \) for some odd prime \( p \geq 13 \). Moreover if \( E(G)/Z(E(G)) \cong A_6 \), the \( G/C_G(E(G)) \cong A_6 \) or \( S_6 \).
Historically the first result in the specific direction of this dissertation was the following theorem of Iwasawa [Iw].

**Theorem 1.1.** A finite group $G$ is supersolvable if and only if all unrefinable subgroup chains in $G$ have the same length.

Fifty years after Iwasawa proved his result Brewster, Ward, and Zimmermann published an article entitled “Finite simple groups of chain difference one” [BWZ]. This article investigated groups in which the lengths of any two unrefinable subgroup chains differ by at most one. This proof relied on the Classification of the Finite Simple Groups. They proved the only such simple groups are $A_6$ and $PSL_2(p)$ for certain primes $p$. In particular all simple groups of chain difference one have dihedral Sylow 2 subgroups. This dissertation establishes this result in an elementary way and then invokes the deep classification theorem by Gorenstein and Walter [GW] to complete the identification of $G$.

Brewster, Ward, and Zimmermann prove the following theorem.

**Theorem 1.2.** Any nonabelian simple group of chain difference one is isomorphic to $PSL_2(q)$ for some prime power $q$. Moreover $PSL_2(q)$ has chain difference one if and only if

1. $q = 4$ or 9, or
2. $q$ is an odd prime, $5|q^2 - 1$ or $16|q^2 - 1$, and $3 \leq \Omega(q \pm 1) \leq 4$, or
3. $q$ is an odd prime, and the conditions $5 \nmid q^2 - 1$ and $16 \nmid q^2 - 1$ hold (equivalently, $q \equiv 3, 13, 27, 37(\text{mod} 40)$, and $2 \leq \Omega(q \pm 1) \leq 3$.

Their approach was to analyze simple groups family by family relying on the classification of finite simple groups.
In this dissertation we take a different approach to the classification of groups of chain difference one. The proof divides into two cases: $G$ solvable and $G$ nonsolvable. In the case $G$ is a solvable group, we use elementary group theory techniques to prove the following theorem.

**Theorem 1.3.** Let $G$ be a finite solvable group of chain difference one. Let $p$ be the smallest prime divisor of $G$. Then either $G$ is $p$-nilpotent or $p = 2$ and the following hold.

1. There exists $U \trianglelefteq G$ with $U \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $U \cong Q_8$;
2. $G/U$ is supersolvable; and
3. $O^2(G) = \langle U \times O_2'(G), x \rangle$ with $x^3 \in O_2'(G)$ and $U = [U, x]$.

In the case $G$ is nonsolvable, an inductive proof is used to prove the following.

**Theorem 1.4.** Let $G$ be a finite group of chain difference one. Then one of the following holds.

1. $G$ is solvable; or
2. $E(G)/Z(E(G))$ is isomorphic to $A_6$ or $PSL_2(p)$ for some prime $p$ satisfying the conditions of Theorem 1.2 and $G/E(G)$ is supersolvable.

In fact an elementary proof, relying primarily on Alperin’s Fusion Theorem establishes the following principal reduction.

**Proposition 1.5.** Let $G$ be a minimal counterexample to Theorem 1.4. Then $G$ is a simple group with a dihedral or semidihedral Sylow $2$-subgroup.

At this point deep results of Gorenstein-Walter [GW] and Alperin-Brauer-Gorenstein [ABG] permit the final identification of $G$. We remark that Bender
and Glauberman have obtained a comparatively short proof of the Dihedral Theorem [BG] [Be].

Notation and Preliminaries.

Let $G$ be a finite group and let $K \leq H \leq G$. We call $C : K = K_0 \leq K_1 \leq \cdots \leq K_n = H$ a chain of subgroups from $K$ to $H$ if each $K_i$ is a subgroup of $H$ and $K_i \neq K_j$ for all $i \neq j$. If $C$ is a chain from $< 1 >$ to $G$, we call $C$ a chain in $G$. For $C$ as above we say that the length $l(C)$ of the chain $C$ is $n$. A chain is said to be unrefinable if $K_i$ is maximal in $K_{i+1}$ for each $i$, $0 \leq i \leq n-1$.

**Definition 1.6.** Let $C^* = \{ C : C$ is an unrefinable chain in $G. \}$. We denote by $\lambda(G)$ the minimum value of $l(C)$ for $C \in C^*$; and we denote by $l(G)$ the maximum value of $l(C)$ for $C \in C^*$.

**Definition 1.7.** Let $n$ be a positive integer. We denote by $\Omega(n)$ the number of prime divisors of $n$ counting multiplicities. For a finite group $G$, $\Omega(|G|)$ is the number of primes dividing the order of $G$ counting multiplicities.

**Definition 1.8.** Let $G$ be a finite group. Then

$$\pi(G) = \{ p : p \text{ is a prime} \ | \ |G| \}.$$ 

We remark for any finite group $G$, we have that $l(G) \leq \Omega(|G|)$.

**Definition 1.9.** If $G$ is a finite group, then $cd(G) = l(G) - \lambda(G)$.

**Definition 1.10.** A subnormal series of a group $G$ is a chain of subgroups

$$G = G_0 > G_1 > \cdots > G_n = 1$$
such that $G_{i+1}$ is normal in $G_i$ for $0 \leq i \leq n$. The factors of the series are the quotient groups $G_i/G_{i+1}$. A subnormal series such that $G_i$ is normal in $G$ for all $i$ is said to be a normal series of $G$.

**Definition 1.11.** A subnormal series is said to be a composition series if each factor $G_i/G_{i+1}$ is a simple group. A normal series is said to be a chief series if $G_{i+1}$ is a maximal $G$-invariant subgroup of $G_i$ for $0 \leq i \leq n - 1$.

In a solvable group, composition factors are abelian [Go;2.4.2]. Since they are simple, this implies all composition factors are cyclic of prime order. Thus in the solvable case $l(G) = \Omega(|G|)$.

**Definition 1.12.** Two subnormal series $S$ and $T$ of a group $G$ are equivalent if there is a one-to-one correspondence between the nontrivial factors of $S$ and the nontrivial factors of $T$ such that the corresponding factors are isomorphic groups.

**Definition 1.13.** Let $G$ be a finite group. Let

$$G : G = G_o \supset G_1 \supset \cdots \supset G_n = 1$$

be a subnormal series. A refinement of $G$ is a subnormal series

$$\mathcal{H} : \mathcal{H}_o = G \supset H_1 \supset \cdots \supset H_n = 1$$

such that, for every $i = 1, 2, \cdots, r$, the term $G_i$ appears as a term in $\mathcal{H}$.

A subnormal series is a composition series if and only if it has no proper refinements.
In our analysis of solvable group $G$, we consider the isomorphism types of the quotient groups in both a chief series and a composition series. The following are useful tools in this analysis.

**Theorem 1.14 (Jordan-Holder Theorem).** Let $G$ be a group with a composition series

$$G : G = G_0 \supset G_1 \supset \cdots \supset G_r = 1.$$ 

Let

$$\mathcal{H} : H_0 = G \supset H_1 \supset \cdots \supset H_s = 1$$

be any composition series of $G$. Then the length of $\mathcal{H}$ is equal to the length of $G$; that is, $r = s$. Furthermore, there is a one-to-one correspondence between the composition factors of $G$ and the composition factors of $\mathcal{H}$ such that the corresponding factor groups are isomorphic.

**Proof.** This theorem is proved in [Su1:1.5.9].

□

**Definition 1.15.** A pair $(\Omega, G)$ consisting of a set $\Omega$ and a group $G$ is said to be a group with an operator domain $\Omega$ if there is a function $\Theta$ from $\Omega$ into $\text{End}(G)$, the set of endomorphisms of $G$. The function $\Theta$ is said to be an action of $\Omega$ on $G$; each element of $\Omega$ is called an operator.

An $\Omega$ subnormal series is a subnormal series of $G$ which is $\Omega$ invariant. An $\Omega$-subnormal series with no proper refinements is an $\Omega$-composition series. If $\Omega = 1$, then an $\Omega$-composition series is a composition series for $G$. If $\Omega = G$, then an $\Omega$ composition series is a chief series for $G$. 
**Theorem 1.16** (Schreier). Let $G$ and $H$ be two $\Omega$-subnormal series of an $\Omega$-group $G$. Then there are refinements $G'$ and $H'$ of $G$ and $H$, respectively, such that $G'$ and $H'$ are isomorphic.

**Proof.** This theorem is proved in [Su1;3.3.8].

□

**Corollary 1.17.** The Jordan-Holder Theorem holds for $\Omega$-composition series.

**Proof.** The proof is contained in [Su1;3.3.8.1]

□

**Definition 1.18.** Let $G$ be a finite group. Let $K \leq H \leq G$.

\[
\begin{align*}
    cf(G) &= \text{number of chief factors of } G \\
    cf_G(H/K) &= \text{the number of } G\text{-composition factors of } H/K. \\
    cp(G) &= \text{number of composition factors of } G
\end{align*}
\]

We note $H/K$ is a $G$ group. Thus by Corollary 1.17, the Jordan-Holder Theorem holds for a $G$-composition series of $H/K$. Thus $cf_G(H/K)$ is invariant. The other numbers are invariant by the Jordan-Holder Theorem and the Schreier Refinement Theorem.
In Chapter 2 we give some preliminary theorems and definitions to be used in the proof of Theorem 1.3 and Theorem 1.4. The proof of Theorem 1.4 will be by induction. We use three deep results in the identification of simple groups of chain difference one, the first of which is a theorem of Brauer and Suzuki [BS]. After an initial reduction to the case of a minimal counterexample, this theorem eliminates the case $m_2(G) = 1$.

**Theorem 2.1 (Brauer-Suzuki).** Let $G$ be a group with a generalized quaternion Sylow 2-subgroup $S$ and let $K = O_{2'}(G)$. Then $G = O_{2'}(G)C_G(z)$ for $z$ an involution in $G$.

**Proof.** This is the main theorem in a paper by Brauer and Suzuki [BS].

□

The Brauer-Suzuki Theorem establishes the existence of a fours-group in our minimal counterexample. In Chapters 5 we define a special family of fours groups called fully fused fours-groups. Chapters 8 and 9 are devoted to establishing the
existence of a self-centralizing fully fused fours-group in our minimal counterexample. Thus we have reduced to the case that a Sylow 2-subgroup of $G$ is either dihedral or semidihedral. We then quote established results of Gorenstein-Walter [GW] and Alperin-Brauer-Gorenstein [ABG] to make a final identification of $G$.

**Theorem 2.2** (Gorenstein-Walter). Let $G$ be a simple group with a dihedral Sylow 2-subgroup. Then $G \cong \text{PSL}_2(q)$ or $A_7$.

*Proof.* This theorem is the main result of the paper of Gorenstein and Walter [GW].

\[\square\]

**Theorem 2.3** (Alperin-Brauer-Gorenstein). Let $G$ be a simple group with a semidihedral Sylow 2-subgroup. Then $G \cong M_{11}$, $\text{PSL}_3(q)$, or $\text{PSU}_3(q)$.

*Proof.* This result is proved in a paper by Alperin, Brauer, and Gorenstein [ABG].

\[\square\]

To establish the existence of a fully fused self-centralizing fours-group, we study fusion of involutions in 2-local subgroups of $G$. We define fusion and then state some theorems which give us information about local fusion of $p$-elements in a group $G$.

**Definition 2.4.** Let $P$ be a Sylow $p$-subgroup of $G$. We say two elements $x$ and $y$ of $P$ are fused in $G$ if $x$ and $y$ are conjugate in $G$. More generally two subsets $X$ and $Y$ of $P$ are said to be fused in $G$ if they are conjugate in $G$. 
It may be the case that $x$ and $y$ lie in a $p$-subgroup $P_1 \leq P$ with $x$ conjugate to $y$ in $N_G(P_1)$. This is clearly a reflexive and symmetric relation. The transitive extension of this relation is called local fusion or local conjugacy.

**Definition 2.5.** Let $G$ be a group, let $H$ be a subgroup of $G$, let $T$ be a Sylow $p$-subgroup of $H$. We say $H$ controls $G$-fusion in $T$ (or $p$-fusion in $G$) if and only if any two subsets of $T$ that are conjugate in $G$ are also conjugate in $H$. Similarly if $A$ is a subset of $T$, we say $H$ controls the $G$ fusion of $A$ in $T$ if and only if whenever the subset $B$ of $T$ is $G$ conjugate to $A$, then $B$ is also $H$ conjugate to $A$.

Our principal tool is the following refinement by David Goldschmidt of Alperin's celebrated theorem on local control of fusion.

**Theorem 2.6 (Alperin-Goldschmidt).** Let $X$ be a finite group. Let $p$ be a prime divisor of $|G|$ and $P$ a Sylow $p$-subgroup of $X$. Let $D$ be the set of all subgroups $D$ of $P$ such that

(a) $N_P(D) \in Syl_2(N_X(D))$; and

(b) $C_X(D) \leq O_{p'^p} N_X(D)$

(c) $O_{p'^p} N_X(D) = O_{p'}(N_X(D)) \times D$.

Suppose $A$ and $B$ are subsets of $P$ and $A^x = B$ for some $x \in X$. Then there exist subsets $A = A_1, A_2, \cdots, A_n = B$ of $P$, elements $D_i \in D$, elements $x_i \in N_X(D_i)$, $1 \leq i \leq n - 1$ and $c \in C_X(A)$ satisfying the following conditions:

1. $x = c x_1 x_2 \cdots x_{n-1}$;

2. $(A_i, A_{i+1}) \leq D_i$; and

3. $A_i^{x_i} = A_{i+1}$. 

Proof. The proof of this result can be found in [HB;X.4.8,X.4.12].

□

We now give state some additional basic results about groups. The following theorem is a consequence of Sylow’s theorem, sometimes called the Frattini argument.

**Theorem 2.7.** If $H \trianglelefteq G$ and $P$ is a Sylow $p$-subgroup of $H$, then $G = N_G(P)H$.

**Proof.** A proof of this theorem is included in [Go;1.3.7]. □

**Theorem 2.8.** Let $G$ be a finite group. If $G = \Phi(G)H$, then $H = G$.

**Proof.** A proof of the theorem is contained in [Go;5.1.1]. □

The next result is Philip Hall’s Three Subgroup Theorem. It is an immediate consequence of Witt’s “Jacobi identity” for groups.

**Theorem 2.9** (Philip Hall). Let $H$, $K$, and $L$ be three subgroups of a group $G$. If $[H, K, L] = [K, L, H] = 1$, then we have $[L, H, K] = 1$.

**Proof.** The proof of this result can be found in [Go;2.2.3]. □

**Definition 2.10.** Let $G$ be a finite group. The unique maximal nilpotent normal subgroup of $G$ is called the Fitting subgroup of $G$ and is denoted by $F(G)$. 

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Theorem 2.11 (Philip Hall). If $G$ is a solvable group, then $C_G(F(G)) \leq F(G)$.

Proof. The proof of the statement is found in [Go;6.1.3].

Definition 2.12. Let $G$ be a finite group. Then $E(G)$ is the maximal semisimple normal subgroup of $G$. We can write $E(G) = Q_1Q_2\cdots Q_n$ uniquely as a central product of some quasisimple groups $Q_1, Q_2, \cdots Q_n$. We call $Q_i$ a component of $G$.

Definition 2.13. Let $G$ be a group and $F(G)$ be the Fitting subgroup of $G$. We set

$$F^*(G) = E(G)F(G)$$

and call $F^*(G)$ the generalized Fitting subgroup of $G$.

Theorem 2.14 (Bender). For any group $G$, we have $C_G(F^*(G)) = Z(F(G)) \leq F^*(G)$.

Proof. The proof of the theorem is contained in [Su2;6.6.2].

Definition 2.15. We say $G$ has a normal $p$-complement or $G$ is $p$-nilpotent if $G = O_{p'}(G)P$, where $P \in Syl_p(G)$.

The following theorem of Frobenius gives a necessary and sufficient condition for a group to have a normal $p$-complement.
Theorem 2.16 (Frobenius). $G$ possesses a $p$-complement if and only if one of the following conditions holds:

1. $N_G(H)/C_G(H)$ is a $p$-group for every nonidentity $p$-subgroup $H$ of $G$.
2. $N_G(H)$ has a normal $p$-complement for every nonidentity $p$-subgroup $H$ of $G$.

Proof. This theorem of Frobenius is proven in [Go;7.4.5].

Definition 2.17. If $\pi$ is a set of primes, a subgroup $H$ of $G$ will be called an $S_\pi$-subgroup of $G$ provided $H$ is a $\pi$-group and $|G:H|$ is divisible by no primes in $\pi$.

Such a subgroup is called a Hall subgroup of $G$. The Schur-Zassenhaus theorem gives an important sufficient condition for the existence and conjugacy of $S_\pi$-subgroups in $G$.

Theorem 2.18 (Schur-Zassenhaus). Let $H$ be a normal $S_\pi$-subgroup of $G$. Then we have

1. $G$ possesses an $S_{\pi'}$-subgroup $K$ which is a complement to $H$ in $G$.
2. If either $H$ of $G/H$ is solvable, then any two $S_{\pi'}$-subgroups of $G$ are conjugate in $G$.

Proof. A proof of the Schur-Zassenhaus theorem is contained in [Go;6.2.1].

Corollary 2.19. Let $A$ be a $\pi'$-group of automorphisms of the $\pi$-group $G$, and
suppose $G$ or $A$ is solvable. Then for each prime $p$ in $\pi$, we have

1. $A$ leaves invariant some Sylow $p$-subgroup of $G$.
2. Any two $A$ invariant Sylow $p$-subgroups of $G$ are conjugate by an element of $C_G(A)$.
4. If $H$ is any $A$ invariant normal subgroup of $G$, then $C_{G/H}(A)$ is the image of $C_G(A)$ in $G/H$.

Proof. The proof of the corollary is also contained in [Go;6.6.6].

□

The following lemma gives us a lower bound for $\lambda(G)$ if $G$ is a non-abelian simple group.

Lemma 2.20. Let $G$ be a finite group. If $\lambda(G) < 2$, then for some prime $p$, $G = A \times P$ with $P \cong Z_p$ and $A$ an elementary abelian minimal normal subgroup of $G$ or $G \cong Z_{p^m}$ for some $m \leq 2$. In particular if a group is a nonabelian simple group, then $\lambda(G) \geq 3$.

