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IDENTIFICATION OF MULTI-DEGREE-OF-FREEDOM NONLINEAR SYSTEMS WITH FOCUS ON
SPECTRAL METHODS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University

By
Christopher M. Richards, B.S.M.E., M.S.M.E.

*****

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Linear system methods, such as modal analysis techniques, are commonly employed to extract system parameters such as natural frequencies, damping ratios and mode shapes from measured frequency response functions. However, conventional “H₁” and “H₂” frequency response function estimates may be contaminated by the presence of nonlinearities and hence make it difficult to extract underlying linear system properties. To overcome this deficiency, a new spectral approach for identifying the parameters of a multi-degree-of-freedom nonlinear system is introduced which is based on a “reverse path” formulation as available in the literature for a single-degree-of-freedom nonlinear system. Certain analytical modifications are made in this dissertation for a multi-degree-of-freedom “reverse path” formulation that utilizes multiple-input/multiple-output data from a nonlinear system when excited by Gaussian random excitation. Conditioned “H₁” and “H₂” frequency response estimates now yield the underlying linear properties without contaminating effects from the nonlinearities. Once the conditioned frequency response functions have been estimated, the nonlinearities, which are described by analytical functions, are also identified by estimating the coefficients of these functions. Identification of the local or distributed nonlinearities which exist at or away from the excitation locations is possible.

System identification techniques, such as the “Reverse Path” Spectral and Restoring Force temporal methods, require a priori knowledge of the nature and mathematical form of the nonlinearities. Without this information, inaccurate estimation of system parameters may result. Unfortunately, this information is not always available for “real” nonlinear systems. To tackle this problem, experimental and analytical characterization can be an initial step in the identification process where several experiments are conducted to investigate the behavior of nonlinear systems under different excitations or system configurations. Analytical models are then applied to describe the systems’ dynamic behavior. If
successful, the characterization may give insight into the nature of the existing nonlinearities. This
procedure is demonstrated via experimental studies off three rubber isolators. Concurrently, under
experimental conditions, some means of quantifying the amount of uncorrelated measurement noise present
in the identification process must also be obtained. To resolve this issue, coherence functions are
introduced which are based on the "Reverse Path" Spectral Approach. These coherence functions, as
calculated from conditioned spectra, indicate the extent of uncorrelated noise present. In addition, the
coherence functions indicate the accuracy of assumed mathematical models employed for describing the
nonlinear systems. Various aspects of this problem are discussed using simulation examples.
Dedicated to my wife and our families
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgment</td>
<td>v</td>
</tr>
<tr>
<td>Vita</td>
<td>vi</td>
</tr>
<tr>
<td>List of Tables</td>
<td>x</td>
</tr>
<tr>
<td>List of Figures</td>
<td>xii</td>
</tr>
<tr>
<td>List of Symbols</td>
<td>xviii</td>
</tr>
</tbody>
</table>

## Chapters:

1. Introduction .......................................................................................................................................... 1
   1.1. Motivation ..................................................................................................................................... 1
   1.2. Problem Statement .......................................................................................................................... 2
   1.3. Literature Review ........................................................................................................................... 4
   1.4. Scope and Objectives ....................................................................................................................... 5
   List of References for Chapter 1 ........................................................................................................... 8

2. A study of dynamic behavior of rubber isolator nonlinearities using experimental and analytical approaches .............................................................................................................................................. 11
   2.1. Introduction .................................................................................................................................. 11
   2.2. Problem Formulation ......................................................................................................................... 12
   2.3. Experimental Methods ...................................................................................................................... 14
   2.4. Experimental Characterization of Isolators ...................................................................................... 16
       2.4.1. Static behavior .......................................................................................................................... 16
       2.4.2. Dynamic behavior based on single-degree-of-freedom experiments ........................................... 22
       2.4.3. Multi-degree-of-freedom experiment ......................................................................................... 28
       2.4.4. Comparison of Results .............................................................................................................. 39
   2.5. Analytical Characterization ............................................................................................................. 40
       2.5.1. Quasi-linear model using continuous system theory .................................................................... 40
       2.5.2. Nonlinear model using discrete system theory ............................................................................ 48
   2.6. Conclusion ..................................................................................................................................... 53
   List of References for Chapter 2 ............................................................................................................ 55

vii
## Table of Contents

3. Identification of multi-degree-of-freedom nonlinear systems under random excitations by the “reverse path” spectral method ...............................................................56
   3.1. Introduction ..................................................................................56
   3.2. Problem Formulation ......................................................................58
      3.2.1. Physical Systems ....................................................................58
      3.2.2. Frequency Response ...............................................................64
   3.3. “Reverse Path” Formulation ..............................................................67
   3.4. Conditioned “Reverse Path” Formulation ............................................74
   3.5. Estimation of Power Spectral Density Matrices .................................78
   3.6. Identification of the Coefficients of the Nonlinear Function Vectors ..............................................................................................................81
   3.7. Results ...........................................................................................86
   3.8. Conclusion .....................................................................................108
   List of References for Chapter 3 ................................................................110

4. Feasibility of identifying nonlinear vibratory systems consisting of unknown polynomial forms .....112
   4.1. Introduction ..................................................................................112
   4.2. Problem Formulation ......................................................................113
      4.2.1. Scope ....................................................................................113
      4.2.2. Methodology ..........................................................................120
   4.3. Coherence Functions Based on Conditioned Spectra .......................123
   4.4. Preliminary Results .........................................................................126
   4.5. Inclusion of Uncorrelated Noise ......................................................134
      4.5.1. Formulation ............................................................................134
      4.5.2. Results ..................................................................................136
   4.6. Examination of Alternative Nonlinear Models .................................145
      4.6.1. Identification of Example I .......................................................145
      4.6.2. Identification of Example II .....................................................149
   4.7. Nonlinearity with Non-integer Exponent ..........................................155
   4.8. Conclusion .....................................................................................158
   List of References for Chapter 4 ................................................................159

5. Identification methods applied to experimental systems .................................................161
   5.1. Introduction ..................................................................................161
   5.2. Problem Formulation ......................................................................162
      5.2.1. Presence of static loads and displacements ...............................162
      5.2.2. Application to experimental systems ...........................................162
   5.3. Modified restoring force method ......................................................164
      5.3.1. Formulation of existing method .................................................164
      5.3.2. Identification issues ...............................................................166
      5.3.3. Proposed modification ...........................................................167
   5.4. Feasibility studies of an analytical model .........................................168
   5.5. Feasibility studies of experimental systems .......................................172
   5.6. Identification of isolator properties using single-degree-of-freedom experiments ..........................................................178
      5.6.1. Experimental data and identification using a quasi-linear model ..........................................................................................178
      5.6.2. Identification using polynomial type models ...............................178
      5.6.3. New nonlinear model .............................................................182
   5.7. Conclusion .....................................................................................187
   List of References for Chapter 5 ................................................................188

6. Conclusion ..........................................................................................190
6.1. Summary and Contributions ................................................................................................................ 190
6.2. Future Work ..............................................................................................................................................193
List of References for Chapter 6 .................................................................................................................. 195

Bibliography ................................................................................................................................................................197
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1. Instrumentation</td>
<td>17</td>
</tr>
<tr>
<td>2.2. Summary of dynamic experiments</td>
<td>18</td>
</tr>
<tr>
<td>2.3. Least Squares estimates of static stiffness curves</td>
<td>23</td>
</tr>
<tr>
<td>2.4. Modal parameters of single-degree-of-freedom configurations</td>
<td>27</td>
</tr>
<tr>
<td>2.5. Modal parameters of multi-degree-of-freedom configuration</td>
<td>35</td>
</tr>
<tr>
<td>2.6. Parametric study of a quasi-linear continuous system model with isolator 1 and lowest random force level, $</td>
<td>f_{llm}</td>
</tr>
<tr>
<td>2.7. Parametric study of the quasi-linear continuous system model with parameters from case 6 of Table 2.6</td>
<td>45</td>
</tr>
<tr>
<td>2.8. Parametric study of the nonlinear discrete system model with isolator 1 and parameters from case 6 of Table 2.6</td>
<td>52</td>
</tr>
<tr>
<td>3.1. Linear modal properties of example systems shown in Figure 3.1</td>
<td>62</td>
</tr>
<tr>
<td>3.2. Linear and nonlinear elastic force coefficients of example systems</td>
<td>63</td>
</tr>
<tr>
<td>3.3. Simulation and signal processing parameters: total number of samples = $2^{14} \eta$, $\Delta t = 0.5 \text{ ms}$, total period = $2^{13} \eta \text{ ms}$, Hanning window, $2^{13}$ samples/average, $2\eta$ averages</td>
<td>65</td>
</tr>
<tr>
<td>3.4. Estimated modal properties using conditioned “$H_{12}$” estimates</td>
<td>90</td>
</tr>
<tr>
<td>3.5. Error and MAC between actual and estimated modal properties of Tables 1 and 4, respectively</td>
<td>91</td>
</tr>
<tr>
<td>3.6. Coefficients of nonlinear elastic force terms</td>
<td>94</td>
</tr>
<tr>
<td>4.1. True linear system properties of Examples I, II and IV given $d_n(x(t)) = {0}$</td>
<td>116</td>
</tr>
<tr>
<td>4.2. True nonlinear elastic force $f_{12}^n(t)$ of Examples I-IV</td>
<td>117</td>
</tr>
<tr>
<td>4.3. Estimated physical properties of Example I by the Temporal Method</td>
<td>127</td>
</tr>
</tbody>
</table>
4.4. Estimated modal properties of Example I by the Temporal Method ..........................................128
4.5. Estimated properties of Example I by the Spectral Method .........................................................130
4.6. Models used in the estimation of Example II ...................................................................................151
4.7. Estimated properties of Example II by the Spectral Method in the absence of noise ..........152
4.8. Estimated properties of Example II by the Spectral Method in the absence of noise ..........154
5.1. Results from analytical case studies conducted using the modified Restoring Force Method .................................................................................................................................................170
5.2. Effective linearized parameters estimated from dynamic experiments of Figure 5.1 ..........180
5.3. Models for describing the g(x(t)) function of isolator depicted in Figure 5.1 ......................... 180
5.4. Parameters of Models A, B and C based on static and dynamic experiments ......................180
5.5. Analytical results from Model D .......................................................................................................183
5.6. Estimates for Model D using Dynamic Experiments .................................................................184
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1.</td>
<td>Problem statement. (a) Multi-degree-of-freedom vibratory system. (b) Strategy for identifying nonlinear systems</td>
<td>3</td>
</tr>
<tr>
<td>2.1.</td>
<td>Problem formulation via a multi-degree-of-freedom vibration system with a nonlinear elastomeric isolator. (a) Conceptual system. (b) Experimental implementation</td>
<td>13</td>
</tr>
<tr>
<td>2.2.</td>
<td>Isolator characterization experiments. (a) Static Stiffness Test. (b) Static Stiffness Model. (c) Single-degree-of-freedom dynamic test. (d) Single-degree-of-freedom dynamic model</td>
<td>15</td>
</tr>
<tr>
<td>2.3.</td>
<td>Isolator 1. (a) schematic of isolator. (b) Static stiffness test results. Arrows indicate history of the applied static load</td>
<td>19</td>
</tr>
<tr>
<td>2.4.</td>
<td>Isolator 2. (a) schematic of isolator. (b) Static stiffness test results. Arrows indicate history of the applied static load</td>
<td>20</td>
</tr>
<tr>
<td>2.5.</td>
<td>Isolator 3. (a) schematic of isolator. (b) Static stiffness test results. Arrows indicate history of the applied static load</td>
<td>21</td>
</tr>
<tr>
<td>2.6.</td>
<td>Magnitude of measured accelerance functions from SDOF configuration under random excitation. Numbers indicate isolator used in single-degree-of-freedom configuration. Key: — lowest force level</td>
<td>24</td>
</tr>
<tr>
<td>2.7.</td>
<td>Normalized effective (linearized) parameters of Isolators. (a) Normalized effective natural frequencies. (b) Normalized effective damping ratios. Key: o-o-o Isolator 1, <em>-</em>-* Isolator 2, +++++ Isolator 3</td>
<td>26</td>
</tr>
<tr>
<td>2.8.</td>
<td>Ratio $\alpha_1$ of dynamic to static stiffness coefficients as a function of rubber durometer. (a) Key: x data and - - - curve given by reference [2.1], — range given by reference [2.2] in compression, o Isolator 1, * Isolator 2 and + Isolator 3 with $m_i = 1.7$ kg. (b) Key: x data and - - - curve given by reference [2.1], — range given by reference [2.2] in shear, o Isolator 1, * Isolator 2 and + Isolator 3 with $m_i = 1.33$ kg</td>
<td>29</td>
</tr>
<tr>
<td>2.9.</td>
<td>Isolator 1 in the single-degree-of-freedom configuration. (a) Accelerance functions. Key: — swept sine excitation with positive sweep, — — — swept sine excitation with negative sweep, — — — random excitation. (b) Auto-power spectrum of response from 36 Hz sinusoidal excitation</td>
<td>30</td>
</tr>
</tbody>
</table>
2.10. Magnitude of accelerance functions from the multi-degree-of-freedom configuration under random excitation. (a) Isolator 1 (b) Isolator 2 (c) Isolator 3. Key: — lowest force level, — — highest force level .........................................................31

2.11. Normalized effective (linearized) natural frequencies. (a) first effective natural frequency. (b) second effective natural frequency. Key: o-o-o Isolator 1, *-*-* Isolator 2, +---+ Isolator 3 ................................................................................................................................33

2.12. Normalized effective damping ratios. (a) effective damping ratio for 1\textsuperscript{st} mode. (b) effective damping ratio for 2\textsuperscript{nd} mode. Key: o-o-o Isolator 1, ♦-♦-♦ Isolator 2, +-+-+ Isolator 3 ................................................................................................................................................34

2.13. Experimentally obtained effective mode shapes of system of Figure 1(b). (a) Static mode. (b) Effective mode 1: \( m_1 \) in of phase with first bending of beam. (c) Effective mode 2: \( m_i \) out of phase with first bending mode of beam. (d) Effective mode 3: second bending of beam, \( m_1 \) stationary ........................................36

2.14. Magnitude of accelerance functions from MDOF configuration. (a) Isolator 1 (b) Isolator 2 Key: — swept sine excitation, positive sweep, — — swept sine excitation, negative sweep, — — random excitation .....................................................................38

2.15. Approximate linear relationship between \( m_1 \) and percentage errors \( \varepsilon_1 \) and \( \varepsilon_2 \). Key: — \( \varepsilon_1(m_1) \), — — \( \varepsilon_2(m_1) \) .............................................................................................................44

2.16. Differences between experimental and analytical effective natural frequencies. (a) first effective natural frequency. (b) second effective natural frequency. Key: o-o-o isolator 1, *-*-* isolator 2, +---+ isolator 3 .......................................................................................................................................47

3.1. Example cases. (a) I: three degree of freedom system with a local asymmetric quadratic-cubic nonlinearity \( f_1^2(t) \) and one excitation \( f_1(t) \). (b) II: three degree of freedom system with distributed cubic nonlinearities \( f_2^3(t) \), \( f_3^3(t) \), \( f_5^3(t) \), and one excitation \( f_1(t) \). (c) III: five degree of freedom system with a local cubic nonlinearity \( f_2^3(t) \), a local asymmetric quadratic-fifth order nonlinearity \( f_3^5(t) \) and two excitations \( f_1(t) \) and \( f_2(t) \). All excitations are Gaussian random .................................................................60

3.2. Dynamic compliance estimates of Example I. Key: — “\( H_1 \)” estimation, --- true linear dynamic compliance function synthesized from the underlying linear system’s modal parameters listed in Table 3.1. a. Magnitude of \( H_{21} \). b. Phase of \( H_{21} \) ........................................66

3.3. Dynamic compliance estimates of Example II. Key: — “\( H_1 \)” estimation, --- true linear dynamic compliance function synthesized from the underlying linear system’s modal parameters listed in Table 3.1. a. Magnitude of \( H_{21} \). b. Phase of \( H_{21} \) ........................................68

3.4. Dynamic compliance estimates of Example III. Key: — “\( H_1 \)” estimation, --- true linear dynamic compliance function synthesized from the underlying linear system’s modal parameters listed in Table 3.1. a. Magnitude of \( H_{11} \). b. Phase of \( H_{11} \) ........................................69

3.5. “Reverse path” formulation of the equations of motion in the frequency domain ...............70

3.6. Component representation of “reverse path” system’s inputs. (a) second nonlinear function vector; (b) third nonlinear function vector; (c) total response vector ....................75
3.7. Systems with uncorrelated inputs: (a) "Reverse path" system with uncorrelated multiple input vectors, (b) "Forward path" for the underlying linear system .................. 77
3.8. Illustration of the recursive algorithm given by (3.23) for \( r = 3 \) .......................................................... 80
3.9. Linear dynamic compliance estimates of Example I. Key: --- "\( H_1 \)" estimate, — conditioned "\( H_{e1} \)" estimate, o o o true linear dynamic compliance function. (a) Magnitude of \( H_{11} \). (b) Phase of \( H_{11} \) ......................................................... 88
3.10. Linear dynamic compliance estimates of Example I. Key: --- conditioned "\( H_{e1} \)" estimate, — conditioned "\( H_{e2} \)" estimate, o o o true linear dynamic compliance function. (a) Magnitude of \( H_{11} \). (b) Phase of \( H_{11} \) ......................................................... 89
3.11. Estimation of the nonlinear elastic force coefficient \( \alpha_2 \) of Example I. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of \( \alpha_2 \). (b) imaginary part of \( \alpha_2 \) ........................................................................................................ 92
3.12. Estimation of the nonlinear elastic force coefficient \( \beta_2 \) of Example I. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of \( \beta_2 \). (b) imaginary part of \( \beta_2 \) ........................................................................................................ 93
3.13. Estimation of the nonlinear elastic force coefficient \( \beta_2 \) of Example I with the employment of frequency response synthesis. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of \( \beta_2 \). (b) imaginary part of \( \beta_2 \) ........................................................................................................ 96
3.14. Linear dynamic compliance estimates of Example II. Key: --- "\( H_1 \)" estimate, — conditioned "\( H_{e2} \)" estimate, o o o true linear dynamic compliance function. (a) Magnitude of \( H_{11} \). (b) Phase of \( H_{11} \) ......................................................... 97
3.15. Linear dynamic compliance estimates of Example II. Key: --- conditioned "\( H_{e1} \)" estimate, — conditioned "\( H_{e2} \)" estimate, o o o true linear dynamic compliance function. (a) Magnitude of \( H_{31} \). (b) Phase of \( H_{31} \) ......................................................... 98
3.16. Estimates of the nonlinear elastic force coefficient \( \beta_1 \) of Example II. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of \( \beta_1 \). (b) imaginary part of \( \beta_1 \) ........................................................................................................ 100
3.17. Estimates of the nonlinear elastic force coefficient \( \beta_2 \) of Example II. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of \( \beta_2 \). (b) imaginary part of \( \beta_2 \) ........................................................................................................ 101
3.18. Estimates of the nonlinear elastic force coefficient \( \beta_3 \) of Example II. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of \( \beta_3 \). (b) imaginary part of \( \beta_3 \) ........................................................................................................ 102
3.19. Linear dynamic compliance estimates of Example III. Key: --- "\( H_1 \)" estimate, — conditioned "\( H_{e2} \)" estimate, o o o true linear dynamic compliance function. (a) Magnitude of \( H_{44} \). (b) Phase of \( H_{44} \) ......................................................... 103
3.20. Linear dynamic compliance estimate of Example III. Key: --- conditioned "\( H_{e1} \)" estimate, — conditioned "\( H_{e2} \)" estimate, o o o true linear dynamic compliance function. (a) Magnitude of \( H_{31} \). (b) Phase of \( H_{31} \) ......................................................... 104
3.21. Estimate of the nonlinear elastic force coefficient $\alpha_3$ of Example III. Key: —
estimation by (3.31), o o o true value of coefficient. (a) real part of $\alpha_3$, (b) imaginary
part of $\alpha_3$ ..............................................................................................................................105

3.22. Estimate of the nonlinear elastic force coefficient $\beta_6$ of Example III. Key: —
estimation by (3.31), o o o true value of coefficient. (a) real part of $\beta_6$, (b) imaginary
part of $\beta_6$ ..............................................................................................................................106

3.23. Estimate of the nonlinear elastic force coefficient $\gamma_6$ of Example III. Key: —
estimation by (3.31), o o o true value of coefficient. (a) real part of $\gamma_6$, (b) imaginary
part of $\gamma_6$ ..............................................................................................................................107

4.1. Simulation examples. (a) Two degree of freedom system with a nonlinear spring
element of elastic force $f_{12}(t)$. (b) Plots of nonlinear elastic forces. Key: — $f_{12}(t)$
of Example I, — $f_{12}(t)$ of Example II, ... linear component of $f_{12}(t)$, i.e. $k_1\Delta x_{12}(t)$ .........114

4.2. Dynamic compliance spectra of Example I. (a) Magnitude of $H_{21}(\omega)$. (b) Phase of
$H_{21}(\omega)$. Key: — $\hat{H}_{21}(\omega)$, — $H_{21}(\omega)$......................................................................................119

4.3. “Reverse path” spectral model. (a) Model with correlated inputs, equation (4.10). (b)
Conditioned model with uncorrelated inputs. (c) “Forward path” of the underlying
linear sub-system ............................................................................................................................122

4.4. Dynamic compliance spectra of Example I using Model A_1 for “$H_{22}$” estimate. (a)
Magnitude of $H_{11}(\omega)$. (b) Phase of $H_{11}(\omega)$. Key: — $\hat{H}_{11}^{[1]}(\omega)$, — $\hat{H}_{11}^{[1]}(\omega)$, o o o
$H_{11}(\omega)$ ..................................................................................................................................129

4.5. Spectrum of estimated coefficient $\hat{a}_3(\omega)$ of Model A_1. (a) Re[$\hat{a}_3(\omega)$]. (b) Im[$\hat{a}_3(\omega)$].
Key: — $\hat{a}_3(\omega)$, o o o true coefficient $\beta_3$ ..................................................................................131

4.6. Coherence functions of Model A_1. (a) Cumulative coherence function $\hat{\gamma}_{x^{-}}^{[2]}(\omega)$.
(b) Coherence functions $\hat{\gamma}_{X,Y}^{[2]}(\omega)$ and $\hat{\gamma}_{Y,X}^{[2]}(\omega)$. Key: — $\hat{\gamma}_{x^{-}}^{[2]}(\omega)$, — $\hat{\gamma}_{X,Y}^{[2]}(\omega)$ ..........133

4.7. Auto-power spectra of noise-free data, moderate and high noise levels. (a) Response
auto-power spectra. (b) Excitation auto-power spectra. Key: — noise-free data, — high noise level, — moderate noise level .................................................................137

4.8. Magnitude of dynamic compliance spectra of Example I using Model A_1 for “$H_{22}$”
estimate. (a) Moderate noise case. (b) High noise case. Key: — $\hat{H}_{11}^{[2]}(\omega)$, — $\hat{H}_{11}^{[1]}(\omega)$, o o o
$H_{11}(\omega)$ ..................................................................................................................................139

4.9. Cumulative coherence functions $\hat{\gamma}_{M1}^{[2]}(\omega)$ of Model A_1. (a) Moderate noise case. (b)
High noise case ..................................................................................................................................140

4.10. Magnitude of dynamic compliance spectra of Example III using Model A_III for “$H_{22}$”
estimate. (a) Key: — $\hat{H}_{11}^{[2]}(\omega)$, — $\hat{H}_{11}^{[1]}(\omega)$, o o o $H_{11}(\omega)$. (b) Key: — $\hat{H}_{11}^{[1]}(\omega)$, — $\hat{H}_{11}^{[1]}(\omega)$, o o o $H_{11}(\omega)$......................................................................................141
4.11. Partial coherence function $\hat{\xi}_{XX}^2(t_{-m})(\omega)$ of Model A

4.12. Magnitude of dynamic compliance spectra of Example III using Model A

4.13. Spectra for Example I using Model B

4.14. Spectra for Example I using Model C

4.15. Spectra for Example I using Model D

4.16. Magnitude of Dynamic compliance functions of Example IV. (a) Model A used for “$H_{c2}$” estimate. Key: --- $\hat{H}_{c2}^1(\omega)$, --- $\hat{H}_{c2}^2(\omega)$, $\ldots$

4.17. Cumulative coherence and stiffness estimates of Example IV. (a) Cumulative coherence of Model B

5.1. Identification experiments: (a) mount specimen, (b) static experimental setup, (c) dynamic experimental setup

5.2. Feasibility study for the modified Restoring Force Method. (a) Isolation system considered for identification. (b) Five degree of freedom model of isolation system for analytical case studies

5.3. Experimental single-degree-of-freedom system results. (a) Measured accelerance. (b) Identified mass versus stiffness curves. Key: --- 48 Hz, --- 60 Hz

5.4. Experimental two-degree-of-freedom system results. (a) Measured accelerance at the driving point. (b) Discrete linear system model

5.5. Results for the two-degree-of-freedom system experiment of Figure 5.4. (a) Identified stiffness $k_1$ versus stiffness $k_2$ curves. (b) Identified mass $m_1$ versus stiffness $k_2$ curves. Key: --- 32 Hz, --- 48 Hz

xvi
5.6. Experimental Results for the isolator of Figure 5.1: (a) stiffness curve from static experiment, (b) frequency response functions from dynamic experiments ........................................... 179

5.7. Synthesized Results for the isolator experiment: (a) stiffness curves, (b) frequency response functions .............................................................................................................................. 186
LIST OF SYMBOLS FOR CHAPTER 2

Bold characters indicate matrices and vectors.

a coefficient vector of nonlinear functions
a location of isolator on beam
b intersect
c linear damping coefficient
C linear damping matrix
E elastic modulus of beam
f force
f(t) generalized excitation vector
f frequency (Hz)
g nonlinear dynamic elastic force
G dynamic restoring force
h height
H(ω) accelerance
i \(\sqrt{-1}\)
l \(2^{nd}\) moment of inertia of beam
k linear stiffness element
K linear stiffness matrix
L length of beam
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>mass</td>
</tr>
<tr>
<td>M</td>
<td>mass matrix</td>
</tr>
<tr>
<td>n</td>
<td>number of types of nonlinearities</td>
</tr>
<tr>
<td>r</td>
<td>radius</td>
</tr>
<tr>
<td>s</td>
<td>slope</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
</tr>
<tr>
<td>T</td>
<td>time window</td>
</tr>
<tr>
<td>u</td>
<td>transverse displacement of beam</td>
</tr>
<tr>
<td>V</td>
<td>shear force</td>
</tr>
<tr>
<td>x</td>
<td>displacement</td>
</tr>
<tr>
<td>x(t)</td>
<td>generalized displacement vector</td>
</tr>
<tr>
<td>y</td>
<td>axis along length of beam</td>
</tr>
<tr>
<td>y(t)</td>
<td>nonlinear function</td>
</tr>
<tr>
<td>α</td>
<td>ratio of dynamic to static stiffness</td>
</tr>
<tr>
<td>ε</td>
<td>error</td>
</tr>
<tr>
<td>φ</td>
<td>amplitude of beam</td>
</tr>
<tr>
<td>Φ</td>
<td>amplitude of m₁</td>
</tr>
<tr>
<td>Δt</td>
<td>time step for numerical simulation</td>
</tr>
<tr>
<td>Δx₁₂(t)</td>
<td>relative displacement = x₁(t) - x₂(t)</td>
</tr>
<tr>
<td>ρ'</td>
<td>mass per unit length</td>
</tr>
<tr>
<td>ω</td>
<td>frequency or natural frequency (rad/s)</td>
</tr>
<tr>
<td>ζ</td>
<td>damping ratio</td>
</tr>
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</table>
subscripts

d       dynamic
L       lumped model
m       isolator index
s       static
S       sample rate
t       total

superscripts

b       beam
cr      coordinate reduction

embellishments

*       first derivative with respect to time
**      second derivative with respect to time
LIST OF SYMBOLS FOR CHAPTER 3

*Bold characters indicate matrices and vectors.*

A \( A^{\text{e}} \) \( A^{\text{f}} \) coefficient matrix of nonlinear function vectors.
B \( B \) linear dynamic stiffness matrix.
C \( C \) linear damping matrix.
f \( f \) elastic force.
f \( f \) generalized excitation vector with Gaussian time history.
F \( F \) spectra of \( f(t) \).
G \( G \) single-sided cross-spectral density matrix.
H \( H \) linear dynamic compliance matrix.
i \( i \) \( \sqrt{-1} \).
k \( k \) linear stiffness element.
K \( K \) linear stiffness matrix.
L \( L \) frequency response function of conditioned “reverse path” model.
M \( M \) mass matrix.
M \( M \) number of measured excitations.
MAC \( MAC \) modal assurance criterion.
N \( N \) dimension of system.
n \( n \) number of types of nonlinearities.
p \( p \) total number of nonlinearities.
PSD \( \text{PSD} \) power spectral density.
qu \( q \) number of locations an unique nonlinearity exists.
t \( t \) time.
T \( T \) time window.
x \( x \) generalized displacement vector.
X \( X \) spectra of \( x \).

xxi
\( y \) nonlinear function vector
\( \mathbf{Y} \) Spectra of \( y \)
\( \alpha \) coefficient of quadratic nonlinear stiffness terms
\( \beta \) coefficient of cubic nonlinear stiffness terms
\( \Delta t \) time step for numerical simulation
\( \Gamma \) matrix of single-sided spectral density matrices involving the response, the \( n \) nonlinear function vectors and the excitation
\( \gamma \) coefficient of fifth order nonlinear stiffness terms
\( \Delta x \) relative displacement
\( \omega \) frequency
\( \Xi \) matrix of single-sided spectral density matrices involving the response and the \( n \) nonlinear function vectors
\( \Psi \) matrix consisting of the dynamic stiffness matrix and the coefficient matrices
\( \mathbf{0} \) null matrix
\( ? \) unmeasured dynamic compliance functions

operators
\( E[.] \) expected value
\( F[.] \) Fourier transform
\( \text{Im}[.] \) imaginary part
\( L[.] \) linear operator
\( N[.] \) non-linear operator
\( \text{Re}[.] \) real part
\( \langle \cdot \rangle_{\omega} \) spectral mean
subscripts

1, 2  conventional estimates of $\mathbf{H}$
c1, c2 conditioned estimates of $\mathbf{H}$
F excitation vector
j jth nonlinear function
k kth junction
(+j) correlated with the jth nonlinear function
(-1:j) uncorrelated with the 1st through the jth nonlinear function
(-1:n) uncorrelated with the 1st through the nth nonlinear function, i.e. linear component
X response vector

superscripts

• complex conjugate
H Hermitian transpose
m exponent of nonlinearity
r determined from reciprocity
T transpose
-1 inverse

embellishments

~ measured quantity
• first derivative with respect to time
** second derivative with respect to time
$-$ spectral mean
LIST OF SYMBOLS FOR CHAPTER 4

*Bold characters indicate matrices and vectors.*

a  coefficient vector of nonlinear functions
a  coefficient nonlinear functions
\( B(\omega) \)  linear dynamic stiffness matrix
c  linear damping coefficient
\( C \)  linear damping matrix
\( d(x(t), \dot{x}(t)) \)  vector of motion dependent restoring force functions
\( f_{12}^*(t) \)  elastic force acting on \( m_1 \)
f(t)  generalized excitation vector with Gaussian time history
\( F(\omega) \)  spectra of f(t)
\( G(\omega) \)  single-sided cross-spectral density matrix
\( H(\omega) \)  linear dynamic compliance matrix
i  \( \sqrt{-1} \)
k  linear stiffness element
\( K \)  linear stiffness matrix
\( L(\omega) \)  frequency response function between nonlinear function \( Y(\omega) \) and \( F(\omega) \)
m  mass
\( M \)  mass matrix
n  number of types of nonlinearities
\( n(t) \)  uncorrelated noise vector
N  dimension of system
\( N(\omega) \)  vector of uncorrelated noise spectra
\( p_{12}^s(t) \)  assumed model of \( f_{12}^s(t) \)
t  time
\( \tau \)  
- time window

\( x(t) \)  
- generalized displacement vector

\( X(\omega) \)  
- spectra of \( x(t) \)

\( y(\Delta x_{12}(t)) \)  
- nonlinear function

\( Y(\omega) \)  
- Spectra of \( y(\Delta x_{12}(t)) \)

\( z(x(t), \dot{x}(t)) \)  
- vector of the assumed form of the system's constraint forces

\( \beta \)  
- coefficient of polynomial term describing \( f_{12}^\beta(t) \) for Examples I-III

\( \phi \)  
- mode shape

\( \Delta t \)  
- time step for numerical simulation

\( \Delta x_{12}(t) \)  
- relative displacement = \( x_1(t) - x_2(t) \)

\( \tilde{\gamma}^2(\omega) \)  
- ordinary coherence function

\( \Gamma(\omega) \)  
- Fourier transform of \( d(x(t), \dot{x}(t)) \)

\( \eta \)  
- coefficient of polynomial term describing \( f_{12}^\eta(t) \) for Example IV

\( \omega \)  
- frequency or natural frequency (rad/s)

\( \tilde{\xi}^2(\omega) \)  
- partial coherence function

\( \zeta \)  
- damping ratio

\( \{0\} \)  
- null vector and matrix

**operators**

\( \mathbb{E}[. \]  
- expected value

\( \mathcal{F} . \)  
- Fourier transform

\( \text{Im}[.] \)  
- imaginary part

\( \text{Re}[.] \)  
- real part

\( \langle . \rangle_\omega \)  
- spectral mean
subscripts

c1, c2  conditioned estimates of H

e  effective matrix determined from linearization

f  excitation vector

i  i\textsuperscript{th} mass location

j  j\textsuperscript{th} nonlinear function

(*)j  correlated with the j\textsuperscript{th} nonlinear function

(-1)j  uncorrelated with the 1\textsuperscript{st} through the j\textsuperscript{th} nonlinear function

(-1,n)  uncorrelated with the 1\textsuperscript{st} through the n\textsuperscript{th} nonlinear function, i.e., linear component

m  signifies cumulative coherence function

n  contains only nonlinear restoring force terms

v  uncorrelated noise

r  r\textsuperscript{th} mode

x  response vector

y  coherence function which indicates contribution from nonlinearities

superscripts

*  complex conjugate

T  transpose

-1  inverse

[1]  "H\textsubscript{1}" estimate

[e1]  "H\textsubscript{e1}" estimate

[e2]  "H\textsubscript{e2}" estimate

xxvi
embellishments

- quantity contaminated by noise
- estimated quantity
- first derivative with respect to time
- second derivative with respect to time
LIST OF SYMBOLS FOR CHAPTER 5

\( a_{ij} \)  \hspace{1em} generic coefficient of the polynomial expansion representing the restoring force
\( a_3 \)  \hspace{1em} coefficient of cubic stiffness term
\( c \)  \hspace{1em} damping coefficient
\( k \)  \hspace{1em} stiffness coefficient
\( f \)  \hspace{1em} static load
\( f'(t) \)  \hspace{1em} dynamic excitation
\( f(t) \)  \hspace{1em} total force
\( g\{\cdot\} \)  \hspace{1em} vector function representing the restoring forces
\( m_i \)  \hspace{1em} mass of \( i^{th} \) degree of freedom
\( M \)  \hspace{1em} mass matrix
\( N \)  \hspace{1em} number of time samples
\( T \)  \hspace{1em} sample period
\( x(t) \)  \hspace{1em} dynamic displacement
\( x_{ij}(t) \)  \hspace{1em} dynamic relative displacement, \( x_i(t) - x_j(t) \)
\( \dot{x}_{ij}(t) \)  \hspace{1em} relative velocity, \( \dot{x}_{ij}(t) = \dot{x}_i(t) - \dot{x}_j(t) \)
\( y(t) \)  \hspace{1em} total displacement, \( y(t) = x(t) + \delta \)
\( y_{ij}(t) \)  \hspace{1em} total relative displacement, \( y_i(t) - y_j(t) \)
\( \dot{y}_{ij}(t) \)  \hspace{1em} relative velocity, \( \dot{y}_{ij}(t) = \dot{y}_i(t) - \dot{y}_j(t) \)
\( Y_i \)  \hspace{1em} matrix of known time samples of the \( i^{th} \) degree of freedom
\( \alpha_i \)  \hspace{1em} vector of unknown parameters of the \( i^{th} \) degree of freedom
\( \beta_{ij} \)  \hspace{1em} equation of unknown parameters at the \( i^{th} \) degree of freedom as a result of expanding the nonlinear terms
\( \beta_i \)  \hspace{1em} vector of the equations \( \beta_{ij} \)
\( \delta \) static displacements

\( \delta_{ij} \) relative static displacements, \( \delta_i - \delta_j \)
CHAPTER I

INTRODUCTION

1.1. MOTIVATION

Common methods for vibration testing and parameter identification are based on linear system theory even though many mechanical and structural systems contain some nonlinear elements. For example, isolators used in machinery, appliance and vehicles exhibit some nonlinearities, especially under strong dynamic loads [1.1.2]. Also, in some cases, nonlinear behavior is preferred since the same isolator may satisfy conflicting performance requirements [1.3]. Liquid filled elastomeric isolators used for automotive and aircraft applications are chief examples of such devices: isolator stiffness and damping rates significantly vary with excitation amplitude and frequency [1.4]. Typical static experiments and dynamic stiffness type experiments (at a single frequency with given excitation amplitude) [1.5] are conducted to evaluate isolator behavior. However, true nonlinear nature is not often revealed and hence large discrepancies may be observed between component tests and system performance. Consequently, there is a need to develop suitable system-oriented experimental and digital signal processing methods that can characterize and, if possible, accurately identify parameters of nonlinear systems which are useful in developing mathematical models for design and analysis. These parameters can also provide useful information for control algorithms used for active noise and vibration isolation applications which include inherent nonlinearities.
1.2. PROBLEM STATEMENT

Consider the multi-degree-of-freedom vibratory system of Figure 1.1(a) where each lumped mass is connected by a linear or nonlinear visco-elastic element. The motion of this system is described by the following set of coupled differential equations

\[ M \ddot{x}(t) + d(x(t), \dot{x}(t)) = f(t) \]  

where \( M \) is the time-invariant mass matrix, \( x(t) \) and \( f(t) \) are the generalized displacement and force vectors, and \( d(x(t), \dot{x}(t)) \) is a vector of motion dependent restoring force functions. Decompose \( d(x(t), \dot{x}(t)) \) as follows

\[ d(x(t), \dot{x}(t)) = C \dot{x}(t) + K x(t) + d''(x(t), \dot{x}(t)) \]

where \( C \) and \( K \) are the time-invariant linear damping and linear stiffness matrices, and \( d''(x(t), \dot{x}(t)) \) is a vector consisting of only nonlinear terms. To identify the parameters of such a system, consider a laboratory experiment where single or multiple force excitations \( f(t) \) of measurable time histories are applied by electro-dynamic shakers and signal generators. System responses \( \ddot{x}(t) \) are measured, typically by accelerometers, and sampled by a multi-channel dynamic signal analyzer. Digital signal processing algorithms may then be utilized to calculate frequency response functions and other spectral functions.

