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VIBRATION ANALYSIS OF BEAMS AND RECTANGULAR PLATES WITH MULTIPLE CONSTRAINED LAYER DAMPING PATCHES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for

the Degree Doctor of Philosophy in the

Graduate School of the Ohio State University

By

Shih-Wei Kung, B.S.M.E., M.S.M.E.

* * * * *

The Ohio State University

1998

Dissertation Committee:

Dr. Rajendra Singh, Adviser

Dr. Donald R. Houser

Dr. Chia-Hsiang Menq

Dr. Robert G. Parker

Approved by

Advisor

Department of Mechanical Engineering
ABSTRACT

This dissertation examines the damped vibration characteristics of beams and rectangular plates treated with multiple constrained-layer viscoelastic patches. Each damping patch consists of a metallic constraining layer and an adhesive viscoelastic layer with spectrally-varying shear modulus and loss factor properties. A new analytical model based on the Rayleigh-Ritz scheme is developed first. Motion variables of all layers are expressed in terms of the flexural displacement of the base structure (beam or plate). Then the flexural shape function sets are incorporated in the Rayleigh-Ritz minimization scheme to obtain a complex eigenvalue problem. This method allows for the visualization of complex modes of all deformation variables including shear deformation of the viscoelastic core that is the major contributor to the overall energy dissipation. Results of the proposed method compare well with those reported in the literature on simply supported sandwich beams and plates with single patches. Analytical predictions of natural frequencies, modal loss factors and complex modes are in excellent agreement with modal measurements. Effect of patch boundary conditions, patch cutouts and locations, and mismatched patch combinations are also analytically and experimentally
examined. A refined technique to estimate unknown spectrally-varying properties of viscoelastic materials, by combining theory and experiment, has also been developed.

Subsequently, three simpler analytical formulations are successfully developed without explicitly solving high order differential equations or complex eigenvalue problems. Approximate Method I is developed for sandwich beams assuming that damped mode shapes are given by the superposition of Euler beam eigenfunctions. In Method II, the formulation is further simplified with the assumption of a very compliant viscoelastic core. Finally, Method III considers a compact patch problem and modal loss factor is expressed as a product of terms related to material properties, layer thickness, patch size and location. Approximate Methods II and III are also extended to rectangular plates. These methods yield reasonably accurate results in a computationally efficient manner while providing much insight into patch damping design concepts. Approximate formulations have been verified by comparing results with modal measurements.
Dedicated to my wife
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VITA

August 1966 ..................................... Born - Taipei, Taiwan, R.O.C.

June 1988 ...................................... B.S. in Mechanical Engineering,
National Taiwan University,
Taipei, Taiwan, R.O.C.

1990 - 1991 ................................... Teaching Assistant,
National Taiwan University,
Taipei, Taiwan, R.O.C.

1992 - present ................................. Graduate Research Associate,
Department of Mechanical Engineering,
The Ohio State University, Columbus, Ohio.

1994 ............................................. M.S. in Mechanical Engineering,
The Ohio State University, Columbus, Ohio.

December 1996 - March 1997 .......... Graduate Engineering Intern,
Nissan Technical Center, Japan.

PUBLICATIONS

1. S.-W. Kung 1994 M.S. Thesis, The Ohio State University. Finite element and
experimental modal analyses of automotive brake pad and insulator.

SAE Paper No. 951243. Evaluation of damping material at higher frequencies
with application to automotive systems including brakes.

95-WA/NCA-28. Determination of joint stiffness through vibration analysis of
beam assemblies.
FIELDS OF STUDY

Major Field: Mechanical Engineering

Specialty Areas: Vibrations and noise control, damping patch design.
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LIST OF SYMBOLS

List of Symbols for Chapter 1

\( a \) acceleration
\( f \) force
\( G \) shear modulus
\( \Delta L \) insertion loss in dB
\( \vec{r} \) position vector
\( T \) temperature
\( p \) sound pressure
\( \eta \) modal loss factor
\( \eta_m \) material loss factor
\( \omega \) frequency

Operators
\[ \| \| \] absolute value

Subscripts
\( 0 \) undamped case
\( a \) acceleration
\( d \) case with damping material attached
\( p \) sound pressure
\( \eta \) modal loss factor
List of Symbols for Chapter 2

\( A \) cross-sectional area
\( \mathbf{A} \) governing equation matrix
\( \mathbf{a}, \mathbf{b}, \mathbf{c} \) coefficient vectors
\( a, b, c \) coefficients
\( \mathbf{B} \) governing equation vector
\( \mathbf{C} \) coefficient vector
\( C \) thickness parameter \((h_1 + 2h_2 + h_3)/2\)
\( d \) spatial matrix
\( \mathbf{d} \) spatial constant
\( \mathbf{E} \) elasticity matrix
\( E \) Young’s modulus
\( e \) elasticity ratio \(E_y A_y/E_1 A_1\)
\( f \) frequency (Hz)
\( \mathbf{G} \) shear modulus
\( \mathbf{H} \) inertia matrix
\( h \) thickness
\( I \) area moment of inertia
\( i \) \(\sqrt{-1}\)
\( \mathbf{K} \) stiffness matrix
\( l \) length
\( \mathbf{M} \) mass matrix
\( N \) total number of patches
\( n \) total number of shape functions
\( \mathbf{q} \) generalized displacement vector
\( q \) generalized displacement
\( \mathbf{r} \) deformation vector
\( S \) admissible shape function matrix
\( T \) kinetic energy
\( u \) in-plane or longitudinal displacement
\( \mathbf{U} \) potential or strain energy
\( \mathbf{V} \) transfer matrix
\( w \) flexural displacement
\( x, z \) spatial coordinates
\( \gamma \) shear deformation
\( \eta \) loss factor
\( \kappa \) shear correction factor
\( \lambda \) eigenvalue
\( \xi \) rotational shape function vector
\( \xi \) rotational shape function
\( \rho \) mass density
\begin{align*}
\varsigma & \quad \text{longitudinal shape function vector} \\
\zeta & \quad \text{longitudinal shape function} \\
\Phi & \quad \text{reduced admissible shape matrix} \\
\varphi & \quad \text{flexural shape function vector} \\
\phi & \quad \text{flexural shape function} \\
\chi & \quad \text{trial function vector for longitudinal displacement} \\
\psi & \quad \text{rotation vector} \\
\nu & \quad \text{rotation} \\
\omega & \quad \text{frequency (rad/s)}
\end{align*}

\textbf{Operators}
\begin{align*}
\mathcal{D} & \quad \text{differential operator matrix} \\
\text{Im} & \quad \text{imaginary part} \\
\text{Re} & \quad \text{real part} \\
\mathcal{J} & \quad \text{spatial operator} \\
\partial & \quad \text{differential operator}
\end{align*}

\textbf{Superscripts}
\begin{align*}
p & \quad \text{patch number} \\
T & \quad \text{transpose} \\
\approx & \quad \text{complex valued} \\
\wedge & \quad \text{normalized quantity}
\end{align*}

\textbf{Subscripts}
\begin{align*}
A, B & \quad \text{type of damping patch} \\
b & \quad \text{measurement of the benchmark case} \\
I & \quad \text{inherent damping} \\
i & \quad \text{layer number} \\
k & \quad \text{admissible function number} \\
r & \quad \text{modal index} \\
1 & \quad \text{layer 1 (elastic constraining layer)} \\
2 & \quad \text{layer 2 (viscoelastic constrained layer)} \\
3 & \quad \text{layer 3 (base structure: beam)}
\end{align*}
List of Symbols for Chapter 3

A \quad \text{governing equation matrix}

a, b, c \quad \text{coefficient vectors}

a, b, c \quad \text{coefficients}

B \quad \text{governing equation vector}

C \quad \text{coefficient vector}

d \quad \text{spatial matrix}

d \quad \text{spatial constant}

E \quad \text{elasticity matrix}

E \quad \text{Young’s modulus}

e \quad \text{elasticity ratio } E_y A_y / E_x A_x

F \quad \text{function used to normalize modes}

f \quad \text{frequency (Hz)}

G \quad \text{Shear modulus}

H \quad \text{inertia matrix}

H \quad \text{thickness parameter } (h_1 + 2h_2 + h_3)/2

h \quad \text{thickness}

i \quad \sqrt{-1}

K \quad \text{stiffness matrix}

l \quad \text{length}

M \quad \text{mass matrix}

N_p \quad \text{total number of patches}

N_x, N_y \quad \text{total number of shape functions}

q \quad \text{generalized displacement vector}

q \quad \text{generalized displacement}

q \quad \text{eigenvector}

r \quad \text{deformation vector}

S \quad \text{admissible shape function matrix}

T \quad \text{kinetic energy}

u, v \quad \text{in-plane or longitudinal displacement}

U \quad \text{potential or strain energy}

V \quad \text{transfer matrix}

w \quad \text{flexural displacement}

X, Y \quad \text{shape function vectors in } x \text{ and } y
$x, y, z$ spatial coordinates
$\alpha, \beta, \iota$ modal weighting factors
$\Phi$ shape function vector
$\phi$ shape function
$\gamma$ shear deformation
$\lambda$ eigenvalue
$\omega$ frequency (rad/s)
$\eta$ loss factor
$\rho$ mass density
$\Psi$ trial function vector
$\psi$ rotation
$\theta$ rotation angle used to normalize complex mode

ABBREVIATIONS

$A, B, C, D, E$ damping cases of benchmark experiments
$A, B$ type of damping patch
$F$ bree boundary

OPERATORS

$\text{Col}$ matrix operator
$\mathcal{D}$ differential operator matrix
$\text{Im}$ imaginary part
$\text{Re}$ real part
$\partial$ differential operator
$||$ absolute value

SUPERSCRIPTS

$p$ patch number
$T$ transpose
$\sim$ complex valued
$^\wedge$ normalized quantity

SUBSCRIPTS

$A, B$ type of damping patch
C          damping Case C
I          inherent damping
i          layer number
k, m, n    admissible function number
max        maximum
min        minimum
r          modal index
u,v        in plane motion
x, y, z    spatial coordinates
w          flexural motion
γ          shear deformation
ψ          rotation
1          layer 1 (elastic constraining layer)
2          layer 2 (viscoelastic constrained layer)
3          layer 3 (base structure)
List of Symbols for Chapter 4

\( a, b, c, d \) coefficients
\( C \) thickness parameter \( (h_1+2h_2+h_3)/2 \)
\( E \) Young’s modulus
\( e \) elasticity ratio \( E_i h_i/E_3 h_3 \)
\( F \) axial force
\( f \) frequency (Hz)
\( G \) relative stiffness
\( G \) Shear modulus
\( H \) patch thickness index
\( h \) thickness
\( I \) area moment of inertia
\( L \) patch size parameter
\( l \) length or width
\( N_p \) total number of patches
\( N \) total number of shape functions
\( \Phi \) patch performance index
\( q \) coefficient
\( r \) trial eigenfunction number
\( s \) spatial constant
\( T \) kinetic energy
\( u \) in-plane or longitudinal displacement
\( U \) potential or strain energy
\( w \) flexural displacement
\( x, y, z \) spatial coordinates
\( Y \) a stiffness and dimension parameter
\( \phi \) shape function
\( \gamma \) shear deformation
\( \eta_k \) modal loss factor of mode \( k \)
\( \eta_{m,2} \) material loss factor of layer 2
\( \eta_p \) modal loss factor contributed by patch \( p \)
\( \lambda \) frequency parameter
\( \mu \) a constant related to stiffness, dimensions and frequency parameter
\( \nu \)  
Poisson's ratio

\( \rho \)  
mass density

\( \tau \)  
shear stress

\( \omega \)  
frequency (rad/s)

**Operators**

\( d \)  
differential operator

**Superscripts**

\( ^- \)  
non-dimensionalized valued

\( ^\wedge \)  
normalized quantity

\( p \)  
patch number

\( k \)  
modal index

**Subscripts**

\( a, b \)  
patch ends

\( 0 \)  
reference value for normalization

\( 1 \)  
layer 1 (elastic constraining layer)

\( 2 \)  
layer 2 (viscoelastic constrained layer)

\( 3 \)  
layer 3 (base structure)

\( \text{max} \)  
maximum

\( \text{min} \)  
minimum

\( \text{opt} \)  
opimum value

\( \text{total} \)  
total value

\( u \)  
in plane motion

\( x, y \)  
spatial coordinates

\( w \)  
flexural motion

\( \gamma \)  
shear deformation
CHAPTER 1

INTRODUCTION

1.1. Background and Literature Review

There is a substantial body of literature on constrained layer damping as evident from the studies described in two books on vibration damping by Nashif et al. [1.1] and Sun and Lu [1.2]. Much of the prior research is applicable to beams and rectangular plates with full coverage, i.e. damping material is uniformly distributed over the vibrating surface. However, in many practical plate and machinery casing structures, it may be even difficult to treat the whole surface with constrained layer viscoelastic material. Non-uniform and/or partial damping treatment may be necessary because of material, thermal, packaging, weight or cost constraints [1.1]. Further, it may indeed be desirable to selectively apply one or more damping patches to control certain resonances. However, there is a limited body of scientific literature of this topic and it therefore remains a somewhat ill-understood and empirical technique. Single or double constrained layer patches have been computationally examined by using higher order differential equation theory [1.3-7], Rayleigh-Ritz method [1.1-4, 1.8-9], or finite element procedure [1.10-12]. Kung [1.13] conducted vibration forced response analysis of damped structures using ABAQUS [1.14], which is the only commercial finite element code available for the type of viscoelastic material analysis discussed in this study. Experimental methods
of investigation have included modal testing [1.1-4] and structural intensity mapping [1.15].

Nokes and Nelson [1.16] were among the earliest investigators to provide an analytical solution to the problem of a partially covered sandwich beam. In their formulation, damped mode shapes are assumed to be the same as the undamped eigenvectors, and the modal loss factor was evaluated by calculating the ratio of energy dissipated to the total modal strain energy. A more thorough analytical study was carried out by Lall et al. [1.8]. In their Rayleigh-Ritz approach, both flexural and longitudinal shape functions were incorporated in the eigenvalue problem for a beam with a single damping patch. The formulation was extended to a simply supported plate with a single damping patch [1.9]. A parametric study on the patch size and location was reported. However, in these partial coverage studies, boundary conditions were limited to simple supports and only one patch analyzed. In addition, no experimental results were provided to support the analytical models. Consequently, a clear need exists for a more refined analysis which this research attempts to fulfill.

1.2. Problem Formulation

1.2.1. Preliminary vibro-acoustic experiment

A preliminary experimental study is conducted using a rectangular plate (342.9mm x 266.7mm) with free edges. Vibro-acoustic measurements are taken to investigate the effect of full or partial damping coverage on both acceleration ($a$) and sound pressure ($p$) levels. Figure 1.1 shows the experimental setup and Figure 1.2 shows three viscoelastic damping treatments of interest: undamped plate, partial coverage (20%) with one
Figure 1.1 Preliminary vibro-acoustic experiment.
Figure 1.2  Rectangular plate example with free edges. (a) Undamped, (b) partial coverage with one damping patch, (c) full damping coverage.
damping patch, and full damping coverage. Composition of the viscoelastic material will be discussed in the next section.

The plate is suspended freely in the anechoic chamber and excited with an impulse hammer. Structural acceleration $a(\vec{r}_s, \omega)$ is measured via a compact accelerometer (of weight 1 gram) attached near the corner of the plate. Sound pressure $p(\vec{r}_p, \omega)$ is recorded via a microphone located 0.5 meter away from the plate. Response signals are fed to the analyzer along with the excitation force $f(\vec{r}_i, \omega)$. Here $\vec{r}$ is the position vector; $\vec{r}_s$ and $\vec{r}_i$ are on the plate and $\vec{r}_p$ is in the acoustic field. Sinusoidal transfer functions in terms of $a/f$ and $p/f$ are obtained and shown in Figures 1.3 (a) and (b) respectively. Comparing results of the undamped and full coverage cases, it is seen that the placement of damping material causes significant changes in the dynamics of the plate in addition to drastic reduction in resonant amplitudes. This demonstrates a heavily damped case along with significant mass loading. However, when a damping strip covering only 20% of the plate surface is applied as shown in Figure 1.2 (b), resonant amplitudes are substantially reduced while transfer functions are closer to those measured for the undamped plate. Measured responses can be analyzed in terms of undamped and damped model properties, which is the focus of this study.

Next, loss factors $\eta$ of selected modes are extracted from the structural transfer function ($a/f$) for each case using the half-power bandwidth method [1.1]. Modal insertion losses ($\Delta L$) in acceleration ($\Delta L_a$) and sound pressure ($\Delta L_p$) levels are then obtained from measured transfer functions as: $\Delta L$ (dB) = level for undamped plate − level
Figure 1.3 Measured transfer functions for the rectangular plate example. (a) $|a/f|$, (b) $|p/f|$. Key: _____ Undamped, — partial coverage with one damping patch, ooo full damping coverage. Refer to Figures 1.1 and 1.2.
with given damping coverage. Another quantifier, called here as the modal damping level increase \( (\Delta L_\eta) \), is defined as

\[
\Delta L_\eta (\text{dB}) = 20 \log_{10} (\eta_d / \eta_0)
\]

where \( \eta_d \) is the modal loss factor for the full or partial coverage case and \( \eta_0 \) is that of the undamped plate. Ideally, modal insertion losses \( (\Delta L_a \text{ and } \Delta L_p) \) should be equal to the modal damping level increase \( \Delta L_\eta \) as indicated by Figure 1.4 where the results of Table 1.1 are plotted for selected modes. For the full coverage case, high modal loss factors \( \eta_d \) are observed and measured \( \Delta L_a \) and \( \Delta L_p \) for various modes are around 30 to 40 dB. For the partial coverage case, insertion losses for \((0,2)\) and \((2,2)\) modes are as high as 25 to 35 dB. However, for \((2,0)\) mode, reduction is only 5 to 7 dB. This shows that the effectiveness of selective damping treatment is location and vibration mode dependent. This issue will be analytically and experimentally examined in this dissertation.

### 1.2.2. Material property issue

The scope of this study is limited to eigensolutions of beams and rectangular plates with multiple patches of different lengths and arbitrary boundary conditions. Figure 1.5 illustrates the damping material which is used for experimental studies. It consists of a metallic constraining layer (#1) and an adhesive layer (#2) capable of dissipating vibration of the base structure (#3). The material properties of the viscoelastic core are assumed to be governed by the linear system theory and defined in frequency domain [1.1] as
Figure 1.4  Relationship between modal insertion loss ($\Delta L$) and modal damping level increase ($\Delta L_\eta$). Key: o insertion loss in acceleration ($\Delta L_a$), x insertion loss in sound pressure ($\Delta L_p$), --- simplified relationship assuming $\Delta L = \Delta L_\eta$.

Figure 1.5  Typical composition of the damping treatment. This material is used for most studies of the dissertation.
### Table 1.1 Modal loss factors and insertion losses for a rectangular plate of Figure 1.1. $\Delta L_a = \text{modal vibration (acceleration) level reduction (based on } a/f)$. $\Delta L_p = \text{modal sound pressure level reduction (based on } p/f)$.

<table>
<thead>
<tr>
<th>Modality</th>
<th>Mode (2,0)</th>
<th>Mode (0,2)</th>
<th>Mode (3,0)</th>
<th>Mode (2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Undamped (Figure 1.2a)</td>
<td>0.15%</td>
<td>0.07%</td>
<td>0.07%</td>
<td>0.04%</td>
</tr>
<tr>
<td>Partial Coverage:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>with one damping patch</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Figure 1.2b)</td>
<td>0.24%</td>
<td>1.34%</td>
<td>0.5%</td>
<td>1.2%</td>
</tr>
<tr>
<td>Modal loss factor $\eta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta L_a$ (dB)</td>
<td>5</td>
<td>26</td>
<td>15</td>
<td>34</td>
</tr>
<tr>
<td>$\Delta L_p$ (dB)</td>
<td>7</td>
<td>24</td>
<td>14</td>
<td>26</td>
</tr>
<tr>
<td>Full damping coverage</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Figure 1.2c)</td>
<td>17%</td>
<td>13%</td>
<td>11%</td>
<td>17%</td>
</tr>
<tr>
<td>Modal loss factor $\eta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta L_a$ (dB)</td>
<td>26</td>
<td>40</td>
<td>36</td>
<td>44</td>
</tr>
<tr>
<td>$\Delta L_p$ (dB)</td>
<td>32</td>
<td>39</td>
<td>33</td>
<td>41</td>
</tr>
</tbody>
</table>
\[ \bar{G} = G(1 + i\eta_m); \]
\[ G = G(\omega, T), \quad \eta_m = \eta_m(\omega, T), \quad i = \sqrt{-1} \quad (1.1) \]

where \( G \) and \( \eta_m \) are material shear modulus and loss factor respectively. They are both frequency \((\omega)\) and temperature \((T)\) dependent [1.1]. A reduced-frequency plot or dimensionless formulation can be obtained empirically by using the well-known temperature-frequency equivalence principle [1.17]. The standard beam test results of sample damping material are shown in Figure 1.6. It is observed that this material has very high damping performance at room temperature. A slight frequency dependency is also observed by comparing results between modes 2 and 3. Experiments are restricted to a material of the type shown in Figure 1.5, at room temperature only. However, theory can accommodate any damping treatment at a temperature of interest.

1.2.3. Objectives and assumptions

Chief purpose of this study is to develop new analytical methods for sandwich beams and rectangular plates with multiple damping patches. Analytical methods and approximations are based on the Rayleigh-Ritz method. Flexural, longitudinal, rotational and shear deformations are considered in all layers of the sandwich. In addition to the assumptions stated for the viscoelastic material in the previous section, the following assumptions are made: all displacements are small and linear elastic theory holds; all layers are isotropic and homogenous; transverse displacements are the same for all three layers; planes remain intact after deformation; and perfect interfacial continuity between layers is maintained and no slip between layers occurs during deformation. Specific objectives are as follows; these are also summarized in Table 1.2.
Figure 1.6 Measured ASTM beam test results for damping material of Figure 1.5 [1.18]. Key: -o- mode 2, -x- mode 3.
<table>
<thead>
<tr>
<th>Structure</th>
<th>Accomplishments</th>
<th>Theory</th>
<th>Approximate Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam</td>
<td>Development of analytical models</td>
<td>Rayleigh-Ritz Method</td>
<td>I</td>
</tr>
<tr>
<td></td>
<td>Validation</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Literature (simple supports)</td>
<td>√</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Experiment (cantilever)</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td></td>
<td>Material estimation procedure (cantilever)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Design concepts (cantilever and simple supports)</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Plate</td>
<td>Development of analytical models</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Validation</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Literature (simply supported)</td>
<td>√</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Experiment (free-free)</td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td></td>
<td>Design concepts (free-free and simple supports)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1.2 Research objectives and scope.
1. Develop analytical models that predict natural frequencies, loss factors, and mode shapes for both beam and plate structures with multiple damping patches given arbitrary boundary conditions (Chapters 2 and 3).

2. Validate the proposed method by comparing results with experimental measurements and with those published in the literature for simple supports (Chapters 2 and 3).

3. Develop a material property estimation technique by combining analytical predictions and measured modal results to estimate unknown material properties of viscoelastic materials (Chapter 2).

4. Develop patch damping design concepts by using three approximate analytical methods that are computationally efficient for rapid parametric design studies. Employ Rayleigh energy method and modal strain energy technique and study compliant core and compact patch analysis issues (Chapter 4).

Each chapter of this dissertation is self-sufficient, containing its own analytical formulation, objectives, results, conclusion, and a list of references. A common bibliography is included at the end of this dissertation.
References for Chapter 1


1.15 A. B. Spalding and J. A. Mann III 1995 *Journal of the Acoustical Society of America* 97, 3617-3624. Placing small constrained layer damping patches on a plate to attain global or local velocity changes.