Proof. Let $\mathcal{C}$ be an unrefinable chain in $G$ with $l(\mathcal{C}) = \lambda(G)$. Let $K$ be the maximal subgroup in this chain. If $K = 1$, then $G$ is of prime order and the conclusion holds. If not, $K$ is a proper nonidentity subgroup of $G$ with $\lambda(K) = 1$. Thus $K \cong Z_p$ for some prime $p$. Let $P \in Syl_p(G)$. If $K \neq P$ then $N_P(K) > K$. Since $K$ is maximal we have $N_P(K) = G$ and $G$ is a $p$ group of order $p^2$ and the result follows.
Therefore we may assume that \( K \in Syl_p(G) \). Since \( K \) is maximal we have that \( N_G(K) = K \) or \( G \). If \( N_G(K) = G \) then \( G/K \cong \mathbb{Z}_q \) for some prime \( q \) as \( \lambda(G/K) = 1 \) and the conclusion holds with \( K = A \). If \( N_G(K) = K \) then by Theorem 2.16, \( G \) is \( p \)-nilpotent. Thus there exists \( A \leq G \) with \( G/A \cong K \). By Corollary 2.19 for each prime divisor \( q \) of \( |A| \), there is a \( K \)-invariant Sylow \( q \)-subgroup \( Q \) of \( A \). Since \( K \) is maximal, \( QK = G \) Thus \( A \) must be a \( q \) group. As \( A \) has no proper characteristic subgroups \( A \) must be elementary abelian. Thus \( G = AK \) and since \( K \) is maximal we have \( K \) acts irreducibly on \( A \), whence \( A \) is a minimal normal subgroup of \( G \).

\[ \Box \]

The following sequence of lemmas concerns the action of groups on \( p \) groups.

**Definition 2.21.** Let \( A \) be a subgroup of \( Aut(G) \) and let

\[
1 = G_n \leq G_{n-1} \cdots \leq G_0 = G
\]

be a subnormal series for \( G \). We say that \( A \) stabilizes the given series if each \( G_i \) is \( A \) invariant and \( A \) acts trivially on each factor \( G_i/G_{i-1}, 1 \leq i \leq n \).

**Lemma 2.22.** Let \( A \) be a \( p' \)-group of automorphisms of the \( p \)-group \( P \) which stabilizes some subnormal series of \( P \). Then \( A = 1 \).

**Proof.** The proof for the lemma can be found in [Go;5.3.2].

\[ \Box \]

**Lemma 2.23.** Let \( G \) be a group and let \( U \in Syl_p(G) \) with \( U \leq G \). Let

\[
1 = U_n \leq U_{n-1} \leq \cdots \leq U_1 = U
\]
be a $G$-chief series for $U$. Then $G/UC_G(U)$ embeds into $\prod_{i=0}^{r-1} Aut(U_i/U_{i+1})$.

Proof. As $U \leq G$, we have $G/C_G(U)$ acts on $\prod_{i=0}^{r-1}(U_i/U_{i+1})$ by conjugation. By Lemma 2.22, the kernel of this action is a $p$ group. By Theorem 2.18, $G = U \times H$ with $H$ a $p'$-group. Thus $H/C_H(U)$ embeds into $\prod_{i=0}^{r-1} Aut(U_i/U_{i+1})$. We have $H/C_H(U) \cong G/UC_G(U)$ and the result follows.

Lemma 2.24. Let $A$ be a $p$ group and let $\alpha \in Aut(A)$ with the order of $\alpha$ relatively prime to $p$. Then


with $[A, \alpha] \trianglelefteq A$. In particular if $A$ is abelian, then

$$A = [A, \alpha] \oplus C_A(\alpha).$$

Proof. A proof for this result is found in [Go;5.2.3].

Definition 2.25. Let $T$ be a subgroup of $G$. A subset $A$ of $T$ is said to be weakly closed in $T$ with respect to $G$, if the following condition is satisfied: whenever $A^g \leq T$ for some element $g$ of $G$, we have $A^g = A$.

Definition 2.26. Let $T \leq H \leq G$ with $T \in Syl_p(G)$. Then $H$ controls $p$-transfer in $G$ if and only if $T \cap [H, H] = T \cap [G, G]$.

As an immediate consequence of the Focal Subgroup Theorem [Su2;5.2.8], we have:
Lemma 2.27. Suppose that $T \leq H \leq G$ with $T \in \text{Syl}_p(G)$ and suppose $H$ controls $p$-fusion in $G$. Then

1. $H$ controls $p$-transfer in $G$;
2. $G = O^p(G)$ if and only if $H = O^p(H)$.

Proof. The proof of the lemma is contained in [GLS;15.10].

□

Lemma 2.28. If $S \in \text{Syl}_p(G)$ and $A$ is a subgroup of $S$, which is weakly closed with respect to $G$, then $N_G(A)$ controls the $G$-fusion of all subsets of $C_S(A)$. In particular if $A \leq Z(S)$, then $N_G(A)$ controls $G$-fusion in $S$.

Proof.

The proof for the lemma is found in [GLS;16.9].

□

Theorem 2.29 (Burnside). If $P$ is a Sylow $p$-subgroup of $G$, then two normal subsets of $P$ are conjugate in $G$ if and only if they are conjugate in $N_G(P)$.

Proof. The proof of the theorem by Burnside is contained in [Go;7.11].

□

The next lemma is referred to as the Thompson Transfer Lemma.

Lemma 2.30 (Thompson). Assume that a group $G$ of even order contains no normal subgroup of index 2. Let $M$ be a maximal subgroup of a Sylow 2-subgroup $S$ of $G$. Then, any involution of $S$ is conjugate to an element of $M$. 
Proof. Thompson's lemma is proven in [Su2;5.1.8].

□

After reducing to the case that $G$ has a self centralizing fours group the following lemmas prove a Sylow 2-subgroup of $G$ is either dihedral or semidihedral.

**Lemma 2.31.** Let $S$ be a group of order $2^n$ with $n \geq 2$. If there exists $x \in S$ such that $|C_S(x)| = 4$, then $S$ has maximal class.

Proof. The proof is by induction on $n$. The case $n = 2$ is clear, so we may assume $|S| \geq 8$. Let $x \in S$ with $|C_S(x)| = 4$. As $Z(S) \leq C_S(x)$, we have $|Z(S)| = 2$. Let $\overline{S} = S/Z(S)$ with $|\overline{S}| = 2^{n-1}$. As $[ , x]$ is a homomorphism of $C_S(x)$ into $Z(S)$ with kernel $C_S(x) = < x >$, we have $|\overline{C_S(x)}| \leq 4$. Thus by induction $\overline{S}$ has class $n - 2$. This implies $S$ has class $n - 1$ and the result follows.

□

**Lemma 2.32.** Let $S$ be a group of order $2^{n+1}$ with $E \cong E_4 \leq S$ such that $C_S(E) = E$. Then $S$ is dihedral or semidihedral.

Proof. As $Z(S) \leq C_S(E) = E$, we have $C_S(E) = C_S(x)$ for some $x \in E$. Thus $S$ has maximal class by Lemma 2.31. Thus $S$ is either quaternion, dihedral, semidihedral, or cyclic of order 4. [Go;5.4.5]. Since $S$ contains a fours group $S$ is not quaternion or cyclic hence the result follows.

□

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CHAPTER 3

SOLVABLE GROUPS AND GROUPS
OF CHAIN DIFFERENCE ZERO

Basic Results.

In Chapter 3 after proving basic results on properties of chain differences which inherit to subgroups and quotient groups, we shall show that for a solvable group
\[ l(G) = cp(G) \] and \[ \lambda(G) = cf(G). \] In particular for a solvable group \( G \) we have \[ cd(G) = cp(G) - cf(G). \]

Lemma 3.1. *If \( G \) is a finite group and \( H \leq G \), then \( cd(H) \leq cd(G) \).*

Proof. Let \( C_1 \) be an unrefinable chain in \( H \) of length \( l(H) \). Let \( C_2 \) be an unrefinable chain in \( H \) of length \( \lambda(H) \). Let \( C_3 \) be any unrefinable chain from \( H \) to \( G \). Clearly
\[ l(G) \geq l(C_1 \cup C_3) = l(C_1) + l(C_3). \]

Also
\[ \lambda(G) \leq l(C_2 \cup C_3) = l(C_2) + l(C_3). \]

Thus we have,
\[ cd(G) = l(G) - \lambda(G) \geq l(C_1 \cup C_3) - l(C_2 \cup C_3) = l(C_1) - l(C_2) = cd(H). \]

\[ \square \]

**Lemma 3.2.** Let \( G \) be a finite group. Let \( N \leq G \). Then \( cd(N) + cd(G/N) \leq cd(G) \).

**Proof.** Let \( 1 = N_1 \leq N_2 \leq \cdots \leq N_{k+1} = N \) be an unrefinable chain in \( N \) of length \( k = l(N) \). Let \( 1 = G_1/N \leq G_2/N \leq \cdots \leq G_{j+1}/N = G/N \) be a unrefinable chain in \( G/N \) of length \( j = l(G/N) \). Then \( 1 = N_1 \leq N_2 \leq \cdots \leq N = G_1 \leq \cdots \leq G_{j+1} = G \) is an unrefinable chain of subgroups of \( G \) of length \( k + j \). Similarly, by taking shortest unrefinable chains instead of longest ones we can construct an unrefinable chain of subgroups of \( G \) of length \( r + s \) where \( r = \lambda(N) \) and \( s = \lambda(G/N) \). Thus

\[ cd(G) \geq k + j - (r + s) = (k - r) + (j - s) = cd(N) + cd(G/N). \]

\[ \square \]

**Lemma 3.3.** If \( G \) is a finite group, \( \Phi(G) \leq A \leq G \) and \( A/\Phi(G) \) is nilpotent, then \( A \) is nilpotent.

**Proof.** Let \( P \in Syl_p(A) \). Thus \( \Phi(G)P/\Phi(G) \in Syl_p(A/\Phi(G)) \). Since \( A/\Phi(G) \) is nilpotent, \( \Phi(G)P/\Phi(G) \) is characteristic in \( A/\Phi(G) \). Hence \( P\Phi(G) \leq G \) and \( P \in Syl_p(\Phi(G)P) \). By Theorem 2.7 \( G = N_G(P)\Phi(G) \). Hence \( G = N_G(P) \), by Theorem 2.8. In particular, \( P \leq A \). Since all Sylow subgroups of \( A \) are normal, we have \( A \) is nilpotent as claimed.
**Corollary 3.4.** If $G$ is a non-identity finite solvable group, then $\Phi(G) \leq F(G)$. Thus $G = MF(G)$ for some maximal subgroup $M$ of $G$.

**Proof.** By definition of $\Phi(G)$, $\Phi(G) \neq G$. Let

$$1 = G_0 \leq G_1 \leq \cdots \leq \Phi(G) \leq G_i \leq \cdots \leq G$$

be a chief series for $G$ through $\Phi(G)$. We have $G_i/\Phi(G)$ nilpotent, indeed abelian. Hence $G_i$ is nilpotent and normal in $G$ by Lemma 3.3. This implies $G_i \leq F(G)$. In particular $\Phi(G) \leq F(G)$.

If $F(G)$ is contained in all maximal subgroups $M$ of $G$, then $F(G) \leq \Phi(G)$ by definition of $\Phi(G)$. Hence $F(G) = \Phi(G)$, a contradiction to the above. Thus we have $M$ such that $F(G) \neq M$ and $G = MF(G)$.

\[\square\]

**Lemma 3.5.** Let $G$ be a finite group. Then $C_G(F(G))$ contains every minimal normal subgroup of $G$.

**Proof.** Let $L$ be a minimal normal subgroup of $G$. If $L \not\leq F(G)$ then, since $F(G) \cap L \leq F(G)$ and $F(G) \cap L \leq G$, $F(G) \cap L = 1$. Thus $[F(G), L] = 1$ and $L \leq C_G(F(G))$.

On the other hand if $L \leq F(G)$ then, since $1 \leq L \leq F(G)$ and $F(G)$ is nilpotent we have $Z(F(G)) \cap L \neq 1$. Since $Z(F(G)) \cap L \leq G$, $Z(F(G)) \cap L = L$. Hence $L \leq C_G(F(G))$.

\[\square\]
Lemma 3.6. Let $G$ be a nontrivial finite group. Then, for any chief series of $G$, say

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G,$$

we have that

$$F(G) = \bigcap_{i=1}^{n} C_G(G_i/G_{i-1}).$$

Proof. Let $L = \bigcap_{i=1}^{n} C_G(G_i/G_{i-1}).$ Then $L \leq G$. As $L \leq C_G(G_i/G_{i-1})$ we have $[L, G_i] \leq G_{i-1}$. Hence $[L \cap G_i, L] \leq L \cap G_{i-1}$. Therefore, the series

$$1 \leq (L \cap G_0) \leq (L \cap G_1) \leq \cdots \leq (L \cap G_n) = L$$

is a central series of $L$, and so $L$ is nilpotent. Hence $L \leq F(G)$.

We prove the reverse inclusion by induction on $n$. We have

$$F(G)G_1/G_1 \cong F(G)/F(G) \cap G_1,$$

and so $F(G)G_1/G_1$ is nilpotent and normal in $G/G_1$. Hence

$$F(G)G_1/G_1 \leq F(G/G_1) = \bigcap_{i=2}^{n} C_{G/G_1}(G_i/G_1/G_{i-1}/G_1) = \bigcap_{i=2}^{n} C_{G/G_1}(G_i/G_{i-1})$$

by induction. As $G_1 \leq C_G(G_i/G_{i-1})$ for $2 \leq i \leq n$, it follows that $F(G) \leq C_G(G_i/G_{i-1})$ for $2 \leq i \leq n$, and it suffices to show that $F(G)$ centralizes $G_1$. As $G_1$ is a minimal normal subgroup of $G$, this follows from Lemma 3.5.

Therefore we have $F(G) = L$ and the lemma is proved.

$\Box$

The next lemma and its proof are essentially due to Kohler in [Ko].
**Lemma 3.7.** Let $G$ be a finite group and $M$ a maximal subgroup of $G$. Then

$$cf(G) \leq cf(M) + 1,$$

and equality holds if $G = MF(G)$.

**Proof.** Let $M$ be any maximal subgroup of $G$. Let $K$ be the largest normal subgroup of $G$ contained in $M$. Choose $A \subseteq G$ with $A/K$ a chief factor of $G$. As $M$ is maximal in $G$, we have $G = MA$. Furthermore $G/A \cong M/A \cap M$. Hence we have

$$cf(G) = cf(G/A) + 1 + cf_G(K)$$

$$= cf(M/A \cap M) + 1 + cf_G(K).$$

We have that

$$cf(M) = cf(M/K) + cf_M(K).$$

Clearly any $G$-chief series of $K$ is a normal $M$-series of $K$, and hence is refinable to an $M$-chief series for $K$ by Theorem 2.18. Thus $cf_G(K) \leq cf_M(K)$.

We have that

$$cf(G/A) = cf(M/A \cap M) \leq cf(M/A \cap M) + cf(A \cap M/K) = cf(M/K).$$

Thus

$$cf(G) = cf(M/A \cap M) + 1 + cf_G(K) \leq cf(M/K) + 1 + cf_M(K) = cf(M) + 1.$$

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If $G = MF(G)$ we argue that if $B/C$ is a $G$ chief factor then $B/C$ is also an $M$ chief factor. Suppose $B/C$ is a $G$ chief factor and $M$ normalizes $D$ with $C \leq D \leq B$. As $F(G)$ centralizes any chief factor of $G$ by Lemma 3.6, we have $F(G)$ centralizes $B/C$. As $G = MF(G)$ this implies that $D \leq G$, whence $D = C$ or $D = B$. Thus every $G$-chief factor of $M$ is an $M$-chief factor. In particular $cf_G(K) = cf_M(K)$. Also $A/K$ is a $G$-chief factor and $K \leq A \cap M \leq M$, whence $A \cap M = K$. Thus

$$cf(G) = cf(M/M \cap A) + 1 + cf_G(K) = cf(M/K) + 1 + cf_M(K) = cf(M) + 1.$$ 

Lemma 3.8. Let $G$ be a finite solvable group. Then $l(G) = cp(G)$.

Proof. Since $G$ is solvable, there is a composition series $C$ for $G$ such that all factors are cyclic of prime order. This is clearly an unrefinable chain. Furthermore the length of $C$ is $\Omega(|G|)$, thus $C$ is of maximal length. Thus $l(G) = cp(G)$ as claimed.

The following result is due to Kohler [Ko]. We include a proof for completeness.

Lemma 3.9 (Kohler). Let $G$ be a finite solvable group. Then we have that $\lambda(G) = cf(G)$ and $cd(G) = cp(G) - cf(G)$.

Proof. We proceed by induction on $|G|$. By Corollary 3.4 we have the existence of a maximal subgroup $M_1$ such that $G = M_1F(G)$. Thus by Lemma 3.7 and
induction
\[ cf(G) = cf(M_1) + 1 = \lambda(M_1) + 1 \geq \lambda(G) \]

Let \( C \) be an unrefinable chain in \( G \) of minimal length. Suppose \( M \) is the maximal subgroup in this chain. Then by Lemma 3.7 and induction we have
\[ cf(G) \leq cf(M) + 1 = \lambda(M) + 1 = \lambda(G). \]

Therefore \( cf(G) = \lambda(G) \). Now by Lemma 3.8, \( cd(G) = cp(G) - cf(G) \).

\( \square \)

**Definition 3.10.** A group is said to be supersolvable if it has a normal series all of whose factors are cyclic.

**Lemma 3.11.** A finite group \( G \) is supersolvable if and only if \( G \) is solvable and \( cd(G) = 0 \).

**Proof.** If \( G \) is supersolvable then by definition it has a normal series all of whose factors are cyclic, which can be refined to a normal series all of whose factors are cyclic of prime order. This normal series is a \( G \)-chief series. Furthermore as it is unrefinable the series is also a composition series. Hence \( cd(G) = 0 \) by Lemma 3.9.

A composition series of a solvable group \( G \) has factors of prime order. If \( cd(G) = 0 \) then this composition series is also a chief series, by Lemma 3.9. Hence \( G \) has a normal series all of whose factors have prime order. Hence \( G \) is supersolvable.

\( \square \)
Lemma 3.12. Let $G$ be a supersolvable group. Then all subgroups and quotients of $G$ are supersolvable.

Proof. Let $G$ be a supersolvable group. By Lemma 3.11 $G$ is solvable thus all subgroups and quotient groups of $G$ are solvable. By Lemma 3.1 and 3.2 all subgroups and quotient groups of $G$ have chain difference zero. Thus by Lemma 3.11 they are all supersolvable.

□

Lemma 3.13. Let $G$ be a finite supersolvable group. Let $p$ be the smallest prime divisor of $|G|$. Then $G$ is $p$-nilpotent.

Proof. Suppose the lemma is not true and let $G$ be a minimal counterexample. We argue that $O_{p'}(G) = 1$. If not then consider $G/O_{p'}(G)$ which is also supersolvable. Thus by induction $G/O_{p'}(G)$ is $p$-nilpotent but then so is $G$, a contradiction. Therefore $O_{p'}(G) = 1$ and $F(G) = O_p(G)$. Since $G$ is not a $p$ group, we may choose $\alpha \in G$ with the order of $\alpha$ equal to $q \neq p$, $q$ a prime. We have that $\alpha$ does not centralize $F(G)$, as $C_G(F(G)) \leq F(G)$ by Theorem 2.11.