When the system is linear, i.e. \( d''(x(t), \dot{x}(t)) = 0 \), or linearized to yield effective damping \( C_\nu \) and stiffness \( K_\nu \) matrices, identification methods can estimate parameters from measured time or frequency domain data in the form of natural frequencies \( \omega_n \), mode shapes \( \phi_n \), and damping ratios \( \zeta_n \) [1.6]. However, complexities such as high modal density, heavily damped modes and measurement noise complicate the accurate determination of these parameters. These complications may be alleviated using mode indicator functions to determine valid modes [1.6] and proper frequency response estimators, such as "\( H_1 \)" or "\( H_2 \)". to minimize uncorrelated measurement noise [1.7, 1.8]. Nonetheless, it is difficult to construct \( M, C \) and \( K \) unless a computational model is available.

To worsen the problem of identification, the effects of \( d''(x(t), \dot{x}(t)) \) for many physical nonlinear systems may substantially influence the dynamic response \( x(t) \) [1.9-1.11]. Consequently, modal testing and similar linear system based methods are no longer valid. Under these circumstances identification methods
Figure 1.1. Problem statement. (a) Multi-degree-of-freedom vibratory system. (b) Strategy for identifying nonlinear systems.
for nonlinear systems must be employed. In contrast to linear techniques which can often be applied
directly to measured time or frequency domain data, the identification of nonlinear systems requires
additional analytical and signal processing techniques. A flow chart illustrating one possible scheme for
identifying nonlinear systems is proposed in Figure 1.1(b). Elements of this scheme are followed in this
study and they will be discussed in detail later. Vibration experiments may be conducted and data collected
in the same manner as done with linear systems [1.12]. However, more advanced digital signal processing
and data conditioning algorithms are necessary for the identification schemes. This data will be analyzed to
characterize the types of nonlinearities present and mathematical models will be chosen to describe the
nonlinearities. Then, based on the assumed mathematical models, identification schemes will be utilized to
extract system parameters.

1.3. LITERATURE REVIEW

To accommodate for the presence of nonlinearities, several researchers have developed time
domain methods to identify nonlinear systems. A primary method is the Force State Mapping or Restoring
Force Method which was initially developed by Masri and Caughey [1.13] for a single-degree-of-freedom
system where the restoring force was described by Chebyshev polynomials. This method has also been
extended to multi-degree-of-freedom systems [1.14]. The use of Chebyshev polynomials required an
iterative algorithm until Yang and Ibrahim modified the method to describe the nonlinear restoring forces
by power series expansions [1.15]. Data processing and experimental design issues for the Restoring Force
Method have been investigated by Worden [1.16-1.17]. Wright and Al-Hadid have investigated the
sensitivity of the method to experimental errors [1.18] and have applied the method to estimate mass in
both the physical and modal domains [1.19]. The method has been applied to a number of experimental
systems [1.20-1.22] including automobile shock absorbers [1.23, 1.24]. A chief limitation of the method is
the accumulation of estimation error resulting from measurement noise, signal processing errors and time
integration problems. Additional discussion on this topic will be included later.

Several frequency domain methods also exist [1.25-1.30]. For example, the functional Volterra
series approach for estimating higher order frequency response functions of nonlinear systems has gained
recognition [1.25]. This method has been used by Gifford and Tomlinson to estimate first and second order frequency response functions of a nonlinear beam subjected to random excitation [1.26], where curve fitting techniques were used for parametric estimation of an analytical model. However, the method is very computationally intensive and estimation of third and higher order frequency response functions has been unsuccessful. To alleviate this problem, Storer and Tomlinson used sinusoidal excitation to estimate only the diagonal second and third order frequency response functions of the Volterra series [1.27]. Other higher order spectral techniques have also been employed for the analysis of nonlinear systems [1.28]. For instance, the bi-coherence function has been used by Shyu to detect the second order nonlinear behavior present in a system [1.29]. Also, Sadasivan et al. studied the sub-harmonic responses of a high speed rotor using bi-spectral and tri-spectral techniques [1.30].

An alternative spectral approach has recently been developed by Bendat et al. [1.31-1.34] for single-input/single-output systems which identifies a “reverse path” system model. A similar approach has been used for the identification of two-degree-of-freedom nonlinear systems where each response location is treated as a single degree of freedom mechanical oscillator [1.35]. Single degree of freedom techniques are then used to identify system parameters [1.12]. However, this approach requires excitations to be applied at every response location and it also inhibits the use of preferred higher dimensional parameter estimation techniques that are commonly used for the modal analysis of linear systems [1.6].

1.4. SCOPE AND OBJECTIVES

Many research issues remain unresolved before the available identification methods can be applied to nonlinear problems of the type illustrated in Figure 1.1(a). Selected key questions which will be addressed in this study are as follows, roughly in the given order:

1. Are there unique experimental characterization methods which lead to a better understanding of nonlinear elements and quantification of their parameters?

2. Can the “reverse path” system model of reference [1.31-1.34] be analytically extended to multi-degree-of-freedom systems?
3. Should there be an *a priori* knowledge of the nature and mathematical form of $d_n(x(t), \dot{x}(t))$, as given in (1.2), before the identification process is initiated? If not, will this lead to difficulties in the identification of $d_n(x(t), \dot{x}(t))$?

4. Is identification for a nonlinear system compounded by the presence of uncorrelated measurement noise?

5. Can coherence type techniques [1.36] be used to facilitate the nonlinear system identification process?

The scope of this work is limited to the following methods and examples. Although both the Restoring Force and “Reverse Path” Spectral Methods are investigated, emphasis is placed on the spectral method. Performance of each method is studied computationally using multi-degree-of-freedom nonlinear systems whose responses are determined from a well known time integration algorithm. The analytical examples include 2, 3 and 5 degree of freedom systems consisting of only the continuous type stiffness nonlinearities. Three elastomeric isolators are also experimentally studied under single and multi-degree-of-freedom configurations. All studies are conducted in the 0-130 Hz frequency range and typical excitations are band-limited random and sinusoidal (harmonic and sine sweep). Specific objectives of this dissertation are as follows:

1. Apply experimental techniques to characterize three different elastomeric isolators (Chapter 2).

2. Introduce a new multi-degree-of-freedom spectral approach based on a “reverse path” system model (Chapter 3).

3. Investigate the performance of identification methods when incorrect or approximate mathematical equations are used to model nonlinearities (Chapter 4).

4. Assess the effects of measurement noise on the identification processes with focus on the spectral method (Chapter 4).

5. Develop new coherence function estimates to assist in determining accurate nonlinear models (Chapter 4).
6. Apply identification methods to the experimental study of isolators and propose refinements to the Restoring Force Method to accommodate for the presence of static loads and displacements (Chapter 5).

Each chapter of this dissertation is self-sufficient, containing its own analytical formulation, objectives, results, conclusion, and a list of references. A common bibliography is included at the end of this dissertation.
<table>
<thead>
<tr>
<th>Reference</th>
<th>Title</th>
<th>Author(s)</th>
<th>Year</th>
<th>Notes</th>
</tr>
</thead>
</table>


1.27. D. M. Storer and G. R. Tomlinson 1993 *Mechanical Systems and Signal Processing* *7*, 173-189. Recent developments in the measurement and interpretation of higher order transfer functions from non-linear structures.


CHAPTER 2

A STUDY OF DYNAMIC BEHAVIOR OF RUBBER ISOLATOR NONLINEARITIES USING
EXPERIMENTAL AND ANALYTICAL APPROACHES

2.1. INTRODUCTION

Experimental testing of rubber isolators is often conducted to determine static and dynamic
stiffness characteristics and damping properties. Static stiffness data may be useful for mean load
requirements; but inevitably, dynamic stiffness is different from that obtained from the static tests [2.1].
Trends have been found relating static to dynamic stiffness through durometer [2.1, 2.2]. Also, dynamic
experiments have been useful for determining dynamic stiffness directly where sinusoidal displacement
excitations are applied in single or multiple directions and forces and displacements are measured to obtain
complex stiffness data at fixed frequencies and amplitudes [2.3]. However, do these techniques reveal the
dynamic behavior of the isolators when they are placed into operating conditions which may be quite
different from test conditions? For instance, do isolators exhibit different behavior when subjected to
excitations different from sinusoidal? Also, can the dynamics of the system in which the isolator is
employed affect the behavior of the isolator itself? If rubber isolators exhibited linear dynamic behavior,
the answers would be no for both of these questions. However, many isolators are nonlinear in nature,
therefore, it may not be reliable to assume that characteristics found in an experimental test will accurately
describe the dynamic behavior of an isolator in another circumstance. In this study, static and dynamic
tests are designed to experimental characterize three rubber isolators under single and multi-degree-of-
freedom configurations and the issues raised here are investigated. In addition, analytical models are
employed to describe the dynamic behavior of the isolators based on the experimental data.
2.2. PROBLEM FORMULATION

Consider the system shown in Figure 2.1(a) where a lumped mass \( m_i \) is mounted to a flexible beam from the bottom at \( y = a \) via an isolator. A second lumped mass is also rigidly connected to the top of the beam at \( y = a \). This generalized arrangement could represent a variety of practical systems, e.g. an engine or gearbox isolated from the chassis of an automobile or the fuselage of an aircraft. For the purposes of analysis, experimental and mathematical system models are formulated to examine the dynamic behavior of this system. For experimental studies, the system in Figure 2.1(b) is constructed and analyzed first using the linear system theory. The flexible beam is described by Euler's beam equation [2.4]. Due to shear force loading from the isolator, the dynamics of the beam must be given by two separate equations. These equations along with the equation of motion of \( m_i \) are:

\[
\frac{\partial^2}{\partial y^2} \left[ EI(y) \frac{\partial^2 u_j(y,t)}{\partial y^2} \right] = -p'(y) \frac{\partial^2 u_j(y,t)}{\partial t^2}; \quad j = 1, \ 0 \leq y \leq a \ \\
\frac{\partial^2}{\partial y^2} \left[ -\rho(y) \frac{\partial^2 u_j(y,t)}{\partial y^2} \right] = -m \ddot{x}_i(t) - k_m (x_i(t) - u_j(a,t)) = 0
\]

where \( E \) is the elastic modulus of the homogeneous beam, \( I(y) \) is the 2nd moment of inertia about the x-axis (perpendicular to the plane shown), \( p'(y) \) is the mass per unit length, \( u_j(y,t) \) are the transverse displacements of the beam, \( x_i(t) \) is the displacement of \( m_i \), and \( k_m \) is the linear stiffness coefficient of the isolator. Also refer to the List of Symbols for chapter 2 for the identification of symbols. Assuming harmonic solutions \( u_j(y,t) = \phi_j(y)e^{\omega t} \) and \( x_i(t) = \Phi e^{\omega t} \), results in the following eigenvalue problem:

\[
\frac{\partial^2}{\partial y^2} \left[ EI(y) \frac{\partial^2 \phi_j(y)}{\partial y^2} \right] - \omega^2 \rho(y) \phi_j(y) = 0; \quad j = 1, \ 0 \leq y \leq a \ \\
- \omega^2 m \dot{\Phi} + k_m (\Phi - \phi_j(a)) = 0 \quad j = 2, \ a \leq y \leq L
\]

which is solved after applying the appropriate boundary conditions existing at \( y = 0 \) and \( L \) and satisfying continuity, slope, force and moment loading conditions at \( y = a \).

The objective of this study is to experimentally characterize three rubber isolators when subjected to random, sinusoidal and swept sine force excitations as well as static time-invariant loads. Initially, single-degree-of-freedom system experiments will be performed to obtain viscous damping and nonlinear stiffness properties for the isolators. The isolators are then placed into the multi-degree-of-freedom
Figure 2.1. Problem formulation via a multi-degree-of-freedom vibration system with a nonlinear elastomeric isolator. (a) Conceptual system. (b) Experimental implementation.
experimental system of Figure 2.1(b) and tests conducted to characterize nonlinear isolator and other system parameters. Equation (2.1a-c) will be used to describe the linearized system responses of the experimental system in the 0-100 Hz frequency range. Finally, discrete nonlinear models will be considered where nonlinear functions are employed to describe the elastic forces of the isolators.

Numerical simulation is used to calculate the nonlinear response of the system models to be compared with the experimental data.

2.3. EXPERIMENTAL METHODS

As mentioned above, one objective of this study is to characterize the isolators' static elastic forces \( f_s = h_m(x_s) \). Therefore, each isolator is first subjected to the static stiffness experiment illustrated in Figure 2.2(a) where static loads \( f_s \) are applied in both tension and compression by adding known masses to the weight containers. After a period of 10 seconds, the static displacement \( x_s \) of the isolator is recorded from the dial indicator. The assumed model for this experiment is shown in Figure 2.2(b) where \( h_m(x_s) \) may be nonlinear.

Next, the isolators are tested in the single-degree-of-freedom dynamic experiment of Figure 2.2(c) where a cylindrical block \( m_1 \) (1.4 kg) is isolated from the center of a steel beam via each isolator. The beam, whose ends are clamped to massive supports, has the following dimensions: unconstrained length = 540 mm, width = 100 mm and thickness = 25.40 mm. As a result, the first beam natural frequency is 468 Hz. This is more than 4 times the largest frequency of interest (\( f_{max} = 100 \) Hz). Therefore, the single-degree-of-freedom model of Figure 2.2(d) can be assumed. In this model, the dynamic restoring force of each isolator in the normal direction is given as \( G_m(\Delta \dot{x}(t), \Delta x(t)) \) where \( m \in [1,3] \) is the index indicating which isolator the dynamic restoring force describes, and \( \Delta \dot{x}(t) \) and \( \Delta x(t) \) are the relative velocity and displacement across the isolator, respectively. Assuming linear viscous damping, the restoring force is written as \( G_m(\Delta \dot{x}(t), \Delta x(t)) = c_m \Delta \dot{x}(t) + g_m(\Delta x(t)) \) where \( c_m \) is the linear damping coefficient and \( g_m(\Delta x(t)) \) is the nonlinear dynamic elastic force of the \( m \)th isolator. Experimental characterization of the isolators under this configuration involves measuring of the acceleration \( \Delta \ddot{x}(t) \) by a piezoelectric
Figure 2.2. Isolator characterization experiments. (a) Static Stiffness Test. (b) Static Stiffness Model. (c) Single-degree-of-freedom dynamic test. (d) Single-degree-of-freedom dynamic model.
acceleration transducer and measuring the excitation $f(t)$ by a piezoelectric force transducer. The excitation $f(t)$ is provided by an electro-dynamic shaker. See Table 2.1 for a list of instrumentation. Note that channels 3-7 listed in Table 2.1 are not used for this configuration, but will be used for the multi-degree-of-freedom configuration discussed next.

For characterization of the isolators under multi-degree-of-freedom configuration, the experimental system of Figure 2.1(b) is considered where the block $m_1$ is isolated from an aluminum beam via a rubber isolator. Note $m_1$ is the same cylindrical block used in the single-degree-of-freedom configuration. The aluminum beam, whose ends are clamped to massive supports, has the following dimensions: unconstrained length = 670 mm, width = 100 m, thickness = 9.56 mm. The isolator is located at the center of the beam where a second block $m_2$ (6.7 kg) is also connected. Experimental characterization involves the measurement of the responses at multiple locations and the applied excitation. As with the single-degree-of-freedom configuration, a piezoelectric force transducer measures the excitation $f(t)$ of $m_1$ provided by an electro-dynamic shaker. However, for this configuration, accelerations $\ddot{x}_i(t)$ at locations $i, i \in [1,6]$, are measured by piezoelectric acceleration transducers.

For each isolator under each of the dynamic configurations, 3 different types of excitations are applied to the isolated mass $m_1$: random, sinusoidal and swept sine. In addition, each type of excitation is applied at different levels, and for the sinusoidal excitation at different frequencies. These cases are summarized in Table 2.2. For all of the experiments, the excitations and responses are sampled at $f_s = 2048$ Hz for 86 seconds. Before each isolator is tested, a “break-in” period is initially conducted where $m_1$ is subjected to random excitation from 0 – 100 Hz at 5 different excitation levels for 1 minute.

2.4. EXPERIMENTAL CHARACTERIZATION OF ISOLATORS

2.4.1. STATIC BEHAVIOR

Shown in Figures 2.3-2.5 are the rubber isolators considered for this analysis. Isolator 1 of Figure 2.3(a) is composed of a neoprene rubber cylinder with 2 aluminum mounting plates fastened to each end. The neoprene has a Shore A hardness of 45. Figure 2.3(b) illustrates this isolator’s stiffness curve from the
<table>
<thead>
<tr>
<th>Transducer</th>
<th>Location in Figure 1(b)</th>
<th>Model #</th>
<th>Calibration</th>
<th>Mass (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force</td>
<td>1</td>
<td>PCB 208 A03</td>
<td>2.58 mV/N</td>
<td>23.0</td>
</tr>
<tr>
<td>Acceleration</td>
<td>1</td>
<td>PCB U352 B65</td>
<td>11.65 mV/m/s²</td>
<td>5.0</td>
</tr>
<tr>
<td>Acceleration</td>
<td>2</td>
<td>PCB U352 B65</td>
<td>11.06 mV/m/s²</td>
<td>5.0</td>
</tr>
<tr>
<td>Acceleration</td>
<td>3</td>
<td>PCB A353 B66</td>
<td>10.63 mV/m/s²</td>
<td>2.0</td>
</tr>
<tr>
<td>Acceleration</td>
<td>4</td>
<td>PCB A353 B66</td>
<td>10.11 mV/m/s²</td>
<td>2.0</td>
</tr>
<tr>
<td>Acceleration</td>
<td>5</td>
<td>PCB A353 B66</td>
<td>9.80 mV/m/s²</td>
<td>2.0</td>
</tr>
<tr>
<td>Acceleration</td>
<td>6</td>
<td>PCB A353 B66</td>
<td>10.00 mV/m/s²</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Using drop calibration method: 5 averages, 150 g mass used for force transducer, g = 9.81 m/s².

Shaker: Vibration Test Systems (VTS) VG 100-6, dynamic mass = 299.4 g, stinger mass = 12.0 g

Table 2.1. Instrumentation.
<table>
<thead>
<tr>
<th>System Configuration</th>
<th>Excitation Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDOF Figure 2(c)</td>
<td>Random 0-100 Hz – 5 levels of force</td>
</tr>
<tr>
<td></td>
<td>Sinusoidal – 4 levels at 5 Hz, peak frequency and 80 Hz</td>
</tr>
<tr>
<td></td>
<td>Sine sweep – 4 levels for both positive and negative sweep rates</td>
</tr>
<tr>
<td>MDOF Figure 1(b)</td>
<td>Random 0-100 Hz – 4 levels of force</td>
</tr>
<tr>
<td></td>
<td>Sinusoidal – 4 levels at 5, 26, 35, 47 and 80 Hz</td>
</tr>
<tr>
<td></td>
<td>Sine sweep – 4 levels for both positive and negative sweep rates</td>
</tr>
</tbody>
</table>

Table 2.2. Summary of dynamic experiments.
Figure 2.3. Isolator 1. (a) schematic of isolator. (b) Static stiffness test results. Arrows indicate history of the applied static load.
Figure 2.4. Isolator 2. (a) schematic of isolator. (b) Static stiffness test results. Arrows indicate history of the applied static load.
Figure 2.5. Isolator 3. (a) schematic of isolator. (b) Static stiffness test results. Arrows indicate history of the applied static load.
static experiment where the arrows indicate the history of the applied static load, i.e. first compression loads are applied, then released. Next, tension loads are applied, then released. Notice the presence of hysteresis from the open-looped curve. Also, notice a slight softening stiffness curve in tension and a slight hardening stiffness in compression. Isolator 2 of Figure 2.4(a) is made from natural rubber with a Shore A hardness of 35. Two square aluminum mounting plates are bonded to the ends of the rubber block. Figure 2.4(b) illustrates the resulting stiffness curve. Again, hysteresis is apparent. Finally, isolator 3 of Figure 2.5(a) is a neoprene (Shore A hardness of 30) bubble-type isolator with a hollow cavity. Figures 2.5(b) illustrate this isolator’s stiffness curve which matches the stiffness curve provided by the manufacture of this isolator [2.5]. In addition to hysteresis, the compression segment of the static stiffness curve of this isolator exhibits a significant amount of nonlinearity. It is believed that the initial softening stiffness is due to the collapsing of the isolator. Then, once in a collapsed state, hardening stiffness results when the isolator’s walls come into contact.

The static stiffness data given with the isolators considered here, is the type of data often provided in product catalogs. However, questions remain as to whether such data accurately describe the actual dynamic elastic force $g_w(x(t))$. Table 2.3 lists linear, quadratic and cubic equations whose coefficients are calculated from least squares curve fits of the static stiffness data. These equations will be used in the preceding analysis for comparison to the experimental system dynamic response to help answer this question.

2.4.2. DYNAMIC BEHAVIOR BASED ON SINGLE-DEGREE-OF-FREEDOM EXPERIMENTS

Figure 2.6 illustrates accelerance functions $H_{11}^H(\omega) \approx \ddot{x}_1(\omega)/f_1(\omega)$ for the three isolators estimated from the data of the random excitation experiments by the “$H_1$” frequency response estimator [2.6, 2.7], where superscript [1] signifies an “$H_1$” estimate, the first subscript indicates which response is involved in the estimate and the second subscript indicates which excitation. Obviously, for this single-degree-of-freedom configuration, this is the only frequency response function to measure. Only the accelerance
Table 2.3. Least Squares estimates of static stiffness curves.

<table>
<thead>
<tr>
<th>Isolator</th>
<th>$k_A x(t)$</th>
<th>$k_B x(t) + \beta_{B1} x(t)^2$</th>
<th>$k_C x(t) + \beta_{C1} x(t)^2 + \beta_{C2} x(t)^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Figure 3(a)</td>
<td>$k_A = 50.9$ kN/m</td>
<td>$k_B = 51.7$ kN/m</td>
<td>$k_C = 51.2$ kN/m, $\beta_{C1} = -2.3$ MN/m$^2$, $\beta_{C2} = 203.6$ MN/m$^3$</td>
</tr>
<tr>
<td>2 Figure 4(a)</td>
<td>$k_A = 129.5$ kN/m</td>
<td>$k_B = 131.3$ kN/m</td>
<td>$k_C = 134.7$ kN/m, $\beta_{C1} = 13.0$ MN/m$^2$, $\beta_{C2} = -7.5$ GN/m$^3$</td>
</tr>
<tr>
<td>3 Figure 5(a)</td>
<td>$k_A = 4.9$ kN/m</td>
<td>$k_B = 5.3$ kN/m</td>
<td>$k_C = 5.6$ kN/m, $\beta_{C1} = 116.4$ kN/m$^2$, $\beta_{C2} = -5.8$ MN/m$^3$</td>
</tr>
</tbody>
</table>
Figure 2.6. Magnitude of measured accelerance functions from SDOF configuration under random excitation. Numbers indicate isolator used in single-degree-of-freedom configuration. Key: — lowest force level $|f|_{\text{rms}}$, - - highest force level $|f|_{\text{rms}}$. 
functions resulting from the maximum and minimum excitation levels are given. The rest of the
accelerance functions lie somewhere between these two. Observe, for isolator 1 and 3, the peak
frequencies decrease with increasing $|f(t)|$, a characteristic similar to that of a nonlinear softening spring.

The accelerance functions are used in a modal analysis software [2.8] to determine the effective
natural frequencies and effective damping ratios for each random excitation level. The estimates as shown
in Figure 2.7 are normalized by the estimate from the lowest excitation data. Absolute values are given in
Table 2.4. For all three isolators, a similar trend can be seen in the change of the natural frequencies.
However, the overall changes are not equivalent: 11.4%, 0.5%, 10.7% for isolators 1, 2 and 3, respectively.
This illustrates the relatively linear elastic behavior of isolator 2 when subjected to random excitations. For
the damping ratio the overall changes are: 7.2%, 11.3%, 10.1% for isolators 1, 2 and 3, respectively.
Interestingly, the effective damping ratio of isolator 2 changes the most. It is possible that isolator 2 has
nonlinear damping but linear stiffness.

A final observation can be made about the isolators from the data presented thus far. This is the
ratio between their dynamic ($k_d$) and static stiffness ($k_s$) coefficients, i.e. $\alpha_s = k_d/k_s$. Reference [2.1]
suggests that this ratio increases with increasing durometer. In order to calculate this ratio, the dynamic
stiffness coefficients need to be calculated. Since a single-degree-of-freedom configuration is assumed,
calculation of $k_d$ is possible once the contributing dynamic mass ($m_d$) is known. Calculating the total
physical mass which contributes to the dynamics of the experiment by:

$$m_s = m_i + m(\text{force transducer}) + m(\text{accelerometer}) + m(\text{shaker dynamic mass}) + m(\text{stinger}) + 0.5m(\text{isolator}) \approx 1.7 \text{ kg.} \quad (2.3)$$

where $m(\cdot)$ indicates the mass of the item in parenthesis which are given in Table 2.1. Recall from section
2.3 that $m_i = 1.4 \text{ kg}$. However, an alternative approach is to determine the dynamic mass from the
accelerance functions' mass line well beyond the peak frequency. This is possible since the Nyquist
frequency is 1000 Hz. An effective mass of $m_s \approx 1.33 \text{ kg}$ was determined for all three isolators. Knowing
$m_s$, the effective dynamic stiffness coefficients ($k_d$) for the three isolators are calculated for the five
excitation levels. The static stiffness is also approximated from the linear least squares curve-fits of the
Figure 2.7. Normalized effective (linearized) parameters of Isolators. (a) Normalized effective natural frequencies. (b) Normalized effective damping ratios. Key: o-o-o Isolator 1, *-*-* Isolator 2, ++-++ Isolator 3.
| Isolator | Level $|f_{max}|$ (mN) | natural frequency (Hz) | damping ratio (%) |
|----------|----------------|------------------------|-------------------|
| 1        | 12.8           | 39.3                   | 6.8               |
|          | 86.2           | 38.0                   | 7.2               |
|          | 157.7          | 37.4                   | 7.3               |
|          | 293.7          | 36.6                   | 7.3               |
|          | 884.2          | 34.8                   | 7.1               |
| 2        | 13.1           | 53.2                   | 2.9               |
|          | 89.3           | 53.0                   | 3.0               |
|          | 164.3          | 53.0                   | 3.0               |
|          | 305.8          | 53.0                   | 3.1               |
|          | 915.2          | 52.9                   | 3.3               |
| 3        | 15.2           | 15.3                   | 6.9               |
|          | 97.9           | 14.9                   | 6.9               |
|          | 180.7          | 14.7                   | 7.2               |
|          | 334.3          | 14.4                   | 7.5               |
|          | 1003.8         | 13.7                   | 7.6               |

Table 2.4. Modal parameters of single-degree-of-freedom configurations.
static stiffness data, Table 2.2. Therefore, $\alpha_k$ is calculated for two values of $m_i$ and for the five force levels. Results are plotted versus durometer in Figure 2.8(a) for $m_i = 1.7$ and in Figure 2.8(b) for $m_i = 1.3$. The five data points given for each isolator and for each value of $m_i$ correspond to each force level. Also plotted are data points provided in [2.1] as designated by 'x' and a dashed curve, also provided in [2.1], which passes through five of these points. Reference [2.2] provides $\alpha_k$ for compression as plotted in Figure 2.8(a), and for shear as plotted in Figure 2.8(b). As shown, with $m_i = 1.7$ the data for the isolators considered in this study do not correlate well with what is provided in the literature [2.1, 2.2]. However, with $m_i = 1.3$, $\alpha_k$ does match well for isolators 1 and 2. Possibly shape factor is the reason for the large discrepancies with the data provided for isolator 3. Recall Figure 2.5 which illustrates a more complex cross-section for isolator 3.

The next type of excitation applied to the isolators under single-degree-of-freedom configuration is swept-sine which occurs between 5-105 Hz in 1 Hz steps. The excitation swept both up and down this frequency range to identify any nonlinear behavior by the differences in the corresponding accelerance functions. Shown in Figure 2.9(a) are results for isolator 1 which are compared with an accelerance function from random excitation. All three results are from the highest excitation levels. Notice, the curves do not overlap, indicating nonlinear system behavior. The final excitation applied to $m_i$ is sinusoidal excitation. The auto-power spectrum of the response is given in Figure 2.9(b) for isolator 1. Superharmonics are present, however they are 3 orders of magnitude below the fundamental harmonic. This is not significant enough to assume strong nonlinear behavior under sinusoidal excitations.

2.4.3. MULTI-DEGREE-OF-FREEDOM EXPERIMENT

Figures 2.10 illustrates accelerance functions $H_{\omega}^{(1)}(\omega) = \tilde{x}_4(\omega)/f_1(\omega)$ estimated from the data of the random excitation experiments. However, the results with only the minimum and maximum excitation levels $|f(t)|$ are shown. Effective (linearized) natural frequencies, damping ratios and modes shapes are estimated from the accelerance functions using the modal analysis software [2.8]. The third peak is not
Figure 2.8. Ratio $\alpha_n$ of dynamic to static stiffness coefficients as a function of rubber durometer. (a) Key: $\times$ data and --- curve given by reference [2.1]. --- range given by reference [2.2] in compression, o Isolator 1, * Isolator 2 and + Isolator 3 with $m_i = 1.7$ kg. (b) Key: $\times$ data and --- curve given by reference [2.1]. --- range given by reference [2.2] in shear, o Isolator 1, * Isolator 2 and + Isolator 3 with $m_i = 1.33$ kg.
Figure 2.9. Isolator 1 in the single-degree-of-freedom configuration. (a) Accelerance functions. Key: — swept sine excitation with positive sweep, —— swept sine excitation with negative sweep, ---- random excitation. (b) Auto-power spectrum of response from 36 Hz sinusoidal excitation.
Figure 2.10. Magnitude of accelerance functions from the multi-degree-of-freedom configuration under random excitation. (a) Isolator 1 (b) Isolator 2 (c) Isolator 3. Key: — lowest force level, — — — highest force level.
evident in the driving point accelerance functions $H_{11}(\omega)$ of any of the isolators. Because of this, the software is unable to estimate parameters of this peak. Therefore, quadrature fitting is employed to determine its effective mode shape. Effective natural frequencies and damping ratios are not determined for this peak. Effective natural frequencies and damping ratios for the first and second peaks are plotted versus excitation level in Figures 2.11-2.12. These values have been normalized by the values determined from the lowest excitation level data. Absolute values are given in Table 2.5. As observed with the single-degree-of-freedom accelerance functions, the peak frequencies decrease with increasing $|f(t)|$. For isolator 1, the second peak shifts more than the first peak. This is due to larger nonlinear softening spring effect caused by the large relative displacement $\Delta x_{12}(t)$ at the second peak. The effective mode shapes are described in Figure 2.13 where it is evident that $\Delta x_{12}(t)$ is greatest for the second peak. These mode shapes are from isolator 1 data at the highest excitation level. Mode shapes for isolator 2 are similar. As a result of the softening spring characteristics of the isolators, increasing excitation level increases $\Delta x_{12}(t)$.

However, the overall appearance of the mode shapes remains unchanged. Isolator 3 shows some unique characteristics, hence it is discussed separately below.

Another point of interest is the third peak and its effective mode shape, Figure 2.13(c). Since the isolator is at the center of the beam and $m_1$ is stationary, the stiffness of the isolator has no effect on this frequency of the peak. This can be seen by observing that the peak frequency is essentially the same for all three isolators who have different stiffness. However, the peak does shift with amplitude of excitation, suggesting that the nonlinear isolators do play some role in its dynamics. This is possibly due to rocking motion of $m_1$ and hence the nonlinear rotational stiffness of the isolators. Unfortunately, only one accelerometer was used to measure the translation of $m_1$, and therefore, this hypothesis could not be tested.

Future work will involve additional accelerometers to simultaneously measure the translation and rotation of $m_1$. Also, note that since the isolator is located at $a = L/2$, theoretically the third mode should not be excited since $L/2$ is a node of the corresponding mode shape. Practically speaking, $a$ is not exactly equal to $L/2$ and therefore the excitation of this mode is possible. Also, misalignment of the shaker/stinger assembly may produce a moment which possibly excites this mode.
Figure 2.11. Normalized effective (linearized) natural frequencies. (a) first effective natural frequency. (b) second effective natural frequency. Key: o-o-o Isolator 1, *-*-* Isolator 2, +--+ Isolator 3.
Figure 2.12. Normalized effective damping ratios. (a) Effective damping ratio for 1st mode. (b) Effective damping ratio for 2nd mode. Key: o-o-o Isolator 1, *-*-* Isolator 2, +---+ Isolator 3.
| Isolator | | Level | $f_{\text{max}}$ (mN) | natural frequencies | damping ratios |
|----------|----------------|----------------|-------------------|----------------|
|          |                | $f_1$ (Hz) | $f_2$ (Hz) | $\zeta_1$ (%) | $\zeta_2$ (%) |
| 1        |                | 13.4      | 26     | 45.3     | 2.3      | 5.5       |
|          |                | 89.5      | 25.8   | 44.4     | 2.3      | 5.8       |
|          |                | 161.1     | 25.7   | 44       | 2.5      | 5.9       |
|          |                | 299.4     | 25.5   | 43.3     | 2.7      | 5.9       |
| 2        |                | 14.2      | 26.8   | 60.2     | 1.8      | 2.5       |
|          |                | 91.6      | 26.7   | 60.1     | 1.8      | 2.6       |
|          |                | 166.5     | 26.7   | 60       | 1.9      | 2.6       |
|          |                | 309.3     | 26.6   | 59.9     | 2.1      | 2.6       |
| 3        |                | 15.3      | 14.5   | -        | 6.5      | 8.4       |
|          |                | 99.9      | 14.2   | -        | 6.7      | 8.4       |
|          |                | 182.4     | 14     | -        | 6.8      | 13.3      |
|          |                | 337.8     | 13.7   | -        | 7.0      | 6.2       |

Table 2.5. Modal parameters of multi-degree-of-freedom configuration.
Figure 2.13. Experimentally obtained effective mode shapes of system of Figure 1(b). (a) Static mode.
(b) Effective mode 1: $m_1$ in of phase with first bending of beam. (c) Effective mode 2: $m_1$
out of phase with first bending mode of beam. (d) Effective mode 3: second bending of beam, $m_1$ stationary.
Additional discussion concerns the accelerance functions of isolator 3, Figure 2.10(c). First, the third peak is not excited much when compared with isolators 1 and 2. This is attributed to two possible reasons. One reason is that isolator 3 is much more compliant than 1 and 2 in rotation, due to its hollow cavity which allows the isolator’s walls to buckle easily when subjected to rotation. Hence isolator 3 has very little rotational stiffness to allow for excitation of the third mode. The second reason is that the top of isolator 3 which is connected to the beam has much smaller surface area than isolators 1 and 2. Therefore, the influence from the isolator on the beam is more of a point force which is less likely to produce a moment. Additionally, observe in Figure 2.10(c) the second peak appears to contain two closely spaced modes. These peaks are not evident in the driving point $H^H_{11}(\omega)$ and therefore estimated parameters are not available for this mode either.