1.18 Wolverine Gasket Company, 1996, experimental data.
CHAPTER 2

VIBRATION ANALYSIS OF BEAMS WITH MULTIPLE CONSTRAINED LAYER DAMPING PATCHES

2.1 Introduction

Elastic beams with constrained layer viscoelastic material have been analyzed by many investigators, as evident from the studies described in two books on vibration damping by Nashif et al. [2.1] and Sun and Lu [2.2]. However, much of the prior work has been limited to full coverage, i.e., viscoelastic material added to one or both sides of the beam in a uniform manner. Conversely, only a very few publications have dealt with partially covered sandwich beams [2.3-6]. Nokes and Nelson [2.3] were among the earliest investigators to provide an analytical solution to the problem of a partially covered sandwich beam. In their formulation, damped mode shapes are assumed to be the same as the undamped eigenvectors, and the modal loss factor was calculated as the ratio of energy dissipated to the total modal strain energy. A more thorough analytical study was carried out by Lall et al. [2.4]. In their Rayleigh-Ritz approach, both flexural and longitudinal shape functions were incorporated in the eigenvalue problem for a beam with a single damping patch.

In practice, non-uniform and/or partial damping treatment is necessary because of material, thermal, packaging, weight or cost constraints. And in some applications
multiple damping patches at selected locations are more desirable. None of the mathematical models, as available in the literature, appears to be directly applicable to this problem. Consequently a clear need exists for a more refined analysis which this chapter attempts to fulfill. Specific objectives are as follows: (i) develop a new analytical method that considers flexural, longitudinal, rotational and shear deformations in all layers of the sandwich beam, (ii) verify the method by comparing results for a single patch with those reported in the literature by Lall et. al [2.4] and Rao [2.7], (iii) estimate the unknown material properties of viscoelastic material used in the experimental study, (iv) validate the method further by comparing predictions with modal measurements on beams with two mismatched patches, and (v) finally examine critical issues such as the patch boundary conditions, a discontinuity in the material (cutout), and mismatched patch combinations. The method is first described for both thin and thick beams where motion variables for all layers are expressed in terms of the flexural displacement of the base structure (i.e., beam). Then the formulation is reduced to a thin beam by employing a Rayleigh-Ritz minimization scheme and an eigenvalue problem of dimension $n$ is obtained where $n$ is the number of admissible functions. This formulation facilitates efficient calculations of various modal deformations in all layers. It should also lead to an improved understanding of damping system designs.

2.2 Analytical Formulation

2.2.1 Physical example

The structure of interest is shown in Figure 2.1, where $N$ damping patches are attached to the base structure (an elastic beam designated here as layer 3). Each patch $p$
Figure 2.1  Beam with constrained layer damping patches. Key: layer 1 = constraining layer (elastic), layer 2 = constrained layer (viscoelastic), layer 3 = base structure (elastic).
of length \( l^p \) is located at \( x^p \). Layer 1 is a metallic layer while layer 2 is an adhesive capable of dissipating vibratory motions. The viscoelastic nature of the second layer is assumed to be linear and frequency dependent. The complex-valued Young’s modulus \( (\bar{E}) \) and shear modulus \( (\bar{G}) \) of the viscoelastic material in patch \( p \) are represented by

\[
\bar{E}_2^p(\omega) = E_2^p(\omega)(1 + i\eta_2^p(\omega)), \\
\bar{G}_2^p(\omega) = G_2^p(\omega)(1 + i\eta_2^p(\omega)), \quad (2.1 \text{ a, b})
\]

where \( i = \sqrt{-1} \), \( \eta_2^p \) is the material loss factor and \( \omega \) is the frequency in rad/s. Note that each patch \( p \) may be different in size and material properties.

The scope of this chapter is limited to the harmonic vibration analysis of a sandwich beam, as shown in Figure 2.1, with arbitrary boundary conditions. One section of the beam is illustrated in Figure 2.2 with all relevant variables specified including flexural \( (w) \) and longitudinal \( (u) \) displacements as well as rotary \( (\psi) \) and shear angles \( (\gamma) \). However, shear deformations in elastic layers (layers 1 and 3) will be ignored in section 2.3 for the sake of simplification.

### 2.2.2 Energy formulation

The complex-valued strain energy \( (\bar{U}) \) of the system of Figure 2.1 has contributions from flexural displacement \( w \) (same for each layer), longitudinal displacements \( u_1^p, u_2^p \) and \( u_1 \), and shear deformation \( \gamma_1^p, \gamma_2^p \) and \( \gamma_3 \) where superscript \( p=1,\ldots,N \) denotes the patch number for layers 1 and 2. Also, refer to Appendix A for the identification of symbols.
Figure 2.2 Variables in all layers. Key as Figure 2.1.
Note that \( r_1^p, r_2^p, \) and \( r_3 \) are deformation vectors in which rotations \( \psi_1^p, \psi_2^p, \) and \( \psi_3 \) are used instead of shear deformations \( \gamma_1^p, \gamma_2^p, \) and \( \gamma_3 \):

\[
\begin{align*}
  r_1^p &= \begin{bmatrix} w \end{bmatrix}, \quad r_2^p &= \begin{bmatrix} w \\ \psi_2^p \end{bmatrix}, \quad r_3 &= \begin{bmatrix} w \\ \psi_3 \end{bmatrix}; \quad p = 1, \ldots, N. 
\end{align*}
\] (2.3 a-c)

Here \( \mathcal{D} \) is the differential operator matrix defined as

\[
\mathcal{D} = \begin{bmatrix}
  \frac{\partial^2}{\partial x^2} & 0 & 0 \\
  \frac{\partial}{\partial x} & -1 & 0 \\
  0 & 0 & \frac{\partial}{\partial x}
\end{bmatrix}
\] (2.4)

And \( E_1^p, \tilde{E}_1^p \) and \( E_3 \) are elasticity matrices that are defined as

\[
\begin{align*}
  E_1^p &= \begin{bmatrix} E_1^p I_1^p & 0 & 0 \\
  0 & \kappa G_1 A_1^p & 0 \\
  0 & 0 & E_1^p A_1^p
\end{bmatrix}, \\
  \tilde{E}_1^p(\omega) &= \begin{bmatrix} \tilde{E}_1^p(\omega) I_1^p & 0 & 0 \\
  0 & \kappa \tilde{G}_1(\omega) A_1^p & 0 \\
  0 & 0 & \tilde{E}_1^p(\omega) A_1^p
\end{bmatrix}, \\
  E_3 &= \begin{bmatrix} E_3 I_3 & 0 & 0 \\
  0 & \kappa G_3 A_3 & 0 \\
  0 & 0 & E_3 A_3
\end{bmatrix}; \\
  p &= 1, \ldots, N.
\end{align*}
\] (2.5 a-c)
where $\kappa$ is the shear correction factor. The real-valued kinetic energy of the system ($T$) due to flexural, longitudinal and rotary motions is expressed as

$$T = \sum_{p=1}^{N} \int_{a}^{b} \left[ \frac{1}{2} \dot{r}_1^p H_1 \dot{r}_1^p + \frac{1}{2} \dot{r}_2^p H_2 \dot{r}_2^p \right] dx + \int_{a}^{b} \frac{1}{2} \dot{r}_3^p H_3 \dot{r}_3^p dx. \quad (2.6)$$

where

$$H_1^p = \begin{bmatrix}
\rho_1^p A_1^p & 0 & 0 \\
0 & \rho_1^p I_1^p & 0 \\
0 & 0 & \rho_1^p A_1^p
\end{bmatrix},$$

$$H_2^p = \begin{bmatrix}
\rho_2^p A_2^p & 0 & 0 \\
0 & \rho_2^p I_2^p & 0 \\
0 & 0 & \rho_2^p A_2^p
\end{bmatrix},$$

$$H_3^p = \begin{bmatrix}
\rho_3 A_3 & 0 & 0 \\
0 & \rho_3 I_3 & 0 \\
0 & 0 & \rho_3 A_3
\end{bmatrix}. \quad (2.7 \text{ a-c})$$

2.2.3 Rayleigh-Ritz method

To implement the Rayleigh-Ritz minimization scheme, the flexural displacement of the beam $w$ is approximated as

$$w(x,t) = \phi(x)q(t), \quad (2.8)$$

where $q = [q_1, q_2, \ldots, q_k, \ldots, q_n]^T$ is the generalized displacement vector of the system and $\phi = [\phi_1, \phi_2, \ldots, \phi_k, \ldots, \phi_n]$ is the flexural shape function vector in which each term $\phi_k$ is an admissible function that satisfies the essential boundary conditions of the beam.

Recall that energy equations (2.2) and (2.6) contain $4N+3$ unknowns: flexural displacement $w$, rotation $\psi_1^p$, $\psi_2^p$, and $\psi_3$, and longitudinal displacements $u_1^p$, $u_2^p$, and $u_3$. 22
\( u \) for \( p = 1, \ldots, N \). If these unknowns were to be approximated with \( n \) trial functions and to be incorporated in the Rayleigh-Ritz minimization scheme, the resulting eigenvalue problem would be of dimension \( n(4N + 3) \). An alternative is to assume relationships between these unknowns; that is, for each flexural admissible function \( \phi_k(x) \), the corresponding rotational shape functions \( \xi_1^p(x) \), \( \xi_2^p(x) \) and \( \xi_3^p(x) \) as well as the longitudinal shape functions \( \zeta_1^p(x) \), \( \zeta_2^p(x) \) and \( \zeta_3^p(x) \) can be calculated by using these relationships, which will be derived in section 2.3. With the above assumption, deformation vectors can be expressed as

\[
\begin{align*}
\mathbf{r}_1^p &= \mathbf{S}_1^p(x)\mathbf{q}(t), \\
\mathbf{r}_2^p &= \mathbf{S}_2^p(x)\mathbf{q}(t), \\
\mathbf{r}_3 &= \mathbf{S}_3(x)\mathbf{q}(t),
\end{align*}
\tag{2.9 a-c}
\]

where \( \mathbf{S}_1^p \), \( \mathbf{S}_2^p \), and \( \mathbf{S}_3 \) are admissible shape function matrices defined as

\[
\begin{align*}
\mathbf{S}_1^p &= \begin{bmatrix} \Phi \\ \xi_1^p \\ \zeta_1^p \end{bmatrix}, \\
\mathbf{S}_2^p &= \begin{bmatrix} \Phi \\ \xi_2^p \\ \zeta_2^p \end{bmatrix}, \\
\mathbf{S}_3 &= \begin{bmatrix} \Phi \\ \xi_3 \\ \zeta_3 \end{bmatrix}; \text{ for } p = 1, \ldots, N;
\end{align*}
\tag{2.10 a-c}
\]

and \( \xi_1^p = [\xi_{1,1}^p \cdots \xi_{1,k}^p \cdots \xi_{1,n}^p], \) \( \xi_2^p = [\xi_{2,1}^p \cdots \xi_{2,k}^p \cdots \xi_{2,n}^p], \) \( \xi_3 = [\xi_{3,1} \cdots \xi_{3,k} \cdots \xi_{3,n}] \) are the corresponding rotational shape function vectors while \( \zeta_1^p = [\zeta_{1,1}^p \cdots \zeta_{1,k} \cdots \zeta_{1,n}], \) \( \zeta_2^p = [\zeta_{2,1} \cdots \zeta_{2,k} \cdots \zeta_{2,n}], \) \( \zeta_3 = [\zeta_{3,1} \cdots \zeta_{3,k} \cdots \zeta_{3,n}] \) are the corresponding longitudinal shape function vectors. Using equation (2.9), the strain and kinetic energies can be written as

\[
\begin{align*}
\bar{U}(\omega) &= \frac{1}{2} \mathbf{q}^T \mathbf{K}(\omega)\mathbf{q}, \\
T &= \frac{1}{2} \mathbf{q}^T \mathbf{Mq},
\end{align*}
\tag{2.11 a, b}
\]

23
where the frequency-dependent complex-valued stiffness ($\tilde{K}$) and real-valued mass ($M$) matrices of the system are

$$\tilde{K}(\omega) = \sum_{\rho=1}^{N} \int \left[ (\mathcal{D} S_1^\rho)^T E_1 (\mathcal{D} S_1^\rho) + (\mathcal{D} S_2^\rho)^T \bar{E}_2 ( \omega ) \left( \mathcal{D} S_2^\rho \right) \right] dx + \int_0^l (\mathcal{D} S_3)^T E_3 (\mathcal{D} S_3) dx;$$

$$M = \sum_{\rho=1}^{N} \int \left[ S_1^\rho H_1 S_1^\rho + S_2^\rho H_2 S_2^\rho \right] dx + \int_0^l S_3^T H_3 S_3 dx. \quad (2.12 \text{ a, b})$$

The frequency-dependent complex eigenvalue problem of dimension $n$ can be obtained as

$$Mq + \tilde{K}(\omega)q = 0. \quad (2.13)$$

Several approaches are available in the literature [2.9, 2.10] for solving eigenvalue problems of non-proportionally damped systems with frequency-dependent parameters, whose eigenvalues and eigenvectors are complex-valued. Using the method of Rikards et al. [2.10], undamped natural frequencies ($\omega_r$) and composite modal loss factors ($\eta_r$) are related to the complex-valued eigenvalues $\tilde{\lambda}_r$ of equation (2.13) in the following manner where $r$ is the modal index:

$$\omega_r = \sqrt{\Re(\tilde{\lambda}_r)}, \quad \eta_r = \frac{\Im(\tilde{\lambda}_r)}{\Re(\tilde{\lambda}_r)}; \quad r = 1, \ldots, n. \quad (2.14 \text{ a, b})$$

2.3 Admissible Functions for Thin Beams

In section 2.2.3, the fundamental relationships between all $4N+3$ unknowns are assumed in order to obtain an eigenvalue problem of dimension $n$. This section explicitly shows these relationships by deriving the corresponding shape function of each unknown for a given admissible flexural function. For the sake of simplification, only thin elastic layers (1 and 3) are assumed. The following two steps are involved in the variable.
reduction procedure. First, the classic sandwich beam theory [2.7] is employed along with the thin elastic layer assumption to reduce the number of unknowns to \( N+2 \). Second, a secondary minimization scheme is used to further reduce the \( N+2 \) unknowns to one flexural shape function vector \( \varphi \).

### 2.3.1 Variable reduction by using classic sandwich beam theory

The Kerwin's weak core assumption [2.8] is applied to longitudinal shape functions \( \zeta_1^p \) and \( \zeta_3 \) as

\[
E_i A_i^p \frac{\partial \zeta_{1,k}}{\partial x} + E_3 A_3 \frac{\partial \zeta_{3,k}}{\partial x} = 0; \quad p=1, \ldots, N. \tag{2.15}
\]

Integrating both sides with respect to \( x \), the following expression is obtained

\[
\zeta_{1,k}^p = d_k^p - e^p \zeta_{3,k}, \tag{2.16}
\]

where \( e^p = E_3 A_3 / (E_i A_i^p) \) and \( d_k^p \) is the constant that relates admissible shape \( \zeta_{3,k} \) to the corresponding \( \zeta_{1,k}^p \).

Next, observing the kinematic relationship in Figure 2.2, the longitudinal deformation \((\zeta_{2,k}^p)\) and rotation \((\varphi_{2,k}^p)\) of layer 2 are expressed as

\[
\zeta_{2,k}^p = \frac{1}{2} \left[ \left( \zeta_{3,k} - \frac{h_1}{2} \zeta_{3,k} \right) + \left( \zeta_{1,k}^p + \frac{h_1}{2} \zeta_{1,k} \right) \right], \tag{2.17}
\]

\[
\varphi_{2,k}^p = \frac{1}{h_2} \left[ \left( \zeta_{3,k} - \frac{h_3}{2} \zeta_{3,k} \right) - \left( \zeta_{1,k}^p + \frac{h_1}{2} \zeta_{1,k} \right) \right]. \tag{2.18}
\]

Substituting (2.16) into (2.17, 2.18), \( \zeta_{2,k}^p \) and \( \varphi_{2,k}^p \) are rewritten as
For a beam with thin elastic layers whose shear deformations $\gamma^i_1$ and $\gamma_3$ are ignored, rotations of layer 1 and layer 3 are the same as the slope of the beam

$$\varepsilon^i_{3,k} = \frac{\partial \phi_k}{\partial x}, \quad \varepsilon^i_{3,k} = \frac{\partial \phi_k}{\partial x}. \quad (2.21 \text{a, b})$$

Therefore, $\varepsilon^i_{2,k}$ and $\varepsilon^i_{3,k}$ can be rewritten as

$$\varepsilon^i_{2,k} = \frac{h_1^p - h_3^p}{4} \frac{\partial \phi_k}{\partial x} + \frac{1 - e^p}{2} \varepsilon^i_{3,k} + \frac{1}{2} d^p_k, \quad (2.22)$$

$$\varepsilon^i_{3,k} = \frac{-h_1^p + h_3^p}{2h_2^p} \frac{\partial \phi_k}{\partial x} + \frac{1 + e^p}{h_2^p} \varepsilon^i_{3,k} - \frac{1}{h_2^p} d^p_k. \quad (2.23)$$

To reduce the $4N+3$ unknowns to $N+2$, define transfer matrices $V_1$, $V_2$, and $V_3$ as

$$S^p_i = V_i^p \Phi^p, \quad S^p_i = V_i^p \Phi^p, \quad S_3 = V_3 \Phi, \quad (2.24 \text{a-c})$$

where transfer matrices $V_1^p$, $V_2^p$, and $V_3$, as shown below, are derived by using equations (2.16, 2.21-23):

$$V_1^p = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\partial}{\partial x} & 0 & 0 \\ 0 & -e^p & 1 \end{bmatrix}, \quad V_2^p = \begin{bmatrix} 1 & 0 & 0 \\ -\left(\frac{h_1^p + h_3^p}{4} \right) \frac{\partial}{\partial x} & \left(\frac{e^p + 1}{h_2^p} \right) & -\frac{1}{h_2^p} \\ \left(\frac{h_1^p - h_3^p}{2h_2^p} \frac{\partial}{\partial x} & h_2^p \right) & \left(\frac{-e^p}{2} \right) & 1 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 & 0 \\ \frac{\partial}{\partial x} & 0 \\ 0 & 1 \end{bmatrix}. \quad (2.25 \text{a-c})$$

Also, the reduced admissible shape matrices $\Phi^p$ and $\Phi$ of equation (2.24) are defined as
where \( d^p = [d_1^p \cdots d_k^p \cdots d_s^p] \) and each term \( d_k^p \) is a constant that relates the admissible shape \( \zeta_{3,k} \) to its corresponding \( \zeta_{1,k}^p \). Note that this constant \( (d_k^p) \) has been ignored by many prior researchers but it is retained here since it plays an important role in determining the longitudinal boundary conditions [2.11] for each patch. For example, if patch \( p \) is a fixed-end patch as shown in Figure 2.3 (a), \( d^p \) must be zero and accordingly it must be eliminated from \( \Phi^p \). As for a free-end patch of Figure 2.3 (b), \( d^p \) remains undetermined until the secondary minimization scheme is used. The issue of patch boundary conditions will be further examined in section 2.5.1.

2.3.2 Variable reductions by using a secondary minimization scheme

Recall that the number of unknowns has been reduced from \( 4N+3 \) to \( N+2 \) for the \( k \text{th} \) admissible function set in matrices \( \Phi^p \) and \( \Phi \). These \( N+2 \) unknowns are flexural shape functions \( \phi_k \), longitudinal shape functions for the base beam \( \zeta_{3,k} \), and the constants \( d_k^p \).

Since no explicit equations are available to relate these unknowns, a secondary minimization scheme is implemented. First, each admissible function \( \zeta_{3,k} \) is approximated as

\[
\zeta_{3,k} = \chi c_k, \tag{2.27}
\]

where \( c_k \) is a coefficient vector to be determined and \( \chi \) is the row trial function vector whose terms satisfy essential boundary conditions. The real part of total strain energy of
Figure 2.3  Patch boundary conditions at $x = 0$. (a) Fixed-end patch, (b) free-end patch where $\varepsilon \to 0$. 
the beam \( (U_k) \) experiencing the deformations of the \( k \)th of admissible function set \( \Phi_k^p \) and \( \Phi_k \) can be expressed

\[
U_k(\omega) = \frac{1}{2} K_k(\omega) q_k^2, \tag{2.28}
\]

where

\[
K_k(\omega) = \sum_{p=1}^{\infty} \int \left[ (\partial V_1^* \Phi_k^p)^T E_1 (\partial V_1^* \Phi_k^p) + (\partial V_2^* \Phi_k^p)^T \tilde{E}_2^p(\omega) (\partial V_2^* \Phi_k^p) \right] dx
+ \int (\partial V_3^* \Phi_k^p)^T E_3 (\partial V_3^* \Phi_k^p) dx, \tag{2.29}
\]

is the effective stiffness at any \( \omega \) of interest and \( q_k \) is the corresponding generalized displacement. Note that in the above analysis the imaginary part of the complex-valued stiffness is ignored because only kinematic relationships are of interest. By substituting equations (2.25-27, 2.29) into (2.28) and minimizing \( U_k \) with respect to coefficients of \( c_k \) and \( d_k \), where \( d_k = [d_k^1 \ldots d_k^p \ldots d_k^\infty]^T \), the set of governing equations can be summarized in matrix form as

\[
AC_k = B_k, \tag{2.30}
\]

where

\[
A = \begin{bmatrix}
A^{cc} & A^{cd} \\
A^{dc} & A^{dd}
\end{bmatrix}, \quad B_k = \begin{bmatrix} B_k^c \end{bmatrix}, \quad C_k = \begin{bmatrix} c_k \end{bmatrix}.
\tag{2.31 a-c}
\]

Submatrices of \( A \) and sub-vectors of \( B_k \) are obtained as follows

\[
A^{cc} = \int \left( \frac{\partial}{\partial x} \chi \right)^T E_3 A_3 \left( \frac{\partial}{\partial x} \chi \right) dx + \sum_{p=1}^{\infty} \int \left( \frac{\partial}{\partial x} \chi \right)^T E_3 A_3 e^p \left( \frac{\partial}{\partial x} \chi \right) + \chi^T \left( \frac{e^p + 1}{h_3^2} \right) G_3^p A_3 \chi \right] dx,
\]

29
\[ A^{cd} = \begin{bmatrix} \frac{(e^1 + 1)G_1^1 A_2^1}{(h_2^1)^2} \int \chi^T dx & \cdots & \frac{(e^n + 1)G_n^p A_2^p}{(h_2^p)^2} \int \chi^T dx & \cdots & \frac{(e^n + 1)G_n^N A_2^N}{(h_2^N)^2} \int \chi^T dx \end{bmatrix} \]

\[ A^{ce} = \begin{bmatrix} \frac{(e^1 + 1)G_2^1 A_2^1}{(h_2^1)^2} \int \chi^T dx & \cdots & \frac{(e^n + 1)G_n^p A_2^p}{(h_2^p)^2} \int \chi^T dx & \cdots & \frac{(e^n + 1)G_n^N A_2^N}{(h_2^N)^2} \int \chi^T dx \end{bmatrix}^T \]

\[ A^{dd} = \begin{bmatrix} \frac{G_1^1 A_2^1 l_1}{(h_2^1)^2} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \frac{G_n^N A_2^N l_N}{(h_2^N)^2} \end{bmatrix} \]

\[ B_k^e = \sum_{p=1}^{N} \int \frac{C_p (e^p + 1)G_p^p A_2^p}{(h_2^p)^2} \frac{\partial \phi_k}{\partial x} \chi^T dx \]

\[ B_k^f = \begin{bmatrix} \frac{C^1 (e^1 + 1)G_2^1 A_2^1 l_1}{(h_2^1)^2} & \cdots & \frac{C_p (e^p + 1)G_p^p A_2^p l_p}{(h_2^p)^2} & \cdots & \frac{C^N (e^n + 1)G_n^N A_2^N l_N}{(h_2^N)^2} \end{bmatrix}^T \]

(2.32 a-f)

where \( C^p = (h_1^p + 2h_2^p + h_3)/2 \) and the spatial operator \( \mathcal{S}^p \) is defined as

\[ \mathcal{S}^p f(x) = f(x^p + l_p/2) - f(x^p - l_p/2). \]

The coefficients \( c_k \) and \( d_k \) of \( C_k \) can be calculated by

\[ C_k = A^{-1} B_k \]

(2.34)

provided \( |A| \neq 0 \). As a result, reduced admissible shape matrices \( \Phi^p \) and \( \Phi \) can be determined for a given flexural shape function vector \( \phi \).
2.4 Comparison with Literature

To validate the proposed formulations, two specific examples found in the existing literature are analyzed first. Table 2.1 summarizes the system parameters that were used by Rao [2.7] and Lall et al. [2.4]. For our study, analytical solutions are obtained by using 20 admissible functions for the flexural displacement and 20 trial longitudinal shape functions for each flexural shape function.

