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P -ADAPTIVE HYBRID/MIXED FINITE ELEMENT METHOD

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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1998

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ABSTRACT

The purpose of this dissertation was to develop an efficient computational program for accurate two- and three-dimensional stress analysis. The $p$-adaptive hybrid/mixed finite element formulation has been developed for this purpose. The most general Hu-Washizu principle in solid mechanics is used in the hybrid/mixed finite element formulation. The stress, strain and displacement variables are assumed independently. The displacement field is interpolated using hierarchical shape functions for $p$-adaptive purpose. The stress and strain fields are interpolated using the normalized Legendre polynomials so that the formation of the element stiffness matrix has been greatly simplified. Computations show that the computing time for the hybrid/mixed $p$-method is less than that for the displacement based $p$-method for 3-D problem. Higher accuracy can be achieved by increasing polynomial order $p$.

For element with curved boundaries and surfaces, new $p$-order geometric mappings are developed from blending functions using Lagrange hierarchical shape functions. This mapping technique can be easily incorporated into most finite element programs and works very well as seen by numerical test.

A novel Lagrange approach was developed for the 1-D hierarchical shape functions, from which the 2-D and 3-D hierarchical shape functions were constructed. This
approach started directly from the second order shape functions plus higher-order
hierarchical shape functions.

Several 2-D and 3-D numerical examples are given to test the convergence and
accuracy of the $p$-version hybrid/mixed finite element programs. Numerical examples
have shown the successful use of the $p$-version hybrid/mixed finite element method.
Dedicated to my parents
ACKNOWLEDGMENTS

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PUBLICATIONS

Research Publications


**FIELDS OF STUDY**

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CHAPTER 1

INTRODUCTION

1.1 Background of the finite element method

The finite element method, first developed by engineers, has application to many fields of engineering, such as structural mechanics, heat transfer and fluid mechanics. Its development originated in the mid-1950s for the stress analysis of structures [1, 2, 3]. Today, the finite element method has become the most widely used computational tool. Many commercial finite element software packages can give accurate solutions for complex engineering problems. The finite element analysis is an approximation to the differential equations governing the mathematical model. A continuum domain is discretized into a finite number of elements. A field variable such as displacement can be approximated by piecewise polynomials using the values of the nodal variables as unknown coefficients. The nodal values can be obtained by the minimization of a functional and solving the corresponding matrix equations with applied boundary conditions. The traditional finite element method uses a fixed low order interpolation polynomial over the element. The accuracy of the approximate solution is achieved by refining the mesh size. This is known as the $h$-version finite element method. The early $h$-version suffered a serious disorder known as locking which
is due to an excessive stiffness for a particular deformation state caused by a lack of important polynomial terms in the interpolation functions. Some locking examples are shear locking which occurs in bending problems and dilatation locking (Poisson's ratio locking) which occurs as Poisson's ratio approaches one half. The locking effects can be reduced by using reduced integration, bubble functions, selective underintegration or assumed strain. However, side effects of these remedies may also occur. Various topics associated with the locking phenomenon are presented in detail in [4].

In the early 1960s, two types of finite elements were developed for use in structural mechanics. One is the displacement based finite element method which is based on the principle of minimum potential energy, in which only displacement is assumed as the field variable. Another is the hybrid stress finite element method [5] which is based on the principle of minimum complementary energy, in which the equilibrating stress field is assumed inside each element and where the displacement variable is interpolated along the element boundary. This development followed applying the Hellinger-Reissner and the Hu-Washizu functional [6] to the finite element method, in which displacements, stresses or/and strains may be used as independent quantities for the field variables. These multi-field finite elements are called hybrid/mixed finite elements. In computations using the displacement based finite element method, the stresses are obtained from the derivative of the displacement field and are less accurate than the displacements. Therefore a stress recovery technique [7, 8, 9] is needed. In the hybrid/mixed finite element method, displacement, stress or/and strain variables are independently assumed and the minimization process of the variational functional is satisfied in the presence of multi-field variables. The hybrid/mixed approach gives
more accurate results in computing the stress field for nearly incompressible materials \((\nu \to \frac{1}{2})\) and is convenient to formulate thin plate and shell elements. However, the disadvantage in using the hybrid/mixed method is the high computational time in solving the element stiffness matrix.

To achieve higher accuracy in the finite element analysis, it is natural to increase the order of the interpolation polynomials for the field variable. This approach, started in the late 1970s and early 1980s, is called the \(p\)-version finite element method \([10, 11, 12, 13]\). The main application of the \(p\)-version finite element method has been to achieve more accurate results without having to develop a detailed finite element model. The \(p\)-method uses a fixed coarse mesh and convergence is obtained by increasing the order of the polynomial \(p\) of the element. Thus, the time to generate a detailed complicated mesh can be avoided. More importantly, the \(p\)-version finite element method can overcome the locking problem that exists in the \(h\)-method and gives high rates of convergence in the energy norm for smoother solutions. In the \(p\)-method, it is advantageous to use the hierarchical shape functions. In this case, it is only necessary to compute the coefficients in the rows and columns of the stiffness matrix associated with the new enriched degrees of freedom at each order \(p\). Therefore, the \(p\)-method is very important for adaptive finite element analysis and the \(p\)-version matrix structure is suitable for parallel computing as addressed in \([14]\).

Most research on the \(p\)-method is mainly focussed on the displacement based finite element method. The success of the \(p\)-version finite element method can be seen by its incorporation into well-known commercial programs, such as NASTRAN, ANSYS, I-DEAS and NISA, etc. However, there have been few attempts to apply
the hybrid/mixed technique to the $p$-adaptive finite element analysis since computing time is one of the major concerns. Work on higher order hybrid/mixed finite elements can be found in [15, 16, 17] and extending the hybrid-Trefftz element to include $p$-version solutions are given in [18, 19, 20].

Computational needs in engineering are to achieve the desired accuracy of numerical results but with less computational time. The $h$-adaptive, $p$-adaptive and $hp$-adaptive finite element methods were developed for this purpose. The rate of convergence for the $h$-version is at best algebraic in the degrees of freedom. The rate of convergence for the $p$-version is exponential for smoother solutions, while for the $hp$-version it is exponential for a large class of engineering problems. The $hp$-version of the finite element method was first addressed in [21], which is the combination of refining the mesh size and increasing the polynomial order. Today, a fully $hp$-adaptive finite element code known as ProPHLEX [22] is available for solving general linear and non-linear problems in continuum mechanics.

In summary, the $hp$-adaptive finite element method is the trend in modern finite element computation. The $p$-method, which is free of locking and converges more accurately to the solution, is the key for the $hp$-method. Attention should also be given to the hybrid/mixed finite element methods, which are derived from multi-field functionals and may have more advantages [23] in engineering problems.

1.2 Focus of the research

This dissertation is focussed on the hybrid/mixed finite element method combined with the $p$-adaptive technique. The displacement field for this method is assumed using hierarchical shape functions. The Hu-Washizu principle was considered among
various variational functionals in solid mechanics, since the use of the orthogonal stress and strain fields in this principle simplifies the formation of the element stiffness matrix. The achievements expected are as follow:

- Develop a $p$-version hybrid/mixed finite element code for 2-D and 3-D stress analysis.
- Keep the advantages of the displacement based $p$-method, such as the elimination of locking and high rate of convergence.
- A better calculation of stresses for nearly incompressible materials.
- Reasonable computational time when compared with the displacement based $p$-method.
- A suitable means to incorporate into existing finite element software.

1.3 Summary of contents

The variational principles in solid mechanics and their application to the finite element are introduced in Chapter 2. By comparison between various variational principles, the best choice for $p$-adaptive hybrid/mixed finite element method turned out to be the Hu-Washizu principle.

In order to develop a finite element code, the interpolation functions for the field variables needed to be constructed. The displacement field occurs in all the variational principles investigated. The $p$-type shape functions for the displacement field, using three different approaches presented in Chapter 3, were used for both the displacement based $p$-method and the hybrid/mixed $p$-method.
In the \( p \)-method, geometric mapping of elements with curved sides and surfaces is critical for convergence. Chapter 4 introduces a new \( p \)-type geometric mapping using hierarchical shape functions based on the Lagrange approach that was presented in Chapter 3.

Chapter 5 presents a formulation of the displacement based \( p \)-method. This is done since some formulations of the displacement based \( p \)-method are needed for the hybrid/mixed \( p \)-method.

Chapter 6 is the key section of this dissertation. The \( p \)-version hybrid/mixed formulations are developed in this chapter. The orthogonal stress and strain fields are interpolated using the normalized Legendre polynomials so that the formation of the stiffness matrix can be simplified. Hence, computational time can be reduced.

Numerical examples are given in Chapter 7 for 2-D and 3-D analysis. Examples are selected to explore the capabilities of the developed hybrid/mixed \( p \)-version finite element code from several aspects:

- The influence of the element shape
- The convergence of both the energy norm and stress with the influence of nearly incompressible material
- The rate of convergence of the energy norm and stress for \( h \)- and \( p \)-method
- The convergence of stress concentration with curved boundaries
- The convergence of stress concentration at the interface of different materials

Finally, conclusions and contributions of the present work, and recommendations for future research are given in Chapter 8.
CHAPTER 2

FINITE ELEMENT METHOD

The displacement based finite element method in solid mechanics is derived from the principle of minimum potential energy. The displacement is assumed using a polynomial inside each discretized element and is compatible along the interelement boundary. This is an approximation of the conditions of equilibrium and prescribed boundary traction.

In addition, a finite element formulation can also be based on the principle of minimum complementary energy. The equilibrating stress field is assumed inside each element and the displacement variable is interpolated along the element boundary. This is an approximation of the conditions of compatibility and inter-element traction reciprocity. If, in the principle of minimum potential energy, both strain-displacement relations and displacement boundary conditions are introduced as conditions of constraint through Lagrange multipliers, one obtains the Hu-Washizhu principle, which yields the stationary condition of a functional with stresses, strains, and displacements as field variables. On the other hand, if, in the complementary energy principle, the stress equilibrium conditions are enforced through Lagrange multipliers, one is led to the Hellinger-Reissner principle, which yields the stationary condition of a functional
with stresses and displacements as field variables. The Hellinger-Reissner functional may also be obtained directly from the Hu-Washizu functional by eliminating the strain variables using the stress-strain relations. Thus, the Hu-Washizu principle may be considered as the more general variational principle in solid mechanics. The finite element formulation by all above multivariable variational principles may be called the hybrid/mixed finite element method.

In this chapter, the basic formulations of linear elasticity is introduced, and the finite element formulations based on the principle of minimum potential energy, minimum complementary energy, Hellinger-Reissner principle and Hu-Washizu principle are presented. The notations used are defined as follows:

- **u** Displacement vector
- **û** Displacement vector at prescribed boundary
- **ü** Displacement vector at element boundary
- **ε** Strain vector
- **σ** Stress vector
- **T** Traction vector
- **T̅** Traction vector at prescribed boundary
- **F_b** Body force vector
- **q** Nodal displacement variable
- **α** Discretized strain variable
- **β** Discretized stress variable
- **K** Element stiffness matrix
2.1 Elasticity problem in solid mechanics

Consider a continuous elastic body occupying the volume \( \Omega \) and bounded by the surface \( \Gamma \). The external forces are the body forces \( \mathbf{F}_b \) and surface forces \( \mathbf{T} \). The surface forces are prescribed on a portion of the boundary \( \Gamma_t \) and the displacements \( \mathbf{u} \) on the remaining portion of the boundary \( \Gamma_u \). The problem is to determine the unknown displacement, strain and stress components throughout the body. The displacement, strain and stress fields will be defined as:

- displacement vector:

\[
\mathbf{u} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T
\]  \hspace{1cm} (2.1)

- strain vector:

\[
\mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{yz} & \gamma_{zx} & \gamma_{xy} \end{bmatrix}^T
\]  \hspace{1cm} (2.2)

- stress vector:

\[
\mathbf{\sigma} = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{yz} & \tau_{zx} & \tau_{xy} \end{bmatrix}^T
\]  \hspace{1cm} (2.3)
Using force equilibrium conditions, the governing differential equations are given as

$$D^T \sigma = F_b \quad (2.4)$$

where

$$F_b = \{X \ Y \ Z\}^T \quad (2.5)$$

and

$$D = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}
\end{bmatrix} \quad (2.6)$$

For a linear elastic body Hooke's law is employed. The constants for an ideally elastic homogeneous isotropic material are Young's modulus $E$ and Poisson's ratio $\nu$. The stress-strain relation is of the form

$$\sigma = C \epsilon \quad (2.7)$$

where $C$ is the stiffness matrix given by

$$C = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu & \nu & \nu \\
\nu & 1 - \nu & \nu \\
\nu & \nu & 1 - \nu \\
\frac{1 - 2\nu}{2} & \frac{1 - 2\nu}{2} & \frac{1 - 2\nu}{2}
\end{bmatrix} \quad (2.8)$$

The compliance form of Hooke's law is of the form

$$\epsilon = S \sigma \quad (2.9)$$
where $S$ is the compliance matrix. The stiffness and compliance matrices are mutually
inverse, that is

$$ S = C^{-1} \quad (2.10) $$

The strain-displacement relation can be expressed using the notation of equation
(2.6), as

$$ \epsilon = Du \quad (2.11) $$

Traction related to internal stresses are expressed as

$$ T = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z \\ \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z \end{bmatrix} \quad (2.12) $$

where $n_x$, $n_y$ and $n_z$ are the direction cosines that form the normal to the surface.

Finally, the boundary conditions for an elastic body are

$$ T = \bar{T} \quad \text{on } \Gamma_t \quad (2.13) $$

$$ u = \bar{u} \quad \text{on } \Gamma_u \quad (2.14) $$

Solving the above differential equations with prescribed boundary conditions is
equivalent to solving the stationary condition of a functional. Commonly used func­
tionals [24] in solid mechanics are:

- Potential energy functional:

$$ \Pi_{PE} = \int_{\Omega} \left[ \frac{1}{2} \epsilon^T C \epsilon d\Omega - F_h u \right] - \int_{\Gamma_t} \bar{T}^T u d\Gamma \quad (2.15) $$

where $\epsilon = Du$
• Complementary energy functional:

\[ \Pi_{CE} = \int_{\Omega} \frac{1}{2} \sigma^T \sigma \, d\Omega - \int_{\Gamma_u} T^T \bar{u} \, d\Gamma \quad (2.16) \]  

• Hellinger-Reissner functional:

\[ \Pi_{HR} = \int_{\Omega} \left[ -\frac{1}{2} \sigma^T \sigma + \sigma^T (D_u) - F_b u \right] \, d\Omega \]

\[ - \int_{\Gamma_u} T^T (u - \bar{u}) \, d\Gamma - \int_{\Gamma_t} \bar{T}^T u \, d\Gamma \quad (2.17) \]

• Hu-Washizu functional:

\[ \Pi_{HW} = \int_{\Omega} \left[ \frac{1}{2} \varepsilon^T C \varepsilon - \sigma^T (\varepsilon - D_u) - F_b u \right] \, d\Omega \]

\[ - \int_{\Gamma_u} T^T (u - \bar{u}) \, d\Gamma - \int_{\Gamma_t} \bar{T}^T u \, d\Gamma \quad (2.18) \]

These functionals can be applied to the finite element formulation to obtain numerical solutions[25]. In deriving the FE formulations using these different principles, the body force \( F_b \) is not considered for the sake of simplicity.

2.2 FE formulation by potential energy principle

In the finite element analysis, the domain is divided into discrete elements. The potential energy may then be written as the sum of the potential energy of each element

\[ \Pi_{PE} = \sum_e \Pi_{PE}^e \quad (2.19) \]

where

\[ \Pi_{PE}^e = \int_{\Omega_e} \frac{1}{2} \varepsilon^T C \varepsilon \, d\Omega - \int_{\Gamma_{te}} \bar{T}^T u \, d\Gamma \quad (2.20) \]
The nodal displacement vector \( \mathbf{q} \) and the corresponding interpolation functions \( \mathbf{N} \) are assumed, so that the displacement field \( \mathbf{u} \) in each element can be approximated as

\[
\mathbf{u} = \mathbf{Nq}
\]  \hspace{1cm} (2.21)

Then, using the strain-displacement relation (2.11), the strain vector is obtained as

\[
\mathbf{\varepsilon} = \mathbf{Bq}
\]  \hspace{1cm} (2.22)

where

\[
\mathbf{B} = \mathbf{D} \mathbf{(N)}
\]  \hspace{1cm} (2.23)

Substituting equations (2.21) and (2.22) into (2.20) yields the expression for the element potential energy as

\[
\Pi_{PE}^e = \frac{1}{2} \mathbf{q}^T \mathbf{Kq} - \mathbf{F}^T \mathbf{q}
\]  \hspace{1cm} (2.24)

where

\[
\mathbf{K} = \int_{\Omega_e} \mathbf{B}^T \mathbf{C} \mathbf{B} \mathbf{d}\Omega
\]  \hspace{1cm} (2.25)

and

\[
\mathbf{F} = \int_{\Gamma_{te}} \mathbf{N}^T \mathbf{f} \mathbf{d}\Gamma
\]  \hspace{1cm} (2.26)

The stationary condition of \( \Pi_{PE}^e \) with respect to \( \mathbf{q} \) leads to

\[
\mathbf{Kq} = \mathbf{F}
\]  \hspace{1cm} (2.27)

The assembled equations (2.27) can then be solved when displacement boundary conditions \( \mathbf{u} = \bar{\mathbf{u}} \) are prescribed.
2.3 FE formulation by complementary energy principle

In the variational functional using the complementary energy principle, the equilibrated stresses are assumed inside each element. In this case the element boundary displacements \( \tilde{u} \) serve as Lagrange multipliers to enforce the inter-element traction reciprocity. The finite element terminology using this principle is known as the hybrid stress method \([5]\). The variational functional is written as

\[
\Pi_{CE} = \sum_{e} \Pi_{CE}^e
\]  

(2.28)

where

\[
\Pi_{CE}^e = \int_{\Omega_e} \frac{1}{2} \sigma^T \sigma d\Omega - \int_{\partial\Omega_e} \mathbf{T}^T \tilde{u} d\Gamma + \int_{\Gamma_{te}} \mathbf{T}^T \tilde{u} d\Gamma
\]  

(2.29)

The equilibrated element stress vector \( \sigma \) is approximated by an interpolation function \( P \), thus

\[
\sigma = P \beta
\]  

(2.30)

where \( \beta \) is an unknown variable. From equation (2.12), the traction stress is written as

\[
\mathbf{T} = \mathbf{R}\beta
\]  

(2.31)

where

\[
\mathbf{R} = \begin{bmatrix}
n_x & 0 & 0 & 0 & n_x & n_y \\
0 & n_y & 0 & n_z & 0 & n_x \\
0 & 0 & n_z & n_y & n_x & 0
\end{bmatrix} P
\]  

(2.32)

The displacement along the element boundary is interpolated using shape function \( L \) as

\[
\tilde{u} = Lq
\]  

(2.33)
Substituting equations (2.30), (2.31) and (2.33) into (2.29) leads to the element complementary energy as

\[ \Pi_{CE}^e = \frac{1}{2} \beta^T H \beta - \beta^T G q + F^T q \]  

(2.34)

where

\[ H = \int_{\Omega_e} P^T S P d\Omega \]  

(2.35)

\[ G = \int_{\partial\Omega_e} R^T L d\Gamma \]  

(2.36)

and

\[ F = \int_{\Gamma_e} L^T T d\Gamma \]  

(2.37)

The stationary condition of \( \Pi_{CE}^e \) in equation (2.34) requires that

\[ \beta = H^{-1} G q \]  

(2.38)

and

\[ G^T \beta = F \]  

(2.39)

Eliminating \( \beta \) in equation (2.39) yields

\[ K q = F \]  

(2.40)

where the stiffness matrix \( K \) becomes

\[ K = G^T H^{-1} G \]  

(2.41)

Again the assembled equations (2.40) can be solved when the displacement boundary conditions \( \hat{u} = \bar{u} \) have been applied.
2.4 FE formulation by Hellinger-Reissner principle

In the variational functional using the Hellinger-Reissner principle, the displacement and stress fields are assumed to be independent. Hence, the Hellinger-Reissner principle applied to the finite element method [6] can be written as

$$\Pi_{HR} = \sum_e \Pi_{HR}^e$$

(2.42)

where

$$\Pi_{HR}^e = \int_{\Omega_e} \left[ -\frac{1}{2} \sigma^T S \sigma + \sigma^T (Du) \right] d\Omega$$

$$- \int_{\partial \Omega_e} T^T (u - \bar{u}) d\Gamma - \int_{\Gamma_{te}} \tilde{T}^T \tilde{u} d\Gamma$$

(2.43)

If the element displacement vector $u$ is divided into a compatible part $u_q$ and an incompatible part $u_\lambda$, i.e.

$$u = u_q + u_\lambda$$

(2.44)

then equation (2.43) will become

$$\Pi_{HR}^e = \int_{\Omega_e} \left[ -\frac{1}{2} \sigma^T S \sigma + \sigma^T (Du_q) + \sigma^T (Du_\lambda) \right] d\Omega$$

$$- \int_{\partial \Omega_e} T^T u_\lambda d\Gamma - \int_{\Gamma_{te}} \tilde{T}^T u_q d\Gamma$$

(2.45)

Applying the divergence theorem, equation (2.45) becomes

$$\Pi_{HR}^e = \int_{\Omega_e} \left[ -\frac{1}{2} \sigma^T S \sigma + \sigma^T (Du_q) - (D^T \sigma)^T u_\lambda \right] d\Omega$$

$$- \int_{\Gamma_{te}} \tilde{T}^T u_q d\Gamma$$

(2.46)

where the compatible and incompatible displacement fields are assumed as

$$u_q = N_q q$$

(2.47)
and

\[ u_\lambda = N_\lambda \lambda \]  \hspace{1cm} (2.48)

and the stresses are approximated by

\[ \sigma = P \beta \]  \hspace{1cm} (2.49)

Then, \( \Pi_{\text{HR}}^e \) becomes

\[ \Pi_{\text{HR}}^e = -\frac{1}{2} \beta^T H \beta + \beta^T G q - \beta^T R \lambda - F^T q \]  \hspace{1cm} (2.50)

where

\[ H = \int_{\Omega_e} P^T S P d\Omega \]  \hspace{1cm} (2.51)

\[ G = \int_{\Omega_e} P^T (D N_q) d\Omega \]  \hspace{1cm} (2.52)

\[ R = \int_{\Omega_e} (D^T P)^T N_\lambda d\Omega \]  \hspace{1cm} (2.53)

and

\[ F = \int_{\Gamma_{te}} N_q^T \bar{T} d\Gamma \]  \hspace{1cm} (2.54)

The stationary condition of \( \Pi_{\text{HR}}^e \) in equation (2.50) with respect to \( \beta, \lambda \) and \( q \) yields

\[ \beta = H^{-1} (G q - R \lambda) \]  \hspace{1cm} (2.55)

\[ R^T \beta = 0 \]  \hspace{1cm} (2.56)

and

\[ G^T \beta = F \]  \hspace{1cm} (2.57)
Combining equations (2.55) and (2.56) gives

$$\lambda = (R^TH^{-1}R)^{-1}R^TH^{-1}Gq$$  \hspace{1cm} (2.58)

Substituting equation (2.58) into (2.55), and then into (2.57) yields

$$Kq = F$$  \hspace{1cm} (2.59)

in which the element stiffness matrix becomes

$$K = G^TH^{-1}G - G^TH^{-1}R(R^TH^{-1}R)^{-1}R^TH^{-1}G$$  \hspace{1cm} (2.60)

If only compatible displacements ($u_x = 0$ on $\Gamma_e$) are considered in equation (2.45), then the element stiffness matrix can be given as

$$K = G^TH^{-1}G$$  \hspace{1cm} (2.61)

Hence, the set of equations (2.59) can be solved when the displacement boundary conditions $u = \bar{u}$ are applied.