The next excitation applied to $m_1$ is swept-sine which occurs between 5-105 Hz in 1 Hz steps. As with the single-degree-of-freedom experiments, the excitation swept both up and down this frequency range to identify any nonlinear behavior by the differences in the corresponding accelerance functions. For isolator 1, Figure 2.14(a), differences at the anti-resonance exist between the positive and negative sweep rates. Slight differences are also evident around the second peak. Spectrum from the highest excitation random experiment is also shown in Figure 2.14(a). Comparing this curve with the ones obtained from sine sweeps reveals large differences. Similarly, for isolator 2, Figure 2.14(b) illustrates the accelerance functions from positive and negative sweeps and random for the largest excitation levels. Again discrepancies are evident, especially in the first peak which shows some deteriorating effects evident from swept sine. Also, discrepancies are evident between the positive and negative sweeps for the second peak. For isolator 3, results not shown since these types of nonlinear behavior are not as apparent. From the sinusoidal testing, though results also not shown, super-harmonics are present. However, they are 3 orders of magnitude lower than the fundamental harmonic and hence considered irrelevant.

Finally, natural frequencies of the beam alone were measured, i.e. without $m_1$, isolator and $m_2$ attached. The first two natural frequencies were measured to be $f^b_1 = 94$ and $f^b_2 = 257$ Hz. This
Figure 2.14. Magnitude of accelerance functions from MDOF configuration. (a) Isolator 1 (b) Isolator 2
Key: — swept sine excitation with positive sweep, – – – swept sine excitation with negative sweep, – – - random excitation.
information will be useful in section 2.5 where analytical models are correlated with the experimental results of this section.

2.4.4. COMPARISON OF RESULTS

In comparing the isolators with one another under the single and multi-degree-of-freedom configurations, several points can be made. First, the first peak of the isolator 3 multi-degree-of-freedom arrangement shifted more than isolators 1 and 2 as seen in Figure 2.11(a). This is surprising since isolator 1 displays the largest change in stiffness under single-degree-of-freedom configuration as shown in Figure 2.7. One possibility for this is that although the first peak of the isolator 3 multi-degree-of-freedom arrangement has shifted a total of 5.8%, the second peak shifts very little. However, the first and second peaks of the isolator 1 multi-degree-of-freedom arrangement shift 1.9% and 4.6%, respectively. Summing these percentages gives a total shifting of 6.5%. Therefore, if one were to quantify the amount of nonlinearity by the total percentage that all of the peaks shift, isolator 1 is the more nonlinear isolator under both the single and multi-degree-of-freedom arrangements. The importance of “in-situ” testing of isolators is illustrated here. Due to their nonlinear nature, it is not easy to correlate single-degree-of-freedom analysis, as typically done, with how the isolators will perform “in-situ” where the behavior is like the multi-degree-of-freedom system. Isolator 3 shifts the first peak much more than the second, while the opposite is true of isolator 1. This would not be revealed from single-degree-of-freedom testing.

Recall that isolator 2 is the least nonlinear under static and random experiments. However, it displays the most nonlinearity under sine sweep as shown in Figure 2.14. Although, isolator 3 is not portrayed here, it should be noted that it behaves the most linear under this test condition. Overall, the experimental analyses or characterizations of this section along with section 2.3 illustrate that the nonlinear behavior of these elastomeric isolators should be quantified in terms of excitation level, system configuration and excitation type.
2.5. ANALYTICAL CHARACTERIZATION

Analytical models are investigated next, starting with the continuous model discussed in section 2.2. Parameters are determined which best describe the experimental system of Figure 2.1(b). A discrete model is then formulated based on these parameters and nonlinear stiffness terms are included which may describe the rubber isolators. Numerical integration is used to obtain the responses of the discrete model which are compared with the experimental data presented earlier.

2.5.1. QUASI-LINEAR MODEL USING CONTINUOUS SYSTEM THEORY

Fixed boundary conditions are chosen to describe the end conditions of the continuous beam given by (2.2):

\[\begin{align*}
\phi_1(0) &= 0, \quad \phi_2(L) = 0 \\
\phi_{1x}(0) &= 0, \quad \phi_{2x}(L) = 0
\end{align*}\] (2.4a-d)

For the conditions at \( x = L/2 \), continuity and slope of the beam, shear force and moment on the beam and the equation of motion of \( m_2 \) must be satisfied:

\[\begin{align*}
\phi_1(L/2) &= \phi_2(L/2), \\
\phi_{1x}(L/2) &= \phi_{2x}(L/2) \\
EI(\phi_{1xxx}(L/2) - \phi_{2xxx}(L/2)) + k_m(V - \phi_1(L/2)) + \omega^2 m_2 \phi_1(L/2) &= 0 \\
EI(\phi_{1xx}(L/2) - \phi_{2xx}(L/2)) - \omega^2 I_2 \phi_{1x}(L/2) &= 0 \\
-\omega^2 m_1 V + k_m(V - \phi_1(L/2)) &= 0
\end{align*}\] (2.5a-e)

where \( I_2 \) is the mass moment of inertia of mass 2, \( I_2 = \frac{1}{12} m_2 \left(3r_2^2 + h_2^2\right) + m_2 \left(\frac{1}{2}r_2^2\right)^2 = \frac{1}{12} m_2 \left(3r_2^2 + 4h_2^2\right) \).

These conditions are applied to (2.2a-c) to obtain natural frequencies and eigenfunctions. A parametric study is conducted to minimize the percentage errors \( \varepsilon_1 \), \( \varepsilon_2 \) and \( \varepsilon_3 \) between the experimental and analytical natural frequencies of the first, second and third modes, respectively. Although it has been shown in the experimental characterization of section 2.4 that isolator dynamics depend upon system configuration, the isolator stiffness coefficient \( k_m \) used in the continuous model is determined from the single-degree-of-freedom random excitation experiments where effective natural frequencies were obtained. However, the
force levels from the single-degree-of-freedom random experiments do not match those of the multi-
degree-of-freedom random experiments. Therefore, interpolation is necessary to determine the effective
natural frequencies of the experimental single-degree-of-freedom systems at the force levels of the multi-
degree-of-freedom system. Once effective natural frequencies are determined, $k_m$ is calculated based on
values chosen for $m_1$. Note that $I_1$ and hence $r_2$ and $h_2$, do not effect $f_1$ and $f_2$ since the second mass only
translates for these modes. Therefore, for the initial parametric studies the value obtained for $f_1$ is not

Once $e_1$ and $e_2$ have been minimized, $r_2$ and $h_2$ are adjusted to tune $f_1$ solely. Finally, recall that at
the end of section 2.4.4, the natural frequencies of the beam alone $f_1^b$ and $f_2^b$ were experimentally
determined. Errors $e_1^b$ and $e_2^b$ between $f_1^b$ and $f_2^b$ and analytical values determined from the continuous
beam model alone, i.e. $m_1 = m_2 = k_m = 0$, are also considered.

Table 2.6 lists results from a parametric study using values for $k_m$ determined from isolator 1
under single-degree-of-freedom configuration and lowest random force level of the multi-degree-of-
freedom configuration, $|F|_{\text{rms}} = 13.4$ mN. Parameters of case 1 are chosen based on the physical parameters
of the experimental system. Large errors exist in both the beam ($e_1^b$, $e_2^b$) and system ($e_1$, $e_2$, $e_3$) natural
frequencies. Next, parameters are chosen such that the errors $e_1^b$ and $e_2^b$ are minimal. However, in
satisfying this constraint it is difficult to minimize $e_1$, $e_2$ and $e_3$. Case 2 gives the best results. Therefore,
is this constraint is not considered for the resulting cases. Instead, the following cases only focus on
minimizing $e_1$, $e_2$ and $e_3$. For case 3, $m_1$ is chosen equal to that obtained from the mass line of the single-
degree-of-freedom random experiments and the length $L$ and thickness $t$ of the beam are chosen to reduce
the beam stiffness. This results in good approximations of $f_1$ and $f_2$ with maximum error less than 2%.
However, differences between effective natural frequencies from minimum and maximum force levels can
be less than 1%. Therefore, errors less than 1% are desired for $e_1$, $e_2$ and $e_3$. In case 4, an intermediate
value between $m_1 = 1.7$ (physical mass) and 1.33 (obtained from the mass line) is chosen, $m_1 = 1.5$, with all
other parameters unchanged. The result is a decrease in $e_1$ and an increase in $e_2$. To determine a value of
<table>
<thead>
<tr>
<th>Parameters</th>
<th>Exper.</th>
<th>Analytical Cases</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>L (m)</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>t (mm)</td>
<td>9.56</td>
<td>9.56</td>
</tr>
<tr>
<td>E (GPa)</td>
<td>71</td>
<td>71</td>
</tr>
<tr>
<td>$\rho'$ (kN/m^3)</td>
<td>26.6</td>
<td>26.6</td>
</tr>
<tr>
<td>$m_1$ (kg)</td>
<td>1.72</td>
<td>1.72</td>
</tr>
<tr>
<td>$k_m$ (kN/m)</td>
<td>104.64</td>
<td>104.64</td>
</tr>
<tr>
<td>$h_2$ (m)</td>
<td>0.19</td>
<td>0.19</td>
</tr>
<tr>
<td>$r_2$ (mm)</td>
<td>38</td>
<td>38</td>
</tr>
<tr>
<td>$f_1^b$ (Hz)</td>
<td>94</td>
<td>112.02</td>
</tr>
<tr>
<td>$f_2^b$ (Hz)</td>
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<td>308.78</td>
</tr>
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<td>$e_1^b$ (%)</td>
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<td>-19.17</td>
</tr>
<tr>
<td>$e_2^b$ (%)</td>
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<td>-20.15</td>
</tr>
<tr>
<td>$f_1$ (Hz)</td>
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<td>28.13</td>
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<td>$f_3$ (Hz)</td>
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<td>$e_3$ (%)</td>
<td>0.00</td>
<td>3.19</td>
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Table 2.6. Parametric study of a quasi-linear continuous system model with isolator 1 and lowest random force level, $|f|_{\text{max}} = 13.4$ mN.
m, which minimizes both \( \varepsilon_1 \) and \( \varepsilon_2 \) simultaneously, assume linear equations approximate the relationships between \( m_i \) and \( \varepsilon_i \), and \( m_i \) and \( \varepsilon_2 \),

\[
\varepsilon_i(m_i) \approx s_i m_i + b_i, \quad i = 1, 2
\]  

(2.6a,b)

where \( s_i \) is the slope and \( b_i \) is the intersect of \( \varepsilon_i(m_i) \). Knowing the four data \( \varepsilon_i(1.33), \varepsilon_i(1.33), \varepsilon_i(1.5) \) and \( \varepsilon_i(1.5) \) the four unknowns \( s_1, b_1, s_2 \) and \( b_2 \) are determined and \( \varepsilon_i(m_i) \) are plotted in Figure 2.15. Although the relationships are not truly linear, the general trend is illustrated. The intersection of \( \varepsilon_i(m_i) \) and \( \varepsilon_i(m_i) \) is found at \( \varepsilon(1.43) = -0.625\% \). Since (2.6a,b) are approximations, -0.625\% is not assumed as the true error. Instead \( m_1 = 1.43 \) kg is used in case 5. As given in Table 2.6, \( \varepsilon_1 = \varepsilon_2 = -0.62 < 1.0\% \) as desired; and, the values for the errors are close to the error from the linear approximations (2.6a,b). Therefore, the linear approximation (2.6a,b) are reasonably valid.

Now that \( \varepsilon_1 \) and \( \varepsilon_2 \) have been reduced, \( r_2 \) and \( h_2 \) are chosen to reduce \( \varepsilon_1 \). The resulting values are given by case 6. Now, all analytical natural frequencies have less than 1\% error. In order to accomplish this, the parameters of the beam must be assigned values much different from the physical values. Two possibilities explain these discrepancies. First, the clamped beam ends of the experimental system are not truly fixed as modeled by (2.4a-d). A more accurate model results from a distribution of springs and masses to describe such clamping conditions [2.9]. These compliant boundary conditions result in lower natural frequencies as can be seen when comparing the experimental results with those of case 1 where the true physical parameters where applied. To accommodate for the differences between the experimental and analytical natural frequencies, the beam length \( L \) is increased and thickness \( t \) decreased as done in cases 2-6. In addition, the values for the elastic modulus \( E \) and specific weight \( \rho' \) are tabulated values for aluminum [2.10]. However, actual \( E \) and \( \rho' \) for the beam used for the experimental system may deviate from the tabulated values. Additional work is need to improve the modeling of the boundary conditions and to obtain more precise values for \( E \) and \( \rho' \).

The parameters of the beam, \( m_1 \) and \( m_2 \) from case 6, are used to investigate the accuracy of the continuous model at higher force levels and for isolators 2 and 3. Note that only \( k_m \) will change as a function of isolator and force level. Results are summarized in Table 2.7. Since the analytical calculation
Figure 2.15. Approximate linear relationship between $m_1$ and percentage errors $\varepsilon_1$ and $\varepsilon_2$. Key: — $\varepsilon_1(m_1)$. — $\varepsilon_2(m_1)$. 

$\varepsilon_1$, $\varepsilon_2$ (%)
<table>
<thead>
<tr>
<th>Isolator level</th>
<th>$k_m$ (kN/m)</th>
<th>$f_1$ (Hz)</th>
<th>$f_2$ (Hz)</th>
<th>$f_1$ (Hz)</th>
<th>$f_2$ (Hz)</th>
<th>$\varepsilon_1$ (%)</th>
<th>$\varepsilon_2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13.4</td>
<td>86.93</td>
<td>26</td>
<td>45.3</td>
<td>26.16</td>
<td>45.58</td>
<td>-0.62</td>
</tr>
<tr>
<td></td>
<td>89.5</td>
<td>81.31</td>
<td>25.8</td>
<td>44.4</td>
<td>26.45</td>
<td>44.35</td>
<td>-0.78</td>
</tr>
<tr>
<td></td>
<td>161.1</td>
<td>78.63</td>
<td>25.7</td>
<td>44</td>
<td>25.92</td>
<td>43.76</td>
<td>-0.86</td>
</tr>
<tr>
<td></td>
<td>299.4</td>
<td>75.27</td>
<td>25.5</td>
<td>43.3</td>
<td>25.8</td>
<td>43.01</td>
<td>-1.18</td>
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<tr>
<td>2</td>
<td>14.2</td>
<td>159.68</td>
<td>26.8</td>
<td>60.2</td>
<td>27.05</td>
<td>59.75</td>
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<td></td>
<td>91.6</td>
<td>158.56</td>
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<td>60.1</td>
<td>27.04</td>
<td>59.55</td>
<td>-1.27</td>
</tr>
<tr>
<td></td>
<td>166.5</td>
<td>158.47</td>
<td>26.7</td>
<td>60</td>
<td>27.04</td>
<td>59.54</td>
<td>-1.27</td>
</tr>
<tr>
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<td>309.3</td>
<td>158.38</td>
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<td>59.9</td>
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<td>59.52</td>
<td>-1.65</td>
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<td>3</td>
<td>15.3</td>
<td>13.27</td>
<td>14.5</td>
<td>-</td>
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<td>31.35</td>
<td>-2.48</td>
</tr>
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<td></td>
<td>99.9</td>
<td>12.59</td>
<td>14.2</td>
<td>-</td>
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<td>31.29</td>
<td>-2.11</td>
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<td>12.18</td>
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<td>-</td>
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<td>31.25</td>
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<td>-</td>
<td>13.99</td>
<td>31.2</td>
<td>-2.12</td>
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Table 2.7. Parametric study of the quasi-linear continuous system model with parameters from case 6 of Table 2.6.
of $f_2$ is not effected by the $k_m$, it is not included. As given, some values of $e_1$ and $e_2$ are slightly higher than values obtained from case 6 of Table 2.6. However, they are still within 3%. Also, determining the accuracy of the continuous model by percentage error can be misleading since $f_1$ and $f_2$ are not the same between isolators. Therefore, the absolute value of the differences $|\Delta f_1|$ and $|\Delta f_2|$ between the experimental and analytical natural frequencies are plotted in Figure 2.16. Concerning the first effective natural frequency, Figure 2.16(a), $\Delta f_1$ increases with increasing excitation level for isolators 1 and 2, whereas $\Delta f_3$ decreases slightly for isolator 3. For the second effective natural frequency, Figure 2.16(b), $\Delta f_1$ initially decreases then increases for isolator 1, while $\Delta f_2$ initially increases then decreases for isolator 2. Recall, no information is available on the second effective natural frequency for isolator 3.

Overall, the continuous model captures the trends of the isolators' behaviors and the errors are not large over the random force levels considered in this study. This model may be useful for determining intermediate values of effective natural frequencies of the experimental system of Figure 2.1(b) with isolators 1, 2, and 3 for random excitation in the force range given. However, as Figure 2.16 illustrates, some of the errors in the natural frequencies increase with increasing excitation level, so extrapolated values outside of this range may be erroneous. In addition, as illustrated by the experimental characterization of the isolators in section 2.4, the isolators behave differently to different types of excitations. Therefore, using the continuous model with the isolator parameters from the single-degree-of-freedom random excitation experiments may lead to inaccurate forced response prediction for harmonic excitations. Also illustrated by the experimental characterization of section 2.4, behavior of the isolators under single and multi-degree-of-freedom configurations can be quite different. This might account for some discrepancies between the experimental and analytical effective natural frequencies since stiffness values obtained from the single-degree-of-freedom experiments are used in the continuous model. One final point is that the generation of "look-up" tables, such as Table 2.7, are necessary since effective parameters must be used with the continuous model to develop a "quasi-linear" model done in this section. This is not necessary for the discrete system model that is discussed next.
Figure 2.16. Differences between experimental and analytical effective natural frequencies. (a) first effective natural frequency. (b) second effective natural frequency. Key: o-o-o isolator 1, *-*-* isolator 2, +---+ isolator 3.
2.5.2. NONLINEAR MODEL USING DISCRETE SYSTEM THEORY

As discussed, the continuous model of section 2.5.1 can result in an accurate representation of the system if the isolator is characterized as a linear or quasi-linear element. However, if nonlinear functions are used to model the isolators restoring forces, a discrete model of reduced dimension must be used such that time domain integration or other semi-analytical techniques [2.11] can be efficiently performed to obtain the system's response. Therefore, a lumped system model is considered for describing the dynamics of the experimental system in Figure 2.1(b). First, the beam is modeled with finite beam elements [2.12] such that a mass and stiffness matrix can be obtained. Second, a damping matrix is obtained by assuming proportional damping of the beam. Third, the matrices are updated to include the dynamics of the isolator, \( m_1 \) and \( m_2 \). Fourth, nonlinear terms are included to describe the nonlinear stiffness behavior of the isolator. Nonlinear damping is not considered in this study even though it was observed in the experimental characterization of isolator 2. Instead an average damping ratio is calculated from the values given in Table 2.3. An average damping coefficient is then calculated for each isolator. Finally, numerical integration is employed to the following set of coupled equations of motion to obtain the system's response

\[
M_L \ddot{x}(t) + C_L \dot{x}(t) + K_L x(t) + \sum_{j=1}^{n} a_j y_j(t) = f(t)
\]  

(2.7)

where \( M_L, C_L \) and \( K_L \) are the linear mass, damping and stiffness matrices of the lumped model, respectively. \( x(t) \) is the generalized response vector and \( f(t) \) is the generalized force vector. Note that \( M_L \) is non-diagonal (but is symmetric) since rotational degrees of freedom of the beam are included. The summation consists of nonlinear functions \( y_j(t) \) for describing the isolator's restoring force and \( a_j \) are vectors containing the coefficients of these functions. Note that the linear components of the isolators' elastic forces are included in \( K_L \) therefore the summation is only that of nonlinear functions (see chapters 3 and 4 for examples of discrete nonlinear systems described in this form).

A 5th order Runge-Kutta Fehlberg numerical integration routine is used to simulate the response of the discrete nonlinear system. For the routine to remain stable, all of the model's natural frequencies must be less than the Nyquist frequency \( f_N \) of the simulation, i.e. \( f_{\text{max}} < f_N = 1/(2 \cdot \Delta t) \), where \( f_{\text{max}} \) is the maximum
effective natural frequency of the discrete nonlinear model and $\Delta t$ is the numerical integration time step. Unfortunately, the natural frequencies of the beam’s rotational degrees of freedom can be large, requiring a very small $\Delta t$ for stable simulation. This in turn requires a large amount of computation in order to gain good spectral resolution since $\Delta f_r = 1/T = 1/(N_{dp} \cdot \Delta t)$ where $\Delta f_r$ is the spectral resolution and $N_{dp}$ is the number of time points calculated. Increasing the amount of computation is the total number of degrees of freedom $N_r = 2 \cdot (N_{be}-1) + 1$ where $N_{be}$ is the number of beam elements and the +1 outside of the parenthesis accounts for $m_i$. As will be shown, in order for accurate representation of the experimental system, choose $N_{be} \geq 4$ and $N_r \geq 7$. Therefore, large $N_{dp}$ must be calculated for 7 degrees of freedom in order to gain good frequency resolution and an accurate model.

To alleviate the problem of large computation, coordinate reduction is applying to the finite element beam model where the symmetric, non-diagonal mass matrix $M_e$ is replaced by a diagonal lumped mass matrix $M_l^S$ [2.12, 2.13]. The new model eliminates rotational degrees of freedom and reduces the amount of computation in two respects. First, the large natural frequencies resulting from the rotational degrees of freedom no longer exist so the restriction on small $\Delta t$ can be relaxed. Second, the number of degrees of freedom reduces to $N_l = (N_r - 1) + 1$.

The solution proposed above does have one drawback. Without rotational degrees of freedom, the mass moment of inertia $I_z$ cannot be included in the model. Therefore, the third effective natural frequency of the model will be much larger than that of the experimental system. However, recall that for the analytical models, the third mode is not affected by the isolator parameters. This is a result of placing the isolator exactly at $a = L/2$ and modeling with a single spring and damping element (if the isolator were modeled with distributed elements and placed off of center, $a \neq L/2$, the slight shifting seen in the experimental characterization may be accounted for here). Therefore, analysis of the third effective mode will be ignored. Future studies will consider modeling the isolators with distributed elements and rotational degrees of freedom of the beam will be included.

The next step in this study is to determine the number of degrees of freedom which result in an accurate representation of the experimental system. This is done using the parameters listed in Table 2.6.
for case 6 and calculating the natural frequencies of the discrete linear system, i.e. $y_j(t) = 0, j = [1, n]$, for different degrees of freedom, then comparing these natural frequencies with the experimental natural frequencies given in Table 2.6. For $N_{w6} = 4$, natural frequencies values are $f_1 = 26.15$ Hz and $f_2 = 45.58$ Hz. These are similar to those in case 6 of Table 2.6, and since the same parameters are used in the discrete model, the discrete linear model may be assumed to be valid.

Now that an underlying linear model has been obtained, $y_j(t)$ will be included for describing the nonlinear isolator stiffness. For the isolator’s linear stiffness coefficient, the stiffness value calculated from the single-degree-of-freedom random excitation experiments at the lowest excitation levels is chosen, Table 2.7. This is a fair assumption for the value of the linear stiffness coefficient since at the lowest force level, the isolators’ response is approximately linear. The accuracy of the nonlinear models is determined as follows. The discrete model (2.7) with the appropriate parameters and $y_j(t)$ is numerically simulated. The force input $f(t)$ is chosen equal to the respective sampled excitation data from the multi-degree-of-freedom experiments for each isolator. For instance, consider a discrete model with isolator 3. The model parameters would include those listed in case 6 of Table 2.6. However, the linear stiffness coefficient would be equal to $13.27$ kN/m from Table 2.7. Functions $y_j(t)$ are chosen to describe the nonlinear stiffness nature of isolator 3. Four numerical simulations are then conducted. The first simulation uses the lowest force level sampled random data from isolator 3 in the multi-degree-of-freedom configuration, the second simulation uses the second lowest, and so forth. Frequency response functions are then calculated using the numerical data and are compared to the experimental frequency response functions. For the sake of brevity, only natural frequency comparison is given.

The two-term functions employed to describe the nonlinear elastic force of the isolators takes the following form

$$f'(\Delta x_{12}(t)) = k_m^* \Delta x_{12}(t) + \gamma (\Delta x_{12}(t))^\alpha$$  (2.8)

In addition to the type of nonlinear term and hence value for $\alpha$, e.g. 2, 3, etc., chosen for describing the isolators’ nonlinear elastic behavior, an additional variable $\gamma$ in this modeling process is the coefficient of the nonlinear term. This variable is adjusted to vary the strength of the nonlinearity. However, there is a
limit to the largest absolute value $|y|_{\text{m}}$ for the numerical simulation to remain stable. Because the isolator exhibits nonlinear softening spring behavior, i.e. the peaks shift down if frequency with increasing excitation level, $y < 0$. However, this results in a multi-valued stiffness curve, i.e. for a fixed elastic force, more than one value exists for the displacement of the isolator. A curve such as this is physically possible if some kind of break-down occurs in the rubber composition, similar to the plastic deformation. Unfortunately, unstable numerical simulation results. Therefore for each $\alpha$, there is a limit to the strength ($\gamma$) of the nonlinearity for the stable numerical simulation.

For isolator 1 a cubic nonlinearity $\alpha = 3$ is first considered for describing the nonlinear elastic behavior. Numerical simulation was conducted for a range of values $\gamma$. However, effective natural frequencies showed very little change with excitation level. Results from $|y|_{\text{m}} = -7.2$ GN/m$^3$ are given in Table 2.8 for the largest force level. As shown, $f_1$ and $f_2$ have not shifted much from the values for the lowest force level. Therefore a stronger nonlinearity is necessary. One would expect a fifth order nonlinearity, $\alpha = 5$, would be a better next candidate over a quadratic, $\alpha = 2$. However, since $\Delta x_{12}(t)$ assumes values less than unity, $(\Delta x_{12}(t))^5 < (\Delta x_{12}(t))^2 < (\Delta x_{12}(t))^2$. Therefore a quadratic nonlinearity is considered next.

Before considering a quadratic nonlinearity, modification must be made to (2.8) since for certain values of $\gamma$, an unphysical stiffness curve can result where positive values of $\Delta x_{12}(t)$ (elongation of isolator) results in compressive forces on the isolator. Rewrite (2.8) as

$$f^\epsilon(\Delta x_{12}(t)) = k_m \Delta x_{12}(t) - \frac{\Delta x_{12}(t)}{|\Delta x_{12}(t)|} |\Delta x_{12}(t)|^\alpha$$

(2.9)

Table 2.8 lists results from this model for $\alpha = 2$ and $\gamma = |y|_{\text{m}} = -21.0$ MN/m$^2$. This nonlinear model is too strong as indicated by the effective natural frequencies which have shifted lower than those for the experimental system. For $\gamma = -9$ MN/m$^2$, the model is not strong enough, however an intermediate value of $\gamma = 15.0$ MN/m$^2$, the model is fairly accurate. Note that the previous results have been from the largest force level. Results for the other three force levels are also given for $\alpha = 2$ and $\gamma = 15.0$ MN/m$^2$. As shown the model is fairly accurate at these levels as well.

51
Table 2.8. Parametric study of the nonlinear discrete system model with isolator 1 and parameters from case 6 of Table 2.6.
The analysis of isolators 2 and 3 is not considered. The amount of elastic nonlinearity exhibited by Isolator 2 under random excitations is minimal and therefore a linear model would suffice for its description. For isolator 3, the phenomenon observed in the experimental characterization would not be predicted by the model considered in this study. A more complex model which includes a better description of the beam boundary conditions and also includes rotary effects at \( x = L/2 \) would be necessary.

2.6. CONCLUSION

As shown in this study, rubber isolator nonlinearities depend upon several factors and the single-degree-of-freedom analysis may not successfully reveal the dynamic behavior of an isolator under multi-degree-of-freedom configurations. Also, some isolators may be highly dependent upon the type of excitation experienced. Therefore, improved testing and analysis techniques are necessary for characterizing rubber isolators. Arguments made here support the "in-situ" testing concept since it may not be possible to correlate isolator data from single-degree-of-freedom experiments with the dynamic behavior under multi-degree-of-freedom conditions.

A quasi-linear continuous system model has been found to describe the dynamic behavior of an experimental system composed of a rubber isolator. Based on effective linear parameters obtained from single-degree-of-freedom-experiments, the continuous system model's natural frequencies match measured data. However, errors may be a result of the discrepancies found in the experimental characterization between the single and multi-degree-of-freedom configurations. A discrete model which describes the nonlinear elastic behavior of the isolators also showed some initial success. However, additional investigation of this model is necessary and improved modeling of the boundary conditions and isolators is needed.

Although the analytical models have shown some success, the methods employed thus far in this study for determining the parameters of these models are of the "trial-and-error" type. More systematic methods such as the identification techniques discussed in the following chapters are more desirous since they estimate nonlinear system parameters directly from experimental data (see chapter 3). However, the
characterization studies conducted here are essential for the identification techniques which require a priori knowledge of the types of nonlinearities present (see chapter 4). The characterization conducted here serves as an initial step in determining this knowledge for the three rubber isolators (see chapter 5).
LIST OF REFERENCES FOR CHAPTER 2


2.5. Tech Products Corporation 1996 *Noise and Vibration Control Products Catalog*.


CHAPTER 3

IDENTIFICATION OF MULTI-DEGREE-OF-FREEDOM NONLINEAR SYSTEMS UNDER RANDOM EXCITATIONS BY THE "REVERSE PATH" SPECTRAL METHOD

3.1. INTRODUCTION

The properties of multi-degree-of-freedom linear systems are typically identified using time or frequency domain modal parameter estimation techniques [3.1]. For the frequency domain techniques, the algorithms extract modal parameters from measured frequency response functions in the presence of uncorrelated noise by using conventional "H₁" or "H₂" frequency response estimation methods [3.2-3.3]. However, if the system under identification also possesses nonlinearities, conventional methods often yield contaminated frequency response functions from which accurate modal parameters cannot be determined [3.4-3.5]. Such conventional methods are also incapable of identifying the nonlinearities.

To accommodate for the presence of nonlinearities, several researchers have developed methods to improve frequency domain analysis of nonlinear systems [3.6-3.11]. For example, the functional Volterra series approach for estimating higher order frequency response functions of nonlinear systems has gained recognition [3.6]. This method has been used to estimate first and second order frequency response functions of a nonlinear beam subjected to random excitation [3.7], where curve fitting techniques were used for parametric estimation of an analytical model. However, the method is very computationally intensive and estimation of third and higher order frequency response functions has been unsuccessful. To alleviate this problem, sinusoidal excitation was used to estimate only the diagonal second and third order frequency response functions of the Volterra series [3.8]. Other higher order spectral techniques have also been employed for the analysis of nonlinear systems [3.9]. For instance, the bi-coherence function has
been used to detect the second order nonlinear behavior present in a system [3.10]. Also, the sub-harmonic responses of a high speed rotor have been studied using bi-spectral and tri-spectral techniques [3.11].

An alternative approach has recently been developed by Bendat et al. [3.12-3.15] for single-input/single-output systems which identifies a "reverse path" system model. A similar approach has been used for the identification of two-degree-of-freedom nonlinear systems where each response location is treated as a single degree of freedom mechanical oscillator [3.16]. Single degree of freedom techniques are then used to identify system parameters [3.17]. However, this approach requires excitations to be applied at every response location and it also inhibits the use of preferred higher dimensional parameter estimation techniques that are commonly used for the modal analysis of linear systems [3.1].

The literature review reveals that there is clearly a need for frequency domain system identification methods that can identify the parameters of nonlinear mechanical and structural systems. Also, improvements to the frequency response estimation methods such as the "H_1" and "H_2" methods are necessary when measurements are made in the presence of nonlinearities. The primary purpose of this chapter is to introduce an enhanced multi-degree-of-freedom spectral approach based on a "reverse path" system model. Additional discussion is included to justify the need for spectral conditioning and computational results are given to illustrate the performance on several nonlinear systems. However, focus of this chapter is on the mathematical formulation for multi-degree-of-freedom nonlinear systems. Specific objectives include the following: (1) accommodate for the presence of nonlinearities so that improved estimates of the linear dynamic compliance functions can be determined from the input/output data of multi-degree-of-freedom nonlinear systems when excited by Gaussian random excitations; (2) estimate the underlying linear systems' modal parameters from these linear dynamic compliance functions using higher dimensional modal analysis parameter estimation techniques; (3) determine the coefficients of the analytical functions which describe local or distributed nonlinearities at or away from the locations where the excitations are applied; (4) assess the performance of this new method via three computational examples with polynomial type nonlinearities. Comparison of this method with an existing time domain method is in progress, and ongoing research is being conducted to consider both correlated and
uncorrelated noise. Issues such as the spectral variability of coefficient estimates as well as other errors are currently being examined and will be included in future research. However, these issues have been omitted from this chapter so that focus can be kept on introducing an analytical approach to multi-degree-of-freedom systems.

3.2. PROBLEM FORMULATION

3.2.1. PHYSICAL SYSTEMS

The equations of motion of a discrete vibration system of dimension N with localized nonlinear springs and dampers can be described in terms of a linear operator $L[x(t)]$ and a nonlinear operator $N[x(t), \dot{x}(t)]$

$$L[x(t)] + N[x(t), \dot{x}(t)] = f(t)$$
$$L[x(t)] = M\ddot{x}(t) + C\dot{x}(t) + Kx(t)$$
$$N[x(t), \dot{x}(t)] = \sum_{j=1}^{n} A_j y_j(t)$$

where $M$, $C$ and $K$ are the mass, damping and stiffness matrices, respectively. $x(t)$ is the generalized displacement vector and $f(t)$ is the generalized force vector. Also refer to the List of Symbols for the identification of symbols. The nonlinear operator $N[x(t), \dot{x}(t)]$ contains only the nonlinear terms which describe the localized constraint forces and this operator is written as the sum of $n$ unique nonlinear function vectors $y_j(t)$ representing each $j^{th}$ type of nonlinearity present (e.g. quadratic, cubic, fifth order, etc.). Considering only nonlinear elastic forces, each $y_j(t)$ is defined as $y_j(t) = \Delta x_k(t)^m$ where $\Delta x_k(t)$ is the relative displacement across the $k^{th}$ junction where the $j^{th}$ type of nonlinearity exists, and $m$, is the power of the $j^{th}$ type of nonlinearity. These vectors $y_j(t)$ are column vectors of length $q_j$, where $q_j$ is the number of locations the $j^{th}$ type of nonlinearity exists. Note that a single physical junction may contain more than one type of nonlinearity (e.g. a quadratic and cubic); therefore, more than one $y_j(t)$ is necessary to describe the nonlinear constraint force across that particular junction, as illustrated in examples to follow. The coefficient matrices $A_j$ contain the coefficients of the nonlinear function vectors and are of
size N by q. Inserting (3.1b) and (3.1c) into (3.1a), the nonlinear equations of motion take the following form

\[ M \ddot{x}(t) + C \dot{x}(t) + Kx(t) + \sum_{j=1}^{n} A_j y_j(t) = f(t) \]  

(3.2)

From a system identification perspective, it is assumed that the types of the nonlinearities and their physical locations are known. Therefore the n nonlinear function vectors \( y_j(t) \) can be calculated; also, the coefficients of \( y_j(t) \) can be placed in the proper element locations of the coefficient matrices \( A_j \).

This assumption renders limitations on the practical use of this method since various types of nonlinearities at each location are not always known. Therefore, research is currently being conducted to alleviate this limitation. However, it should be noted that this restriction is currently true for any identification scheme when applied to practical nonlinear systems.

Consider several multi-degree-of-freedom nonlinear systems as illustrated in Figure 3.1. The first example as shown in Figure 3.1(a) possesses an asymmetric quadratic-cubic nonlinear stiffness element which exists between the second and third masses and a Gaussian random excitation is applied to the first mass

\[ f_{23}(t) = k_2(x_2(t) - x_3(t)) + \alpha_2(x_2(t) - x_3(t))^2 + \beta_2(x_2(t) - x_3(t))^3 \]

(3.3a,b)

\[ f(t) = \begin{bmatrix} f_1(t) & 0 & 0 \end{bmatrix}^T \]

Assuming that the form of the nonlinear elastic force \( f_{23}(t) \) is known, the nonlinear operator \( \mathcal{N}[x(t), \dot{x}(t)] \), the nonlinear functions \( y_1(t) \) and \( y_2(t) \) and their respective coefficient matrices \( A_1 \) and \( A_2 \) take the following form:

\[ \begin{align*}
\mathcal{N}[x(t), \dot{x}(t)] &= A_1 y_1(t) + A_2 y_2(t), \\
y_1(t) &= (x_2(t) - x_3(t))^2, \\
y_2(t) &= (x_2(t) - x_3(t))^3.
\end{align*} \]  

(3.4a-e)

\[ A_1 = \begin{bmatrix} 0 & \alpha_2 & -\alpha_2 \end{bmatrix}^T, \quad A_2 = \begin{bmatrix} 0 & \beta_2 & -\beta_2 \end{bmatrix}^T \]

Notice, since two types of nonlinearities (quadratic and cubic) exist at a single junction, \( y_1(t) \) and \( y_2(t) \) both contain the same relative displacements. Example II of Figure 3.1(b) has distributed cubic stiffness nonlinearities at every junction and a Gaussian random excitation is applied to the first mass. Therefore.
Figure 3.1. Example cases. (a) I: three degree of freedom system with a local asymmetric quadratic-cubic nonlinearity $f_{23}^e(t)$ and one excitation $f_i(t)$, (b) II: three degree of freedom system with distributed cubic nonlinearities $f_{12}^e(t)$, $f_{13}^e(t)$, and one excitation $f_i(t)$, (c) III: five degree of freedom system with a local cubic nonlinearity $f_3^e(t)$, a local asymmetric quadratic-fifth order nonlinearity $f_{23}^e(t)$ and two excitations $f_i(t)$ and $f_j(t)$. All excitations are Gaussian random.
\[ f_1(t) = k_1(x_1(t) - x_2(t)) + \beta_1(x_1(t) - x_2(t))^3. \]
\[ f_2(t) = k_2(x_2(t) - x_3(t)) + \beta_2(x_2(t) - x_3(t))^3. \]
\[ f_3(t) = k_3x_3(t) + \beta_3x_3(t)^3, \quad f(t) = [f_1(t) \ 0 \ 0]^T. \]
\[ N[x(t), \dot{x}(t)] = A_1y_1(t). \]
\[ y_1(t) = \begin{bmatrix} (x_1(t) - x_2(t))^3 \ (x_2(t) - x_3(t))^3 \ x_3(t)^3 \end{bmatrix}^T, \]
\[ A_1 = \begin{bmatrix} \beta_1 & 0 & 0 \\ -\beta_1 & \beta_2 & 0 \\ 0 & -\beta_2 & \beta_3 \end{bmatrix}. \]

Here a single type of nonlinearity exists at three junctions. Therefore, \( y_1(t) \) is a 3 by 1 column vector.