Rao [2.7] studied a clamped-free beam with a full damping treatment on one side of the beam. The viscoelastic material has a fixed boundary condition at \( x = 0 \). He found the exact solution only for the first mode. Additionally, first three natural frequencies and modal loss factors were calculated by using an approximate formulation [2.7]. It is seen in Table 2.2 that the results obtained from our method are very close to the exact solutions given by Rao. Reasonable agreement is also seen with Rao’s approximations.

Lall et al. [2.4] analyzed a simply supported beam with a single patch. The following parametric studies were carried out: coverage ratios \( l^p/l = 20\%, 40\%, 60\% \) and 100\%; patch locations \( x^p/l = 0.1, 0.2, 0.3, \) and 0.5 respectively. Comparisons of Table 2.3 show an excellent match between Lall’s and our methods. Such results are expected since both methods are based on the Rayleigh-Ritz approach. Chief advantage of the proposed formulation however is the ease with which mode shapes for all types of deformation in any layer can be visualized. Figure 2.4 shows the first three flexural modes of the beam as well as the corresponding longitudinal modes of layers 1 and 3, and shear mode of layer 2 for Rao’s example [2.4].
Table 2.1 System parameters used for examples given in the literature. Refer to Figure 2.1 for nomenclature.

<table>
<thead>
<tr>
<th>Material properties</th>
<th>Rao [2.7]</th>
<th>Lall et al. [2.4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$ and $E_3$ (Pa)</td>
<td>$2.06 \times 10^9$</td>
<td>$2.07 \times 10^9$</td>
</tr>
<tr>
<td>$G_2$ (Pa)</td>
<td>$9.8 \times 10^9$</td>
<td>$2.615 \times 10^5$</td>
</tr>
<tr>
<td>$\eta_2$</td>
<td>0.1</td>
<td>0.38</td>
</tr>
<tr>
<td>$\rho_1$ and $\rho_3$ (kg/m$^3$)</td>
<td>7850</td>
<td>7800</td>
</tr>
<tr>
<td>$\rho_2$ (kg/m$^3$)</td>
<td>2600</td>
<td>2000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dimensions (mm)</th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_1$</td>
<td>3</td>
<td>0.5</td>
</tr>
<tr>
<td>$h_2$</td>
<td>5</td>
<td>2.5</td>
</tr>
<tr>
<td>$h_3$</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>$l$</td>
<td>100</td>
<td>300</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Beam boundary conditions</th>
<th>Clamped-free</th>
<th>Simply supported on both sides</th>
</tr>
</thead>
<tbody>
<tr>
<td>Patch boundary conditions at $x = 0$</td>
<td>fixed end</td>
<td>free end</td>
</tr>
</tbody>
</table>

Table 2.2 Comparison between Rao's published [2.7] and proposed methods. See Table 2.1 for parameters.

<table>
<thead>
<tr>
<th>Natural frequency (Hz)</th>
<th>Modal loss factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>published</td>
</tr>
<tr>
<td>Mode 1</td>
<td>1309</td>
</tr>
<tr>
<td></td>
<td>(1320) *</td>
</tr>
<tr>
<td>Mode 2</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>(6869) *</td>
</tr>
<tr>
<td>Mode 3</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>(16497) *</td>
</tr>
</tbody>
</table>

* solution from an approximate formulation given by Rao [2.7]
<table>
<thead>
<tr>
<th>Coverage</th>
<th>Natural frequency (rad/s)</th>
<th>Modal loss factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>published</td>
<td>proposed</td>
</tr>
<tr>
<td>20%</td>
<td>811</td>
<td>811</td>
</tr>
<tr>
<td>40%</td>
<td>789</td>
<td>788</td>
</tr>
<tr>
<td>60%</td>
<td>759</td>
<td>759</td>
</tr>
<tr>
<td>100%</td>
<td>741</td>
<td>741</td>
</tr>
<tr>
<td></td>
<td>2948</td>
<td>2949</td>
</tr>
<tr>
<td></td>
<td>6630</td>
<td>6630</td>
</tr>
<tr>
<td></td>
<td>11783</td>
<td>11783</td>
</tr>
</tbody>
</table>

Table 2.3 Comparison between Lall et al.'s published [2.4] and proposed methods. See Table 2.1 for parameters.
Figure 2.4  First three mode shapes for Rao's example [2.7]. (a) Flexural modes ($w$) of the beam, (b) longitudinal modes ($u_1$) of layer 1, (c) longitudinal modes ($u_3$) of layer 3, (d) shear modes ($\gamma_2$) of layer 2. Key: --- mode 1, _ mode 2, • mode 3.
2.5 Experimental Studies

In order to further verify the analytical model as well as to investigate various phenomena associated with patch damping, modal tests are carried out on a cantilever beam made of mild steel (Table 2.4). A periodic "chirp" as generated within the signal analyzer is fed to a non-contacting magnetic transducer that excites the beam at the free end, as shown in Figure 2.5. Structural response is measured via a compact accelerometer (of weight 1 gram) near the root. Sinusoidal transfer functions are then obtained. No calibration is necessary since only frequency measurements are needed. First five natural frequencies \( \left( f_1 \right) \) and modal loss factors \( \left( \eta_1 \right) \) are then extracted using the half-power bandwidth method [2.1].

Two types of damping treatment (designated here as Patches A and B) with material properties and layer thickness as specified in Table 2.4 are applied in these studies. However, the material properties of the viscoelastic core are not available. Therefore, a process for estimating the material properties must be developed before analyzing the damped beam structure. An uncertainty study is also carried out to establish error bounds for estimations. Finally, the procedure used for obtaining normalized expressions is explained in this section.

2.5.1 Material property estimation

A material property estimation technique is employed by combining analytical predictions and measured modal results. Of interest here are the properties of layer 2: \( G_2 \) and \( \eta_2 \) since layers 1 and 3 are made of well-known steel. The following procedure is demonstrated with Patch A as an example:
<table>
<thead>
<tr>
<th>Stiffness (N/m²)</th>
<th>Density (kg/m³)</th>
<th>Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Patch A</td>
<td>Layer 1 $E_1 = 180 \times 10^9$</td>
<td>$\rho_1 = 7720$</td>
</tr>
<tr>
<td></td>
<td>Layer 2 $G_2 = 0.25 \times 10^6$</td>
<td>$\rho_2 = 2000$</td>
</tr>
<tr>
<td>Patch B</td>
<td>Layer 1 $E_1 = 180 \times 10^9$</td>
<td>$\rho_1 = 7720$</td>
</tr>
<tr>
<td></td>
<td>Layer 2 $G_2 = 3 \times 10^6$</td>
<td>$\rho_2 = 2000$</td>
</tr>
<tr>
<td>Baseline beam of</td>
<td>Layer 3 $E_3 = 180 \times 10^9$</td>
<td>$\rho_3 = 7350$</td>
</tr>
</tbody>
</table>

Table 2.4 Properties of baseline beam and damping patches.
Figure 2.5  Schematic of experimental setup.
(1) Choose one example and perform an experiment. The example case is a cantilever beam with full damping treatment on one side and free patch boundary at x = 0 as shown in Figure 2.6 (a). Natural frequencies and modal loss factors for the first few modes are then obtained as listed in Table 2.5.

(2) Develop \( f_r-G_2 \) relationships where \( f_r = \omega_r/2\pi \) is the natural frequency in Hz. With the material loss factor \( \eta_2 \) taken as zero, the analytical model is used to predict the variation in \( f_r \) over a range of \( G_2 \) values. A general trend of this relationship can be seen in Figure 2.7, where three distinct regions are observed: very compliant, transition and very stiff. These are similar to those reported in beams with joints [2.12].

(3) Given measured \( f_r \) results, find shear modulus \( G_2 \) from the graphs. In Figure 2.7, a horizontal line is drawn at the measured frequency for each mode. A cross mark represents the intersection of this line and the \( f_r-G_2 \) curve; this yields \( G_2 \) at that frequency. Note that in Figure 2.7 (a), no intersection is found because the measured value is less than the low frequency asymptote of the curve. This is because of the non-ideal clamping boundary conditions [2.13], which especially affect mode 1. Therefore mode 1 is excluded from the shear modulus estimation procedure. Figure 2.7 (b-e) show similar \( G_2 \) values over the range of interest. For this particular case, it is safe to assume a spectrally-invariant \( G_2 \), as listed in Table 2.4.

(4) Develop \( \eta_r-\eta_2 \) relationships and estimate the material loss factor of layer 2 as a function of frequency. With the assumed \( G_2 \) or \( G_1(f) \), the analytical model is again used to predict a general relationship between \( \eta_2 \) and the modal loss factors \( \eta_r \) of the sandwich beam. In Figure 2.8, such \( \eta_r-\eta_2 \) relationships are compared with the measured modal loss
Figure 2.6 Benchmark examples. (a) Full free-end patch (benchmark for damping studies based on experimental measurements), (b) beam without any patch (baseline beam).
(a) Beam with full free-end Patch A

<table>
<thead>
<tr>
<th>Mode</th>
<th>Undamped natural frequency</th>
<th>Modal loss factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experiment*</td>
</tr>
<tr>
<td></td>
<td>$f_r$ (Hz)</td>
<td>$(\beta, \lambda)^2$</td>
</tr>
<tr>
<td>1</td>
<td>34</td>
<td>3.6</td>
</tr>
<tr>
<td>2</td>
<td>219</td>
<td>20.7</td>
</tr>
<tr>
<td>3</td>
<td>580</td>
<td>54.7</td>
</tr>
<tr>
<td>4</td>
<td>1117</td>
<td>105.4</td>
</tr>
<tr>
<td>5</td>
<td>1834</td>
<td>173.1</td>
</tr>
</tbody>
</table>

* Experimental database for the full free-end patch is the benchmark case

(b) Baseline beam (without any patch)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Undamped natural frequency</th>
<th>Modal loss factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experiment</td>
</tr>
<tr>
<td></td>
<td>$f_r$ (Hz)</td>
<td>$(\beta, \lambda)^2$</td>
</tr>
<tr>
<td>1</td>
<td>37</td>
<td>3.5</td>
</tr>
<tr>
<td>2</td>
<td>233</td>
<td>22.0</td>
</tr>
<tr>
<td>3</td>
<td>654</td>
<td>61.7</td>
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<tr>
<td>4</td>
<td>1281</td>
<td>120.9</td>
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<tr>
<td>5</td>
<td>2118</td>
<td>199.9</td>
</tr>
</tbody>
</table>

Table 2.5 Benchmark results for cantilever beams.
Figure 2.7  Predicted $f_r$-$G_2$ relationships and measured $f_r$ values for estimating $G_2$ value. (a) Mode 1, (b) mode 2, (c) mode 3, (d) mode 4, (e) mode 5. Key: — theory, — experiment, x intersection.
Figure 2.8 Predicted $\eta_r-\eta_2$ relationships and measured $\eta_r$ values for estimating $\eta_2$ value. Key: ___ mode 1, ... mode 2, --- mode 3, -.-- mode 4, -- mode 5, x measurements.
factors. Again, each cross mark indicates the \( \eta_r \) value. As a result, a frequency-dependent relationship is obtained for the viscoelastic core of Patch A that is curve fitted to yield:

\[
\eta_{2A}(f) = 8.78 \times 10^{-1} + 3.94 \times 10^{-3} f + 4.34 \times 10^{-2} f^2 - 6.65 \times 10^{-9} f^3 + 2.29 \times 10^{-12} f^4,
\]

where \( f \) is the frequency in Hz. The same procedure is performed on Patch B and the material loss factor is expressed as

\[
\eta_{2B}(f) = 2.39 \times 10^{-1} + 3.54 \times 10^{-4} f - 7.97 \times 10^{-7} f^2 + 1.01 \times 10^{-9} f^3 - 3.32 \times 10^{-13} f^4.
\]

With given material properties as listed in Table 2.4, a specific eigenvalue problem may be constructed for each mode with a particular \( \eta_2 \) value at the natural frequency. An iterative procedure is obviously needed for obtaining eigensolutions [2.9]. Nonetheless, despite their frequency-dependent nature, the material loss factors are assumed to be invariant in the immediate vicinity of an eigenvalue to avoid any iteration [2.10]. Experimental results that will be reported in the next section validate this assumption. Also note that the analytical formulation is again used with 20 admissible functions and 20 trial functions for all example cases.

### 2.5.2 Uncertainty of material property

The determination of \( \eta_2 \) is a key to the success of analytical method of the chapter. However, modal measurements used for the estimation procedure are affected by many factors including inherent beam damping, microscopic friction at the root and non-perfect bindings between layers. In practical structures a significant variation in measured \( \eta_r \) values may be seen. To examine such uncertainties, a \( \pm 20\% \) tolerance in the damping measurement is assumed. The upper and lower bounds of \( \eta_{2A}(f) \) due to this tolerance
are shown in Figure 2.9. These values are applied to the case of Figure 2.10 (a), where a single patch is applied to the cantilever beam from \( x = 0.3l \) to \( l \). A comparison of predicted and measured modal loss factors is shown in Figure 2.10 (b). Note that the error bars on predicted \( \eta_r \) indicate the uncertainty associated with the \( \eta_{3A}(f) \) estimation, while the error bars on experimental results indicate the 20% tolerance in measurements. The overlap of error bars implies excellent agreement between measurements and predictions. Only the mean values of \( \eta_2 \) are used in subsequent studies; however the probabilistic nature of damping values must be considered when viewing all results.

2.5.3 Normalization procedures

Often it is desirable to express modal results in normalized forms. For example, the loss factors of a beam with patch damping (\( \eta_r \)) may be normalized with respect to the case where the beam is fully covered with a viscoelastic material (\( \eta_{r, \text{full}} \))

\[
\hat{\eta}_r = \eta_r / \eta_{r, \text{full}}
\]

in which both values are either predicted or measured. This normalization (given by superscript \(^{\wedge}\)) can be used to describe the effectiveness of patch damping concept. Similarly, natural frequencies may be normalized as follows to indicate the mass loading effect

\[
\hat{\omega}_r = \omega_r / \omega_{r, \text{full}}
\]

However, since different types of patches and boundary conditions will be discussed later, it is more appropriate to use a single set of measured results throughout the chapter
Figure 2.9  Frequency dependent material loss factor $\eta_2$ for layer 2 of patch A. Key: - o- assumed mean, --- upper limit, ... lower limit.
Figure 2.10  Comparisons of measurements and predictions of cantilever sandwich beam with a single patch. (a) Schematic, (b) modal loss factors. Key: ■ experiment, - theory, I variation.
as the base for normalization. The resulting normalized natural frequency $\tilde{\omega}_r$ and modal loss factor $\tilde{\eta}_r$ are defined here as

$$\tilde{\eta}_{r,b} = \eta_r / \eta_{r,b}$$  \hspace{1cm} (2.37)$$

$$\tilde{\omega}_{r,b} = \omega_r / \omega_{r,b}$$  \hspace{1cm} (2.38)$$

where subscript b refers to the measured results of a benchmark case: a cantilever beam with full material A damping treatment and free patch end at $x = 0$ as shown in Figure 2.6 (a).

Yet another expression is used to describe the dimensionless eigenvalue by assuming the damped structure to be an undamped Euler beam. This eigenvalue parameter $(\beta_l)^2$ is defined as

$$(\beta_l)^2 = \omega_r l^2 / \sqrt{E_3 / \rho_3}.$$  \hspace{1cm} (2.39)$$

Measured and predicted modal results for the benchmark case as well as the baseline beam (i.e., undamped beam without any damping patch) of Figure 2.6 are listed in Table 2.5. Measurements show that fairly high inherent damping is present in the first mode of the baseline beam. This may be the result of non-ideal clamping condition at $x = 0$. Consequently, some caution must be exercised when examining the damped beam results especially at the first mode.

2.6 Results and Discussion

2.6.1 Effect of patch boundary

A cantilever beam with a damping patch embedded into the fixed boundary at $x = 0$ is said to have a fixed-end patch (Figure 2.3) as opposed to the free-end patch where the
material is unconstrained at \( x = 0 \). Practically, this boundary condition is achieved by clamping the patch along with the beam at the root. Analytically, a fixed patch end is simulated by forcing the column vector \( \mathbf{d}' \) to be zero as described in section 2.3. Measured and predicted modal characteristics are then normalized and compared with the benchmark case (free-end patch) as shown in Figure 2.11.

Both measurements and predictions indicate that natural frequencies and loss factors of the beam with fixed-end patch are much higher than those of the beam with a free-end patch, especially for mode 1. Nearly coincident flexural mode shapes (Figure 2.12) of these two cases, as predicted by analytical models, provide little explanation for this. However, a closer examination of shear deformation mode shapes of layer 2, as seen in Figure 2.13, yields very distinct characteristics between these two cases. The fixed-end patch, acting as an additional constraint, causes the natural frequencies to increase and forces the deformation \( \gamma_2 \) to be zero at the root. This constraint also results in a higher \( \gamma_2^2 \) value when integrated over the patch length, which implies more energy dissipation. This discrepancy in shear deformation mode shapes is very noticeable for mode 1, but not as significant for modes 2 and 3 as shown in Figure 2.13.

### 2.6.2 Effect of cutouts in damping treatment

A cutout, no matter how small it is, essentially creates a beam with two distinct damping patches. Resulting modal characteristics are investigated by using two example cases. Figure 2.14 (a) shows a cantilever sandwich beam with a small cutout (with 3% of beam length) located at 0.3\( l \) from the root \( (x = 0) \). Figure 2.15 (a) shows a similar beam except that the cutout is 30% of the beam length but still located at 0.3\( l \).
Figure 2.11 Results for a cantilever beam with fixed-end patch. (a) Schematic, (b) normalized eigenvalues, (c) normalized modal loss factors. Key: □ experiment, • theory, -- benchmark (measured value for a free-end patch).
Figure 2.12  Flexural displacement mode shapes. (a) Mode 1, (b) mode 2, (c) mode 3.  
Key: ___ free-end patch, x fixed-end patch, ... baseline beam.
Figure 2.13  Shear deformation mode shapes of layer 2. (a) Mode 1, (b) mode 2, (c) mode 3. Key: ___ free-end patch, × fixed-end patch.
Figure 2.14 Results for a cantilever sandwich beam with a small cutout. (a) Schematic with a free-end patch, (b) normalized eigenvalues, (c) normalized modal loss factors. Key: ▼ experiment, • theory.
Figure 2.15  Results for a cantilever sandwich beam with a large cutout. (a) Schematic with a free-end patch, (b) normalized eigenvalues, (c) normalized modal loss factors. Key: □ experiment, • theory.
Measured and predicted eigenvalues and modal loss factors are normalized and plotted in Figures 2.14 and 2.15. It is observed that the small-cutout case yields more damping than the large-cutout one, as one would intuitively expect. Higher flexural amplitude is found near the cutout location of the large-cutout case, especially for the third mode (Figure 2.16). Figure 2.17 shows shear deformation mode shapes in the core material. Note that a higher $\gamma^2$ value, when integrated over the patch length, indicates increased energy dissipation.

Parametric studies are carried out analytically in order to further investigate the effect of cutout size and cutout location. Figure 2.18 shows normalized loss factors of the first three modes for a sandwich beam with a 3% cutout at various locations $x^p/l$. It is seen that the loss factor value is very sensitive to the cutout location, especially for lower modes. Modal loss factors as affected by cutout size with a given cutout location (0.3/l from the root) are plotted in Figure 2.19. Again, a monotonic decrease in $\eta_r$ is observed as the cutout size is increased.

### 2.6.3 Effect of two mismatched patches

A cantilever beam with two mismatched patches is examined by using three example cases, as shown in Figure 2.20, to see whether the damping patches introduce damping in an additive manner. A beam with a single Patch A of length 0.14/l located at 0.2/l from the root is designated as Case A, and a beam with a single Patch B of 0.22/l at 0.7/l from the root as Case B. Then both patches A and B are applied simultaneously; this is designated as Case C. Measured and predicted eigenvalues and loss factors are normalized and listed in Table 2.6.
Figure 2.16  Flexural displacement mode shapes. (a) Mode 1, (b) mode 2, (c) mode 3. Key: ___ small cutout, × large cutout, — baseline beam.
Figure 2.17  Shear deformation mode shapes of layer 2. (a) Mode 1, (b) mode 2, (c) mode 3. Key: ___ small cutout, × large cutout, -- baseline beam.
Figure 2.18 Effect of cutout locations on normalized loss factors of a sandwich beam. Key: □ mode 1, o mode 2, x mode 3.

Figure 2.19 Effect of cutout size on normalized loss factors of a sandwich beam. Key: □ mode 1, o mode 2, x mode 3.
Figure 2.20  Cantilever beam with mismatched patches.
Table 2.6 Results for a cantilever beam with mismatched patches. Refer to Figure 2.20.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Eigenvalue $(\beta,l)^2$</th>
<th>Normalized loss factor $\tilde{\eta}_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experiment</td>
</tr>
<tr>
<td>(a) Beam with Patch A (Case A)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.6</td>
<td>3.5</td>
</tr>
<tr>
<td>2</td>
<td>21.7</td>
<td>21.5</td>
</tr>
<tr>
<td>3</td>
<td>58.9</td>
<td>58.2</td>
</tr>
<tr>
<td>4</td>
<td>115.1</td>
<td>113.7</td>
</tr>
<tr>
<td>5</td>
<td>193.5</td>
<td>189.0</td>
</tr>
<tr>
<td>(b) Beam with Patch B (Case B)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.4</td>
<td>3.1</td>
</tr>
<tr>
<td>2</td>
<td>22.0</td>
<td>22.0</td>
</tr>
<tr>
<td>3</td>
<td>61.3</td>
<td>61.7</td>
</tr>
<tr>
<td>4</td>
<td>119.3</td>
<td>117.8</td>
</tr>
<tr>
<td>5</td>
<td>195.2</td>
<td>188.1</td>
</tr>
<tr>
<td>(c) Beam with Patches A and B (Case C)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.4</td>
<td>3.2</td>
</tr>
<tr>
<td>2</td>
<td>21.8</td>
<td>21.6</td>
</tr>
<tr>
<td>3</td>
<td>59.1</td>
<td>58.8</td>
</tr>
<tr>
<td>4</td>
<td>114.3</td>
<td>113.8</td>
</tr>
<tr>
<td>5</td>
<td>189.2</td>
<td>184.7</td>
</tr>
</tbody>
</table>
A simple additive effect in modal damping can be expressed as

\[ \eta_{r,C} = \eta_{r,A} + \eta_{r,B}, \tag{2.40} \]

where subscripts A, B and C are the case designations defined earlier. According to the analytical model, equation (2.40) may not work since the resulting mode shapes (Figure 2.21) are not the same because of the mass loading effect. Therefore, a modified expression is introduced to describe the additive effect as

\[ \eta_{r,C} = \alpha_r \eta_{r,A} + \beta_r \eta_{r,B}, \tag{2.41} \]

where \( \alpha_r \) and \( \beta_r \) are the weighting factors for mode \( r \). Note that \( \alpha_r \) and \( \beta_r \) are obtained analytically and sample values are listed in Table 2.7. It is seen that \( \alpha_r \approx 1 \) and \( \beta_r > 1 \). This indicates that Patch B provides more damping in Case C than it does in Case B. This may be explained by looking at the shear deformation mode shapes of layer 2 in Figure 2.22, where \( \gamma_2 \) of Case C has higher absolute values in the Patch B region than that in the Patch B region of Case B.