2.5 FE formulation by Hu-Washizu principle

The variational functional using the Hu-Washizu principle is the most general principle, from which all the other functionals can be derived. In this principle the displacement, strain and stress fields are assumed to be independent. When the Hu-Washizu principle is applied using the finite element method [6], it can be written as

$$\Pi_{HW} = \sum_e \Pi_{HW}^e$$  \hspace{1cm} (2.62)
where

\[
\Pi_{HW} = \int_{\Omega_e} \left[ \frac{1}{2} \varepsilon^T C \varepsilon - \sigma^T \varepsilon + \sigma^T (D \mathbf{u}) \right] d\Omega - \int_{\partial \Omega_e} T^T (u - \bar{u}) d\Gamma - \int_{r_{te}} \tilde{T}^T \bar{u} d\Gamma
\]  

(2.63)

If compatible displacements are considered at the element boundaries, then \( \Pi_{HW} \) can be written as

\[
\Pi_{HW}^e = \int_{\Omega_e} \left[ \frac{1}{2} \varepsilon^T C \varepsilon - \sigma^T \varepsilon + \sigma^T (D \mathbf{u}) \right] d\Omega - \int_{r_{te}} \tilde{T}^T \mathbf{u} d\Gamma
\]  

(2.64)

The displacement can be assumed as

\[
\mathbf{u} = \mathbf{N} \mathbf{q}
\]  

(2.65)

If both \( \sigma \) and \( \varepsilon \) are approximated by the same interpolation function \( P \), then

\[
\varepsilon = P \alpha
\]  

(2.66)

and

\[
\sigma = P \beta
\]  

(2.67)

Hence, \( \Pi_{HW}^e \) becomes

\[
\Pi_{HW}^e = \frac{1}{2} \alpha^T \mathbf{M} \alpha - \beta^T \mathbf{H} \alpha + \beta^T \mathbf{G} \mathbf{q} - \mathbf{F}^T \mathbf{q}
\]  

(2.68)

where

\[
\mathbf{M} = \int_{\Omega_e} \mathbf{P}^T \mathbf{C} \mathbf{P} d\Omega
\]  

(2.69)

\[
\mathbf{H} = \int_{\Omega_e} \mathbf{P}^T \mathbf{P} d\Omega
\]  

(2.70)
\[ G = \int_{\Omega} P^T B d\Omega \quad (2.71) \]

and

\[ F = \int_{\Gamma} N^T \bar{T} d\Gamma \quad (2.72) \]

The stationary condition of \( \Pi_{HW}^e \) in equation (2.68) gives

\[ \alpha = \bar{H}^{-1} G q \quad (2.73) \]

\[ \beta = \bar{H}^{-1} M \alpha \quad (2.74) \]

and

\[ G^T \beta = F \quad (2.75) \]

Elimination of \( \alpha \) and \( \beta \) between equation (2.73), (2.74) and (2.75) yields

\[ K q = F \quad (2.76) \]

where the stiffness matrix is given as

\[ K = G^T \bar{H}^{-1} M \bar{H}^{-1} G \quad (2.77) \]

If the element displacement field \( u \) is divided into a compatible part \( u_q \) and an incompatible part \( u_\lambda \), i.e.

\[ u = u_q + u_\lambda \quad (2.78) \]

then using the same procedure used in deriving equation (2.60), the element stiffness matrix can be expressed as

\[ K = G^T \bar{H}^{-1} G - G^T \bar{H}^{-1} R \left( R^T \bar{H}^{-1} R \right)^{-1} R^T \bar{H}^{-1} G \quad (2.79) \]
where

$$H^{-1} = H^{-1}M\tilde{H}^{-1} \quad (2.80)$$

Therefore, the assembled equations (2.76) can be solved when the displacement boundary conditions $u = \tilde{u}$ are applied.
CHAPTER 3

P-TYPE SHAPE FUNCTIONS

The shape functions constructed for the assumed displacement field can be used for the displacement based finite element and also for the hybrid/mixed finite element. The displacement field $u$ is described by nodal displacement variables $\{u_j\}$, i.e.

$$u = [N_j] \{u_j\}$$

(3.1)

where $[N_j]$ are nodal shape functions with $N_j = 1$ at node $j$ and zero at all other nodes.

For a more generalized approximation of the displacement field, $u$ can be described by nodal displacement variables $\{u_j\}$ and in addition, by the generalized displacement variables $\{a_k\}$, such that

$$u = [N_j] \{u_j\} + [N_k] \{a_k\}$$

(3.2)

In the $p$-version finite element, the $[N_k]$ are hierarchical shape functions [26, 27] and the $\{a_k\}$ are corresponding hierarchical displacement variables. The hierarchical shape functions should be chosen in such a way that the set of shape functions at lower order must be in the set of shape functions at higher order and also must contribute zero value at node $j$. 

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The forming of the shape functions[28] depends on the element geometry. Commonly used elements are as follows: 2-D triangular and 3-D tetrahedral elements, 2-D rectangular and 3-D hexahedral elements, 3-D pentahedral elements, and plate and shell elements. The hierarchical shape functions for the above elements can be formed from the 1-D hierarchical shape functions. In this Chapter, we will introduce 1-D hierarchical shape functions and develop the hierarchical shape functions for the case of 2-D rectangular p-elements and 3-D hexahedral p-elements.

3.1 One-dimensional hierarchical shape functions

In the p-method, the most commonly used hierarchical shape functions for the element displacement are based on Legendre, Chebyshev and Lagrange approaches[29, 30, 31, 32]. For the 1-D problem, the displacement $u(\xi)$ in each element can be expanded using linear shape functions plus higher-order hierarchical shape functions which may be any of the above three methods.

3.1.1 Legendre approach

Let $u_1$ and $u_2$ represent nodal displacement variables, and $a_k$ the hierarchical displacement variables. Then $u(\xi)$ can be written as

$$u(\xi) = \left(\frac{1-\xi}{2}\right) u_1 + \left(\frac{1+\xi}{2}\right) u_2 + \sum_{k=2}^{p} \phi_k(\xi) a_k$$

(3.3)

where the hierarchical shape functions $\phi_k(\xi)$ are derived from Legendre polynomials $L_k(\xi)$ as

$$\phi_k(\xi) = \frac{1}{\sqrt{2(2k-1)}} \left\{L_k(\xi) - L_{k-2}(\xi)\right\}$$

(3.4)
By considering equation (A.4) given in Appendix A, the polynomials \( \phi_k(\xi) \) satisfy the conditions \( \phi_k(\pm1) = 0 \) as shown in Figure 3.1. In addition, from equation (A.6) it is noted that the functions \( \phi'_k(\xi) \) will also satisfy the orthogonality condition:

\[
\int_{-1}^{1} \phi'_m(\xi) \phi'_n(\xi) \, d\xi = \begin{cases} 
0 & m \neq n \\
1 & m = n
\end{cases}
\]  

(3.5)

3.1.2 Chebyshev approach

In the Chebyshev approach, the displacement \( u(\xi) \) is expanded using the same form as the above Legendre approach:

\[
u(\xi) = \left(\frac{1-\xi}{2}\right)u_1 + \left(\frac{1+\xi}{2}\right)u_2 + \sum_{k=2}^{p} \phi_k(\xi) a_k
\]  

(3.6)

where \( \phi_k(\xi) \) are derived from Chebyshev polynomials \( T_k(\xi) \) as

\[
\phi_k(\xi) = T_k(\xi) - T_0(\xi) \quad \text{for } k = 2, 4, 6, \ldots
\]  

(3.7)

\[
\phi_k(\xi) = T_k(\xi) - T_1(\xi) \quad \text{for } k = 3, 5, 7, \ldots
\]  

then from Appendix B, and equation (B.4), we can see that the polynomials \( \phi_k(\xi) \) satisfy the conditions \( \phi_k(\pm1) = 0 \) as shown in Figure 3.2.

3.1.3 Lagrange approach

In the Lagrange approach, the displacement \( u(\xi) \) is described by nodal displacement \( u_1 \) and \( u_2 \), and higher-order derivatives of the displacement, \( u^{(k)}(0) \), i.e.

\[
u(\xi) = \left(\frac{1-\xi}{2}\right)u_1 + \left(\frac{1+\xi}{2}\right)u_2 + \sum_{k=2}^{p} \phi_k(\xi) u^{(k)}(0)
\]  

(3.8)
Figure 3.1: 1-D Legendre hierarchical shape functions
Figure 3.2: 1-D Chebyshev hierarchical shape functions
where

\[
\phi_k (\xi) = \frac{\xi^k - 1}{k!} \quad \text{for } k = 2, 4, 6, \ldots \\
\phi_k (\xi) = \frac{\xi^k - \xi}{k!} \quad \text{for } k = 3, 5, 7, \ldots
\]  

(3.9)

The detailed derivation is given in Appendix C and the polynomials \( \phi_k (\xi) \) satisfy the conditions \( \phi_k (\pm 1) = 0 \) as shown in Figure 3.3.

### 3.1.4 Novel Lagrange approach

Extending the formulation of the Lagrange approach (see Appendix D), we conclude that the displacement \( u (\xi) \) can also be expanded by second order Lagrange shape functions plus higher-order hierarchical shape functions. Thus, the expansion is started directly from \( p = 2 \).

\[
u (\xi) = N_1 u_1 + N_2 u_2 + N_3 u_3 + \sum_{k=3}^{p} \phi_k (\xi) u^{(k)} (0)
\]

(3.10)

where

\[
N_1 = \frac{1}{2} (\xi - 1) \xi \\
N_2 = \frac{1}{2} (\xi + 1) \xi \\
N_3 = 1 - \xi^2
\]

(3.11)

and

\[
\phi_k (\xi) = \frac{\xi^k - \xi}{k!} \quad \text{for } k = 3, 5, 7, \ldots \\
\phi_k (\xi) = \frac{\xi^k - \xi^2}{k!} \quad \text{for } k = 4, 6, 8, \ldots
\]

(3.12)

Here \( u_j (j = 1, 2, 3) \) is the displacement at node \( j \) and \( u^{(k)} (0) \) is a \( k^{th} \) order derivative of \( u (\xi) \) at \( \xi = 0 \). The nodal shape function \( N_j \) has the property, \( N_j = 1 \) at node \( j \).
Figure 3.3: 1-D Lagrange hierarchical shape functions
and zero at all other nodes. The hierarchical shape functions $\phi_k(\xi)$ give zero value at all nodes ($j = 1, 2, 3$). These properties are expressed and plotted in Figure 3.4.

Higher-dimensional hierarchical shape functions can be constructed using Legendre, Chebyshev and Lagrange approaches. As mentioned in [31, 32], the $p$-method using all three approaches gives the same accuracy for linear problems. The Legendre approach gives the lowest conditioning number for the stiffness matrix. The Chebyshev approach gives the second lowest conditioning number and the Lagrange approach gives the highest conditioning number for the stiffness matrix. The Lagrange approach has some advantage over the others in determining the curved element mapping which is given later in Chapter 4. In the development of $p$-version finite element programs, both the Legendre and Lagrange approaches are used.

3.2 Two-dimensional hierarchical shape functions

3.2.1 Four-node quadrilateral $p$-type shape functions

For a two-dimensional problem, the shape functions are constructed from the product of one-dimensional shape functions. Let $(x, y)$ represent the global coordinates and $(\xi, \eta)$ represent the parametric coordinates. Consider the quadrilateral element shown in Figure 3.5. The dot marks represent four vertex nodes. The circular marks at the middle of the element side are side nodes and the cross represents a center node. There are $4(p-1)$ side nodes and $(p-2)(p-3)/2$ center nodes. The side and center nodes are hierarchical nodes.

For $p = 1$, a 4-node quadrilateral element can be expressed as follows:

$$
\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \sum_{i=1}^{4} N_i \begin{bmatrix} u_i \\ v_i \end{bmatrix}
$$

(3.13)
Figure 3.4: 1-D novel Lagrange hierarchical shape functions
where the shape functions at the four nodes are given as

\[
\begin{align*}
N_1 &= \frac{1}{4} (1 - \xi) (1 - \eta) \\
N_2 &= \frac{1}{4} (1 + \xi) (1 - \eta) \\
N_3 &= \frac{1}{4} (1 + \xi) (1 + \eta) \\
N_4 &= \frac{1}{4} (1 - \xi) (1 + \eta)
\end{align*}
\]

For \( p \geq 2 \), the hierarchical shape functions associated with side nodes can be formed from the 1-D hierarchical shape functions \( \phi_k (\xi) \) expressed in equations (3.4), (3.7) and (3.9) and are given as

\[
\begin{align*}
N_k^{(1)} &= \frac{1}{2} (1 - \eta) \phi_k (\xi) \\
N_k^{(2)} &= \frac{1}{2} (1 + \xi) \phi_k (\eta)
\end{align*}
\]
\[ N_k^{(3)} = \frac{1}{2} (1 + \eta) \phi_k (\xi) \]
\[ N_k^{(4)} = \frac{1}{2} (1 - \xi) \phi_k (\eta) \]

\[ k = 2, ..., p \]

where the superscript \(^{(i)}\) of \( N_k^{(i)} \), represents the four sides of the quadrilateral element, and each \( N_k^{(i)} \) makes zero contribution to any side other than side \( i \).

For \( p \geq 4 \), the internal interpolation functions at the center nodes are used for completeness of the polynomials up to order \( p \). These internal functions make zero contribution along all element sides.

\[ N_{ij}^{(0)} = \phi_i (\xi) \phi_j (\eta) \]

\[ i = 2, ..., p - 2 \]
\[ j = 2, ..., p - i \]

### 3.2.2 Eight-node quadrilateral \( p \)-type shape functions

Consider the quadrilateral element shown in Figure 3.6. The dot marks represent the eight nodes. The circular marks at the middle of the element side are the hierarchical side nodes and the cross represents a hierarchical center node. There are \( 4 (p - 2) \) hierarchical side nodes and \( (p - 2) (p - 3) / 2 \) hierarchical center nodes.

For a quadrilateral element, the displacements can be expressed using 8-noded Lagrange shape functions plus higher-order hierarchical shape functions. Starting directly from \( p = 2 \), an 8-node quadrilateral element can be expressed as follow:

\[ \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \sum_{i=1}^{8} N_i \begin{bmatrix} u_i \\ v_i \end{bmatrix} \]

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where the Lagrange shape functions at the eight nodes are

\[
\begin{align*}
N_1 &= \frac{1}{4} (1 - \xi) (1 - \eta) (-1 - \xi - \eta) \\
N_2 &= \frac{1}{4} (1 + \xi) (1 - \eta) (-1 + \xi - \eta) \\
N_3 &= \frac{1}{4} (1 + \xi) (1 + \eta) (-1 + \xi + \eta) \\
N_4 &= \frac{1}{4} (1 - \xi) (1 + \eta) (-1 - \xi + \eta) \\
N_5 &= \frac{1}{2} (1 - \eta) (1 - \xi^2) \\
N_6 &= \frac{1}{2} (1 + \xi) (1 - \eta^2) \\
N_7 &= \frac{1}{2} (1 + \eta) (1 - \xi^2) \\
N_8 &= \frac{1}{2} (1 - \xi) (1 - \eta^2)
\end{align*}
\] (3.18)
For \( p \geq 3 \), the hierarchical shape functions at side nodes are expressed as

\[
\begin{align*}
N_k^{(1)} &= \frac{1}{2} (1 - \eta) \phi_k (\xi) \\
N_k^{(2)} &= \frac{1}{2} (1 + \xi) \phi_k (\eta) \\
N_k^{(3)} &= \frac{1}{2} (1 + \eta) \phi_k (\xi) \\
N_k^{(4)} &= \frac{1}{2} (1 - \xi) \phi_k (\eta)
\end{align*}
\]  

(3.19)

where the \( \phi_k \) should be selected from equations (3.12).

For \( p \geq 4 \), the internal interpolation functions can be selected from Legendre polynomials \( L_n (\xi) \) as

\[
N_{ij}^{(0)} = (1 - \xi^2) (1 - \eta^2) L_i (\xi) L_j (\eta)
\]  

(3.20)

\[
i = 0, ..., p - 4 \\
j = 0, ..., p - 4 - i
\]

where \( N_{ij}^{(0)} \) contribute zero values along all element sides.

### 3.3 Three-dimensional hierarchical shape functions

#### 3.3.1 Eight-node hexahedral \( p \)-type shape functions

For a three-dimensional problem, the shape functions can also be constructed from the product of one-dimensional shape functions. Let \( (x, y, z) \) represent the global coordinates and \( (\xi, \eta, \zeta) \) represent the parametric coordinates. Consider the hexahedral element shown in Figure 3.7. The dot marks represent the eight vertex
Figure 3.7: 8-node hexahedral element with hierarchical degrees of freedom

nodes. The circular marks at the middle of the element edges are the edge nodes and, the cross marks at the center of element faces are the face nodes. The star at the center of the element represents the so called bubble node. There are $12(p-1)$ edge nodes, $3(p-2)(p-3)$ face nodes and $(p-3)(p-4)(p-5)/6$ bubble nodes. The edge, face and bubble nodes are all hierarchical nodes.

For $p = 1$, an 8-node hexahedral element can be expressed as follows:

$$
\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \sum_{i=1}^{8} N_i \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}
$$

(3.21)

where the shape functions at the eight vertex nodes are

$$
N_1 = \frac{1}{8} (1 - \xi)(1 - \eta)(1 - \zeta)
$$

$$
N_2 = \frac{1}{8} (1 - \xi)(1 + \eta)(1 - \zeta)
$$

35
\[ N_3 = \frac{1}{8} (1 - \xi) (1 + \eta) (1 + \zeta) \]
\[ N_4 = \frac{1}{8} (1 - \xi) (1 - \eta) (1 + \zeta) \]
\[ N_5 = \frac{1}{8} (1 + \xi) (1 - \eta) (1 - \zeta) \]
\[ N_6 = \frac{1}{8} (1 + \xi) (1 + \eta) (1 - \zeta) \]
\[ N_7 = \frac{1}{8} (1 + \xi) (1 + \eta) (1 + \zeta) \]
\[ N_8 = \frac{1}{8} (1 + \xi) (1 - \eta) (1 + \zeta) \]  
\[ (3.22) \]

For \( p \geq 2 \), the hierarchical shape functions associated with edge nodes can be formed from the 1-D hierarchical shape functions \( \phi_k (\xi) \) expressed in equations (3.4), (3.7) and (3.9) and are given as

\[ N_k^{(1)} = \frac{1}{4} (1 - \xi) (1 - \zeta) \phi_k (\eta) \]
\[ N_k^{(2)} = \frac{1}{4} (1 - \xi) (1 + \eta) \phi_k (\zeta) \]
\[ N_k^{(3)} = \frac{1}{4} (1 - \xi) (1 + \zeta) \phi_k (\eta) \]
\[ N_k^{(4)} = \frac{1}{4} (1 - \xi) (1 - \eta) \phi_k (\zeta) \]
\[ N_k^{(5)} = \frac{1}{4} (1 - \eta) (1 - \zeta) \phi_k (\xi) \]
\[ N_k^{(6)} = \frac{1}{4} (1 + \eta) (1 - \zeta) \phi_k (\xi) \]
\[ N_k^{(7)} = \frac{1}{4} (1 + \eta) (1 + \zeta) \phi_k (\xi) \]
\[ N_k^{(8)} = \frac{1}{4} (1 - \eta) (1 + \zeta) \phi_k (\xi) \]
\[ N_k^{(9)} = \frac{1}{4} (1 + \xi) (1 - \zeta) \phi_k (\eta) \]
\[ N_k^{(10)} = \frac{1}{4} (1 + \xi) (1 + \eta) \phi_k (\zeta) \]
\[ N_k^{(11)} = \frac{1}{4} (1 + \xi) (1 + \zeta) \phi_k (\eta) \]
where the superscript \(^{(i)}\) of \(N_k^{(i)}\), represents the twelve edges of the hexahedral element.