Example III of Figure 3.1(c) is composed of a cubic nonlinear stiffness element between the second and third masses and an asymmetric nonlinear stiffness element described by a quadratic and fifth order term between the third and fifth masses. Gaussian random excitations are applied to masses 1 and 4 of this system. Therefore,

\[ f_2(t) = k_6(x_2(t) - x_3(t)) + \alpha_6(x_2(t) - x_3(t))^2 + \gamma(x_5(t) - x_3(t))^5. \]
\[ f(t) = [f_1(t) \ 0 \ 0 \ f_4(t) \ 0]^T. \]
\[ N[x(t), \dot{x}(t)] = A_1y_1(t) + A_2y_2(t) + A_3y_3(t). \]
\[ y_1(t) = (x_2(t) - x_3(t))^3. \quad y_2(t) = (x_5(t) - x_3(t))^2. \quad y_3(t) = (x_5(t) - x_3(t))^5. \]
\[ A_1 = (0 \ \beta_1 \ -\beta_3 \ 0 \ 0)^T. \quad A_2 = (0 \ 0 \ -\alpha_6 \ 0 \ \alpha_6)^T. \]
\[ A_3 = (0 \ 0 \ -\gamma_6 \ 0 \ \gamma_6)^T. \]

The modal parameters of the underlying linear systems (i.e. systems with \( A_i = 0 \)) are given in Table 3.1 and the coefficients of the nonlinear elastic forces (i.e. the elements of \( A_i \)) are given in Table 3.2 in terms of \( \alpha, \beta \) and \( \gamma \), where \( \alpha \) is the coefficient of the quadratic nonlinearities, \( \beta \) is the coefficient of the cubic nonlinearities and \( \gamma \) is the coefficient of the fifth order nonlinearity. The linear elastic force coefficients are also given in Table 3.2 for comparison purposes to illustrate the strength of the nonlinearities.
<table>
<thead>
<tr>
<th>Example</th>
<th>mode</th>
<th>natural frequency (Hz)</th>
<th>% damping</th>
<th>eigenvector</th>
</tr>
</thead>
<tbody>
<tr>
<td>I, II 1</td>
<td>22.4</td>
<td>0.7</td>
<td>{1.00, 0.80, 0.45}</td>
<td></td>
</tr>
<tr>
<td>I, II 2</td>
<td>62.8</td>
<td>2.0</td>
<td>{-0.80, 0.45, 1.00}</td>
<td></td>
</tr>
<tr>
<td>I, II 3</td>
<td>90.7</td>
<td>2.9</td>
<td>{-0.45, 1.00, -0.80}</td>
<td></td>
</tr>
<tr>
<td>III 1</td>
<td>11.1</td>
<td>0.7</td>
<td>{0.23, 0.44, 0.61, 0.32, 1.00}</td>
<td></td>
</tr>
<tr>
<td>III 2</td>
<td>30.3</td>
<td>1.9</td>
<td>{0.78, 1.00, 0.49, 0.39, -0.26}</td>
<td></td>
</tr>
<tr>
<td>III 3</td>
<td>44.3</td>
<td>2.8</td>
<td>{-0.57, -0.26, 0.45, 1.00, -0.09}</td>
<td></td>
</tr>
<tr>
<td>III 4</td>
<td>59.0</td>
<td>3.7</td>
<td>{1.00, -0.75, -0.44, 0.59, 0.04}</td>
<td></td>
</tr>
<tr>
<td>III 5</td>
<td>72.3</td>
<td>4.6</td>
<td>{0.28, -0.60, 1.00, -0.47, -0.06}</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1. Linear modal properties of example systems shown in Figure 3.1.
Table 3.2. Linear and nonlinear elastic force coefficients of example systems.

<table>
<thead>
<tr>
<th>Example</th>
<th>linear ( k_1 = 100 \text{ kN/m} )</th>
<th>nonlinear ( \alpha_2 = -8 \text{ MN/m}^2, \beta_2 = 500 \text{ MN/m}^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( k_1 = 100 \text{ kN/m} )</td>
<td>( \beta_1 = \beta_2 = \beta_3 = 1 \text{ GN/m}^3 )</td>
</tr>
<tr>
<td>II</td>
<td>( k_1 = k_2 = k_3 = 100 \text{ kN/m} )</td>
<td>( \beta_1 = \beta_2 = \beta_3 = 1 \text{ GN/m}^3 )</td>
</tr>
<tr>
<td>III</td>
<td>( k_1 = k_4 = 50 \text{ kN/m} )</td>
<td>( \beta_1 = 500 \text{ MN/m}^3, \alpha_2 = -500 \text{ kN/m}^2, \gamma_2 = 10 \text{ GN/m}^3 )</td>
</tr>
</tbody>
</table>
3.2.2. FREQUENCY RESPONSE

The problem statement is as follows: identify the modal parameters of Table 3.1 and the coefficients of the nonlinear elastic force terms of Table 3.2 by a spectral technique. Taking the Fourier transform \( F[.] \) of (3.2)

\[
B(\omega)X(\omega) + \sum_{j=1}^{n} A_j Y_j(\omega) = F(\omega)
\]

\[
X(\omega) = F[x(t)], \quad Y_j(\omega) = F[y_j(t)], \quad F(\omega) = F[f(t)]
\]

(3.7a-c)

The system (3.7a) is composed of a linear dynamic stiffness matrix \( B(\omega) \), and terms representing the nonlinear elastic forces \( A_j Y_j(\omega) \). Using frequency domain based higher dimensional modal parameter estimation techniques [3.1], the modal parameters are extracted from the linear dynamic compliance matrix \( H(\omega) = B(\omega)^{-1} \). Two common methods for estimating the dynamic compliance matrix (i.e. "\( H_1 \)" and "\( H_2 \)" frequency response estimation methods [3.2-3.3]) can be applied directly to multiple-input/multiple-output data from a nonlinear system excited by Gaussian random excitation. However, effects from the presence of the nonlinear elastic forces \( A_j Y_j(\omega) \) can corrupt the underlying linear characteristics of the response causing non-Gaussian output and resulting in estimated dynamic compliance functions which often lead to erroneous results from modal parameter estimation. These nonlinear effects are illustrated using numerically simulated data from the example systems. A 5\(^{th}\) Order Runge-Kutta Fehlberg numerical integration method is used to calculate the response data. The time steps are held constant so that the Fourier transform can be applied to the data. Also, high frequency numerical simulation errors are minimized by choosing a Nyquist frequency eight times greater than the frequency range of interest. Refer to Table 3.3 for simulation and signal processing parameters.

An "\( H_1 \)" estimated dynamic compliance function of Example 1 is shown in Figure 3.2 along with the actual linear dynamic compliance function synthesized from the underlying linear system’s modal parameters of Table 3.1. Comparing these two curves it can be seen that incorrect natural frequencies and damping ratios will result from modal parameter estimation of the first two modes of the "\( H_1 \)" estimated dynamic compliance function and modal information of the third mode is unattainable. Likewise, a
<table>
<thead>
<tr>
<th>Example</th>
<th>$\eta$</th>
<th>magnitude of Gaussian excitation(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>15</td>
<td>5 kN</td>
</tr>
<tr>
<td>II</td>
<td>10</td>
<td>500 N</td>
</tr>
<tr>
<td>III</td>
<td>15</td>
<td>2 kN (both excitations)</td>
</tr>
</tbody>
</table>

Table 3.3. Simulation and signal processing parameters: total number of samples = $2^{14}\eta$, $\Delta t = 0.5$ ms. total period = $2^{11}\eta$ ms. Hanning window, $2^{11}$ samples/average, $2\eta$ averages.
Figure 3.2. Dynamic compliance estimates of Example I. Key: — "H," estimation, --- true linear dynamic compliance function synthesized from the underlying linear system's modal parameters listed in Table 3.1. a. Magnitude of $H_{31}$. b. Phase of $H_{31}$.
Dynamic compliance function of Example II is identified by the "$H_1$" estimation method. Since nonlinearities exist at each junction of this system, every mode is effected dramatically as can be seen in Figure 3.3. Modal parameters cannot be identified from this result. Finally, Figure 3.4 illustrates the effects of the nonlinearities on the "$H_1$" estimate of a dynamic compliance function of Example III.

Although parameter estimation of the first mode may yield modal parameters that are only in slight error, the second, third and fourth modes are "noisy" and have distorted natural frequencies, damping, magnitude and phase characteristics. Parameter estimation of these modes will yield incorrect results and parameters of the fifth mode are unidentifiable.

One might argue that an improved estimate of the linear dynamic compliance functions could simply be obtained by exciting the systems at lower excitation levels, hence minimizing the nonlinear effects. However, reducing excitation levels to minimize the effects of nonlinearities makes the concurrent identification of the nonlinearities even more difficult. Also, for massive structures, large excitation levels may indeed be necessary in order to produce measurable responses at all of the desired output locations. Finally, nonlinear structures should be identified using excitation levels comparable to those experienced under real conditions (which may entail large excitations). To address these issues, a multi-degree-of-freedom "reverse path" method is proposed in this chapter. The examples of Figure 3.1 will be used to illustrate its potential. This method starts with a "reverse path" model as discussed in the following section.

3.3. "REVERSE PATH" FORMULATION

The concept of a "reverse path" model for single degree of freedom systems is adapted from the works of Bendat et al. [3.12-3.15] but it is generalized here for the application to multi-degree-of-freedom systems. A multi-degree-of-freedom "reverse path" model as shown in Figure 3.5 is derived by rearranging (3.7a) with $F(\omega)$ as the output and $X(\omega)$ and $Y_j(\omega)$ as the inputs

$$F(\omega) = B(\omega)X(\omega) + \sum_{j=1}^{n} A_j Y_j(\omega)$$  \hspace{1cm} (3.8)
Figure 3.3. Dynamic compliance estimates of Example II. Key: — "H" estimation, --- true linear dynamic compliance function synthesized from the underlying linear system's modal parameters listed in Table 3.1. a. Magnitude of $H_{21}$. b. Phase of $H_{21}$. 
Figure 3.4. Dynamic compliance estimates of Example III. Key: — "H," estimation, --- true linear dynamic compliance function synthesized from the underlying linear system's modal parameters listed in Table 3.1. a. Magnitude of $H_{11}$, b. Phase of $H_{11}$. 

69
Figure 3.5. "Reverse path" formulation of the equations of motion in the frequency domain.
Observe that the matrices $B(\omega)$ and $A_i$ can be identified directly by measuring $X(\omega)$ and $F(\omega)$ and calculating $Y_i(\omega)$. Recall, it is assumed that the types and locations of the nonlinearities are known, therefore $Y_i(\omega) = F[y_i(t)]$ can be calculated. For the initial derivation, assume that excitations are applied at each response location (i.e. $F(\omega)$ is a fully populated $N$ by 1 column vector). The single-sided power spectral density (PSD) matrices $G_{X\phi}(\omega)$, $G_{IP}(\omega)$, $G_{SI}(\omega)$, . . . , $G_{n\phi}(\omega)$ are defined as follows where the frequency dependence ($\omega$) has been dropped for the sake of brevity

$$G_{X\phi} = \frac{2}{T} E[X^*F^T] = \frac{2}{T} E \left[ X^* (BX^T + \sum_{i=1}^{n} (A_iY_i)^T) \right] = \frac{2}{T} E \left[ X^*X^TB^T + \sum_{i=1}^{n} (X^*Y_i^TA_i^T) \right]$$

$$= G_{XX}B^T + \sum_{j=1}^{n} G_{Xj}A_j^T$$

$$G_{IP} = \frac{2}{T} E[Y_i^*F^T] = \frac{2}{T} E \left[ Y_i^* (BX^T + \sum_{i=1}^{n} (A_iY_i)^T) \right] = \frac{2}{T} E \left[ Y_i^*X^TB^T + \sum_{i=1}^{n} (Y_i^*Y_i^TA_i^T) \right]$$

$$= G_{IX}B^T + \sum_{j=1}^{n} G_{Ij}A_j^T$$

$$i = 1, 2, \ldots, n$$

where $E[.]$ is the expected value and $T$ is the time window. The superscripts * and $^T$ indicate complex conjugate and transpose, respectively. Note, the PSD functions are used here for the mathematical formulation, like Bendat [3.12]. The power spectrum (PS) formulation could have been used as well. This should not effect the performance of the proposed method. The spectral density matrices $G_{X\phi}$ and $G_{XX}$ are $N$ by $N$ matrices, $G_{\phi}$ and $G_{X\phi}$ are $q_i$ by $N$ matrices, $G_{XX}$ are $N$ by $q_i$ matrices, and $G_{\phi}$ are $q_i$ by $q_i$ matrices.

Equations (3.9a-c) can be written in matrix form
Here $\Gamma$ is $N+p$ by $N$, $\Xi$ is $N+p$ by $N+p$ and $\Psi$ is $N$ by $N+p$ where $p = \sum_{j=1}^{n} q_j$. Solving for $\Psi^T$ results in the simultaneous solution of all of the system matrices

$$\Psi^T = \Xi^{-1} \Gamma$$

If, as initially assumed, the excitation vector is fully populated, $\Psi$ will be completely identified by the solution of (3.11), resulting in a full linear dynamic stiffness matrix $B$ and identification of all of the elements of the coefficient matrices $A_i$. Therefore, all of the coefficients of the terms describing the nonlinearities can be identified from the elements of $A_i$. Also, the linear dynamic compliance matrix can be calculated (i.e. $H = B^\top$) and modal parameters can be estimated. However, practically speaking, excitations are not normally applied at every response location. Therefore, measured $\ddot{F}$ will only be a vector of length $M$ where $M$ is the number of applied excitations and typically $M < N$. Note that the superscript $\ddot{\cdot}$ indicates a measured quantity and since the systems of Figure 3.1 represent physical systems for this study, the term "measured" and the superscript $\ddot{\cdot}$ are used to represent quantities which are obtained directly from the numerically simulated input/output data of these systems. Consequently, only the rows of $\ddot{\Psi}$ corresponding to the locations where excitations are applied can be identified. The rest of the elements of $\ddot{\Psi}$ cannot be identified. This is illustrated by application of this approach to Example I. Recall $A_i$ and $A_2$ from (3.4d,e), therefore (3.10d) takes the form
Excitation is applied to the first degree of freedom, i.e. \( \ddot{F} = \ddot{F}_1 \) (notice that this is a location away from the location of the nonlinearity which is between masses 2 and 3). Therefore, only the first column of the measured spectral density matrices involving \( \ddot{F} \) are calculated, resulting in only the first column of \( \tilde{\Gamma} = [\tilde{G}_{XX}, \tilde{G}_{1X}, \tilde{G}_{2X}] \). The measured matrix \( \Xi \) takes the form

\[
\Xi = \begin{bmatrix} \tilde{G}_{XX} & \tilde{G}_{X1} & \tilde{G}_{X1} \\ \tilde{G}_{1X} & \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{2X} & \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix}
\] (3.13)

Solving for \( \tilde{\Psi} \), only the first row is identified

\[
\tilde{\Psi} = [\tilde{\Psi}_{11}, \tilde{\Psi}_{12}, \tilde{\Psi}_{13}, \tilde{\Psi}_{14}, \tilde{\Psi}_{15}] \] (3.14)

Comparing (3.14) and (3.12) notice that only \( B_{11}, B_{12}, \) and \( B_{13} \) are recovered. The elements containing \( \alpha_2 \) and \( \beta_2 \) are not recovered and therefore no information about the coefficients of the nonlinearities is obtained. This is due to the fact that the location of the nonlinearity is away from the applied excitation.

Also note that although \( B_{21} \) and \( B_{11} \) are also determined from reciprocity (i.e. \( B_{21} = B_{12}, B_{11} = B_{13} \)), the resulting \( B \) is singular and therefore \( H = B^{-1} \) cannot be calculated. As a consequence, modal parameter estimation cannot be employed to estimate the underlying linear systems natural frequencies, damping ratios and mode shapes. This illustrates the necessity for the conditioned "reverse path" system approach that will be discussed in the following sections. These refinements are needed to estimate the coefficients of the nonlinearities away from the locations of the applied excitations and also allow for the identification of the linear dynamic compliance matrix \( H \) when an excitation vector of length \( M < N \) is applied.
3.4. CONDITIONED “REVERSE PATH” FORMULATION

As with the “reverse path” model, the concepts of this section are developed from the works of Bendat et al. [3.12-3.15]; but again, these concepts are generalized for the application to multi-degree-of-freedom systems. The problems discussed in section 3.3 are overcome by decomposing the system of Figure 3.5 into uncorrelated paths. This is accomplished by constructing a hierarchy of uncorrelated response components in the frequency domain. To illustrate this, consider Example III. Observe that the spectra of the second nonlinear function vector $Y_2$ can be decomposed into a component which is correlated with the spectra of the first nonlinear function vector $Y_1$, denoted by $Y_{2,1}$, through a frequency response matrix $L_{12}$, and a component which is uncorrelated with the spectra of the first nonlinear function vector, denoted by $Y_{2,-1}$. This is illustrated in Figure 3.6(a). Consequently, the spectral component of the second nonlinear function vector uncorrelated with the spectra of the first nonlinear function vector can be calculated

$$Y_{2(-1)} = Y_2 - Y_{2(+1)}$$

$$= Y_2 - L_{12}Y_1$$

Likewise, the spectra of the third nonlinear function vector $Y_3$ can be decomposed into a component which is correlated with the spectra of the first and second nonlinear function vectors, $Y_{3,-1}$ and $Y_{3,-2}$, respectively, and a component uncorrelated with the spectra of the first and second nonlinear function vectors, $Y_{3,-2}$, as shown in Figure 3.6(b). Note that $Y_{3,-2}$ is a result of $Y_{3,-1}$ and not $Y_3$. The spectral component of the third nonlinear function vector uncorrelated with the spectra of the first and second nonlinear function vectors is calculated by

$$Y_{3(-1,2)} = Y_3 - Y_{3(+1)} - Y_{3(+2)}$$

$$= Y_3 - L_{13}Y_1 - L_{23}Y_{2(-1)}$$

In general

$$Y_{i(-1,-1)} = Y_i - \sum_{j=1}^{[-1]} L_{ji}Y_{j(-1,-1)}$$

The subscripts can be understood as follows. The $i$ outside of the parentheses signifies a spectral component of the $i^{th}$ nonlinear function vector, and $(-1,-1)$ indicates that this component is uncorrelated.
Figure 3.6. Component representation of "reverse path" system's inputs. (a) second nonlinear function vector; (b) third nonlinear function vector; (c) total response vector.
with the spectra of the nonlinear function vectors 1 through \( j-1 \) (the minus signs signify uncorrelated with, while the plus signs such as those in the subscripts of (3.16) signify correlated with). The \( L_j \) are determined using the conditioned "H_{ci}" frequency response function estimation method

\[
L_j^{\mathcal{T}} = G_{ji}^{(-1:j-1)} G_{ji}^{(-1:j-1)}
\]

(3.18)

where \( G_{ji}^{(-1:j-1)} \) and \( G_{ji}^{(-1:j-1)} \) are conditioned PSD matrices involving spectral components of the \( i^{th} \) and \( j^{th} \) nonlinear function vectors uncorrelated with the spectra of the first through the \((j-1)^{th}\) nonlinear function vectors. Calculation of these PSD matrices is delayed until section 3.5. With this hierarchy established, the response vector can now be decomposed into the summation of a linear spectral component and each of the uncorrelated nonlinear spectral components as shown in Figure 3.6(c). The component \( X_{r-(n)} \) is a spectral component from the \( i^{th} \) nonlinear function vector. The relationship between \( X_{r-(i)} \) and the nonlinear spectral component \( Y_{r-(i-1)} \) is through the frequency response matrix \( L_{X} \). The spectral component \( X_{r-(i-1)} \) is the component of the response uncorrelated with the spectra of all \( n \) nonlinear function vectors. In other words, \( X_{r-(i-1)} \) is the linear spectral component of the response. This component is calculated by

\[
X_{r-(i-1)} = X - \sum_{j=1}^{n} L_{X} Y_{r-(i-1-j)}
\]

(3.19)

The \( L_{X} \) are estimated using the conditioned "H_{ci}" frequency response estimation method (3.18) with \( i \) replaced by \( X \).

With the response decomposed into uncorrelated spectral components, Figure 3.5 can be redrawn with \( n+1 \) uncorrelated input vectors as shown in Figure 3.7(a). Comparing these figures, notice that the coefficient matrices between each \( Y_{r-(i-1)} \) and \( F_{r-(i-1)} \) are not the original coefficient matrices, \( A_{r} \), between \( Y_{r} \) and \( F_{r} \). This alteration is necessary in order for the overall output, \( F_{r} \), to remain unchanged. The original coefficient matrices are recovered once \( B \) is identified as covered in section 3.6. However, the path between \( X_{r-(i-1)} \) and \( F_{r-(i-1)} \) remains unchanged. This path is the linear dynamic stiffness matrix \( B \) and its input and output vectors are uncorrelated with all of the spectra of the nonlinear function vectors.
Figure 3.7. Systems with uncorrelated inputs: (a) "Reverse path" system with uncorrelated multiple input vectors, (b) "Forward path" for the underlying linear system.
Therefore, the underlying linear system can be identified without any corruption from the nonlinearities.

Since linear techniques (i.e. modal parameter estimation techniques) normally involve the dynamic compliance matrix $H$, and not the dynamic stiffness matrix $B$, identification of the linear path is conducted by re-reversing the flow of the linear path as illustrated in Figure 3.7(b). Now, any of the conventional frequency response estimation methods can be modified to estimate $H$. For example, the conditioned "$H_{c1}$" and "$H_{c2}$" estimates of the linear dynamic compliance matrix are as follows:

conditioned "$H_{c1}$" estimate:  
$$H^T = G_{FF}^{-1} G_{FX} G_{FX}^{-1}$$  
(3.20a,b)

conditioned "$H_{c2}$" estimate:  
$$H^T = G_{XX}^{-1} G_{FX} G_{FX}^{-1}$$  

Once the conditioned frequency response functions have been estimated, modal analysis techniques can be used to extract natural frequencies, damping ratios and mode shapes without influence of the nonlinearities.

Other modal indicators can also be used to evaluate the number of modes in the frequency range [3.1].

3.5. ESTIMATION OF POWER SPECTRAL DENSITY MATRICES

In order to estimate the linear dynamic compliance matrix by the conditioned estimates (3.20a,b), it is necessary to calculate conditioned PSD matrices. Calculation of these matrices begins with the calculation of unconditioned PSD matrices which are determined directly from the response vector $X$, the excitation vector $F$ and the nonlinear function vectors $Y$,

$$G_{XX} = \frac{2}{T} E[X^* X^T], \quad G_{XF} = \frac{2}{T} E[X^* F^T], \quad G_{FF} = \frac{2}{T} E[F^* F^T], \quad G_{ii} = \frac{2}{T} E[Y_i^* Y_i^T], \quad G_{ij} = \frac{2}{T} E[Y_i^* Y_j^T], \quad G_{ij} = \frac{2}{T} E[Y_i^* F^T]$$  
(3.21a-g)

The conditioned PSD matrices, are more laborious to calculate [3.18]. These PSD matrices involve the response components (e.g. $X_{n-1 ...}$, $Y_{n-1 ...}$). For example, the PSD matrix involving $Y_{2n-1}$ with itself is

$$G_{22} = \frac{2}{T} E[Y_{2n-1}^* Y_{2n-1}^T] = \frac{2}{T} E\left[(Y_{2n-1}^* - Y_{2n-1}^*)^T Y_{2n-1}^T\right]$$  
$$= \frac{2}{T} E\left[Y_{2n-1}^* Y_{2n-1}^T - Y_{12}^* Y_{2n-1}^T - Y_{2n-1}^* Y_{12}^T + Y_{12}^* Y_{12}^T\right] = \frac{2}{T} E\left[Y_{2n-1}^* Y_{2n-1}^T\right]$$  
(3.22)
Note that the second term of the third equality is equal to a \( q_1 \) by \( q_2 \) matrix of zeros since \( Y_{y_{2(-1)}} \) is uncorrelated with \( Y_{y} \) and therefore \( E[L_{12}^* Y_{y}^* Y_{y_{2(-1)}}^T] = 0 \). The result from (3.22) shows that the calculation of \( G_{y_{2(-1)}} \) requires \( G_{22} \), \( G_{21} \) and \( L_{12} \). Calculations of \( G_{22} \) and \( G_{21} \) are given in (3.21d,e), and the frequency response matrix \( L_{12} \) is estimated using the conditioned "Hei" frequency response estimation method (3.18). Therefore, in order to determine \( L_{12} \) and ultimately \( G_{y_{2(-1)}} \), the PSD matrices \( G_{11} \) and \( G_{12} \) must also be calculated, where \( G_{11} \) is determined from (3.21d) and \( G_{12} = G_{21}^H \) (the superscript \( H \) indicates the Hermitian transpose). In general \( r < i, j \),

\[
G_{ij(-r)} = \frac{2}{T} E \left[ Y_{y_i}^* Y_{y_{j-r}}^T \right] = \frac{2}{T} E \left[ Y_{y_i}^* \left( Y_{y_j}^T - \sum_{k=1}^{r} Y_{y_{j-k}}^T \right) \right]
\]

\[
= \frac{2}{T} E \left[ Y_{y_i}^* \left( Y_{y_j}^T - \sum_{k=1}^{r} Y_{y_{j-k+1}}^T L_{kj}^T \right) \right] = G_{ij} - \sum_{k=1}^{r} G_{ik(-k+1)} L_{kj}^T \quad (3.23)
\]

The result from (3.23) reveals a recursive algorithm. Calculation of \( G_{y_{k+1}} \) starts with the computation of \( G_{y_{k-1}} \) from (3.2e). Next, \( i < j \), is calculated

\[
G_{ij(-1)} = G_{ij} - G_{ij} L_{ij}^T \quad (3.24)
\]

Then, \( i < j \), is calculated

\[
G_{ij(-2)} = G_{ij(-1)} - G_{ij(-1)} L_{ij}^T \quad (3.25)
\]

where \( G_{ij(-1)} \) has already been calculated from (3.24). \( G_{ij(-1)} = G_{ij} - G_{ij} L_{ij}^T \), and \( L_{ij} \) and \( L_{ij} \) are estimated using (3.18). This method is continued until the calculation of \( G_{y_{i+1}} \) is reached. The frequency response matrix \( L_{n_{ij}} \) of (3.23) is estimated using (3.18) which requires the PSD matrices \( G_{y_{i+1}} \) and \( G_{y_{i-1}} \).

As another example, consider the calculation of \( G_{ij(-3)} \), \( i, j > 3 \). Equation (3.23) is illustrated graphically in Figure 3.8(a) for \( r = 3 \). Notice that in order to calculate \( G_{ij(-3)} \), \( G_{ij(-1)} \) and \( G_{ij(-2)} \) must first be calculated as illustrated in Figure 3.8(b) and 3.8(c).

79
Figure 3.8. Illustration of the recursive algorithm given by (3.23) for \( r = 3 \).
Note that the subscripts \( i \) and/or \( j \) can be replaced by \( X \) for calculating conditioned PSD matrices involving the response vector. Likewise, \( F \) can replace the subscripts \( i \) and/or \( j \) for calculating the conditioned PSD matrices involving the excitation vector. These substitutions are necessary for the estimation of the linear dynamic compliance matrix by (3.20a,b).

### 3.6. IDENTIFICATION OF THE COEFFICIENTS OF THE NONLINEAR FUNCTION VECTORS

The coefficient matrices \( A_i \) are recovered by re-examining (3.8)

\[
F(\omega) = B(\omega)X(\omega) + \sum_{j=1}^{n} A_j Y_j(\omega) \tag{3.26}
\]

Transposing (3.26) and pre-multiplying by \( Y_{i(-1:i-1)}^*(\omega) \)

\[
Y_{i(-1:i-1)}^*(\omega)F^T(\omega) = Y_{i(-1:i-1)}^*(\omega)X^T(\omega)B^T(\omega) + \sum_{j=1}^{n} Y_{i(-1:i-1)}^*(\omega)Y_j^T(\omega)A_j^T \tag{3.27}
\]

Taking \( \frac{2}{T} E[\cdot] \) yields

\[
G_{i[F(-1:i-1)]}(\omega) = G_{i[X(-1:i-1)]}(\omega)B^T(\omega) + \sum_{j=1}^{n} G_{i[j(-1:i-1)]}(\omega)A_j^T \tag{3.28}
\]

Notice the summation starts at \( i \) since \( E[Y_{i(-1:i-1)}^*(\omega)Y_j^T(\omega)] = 0 \) for all \( j < i \). Pre-multiplying (3.28) by

\[
G_{i[-1:i-1]}^{-1}(\omega)
\]

\[
G_{i[-1:i-1]}^{-1}(\omega)G_{i[F(-1:i-1)]}(\omega) = G_{i[-1:i-1]}^{-1}(\omega)G_{i[X(-1:i-1)]}(\omega)B^T(\omega) + \sum_{j=1}^{n} G_{i[j(-1:i-1)]}(\omega)G_{i[j(-1:i-1)]}(\omega)A_j^T \tag{3.29}
\]

The first term in the summation becomes \( A_i^T \). Therefore,

\[
A_i^T = G_{i[-1:i-1]}^{-1}(\omega)\left( G_{i[F(-1:i-1)]}(\omega) - G_{i[X(-1:i-1)]}(\omega)B^T(\omega) - \sum_{j=i+1}^{n} G_{i[j(-1:i-1)]}(\omega)A_j^T \right) \tag{3.30}
\]

Using (3.30) all of the coefficient matrices \( A_i \) can be identified by starting with the identification of \( A_n \) and working backwards to the identification of \( A_1 \). However, a couple of details about (3.30) need to be discussed. First, (3.30) results in solutions of the elements of \( A_i \), which are frequency dependent since the
dynamic stiffness matrix $B(\omega)$ and all of the PSD matrices are frequency dependent. Therefore, a complex frequency domain curve is obtained for each element of $A_i$. If the identification is accurate, the spectral mean of the curves will be equal to the corresponding real valued coefficients. Second, section 3.4 suggests the calculation $H(\omega)$ rather than $B(\omega)$. The inversion $B(\omega) = H^{-1}(\omega)$ can be performed. However, if $N$ is large, this will be a costly computation. Also, if excitations are not applied at each response location, not all of the columns of measured $\tilde{H}(\omega)$ will be available, and therefore this inversion cannot be performed. As another alternative, frequency response synthesis can be to employ to obtain a fully populated $\tilde{H}(\omega)$ so that the inversion $B(\omega) = H^{-1}(\omega)$ can be calculated [3.17]. However, as will be illustrated in section 3.7, this approach yields poor estimates of the coefficients. To alleviate this problem, (3.30) is post-multiplied by $H^T(\omega)$

$$
A_i^T H^T(\omega) = G_{n-1,i-1}^{-1}(\omega) \left( G_{(F_{i-1},j)}(\omega) H^T(\omega) - G_{(j,i-1,j)}(\omega) - \sum_{j=1}^{n} G_{(j,i-1,j)}(\omega) A_j^T H^T(\omega) \right)
$$

where the left hand side of (3.31) is symbolically multiplied out since $A_i$ is unknown. To illustrate this algorithm consider Example I. Recall, Gaussian random excitation is only applied to mass 1. Therefore, the measured linear dynamic compliance functions estimated by (3.20a) or (3.20b) will only result in the elements of the first column of $\tilde{H}(\omega)$

$$
\tilde{H} = \begin{bmatrix}
H_{11} & ? & ? \\
H_{21} & ? & ? \\
H_{31} & ? & ?
\end{bmatrix}
$$

where the frequency dependence ($\omega$) has again been dropped for the sake of brevity and the symbol ? indicates unmeasured linear dynamic compliance functions. Assuming that the necessary PSD matrices for (3.31) have been calculated, estimations of the coefficient matrices begins with $A_1$ (recall $A_1$ from (3.4e)). Therefore, equation (3.31) becomes
\( A_{2}^{T} \tilde{H}^{T} = \tilde{G}_{22}^{-1}(\tilde{G}_{22} - \tilde{G}_{2X}) \)

\[
\begin{bmatrix}
0 & 0 & 0 \\
-\beta_{2} & -\beta_{2} & -\beta_{2}
\end{bmatrix}
\begin{bmatrix}
\tilde{H}_{11} & \tilde{H}_{21} & \tilde{H}_{31} \\
? & ? & ? \\
? & ? & ?
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{G}_{22} & 0 & 0 \\
0 & \tilde{G}_{2X} & 0 \\
0 & 0 & \tilde{G}_{2X}
\end{bmatrix}
\begin{bmatrix}
\tilde{H}_{11} & \tilde{H}_{21} & \tilde{H}_{31} \\
? & ? & ? \\
? & ? & ?
\end{bmatrix}
\begin{bmatrix}
\tilde{G}_{22} & 0 & 0 \\
0 & \tilde{G}_{2X} & 0 \\
0 & 0 & \tilde{G}_{2X}
\end{bmatrix}
\]

\[(3.33a-d)\]

\[
\beta_{2} = \frac{1}{\tilde{G}_{22}^{-1}}
\begin{bmatrix}
\tilde{G}_{22} & 0 & 0 \\
0 & \tilde{G}_{2X} & 0 \\
0 & 0 & \tilde{G}_{2X}
\end{bmatrix}
\begin{bmatrix}
\tilde{H}_{11} & \tilde{H}_{21} & \tilde{H}_{31} \\
? & ? & ? \\
? & ? & ?
\end{bmatrix}
\begin{bmatrix}
\tilde{G}_{22} & 0 & 0 \\
0 & \tilde{G}_{2X} & 0 \\
0 & 0 & \tilde{G}_{2X}
\end{bmatrix}
\]

Notice, a problem still exists. The coefficient \( \beta_{2} \) cannot be identified since the linear dynamic compliance functions on the left hand side of (3.33d) are unknown. This problem can be alleviated by realizing that since \( H \) represents only the underlying linear system, reciprocity relationships can be applied, i.e. \( H_{x} = H_{y} \).

With this property, additional elements of \( H \) are available which can be used in (3.31) to solve for \( A_{x} \).