Additionally, the issue of inherent damping in the experimental study needs to be resolved. Measured modal loss factors are considered to have contributions from inherent damping of baseline beam and applied patch damping, i.e.,

\[ \eta_{r,AI} = \eta_{r,I} + \eta_{r,A}, \]
\[ \eta_{r,BI} = \eta_{r,I} + \eta_{r,B}, \tag{2.42 a-c} \]
\[ \eta_{r,CI} = \eta_{r,I} + \eta_{r,C}, \]

where the subscript I indicates the inherent damping. The values of inherent damping \( \eta_{r,I} \)
Figure 2.21  Flexural displacement mode shapes. (a) Mode 1, (b) mode 2, (c) mode 3. Key: ... beam with Patch A, ___ beam with Patch B, • beam with Patches A and B.
Table 2.7 Weighting factors $\alpha_r$ and $\beta_r$ for equation (2.41) as derived from analytical models.
Figure 2.22 Shear deformation mode shapes of layer 2. (a) Mode 1, (b) mode 2, (c) mode 3. Key: ... beam with Patch A, ... beam with Patch B, • beam with Patches A and B.
are found from the modal measurements on the baseline beam. Modal damping of Case C is then estimated with weighting factors similar to equation (2.39) as

\[
\eta_{r,Cl} = \eta_{r,1} + \alpha_r \eta_{r,A} + \beta_r \eta_{r,B} = \eta_{r,1} + \alpha_r (\eta_{r,1} - \eta_{r,1}) + \beta_r (\eta_{r,1} - \eta_{r,1}).
\]  

(2.43)

Note that one may also develop a simple additive estimation procedure where \(\alpha_r = \beta_r = 1\). Modal loss factors that specifically exclude inherent damping for Cases A and B are normalized and listed in Table 2.8 (a). The values of \(\eta_{r,Cl}\) are first estimated by using equation (2.41) with \(\alpha_r\) and \(\beta_r\) taken from the analytical results (Table 2.7) and then with \(\alpha_r = \beta_r = 1\). Both sets of estimations (weighted and simple additive) are then compared with measured results of Case C. Table 2.8 (b) shows that both estimation methods compare well with measurements. Similar studies can be carried out on other patch patterns.

2.7 Conclusion

This chapter presents a refined method for analyzing the harmonic response of beams with multiple constrained-layer viscoelastic patches. Initially the method is developed for thick beams, but subsequently it is restricted to thin beams. The classic sandwich beam theory and a secondary minimization scheme are employed to derive kinematic relationships between flexural displacement and other deformations in all layers. This approach requires the inclusion of only flexural shape functions in the complex eigenvalue problem. Nonetheless, eigenvectors can be related to flexural, longitudinal and shear mode shapes, some of which can not be experimentally measured. Most

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(a) Estimation of $\hat{\eta}_{r,A}$ and $\hat{\eta}_{r,B}$

<table>
<thead>
<tr>
<th>Mode $r$</th>
<th>$\hat{\eta}_{r,1}$</th>
<th>$\hat{\eta}<em>{r,A} = \hat{\eta}</em>{r,1} - \hat{\eta}_{r,1}$</th>
<th>$\hat{\eta}<em>{r,B} = \hat{\eta}</em>{r,1} - \hat{\eta}_{r,1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14.9%</td>
<td>8.9%</td>
<td>0.0%</td>
</tr>
<tr>
<td>2</td>
<td>0.5%</td>
<td>0.2%</td>
<td>2.1%</td>
</tr>
<tr>
<td>3</td>
<td>0.3%</td>
<td>1.1%</td>
<td>5.3%</td>
</tr>
<tr>
<td>4</td>
<td>0.5%</td>
<td>2.8%</td>
<td>5.0%</td>
</tr>
<tr>
<td>5</td>
<td>0.6%</td>
<td>2.6%</td>
<td>2.6%</td>
</tr>
</tbody>
</table>

(b) Comparison of estimated and measured $\hat{\eta}_{r,CI}$

<table>
<thead>
<tr>
<th>Mode $r$</th>
<th>$\hat{\eta}<em>{r,CI} = \hat{\eta}</em>{r,1} + \hat{\eta}<em>{r,A} + \hat{\eta}</em>{r,B}$</th>
<th>$\hat{\eta}<em>{r,CI} = \hat{\eta}</em>{r,1} + \alpha \hat{\eta}<em>{r,A} + \beta \hat{\eta}</em>{r,B}$</th>
<th>Measured $\hat{\eta}_{r,CI}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23.8%</td>
<td>23.9%</td>
<td>26.4%</td>
</tr>
<tr>
<td>2</td>
<td>2.8%</td>
<td>3.2%</td>
<td>2.6%</td>
</tr>
<tr>
<td>3</td>
<td>6.8%</td>
<td>8.0%</td>
<td>6.6%</td>
</tr>
<tr>
<td>4</td>
<td>8.3%</td>
<td>10.9%</td>
<td>8.8%</td>
</tr>
<tr>
<td>5</td>
<td>5.8%</td>
<td>6.6%</td>
<td>5.5%</td>
</tr>
</tbody>
</table>

Table 2.8 Comparison between estimated and measured $\hat{\eta}_{r,CI}$ for the example of Figure 2.20.
important of all, the knowledge of shear deformation modes in the viscoelastic core provides an improved understanding of the effect of patch damping.

The proposed model can be applied to either fully or partially covered sandwich beams. It is successfully validated by comparing results with the examples described by Rao [2.7] and Lall et al. [2.4]. Several damping configurations are then experimentally and analytically studied. Excellent agreement between theory and experiment is seen for all examples. Some important patch damping issues have been clarified especially through an examination of modal deformations. In order to identify the unknown properties of the viscoelastic material used in this chapter, an estimation procedure has been proposed. Frequency-dependent material loss factor and stiffness are estimated by combining analytical parametric studies with modal measurements from beam tests. An uncertainty study has also been carried out to establish the error bounds of these estimations.

Future work will extend this formulation to thick beams and plates. Important design issues, including the optimization of patch patterns for improved damping performance, also need to be investigated.
References for Chapter 2


CHAPTER 3

COMPLEX EIGENSOLUTIONS OF RECTANGULAR PLATES WITH DAMPING PATCHES

3.1 Introduction

In many practical plate and machinery casing structures, it is difficult to treat the whole surface with constrained layer viscoelastic material. Further, it may indeed be desirable to selectively apply one or more damping patches to control certain resonances. To study such issues, an efficient analytical method is needed to predict modal properties of a damped structure. Therefore, the research problem is formulated in the context of a rectangular plate with arbitrary boundary condition. Complex eigensolutions are sought for various damping cases including full and partial coverage, as well as the configuration with multiple damping patches. The method proposed in this chapter first relates motion variables for all layers of a sandwich plate in terms of the flexural displacement of the base plate by using well-known equations of sandwich structures [3.1-5] and a minimization scheme. Then the flexural shape function sets constructed by \( N_x \) and \( N_y \) shape functions in \( x \) and \( y \) directions are incorporated in the Rayleigh-Ritz minimization scheme to obtain a complex eigenvalue problem of dimension \( N_x \times N_y \). This formulation yields efficient calculations of various modal deformations in all layers. This work is an
extension of the analytical and experimental methodology we proposed recently for sandwich beams [3.1].

There is a vast body of literature on the dynamic analysis of sandwich beams and plates as evident from the extensive references cited in two books on vibration damping [3.2, 3.3]. Most of the publications consider only the full damping coverage, however, Lall et al. [3.4] have addressed the partial coverage issue. They analyzed a simply supported plate with a single damping patch using the Rayleigh-Ritz method and calculated the natural frequencies and modal loss factors. A parametric study on the patch size and location was reported. No experimental results were provided but the full coverage case was compared with an analysis carried out earlier by Mead, as reported in [3.4]. This simply supported plate case will be studied again in this chapter for the sake of verification. Limiting case results of He and Ma [3.5] will also be included and compared with other predictions. Since no prior experimental results are available, benchmark modal measurements are conducted on a plate with free edges for several damping configurations. Our method is then validated by comparing complex eigensolution predictions with modal measurements. A normalization scheme for viewing complex modes is also developed. Finally, design-oriented loss factor estimation procedures that examine the additive effect of two damping patches are proposed.

3.2 Analytical Formulation

3.2.1 Physical example

The structure of interest is a rectangular plate with \( N_p \) damping patches attached, as shown in Figure 3.1. Each patch \( p \) is of length \( l_x^p \) and width \( l_y^p \) and is located at \( (x^p, y^p) \).
Figure 3.1 Multiple constrained layer damping patches for a rectangular thin plate with arbitrary boundary conditions.
in the Cartesian coordinates. Each patch has two layers: layer 1 is a metallic layer while layer 2 is made of the viscoelastic material. Note that each patch may be different in size, thickness and material property. The base plate is assumed to be undamped and is designated as layer 3. The boundary conditions are specified later when eigensolutions are sought. Figure 3.2 shows sections of the plate in the $xz$ and $yz$ planes. All relevant variables considered include: flexural $w(x,y;t)$ displacement in $z$ direction, and in-plane displacements $u(x,y;t)$ and $v(x,y;t)$ along $x$ and $y$ respectively, and rotary ($\psi$) and shear angles ($\gamma$) in the $xz$ and $yz$ planes. These variables are included in three deformation vectors $r_1^p$, $r_2^p$, and $r_3$ as defined below

$$
\begin{align*}
    r_1^p &= \begin{bmatrix} w \\ u_1^p \\ v_1^p \\ \psi_{x,1}^p \\ \psi_{y,1}^p \end{bmatrix},
    r_2^p &= \begin{bmatrix} w \\ u_2^p \\ v_2^p \\ \psi_{x,2}^p \\ \psi_{y,2}^p \end{bmatrix},
    r_3 &= \begin{bmatrix} w \\ u_3 \\ v_3 \\ \psi_{x,3} \\ \psi_{y,3} \end{bmatrix};
    p = 1, \ldots, N_p .
\end{align*}
$$

where subscripts 1, 2 and 3 denotes the layer numbers. Note that shear deformations are not included here because they can be obtained from the spatial derivatives of $w(x,y;t)$ and the rotation of layer 2. For elastic layers 1 and 3, shear deformations are assumed to be zero for the sake of simplification.

3.2.2 Energy formulation

The strain energy ($U$) of the composite plate with reference to Figures 3.1 and 3.2 is written as
Figure 3.2  Undeformed and deformed segments along with variables in all layers. (a) xz plane, (b) yz plane.
\[ U = \sum_{p=1}^{N_p} \int \left[ \frac{1}{2} (\Omega \mathbf{r}_1^p)^\top \mathbf{E}_1^p (\Omega \mathbf{r}_1^p) + \frac{1}{2} (\Omega \mathbf{r}_2^p)^\top \mathbf{E}_2^p (\Omega \mathbf{r}_2^p) \right] \mathrm{d}x \mathrm{d}y \]

\[ + \int \int \frac{1}{2} (\Omega \mathbf{r}_3^p)^\top \mathbf{E}_3 (\Omega \mathbf{r}_3) \mathrm{d}x \mathrm{d}y. \]  

(3.2)

where \( \Omega \) is the differential operator matrix defined as

\[ \Omega = \begin{bmatrix}
\partial^2 / \partial x^2 & 0 & 0 & 0 \\
0 & \partial^2 / \partial x \partial y & 0 & 0 \\
0 & 0 & \partial^2 / \partial y^2 & 0 \\
0 & \partial / \partial x & 0 & 0 \\
0 & 0 & \partial / \partial y & 0 \\
0 & \partial / \partial y & 0 & 0 \\
\partial / \partial x & 0 & 0 & -1 \\
0 & \partial / \partial y & 0 & 0 & -1
\end{bmatrix}. \]  

(3.3)

And \( \mathbf{E}_1^p, \mathbf{E}_2^p \) and \( \mathbf{E}_3 \) are elasticity matrices that are defined as

\[
\mathbf{E}_i^p = \begin{bmatrix}
\mathbf{E}_{w,i} & 0 & 0 \\
0 & \mathbf{E}_{w,i} & 0 \\
0 & 0 & \mathbf{E}_{r,i}
\end{bmatrix}, \quad p = 1, \ldots, N_p \quad \text{for } i=1,2; \ i=1,2,3
\]  

(3.4)

where

\[
\mathbf{E}_{w,i}^p = \frac{E_i^p (h_i^p)^3}{12 (1 - \nu_i^2)} \begin{bmatrix}
1 & 0 & \nu_i \\
0 & 2(1 - \nu_i) & 0 \\
\nu_i & 0 & 1
\end{bmatrix},
\]

\[
\mathbf{E}_{w,i}^p = \frac{E_i^p (h_i^p)^3}{2 (1 - \nu_i^2)} \begin{bmatrix}
1 & \nu_i & 0 & 0 \\
\nu_i & 1 & 0 & 0 \\
0 & 0 & (1 - \nu_i)/2 & (1 - \nu_i)/2 \\
0 & 0 & (1 - \nu_i)/2 & (1 - \nu_i)/2
\end{bmatrix},
\]

\[
\mathbf{E}_{r,i}^p = G_i^p h_i^p \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
\]

(3.5 a-c)

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and $E, G, \nu$ and $h$ are Young's modulus, shear modulus, Poisson's ratio and thickness respectively. The kinetic energy ($T$) of the composite plate due to flexural, longitudinal and rotary motions is expressed as

$$T = \sum_{p=1}^{N_p} \int \left[ \frac{1}{2} \dot{\mathbf{r}}_1^p \mathbf{H}_1 \dot{\mathbf{r}}_1^p + \frac{1}{2} \dot{\mathbf{r}}_2^p \mathbf{H}_2 \dot{\mathbf{r}}_2^p \right] dx dy + \int_0^l \left[ \frac{1}{2} \dot{\mathbf{r}}_3^p \mathbf{H}_3 \dot{\mathbf{r}}_3^p \right] dx dy,$$

(3.6)

$$\mathbf{H}_i^p = \begin{bmatrix} h_i^p & 0 \\ h_i^p & (h_i^p)^3/12 \\ 0 & (h_i^p)^3/12 \end{bmatrix},$$

(3.7)

$p = 1, \ldots, N_p$ for $i=1,2; \ i = 1,2,3$

### 3.2.3 Ritz shape functions

To implement the Rayleigh-Ritz minimization scheme, deformation vectors are approximated as

$$\mathbf{r}_1^p(x,y,t) = S_1^p(x,y)q(t),$$

$$\mathbf{r}_2^p(x,y,t) = S_2^p(x,y)q(t),$$

$$\mathbf{r}_3^p(x,y,t) = S_3^p(x,y)q(t),$$

(3.8 a-c)

where $\mathbf{q}$ is the column vector of generalized displacements of the system, and $S_1^p, S_2^p$ and $S_3$ are shape function matrices with $N_s$ admissible function sets

$$S_1^p = [S_1^{p,1} \ldots S_1^{p,k} \ldots S_1^{p,N_s}]$$

$$S_2^p = [S_2^{p,1} \ldots S_2^{p,k} \ldots S_2^{p,N_s}]$$

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Note that $S_{1,k}^p$, $S_{2,k}^p$, and $S_{3,k}$, as shown below, are the $k$th shape function set of the deformation variables of layers 1, 2 and 3 respectively.

$$S_3 = \begin{bmatrix} S_{3,1} & \cdots & S_{3,k} & \cdots & S_{3,N} \end{bmatrix} \quad (3.9 \text{a-c})$$

Here flexural shape function $\phi_{w,k}(x,y)$ is the $k$th element of the flexural shape function row vector $\Phi_w = \begin{bmatrix} \phi_{w,1} & \cdots & \phi_{w,k} & \cdots & \phi_{w,N} \end{bmatrix}$. Note that $\Phi_w(x,y)$ is obtained from the following equation where $X_w(x)$ is a column vector of dimension $N_x$ and $Y_w(x)$ is a row vector of dimension $N_y$. Elements of $X_w$ and $Y_w$ must be chosen such that they satisfy the geometrical boundary conditions of the plate in $x$ and $y$ directions, respectively

$$\Phi_w(x,y) = \text{Col}[X_w(x)Y_w(y)] \quad (3.11)$$

where Col is a matrix operator that converts a matrix of dimension $(N_x, N_y)$ to a row vector of dimension $N_z = N_x \times N_y$. This operation assigns element $(m, n)$ of the matrix to element $k$ of the row vector where $k = (m-1)N_y + n$; $m = 1 \cdots N_x$, and $n = 1 \cdots N_y$. The flexural shape function $\phi_{w,k}(x,y)$ is used as the "master" shape function vector while all other shape functions are "slaves" and should be calculated from $\phi_{w,k}(x,y)$. Some of the slave shape functions can be obtained explicitly while the rest must be solved for by a minimization scheme.
3.3 Shape Function Reduction

Some of the shape functions, as defined earlier by (3.10), can be deduced directly from their kinematic relationships with \( \phi_{w,k} \). First, relationships between in-plane shape functions of layers 1 and 3 are employed which can be written by extending the integrated form of the weak core assumption as stated in reference [3.1].

\[
\phi_{w1,k}^p = -e^p \phi_{w3,k} + d_{x,k}^p, \\
\phi_{v1,k}^p = -e^p \phi_{v3,k} + d_{y,k}^p, \\
\phi_{w2}^p = \frac{h_1^p - h_3}{4} \frac{\partial \phi_w}{\partial x} + \frac{1 - e^p}{2} \phi_{w3} + \frac{1}{2} d_x^p, \\
\phi_{v2}^p = \frac{h_1^p - h_3}{4} \frac{\partial \phi_w}{\partial y} + \frac{1 - e^p}{2} \phi_{v3} + \frac{1}{2} d_y^p, \\
\phi_{w2}^p = \frac{h_1^p + h_3}{2h_2^p} \frac{\partial \phi_w}{\partial x} + \frac{1 + e^p}{h_2^p} \phi_{w3} - \frac{1}{h_2^p} d_x^p, \\
\phi_{v2}^p = \frac{h_1^p + h_3}{2h_2^p} \frac{\partial \phi_w}{\partial y} + \frac{1 + e^p}{h_2^p} \phi_{v3} - \frac{1}{h_2^p} d_y^p. 
\]

(3.12 a, b)

Note that \( d_{x,k}^p \) and \( d_{y,k}^p \) are constants that relate deformation shapes \( \phi_{w3,k} \) and \( \phi_{v3,k} \) to the corresponding \( \phi_{w1,k} \) and \( \phi_{v1,k} \) for each patch \( p \). Second, the deformations in layer 2 can be rewritten as follows by observing the kinematic relationship in Figure 3.2.

The \( k \)th component of the strain energy of plate \( (U_k) \) with admissible function set \( S_{1,k}^p, S_{2,k}^p \) and \( S_{3,k} \) is as follows where \( q_k \) is the corresponding generalized displacement.
\[ U_k = \frac{1}{2} K_k q_k^2, \quad (3.14) \]

\[ K_k = \sum_{p=1}^{N_p} \int \int \left[ \left( \frac{\partial S_{1,k}^p}{\partial x} \right)^T E_1 \left( \frac{\partial S_{1,k}^p}{\partial x} \right) + \left( \frac{\partial S_{2,k}^p}{\partial y} \right)^T E_2 \left( \frac{\partial S_{2,k}^p}{\partial y} \right) \right] dx dy \]

\[ + \int \int \left( \frac{\partial S_{3,k}^p}{\partial x} \right)^T E_3 \left( \frac{\partial S_{3,k}^p}{\partial x} \right) dx dy \quad (3.15) \]

Assume that the corresponding displacements \( \phi_{u,k} \) and \( \phi_{v,k} \) are combinations of row trial function vectors \( \Psi_u(x,y) \) and \( \Psi_v(x,y) \)

\[ \phi_{u,k} = \Psi_u(x,y) c_{u,k} \]

\[ \phi_{v,k} = \Psi_v(x,y) c_{v,k} \quad (3.16 \text{a, b}) \]

where \( c_{u,k} \) and \( c_{v,k} \) are column vectors containing coefficients that need to be determined while \( \Psi_u(x,y) \) and \( \Psi_v(x,y) \) are the row trial function vectors. Both \( \Psi_u(x,y) \) and \( \Psi_v(x,y) \) are obtained from their \( x \) components \( X_u(x) \) and \( X_v(x) \) and \( y \) components \( Y_u(x) \) and \( Y_v(x) \) respectively as follows

\[ \Psi_u(x,y) = \text{Col}[X_u(x)Y_u(y)] \]

\[ \Psi_v(x,y) = \text{Col}[X_v(x)Y_v(y)] \quad (3.17 \text{a, b}) \]

where \( X_u \) and \( X_v \) are column vectors of dimension \( N_{ux} \) and \( N_{vx} \), and \( Y_u \) and \( Y_v \) are row vectors of dimension \( N_{uy} \) and \( N_{vy} \) whose elements satisfy essential boundary conditions in \( x \) and \( y \) directions respectively.