For \(p \geq 4\), the higher-order hierarchical shape functions for the face nodes are given as

\[
N_{mn}^{(1)} = \frac{1}{2} (1 - \zeta) \phi_m (\xi) \phi_n (\eta) \\
N_{mn}^{(2)} = \frac{1}{2} (1 + \eta) \phi_m (\zeta) \phi_n (\xi) \\
N_{mn}^{(3)} = \frac{1}{2} (1 + \zeta) \phi_m (\xi) \phi_n (\eta) \\
N_{mn}^{(4)} = \frac{1}{2} (1 - \eta) \phi_m (\zeta) \phi_n (\xi) \\
N_{mn}^{(5)} = \frac{1}{2} (1 + \xi) \phi_m (\eta) \phi_n (\zeta) \\
N_{mn}^{(6)} = \frac{1}{2} (1 - \xi) \phi_m (\eta) \phi_n (\zeta)
\]  \( (3.24) \)

\( m = 2, \ldots, p - 2 \)

\( n = 2, \ldots, p - m \)

where the superscript \(^{(i)}\) of \(N_k^{(i)}\), represents the six faces of the hexahedral element.

For \(p \geq 6\), the internal interpolation functions for the bubble nodes are used to complete the polynomials up to order \(p\). These internal shape functions are constructed as

\[
N_{ijk}^{(0)} = \phi_i (\xi) \phi_j (\eta) \phi_k (\zeta)
\]  \( (3.25) \)
3.3.2 Twenty-node hexahedral $p$-type shape functions

Consider the hexahedral element shown in Figure 3.8. The dot marks represent the twenty nodes. The circular marks at the middle of the element edges are the hexahedral edge nodes, the cross marks at the center of the element faces are the face nodes and the star at the center of the element represents the bubble node. There are $12 \, (p-2)$ edge nodes, $3 \, (p-2) \, (p-3)$ center nodes and $(p-3) \, (p-4) \, (p-5) / 6$ bubble nodes.
For a hexahedral element, the displacements can also be expressed by 20-node Lagrange shape functions plus higher-order hierarchical shape functions. Starting directly from $p = 2$, a 20-node hexahedral element can be expressed as:

$$
\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \sum_{i=1}^{20} N_i \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}
$$

(3.26)

where the Lagrange shape functions at the twenty nodes are listed as:

$$
\begin{align*}
N_1 &= \frac{1}{8} (1 - \xi) (1 - \eta) (1 - \zeta) (-2 - \xi - \eta - \zeta) \\
N_2 &= \frac{1}{8} (1 - \xi) (1 + \eta) (1 - \zeta) (-2 - \xi + \eta - \zeta) \\
N_3 &= \frac{1}{8} (1 - \xi) (1 + \eta) (1 + \zeta) (-2 - \xi + \eta + \zeta) \\
N_4 &= \frac{1}{8} (1 - \xi) (1 - \eta) (1 + \zeta) (-2 - \xi - \eta + \zeta) \\
N_5 &= \frac{1}{8} (1 + \xi) (1 - \eta) (1 - \zeta) (-2 + \xi - \eta - \zeta) \\
N_6 &= \frac{1}{8} (1 + \xi) (1 + \eta) (1 - \zeta) (-2 + \xi + \eta - \zeta) \\
N_7 &= \frac{1}{8} (1 + \xi) (1 + \eta) (1 + \zeta) (-2 + \xi + \eta + \zeta) \\
N_8 &= \frac{1}{8} (1 + \xi) (1 - \eta) (1 + \zeta) (-2 + \xi - \eta + \zeta) \\
N_9 &= \frac{1}{4} (1 - \xi) (1 - \eta^2) (1 - \zeta) \\
N_{10} &= \frac{1}{4} (1 - \xi) (1 + \eta) (1 - \zeta^2) \\
N_{11} &= \frac{1}{4} (1 - \xi) (1 - \eta^2) (1 + \zeta) \\
N_{12} &= \frac{1}{4} (1 - \xi) (1 - \eta) (1 - \zeta^2) \\
N_{13} &= \frac{1}{4} (1 - \xi^2) (1 - \eta) (1 - \zeta) \\
N_{14} &= \frac{1}{4} (1 - \xi^2) (1 + \eta) (1 - \zeta) \\
N_{15} &= \frac{1}{4} (1 - \xi^2) (1 + \eta) (1 + \zeta) \\
N_{16} &= \frac{1}{4} (1 - \xi^2) (1 - \eta) (1 + \zeta)
\end{align*}
$$
\[ N_{17} = \frac{1}{4} (1 + \xi) \left( 1 - \eta^2 \right) (1 - \zeta) \]
\[ N_{18} = \frac{1}{4} (1 + \xi) (1 + \eta) \left( 1 - \zeta^2 \right) \]
\[ N_{19} = \frac{1}{4} (1 + \xi) \left( 1 - \eta^2 \right) (1 + \zeta) \]
\[ N_{20} = \frac{1}{4} (1 + \xi) (1 - \eta) \left( 1 - \zeta^2 \right) \]

(3.27)

For 3-D higher-order hierarchical shape functions, 1-D hierarchical shape functions \( \phi_k(\xi) \) are used which are selected from equations (3.12). In addition \( \phi_2(\xi) \) is defined as \( \frac{1}{2} (1 - \xi^2) \).

For \( p \geq 3 \), the higher-order hierarchical shape functions for the edge nodes are given as

\[ N_k^{(1)} = \frac{1}{4} (1 - \xi) (1 - \zeta) \phi_k(\eta) \]
\[ N_k^{(2)} = \frac{1}{4} (1 - \xi) (1 + \eta) \phi_k(\zeta) \]
\[ N_k^{(3)} = \frac{1}{4} (1 - \xi) (1 + \zeta) \phi_k(\eta) \]
\[ N_k^{(4)} = \frac{1}{4} (1 - \xi) (1 - \eta) \phi_k(\zeta) \]
\[ N_k^{(5)} = \frac{1}{4} (1 - \eta) (1 - \zeta) \phi_k(\xi) \]
\[ N_k^{(6)} = \frac{1}{4} (1 + \eta) (1 - \zeta) \phi_k(\xi) \]
\[ N_k^{(7)} = \frac{1}{4} (1 + \eta) (1 + \zeta) \phi_k(\xi) \]
\[ N_k^{(8)} = \frac{1}{4} (1 - \eta) (1 + \zeta) \phi_k(\xi) \]
\[ N_k^{(9)} = \frac{1}{4} (1 + \xi) (1 - \zeta) \phi_k(\eta) \]
\[ N_k^{(10)} = \frac{1}{4} (1 + \xi) (1 + \eta) \phi_k(\zeta) \]
\[ N_k^{(11)} = \frac{1}{4} (1 + \xi) (1 + \zeta) \phi_k(\eta) \]
\[ N_k^{(12)} = \frac{1}{4} (1 + \xi) (1 - \eta) \phi_k(\zeta) \]

(3.28)
\[ k = 3, \ldots, p \]

For \( p \geq 4 \), the higher-order hierarchical shape functions for the face nodes are given as

\[
N_{mn}^{(1)} = \frac{1}{2} (1 - \zeta) \phi_m (\xi) \phi_n (\eta) \\
N_{mn}^{(2)} = \frac{1}{2} (1 + \eta) \phi_m (\zeta) \phi_n (\xi) \\
N_{mn}^{(3)} = \frac{1}{2} (1 + \zeta) \phi_m (\xi) \phi_n (\eta) \\
N_{mn}^{(4)} = \frac{1}{2} (1 - \eta) \phi_m (\zeta) \phi_n (\xi) \\
N_{mn}^{(5)} = \frac{1}{2} (1 + \xi) \phi_m (\eta) \phi_n (\zeta) \\
N_{mn}^{(6)} = \frac{1}{2} (1 - \xi) \phi_m (\eta) \phi_n (\zeta)
\]  \hspace{1cm} (3.29)

\[ m = 2, \ldots, p - 2 \]
\[ n = 2, \ldots, p - m \]

and for \( p \geq 6 \), the bubble shape functions are expressed as

\[
N_{ijk}^{(0)} = \phi_i (\xi) \phi_j (\eta) \phi_k (\zeta)
\]  \hspace{1cm} (3.30)

\[ i = 2, \ldots, p - 4 \]
\[ j = 2, \ldots, p - 2 - i \]
\[ k = 2, \ldots, p - i - j \]
CHAPTER 4

P-TYPE GEOMETRIC MAPPING

The accurate geometric mapping for elements with curved sides and surfaces is very important in producing accurate numerical results as the order $p$ of the shape functions increases. For example, the stress concentration problems with circular or elliptical curved boundaries will not converge to the maximum stress with increasing order of $p$ when geometric mapping keeps only first order accuracy. However, the convergence of the maximum stress is improved if geometric mapping keeps second order accuracy, such as 8-node quadrilateral or 20-node hexahedral mapping. In order to obtain sufficient accuracy in the finite element solution, a blending function method should be used. This technique was given in [28, 33] for an exact 2-D curved element mapping and was also used for developing a so called quasi-regional mapping for 2-D and 3-D cases [34]. In this chapter, based on the blending function method, a novel $p$-order geometric mapping will be derived using hierarchical shape functions based on a Lagrange approach. The Lagrange approach is used because the hierarchical degrees of freedoms have a physical meaning which can be seen from the 1-D hierarchical shape functions introduced in sections 3.1.3 and 3.1.4. The detailed development of the novel $p$-order geometric mapping is given in the following sections.
4.1 Two-dimensional geometric mapping

4.1.1 Four-node $p$-type geometric mapping

Consider a simple case of a quadrilateral element with a curved boundary segment on side 2 as shown in Figure 4.1. The curve functions are given in parametric form as

\[
\begin{align*}
x &= X_2(\eta) \\
y &= Y_2(\eta)
\end{align*}
\]

(4.1)

and satisfy the conditions

\[
\begin{align*}
X_2(-1) &= x_2 \\
Y_2(-1) &= y_2 \\
X_2(1) &= x_3 \\
Y_2(1) &= y_3
\end{align*}
\]

(4.2)
The exact mapping functions can be obtained using the blending function method given in [28]. These mapping functions are given as

\[
\begin{bmatrix}
  x \\
  y \\
\end{bmatrix}
= \sum_{i=1}^{4} N_i \begin{bmatrix}
  x_i \\
  y_i \\
\end{bmatrix}
+ \sum_{k=2}^{p} \left( \frac{1 + \xi}{2} \right) \phi_k(\eta) \begin{bmatrix}
  X_2^{(k)}(0) \\
  Y_2^{(k)}(0) \\
\end{bmatrix}
\]  

(4.3)

Expanding \( X_2(\eta) \) and \( Y_2(\eta) \) as in equation (3.8) yields a higher-order geometric mapping as

\[
\begin{bmatrix}
  x \\
  y \\
\end{bmatrix}
= \sum_{i=1}^{4} N_i \begin{bmatrix}
  x_i \\
  y_i \\
\end{bmatrix}
+ \sum_{k=2}^{p} \left( \frac{1 + \xi}{2} \right) \phi_k(\eta) \begin{bmatrix}
  X_2^{(k)}(0) \\
  Y_2^{(k)}(0) \\
\end{bmatrix}
\]  

(4.4)

Considering equation (3.15), equation (4.4) becomes

\[
\begin{bmatrix}
  x \\
  y \\
\end{bmatrix}
= \sum_{i=1}^{4} N_i \begin{bmatrix}
  x_i \\
  y_i \\
\end{bmatrix}
+ \sum_{k=2}^{p} N_k^{(2)} \begin{bmatrix}
  X_2^{(k)}(0) \\
  Y_2^{(k)}(0) \\
\end{bmatrix}
\]  

(4.5)

where \( X_2^{(k)}(0) \) and \( Y_2^{(k)}(0) \) are known \( k \)-th order derivatives of \( X_2(\eta) \) and \( Y_2(\eta) \) at \( \eta = 0 \), and \( N_i \) and \( N_k^{(2)} \) are nodal shape functions and hierarchical shape functions based on the Lagrange approach.

For the more general case of a element with four curved sides, the mapping becomes

\[
\begin{bmatrix}
  x \\
  y \\
\end{bmatrix}
= \sum_{i=1}^{4} N_i \begin{bmatrix}
  x_i \\
  y_i \\
\end{bmatrix}
+ \sum_{i=1}^{4} \sum_{k=2}^{p} N_k^{(l)} \begin{bmatrix}
  X_i^{(k)}(0) \\
  Y_i^{(k)}(0) \\
\end{bmatrix}
\]  

(4.6)

This \( p \)-type geometric mapping should be applied to the \( p \)-element described in section 3.2.1.

### 4.1.2 Eight-node \( p \)-type geometric mapping

If the shape functions start from an 8-node quadrilateral element, then using the shape functions in equations (3.18) and (3.19) the geometric mapping using the
blending function approach may be written as

\[
\begin{align*}
\begin{cases}
  x \\
  y 
\end{cases} &= \sum_{i=1}^{8} N_i \begin{cases}
  x_i \\
  y_i
\end{cases} + \begin{cases}
  (X_2(\eta) - N_2x_2 - N_6x_6 - N_3x_3)^{\frac{1+\xi}{2}} \\
  (Y_2(\eta) - N_2y_2 - N_6y_6 - N_3y_3)^{\frac{1+\xi}{2}}
\end{cases} \\
&\quad + \sum_{i=1}^{4} N_i \begin{cases}
  x_i^{(k)}(0) \\
  y_i^{(k)}(0)
\end{cases}
\end{align*}
\] (4.7)

Applying the procedures from equations (4.4) through (4.6), the corresponding mapping functions are developed as

\[
\begin{align*}
\begin{cases}
  x \\
  y 
\end{cases} &= \sum_{i=1}^{8} N_i \begin{cases}
  x_i \\
  y_i
\end{cases} + \sum_{i=1}^{4} \sum_{k=3}^{p} N_i^{(l)} \begin{cases}
  X_i^{(k)}(0) \\
  Y_i^{(k)}(0)
\end{cases} \\
&\quad + \sum_{i=1}^{4} N_i \begin{cases}
  x_i^{(k)}(0) \\
  y_i^{(k)}(0)
\end{cases}
\end{align*}
\] (4.8)

where \( N_i \) and \( N_i^{(l)} \) are nodal shape functions and hierarchical shape functions based on the novel Lagrange approach starting directly from \( p = 2 \). This \( p \)-type geometric mapping should be applied to the \( p \)-element described in section 3.2.2.

### 4.2 Three-dimensional geometric mapping

For a three-dimensional hexahedral element, the curve functions of the element edges and surface functions of the element faces can be given in parametric form as

\[
\begin{align*}
\begin{cases}
  x = X_i(t) \\
  y = Y_i(t) \\
  z = Z_i(t)
\end{cases} \\
\end{align*}
\] (4.9)

and

\[
\begin{align*}
\begin{cases}
  x = \tilde{X}_s(t_1, t_2) \\
  y = \tilde{Y}_s(t_1, t_2) \\
  z = \tilde{Z}_s(t_1, t_2)
\end{cases}
\end{align*}
\] (4.10)

Here \( l \) represents the twelve element edges and \( t \) is used as \( \xi, \eta \) or \( \zeta \) in the curve functions; \( s \) represents the six element faces and \( t_1 \) and \( t_2 \) are used as \( \xi, \eta \) or \( \zeta \) in the surface functions.
For the 3-D mapping of an element with curved edges and surfaces, the ideas of the 2-D mapping methods given in section 4.1 can be extended to the 3-D problem. The higher order 3-D mapping for an 8-node hexahedral element can be written as:

\[
\begin{bmatrix}
  x \\
  y \\
  z
end{bmatrix} = \sum_{i=1}^{8} N_i \begin{bmatrix}
  x_i \\
  y_i \\
  z_i
end{bmatrix} + \sum_{i=1}^{p} \sum_{k=2}^{p} N_{k}^{(l)} \begin{bmatrix}
  X_i^{(k)}(0) \\
  Y_i^{(k)}(0) \\
  Z_i^{(k)}(0)
end{bmatrix} \\
+ \sum_{s=1}^{6} \sum_{m=2}^{p-2} N_{m}^{(s)} \begin{bmatrix}
  \tilde{X}_s^{(m,n)}(0,0) \\
  \tilde{Y}_s^{(m,n)}(0,0) \\
  \tilde{Z}_s^{(m,n)}(0,0)
end{bmatrix}
\]

(4.11)

This can be verified by noting that the \( \phi_k(t) \) used for constructing the hierarchical shape functions \( N_k^{(l)} \) and \( N_{mn}^{(s)} \) satisfy

\[
\frac{\partial^i}{\partial t^i} \phi_k(0) = \begin{cases} 
0 & i \neq k \\
1 & i = k
\end{cases}
\]

(4.12)

where \( N_k^{(l)} \) and \( N_{mn}^{(s)} \) are the 3-D hierarchical shape functions from equations (3.23) and (3.24) based on the Lagrange approach. In equation (4.11), for \( p = 1 \), only the first term is involved; the second term is for \( p \geq 2 \); the third term is for \( p \geq 4 \). Using equations (3.22), (3.23), (3.24) and (3.25) based on the Lagrange approach, we find that the mapping function (4.11) satisfies the followings:

1. for edge 1 at \( \xi = -1 \) and \( \zeta = -1 \)

\[
\begin{align*}
\frac{\partial^k}{\partial \eta^k} x(-1,0,-1) &= X_1^{(k)}(0) \\
\frac{\partial^k}{\partial \eta^k} y(-1,0,-1) &= Y_1^{(k)}(0) \\
\frac{\partial^k}{\partial \eta^k} z(-1,0,-1) &= Z_1^{(k)}(0)
\end{align*}
\]

(4.13)
2. for edge 2 at $\xi = -1$ and $\eta = 1$

\[
\frac{\partial^k}{\partial \zeta^k} x (-1, 1, 0) = X_2^{(k)}(0) \\
\frac{\partial^k}{\partial \zeta^k} y (-1, 1, 0) = Y_2^{(k)}(0) \\
\frac{\partial^k}{\partial \zeta^k} z (-1, 1, 0) = Z_2^{(k)}(0)
\]
(4.14)

3. for edge 3 at $\xi = -1$ and $\zeta = 1$

\[
\frac{\partial^k}{\partial \eta^k} x (-1, 0, 1) = X_3^{(k)}(0) \\
\frac{\partial^k}{\partial \eta^k} y (-1, 0, 1) = Y_3^{(k)}(0) \\
\frac{\partial^k}{\partial \eta^k} z (-1, 0, 1) = Z_3^{(k)}(0)
\]
(4.15)

4. for edge 4 at $\xi = -1$ and $\eta = -1$

\[
\frac{\partial^k}{\partial \zeta^k} x (-1, -1, 0) = X_4^{(k)}(0) \\
\frac{\partial^k}{\partial \zeta^k} y (-1, -1, 0) = Y_4^{(k)}(0) \\
\frac{\partial^k}{\partial \zeta^k} z (-1, -1, 0) = Z_4^{(k)}(0)
\]
(4.16)

5. for edge 5 at $\eta = -1$ and $\zeta = -1$

\[
\frac{\partial^k}{\partial \zeta^k} x (0, -1, -1) = X_5^{(k)}(0) \\
\frac{\partial^k}{\partial \zeta^k} y (0, -1, -1) = Y_5^{(k)}(0) \\
\frac{\partial^k}{\partial \zeta^k} z (0, -1, -1) = Z_5^{(k)}(0)
\]
(4.17)

6. for edge 6 at $\eta = 1$ and $\zeta = -1$

\[
\frac{\partial^k}{\partial \zeta^k} x (0, 1, -1) = X_6^{(k)}(0)
\]
\[
\frac{\partial^k}{\partial \zeta^k} y(0,1,-1) = Y_6^{(k)}(0) \\
\frac{\partial^k}{\partial \zeta^k} z(0,1,-1) = Z_6^{(k)}(0)
\] (4.18)

7. for edge 7 at \( \eta = 1 \) and \( \zeta = 1 \)

\[
\frac{\partial^k}{\partial \xi^k} x(0,1,1) = X_7^{(k)}(0) \\
\frac{\partial^k}{\partial \xi^k} y(0,1,1) = Y_7^{(k)}(0) \\
\frac{\partial^k}{\partial \xi^k} z(0,1,1) = Z_7^{(k)}(0)
\] (4.19)

8. for edge 8 at \( \eta = -1 \) and \( \zeta = 1 \)

\[
\frac{\partial^k}{\partial \xi^k} x(0,-1,1) = X_8^{(k)}(0) \\
\frac{\partial^k}{\partial \xi^k} y(0,-1,1) = Y_8^{(k)}(0) \\
\frac{\partial^k}{\partial \xi^k} z(0,-1,1) = Z_8^{(k)}(0)
\] (4.20)