Applying reciprocity relations to the measured linear dynamic compliance matrix of Example 1

\[
\tilde{H} = 
\begin{bmatrix}
\tilde{H}_{11} & \tilde{H}_{11} & \tilde{H}_{11} \\
\tilde{H}_{21} & \tilde{H}_{21} & \tilde{H}_{21} \\
\tilde{H}_{31} & \tilde{H}_{31} & \tilde{H}_{31}
\end{bmatrix}
\]

\[(3.34)\]

where the superscript ' indicates linear dynamic compliance functions realized from reciprocity. Using this matrix in (3.31)
\[ A^T_2 \tilde{H}^T = \tilde{G}^{-1}_{22(-1)}(\tilde{G}_{2F(-1)} \tilde{H}^T - \tilde{G}_{2X(-1)}) \]

\[
\begin{bmatrix}
0 & \beta_2 \\
-\beta_2 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{H}_{11} \\
\tilde{H}_{21} \\
\tilde{H}_{31}
\end{bmatrix} =
\begin{bmatrix}
\tilde{H}_{12} \\
\tilde{H}_{13}
\end{bmatrix}
\]

\[ \tilde{G}^{-1}_{22(-1)}\begin{bmatrix}
\tilde{G}_{2F(-1)} & 0 \\
0 & \tilde{G}_{2X(-1)}
\end{bmatrix} - \begin{bmatrix}
\tilde{G}_{2X(-1)} \\
\tilde{G}_{2X(-1)}
\end{bmatrix} \]

\[
(3.35a-d)
\]

\[ \beta_2 [H_{12} - H_{13}] = \]

\[ \tilde{G}^{-1}_{22(-1)}\begin{bmatrix}
\tilde{G}_{2F(-1)} \tilde{H}_{11} \\
\tilde{G}_{2F(-1)} \tilde{H}_{21} \\
\tilde{G}_{2F(-1)} \tilde{H}_{31}
\end{bmatrix} - \begin{bmatrix}
\tilde{G}_{2X(-1)} \\
\tilde{G}_{2X(-1)}
\end{bmatrix} \]

\[ \beta_2 \begin{bmatrix}
H_{12} - H_{13} \\
? \\
?
\end{bmatrix} = \tilde{G}^{-1}_{22(-1)}\begin{bmatrix}
\tilde{G}_{2F(-1)} \tilde{H}_{11} \\
\tilde{G}_{2F(-1)} \tilde{H}_{21} \\
\tilde{G}_{2F(-1)} \tilde{H}_{31}
\end{bmatrix} - \begin{bmatrix}
\tilde{G}_{2X(-1)} \\
\tilde{G}_{2X(-1)}
\end{bmatrix} \]

Now \( \beta_2 \) can be determined by using the first equation of (3.35d). The same approach is used for \( A_1 \),

\[ A^T_1 \tilde{H}^T = G^{-1}_{11}(G_{1F} \tilde{H}^T - G_{1X} - G_{12} A^T_2 \tilde{H}^T) \]

\[
\begin{bmatrix}
0 & \alpha_2 \\
-\alpha_2 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{H}_{11} \\
\tilde{H}_{21} \\
\tilde{H}_{31}
\end{bmatrix} =
\begin{bmatrix}
\tilde{H}_{12} \\
? \\
?
\end{bmatrix}
\]

\[ \tilde{G}^{-1}_{11}\begin{bmatrix}
\tilde{G}_{1F} & 0 \\
0 & \tilde{G}_{1X}
\end{bmatrix} - \begin{bmatrix}
\tilde{G}_{1X} \\
\tilde{G}_{1X}
\end{bmatrix} - G_{12}\begin{bmatrix}
0 & \alpha_2 \\
-\alpha_2 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{H}_{12} \\
\tilde{H}_{13}
\end{bmatrix} \]

\[ \alpha_2 [H_{12} - H_{13}] = \]

\[ \tilde{G}^{-1}_{11}\begin{bmatrix}
\tilde{G}_{1F} \tilde{H}_{11} \\
\tilde{G}_{1F} \tilde{H}_{21} \\
\tilde{G}_{1F} \tilde{H}_{31}
\end{bmatrix} - \begin{bmatrix}
\tilde{G}_{1X} \\
\tilde{G}_{1X} \\
\tilde{G}_{1X}
\end{bmatrix} - G_{12}\alpha_2 \begin{bmatrix}
H_{12} - H_{13}
\end{bmatrix} \]

\[ \alpha_2 \begin{bmatrix}
H_{12} - H_{13} \\
? \\
?
\end{bmatrix} = \tilde{G}^{-1}_{11}\begin{bmatrix}
\tilde{G}_{1F} \tilde{H}_{11} \\
\tilde{G}_{1F} \tilde{H}_{21} \\
\tilde{G}_{1F} \tilde{H}_{31}
\end{bmatrix} - \begin{bmatrix}
\tilde{G}_{1X} \\
\tilde{G}_{1X}
\end{bmatrix} - G_{12}\alpha_2 \begin{bmatrix}
H_{12} - H_{13} \\
? \\
?
\end{bmatrix} \]

\[ (3.36a-d) \]
The outcome from the above example reveals the crucial employment of reciprocity relations to the linear dynamic compliance matrix. Without these relationships it is not always possible to calculate the coefficients of the nonlinearity. This emphasizes the need to decompose the system of Figure 3.5 into the uncorrelated sub-systems of Figure 3.7(a). Also, note that equations (3.35d) and (3.36d) yield a one-to-one dependence between the number of equations and the number of unknowns. This is due to the fact that only one excitation is applied to the system. However, when multiple excitations are simultaneously or sequentially applied, additional rows and columns of $H$ are known, resulting in an over-determined set of equations for the coefficients to be solved by least squares.
3.7. RESULTS

To illustrate the performance of the conditioned multi-degree-of-freedom “reverse path” approach, the simulated data used in section 3.2 is also be used here so that direct comparisons can be made between the conventional “H;” estimates shown in Figures 3.2-3.4 and the conditioned “H;e,” and “H;e,” estimates. The results to follow indicate better performance from the conditioned “H;e,” estimates over the conditioned “H;e,” estimates. This is surprising since no measurement noise is present, and therefore equal performance would be expected from both methods. However, it is possible that this discrepancy is due to numerical errors present in the simulation data of a lightly damped nonlinear system. This issue will be re-examined in upcoming studies. For this discussion, conditioned “H;e,” estimates are initially illustrated and compared with the conventional “H;” estimates and the actual linear dynamic compliance functions synthesized from the modal properties of Table 3.1. These results are then followed by illustrations comparing the “H;e,” estimates, “H;e,” estimates and the actual linear dynamic compliance functions. Modal parameter estimation is conducted on the “H;e,” estimated linear dynamic compliance functions to extract modal parameters. All of the modal parameters are estimated using a polynomial curve fitting technique from a modal analysis software [3.19].

The coefficients of the nonlinear elastic force terms are estimated from the algorithm (3.31). Since the “H;e,” estimates are more accurate than the “H;e,” estimates, the “H;e,” estimates are used in (3.31) for H(ω). For all of the example systems, reciprocity relationships are employed to obtain additional elements of H(ω) as illustrated in section 3.6. Also, since solutions of (3.31) result in frequency domain curves for the coefficients of the nonlinear elastic force terms, the coefficients are numerical estimated from the spectral mean of these curves in the vicinity between 0-125 Hz, e.g.

\[ \bar{\alpha}_2 = \langle \alpha_2(\omega) \rangle_{\omega}, \quad 0 < \omega < 125, \]

where the overhead bar indicates the spectral mean.

Example 1, consisting of the asymmetric nonlinearity is first considered. Since two different nonlinear function vectors are present (equation (3.4b,c)), the measured linear dynamic compliance functions are determined from the following “H;e,” estimate

\[ \tilde{H}^T = \tilde{G}_{Xf(-1:2)}^{-1} \tilde{G}_{XX(-1:2)} \]  

(3.37)
where $\tilde{G}^{-1}_{X_F(-1:2)}$ and $\tilde{G}_{XX(-1:2)}$ are calculated from the algorithm (3.23). Since a single excitation is applied to mass 1 (i.e. $F = F_1$), only the first column of $\tilde{H}$ is identified. Also, since $\tilde{G}_{X_F(-1:2)}$ is a 3 by 1 column vector, (3.37) is solved in a least squares sense. A sample is shown in Figure 3.9. Since the true linear dynamic compliance functions have already been shown in Figures 3.2-3.4, only every twentieth spectral line is plotted for these curves to improve the clarity of the figure. This is also true of figures to follow. Notice, a considerable improvement has been made when comparing the "$H_2"$ and "$H_1"$ estimate. Figure 3.10 illustrates a sample "$H_1"$ estimate of the linear dynamic compliance functions given by

$$\tilde{H}^T = G^{-1}_{X_F(-1:2)} G_{FX(-1:2)}$$

(3.38)

Estimation of the first and second modes are not well predicted when compared with the "$H_2"$ estimate. Nonetheless, this is still an improvement over the "$H_1"$ estimate.

The estimates by (3.37) are used in the modal parameter estimation software to estimate the natural frequencies, modal damping, and mode shapes of the underlying linear system. Results are listed in Table 3.4. The percent error in the natural frequencies and damping is also given in Table 3.5. To assess the accuracy of the mode shape estimates, the modal assurance criterion (MAC) is used [3.20]. Since the "$H_2"$ estimates show some degeneracy in the magnitude of the first mode, this is reflected upon the estimated damping which is in error by 70%. Nonetheless, overall the modal parameters are well predicted.

The coefficients of the nonlinear elastic force terms are identified via (3.31). As mentioned before, reciprocity as illustrated in the example of section 3.6 is employed to identify these coefficients, (equations (3.35a-d) and (3.36a-d)). The resulting solutions are shown in Figures 3.11-3.12. It is not evident at this point why large deviations from the actual values occur. However, these deviations tend to be largest in the vicinity of the modes. An attempt to rectify answers to this phenomenon will be addressed in subsequent research. Averaging the values from these curves at each sample frequency, the spectral means of the coefficients are calculated and are listed in Table 3.6. The resulting spectral means for the coefficients are complex valued, indicating errors in the estimates since the actual coefficients are real.
Figure 3.9. Linear dynamic compliance estimates of Example I. Key: "H₁" estimate, "H₂" estimate, o o o true linear dynamic compliance function. (a) Magnitude of $H₁$. (b) Phase of $H₁$. 

88
Figure 3.10. Linear dynamic compliance estimates of Example I. Key: --- conditioned \( H_{\text{cl}}' \) estimate, — conditioned \( H_{\text{oc}}' \) estimate, o o o true linear dynamic compliance function. (a) Magnitude of \( H_{\text{nt}} \). (b) Phase of \( H_{\text{nt}} \).
<table>
<thead>
<tr>
<th>Example</th>
<th>mode</th>
<th>natural frequency, (Hz)</th>
<th>% damping</th>
<th>eigenvector</th>
</tr>
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<td>I</td>
<td>1</td>
<td>22.3</td>
<td>0.2</td>
<td>{ 1.00, 0.81+0.01i, 0.45 }</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>62.8</td>
<td>2.0</td>
<td>{-0.79, 0.41+0.02i, 1.00 }</td>
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<tr>
<td></td>
<td>3</td>
<td>90.8</td>
<td>3.1</td>
<td>{-0.48+0.06i, 1.00, -0.82-0.05i }</td>
</tr>
<tr>
<td>II</td>
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<td>0.6</td>
<td>{ 1.00, 0.81, 0.45 }</td>
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<td>62.7</td>
<td>1.9</td>
<td>{-0.79, 0.41+0.02i, 1.00 }</td>
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<tr>
<td></td>
<td>3</td>
<td>90.7</td>
<td>3.1</td>
<td>{-0.49+0.06i, 1.00, -0.81-0.05i }</td>
</tr>
<tr>
<td>III</td>
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<td>0.0</td>
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</tr>
<tr>
<td></td>
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<td>1.9</td>
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</tr>
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<td></td>
<td>3</td>
<td>44.2</td>
<td>2.7</td>
<td>{-0.55-0.02i, -0.24+0.01i, 0.43+0.01i, 1.00, -0.08-0.01i }</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>58.9</td>
<td>3.6</td>
<td>{ 1.00, -0.73-0.2i, -0.65-0.08i, 0.65, 0.06 }</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>67.1</td>
<td>4.5</td>
<td>{ 0.29+0.70i, -0.84+0.31i, 1.00, -0.43-0.15i, -0.06-0.01i }</td>
</tr>
</tbody>
</table>

Table 3.4. Estimated modal properties using conditioned "H_\text{Lc}" estimates.
\[
\text{% error} = \frac{\text{estimated} - \text{actual}}{\text{actual}} \cdot 100
\]

<table>
<thead>
<tr>
<th>Example</th>
<th>mode</th>
<th>natural frequency</th>
<th>% damping</th>
<th>MAC</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td>0.3</td>
<td>70.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.1</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.1</td>
<td>8.4</td>
<td>1.0</td>
</tr>
<tr>
<td>II</td>
<td>1</td>
<td>0.1</td>
<td>10.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0</td>
<td>2.5</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.1</td>
<td>9.8</td>
<td>1.0</td>
</tr>
<tr>
<td>III</td>
<td>1</td>
<td>1.1</td>
<td>95.7</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.4</td>
<td>2.1</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.1</td>
<td>2.2</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.0</td>
<td>3.0</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>7.2</td>
<td>0.9</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Table 3.5.  Error and MAC between actual and estimated modal properties of Tables 1 and 4, respectively.
Figure 3.11. Estimation of the nonlinear elastic force coefficient $\alpha_z$ of Example 1. Key: — estimation by (31), o o o true value of coefficient. (a) real part of $\alpha_z$, (b) imaginary part of $\alpha_z$. 
Figure 3.12. Estimation of the nonlinear elastic force coefficient $\beta_2$ of Example 1. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of $\beta_2$. (b) imaginary part of $\beta_2$. 

93
<table>
<thead>
<tr>
<th>Example</th>
<th>Coefficient</th>
<th>Spectral Mean</th>
<th>Actual Value</th>
<th>% Error of Real Part</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$\alpha_2$</td>
<td>-7.96 - 1.46E-2i MN/m$^2$</td>
<td>-8.00 MN/m$^2$</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>$\beta_2$</td>
<td>500.26 + 1.02i MN/m$^3$</td>
<td>500.00 MN/m$^3$</td>
<td>0.1</td>
</tr>
<tr>
<td>II</td>
<td>$\beta_1$</td>
<td>1.03 + 2.41E-2i GN/m$^3$</td>
<td>1.00 GN/m$^3$</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>$\beta_2$</td>
<td>0.99 - 3.30E-3i GN/m$^3$</td>
<td>1.00 GN/m$^3$</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>$\beta_3$</td>
<td>1.00 - 8.71E-4i GN/m$^3$</td>
<td>1.00 GN/m$^3$</td>
<td>0.0</td>
</tr>
<tr>
<td>III</td>
<td>$\alpha_6$</td>
<td>-487.81 + 9.28E-1i kN/m$^2$</td>
<td>-500.00 kN/m$^2$</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>$\gamma_6$</td>
<td>9.88 + 1.72E-2i GN/m$^4$</td>
<td>10.00 GN/m$^4$</td>
<td>1.2</td>
</tr>
</tbody>
</table>

Table 3.6. Coefficients of nonlinear elastic force terms.
valued. However, the imaginary parts of the spectral means are several orders of magnitude less than the real parts, and the percent errors of the real parts of the spectral means from the actual real valued coefficients are all within 1%.

Recall from section 3.6 where it was suggested to employ frequency response synthesis in order to obtain a fully populate linear dynamic compliance matrix for the solution of the coefficients by (3.30). This approach is illustrated here for the estimation of $\beta_2$. The second and third columns of the linear dynamic compliance functions are synthesized from the estimated modal properties. Note that since the linear dynamic compliance matrix is now fully populated, there are three equations for one unknown $\beta_2$ which is solved for in a least squares sense. The resulting spectral curve for $\beta_2$ is shown in Figure 3.13. $\tilde{\beta}_2 = 521.92 - 37.631 \text{ MN/m}^2$ and the error in the real part is 4.38%. Although the error is not large, the curve in Figure 3.13 may mislead one to believe that $\beta_2$ is not independent of frequency. Similar results were obtained for the other examples.

Example II is next considered for identification where a single nonlinear function vector exists (equation (3.5f)), therefore the measured linear dynamic compliance functions are determined from the following “$H_{e2}$” estimate

$$\tilde{H}^T = \tilde{G}_{XF(-1)}^{-1} \tilde{G}_{XX(-1)}$$

(3.39)

As with Example I, a least squares estimation is used for the solution of (3.39) since the measured PSD matrix $\tilde{G}_{XF(-1)}$ is a 3 by 1 column vector. A sample estimate is shown in Figure 3.14. Likewise, the “$H_{e1}$” estimate of the linear dynamic compliance functions is

$$\tilde{H}^T = G_{FF(-1)}^{-1} G_{FX(-1)}$$

(3.40)

and a sample is illustrated in Figure 3.15. Significant improvements are obtained from both of these results when compared to the “$H_1$” estimate. Only a slight discrepancy can be seen where the “$H_{e1}$” method underestimates the magnitude of the first mode. Estimated modal parameters from (3.39) are listed in Table 3.4 and an error analysis is in Table 3.5. Like Example I, the modal parameters of the underlying linear system are well predicted.
Figure 3.13. Estimation of the nonlinear elastic force coefficient $\beta_2$ of Example 1 with the employment of frequency response synthesis. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of $\beta_2$. (b) imaginary part of $\beta_2$. 
Figure 3.14. Linear dynamic compliance estimates of Example II. Key: --- "H_1" estimate, — conditioned "H_{12}" estimate, o o o true linear dynamic compliance function. (a) Magnitude of H_{11}. (b) Phase of H_{11}.
Figure 3.15. Linear dynamic compliance estimates of Example II. Key: --- conditioned "H_{ij}" estimate, — conditioned "H_{ij}" estimate, o o o true linear dynamic compliance function. (a) Magnitude of \(H_{ij}\). (b) Phase of \(H_{ij}\).
Using the results from (3.39) in (3.31) the coefficients of the nonlinear elastic force terms are identified. Since a single excitation is applied to mass 1, reciprocity relations are necessary in order to identify the coefficients. Results are shown in Figures 3.16-3.18, and Table 3.6 lists the numerical estimates of the coefficients, where the imaginary parts are at least two orders of magnitude less than the real parts and the percent error of the real parts from the actual real valued coefficients is 3% and less. Overall, the system has been well identified.

Example III, the five degree of freedom system, is finally considered. Three nonlinear function vectors exist (equation (3.6e-g)), therefore the "H_{ij}" estimate is

\[ \tilde{H}^T = \tilde{G}^{-1}_{XX(-1:3)} \tilde{G}_{XX(-1:3)} \]  

A sample estimate is illustrated in Figure 3.19. A considerable improvement has been made in estimating this linear dynamic compliance function. Likewise, the "H_{ij}" estimate is

\[ \tilde{H}^T = \tilde{G}^{-1}_{FF(-1:3)} \tilde{G}_{FX(-1:3)} \]  

and a sample estimate is show in Figure 3.20. Like Example I, the "H_{ij}" method does not estimate the linear dynamic compliance function as well as the "H_{ij}" method. Estimated natural frequency, modal damping and mode shapes from (3.41) are given in Table 3.4 with an error assessment in Table 3.5. Aside from the estimated damping of the first mode and the estimated natural frequency of the last mode, the modal parameters are well predicted.

Shown in Figures 3.21-3.23 are estimates of the coefficients of the nonlinear elastic force terms via (3.31) using the results from (3.41) and the employment of reciprocity relations to obtain additional elements of \( \tilde{H} \). The numerical estimates are listed in Table 3.6 along with the true values and percent error in the real parts. The imaginary parts of the spectral means are again several orders of magnitude less than the real parts and the errors of the real parts from the actual values are all less than 3%. Overall, this system has been well identified.

99
Figure 3.16. Estimates of the nonlinear elastic force coefficient $\beta$, of Example II. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of $\beta$. (b) imaginary part of $\beta$. 
Figure 3.17. Estimates of the nonlinear elastic force coefficient $\beta_2$ of Example II. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of $\beta_2$. (b) imaginary part of $\beta_2$. 

101
Figure 3.18. Estimates of the nonlinear elastic force coefficient $\beta_1$ of Example II. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of $\beta_1$, (b) imaginary part of $\beta_1$. 
Figure 3.19. Linear dynamic compliance estimates of Example III. Key: — "H₁" estimate, —
conditioned "H₂" estimate, o o o true linear dynamic compliance function. (a) Magnitude of
$H_{44}$. (b) Phase of $H_{44}$. 
Figure 3.20. Linear dynamic compliance estimate of Example III. Key: --- conditioned "H_{cl}" estimate, — conditioned "H_{c1}" estimate, o o o true linear dynamic compliance function. (a) Magnitude of $H_{31}$. (b) Phase of $H_{31}$.
Figure 3.21. Estimate of the nonlinear elastic force coefficient $\alpha_1$ of Example III. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of $\alpha_1$. (b) imaginary part of $\alpha_1$. 

105
Figure 3.22. Estimate of the nonlinear elastic force coefficient $\beta_0$ of Example III. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of $\beta_0$. (b) imaginary part of $\beta_0$. 
Figure 3.23. Estimate of the nonlinear elastic force coefficient $\gamma_e$ of Example III. Key: — estimation by (3.31), o o o true value of coefficient. (a) real part of $\gamma_e$. (b) imaginary part of $\gamma_e$. 
3.8. CONCLUSION

It has been shown in this chapter that conventional frequency response estimation methods such as the \( H_1 \) and \( H_2 \) estimates are often inadequate for accurately estimating the linear dynamic compliance functions of multi-degree-of-freedom nonlinear systems when excited by Gaussian random excitations. Therefore, a new spectral approach has been developed based on a "reverse path" formulation as available in the literature for single-degree-of-freedom nonlinear systems \([3.12]\), with emphasis on the mathematical development for application to multi-degree-of-freedom systems. With new formulation, conditioned \( H_{c1} \) and \( H_{c2} \) estimates of linear dynamic compliance functions can now be obtained which drastically reduce, or even eliminate in some cases, the contamination introduced by nonlinearities. This allows for the identification of the modal parameters of the underlying linear system without any undue influences caused by nonlinearities. The coefficients of analytical functions which describe the nonlinearities are also estimated by this new method. These nonlinearities may be local or distributed and they may exist at or away from the locations of the excitations.

This new spectral approach has been tested on three example systems with polynomial nonlinearities. These systems were excited by Gaussian random excitations applied at either one or two locations. The multiple-input/multiple-output data from these systems have been successfully used and the results illustrate benefits of this approach. However, further refinements are necessary before the method can be applied to the measured input/output data of "real" nonlinear systems. For instance, modifications need to be made to accommodate for uncorrelated measurement noise and restrictions imposed by the types of sensors and actuators used in the experiments. Also, since coherence functions are often used as a means to determine the validity of spectral measurements, development of similar quantifiers for the multi-degree-of-freedom "reverse path" formulation are also necessary. However, calculation of these functions is much more extensive for conditioned systems; therefore, this formulation has been reserved for additional studies. Finally, as pointed out earlier in this chapter, the nature or type of nonlinearities \( y(t) \) present must be known \textit{a priori} for the method to be successful when applied to practical nonlinear systems. However,
this information may not be available under “real” conditions. Hence, research is currently being conducted to alleviate this limitation and progress will be reported in subsequent research.
LIST OF REFERENCES FOR CHAPTER 3


CHAPTER 4

FEASIBILITY OF IDENTIFYING NONLINEAR VIBRATORY SYSTEMS CONSISTING OF UNKNOWN POLYNOMIAL FORMS

4.1. INTRODUCTION

Consider a nonlinear structural or mechanical system described by the following set of N coupled differential equations

\[ M\ddot{x}(t) + d(x(t),\dot{x}(t)) = f(t) \]  

(4.1)

where \( M \) is the time-invariant mass matrix, \( x(t) \) and \( f(t) \) are the generalized displacement and force vectors, and \( d(x(t),\dot{x}(t)) \) is a vector of motion dependent restoring force functions. Decompose \( d(x(t),\dot{x}(t)) \) as follows

\[ d(x(t),\dot{x}(t)) = Cx(t) + Kx(t) + d''(x(t),\dot{x}(t)), \]

\[ M\ddot{x}(t) + C\dot{x}(t) + Kx(t) + d''(x(t),\dot{x}(t)) = f(t) \]  

(4.2a,b)

where \( C \) and \( K \) are the time-invariant linear damping and linear stiffness matrices, and \( d''(x(t),\dot{x}(t)) \) is a vector consisting of only nonlinear terms. When the system is linear, i.e. \( d''(x(t),\dot{x}(t)) = \{0\} \), or linearized to yield effective damping \( C \), and stiffness \( K \), identification methods can estimate parameters from measured time or frequency domain data in the form of natural frequencies \( \omega_n \), mode shapes \( \phi \), and damping ratios \( \zeta \) [4.1]. However, complexities such as high modal density, heavily damped modes and measurement noise complicate the accurate determination of these parameters. These complications may be alleviated using mode indicator functions to determine valid modes [4.1] and proper frequency response estimators, such as "H_1" or "H_2" to minimize uncorrelated measurement noise [4.2,4.3]. Nonetheless, it is difficult to construct \( M, C \) and \( K \) unless a computational model is available.
To worsen the problem of identification, the effects of $d_n(x(t), \dot{x}(t))$ for many physical nonlinear systems may substantially influence the dynamic response $x(t)$ [4.4-4.6]. Consequently, modal testing and similar methods are no longer valid. Under these circumstances identification methods for nonlinear systems must be employed. Unfortunately, literature on such techniques is rather sparse as discussed in [4.7]. A temporal method known as the Restoring Force or Force State Mapping Method has been developed [4.8-4.9] and investigated [4.10-4.11]. Likewise, a spectral method based upon a "reverse path" analysis has been formulated for single degree of freedom systems [4.12-4.15] and recently modified for multi-degree-of-freedom systems [4.7]. However, many issues remain unresolved before such methods can be applied to practical problems. Three key questions follow. 1. Should there be an \textit{a priori} knowledge of the nature and mathematical form of $d_n(x(t), \dot{x}(t))$ before the identification process is initiated? If not, will an appropriate model result from the approximation of $d_n(x(t), \dot{x}(t))$? 2. Is the identification problem compounded by the presence of measurement noise? 3. Can coherence techniques be used to facilitate the identification process? These issues are addressed in this chapter via several simulation examples. Only continuous nonlinearities will be considered with emphasis on polynomial forms. The "Reverse Path" Spectral Method for multi-degree-of-freedom systems [4.7] will be the chief method of evaluation. However, the Temporal Method [4.8-4.11] will also be utilized for one example to illustrate whether the issues raised here are method dependent. The effects of measurement noise on the estimates are examined for moderate and high levels of uncorrelated noise. Also, the performance of the Spectral Method and the associated coherence functions will be critically assessed under conditions that the nature or shape of the nonlinearities is unknown but approximated by alternative mathematical functions.

4.2. PROBLEM FORMULATION

4.2.1. SCOPE

Consider a two degree of freedom system of Figure 4.1(a) consisting of only one nonlinear spring element with elastic force $f_{15}(t)$. Therefore define $d_n(x(t), \dot{x}(t)) = d_4(x(t))$ and (4.2b) becomes
Figure 4.1. Simulation examples. (a) Two degree of freedom system with a nonlinear spring element of elastic force $f^{e}_{12}(t)$. (b) Plots of nonlinear elastic forces. Key: -- $f^{e}_{12}(t)$ of Example I. --- $f^{e}_{12}(t)$ of Example II. ... linear component of $f^{e}_{12}(t)$, i.e. $k_{1}\Delta x_{12}(t)$. 
\[ M \ddot{x}(t) + C \dot{x}(t) + Kx(t) + d_n(\dot{\text{x}}(t)) = f(t) \] (4.3)

Masses, linear damping and linear stiffness coefficients are listed in Table 4.1 along with the system's modal properties determined by assuming that \( d_n(x(t)) = \{0\} \). Also refer to the List of Symbols for the identification of symbols. Several examples will be examined where the nonlinear spring stiffness is described by different mathematical forms. Examples I and II are discussed first where elastic force \( f_{12}^*(t) \) of Example I is described by linear and cubic terms and \( f_{12}^*(t) \) of Example II is described by linear, quadratic and 5th order terms as listed in Table 4.2. Therefore,

**Example I:**

\[ d_n(x(t)) = \begin{bmatrix} \beta_3 \Delta x_{12}(t)^3 \\ -\beta_3 \Delta x_{12}(t)^3 \end{bmatrix} \]

**Example II:**

\[ d_n(x(t)) = \begin{bmatrix} \beta_2 \Delta x_{12}(t)^2 + \beta_3 \Delta x_{12}(t)^5 \\ -\beta_2 \Delta x_{12}(t)^2 - \beta_3 \Delta x_{12}(t)^5 \end{bmatrix} \] (4.4a,b)

where \( \Delta x_{12}(t) = x_1(t) - x_2(t) \) and \( \beta_2, \beta_3 \) and \( \beta_3 \) are coefficients of the polynomial terms describing \( f_{12}^*(t) \). By applying a synthesized Gaussian random excitation \( f(t) \) with \( |f(t)| = 60 \text{ N-rms}, \text{ mean} = 0 \) and variance = 1. \( x(t) = [x_1(t) \ x_2(t)]^T \), \( \dot{x}(t) \) and \( \ddot{x}(t) \) for both examples are calculated using a 5th Order Runge-Kutta Fehlberg numerical integration method. The time steps (\( \Delta t \)) are held constant so that the Fast Fourier Transform (FFT) can be applied to the data, and high frequency numerical simulation errors are minimized by choosing a Nyquist frequency eight times greater than the highest frequency of interest. The following numerical simulation parameters are used: \( \Delta t = 0.5 \text{ ms}, \) number of samples = 15 (\( 2^{14} \)), total period = 15 (\( 2^{13} \)) ms. Corresponding stiffness curves are illustrated in Figure 4.1(b) where \( (\Delta x_{12}(t))_m \) and \( (f_{12}^*(t))_m \) are the maximum \( \Delta x_{12}(t) \) and \( f_{12}^*(t) \) experienced by Examples I and II in the numerical simulations.

Under experimental testing conditions, the modal properties of Table 4.1 are typically identified from the input/output data of Examples I and II using well accepted modal analysis or comparable system identification techniques. However, erroneous modal parameters may result, as illustrated in the frequency domain to follow. First, take the Fourier transform \( F_{12}^* \) of (4.3)
In Example III $c_i = 100 \text{ N s/m}$, other physical parameters remain the same.

Table 4.1. True linear system properties of Examples I, II and IV given $d_a(x(t)) = \{0\}$.  

<table>
<thead>
<tr>
<th>DOF, $i$</th>
<th>$m_i$ (kg)</th>
<th>$c_i$ (N s/m)</th>
<th>$k_i$ (kN/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Physical</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.0</td>
<td>10.0</td>
<td>100.0</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>10.0</td>
<td>100.0</td>
</tr>
<tr>
<td>Modal</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>31.1</td>
<td>1.0</td>
<td>${1.0, 0.6}$</td>
</tr>
<tr>
<td>2</td>
<td>81.4</td>
<td>2.6</td>
<td>${-0.6, 1.0}$</td>
</tr>
</tbody>
</table>
Example | $f_{12}^*(t)$ | Coefficients
---|---|---
I, III | $k_1\Delta x_{12}(t) + \beta_2 \Delta x_{12}'(t)^2$ | $\beta_2 = 500.0 \text{ MN/m}^3$
II | $k_1\Delta x_{12}(t) + \beta_2 \Delta x_{12}'(t)^2 + \beta_3 \Delta x_{12}'(t)^3$ | $\beta_2 = -1.0 \text{ MN/m}^2, \beta_3 = 10.0 \text{ GN/m}^4$
IV | $k_1\Delta x_{12}(t) + \eta \text{ sgn}(\Delta x_{12}(t))|\Delta x_{12}(t)|^{\frac{3}{2}}$ | $\eta = 1.0 \text{ MN/m}^{1.5}$

Table 4.2. True nonlinear elastic force $f_{12}^*(t)$ of Examples I-IV.
\[ \mathbf{B}(\omega)X(\omega) + \Gamma_{n}(\omega) = \mathbf{F}(\omega), \]
\[ X(\omega) = \mathbf{F}[x(t)]\Gamma_{n}(\omega) = \mathbf{F}[\mathbf{a}_{n}(x(t))], \quad \mathbf{F}(\omega) = \mathbf{F}[f(t)] \]
\( (4.5a-f) \)
\[ \mathbf{B}(\omega) = -\omega^{2}\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}, \quad \mathbf{B}(\omega) = \mathbf{H}(\omega)^{-1} \]

where \( \mathbf{B}(\omega) \) and \( \mathbf{H}(\omega) \) are the linear dynamic stiffness and compliance matrices, respectively.

Contamination of \( \mathbf{H}(\omega) \) results due to the presence of \( \Gamma_{n}(\omega) \). Figure 4.2 illustrates a sample result in terms of \( \mathbf{H}_{11}(\omega) \) that is estimated from the data of Example I by the "\( \mathbf{H}_{1} \)" frequency response estimator [4.2-4.3], where \(^{\wedge}\) signifies estimated, superscript \(^{[1]}\) signifies an "\( \mathbf{H}_{1} \)" estimate, and subscripts \(_{2}\) and \(_{3}\) signify that \( \mathbf{H}_{23}(\omega) \) is a cross-point function between \( x_{2}(t) \) and \( f_{2}(t) \). The following procedure is used for all spectral calculations. The sampled data are first divided into 30 averages consisting of \( 2^{13} \) samples per average. Since the Nyquist frequency is much greater than the highest frequency of interest, an eighth order Chebyshev type I low pass filter with a cut-off frequency at 100 Hz is next applied. The data are then re-sampled at a new \( \Delta t' = 8\Delta t \) and a Hanning window is employed to minimize leakage errors. Also shown in Figure 4.2 is a true linear \( \mathbf{H}_{21}(\omega) \) that is synthesized from the modal parameters of Table 4.1. As illustrated, the first mode of \( \mathbf{H}_{21}(\omega) \) has shifted up in frequency, as expected due to the presence of the hardening spring nonlinearity located between \( m_{1} \) and \( m_{2} \). Worse is the effect on the second mode which is highly corrupted by the nonlinearity. Frequency domain modal parameter estimation techniques would lead to erroneous results for the natural frequency of the first mode and modal parameters of the second mode would be unattainable. The same procedure is also applied to the data of Example II and problems similar to Example I are encountered. Also, comparable problems are encountered when time domain techniques [4.1] are used to estimate the parameters of Examples I and II. Therefore, appropriate temporal or spectral nonlinear system identification methods must be employed in order to determine the true parameters of these examples. However, this initial analysis may serve as a strategy for locating nonlinearities. First, observe from Table 4.1 that the mode shape of the second mode \( \phi_{2} \) has a larger relative displacement between \( m_{1} \) and \( m_{2} \) than the first mode \( \phi_{1} \). This larger displacement amplifies the nonlinear behavior of the elastic force \( f_{12}^{e}(t) \) causing mode 2 to be more corrupted. Consequently, by studying frequency response functions at various excitation levels and examining the mode shapes of the
Figure 4.2. Dynamic compliance spectra of Example 1. (a) Magnitude of $H_{21}(\omega)$. (b) Phase of $H_{21}(\omega)$.

Key: — $H_{21}(\omega)$, --- $\tilde{H}_{21}(\omega)$.
modes most corrupted by the nonlinear response, nonlinearities may be located where the relative
displacements of these mode shapes are largest.

4.2.2. METHODOLOGY

The identification schemes for nonlinear systems discussed in this chapter estimate the linear
properties and nonlinear elastic forces by fitting a mathematical model of the following form to the
measured or simulated excitation and response data

\[ \dot{M}\ddot{x}(t) + \ddot{z}(x(t), \dot{x}(t)) = f(t) \]  \hspace{1cm} (4.6)

where \( \dot{M} \) is the estimated mass matrix and \( \ddot{z}(x(t), \dot{x}(t)) \) is a vector of motion dependent functions which estimate the system's constraint forces. Considering only Examples I and II with linear viscous damping and a nonlinear elastic force \( f_{12}^e(t) \), write \( \ddot{z}(x(t), \dot{x}(t)) \) as

\[
\ddot{z}(x(t), \dot{x}(t)) = \dot{C}\dot{x}(t) + \dot{K}x(t) + \ddot{z}_n(x(t)),
\]

\[
\ddot{z}_n(x(t)) = \sum_{j=1}^{n} \hat{a}_j y_j(\Delta x_{12}(t))
\]  \hspace{1cm} (4.7a,b)

where \( \dot{C} \) and \( \dot{K} \) are estimated linear damping and stiffness matrices and \( \ddot{z}_n(x(t)) \) contains only the nonlinear terms of an assumed model \( p_{12}^e(t) \) for describing the true elastic force \( f_{12}^e(t) \). The \( n \) unique nonlinear functions \( y_j(\Delta x_{12}(t)) \) transform the relative displacement \( \Delta x_{12}(t) \) into the form of each nonlinear term of \( p_{12}^e(t) \) and \( \hat{a}_j \) are coefficient vectors containing estimates of the coefficients of the nonlinear terms, i.e. \( \hat{a}_j = [\hat{a}_j - \hat{a}_j]^T \). Therefore, (4.6) becomes

\[ \dot{M}\ddot{x}(t) + \dot{C}\dot{x}(t) + \dot{K}x(t) + \sum_{j=1}^{n} \hat{a}_j y_j(\Delta x_{12}(t)) = f(t) \]  \hspace{1cm} (4.8)

The Restoring Force Method is a temporal method which estimates the masses, linear damping and linear
stiffness coefficients as well as the coefficients of the nonlinear elastic force terms by fitting the model
(4.8) to the measured or simulated excitation and response data in the time domain [4.11-4.12]. The
system of equations (4.8) are not fit to the data simultaneously; rather, the method begins with the
nonhomogeneous equation describing the motion of the forced degree of freedom \( m_1 \) and then iterates to the equation describing the motion of the adjacent degree of freedom \( m_j \). Alternatively, the Multi-Degree-of-Freedom “Reverse Path” Spectral Method, referred to as the Spectral Method from this point forth, is a frequency domain system identification approach proposed by us in a recent article [4.7]. Applying the Fourier transform \( F[\cdot] \) to (4.8)

\[
\hat{B}(\omega)X(\omega) + \sum_{j=1}^{n} \hat{a}_j Y_j(\omega) = F(\omega),
\]

\[
X(\omega) = F[x(t) + \omega f(t)], Y_j(\omega) = F[y_j(\Delta x_{12}(t))], F(\omega) = F[f(t)],
\]

\[
(4.9a-e)
\]

where \( \hat{B}(\omega) \) is an estimate of \( B(\omega) \). For the “reverse path” analysis, the excitation \( F(\omega) \) is treated as an output to the model and the total response \( X(\omega) \) along with the nonlinear functions \( Y_j(\omega) \) are treated as inputs to the model [4.7]. Therefore, rewrite (4.9a) as follows,

\[
F(\omega) = \hat{B}(\omega)X(\omega) + \sum_{j=1}^{n} \hat{a}_j Y_j(\omega)
\]

(4.10)

Equation (4.10) is also shown graphically in Figure 4.3(a). By applying spectral conditioning techniques [4.7,4.16], Figure 4.3(a) can be redrawn as a conditioned model with uncorrelated inputs as shown in Figure 4.3(b), where \( Y_{j-1,1}(\omega) \) are conditioned spectra of the nonlinear functions \( Y_j(\omega) \), and \( X_{1,0}(\omega) \) contains only the linear spectral components of the total response \( X(\omega) \). Therefore, the linear path \( \hat{B}(\omega) \) can now be estimated without any influence from the nonlinearities. Since frequency domain identification techniques typically identify parameters associated with the linear dynamic compliance matrix \( H(\omega) = \hat{B}(\omega)^{-1} \), or derivatives thereof, the linear path is re-reversed as shown in Figure 4.3(c).