By substituting equations (3.12-13, 3.16) into (3.15) and minimizing \( U_k \) with respect to the coefficients of \( c_{u,k} \), \( c_{v,k} \), \( d_{x,k}^p \) and \( d_{y,k}^p \), a set of governing equations can be formed in matrix form as

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where

\[
A = \begin{bmatrix}
A^{uu} & A^{u
 \nu} & A^{u \nu} & 0 \\
A^{\nu u} & A^{\nu \nu} & 0 & A^{\nu}
\end{bmatrix}_{(A^*)^T},
B_k = \begin{bmatrix}
B_k^u \\
B_k^{\nu u} \\
\vdots \\
B_k^{\nu
 \nu}
\end{bmatrix},
C_k = \begin{bmatrix}
c_{u,k} \\
c_{\nu u,k} \\
\vdots \\
d_{\nu \nu,k}
\end{bmatrix},
\]

and

\[
d_{x,k} = \begin{bmatrix}
d_{x,k}^1 & \cdots & d_{x,k}^{p-1} & \cdots & d_{x,k}^{n_x}
\end{bmatrix}^T,
\]

\[
d_{y,k} = \begin{bmatrix}
d_{y,k}^1 & \cdots & d_{y,k}^{p-1} & \cdots & d_{y,k}^{n_y}
\end{bmatrix}^T.
\]

Submatrices of \( A \) and sub-vectors of \( B_k \) are obtained from the following

\[
A^{uu} = \sum_{p=1}^{n_x} \frac{E_3 h_3 e^p}{1-(\nu_3)^2} \int_T \left[ \left( \frac{\partial}{\partial x} \Psi_u \right)^T \left( \frac{\partial}{\partial x} \Psi_u \right) + \frac{1-\nu_1}{2} \left( \frac{\partial}{\partial y} \Psi_u \right)^T \left( \frac{\partial}{\partial y} \Psi_u \right) \right] dx dy
\]

\[
+ \sum_{p=1}^{n_x} \frac{G_2}{h_3^p} \left( 1+e^p \right) \int_T \left[ (\Psi_u)^T (\Psi_u) + \frac{1-\nu_1}{2} (\Psi_u)^T (\Psi_u) \right] dx dy
\]

\[
+ \frac{E_3 h_3}{1-(\nu_3)^2} \int_T \int_T \left[ \left( \frac{\partial}{\partial x} \Psi_u \right)^T \left( \frac{\partial}{\partial x} \Psi_u \right) + \frac{1-\nu_3}{2} \left( \frac{\partial}{\partial y} \Psi_u \right)^T \left( \frac{\partial}{\partial y} \Psi_u \right) \right] dx dy
\]

\[
A^{\nu u} = \sum_{p=1}^{n_x} \frac{E_3 h_3 e^p}{1-(\nu_3)^2} \int_T \left[ \left( \frac{\partial}{\partial x} \Psi_u \right)^T \left( \frac{\partial}{\partial x} \Psi_{\nu} \right) + \frac{1-\nu_1}{2} \left( \frac{\partial}{\partial y} \Psi_u \right)^T \left( \frac{\partial}{\partial y} \Psi_{\nu} \right) \right] dx dy
\]

\[
+ \sum_{p=1}^{n_x} \frac{G_2}{h_3^p} \left( 1+e^p \right) \int_T \left[ (\Psi_u)^T (\Psi_{\nu}) + \frac{1-\nu_1}{2} (\Psi_u)^T (\Psi_{\nu}) \right] dx dy
\]

\[
+ \frac{E_3 h_3}{1-(\nu_3)^2} \int_T \int_T \left[ \left( \frac{\partial}{\partial x} \Psi_u \right)^T \left( \frac{\partial}{\partial x} \Psi_{\nu} \right) + \frac{1-\nu_3}{2} \left( \frac{\partial}{\partial y} \Psi_u \right)^T \left( \frac{\partial}{\partial y} \Psi_{\nu} \right) \right] dx dy
\]
\[ A^w = \sum_{p=1}^{N_x} E_3 h_3 e^p \int \int [v_1 \left( \frac{\partial}{\partial x} \Psi_u \right) \left( \frac{\partial}{\partial x} \Psi_v \right)^T + \frac{1-v_1}{2} \left( \frac{\partial}{\partial y} \Psi_u \right) \left( \frac{\partial}{\partial y} \Psi_v \right)^T] \, dx \, dy \]

\[ + \frac{E_3 h_3}{1-(\nu_3)^2} \int \int [v_3 \left( \frac{\partial}{\partial x} \Psi_u \right) \left( \frac{\partial}{\partial x} \Psi_v \right)^T + \frac{1-v_3}{2} \left( \frac{\partial}{\partial y} \Psi_u \right) \left( \frac{\partial}{\partial y} \Psi_v \right)^T] \, dx \, dy \]

\[ A^w = \sum_{p=1}^{N_x} E_3 h_3 e^p \int \int [v_1 \left( \frac{\partial}{\partial y} \Psi_u \right) \left( \frac{\partial}{\partial y} \Psi_v \right)^T + \frac{1-v_1}{2} \left( \frac{\partial}{\partial x} \Psi_u \right) \left( \frac{\partial}{\partial x} \Psi_v \right)^T] \, dx \, dy \]

\[ + \frac{E_3 h_3}{1-(\nu_3)^2} \int \int [v_3 \left( \frac{\partial}{\partial y} \Psi_u \right) \left( \frac{\partial}{\partial y} \Psi_v \right)^T + \frac{1-v_3}{2} \left( \frac{\partial}{\partial x} \Psi_u \right) \left( \frac{\partial}{\partial x} \Psi_v \right)^T] \, dx \, dy \]

\[ A^u = \begin{bmatrix} (A^u)_1 & \cdots & (A^u)_p & \cdots & (A^u)_{N_x} \end{bmatrix}; \quad (A^u)_p = \frac{(e^p+1)G_2^p}{h_3^p} \int \Psi_u \, dx \, dy \]

\[ A^r = \begin{bmatrix} (A^r)_1 & \cdots & (A^r)_p & \cdots & (A^r)_{N_x} \end{bmatrix}; \quad (A^r)_p = \frac{(e^p+1)G_2^p}{h_3^p} \int \Psi_v \, dx \, dy \]

\[ \mathbf{A}^x = \begin{bmatrix} (A^x)_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (A^x)_{N_x} \end{bmatrix} \quad ; \quad (A^x)_p = \frac{G_2^p \int \Psi_x^1 \, dx \, dy}{h_3^p} \]

\[ \mathbf{A}^y = \begin{bmatrix} (A^y)_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (A^y)_{N_x} \end{bmatrix} \quad ; \quad (A^y)_p = \frac{G_2^p \int \Psi_y^1 \, dx \, dy}{h_3^p} \]

\[ \mathbf{B}^x = \sum_{p=1}^{N_x} H_p (e^p+1)G_2^p \int \int \left( \frac{\partial \Phi}{\partial x} \right) \Psi_u^T \, dx \, dy \]

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\[ B_f^* = \sum_{p=1}^{N_f} \frac{H^p (e^p + 1) G_{2,2}^p}{h^2_p} \int \left( \frac{\partial \phi_{w,k}}{\partial y} \right) \psi_r^T \, dx \, dy, \]

\[ B_f^\phi = \begin{bmatrix} (B_f^\phi)_1 & \ldots & (B_f^\phi)_p & \ldots & (B_f^\phi)_{N_f} \end{bmatrix}^T; \quad (B_f^\phi)_p = -\frac{H^p (e^p + 1) G_{2,2}^p}{h^2_p} \int \left( \frac{\partial \phi_{w,k}}{\partial x} \right) \, dx \, dy \]

where \( H^p = \frac{h_1^p + 2h_2^p + h_3}{2} \). The coefficients of \( C_k \) can be calculated by

\[ C_k = A^{-1} B_k \]

(3.22)

provided \(|A| \neq 0\). As a result, all entries in the \( k \)th shape function set \( S_{1,k}^p, S_{2,k}^p \), and \( S_{3,k} \) can be determined for a given flexural shape function \( \phi_{w,k} \). This process needs to be repeated \( N_f \) times and then \( C \) is obtained as

\[ C = \begin{bmatrix} c_s \ldots c_{s,k} \ldots c_{s,N_f} \\ c_r \ldots c_{r,k} \ldots c_{r,N_f} \\ d_x \ldots d_{x,k} \ldots d_{x,N_f} \\ d_y \ldots d_{y,k} \ldots d_{y,N_f} \end{bmatrix}. \]

(3.23)

Then the entire shape function matrices \( S_1^p, S_2^p \), and \( S_3 \) are determined.

### 3.4 Complex Eigenvalue Formulation

For harmonic vibration analysis at frequency \( \omega \), the complex-valued Young’s modulus and shear modulus of the viscoelastic material in layer 2 of patch \( p \) are represented by

\[ \tilde{E}_2^p(\omega) = E_2^p \left(1 + i \eta_2^p(\omega) \right), \]

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\[ G_0^\tau(\omega) = G_0^\tau(1 + i\eta_0^\tau(\omega)), \quad (3.24 \ a, b) \]

where \( i = \sqrt{-1} \), and \( \eta_0^\tau \) is the material loss factor. Using equations (3.2, 3.6, 3.8), rewrite strain and kinetic energies in the complex-valued form as

\[ \bar{U} = \frac{1}{2} q^T \bar{K} q, \quad T = \frac{1}{2} \dot{q}^T M \dot{q}, \quad (3.25 \ a, b) \]

where the complex-valued stiffness (\( \bar{K} \)) and real-valued mass (\( M \)) matrices of the system are defined as follows from equations (3.2) and (3.6)

\[ \bar{K} = \sum_{p=1}^{N_s} \int \int \left[ (\partial S_p^\tau)^T E_1 (\partial S_p^\tau) + (\partial S_p^\tau)^T \bar{E}_2 (\partial S_p^\tau) \right] dx dy + \int_0^1 \int_0^1 (\partial S_3^\tau)^T E_3 (\partial S_3) dx dy; \]

\[ M = \sum_{p=1}^{N_s} \int \int [S_p^\tau S_p + S_3^\tau H_3 S_3] \] 

The complex eigenvalue problem of dimension \( N_s \) is then formulated for the following matrix form of governing equations

\[ M \ddot{q} + \bar{K} q = 0. \quad (3.27) \]

Eigenvalues \( (\tilde{\lambda}_r) \) and eigenvectors \( (\bar{q}_r) \) of this non-proportionally damped system problem are complex-valued where \( r \) is the modal index. The undamped natural frequencies \( (\omega_r) \) and composite modal loss factors \( (\eta_r) \) are related to \( \tilde{\lambda}_r \) of equation (3.27) in the following manner [3.1]

\[ \omega_r = \sqrt{\Re(\tilde{\lambda}_r)}, \quad \eta_r = \frac{\Im(\tilde{\lambda}_r)}{\Re(\tilde{\lambda}_r)}; \quad r = 1, \ldots, N_s. \quad (3.28 \ a, b) \]

Complex mode shapes of relevant deformation variables can be calculated from the resulting eigenvectors \( \bar{q}_r \) of (3.27). For example, the complex flexural mode shape is
Complex in-plane mode shapes of layer 3 in $x$ and $y$ directions are

$$\tilde{\vec{u}}_{3,r}(x,y) = \Phi_u(x,y) \vec{q}_r,$$

$$\tilde{\vec{v}}_{3,r}(x,y) = \Phi_v(x,y) \vec{q}_r.$$  \hspace{1cm} (3.30 a, b)

Complex shear deformation mode shapes of layer 2 in $xz$ and $yz$ planes are

$$\tilde{\vec{\gamma}}_{xz2,r}(x,y) = \frac{1}{h_x^p} \left[ H^p \frac{\partial \Phi_u(x,y)}{\partial x} - (1 + e^p) \Psi_u(x,y) \vec{c}_x + \vec{d}_x \right] \vec{q}_r,$$

$$\tilde{\vec{\gamma}}_{yz2,r}(x,y) = \frac{1}{h_y^p} \left[ H^p \frac{\partial \Phi_v(x,y)}{\partial y} - (1 + e^p) \Psi_v(x,y) \vec{c}_y + \vec{d}_y \right] \vec{q}_r.$$  \hspace{1cm} (3.31 a, b)

3.5 Comparison with Literature

To validate the proposed theory, consider a rectangular plate (of dimension $l_x$ and $l_y$) with simple supports along all edges. This case with full surface damping has been analyzed by Lall et al. [3.4], Mead (as reported in reference [3.4]) and He and Ma [3.5]. In particular, He and Ma [3.5] have provided a closed form equation for full coverage. System parameters as defined by Lall et al. [3.4] are summarized in Table 3.1. For our study, analytical solutions are obtained by using four flexural shapes of sine functions in each $X_w$ or $Y_w$. Also the in-plane trial functions in $u$ and $v$ are constructed by using four trial functions for each direction $x$ or $y$, resulting in matrix $A$ of equation (3.19) of size 32. For each flexural shape function $\phi_{w,k}(x,y)$, vector $B_k$ is constructed and coefficients in vector $C_k = A^{-1}B_k$ are calculated to determine the $k$th shape function set $S_{1,k}$, $S_{2,k}$ and $S_{3,k}$. Then the entire shape function matrices $S_1$, $S_2$ and $S_3$ set are incorporated in the
Table 3.1 System parameters used for rectangular plate example as described in the literature [3.4]. All edges assume simple supports. Refer to Figure 3.1 for nomenclature.
Rayleigh-Ritz minimization scheme to obtain the complex-valued eigenvalue problem of size 16. Natural frequencies and modal loss factors are then obtained by using equation (3.28) after solving the eigenvalue problem.

First consider a simply supported square plate with full coverage, labeled here as Example I in Table 3.1. Natural frequencies and modal loss factors of the first four modes obtained from the literature [3.4, 3.5] are compared with results of our method. It is seen in Table 3.2 that our predictions are nearly identical to the results of Mead’s [3.4] and He and Ma’s [3.5]. They also match the results reported by Lall et al. [3.4] even though minor discrepancies are however seen. All methods essentially predict same results.

For Example II of Table 3.1, the base plate is kept the same but a square damping patch of varying size ($l_1^p = l_2^p = l^p$) as shown in Figure 3.3 is applied. This case was specifically studied by Lall et al. [3.4]. Both layers (1 and 2) of the patch are thinner and the viscoelastic core is stiffer than described in Example I. Parametric studies of patch size ($l^p$) variation for the first natural frequency and modal loss factor as reported by Lall et al. [3.4] are carried out and results are in Figure 3.4. Poor agreement between two methods is seen. Next, He and Ma’s closed form solution [3.5] is used again for the full treatment case. It is seen that our results indeed match the closed form solution for the limiting case of $l^p = l$. Further validation is necessary and it is established through a series of modal experiments as presented in the next section.

<table>
<thead>
<tr>
<th>Natural frequency $\omega$ (rad/s)</th>
<th>Lall et al. [3.4]</th>
<th>Mead [3.4]</th>
<th>He and Ma [3.5]</th>
<th>proposed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode (1,1)</td>
<td>975.17</td>
<td>975.00</td>
<td>975.00</td>
<td>974.91</td>
</tr>
<tr>
<td>Mode (1,2)</td>
<td>2350.79</td>
<td>2350.83</td>
<td>2350.80</td>
<td>2350.80</td>
</tr>
<tr>
<td>Mode (2,1)</td>
<td>2350.79</td>
<td>2350.83</td>
<td>2350.80</td>
<td>2350.80</td>
</tr>
<tr>
<td>Mode (2,2)</td>
<td>3725.33</td>
<td>3725.60</td>
<td>3725.60</td>
<td>3725.60</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Loss factor $\eta$</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode (1,1)</td>
<td>4.431 %</td>
<td>4.385 %</td>
<td>4.385 %</td>
<td>4.386 %</td>
</tr>
<tr>
<td>Mode (1,2)</td>
<td>1.918 %</td>
<td>1.911 %</td>
<td>1.911 %</td>
<td>1.911 %</td>
</tr>
<tr>
<td>Mode (2,1)</td>
<td>1.918 %</td>
<td>1.911 %</td>
<td>1.911 %</td>
<td>1.911 %</td>
</tr>
<tr>
<td>Mode (2,2)</td>
<td>1.224 %</td>
<td>1.221 %</td>
<td>1.221 %</td>
<td>1.221 %</td>
</tr>
</tbody>
</table>

Table 3.2 Comparison between published [3.4, 3.5] and proposed methods for Example I. See Table 3.1 for rectangular plate parameters.
Figure 3.3 Patch size and location for Example II, as by Lall et al. [3.4]. The base plate is simply supported on all edges. Here $l_p$ varies from 0 to 0.4m. Refer to Table 3.1 for more details.
Figure 3.4  Comparison between published and proposed methods for Example II. (a) First natural frequency, (b) first modal loss factor. Key: o proposed method, x He and Ma [3.5], _Lall et al. [3.4].
3.6 Experimental Verification

3.6.1 Example cases

A rectangular plate \((l_x = 342.9\, \text{mm}, l_y = 266.7\, \text{mm})\) under the F-F-F-F boundary conditions is chosen as the benchmark example to experimentally verify our analytical model where F denotes a free boundary. The plate is suspended freely and excited with an impulse hammer. Structural acceleration is measured via a compact accelerometer (of weight 1 gram) that is attached near one corner of the plate. Response signals are fed to the analyzer along with the excitation force and sinusoidal transfer functions are obtained. First eight natural frequencies \((f_r)\) and modal loss factors \((\eta_r)\) are then extracted using the half-power bandwidth method [1]. Two types of damping patches (designated here as Patches A and B) with material properties and layer thickness, as specified in Table 3, are applied in this study. Material properties of the viscoelastic core are obtained by adopting the material property estimation technique we had developed earlier in section 2.5.1. Frequency-dependent material loss factor \(\eta_2(\omega)\) and shear modulus \(G_2(\omega)\) are obtained for each damping patch. Since the variation in \(G_2\) is very small in the frequency range of interest, \(G_2\) may be safely assumed to be spectrally-invariant in order to avoid solving the eigenvalue problem with a frequency-dependent stiffness matrix.

Five damping cases for the rectangular plate including limiting cases are considered here as illustrated schematically in Figure 3.5. Case A denotes the plate with a single patch (patch A) attached away from all edges. Case B is the plate with patch B attached at the edge. The plate with both patches A and B applied simultaneously is designated as
<table>
<thead>
<tr>
<th>Layer</th>
<th>Stiffness $E_i$ (N/m²)</th>
<th>Density $\rho_i$ (kg/m³)</th>
<th>Thickness $h_i$ (mm)</th>
<th>Material loss factor $\eta_i(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base layer</td>
<td>$E_3=180\times10^9$</td>
<td>$\rho_3 = 7350$</td>
<td>$h_3 = 2.4$</td>
<td></td>
</tr>
<tr>
<td>Patch A*</td>
<td>Layer 1 $E_1=180\times10^9$</td>
<td>$\rho_1 = 7720$</td>
<td>$h_1 = 0.79$</td>
<td>$\eta_{A}(f)$</td>
</tr>
<tr>
<td></td>
<td>Layer 2 $G_2 = 0.25\times10^6$</td>
<td>$\rho_2 = 2000$</td>
<td>$h_2 = 0.051$</td>
<td>$= 4.07 \times 10^{-3} f - 0.112$</td>
</tr>
<tr>
<td>Patch B*</td>
<td>Layer 1 $E_1=180\times10^9$</td>
<td>$\rho_1 = 7720$</td>
<td>$h_1 = 0.43$</td>
<td>$\eta_{B}(f)$</td>
</tr>
<tr>
<td></td>
<td>Layer 2 $G_2 = 0.5 \times 10^6$</td>
<td>$\rho_2 = 2000$</td>
<td>$h_2 = 0.051$</td>
<td>$= 1.06 \times 10^{-3} f + 0.131$</td>
</tr>
</tbody>
</table>

* Material is provided by the Wolverine Gasket Company. Codes for the patches are: A = WXP-1828, B = WXP-18070.

Table 3.3 Properties of sandwich plate used for benchmark experiments.
Figure 3.5  Damping patch cases for the benchmark plate used for experiments studies. (a) Case A, (b) Case B, (c) Case C, (d) Case D, (e) Case E. All edges are free.
Case C. Two limiting cases constitute the baseline studies: Case D is the undamped base
plate without any damping patch, and Case E is the full coverage case when the plate is
fully covered on one side with Patch A material only. Many other damping
configurations are possible but these five cases are believed to be necessary and hence
only these results are presented here.

3.6.2 Modal results

Natural frequencies and modal loss factor for the first eight modes have been
measured. Corresponding predictions are then obtained by using the analytical procedure
as discussed earlier. The number of shape functions is increased from four to eight and
free-free beam mode shapes are used as flexural trial shapes in both $x$ and $y$ directions.
The material loss factor of the viscoelastic core is assumed to be constant at the vicinity
of a mode and only one eigenvalue problem is solved for each mode. In addition, mode
shapes are obtained from eigenvectors $\tilde{q}$, by using equations (3.29-31). Modal indices
that described nodal lines along $x$ and $y$ directions are listed in Table 3.4. Comparisons of
modal results as listed in Tables 3.5 and 3.6 shows excellent agreement between
experiment and theory for all cases.

One advantage of our method is the ease with which mode shapes of all deformation
variables may be viewed. Of importance here are the shear deformation modes of the
viscoelastic core in the $xz$ and $yz$ planes since they are the major contributors to the total
energy dissipation; note that these modes can not be measured. Figure 3.6 shows the
predicted flexural and shear deformation shapes of mode $(1, 2)$ for Case C; compare these
with the results of Figure 3.7 for the full coverage case (E). It is observed that shear

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<table>
<thead>
<tr>
<th>Mode ( r )</th>
<th>Natural frequency range (Hz) from theory</th>
<th>Modal index*</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>83–94</td>
<td>(2, 2)</td>
</tr>
<tr>
<td>2</td>
<td>109–124</td>
<td>(2, 0)</td>
</tr>
<tr>
<td>3</td>
<td>184–190</td>
<td>(0, 2)</td>
</tr>
<tr>
<td>4</td>
<td>199–207</td>
<td>(2, 1)</td>
</tr>
<tr>
<td>5</td>
<td>243–255</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>6</td>
<td>291–308</td>
<td>(3, 0)</td>
</tr>
<tr>
<td>7</td>
<td>377–406</td>
<td>(3, 1)</td>
</tr>
<tr>
<td>8</td>
<td>385–411</td>
<td>(2, 2)</td>
</tr>
</tbody>
</table>

* Nodal lines along \( x \) and \( y \)

Table 3.4 Modal indices for the benchmark plate example.
### Table 3.5 Measured and predicted modal results for the benchmark plate with damping patches as shown in Figure 3.5.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Case A (with single Patch A)</th>
<th></th>
<th>Case B (with single Patch B)</th>
<th></th>
<th>Case C (with Patches A and B)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Natural frequency $f_r$ (Hz)</td>
<td>Modal loss factor $\eta_r$</td>
<td>Natural frequency $f_r$ (Hz)</td>
<td>Modal loss factor $\eta_r$</td>
<td>Natural frequency $f_r$ (Hz)</td>
</tr>
<tr>
<td></td>
<td>Experiment</td>
<td>Theory</td>
<td>Experiment</td>
<td>Theory</td>
<td>Experiment</td>
</tr>
<tr>
<td>(1,1)</td>
<td>90</td>
<td>84</td>
<td>0.32%</td>
<td>0.22%</td>
<td>86</td>
</tr>
<tr>
<td>(2,0)</td>
<td>109</td>
<td>111</td>
<td>0.38%</td>
<td>0.18%</td>
<td>106</td>
</tr>
<tr>
<td>(0,2)</td>
<td>199</td>
<td>184</td>
<td>0.51%</td>
<td>0.67%</td>
<td>204</td>
</tr>
<tr>
<td>(2,1)</td>
<td>220</td>
<td>207</td>
<td>0.43%</td>
<td>0.44%</td>
<td>214</td>
</tr>
<tr>
<td>(1,2)</td>
<td>259</td>
<td>253</td>
<td>0.34%</td>
<td>0.41%</td>
<td>259</td>
</tr>
<tr>
<td>(3,0)</td>
<td>329</td>
<td>305</td>
<td>1.68%</td>
<td>1.05%</td>
<td>328</td>
</tr>
<tr>
<td>(3,1)</td>
<td>422</td>
<td>405</td>
<td>0.39%</td>
<td>0.32%</td>
<td>416</td>
</tr>
<tr>
<td>(2,2)</td>
<td>445</td>
<td>414</td>
<td>0.40%</td>
<td>0.52%</td>
<td>447</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode</th>
<th>Case A (with single Patch A)</th>
<th></th>
<th>Case B (with single Patch B)</th>
<th></th>
<th>Case C (with Patches A and B)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Natural frequency $f_r$ (Hz)</td>
<td>Modal loss factor $\eta_r$</td>
<td>Natural frequency $f_r$ (Hz)</td>
<td>Modal loss factor $\eta_r$</td>
<td>Natural frequency $f_r$ (Hz)</td>
</tr>
<tr>
<td></td>
<td>Experiment</td>
<td>Theory</td>
<td>Experiment</td>
<td>Theory</td>
<td>Experiment</td>
</tr>
<tr>
<td>(1,1)</td>
<td>86</td>
<td>82</td>
<td>0.32%</td>
<td>0.25%</td>
<td>86</td>
</tr>
<tr>
<td>(2,0)</td>
<td>106</td>
<td>109</td>
<td>0.24%</td>
<td>0.01%</td>
<td>106</td>
</tr>
<tr>
<td>(0,2)</td>
<td>204</td>
<td>188</td>
<td>1.34%</td>
<td>1.05%</td>
<td>204</td>
</tr>
<tr>
<td>(2,1)</td>
<td>214</td>
<td>206</td>
<td>0.58%</td>
<td>0.54%</td>
<td>214</td>
</tr>
<tr>
<td>(1,2)</td>
<td>259</td>
<td>255</td>
<td>1.52%</td>
<td>1.36%</td>
<td>259</td>
</tr>
<tr>
<td>(3,0)</td>
<td>328</td>
<td>304</td>
<td>0.50%</td>
<td>0.09%</td>
<td>328</td>
</tr>
<tr>
<td>(3,1)</td>
<td>416</td>
<td>405</td>
<td>0.58%</td>
<td>0.66%</td>
<td>416</td>
</tr>
<tr>
<td>(2,2)</td>
<td>447</td>
<td>414</td>
<td>1.20%</td>
<td>0.92%</td>
<td>447</td>
</tr>
</tbody>
</table>

Table 3.5 Measured and predicted modal results for the benchmark plate with damping patches as shown in Figure 3.5.
### (a) Case D—Baseline case (base plate without any patch)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Experiment $f_r$ (Hz)</th>
<th>Theory $f_r$ (Hz)</th>
<th>Experiment Modal loss factor $\eta_r$</th>
<th>Theory Modal loss factor $\eta_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>89</td>
<td>83</td>
<td>0.32%</td>
<td>0%</td>
</tr>
<tr>
<td>(2,0)</td>
<td>107</td>
<td>112</td>
<td>0.15%</td>
<td>0%</td>
</tr>
<tr>
<td>(0,2)</td>
<td>202</td>
<td>185</td>
<td>0.07%</td>
<td>0%</td>
</tr>
<tr>
<td>(2,1)</td>
<td>218</td>
<td>207</td>
<td>0.13%</td>
<td>0%</td>
</tr>
<tr>
<td>(1,2)</td>
<td>257</td>
<td>254</td>
<td>0.07%</td>
<td>0%</td>
</tr>
<tr>
<td>(3,0)</td>
<td>336</td>
<td>308</td>
<td>0.07%</td>
<td>0%</td>
</tr>
<tr>
<td>(3,1)</td>
<td>420</td>
<td>406</td>
<td>0.06%</td>
<td>0%</td>
</tr>
<tr>
<td>(2,2)</td>
<td>447</td>
<td>417</td>
<td>0.04%</td>
<td>0%</td>
</tr>
</tbody>
</table>

### (b) Case E—Fully covered case (plate with Patch A material)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Experiment $f_r$ (Hz)</th>
<th>Theory $f_r$ (Hz)</th>
<th>Experiment Modal loss factor $\eta_r$</th>
<th>Theory Modal loss factor $\eta_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>89</td>
<td>94</td>
<td>13%</td>
<td>10%</td>
</tr>
<tr>
<td>(2,0)</td>
<td>111</td>
<td>124</td>
<td>17%</td>
<td>12%</td>
</tr>
<tr>
<td>(0,2)</td>
<td>196</td>
<td>190</td>
<td>13%</td>
<td>18%</td>
</tr>
<tr>
<td>(2,1)</td>
<td>200</td>
<td>199</td>
<td>10%</td>
<td>10%</td>
</tr>
<tr>
<td>(1,2)</td>
<td>---*</td>
<td>243</td>
<td>---*</td>
<td>11%</td>
</tr>
<tr>
<td>(3,0)</td>
<td>314</td>
<td>291</td>
<td>11%</td>
<td>15%</td>
</tr>
<tr>
<td>(3,1)</td>
<td>393</td>
<td>377</td>
<td>10%</td>
<td>14%</td>
</tr>
<tr>
<td>(2,2)</td>
<td>423</td>
<td>385</td>
<td>17%</td>
<td>13%</td>
</tr>
</tbody>
</table>

* Not found in the experiment

Table 3.6 Measured and predicted modal results for limiting cases of benchmark plate as shown in Figure 3.5.
Figure 3.6  Real part of deformation mode shapes of Case C for mode (1,2). (a) Flexural mode, (b) shear mode of layer 2 in the xz plane, (c) shear mode of layer 2 in the yz plane.
Figure 3.7  Real part of deformation mode shapes of the full coverage case (case E) for mode (1,2). (a) Flexural mode, (b) shear mode of layer 2 in the $xz$ plane, (c) shear mode of layer 2 in the $yz$ plane.
modes of layer 2 in Case C show only segments of the full shapes corresponding to Case E. Also, the amplitude of shear modes in the $xz$ plane ($\gamma_{xz}$) is lower than that in the $yz$ plane ($\gamma_{yz}$). This is can be explained simply by examining curvatures of the flexural mode in $x$ and $y$ directions. Since the flexural mode is close to a straight line along $y$ direction, the viscoelastic core experiences small shear deformations in the $xz$ plane ($\gamma_{xz}$). Conversely, the large curvature in $y$ direction causes large shear deformations in the $yz$ plane ($\gamma_{yz}$).