Let

$$1 = F_0 \leq F_1 \leq \cdots \leq F_n = F(G)$$

be a $G$ invariant chief series for $F(G)$. As $G$ is supersolvable we have $F_i/F_{i-1} \cong Z_p$ for all $1 \leq i \leq n$. By Lemma 2.22 $\alpha$ acts nontrivially on $F_i/F_{i-1}$ for some $1 \leq i \leq n$. In particular $\alpha$ maps isomorphically into $\text{Aut}(F_i/F_{i-1}) \cong Z_{p-1}$. But by assumption $q \geq p$; thus $q \not| p - 1$, a contradiction. Hence $G$ is $p$-nilpotent as claimed.

□
**Lemma 3.14.** Let $H$ be a finite group. Then $O_{2',2}(H)$ is the unique maximal normal 2-nilpotent subgroup of $H$.

*Proof.* Let $M$ be any normal 2-nilpotent subgroup of $H$. Since $M \leq H$,

$$O_{2'}(M) \leq O_{2'}(H).$$

Then

$$O_{2',2}(M) \leq O_{2'}(H)O_{2',2}(M) \leq H$$

and $O_{2'}(H)O_{2',2}(M)/O_{2'}(H)$ is a normal 2-subgroup of $H/O_{2'}(H)$. Hence

$$M = O_{2',2}(M) \leq O_{2'}(H)O_{2',2}(M) \leq O_{2',2}(H).$$

Thus $O_{2',2}(H)$ contains every normal 2-nilpotent subgroup of $H$, hence is the unique maximal normal 2-nilpotent subgroup of $H$.

□

**Lemma 3.15.** Let $G$ be a finite group all of whose proper subgroups are supersolvable. Then $G$ is solvable.

*Proof.* Suppose the lemma is not true. Let $G$ be a minimal counterexample. Suppose there exists $N \leq G$ with $N \neq 1$. Then by induction we have $G/N$ and $N$ are solvable. Therefore $G$ is solvable a contradiction. Hence $G$ is simple. Suppose $p$ is the smallest prime dividing the order of $G$. Let $D$ be any nonidentity $p$-subgroup of $G$. We have that $N_G(D)$ is proper in $G$ as $G$ is simple. Therefore by hypothesis $N_G(D)$ is supersolvable and so by Lemma 3.13 $N_G(D)$ is $p$-nilpotent. But then $G$ is $p$-nilpotent by Theorem 2.16. Therefore $G = O_{p'}(G)P$ and so $G \cong Z_p$, as $G$ is simple. Again $G$ is solvable a contradiction.

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The following theorem was proved by Iwasawa.

**Theorem 3.16.** (Iwasawa [Iw]) Let $G$ be a finite group. Then $G$ is supersolvable if and only if $G$ has chain difference 0.

**Proof.** By Lemma 3.11 it remains to show that a group of chain difference 0 is solvable. Suppose not and let $G$ be a minimal counterexample. Let $N$ be a proper subgroup of $G$. By induction and Lemma 3.1 $N$ is supersolvable. By Lemma 3.15 since all proper subgroups of $G$ are supersolvable, $G$ is solvable.

□
In Chapter 4 we prove some general results about groups of chain difference one. The goal is to show that in groups of chain difference one \( p \)-constraint is equivalent to solvability. We include two different proofs of this equivalence. In the first we prove a general lemma not specific to groups of chain difference one. We show that if \( cd(G) = 1 \) and \( E(G) = 1 \) then in fact \( G \) must be solvable. In particular if \( G \) is \( p \)-constrained, \( E(G) = 1 \), thus for groups of chain difference one \( p \) constraint is equivalent to solvability.

In the second proof we observe if \( G \) is a group of chain difference one and \( F^*(G) = O_2(G) \) then \( |G| = 2^a3 \). This result combined with a theorem proving that a group of odd order and of chain difference one is solvable implies that if \( G \) is \( p \)-constrained, then \( G \) is solvable.

**Lemma 4.1.** Let \( q \) be a prime, \( V \) a vector space over a field of characteristic \( q \), and \( A \) an abelian subgroup of \( GL(V) \) of exponent \( d \) dividing \( q - 1 \). Then \( A \) is diagonalizable. Thus every irreducible \( A \)-submodule of \( V \) is 1-dimensional.

**Proof.** Let \( A \) be an abelian subgroup of \( GL(V) \) of exponent \( d \) dividing \( q - 1 \). Since \( V \) is a vector space over a field of characteristic \( q \), all elements of \( GL(V) \)
of order $d$ are diagonalizable, since $x^d - 1$ splits into distinct linear factors over $F_q$. If $A$ acts on $V$ as scalar matrices then the lemma is trivially true. Thus we may choose $\Phi \in A$ with $\Phi$ not a scalar matrix. Let $U$ be a proper eigenspace for $\Phi$. Let $W$ be the sum of the remaining eigenspaces for $\Phi$. Since $A$ is abelian, $U$ and $W$ are $A$ invariant. We have $V = U \oplus W$. By induction $A$ is diagonalizable on $U$ and $W$, and therefore $A$ is diagonalizable on $V$. Thus every irreducible $A$-submodule of $V$ is one dimensional.

Lemma 4.2. Let $S$ be a finite group. Let $H$ be a subgroup of $\text{Aut}(S)$ and let $K$ be the kernel of the $H$ action on $F^*(S)$. Let $F(S)$ be the Fitting subgroup of $S$. Then $K$ is a $\pi$ group where $\pi = \pi(|Z(F(S))|)$.

Proof. Let $K$ be the kernel of the $H$ action on $F^*(S)$. Let $X$ be a $\pi'$-subgroup of $K$. Let $S_X$ be the semidirect product. We have $C_{S_X}(F^*(S)) = C_S(F^*(S))X = Z(F(S)) \times X$. Thus $X = O_{\pi'}(C_{S_X}(F^*(S))) \leq S_X$. This implies that $[S, X] \leq S \cap X = 1$. Hence $X = 1$. Thus $K$ is a $\pi$ group.

□

Definition 4.3.

Let $S$ be a finite group and $H$ a subgroup of $\text{Aut}(S)$. An $H$-chief factor $L/K$ of $S$ is said to be $H$-central if $[L, H] \leq K$. Otherwise it is called $H$-noncentral.

Lemma 4.4. Let $S$ be a finite group and let $H$ be a subgroup of $\text{Aut} S$. Suppose every $H$-chief factor of $S$ is either $H$ central or of prime order. Then $H$ is supersolvable.
Proof. Choose a counterexample with $|H|$ minimal and subject to that with $|S|$ minimal. By induction every proper subgroup of $H$ is supersolvable and so by Lemma 3.15, $H$ is solvable. Let $Z = Z(F(H))$. We have $H/Z$ is isomorphic to a subgroup of $Aut(F(H))$ as $C_H(F(H)) = Z$. Suppose every $H/Z$ chief factor of $F(H)$ has prime order. Then by induction $H/Z$ is supersolvable. Furthermore as every $H$-chief factor of $Z$ has prime order, $H$ is supersolvable.

Thus we may assume that there is a prime $q$ and a normal $q$ subgroup $Q$ of $H$ minimal subject to some $H$-chief factor of $Q$ being noncyclic. By minimality $Q/Q_1$ is a noncyclic $H$-chief factor for some $Q_1 \leq Q$. We claim that $Q_1$ is the unique maximal $H$-invariant subgroup of $Q$. For if $Q_2$ is another maximal $H$-invariant subgroup of $Q$, we have $Q = Q_1 Q_2$ and $Q_2/Q_1 \cap Q_2 \cong Q/Q_1$. Thus in particular $Q_2 \leq H$ and $Q_2/Q_2 \cap Q_1$ is a noncyclic $H$-chief factor contrary to the minimal choice of $Q$.

Suppose $N \leq H$ with $H/N$ supersolvable. We claim $Q \leq N$. If not since $Q \cap N \leq H$ and properly contained in $Q$, $Q \cap N$ is contained in $Q_1$, the unique maximal $H$-invariant subgroup of $Q$. This implies $Q/Q_1$ is isomorphic to a noncyclic chief factor of $H/N$, a contradiction. Therefore $Q \leq N$.

We shall show that $F(S)$ is a $q$ group. Suppose for $r \neq q$, $O_r(S) \neq 1$. $H$ acts on $O_r(S)$ and $S/O_r(S)$. Suppose $N_1$ and $N_2$ are the kernels of the respective actions. By induction both $H/N_1$ and $H/N_2$ are supersolvable. By the previous argument $Q \leq N_1 \cap N_2$, hence $Q$ stabilizes the chain $1 \leq O_r(S) \leq S$. As $Q$ fixes the coset $gO_r(S)$ for every $g \in S - O_r(S)$, $Q$ must fix an element of each coset $gO_r(S)$. But $[Q, O_r(S)] = 1$ and so $Q$ fixes every element of each coset $gO_r(S)$. Therefore $[Q, S] = 1$, a contradiction. This implies that $O_r(S) = 1$ for all $r \neq q$;
that is $F(S)$ is a $q$ group.

Let $K$ be an $H$-chief factor of $E(S)/Z(E(S))$. Thus $K$ is a direct product of nonabelian simple groups. By hypothesis $H$ centralizes $K$ since $K$ is not of prime order. Hence $H$ centralizes $E(S)/Z(E(S))$. This implies $[E(S), H, E(S)] = [H, E(S), E(S)] = 1$. By Theorem 2.9, we have $[E(S), E(S), H] = [E(S), H] = 1$. Thus $C_H(F^*(S)) = C_H(F(S))$. By Lemma 4.2 $C_H(F(S))$ is a $q$ group since $Z(F(S))$ is a $q$ group.

Since the $H$ action on $F(S)$ completely determines the $H$ action on $F^*(S)$, let

$$1 = F_r \leq \cdots \leq F_0 = F(S)$$

be an $H$ chief series for $F(S)$ with $F_i / F_{i+1} = \tilde{F}_i \cong Z_q$, as by hypothesis every $H$ chief factor of $F(S)$ has prime order. The $H$-action on $F(S)$ defines a homomorphism $\rho$ from $H$ into $\prod Aut(\tilde{F}_i)$. Let $K^* = \ker(\rho)$ and let $K = C_H(F(S))$, the kernel of the $H$ action on $F(S)$. Then $K \leq K^*$ and $K^*/K$ is a $q$ group by Lemma 2.22. As $K$ is a $q$ group, we have $K^*$ is a $q$ group.

Finally we have that $H/K^*$ embeds into $\prod Aut\tilde{F}_i$, a homocyclic abelian group of exponent $q - 1$. Let $V$ be an $H$-chief factor in $K^*$. As $K^*$ is nilpotent, $K^*$ acts trivially on $V$. We claim $V$ is cyclic which implies $H$ is supersolvable. Let $\varphi$ be the homomorphism of $H$ into $GL(V) \cong GL(n, q)$. Then $\varphi(H)$ is isomorphic to a quotient of $H/K^*$. So $\varphi(H)$ is an abelian subgroup of $GL(V)$ of exponent dividing $q - 1$. As $V$ is an irreducible $\varphi(H)$ submodule, $V$ is 1-dimensional by Lemma 4.1, and the lemma is proved.
Lemma 4.5. If $H$ is solvable with $cd(H) = 1$ and $F \leq H$ with $cd(H/F) = 1$, then every $H$ chief factor of $F$ has prime order. In particular a solvable group of chain difference one has only one noncyclic chief factor and this chief factor has order $p^2$ for some prime $p$.

Proof. Since $H$ is solvable, Lemmas 3.9 and 3.8 imply that $\lambda(H) = cf(H)$ and $l(H) = cp(H)$. Since $cd(H) = 1$, we have $cp(H) = cf(H) + 1$. The factors in a composition series for a solvable group are all of prime order. Thus $H$ must have only one noncyclic chief factor and it must be of order $p^2$. If $cd(H/F) = 1$, $H/F$ has a noncyclic chief factor. Since $cd(H) = 1$, every $H$ chief factor of $F$ has prime order.

\square

Lemma 4.6. Let $H$ be a finite group of chain difference one. Let $F$ be a nilpotent normal subgroup of $H$ such that $C_H(F) \subseteq F$. Then $H$ is solvable and $H/F$ is supersolvable. Indeed either $H$ is supersolvable or for some prime $p$, there is a noncyclic $H$-chief factor $L/K$ of $O_p(H)$ and $H/L$ is supersolvable.

Proof. We proceed by induction on $|H|$. Let $K$ be any proper subgroup of $H$ such that $F \subseteq K$. Then $K$ satisfies the hypothesis of the lemma. Thus by induction we have $K$ solvable and $K/F$ supersolvable. In particular all proper subgroups of the group $H/F$ are supersolvable. Therefore by Lemma 3.15 we have $H/F$ is solvable and thus $H$ is solvable.

It remains to show that $H/F$ is supersolvable. If not then by Lemma 3.15 $cd(H/F) = 1$. As $cd(H) = 1$ Lemma 4.5 implies that $H$-chief factors of $F$ must be cyclic of prime order. Thus by Lemma 4.4 $H/C_H(F)$ is supersolvable. Hence
$H/F$ is supersolvable as claimed.

If every $H$-chief factor of $F(H)$ is of prime order, then by Lemma 4.4 $H/C_H(F(H))$ is supersolvable and, since \( C_H(F(H)) \leq F(H) \) and all $H$-chief factors of $F(H)$ are of prime order, Lemma 3.11 implies that $H$ is supersolvable. If not, then for some prime $p$, there is a non-cyclic $H$-chief factor $L/K$ of $O_p(H)$. Then by Lemma 4.5, $cd(H/L) = 0$ and so $H/L$ is supersolvable by Lemma 3.11.

\[\Box\]

**Theorem 4.7.** Let $H$ be a finite group with $cd(H) \leq 1$. Then either

1. $E(H) = L$ is quasisimple and $H/L$ is supersolvable, or
2. $H$ is solvable.

**Proof.** Suppose $E(H) \neq 1$ and let $L$ be a component of $H$. Then $cd(L) = 1$ and so by Lemma 3.2 \( cd(N_H(L)/L) = 0 \). Hence by Lemma 3.17, $N_H(L)/L$ is supersolvable. As $L \leq E(H)$, we conclude that $E(H) = L$. Then $L \leq H$ and $H/L$ is supersolvable.

Next suppose that $F^*(H) = F(H)$. Then $F(H)$ is a normal nilpotent subgroup of $H$ such that $C_H(F(H)) \subseteq F(H)$. Thus by Lemma 4.6, $H$ is solvable. Thus the theorem is proved.

\[\Box\]

We have shown that a group of chain difference one either contains a unique component or is solvable. We now prove Theorem 1.3 about the structure of solvable groups of chain difference one.
Theorem 1.3. Let $G$ be a finite solvable group with $cd(G) = 1$. Let $p$ be the smallest prime divisor of $G$. Then either $G$ is $p$-nilpotent or $p = 2$ and the following hold.

1. There exists $U \leq G$ with $U \cong Z_2 \times Z_2$ or $U \cong Q_8$;
2. $G/U$ is supersolvable; and
3. $O^2(G) = \langle U \times O_2'(G), x \rangle$ with $x^3 \in O_2'(G)$ and $U = [U, x]$.

Proof. Suppose that $G$ is not $p$-nilpotent. By Lemmas 4.5 and 4.6, there is a prime $q$ and a non-cyclic $G$-chief factor $U/U_1$ with $U \leq O_q(G)$, $U/U_1 \cong Z_q \times Z_q$ and $G/U$ supersolvable. We choose a minimal such $U \leq G$. Thus $G/U$ is $p$-nilpotent, by Lemma 3.13. If $q \neq p$, then $G$ is $p$-nilpotent, contrary to assumption. Hence $q = p$ and $U$ is a normal Sylow $p$-subgroup of $O_p(G) = H$. Note that $H$ is not $p$-nilpotent. Now as $U \in Syl_p(H)$ and $U \leq H$, we have $UC_H(U) = U \times O_p'(H)$. Thus, as $H$ is not $p$-nilpotent, $H \neq UC_H(U)$.

Let $U = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_r = \langle 1 \rangle$ be a $G$-chief series for $U$. As $cd(G/U_1) = 1$, every $G$-chief factor of $U_1$ is cyclic of order $p$ by Lemma 4.5. Thus $U_i/U_{i+1} \cong Z_p$ for all $i \geq 1$. By Lemma 2.12, $H/UC_H(U)$ embeds into $\prod_{i=0}^{r-1} Aut(U_i/U_{i+1})$. On the other hand $Aut(Z_p) \cong Z_{p-1}$ and $p$ is the smallest prime divisor of $|G|$. As $H/U$ is a $p'$-group, we have $H = UC_G(U_1)$. Moreover by Lemma 4.5, $H/UC_H(U)$ embeds into $Aut(U/U_1) \cong GL_2(p)$. Again as $p$ is the smallest prime divisor of $|G|$ and since $|GL_2(p)| = (p+1)(p-1)^2p$, we conclude that $p+1$ is a prime, that is $p = 2$ and $H/UC_H(U) \cong Z_3$. Thus $H = \langle U \times O_2'(G), x \rangle$ with $x^3 \in O_2'(G)$ and $[U, x] \neq 1$. Indeed $H_o = \langle [U, x] \times O_2'(G), x \rangle \leq H$ with $H/H_o$ a 2-group and so $H_o \geq O^2(H) = H$. Thus $U = [U, x]$.

As $H = UC_H(U_1)$ and $U_1 \leq H$, we have that $C_H(U_1) \geq O^2(H) = H$. In
particular $U_1 \leq Z(U)$ and so if $U = \langle a, b, U \rangle$, then $[U, U] = \langle [a, b] \rangle$ with $[a, b]^2 = [a^2, b] = 1$. Let $\overline{U} = U/[U, U]$. By Lemma 2.23, $\overline{U} = [\overline{U}, x] \oplus C_{\overline{U}}(x)$ with $[\overline{U}, x] \cong U/U_1$. As $U = [U, x]$, it follows that $\overline{U} = [\overline{U}, x]$ and so either $U \cong Z_2 \times Z_2$ or $U$ is nonabelian of order 8. In the latter case, as $U = [U, x]$, clearly $U \cong Q_8$, as claimed.

\[ \square \]

**Corollary 4.8.** Under the hypothesis of Theorem 1.3 if $U = O_2(O^2(G))$ with $U \cong Z_2 \times Z_2$, then $C_G(U) = O_{2',2}(G)$.

**Proof.** By Theorem 1.3, $O^2(G) = \langle U \times O_{2'}(G), x \rangle$, thus $O^2(G) \cap C_G(U) = U \times O_{2'}(G)$. We have

$$C_G(U)/O^2(G) \cap C_G(U) \cong C_G(U)O^2(G)/O^2(G)$$

which is a 2 group. Thus $C_G(U)$ is a normal 2-nilpotent subgroup of $G$ and is contained in $O_{2',2}(G)$ the maximal normal 2-nilpotent subgroup of $G$. If $C_G(U) < O_{2',2}(G)$, then $G/C_G(U) \leq S_3$ contains a normal 2-subgroup, a contradiction. Thus $C_G(U) = O_{2',2}(G)$.

\[ \square \]

The following sequence of lemmas provides a proof, independent of Lemma 4.4, that $p$-constraint is equivalent to solvability for a group of chain difference one. The chapter concludes with a proof that any odd order group of chain difference one is solvable. Of course this is a corollary of the Odd Order Theorem of Feit and Thompson. However in our context an elementary proof is available.