Conditioned frequency response function estimators “\( H_{c1} \)” and “\( H_{c2} \)” are now defined as

conditioned “\( H_{c1} \)” estimate:

\[
\hat{H}_{c1}^{[1]}(\omega)^T = G_{F(-1n)}^{-1}(\omega)G_{FX(-1n)}(\omega)
\]

(4.11a,b)

conditioned “\( H_{c2} \)” estimate:

\[
\hat{H}_{c2}^{[2]}(\omega)^T = G_{F(-1n)}^{-1}(\omega)G_{XX(-1n)}(\omega)
\]
Figure 4.3. "Reverse path" spectral model. (a) Model with correlated inputs, equation (4.10). (b) Conditioned model with uncorrelated inputs. (c) "Forward path" of the underlying linear subsystem.
where $G_{FF^{-1}n}^{\text{at}}(o)$, $G_{FX^{-1}n}^{\text{at}}(o)$ and $G_{XX^{-1}n}^{\text{at}}(o)$ are conditioned spectral density matrices. Calculation of these matrices is discussed by Richards and Singh [4.7] which is a higher dimensional derivation of the calculation proposed earlier by Bendat and Piersol [4.6, 4.16]. Now, modal parameters can be determined from the conditioned estimates (4.11a) or (4.11b) without any influence from the nonlinearities. The coefficients $\hat{a}_j$ are also recovered as a function of frequency [4.7], i.e. $\hat{a}_j = \hat{a}_j(o)$. Since the true coefficients $a_j$ are constants for Examples I and II, accurate estimates should lead to $\langle \hat{a}_j(o) \rangle_h = a_j$, where $\langle \rangle_h$ signifies spectral mean.

As will be illustrated in section 4.4, both Temporal and Spectral Methods successfully estimate the linear properties of Table 4.1 and the coefficients of the nonlinear elastic force terms of Table 4.2 when the simulated data set is noise free and the correct model $p^{\text{at}}_{12}(t)$ is chosen to describe the true nonlinear elastic force $f^{\text{at}}_{12}(t)$, i.e. $p^{\text{at}}_{12}(t) = f^{\text{at}}_{12}(t)$. However, in practice, experimental data is corrupted by measurement noise, and the nature and mathematical form of $f^{\text{at}}_{12}(t)$ are rarely known. As a result, errors occur in the estimates, especially when measurement noise is significant or when $f^{\text{at}}_{12}(t)$ is poorly represented by $p^{\text{at}}_{12}(t)$; both issues will be discussed in sections 4.5 and 4.6. Consequently, two of the specific objectives of this study are to determine when noise levels are tolerable and when an accurate model has been chosen. From the Spectral Method one may take advantage of coherence concepts for measuring the "cause-effect" relationships. These coherence functions may also indicate the frequencies corrupted by the nonlinearities, which itself may be useful for designing nonlinear elements such as rubber and hydraulic engine mounts [4.17].

4.3. COHERENCE FUNCTIONS BASED ON CONDITIONED SPECTRA

Recall the conditioned "reverse path" model of Figure 4.3(b). Ordinary coherence functions between the conditioned spectra $Y_{\eta^{-1}n}^{\text{at}}(o)$ and excitation $F_{\xi}(o)$ of $F(o)$ are given as
where $G_{jF_i(-j-l)}(\omega)$ is the conditioned cross-power spectral density function between $Y_{i-I-j-l}(\omega)$ and $F_i(\omega)$, $G_{iF_i(-j-l)}(\omega)$ is the conditioned auto-power spectral density function of $Y_{i-I-j-l}(\omega)$ and $G_{F_i}(\omega)$ is the unconditioned auto-power spectral density function of the excitation $F_i(\omega)$. Similarly, ordinary coherence functions between each element of $X_{i-n}(\omega) = [X_{n-I-n}(\omega) X_{2n-I-n}(\omega)]^t$ and excitation $F_i(\omega)$ can be calculated by

$$\hat{\gamma}_{Y_{X_{i-n}}(\omega)}^2 = \frac{|G_{X_{i-n}(-i-n)}(\omega)|^2}{G_{X_{i-n}(-i-n)}(\omega)G_{F_i}(\omega)}, \quad i = 1, 2$$

(4.13)

where $G_{X_{i-n}(-i-n)}(\omega)$ is the conditioned cross-power spectral density function between $X_{i-n}(\omega)$ and $F_i(\omega)$, and $G_{X_{i-n}(-i-n)}(\omega)$ is the conditioned auto-power spectral density function of $X_{i-n}(\omega)$. The coherence functions of (4.12) and (4.13) are scalar values between 0 and 1 at each frequency and they indicate the amount of contribution from each respective input to the model of Figure 4.3(b).

Notice in Figure 4.3(b) that no conditioning is used to uncorrelate the elements $X_{n-I-n}(\omega)$ and $X_{2n-I-n}(\omega)$ of $X_{i-n}(\omega)$, i.e. the linear component of the response $X_{i-n}(\omega)$ remains a vector input to the model. Therefore, a multiple coherence function cannot be defined in its conventional form [4.6, 4.16] for this model. However, cumulative coherence functions $\hat{\gamma}^2_{Mi}(\omega)$ which include only one of the coherence functions of (4.13) in the summation are defined here as

$$\hat{\gamma}^2_{Mi}(\omega) = \gamma^2_{X_{F_i}(-i-n)}(\omega) + \gamma^2_{Y_{F_i}}(\omega), \quad i = 1, 2;$$

(4.14a,b)

$$\hat{\gamma}^2_{Y_{F_i}}(\omega) = \sum_{j=1}^{n} \gamma^2_{F_i(-j-l)}(\omega)$$

which are also scalar values between 0 and 1 at each frequency and may be considered as a measure of the accuracy of the entire model, Figure 4.3(b). Each coherence function $\hat{\gamma}^2_{Mi}(\omega)$ is given as the sum of two terms. The first term $\gamma^2_{X_{F_i}(-i-n)}(\omega)$ indicates contribution from linear spectral component of the response of
the \(i^{th}\) mass, and the second term \(\hat{\gamma}_{YF}^2(\omega)\) indicates contribution from the nonlinearities. Analysis of 

\[ \hat{\gamma}_{X_{i},Y_{i}}^2(\omega) \] 

and \(\hat{\gamma}_{YF}^2(\omega)\) may be useful when it is desired to have nonlinearities contributing at certain frequencies, by design [4.17].

Finally, partial coherence functions are defined for each path of the model of Figure 4.3(b). However, only the partial coherence functions between the elements of \(X_{i-1,n}(\omega)\) and \(F_{i-1,n}(\omega)\) are given here as

\[
\hat{\gamma}_{X_{i},F_{i}(-1:n)}(\omega) = \frac{\left| G_{X_{i},F_{i}(-1:n)}(\omega) \right|^2}{G_{X_{i},X_{i}(-1:n)}(\omega)G_{F_{i},F_{i}(-1:n)}(\omega)} \quad i = 1, 2
\]  

(4.15)

The partial coherence functions (4.15) are similar to the ordinary coherence functions (4.13) with the exception that the denominator of (4.15) contains the conditioned auto-power spectral density function \(G_{F_{i},F_{i}(-1:n)}(\omega)\). Consequently, \(\hat{\gamma}_{X_{i},F_{i}(-1:n)}(\omega)\) are the ordinary coherence functions for the linear sub-system of Figure 4.3(c). These functions indicate the accuracy of (4.11a,b), and

\[
\hat{\gamma}_{X_{i},F_{i}(-1:n)}(\omega) = \frac{\left| H_{i1}^{[e1]}(\omega) \right|^2}{\left| H_{i1}^{[e2]}(\omega) \right|^2} \quad i = 1, 2
\]  

(4.16)

where the superscripts \([e1]\) and \([e2]\) signify "H_{e1}" and "H_{e2}" estimates, respectively. Note, equation (4.16) is an approximation since \(F(\omega) = [F_{i}(\omega) 0]^{T}\) for Examples I and II, and therefore \(G_{X_{i}F_{i}(-1:n)}(\omega)\) of (4.11b) is a column vector of dimension 2. Accordingly, this leads to a pseudo-inverse to solve for

\[
\begin{bmatrix} H_{e2}^{[e2]}(\omega) \end{bmatrix}^{T} = \begin{bmatrix} H_{11}^{[e1]}(\omega) & H_{12}^{[e2]}(\omega) \end{bmatrix} \begin{bmatrix} H_{11}^{[e2]}(\omega) \end{bmatrix}.
\]
4.4. PRELIMINARY RESULTS

Consider the simulated input/output data of Example 1 in the absence of uncorrelated noise and assume that the correct Model \( A_1 \) of the elastic force \( f_{12}^x(t) \) has been chosen

\[
\text{Model } A_1: \quad p_{12}^x(t) = \dot{k}_1 \Delta x_{12}(t) + \dot{a}_3 y_3(\Delta x_{12}(t)) = \dot{k}_1 \Delta x_{12}(t) + \dot{a}_3 \Delta x_{12}(t)^3
\] (4.17)

From the Temporal Method, the estimated mass, damping and stiffness coefficients and the coefficient \( \dot{a}_3 \) of the nonlinear function \( y_3(\Delta x_{12}(t)) \) are listed in Table 4.3 along with the percent error of the estimated to true values. As can be seen, the method accurately estimates the system properties. For the sake of comparison, since modal parameters are estimated from (4.11a) or (4.11b) for the Spectral Method, Table 4.4 lists modal parameters which are calculated from the mass, damping and stiffness coefficients of Table 4.3.

Next, the Spectral Method is employed to the data using Model \( A_1 \) of (4.17). A sample conditioned "\( H_{c2} \)" estimate from equation (4.11b), designated as \( \hat{\tilde{H}}_{11}^{[c2]}(\omega) \), is illustrated in Figure 4.4. Also shown are \( \hat{\tilde{H}}_{11}^{[1]}(\omega) \) and \( H_{11}(\omega) \). The magnitude and phase are both well estimated by \( \hat{\tilde{H}}_{11}^{[c2]}(\omega) \). The underlying linear system's modal parameters are calculated from the "\( H_{c2} \)" spectra using a modal parameter estimation software [4.18]. In Table 4.5, results and errors between the estimated and true values are listed. As shown, this method has successfully recovered the true linear parameters. Illustrated in Figure 4.5, is the complex valued \( \dot{a}_3(\omega) \) whose trend is constant with frequency. The spectral mean \( \langle \dot{a}_3(\omega) \rangle_{\omega} \), listed in Table 4.5 suggests some error in the estimate since the true coefficient \( \beta_3 \) is a real valued constant. However, the real part of the spectral mean is three orders of magnitude greater than the imaginary part and the percent error between \( \left| \langle \dot{a}_3(\omega) \rangle_{\omega} \right| \) and \( \beta_3 \) is less than 1%. For the sake of brevity, the phase spectra of \( H(\omega) \) and \( \hat{\tilde{H}}(\omega) \) as well as the spectra of the estimated coefficients \( \dot{a}_j(\omega) \) will be excluded from the remaining results. Unless otherwise mentioned, assume that the results are similar to those presented in Figures 4.4 and 4.5.
<table>
<thead>
<tr>
<th>Model, Noise level</th>
<th>i</th>
<th>$\hat{m}_i$ (kg)</th>
<th>$\hat{c}_i$ (N·s/m)</th>
<th>$\hat{k}_i$ (kN/m)</th>
<th>Estimated coefficients of $y_i(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{ii}$ None</td>
<td>1</td>
<td>1.0 (0.0)</td>
<td>10.0 (0.0)</td>
<td>100.0 (0.0)</td>
<td>$\hat{a}_3(\omega) = 500.0 \text{ MN/m}^2 (0.0)$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.0 (0.0)</td>
<td>10.0 (0.0)</td>
<td>100.0 (0.0)</td>
<td></td>
</tr>
<tr>
<td>$A_{ii}$ Moderate</td>
<td>1</td>
<td>1.0 (0.0)</td>
<td>10.0 (0.0)</td>
<td>100.2 (0.2)</td>
<td>$\hat{a}_3(\omega) = 497.4 \text{ MN/m}^2 (0.5)$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1.0 (0.0)</td>
<td>10.0 (0.0)</td>
<td>99.7 (0.3)</td>
<td></td>
</tr>
<tr>
<td>$A_{ii}$ High</td>
<td>1</td>
<td>0.8 (20.0)</td>
<td>13.8 (38.0)</td>
<td>118.4 (18.4)</td>
<td>$\hat{a}_3(\omega) = 318.1 \text{ MN/m}^2 (36.4)$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.7 (30.0)</td>
<td>5.9 (41.0)</td>
<td>75.6 (24.4)</td>
<td></td>
</tr>
<tr>
<td>$B_{ii}$ None</td>
<td>1</td>
<td>0.9 (10.0)</td>
<td>10.9 (9.0)</td>
<td>257.7 (157.7)</td>
<td>$\hat{a}_5(\omega) = 252.7 \text{ GN/m}^2 (157.7)$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.9 (10.0)</td>
<td>8.6 (14.0)</td>
<td>93.7 (6.3)</td>
<td></td>
</tr>
</tbody>
</table>

$\% \text{ error} = (\frac{\text{estimated}-\text{true}}{\text{true}})\cdot 100 \text{ given in parenthesis.}$

Table 4.3. Estimated physical properties of Example 1 by the Temporal Method.
<table>
<thead>
<tr>
<th>Model, noise level</th>
<th>mode, ( r )</th>
<th>( \hat{\omega}_r ) (Hz)</th>
<th>( \hat{\zeta}_r ) (%)</th>
<th>( \hat{\phi}_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_t ), None</td>
<td>1</td>
<td>31.1 (0.0)</td>
<td>1.0 (0.0)</td>
<td>{1.0, 0.6}</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>81.4 (0.0)</td>
<td>2.6 (0.0)</td>
<td>{-0.6, 1.0}</td>
</tr>
<tr>
<td>Moderate</td>
<td>1</td>
<td>31.1 (0.0)</td>
<td>1.0 (0.0)</td>
<td>{1.0, 0.6}</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>81.6 (0.2)</td>
<td>2.6 (0.0)</td>
<td>{-0.6, 1.0}</td>
</tr>
<tr>
<td>( A_t ), High</td>
<td>1</td>
<td>32.0 (2.9)</td>
<td>0.9 (10.0)</td>
<td>{1.0, 0.7}</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>96.2 (18.2)</td>
<td>3.3 (26.9)</td>
<td>{-0.7, 1.0}</td>
</tr>
<tr>
<td>( B_t ), None</td>
<td>1</td>
<td>33.9 (9.0)</td>
<td>0.9 (10.0)</td>
<td>{1.0, 0.8}</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>123.8 (52.1)</td>
<td>1.8 (30.8)</td>
<td>{-0.8, 1.0}</td>
</tr>
</tbody>
</table>

\( \% \text{ error} = \left( \frac{\text{estimated} - \text{true}}{\text{true}} \right) \times 100 \) given in parenthesis.

Table 4.4. Estimated modal properties of Example I by the Temporal Method.
Figure 4.4. Dynamic compliance spectra of Example 1 using Model A, for $H_{12}$ estimate. (a) Magnitude of $H_{11}(\omega)$. (b) Phase of $H_{11}(\omega)$. Key: — $H_{11}(\omega)$, $\cdot \cdot \cdot H_{11}(\omega)$, $\circ \circ \circ H_{11}(\omega)$. 
Table 4.5. Estimated properties of Example I by the Spectral Method.

<table>
<thead>
<tr>
<th>Model, Noise level</th>
<th>mode, ( r )</th>
<th>( \omega_r ) (Hz)</th>
<th>( \zeta_r ) (%)</th>
<th>( \phi_r )</th>
<th>Spectral mean of estimated coefficients of ( y_i(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 ), None</td>
<td>1</td>
<td>31.2 (0.3)</td>
<td>1.0 (0.0)</td>
<td>1.0, 0.6</td>
<td>( \langle \hat{a}_3(\omega) \rangle_w = 501.4 + 0.2i \text{MN/m}^2 ) (0.3)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>81.6 (0.2)</td>
<td>2.6 (0.0)</td>
<td>-0.6, 1.0</td>
<td></td>
</tr>
<tr>
<td>( A_1 ), Moderate</td>
<td>1</td>
<td>31.1 (0.0)</td>
<td>1.1 (10.0)</td>
<td>1.0, 0.6</td>
<td>( \langle \hat{a}_3(\omega) \rangle_w = 496.0 - 8.4i \text{MN/m}^2 ) (1.9)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>82.1 (0.9)</td>
<td>1.8 (30.8)</td>
<td>-0.6, 1.0</td>
<td></td>
</tr>
<tr>
<td>( A_1 ), High</td>
<td>1</td>
<td>32.5 (4.5)</td>
<td>0.1 (90.0)</td>
<td>1.0, 0.8</td>
<td>( \langle \hat{a}_3(\omega) \rangle_w = 160.8 + 59.7i \text{MN/m}^2 ) (68.9)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>--- (---)</td>
<td>--- (---)</td>
<td>---</td>
<td></td>
</tr>
<tr>
<td>( B_1 ), None</td>
<td>1</td>
<td>34.0 (9.3)</td>
<td>1.2 (20.0)</td>
<td>1.0, 0.8</td>
<td>( \langle \hat{a}_5(\omega) \rangle_w = 251.9 + 0.3i \text{GN/m}^2 ) (---)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>118.5 (45.6)</td>
<td>1.6 (38.5)</td>
<td>-0.8, 1.0</td>
<td></td>
</tr>
</tbody>
</table>

\( \% \text{error} = (|\text{estimated-true}|/\text{true}) \times 100 \) given in parenthesis.
Figure 4.5. Spectrum of estimated coefficient $\hat{a}_3(\omega)$ of Model $A_1$. (a) Re[$\hat{a}_1(\omega)$]. (b) Im[$\hat{a}_1(\omega)$]. Key: $\hat{a}_3(\omega)$, $\circ \circ \circ$ true coefficient $\beta_3$. 

131
The cumulative coherence function $\hat{\gamma}^2_{M1}(\omega)$ illustrated in Figure 4.6(a) indicates the overall certainty of the assumed mathematical model. Aside from a minor drop in coherence at the frequency of the first resonance, $\hat{\gamma}^2_{M1}(\omega)$ is unity indicating that an accurate model has been chosen. This is expected since uncorrelated noise is absent and $p_{12}^2(t) = p_{12}^2(t)$ for Model A. The coherence functions $\hat{\gamma}^2_{YF}(\omega)$ and $\hat{\gamma}^2_{X,F_1(-1)}(\omega)$ shown in Figure 4.6(b) illustrate the contribution of the separate paths, i.e. $\hat{\gamma}^2_{YF}(\omega)$ shows the contribution of $Y_j(\omega)$ since $y_j(\Delta x_{12}(t))$ is the only nonlinear function included in this model, and $\hat{\gamma}^2_{X,F_1(-1)}(\omega)$ shows contribution of linear component $X_{11,-1}(\omega)$ of the response $X_1(\omega)$. The coherence function $\hat{\gamma}^2_{X,F_1(-1)}(\omega)$ peaks in the 50 Hz frequency range and at the upper end of the spectrum, while the coherence function $\hat{\gamma}^2_{YF}(\omega)$ of the cubic nonlinear function $Y_1(\omega)$ does the opposite. This may be explained as follows. First, $\hat{\gamma}^2_{YF}(\omega)$ is high at frequencies corresponding to the peaks of $\hat{H}^{[\text{c2}]}_{11}(\omega)$ of Figure 4.4(a). This suggests that the cubic nonlinearity dominates at these frequencies which is understandable since dominant nonlinear behavior is expected at frequencies where large amplitudes occur.

In contrast, the linear path $X_{11,-1}(\omega)$ is not as apparent at these frequencies as illustrated by $\hat{\gamma}^2_{X,F_1(-1)}(\omega)$. However, $\hat{\gamma}^2_{X,F_1(-1)}(\omega)$ increases in the 50 Hz range as mentioned. This is also understood by examining Figure 4.4(a). In the 50 Hz range the two estimates $\hat{H}^{[\text{c1}]}_{11}(\omega)$ and $\hat{H}^{[\text{c2}]}_{11}(\omega)$ coincide with $H_{11}(\omega)$. Therefore, $\hat{H}^{[\text{c1}]}_{11}(\omega)$ is accurate in this frequency range. However, since the conventional "$H_i$" estimator does not account for nonlinearities, the response in this range must be close to the linear behavior. Hence the reason for an increase in $\hat{\gamma}^2_{X,F_1(-1)}(\omega)$ values in this range. Although both $\hat{\gamma}^2_{M1}(\omega)$ and $\hat{\gamma}^2_{M2}(\omega)$ are examined for each model, $\hat{\gamma}^2_{M2}(\omega)$ is excluded since both are similar. The ordinary coherence $\hat{\gamma}^2_{X,F_1(-1)}(\omega)$ which is similar to $\hat{\gamma}^2_{YF}(\omega)$ is also excluded for the same reason. The results of this section serve as the starting point as other complications are included in the identification scheme.
Figure 4.6. Coherence functions of Model A. (a) Cumulative coherence function $\hat{\gamma}^2_{M_1}(\omega)$. (b) Coherence functions $\hat{\gamma}^2_{X_{i(t-1)}}(\omega)$ and $\hat{\gamma}^2_{Y_{F_1}}(\omega)$. Key: — $\hat{\gamma}^2_{X_{i(t-1)}}(\omega)$, --- $\hat{\gamma}^2_{Y_{F_1}}(\omega)$. 
4.5. INCLUSION OF UNCORRELATED NOISE

4.5.1. FORMULATION

In the presence of uncorrelated noise, the response and excitation vectors are modified as

\[ x(t) = \tilde{x}(t) + n_x(t), \quad f(t) = \tilde{f}(t) + n_f(t) \]  

(4.18a,b)

where \( \tilde{x}(t) \) is the contaminated response vector of the true response vector \( x(t) \) by uncorrelated noise vector \( n_x(t) \). Likewise, \( \tilde{f}(t) \) is the contaminated excitation vector of the true excitation vector \( f(t) \) by uncorrelated noise vector \( n_f(t) \). The Fourier transform \( \mathcal{F} \) of (4.18a,b) leads to

\[ \tilde{X}(\omega) = X(\omega) + N_X(\omega), \quad \tilde{F}(\omega) = F(\omega) + N_F(\omega) \]  

(4.19a,b)

and since \( N_X(\omega) \) and \( N_F(\omega) \) are uncorrelated noise spectra

\[ G_{NNx}(\omega) = \frac{1}{2} E[X(\omega)^* \cdot N_X(\omega)^T] = 0, \quad G_{NNf}(\omega) = \frac{1}{2} E[F(\omega)^* \cdot N_F(\omega)^T] = 0 \]  

(4.20a-e)

where \( 0 \) is a square null matrix of dimension 2. However,

\[ G_{N>Nx}(\omega) = \frac{1}{2} E[N_X(\omega)^* \cdot N_X(\omega)^T] \neq 0, \quad G_{N>Nf}(\omega) = \frac{1}{2} E[N_F(\omega)^* \cdot N_F(\omega)^T] \neq 0 \]  

(4.21a,b)

where \( G_{N>Nx}(\omega) \) and \( G_{N>Nf}(\omega) \) are square diagonal matrices of dimension 2. When conventional frequency response estimators are employed, noise only exists in the auto-power spectra of \( \tilde{X}(\omega) \) and \( \tilde{F}(\omega) \). However, when calculating the conditioned estimates of (4.11a,b), noise exists in additional spectra, as follows. The noise \( n_x(t) \) contaminates the \( n \) unique nonlinear functions \( y_j(\Delta x_{12}(t)) \) since they are calculated directly from the contaminated response \( \tilde{x}(t) \), i.e.

\[ \varepsilon_j = \tilde{y}_j(\Delta \tilde{x}_{12}(t)) \]

\[ \Delta \tilde{x}_{12}(t) = \tilde{x}_1(t) - \tilde{x}_2(t) = x_1(t) + n_{x_1}(t) - x_2(t) - n_{x_2}(t) = \Delta x_{12}(t) + \Delta n_x(t) \]  

(4.22a-c)

Since the functions \( \varepsilon_j(\Delta \tilde{x}_{12}(t)) \) are of the polynomial form, they can be written as
\[
\tilde{y}_j(\Delta x_{12}(t)) = y_j(\Delta x_{12}(t)) + n_{y_j}(t),
\]
\[
y_j(\Delta x_{12}(t)) = y_j(x(t)),
\]
\[
n_{y_j}(t) = n_{y_j}(x(t), n_x(t))
\]

(4.23a-c)

where \( \tilde{y}_j(\Delta x_{12}(t)) \) is grouped into a term which is only a function of \( x(t) \) without noise and a term which is a function of \( x(t) \) and \( n_x(t) \). To assimilate (4.23a-c) consider a cubic function \( y_3(\Delta x_{12}(t)) = \Delta x_{12}(t)^3 \)
calculated from contaminated response \( \tilde{x}(t) \)

\[
\tilde{y}_3(\Delta x_{12}(t)) = (\Delta x_{12}(t))^3 = (\Delta x_{12}(t) + \Delta n_x(t))^3
\]
\[
= \Delta x_{12}(t)^3 + 3\Delta x_{12}(t)^2 \Delta n_x(t) + 3\Delta x_{12}(t)\Delta n_x(t)^2 + \Delta n_x(t)^3
\]
\[
y_3(\Delta x_{12}(t)) + n_{y_3}(t),
\]
\[
n_{y_3}(t) = 3\Delta x_{12}(t)^2 \Delta n_x(t) + 3\Delta x_{12}(t)\Delta n_x(t)^2 + \Delta n_x(t)^3
\]

(4.24a,b)

As shown, \( \tilde{y}_3(\Delta x_{12}(t)) \) is arranged into a term which is only a function of \( x(t) \) without noise and a term which is a function of \( x(t) \) and \( n_x(t) \). This procedure can be extended to \( \tilde{y}_j(\Delta x_{12}(t)) \) of any polynomial order \( j \). The Fourier transform \( F[\cdot] \) of (4.23a) leads to

\[
\tilde{Y}_j(\omega) = Y_j(\omega) + N_j(\omega), \quad Y_j(\omega) = F[y_j(\Delta x_{12}(t))], \quad N_j(\omega) = F[n_{y_j}(t)]
\]

(4.25a-c)

and as a result of (4.23b) and (4.23c)

\[
G_{X_j}(\omega) = \frac{1}{\pi} E[X(\omega)^* \cdot Y_j(\omega)] \neq \{0\}, \quad G_{N_jX_j}(\omega) = \frac{1}{\pi} E[X(\omega)^* \cdot N_j(\omega)] \neq \{0\},
\]
\[
G_{N_jN_j}(\omega) = \frac{1}{\pi} E[N_j(\omega)^* \cdot N_j(\omega)] \neq 0, \quad G_{Y_jN_j}(\omega) = \frac{1}{\pi} E[Y_j(\omega)^* \cdot N_j(\omega)] = 0
\]

(4.26a-f)

where \( \{0\} \) are null column vectors of dimension 2. Note that (4.26a-f) are true in a general sense; however, depending on the form of \( y_j(\Delta x_{12}(t)) \), some of the spectral density functions may be zero.

Therefore, (4.26a-f) give the worst possible case. Also, in general components of \( \tilde{Y}_j(\omega) \) and \( \tilde{Y}_i(\omega) \) may be correlated depending on the form of \( \tilde{y}_j(\Delta x_{12}(t)) \) and \( \tilde{y}_i(\Delta x_{12}(t)) \),

\[
G_{ij}(\omega) = \frac{1}{\pi} E[Y_i(\omega)^* \cdot Y_j(\omega)] \neq 0, \quad G_{IN_i}(\omega) = \frac{1}{\pi} E[Y_i(\omega)^* \cdot N_i(\omega)] \neq 0;
\]
\[
G_{N_iN_j}(\omega) = \frac{1}{\pi} E[N_i(\omega)^* \cdot N_j(\omega)] \neq 0, \quad G_{N_jN_j}(\omega) = \frac{1}{\pi} E[N_j(\omega)^* \cdot N_j(\omega)] = 0
\]

(4.27a-d)

Consequently, from (4.20a-e), (4.21a,b), (4.26a-f) and (4.27a-d),
\[ \tilde{G}_{XX}(\omega) = G_{XX}(\omega) + \tilde{G}_{N_xN_x}(\omega) \]
\[ \tilde{G}_{XF}(\omega) = G_{XF}(\omega) \]
\[ \tilde{G}_{Xj}(\omega) = G_{Xj}(\omega) + \tilde{G}_{Xj}(\omega) + G_{N_xN_j}(\omega) \]
\[ \tilde{G}_{FF}(\omega) = G_{FF}(\omega) + \tilde{G}_{N_xN_x}(\omega) \]
\[ \tilde{G}_{Fj}(\omega) = G_{Fj}(\omega) \]
\[ \tilde{G}_{ij}(\omega) = G_{ij}(\omega) + 2 \text{Re}[G_{ij}(\omega)] + \tilde{G}_{N_xN_j}(\omega) \]
\[ \tilde{G}_{ij}(\omega) = G_{ij}(\omega) + G_{ij}(\omega) + G_{N_xN_j}(\omega) + G_{N_xN_j}(\omega) \]

As given by (4.28a-g), noise corrupts most of these unconditioned spectral functions and matrices. As a result, noise also contaminates conditioned spectral density functions used to calculate “\(H_{el}\)” and “\(H_{el}\)” estimates of (4.11a,b) and the coherence functions of section 4.3 since conditioned spectra are calculated from 4.28(a-g). Additional research is needed in order to minimize the presence of noise in (4.28a-g) by considering a reference approach [4.19] which yields an improved estimate of auto-power spectral density functions.

For the Temporal Method, similar problems occur when uncorrelated noise is present since the nonlinear functions \( \tilde{y}(JAx_1(t)) \) are calculated directly from the contaminated response \( \tilde{x}(t) \). Increasing the length of time series utilized by the Temporal Method minimizes contamination of uncorrelated noise since the method is based on a least squares solution. Alternatively, time domain averaging operations may be performed or cross-correlation techniques may be adapted. This is suggested as a topic for future research.

4.5.2. RESULTS

Both methods are evaluated next by adding “white” uncorrelated noise to the simulated excitation and response data. The auto-power spectra of \( X_1(\omega) \), \( F_1(\omega) \), \( N_{X_1}(\omega) \) and \( N_{F_1}(\omega) \) are shown in Figure 4.7; those of \( X_2(\omega) \) and \( N_{X_2}(\omega) \) are not shown since they are similar to that of \( X_1(\omega) \) and \( N_{X_1}(\omega) \). These noise levels are comparable to those used by Mohammad et. al. [4.10] and Yang and Ibrahim [4.11] in their studies based on the Temporal Method. The correct nonlinear form is assumed for this study, i.e.
Figure 4.7. Auto-power spectra of noise-free data, moderate and high noise levels. (a) Response auto-power spectra. (b) Excitation auto-power spectra. Key: — noise-free data, — high noise level, - - - moderate noise level.
equation (4.17). Shown in Figure 4.8 are sample dynamic compliance functions $\hat{H}^{(c2)}_{21}(\omega)$, $\hat{H}^{(c1)}_{11}(\omega)$ and $H_{11}(\omega)$. For the moderate noise case the second resonant peak of $\hat{H}^{(c2)}_{21}(\omega)$ reveals some noise corruption as shown in Figure 4.8(a). Estimated parameters are listed in Tables 4.3-4.5 based on both methods. As shown, not much error has occurred in these estimates with the exception of $\zeta_2$ by the Spectral Method. However, for the high noise case, estimates are rather poor as shown in Figure 4.8(b) and listed in Tables 4.3-4.5. In fact, since the second mode of the $H^{(c2)}_2$ estimates are undetectable, no parameters can be determined for this mode by the Spectral Method. Illustrated in Figure 4.9 are $\hat{\gamma}^2_{M1}(\omega)$ for the two noise levels. The large amount of noise responsible for the inaccurate identification in the high noise case is indicated by $\hat{\gamma}^2_{M1}(\omega)$ of Figure 4.9(b). This suggests that the coherence techniques can in fact aid in determining when noise levels are intolerable.

To this point no results for $H^{(c2)}_2$ estimates based on equation (4.11a) or partial coherence functions as defined by equation (4.15) have been shown. These estimates can also be investigated; however, they may be corrupted by the numerical errors introduced by simulating the lightly damped systems, i.e. Examples I and II. To minimize these errors, the damping coefficients of Example I are increased by a factor of 10 to form a new Example III. Applying the Spectral Method with Model $A_{III}$ which takes the same form as $A_1$ of (4.17), conditioned $H^{(c1)}_1$ and $H^{(c2)}_2$ estimates are calculated in the absence of uncorrelated noise. Shown in Figures 4.10 and 4.11 are $\hat{H}^{(c2)}_{11}(\omega)$, $\hat{H}^{(c1)}_{11}(\omega)$, $\hat{H}^{(c1)}_{11}(\omega)$, $H_{11}(\omega)$ and $\hat{\gamma}^2_{X,F_{1,-10n}}(\omega)$. Some numerical errors still exist since $\hat{H}^{(c2)}_{11}(\omega) = \hat{H}^{(c1)}_{11}(\omega)$ and $\hat{\gamma}^2_{X,F_{1,-10n}}(\omega)$ drops below unity at the first resonance peak. However, in the absence of noise both $\hat{H}^{(c2)}_{11}(\omega)$ and $\hat{H}^{(c1)}_{11}(\omega)$ are accurate estimates.

The conditioned $H^{(c1)}_1$ and $H^{(c2)}_2$ estimates, as proposed in equation (4.11a,b), are analogous to the conventional $H^{(c1)}_1$ and $H^{(c2)}_2$ estimates currently used for linear systems [4.2, 4.3]. Therefore, intuition may lead one to expect the $H^{(c2)}_2$ estimate to perform better than the $H^{(c1)}_1$ estimate in the presence of uncorrelated noise only in the excitation and none in the response. Likewise, one would expect the $H^{(c1)}_1$ estimate to perform better than the $H^{(c2)}_2$ estimate in the presence of uncorrelated noise only in the response.
Figure 4.8. Magnitude of dynamic compliance spectra of Example 1 using Model $A_i$ for "$H_{22}$" estimate.
(a) Moderate noise case. (b) High noise case. Key: $\tilde{H}_{11}^{[2]1}(\omega)$, $\tilde{H}_{11}^{[1]1}(\omega)$, $H_{11}(\omega)$. 

139
Figure 4.9. Cumulative coherence functions $\hat{\gamma}_m^2(\omega)$ of Model A. (a) Moderate noise case. (b) High noise case.
Figure 4.10. Magnitude of dynamic compliance spectra of Example III using Model A_{III} for "H_{2s}" estimate. (a) Key: \( \hat{H}_{11}^{[2]}(\omega), \quad \hat{H}_{11}^{[1]}(\omega), \quad \circ \circ \circ \ H_{11}(\omega) \). (b) Key: \( \hat{H}_{11}^{[2]}(\omega), \quad \hat{H}_{11}^{[1]}(\omega), \quad \circ \circ \circ \ H_{11}(\omega) \).
Figure 4.11. Partial coherence function $\hat{\gamma}_{X_iX_j}^2(\omega)$ of Model AIII.
and none in the excitation. To illustrate this, \( \hat{H}_{11}^{[c2]}(\omega) \), \( \hat{H}_{11}^{[c1]}(\omega) \) and \( H_{11}(\omega) \) are illustrated in Figure 4.12(a) for a high level of uncorrelated noise in the excitation only. As expected, although the results are still "noisy", \( \hat{H}_{11}^{[c2]}(\omega) \) is the better estimate. Figure 4.12(b) illustrates the same estimates for high levels of uncorrelated noise in the responses only. Unlike what was expected, the "H_{e2}" method is still a slightly better estimate. This outcome may be a result of the conditioning required to calculate the "H_{e1}" and "H_{e2}" estimates. However, this issue requires additional examination in future experimental studies.
Figure 4.12. Magnitude of dynamic compliance spectra of Example III using Model \( A_{\text{III}} \) for Spectral Method calculations. (a) High noise level in excitation only. (b) High noise level in response only. Key: \( \cdot \cdot \cdot \cdot H_{11}^{[c]}(\omega) \), \( \cdot \cdot \cdot \cdot H_{11}^{[c]}(\omega) \), \( \cdot \cdot \cdot \cdot \cdot H_{11}(\omega) \).
4.6. EXAMINATION OF ALTERNATIVE NONLINEAR MODELS

The following results illustrate the consequences of identification when alternate formulae are chosen to describe the elastic behavior of the nonlinear spring element, i.e. either correct \( p^e_{12}(t) = f^e_{12}(t) \), incorrect \( p^e_{12}(t) \neq f^e_{12}(t) \) or approximate \( p^e_{12}(t) \approx f^e_{12}(t) \) mathematical models will be employed. Due to the heavy damping of the second mode, Example III of section 4.5.2 will not be evaluated in the following results since it is difficult to determine how well the second mode is estimated by the Spectral Method. Therefore, only Examples I and II will be investigated. As a consequence, since numerical errors resulting from the simulations of Examples I and II corrupt the "H_c1" estimates of equation (4.11a) and partial coherence functions of equation (4.15), only "H_c2" estimates of equation (4.11b), \( \hat{\gamma}^2_{M1}(\omega) \) and \( \hat{\gamma}^2_{YR}(\omega) \) of equation (4.14a,b) will be shown. Finally, since the effects of noise on both Temporal and Spectral Methods have already been examined, data for the following studies are noise free for the sake of maintaining a clear focus on results with alternative models.