3.6.3 Complex mode shapes

Since the system is non-proportionally damped, complex modes need to be normalized [3.6] before making any comparison between theory and experiment. The normalization has to be carried out in the spatial domain on flexural mode shapes $\tilde{w}_r(x, y)$ instead of considering a column eigenvectors ($\tilde{q}_r$). The normalized expression of the flexural mode $\tilde{w}_r$ in terms of phase angle and amplitude are

$$\tilde{w}_r(x, y) = \frac{\tilde{w}_r(x, y)e^{i\theta_{\text{min}}}}{|\tilde{w}_r(x_{\text{max}, y_{\text{max}}})|}$$

where $^\wedge$ denotes normalized values, $|\ |$ is the operator for absolute value, $\theta_{\text{min}}$ is the rotating angle used for normalization, $(x_{\text{max}}, y_{\text{max}})$ indicates the location of the largest flexural amplitude. The normalization of amplitude is straight forward but the rotating angle $\theta_{\text{min}}$ need to be calculated such the norm of the imaginary part of the normalized flexural mode $\text{Im}(\hat{w}e^{i\theta})$ is minimum.

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Define a function \( F(\theta) \) that is the spatial mean-square value of \( \text{Im}(\bar{w} e^{i\theta}) \) when integrated over the plate surface, where \( \theta \) is an arbitrary angle of rotation

\[
F(\theta) = \iint (\text{Im}(\bar{w} e^{i\theta}))^2 \, dx \, dy = \iint (\text{Re}(\bar{w}) \sin \theta + \text{Im}(\bar{w}) \cos \theta)^2 \, dx \, dy
\]  

(3.33)

Define \( \theta_{\text{min}} \) as the angle when \( F(\theta) \) reaches its minimum value. A necessary condition derived from \( \frac{\partial F(\theta)}{\partial \theta} = 0 \) is

\[
\theta_{\text{min}} = \frac{1}{2} \tan^{-1} \left[ -\frac{2 \iint \text{Re}(\bar{w}) \text{Im}(\bar{w}) \, dx \, dy}{\iint (\text{Re}(\bar{w})^2 - \text{Im}(\bar{w})^2) \, dx \, dy} \right]
\]  

(3.34)

The angle has to be examined by the following expression to ensure that \( F(\theta_{\text{min}}) \) is indeed the minimum value instead of being maximum

\[
\frac{\partial^2 F(\theta)}{\partial \theta^2} = 2 \iint \left( (\text{Re}(\bar{w}) \cos \theta - \text{Im}(\bar{w}) \sin \theta)^2 - (\text{Re}(\bar{w}) \sin \theta + \text{Im}(\bar{w}) \cos \theta)^2 \right) \, dx \, dy > 0.
\]  

(3.35)

Otherwise, \( \theta_{\text{min}} \) must to be rotated by \( \pi/2 \). Note that when the plate surface mode shape is described in terms of discrete points, for example in measured dataset, the integration operator of equations (3.33-35) is replaced by a double summation.

After normalization, the real part (\( \text{Re} \)) and the imaginary part (\( \text{Im} \)) of predicted mode shapes can be compared with those measured. Figure 3.8 shows the real part of the normalized mode shape for mode (1,1). Note that the real parts of the first mode shapes are almost the same for all five cases. Figures 3.9-13 show the imaginary part of the measured and predicted mode shapes for all five cases. It seems that the imaginary parts vary slightly depending on the damping configuration. Comparison between predictions and measurements again shows excellent agreement for each case.
Figure 3.8  Real part of flexural mode shapes of mode (1,1) for all five cases. (a) Prediction, (b) measurement.
Figure 3.9 Imaginary part of flexural mode shapes of mode (1,1) for Case A. (a) Prediction, (b) measurement.
Figure 3.10 Imaginary part of flexural mode shapes of mode (1,1) for Case B. (a) Prediction, (b) measurement.
Figure 3.11  Imaginary part of flexural mode shapes of mode (1,1) for Case C. (a) Prediction, (b) measurement.
Figure 3.12 Imaginary part of flexural mode shapes of mode (1,1) for Case D. (a) Prediction, (b) measurement.
Figure 3.13  Imaginary part of flexural mode shapes of mode (1,1) for Case E. (a) Prediction, (b) measurement.
3.7 Design Study: Additive Effect of Patches

Cases A, B and C are further examined to see how the damping patches add modal damping to the plate. Since Case C is a combination of Cases A and B, modal loss factors of Case C may be estimated by using the results of Cases A and B in an weighted additive manner; we had introduced this procedure earlier in reference [3.1] with application to beams.

\[ \eta_{r,C} = \alpha_r \eta_{r,A} + \beta_r \eta_{r,B} \]  

(3.36)

where subscripts A, B and C are the case designations defined earlier and \( \alpha_r \) and \( \beta_r \) are the weighting factors for mode \( r \). Note that \( \alpha_r \) and \( \beta_r \) can be obtained using the analytical method and sample values are listed in Table 3.7. However practically it is more convenient to use the simple additive estimation procedure where \( \alpha_r = \beta_r = 1 \). Table 3.8 shows the comparison of both weighted and simple additive estimations with calculated values of \( \eta_{r,C} \). Errors introduced by the simple additive estimation method are less than 10\% for most of the modes.

Both estimation procedures are next applied to our benchmark experimental study but the weighting factors are again obtained from the analytical model. Table 3.9 (a) compares both estimations with measurements. It is seen that both methods provide reasonable estimates except for the first two modes. It may be explained from the fact that base plate used for the experimental studies (Case D in Figure 3.5 and Table 3.6) has high inherent damping, especially at \( r = 1, 2 \). Therefore, the two estimation methods are modified and the inherent damping must be taken into account in an additive manner. Measured modal loss factors are considered to have contributions from inherent (I)
Table 3.7 Weighting factors $\alpha_r$ and $\beta_r$ for equation (3.38) as derived from analytical models.

<table>
<thead>
<tr>
<th>Mode</th>
<th>$\alpha_r$</th>
<th>$\beta_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.14</td>
<td>1.11</td>
</tr>
<tr>
<td>2</td>
<td>0.98</td>
<td>1.08</td>
</tr>
<tr>
<td>3</td>
<td>1.14</td>
<td>1.00</td>
</tr>
<tr>
<td>4</td>
<td>0.87</td>
<td>1.02</td>
</tr>
<tr>
<td>5</td>
<td>0.81</td>
<td>1.09</td>
</tr>
<tr>
<td>6</td>
<td>1.00</td>
<td>0.95</td>
</tr>
<tr>
<td>7</td>
<td>1.14</td>
<td>1.07</td>
</tr>
<tr>
<td>8</td>
<td>0.94</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 3.8 Comparison between calculated and estimated modal loss factors of Case C, based on the analytical method.

<table>
<thead>
<tr>
<th>Mode</th>
<th>calculated</th>
<th>weighted additive method</th>
<th>$\eta_{r,c}$</th>
<th>simple additive method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>estimated</td>
<td>error</td>
<td>estimated</td>
<td>error</td>
</tr>
<tr>
<td>1</td>
<td>0.52%</td>
<td>0.52%</td>
<td>0%</td>
<td>0.46%</td>
</tr>
<tr>
<td>2</td>
<td>0.19%</td>
<td>0.19%</td>
<td>0%</td>
<td>0.19%</td>
</tr>
<tr>
<td>3</td>
<td>1.81%</td>
<td>1.81%</td>
<td>0%</td>
<td>1.72%</td>
</tr>
<tr>
<td>4</td>
<td>0.93%</td>
<td>0.93%</td>
<td>0%</td>
<td>0.98%</td>
</tr>
<tr>
<td>5</td>
<td>1.81%</td>
<td>1.81%</td>
<td>0%</td>
<td>1.77%</td>
</tr>
<tr>
<td>6</td>
<td>1.13%</td>
<td>1.13%</td>
<td>0%</td>
<td>1.13%</td>
</tr>
<tr>
<td>7</td>
<td>1.07%</td>
<td>1.07%</td>
<td>0%</td>
<td>0.98%</td>
</tr>
<tr>
<td>8</td>
<td>1.41%</td>
<td>1.41%</td>
<td>0%</td>
<td>1.44%</td>
</tr>
</tbody>
</table>
(a) Estimation of $\eta_{r,c}$ without considering inherent damping of base plate

<table>
<thead>
<tr>
<th>Mode</th>
<th>Measured</th>
<th>Calculated*</th>
<th>Estimated</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Weighted</td>
<td>Simple</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.52%</td>
<td>0.52%</td>
<td>0.72%</td>
</tr>
<tr>
<td>2</td>
<td>0.52%</td>
<td>0.19%</td>
<td>0.63%</td>
</tr>
<tr>
<td>3</td>
<td>1.97%</td>
<td>1.81%</td>
<td>1.91%</td>
</tr>
<tr>
<td>4</td>
<td>0.88%</td>
<td>0.93%</td>
<td>0.96%</td>
</tr>
<tr>
<td>5</td>
<td>1.93%</td>
<td>1.81%</td>
<td>1.93%</td>
</tr>
<tr>
<td>6</td>
<td>2.30%</td>
<td>1.13%</td>
<td>2.17%</td>
</tr>
<tr>
<td>7</td>
<td>1.00%</td>
<td>1.07%</td>
<td>1.07%</td>
</tr>
<tr>
<td>8</td>
<td>1.97%</td>
<td>1.41%</td>
<td>1.58%</td>
</tr>
</tbody>
</table>

* From Table 3.8

(b) Estimation of $\eta_{r,CI}$ with inherent damping measured in base plate

<table>
<thead>
<tr>
<th>Mode</th>
<th>Measured</th>
<th>Calculated†</th>
<th>Estimated</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Weighted</td>
<td>Simple</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.52%</td>
<td>0.84%</td>
<td>0.31%</td>
</tr>
<tr>
<td>2</td>
<td>0.52%</td>
<td>0.34%</td>
<td>0.47%</td>
</tr>
<tr>
<td>3</td>
<td>1.97%</td>
<td>1.88%</td>
<td>1.83%</td>
</tr>
<tr>
<td>4</td>
<td>0.88%</td>
<td>1.06%</td>
<td>0.85%</td>
</tr>
<tr>
<td>5</td>
<td>1.93%</td>
<td>1.88%</td>
<td>1.87%</td>
</tr>
<tr>
<td>6</td>
<td>2.30%</td>
<td>1.20%</td>
<td>2.10%</td>
</tr>
<tr>
<td>7</td>
<td>1.00%</td>
<td>1.13%</td>
<td>1.00%</td>
</tr>
<tr>
<td>8</td>
<td>1.97%</td>
<td>1.45%</td>
<td>1.54%</td>
</tr>
</tbody>
</table>

† Table 3.8 value plus inherent damping found in experiments

Table 3.9 Comparison between measured, calculated and estimated modal loss factors of Case C, based on the experimental study.
damping of base plate and applied patch damping (A, B or C) based on simple additive estimation method.

\[ \eta_{r,A1} = \eta_{r,1} + \eta_{r,A}, \]
\[ \eta_{r,B1} = \eta_{r,1} + \eta_{r,B}, \]
\[ \eta_{r,C1} = \eta_{r,1} + \eta_{r,C}, \]

The values of inherent damping \( \eta_{r,1} \) are found from the modal measurements on the baseline case. A refined estimate for the modal damping of Case C is introduced based on the weighted additive method as

\[ \eta_{r,C1} = \eta_{r,1} + \alpha_r \eta_{r,A} + \beta_r \eta_{r,B} \]
\[ = \eta_{r,1} + \alpha_r (\eta_{r,Al} - \eta_{r,1}) + \beta_r (\eta_{r,B1} - \eta_{r,1}), \]

where \( \alpha_r \) is the weighting factor for inherent damping whose value is taken as unity here.

Table 3.9 (b) shows estimates with inherent damping considered by both weighted and simple additive methods. In comparison with measurements, it is seen that, unlike the methods without inherent damping, both methods underestimate modal loss factors especially for the first two modes. Note that in Table 3.9 (b) the calculated values from the analytical method also include inherent damping found in measurements. Table 3.9 shows that the estimate of second mode is considerably improved when the inherent damping is considered. Minor changes in other modes are seen. Overall, it is concluded that the additive estimation is a reasonable design prediction scheme but suitable weighting factors should be needed for improvements.
3.8 Conclusion

A new analytical model of a rectangular plate with multiple constrained layer damping patches has been developed to predict complex eigensolutions. Comparison with the work of three prior investigators [3.4, 3.5] on a simply supported plate validates the model for limiting case of full coverage. Analytical predictions of natural frequencies, modal loss factors and complex modes for a rectangular plate with one or two patches are in excellent agreement with modal measurements. In addition, a normalization scheme for complex mode shapes has been developed for comparing measured and predicted mode shapes. A parametric design study has been also performed for a plate with two damping patches. Again, good agreement is seen between theory, experiment and simplified estimation methods.

The method proposed in this chapter illustrates the importance of the kinematic relationships between flexural deformation shapes and other deformation variables including shear deformations of the viscoelastic core which are the major contributors to the overall energy dissipation. The visualization of flexural mode and its associated shear deformations modes may explain why a damping patch at a certain location results in higher damping performance for a certain mode than at other patch locations. Further work is needed to optimize the selective damping treatment concept [3.7]. Also, the availability of the complex mode shapes will assist in the calculation of the structural intensities [3.8-9].
References for Chapter 3


3.8 A. B. Spalding and J. A. Mann III 1995 Journal of the Acoustical Society of America 97, 3617-3624. Placing small constrained layer damping patches on a plate to attain global or local velocity changes.

3.9 A. Nejade and R. Singh 1997 personal communication regarding structural intensity measurements.
CHAPTER 4

DEVELOPMENT OF APPROXIMATE METHODS FOR THE ANALYSIS OF PATCH DAMPING DESIGN CONCEPTS

4.1 Introduction

Patch damping design is an efficient and cost effective concept for solving noise and vibration problems [4.1-4]. However, there is a limited body of scientific literature on this topic and it therefore remains a somewhat ill-understood and empirical technique. Single or double constrained layer patches have been computationally examined by using higher order differential equation theory [4.3-7], Rayleigh-Ritz method [4.1-4, 4.8-9], or finite element procedure [4.10-12]. Experimental methods of investigation have included modal testing [4.1-4] and structural intensity mapping [4.13]. A comprehensive study of this topic has been undertaken and a new computational scheme has been proposed, based on the Rayleigh-Ritz method, for beams and plates with multiple patches of arbitrary properties [4.1, 4.2]. Our calculation methods have been verified by comparing results with measured modal data or data available in the literature.

This chapter extends Chapters 2 and 3 by proposing three approximate analytical methods. An attempt is being made to seek some insight into the patch damping design process. Tractable formulations that identify the role of the following parameters are developed: material properties, viscoelastic layer thickness, patch size, number of patches
and their locations. Like the previous chapters, cantilever beams and rectangular plates (with free-free or simply supported edges) serve as prime examples, and experimental modal analysis and Rayleigh-Ritz method are used to examine the validity of each method. Key assumptions which form the basis of each method are: (i) damped mode shape of a sandwich beam may be described by the eigenfunctions of undamped Euler's beam [4.14], (ii) viscoelastic core layer is very compliant, and (iii) only a single compact patch is applied. The last two assumptions are also applied to the plate example. Approximate methods are expected to be computationally efficient and suitable for rapid parametric design studies.

4.2 Approximate Method I for Sandwich Beams

4.2.1 Euler beam modes

The structure of interest is an elastic beam (designated as layer 3) with \( N_p \) damping patches attached, as shown in Figure 4.1. Each patch \( p \) of length \( l_p \) is located at \( x^p \), and has two layers: layer 1 is an elastic layer while layer 2 is made of viscoelastic material. Note that each patch may be different in size, thickness and material property. The analytical approach, to be developed in this section, is called Approximate Method I. One more assumption in addition to those usually made for a sandwich beam [4.5] is made. Specifically, the \( k \)th damped mode shape of the sandwich beam is approximated by the eigenfunction \( w_k \) of undamped Euler beam that is described as follows where \( \lambda_k \) is the frequency parameter of mode \( k \) and \( \bar{x} = x/l_x \).

\[
\frac{d^4 w_k}{dx^4} = \lambda_k^4 w_k, \quad (4.1 \text{ a, b})
\]
Figure 4.1  Beam with multiple constrained layer damping patches. Key: layer 1 = constraining layer (elastic), layer 2 = constrained layer (viscoelastic), layer 3 = base structure (elastic).
\[ w_k(x) = a_k \sin(\lambda_k x) + b_k \cos(\lambda_k x) + c_k \sinh(\lambda_k x) + d_k \cosh(\lambda_k x), \]

where \( a_k, b_k, c_k \) and \( d_k \) are coefficients whose sample values for selected boundary conditions are listed in Table 4.1, for the sake of completeness. Refer to [4.15] for other cases.

Taking advantage of equation (4.1), corresponding longitudinal \( (u_k) \) and shear deformation \( (\gamma_k) \) mode shapes will be derived. The Rayleigh method along with the modal strain energy technique are then used to explicitly obtain the undamped natural frequency \( (\omega_k) \) and modal loss factor \( (\eta_k) \) without solving for a complex eigenvalue problem.

### 4.2.2 Kinematic relationships

Other deformation variables are longitudinal shapes \( u_{1,k} \) and \( u_{3,k} \) of layers 1 and 3, and shear deformation shape \( \gamma_{1,k} \) of layer 2 for \( p = 1 \cdots N_p \). To express these in terms of the flexural shape \( w_k \), some kinematic relationships must be used. First, the relationship between \( u_{1,k} \) and \( u_{3,k} \) can be written by extending the integrated form of the weak core assumption [4.1].

\[ u_{3,k} = -e^\rho u_{1,k} + s_k^\rho, \]  

where \( e^\rho = \frac{E^\rho h^\rho}{E^s h^s} \); \( p = 1, \cdots, N_p \).

Note that \( s_k^\rho \) is the constant that relates deformation shape \( u_{1,k} \) to the corresponding \( u_{3,k} \).
### A. Simply-supported beam

<table>
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<th>$b_k$</th>
<th>$c_k$</th>
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### B. Clamped-free beam

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<th>$b_k$</th>
<th>$c_k$</th>
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<td>...</td>
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</tr>
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<td>-1</td>
<td>-1</td>
<td>1</td>
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</tbody>
</table>

### C. Free-free beam

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<th>$b_k$</th>
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<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>$(2n+1)\pi/2$</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4.1  Euler beam eigenfunction coefficients for selected boundary conditions.
for each patch \( p \). Second, by observing Figure 4.2, the relationship between the shear deformation shape \( \gamma_{z,k}^p \) and longitudinal displacements \( u_{l,k}^p \) and \( u_{3,k}^p \) is

\[
\gamma_{z,k}^p = \frac{1}{h_2} \left( u_{l,k}^p - u_{3,k}^p + \frac{C^p}{l_x} \frac{d\omega_k}{d\bar{x}} \right) \tag{4.4}
\]

where \( C^p = (h_1^p + 2h_2^p + h_3^p)/2 \). Also, from equations (4.2) and (4.4), \( \gamma_{z,k}^p \) can be expressed as

\[
\gamma_{z,k}^p = \frac{1}{h_2} \left( (1 + e^p) u_{l,k} - \frac{C^p}{l_x} \frac{d\omega_k}{d\bar{x}} - s_k^p \right) \tag{4.5}
\]

Finally, consider the free body diagram of a beam section of length \( d\bar{x} \) as illustrated in Figure 4.3. Relationships between the axial force \( F_{z,k} \), longitudinal displacement \( u_{l,k}^p \), and the shear stress \( \tau_{z,k}^p \) and strain \( \gamma_{z,k}^p \) are

\[
\tau_{z,k}^p d\bar{x} = dF_{z,k} ; \quad F_{z,k} = E_t^p h_1^p \frac{du_{l,k}^p}{d\bar{x}} ; \quad \tau_{z,k}^p = G_z \gamma_{z,k}^p \tag{4.6 a-c}
\]

Combining Equations (4.5) and (4.6), a differential equation in terms of \( u_{3,k} \) is found to be as follows

\[
\frac{d^2 u_{l,k}^p}{d\bar{x}^2} = \frac{G_z^p l_x}{E_t^p h_1^p h_2^p} \left[ (e^p + 1) u_{l,k}^p + \frac{C^p}{l_x} \frac{d\omega_k}{d\bar{x}} - s_k^p \right] \tag{4.7}
\]

Corresponding shapes are derived in the next two sections using equations (4.5) and (4.7).
Figure 4.2 Undeformed and deformed segments along with variables in all layers. Key as Figure 4.1.
Figure 4.3  Axial force and shear stress relationships for a segment of the sandwich beam. Key as Figure 4.1.
4.2.3 Corresponding longitudinal shapes

To solve the second order differential equation (4.7) in terms of \( u_{l,k}^{p} \), ignore constant \( s_{k}^{p} \) for the sake of simplification. This assumption is reasonable since it does not affect the longitudinal strain energy. Equation (4.7) is rewritten as

\[
\frac{d^{2}u_{l,k}^{p}}{dx^{2}} - Y^{p}u_{l,k}^{p} = \frac{Y^{p}C^{p}}{l_{x}(e^{p} + 1)} \frac{dw_{k}}{dx}, \tag{4.8 a, b}
\]

\[
Y^{p} = G_{l}^{p}l_{x}^{2}(e^{p} + 1)/E_{l}^{p}h_{l}^{p}h_{l}^{p}.
\]

Taking advantage of equation (4.1), the solution to equation (4.8) is obtained as

\[
u_{l,k}^{p}(\bar{x}) = A_{k}^{p}\sinh(\sqrt{Y^{p}} \bar{x}) + B_{k}^{p}\cosh(\sqrt{Y^{p}} \bar{x}) + \mu_{k}^{p}Y^{p} \frac{dw_{k}}{dx} + \mu_{k}^{p} \frac{d^{3}w_{k}}{dx^{3}} \tag{4.9}
\]

where

\[
\mu_{k}^{p} = \frac{C^{p}Y^{p}}{l_{x}(e^{p} + 1)\left[\lambda_{k}^{4} - (Y^{p})^{2}\right]} \tag{4.10 a, b}
\]

Constants \( A_{k}^{p} \) and \( B_{k}^{p} \) of equation (4.9) must be determined in accordance with the longitudinal boundary conditions of layer 1. For a damping patch with free ends at \( \bar{x}_{a}^{p} = \bar{x}^{p} - 0.5\bar{l}_{x}^{p} \) and \( \bar{x}_{b}^{p} = \bar{x}^{p} + 0.5\bar{l}_{x}^{p} \), \( A_{k}^{p} \) and \( B_{k}^{p} \) can be calculated as follows by applying boundary conditions \( \frac{du_{l,k}^{p}(\bar{x}_{a}^{p})}{dx} = 0 \) and \( \frac{du_{l,k}^{p}(\bar{x}_{b}^{p})}{dx} = 0 \) to equation (4.9)

\[
\begin{bmatrix}
A_{k}^{p} \\
B_{k}^{p}
\end{bmatrix} = -\frac{\mu_{k}^{p}}{\sqrt{Y^{p}}} \begin{bmatrix}
\cosh(\bar{x}_{a}^{p}) & \sinh(\bar{x}_{a}^{p}) \\
\cosh(\bar{x}_{b}^{p}) & \sinh(\bar{x}_{b}^{p})
\end{bmatrix}^{-1} \begin{bmatrix}
Y^{p} \frac{d^{2}w_{k}(\bar{x}_{a}^{p})}{dx^{2}} + (\lambda_{k})^{4} w_{k}(\bar{x}_{a}^{p}) \\
Y^{p} \frac{d^{2}w_{k}(\bar{x}_{b}^{p})}{dx^{2}} + (\lambda_{k})^{4} w_{k}(\bar{x}_{b}^{p})
\end{bmatrix}. \tag{4.11}
\]
Now given the expression of $u_{t,k}^p$, longitudinal shape $u_{l,k}$ of layer 3 may also be obtained from equation (4.2).