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Lemma 4.9. Let $G$ be a group with $cd(G) = 1$ and let $p$, $q$ be primes with $p < q$. Suppose $F^*(G) = O_p(G) = P$ and $G = PQ$ with $Q \in Syl_q(G)$. Then $p = 2$, $|Q| = 3$ and $O^2(G) \cong A_4$ or $SL_2(3)$.

Proof. As $G$ is solvable but not nilpotent, this is immediate from Theorem 1.3. 

□

Theorem 4.10. Let $G$ be a group with $cd(G) = 1$. Let $p$ be the smallest prime divisor of $|G|$ and suppose $G$ is $p$-constrained. Then either $G$ is $p$-nilpotent or $p = 2$ and $G/O_2(G)$ is solvable of order $2^a 3^b$.

Proof. The proof is by induction. Let $G$ be a minimal counterexample. Suppose $O_{p'}(G) \neq 1$. Then $G/O_{p'}(G)$ has chain difference one and $p$ is the smallest prime dividing the order of $G$. Since $G$ is $p$ constrained so is $G$. Then by induction either $p = 2$ and the theorem holds or $G$ is $p$-nilpotent, whence so is $G$. Therefore we may assume $O_{p'}(G) = 1$ and $G$ is not $p$-nilpotent. Thus $F^*(G) = P$ is a $p$-group and $P \neq G$. Let $q$ be a prime with $q||G|$, $q > p$ and let $Q \in Syl_q(G)$. Since $PQ$ is a solvable group, $F^*(PQ) = F(PQ)$ and $P = O_p(PQ)$. Let $X = F^*(PQ) \cap Q$. Then $X = O_q(PQ)$, thus $[P, X] = 1$. This implies $X \leq F^*(G) \cap Q$, hence $X = 1$. Therefore $F^*(PQ) = P$ and so by Lemma 4.9, $p = 2$ and $|Q| = 3$. As $q$ was arbitrary, we can conclude $|G| = 2^a 3^b$. Then $G$ has a permutation representation of degree 3 with kernel a 2-group. Hence $G$ is solvable as claimed. 

□

Theorem 4.11. Let $G$ be a group of odd order with $cd(G) = 1$. Then $G$ is solvable.
Proof. Let $G$ be a minimal counterexample and let $p$ be the smallest prime divisor of $|G|$. Then $G$ is simple and all proper subgroups are solvable. Thus by Theorem 4.10, every $p$-local is $p$-nilpotent. So $G$ is $p$-nilpotent by Theorem 2.16. So $G \cong Z_p$.

\[ \square \]

**Corollary 4.12.** Let $G$ be a 2-constrained group with $cd(G) = 1$. Then $G$ is solvable.

Proof. By Theorem 4.10, $G/O_2'(G)$ is solvable and by Theorem 4.11 $O_2'(G)$ is solvable, thus $G$ is solvable.

\[ \square \]
CHAPTER 5

THE FAMILY OF FULLY FUSED FOURS GROUPS

In this section we introduce and study the important family of fully fused fours groups.

Definition 5.1. Let $H$ be a subgroup of $G$. We define

$$\mathcal{F}_H = \{U \leq H : U \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ and } 3||N_H(U)/C_H(U)||\}$$

In particular we let $\mathcal{F} = \mathcal{F}_G$.

If $U \in \mathcal{F}$, we say $U$ is a fully fused fours group. If $U \in \mathcal{F}_H$, we say $U$ is fully fused in $H$.

Lemma 5.2. Let $H \leq G$. Let $U \leq H$ with $U \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $U \in \mathcal{F}_H$. Then $U \leq O^2(H)$.

Proof. Suppose that we have $U \leq H$ with $U \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $U \in \mathcal{F}_H$. Since $U \in \mathcal{F}_H$, there exists an $x \in N_H(U)$ of odd order such that $[U, x] = U$. However $x \in O^2(H)$ and so $[U, x] \leq O^2(H)$. Therefore $U \leq O^2(H)$.

\[\square\]
Lemma 5.3. Let $G$ be a finite group of chain difference one. Let $H \leq G$. Let $U \leq H$ with $U \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $U \in \mathcal{F}_H$. If $H$ is solvable, then $U \leq H$ and $U \in \text{Syl}_2(O^2(H))$. Otherwise $U \leq E(H)$.

Proof. Suppose $H$ is solvable. By Lemma 5.2 $U \leq O^2(H)$. Thus $H$ is not 2-nilpotent. Therefore by Theorem 1.3, there exists $V \leq H$ with $V \cong Q_8$ or $E_4$ and $O^2(H) = \langle V \times O_2'(H), x \rangle$. Since $U \leq O^2(H)$, $U \leq V$. As $U \cong E_4$, we have $U = V$ and $U \leq H$.

If $H$ is not solvable then by Theorem 4.7 $E(H)$ is a quasisimple group and $H/E(H)$ is supersolvable. Hence $O^2(H/E(H))$ has odd order by Lemma 3.13. As $U \leq O^2(H)$ by Lemma 5.2, it follows that $U \leq E(H)$.

\[\square\]

Lemma 5.4. Let $G$ be a finite group of chain difference one. Suppose $U$ and $V$ are fully fused fours groups with $U \neq V$. Then $[U,V] \neq 1$.

Proof. Suppose $U$ and $V$ are fully fused fours groups which commute. Let $N = N_G(U)$ and let $S \in \text{Syl}_2(N_G(U))$ with $UV \leq S$. Let $T = C_S(U) \in \text{Syl}_2(C_G(U))$. By Theorem 2.7, we have $N = C_G(U)N_N(T)$. Therefore there exist $x \in N_N(T)$ of odd order with $[U,x] = U$. Let $T^* = T < x >$, a solvable group. As $U \in \mathcal{F}_{T^*}$, $U \in \text{Syl}_2(O^2(T^*))$ by Lemma 5.3. Since $[T,x] \leq O^2(T^*)$, we have $[T,x] = U$. In particular $[UV,x] = U$. By the same argument there exists $y \in N_G(V)$ of odd order with $[UV,y] = V$.

Let $H = N_G(UV)$. We will show $cd(H) = 1$ implies that $U = V$. Let $p$ be the largest prime dividing the order of $H/C(UV) = \overline{H}$. Let $P \in \text{Syl}_p(H)$. Then
$H_0 = UVP$ is solvable. If $H_0$ is 2-nilpotent, then $[UV, P] \leq UV \cap O_2'(H_0) = 1$, contrary to the fact that $P \not\leq C_G(UV)$. Thus by Theorem 1.3 and since $UV \leq H_0$, $H_0 = \langle UV \times O_2'(H_0), h \rangle$ with $h^3 \in O_2'(H_0)$. Since $O_2'(H_0) \leq C_H(UV) = C$, $|P| = 3^b$ and $|\overline{H}| = 2^a 3$. Let $C \leq K \leq H$ with $K = O_2(\overline{H})$ and let $Q \in Syl_2(K)$. By Theorem 2.7 $H = KN_H(Q)$ and as $K = CQ$, $H = CN_H(Q)$.

Let $X \in Syl_3(N_H(Q))$. As $QX = O^2(\overline{H})$, $\overline{x} \in QX$ and $\overline{y} \in QX$. Hence $U = [UV, x] \leq O^2(QX)$ and $V = [UV, y] \leq O^2(QX)$. However $QX$ is a solvable group with $cd(QX) = 1$. So by Theorem 1.3, a Sylow 2-subgroup of $O^2(QX)$ is isomorphic to $E_4$ or $Q_8$. So $U = V$, as claimed.

\[ \square \]

**Theorem 5.5.** Let $G$ be a finite group of chain difference 1. Let $S \in Syl_2(G)$ and suppose $z \in I(Z(S))$ with $z^G \cap S \neq \{z\}$. Then there exists $U \leq D \leq S$ and $g \in N_G(D)$ such that:

1. $D = C_S(U)$; and
2. $N_S(D) = N_S(U) \in Syl_2(N_G(D)) \leq Syl_2(N_G(U))$; and
3. $U \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $U \leq N_G(D)$; and
4. $g^2 \in O_2(N_G(D))$ and $[D, g] = U$; and
5. $z \in D$ and $z^g = s \neq z$.

**Proof.** Let $S \in Syl_2(G)$ and suppose $z \in I(Z(S))$ with $z^G \cap S \neq \{z\}$. By Theorem 2.6, there exists $D \leq S$ with $z \in D$ satisfying

(a) $N_S(D) \in Syl_2(N_G(D))$; and
(b) $C_G(D) \leq O_{2'}N_G(D) = O_{2'}(N_G(D)) \times D$; and
(c) $z^{N_G(D)} \neq \{z\}$.

Let $N = N_G(D)$. As $N = O^2(N)N_S(D)$ and $z \in Z(S)$, we have $[z, O^2(N)] \neq 1$.
and so we may choose \( g \in N_G(D) \) of odd order such that \( z^g \neq z \). Set \( s = z^g \). Then \([z, g] = zs \in O^2(N)\) and so \( O^2(N)\) has even order and \( N\) does not have a normal 2-complement. Property (b) implies that \( N\) is 2-constrained, hence solvable by Theorem 4.7. Thus by Theorem 1.3, \( O^2(N) = (U \times O_{2'}(N), x)\) with \( U = [U, x] \cong Z_2 \times Z_2\) or \( Q_8\) and with \( x^3 \in O_{2'}(N)\). As \( O_{2'}(N) \leq C_N(D)\), \( g \notin O_{2'}(N)\) thus we may assume \( x = g\). Then \([D, g] \leq D \cap O^2(N) = U\) and so \([D, g] = U\). If \( U \cong Q_8\), then \(<zs> = Z(U)\) and \((zs)^g = zs\). But then \(s^g = z^g(zs)^g = z\) and so \(z^{g^2} = z\), a contradiction as \(<g^2> \neq <g>\). Hence \(U \cong Z_2 \times Z_2\). Thus we have proved (3), (4) and (5).

As \([U, D]\) is a proper \(<g>-invariant subgroup of \(U\), \([U, D] = 1\). Thus \(D = [D, g]C_D(g) = U \times C_D(g)\) by Lemma 2.24. Now \(U \leq N\) and so \(N \leq M = N_G(U)\). Let \(D^* = N_S(D) \leq S^* \in Syl_2(M)\). Again by Corollary 4.8, \(C_N(U) = D \times O_{2'}(N)\). In particular, \(D = C_{D^*}(U)\). As \(N_{S^*}(D^*)\) normalizes \(D^*\) and \(U\), \(N_{S^*}(D^*)\) normalizes \(D = C_{D^*}(U)\) and so \(N_{S^*}(D^*) = D^*\). Thus \(S^* = D^* \in Syl_2(M)\). In particular \(D^* = N_S(U)\) and \(D = C_S(U)\), proving (1) and (2).

\(\Box\)

**Lemma 5.6.** Let \(G\) be a finite group with \(cd(G) = 1\). Let \(P \in Syl_2(G)\). Then one of the following holds:

1. Distinct involutions of \(Z(P)\) are not \(G\)-conjugate; or
2. \(Z(P)\) contains the unique fully fused fours subgroup of \(P\).

**Proof.** Suppose that conclusion (1) fails. Then there exists \(u, v \in Z(P)\) with \(u \neq v\). We have \(v \in u^{N_G(P)}\) by Theorem 2.29. Thus there exists \(x \in \)
of odd order with $u^2 = v$. By Lemma 2.24 $Z(P) = [Z(P), x] \times C_{Z(P)}(x)$. Since $uv^2 \in [Z(P), x]$, we have $[Z(P), x] \neq 1$. So by Theorem 1.3 applied to the solvable group $Z(P) < x >$, we have $Z = [Z(P), x] \cong Z_2 \times Z_2$. Clearly $Z$ is a fully fused fours group and by Lemma 5.4 $Z$ is the only fully fused 4-subgroup of $P$.

□
In Chapter 6 we investigate known simple groups with dihedral and semidihedral Sylow 2-subgroups and their chain differences. We begin with a discussion of properties of the family of groups \( PSL_2(q) \) and \( PGL_2(q) \) where \( q = p^n \) for some prime \( p \) and \( F_q \) is the finite field of order \( q \).

We regard \( GL_2(q) \) as the group of linear transformations of a vector space \( V \) of dimension 2 over a finite field of order \( q \) or equivalently the group of \( 2 \times 2 \) invertible matrices with entries from the field of order \( q \). The subgroup of \( GL_2(q) \) which consists of the matrices of determinant one is called the special linear group and denoted \( SL_2(q) \). The center of \( GL_2(q) \) consists of scalar matrices and the factor group \( GL_2(q)/Z(GL_2(q)) \) is \( PGL_2(q) \). Finally the image of \( SL_2(q) \) in \( PGL_2(q) \) is called the projective special linear group and is denoted by \( PSL_2(q) \)

Henceforth \( p \) is an odd prime and \( q = p^n \) for some \( n \geq 1 \).

**Lemma 6.1.** \( |PSL_2(q)| = q(q-1)(q+1)/2 \). \( |PGL_2(q)| = (q^2 - 1)q \).

**Proof.** The proof for the lemma is contained in [Go;2.8.1].

\( \square \)
**Theorem 6.2.** $PSL_2(q)$ is a simple group for $q \geq 5$. For $q$ odd, we have $PSL_2(q) \leq PGL_2(q)$ of index 2.

*Proof.* The proof that $PSL_2(q)$ is a simple group for $q \geq 5$ can be found in [Go;2.8.3]. By the order formulas given in 6.1, we have $|PGL_2(q) : PSL_2(q)| = 2$.

□

**Theorem 6.3 (Moore-Wiman).** Let $G = PSL_2(q)$ where $q = r^a \geq 5$ and $r$ is an odd prime. Then $G$ has subgroups of the following isomorphism types (in the indicated cases) and every subgroup of $G$ is isomorphic to one of the following groups.

1. The dihedral groups of order $(q \pm 1)$ and their subgroups;
2. The Borel subgroups of $G$, which are Frobenius groups of order $q(q-1)/2$ and their subgroups.
3. The alternating group $A_5$, if $5 || |G|$.
4. The groups $PSL_2(r^b)$ (if $b$ is a proper divisor of $a$) and $PGL_2(r^b)$ (if $2b$ divides $a$).
5. The symmetric group $S_4$, if $8$ divides $|G|$; and
6. The alternating group $A_4$.

*Proof.* This is stated as Theorem 6.5.1 in [GLS2]. It is an immediate consequence of the version of "Dickson's Theorem" presented in [Hu;II.8.27].

□

We note in particular that for $p$ odd, $PGL_2(p) \leq PSL_2(p^2)$. By order considerations there is no proper subgroup that could properly contain $PGL_2(p)$ in
Thus $PGL_2(p)$ is maximal in $PSL_2(p^2)$. Also trivially $PGL_2(2) = PSL_2(2)$ is contained as a maximal subgroup of $PSL_2(4)$.

**Lemma 6.4.** Suppose $G \cong PSL_2(p)$ with $p$ an odd prime. Then a maximal subgroup of $G$ is one of the following groups.

1. A dihedral group of order $p \pm 1$; or
2. A solvable group of order $p(p - 1)/2$; or
3. $A_4$ when $16 \mid p^2 - 1$ and $5 \mid p^2 - 1$ and $p > 3$; or
4. $S_4$ if $p \equiv \pm 1 (\text{mod} 8)$; or
5. $A_5$ if $5 \mid p^2 - 1$.

Every such subgroup of type (2)-(5) is maximal. Moreover if $p > 11$, then subgroups of type (1) are also maximal.

**Proof.** By Theorem 6.3 every maximal subgroup of $G$ is on the list and it suffices to consider containments of one in another. Clearly only subgroups of type (2) have order divisible by $p$. Hence they are always maximal. A dihedral subgroup of $S_4$ or $A_5$ has order at most 10. Hence by order considerations, subgroups of type (1) are maximal whenever $p > 11$. Also if $M$ is a subgroup of type (1) or (2), then $[M, M]$ is cyclic. So the only possible containment of a subgroup of type (3), (4), or (5), in a larger proper subgroup of $G$ are: $A_4 \leq S_4$ or $A_4 \leq A_5$. Clearly this is impossible if $16 \mid p^2 - 1$ and $5 \mid p^2 - 1$. On the other hand let $U \leq M \cong A_4$ with $U \cong Z_2 \times Z_2$ and $U \leq S \in Syl_2(G)$. If $16 \mid p^2 - 1$, $U \neq S$ and so $U \neq N_S(U)$. Hence $M$ is properly contained in $N_G(U) \cong S_4$. If $16 \mid p^2 - 1$, but $5 \mid p^2 - 1$, then $G$ contains a subgroup $M^* \cong A_5$ by Theorem 6.3, and $M^* \geq M_1 = N_G(U_1) \cong A_4$, with $U_1 \cong Z_2 \times Z_2$. As $U \in Syl_2(G)$, $U = U_1^g$ for some $g \in G$ and so $M = M_1^g \leq (M^*)^g$. Thus $M$ is maximal in $G$ if and only
Lemma 6.5. Let $G \cong PSL_2(q)$. Then a Sylow 2-subgroup of $G$ is dihedral.

Proof. Let $q \equiv e \pmod{4}$, $e = \pm 1$. Then by Theorem 6.3, $G$ contains a dihedral subgroup $D$ of order $q - e$ and since $q(q + e)/2$ is odd, $D$ contains a Sylow 2-subgroup $S$ of $G$. As $q - e$ is divisible by 4, $S$ is a dihedral 2-group. □

We now discuss properties of a dihedral group of order $2^n$.

Lemma 6.6. Let $D$ be a dihedral 2-group with $|D| > 4$. Let $a$ and $b$ be involutions in $D$ such that $<a, b> = D$. Then $D$ has 3 conjugacy classes of involutions represented by $a$, $b$, and $z$ where $<z> = Z(D)$. Conversely if $a'$ and $b'$ are non-central involutions of $D$ such that $a'$ is not $D$-conjugate to $b'$, then $<a', b'> = D$. Furthermore let $T_0 = <a, z>$ and $T_1 = <z, b>$. Then $T_1$ is not conjugate to $T_0$ and any four subgroup of $D$ is conjugate to $T_0$ or $T_1$.