4.6.1. IDENTIFICATION OF EXAMPLE I

Recall, in section 4.4 where both Temporal and Spectral Methods successfully identify Example I using Model A, of equation (4.17) which correctly models \( f^e_{12}(t) \). Now assume that the form of the nonlinearity \( f^e_{12}(t) \) is incorrectly assumed to be modeled as a linear and fifth order polynomial

\[
\text{Model B}_1: \quad p^e_{12}(t) = k_1 \Delta x_{12}(t) + a_5 \Delta x_{12}(t) = k_1 \Delta x_{12}(t) + a_5 \Delta x_{12}(t)
\]

The procedures of section 4.4 are again carried through here for both methods and results are given in Tables 4.3-4.5 and in Figure 4.13(a). As shown, the estimates are rather poor. However, without knowing the true parameters, one may not be able to conclude that Model \( B_1 \) is inaccurate. This acknowledges the importance of diagnostic tools to determine the validity of the models, such as the cumulative coherence function \( \hat{\gamma}^2_{M1}(\omega) \) of Figure 4.13(b). As \( \hat{\gamma}^2_{M1}(\omega) \) indicates, \( B_1 \) is an inaccurate model above 70 Hz.

However, \( \hat{\gamma}^2_{YR}(\omega) \) is near unity elsewhere. This may lead one to believe \( B_1 \) is accurate below 70 Hz, when
Figure 4.13. Spectra for Example 1 using Model B, for Spectral Method calculations. (a) Magnitude of $H_{11}(\omega)$. Key: $-\hat{H}_{11}^{[21]}(\omega)$, $-\hat{H}_{11}^{[11]}(\omega)$, $0\ 0\ 0\ H_{11}(\omega)$. (b) Cumulative coherence function $\hat{\gamma}^2_{M1}(\omega)$. 

146
if fact the first mode of $\hat{H}_{31}^{[c2]}(\omega)$ is shifted to a higher frequency than the true natural frequency. Therefore, in order to prevent this misinterpretation and to ensure that the best possible model is chosen, cumulative coherence functions $\hat{\gamma}_{\text{MI}}^{2}(\omega)$ of several models should be compared. Doing so with $\hat{\gamma}_{\text{MI}}^{2}(\omega)$ of Models A$_1$ and B$_1$ indicates that A$_1$ is more accurate.

Observe that both Temporal and Spectral Methods give similar results, suggesting that erroneous estimates resulting from an incorrect nonlinear model are not method dependent. Subsequently, results to follow will include only those from the Spectral Method, but any dramatic differences between estimates obtained from the two methods will be mentioned.

Next consider Model C$_1$ which assumes that $f_{12}^c(t)$ contains a quadratic term

$$\text{Model C}_1: \quad p_{12}^c(t) = k_1\Delta x_{12}(t) + a_2y_2(\Delta x_{12}(t)) = k_1\Delta x_{12}(t) + a_2\Delta x_{12}^2(t) \quad (4.30)$$

Physically, this type of nonlinearity represents a spring with hardening stiffness in tension and softening stiffness in compression when $a_2 > 0$, and vice versa when $a_2 < 0$. Estimates $\hat{H}_{11}^{[1]}(\omega)$, $\hat{H}_{11}^{[c2]}(\omega)$ and true $H_{11}(\omega)$ are shown in Figure 4.14(a). The two estimates are similar but deviate from the true $H_{11}(\omega)$ suggesting that Model C$_1$ does not improve the identification over a similar but linear model. The cumulative coherence $\hat{\gamma}_{\text{MI}}^{2}(\omega)$ exceeds 0.85 up to approximately 70 Hz then it drops off as shown in Figure 4.14(b). Comparing $\hat{\gamma}_{\text{MI}}^{2}(\omega)$ of Models A$_1$, B$_1$ and C$_1$, indicates that C$_1$ is the least accurate model chosen thus far, and the coherence function $\hat{\gamma}_{\text{VE}}^{2}(\omega)$ also shown in Figure 4.14(b) is poor at all frequencies, indicating that the quadratic term adds zero contribution to the model. Hence the reason why $\hat{H}_{11}^{[1]}(\omega)$ and $\hat{H}_{11}^{[c2]}(\omega)$ are similar. However, in the small band between 45-55 Hz the cumulative coherence is close to unity. This is due to the fact that $\hat{H}_{11}^{[c2]}(\omega)$ and $H_{11}(\omega)$ coincide in this range as shown in Figure 4.14(a). Therefore, even such an incorrect model yields an accurate representation in this frequency range.

The next Model D$_1$ includes both a quadratic and cubic term

$$\text{Model D}_1: \quad p_{12}^c(t) = k_1\Delta x_{12}(t) + a_2y_2(\Delta x_{12}(t)) + a_3y_3(\Delta x_{12}(t)) = k_1\Delta x_{12}(t) + a_2\Delta x_{12}^2(t) + a_3\Delta x_{12}^3(t) \quad (4.31)$$
Figure 4.14. Spectra for Example 1 using Model C, for Spectral Method calculations. (a) Magnitude of $H_{11}(\omega)$. Key: — $\hat{H}_{11}^{(c2)}(\omega)$, --- $\hat{H}_{11}^{(l)}(\omega)$, o o o $H_{11}(\omega)$. (b) Coherence functions $\hat{\gamma}_M^2(\omega)$ and $\hat{\gamma}_{Y_F}^2(\omega)$. Key: — $\hat{\gamma}_M^2(\omega)$, --- $\hat{\gamma}_{Y_F}^2(\omega)$. 
Estimates $\hat{H}_{11}(\omega)$ and $\hat{H}_{11}^{[2]}(\omega)$ are illustrated in Figure 4.15(a). As shown, $\hat{H}_{11}^{[2]}(\omega)$ closely matches true $H_{11}(\omega)$. Indication that $D_1$ is an accurate model is given by $\hat{\gamma}^2_{M1}(\omega)$ of Figure 4.15(b). Since both $D_1$ and $A_1$ are accurate, this illustrates that the identification process may not yield an unique model. The coherence function $\hat{\gamma}^2_{RF}(\omega)$ also shown in Figure 4.15(b) is similar to $\hat{\gamma}^2_{RF}(\omega)$ of Model $A_1$, Figure 4.6(b), suggesting that the two models are similar. The spectral means of $\hat{a}_2(\omega)$ and $\hat{a}_3(\omega)$ are $\langle \hat{a}_2(\omega) \rangle_\omega = 15.57 - 11.22i \text{kN/m}^2$, $\langle \hat{a}_3(\omega) \rangle_\omega = 501.28 + 0.47i \text{MN/m}^2$. The spectral mean $\langle \hat{a}_3(\omega) \rangle_\omega$ of the cubic nonlinearity is an accurate estimate with an imaginary part three orders of magnitude less than the real part. However, the spectral mean $\langle \hat{a}_2(\omega) \rangle_\omega$ of the quadratic coefficient has an imaginary part of the same magnitude as the real part. This may serve as an indication that the quadratic term is not present in $f_{12}(t)$.

The cumulative study of Models $A_1$, $B_1$, $C_1$ and $D_1$ suggest a possible strategy one may employ for determining an accurate model for describing Example I. Although it is a trial-and-error type method and realistic systems may be more difficult to evaluate, coherence functions as illustrated here can provide significant clues and insight into the validity of models chosen. A final note concerning the presence of uncorrelated noise. As shown in this section, the cumulative coherence functions indicate when an inaccurate model is being employed by assuming values less than unity. And, as shown in section 4.5, the same is true when uncorrelated measurement noise is present in the excitation and response data. So, it is not possible to differentiate between the two errors when both are simultaneously present. Therefore, it is important to ensure, whenever possible, that measurement signals are noise free, or since this is not realistic, that noise levels are kept as low as possible.

4.6.2. IDENTIFICATION OF EXAMPLE II

Models given in Table 4.6 are used next to identify Example II by the Spectral Method. Model $A_{II}$ is composed of the correct terms to model $f_{12}(t)$: and, as listed in Table 4.7, the resulting estimates are accurate. Models $B_{II}$ and $C_{II}$ contain only a fifth order and only a quadratic term, respectively. These models were chosen to illustrate the consequences of leaving out a term from $p_{12}(t)$ to describe $f_{12}(t)$. 

149
Figure 4.15. Spectra for Example 1 using Model D for Spectral Method calculations. (a) Magnitude of $H_{ii}(\omega)$. Key: $\hat{H}_{ii}^{[c1]}(\omega)$, $\hat{H}_{ii}^{[c1]}(\omega)$, $\hat{H}_{ii}^{[c1]}(\omega)$. (b) Coherence functions $\gamma_{M1}^{2}(\omega)$ and $\gamma_{V_F}^{2}(\omega)$. Key: $\gamma_{M1}^{2}(\omega)$, $\gamma_{V_F}^{2}(\omega)$.
Table 4.6. Models used in the estimation of Example II.

<table>
<thead>
<tr>
<th>Model</th>
<th>( p_{12}^e(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{12} )</td>
<td>( k_1 \Delta x_{12}(t) + a_2 y_2(\Delta x_{12}(t)) + a_3 y_3(\Delta x_{12}(t)) = k_1 \Delta x_{12}(t) + a_2 \Delta x_{12}^2(t) + a_3 \Delta x_{12}^3(t) )</td>
</tr>
<tr>
<td>( B_{12} )</td>
<td>( k_1 \Delta x_{12}(t) + a_2 y_2(\Delta x_{12}(t)) = k_1 \Delta x_{12}(t) + a_2 \Delta x_{12}^2(t) )</td>
</tr>
<tr>
<td>( C_{12} )</td>
<td>( k_1 \Delta x_{12}(t) + a_2 y_2(\Delta x_{12}(t)) = k_1 \Delta x_{12}(t) + a_2 \Delta x_{12}^2(t) )</td>
</tr>
<tr>
<td>( D_{12} )</td>
<td>( k_1 \Delta x_{12}(t) + a_2 y_2(\Delta x_{12}(t)) = k_1 \Delta x_{12}(t) + a_2 \Delta x_{12}^2(t) )</td>
</tr>
<tr>
<td>( E_{12} )</td>
<td>( k_1 \Delta x_{12}(t) + \sum_{j=2}^{5} a_j y_j(\Delta x_{12}(t)) = k_1 \Delta x_{12}(t) + \sum_{j=2}^{5} a_j \Delta x_{12}^j(t) )</td>
</tr>
<tr>
<td>( F_{12} )</td>
<td>( k_1 \Delta x_{12}(t) + \sum_{j=2}^{5} a_j y_j(\Delta x_{12}(t)) = k_1 \Delta x_{12}(t) + \sum_{j=2}^{5} a_j \Delta x_{12}^j(t) )</td>
</tr>
<tr>
<td>( G_{12} )</td>
<td>( k_1 \Delta x_{12}(t) + \sum_{j=2}^{5} a_j y_j(\Delta x_{12}(t)) = k_1 \Delta x_{12}(t) + \sum_{j=2}^{5} a_j \Delta x_{12}^j(t) )</td>
</tr>
<tr>
<td>( H_{12} )</td>
<td>( k_1 \Delta x_{12}(t) + \sum_{j=2}^{5} a_j y_j(\Delta x_{12}(t)) = k_1 \Delta x_{12}(t) + \sum_{j=2}^{5} a_j \Delta x_{12}^j(t) )</td>
</tr>
<tr>
<td>( I_{12} )</td>
<td>( k_1 \Delta x_{12}(t) + \sum_{j=2}^{5} a_j y_j(\Delta x_{12}(t)) = k_1 \Delta x_{12}(t) + \sum_{j=2}^{5} a_j \Delta x_{12}^j(t) )</td>
</tr>
<tr>
<td>Model</td>
<td>mode, r</td>
</tr>
<tr>
<td>-------</td>
<td>---------</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
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<tr>
<td>( B_1 )</td>
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<td>( C_1 )</td>
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<tr>
<td>( D_1 )</td>
<td>1</td>
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<td></td>
<td>2</td>
</tr>
</tbody>
</table>

% error = \((\text{estimated-true}/\text{true})\times100\) given in parenthesis.

Table 4.7. Estimated properties of Example II by the Spectral Method in the absence of noise.
As a result, neither model produces accurate estimates. However, comparison shows that inclusion of only
the fifth order term, Model B₁₁, yields a better estimate. Model D₁₁ represents \( f_{12}^5(t) \) by a cubic
nonlinearity. Modeling nonlinearities by a cubic polynomial is a common assumption [4.12-4.15, 4.20].
However, as shown for this system, the model results in poor estimates. In fact, the estimated damping for
the first mode is negative.

One possible strategy for modeling continuous nonlinearities is by polynomial expansions.
Models E₁₁ and F₁₁ represent \( f_{12}^5(t) \) by expansions including linear through fifth order and linear through
seventh order terms, respectively. As Table 4.8 shows, such models result in accurate estimates of the
modal parameters. Notice that the spectral means of the coefficients of the quadratic \( \hat{\alpha}_2(\omega) \) and fifth
order \( \hat{\alpha}_5(\omega) \) terms are well estimated with negligible imaginary parts, where as the other coefficients
have spectral means with imaginary parts comparable to their real parts. This may serve as an indicator of
which polynomial orders are truly present.

Models E₁₁ and F₁₁ both contain the correct terms of \( f_{12}^5(t) \) which may be a reason why they result
in accurate estimates. Therefore, the next models considered, G₁₁, H₁₁ and I₁₁, which are polynomials of the
seventh, ninth and twelfth order, respectively, explicitly exclude these terms. Although more error exists in
the modal estimates using these models when compared with the modal estimates using Models E₁₁ and F₁₁,
the error tends to decrease with increasing terms. This suggests that polynomial expansions may be
successful for estimating nonlinearities. It should however be noted, that an increase in the number of
terms in the polynomial expansion also increases the computation of the conditioned spectral density
matrices necessary for the Spectral Method. The same is true of the Temporal Method since the increased
computation entails the inversion of an increasingly larger system matrix. Also, note that the use of higher
order polynomial expansions may not lead to one unique model since many models including A₁₁, E₁₁, F₁₁,
G₁₁, H₁₁ and I₁₁ produce relatively accurate results.
| \(E_{ii}\) | \(1\) | 31.1 (0.0) | 0.9 (10.0) | (1.0, 0.6) | \(\hat{a}_2(\omega)_{\omega} = -1.0 + 3.3e^{-3i}\) MN/m²
|  | 2 | 81.3 (0.1) | 2.6 (0.0) | (-0.6, 1.0) | \(\hat{a}_2(\omega)_{\omega} = 494.8 - 114.5i\) kN/m³
| \(F_{ii}\) | \(1\) | 31.0 (0.3) | 0.3 (70.0) | (1.0, 0.6) | \(\hat{a}_2(\omega)_{\omega} = -1.0 + 2.4e^{-3i}\) MN/m²
|  | 2 | 81.3 (0.1) | 2.6 (0.0) | (-0.6, 1.0) | \(\hat{a}_2(\omega)_{\omega} = 3.0 - 0.2i\) MN/m²
| \(G_{ii}\) | \(1\) | 30.0 (3.5) | 0.2 (80.0) | (1.0, 0.6) | \(\hat{a}_3(\omega)_{\omega} = 26.1 - 0.9i\) MN/m³
|  | 2 | 77.0 (5.4) | 1.7 (34.6) | (-0.5, 1.0) | \(\hat{a}_3(\omega)_{\omega} = -466.5 + 9.7i\) MN/m³
| \(H_{ii}\) | \(1\) | 30.6 (1.6) | 0.5 (50.0) | (1.0, 0.6) | \(\hat{a}_4(\omega)_{\omega} = 19.2 - 0.4i\) MN/m³
|  | 2 | 78.6 (3.4) | 2.0 (30.0) | (-0.6, 1.0) | \(\hat{a}_4(\omega)_{\omega} = 1815.4 + 22.9i\) GN/m⁷
| \(I_{ii}\) | \(1\) | 30.6 (1.6) | 0.5 (50.0) | (1.6, 1.0) | \(\hat{a}_5(\omega)_{\omega} = 12.2 + 1.6e^{-2i}\) MN/m²
|  | 2 | 80.4 (1.2) | 2.5 (3.8) | (1.0, -1.6) | \(\hat{a}_5(\omega)_{\omega} = -526.3 + 25.8i\) TN/m²

% error = \(|\text{estimated-true}/\text{true}|\) · 100 given in parenthesis.

Table 4.8. Estimated properties of Example II by the Spectral Method in the absence of noise.
4.7. NONLINEARITY WITH NON-INTEGER EXPONENT

The final example is considered to examine the difficulties of identifying a system with a non-polynomial but continuous type nonlinearity. This system maintains the identical linear parameters of Examples I and II, however $f^{12}_{12}(t)$ now contains a term with a non-integer exponent, similar to nonlinearities caused by Hertzian contact forces [4.21]. This is designated as Example IV,

$$f^{12}_{12}(t) = k_{11} \Delta x^{2}_{12}(t) + \eta \cdot \text{sgn}(\Delta x^{2}_{12}(t)) |\Delta x^{2}_{12}(t)|^{1.8}, \quad \eta = 1.0 \text{ MN/m}^{1.8} \quad (4.32a,b)$$

where $\text{sgn}(\Delta x^{2}_{12}(t)) = \Delta x^{2}_{12}(t)/|\Delta x^{2}_{12}(t)|$. First, to illustrate that this type of system can in fact be identified given the correct nonlinear mathematical form, Model $A_{IV}$ is employed that assumes the true form as

Model $A_{IV}$: $p^{12}_{12}(t) = k_{11} \Delta x^{2}_{12}(t) + a \cdot \gamma(t) = k_{11} \Delta x^{2}_{12}(t) + a \cdot \text{sgn}(\Delta x^{2}_{12}(t)) |\Delta x^{2}_{12}(t)|^{1.8} \quad (4.33)$

As shown in Figure 4.16(a), $\hat{\tilde{H}}^{[2]}_{11}(\omega)$ accurately estimates the true $H^{11}(\omega)$. However, in practice, as with the other examples, the a priori knowledge of the mathematical form of $f^{12}_{12}(t)$ may be unknown. Therefore, as executed with Example II, a more plausible strategy is to estimate (4.32a) with a polynomial expansion given by

Model $B_{IV}$: $p^{12}_{12}(t) = k_{11} \Delta x^{2}_{12}(t) + \sum_{i=2}^{n} a_{i} \gamma_{i}(\Delta x^{2}_{12}(t)) = k_{11} \Delta x^{2}_{12}(t) + \sum_{i=2}^{n} a_{i} \Delta x^{2}_{12}(t) \quad (4.34)$

Model $B_{IV}$ is applied to Example IV for values of $n = 5$ and 10. Resulting $\hat{\tilde{H}}^{[2]}_{11}(\omega)$ are shown in Figure 4.16(b). Note that $n = 10$ is the largest possible value that could be chosen without numerical conditioning errors to arise. As can be seen from Figure 4.16(b), $\hat{\tilde{H}}^{[2]}_{11}(\omega)$ improves with increasing values of $n$. This increase in accuracy is also indicated by $\hat{\gamma}_{M1}^{2}(\omega)$ of Figure 4.17(a). Notice the y-axis is displayed only from 0.88 to unity in order to illustrate the differences between the two curves. Shown in Figure 4.17(b) are estimated stiffness curves for $n = 5$ and 10 along with the true stiffness curve. Note however, the Spectral Method estimates linear modal parameters and not physical parameters such as $k_{11}$. Hence, these curves do not include the linear stiffness term and accordingly one should not interpret these plots as physical stiffness curves. As shown, the estimated nonlinear stiffness approaches the true with increasing $n$. This example suggests that the employment of polynomial expansions to describe the unknown
Figure 4.16. Magnitude of Dynamic compliance functions of Example IV. (a) Model $A_N$ used for $H_{c2}$ estimate. Key: — $\hat{H}_{11}^{[c2]}(\omega)$, --- $\hat{H}_{11}^{[1]}(\omega)$, o o o $H_{21}(\omega)$. (b) Model $B_N$ used for $H_{c2}$ estimate. Key: — $\hat{H}_{11}^{[c2]}(\omega)$ with $n = 10$, --- $\hat{H}_{11}^{[c2]}(\omega)$ with $n = 5$, o o o $H_{21}(\omega)$. 
Figure 4.17. Cumulative coherence and stiffness estimates of Example IV. (a) Cumulative coherence of Model B IV. Key: $-\tilde{\gamma}_{M1}^2(\omega)$ with $n = 10$, $-\tilde{\gamma}_{M1}^2(\omega)$ with $n = 5$. (b) Nonlinear components of stiffness estimates. Key: $-\hat{p}_{12}(t)$ with $n = 10$, $-\hat{p}_{12}(t)$ with $n = 5$, o o o $f_{12}^e(t)$. 
nonlinearities may be a successful strategy once numerical conditioning errors are eliminated so that additional terms can be included in the model. Further examination of non-integer exponent type nonlinearities as given by equation (4.32a) is needed since literature on this subject is sparse.

4.8. CONCLUSION

When identifying nonlinear mechanical and structural systems, some *a priori* knowledge of the nature and mathematical form of the nonlinearities is necessary. Nonetheless, an unique model is still not guaranteed since this knowledge may be limited and different mathematical formulae for describing the nonlinearities may result in reasonably accurate estimates. Also, the presence of uncorrelated noise often corrupts the response and excitation data. These problems have been illustrated here by four simulation systems, and for two of these examples, in the presence of moderate and high measurement noise levels. Both the Restoring Force Temporal Method [4.10, 4.11] and the “Reverse Path” Spectral Method [4.7] have been employed to illustrate that these problems are not method dependent; instead they must be addressed regardless of which technique is used. Coherence functions have been developed based on the Spectral Method and their application has been demonstrated. These coherence functions indicate the level of uncorrelated noise present in the data and when reasonably accurate models have been chosen to describe the nonlinear systems. Therefore, parameters estimated by techniques such as the ones discussed here can be assessed with some level of confidence.

With the capability of now being able to indicate the accuracy of the mathematical models chosen to describe nonlinear systems, ongoing and future research will extend the identification techniques examined here to other complex problems including damping nonlinearities. The next promising area appears to be the non-integer nonlinearity which has sparsely been addressed in the scientific literature [4.21]. Future research will also focus on enhancing conditioned and unconditioned spectral density functions that are corrupted by uncorrelated measurement noise. A reference method [4.19] will act as a starting point for such an investigation. Also, time domain averaging operations and cross-correlation techniques may be adapted to improve the Restoring Force Method in the presence of uncorrelated measurement noise.
LIST OF REFERENCES FOR CHAPTER 4


CHAPTER 5

IDENTIFICATION METHODS APPLIED TO EXPERIMENTAL SYSTEMS

5.1. INTRODUCTION

The identification methods of chapters 3 and 4 will be applied to several experimental systems, including the isolators studied in chapter 2. The "Reverse Path" Spectral and Restoring Force time domain methods will be employed. The Force-State Mapping or Restoring Force Method for identifying linear and nonlinear systems has been studied extensively [5.1-5.8]. The method was initially developed for the identification of nonlinear single-degree-of-freedom nonlinear systems [5.1] and later extended for multi-degree-of-freedom systems [5.2]. Chebyshev polynomials were used to describe the restoring forces and the masses were assumed to be known. Resulting plots were obtained revealing the nature of the nonlinearities present. The method has since been enhanced where the restoring forces are modeled by a polynomial form and estimation of the masses is included [5.3-5.6]. Experimental application of the method has been demonstrated for the identification of single, two and three degree of freedom systems [5.5.5.7.5.8].

The Restoring Force Method utilizes simultaneously measured excitation and response data from an experimental system to obtain a discrete nonlinear vibration model. Experimental data are typically collected using piezoelectric transducers. These transducers are insensitive to static loads and displacements, and without this information the identification of nonlinear systems invariably produces incorrect results. To overcome this deficiency, a modified restoring force method is proposed in this chapter which includes static loads and displacements as unknown parameters to be determined. Initial studies of the modified method will be conducted with application to a discrete nonlinear isolation system, under both static and dynamic loads. In order to be consistent with measured data using piezoelectric
transducers, the static forces and displacements are removed. Measurement noise is also added for some cases. Experimental application of the Restoring Force and the “Reverse Path” Spectral Methods are conducted for single and multi-degree-of-freedom linear and nonlinear systems. For the nonlinear systems, three elastomeric isolators will be utilized; recall initial results of chapter 2.

5.2. PROBLEM FORMULATION

5.2.1. PRESENCE OF STATIC LOADS AND DISPLACEMENTS

When externally applied static and dynamic loads are simultaneously applied to a dynamic system, static displacements and dynamic motions result. However, unlike linear systems where the static elastic forces cancel the external static loads, this is not the case for nonlinear systems because of coupling terms which arise from the nonlinearity. Consider the Duffing’s oscillator in the presence of a static \((f^o)\) and a dynamic \((f^d)\) load

\[
\begin{align*}
\ddot{x}(t) + c\dot{x}(t) + k(x(t) + \delta) + a_3(x(t) + \delta)^3 &= f^o + f^d(t) \\
\ddot{x}(t) + c\dot{x}(t) + k(x(t) + \delta) + a_3(x(t)^3 + 3\delta x(t)^2 + 3\delta^2 x(t) + \delta^3) &= f^o + f^d(t) \\
\ddot{x}(t) + c\dot{x}(t) + kx(t) + a_3(x(t)^3 + 3\delta x(t)^2 + 3\delta^2 x(t)) &= f^o - k\delta - a_3\delta^3 + f^d(t)
\end{align*}
\]

The coupled terms involving the static and dynamic displacements prevent the cancellation of static terms. If the static loads and displacements are measurable, then they can be added to the dynamic forces and displacements to calculate the total force, \(f(t) = f^o + f^d(t)\), and displacement, \(y(t) = x(t) + \delta\). However, conventional vibration instrumentation such as piezoelectric accelerometers and force transducers are incapable of static measurements. Therefore, enhancements to the Restoring Force Method are necessary to accommodate for these unknowns.

5.2.2. APPLICATION TO EXPERIMENTAL SYSTEMS

Consider a single-degree-of-freedom (SDOF) system consisting of a typical rubber isolator illustrated as in Figure 5.1(a). This isolator is evaluated using both static and dynamic experiments with loads acting in the normal mode. For the static experiment shown in Figure 5.1(b), the governing equation
Figure 5.1. Identification experiments: (a) mount specimen, (b) static experimental setup, (c) dynamic experimental setup.
for the elastic force of the isolator is \( h(x_j) = f \), where \( h(x_j) \) is the elastic nonlinear relationship between the static displacement \( x_j \), measured by a dial indicator, and the static force \( f \), applied in both compression and tension. For the dynamic experiment illustrated in Figure 5.1(c), the isolator supports a steel cylindrical mass \( m = 1.4 \text{ kg} \) whose motion is governed by \( m \ddot{x}(t) + G(\dot{x}(t), x(t)) = f(t) \) where \( G(\dot{x}(t), x(t)) \) is the dynamic restoring force of the mount and \( f(t) \) is the applied random excitation. A piezoelectric force transducer measures \( f(t) \) provided by an electro-dynamic shaker, and the acceleration of the mass \( \ddot{x}(t) \) is measured by a piezoelectric accelerometer. Random excitation is chosen so that the "Reverse Path" Spectral Method \([5.9, 5.10]\) which is based on random data analysis can be employed. Also employed is the Restoring Force Method \([5.1-5.8]\), however this method does not require random excitation. These dynamic experimental methods are chosen over other methods such as the dynamic stiffness method \([5.11]\) which only includes the first harmonic of the response and provides only pseudo-linear properties.

Assuming linear viscous damping, the restoring force is written as \( G(\dot{x}(t), x(t)) = c\dot{x}(t) + g(x(t)) \) where \( c \) is the linear damping coefficient and \( g(x(t)) \) is the nonlinear elastic force of the mount. The objective is to employ experimental and analytical techniques to the static and dynamic data to determine a model \( \hat{g}(x(t)) \) which best describes \( g(x(t)) \). Certain criteria are introduced which should determine when a correct model has been estimated. The correlation of \( h(x_j) \) with \( g(x(t)) \) is also examined.

5.3. MODIFIED RESTORING FORCE METHOD

5.3.1. FORMULATION OF EXISTING METHOD

For the sake of completeness, the Restoring Force Method as currently interpreted in the literature \([5.1-5.8]\) will be introduced. Assume that experimental vibration testing of the system under consideration yields simultaneous measurements of the accelerations, velocities and displacements at all degrees of freedom in the frequency range of interest. If only acceleration measurements are available, as would be the case when using piezoelectric accelerometers, time integration of the signals is necessary to calculate the velocities and displacements. In addition, assume that static displacements, if present, are known.

Then the total response vectors \( \ddot{y}(t), \dot{y}(t) \) and \( y(t) = x(t) + \delta \) are available. Refer to the List of Symbols for
chapter 5 for the identification of symbols. Also assume that all excitations applied to the system are measured and static loads are known. Then the excitation vector \( f(t) = f^o + f^d(t) \) is available. Many types of excitations have been used for the restoring force method [5.12]. Yang and Ibrahim [5.3] have shown how the method can be used for impact or initial condition testing where the entire system is scaled to the total mass of the system. Random excitation has been used for the identification of automotive shock absorbers where possible frequency dependent nonlinearities exist [5.7]. However, random data imposes practical difficulties when integration is necessary to calculate velocities and displacements from measured accelerations. Sinusoidal excitation has been found to be the most practical for this purpose.

Once measurements have been made, the Restoring Force Method estimates the mass, damping and stiffness elements of a set of nonlinear differential equations chosen to model the system. These elements are estimated in a least squares sense by satisfying the nonlinear differential equations at every time sample in the acquisition period. The system under study (i.e. the system whose parameters are to be identified) is modeled by the following system of equations:

\[
M\ddot{y}(t) + g\{y(t),\dot{y}(t)\} = f(t) \tag{5.2}
\]

With at least one \( f_i(t) \) non-zero, the method begins with the \( i \)th equation of motion

\[
m_i\ddot{y}_i(t) + g_i\{y(t),\dot{y}(t)\} = f_i(t) \tag{5.3}
\]

Modeling the restoring force as a polynomial expansion of the relative displacements and relative velocities

\[
g_i\{y(t),\dot{y}(t)\} = \sum_{j=1}^r \sum_{k=0}^a \sum_{l=0}^m a_{ijkl} y_{ij}(t)^k \dot{y}_{ij}(t)^l \tag{5.4}
\]

equation (5.3) becomes

\[
m_i\ddot{y}_i(t) + \sum_{j=1}^r \sum_{k=0}^a \sum_{l=0}^m a_{ijkl} y_{ij}(t)^k \dot{y}_{ij}(t)^l = f_i(t) \tag{5.5}
\]

Writing the \((r \cdot n \cdot m)+1\) unknowns as a vector to form an over-determined set of \( N \) equations where \( N \) is the number of time samples over a time window \((T=N \Delta t)\)

\[
Y_i \alpha = \bar{f}_i \tag{5.6}
\]
where

\[
Y_i = \begin{bmatrix}
\ddot{y}_i & \ddot{y}_1 & \ldots & \ddot{y}_q & \ldots & \ddot{y}_r
\end{bmatrix}
\]

\[
\ddot{y}_q = \begin{bmatrix}
\ddot{y}_{q1} & \ddot{y}_q^1 & \ldots & \ddot{y}_q^n & \ddot{y}_q & \ldots & \ddot{y}_q^n & \ddot{y}_q & \ddot{y}_q^n & \ddot{y}_q & \ddot{y}_q^n
\end{bmatrix}
\]

(5.7a,b)

The overhead bar signifies a column vector whose elements are the $N$ sampled instances in time, e.g.

\[
\ddot{y}_i = \begin{bmatrix}
\ddot{y}_i(t_1) & \ddot{y}_i(t_2) & \ldots & \ddot{y}_i(t_{N-1}) & \ddot{y}_i(t_N)
\end{bmatrix}^T
\]

\[
\alpha_i = \begin{bmatrix}
m_i & \tilde{a}_{i1} & \ldots & \tilde{a}_{iq} & \ldots & \tilde{a}_{ir}
\end{bmatrix}^T
\]

(5.8a,b)

\[
\tilde{a}_{iq} = \begin{bmatrix}
a_{iq00} & a_{iq01} & \ldots & a_{iq0m} & a_{iq10} & \ldots & a_{iq11} & \ldots & a_{iq1m} & a_{iq20} & \ldots & a_{iq21} & \ldots & a_{iq2m} & \ldots & a_{iqm1} & a_{iqm2} & \ldots & a_{iqmm}
\end{bmatrix}
\]

(5.7a,b)

In general, the dimensions of (5.6) are

\[
[Y_i]_{(N,r,m)}(\alpha_i)_{(r,m,1)} = \{f_i\}_{(N,1)}
\]

(5.9)

5.3.2. IDENTIFICATION ISSUES

The system of equations (5.9) is solved by least squares. If there is any a priori knowledge about the types of nonlinearities present at the $i^{th}$ degree of freedom, terms can be left out of the summation (5.4) to improve both computational efficiency and accuracy of the estimation. Also, if any elements of (5.3) are known prior to the identification, the terms containing the parameters of the known elements are moved to the right hand side of the equation. Once the unknown elements for the $i^{th}$ equation are determined, the parameters of the equations adjacent to the $i^{th}$ degree of freedom are identified. For example, assume that the $p^{th}$ degree of freedom is connected to the $i^{th}$ degree of freedom and that the elements of the $i^{th}$ degree of freedom have been identified. The terms with the known elements of the $p^{th}$ degree of freedom are moved to the right hand side of the equation. At least one of these elements will be known since the two degrees of freedom are connected. The unknowns of the $p^{th}$ degree of freedom are then computed. This process is continued until all of the elements are estimated for all of the degrees of freedom.

The formulation above suggests that the method starts with the nonhomogeneous equations of motion (i.e. the forced degrees of freedom). However, it is feasible to start with any one of the equations.
of motion so long as one of the parameters is brought over to the right hand side of the equation making this initial equation in the procedure nonhomogeneous. Then all of the equations are divided by this parameter scaling the system to unit value of that parameter. Identification as illustrated is then performed. If the system is forced, when the forced equation or equations of motion are identified, this scale factor can be determined and the entire system's elements can be scaled to their true values.

There are some inherent difficulties with the restoring force method. Because relative motions are calculated, simultaneous measurements must be made to eliminate phase errors. Simultaneous measurements at all of the response degrees of freedom is not necessary. However, the motion of the $i^{th}$ degree of freedom and the motion of all of the adjacent degrees of freedom must be made simultaneously. This imposes a rather strict requirement on instrumentation. A multi-channel data acquisition system is necessary that can collect as many simultaneous channels of data as the number of equations coupled to the most coupled equation of motion. Also, because the method identifies the system by progressing through the system of equations, errors in the identification of one equation of motion propagate to the next equation of motion. Therefore, accuracy degenerates as the identification proceeds away from the initial degree of freedom identified. This difficulty is worsened by uncorrelated measurement noise, signal processing errors, and time integration problems. Finally, the method does not account for unmeasurable static loads and displacements. This problem is addressed next.

5.3.3. PROPOSED MODIFICATION

To accommodate for the unmeasurable static loads and displacements, they must be included in the model as unknowns. Rewriting the system model as:

$$M\ddot{x}(t) + g\{x(t) + \delta, \dot{x}(t)\} = f^o + f^d(t)$$  \hspace{1cm} (5.10)

where $\ddot{x}(t)$ and $f^d(t)$ are measured and $\dot{x}(t)$ and $x(t)$ are carefully determined using integration. Assuming that at least one $f^d_i(t)$ is non-zero, the method begins with the $i^{th}$ equation of motion with $f^o_i$ moved to the left hand side as an unknown.
\[ m_i \ddot{x}_i(t) + g_i \{ x(t) + \delta \dot{x}(t) \} = f_i^d(t) \]  \hspace{1cm} (5.11)

Modeling the restoring force as a polynomial expansion of the relative displacements and relative velocities

\[ g_i \{ x(t) + \delta \dot{x}(t) \} = \sum_{j=1}^{r} \sum_{k=0}^{n} \sum_{l=0}^{m} a_{ijkl} (x_{ij}(t) + \delta_{ij})^k \dot{x}_{ij}(t)^l \]  \hspace{1cm} (5.12)

Equation (5.11) becomes

\[ m_i \ddot{x}_i(t) + \sum_{j=1}^{r} \sum_{k=0}^{n} \sum_{l=0}^{m} a_{ijkl} (x_{ij}(t) + \delta_{ij})^k \dot{x}_{ij}(t)^l - f_i^o = f_i^d(t) \]  \hspace{1cm} (5.13)

Equation (5.13) is similar to (5.5) with the exception of the additional unknowns \( \delta_{ij} \) and \( f_i^o \). To identify the parameters of (5.3) the summation is expanded and terms are collected containing the combination of the response variables \( \ddot{x}_i(t), \dot{x}_i(t) \) and \( x_i(t) \). The coefficients in front of each term designated as \( \beta_i = \{ \beta_{ij} \} \), \( p = (r m n)+2 \), are determined by least squares. These coefficients form a set of simultaneous nonlinear algebraic equations resulting from the expansion of the summation and collection of the response variables. They contain the \( (r m n)+r+2 \) unknowns in terms of \( m, a_{ijkl}, \delta_{ij}, \) and \( f_i^o \). Notice that this is an underdetermined problem. However, careful choice of the expansion allows for the determination of all the system parameters, \( m \), and \( a_{ijkl} \). This is illustrated by example in the following section.