### 4.2.4 Corresponding shear deformation shape

Recall equation (4.5) where the relationship between longitudinal and shear deformations is defined. Since $u_{t,k}^p$ is available now, the only unknown left is the constant $s_k^p$. Note that because $s_k^p$ affects the shear strain energy, it is of importance here and it should be obtained from the energy viewpoint. The shear strain energy $U_{r_{2,k}}^p$ of patch $p$ in layer 2 is as follows where integration is carried out over the patch length from $\bar{x}_a^p$ to $\bar{x}_b^p$:

$$U_{r_{2,k}}^p = \frac{1}{2} G_2 h_2 l_y \int_{\bar{x}_a^p}^{\bar{x}_b^p} \left( \gamma_{r_{2,k}}^p \right)^2 dx$$  \hspace{1cm} (4.12)

where $G_2$ is the shear modulus of layer 2, and $l_y$ is the width of the beam. Substitute equations (4.5) and (4.9) into (4.12), minimize $U_{r_{2,k}}^p$ by setting $\frac{\partial U_{r_{2,k}}^p}{\partial s_k^p} = 0$ and subsequently find $s_k^p$ as

$$s_k^p = \left( \frac{e^p + 1}{l_z} \right) \left\{ \frac{A_k^p}{\sqrt{Y^p}} \left[ \cosh \left( \sqrt{Y^p} \bar{x}_b^p \right) - \cosh \left( \sqrt{Y^p} \bar{x}_a^p \right) \right] \ight.$$  

$$+ \frac{B_k^p}{\sqrt{Y^p}} \left[ \sinh \left( \sqrt{Y^p} \bar{x}_b^p \right) - \sinh \left( \sqrt{Y^p} \bar{x}_a^p \right) \right] \ight.$$  

$$+ \left( \frac{C^p}{l_z (e^p + 1) Y^p} + \mu_k^p Y^p \right) \left[ w_k(\bar{x}_b^p) - w_k(\bar{x}_a^p) \right]$$  

$$+ \mu_k^p \left[ \frac{d^3 w_k(\bar{x}_b^p)}{d\bar{x}^3} - \frac{d^3 w_k(\bar{x}_a^p)}{d\bar{x}^3} \right] \}.$$  \hspace{1cm} (4.13)
Now, corresponding shear strain $\gamma_{2, k}^{p}$ of layer 2 is also available by combining equations (4.5) and (4.13).

### 4.2.5 Natural frequency and modal loss factor calculations

With the availability of all deformation shapes, energy formulation is used to estimate modal parameters of the sandwich beam for the $k$th mode of interest. Modal loss factor $\eta_k$ is obtained as follows by using the modal strain energy method

$$
\eta_k = \frac{\sum_{p=1}^{N} \eta_{m,2}^p U_{r,2,k}^p}{U_{\text{total},k}} 
$$

(4.14)

where $\eta_{m,2}^p$ is the material loss factor of the viscoelastic core of patch $p$ and $U_{\text{total},k}$ is the total modal strain energy that is approximated as follows

$$
U_{\text{total},k} = U_{w,3,k} + \sum_{p=1}^{N} \left(U_{w,1,k}^p + U_{w,3,k}^p + U_{r,2,k}^p\right).
$$

(4.15)

Note that

$$
U_{w,3,k} = \frac{1}{2l_x^3} \int E_3 I_3 \left(\frac{d^2 w_k}{dx^2}\right)^2 d\bar{x},
$$

(4.16)

$$
U_{w,1,k}^p = \frac{1}{2l_x^3} \int_{\rho} E_1^p I_1^p \left(\frac{d^2 w_k}{dx^2}\right)^2 d\bar{x}
$$

(4.17)

are strain energies of layers 1 and 3 due to flexural motion, and

$$
U_{w,13,k}^p = \frac{1}{2l_x} \int_{\rho} E_1^p h_1^p I_1^p \left(1 + e^p \right) \left(\frac{du_{1,\lambda}^p}{d\bar{x}}\right)^2 d\bar{x},
$$

(4.18)
is the strain energy of layers 1 and 3 due to longitudinal motions. In addition, the undamped natural frequency is obtained by using the Rayleigh method

\[ \omega_k = \sqrt{\frac{U_{\text{total},k}}{T_{\text{total},k}}} \]  

(4.19)

where \( T_{\text{total},k} \) is the total kinetic energy of mode \( k \). Again, \( T_{\text{total},k} \) is approximated based on flexural motion of all layers as

\[ T_{\text{total},k} \approx T_{w,k} = \frac{1}{2} l_x \int \rho_3 I_3 (w_k)^2 d\xi + \frac{1}{2} l_x \sum_{p=1}^{\infty} \left( \rho_p^0 I_p^0 + \rho_p^1 I_p^1 \right) (w_k)^2 d\xi . \]  

(4.20)

### 4.3 Examination of Approximate Method I

#### 4.3.1 Procedure using measured or computed mode shapes

A practical beam structure may have mode shapes different from the ideal Euler beam eigenfunctions of section 4.2.1 due to non-classical boundary conditions, non-uniform geometry, and mass loading effect introduced by damping patches. In such cases, predicted (say from a finite element code) or measured flexural mode shapes of a sandwich beam may be obtained and then discrete spatial data are curve-fitted to yield a continuous function \( \phi_{w,k}(\xi) \). In order to apply our method to arbitrary mode shapes, \( \phi_{w,k}(\xi) \) is approximated by a superposition of undamped Euler beam eigenfunctions \( w_r(\xi) \) which are obtained by satisfying appropriate boundary conditions

\[ \phi_{w,k}(\xi) = q_1 w_1(\xi) + q_2 w_2(\xi) + \cdots q_r w_r(\xi) + \cdots \approx \sum_{r=1}^{N} q_r w_r(\xi) \]  

(4.21)

where \( N \) is the number of eigenfunctions included and \( q_r \) is the curve-fit coefficient.
obtained as

\[ q_r = \frac{\int \phi_{w,k} w_r \, dx}{\int (w_r)^2 \, dx}. \]  

(4.22)

Following the superposition principle, longitudinal \( \phi_{w,k} \) and shear \( \phi_{r_{2,k}} \) deformation are obtained as follows using equations (4.9) and (4.5)

\[ \phi_{w,k}^i(x) = \sum_{r=1}^{N} q_r u_{r,r}^i(x), \]  

(4.23)

\[ \phi_{r_{2,k}}^i(x) = \sum_{r=1}^{N} q_r r_{r_{2,r}}^i(x). \]  

(4.24)

Replacing \( w_k, u_{r,r}^i \) and \( r_{r_{2,r}}^i \) with \( \phi_{w,k}, \phi_{w,k}^i \) and \( \phi_{r_{2,k}}^i \) respectively into energy equations (4.12, 4.16-18, 4.20), natural frequency \( \omega_k \) and modal loss factor \( \eta_k \) associated with a specific mode shape \( \phi_{w,k} \) is obtained by using equations (4.14, 4.19).

### 4.3.2 Validation

A cantilever beam is used as an example to verify Approximate Method I in comparison with the analytical Rayleigh-Ritz method described earlier in Chapter 2. Beam and damping material parameters are summarized in Table 4.2. First, the full coverage case of Figure 4.4 (a) is examined. The first five Euler beam eigenfunctions are assumed for the flexural displacements. Corresponding longitudinal and shear displacements are found, and modal strain and kinetic energies are obtained. Natural frequencies \( \omega_k \) and modal loss factors \( \eta_k \) are then calculated using Approximate Method I. Predictions are listed in Table 4.3 in comparison with those obtained from the Rayleigh-Ritz method. It is seen that discrepancies are very small especially for higher
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<th>Material</th>
<th>Layer 1</th>
<th>Layer 2</th>
<th>Layer 3</th>
</tr>
</thead>
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<td>$G_2 = 0.25 \times 10^6$</td>
<td>$E_3 = 180 \times 10^9$</td>
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<td>varies</td>
<td>$l = 177.8$</td>
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<tr>
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<td>$h_3 = 1.47$</td>
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Table 4.2  System parameters for the beam example.

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<th>Approximation I (with Euler beam mode description)</th>
<th>Rayleigh-Ritz Method</th>
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<td>15.5%</td>
</tr>
</tbody>
</table>

Table 4.3  Comparison between Rayleigh-Ritz method and Approximate Method I for the full coverage case. See Table 4.2 for parameters.
Figure 4.4  Cantilever beam examples. (a) Full coverage, (b) partial coverage ($\bar{x}^p = 0.2$, $\bar{l}^p = 0.14$). Refer to Table 4.2 for parameters.
modes. Discrepancies in modes 1 and 2 are however larger because the actual mode shapes are different from the Euler beam eigenfunctions. Thus, this method yields a good prediction only when the mode shape is correct.

Next, the same cantilever beam with a single patch of length $t^P = 0.14$ (Figure 4.4 b) is examined and results are compared in Table 4.4. Due to the mass loading effect, actual mode shapes are expected to deviate more from the Euler beam eigenfunctions than those observed in the full coverage case. Therefore, larger discrepancies are observed. Finally the mode shapes are corrected by incorporating the mass loading effect and by expressing each mode as a superposition of first ten Euler beam eigenfunctions, using the formulation in section 4.3.1; see Table 4.5 for typical values of curve-fit coefficients. Approximate Method I is now in excellent agreement with the Rayleigh-Ritz method, as evident from Table 4.4.

4.4 Approximate Method II for a Compliant Viscoelastic Core

4.4.1 Formulation

In many practical damping treatments, the viscoelastic layer is often very compliant compared with metallic layers (1 and 3). Consequently longitudinal deformations of elastic layers become negligible and Approximate Method I may be further simplified to yield Approximate Method II which calculates modal loss factors for a sandwich beam given arbitrary mode shapes. Recall equation (4.5) where $\gamma_{i,k}^p$ is determined from $u_{i,k}^p$, $w_k$, and $s_k^p$. The effect of $u_{i,k}^p$ on $\gamma_{i,k}^p$ is small when $G_2 << E_1$, compared with that of
### Table 4.4
Comparison between Rayleigh-Ritz method and Approximate Method I for the partial coverage (14%) case using only one damping patch. See Table 4.2 for parameters.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
<th>Mode 4</th>
<th>Mode 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.20%</td>
<td>0.01%</td>
<td>0.01%</td>
<td>0.01%</td>
<td>0.16%</td>
</tr>
<tr>
<td></td>
<td>0.16%</td>
<td>0.25%</td>
<td>0.26%</td>
<td>0.25%</td>
<td>0.16%</td>
</tr>
<tr>
<td></td>
<td>0.88%</td>
<td>0.72%</td>
<td>0.72%</td>
<td>0.72%</td>
<td>0.72%</td>
</tr>
</tbody>
</table>

* Superposition of Euler beam modes

### Table 4.5
Curve-fit coefficients $q_r$ for expressing mode shapes of the partially covered beam of Figure 4.4 (b) in terms of superposition of the undamped Euler beam eigenfunctions.

<table>
<thead>
<tr>
<th>$q_1$</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
<th>Mode 4</th>
<th>Mode 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000</td>
<td>-0.0067</td>
<td>-0.0147</td>
<td>-0.0137</td>
<td>-0.0078</td>
<td></td>
</tr>
<tr>
<td>-0.0007</td>
<td>0.9999</td>
<td>-0.0691</td>
<td>-0.0604</td>
<td>-0.0356</td>
<td></td>
</tr>
<tr>
<td>0.0012</td>
<td>0.0082</td>
<td>0.9971</td>
<td>-0.1420</td>
<td>-0.0748</td>
<td></td>
</tr>
<tr>
<td>0.0010</td>
<td>0.0026</td>
<td>0.0260</td>
<td>0.9874</td>
<td>0.0977</td>
<td></td>
</tr>
<tr>
<td>0.0005</td>
<td>0.0011</td>
<td>0.0042</td>
<td>0.0323</td>
<td>0.0287</td>
<td></td>
</tr>
<tr>
<td>0.0002</td>
<td>0.0009</td>
<td>0.0049</td>
<td>0.0010</td>
<td>0.0042</td>
<td></td>
</tr>
<tr>
<td>0.0000</td>
<td>0.0011</td>
<td>0.0141</td>
<td>0.0014</td>
<td>0.0008</td>
<td></td>
</tr>
<tr>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0008</td>
<td>0.0016</td>
<td>0.0008</td>
<td></td>
</tr>
<tr>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0003</td>
<td>0.0010</td>
<td>0.0011</td>
<td></td>
</tr>
</tbody>
</table>
Note that $s^p_k$ still needs to be determined by minimizing energy $U^p_{r2,k}$ irrespective of $u^p_{l1,k}$. Then shear deformation $\gamma^p_{r2,k}$ is expressed as follows in terms of $w_k$:

$$
\gamma^p_{r2,k} = \frac{C^p}{h_x^p l_x} \left[ \frac{dw_k(x)}{dx} - \frac{1}{l_x^p} w_k(x^p_a) + \frac{1}{l_x^p} w_k(x^p_b) \right].
$$

(4.25)

For a beam with arbitrary mode shape $\phi_{w,k}$, shear deformation $\phi^p_{r2,k}$ is expressed as follows by using equations (4.21, 4.24-25):

$$
\phi^p_{r2,k} = \frac{C^p}{h_x^p l_x} \left[ \frac{d\phi_{w,k}(x)}{dx} - \frac{1}{l_x^p} \phi_{w,k}(x^p_a) + \frac{1}{l_x^p} \phi_{w,k}(x^p_b) \right].
$$

(4.26)

Accordingly, the resulting shear strain energy is

$$
U^p_{r2,k} = \frac{G_x l_x (C^p)^2}{2 h_x^p l_x} \int \left( \frac{d\phi_{w,k}(x)}{dx} - \frac{1}{l_x^p} \phi_{w,k}(x^p_a) + \frac{1}{l_x^p} \phi_{w,k}(x^p_b) \right)^2 dx.
$$

(4.27)

Replace $w_k$ of equation (4.16) with $\phi_{w,k}$, ignore $U^p_{u13,k}$ in equation (4.15), and then apply equations (4.19) and (4.14) again to yield $\omega_k$ and $\eta_k$.

### 4.4.2 Validation

Approximate Method II is validated by re-examining examples of Table 4.2 and Figure 4.4. For the full coverage case, loss factors for modes 2 and 3 are calculated over a range of $G_2$ values by using Methods I and II; comparisons are shown in Figure 4.5. It is observed that modal loss factors obtained from both methods coincide only when $G_2$ is very small. For the second mode when $G_2$ is greater than $10^5$ Pa, Method II starts to overestimate and finally the result reaches asymptotically 0.1, which is equal to the material loss factor of layer 2. Conversely, lost factor prediction from Method I reaches a
Figure 4.5 Modal loss factor predictions as a function of $G_2$ for the full coverage case. Key: ... Approximate Method I, _._._._. Approximate Method II, o mode 2, and x mode 3.
maximum value at \( G_2 = G_{2,\text{opt}} = 10^6 \) Pa and then reduces as \( G_2 \) is increased. Note that \( G_{2,\text{opt}} \) is designated here as an optimum \( G_2 \) value that results in maximum possible loss factor for a particular mode. Similar results are seen for mode 3 except that \( G_{2,\text{opt}} \) now is about \( 3 \times 10^6 \) Pa and Method II is valid up to \( G_2 \approx 3 \times 10^6 \) Pa using a 10% error criterion.

Next, consider the partial coverage case \( (\tilde{I}^p = 0.14) \) of Figure 4.4 (b). Loss factors predicted for mode 2 are plotted in Figure 4.6. It is seen that \( G_{2,\text{opt}} \) shifts to a higher value and Method II is now valid up to \( G_2 = 5 \times 10^6 \) Pa. This suggests that Method II may be used up to higher \( G_2 \) values for even smaller patches. Another partial coverage case with a compact patch of \( \tilde{I}^p = 0.02 \) applied at the same location shows that Method II is indeed valid up to \( G_2 = 2 \times 10^8 \) Pa. Consequently the very compliant core assumption of Method II is not necessary when a very compact patch is applied.

### 4.5 Approximate Method III for a Compact Patch

#### 4.5.1 Formulation

Consider a compact patch of very small patch length \( \tilde{l}^p \). Expand \( \frac{d\phi_{w,k}(\tilde{x})}{d\tilde{x}} \) using Taylor series in the vicinity of the patch center \( \tilde{x}_p \), and obtain the following expression by ignoring higher order terms

\[
\frac{d\phi_{w,k}(\tilde{x})}{d\tilde{x}} \approx \frac{d\phi_{w,k}(\tilde{x}^p)}{d\tilde{x}} + \frac{d^2\phi_{w,k}(\tilde{x}^p)}{d\tilde{x}^2}(\tilde{x} - \tilde{x}^p). \tag{4.28}
\]

Also, observe that

\[
\frac{d\phi_{w,k}(\tilde{x}^p)}{d\tilde{x}} \approx \frac{1}{\tilde{l}^p} \left[ \phi_{w,k}(\tilde{x}_b^p) - \phi_{w,k}(\tilde{x}_a^p) \right] \tag{4.29}
\]
Figure 4.6  Loss factor predictions for mode 2 as a function of $G_2$ for partial coverage cases. Key: ... Approximate Method I, ____ Approximate Method II, o patch size $\tilde{l}^p = 0.14$, and x patch size $\tilde{l}^p = 0.02$. 
Substituting equations (4.28, 4.29) into (4.27), the resulting shear strain energy for the $k$th mode is

$$U_{r,k} = \frac{G_z l_y}{2h_2^3 l_x} \left( \frac{d^2 \phi_{w,k} (\bar{x})}{d\bar{x}^2} \right)^2 \int_{\bar{x}_1}^{\bar{x}_2} \left( \bar{x} - \bar{x}_p \right)^2 d\bar{x}$$

(4.30)

For a compact patch, total strain energy may be further approximated by considering only the flexural strain energy of the base beam, i.e. $U_{\text{total},k} = U_{w,3,k}$. Modal loss factor $\eta_k^p$ contributed by patch $p$ is then proposed as follows

$$\eta_k^p = \eta_{m,2} G^p \mathcal{H}^p \mathcal{L}^p \Phi(\bar{x}_p)$$

(4.31)

where

$$G^p = G_z r / E_3 = \text{relative stiffness term,}$$

(4.32 a-f)

$$\mathcal{H}^p = \left( C^p \right)^2 / \left( h_1^3 h_2^3 \right) = \text{thickness parameter}$$

$$\mathcal{L}^p = \left( \bar{I}_p \right)^3 = \text{patch size index}$$

$$\Phi(\bar{x}) = \left( \frac{d^2 \phi_{w,k} (\bar{x})}{d\bar{x}^2} \right)^2 = \text{patch performance index}$$

$$\Phi(\bar{x}) = \frac{\Phi(\bar{x})}{\Phi_0} = \text{normalized } \Phi(\bar{x})$$

$$\Phi_0 = \int_{\bar{x}} \left( \frac{d^2 \phi_{w,k} (\bar{x})}{d\bar{x}^2} \right)^2 d\bar{x}.$$
4.5.2 Validation studies

As described in the derivation of Method III, equations (4.31-32) can now be used to conduct parametric design studies and to determine optimum patch locations. The procedure is demonstrated here using the same beam example of Table 4.2.

First, examine the second mode (Figure 4.7 a) of the beam with simply supported boundary conditions. Normalized patch index $\hat{P}$ is calculated for mode 2 as a function of $\bar{x}$ as shown in Figure 4.7 (b). High values of $\hat{P}$ ($\bar{x}_{\text{max}} = 0.25$ and $\bar{x}_{\text{max}} = 0.75$) suggest best patch locations that should result in high modal damping. This is in agreement with an empirical design concept: place damping patches at anti-nodes. To verify this, a compact patch of length $l^p = 0.1$ is placed at various $\bar{x}$ along the beam. Modal loss factors are calculated using the Rayleigh-Ritz method [4.1] and then normalized with respect to the full coverage case. Observe excellent agreement between patch performance index and the Rayleigh-Ritz method. This demonstrates that the patch locations suggested by Method III are appropriate. Further, apply a larger patch of length $l^p = 0.3$ and calculate normalized modal loss factors of the beam with varying patch location using the Rayleigh-Ritz method. Figure 4.7 (c) again shows excellent agreement except for the case when the patch is placed at $\bar{x}_{\text{min}} = 0.5$. This discrepancy is due to the compact patch assumption of Method III and $\hat{P}(\bar{x}_{\text{min}})$ is virtually zero; however, modal loss factor is non-zero for a larger patch even when the patch is located at $\bar{x}_{\text{min}}$. Therefore, Method III should be viewed as an analytical tool for initiating design.

Next, the second mode of the cantilever beam is examined as shown in Figure 4.8 (a). Normalized patch index $\hat{P}$ is calculated for mode 2 as a function of $\bar{x}$ as shown in
Figure 4.7  Patch performance for the second mode of a simply supported beam when a compact patch of length $\bar{l}^p = 0.14$ is placed at various $\bar{x}$. (a) Mode shape, (b) $\bar{l}^p = 0.1$, (c) $\bar{l}^p = 0.3$. Here $\bar{x}_{max}$ is the location that yields maximum possible $\bar{\eta}_2$. Key: ___ normalized patch performance index $\hat{\Phi}$, ■ predicted $\bar{\eta}_2$. 