9. for edge 9 at \( \xi = 1 \) and \( \zeta = -1 \)

\[
\frac{\partial^k}{\partial \eta^k} x(1,0,-1) = X_9^{(k)}(0) \\
\frac{\partial^k}{\partial \eta^k} y(1,0,-1) = Y_9^{(k)}(0) \\
\frac{\partial^k}{\partial \eta^k} z(1,0,-1) = Z_9^{(k)}(0)
\] (4.21)

10. for edge 10 at \( \xi = 1 \) and \( \eta = 1 \)

\[
\frac{\partial^k}{\partial \zeta^k} x(1,1,0) = X_{10}^{(k)}(0) \\
\frac{\partial^k}{\partial \zeta^k} y(1,1,0) = Y_{10}^{(k)}(0) \\
\frac{\partial^k}{\partial \zeta^k} z(1,1,0) = Z_{10}^{(k)}(0)
\] (4.22)
11. for edge 11 at $\xi = 1$ and $\zeta = 1$

\[
\begin{align*}
\frac{\partial^k}{\partial \eta^k} x(1,0,1) &= X_{11}^{(k)} (0) \\
\frac{\partial^k}{\partial \eta^k} y(1,0,1) &= Y_{11}^{(k)} (0) \\
\frac{\partial^k}{\partial \eta^k} z(1,0,1) &= Z_{11}^{(k)} (0) \\
\end{align*}
\]

(4.23)

12. for edge 12 at $\xi = 1$ and $\eta = -1$

\[
\begin{align*}
\frac{\partial^k}{\partial \zeta^k} x(1,-1,0) &= X_{12}^{(k)} (0) \\
\frac{\partial^k}{\partial \zeta^k} y(1,-1,0) &= Y_{12}^{(k)} (0) \\
\frac{\partial^k}{\partial \zeta^k} z(1,-1,0) &= Z_{12}^{(k)} (0) \\
\end{align*}
\]

(4.24)

It is also noted that the mapping (4.11) satisfies the following conditions

1. for face 1 at $\zeta = -1$

\[
\begin{align*}
\frac{\partial^m}{\partial \zeta^m} \frac{\partial^n}{\partial \eta^n} x(0,0,-1) &= \tilde{X}_1^{(m,n)} (0,0) \\
\frac{\partial^m}{\partial \zeta^m} \frac{\partial^n}{\partial \eta^n} y(0,0,-1) &= \tilde{Y}_1^{(m,n)} (0,0) \\
\frac{\partial^m}{\partial \zeta^m} \frac{\partial^n}{\partial \eta^n} z(0,0,-1) &= \tilde{Z}_1^{(m,n)} (0,0) \\
\end{align*}
\]

(4.25)

2. for face 2 at $\eta = 1$

\[
\begin{align*}
\frac{\partial^m}{\partial \zeta^m} \frac{\partial^n}{\partial \xi^n} x(0,1,0) &= \tilde{X}_2^{(m,n)} (0,0) \\
\frac{\partial^m}{\partial \zeta^m} \frac{\partial^n}{\partial \xi^n} y(0,1,0) &= \tilde{Y}_2^{(m,n)} (0,0) \\
\frac{\partial^m}{\partial \zeta^m} \frac{\partial^n}{\partial \xi^n} z(0,1,0) &= \tilde{Z}_2^{(m,n)} (0,0) \\
\end{align*}
\]

(4.26)
3. for face 3 at $\zeta = 1$

\[
\begin{align*}
\frac{\partial^m}{\partial \xi^m} \frac{\partial^n}{\partial \eta^n} x (0, 0, 1) &= X_3^{(m,n)} (0, 0) \\
\frac{\partial^m}{\partial \xi^m} \frac{\partial^n}{\partial \eta^n} y (0, 0, 1) &= Y_3^{(m,n)} (0, 0) \\
\frac{\partial^m}{\partial \xi^m} \frac{\partial^n}{\partial \eta^n} z (0, 0, 1) &= Z_3^{(m,n)} (0, 0)
\end{align*}
\] (4.27)

4. for face 4 at $\eta = -1$

\[
\begin{align*}
\frac{\partial^m}{\partial \xi^m} \frac{\partial^n}{\partial \eta^n} x (0, -1, 0) &= X_4^{(m,n)} (0, 0) \\
\frac{\partial^m}{\partial \xi^m} \frac{\partial^n}{\partial \eta^n} y (0, -1, 0) &= Y_4^{(m,n)} (0, 0) \\
\frac{\partial^m}{\partial \xi^m} \frac{\partial^n}{\partial \eta^n} z (0, -1, 0) &= Z_4^{(m,n)} (0, 0)
\end{align*}
\] (4.28)

5. for face 5 at $\xi = 1$

\[
\begin{align*}
\frac{\partial^m}{\partial \eta^m} \frac{\partial^n}{\partial \zeta^n} x (1, 0, 0) &= X_5^{(m,n)} (0, 0) \\
\frac{\partial^m}{\partial \eta^m} \frac{\partial^n}{\partial \zeta^n} y (1, 0, 0) &= Y_5^{(m,n)} (0, 0) \\
\frac{\partial^m}{\partial \eta^m} \frac{\partial^n}{\partial \zeta^n} z (1, 0, 0) &= Z_5^{(m,n)} (0, 0)
\end{align*}
\] (4.29)

6. for face 6 at $\xi = -1$

\[
\begin{align*}
\frac{\partial^m}{\partial \eta^m} \frac{\partial^n}{\partial \zeta^n} x (-1, 0, 0) &= X_6^{(m,n)} (0, 0) \\
\frac{\partial^m}{\partial \eta^m} \frac{\partial^n}{\partial \zeta^n} y (-1, 0, 0) &= Y_6^{(m,n)} (0, 0) \\
\frac{\partial^m}{\partial \eta^m} \frac{\partial^n}{\partial \zeta^n} z (-1, 0, 0) &= Z_6^{(m,n)} (0, 0)
\end{align*}
\] (4.30)

The mapping function (4.11) should be applied to the $p$-element described in section 3.3.1. This mapping is actually a generalized hierarchical isoparametric mapping for the $p$-method.
For a 20-node hexahedral element with curved element edges and surfaces, the generalized hierarchical isoparametric mapping for the p-method can then be written as:

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \sum_{i=1}^{20} N_i \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} + \sum_{i=1}^{12} \sum_{k=3}^{p} N_i^{(k)} \begin{bmatrix} X_i^{(k)}(0) \\ Y_i^{(k)}(0) \\ Z_i^{(k)}(0) \end{bmatrix} + \sum_{s=1}^{6} \sum_{m=2}^{p-2} \sum_{n=2}^{p-m} N_{mn}^{(s)} \begin{bmatrix} \bar{X}_{s}^{(m,n)}(0,0) \\ \bar{Y}_{s}^{(m,n)}(0,0) \\ \bar{Z}_{s}^{(m,n)}(0,0) \end{bmatrix}
\] (4.31)

Here \(N_i^{(k)}\) and \(N_{mn}^{(s)}\) are the 3-D hierarchical shape functions from equations (3.28) and (3.29) based on the novel Lagrange approach. The mapping function (4.31) should be applied to the p-element described in section 3.3.2.
CHAPTER 5

THE DISPLACEMENT BASED P-METHOD

The displacement based finite element method is based on the minimization of the potential energy principle in which only the displacement field is interpolated in each element. In the $p$-method, shape functions for the element displacement are assumed to be hierarchical (see Chapter 3). The displacement in each element can be written as

$$ u = [N_j] \{u_j\} + [N_k] \{a_k\} $$  \hspace{1cm} (5.1)

where $u_j$ is the displacement at node $j$ and $a_k$ is a generalized variable. The nodal shape function $N_j$ has the property, $N_j = 1$ at node $j$ and $N_j = 0$ at all other nodes. The hierarchical shape function $N_k$ must make zero contribution at the nodes. Substituting these approximations into the formulation of the potential energy principle leads to a set of linear equations which can be solved numerically. The important step in this process is how to assume the shape functions, construct the stiffness matrix and form the load vector. Many formulations presented in this chapter are used for the hybrid/mixed $p$-version finite element method in Chapter 6. The programs using the displacement based $p$-method are also written for comparison with the hybrid/mixed $p$-version finite element programs.
In the displacement based \( p \)-method, the element stiffness matrix is computed from the hierarchical shape functions so that the programming can be designed adaptively. This is known as the \( p \)-adaptive finite element method. From the derivation in section 2.2, the assembled global stiffness matrix equation was given as:

\[
\sum_c K q = \sum_c F \quad (5.2)
\]

where

\[
K = \int_{\Omega_e} B^T C B \, d\Omega \quad (5.3)
\]

and

\[
F = \int_{\Gamma_e} N^T T \, d\Gamma \quad (5.4)
\]

### 5.1 Two-dimensional stiffness matrix

When the geometric mapping between coordinates \( (x, y) \) and \( (\xi, \eta) \) is determined, the Jacobian matrix for a 2-D problem can be computed from

\[
J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad (5.5)
\]

Using the chain rule, the relation between the differentiation with coordinates \( (x, y) \) and the differentiation with \( (\xi, \eta) \) can be expressed as

\[
\left\{ \frac{\partial}{\partial x} \right\} = J^{-1} \left\{ \frac{\partial}{\partial \xi} \right\} = \frac{1}{|J|} \left[ \frac{\partial y}{\partial \xi} \, \frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \eta} \, \frac{\partial}{\partial \xi} \right] \left\{ \frac{\partial}{\partial \xi} \right\} \quad (5.6)
\]

Therefore,

\[
\frac{\partial N_i}{\partial x} = \frac{1}{|J|} \left( \frac{\partial y}{\partial \eta} \frac{\partial N_i}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial N_i}{\partial \eta} \right) \quad (5.7)
\]
\[
\frac{\partial N_i}{\partial y} = \frac{1}{|J|} \left( -\frac{\partial x}{\partial \eta} \frac{\partial N_i}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial N_i}{\partial \eta} \right) \tag{5.8}
\]

The B matrix in the finite element method is used to relate the strain field $\varepsilon$ to the discretized nodal displacement vector $q$. In the 2-D problem, $B = \{B_1, B_2, ..., B_n\}$ with $B_i$ given as

\[
B_i = \begin{bmatrix}
\frac{\partial N_i}{\partial x} & 0 \\
0 & \frac{\partial N_i}{\partial y}
\end{bmatrix}
\tag{5.9}
\]

Using equations (5.7) and (5.8) and separating out the Jacobian determinant from equation (5.9) yields

\[
B = \frac{1}{|J|} B^* (\xi, \eta) \tag{5.10}
\]

where $B^* = \{B_1^*, B_2^*, ..., B_n^*\}$ with $B_i^*$ given as

\[
B_i^* = \begin{bmatrix}
b_{i_1}^* & 0 \\
0 & b_{i_2}^*
\end{bmatrix}
\tag{5.11}
\]

and

\[
b_{i_1}^* = \frac{\partial y}{\partial \eta} \frac{\partial N_i}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial N_i}{\partial \eta} \tag{5.12}
\]

\[
b_{i_2}^* = -\frac{\partial x}{\partial \eta} \frac{\partial N_i}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial N_i}{\partial \eta} \tag{5.13}
\]

The element stiffness matrix (5.3) then becomes

\[
K = \int_{-1}^{1} \int_{-1}^{1} B^T C B^* \frac{1}{|J|} \xi d\xi d\eta \tag{5.14}
\]

Finally the element stiffness matrix can be computed and expressed in detail as

\[
K = \begin{bmatrix}
\cdots & \cdots & \cdots \\
\cdots & k_{ij}^{11} & k_{ij}^{12} \\
\cdots & k_{ij}^{21} & k_{ij}^{22} \\
\cdots & \cdots & \cdots
\end{bmatrix} \tag{5.15}
\]
where the subscripts \( i \) and \( j \) represent the nodal or hierarchical nodal numbers in element level and

\[
\begin{align*}
k_{ij}^{11} &= c_{11} b_{ij}^{11} + c_{33} b_{ij}^{22} \\
k_{ij}^{12} &= c_{12} b_{ij}^{12} + c_{33} b_{ij}^{21} \\
k_{ij}^{21} &= c_{12} b_{ij}^{21} + c_{33} b_{ij}^{12} \\
k_{ij}^{22} &= c_{22} b_{ij}^{22} + c_{33} b_{ij}^{11}
\end{align*}
\]

(5.16)

where

\[
\begin{align*}
b_{ij}^{11} &= \int_{-1}^{1} \int_{-1}^{1} b_i^* b_j^* \frac{1}{|J|} d\xi d\eta \\
b_{ij}^{12} &= \int_{-1}^{1} \int_{-1}^{1} b_i^* b_j^* \frac{1}{|J|} d\xi d\eta \\
b_{ij}^{21} &= \int_{-1}^{1} \int_{-1}^{1} b_i^* b_j^* \frac{1}{|J|} d\xi d\eta \\
b_{ij}^{22} &= \int_{-1}^{1} \int_{-1}^{1} b_i^* b_j^* \frac{1}{|J|} d\xi d\eta
\end{align*}
\]

(5.17)

5.2 Three-dimensional stiffness matrix

Consider the set of local coordinates \((\xi, \eta, \zeta)\) and the set of global coordinates \((x, y, z)\). The transformation between global and local coordinates is written in general as:

\[
\begin{align*}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x(\xi, \eta, \zeta) \\ y(\xi, \eta, \zeta) \\ z(\xi, \eta, \zeta) \end{bmatrix} \\
\end{align*}
\]

(5.18)

The Jacobian matrix transforming between differential volumes in the two coordinate systems is defined as

\[
\begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\
\frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta}
\end{bmatrix}
\]

(5.19)
Using the chain rule, the relation between the differentiation with coordinates \((x, y, z)\) and the differentiation with \((\xi, \eta, \zeta)\) can be expressed as

\[
\left\{ \begin{array}{c} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{array} \right\} = J^{-1} \left\{ \begin{array}{c} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{array} \right\}
\] (5.20)

If \(J^*\) represents the adjoint of the Jacobian matrix, the inverse of the Jacobian matrix can be written as:

\[
J^{-1} = \frac{J^*}{|J|} = \frac{1}{|J|} \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* \\ J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* \end{bmatrix}
\] (5.21)

where

\[
\begin{align*}
J_{11}^* &= J_{22}J_{33} - J_{23}J_{32} \\
J_{21}^* &= J_{32}J_{13} - J_{33}J_{12} \\
J_{31}^* &= J_{12}J_{23} - J_{13}J_{22} \\
J_{12}^* &= J_{23}J_{31} - J_{21}J_{33} \\
J_{22}^* &= J_{33}J_{11} - J_{31}J_{13} \\
J_{32}^* &= J_{13}J_{21} - J_{11}J_{23} \\
J_{13}^* &= J_{21}J_{32} - J_{22}J_{31} \\
J_{23}^* &= J_{31}J_{12} - J_{32}J_{11} \\
J_{33}^* &= J_{11}J_{22} - J_{12}J_{21}
\end{align*}
\] (5.22)

Therefore,

\[
\frac{\partial}{\partial x} = \frac{1}{|J|} \left( J_{11}^* \frac{\partial}{\partial \xi} + J_{21}^* \frac{\partial}{\partial \eta} + J_{31}^* \frac{\partial}{\partial \zeta} \right)
\] (5.23)

\[
\frac{\partial}{\partial y} = \frac{1}{|J|} \left( J_{12}^* \frac{\partial}{\partial \xi} + J_{22}^* \frac{\partial}{\partial \eta} + J_{32}^* \frac{\partial}{\partial \zeta} \right)
\] (5.24)
\[
\frac{\partial}{\partial z} = \frac{1}{|J|} \left( J_{11} \frac{\partial}{\partial \xi} + J_{21} \frac{\partial}{\partial \eta} + J_{31} \frac{\partial}{\partial \zeta} \right) \quad (5.25)
\]

For a 3-D problem, \( B = \{ B_1, B_2, \ldots, B_n \} \) with \( B_i \) given as

\[
B_i = \begin{bmatrix}
\frac{\partial N_i}{\partial x} & 0 & 0 \\
0 & \frac{\partial N_i}{\partial y} & 0 \\
0 & 0 & \frac{\partial N_i}{\partial z} \\
\frac{\partial N_i}{\partial \xi} & 0 & \frac{\partial N_i}{\partial \eta} \\
\frac{\partial N_i}{\partial \eta} & \frac{\partial N_i}{\partial \xi} & 0
\end{bmatrix} 
\]

(5.26)

Using relations (5.23), (5.24) and (5.25) and separating out the Jacobian determinant in the \( B \) matrix gives

\[
B = \frac{1}{|J|} B^* (\xi, \eta, \zeta) \quad (5.27)
\]

where \( B^* = \{ B_1^*, B_2^*, \ldots, B_n^* \} \) with \( B_i^* \) given as

\[
B_i^* = \begin{bmatrix}
b_{i1}^* & 0 & 0 \\
0 & b_{i2}^* & 0 \\
0 & 0 & b_{i3}^* \\
b_{i1}^* & 0 & b_{i2}^* \\
b_{i2}^* & b_{i1}^* & 0
\end{bmatrix} 
\]

(5.28)

and

\[
b_{i1}^* = J_{11}^* \frac{\partial N_i}{\partial \xi} + J_{21}^* \frac{\partial N_i}{\partial \eta} + J_{31}^* \frac{\partial N_i}{\partial \zeta} 
\]

(5.29)

\[
b_{i2}^* = J_{12}^* \frac{\partial N_i}{\partial \xi} + J_{22}^* \frac{\partial N_i}{\partial \eta} + J_{32}^* \frac{\partial N_i}{\partial \zeta} 
\]

(5.30)

\[
b_{i3}^* = J_{13}^* \frac{\partial N_i}{\partial \xi} + J_{23}^* \frac{\partial N_i}{\partial \eta} + J_{33}^* \frac{\partial N_i}{\partial \zeta} 
\]

(5.31)

The stiffness matrix (5.3) then becomes

\[
K = \int_{-1}^{1} \int_{-1}^{1} B^T C B^* \frac{1}{|J|} d\xi d\eta d\zeta \quad (5.32)
\]
Finally the element stiffness matrix can be expressed in detail as

\[
\mathbf{K} = \begin{bmatrix}
  
  \vdots & \vdots & \vdots \\
  k_{ij}^{11} & k_{ij}^{12} & k_{ij}^{13} \\
  k_{ij}^{21} & k_{ij}^{22} & k_{ij}^{23} \\
  k_{ij}^{31} & k_{ij}^{32} & k_{ij}^{33} \\
  \vdots & \vdots & \vdots
\end{bmatrix}
\] (5.33)

where the subscripts \( i \) and \( j \) represent the nodal or hierarchical nodal numbers in element level and

\[
k_{ij}^{11} = c_{11}bb_{ij}^{11} + c_{55}bb_{ij}^{33} + c_{66}bb_{ij}^{22}
\]

\[
k_{ij}^{12} = c_{12}bb_{ij}^{12} + c_{66}bb_{ij}^{21}
\]

\[
k_{ij}^{13} = c_{13}bb_{ij}^{13} + c_{55}bb_{ij}^{31}
\]

\[
k_{ij}^{21} = c_{12}bb_{ij}^{21} + c_{66}bb_{ij}^{12}
\]

\[
k_{ij}^{22} = c_{22}bb_{ij}^{22} + c_{44}bb_{ij}^{33} + c_{66}bb_{ij}^{11}
\]

\[
k_{ij}^{23} = c_{23}bb_{ij}^{23} + c_{44}bb_{ij}^{32}
\]

\[
k_{ij}^{31} = c_{13}bb_{ij}^{31} + c_{55}bb_{ij}^{13}
\]

\[
k_{ij}^{32} = c_{23}bb_{ij}^{32} + c_{44}bb_{ij}^{23}
\]

\[
k_{ij}^{33} = c_{33}bb_{ij}^{33} + c_{44}bb_{ij}^{22} + c_{55}bb_{ij}^{11}
\] (5.34)

where

\[
bb_{ij}^{11} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} b_{11}^* b_{j1}^* \frac{1}{|J|} d\xi d\eta d\zeta
\]

\[
bb_{ij}^{12} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} b_{11}^* b_{j2}^* \frac{1}{|J|} d\xi d\eta d\zeta
\]

\[
bb_{ij}^{13} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} b_{11}^* b_{j3}^* \frac{1}{|J|} d\xi d\eta d\zeta
\]

\[
bb_{ij}^{21} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} b_{12}^* b_{j1}^* \frac{1}{|J|} d\xi d\eta d\zeta
\]
In equation (5.32), \( B^* \) is computed from the hierarchical shape functions. Therefore, the element stiffness matrix is also hierarchical and can be expressed as

\[
[K^e] = \begin{bmatrix} K^e_{mm} & K^e_{md} \\ K^e_{md} & K^e_{dd} \end{bmatrix}
\]  

(5.36)

where \([K^e_{mm}]\) is the lower order element stiffness matrix and \([K^e_{nn}]\) is the higher order element stiffness matrix.