Proof. Let $D$ be a dihedral 2-group. Suppose $D = <a, b>$ with $x = ab$ of order $2^n$, then $x^a = x^b = x^{-1}$. Now $|a^D| = |D : C_D(a)| = 1/4|D|$. Also $a^{xj} = x^{-j}ax^j = ax^{2j}$ and so $a^D \geq \{ax^{2j} : 1 \leq j \leq 2^{n-1}\}$. So $a^D = \{ax^{2j} : 1 \leq j \leq 2^{n-1}\}$. Likewise $b^D = \{bx^{2j} : 1 \leq j \leq 2^{n-1}\}$. Note that $b = ax$ and so $bx^{2j} = ax^{2j+1}$. As every noncentral involution of $D$ lies in $a < x >$, every noncentral involution is $D$-conjugate either to $a$ or to $b$, proving the first part of the lemma. Conversely if $a'$ and $b'$ are noncentral involutions in $D$ which are not conjugate, by the above we may assume $a' = ax^{2j}$ and $b' = ax^{2k+1}$. Thus $a'b' = ax^{2j}ax^{2k+1} = x^{2(k-j)+1}$. Since $2(k - j) + 1$ is odd, $<x^{2(k-j)+1} >= <x>$
thus $a'b'$ is a generator of the maximal cyclic subgroup of $D$ thus $\langle a'b',a' \rangle = \langle a',b' \rangle = D$.

Now let $T_0 = \langle a,z \rangle$ and $T_1 = \langle b,z \rangle$. Then $az \in a^D$. Thus $T_0$ is not conjugate to $T_1$, since $a$ is not conjugate to $b$. Since every fours group in $D$ is of the form $\langle c,z \rangle$ where $c \in D - \langle x \rangle$, and $c$ is $D$-conjugate to either $a$ or $b$ with $z \in Z(D)$, we have $\langle c,z \rangle$ is conjugate to either $T_0$ or $T_1$.

\[ \square \]

**Lemma 6.7.** Let $G \cong PSL_2(q)$, $q$ odd. Then $G$ has one conjugacy class of involutions.

*Proof.* The result is clear if $G \cong PSL_2(3) \cong A_4$. By Theorem 6.2, $G$ is simple if $q \geq 5$. By Lemma 6.5 if $D \in Syl_2(G)$, then $D$ is dihedral. Let $C$ be a maximal cyclic subgroup of $D$. By Theorem 2.30, every involution of $D$ is $G$-conjugate to the unique involution $z \in C$ and the result follows. \[ \square \]

**Lemma 6.8.** Let $L \cong PSL_2(p)$ with $p > 11$ and let $z$ be an involution of $L$. Then $C_L(z) = D \cong D_{p-\epsilon}$ where $p \equiv \epsilon(\text{mod}4)$, and $D$ is maximal in $L$.

*Proof.* By Lemma 6.4, $L$ contains a maximal subgroup $D \cong D_{p-\epsilon}$ and $Z(D) = \langle z \rangle$ for some involution $z$ of $L$. As $D$ is maximal and $Z(L) = 1$, $D = C_L(z)$. As all involutions of $L$ are conjugate, we are done.

\[ \square \]

**Lemma 6.9.** Let $L \cong PSL_2(p)$ with $p > 11$ and let $U$ be a 4-subgroup of $L$. Then $C_L(U) = U$ and $N_L(U) \cong A_4$ or $S_4$.
Proof. By Lemma 6.8, if \( u \in U \), then \( C_L(u) = D \cong D_{p-\epsilon} \). Thus \( C_L(U) = C_D(U) = U \). If \( U \in Syl_2(L) \) then as \( L \) has one class of involutions by Lemma 6.7, \( N_L(U) \cong A_4 \) by Theorem 2.29. If not then as all involutions in \( L \) are \( L \)-conjugate, for each \( u \in U \), \( U \leq D_u \in Syl_2(C_L(u)) \) with \( D_u \) dihedral of order at least 8. Thus for each \( u \in U \), \( U \leq R_u \leq D_u \) with \( R_u \cong D_8 \) and \( Z(R_u) = \langle u \rangle \). As \( U = C_L(U) \), it is immediate that \( N_L(U) \cong S_4 \).

\[ \square \]

Lemma 6.10. Let \( G \cong PGL_2(p) \) for \( p \) an odd prime with \( p > 11 \) and \( p \equiv \epsilon \,(\text{mod}4), \epsilon = \pm 1 \). Let \( L = [G,G] \cong PSL_2(p) \). Then

(1) There are 2 conjugacy classes of involutions in \( G \).

(2) Let \( v \) be an involution in \( G - L \). Then \( C_L(v) \cong D_{p+\epsilon} \).

Proof. By Lemma 6.3,

\[ L \leq G \leq L^* = PSL_2(p^2). \]

Since \( L \) and \( L^* \) have dihedral Sylow 2-subgroups, it is immediate that \( G \) has a dihedral Sylow 2-subgroup \( S \). By Lemma 6.7, \( S \) has 3 conjugacy classes of involutions \( \{z\} \cup a^S \cup b^S \). Since \( S \cap L \) contains a four-group and \( L \) has only one conjugacy class of involutions by Lemma 6.7, we may assume that \( \{z\} \cup a^S \leq L \leq z^G \). If \( b \in L \), then \( S = \langle a, b \rangle \leq L \), a contradiction. So \( b \in G - L \) and so \( z^G \) and \( b^G \) are the two \( G \)-classes of involutions. Suppose \( p \equiv \epsilon \,(\text{mod}4) \). By Lemma 6.3 \( L \) contains a dihedral group \( D \) of order \( p + \epsilon \) with \( Z_{p+\epsilon/2} \cong T \leq D \).

Let \( r \) be a prime divisor of \( |T| \) and let \( R \in Syl_r(T) \). As \( \gcd(p(p - \epsilon), (p + \epsilon)/2) = 1, R \in Syl_r(G) \). Also \( R \leq D \) and as \( D \) is maximal in \( L \) by Lemma 6.4, \( D = N_L(R) \) and \( T = C_L(R) \). By Theorem 2.7, \( G = N_G(R)L \). Thus if

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\( D^* = N_G(R) \), then \(|D^* : D| = 2 \) and \( T \leq D^* \).

As \( D^* \) is a nonabelian subgroup of \( L^* \) of order \( 2(p + e) \) with a cyclic normal subgroup of order \( (p+e)/2 \geq 7 \), it follows by inspection of the list of subgroups of \( \text{PSL}_2(q) \) in Theorem 6.3 that \( D^* \) is contained in a dihedral subgroup \( Q \) of \( L^* \) of order \( p^2 - 1 \) since \( \gcd(p^2 + 1, p^2 - 1) = 2 \). As \( Q \) is dihedral, so is \( D^* \) and a Sylow 2-subgroup \( U \) of \( D^* \) is a Klein 4-group. Thus for some \( v \in U \), \( <v> = Z(Q) \) and so \( <v> = Z(D^*) \) and \( D^* = D \times <v> \). Thus \( v \in G - L \) and \( D^* \leq C_G(v) \). As \( D \) is maximal in \( L \) by Proposition 6.4, \( D^* = C_G(v) \) and \( D = C_L(v) \), as claimed.

\( \square \)

**Lemma 6.11.** Let \( A \) be the group of automorphisms of \( \text{PSL}_2(F) \) over any field \( F \). Then there is a normal subgroup \( G_o \) such that

\[ A \triangleright G_o \cong \text{PGL}_2(F) \text{ and } A/G_o \cong \text{Aut}(F). \]

**Proof.** The proof of the lemma can be found in [Su2:8.8] \( \square \)

**Lemma 6.12.** Let \( L \cong \text{PSL}_2(p) \) with \( p \) a prime and \( p > 11 \) and let \( L \leq L^* \cong \text{PGL}_2(p) \). Let \( Y \leq L^* \) with \( Y \cong E_4 \). Then we have

\[ L = \langle C_L(y) : y \in Y^\# \rangle. \]

**Proof.** Let \( p \equiv e(\text{mod} 4) \) and let \( z \in Y^\# \cap L \). By Lemma 6.8, \( C_L(z) = D \cong D_{p-e} \) with \( D \) maximal in \( L \) and \( <z> = Z(D) \). Let \( Y = <z,y> \). If \( y \in L \), then by Lemma 6.7 \( y \in z^L \) and so \( C_L(y) \cong D \). As \( y \not\in Z(D) \), \( C_L(y) \not\leq D \) and so
\[ L = \langle C_L(z), C_L(y) \rangle, \] as claimed. If \( y \notin L \), then by Lemma 6.10, \( C_L(y) \cong D_{p+e} \) and so \( C_L(y) \not\leq D \). Again \( L = \langle C_L(z), C_L(y) \rangle, \) as claimed.

\[ \square \]

**Lemma 6.13.** *If \( cd(PSL_2(q)) = 1 \) then \( q = p \) or \( p^2 \) for some prime \( p \).*

**Proof.** Let \( T = \langle \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \rangle \) with \( \lambda \in F \), where \( F \) is the field of order \( q \). We have \( T \cong F^+ \). Let \( D = \langle \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \rangle \) with \( \omega \in F^* \). We have \( D \cong F^* \). We have \( T \) and \( D \) are subgroups of \( SL_2(q) \). Let \( \overline{T} \) and \( \overline{D} \) be the images of \( T \) and \( D \) respectively in \( PSL_2(q) \). It is clear that \( \overline{D} \) normalizes \( \overline{T} \) and \( \overline{T} \cap \overline{D} = 1 \). Hence \( \overline{B} = \overline{T} \times \overline{D} \) is a subgroup of \( PSL_2(q) \). We argue that \( \overline{D} \) acts irreducibly on \( \overline{T} \).

Let \( \overline{T}_{\lambda_1} = \begin{pmatrix} 1 & 0 \\ \lambda_1 & 1 \end{pmatrix} \). Let \( \overline{T}_{\lambda_2} = \begin{pmatrix} 1 & 0 \\ \lambda_2 & 1 \end{pmatrix} \). Let \( \overline{D}_\omega = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \). Then

\[ \overline{D}_\omega \overline{T}_{\lambda_1} \overline{T}_{\lambda_2}^{-1} = \overline{T}_{\lambda_1 \omega^{-2}}. \]

Thus \( \overline{T}_{\lambda_1} \) is conjugate to \( \overline{T}_{\lambda_2} \) by an element of \( \overline{D} \) if and only if there exists an \( \omega \in F \) such that \( \lambda_1 \omega^{-2} = \lambda_2 \). Thus \( \overline{D} \) acts transitively on \( \overline{T}_D \) if \( p = 2 \), while \( \overline{D} \) acts on \( \overline{T}_D \) with two orbits of size \( (q-1)/2 \) if \( q \) is odd. If \( \overline{D} \) acts reducibly on \( \overline{T} \), then \( \overline{T} \) contains an \( \overline{D} \) invariant subgroup of order \( p^m \) with \( 0 < m < n \). But \( p^m - 1 < p^n - 1/2 \), a contradiction. Therefore \( \overline{D} \) acts irreducibly on \( \overline{T} \).

Now let \( \overline{B} = \overline{T D} \). We have \( \overline{B} \) is a solvable group with \( cd(\overline{B}) \leq 1 \), thus by Lemma 4.5 any noncyclic chief factor of \( \overline{B} \) has order at most \( p^2 \) for some prime \( p \). Since \( \overline{D} \) acts irreducibly on \( \overline{T} \), \( \overline{T} \) is a chief factor of \( \overline{B} \), hence has order \( p \) or \( p^2 \).
The following lemmas give bounds on the lengths of chains in the group $PSL_2(p)$.

**Corollary 6.14.**

(1) $\lambda(PSL_2(p)) \leq 1 + \min \{\Omega(p - 1), \Omega(p + 1)\}$.  
(2) $\lambda(PSL_2(p^2)) \leq 3 + \min \{\Omega(p - 1), \Omega(p + 1)\}$.

**Proof.** By Lemma 6.4 we see that for $p > 11$, $PSL_2(p)$ contains maximal dihedral subgroups of order $p \pm 1$. Since dihedral groups are solvable they have a maximal chain of length $\Omega(p \pm 1)$. Thus for $p > 11$, we can construct maximal chains in $PSL_2(p)$ of length $\Omega(p \pm 1) + 1$. In particular $\lambda(PSL_2(p)) \leq 1 + \min \{\Omega(p + 1), \Omega(p - 1)\}$.

For $p \in \{2, 3, 5, 7, 11\}$, it is easy to check that the Borel subgroups of order 2, 3, 10, 21, and 55 respectively are maximal of length 1, 1, 2, 2, and 2, as desired. Hence for all primes $p$ the inequality holds and (1) is proved.

For (2) we take a maximal chain of $PSL_2(p)$ of length $\lambda(PSL_2(p))$ and extend it to a maximal chain of $PSL_2(p^2)$, passing through $PGL_2(p)$ which is maximal in $PSL_2(p^2)$ by Theorem 6.3 and the remark which follows it. This implies by (1) that $\lambda(PSL_2(p^2)) \leq 3 + \min\{\Omega(p + 1), \Omega(p - 1)\}$.

□
Corollary 6.15. We have \( l(\text{PSL}_2(p^2)) \geq 2 + \Omega(p - 1) + \Omega(p + 1) \).

Proof. For any odd prime \( p \), \( \text{PSL}_2(p^2) \) contains a solvable subgroup \( B \) of order \( p^2(p^2 - 1)/2 \) by Theorem 6.3. We have

\[
l(B) = 1 + \Omega(p + 1) + \Omega(p - 1).
\]

Therefore we have a chain in \( \text{PSL}_2(p^2) \) of length

\[
2 + \Omega(p + 1) + \Omega(p - 1).
\]

This gives the desired result for odd \( p \). It is easy to check for \( \text{PSL}_2(4) \).

\[\square\]

Proposition 6.16. \( cd(\text{PSL}_2(p^2)) = 1 \) if and only if \( p \leq 3 \).

Proof. Assume \( cd(\text{PSL}_2(p^2)) = 1 \). From Corollary 6.14 and Corollary 6.15 we have the following inequality:

\[
\begin{align*}
\text{cd}(\text{PSL}_2(p^2)) &\geq 2 + \Omega(p - 1) + \Omega(p + 1) - (3 + \min\{\Omega(p + 1), \Omega(p - 1)\}) \\
&= \max\{\Omega(p + 1), \Omega(p - 1)\} - 1.
\end{align*}
\]

This implies \( \max\{\Omega(p - 1), \Omega(p + 1)\} \leq 2 \). Since 4 divides either \( p + 1 \) or \( p - 1 \) if \( p \) is odd, we conclude that \( p \pm 1 \leq 4 \). Therefore \( p \leq 5 \). In \( \text{PSL}_2(5^2) \) there is a dihedral subgroup of order 24. Hence \( l(\text{PSL}_2(5^2)) \geq 5 \). There is also a chain of
length 3 through the dihedral subgroup $D$ of order 26. By order considerations, none of the proper subgroups of $PSL_2(5^2)$ have order divisible by 26 except $D_{26}$. Thus $D$ is maximal in $PSL_2(5^2)$. Therefore we have the $cd(PSL_2(5^2)) > 1$. Hence $p \leq 3$ as claimed.

It remains to show that $cd(PSL_2(4)) = cd(PSL_2(9)) = 1$. Each maximal subgroup of $PSL_2(9)$ is isomorphic to one of the following: $A_5$, $S_4$, or $B$, a solvable group of order 36. By inspection every maximal chain in $PSL_2(9)$ has length 4 or 5. Hence $PSL_2(9)$ has chain difference 1, as claimed. The maximal subgroups of $PSL_2(4)$ are isomorphic to $S_3$, $A_4$, and $D_{10}$. By inspection every maximal chain in $PSL_2(4)$ has length 3 or 4. Hence $PSL_2(4)$ has chain difference one as well.

$\square$

We now prove a result which in particular proves the second part of Theorem 1.2

**Lemma 6.17.** $PSL_2(q)$ has chain difference one if and only if

1. $q = 4$ or $9$, or
2. $q$ is an odd prime, $5|q^2 - 1$ or $16|q^2 - 1$, and $3 \leq \Omega(q \pm 1) \leq 4$, or
3. $q$ is an odd prime, and the conditions $5 \mid q^2 - 1$ and $16 \mid q^2 - 1$ hold (equivalently, $q \equiv 3, 13, 27, 37(\mod 40)$), and $2 \leq \Omega(q \pm 1) \leq 3$.

**Proof.** Let $G \cong PSL_2(q)$ with $cd(G) = 1$. If $q = 2^n$, then $q = 4$ by Lemma 6.13.

Thus we may assume $q$ is odd. By Lemma 6.13 and Proposition 6.16, either $q = 9$ or $q$ is a prime. Maximal subgroups of type (1) or (2) from Lemma 6.4 are supersolvable groups $\mathcal{M}$ with $\lambda(\mathcal{M}) = l(\mathcal{M}) = \Omega(q \pm 1)$. In particular if
If $q \in \{7, 11\}$, then $\lambda(G) = 3$. If $N \cong S_4$ or $A_5$, then $\lambda(N) = 3$ and $l(N) = 4$. Thus in particular $cd(PSL_2(q)) > 1$ for $q \in \{7, 11\}$. Henceforth we may assume $q > 11$, whence dihedral subgroups of order $p \pm 1$ are maximal by Lemma 6.4. Thus if either $5|q^2 - 1$ or $16|q^2 - 1$, then $cd(G) = 1$ if and only if $3 \leq \Omega(q \pm 1) \leq 4$. This yields (2). If neither 16 nor 5 divides $q^2 - 1$, then $N \cong A_4$ is maximal in $G$ with $\lambda(N) = 2$ and $l(N) = 3$. This yields (3).

□

Lemma 6.18. Let $L \cong PSL_2(9)$. Then $|Aut(L) : L| = 4$ and $Aut(L)$ has three subgroups of index 2, only one of which is isomorphic to $S_6$.

Proof. By Lemma 6.10, $|Aut(L)/L| = 4$. Since $L \cong A_6$, it is clear $Aut(L)$ contains a subgroup of index 2 containing $L$ isomorphic to $S_6$. Furthermore $S_6$ contains involutions not in $L$. By Lemma 6.11, $Aut(L)$ contains a subgroup of index 2 isomorphic to $PGL_2(9)$ containing $L$ which also contains involutions outside $L$. Thus we have $Aut(L)/L \cong Z_2 \times Z_2$. Let $S \in Syl_2(Aut(L))$ and let $D = S \cap L$ with $D \cong D_6$. Then $S \sigma S = D \times < \sigma >$ for some transposition $\sigma$. Also by Lemma 6.10, $S \cap PGL_2(9) = D_1$ is dihedral of order 16. So $PGL_2(9) \not\cong S_6$. Let $M$ denote the third subgroup with $|M : L| = 2$. If $M \cong S_6$, then $M \sigma S = D \times < \tau >$ for some involution $\tau$. But then $\sigma \tau \in S \cap PGL_2(9)$ with $[D, \sigma \tau] = 1$ and $\sigma \tau \notin D$, contrary to the fact that $D \cong D_{16}$. So $M \not\cong S_6$, proving the lemma. □

Lemma 6.19. If $A_6 \leq G \leq AutA_6$ with $cd(G) = 1$, then $G = A_6$ or $S_6$. 55
Proof. It is easy to check that $cd(S_6) = 1$. Now suppose that $A_6 \leq G \leq AutA_6$ with $[G : A_6] = 2$ and $G \neq S_6$. There exists an unrefinable chain of length 6:

$$G > A_6 > A_5 > A_4 > V_4 > Z_2 > 1.$$ 

Let $P \in Syl_5(G)$ and $N = N_G(P)$. We have $|N| = 20$. We argue $N$ is maximal in $G$, whence $G$ has an unrefinable chain of length 4. Suppose that $N$ is properly contained in $M$ a maximal subgroup of $G$. Since $N = N_G(P)$, $M$ must contain at least two Sylow 5-subgroups, thus must contain at least 6 Sylow 5-subgroups by Sylow's Theorem. Hence $|M| \geq 120$, whence $|G : M| \leq 6$. Since $A_6$ is the unique proper normal subgroup of $G$, $G$ embeds in $S_6$, a contradiction to Lemma 6.18.

\[\square\]

Lemma 6.20. Let $L \cong A_6$ with $L \leq L^* \cong S_6$. Then $L$ has two orbits $\{U^L, V^L\}$ on the 4-subgroups of $L$. Moreover $U^L = U^{L^*}$ and $V^L = V^{L^*}$.