135
Figure 4.8 Comparison between patch performance index and Rayleigh-Ritz predictions for the second mode of a cantilever beam when a single patch is placed at various $\bar{x}$. Here $\bar{x}_{\text{max}}$ is the location that yields maximum possible $\bar{\eta}_2$. (a) Mode shape, (b) comparison between Methods I, III, and experiment, (c) Rayleigh-Ritz predictions. Key: ___ normalized patch performance index $\hat{\phi}$, □ measured $\bar{\eta}_2$, ■ predicted $\bar{\eta}_2$. 
Figure 4.8 (b). It is then compared with measured and predicted (using Method I) $\tilde{f}_2$ for the cantilever beam with a patch of length $\tilde{l}_p = 0.14$ placed at various $\tilde{x}$. Excellent agreement is again observed between Methods I, III and modal measurements. But note that the empirical design concept of placing damping patches at anti-nodes is not exactly valid for this case. It is seen in Figure 4.8 (a) that the anti-node of mode 2 is located at $\tilde{x}_{\text{max}} = 0.48$. However, according to the patch index, one of the relative maxima of $\tilde{\phi}$ is at $\tilde{x}_{\text{max}} = 0.53$. It is then verified by using the Rayleigh-Ritz method. Placement of damping patch at $\tilde{x} = 0.53$ gives higher modal damping than at $\tilde{x} = 0.48$ (Figure 4.8 c).

For higher modes, the high $\tilde{\phi}$ point is closer to an anti-node since such modes are less affected by the precise nature of boundary conditions. Hence insight based on simple supports may be a good starting point for design, but then Methods I or II may be used for detailed studies. Also note that another relative maximum at $\tilde{x}_{\text{max}} = 0$ is observed in Figure 4.8 (b). Since it is impossible to place a patch of finite length $\tilde{l}_p$ at $\tilde{x} = 0$ without exceeding the beam, the patch may be located near the root to obtain high damping performance.

4.6 Approximate Methods for Rectangular Plates

4.6.1 Approximate Method II given measured or computed mode shapes

Approximate Method II for beams with compliant core layer assumption (as presented in section 4.4) is now extended to a 2-D case. The structure of interest is a rectangular plate with multiple damping patches as illustrated in Figure 4.9. Assume that
Figure 4.9  Multiple constrained layer damping patches for a rectangular plate.
the measured or computed mode shape $\phi_{w,k}(x,y)$ is available. Shear strain $\phi_{\tau_{2,k}}$ in the $xz$ plane of layer 2 is approximated as follows when longitudinal motion is ignored

$$
\phi_{\tau_{2,k}}^p = -\frac{C_p}{h_1}\left[ \frac{d\phi_{w,k}(x,y)}{dx} - \frac{1}{L_x} \int_{y} d\phi_{w,k}(x,y)dx dy \right].
$$

Then strain energy $U_{\tau_{2,k}}^p$ due to $\phi_{\tau_{2,k}}^p$ is obtained as

$$
U_{\tau_{2,k}}^p = \frac{G_s l_y (C_p)^2}{2h^2 l_x} \left\{ \int_{x} \left[ \frac{d\phi_{w,k}(x,y)}{dx} \right]^2 dx dy - \frac{1}{L_y} \int_{y} \left[ \int_{x} \frac{d\phi_{w,k}(x,y)}{dx} dx dy \right]^2 \right\}.
$$

Similarly, shear strain $\phi_{\tau_{2,k}}^p$ in the $yz$ plane of layer 2 and its resulting strain energy $U_{\tau_{2,k}}^p$ are

$$
\phi_{\tau_{2,k}}^p = -\frac{C_p}{h_1}\left[ \frac{d\phi_{w,k}(x,y)}{dy} - \frac{1}{L_y} \int_{x} d\phi_{w,k}(x,y)dy dx \right],
$$

$$
U_{\tau_{2,k}}^p = \frac{G_s l_x (C_p)^2}{2h^2 l_y} \left\{ \int_{y} \left[ \frac{d\phi_{w,k}(x,y)}{dy} \right]^2 dy dx - \frac{1}{L_x} \int_{x} \left[ \int_{y} \frac{d\phi_{w,k}(x,y)}{dy} dy dx \right]^2 \right\}.
$$

Again, the total strain energy $U_{\text{total},k}$ is approximated as

$$
U_{\text{total},k} = U_{w_{1,k}} + \sum_{p=1}^{N} \left( U_{\tau_{1,k}}^p + U_{\tau_{2,k}}^p + U_{\tau_{2,k}}^p \right)
$$

where $U_{w_{1,k}}$ and $U_{w_{3,k}}$ are strain energies due to the flexural motion, defined as

$$
U_{w_{1,k}} = \frac{1}{2} \int_{x} \frac{E_p}{l_y} \left[ \frac{d^2 \phi_{w,k}}{dx^2} \right] \left[ \frac{d^2 \phi_{w,k}}{dx^2} \right] d^2 \phi_{w,k}
$$

$$
+ \frac{2
\nu_i}{l_x l_y} \left[ \frac{d^2 \phi_{w,k}}{dx^2} \right] \left[ \frac{d^2 \phi_{w,k}}{dy^2} \right] dx dy
$$

$$
+ \frac{2(1-\nu_p)}{l_x l_y} \left[ \frac{d^2 \phi_{w,k}}{dx^2} \right] \left[ \frac{d^2 \phi_{w,k}}{dy^2} \right] dx dy
$$

(4.37)
Finally, $\eta_k$ is obtained by using the modal strain energy method per equation (4.14).

### 4.6.2 Approximate Method III

Similar to the discussion of section 4.5.1, compact patches of very small lengths $l_x^p$ and widths $l_y^p$ are assumed, and $\frac{d\phi_{w,k}(\bar{x},\bar{y})}{dx}$ is again expanded using Taylor series where the higher order terms are ignored

$$\frac{d\phi_{w,k}(\bar{x},\bar{y})}{dx} = \frac{d\phi_{w,k}(\bar{x}^p,\bar{y})}{dx} + \frac{d^2\phi_{w,k}(\bar{x}^p,\bar{y})}{dx^2} (\bar{x} - \bar{x}^p)$$

(4.39)

Also observing that

$$\frac{d\phi_{w,k}(\bar{x}^p,\bar{y})}{dx} = \frac{1}{l_x^p l_y^p} \int d\phi_{w,k}(\bar{x},\bar{y}) d\bar{x} d\bar{y}$$

(4.40)

the resulting shear strain energy $U_{p\neq2,k}^p$ is approximated as

$$U_{p\neq2,k}^p = \frac{G_x l_y (C^p)^2}{2 h_x^p l_x^p} \left[ \frac{d^2\phi_{w,k}(\bar{x}^p,\bar{y}^p)}{dx^2} \right]^2 \frac{l_x^p (\bar{r}^p)^3}{12}$$

(4.41)

Similarly, shear strain energy $U_{p\neq2,k}^p$ is approximated as

$$U_{p\neq2,k}^p = \frac{G_x l_x (C^p)^2}{2 h_x^p l_x^p} \left[ \frac{d^2\phi_{w,k}(\bar{x}^p,\bar{y}^p)}{dy^2} \right]^2 \frac{l_x^p (\bar{r}^p)^3}{12}$$

(4.42)

Ignoring flexural strain energy due to the damping patch, a general expression of loss
factor is proposed as follows, like equation (4.31) for beams

\[
\eta_k^p = \eta_{m,2}^p G^p \mathcal{H}^p \left[ \mathcal{L}_{x}^p \hat{\phi}_x(\bar{x}, \bar{y}) + \mathcal{L}_{y}^p \hat{\phi}_y(\bar{x}, \bar{y}) \right]
\]  

(4.43)

where \( \mathcal{L}_x^p \) is the patch size index, and \( \hat{\phi}_x \) is the normalized patch performance index associated with shear strains \( \phi_{x,l,k}^p \). Similarly indices \( \mathcal{L}_y^p \) and \( \hat{\phi}_y \) are associated with shear strains \( \phi_{y,l,k}^p \). In addition, thickness parameter \( \mathcal{H}^p = \left( C^p \right)^2 / \left( h_3^2 h_2^p \right) \) is the same as in section 4.5.1 while relative stiffness \( G^p \) term is now equal to \( G_2^p (1 - \nu_3^2) / E_3 \). Patch size indices, \( \mathcal{L}_x^p \) and \( \mathcal{L}_y^p \), are obtained as follows

\[
\mathcal{L}_x^p = \bar{I}_x^{p} \left( \bar{I}_x^p \right)^2, \quad \mathcal{L}_y^p = \bar{I}_y^{p} \left( \bar{I}_y^p \right)^2.
\]  

(4.44)

Patch performance indices and their normalized forms are

\[
\varphi_x(\bar{x}, \bar{y}) = \left( \frac{d^2 \phi_{x,k} (\bar{x}, \bar{y})}{l_x^2 \frac{d^2}{dx^2}} \right)^2; \quad \bar{\varphi}_x(\bar{x}, \bar{y}) = \left( \frac{d^2 \phi_{x,k} (\bar{x}, \bar{y})}{l_x^2 \frac{d^2}{dy^2}} \right)^2
\]  

(4.45 a-d)

\[
\hat{\varphi}_x(\bar{x}, \bar{y}) = \frac{\varphi_x(\bar{x}, \bar{y})}{\varphi_0}; \quad \hat{\varphi}_y(\bar{x}, \bar{y}) = \frac{\varphi_y(\bar{x}, \bar{y})}{\varphi_0}
\]

where

\[
\varphi_0 = \int_0^1 \int_0^1 \left[ \frac{1}{l_x^4} \left( \frac{d^2 \phi_{x,k}}{dx^2} \right)^2 + \frac{1}{l_y^4} \left( \frac{d^2 \phi_{y,k}}{dy^2} \right)^2 \right. \\
\left. + \frac{2\nu_3}{l_x^2 l_y^2} \left( \frac{d^2 \phi_{x,k}}{dx^2} \frac{d^2 \phi_{y,k}}{dy^2} \right) + \frac{2(1 - \nu_3)}{l_x^2 l_y^2} \left( \frac{d^2 \phi_{x,k}}{dx dy} \right)^2 \right] d\bar{x} d\bar{y}.
\]  

(4.46)

Finally, define combined patch performance index \( \varphi \) and its normalized value \( \hat{\varphi} \) as follows to quantify the global effect of damping but only for a specific mode, over the entire surface of vibrating plate.
\[ \varphi(\tilde{x}, \tilde{y}) = \varphi_x(\tilde{x}, \tilde{y}) + \varphi_y(\tilde{x}, \tilde{y}). \] \hspace{1cm} (4.47 a. b)

\[ \hat{\varphi}(\tilde{x}, \tilde{y}) = \hat{\varphi}_x(\tilde{x}, \tilde{y}) + \hat{\varphi}_y(\tilde{x}, \tilde{y}). \]

### 4.7 Examination of Plate Formulations

#### 4.7.1 Validation studies

A rectangular plate with free boundary at all edges is considered here for experimental verification. System parameters of the plate and the damping patches are summarized in Table 4.6. The undamped mode shapes are calculated using a Rayleigh-Ritz method of Chapter 3 prior to the application of Method III. In particular, mode (2.1) as shown in Figure 4.10 is considered for the sake of illustration. As seen in Figure 4.11 (a) and (b), patch performance indices \( \varphi_x \) and \( \varphi_y \) are calculated over the plate surface respectively from the undamped mode shape. Note that the dark areas imply high \( \varphi_x \) or \( \varphi_y \) values; these suggest locations that should result in high damping performance. Since \( \varphi_x \) is about one order of magnitude higher than \( \varphi_y \), the combined index \( \varphi \) of Figure 4.11 (c) is dominated by \( \varphi_x \). Consider a preliminary design: cover the dark areas of \( \varphi \) by applying two identical damping patches of \( \tilde{l}_x^p = 0.4 \) and \( \tilde{l}_y^p = 0.25 \) and locate them at \((\tilde{x}_x, \tilde{y}_y) = (0.5, 0.125) \) and \((0.5, 0.785) \) as shown in Figure 4.12 (a). This configuration is designated as Pattern A. Two alternate configurations, designated as Patterns B and C, are also considered in Figure 4.12 (b) and (c) respectively. Predicted modal loss factors for all three cases are then obtained using Approximate Methods II and III. These are compared with modal measurements and Rayleigh-Ritz predictions in Table 4.7. Results indicate that Pattern A indeed yields the highest system damping for this particular mode.
### Table 4.6 System parameters for the rectangular plate example.

<table>
<thead>
<tr>
<th>Material</th>
<th>Layer 1</th>
<th>Layer 2</th>
<th>Layer 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stiffness (N/m²)</td>
<td>$E_1 = 180 \times 10^9$</td>
<td>$G_2 = 0.25 \times 10^6$</td>
<td>$E_3 = 180 \times 10^9$</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu_1 = 0.3$</td>
<td>$\nu_2 = 0.45$</td>
<td>$\nu_3 = 0.3$</td>
</tr>
<tr>
<td>Density (kg/m³)</td>
<td>$\rho_1 = 7720$</td>
<td>$\rho_2 = 2000$</td>
<td>$\rho_3 = 7350$</td>
</tr>
<tr>
<td>Material loss factor</td>
<td>$\eta_{m,1} &lt;&lt; \eta_{m,2}$</td>
<td>$\eta_{m,2} = 0.8$</td>
<td>$\eta_{m,3} &lt;&lt; \eta_{m,2}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dimensions</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (mm)</td>
<td>varies</td>
<td>varies</td>
<td>$l_x = 342.9$</td>
</tr>
<tr>
<td>Width (mm)</td>
<td>varies</td>
<td>varies</td>
<td>$l_y = 266.7$</td>
</tr>
<tr>
<td>Thickness (mm)</td>
<td>$h_1 = 0.79$</td>
<td>$h_2 = 0.051$</td>
<td>$h_3 = 2.4$</td>
</tr>
</tbody>
</table>

### Table 4.7 Comparison between measured and predicted modal loss factors for the rectangular plate of Figure 4.10 with free boundaries. See Table 4.6 for parameters.

<table>
<thead>
<tr>
<th>Damping patch</th>
<th>Experiment</th>
<th>Theory Rayleigh-Ritz Method</th>
<th>Approximate Method II</th>
<th>Approximate Method III</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pattern A</td>
<td>1.9%</td>
<td>1.5%</td>
<td>1.7%</td>
<td>2.0%</td>
</tr>
<tr>
<td>Pattern B</td>
<td>1.3%</td>
<td>1.2%</td>
<td>1.4%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Pattern C</td>
<td>0.7%</td>
<td>0.7%</td>
<td>0.9%</td>
<td>0.0%</td>
</tr>
</tbody>
</table>
Figure 4.10 Deformation shape of mode (2, 1) of a rectangular plate with free boundaries. Dotted lines represent nodal lines. (a) 2-D plot, (b) 3-D plot.
Figure 4.11 The (2,1) mode shape and corresponding patch performance indices for the plate example of Table 4.6 with free boundaries. (a) Flexural mode \( \phi_w \), (b) \( P_x \), (c) \( P_y \), (d) \( P \).
Figure 4.12 Damping patch configurations studied for the rectangular plate example of Table 4.6 with free boundaries. (a) Pattern A, (b) Pattern B, (c) Pattern C.
Observe an excellent agreement between theory and experiment which confirms the utility of Method II. Also note that Table 4.7 suggests that modal loss factors for Patterns B and C are zero only when Method III is used for prediction. This is because that both $\varphi_x$ and $\varphi_y$ are zero for patches located along the line of $l_y = 0.5$; this is similar to the discussion of $\varphi(x_{in}) = 0$ in section 4.5.2.

### 4.7.2 Parametric studies

Effects of patch lengths and widths are investigated next by using Approximate Method II. Consider only one patch and locate it at $(\bar{x}, \bar{y}) = (0.5, 0.125)$. Fix width as $\bar{l}_y = 0.25$ and vary axial length $\bar{l}_x$ from 0 to 1. It is seen from Figure 4.13 (a) that loss factor for mode $(2, 1)$ increases as the patch length is increased and the relationship gradually becomes linear as the asymptotic curve suggests. Then, fix length $\bar{l}_x = 0.4$ and vary patch width $\bar{l}_x$ from 0 to 1. Figure 4.13 (b) shows that $\eta$ varies linearly first, then settles down when $\bar{l}_y$ is greater than 0.3, and finally increases for $\bar{l}_y > 0.7$. Three asymptotic lines approximate the curve rather well. The flat asymptotic line also reflects the low $\varphi$ region of Figure 4.11 (c) at the center of the plate; this is an inefficient location of damping patch.

Such asymptotic results are rather problem specific, as evident from yet another example of Figure 4.14 where a simply supported square plate is considered. Material properties of the patches and plate of Table 4.6 are the same as the previous plate with free boundaries. Loss factor of mode $(3,3)$ is calculated using Method II when patch
Figure 4.13  Effect of patch dimensions for the rectangular plate of Table 4.6 with free boundaries. (a) Varying patch length, (b) varying patch width. Key: ___ Prediction, — asymptote.
Figure 4.14 Effect of patch dimensions for the rectangular plate of Table 4.6 with simply supported edges. (a) Varying patch length, (b) varying patch width. Key: ___ Prediction, --- asymptote.
width $\bar{l}_y^p$ is fixed as 0.2 and axial length $\bar{l}_x^p$ is varying from 0 to 1. It is seen in Figure 4.14 (a) that loss factor increases with some fluctuations when $\bar{l}_x^p$ is increased. Similar results are seen in Figure 4.14 (b) when $\bar{l}_x^p$ is fixed at 0.2 and $\bar{l}_y^p$ is varying from 0 to 1. Again, these fluctuations can be explained using the patch performance index $\Phi$ or the anti-node design concept.

The final parametric study is used to verify the patch performance index that suggests the best patch location(s). Once again consider the plate with free boundaries. Figure 4.15 shows the possible locations for a patch of size $\bar{l}_x^p = 0.4$ and $\bar{l}_y^p = 0.25$. Method II is again used to calculate loss factor for mode (2, 1) over the region of possible patch locations. The 3-D loss factor map of Figure 4.16 shows that $(\bar{x}, \bar{y}) = (0.5, 0.125)$ and $(0.5, 0.785)$ are indeed the best locations for damping patch of the size mentioned above. But locations $(\bar{x}, \bar{y}) = (0.2, 0.5)$ and $(0.8, 0.5)$ may also be considered as candidates when four damping patches are to be applied. Similar studies may be carried out for other modes, using Methods II and III.

4.8 Conclusion

This chapter has developed, examined, and validated three approximate methods for damping patch design studies. Given eigenfunctions of an undamped Euler beam, Method I avoids solving complex eigenvalue problems but still yields accurate results. Also, it is more than 50 times faster than Rayleigh-Ritz methods of Chapter 2 depending on the number of trial functions used. Method II further assumes a very compliant core and is computationally much faster than Method I. Method III is limited to a single
Figure 4.15 Possible patch locations on the rectangular plate with free boundaries for a detailed parametric study.
Figure 4.16  The 3-D loss factor map for mode (2, 1) of the rectangular plate of Table 4.6 and Figure 4.15 with free boundaries.
compact patch application and the resulting patch index calculation is based on only one algebraic equation. Finally, Methods II and III are successfully extended to rectangular plates and the advantage in computational speed over the Rayleigh-Ritz method of Chapter 3 is even more significant.

Based on the material presented in this chapter, the following design procedure is suggested.

1. Select a particular mode of beam or plate. Predict and plot patch performance indices using Method III over the entire surface. Placing a damping patch at or near anti-nodes as a starting point. Choose a preliminary design concept including the number of patches and their locations in accordance with predictions.

2. Using Method II for a plate and Methods I or II for a beam, perform parametric design studies on the best possible locations for preliminary patches.

3. Modify design if necessary and perform parametric studies. Achieve desirable damping value for a given mode of interest.

4. Iterate procedures for other modes of interest and determine the best design, using these methods repeatedly, that provides most damping over the frequency range of interest.

5. Use Rayleigh-Ritz method to confirm the final design concept. Validate design by conducting modal or vibro-acoustic measurements.

Chief contribution of this chapter is the development of analytically simpler formulations that yield reasonable accurate results, in a computationally efficient manner, while providing much insight into the patch damping design concepts. Future work may
include the development of an optimization scheme that considers several modes. Dynamic scaling issues for viscoelastically damped structures will also be addressed using the explicit form of modal loss factor, as described by Method III.
References for Chapter 4


CHAPTER 5

CONCLUSION

5.1 Summary and Contributions

This study has led to the development of several new analytical methods for beams and rectangular plates treated with multiple constrained-layer viscoelastic patches. These methods include a comprehensive analytical model based on Rayleigh-Ritz scheme for detailed analyses and three simplified models based on the modal strain energy method for examining patch damping design concepts. In the Rayleigh-Ritz approach, kinematic relationships are first derived between flexural displacement and other deformations in all layers. Then only flexural shape functions are needed in the complex eigenvalue problem. Nonetheless, eigenvectors can be related to flexural, longitudinal and shear mode shapes, some of which can not be experimentally measured. The visualization of shear deformations of the viscoelastic core, which are the major contributors to the overall energy dissipation, provides an improved understanding of the effect of patch damping. Subsequently, three simpler analytical formulations are developed without explicitly solving high order differential equations or complex eigenvalue problems. These methods yield reasonably accurate results in a computationally efficient manner while they provide much insight into the patch damping design concepts. Overall, the major contributions of this dissertation are:
1. New analytical models of beams and rectangular plates with multiple constrained layer damping patches have been developed to predict natural frequencies, loss factors, and mode shapes.

2. The proposed methods have been validated by comparing results with experimental measurements and with those published in the literature for simple supports. Excellent agreement between theory and experiment is seen for all examples. Some important patch damping issues have been clarified especially through an examination of modal deformations.

3. A refined estimation technique has been developed to obtain unknown material properties of viscoelastic materials. Frequency-dependent material loss factor and stiffness are estimated by combining analytical parametric studies with modal measurements from beam tests. An uncertainty study has also been carried out to establish the error bounds of these estimations.

4. Three approximate analytical methods have been developed for patch damping design concepts. These are computationally efficient and suitable for rapid parametric design studies. Significant improvement in computational speed over the Rayleigh-Ritz method is seen, especially for plate problems.

5.2 Future research

This dissertation develops an analytical framework for dealing with multiple damping patches. Many research issues still remain unsolved. For example, the thick beam or plate effect that was neglected during the derivation of governing equations should be
examined in a future investigation. Some other issues, as listed below, should be addressed:

1. Extend analytical formulations, including Rayleigh-Ritz approach and three simpler formulations, to annular plate-like structures that simulate many "real-life" machine elements such as gears, disks and clutches [5.1].

2. Calculate forced harmonic responses of structures with multiple damping patches.

3. Develop dynamic scaling concepts [5.2] for viscoelastically damping structures. This may be achieved by using the explicit form of modal loss factor, as described in Chapter 4.

4. Develop a numerical scheme that suggests an optimum number of patches, their locations, dimensions, and material properties for best possible damping performance a frequency range of interest covering over several modes.

5. Calculate structural intensities [5.2, 5.3] for viscoelastically damped structures by incorporating the complex mode shapes obtained from the proposed formulations.

6. Extend Approximate Method I of Chapter 4 to structures with more complicated geometry by incorporating measured or finite element predicted mode shapes.
Reference for Chapter 5

5.1 H. Vinayak and R. Singh 1996 *Journal of Sound and Vibration* 192, 741-769. Eigensolutions of annular-like elastic disks with intentionally removed or added material.


5.3 A. B. Spalding and J. A. Mann III 1995 *Journal of the Acoustical Society of America* 97, 3617-3624. Placing Small Constrained layer Damping Patches on a Plate to Attain Global or Local Velocity Changes.

5.4 A. Nejadi and R. Singh 1997 personal communication regarding structural intensity measurements.


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