### 5.3 Two-dimensional load vector

A two-dimensional load vector is formed for the p-method. Given the geometric mapping between coordinates \((x, y)\) and \((\xi, \eta)\), we have

\[
dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta = J_{11} d\xi + J_{21} d\eta
\]

(5.37)

\[
dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta = J_{12} d\xi + J_{22} d\eta
\]

(5.38)

where \(J_{11}, J_{12}, \text{ etc.}\) are elements of the Jacobian matrix. For element sides \(\xi = \pm 1\),

\[
dx = J_{21} (\pm 1, \eta) d\eta
\]

\[
dy = J_{22} (\pm 1, \eta) d\eta
\]

(5.39)
For element sides $\eta = \pm 1$,

$$
\begin{align*}
\frac{dx}{d\eta} &= J_{11}(\xi, \pm 1) d\eta \\
\frac{dy}{d\eta} &= J_{12}(\xi, \pm 1) d\eta 
\end{align*}
$$

(5.40)

In equation (5.4), $\mathbf{F}$ is a load vector, $\mathbf{N}$ is the shape function and $\mathbf{T}$ is the load applied on the boundary $\Gamma_e$ of the element. Let $\mathbf{n}$ be the unit normal vector, $\mathbf{\tau}$ the unit tangential vector and $\alpha$ the angle between the normal $\mathbf{n}$ direction and the $x$ axis.

We have

$$
\mathbf{T} = T_n \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix} + T_\tau \begin{bmatrix} -\sin \alpha \\ \cos \alpha \end{bmatrix}
$$

(5.41)

and

$$
\sin \alpha = -\frac{dx}{d\Gamma}, \quad \cos \alpha = \frac{dy}{d\Gamma}
$$

(5.42)

Hence, equation (5.4) becomes

$$
\mathbf{F} = \int_{\Gamma_e} \mathbf{N}^T T_n \begin{bmatrix} \frac{dy}{d\eta} \\ -\frac{dx}{d\eta} \end{bmatrix} + \int_{\Gamma_e} \mathbf{N}^T T_\tau \begin{bmatrix} \frac{dx}{d\eta} \\ \frac{dy}{d\eta} \end{bmatrix}
$$

(5.43)

For a load applied on element sides $\xi = \pm 1$, using equation (5.39) we have

$$
\mathbf{F}_i = \int_{-1}^{1} N_i(\pm 1, \eta) \begin{bmatrix} J_{22}(\pm 1, \eta) \\ -J_{21}(\pm 1, \eta) \end{bmatrix} T_n d\eta + \int_{-1}^{1} N_i(\pm 1, \eta) \begin{bmatrix} J_{21}(\pm 1, \eta) \\ J_{22}(\pm 1, \eta) \end{bmatrix} T_\tau d\eta
$$

(5.44)

For a load applied on element sides $\eta = \pm 1$, using equation (5.40) we have

$$
\mathbf{F}_i = \int_{-1}^{1} N_i(\xi, \pm 1) \begin{bmatrix} J_{12}(\xi, \pm 1) \\ -J_{11}(\xi, \pm 1) \end{bmatrix} T_n d\xi + \int_{-1}^{1} N_i(\xi, \pm 1) \begin{bmatrix} J_{11}(\xi, \pm 1) \\ J_{12}(\xi, \pm 1) \end{bmatrix} T_\tau d\xi
$$

(5.45)

where

$$
\mathbf{F}_i = \begin{bmatrix} F_{xi} \\ F_{yi} \end{bmatrix}
$$

(5.46)
5.4 Three-dimensional load vector for distributed pressure

In the p-method, the nodal load vector should be computed at each order $p$ by substituting nodal shape functions and hierarchical shape functions into equation (5.4). Formulation of the load vector in the 3-D p-method is derived as follow.

For the element surface $\zeta = \pm 1$,

\[ \mathbf{n} \, d\Gamma = d\xi \times d\eta = \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{bmatrix} \times \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{bmatrix} \, d\xi d\eta \quad (5.47) \]

\[ \mathbf{n} \, d\Gamma = \begin{bmatrix} \frac{\partial y}{\partial \zeta} \frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \zeta} \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \zeta} \frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \zeta} \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \zeta} \frac{\partial x}{\partial \eta} \end{bmatrix} \, d\xi d\eta \quad (5.48) \]

Let $\mathbf{T} = T_{n} \mathbf{n}$, where $\mathbf{n}$ is a unit normal vector and $T_{n}$ is the magnitude of surface pressure. We have

\[ \mathbf{T} \, d\Gamma = T_{n} \mathbf{n} \, d\Gamma \quad (5.49) \]

The load vector for $\zeta = \pm 1$ becomes

\[ \mathbf{F} = \int_{\Gamma} N^{T} \begin{bmatrix} \frac{\partial y}{\partial \zeta} \frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \zeta} \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \zeta} \frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \zeta} \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \zeta} \frac{\partial x}{\partial \eta} \end{bmatrix} \, d\xi d\eta \quad (5.50) \]

Taking notice of equations (5.19) and (5.22), the load distributed at each node or hierarchical node $i$ can be written as

\[ \mathbf{F}_{i} = \int_{-1}^{1} \int_{-1}^{1} N_{i}(\xi, \eta, \pm 1) \begin{bmatrix} J_{31}^{*} (\xi, \eta, \pm 1) \\ J_{32}^{*} (\xi, \eta, \pm 1) \\ J_{33}^{*} (\xi, \eta, \pm 1) \end{bmatrix} T_{n} \, d\xi d\eta \quad (5.51) \]

For the element surface $\xi = \pm 1$, the load vector is written as

\[ \mathbf{F}_{i} = \int_{-1}^{1} \int_{-1}^{1} N_{i}(\pm 1, \eta, \xi) \begin{bmatrix} J_{11}^{*} (\pm 1, \eta, \xi) \\ J_{12}^{*} (\pm 1, \eta, \xi) \\ J_{13}^{*} (\pm 1, \eta, \xi) \end{bmatrix} T_{n} \, d\eta d\xi \quad (5.52) \]
For the element surface $\eta = \pm 1$, the load vector is written as

$$F_i = \int_{-1}^{1} \int_{-1}^{1} N_i (\xi, \pm 1, \zeta) \left\{ J_{21}^{\eta} (\xi, \pm 1, \zeta) \\ J_{22}^{\eta} (\xi, \pm 1, \zeta) \right\} T \, d\zeta \, d\xi \quad (5.53)$$

where

$$F_i = \begin{bmatrix} F_{xi} \\ F_{yi} \end{bmatrix} \quad (5.54)$$

The normal pressure distribution acting on an element surface may be defined by specifying intensities $p_1, p_2, p_3$ and $p_4$ at the corner nodes. The variation of pressure over the element face is then given as:

$$p (s, t) = p_1 \times h_1 + p_2 \times h_2 + p_3 \times h_3 + p_4 \times h_4 \quad (5.55)$$

where

$$h_1 = \frac{1}{4} (1 + s) (1 + t)$$

$$h_2 = \frac{1}{4} (1 - s) (1 + t)$$

$$h_3 = \frac{1}{4} (1 - s) (1 - t)$$

$$h_4 = \frac{1}{4} (1 + s) (1 - t) \quad (5.56)$$

and $(s, t)$ are quadrilateral natural face coordinates.

For certain pressure distribution $p (s, t)$, $p_1, p_2, p_3$ and $p_4$ are specified. If the pressure is a uniform distribution, $p_1, p_2, p_3$ and $p_4$ are the same.

Let us take as an example, a linear pressure distribution:

$$p (s, t) = 2 + 3s + 4t \quad (5.57)$$

First, expand equation (5.55) as:

$$p (s, t) = c_1 + c_2 \times s + c_3 \times t + c_4 \times s \times t \quad (5.58)$$
where

\[ 4c_1 = p_1 + p_2 + p_3 + p_4 \]
\[ 4c_2 = p_1 - p_2 - p_3 + p_4 \]
\[ 4c_3 = p_1 + p_2 - p_3 - p_4 \]
\[ 4c_4 = p_1 - p_2 + p_3 - p_4 \]  \hspace{1cm} (5.59)

comparing equation (5.57) with equation (5.58) yields

\[ p_1 + p_2 + p_3 + p_4 = 8 \]
\[ p_1 - p_2 - p_3 + p_4 = 12 \]
\[ p_1 + p_2 - p_3 - p_4 = 16 \]
\[ p_1 - p_2 + p_3 - p_4 = 0 \]  \hspace{1cm} (5.60)

The results of equation (5.60) can be given as

\[ p_1 = 9 \quad p_2 = 3 \quad p_3 = -5 \quad p_4 = 1 \]  \hspace{1cm} (5.61)

Then the linear pressure distribution can be specified as

\[ p(s, t) = 2 + 3s + 4t \]
\[ = 9h_1 + 3h_2 - 5h_3 + h_4 \]  \hspace{1cm} (5.62)

By following the specified order of the surface numbers, \( h_1, h_2, h_3 \) and \( h_4 \) can be given for the six faces in a hexahedral element as:

1. for face 1 at \( \zeta = -1 \)

\[ h_1 = N_1(\xi, \eta, -1) \]

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\[ h_2 = N_5 (\xi, \eta, -1) \]
\[ h_3 = N_6 (\xi, \eta, -1) \]
\[ h_4 = N_2 (\xi, \eta, -1) \]  
\[ (5.63) \]

2. for face 2 at \( \eta = 1 \)

\[ h_1 = N_2 (\xi, 1, \zeta) \]
\[ h_2 = N_3 (\xi, 1, \zeta) \]
\[ h_3 = N_7 (\xi, 1, \zeta) \]
\[ h_4 = N_6 (\xi, 1, \zeta) \]  
\[ (5.64) \]

3. for face 3 at \( \zeta = 1 \)

\[ h_1 = N_4 (\xi, \eta, 1) \]
\[ h_2 = N_8 (\xi, \eta, 1) \]
\[ h_3 = N_7 (\xi, \eta, 1) \]
\[ h_4 = N_3 (\xi, \eta, 1) \]  
\[ (5.65) \]

4. for face 4 at \( \eta = -1 \)

\[ h_1 = N_1 (\xi, -1, \zeta) \]
\[ h_2 = N_4 (\xi, -1, \zeta) \]
\[ h_3 = N_8 (\xi, -1, \zeta) \]
\[ h_4 = N_5 (\xi, -1, \zeta) \]  
\[ (5.66) \]
5. for face 5 at $\xi = 1$

\[
\begin{align*}
  h_1 &= N_5(1, \eta, \zeta) \\
  h_2 &= N_6(1, \eta, \zeta) \\
  h_3 &= N_7(1, \eta, \zeta) \\
  h_4 &= N_8(1, \eta, \zeta)
\end{align*}
\] (5.67)

6. for face 6 at $\xi = -1$

\[
\begin{align*}
  h_1 &= N_1(-1, \eta, \zeta) \\
  h_2 &= N_2(-1, \eta, \zeta) \\
  h_3 &= N_3(-1, \eta, \zeta) \\
  h_4 &= N_4(-1, \eta, \zeta)
\end{align*}
\] (5.68)

where $N_i(\xi, \eta, \zeta)$ are shape functions from an eight-node hexahedral element.

5.5 Stress recovery techniques

Stresses computed directly from displacement fields are continuous inside the element and discontinuous on the inter-element boundaries. References [7, 8] show that the continuous stress field is more accurate than the piecewise continuous stress field. The basic idea of constructing a continuous stress field is similar to forming a continuous displacement field in the finite element. The stresses are interpolated by the same shape functions $N$ used for the displacement field, i.e.

\[ \sigma = N\sigma^* \] (5.69)
where $\sigma$ is the continuous stress field, $\sigma^*$ is the nodal stress vector. The nodal stress vector $\sigma^*$ may be determined by the minimization of the global $L_2$ norm as follows:

$$\frac{\partial}{\partial \sigma^*} \left[ \int_{\Omega} (\sigma - \sigma_h)^T (\sigma - \sigma_h) \, d\Omega \right] = 0 \quad (5.70)$$

this leads to the expression

$$\int_{\Omega} N^T (\sigma - \sigma_h) \, d\Omega = 0 \quad (5.71)$$

Substituting equation (5.69) into (5.71) yields

$$\sigma^* = (\int_{\Omega} N^T N \, d\Omega)^{-1} \int_{\Omega} N^T \sigma_h \, d\Omega \quad (5.72)$$

where $\sigma_h$ is calculated directly from the displacement field as

$$\sigma_h = CBq \quad (5.73)$$

The above smoothing procedure may be carried out over the entire finite element domain and is referred to as global stress recovery\[9]. Alternatively, the smoothing procedure may be performed separately over each element and this technique is called local stress recovery\[9]. To further improve the accuracy and reduce the computational time, Zienkiewicz and Zhu developed a superconvergent patch recovery method (SPR) \[35, 36, 37, 38\]. In this method, the stresses at a node are calculated from a superconvergent patch which is composed of several elements sharing a node. The stresses obtained at this node are superconvergent. The continuous stress field for each patch is assumed as equation (5.69) in a local coordinate system. A least square technique is applied to fit the stress field with the stresses at the Gauss points in the superconvergent patch, that is

$$\frac{\partial}{\partial \sigma^*} \sum_{j=1}^{n} [(\sigma)_j - (\sigma_h)_j]^2 = 0 \quad (5.74)$$
Here \( j \) is the Gauss point and \( n \) is the total number of Gauss points in the superconvergent patch. When the superconvergent nodal stresses are obtained the stress field can be expressed by equation (5.69).

Recently, Zienkiewicz introduced an alternative procedure called recovery by equilibrium in patches (REP)\[39\]. In a patch \( \Omega_p \), equilibrium is satisfied as

\[
\int_{\Omega_p} \mathbf{B}^T \sigma \text{d}\Omega = \mathbf{F}_p \tag{5.75}
\]

where the continuous stress field in the patch is assumed as

\[
\sigma = \mathbf{P} \beta \tag{5.76}
\]

The unknown parameter \( \beta \) is obtained from minimizing with respect to \( \beta \) the following functional:

\[
\Pi = (\mathbf{H} \beta - \mathbf{F}_p)^T (\mathbf{H} \beta - \mathbf{F}_p) \tag{5.77}
\]

where

\[
\mathbf{H} = \int_{\Omega_p} \mathbf{B}^T \mathbf{P} \text{d}\Omega \tag{5.78}
\]

Minimization of equation (5.77) results in

\[
\beta = [\mathbf{H}^T \mathbf{H}]^{-1} \mathbf{H}^T \mathbf{F}_p \tag{5.79}
\]

5.6 Newton’s method

Usually, the stresses are expressed in local coordinates,

\[
\sigma = \sigma(\xi, \eta, \zeta) \tag{5.80}
\]
If we want to calculate the stresses at a point \((x_g, y_g, z_g)\) in global coordinates, we need to first find out which element this point belongs to and then determine the local coordinates \((\xi_g, \eta_g, \zeta_g)\) corresponding to \((x_g, y_g, z_g)\) from the mapping function of the element. This means we need to find \((\xi_g, \eta_g, \zeta_g)\) satisfying the following nonlinear system equations

\[
f(X) = \begin{cases} f_1 & = x(\xi, \eta, \zeta) - x_g \\ f_2 & = y(\xi, \eta, \zeta) - y_g \\ f_3 & = z(\xi, \eta, \zeta) - z_g \end{cases} = 0 \tag{5.81}
\]

where

\[
X = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{5.82}
\]

Newton’s method for solving nonlinear equations is one of the most well-known and powerful procedures in numerical analysis. It converges if the initial approximation is sufficiently close to the solution and converges quadratically. Newton’s method is selected for obtaining \((\xi_g, \eta_g, \zeta_g)\) and is presented as

\[
\Delta X^{(n)} = -J(X^{(n)}) f(X^{(n)}) \tag{5.83}
\]

where \(J(X)\) is the Jacobian matrix from geometric mapping of the element as

\[
J(X) = \left[ \frac{\partial f_i}{\partial x_i} \right] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \tag{5.84}
\]

and

\[
X^{(n+1)} = X^{(n)} + \Delta X^{(n)} \tag{5.85}
\]
The mapping functions are presented in detail in Chapter 4. The starting value is taken as the center point of the element, i.e.

\[
\mathbf{X}^{(0)} = \begin{bmatrix}
  \xi^{(0)} \\
  \eta \\
  \zeta
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}
\]

(5.86)
CHAPTER 6

THE HYBRID/MIXED P-METHOD

In this Chapter, the hybrid/mixed p-version finite element formulation has been developed. The hierarchical shape functions and the orthogonal stress and strain interpolations are applied using the Hu-Washizu principle. It is anticipated that this method will reduce the computational time in computing the element stiffness matrix since the inverse of the $H$ matrix is avoided as given later in section 6.2.

6.1 Formulation by Hu-Washizu principle

Consider an arbitrary element with domain $\Omega_e$ and boundary $\Gamma_e$. The mixed variational functional using the Hu-Washizu principle for a compatible element can be expressed as

$$
\Pi_{\text{HW}} = \int_{\Omega_e} \left[ \frac{1}{2} \epsilon^T C \epsilon - \sigma^T \epsilon + \sigma^T (D u) \right] \, d\Omega - \int_{\Gamma_{te}} \bar{T}^T u \, d\Gamma \tag{6.1}
$$

where $u$, $\epsilon$ and $\sigma$ are the displacement, strain and stress vectors; $C$ is a constant material matrix and $D$ is the differential operator matrix relating the strain vector to the displacement vector. The known boundary traction $\bar{T}$ is applied on the element traction boundary $\Gamma_{te}$. 

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In the hybrid/mixed finite element formulation, the displacement can be interpolated by the shape function $N_i$; the strains and stresses are approximated independently by interpolation functions $P'$ and $P$, thus

$$u = Nq \quad (6.2)$$

$$\epsilon = P'\alpha \quad (6.3)$$

and

$$\sigma = P\beta \quad (6.4)$$

where $q$, $\alpha$ and $\beta$ are unknown variables. Hence, $\Pi_{HW}$ can be written as

$$\Pi_{HW}^e = \frac{1}{2} \alpha^T M \alpha - \beta^T \tilde{H} \alpha + \beta^T G q - F^T q \quad (6.5)$$

where

$$M = \int_{\Omega_e} P^T CP' d\Omega \quad (6.6)$$

$$\tilde{H} = \int_{\Omega_e} P^T P' d\Omega \quad (6.7)$$

$$G = \int_{\Omega_e} P^T (DN) d\Omega \quad (6.8)$$

and

$$F = \int_{\Gamma_e} N^T \mathbf{T} d\Gamma \quad (6.9)$$

The stationary condition of equation (6.5) with respect to $q$, $\alpha$ and $\beta$ leads to

$$\alpha = \tilde{H}^{-1} G q \quad (6.10)$$
\[
\beta = \tilde{H}^{-T}M\alpha \tag{6.11}
\]

and

\[
G^T\beta = F \tag{6.12}
\]

from which the stiffness matrix can be obtained as

\[
K = G^T\tilde{H}^{-T}M\tilde{H}^{-1}G \tag{6.13}
\]

The necessary condition for rank sufficiency of the stiffness matrix \(K\) is \(n_\alpha = n_\beta \geq n_q - l\), where \(n_\alpha\) and \(n_\beta\) are the numbers of the \(\alpha\) and \(\beta\) coefficients, \(n_q\) is the number of degrees of freedom for the generalized displacement \(q\), and \(l\) is the number of rigid body modes \([40]\).

The problem can be solved with prescribed boundary conditions from the assembled global stiffness matrix equation:

\[
\sum_e Kq = \sum_e F \tag{6.14}
\]

The calculation of the load vector in the hybrid/mixed finite element method is the same as that used in the displacement based method (see sections 5.3 and 5.4). However, the calculation of element stiffness matrix is different.

### 6.2 Orthogonal stress and strain interpolations

In the stiffness matrix equation (6.13) the computation involves the inverse of the \(\tilde{H}\) matrix. The order of the \(\tilde{H}\) matrix will be increased with interpolation polynomial order \(p\) due to the stress and strain fields. It is time consuming to compute the inverse of the \(\tilde{H}\) matrix for the higher-order hybrid/mixed method. The orthogonal
stress and strain interpolations may be used to overcome this difficulty. In order to achieve orthogonal stress and strain interpolations, the interpolation functions $P$ for the stress field and $P'$ for the strain field should be constructed to satisfy the following condition:

$$\int_{\Omega^*} P^T P' d\Omega = I$$  \hspace{1cm} (6.15)

Two approaches can be used. Approach one is to select $P' = P$, then using the Gram-Schmidt procedure, obtain the orthogonal polynomial functions for the $P$ matrix (see Appendix E). This approach is equivalent to the orthogonal procedure given in [41, 42]. However, the Gram-Schmidt procedure for higher-order $p$ is time consuming. The second approach is to select $P' = \frac{P}{|J|}$ where the $P$ matrix is constructed from Legendre polynomials $L_n(\xi)$. The detailed procedures are presented in the following sections. The second approach is preferred and used for the hybrid/mixed $p$-method.

### 6.2.1 Two-dimensional case

In the hybrid/mixed finite element formulation, the hierarchical shape functions are applied to the displacement field. For a 2-D case, the stress and the strain fields are assumed as

$$\sigma = P^* \beta$$  \hspace{1cm} (6.16)

and

$$\varepsilon = \frac{P^*}{|J|} \alpha$$  \hspace{1cm} (6.17)
where $|J|$ is the determinant of the Jacobian matrix and the $P^*$ matrix is expressed as

$$P^* = \begin{bmatrix} P^* & P^* \\ P^* & P^* \end{bmatrix}$$  \hspace{1cm} (6.18)

where $P^*$ is assumed as the product of the normalized Legendre polynomials $\ell_n (\xi) = \sqrt{n + \frac{1}{2}} L_n (\xi)$,

$$P^* = \{ \ell_0 (\xi) \ell_0 (\eta), \ell_1 (\xi) \ell_0 (\eta), \ell_0 (\xi) \ell_1 (\eta), \ldots, \ell_p (\xi) \ell_q (\eta) \}$$  \hspace{1cm} (6.19)

The element invariance can be assured as addressed in reference [43, 44] even if the expansion of $P^*$ is an incomplete polynomial.