### 5.4. FEASIBILITY STUDIES ON AN ANALYTICAL MODEL

Consider the identification of the isolation system, Figure 5.2(a). For the three analytical case studies to follow, the subsequent information applies. A five degree of freedom analytical model of the isolation system was analyzed, Figure 5.2(b). The true values of the elements are given in Table 5.1. The restoring force of the mount was given by the following equation to represent a linear damping and cubic hardening spring isolator

\[ f_{53}(t) = c_5 \ddot{x}_{53}(t) + k_5 (x_{53}(t) + \delta_{53}) + a_5 (x_{53}(t) + \delta_{53})^3 \]  \hspace{1cm} (5.14)

A static load and excitation are applied to the fifth mass

\[ f_s(t) = f_s^o + f_s^d(t) \]  \hspace{1cm} (5.15)
Figure 5.2. Feasibility study for the modified Restoring Force Method. (a) Isolation system considered for identification. (b) Five degree of freedom model of isolation system for analytical case studies.
<table>
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<tr>
<th>Elements of Figure 5.2(b)</th>
<th>true values</th>
<th>original method (no noise)</th>
<th>modified method with 60 dB s/n</th>
<th>modified method with 40 dB s/n</th>
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<td>0.8</td>
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<td>0.6</td>
<td>1.0</td>
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</tr>
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<td>5.0</td>
<td>1</td>
<td>5.1</td>
<td>6.0</td>
</tr>
<tr>
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<td>5.1</td>
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</tr>
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<td>5.1</td>
<td>3.7</td>
</tr>
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<td>5.0</td>
<td>4.1</td>
</tr>
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</tr>
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<tr>
<td>isolated mass, (kg):</td>
<td>4.0</td>
<td>2.5</td>
<td>4.0</td>
<td>3.3</td>
</tr>
<tr>
<td>isolation path:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>damping (N·s/m)</td>
<td>5.0</td>
<td>26.7</td>
<td>5.9</td>
<td>35.8</td>
</tr>
<tr>
<td>linear stiffness, $k_1$ (kN/m)</td>
<td>50.0</td>
<td>1409.2</td>
<td>49.2</td>
<td>-27.4</td>
</tr>
<tr>
<td>cubic stiffness, $a_0$ (GN/m$^3$)</td>
<td>9.0</td>
<td>-7.2</td>
<td>5.0</td>
<td>3.7</td>
</tr>
</tbody>
</table>

Table 5.1. Results from analytical case studies conducted using the modified Restoring Force Method.
A 5th order Runge-Kutta Fehlberg integration method was used to simulate 4096 time points at 2000 samples/second resulting in 2.0475 seconds of data. From this \( y(t), \dot{y}(t), \ddot{y}(t) \) and \( f(t) \) were realized. To emulate vibration testing using piezoelectric transducers, static loads and displacements were removed from the data.

Identification begins with the fifth equation of motion

\[
m_s \dddot{x}_s(t) + c_s \dddot{x}_s(t) + k_s \left( x_s(t) + \delta_{s3} \right) + a_3 \left( x_s(t) + \delta_{s3} \right)^3 = f_s^o + f_s^d(t)
\]

where \( \dddot{x}_s(t) \), \( \dddot{x}_s(t) \), and \( x_s(t) \) are known at the 4096 sampled times and \( m_s, c_s, k_s \), and \( a_3 \) are to be determined. For the modified restoring force method, the nonlinear term is expanded and terms are collected in terms of the known quantities

\[
m_s \dddot{x}_s(t) + c_s \dddot{x}_s(t) + k_s \left( x_s(t) + \delta_{s3} \right) + a_3 \left( x_s(t) + 3 \delta_{s3} x_s^2(t) + 3 \delta_{s3} x_s^2(t) + \delta_{s3} \right) - f_s^o = f_s^d(t)
\]

\[
m_s \dddot{x}_s(t) + c_s \dddot{x}_s(t) + a_3 x_s^3(t) + 3a_3 \delta_{s3} x_s^3(t) + \left( \dot{k}_s + 3a_3 \delta_{s3} \right) x_s(t) + \left( k_s \delta_{s3} + a_3 \delta_{s3}^2 - f_s^o \right) = f_s^d(t)
\]

Define the following and rewrite (5.18) as

\[
\begin{align*}
\beta_{s1} &= m_s \\
\beta_{s2} &= c_s \\
\beta_{s3} &= a_3 \\
\beta_{s4} &= 3a_3 \delta_{s3} \\
\beta_{s5} &= \left( k_s + 3a_3 \delta_{s3} \right) \\
\beta_{s6} &= \left( k_s \delta_{s3} + a_3 \delta_{s3}^2 - f_s^o \right)
\end{align*}
\]

\[
\beta_{s1} \dddot{x}_s + \beta_{s2} \dddot{x}_s + \beta_{s3} x_s^3 + \beta_{s4} x_s^2 + \beta_{s5} x_s^3 + \beta_{s6} = f_s
\]

This equation is solved for \( \beta_s = [ \beta_{s1} \beta_{s2} \beta_{s3} \beta_{s4} \beta_{s5} \beta_{s6} ]^T \) by least squares. Equations (5.19a-f) are six equations with six unknowns which are solved directly. Notice, although quadratic nonlinearities are absent, a quadratic and constant term were included in the least squares solution to accommodate for the unknown static displacement. Once \( m_s, c_s, k_s \) and \( a_3 \) are identified the third equation of motion is considered with the terms including the now known quantities \( c_s, k_s \) and \( a_3 \) moved to the right hand side of the equation. This is continued until all of the parameters of the isolation system are identified.

Notice, in the absence of the static load, \( f_s^o \), (5.18) reduces to

\[
m_s \dddot{x}_s(t) + c_s \dddot{x}_s(t) + a_3 x_s^3(t) + k_s x_s(t) = f_s^d(t)
\]

and the modified and unmodified versions of the restoring force method are identical.
To illustrate the performance of the proposed method, a case was studied where no static load is present. The excitation was chosen to be $f_2(t) = 1000\sin(2\pi 50t)$ N. The modified version of the restoring force method was applied to the simulated data. The exact values of all of the parameters were estimated. This is of no surprise since the only error involved in the calculations is the numerical integration error associated with discretization.

Next, a static load is applied to the fifth mass, $f_5(t) = -10e3 + 1000\sin(2\pi 50t)$ N, resulting in the following static displacements: $\delta = [-79.7, -159.4, -239.2, -119.6, -251.3]$ mm. As mentioned, the static load and static displacements were removed from the force and response data to emulate experimentally measured vibration data using piezoelectric transducers. The resulting data was then used to study the performance of the restoring force method and the modified restoring force method. Again, the modified version yielded exact results. However, the effects of unmeasurable static loads and displacements is revealed in the results of the unmodified version as seen in Table 5.1. The unmodified method is unsuccessful at identifying the elements of the system in the presence of unmeasurable static loads and displacements.

The final test is to examine the performance of the proposed method when measurement noise is present in excitation and response data. This was done by adding Gaussian distributed random noise to the case 2 data with signal-to-noise levels of 60 dB and 40 dB. Results of Table 5.1 reveal the sensitivity of the method which comes from the fact that the time domain data is not averaged to reduce noise. Note that the unmodified restoring force method is as sensitive to noise as the modified version.

5.5. FEASIBILITY STUDIES ON EXPERIMENTAL SYSTEMS

The restoring force method has been applied to the identification of a single and a two degree of freedom linear experimental system under sinusoidal excitation. PCB piezoelectric accelerometers and a piezoelectric force transducer were used to make acceleration and excitation measurements. Data was collected using a Hewlett Packard 3566A multi-channel front end and signal conditioner. The device was
controlled by a PC based HP3566A software, and $N = 8192$ time records were collected at a sample rate of 32786 samples/second resulting in $T = 0.25$ seconds of data.

To calculate the velocities and displacements, integration of the measured acceleration signals is necessary. There are many approaches for integrating signals [5.13]. Here a frequency domain integration approach is used. The Fast Fourier Transform is employed to convert the data to the frequency domain.

Each resulting frequency bin is then divided by $\left(\frac{2\pi}{T}\right)ki$ where $T$ is the sample period ($T = 0.25$ seconds), $k$ is the bin number and $i = \sqrt{-1}$. The resulting spectrum is inverse Fast Fourier Transformed to obtain the integrated signal in the time domain. To minimize leakage error, the excitation and resulting acceleration response are chosen to be a multiple of $1/T = 4$ Hz. Note that this integration procedure will be more difficult for a nonlinear system, especially when the response includes sub- and super-harmonics.

A modification must be made to the Restoring Force Method when identifying linear systems because each mass is essentially proportional to the linear stiffness coefficient. This effectively reduces the number of unique solutions by one resulting in non-unique mass and stiffness values. By identifying the system at different excitation frequencies, unique solutions can be recovered by examining mass versus stiffness curves.

First, a single degree of freedom linear system experiment is constructed of a simply supported aluminum beam which behaves as an elastic supporting structure for a concentrated 0.5 kg cylindrical mass located at the center of the beam. The aluminum beam is 500 mm long and 6 mm thick resulting in a center supporting stiffness of 50 kN/m and a mass of 0.8 kg. Using a total mass of $0.5 + 0.8 = 1.3$ kg, the resulting natural frequency is 31.2 Hz. However, a broadband excitation was used to obtain the frequency response function shown in Figure 5.3(a) which reveals a natural frequency of 38.5 Hz. Either the stiffness of the beam is higher than calculated or not all of the mass from the beam is contributing to the system dynamics.

To alleviate the non-unique identification problem, the single-degree-of-freedom system is identified at two separate excitation frequencies: 48 Hz and 60 Hz. At each frequency the mass estimate is
Figure 5.3. Experimental single-degree-of-freedom system results. (a) Measured accelerance. (b) Identified mass versus stiffness curves. Key: — 48 Hz, — 60 Hz.
varied and the stiffness is identified to reveal a linear relation between the two quantities. The linear relations are plotted in Figure 5.3(b) and the intersection of the two lines identifies the mass and stiffness. The extracted values of the mass and stiffness are 0.8958 kg and 52.41 kN/m. This results in a natural frequency of 38.5 Hz which matches that of the frequency response function.

Next, the same aluminum beam is modified and a two degree of freedom system experiment is constructed. The 0.5 kg mass, denoted here as mass 1, is move to 125 mm from on end of the beam and a 0.38 kg mass, denoted as mass 2, is added 125 mm from the other end. An accelerance frequency response function of the resulting system is shown in Figure 5.4(a). The natural frequencies are at 39.5 Hz and 132 Hz. A schematic of the discrete system to be identified is shown in Figure 5.4(b).

Harmonic excitation at 32 Hz is applied to mass 1. Identifying the first equation of motion (i.e. identifying \(m_1, k_1\) and \(k_2\)) results in a non-unique solution since \(\ddot{y}_1(t), \dot{y}_1(t)\) and \(y_1(t)\) are proportional. To alleviate this problem, \(m_1\) is chosen to be 1 kg, \(k_2\) is varied and \(k_1\) is identified to reveal a linear relation between the two stiffness rates. The same procedure is done with a 48 Hz excitation and the two lines are graphed in Figure 5.5(a). Values of \(k_1 = 73.5\text{ kN/m}\) and \(k_2 = 262.3\text{ kN/m}\) are obtained. With \(k_2\) now known, identification of the second equation of motion (i.e. identifying \(m_2, k_2\)) can proceed. Here the mass is varied and \(k_1\) is identified. This is conducted at an excitation of 32 and 48 Hz, Figure 5.5(b), resulting in \(m_2 = 0.83\text{ kg}\) and \(k_2 = 39.2\text{ kN/m}\). An eigen-solution of this system results in natural frequencies of 39.4 Hz and 127.1 Hz. In comparison with the natural frequencies obtained from the frequency response function, there is a 0.25% error in the estimation of the first mode and a 3.71% error in the second mode. Identified parameters from the one and two degree of freedom systems have yielded natural frequency calculations that match well with the measured natural frequencies of these systems.
Figure 5.4. Experimental two-degree-of-freedom system results. (a) Measured accelerance at the driving point. (b) Discrete linear system model.
Figure 5.5. Results for the two-degree-of-freedom system experiment of Figure 5.4. (a) Identified stiffness $k_2$ versus stiffness $k_1$ curves. (b) Identified mass $m_2$ versus stiffness $k_3$ curves. Key: — 32 Hz, --- 48 Hz.
5.6. IDENTIFICATION OF ISOLATOR PROPERTIES USING SINGLE-DEGREE-OF-FREEDOM EXPERIMENTS

5.6.1. EXPERIMENTAL DATA AND IDENTIFICATION USING A QUASI-LINEAR MODEL

Figure 5.6(a) illustrates the stiffness curve obtained from the static experiment and Figure 5.6(b) illustrates the resulting magnitudes of frequency response functions (FRFs) measured from three dynamic experiments with different random excitation amplitudes $|f(t)|$ as listed in Table 5.2. Also listed in Table 5.2 are effective linear system parameters determined by curve fitting each FRF using a modal analysis technique. Note, throughout this chapter subscripts I, II and III indicate variables pertaining to Dynamic Experiments I, II and III, respectively. For Dynamic Experiment I, the excitation $f_1(t)$ is small and therefore it is assumed that nonlinear effects are negligible. Observe that the peak frequencies decrease with increasing $|f(t)|$, a characteristic similar to that of a nonlinear softening spring. Therefore, $k^*$ decreases with increasing excitation level. This quasi-linear model is useful for examining trends in the variations of the parameters. However, the generation of “look-up” tables is necessary for all operating conditions. A continuous model would be more desirable that captures the system response over a broad range. Also from the static stiffness curve, notice that a slight softening spring stiffness is evident in tension, but a slight hardening spring stiffness is found in compression. This observation makes it difficult to correlate $h(x_i)$ with $g(x(t))$ since hardening and softening effects should tend to cancel each other, but a softening effect is obvious from the dynamic experiments.

5.6.2. IDENTIFICATION USING POLYNOMIAL TYPE MODELS

Three initial polynomial models of $z(x(t))$ consisting of quadratic and cubic terms are chosen to estimate $h(x_i)$. They are Models A, B & C listed in Table 5.3. Such nonlinear models are widely discussed in scientific literature as suitable models for describing many physical systems [5.9]. Assuming that the hysteresis evident in the measured $h(x_i)$ curve is negligible, these models are fit to the static stiffness curve in a least squares sense. The resulting coefficients are listed in Table 5.4. To evaluate how well these estimates describe $g(x(t))$, numerical simulations of the model $m_c \ddot{x}(t) + c_m \dot{x}(t) + z(x(t)) = f(t)$ are
Figure 5.6. Experimental Results for the isolator of Figure 5.1: (a) stiffness curve from static experiment, (b) frequency response functions from dynamic experiments.
<table>
<thead>
<tr>
<th>Dynamic Experiment</th>
<th>Excitation Level</th>
<th>effective natural frequency, $\Omega'$ (Hz)</th>
<th>effective linear damping, $c'$ (N·s/m)</th>
<th>effective linear stiffness, $k'$ (kN/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>9</td>
<td>39.6</td>
<td>50</td>
<td>87</td>
</tr>
<tr>
<td>II</td>
<td>91</td>
<td>37.7</td>
<td>51</td>
<td>78</td>
</tr>
<tr>
<td>III</td>
<td>443</td>
<td>35.5</td>
<td>48</td>
<td>70</td>
</tr>
</tbody>
</table>

Table 5.2. Effective linearized parameters estimated from dynamic experiments of Figure 5.1.

<table>
<thead>
<tr>
<th>Model</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ddot{z}(t)$</td>
<td>$k_A \ddot{x}(t) + \beta_{B_1} \dot{x}(t)^2$</td>
<td>$k_C \ddot{x}(t) + \beta_{C_1} \dot{x}(t)^2 + \beta_{C_2} \dot{x}(t)^4$</td>
<td>$k_D \ddot{x}(t) + \beta_{D \text{sgn}(x(t))}</td>
<td>x(t)</td>
</tr>
</tbody>
</table>

Table 5.3. Models for describing the $g(x(t))$ function of isolator depicted in Figure 5.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Static Experiment</th>
<th>Temporal I</th>
<th>Temporal II</th>
<th>Temporal III</th>
<th>Spectral I</th>
<th>Spectral II</th>
<th>Spectral III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_A$ (kN/m)</td>
<td>51</td>
<td>85</td>
<td>77</td>
<td>68</td>
<td>87</td>
<td>78</td>
<td>70</td>
</tr>
<tr>
<td>$k_B$ (kN/m)</td>
<td>52</td>
<td>85</td>
<td>77</td>
<td>68</td>
<td>87</td>
<td>78</td>
<td>70</td>
</tr>
<tr>
<td>$\beta_B$ (MN/m$^2$)</td>
<td>-2</td>
<td>-5</td>
<td>2</td>
<td>2</td>
<td>10</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$k_C$ (kN/m)</td>
<td>51</td>
<td>85</td>
<td>77</td>
<td>69</td>
<td>87</td>
<td>78</td>
<td>70</td>
</tr>
<tr>
<td>$\beta_{C_1}$ (MN/m$^2$)</td>
<td>-2</td>
<td>-4</td>
<td>2</td>
<td>2</td>
<td>8</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$\beta_{C_2}$ (GN/m$^5$)</td>
<td>200</td>
<td>1191</td>
<td>-42</td>
<td>-3</td>
<td>-1570</td>
<td>-64</td>
<td>-1</td>
</tr>
</tbody>
</table>

Table 5.4. Parameters of Models A, B and C based on static and dynamic experiments.
conducted where \( m \) is the mass of the steel cylinder, \( c_{\text{avg}} \) is the average of the effective damping coefficients listed in Table 5.2, and \( \ddot{x}(x(t)) \) is one of the Models A, B or C along with their estimated coefficients of Table 5.4. The measured excitations from the dynamic experiments are used separately as inputs to the simulations, i.e. \( f(t) = f_1(t), f_2(t) \) or \( f_3(t) \). The FRFs are then calculated and their effective natural frequencies are compared with the respective \( \Omega^e \) of Table 5.2. For \( f(t) = f_1(t) \) the effective natural frequencies are 30.7, 30.8 and 30.8 Hz for Models A, B and C, respectively. Comparing these effective natural frequencies with \( \Omega^e \) indicates that the mount has a substantially larger dynamic stiffness than static stiffness. Similar conclusions are drawn from the numerical simulations using \( f(t) = f_2(t) \) and \( f(t) = f_3(t) \).

Therefore, estimates \( \dot{z}(x(t)) \) of \( h(x, \omega) \) do not correlate well with \( g(x(t)) \). In addition to these studies, numerical simulations are also conducted by choosing various other values for the coefficients of these models. However, no values can be found for these coefficients which result in effective natural frequencies that match those from the dynamic experiments. This leads to the conclusion that Models A, B and C may not be correct models for estimating \( g(x(t)) \).

Next, the function \( g(x(t)) \) is estimated using two identification methods: a Temporal Method which fits a model of the form \( m\ddot{x}(t) + c\dot{x}(t) + \ddot{x}(x(t)) = f(t) \) to measured excitation and response data in the time domain, and a Spectral Method which fits the same model, however, to measured excitation and response data in the frequency domain. The velocity \( \dot{x}(t) \) and displacement \( x(t) \) are determined from frequency domain integration of the measured acceleration \( \ddot{x}(t) \) [5.13]. Using Models A, B and C of Table 5.3 for \( \ddot{x}(x(t)) \), these rigorous methods are employed to the dynamic data to estimate the coefficients of these models. As with the static stiffness results, criteria are needed to indicate if an accurate model has been estimated. Two plausible criteria are as follows. *First Criterion*: compare estimated values for stiffness coefficients at the different excitation levels. Since the model is a nonlinear differential equation with constant coefficients, estimates should not vary much with excitation level. *Second Criterion*: since it is assumed that nonlinear effects are negligible for Dynamic Experiment I, then it can also be assumed that \( k^e_1 \) of Table 5.2 is a good estimate of the system's linear stiffness coefficient, and therefore estimates for
$k_a$, $k_b$ or $k_c$ should approximately equal $k_f^e$ at all excitation levels. Other criteria for indicating an accurate model are also possible.

Results for the Temporal and Spectral Methods are listed in Table 5.4. Note, the methods also estimate the damping coefficient, however, these results are not included. Large variations in the coefficients over the three excitation levels are evident. This fails to satisfy the First Criterion. The estimates satisfy the Second Criterion within 2% error using the data from Dynamic Experiment I. However, this does not indicate an accurate model of $g(x(t))$ over the entire excitation range. Therefore, as with the numerical simulation studies, these studies also suggest that Models A, B and C are not successful choices of $\hat{z}(x(t))$ for describing $g(x(t))$. Additional polynomial forms have also been investigated, however, they also fail to satisfy these criteria.

5.6.3. NEW NONLINEAR MODEL

A refined Model D is proposed as listed in Table 5.3 where the exponent $\alpha$ may have a non-integer value and $\text{sgn}(x(t)) = x(t)/|x(t)|$. This model for $g(x(t))$ is used in

$$m_c \ddot{x}(t) + c_{av} \dot{x}(t) + z_D(x(t)) = f(t)$$

and numerical simulations are repeated using $f(t) = f_D(t)$ and $f(t) = f_{in}(t)$. The value chosen for the linear stiffness coefficient is fixed at $k_p = 86.8$ kN/m to minimize the number of variables. Note that $k_p = k_f^e$ since it is assumed that $k_f^e$ is an accurate value for the linear stiffness coefficient. Table 5.5 lists effective natural frequencies $\Omega_5$ and effective linear stiffness coefficients $k_5^e$ for various values of $\alpha$ and $\beta_D$. For $\alpha = 1.5, 1.6, 1.7, 1.8, 2.0$ and $2.1$, the effective parameters closely match those of the dynamic experiments, indicating that a model of this form may be a better choice for describing $g(x(t))$. Note however that no analytical studies are conducted using $f(t) = f_{in}(t)$.

The improved results obtained from the numerical simulations lead to a hypothesis that the Temporal and Spectral Methods might successfully estimate an accurate model $\hat{z}_D$ for describing $g(x(t))$. Estimates by these methods are listed in Table 5.6 for the values of $\alpha$ in Table 5.5. However, the estimates
<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta_D$ (kN/m$^2$)</th>
<th>effective natural frequency, $\Omega_{si}$ (Hz)</th>
<th>effective linear stiffness, $k_{si}$ (kN/m)</th>
<th>effective natural frequency, $\Omega_{sum}$ (Hz)</th>
<th>effective linear stiffness, $k_{sum}$ (kN/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>-35</td>
<td>37.0</td>
<td>76</td>
<td>36.0</td>
<td>72</td>
</tr>
<tr>
<td>1.3</td>
<td>-200</td>
<td>38.5</td>
<td>82</td>
<td>36.0</td>
<td>72</td>
</tr>
<tr>
<td>1.5</td>
<td>-870</td>
<td>39.25</td>
<td>85</td>
<td>36.0</td>
<td>72</td>
</tr>
<tr>
<td>1.6</td>
<td>-2000</td>
<td>39.5</td>
<td>86</td>
<td>36.0</td>
<td>72</td>
</tr>
<tr>
<td>1.7</td>
<td>-4500</td>
<td>39.5</td>
<td>86</td>
<td>36.0</td>
<td>72</td>
</tr>
<tr>
<td>1.8</td>
<td>-10,000</td>
<td>39.6</td>
<td>87</td>
<td>36.0</td>
<td>72</td>
</tr>
<tr>
<td>2.0</td>
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<td>87</td>
<td>36.0</td>
<td>72</td>
</tr>
<tr>
<td>2.1</td>
<td>-110,000</td>
<td>39.75</td>
<td>87</td>
<td>36.0</td>
<td>72</td>
</tr>
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</table>

Table 5.5. Analytical results from Model D.
<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Coefficient</th>
<th>Temporal</th>
<th>Spectral</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k_D$ (kN/m)</td>
<td>I</td>
<td>II</td>
</tr>
<tr>
<td>1.1</td>
<td>84</td>
<td>81</td>
<td>76</td>
</tr>
<tr>
<td></td>
<td>$\beta_D$ (kN/m$^a$)</td>
<td>5</td>
<td>-11</td>
</tr>
<tr>
<td>1.3</td>
<td>85</td>
<td>78</td>
<td>71</td>
</tr>
<tr>
<td></td>
<td>$\beta_D$ (kN/m$^a$)</td>
<td>18</td>
<td>-26</td>
</tr>
<tr>
<td>1.5</td>
<td>85</td>
<td>78</td>
<td>70</td>
</tr>
<tr>
<td></td>
<td>$\beta_D$ (kN/m$^a$)</td>
<td>115</td>
<td>-108</td>
</tr>
<tr>
<td>1.6</td>
<td>85</td>
<td>78</td>
<td>70</td>
</tr>
<tr>
<td></td>
<td>$\beta_D$ (kN/m$^a$)</td>
<td>113</td>
<td>-136</td>
</tr>
<tr>
<td>1.7</td>
<td>85</td>
<td>78</td>
<td>70</td>
</tr>
<tr>
<td></td>
<td>$\beta_D$ (kN/m$^a$)</td>
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</tr>
<tr>
<td>1.8</td>
<td>85</td>
<td>78</td>
<td>70</td>
</tr>
<tr>
<td></td>
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<td>69</td>
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<tr>
<td></td>
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<tr>
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</tr>
<tr>
<td></td>
<td>$\beta_D$ (kN/m$^a$)</td>
<td>61800</td>
<td>-15477</td>
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</table>

Table 5.6. Estimates for Model D using Dynamic Experiments.
fail to satisfy the First Criterion since large variations of $k_o$ and $\beta_o$ exist over the three excitation levels. The best results are for $\alpha = 1.3$, where the Spectral Method estimates values for $k_{d1} = 89$ and $k_{dII} = 85$ kN/m$^3$. Both of these values satisfy the Second Criterion within a 3% error of $k^e_f$. Unfortunately, the value for $\beta_o$ more than doubles between Dynamic Test I and Dynamic Test II, and as mentioned fails to satisfy the First Criterion. Illustrated in Figure 5.7 are synthesized curves using the estimated parameters by the Spectral Method with $\alpha = 1.3$. Figure 5.7(a) contains synthesized stiffness curves where the displacement range shown corresponds to $\pm|x_{mI}|$. As can be seen, these curves appear linear. In fact, when $\hat{z}_{dI}(x)$ and the effective linear stiffness curve $\hat{z}^e_f(x) = k^e_f x$ are plotted over this displacement range, the two curves cannot be distinguished. However, $\hat{z}_{dII}$ has a less steep slope due in part to the underestimated linear term, but also due to the larger softening nonlinear term. If these curves were plotted over a larger displacement range, the nonlinear softening spring stiffness characteristics would be evident. Figure 5.7(b) illustrates synthesized FRFs using the mass of the steel cylinder $m_c$, the estimated linear stiffness coefficients $k_{dI}$ and $k_{dII}$ and the estimated linear damping coefficients $c_{dI} = 45$ N·s/m and $c_{dII} = 55$ N·s/m. Accurate estimates are indicated by synthesized FRFs which overlapped the FRF from Dynamic Test I since the response $x_c(t)$ is small and therefore nonlinearities are assumed to have negligible effects. As illustrated, Synthesis I matches more closely than Synthesis II. This is due to a over estimated damping coefficient $c_{dII}$. These results show that using the data from Dynamic Experiment I, the Spectral Method estimates a Model D with $\alpha = 1.3$ whose synthesized stiffness curve and FRF match closely to the linear stiffness curve and measured FRF from Dynamic Experiment I. This indicates that this model may be valid for random excitation levels of amplitude $|f_c(t)|$. However, this does not provide sufficient evidence that this model is valid at other excitation levels. It is essential that nonlinear methods estimate accurate models over a variety of excitation levels. This is indeed one of the difficulties faced when estimating models for nonlinear systems. It is, however, quite possible that the exponent is a function of the response or excitation amplitude, i.e. $\alpha = \alpha(|x(t)|)$. If this were true, models with different values of $\alpha$ for each excitation amplitude $|f(t)|$ would be required to characterize the isolator.

185
Figure 5.7. Synthesized Results for the isolator experiment: (a) stiffness curves, (b) frequency response functions.
5.7. CONCLUSION

Some progress has been made to account for static loads and displacements which cannot be measured by piezoelectric force and acceleration transducers. The Restoring Force Methods has been modified and has shown to successfully identify the system parameters of an analytical system in the presence of static loads.

Experimental and analytical results illustrate that linear parameter estimation schemes, such as modal analysis methods, when applied to the isolation mount of Figure 5.1, are incapable of identifying continuous mathematical models that are valid over a broad excitation range. Also, static stiffness models do not capture the dynamic elastic behavior of the mount. New or improved experimental identification schemes are needed that estimate nonlinear models which capture the behavior of isolation mounts over broad excitation levels. Two such methods, a Temporal and Spectral Method, have been studied in this chapter. Unfortunately, even these methods may lead to erroneous estimates since the types of models chosen to describe the restoring forces must be known \textit{a priori}. Perhaps the most surprising result has been the failure of polynomial type models; virtually all of the scientific literature discusses such models. Conversely, the non-integer exponent type model has been more successful but there is no literature on such formulations.
LIST OF REFERENCES FOR CHAPTER 5


CHAPTER 6

CONCLUSION

6.1. SUMMARY AND CONTRIBUTIONS

Before addressing the contributions of this dissertation, an overview of the existing state of the knowledge is provided. The available literature on the identification of nonlinear systems is rather sparse. Advances have been made as cited in the bibliography, however, more research is needed. Although Bendat et. al. [6.1-6.4] had developed a spectral method for single-degree-of-freedom systems which gives promising results, refinements were necessary for the application to multi-degree-of-freedom systems. Also, most of the current methods require a priori knowledge of the mathematical form of the nonlinearities present. Yet, literature does not address the questions raised when this knowledge is unavailable. These questions include (1) what happens when mathematical models different from the actual nonlinearities are employed in the identification? (2) can approximations be used to describe the unknown nonlinearities? (3) and will unique models result from the identification process? In addition, experimental data is typically corrupted by uncorrelated measurement noise. However, this issue is only addressed in selected literature [6.7, 6.10, 6.17] and no prior work proposes means to quantify errors due to measurement noise. Finally, although the Restoring Force Method has been applied to nonlinear experimental systems, these systems are typically contrived to produce nonlinearities of a known type [6.13] or they only consist of a single-degree-of-freedom [6.12, 6.15, 6.16]; and, no experimental applications of the "Reverse Path" Spectral Method have been conducted.

This dissertation addresses most of the issues raised above. As a first step, Chapter 2 covers experimental characterization of three rubber isolators. This study has been conducted to provide insight into the isolators' nonlinear dynamic behavior. The isolators have been subjected to static stiffness
experiments and dynamic experiments under both single and multi-degree-of-freedom system configurations. Nonlinear behavior is observed that may not have been uncovered by traditional dynamic stiffness type methods. This suggests that these new experimental techniques may be useful for evaluating true isolator behavior under system configurations. Also, to assist in quantifying the dynamic behavior of the isolators under multi-degree-of-freedom configuration, analytical models were formulated. A quasi-linear continuous system model gives accurate effective natural frequencies; and, a nonlinear discrete model is employed such that nonlinear terms could be used to describe the nonlinear nature of the isolators.

It has been shown in Chapter 3 that conventional frequency response estimation methods such as the “H₁” and “H₂” estimates are often inadequate for accurately estimating the linear dynamic compliance functions of multi-degree-of-freedom nonlinear systems when excited by Gaussian random excitations. Therefore, a new spectral approach has been developed based on a “reverse path” formulation as available in the literature for single-degree-of-freedom nonlinear systems [6.1-6.4], with emphasis on the mathematical development for application to multi-degree-of-freedom systems. With new formulation, conditioned “H₁” and “H₂” estimates of linear dynamic compliance functions can now be obtained which drastically reduce, or even eliminate in some cases, the contamination introduced by nonlinearities. This allows for the identification of the modal parameters of the underlying linear system without any undue influences caused by nonlinearities. The coefficients of analytical functions which describe the nonlinearities are also estimated by this new method. These nonlinearities may be local or distributed and they may exist at or away from the locations of the excitations.

When identifying nonlinear mechanical and structural systems, some \textit{a priori} knowledge of the nature and mathematical form of the nonlinearities is necessary. Nonetheless, an unique model is still not guaranteed since this knowledge may be limited and different mathematical formulae for describing the nonlinearities may result in reasonably accurate estimates. Also, the presence of uncorrelated noise often corrupts the response and excitation data. These problems have been illustrated in Chapter 4, and in some cases, in the presence of moderate and high measurement noise levels. Both the Restoring Force Temporal Method [6.5-6.16] and the “Reverse Path” Spectral Method [6.1-6.4] have been employed to illustrate that
these problems are not method dependent; instead they must be addressed regardless of which technique is used. Coherence functions have been developed based on the Spectral Method and their application has been demonstrated. These coherence functions indicate the level of uncorrelated noise present in the data and when reasonably accurate models have been chosen to describe the nonlinear systems. Therefore, parameters estimated by techniques such as the ones discussed here can be assessed with some level of confidence.

In Chapter 5 it is shown that static forces and displacements which exist in mechanical and structural systems, including isolation configurations, can alter the dynamic response characteristics of nonlinear systems. Also, since typical vibration data is collected with piezoelectric transducers which are insensitive to static loads and displacements, erroneous estimates of system parameters can result from identification techniques. Therefore, a modified Restoring Force temporal method has been developed which accounts for these static loads and displacements. In addition, one of the isolators considered earlier in this study was critically examined and identified using the Reverse Path Spectral and Restoring force Methods. Analytical models were also used to describe single-degree-of-freedom system behavior. Initial results were promising, however many of the issues raised throughout this study must be investigated further in order to improve the identification of nonlinear systems.

In conclusion, the specific contributions of this dissertation are as follows:

1. Enhanced experimental techniques have been applied to characterize three different elastomeric isolators.
2. A new multi-degree-of-freedom spectral approach based on a "reverse path" system model has been developed and has successfully been applied to several example systems with different nonlinearities.
3. The performance of two identification methods has been investigated when incorrect or approximate mathematical equations are used to model nonlinearities.
4. The effects of measurement noise on the identification processes have been assessed.
5. Improved coherence function estimates have been developed to assist in determining accurate nonlinear models.
Identification methods have been applied to the experimental study of isolators and refinements to the Restoring Force Method have been made to accommodate for the presence of static loads and displacements.

6.2. FUTURE WORK

Further refinements of the identification methods discussed in this dissertation are necessary. For instance, as illustrated in Chapter 4, uncorrelated measurement noise can have an impact on the accuracy of the identification schemes. Therefore, for the “Reverse Path” Spectral method, enhancements need to be made to conditioned and unconditioned spectral density functions that are corrupted by uncorrelated measurement noise. A reference method [6.18] may act as a starting point for such an investigation. Also, time domain averaging operations and cross-correlation techniques may be adapted to improve the Restoring Force Method in the presence of uncorrelated measurement noise.

It has been illustrated that the nature of the mathematical models for describing the nonlinearities must be known \textit{a priori} for the identification methods to be successful. Unfortunately, this information is not always available under “real” conditions. Therefore, additional research is necessary to overcome this problem. The next promising area appears to be the non-integer nonlinearity which has sparsely been addressed in the scientific literature [6.19]. Another possibility is the employment of different polynomial or orthogonal function expansions for approximating nonlinearities of unknown form. These should include terms considering both displacement and velocity nonlinearities, as well as cross terms. Nonlinear damping and hysteresis should especially be considered when identifying elastomeric isolators such as those studied in chapters 2 and 5. Obviously, the inclusion of additional terms to describe the possible nonlinearities present complicates the possibility of determining an accurate and unique mathematical model. However, improvements to experimental characterization may help to alleviate this complication by determining the types of nonlinearities that might be present and hence the terms for describing these nonlinearities.
Although, it may be desirous for some applications to have a mathematical model which is valid over a broad operating range, it is quite possible that for some nonlinear systems this cannot be achieved. A single model may be necessary for each operating condition. Also, under circumstances where the mathematical form of the nonlinearities is unknown, an unique model may not result. Therefore, criteria are necessary for determining the best model out of a set of non-unique models and establish the operating range for which this model is valid.

Once identification schemes have been improved and lead to accurate mathematical models over the desired operating ranges, these models may be useful for some active control algorithms requiring knowledge of certain transfer paths, specifically nonlinear paths. For time invariant systems, off-line identification prior to control would suffice since the identified model would be valid during controlled operation. However, time invariance is common, especially among systems consisting of elastomeric isolators where stiffness and damping properties are highly dependent upon temperature and other environmental effects. Under these circumstances, identification schemes need to be modified for on-line applications. These applications will require not only accuracy, but computational speed will become an issue.
LIST OF REFERENCES FOR CHAPTER 6


197