Proof. Let $U = \langle (12)(34), (13)(24) \rangle$ and $V = \langle (12)(34), (12)(56) \rangle$. Clearly $U$ and $V$ are 4-subgroups of $L$ with $UV \in Syl_2(L)$ and $V$ not $L^*$ conjugate to $U$. As $U$ and $V$ are the only 4-subgroups of $UV$, Sylow's Theorem implies that $U^L$ and $V^L$ are the two $L$-orbits on the 4-subgroup of $L$. Moreover $U^L = U^{L^*}$, $V^L = V^{L^*}$.

\[\square\]


Proof. As $PSL_2(7)$ contains a subgroup isomorphic to $S_4$ of index 7, we see that $PSL_2(7)$ is a subgroup of $A_7$. Thus as $cd(PSL_2(7)) > 1$ by Lemma 6.17, we have that $cd(A_7) > 1$. 

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The remainder of the Chapter is a discussion of 3-dimensional linear groups and unitary groups. The following result is a special case of a theorem in [GLS2;6.5.3] which we will need.

**Theorem 6.22.** Let $K = SU_3(p) = SU_3(V)$ where $p$ is an odd prime and $V$ is the natural module for $K$. Then every subgroup of $K$ is isomorphic to a subgroup of:

1. a Borel subgroup, i.e. a solvable group of upper triangular matrices;
2. the stabilizer of a nonsingular line in $V$, which is isomorphic to $GU_2(p)$;
3. an extension of an abelian group of order $(p+1)^2$ by $S_3$;
4. a Frobenius group with kernel of order $p^2 - p + 1$ and complement of order 3;
5. the group $O_3(p) \cong PGL_2(p)$;

and for certain values of $p$:

6. a section of the solvable group $GU_3(2)$;
7. the group $L_3(2)$;
8. the group $A_5$;
9. the groups $M_{10}$ and $A_7$ (for $p = 5$ only).

**Proof.** As stated above this is a special case of a theorem proved in [GLS2;6.5.3].

**Theorem 6.23.** Let $q = p^n$ for $p$ an odd prime. The group $PSL_3(q)$ has chain difference greater than one.
Proof. We have $PSL_3(q)$ contains $PSL_3(p)$ as a subgroup so it suffices to show $SL_3(p)$ has chain difference greater than 1. Let $M$ be a maximal parabolic subgroup of $G = SL_3(p)$ and let $P = O_p(M)$. We have $P \cong Z_p \times Z_p$ and $C_M(P) \cong P \times Z(G)$. Thus $C_M(P) \leq O_{p,p}(M)$, whence $M$ is $p$-constrained. However $M$ is nonsolvable if $p > 3$, whence $cd(M) > 1$ by Lemma 4.6. If $p = 3$, then $O^2(P) \cong (Z_3 \times Z_3)SL_2(3)$ does not have a normal Sylow 2-subgroup. So again $cd(P) > 1$ by Theorem 1.3 and the result holds. □

The following argument appears in [BWZ].

Theorem 6.24. Let $G \cong U_3(q)$ for $q = p^n$, $p$ a prime, $q \equiv 1 (mod 4)$. Then $cd(G) > 1$.

Proof. If $t$ is any involution in $G$, then the matrix for $t$ with respect to a suitable orthonormal basis for the natural module $V$ is

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
$$

We have $C = C_G(t) \cong GU_2(q)$ and contains a normal subgroup $K$ isomorphic to $SU_2(q) \cong SL_2(q)$. Thus $G$ has a section isomorphic to $PSL_2(q)$. Since $G$ has chain difference one we may assume $q = p$ for some prime $p$ with $p \geq 5$ or $q = 9$. The group $U_3(5)$ contains a subgroup isomorphic to $A_7$ by Theorem 6.22 and $A_7$ has chain difference greater than 1 by Lemma 6.21. We argue $q \neq 9$. We may assume $t \in Z(S)$ for some $S \in Syl_2(G)$. Hence $K/ < t > \cong PSL_2(9)$ and $|KS/ < t > : K/ < t > | = 2$. Thus $KS/ < t >$ is isomorphic to a subgroup of $58$. 


index 2 in $Aut(PSL_2(9))$. The only subgroup of index 2 in $Aut(PSL_2(9))$ with chain difference one is $S_6$. The Sylow 2-subgroup of $S_6 \cong D_8 \times Z_2$. However $S$ is semidihedral of order 32; thus $S/\langle t \rangle$ has a maximal cyclic subgroup of order 8, a contradiction. Thus we may assume $p \geq 11$.

By Theorem 6.24, $G$ contains $M \cong PGL_2(p)$. We argue that $M$ is maximal in $G$. Since $M$ is nonsolvable, $M$ cannot be contained in a subgroup of type (1), (3), (4), or (6). Since $p \geq 11$, $M$ is not contained in a subgroup of type (7) or (8). Finally if $M_1$ is of type (2), then $M_1' \cong SL_2(p)$. So $M_1'$ contains a unique involution, whereas $M' \cong PSL_2(p)$. Thus $M$ is maximal in $G$ and

$$
\lambda(G) \leq 1 + \lambda(M) \leq 2 + \lambda(PSL_2(p)).
$$

On the other hand

$$
l(G) \geq 1 + l(KS) \geq 3 + l(K/\langle t \rangle) = 3 + l(PSL_2(p)) \geq 4 + \lambda(PSL_2(p)).
$$

Thus

$$
cd(G) = l(G) - \lambda(G) \geq 2,
$$

as claimed.
In Chapter 7 we begin the inductive proof of Theorem 1.4. We first reduce to a minimal counterexample, then we show \( F \neq \emptyset \), giving the existence of a fully fused fours group in \( G \). For completeness we restate Theorem 1.4.

**Theorem 1.4.** Let \( G \) be a finite group of chain difference one. Then one of the following holds.

1. \( G \) is solvable; or
2. \( E(G)/Z(E(G)) \) is isomorphic to \( A_6 \) or \( PSL_2(p) \) for some prime \( p \) satisfying the conditions of Theorem 1.1 and \( G/E(G) \) is supersolvable.

The proof is by induction on the order of the group. Thus for the remainder of this dissertation we assume that \( G \) is a minimal counterexample to Theorem 1.4.

**Lemma 7.1.** \( G \) is simple.

**Proof.** As \( G \) is a counterexample, \( G \) is not solvable. Hence by Theorem 4.7, \( E(G) \) is a quasisimple group with \( G/E(G) \) supersolvable. If \( G \neq E(G) \), then
\( E(G) \) is a proper subgroup of \( G \), hence by induction, \( E(E(G))/Z(E(E(G))) \) is on our list of conclusions. However \( E(E(G)) = E(G) \); thus the theorem is true for \( G \). Thus we have \( G = E(G) \). If \( G \) is not simple, that is if \( Z(G) \neq 1 \), then \( |G/Z(G)| < |G| \). In particular since \( G/Z(G) \) is a simple group, by induction it is on the list of conclusions. Therefore we have \( G = E(G) \) and \( Z(G) = 1 \), hence the minimal counterexample is a simple group. □

The following is a structure theorem for proper nonsolvable subgroups of \( G \).

**Theorem 7.2.** Suppose \( H < G \) and \( H \) is not solvable. We have that \( E(H) = L \) is isomorphic to one of the following quasisimple groups: \( SL_2(p) \), \( PSL_2(p) \), \( A_6 \), \( 2A_6 \), \( 3A_6 \) or \( 6A_6 \). We have \( L \trianglelefteq H \), with \( H/L \) supersolvable and \( CH(L) = O_{2',2}(H) \). Moreover \( [O^2(H) : L] \) is odd and \( H \neq N_G(E) \) for any elementary 2-subgroup \( E \) of \( H \) with \( U \leq E \) for some \( U \in \mathcal{F}_H \).

**Proof.** Since \( H \) is properly contained in \( G \) with \( G \) a minimal counterexample, we have \( L \) is isomorphic to one of the following; \( SL_2(p) \), \( PSL_2(p) \), \( A_6 \), \( 2A_6 \), \( 3A_6 \) or \( 6A_6 \). By Theorem 4.7 we have \( L \trianglelefteq H \) with \( L \) quasisimple and \( H/L \) supersolvable.

Next we prove \( CH(L) = O_{2',2}(H) \). We have that \( CH(L) \ast L/L \cong CH(L)/Z(L) \) embeds in \( H/L \), thus is supersolvable. Therefore by Lemma 3.13, \( CH(L) \) is 2-nilpotent. Since \( CH(L) \trianglelefteq H \) we have \( CH(L) \leq O_{2',2}(H) \) by Lemma 3.14. We argue \( O_{2',2}(H) \leq CH(L) \). We have \( O_{2',2}(H) \cap L \leq Z(L) \). Thus

\[
[L, O_{2',2}(H), L] = 1
\]

and

\[
\]
By Theorem 2.9,

\[[L, L, O_{2'}(H)] = [L, O_{2'}(H)] = 1.\]

Therefore \(O_{2'}(H) \leq C_H(L)\). Therefore \(C_H(L) = O_{2'}(H)\).

As \(H/L\) is supersolvable, \(O^2(H/L)\) has odd order, whence \([O^2(H) : L]\) is odd.

If \(U \in \mathcal{F}_H\), then by Lemma 5.2, \(U \leq O^2(H)\) and so \(U \leq L\). If \(U \leq E \leq H\) then \(U \leq E \cap L \leq L\) and so \(U \leq Z(L)\). But \(Z(L)\) is cyclic, a contradiction.

\[\Box\]

The next theorem in this chapter guarantees the existence of a fully fused fours group in \(G\). It is of course an instance of Glauberman's celebrated \(Z^*-\)Theorem. In our context an easy argument reduces us to the Brauer-Suzuki case.

**Theorem 7.3.** Let \(S \in Syl_2(G)\) and let \(z \in Z(S)\) of order 2. Then \(z^G \cap S \neq \{z\}\).

**Proof.** Let \(S \in Syl_2(G)\) and let \(z \in Z(S)\) of order 2. As \(G\) is simple, we have \(C_G(z)\) is a proper subgroup of \(G\) with \(S \leq C_G(z)\). Since \(z^G \cap S = \{z\}\), \(<z>\) is weakly closed in \(S\). By Lemma 2.28 since \(<z> \leq Z(S)\), we have \(C_G(z)\) controls the 2-fusion in \(G\). Then since \(G = O^2(G)\), \(C_G(z) = O^2(C_G(z))\), by Lemma 2.28. Hence by Theorem 7.1 either \(C = E(C) \ast O_{2'}(C)\) with \(E(C) \cong SL_2(p)\) or \(2A_6\) or \(6A_6\), or \(C = (Q \times O_{2'}(C), x)\) with \(Q \cong Q_8 \in Syl_2(C)\). In either case \(m_2(C) = m_2(G) = 1\). However by Theorem 2.1, \(m_2(G) > 1\), a contradiction.

\[\Box\]

**Corollary 7.4.** \(\mathcal{F} \neq \emptyset\).

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Proof. Let $S \in Syl_2(G)$ and $z$ be an involution in $Z(S)$. By Theorem 7.3, $z^G \cap S \neq \{z\}$. Then by Theorem 5.5, there exists $D \leq S$ with $z \in D$ and $z^N \neq \{z\}$, where $N = N_G(D)$. Moreover there exists $V \in \mathcal{F}_N \leq \mathcal{F}$. So $\mathcal{F} \neq \emptyset$, as claimed

For completeness we restate Theorem 1.3 which gives the structure of solvable subgroups of $G$.

**Theorem 1.3.** Suppose $H < G$ and $H$ is solvable. Then either

(1) $H$ is 2-nilpotent; or

(2) There exists $U \leq H$ such that $U \cong Z_2 \times Z_2$ or $Q_8$. Furthermore there exists $x \in H$ with $U = [U, x]$ and $x^3 \in O_2'(H)$ and $(U \times O_2'(H), x) = O^2(H)$.

The following theorem is a structure theorem for subgroups of $G$ which are normalizers of fully fused fours groups.

**Theorem 7.5.** Suppose $N \leq G$ with $V \in \mathcal{F}_N$. Suppose that either $V \leq N$ or $N$ is solvable. Let $T \in Syl_2(N)$ and let $T \cap C_N(V) = T_o \in Syl_2(C_N(V))$. Then there exists $x \in N$ with $V = [V, x]$ and $x^3 \in O_2'(N)$ and $(V \times O_2'(N), x) = O^2(N)$. Furthermore, $T_o \in Syl_2(O_2'(N))$ and we can choose $x \in N_N(T_o)$ with $T_o = [T_o, x] \times C_{T_o}(x)$ and $[T_o, x] = V$. We have $\mathcal{F}_N = \{V\}$ and $V \leq N$.

**Proof.** By Theorem 7.2 $N_G(V)$ is solvable and so in any case $N$ is solvable.

By Theorem 1.3, there exists $U \leq N$ with $U \cong Z_2 \times Z_2$ or $Q_8$ and with $N/U$ supersolvable. Again as $V$ is an $N$-chief factor, $V \leq U$. Hence $U = V \leq N$ and...
\[ \mathcal{F}_N = \{ V \}. \]

By Lemma 4.8, we have \( C_N(V) = O_{2',2}(N) \) and \( T_o \in Syl_2(C_N(V)) \). By Theorem 2.7 \( N = O_{2',2}(N)N(T_o) = O_{2'}(N)N(T_o) \). This implies \( V \) is fully fused in \( N(N(T_o)) \). Therefore by the first paragraph there exists an \( x \in N(N(T_o)) \) with \([V, x] = V \) and \( x^3 \in O_{2'}(N) \). We have \( T_o = [T_o, x]C_{T_o}(x) \) by Lemma 2.24. We argue \([T_o, x] = V \). By Theorem 1.3, we have \( O^2(N) = \langle V \times O_{2'}(N), x \rangle \).

Since \([T_o, x] \leq O^2(N)\) and \([T_o, x] \leq T_o \) a 2-group and \( V \) is the unique 2-group in \( O^2(N) \), we have \([T_o, x] \leq V \). Since \([V, x] = V \), \([T_o, x] = V \) and \( T_o \) is the semidirect product of \([T_o, x] \) and \( C_{T_o}(x) \). Since \( V \leq Z(T_o) \), we have \( T_o = [T_o, x] \times C_{T_o}(x) \).

\[ \square \]

**Lemma 7.6.** Let \( N = N_G(V) \) for \( V \in \mathcal{F} \). Let \( T \in Syl_2(N_G(V)) \) and let \( T_o \) and \( x \) be as in Theorem 7.5. Let \( C_{T_o}(x) = S_o \) and let \( s \in Z(T) \cap S_o \). Then

1. \( S_o \trianglelefteq T \) and \( Z(T) \cap S_o \neq 1 \); and
2. \( s^N \cap T = \{ s \} \) for every involution in \( Z(T) \cap S_o \).

**Proof.** Let \( \overline{N} = N/O_{2'}(N) \). Then \( \overline{T_o} = \overline{V} \times \overline{S_o} = O_2(N) \) and \( \overline{V} < \overline{s} > = O^2(N) \). Thus \( \overline{S_o} = C_{\overline{N}}(O^2(\overline{N})) \) and so \( S_o = T \in Syl_2(C_{\overline{N}}(O^2(\overline{N}))) \). As \( C_{\overline{N}}(O^2(\overline{N})) \leq N \), we have that \( S_o = T \cap C_{\overline{N}}(O^2(\overline{N})) \) which is normal in \( T \). Hence \( Z(T) \cap S_o \neq 1 \).

Let \( s \) be an involution in \( Z(T) \cap S_o \). Let \( \overline{s} \) be the image of \( s \) in \( \overline{N} \). By Theorem 7.5, \( O^2(N) = \langle V \times O_{2'}(N), x \rangle \). Hence \( \overline{N} = O^2(\overline{N})\overline{T} = (\overline{T}, \overline{x}) \). We have \( \overline{s} \in Z(\overline{T}) \cap \overline{S_o} \) and \([\overline{s}, \overline{x}] = 1 \). Hence \( \overline{s} \in Z(\overline{N}) \). In particular \( \overline{s}^N \cap \overline{T} = \{ \overline{s} \} \). This implies \( s^N \cap T \) is contained in the coset \( sO_{2'}(N) \). However \( | < s > O_{2'}(N)|_2 = 2 \). Thus \( < s > \in Syl_2(< s > O_{2'}(N)) \), whence \( s^N \cap T = \{ s \} \).

\[ \square \]

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We conclude with a technical lemma which will be applied both in Sections 8 and 9.

**Lemma 7.7.** $G$ does not have a a Sylow 2-subgroup $S$ with a maximal subgroup $T$ satisfying the conditions:

(a) $T = D_1 \times < s >$ with $D_1$ nonabelian dihedral with center $< u >$ and $s^2 = 1 \neq s$;
(b) $D_1 \leq S$;
(c) $T \in Syl_2(C_G(s))$;
(d) Every involution of $D_1$ is in $u^G$.
(e) If $U$ is a 4-subgroup of $D_1$ and $y \in S - T$, then $U^y$ is not $T$-conjugate to $U$.

**Proof.** As $D_1 \leq S$, we have $u \in Z(S)$ and $|S/D_1| = 4$. If $S/D_1$ is cyclic, then by Lemma 2.30, we have that $s$ is $G$-conjugate to an element of $D_1$ which implies $s$ is $G$-conjugate to $u$. However $|C_G(s)|_2 = |T| < |S| = |C_G(u)|_2$, by (c), a contradiction. Therefore $S/D_1$ is a fours group and contains 3 maximal subgroups.

We have that $S = D_1 \cup sD_1 \cup yD_1 \cup ysD_1$ for some $y \in S - T$. We argue each coset except $D_1$ must contain an involution conjugate to $s$. The subgroup $< D_1, y >$ has index 2 in $S$, thus by Lemma 2.30 $s$ must be conjugate to an element of $< D_1, y >$. As $s$ is not conjugate to any element of $D_1$ it is the case that $s$ is conjugate to an element of $yD_1$. This implies the coset $yD_1$ contains an involution; hence we may assume $y$ is an involution. By the same argument
the coset $ysD_1$ contains an involution conjugate to $s$. Let $ysd \in ysD_1$ with

$$ysdysd = 1.$$ 

Then

$$suydysd = 1$$

since $ysy^{-1} = su$, and so

$$y^{-1}dy = usd^{-1}s = d^{-1}u$$

since $y = y^{-1}$, $s^2 = 1$ and $<u, s> \leq Z(T)$.