The $M$ and $H$ matrices in equations (6.6) and (6.7) become

$$M = \int_{-1}^{1} \int_{-1}^{1} P^T C P^* \frac{1}{|J|} d\xi d\eta$$  \hspace{1cm} (6.20)

$$H = \int_{-1}^{1} \int_{-1}^{1} P^T P^* d\xi d\eta = I$$  \hspace{1cm} (6.21)

and the $G$ matrix in equation (6.8) becomes

$$G = \int_{-1}^{1} \int_{-1}^{1} P^T (D N) |J| d\xi d\eta$$  \hspace{1cm} (6.22)

Because of the orthogonal properties of Legendre polynomials, the $H$ matrix becomes the identity matrix so that equation (6.10), (6.11), (6.12) are simplified to

$$\alpha = Gq$$  \hspace{1cm} (6.23)

$$\beta = MGq$$  \hspace{1cm} (6.24)

$$G^T \beta = F$$  \hspace{1cm} (6.25)
and the stiffness matrix equation (6.13) becomes

\[
K = G^T M G
\]  
(6.26)

By comparison between equation (6.13) and equation (6.26), the computing of the simplified stiffness matrix (6.26) does not involve the inverse of the \( H \) matrix. Namely, the computation of \( H^{-1}G \) is avoided. It is evident that there is a significant time-saving using the above orthogonal stress and strain interpolations for the higher-order hybrid/mixed method.

Furthermore, the \( M \) matrix in equation (6.20) can be computed from

\[
M = \begin{bmatrix}
    c_{11} \bar{M} & c_{12} \bar{M} \\
    c_{12} \bar{M} & c_{22} \bar{M} \\
    & c_{33} \bar{M}
\end{bmatrix}
\]  
(6.27)

where

\[
\bar{M} = \int_{-1}^{1} \int_{-1}^{1} P^* P \frac{1}{|J|} d\xi d\eta
\]  
(6.28)

If the mapping of the element is a linear triangular mapping, the determinant of the Jacobian matrix \(|J|\) becomes a constant and equation (6.28) will become \( \bar{I}/|J| \) because of the orthogonal properties of \( P^* \).

The \( G \) matrix in equation (6.22) can be further simplified as

\[
G = \int_{-1}^{1} \int_{-1}^{1} P^* B^* d\xi d\eta
\]  
(6.29)

where if \( G = \{G_1, G_2, ..., G_n\} \) then

\[
G_i = \begin{bmatrix}
    \bar{g}_{i1} & 0 \\
    0 & \bar{g}_{i2} \\
    \bar{g}_{i2} & \bar{g}_{i1}
\end{bmatrix}
\]  
(6.30)

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and

\[ \ddot{g}_{i1} = \int_{-1}^{1} \int_{-1}^{1} P^* b^*_i \, d\xi \, d\eta \]  
(6.31)

\[ \ddot{g}_{i2} = \int_{-1}^{1} \int_{-1}^{1} P^* b^*_i \, d\xi \, d\eta \]  
(6.32)

It should be noted that \( P^* \) is constructed from Legendre polynomials \( L_k(\xi) \) as in equation (6.19). The integration \( \int_{-1}^{1} L_k(\xi) \xi^l \, d\xi = 0 \) if \( k > l \). Therefore, the polynomial terms of \( P^* \) are selected to be the same as the polynomial terms of the shape functions.

Finally the element stiffness matrix can be computed by substituting equations (6.27) and (6.30) into equation (6.26). This is expressed in detail as

\[
K = \begin{bmatrix}
\vdots & \vdots & \vdots \\
\ldots & k^1_{ij} & k^{12}_{ij} \\
\ldots & k^2_{ij} & k^{22}_{ij} \\
\vdots & \vdots & \vdots 
\end{bmatrix}
\]

(6.33)

where the subscripts \( i \) and \( j \) represent the nodal or hierarchical nodal numbers of the element and

\[
k^1_{ij} = c_{11} \left( \ddot{g}_{i1}^T \dot{M} \ddot{g}_{j1} \right) + c_{33} \left( \ddot{g}_{i2}^T \dot{M} \ddot{g}_{j2} \right)
\]

\[
k^{12}_{ij} = c_{12} \left( \ddot{g}_{i1}^T \dot{M} \ddot{g}_{j2} \right) + c_{33} \left( \ddot{g}_{i2}^T \dot{M} \ddot{g}_{j1} \right)
\]

\[
k^{21}_{ij} = c_{12} \left( \ddot{g}_{i2}^T \dot{M} \ddot{g}_{j1} \right) + c_{33} \left( \ddot{g}_{i1}^T \dot{M} \ddot{g}_{j2} \right)
\]

\[
k^{22}_{ij} = c_{22} \left( \ddot{g}_{i2}^T \dot{M} \ddot{g}_{j2} \right) + c_{33} \left( \ddot{g}_{i1}^T \dot{M} \ddot{g}_{j1} \right)
\]

(6.34)
6.2.2 Three-dimensional case

In the hybrid/mixed finite element formulation, the displacement can be interpolated by the hierarchical shape function matrix \( N \); the stress and strain are approximated by the \( P^* \) matrix which is expressed as

\[
P^* = \begin{bmatrix}
P^* \\
P^* \\
P^* \\
P^* \\
P^*
\end{bmatrix}
\]  
(6.35)

Here \( P^* \) is assumed to be the product of the normalized Legendre polynomials \( \ell_n(\xi) = \sqrt{n + \frac{1}{2}} L_n(\xi) \),

\[
P^* = \{ \ell_0(\xi) \ell_0(\eta) \ell_0(\zeta), \cdots, \ell_p(\xi) \ell_q(\eta) \ell_r(\zeta) \} 
\]  
(6.36)

Hence, the \( M, \bar{H} \) and \( G \) matrices in equations (6.6), (6.7) and (6.8) become

\[
M = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} P^*^T C P^* \frac{1}{|J|} d\xi d\eta d\zeta 
\]  
(6.37)

\[
\bar{H} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} P^*^T P^* d\xi d\eta d\zeta = I 
\]  
(6.38)

and

\[
G = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} P^*^T B^* d\xi d\eta d\zeta 
\]  
(6.39)

For the 3-D case, the \( M \) matrix in equation (6.37) becomes

\[
M = \begin{bmatrix}
c_{11}\bar{M} & c_{12}\bar{M} & c_{13}\bar{M} \\
c_{12}\bar{M} & c_{22}\bar{M} & c_{23}\bar{M} \\
c_{13}\bar{M} & c_{23}\bar{M} & c_{33}\bar{M} \\
c_{44}\bar{M} & c_{55}\bar{M} & c_{66}\bar{M}
\end{bmatrix} 
\]  
(6.40)
where
\[
\mathbf{M} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \mathbf{P}^* \mathbf{P}^T \frac{1}{|J|} d\xi d\eta d\zeta \tag{6.41}
\]

If the mapping of the element is a linear tetrahedral mapping, the determinant of the Jacobian matrix $|J|$ becomes a constant and the equation (6.41) will become $\mathbf{I}/|J|$ because of the orthogonal properties of $\mathbf{P}^*$.

For the $\mathbf{G}$ matrix, if $\mathbf{G} = \{G_1, G_2, \ldots, G_n\}$ then

\[
\mathbf{G}_i = \begin{bmatrix}
\tilde{g}_{i1} & 0 & 0 \\
0 & \tilde{g}_{i2} & 0 \\
0 & 0 & \tilde{g}_{i3} \\
\end{bmatrix}
\tag{6.42}
\]

where
\[
\tilde{g}_{i1} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \mathbf{P}^* \mathbf{b}_{i1}^* d\xi d\eta d\zeta \tag{6.43}
\]
\[
\tilde{g}_{i2} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \mathbf{P}^* \mathbf{b}_{i2}^* d\xi d\eta d\zeta \tag{6.44}
\]
\[
\tilde{g}_{i3} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \mathbf{P}^* \mathbf{b}_{i3}^* d\xi d\eta d\zeta \tag{6.45}
\]

The polynomial terms of $\mathbf{P}^*$ are selected using the same polynomial term used in the 3-D shape functions.

Finally the element stiffness matrix can be expressed in detail as

\[
\mathbf{K} = \begin{bmatrix}
\vdots & \vdots & \vdots \\
\kappa_{ij}^{11} & \kappa_{ij}^{12} & \kappa_{ij}^{13} & \kappa_{ij}^{21} \kappa_{ij}^{22} \kappa_{ij}^{23} & \kappa_{ij}^{31} \kappa_{ij}^{32} \kappa_{ij}^{33} \\
\end{bmatrix}
\tag{6.46}
\]
where the subscripts $i$ and $j$ represent the nodal or hierarchical nodal numbers of the element and

\[
\begin{align*}
    k_{ij}^{11} &= c_{11} \left( \tilde{g}_{i1}^T \tilde{M}_{ij} \tilde{g}_{j1} \right) + c_{55} \left( \tilde{g}_{i3}^T \tilde{M}_{ij} \tilde{g}_{j3} \right) + c_{66} \left( \tilde{g}_{i2}^T \tilde{M}_{ij} \tilde{g}_{j2} \right) \\
    k_{ij}^{12} &= c_{12} \left( \tilde{g}_{i1}^T \tilde{M}_{ij} \tilde{g}_{j2} \right) + c_{66} \left( \tilde{g}_{i2}^T \tilde{M}_{ij} \tilde{g}_{j1} \right) \\
    k_{ij}^{13} &= c_{13} \left( \tilde{g}_{i1}^T \tilde{M}_{ij} \tilde{g}_{j3} \right) + c_{55} \left( \tilde{g}_{i3}^T \tilde{M}_{ij} \tilde{g}_{j1} \right) \\
    k_{ij}^{21} &= c_{12} \left( \tilde{g}_{i2}^T \tilde{M}_{ij} \tilde{g}_{j1} \right) + c_{66} \left( \tilde{g}_{i1}^T \tilde{M}_{ij} \tilde{g}_{j2} \right) \\
    k_{ij}^{22} &= c_{22} \left( \tilde{g}_{i2}^T \tilde{M}_{ij} \tilde{g}_{j2} \right) + c_{44} \left( \tilde{g}_{i3}^T \tilde{M}_{ij} \tilde{g}_{j3} \right) + c_{66} \left( \tilde{g}_{i1}^T \tilde{M}_{ij} \tilde{g}_{j1} \right) \\
    k_{ij}^{23} &= c_{23} \left( \tilde{g}_{i2}^T \tilde{M}_{ij} \tilde{g}_{j3} \right) + c_{44} \left( \tilde{g}_{i3}^T \tilde{M}_{ij} \tilde{g}_{j2} \right) \\
    k_{ij}^{31} &= c_{13} \left( \tilde{g}_{i3}^T \tilde{M}_{ij} \tilde{g}_{j1} \right) + c_{55} \left( \tilde{g}_{i1}^T \tilde{M}_{ij} \tilde{g}_{j3} \right) \\
    k_{ij}^{32} &= c_{23} \left( \tilde{g}_{i3}^T \tilde{M}_{ij} \tilde{g}_{j2} \right) + c_{44} \left( \tilde{g}_{i2}^T \tilde{M}_{ij} \tilde{g}_{j3} \right) \\
    k_{ij}^{33} &= c_{33} \left( \tilde{g}_{i3}^T \tilde{M}_{ij} \tilde{g}_{j3} \right) + c_{44} \left( \tilde{g}_{i2}^T \tilde{M}_{ij} \tilde{g}_{j2} \right) + c_{55} \left( \tilde{g}_{i1}^T \tilde{M}_{ij} \tilde{g}_{j1} \right)
\end{align*}
\]

(6.47)

6.3 Condensation of internal shape functions

In the hybrid/mixed $p$-version finite element, the hierarchical shape functions, and the orthogonal stress and strain interpolations are used. In equation (6.1), the displacement field can be divided into two parts such as

\[
\begin{align*}
    u &= u_q + u_\lambda \\
    &= N_q q + N_\lambda \lambda
\end{align*}
\]

(6.48)

where $u_q$ is interpolated by the external shape functions associated with nodes, sides and faces of the element and $u_\lambda$ is associated with the internal shape functions ($u_\lambda = 0$ at element boundary). The orthogonal stress and strain interpolation functions are
defined as

\[ \sigma = P\beta, \quad \epsilon = P'\alpha \]  \hspace{1cm} (6.49)

with orthogonal property

\[ \int_{\Omega_e} P^TP'd\Omega = I \]  \hspace{1cm} (6.50)

Hence, equation (6.1) becomes

\[ \Pi_{HW}' = \int_{\Omega_e} \left[ \frac{1}{2} \epsilon^T C \epsilon - \sigma^T \epsilon + \sigma^T (D u_q) + \sigma^T (D u_\lambda) \right] d\Omega 
- \int_{\Gamma_{te}} \hat{T}^T u_q d\Gamma \]  \hspace{1cm} (6.51)

With \( q, \lambda, \alpha \) and \( \beta \) as unknown variables, \( \Pi_{HW}' \) can be written as

\[ \Pi_{HW}' = \frac{1}{2} \alpha^T M \alpha - \beta^T \alpha + \beta^T G_q q + \beta^T G_\lambda \lambda - F^T q \]  \hspace{1cm} (6.52)

where

\[ M = \int_{\Omega_e} P^T C P' d\Omega \]  \hspace{1cm} (6.53)

\[ G_q = \int_{\Omega_e} P^T (D N_q) d\Omega \]  \hspace{1cm} (6.54)

\[ G_\lambda = \int_{\Omega_e} P^T (D N_\lambda) d\Omega \]  \hspace{1cm} (6.55)

and

\[ F = \int_{\Gamma_{te}} N_q^T \hat{T} d\Gamma \]  \hspace{1cm} (6.56)

The stationary condition of equation (6.52) with respect to \( q, \lambda, \alpha \) and \( \beta \) leads to

\[ \alpha = G_q q + G_\lambda \lambda \]  \hspace{1cm} (6.57)
\[ \beta = M\alpha \] (6.58)

\[ G^T_\alpha \beta = 0 \] (6.59)

and

\[ G^T_q \beta = F \] (6.60)

From equations (6.57), (6.58) and (6.59), we obtain

\[ \lambda = - \left( G^T_\alpha M G_\alpha \right)^{-1} G^T_\alpha M G_q q \] (6.61)

and the stiffness matrix becomes

\[ K = G^T_q M G_q - G^T_\alpha M G_\alpha \left( G^T_\alpha M G_\alpha \right)^{-1} G^T_\alpha M G_q \] (6.62)

The internal shape functions can be condensed out at the element level, so that the order of the global stiffness matrix will be reduced significantly at higher-order \( p \) values.

Applying the divergence theorem, equation (6.51) becomes

\[ \Pi_{IIW} = \int_{\Omega_t} \left[ \frac{1}{2} \epsilon^T C \epsilon - \sigma^T \epsilon - \left( D^T \sigma \right)^T u_q - \left( D^T \sigma \right)^T u_\lambda \right] d\Omega \\
+ \int_{\partial \Omega_t} \bar{T}^T u_q d\Gamma - \int_{\Gamma_t} \bar{T}^T u_q d\Gamma \] (6.63)

where \( u_q \) is a Lagrange multiplier which is used to enforce the inter-element traction reciprocity and \( u_\lambda \) is a Lagrange multiplier used to enforce stress equilibrium inside the element. As the polynomial order of \( p \) increases, the internal shape functions do not appear in the assumed displacement field until \( p \geq 4 \) for the 2-D problem and \( p \geq 6 \) for the 3-D problem. The presence of the internal shape functions is very important if accurate results are to be obtained.
6.4 Error analysis

Computing the error is very important in numerical analysis, especially combined with adaptive techniques. In the $p$-version finite element method, the adaptive analysis is based on the error energy norm calculated for each order $p$. Some important concepts and equations are defined and introduced as follows:

- **The principle of virtual work**

\[ B(u_{EX}, v) = F(v) \]  \hspace{1cm} (6.64)

where $B(u, v)$ is a bilinear form and $F(v)$ is a linear functional

- **Strain energy** is defined as

\[ U(u) = \frac{1}{2} B(u, u) \]  \hspace{1cm} (6.65)

- **The potential energy function** is defined as

\[ \Pi(u) = \frac{1}{2} B(u, u) - F(u) \]  \hspace{1cm} (6.66)

- **The energy norm** is defined

\[ \|u\| = \sqrt{U(u)} \]  \hspace{1cm} (6.67)

- **The error in the energy norm** is

\[ \|e\| = \|u_{FE} - u_{EX}\| \]  \hspace{1cm} (6.68)

- **The relative error in the energy norm**

\[ e_r = \frac{\|u_{FE} - u_{EX}\|}{\|u_{EX}\|} \]  \hspace{1cm} (6.69)
For an arbitrary function $w$, the potential energy is given as

$$
\Pi (u_{EX} + w) = \frac{1}{2} \mathbf{B} (u_{EX}, u_{EX}) + \mathbf{B} (u_{EX}, w) + \frac{1}{2} \mathbf{B} (w, w) - F (u_{EX}) - F (w)
$$

Using the principle of virtual work, such that $\mathbf{B} (u_{EX}, w) - F (w) = 0$, yields

$$
\Pi (u_{EX} + w) - \Pi (u_{EX}) = \frac{1}{2} \mathbf{B} (w, w) = U (w)
$$

Hence, from equations (6.67) and (6.71), the error of the energy norm becomes

$$
||e||^2 = \Pi (u_{FE}) - \Pi (u_{EX}) = |U (u_{FE}) - U (u_{EX})|
$$

and from equations (6.69) and (6.72), the relative error of the energy norm is

$$
e_r = \sqrt{\frac{|U (u_{FE}) - U (u_{EX})|}{U (u_{EX})}}
$$

where the strain energy is obtained from one of the following equations:

$$
U = \int_\Omega \frac{1}{2} \sigma^T \varepsilon \, d\Omega
$$

$$
U = \int_\Omega \frac{1}{2} \varepsilon^T C \varepsilon \, d\Omega
$$

$$
U = \int_\Omega \frac{1}{2} \sigma^T S \sigma \, d\Omega
$$

$$
U = \int_\Omega \frac{1}{2} (D \varepsilon)^T C (D \varepsilon) \, d\Omega
$$

Substituting equations (6.3) and (6.4) into equation (6.74) yields for the hybrid/mixed $p$-method

$$
U_{FE} = \sum_e U_e
$$
where

\[ U_e = \int_{\Omega_e} \frac{1}{2} (P\beta^T (P'\alpha) \, d\Omega \]  

(6.79)

Using the orthogonal stress and strain interpolations, the strain energy becomes

\[ U_e = \frac{1}{2} \beta^T \alpha \]  

(6.80)

and using the relations among \( \alpha, \beta \) and \( q \), the strain energy can also be written in the following forms:

\[ U_e = \frac{1}{2} \alpha^T M \alpha \]  

(6.81)

\[ U_e = \frac{1}{2} \beta^T M^{-1} \beta \]  

(6.82)

\[ U_e = \frac{1}{2} q^T K q \]  

(6.83)

### 6.5 The computational time for \( p \)-method

The displacement based \( p \)-method was introduced in Chapter 5 and the new formulation of the hybrid/mixed \( p \)-method was presented in this chapter. The hierarchical shape functions, geometric mapping technique, computation of load vector and equation solver can be applied alternatively to both methods. In the displacement based \( p \)-method only the displacement field is introduced as a field variable, and in the hybrid/mixed \( p \)-method displacement, stress and strain are assumed as field variables. This leads to a different formulation in computing the element stiffness matrix and the stress field. For comparison, key formulations are presented as follows:

1. The displacement based \( p \)-method
• The assembly of the element equations

\[ \sum_e K q = \sum_e F \]  \hspace{1cm} (6.84)

where

\[ K = \int_{\Omega_e} B^T C B d\Omega \]  \hspace{1cm} (6.85)

and

\[ F = \int_{\Gamma_e} N^T \mathbf{T} d\Gamma \]  \hspace{1cm} (6.86)

• The element stiffness matrix

\[ K = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} B^T C B^* \frac{1}{|J|} d\xi d\eta d\zeta \]  \hspace{1cm} (6.87)

• Stresses computed from the displacement field

\[ \sigma_h = C B q \]  \hspace{1cm} (6.88)

• Local stress recovery

\[ \sigma = N \sigma^* \]  \hspace{1cm} (6.89)

where

\[ \sigma^* = \left( \int_{\Omega_e} N^T N d\Omega \right)^{-1} \int_{\Omega_e} N^T \sigma_h d\Omega \]  \hspace{1cm} (6.90)

2. The hybrid/mixed p-method

• The assembly of the element equations

\[ \sum_e K q = \sum_e F \]  \hspace{1cm} (6.91)
where

\[ F = \int_{\Gamma_e} N^T \bar{T} \, d\Gamma \]  

(6.92)

- Orthogonal property

\[ \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} P^T P^* d\xi d\eta d\zeta = I \]  

(6.93)

- The element stiffness matrix

\[ K = G^T M G \]  

(6.94)

where

\[ M = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} P^T C P^* \frac{1}{|J|} d\xi d\eta d\zeta \]  

(6.95)

and

\[ G = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} P^T B^* d\xi d\eta d\zeta \]  

(6.96)

- The stress in the element

\[ \sigma = P^* \beta \]  

(6.97)

where

\[ \beta = MGq \]  

(6.98)

In the displacement based \( p \)-method the element stiffness matrix is computed from equation (6.87) and in the hybrid/mixed \( p \)-method from equations (6.94), (6.95) and (6.96). The computational time at higher-order \( p \) using equation (6.87) is higher than using equations (6.94), (6.95) and (6.96).
For a 2-D 30-element mesh, the computational time of the element stiffness matrix for both methods is given in Table 6.1. It can be seen that the hybrid/mixed $p$-method used a little bit more time than the displacement based $p$-method. However, for a 3-D problem, the time computed for the element stiffness matrix using a 1-element, a 3-element and a 7-element mesh is given in Table 6.2, Table 6.3 and Table 6.4. The hybrid/mixed $p$-method (HMPM) used less time than the displacement based $p$-method (DBPM) in determining both stresses and displacements. All the programs were run on SGI workstation.
<table>
<thead>
<tr>
<th>order p</th>
<th>( u(\text{DBPM}) )</th>
<th>( u(\text{HMPM}) )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
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<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>22</td>
<td>27</td>
</tr>
</tbody>
</table>

Table 6.1: Computational time(seconds) for a 2-D 30-element mesh

<table>
<thead>
<tr>
<th>order p</th>
<th>( u(\text{DBPM}) )</th>
<th>( \sigma(\text{DBPM}) )</th>
<th>( u(\text{HMPM}) )</th>
<th>( \sigma(\text{HMPM}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
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<td>21</td>
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</tr>
<tr>
<td>7</td>
<td>87</td>
<td>5</td>
<td>58</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>225</td>
<td>13</td>
<td>140</td>
<td>0</td>
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</tbody>
</table>

Table 6.2: Computational time(seconds) for a 3-D 1-element mesh
<table>
<thead>
<tr>
<th>order</th>
<th>p</th>
<th>u(DBPM)</th>
<th>u(HMPM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>22</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>66</td>
<td>56</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>173</td>
<td>147</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>418</td>
<td>352</td>
</tr>
</tbody>
</table>

Table 6.3: Computational time(seconds) for a 3-D 3-element mesh

<table>
<thead>
<tr>
<th>order</th>
<th>p</th>
<th>u(DBPM)</th>
<th>u(HMPM)</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>22</td>
<td>18</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>78</td>
<td>61</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>237</td>
<td>170</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>602</td>
<td>392</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>1434</td>
<td>926</td>
</tr>
</tbody>
</table>

Table 6.4: Computational time(seconds) for a 3-D 7-element mesh
CHAPTER 7

NUMERICAL EXAMPLES

In this chapter, different examples are given to test the hybrid/mixed p-version finite element programs for both 2-D and 3-D problems. To verify the advantages of the finite element programs, the examples should show the influence of arbitrary element shapes, the influence of the element with curve boundaries and surfaces, convergence of both displacement and stress, and Poisson's ratio effect for nearly incompressible material. The examples are presented as follows:

7.1 Pure bending of a patch

A 2.4 \times 1.2\text{-inch} unit thickness plate with arbitrary shaped elements is shown in Figure 7.1. The left side of the patch is constrained and the right side is subjected to a linear distributed bending load. Young's modulus is \( E = 1.0 \text{ psi} \) and Poisson's ratio is \( \nu = 0.25 \). Using the hybrid/mixed p-element with second order element mapping, the stresses at point A are given in Table 7.1. If, in Figure 7.1, the element side passing point B is a curve (this can be done by giving point B a new coordinate value in finite element input data), then the stresses at point A are given in Table 7.2. The analytical results of stresses at point A are \( \sigma_x = 5.0 \text{ psi}, \sigma_y = \tau_{xy} = 0 \). The results
show that the numerical results become the same as the analytical solution for $p \geq 4$
using linear side elements and for $p \geq 6$ using curved side element. The element
shape influences the accuracy of the finite element solution.

### 7.2 A concentric cylinder under internal pressure

A concentric cylinder under internal pressure was considered by Szabo [45] for
testing the rate of convergence of the energy norm using the displacement based
$p$-method. The same problem is analyzed here using the hybrid/mixed $p$-method.
Assume a plane strain conditions with Poisson's ratio $\nu$ ranging from 0.0 to 0.49999.
Table 7.1: Pure bending of patch with linear element side

Young's modulus $E$ is taken to be 1.0 psi. The inner and the outer radii are $a = 1.0$ inch and $b = 2.0$ inch. The internal pressure $p_a$ is assumed to be 1.0 psi. The finite element analysis is performed on a three-element mesh for a quarter of the cylinder as shown in Figure 7.2. The analytical solution for the stresses are given as

$$\sigma_r = -\frac{b^2 - 1}{a^2 - 1} p_a = -\frac{1}{3} \left( \frac{4 - r^2}{r^2} \right)$$  \hspace{1cm} (7.1)

$$\sigma_\theta = \frac{b^2 + 1}{a^2 - 1} p_a = \frac{1}{3} \left( \frac{4 + r^2}{r^2} \right)$$  \hspace{1cm} (7.2)

$$r_{r\theta} = 0$$  \hspace{1cm} (7.3)

The maximum circumferential stress at the inner boundary is

$$(\sigma_\theta)_{\text{max}} = \frac{5}{3} = 1.66666\ldots \text{ (psi)}$$  \hspace{1cm} (7.4)
Figure 7.2: 3-element mesh for a quarter of the cylinder under internal pressure
The exact strain energy of one quarter of the cylinder can be expressed in polar coordinates as

\[ U = \frac{1 + \nu}{2E} \int_0^\frac{\pi}{4} \int_0^b \left[ (1 - \nu) \left( \sigma_r^2 + \sigma_\theta^2 \right) - 2\nu \sigma_r \sigma_\theta + 2\tau_r^2 \right] r \, dr \, d\theta \] (7.5)

Substituting values from equations (7.1), (7.2) and (7.3) into equation (7.5) yields

\[ U = \frac{\pi}{12} (1 + \nu) (5 - 2\nu) \] (7.6)

For Poisson’s ratio \( \nu \) ranging from 0.0 to 0.49999, the values of strain energy are calculated and given in Table 7.3.

In order to test the accuracy of the numerical analysis, the relative error in the energy norm \( e_U \) and the relative error in the stress \( e_\sigma \) are used and are computed from the expressions

\[ e_U = \sqrt{\frac{U - U_p}{U}} \] (7.7)

Table 7.2: Pure bending of patch with curved element side

<table>
<thead>
<tr>
<th>order p</th>
<th>( \sigma_x )</th>
<th>( \sigma_y )</th>
<th>( \tau_{xy} )</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>0.3408519769E+01</td>
<td>-0.3443031303E+01</td>
<td>+0.2109577033E-01</td>
</tr>
<tr>
<td>3</td>
<td>0.4047173801E+01</td>
<td>-0.2874366030E+01</td>
<td>-0.5097684464E-01</td>
</tr>
<tr>
<td>4</td>
<td>0.5155835591E+01</td>
<td>+0.2471379842E-01</td>
<td>+0.1405670560E-01</td>
</tr>
<tr>
<td>5</td>
<td>0.5004217123E+01</td>
<td>+0.1059542888E-01</td>
<td>+0.4130253244E-03</td>
</tr>
<tr>
<td>6</td>
<td>0.5000000000E+01</td>
<td>-0.1562660514E-13</td>
<td>-0.2597126039E-13</td>
</tr>
<tr>
<td>7</td>
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<td>-0.1650976264E-11</td>
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<td>Poisson's ratio $\nu$</td>
<td>Strain energy $U$</td>
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<td></td>
</tr>
<tr>
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<td>----------------------------------</td>
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<tr>
<td>0.5</td>
<td>$\pi/2$</td>
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</tr>
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</table>

Table 7.3: Strain energy for different Poisson’s ratio

and

$$e_\sigma = \left| \frac{\sigma - \sigma_p}{\sigma} \right|$$  \hspace{1cm} (7.8)

where $U$ and $\sigma$ are the strain energy and stress obtained from the analytical solution. $U_p$ and $\sigma_p$ are the strain energy and stress obtained from finite element solution of order $p$.

The hybrid/mixed $p$-method combined with $p$-type geometric mapping introduced in Chapter 4 is used to solve the problem. The order $p$ is increased from 1 to 10. The convergence and the effect of increasing Poisson’s ratio are shown in Figure 7.3. Figure 7.3 (a) shows the relative error of the energy norm versus the number of degrees of freedom plotted on a log-log scale and Figure 7.3 (b) shows the relative error of the maximum circumferential stress versus the number of degrees of freedom also plotted on a log-log scale. The rate of convergence is exponential for both the energy norm and the maximum circumferential stress for this problem. The convergence can also be seen as Poisson’s ratio approaches 0.5 for $p \geq 4$.  

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Figure 7.3: A quarter of the cylinder under internal pressure: (a) relative error of the energy norm vs. number of degrees of freedom; (b) relative error of the maximum circumferential stress vs. number of degrees of freedom
7.3 Rectangular plate with parabolic tension

A 2 \times 2\text{-inch} plate of unit thickness under a parabolic distributed tensile force is considered. This problem has been solved by the principle of complementary virtual work using the theory of elasticity. The stress function is taken as

\begin{equation}
\phi = \phi_0 + \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \ldots
= \frac{p}{2} y^2 \left( 1 - \frac{y^2}{6b^2} \right) + \left( x^2 - a^2 \right)^2 \left( y^2 - b^2 \right)^2 \left( \alpha_1 + \alpha_2 x^2 + \alpha_3 y^2 + \ldots \right) \quad (7.9)
\end{equation}

The stress function \( \phi \) always satisfies the equilibrium conditions and also is designed to satisfy the traction boundary condition. The size of the rectangular domain is \( a = b = 1 \). If, in applying the stress function equation (7.9), one term, two terms or four terms are taken as approximation, the tensile stress at center of the plate can be obtained as \( 1.000p \), \( 0.830p \) and \( 0.862p \), respectively. Accurate stresses for a sufficient number of digits is not available. This problem is also solved using the hybrid/mixed p-method. A quarter of the plate is analyzed by applying symmetric boundary conditions and meshed as shown in Figure 7.4 (a). The mesh size will be discretized each time by half at the element side up to 32 sections at each side as shown in Figure 7.4 (b). The strain energy is computed and shown in Table 7.4 and the corresponding tensile stress at center of the plate is computed and given in Table 7.5. It is seen that both the strain energy and stress converge with increasing polynomial order \( p \) and refinement of the element mesh size. In order to compute the relative error of the energy norm and stress, the numerical values at \( p = 11 \) and \( h = \frac{1}{8} \) in Table 7.4 and Table 7.5 are used to replace the exact solution. The corresponding relative error of \( p \)-extension and \( h \)-extension are shown in Figure 7.5. Figure 7.5 (a)
shows the relative error of the energy norm versus the number of degrees of freedom plotted on a log-log scale. The p-convergence is much faster than h-convergence. For strain energy, the coarse mesh gives the optimum convergence. Figure 7.5 (b) shows the relative error of the tensile stress at the center of the plate versus the number of degrees of freedom plotted on a log-log scale. For the stress, the convergence is significantly improved for \( p \geq 4 \). The optimum convergence for stress is not the coarse mesh but some where in between the coarse and fine mesh.

### 7.4 A plate with a circular hole under tension

An 80 \( \times \) 20-inch unit thickness plate with a 10 inch diameter hole in the plate center is subjected to a uniform tension of 100 psi as shown in Figure 7.6. Only a quarter of the plate is discretized using 3 elements, 9 elements, 14 elements and
Figure 7.4: A quarter of the square plate with parabolic tension: (a) 1-element mesh; (b) 1024-element mesh
Figure 7.5: A quarter of the rectangular plate with parabolic tension: (a) relative error of the energy norm vs. number of degrees of freedom; (b) relative error of the tensile stress at the center of the plate vs. number of degrees of freedom.
Table 7.5: Stresses for different \( h \) and \( p \)

30 elements as shown in Figure 7.7 (a), (b), (c) and (d). The circular boundary is divided by 2 sections for a 3-element mesh, 3 sections for a 9-element mesh, 4 sections for a 14-element mesh and 6 sections for a 30-element mesh. The \( p \)-order geometric mapping for the circular boundary is applied in the hybrid/mixed \( p \)-method. Young’s modulus is \( E = 3.0 \times 10^7 \) psi and Poisson’s ratio is taken to be \( \nu = 0.3 \). The values of the maximum stress are listed in Table 7.6. The results converge with increasing polynomial order of \( p \) from 1 to 12. The accuracy of the maximum stress is increased with increasing order of \( p \) and the mesh size. The value of the maximum stress for a 30-element mesh with \( p = 12 \) gives converged value to 8 digits as 434.76041 psi. Peterson’s result [48, 49] of this example is \( \sigma_{max} = 433 \) psi.
7.5 **A plate with an elliptical hole under tension**

A plate with an elliptical hole of semimajor axis $a = 2$ inch and semiminor axis $b = 1$ inch is subjected to a uniform tension of 100 psi as shown in Figure 7.8, and is discretized using 12 elements with $w/L = 0.4$ as shown in Figure 7.9. Young's modulus is $E = 3.0 \times 10^7$ psi and Poisson's ratio is $\nu = 0.3$. Varying the ratio of semimajor axis $a$ over the width $w$ yields the maximum stress values versus the polynomial order $p$ given in Figure 7.10 (a). The numerical results converge for each $a/w$ and the converged values are compared with Peterson's results [48, 49] as shown in Figure 7.10 (b). This solution is obtained using the hybrid/mixed $p$-method with a $p$-type geometric mapping of the elliptical hole. Numerical results agree very well with Peterson's results.

7.6 **A plate with a circular inclusion under tension**

An infinite plate of unit thickness with a circular inclusion at the plate center and subjected to a uniform tension of 10 psi is shown in Figure 7.11. A $40 \times 20$-inch
Figure 7.7: Stress concentration for a plate with a circular hole under tension: (a) 3-element mesh; (b) 9-element mesh; (c) 14-element mesh; (d) 30-element mesh
Figure 7.8: A plate with an elliptical hole under tension

Figure 7.9: 12-element mesh for a plate with an elliptical hole
Figure 7.10: Stress concentration for a plate with an elliptical hole under tension: (a) maximum stress vs. order $p$; (b) maximum stress vs. $a/w$
Table 7.6: Uniform tension of the plate with a circular hole

<table>
<thead>
<tr>
<th>order $p$</th>
<th>3-element</th>
<th>9-element</th>
<th>14-element</th>
<th>30-element</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>246.3334839</td>
<td>327.5279403</td>
<td>362.3900178</td>
<td>401.4900496</td>
</tr>
<tr>
<td>2</td>
<td>411.9515928</td>
<td>437.2264859</td>
<td>441.2777975</td>
<td>441.1896963</td>
</tr>
<tr>
<td>3</td>
<td>456.8660370</td>
<td>450.3759879</td>
<td>440.8478815</td>
<td>435.9006700</td>
</tr>
<tr>
<td>4</td>
<td>439.1523534</td>
<td>439.6986968</td>
<td>434.8172488</td>
<td>432.9668805</td>
</tr>
<tr>
<td>5</td>
<td>431.9983460</td>
<td>431.0712691</td>
<td>434.1475580</td>
<td>434.4226108</td>
</tr>
<tr>
<td>6</td>
<td>437.8895508</td>
<td>433.9106925</td>
<td>434.8466259</td>
<td>434.7970303</td>
</tr>
<tr>
<td>7</td>
<td>433.5265890</td>
<td>436.2481292</td>
<td>434.8570383</td>
<td>434.7615168</td>
</tr>
<tr>
<td>8</td>
<td>432.7754639</td>
<td>435.2542682</td>
<td>434.7701109</td>
<td>434.7587707</td>
</tr>
<tr>
<td>9</td>
<td>434.8425524</td>
<td>434.6923703</td>
<td>434.7575147</td>
<td>434.7604009</td>
</tr>
<tr>
<td>10</td>
<td>434.9734594</td>
<td>434.7373342</td>
<td>434.7600586</td>
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<td>11</td>
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<td>434.7761272</td>
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</tr>
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<td>434.7637403</td>
<td>434.7656099</td>
<td>434.7602666</td>
<td>434.7604137</td>
</tr>
</tbody>
</table>

plate is considered to approximate the infinite plate with a 2-inch diameter circular inclusion. Only a quarter of the plate is discretized as shown in Figure 7.12. Young’s modulus and Poisson’s ratio for the inclusion material are $E_i = 410$ psi and $\nu_i = 0.2$, $E_m = 69$ psi and $\nu_m = 0.33$ for the matrix material. The tensile stress along the center of the inclusion is plotted in Figure 7.13 and 7.14. The plots show that the numerical results (circle mark) converge to the analytical solution (solid line) when increasing the polynomial order $p$. The analytical solution is given in Appendix F and its numerical solution was solved using the hybrid/mixed $p$-method with second order geometric mapping for the circular inclusion.
Figure 7.11: A plate with a circular inclusion under tension

Figure 7.12: 22-element mesh for a plate with a circular inclusion under tension
Figure 7.13: Tensile stress along the center of the inclusion for $p = 2, 3, 4, 5$
Figure 7.14: Tensile stress along the center of the inclusion for $p = 6, 7, 8, 9$
<table>
<thead>
<tr>
<th>Order $p$</th>
<th>Higher-order mapping</th>
<th>Second-order mapping</th>
<th>Linear mapping</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>228.85</td>
<td>........</td>
<td>228.85</td>
</tr>
<tr>
<td>2</td>
<td>405.94</td>
<td>406.09</td>
<td>416.34</td>
</tr>
<tr>
<td>3</td>
<td>460.07</td>
<td>456.08</td>
<td>485.79</td>
</tr>
<tr>
<td>4</td>
<td>441.22</td>
<td>438.78</td>
<td>496.68</td>
</tr>
<tr>
<td>5</td>
<td>431.03</td>
<td>429.36</td>
<td>541.94</td>
</tr>
<tr>
<td>6</td>
<td>439.16</td>
<td>438.73</td>
<td>623.07</td>
</tr>
<tr>
<td>7</td>
<td>434.99</td>
<td>436.01</td>
<td>686.89</td>
</tr>
<tr>
<td>8</td>
<td>433.12</td>
<td>435.55</td>
<td>741.77</td>
</tr>
</tbody>
</table>

Table 7.7: Stress concentration around hole

7.7 **A 3-D plate with a circular hole under tension**

An 80 × 20-inch plate of unit thickness with a 10 inch diameter hole in the plate center is subjected to a uniform tension of 100 psi. One quarter of the plate is modeled by three hexahedral elements as shown in Figure 7.15. Young's modulus is $E = 3.0 \times 10^7$ psi and Poisson's ratio is taken to be $\nu = 0$. This problem is solved by the hybrid/mixed $p$-method combined with higher-order, second order and linear geometric mapping. The values of the maximum stress are given in Table 7.7. Peterson's result [48, 49] for this example is $\sigma_{max} = 433$ psi. Table 7.7 shows that the maximum stress does not converge using linear mapping, but converges using second order and higher-order mapping. To achieve higher-order accuracy, the $p$-method with higher-order mapping should be used. However, finite element input data can be easily prepared using the $p$-method with second order mapping.
Figure 7.15: Three-element mesh for a hole within a plate
7.8 Bending of a circular curved bar

A quarter of a circular curved bar with rectangular cross section is discretized using 20-node hexahedral elements as shown in Figure 7.16. The curved bar is constrained at the upper end and bent by a force $P = 1000$ pounds at the lower end in the radial direction. Young’s modulus is $E = 3.0 \times 10^7$ psi and Poisson’s ratio is $\nu = 0$. The inner and the outer radii are $a = 3.0$ inch and $b = 5.0$ inch. The thickness is 1.0 inch. The elasticity solution of bending stress at the constrained end is

$$\sigma_{\theta} = \frac{3r - \frac{a^2 + b^2}{r} - \frac{a^2 b^2}{r^2}}{(a^2 + b^2) \ln \frac{b}{a} + a^2 - b^2} P$$  \quad (7.10)$$

The polynomial order $p$ starts directly from $p = 2$ with a 20-node geometric mapping. The bending stresses along the cross section of the upper end is plotted in Figure 7.17. It shows that the numerical values (circular mark) is almost identical to the elasticity solution (solid line) for $p \geq 3$.

7.9 Bending of a T-shaped bar

A T-shaped bar of unit thickness with shoulder fillets is shown in Figure 7.18, in which $H = 3.0$ inch, $h = 2.0$ inch, the radius of the fillet $r = 0.25$ inch, $a = 1.0$ inch and $b = 3.25$ inch. The T-shaped bar can be meshed as in Figure 7.19 with a concentrated load 1000 pounds. Because of the symmetric condition of the problem, the mesh in Figure 7.20 with symmetric boundary constrain and a concentrated load 500 pounds are used for the computation. The maximum stress concentration at fillet is computed by the hybrid/mixed $p$-method with second order element mapping and ANSYS $p$-element (only second order geometric mapping is incorporated in ANSYS).
Figure 7.16: 12-element mesh for the curved bar
Figure 7.17: Bending stresses along the cross section of the fixed end
Table 7.8: Maximum stress at fillet of T-section

The result becomes close at higher order, given in Table 7.8. Peterson's result [48, 49] of this problem is $\sigma_{\text{max}} = 5040$ psi.
Figure 7.18: T-shaped bar with fillets problem
Figure 7.19: Finite element mesh for T-shaped bar
Figure 7.20: Finite element mesh for T-shaped bar with symmetric boundary constraint
CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

8.1 Conclusions

The purpose of this dissertation was to develop an efficient computational code for accurate two- and three-dimensional stress analysis. The $p$-adaptive hybrid/mixed finite element formulation has been developed for this purpose. It is proved, in this dissertation, that the hybrid/mixed $p$-method developed is superior to the displacement based $p$-method in terms of efficiency and accuracy for computing both displacement and stress fields. Using the hierarchical shape functions for the displacement field and using Legendre polynomials for interpolating the stress and strain fields has simplified the hybrid/mixed formulation. This work made significant contributions to the higher-order hybrid/mixed finite element technique. In addition, a new $p$-order geometric mapping formulation is developed and can easily be incorporated into most finite element programs. The $p$-method starting directly from second order isoparametric element using novel Lagrange hierarchical shape functions is another contribution.