In particular we claim this implies $d$ is not an involution. If $d$ is an involution then $<d, u>$ is normalized by $y$. But $y$ doesn’t normalize any fours subgroup of $D_1$ by (e). Likewise if $v$ is an involution of $D_1 – <u>$, then $v^y$ is not $D_1$-conjugate to $v$ and so $D_1 = <v, v^y>$ by Lemma 6.6. Therefore we have $d$ is an element of the maximal cyclic subgroup $<vv^y>$ of $D_1$. We look at the action of $y$ on $vv^y$. As $y$ is an involution, we have

$$y^{-1}vv^yy = v^yy = (vv^y)^{-1}.$$ 

Thus $y$ inverts every element of the maximal cyclic subgroup. But this is a contradiction as $d^{-1} \neq d^{-1}u$. Hence the coset $ysD_1$ does not contain an involution, a contradiction. Thus $G$ does not contain a Sylow 2-subgroup $S$ with the properties above.

\[\Box\]
CHAPTER 8

THE NONSOLVABLE CASE

We have established the existence of a fully fused fours-group in our minimal counterexample. We now investigate non-solvable 2 locals containing a fully fused fours-group. In this section we severely limit the possibilities for such non-solvable 2-local subgroups of $G$.

Definition 8.1.

$$\mathcal{E} = \{L \leq G : L \cong PSL_2(p), |L_2| > 4\}.$$  

$$\mathcal{H} = \{H \leq G : |O_{2^r,3}(H)| \text{ is even and } E(H) \in \mathcal{E}\}.$$  

$$\mathcal{H}^* = \{H \in \mathcal{H} : H \text{ is maximal in } \mathcal{H} \text{ with respect to inclusion}\}.$$  

We remark since $cd(PSL_2(7)) > 1$ by Lemma 6.17 and $|PSL_2(p)|_2 = 4$ for $p = 5, 11$ or 13, we have $p > 17$ for any $L \in \mathcal{E}$.

Lemma 8.2. If $L \leq G$ with $L \in \mathcal{E}$, then $N_G(L)$ is the unique maximal subgroup of $G$ containing $L$.

Proof. Let $M$ be a maximal subgroup of $G$ containing $L$. By induction since $M$ is not solvable, there exists $L_o \leq M$ with $L_o = E(M)$, $M/L_o$ supersolvable.
and $L_0/Z(L_0)$ isomorphic to $A_6$ or $L_2(p)$. Since $M/L_0$ is supersolvable, we have $L \leq L_0$. As $p > 5$, it follows from Theorem 4.7 that $L = L_0$. Thus $M = N_G(L)$ as claimed.

\[ \square \]

**Lemma 8.3.** Suppose $H \in \mathcal{H}^*$. Then $H = N_G(L)$ where $L = E(H) \cong PSL_2(p)$. Furthermore $H$ is maximal in $G$ and is the unique maximal subgroup of $G$ containing $L$.

**Proof.** By hypothesis there exists $L \leq H$, $L \cong L_2(p)$ $p \geq 17$. We have $H \leq N_G(L)$; furthermore if the containment is proper, then as $H$ is maximal in $\mathcal{H}$, we have that $N_G(L) \not\in \mathcal{H}$. This implies that $|O_{2',2}(N_G(L))|$ is odd. However by Theorem 7.2,

$$O_{2',2}(H) = C_G(L) \leq O_{2',2}(N_G(L)).$$

Therefore $|O_{2',2}(N_G(L))|$ is even. Hence $N_G(L) \in \mathcal{H}^*$ and $H = N_G(L)$. By Lemma 8.2 $N_G(L)$ is the unique maximal subgroup containing $L$, and the conclusion holds.

\[ \square \]

**Corollary 8.4.** Let $H \in \mathcal{H}^*$ and let $X \leq O_{2',2}(H)$, $X \neq 1$. Then we have $N_G(X) \leq H$.

**Proof.** By Lemma 8.3 $H = N_G(L)$ for some $L \in \mathcal{E}$. Since $X \leq O_{2',2}(H)$, we have $L \leq C_G(X)$ by Theorem 7.2. Hence by Lemma 8.2 we have that $N_G(X)$ is contained in $N_G(L)$, the unique maximal subgroup of $G$ containing $L$.

\[ \square \]
Lemma 8.5. Let \( H \in \mathcal{H}^* \) with \(|H|_2 < |G|_2\). Let \( E \cong E_8 \) be a subgroup of \( E(H) \times C_H(E(H)) \) with \(|E(H) \cap E| = 4\). Then \( N_G(E) \leq H \).

Proof. By hypothesis \( E(H) \cong PSL_2(p) \) for some \( p \geq 17 \). Let \( E = E_o \times < s > \) where \( E_o = E \cap E(H) \), \( s \in C_H(E(H)) \). By Lemma 6.9, there exists \( x \in E(H) \) with \( x^3 = 1 \), \( E_o < x > \cong A_4 \). Thus \( x \in N = N_G(E) \) and \( E_o \in \mathcal{F}_N \). By Lemma 5.3 and Theorem 7.2, \( N \) is solvable and \( E_o = [E_o, x] \leq O^2(N) \). Now \( O^2(N)E \leq N \) and by Theorem 7.5 \( O^2(N) = \langle E_o \times O_2'(N), x \rangle \) with \( O_2'(N) \leq C_G(E) \). Then \( O^2(N)E \leq C_G(s) \) and \( < s > = O_2(Z(O^2(N)E)) \leq N \). Thus \( N \leq C_G(s) \leq H \) by Corollary 8.4.

\[ \square \]

Now we are ready to prove the first main result of this section.

Theorem 8.6. If \( H \in \mathcal{H}^* \) then \(|H|_2 < |G|_2\).

Proof. Suppose not. Choose \( H \in \mathcal{H}^* \) with \(|H|_2 = |G|_2\). By Lemma 8.3 \( H = N_G(L) \) for some \( L \) with \( L \cong PSL_2(p) \), \( p \geq 17 \). Let \( C = C_H(L) \). Let \( T \) be a Sylow 2-subgroup of \( H \). Since \( C \leq H \), we have \( S_o = T \cap C \in Syl_2(C) \). Furthermore \( S_o \leq T \). Thus \( S_o \cap Z(T) \neq 1 \). Let \( s \) be an involution in \( S_o \cap Z(T) \). By Theorem 7.2 \( s^G \cap T \neq \{s\} \). By Theorem 5.5 there exists \( D \leq T \) with \( N_T(D) \in Syl_2(N_G(D)) \) and \( g \in N_G(D) \) \( g \) of odd order with \( s^g = t \neq s \). Also by Theorem 5.5, we have \( D = V \times D_o = C_T(V) \) where \( V \) is a fully fused fours group in \( N_G(D) \). We have \( V \times Z(D_o) = Z(D) \). We argue the following conclusions about the structure of \( D \).

(1) \( \Omega_1(D \cap S_o) = < s > \).
(2) There exists an element $v \in V$, with $v$ acting as an outer automorphism on $L$.

(3) $D = < v, u, s >$ where $< u > = Z(T \cap L)$.

(4) $S_o = < s >$ and $G$ has at least three conjugacy classes of involutions.

We first prove $\Omega_1(D \cap S_o) = < s >$. We have $D \leq T \leq H$ with $D = D^g \leq T^g \leq H^g$; therefore $D \leq H^g$. Suppose $\Omega_1(D \cap S_o) \neq < s >$. Then $D \cap S_o$ contains a fours group $E = < x_1, x_2 >$. By Corollary 8.4 $C_G(x) \leq H$ for any $x \in E^2$. Let $C^* = < C(x_1), C(x_2), C(x_1x_2) >$.

Then we have $C^* \leq H$. Furthermore $E \leq N(L^g)$ as $E \leq D \leq N(L^g)$. Therefore by Lemma 6.11 we see $L^g \leq C^* \leq H$. Hence $L^g \leq H^\infty = L$. So $L = L^g$ and $g \in H$. Since $g$ has odd order, by Theorem 5.5 we have $g \in O^2(H) = L \times O_2(H)$. Then $[s, g] \in O_2(H) \cap D = 1$, a contradiction. Thus $g \notin H$ and $\Omega_1(D \cap S_o) = < s >$ as claimed in (1).

We argue $V \notin L$. Suppose $V \leq L$, then $s \notin V$. Let $N_1 = N_G(V)$ and let $x \in N_L(V)$ of order 3. Then by Lemma 7.6, $s^{N_1} \cap D = \{s\}$, a contradiction.

Next suppose $V \leq (L \times S_o) - L$. Let $W$ be the projection of $V$ into $L$. Then $C_L(W) = C_L(V)$ and $C_L(W)$ contains $U$ a fours group and by Lemma 6.9, every fours group in $L$ is fully fused in $L$. By Lemma 5.4 if $U$ and $V$ are fully fused fours groups with $U \neq V$, then $[U, V] \neq 1$, a contradiction. Therefore some element of $V$, call it $v$, acts as an outer automorphism on $L$, proving (2).

We have $V \leq Z(D)$. Thus $D \leq C_T(v)$. By Lemma 6.10, $C_L(v)$ is a dihedral group of order $p + \epsilon$ with $(p + \epsilon)/2$ odd. Thus $|C_L(v)|_2 = 2$. There exists $u \in Z(T) \cap C_L(v)$ with $< u > \in Syl_2(C_L(v))$. We have $C_T(v) = < v, u >$.
\( \times C_{S_o}(v) \). As \( D \leq C_T(v) \) and \( \Omega_1(D \cap S_o) = < s > \) we have \( \Omega_1(D) = < v, u, s > \). Furthermore we argue \( \Omega_1(D) = D \). We have \( D = V \times C_D(g) \) by Theorem 7.5. As \( s \in D - C_D(g) \), we have that \( s \) is not a square in \( D \). However if \( \Omega_1(D) \neq D \) then \( |S_o \cap D| > 2 \). In particular as \( \Omega_1(D \cap S_o) = < s > \) we have \( S_o \cap D \) is either cyclic or quaternion. This implies \( s \) is a square in \( D \), a contradiction. Therefore we have \( D = < v, u, s > \), proving (3).

We now argue that \( |S_o| = 2 \). We have that \( C_T(D) \leq D \). We have equality as \( D \) is abelian. Since \( < u, s > \leq Z(T) \), we have \( C_T(D) = C_T(v) = D \). If \( S_o \neq < s > \) then consider the group \( S_o/ < s > = \overline{S}_o \). In particular \( v \) must centralize some involution \( r \) of \( \overline{S}_o \). Since \( v \) does not centralize \( r \), we have \( r^v = rs \). Hence \( < r, v, s > \) forms a dihedral group with \( v \) conjugate to \( vs \). By Lemma 6.10 involutions in \( L < v > \) are \( L \)-conjugate. Hence \( v \) is \( L \)-conjugate to \( uv \). Thus \( v^G \cap D \geq \{v, uv, vs, uvs\} \). We have \( < s, u > \leq Z(T) \). By Lemma 5.6, since \( < u, s > \) is not the unique fully fused fours group in \( T \), we have that distinct elements of \( Z(T) \) are not \( G \) conjugate. In particular \( D \) must meet at least 3 \( G \)-conjugacy classes of involutions. On the other hand \( N_G(D) \) has exactly three orbits on \( D^2 \) of lengths 1, 3, and 3. One of these must be \( v^G \cap D \), a contradiction. Therefore we have \( |S_o| = 2 \) as claimed in (4).

We can now quickly complete the proof. Let \( T_o \in Syl_2(L < v >) \). Then \( T_o \) is maximal in \( T \). By Lemma 2.30, every involution of \( T \) is conjugate to an involution in \( L < v > \). However \( L < v > \) has exactly 2 conjugacy classes of involutions, a contradiction, as \( T \) has at least 3 \( G \)-classes of involutions. Hence \( |H|_2 < |G|_2 \).

\( \square \)
We can now complete our analysis of the non-solvable 2-local subgroups of $G$.

**Theorem 8.7.** We have that $\mathcal{H}^* = \emptyset$

**Proof.** We assume $\mathcal{H}^* \neq \emptyset$ and choose $H \in \mathcal{H}^*$. By Lemma 8.3 $H = N_G(L)$, where $L = E(H)$ with $L \cong PSL_2(p)$ for some $p \geq 17$. Let $T \in Syl_2(H)$ with $T \leq S$ for $S \in Syl_2(G)$. By Theorem 8.6 $T < S$. Let $C = C_H(L)$. Let $T_0 = T \cap (L \times C)$, $D = T \cap L$, and $S_0 = T \cap C$. Clearly $T_0$, $D$, and $S_0$ are normal in $T$. In the first part of the proof we argue the following claims.

1. Let $g \in N_S(T) - T$ with $g^2 \in T$. Then $S_0 \cap S_0^g = 1$.
2. $S_0$ is cyclic and $T/S_0$ is nonabelian dihedral.
3. $S_0 = \langle s \rangle \cong \mathbb{Z}_2$.
4. $T \in Syl_2(C_G(s))$, $\langle u, s \rangle = Z(T)$ and $\langle u \rangle = Z(S)$.
5. $|N_S(T) : T| = 2$.

Since $T < S$ we may choose $g \in N_S(T) - T$ with $g^2 \in T$. We argue that $S_0 \cap S_0^g = 1$. If not, let $S_0 \cap S_0^g = F$. Then $g \in N_G(F) \leq H$ by Corollary 8.4., a contradiction, proving (1).

Next we shall prove $m_2(S_0) = 1$ and furthermore $S_0$ is cyclic. Let

$$N = \langle N_L(Y) : 1 \neq Y \leq S_0^g \rangle.$$ 

We comment that if $Y \leq S_0^g$, then $Y^{g^{-1}} \leq S_0$. Hence we have $L \leq N_G(Y^{g^{-1}})$ by Corollary 8.4, which implies $L^g \leq N_G(Y)$ for any $1 \neq Y \leq S_0^g$. Hence $N \leq N_G(L^g)$. If the 2 rank of $S_0$ is greater than 1, we have a subgroup $Y \leq S_0^g$ with $Y \cong E_4$. Therefore by Lemma 6.12 $L \leq \langle C_L(y) : y \in Y^2 \rangle \leq N$, which

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implies $L = L^g$ a contradiction. Thus $m_2(S_0) = 1$. Since $S_0 \cap S_0^g = 1$, $S_0^g$ embeds in $T/S_0 = \overline{T} \in Syl_2(N_G(L)/C_G(L))$ with $\overline{T}$ a nonabelian dihedral group by Lemma 6.5. Therefore $S_0$ is cyclic, proving (2).

We argue $S_0 \cong Z_2$. Suppose to the contrary $|S_0| > 2$ and let $S_1$ be the cyclic subgroup of $S_0$ of order 4. If $S_1 \leq Z(T)$, then $S_1^g$ embeds in $Z(T)S_0/S_0 \leq Z(T/S_0)$. But $T/S_0$ is non-abelian dihedral. Hence $|Z(T/S_0)| = 2$, a contradiction. Hence $|S_0 \cap Z(T)| = 2$. Thus $C_T(S_1) = C_T(S_0) = D \times S_0$ and there exists $x \in T - (D \times S_0)$ with $|C_{S_0}(x)| = 2$. Indeed, since $S_0 \leq Z(D \times S_0)$, $|C_{S_0}(x)| = 2$ for all $x \in T - (D \times S_0)$. Moreover $S_0^g \leq C_T(S_0) = D \times S_0$ since $[S_0, S_0^g] \leq S_0 \cap S_0^g = 1$. Let $P_0$ denote the image of $S_0^g$ under the projection map $\pi : (D \times S_0) \to D$. As $S_0^g \cap Ker(\pi) = S_0 \cap S_0^g = 1$, $P_0 = \pi(S_0^g) \cong S_0^g$.

Now suppose $x \in T - (D \times S_0)$ with $x^2 = 1$. By Lemma 6.10,

$$C_{D \times S_0}(x) = Z(D) \times \Omega_1(S_0) \cong Z_2 \times Z_2.$$

Thus $\Omega_1(C_T(P_0)) \leq D \times S_0$. Hence

$$\Omega_1(C_T(S_0^g)) = \Omega_1(C_T(P_0)) = \Omega_1(C_{D \times S_0}(P_0)) = Z(D) \times \Omega_1(S_0) \cong Z_2 \times Z_2.$$

However

$$\Omega_1(C_T(S_0^g)) = \Omega_1(C_T(S_0))^g = (D \times \Omega_1(S_0))^g \cong Z_2 \times Z_2,$$

a contradiction. Hence $S_0 \cong Z_2$ proving (3). We set $S_0 = \langle s \rangle$. By Corollary 8.4, $C_G(s) \leq N_G(L) = H$. Hence $T \in Syl_2(C_G(s))$. Moreover $< U, s > = Z(T)$ and $< u > = Z(T) \cap [T, T] \leq N_S(T)$. So $< u > = Z(S)$, proving (4).

Our next goal is to prove $[N_S(T) : T] = 2$ and to determine the structure of $T$. Set $T_0 = D \times < s >$. By hypothesis $|D| \geq 8$, thus $D$ has two $D$-conjugacy classes
of \(E_4\)'s and so \(T_o\) has two \(T_o\)-conjugacy classes of \(E_6\)'s. The first case to consider is when \(J(T) = T_o\). Hence \(T_o\) is characteristic in \(T\). It follows \(T_o \leq N_S(T)\); thus \(g\) normalizes \(T_o\). Let \(E \cong E_8 \leq T_o\). If \(E^g = E^t\) for some \(t \in T\) then \(E^{st^{-1}} = E\) and \(gt^{-1} \in N_G(E) \leq H\) by Lemma 8.5. Thus \(g \in H\), a contradiction. It follows \(g\) interchanges the two conjugacy classes of \(E_8\)'s as claimed. Hence \(N_S(T)\) acts on a set of size two with \(Stab_{N_S(T)}(E^T) = T\) and \(|N_S(T) : T| = 2\), proving (5). Indeed our argument shows that the two \(T_o\)-classes of \(E_8\)'s are not \(T\)-conjugate and so \(T_o = T\) in this case.

The other case is that there exists \(x \in T - T_o\), \(x\) an involution. It follows from Lemma 6.11 that \(L < x > \cong PGL_2(p)\). We have \(T = D < x > \times < s >\) with \(D < x >\) dihedral of order greater than \(8\). By Lemma 6.6, \(D < x >\) has two \(T\)-classes of fours groups; thus \(T\) has two \(T\)-conjugacy classes of \(E_8\)'s. By the argument above \(g\) interchanges these two \(T\)-conjugacy classes of \(E_8\)'s and we have \(|N_S(T) : T| = 2\) proving (5).