In Chapter 2, several important variational principles in solid mechanics were introduced, including the principle of minimum potential energy, the principle of
minimum complementary energy, Hellinger-Reissner principle and Hu-Washizu principle. The finite element formulations based on these principles can be expressed as the stiffness matrix multiplied by the displacement vector equals the load vector. The main difference between the finite element formulations using these principles is the formation of the element stiffness matrix. At the beginning of the research, formulating the hybrid/mixed p-method was influenced by the methods given in papers [15, 16, 41, 42]. The Hu-Washizu principle was selected for developing the new p-method because in this principle, using orthogonal stress and strain interpolation can help reduce the computational time of generating the element stiffness matrix.

In the p-method, the hierarchical shape functions are chosen for the interpolation of the displacement field so that the element stiffness matrix is also hierarchical. In Chapter 3, the p-type hierarchical shape functions are introduced. The shape functions start from first-order plus higher-order hierarchical shape functions. With the development of one-dimensional hierarchical shape functions using Legendre, Chebyshev and Lagrange approaches, the higher-dimensional hierarchical shape functions were constructed. The p-method using the three different approaches gives identical results in accuracy. Among the three approaches, only the hierarchical degrees of freedom using a Lagrange approach have physical meaning. Therefore, the Lagrange approach was used to develop the p-order geometric mapping. As a by-product, a novel Lagrange approach starting directly from second-order Lagrange shape functions plus higher-order hierarchical shape functions was developed which is suitable for the p-type shape functions.
In Chapter 4, a new development of geometric mapping for the $p$-method was presented. For the element with curved boundaries or curved surfaces, the mapping functions were represented by a $p$-order polynomial using the Lagrange hierarchical shape functions. The coefficients corresponding to the shape functions are the higher-order derivatives of the curve and surface functions and should be calculated in advance so that the curved sides and surfaces can be approximated by known polynomials. The accuracy of the mapping can be achieved by increasing the polynomial order of $p$. This technique can easily be incorporated into the $p$-version finite element programs. The higher-order derivatives of the curve and surface functions can be calculated using MAPLE software with enough numbers of digits for accuracy.

In Chapter 5, the formulation of the displacement based $p$-version finite element method was presented. The stiffness matrix and load vector are hierarchical since the hierarchical shape functions are applied to the displacement field. The formulations were presented for both the 2-D and 3-D case. Part of the formulations in this chapter can be used to develop the $p$-version hybrid/mixed finite element equations. 2-D and 3-D displacement based $p$-version finite element programs have been written for comparison with the $p$-version hybrid/mixed finite element programs.

In Chapter 6, the development of the $p$-version hybrid/mixed finite element formulation was presented. The combination of using hierarchical shape functions and the orthogonal stress and strain interpolations simplifies the formation of the stiffness matrix and makes it possible for the $p$-type hybrid/mixed finite element formulation to be efficiently applied to real engineering problems. Because of the use of the
hierarchical shape functions, programming can be designed adaptively with increasing polynomial order $p$. This is known as $p$-adaptive hybrid/mixed finite element method. In order to show convergence of this technique, the error of the energy norm is computed at each $p$ level. The linear system equations are solved using the wave-front solution method [46]. The computational time between the displacement based $p$-method (DBPM) and the hybrid/mixed $p$-method (HMPM) is almost the same for lower order $p$. However, for $p \geq 5$, HMPM requires more time than DBPM for 2-D programs and HMPM requires much less time than DBPM for 3-D programs. In other words, the 3-D hybrid/mixed $p$-version finite element program is more efficient than the 3-D displacement based $p$-version finite element program.

In Chapter 7, nine numerical examples are given to test the $p$-version hybrid/mixed finite element code. The results of examples show that $p$-method gives much higher convergence rate than the regular $h$-method. In the $p$-version technique, the hybrid/mixed method has advantages over the displacement based method in computing the stress field, especially for nearly incompressible material. Both methods give the same accuracy and convergence rate for strain energy. All numerical examples have shown the successful use of the $p$-version hybrid/mixed finite element method.

8.2 Contributions

A significant contribution is made to $p$-version finite element analysis using the hybrid/mixed method. By using hierarchical shape functions and orthogonal stress and strain interpolations in the Hu-Washizu principle, computing time for the hybrid/mixed $p$-method is less than that for the displacement based $p$-method for 3-D
problems. In addition, the hybrid/mixed p-method gives more accurate stress value than the displacement based p-method, especially for nearly incompressible material.

Based on a blending function method, a new p-type geometric mapping was developed for elements with curved boundaries and surfaces and successfully used in the hybrid/mixed p-method. This mapping technique can easily be incorporated into finite element programs and works very well as seen by numerical examples given in sections 7.2 and 7.4.

A novel Lagrange approach was developed for the 1-D hierarchical shape functions, from which the 2-D and 3-D hierarchical shape functions were constructed. This approach started directly from the second order shape functions plus higher-order hierarchical shape functions which can be used to formulate the eight- or nine-noded p-version shell elements[47].

The new developed hybrid/mixed p-method is competitive with the displacement based p-method used in FEA software programs.

8.3 Recommendations for future research

According to the research and development of the hybrid/mixed p-method and the results of the numerical examples, some recommendations for future research are as follows:

• Develop the hp-adaptive hybrid/mixed finite element programs for both 2-D and 3-D stress analysis.

• Develop the hybrid/mixed p-method to include shell element.

• Extend the hybrid/mixed p-method to non-linear problems.
APPENDIX A

LEGENDRE POLYNOMIAL

Legendre polynomials $L_n (\xi)$ are expressed as

$$L_n (\xi) = \frac{1}{2^n n!} \frac{d^n}{d\xi^n} (\xi^2 - 1)^n$$  \hspace{1cm} (A.1)

Orthogonality of Legendre polynomials:

$$\int_{-1}^{1} L_m (\xi) L_n (\xi) d\xi = 0 \hspace{1cm} m \neq n$$

$$\int_{-1}^{1} L_n (\xi) L_n (\xi) d\xi = \frac{2}{2n + 1}$$  \hspace{1cm} (A.2)

Orthogonal series of Legendre polynomials:

$$f (\xi) = A_0 L_0 (\xi) + A_1 L_1 (\xi) + A_2 L_2 (\xi) + ...$$

where

$$A_k = \frac{2k + 1}{2} \int_{-1}^{1} f (\xi) L_k (\xi) d\xi$$  \hspace{1cm} (A.3)

Special results involving Legendre polynomials:

$$L_n (1) = 1 \hspace{1cm} L_n (-1) = (-1)^n$$  \hspace{1cm} (A.4)

$$L_{n+1} (\xi) - L_{n-1} (\xi) = (2n + 1) \int_{-1}^{\xi} L_n (s) ds$$  \hspace{1cm} (A.5)
\[ L'_{n+1} (\xi) - L'_{n-1} (\xi) = (2n + 1) L_n (\xi) \]  
(A.6)

\[ \int_{-1}^{1} \xi^i L_k (\xi) d\xi = 0 \quad for \quad k > l. \]  
(A.7)
APPENDIX B

CHEBYSHEV POLYNOMIAL

Chebyshev polynomials $T_n(\xi)$ are expressed as

$$ T_n(\xi) = \cos(n \cos^{-1} \xi) $$

$$ T_{n+1}(\xi) - 2\xi T_n(\xi) + T_{n-1}(\xi) = 0 \quad (B.1) $$

Orthogonality of Chebyshev polynomials:

$$ \int_{-1}^{1} \frac{T_m(\xi) T_n(\xi)}{\sqrt{1 - \xi^2}} d\xi = 0 \quad m \neq n $$

$$ \int_{-1}^{1} \frac{T_n(\xi)}{\sqrt{1 - \xi^2}} d\xi = \left\{ \begin{array}{ll} \pi & n = 0 \\ \pi/2 & n > 0 \end{array} \right. \quad (B.2) $$

Orthogonal series of Chebyshev polynomials:

$$ f(\xi) = \frac{1}{2} A_0 T_0(\xi) + A_1 T_1(\xi) + A_2 T_2(\xi) + ... $$

where

$$ A_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(\xi) T_k(\xi)}{\sqrt{1 - \xi^2}} d\xi \quad (B.3) $$

Special results involving Chebyshev polynomials:

$$ T_n(1) = 1 \quad T_n(-1) = (-1)^n \quad (B.4) $$

$$ T_{2n}(0) = (-1)^n \quad T_{2n+1}(0) = 0 \quad (B.5) $$
A function $u(\xi)$ can be expanded by linear Lagrange shape functions plus higher-order hierarchical shape functions. Define the following function

$$f(\xi) = u(\xi) - \left(\frac{1-\xi}{2}\right)u_1 - \left(\frac{1+\xi}{2}\right)u_2$$  \hspace{1cm} (C.1)

where $-1 \leq \xi \leq 1$, and $u_1 = u(-1)$ and $u_2 = u(1)$.

The Taylor expansion of the function $f(\xi)$ is given as

$$f(\xi) = f(0) + f'(0)\xi + \sum_{n=2}^{\infty} f^{(n)}(0)\frac{\xi^n}{n!}$$  \hspace{1cm} (C.2)

where

$$f(0) = u(0) - \frac{u_1 + u_2}{2}$$
$$f'(0) = u'(0) + \frac{u_1 - u_2}{2}$$  \hspace{1cm} (C.3)
$$f^{(n)}(0) = u^{(n)}(0)$$

The Taylor expansion of function $u(\xi)$ is given as

$$u(\xi) = u(0) + u'(0)\xi + \sum_{n=2}^{\infty} u^{(n)}(0)\frac{\xi^n}{n!}$$  \hspace{1cm} (C.4)
and the function \( u \) at \( \xi = -1 \) and \( \xi = 1 \) becomes

\[
\begin{align*}
  u_1 &= u(-1) = u(0) - u'(0) + \sum_{n=2}^{\infty} u^{(n)}(0) \frac{(-1)^n}{n!} \\
  u_2 &= u(1) = u(0) + u'(0) + \sum_{n=2}^{\infty} u^{(n)}(0) \frac{1}{n!}
\end{align*}
\]  
(C.5)  
(C.6)

Equation (C.5) plus equation (C.6) gives

\[
f(0) = u(0) - \frac{u_1 + u_2}{2} = -\sum_{n=2,4,6}^{\infty} \frac{u^{(n)}(0)}{n!}
\]  
(C.7)

and equation (C.5) minus equation (C.6) gives

\[
f'(0) = u'(0) + \frac{u_1 - u_2}{2} = -\sum_{n=3,5,7}^{\infty} \frac{u^{(n)}(0)}{n!}
\]  
(C.8)

Substituting equation (C.7) and equation (C.8) into equation (C.2) and then into equation (C.1), gives

\[
u(\xi) = \left(\frac{1-\xi}{2}\right) u_1 + \left(\frac{1+\xi}{2}\right) u_2 + \sum_{n=2}^{\infty} \phi_n(\xi) u^{(n)}(0)
\]  
(C.9)

where

\[
\phi_n(\xi) = \frac{\xi^n - 1}{n!} \quad \text{for } n = 2, 4, 6, ... 
\]  
(C.10)

\[
\phi_n(\xi) = \frac{\xi^n - \xi}{n!} \quad \text{for } n = 3, 5, 7, ...
\]  
(C.11)
APPENDIX D

NOVEL LAGRANGE HIERARCHICAL EXPANSION

A function $u(\xi)$ can be expanded by second order Lagrange shape functions plus higher-order hierarchical shape functions [51]. Define the following function

$$f(\xi) = u(\xi) - N_1 u_1 - N_2 u_2 - N_3 u_3$$  \hspace{1cm} (D.1)

where

$$N_1 = \frac{1}{2}(\xi - 1)\xi, \quad N_2 = \frac{1}{2}(\xi + 1)\xi, \quad N_3 = 1 - \xi^2$$  \hspace{1cm} (D.2)

and

$$u_1 = u(-1), \quad u_2 = u(1), \quad u_3 = u(0)$$  \hspace{1cm} (D.3)

The Taylor expansion of the function $f(\xi)$ is

$$f(\xi) = f(0) + f'(0)\xi + f''(0)\frac{\xi^2}{2} + \sum_{n=3}^{\infty} f^{(n)}(0)\frac{\xi^n}{n!}$$  \hspace{1cm} (D.4)

where

$$f(0) = 0$$

$$f'(0) = u'(0) + \frac{u_1 - u_2}{2}$$

$$f''(0) = u''(0) - u_1 - u_2 + 2u_3$$

$$f^{(n)}(0) = u^{(n)}(0)$$  \hspace{1cm} (D.5)
The Taylor expansion of function \( u(\xi) \) is given as

\[
u(\xi) = u(0) + u'(0) \xi + u''(0) \frac{\xi^2}{2} + \sum_{n=3}^{\infty} u^{(n)}(0) \frac{\xi^n}{n!}
\] (D.6)

and the function \( u \) at \( \xi = -1, \xi = 0 \) and \( \xi = 1 \) becomes

\[
u_1 = u(-1) = u(0) - u'(0) + u''(0) \frac{1}{2} + \sum_{n=3}^{\infty} u^{(n)}(0) \frac{(-1)^n}{n!}
\] (D.7)

\[
u_2 = u(1) = u(0) + u'(0) + u''(0) \frac{1}{2} + \sum_{n=3}^{\infty} u^{(n)}(0) \frac{1}{n!}
\] (D.8)

\[
u_3 = u(0)
\] (D.9)

Equation (D.7) minus equation (D.8) gives

\[
\frac{f'(0)}{2} = u'(0) + \frac{u_1 - u_2}{2} = - \sum_{n=3,5,7}^{\infty} u^{(n)}(0) \frac{(-1)^n}{n!}
\] (D.10)

and equation (D.7) plus equation (D.8) gives

\[
\frac{f''(0)}{2} = \frac{u''(0)}{2} + u_3 - \frac{u_1 + u_2}{2} = - \sum_{n=4,6,8}^{\infty} u^{(n)}(0) \frac{1}{n!}
\] (D.11)

Substituting equation (D.10) and equation (D.11) into equation (D.4) and then into equation (D.1), gives

\[
u(\xi) = N_1 u_1 + N_2 u_2 + N_3 u_3 + \sum_{n=3}^{\infty} \phi_n(\xi) u^{(n)}(0)
\] (D.12)

where

\[
\phi_n(\xi) = \frac{\xi^n - \xi}{n!} \quad \text{for } n = 3, 5, 7, ...
\] (D.13)

\[
\phi_n(\xi) = \frac{\xi^n - \xi^2}{n!} \quad \text{for } n = 4, 6, 8, ...
\] (D.14)
APPENDIX E

GRAM-SCHMIDT PROCEDURE

If, in Hu-Washizu principle from section 6.1, select $P' = P$ then the orthogonal stress and strain modes $P_k^* = \{p_1^*, p_2^*, p_3^*, \ldots\}$ can be obtained from Gram-Schmidt procedure [41, 42]. The procedure starts from a base interpolation polynomials $P_k = \{p_1, p_2, p_3, \ldots\}$ as

$$ u_i = \begin{cases} p_i & i = 1 \\ p_i - \sum_{j=1}^{i-1} \langle p_j^*, p_i \rangle p_j^* & i > 1 \end{cases} \quad (E.1) $$

where we define

$$ \langle u, v \rangle = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} u_i v_j |J| d\xi d\eta d\zeta \quad (E.3) $$

so that

$$ \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} P_k^T P_k^* |J| d\xi d\eta d\zeta \equiv I \quad (E.4) $$

The polynomial operation can be performed using procedure in [50]
Thus, based on this approach the finite element formulation in Hu-Washizu principle becomes simplified. Equations (6.6), (6.7) and (6.8) become

\[ M = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \mathbf{P}^* \mathbf{C} \mathbf{P}^* |\mathbf{J}| d\xi d\eta d\zeta \]

\[ = \begin{bmatrix}
    c_{11} \mathbf{I} & c_{12} \mathbf{I} & c_{13} \mathbf{I} \\
    c_{12} \mathbf{I} & c_{22} \mathbf{I} & c_{23} \mathbf{I} \\
    c_{13} \mathbf{I} & c_{23} \mathbf{I} & c_{33} \mathbf{I}
\end{bmatrix}
\]

\[ \mathbf{C}_{\text{ijkl}} \]

(E.5)

\[ \mathbf{H} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \mathbf{P}^* \mathbf{P}^* |\mathbf{J}| d\xi d\eta d\zeta = \mathbf{I} \]  

(E.6)

and

\[ \mathbf{G} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \mathbf{P}^* \mathbf{B} |\mathbf{J}| d\xi d\eta d\zeta \]  

(E.7)

where \( c_{ij} \) is a material constant, and \( \mathbf{I} \) is the identity matrix. Finally, equations (6.10), (6.11) and (6.12) are simplified as

\[ \alpha = \mathbf{G} \mathbf{q} \]  

(E.8)

\[ \beta = \mathbf{M} \mathbf{G} \mathbf{q} \]  

(E.9)

and

\[ \mathbf{K} = \mathbf{G}^T \mathbf{M} \mathbf{G} \]  

(E.10)
APPENDIX F

CYLINDRICAL INCLUSION IN AN INFINITE PLATE UNDER UNIFORM TENSION

The analytical solution on stress concentration around a cylindrical inclusion in a field of uniaxial tension is presented in this appendix.

Equilibrium equations in polar coordinates \((r, \theta)\) are given as

\[
\begin{align*}
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + K_r &= 0 \\
\frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + K_\theta &= 0
\end{align*}
\]

For plane strain problem, the stress-strain relation can be written as

\[
\begin{align*}
\epsilon_r &= \frac{1 - \nu^2}{E} \left( \sigma_r - \frac{\nu}{1 - \nu} \sigma_\theta \right) \\
\epsilon_\theta &= \frac{1 - \nu^2}{E} \left( \sigma_\theta - \frac{\nu}{1 - \nu} \sigma_r \right) \\
\gamma_{r\theta} &= \frac{1}{\mu} \tau_{r\theta}
\end{align*}
\]  

where the relation among the shear modulus \(\mu\), Young's modulus \(E\) and Poisson's ratio \(\nu\) is expressed as

\[
\mu = \frac{E}{2(1 + \nu)}
\]

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The strain-displacement relation is expressed as

\[
\begin{align*}
\varepsilon_r &= \frac{\partial u_r}{\partial r} \\
\varepsilon_\theta &= \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \\
\gamma_{r\theta} &= \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}
\end{align*}
\]  

(F.4)

In two-dimensional elasticity, the problem is usually solved by assuming stress function. The stress function in polar coordinates is given as:

\[
\nabla^4 \Phi = 0
\]

(F.5)

where

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}
\]

(F.6)

Equilibrated stresses expressed by stress functions in polar coordinate are:

\[
\begin{align*}
\sigma_r &= \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \\
\sigma_\theta &= \frac{\partial^2 \Phi}{\partial r^2} \\
\tau_{r\theta} &= -\frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial \Phi}{\partial \theta}
\end{align*}
\]

(F.7)

Consider a cylindrical inclusion within an infinite exterior region. The exterior is subject to uniaxial tension \( p \). The origin of the polar coordinate is selected at the center of the inclusion. Denote the shear modulus and Poisson’s ratio of the inclusion by \( \mu_1, \nu_1 \) and those of the exterior region by \( \mu_2, \nu_2 \), respectively.

The boundary conditions at inclusion boundary \( r = a \) satisfy

\[
\sigma_r^1(a, \theta) = \sigma_r^2(a, \theta)
\]

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Stress functions satisfying equation (F.5) may be taken as:

- For the inclusion

\[
\Phi = Ar^2 + Br^2 \cos 2\theta + Ca^{-2}r^2 \cos 2\theta \tag{F.9}
\]

- For the exterior

\[
\Phi = \frac{p}{4}r^2 (1 - \cos 2\theta) + Fa^2 \log r + Ha^4r^{-2} \cos 2\theta + Ma^2 \cos 2\theta \tag{F.10}
\]

Substituting (F.9) and (F.10) into (F.7), we can verify that the stresses satisfy the condition of uniaxial tension field away from inclusion and then using boundary conditions (F.8) at \( r = a \) results in six equations and six unknowns \( A, B, C, F, H \) and \( M \). These equations are:

\[
2A - F = \frac{1}{2}
\]

\[
2 (1 - 2\nu_1) A + gF = \left( \frac{1}{2} - \nu_2 \right) g \tag{F.11}
\]

and

\[
-B + 3H + 2M = \frac{1}{4}
\]

\[
B + 3C + 3H + M = \frac{1}{4}
\]

\[
-B - 2\nu_1C - gH - 2 (1 - \nu_2) gM = \frac{1}{4}g
\]

\[
B + (3 - 2\nu_1) C - gH + (1 - 2\nu_2) gM = -\frac{1}{4}g \tag{F.12}
\]

where \( g = \mu_1/\mu_2 \). The solution can be obtained by solving above equations.
BIBLIOGRAPHY