Let \(U = < u, v >\) be a fours subgroup of \(D\) and let \(D_1 = < v, v^g >\). Thus \(D_1\) is a dihedral subgroup of \(T\). Moreover as \(E = < u, v, s >\) is not \(T\)-conjugate to \(E^g, v\) and \(v^g\) are not conjugate in \(T = T/ < s >\). As \(T\) is dihedral, we infer from Lemma 6.6 that \(T = < v, v^g >\) and so \(T = D_1 \times < s >\). In particular \([T, T] \leq D_1\) and so \(v^{-1}v^g \in D_1\). Thus \(D_1 \leq N_S(T)\). Finally all involutions of \(D_1\) are \(D_1\) conjugate to \(u, v, \) or \(v^g\). Hence as \(v \in u^G\), all involutions of \(D_1\) lie in \(u^G\). We have established conditions (a), (c), and (d) of Lemma 7.7 hold. Indeed it remains to show that \(S = N_S(T)\).

Now suppose \(S \neq N_S(T)\). If \(T = J(N_S(T))\), then \(T \leq N_S(N_S(T))\) and so \(S = N_S(T)\), contrary to assumption. Hence \(N_S(T)\) contains and elementary
abelian subgroup $A$ with $m(A) \geq 3$ and $A \not\subseteq T$. As $A$ does not normalize any $E_8$ subgroup of $T$ by Lemma 8.5, but $A$ normalizes $A \cap T$, we have $|A| = 8$ and $A \cap T = Z(T) = \langle u, s \rangle$. However $C_g(s) \leq H$ and $A \not\subseteq H$, a contradiction. So $N_S(T) = S$. Thus $\langle T, g \rangle = S$, proving (6).

Now hypothesis (b) and (e) of Lemma 7.7 have been established. Thus $G$ has a Sylow 2-subgroup $S$ with a maximal subgroup $T$ with all the properties listed in Lemma 7.7, a contradiction. Thus $H^* = \emptyset$. 

□
CHAPTER 9

THE SOLVABLE CASE

In Chapter 9 we investigate solvable 2locals containing a fully fused fours-group $U$. We establish that $U \in Syl_2(C_G(U))$, whence $G$ has a dihedral or semidihedral Sylow 2-subgroup by Lemma 2.32.

Definition.

\[ S = \{ H \leq G : H \text{ is solvable and } \mathcal{F}_H \neq \emptyset \}. \]

\[ S^* = \{ H \in S : H \text{ is maximal with respect to inclusion } \}. \]

Lemma 9.1. Let $H \in S$. Then $|\mathcal{F}_H| = 1$ and $H \leq N_G(U)$; where $\mathcal{F}_H = \{ U \}$.

Proof. This is a restatement of Theorem 7.5

\[ \square \]

Lemma 9.2. $S^* = \{ N_G(U) : U \in \mathcal{F} \}$ and $S^* \neq \emptyset$.

Proof. By Corollary 7.4, $\mathcal{F} \neq \emptyset$. Let $U \in \mathcal{F}$ and let $H = N_G(U)$. Then $H$ is solvable by Theorem 7.2, and thus $H \in S$. 

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Now let $H \in S$. So by Lemma 9.1, $\mathcal{F}_H = \{U\}$ and $H \leq N_G(U)$ with $N_G(U) \in S$ by Proposition 7.2. Hence $S^*$ is as claimed. Then since $\mathcal{F} \neq \emptyset$, we have $S^* \neq \emptyset$. □

**Lemma 9.3.** Suppose $U \in \mathcal{F}$. Let $H = N_G(U)$ and let $T \cap C_G(U) = T_o \in Syl_2(C_G(U))$. Let $x \in N_H(T_o)$ with $x^3 \in O_{2'}(N_H(T_o))$ and with $T_o = [T_o, x] \times C_{T_o}(x)$. Let $S_o = C_{T_o}(x)$ and let $1 \neq Y \leq S_o$. Let $N = N_G(Y)$. Then the following conclusions hold:

1. $U \in \mathcal{F}_N$
2. Either $U \leq N$ or $U \leq E(N)$.
3. $|H|_2 \geq |N|_2$.

**Proof.** By Theorem 7.4, we may choose $T_o$ and $x$ as stated. For $1 \neq Y \leq S_o$, we have $(U, x) \leq C_G(Y) \leq N$ which implies $U \in \mathcal{F}_N$, proving (1). By Lemma 5.3 either $U \leq N$ or $U \leq E(N)$, proving (2).

Now suppose $|N|_2 > |H|_2$. If $U \leq N$, then $N \leq H$, a contradiction. Therefore $U \leq E(N) = L$. As $L$ contains a fours group, by Theorem 7.2, $L$ is isomorphic to $PSL_2(p)$ with $p = 5$ or $p \geq 13$ or $L/O_{2'}(L) \cong A_6$. As $1 \neq Y \leq O_2(N)$ and $\mathcal{H}^* = \emptyset$, we have that either $|L|_2 = 4$ or $L/O_{2'}(L) \cong A_6$.

Suppose $|L|_2 = 4$. We have $L \leq N$ and $U \in Syl_2(L)$. By Theorem 2.7 we have $N = LN_N(U)$. Thus $|N : N_N(U)| = |L : N_L(U)|$ is odd. But by hypothesis $|N|_2 > |H|_2$, a contradiction. Thus $|L|_2 > 4$ and so $L \cong A_6$ or $3A_6$.

We have $N/C_N(L)$ embeds in $Aut(L)$. As $cd(N/C_N(L)) = 1$, we have $N/C_N(L) \cong A_6$ or $S_6$, by Lemma 6.18. Let $\{U^L, V^L\}$ be the set of conjugacy classes of fours groups in $L$. By Lemma 6.19, $S_6$ is in the kernel of the
action of $Aut(L)$ on this set. Therefore $U^L = U^N$. Thus by Theorem 2.7, $N = LN_N(U)$. As $U$ is normal in a Sylow 2-subgroup of $L$, we again have that $|N : N_N(U)| = |L : N_L(U)|$ is odd, a final contradiction.

□

**Theorem 9.4.** If $H \in S^*$ then $|H|_2 = |G|_2$, or $S \in Syl_2(G)$ with $S$ dihedral or semidihedral.

**Proof.** Suppose $H \in S^*$ with $|H|_2 < |G|_2$. By Lemma 9.2 $H = N_G(U)$ for some $U \in F$. Let $U \leq T \in Syl_2(H)$ with $T < S \in Syl_2(G)$. Since $T < S$, there exists $y \in N_S(T)$ with $y^2 \in T$ and $y \notin T$. Let $T \cap C_G(U) = T_o \in Syl_2(C_G(U))$. By Theorem 7.4, there exists $x \in N_G(T_o)$ with $U = [T_o, x]$, $x^3 \in C_G(T_o)$ and $T_o = U \times C_{T_o}(x)$. We set $S_o = C_{T_o}(x)$. If $S_o = 1$, then $U$ is self-centralizing in $S$ and so $S$ is dihedral or semidihedral by Lemma 2.32, as desired. Hence we may assume $S_o \neq 1$. We argue the following

1. $S_o \cap S^y_o = 1$
2. $S^y_o \cap U = 1$ and $UU^y \cong D_8$.
3. $S_o \cong Z_2$.

Suppose that $S_o \cap S^y_o = Y \neq 1$. Since $S_o \leq T$, we have $Y \leq T$. Thus $T \leq N_G(Y) = N$. Since also we have $y \in N_G(Y) - T$, $|N|_2 > |H|_2$, contrary to Lemma 9.3. This proves (1).

We claim $S^y_o \cap U = 1$. It is enough to show $U^y \cap S_o = 1$. By Lemma 5.4, since $y \notin H$, $1 \neq [U, U^y] \leq U \cap U^y$. If also, $S_o \cap U^y \neq 1$, then $U^y \leq U \times S_o \leq C_T(U)$, a contradiction. Thus $S^y_o \cap U = (S_o \cap U^y)^y = 1$. Moreover we observe that $U^y U \cong D_8$. This proves (2).
We assume \(|S_o| > 2\). We have \([S_o^y, U] \leq S_o^y \cap U = 1\). Therefore \(S_o^y \leq C_S(U) = U \times S_o\). However \(S_o^y \cap S_o = 1\). Hence \(S_o^y\) is isomorphic to \(Z_2 \times Z_2\). If \(S_o \leq Z(T)\), then \(S_o^y \leq Z(UU^y) \times S_o\). Again as \(S_o \cap S_o^y = 1\), \(S_o \cong S_o^y \cong Z_2\). Therefore we may assume \(S_o\) is not contained in \(Z(T)\). Thus there exists \(x \in T - (U \times S_o)\) with \(< x, S_o >\) a dihedral group. We have \(C_{U\times S_o}(x) \cong Z_2 \times Z_2\), therefore \(U \times S_o = E \cong E_{16}\) is characteristic in \(T\). Hence \(E^y = E\). On the other hand this implies \(U^y \leq E = C_T(U)\), a contradiction to (2). Therefore \(S_o \cong Z_2\), proving (3).

Thus we have reduced to the case \(T = D_1 \times < s >\) with \(D_1 = UU^y \cong D_8\). We argue the following:

(4) Let \(E = U \times < s >\). Then \(N_S(E) = T = N_S(E^y)\).

(5) \(N_S(T) = < T, y >\) and \(s^y \neq s\).

(6) We have \(S = < T, y >\).

Suppose \(w \in N_S(E) - T\). We have \(U \leq E\), hence \(U^w \leq E\). Since \(E\) is elementary abelian \([U, U^w] = 1\), a contradiction to Lemma 5.4. Thus \(N_S(E) = T\). Then \(N_S(E^y) = N_S(E^y) = T\), proving (4).

Let \(x \in N_S(T) - T\). Since \(E^x \neq E\), \(E\) and \(E^x\) are distinct normal elementary abelian subgroups of \(T\). Furthermore these are the only subgroups of \(T\) isomorphic to \(E_8\). Hence \(N_S(T)\) acts on this set of size 2 with \(Stab_{N_S(T)}(E) = T\). Thus \(|N_S(T) : T| = 2\) and so \(N_S(T) = < T, y >\). Moreover by (1), \(s^y \neq s\), proving (5).

Suppose \(N_S(T) \neq S\). If \(T = J(N_S(T))\), then \(T \leq N_S(N_S(T))\) and so \(S = N_S(T)\), contrary to assumption. Hence \(N_S(T)\) contains an elementary abelian subgroup \(A\) with \(m(A) \geq 3\) and \(A \nleq T\). By (4),

\[A \cap T \leq E \cap E^y = Z(T) = < u, s >\]
Thus

\[ A \cap T \leq C_{Z(T)}(A) = \langle u \rangle \]

by (5), since \( A \nsubseteq T \). But then \( m(A) \leq 2 \), a contradiction. Hence \( T = J(N_S(T)) \) and \( S = N_S(T) = (T, y) \), proving (6).

If we take \( D = U \) we have \( T \in Syl_2(C_G(s)) \) by Lemma 9.3(3), whence \( \langle u \rangle = [T, T] = Z(S) \cap T = Z(S) \). We note that we have now verified every condition of Lemma 7.7 for \( G \) with \( S \) a Sylow 2-subgroup and \( T \) a maximal subgroup of \( S \). This gives a final contradiction.

\[ \square \]

**Theorem 9.5.** Let \( S \in Syl_2(G) \). Then \( S \) is dihedral or semidihedral.

**Proof.** Suppose not. By Theorem 9.4 we may assume that \( |H|_2 = |G|_2 \) for all \( H \in S^* \). Choose \( H \in S^* \). By Lemma 9.2 \( H = N_G(U) \) for some \( U \in \mathcal{F} \). Let \( U \leq S \in Syl_2(H) \) Let \( S \cap C_G(U) = T \in Syl_2(C_G(U)) \). By Theorem 7.4, there exists \( x \in N_G(T) \) with \( U = [T, x], x^3 \in C_G(T) \) and \( T = U \times C_T(x) \). We set \( S_o = C_T(x) \) and we may assume \( S_o \neq 1 \) as \( S \nsubseteq D_8 \) or \( D_4 \). Let \( N = N_H(T) \). Then \( \langle O_2^2(N), T \rangle \leq O_2^2(N) \cap T = 1 \) and so \( C_T(\langle O_2^2(N), x \rangle) = C_T(x) = S_o \). On the other hand \( O_2^2(N) = \langle U \times O_2^2(N), x \rangle \) and so \( S_o = C_T(O_2^2(N)) \leq N \). In particular \( S_o \lneq S \).

Choose \( s \in S_o \cap Z(S) \) with \( s^2 = 1 \neq s \) and with \( s \in \Phi(S_o) \) if \( \Phi(S_o) \neq 1 \). By Theorem 7.5, \( S^G \cap S \neq \{ s \} \). Thus by Theorem 5.10 there exists \( V \leq D \leq S \) and \( h \in N_G(D) \) such that:

(a) \( D = C_S(V) \); and

(b) \( N_S(D) = N_S(V) \in Syl_2(N_G(D)) \leq Syl_2(N_G(V)) \); and
(c) \( V \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( V \leq N_G(D) \); and
(d) \( h^3 \in O_2'(N_G(D)) \) with \([D, h] = V\); and
(e) \( s \in D \) and \( s^h = t \neq s \).

We argue the following:

(1) \( S \leq N_G(V) \); and
(2) \( D = V \times S_o \); and
(3) \( S_o \) is elementary abelian and \( S = UV \times S_o \) with \( UV \cong D_8 \).

Since \( N_G(V) \in S^* \) by Lemma 9.2 for any \( V \in \mathcal{F} \), and \( |H|^2 = |G|^2 \) for all \( H \in S^* \) by Theorem 9.4, we have \( |N_G(V)|_2 = |G|^2 \). Now by (b) above, we have \( S \leq N_G(V) \), proving (1).

We argue \( D = V \times S_o \). By (a) \( C_S(V) = D \). Thus \( V \leq Z(D) \). In \( H \), \( s^H \cap T = \{s\} \) by Lemma 7.2. If \( U = V \), then \( s^{N_G(V)} \cap T = \{s\} \), a contradiction as \( s^h = t \neq s \) for \( h \in N_G(V) \). Hence \( U \neq V \). By Lemma 5.4, \( 1 \neq [U, V] \) thus \([U, V] = U \cap V = < u > \), where \( < u > = U \cap Z(S) \). Thus \( U \nleq D = C_S(V) \) and likewise \( V \nleq T = C_S(U) \). Hence we may choose \( x \in S - T \) such that \([U, x] = < u > \) and \( V = \{x, u\} \). As \( S_o \leq S \), \( [V, S_o] \leq V \cap S_o = 1 \). Hence \( S_o \leq D = C_S(V) \) and so \( D = V \times S_o \), proving (2).

Now \( h \in N_G(V) \) with \([V, h] = V \) and \( D = V \times S_1 \), where \( S_1 = C_D(h) \). By a similar argument to the one which showed \( S_o \leq S \), we have \( S_1 \leq S \) and thus \([U, S_1] = 1 \). Clearly \( \Phi(D) = \Phi(V \times S_1) = \Phi(S_1) \). Likewise \( \Phi(D) = \Phi(S_o) \). However \( s \notin S_1 \) as \( s^h \neq s \); consequently \( s \notin \Phi(S_o) \) and so \( \Phi(S_o) = 1 \) by choice of \( s \). Thus \( S_o \) is elementary abelian and \( S = UV \times S_o \) with \( UV \cong D_8 \), proving (3).

We now quickly complete the proof. Let \( a = u_1z_o \) with \( u_1 \in V \), \( z_o \in S_o \). Then for some \( x_1 \in < x > \), \( a^{x_1} = uz_o \) or \( z_o \). In any case \( a^{x_1} \in < u > z_o \). Likewise
if $a_1 = u_1z_1$ with $u_1 \in U$ and $z_1 \in S_1$, then $a_1^G \cap <u> \neq \emptyset$. As $S_o$ and $S_1$ are distinct hyperplanes of $Z(S)$, $S_o \cap S_1$ is a hyperplane of both. Thus $M = UV \times (S_o \cap S_1)$ is a maximal subgroup of $S$, in which every involution is conjugate to an element of $Z_o = <u> \times (S_o \cap S_1)$. Now let $s_o \in S_o - (S_o \cap S_1)$. By Lemma 2.30, $s_o^G \cap M \neq \emptyset$, and so $s_o^G \cap Z_o \neq \emptyset$. Let $s_o^g \in Z_o$. Then $s_o$ and $s_o^g$ are distinct $G$-conjugate elements of $Z(S)$. Since $U \in F$ and $U \not\leq Z(S)$, this contradicts Lemma 5.6. Thus $S_o = 1$ and the result follows.

□

We conclude this section with a restatement and proof of Proposition 1.5.

**Proposition 1.5.** Let $G$ be a minimal counterexample to Theorem 1.4. Then $G$ is a simple group with a dihedral or semidihedral Sylow 2-subgroup.

**Proof.** Let $G$ be a minimal counterexample to Theorem 1.4. In Lemma 7.1 we prove $G$ is a simple group. Let $S \in Syl_2(G)$. By Theorem 9.5, $S$ is dihedral or semidihedral.

□
In Chapter 10, we complete the proof of Theorem 1.4. For completeness we restate Theorem 1.4.

**Theorem 1.4.** Let $G$ be a finite group of chain difference one. Then one of the following holds.

1. $G$ is solvable; or
2. $E(G)/Z(E(G))$ is isomorphic to $A_6$ or $PSL_2(p)$ for some prime $p$ satisfying the conditions of Theorem 1.1 and $G/E(G)$ is supersolvable.

**Proof.** The proof is by induction. We may assume $G$ is not solvable. By Proposition 1.5, $G$ is a simple group with a dihedral or semidihedral Sylow 2-subgroup. If $S$ is dihedral then by Theorem 2.2 $G \cong PSL_2(q)$ for $q = p^n$, where $p$ is an odd prime or $G \cong A_7$. In the former case we have $q = p$ a prime satisfying the conditions of Theorem 1.3 or $q = 9$, in which case $G \cong PSL_2(9) \cong A_6$. By Lemma 6.21 we have $A_7$ does not have chain difference one.

If $S$ is semidihedral, then by Theorem 2.3 $G \cong M_{11}, PSL_3(q)$, or $PSU_3(q)$ for $q$ odd. By Theorem 6.24 $cd(PSU_3(q)) > 1$ for all odd $q$, and by Theorem...
6.23 $\text{cd}(\text{PSL}_3(q)) > 1$, for all $q$ odd. Thus $G$ is isomorphic to $M_{11}$. Since $M_{10}$ is the one point stabilizer in $M_{11}$, $\text{cd}(M_{11}) \geq \text{cd}(M_{10})$. Now $M_{10}$ contains $A_6$ as a subgroup of index two and has a semidihedral Sylow 2-subgroup. Hence $M_{10} \not\cong S_6$ or $A_6 \times Z_2$, and so $\text{cd}(M_{10}) > 1$ by Lemma 6.13. This complete the proof.

□
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