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FREQUENCY RESPONSE OF LAMINATED COMPOSITE PLATES
AND SHELLS WITH MATRIX CRACKS TYPE OF DAMAGE MODE

DISSERTATION

Presented In Partial Fulfillment Of The Requirements
For The Degree Doctor Of Philosophy In The Graduate
School Of The Ohio State University

By

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1998

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ABSTRACT

The present study has been designed to tackle a new set of problems for structural composites, as these materials are finding new applications in civil engineering field. An attempt has been made to study the frequency response of laminated polymer composite plates and shallow shells containing matrix cracks type of damage with arbitrary support conditions and free vibratory motions. The shell governing equations are derived using a simplified shallow shell theory based on a first order shear deformation field. The continuum damage mechanics approach has been used to model the matrix cracks in a damaged region within the plates and shallow shells. In such approach, the damage is accounted for in the laminate constitutive equations by using a set of second order tensor internal state variables which are strain-like quantities. The simplified damage model was then used to study the changes in frequency response of laminated composite plates and shallow cylindrical shells. The Ritz method and a finite element method have been proposed and developed as approximate solution procedures to quantify the change in the free vibration frequencies due to matrix cracks type of damage under both material as well as geometrical variables such as size, shape and extent of damage, degree of curvature, ratio of orthotropy, thickness ratio as well as support conditions. The analysis of various
plates and shells with a centrally located damaged-zone depicts a typical trend of reduction in the vibration frequencies. This reduction is more pronounced for higher frequency modes and it shows greater sensitivity toward the size of the damaged region and density of cracks. The results also show that the changes in the frequency, especially for the fundamental mode, appear to be less sensitive to the shell boundary conditions as well as small values of curvature.

The investigation of various undamaged plates and shallow shells demonstrates the importance of a first-order shear deformation theory. The present treatment of the effect of matrix cracks is more realistic and economical than the standard ply-discount strategy which under-estimates the residual stiffness of damaged plates and shells.
To those whom I love from the bottom of my heart and they know it!

iv
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There are few moments in life into which one can not express his true feeling by words. For me to write a word of acknowledgment is one of these moments. During the long journey of this research work, the support, the encouragement, the guidance of my advisor, Dr. Shive K. Chaturvedi, were overwhelming. He did all of with kindness, patience and understanding. Being grateful is not enough for such rare qualities in dealing and interaction. His help and inspiration throughout the development of this work will always be in mind for the rest of my life.

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# TABLE OF CONTENTS

**ABSTRACT** ............................................................................................................................ ii

**DEDICATION** ...................................................................................................................... iv

**ACKNOWLEDGMENTS** ..................................................................................................... v

**VITA** .................................................................................................................................... vii

**LIST OF TABLES** ............................................................................................................. xiv

**LIST OF FIGURES** ............................................................................................................. xvii

**NOMENCLATURE** ............................................................................................................ xxi

## CHAPTERS

1. **INTRODUCTION** ........................................................................................................... 1
   - 1.1 General .................................................................................................................... 1
   - 1.2 Motivation of the Present Work ........................................................................ 2
   - 1.3 Primary Objectives and Scope .......................................................................... 5
   - 1.4 Present Work Outlines .................................................................................... 6

2. **LITERATURE REVIEW** ................................................................................................. 12
   - 2.1 Introduction ......................................................................................................... 12
   - 2.2 The Undamaged Zone ....................................................................................... 14
       - 2.2.1 Introduction ............................................................................................... 14
       - 2.2.2 The Governing Equations ......................................................................... 19
           - 2.2.2.1 Laminated Composite Shell ................................................................. 19
           - 2.2.2.2 Laminated Composite Plate ................................................................. 28
   - 2.3 Solution Procedure ............................................................................................ 32
       - 2.3.1 General ........................................................................................................ 32
1. Introduction ........................................................................................................... 1
2. The Boundary Conditions ..................................................................................... 2
3. The Continuity Conditions ................................................................................... 3
4. The Damaged Zone Modeling .............................................................................. 3
5. Summary of damage modeling ............................................................................ 4
6. Concluding Remarks ........................................................................................... 5

3. MODELING OF DAMAGE .................................................................................. 23

3.1 General ............................................................................................................. 23
3.2 Matrix Cracks Modeling .................................................................................. 24
3.3 Expressions for Elastic Constants ...................................................................... 26

4. ENERGY FORMULATION FOR LAMINATED COMPOSITE SHEAL SHELLS ....... 33

4.1 Introduction ..................................................................................................... 33
4.2 Strain Energy .................................................................................................. 34
4.3 Kinetic Energy ................................................................................................ 38
4.4 Specializations ................................................................................................ 39
4.4.1 Simplified Shallow Shell .......................................................................... 39
4.4.2 Spherical Shell ......................................................................................... 42
4.4.3 Cylindrical Shell ..................................................................................... 43
4.4.4 Laminated Composite Plate .................................................................... 44
4.5 The Governing Equations of Motion .............................................................. 47
4.5.1 Shallow Shell .......................................................................................... 47
4.5.2 Simplified Shallow Shell ......................................................................... 53
4.5.3 Flat Plate ................................................................................................. 56
4.6 Closure ............................................................................................................ 57

5. RITZ METHOD FOR LAMINATED SHALLOW SHELLS AND PLATES WITH MATRIX CRACKS ................................................................................ 71

5.1 Introduction .................................................................................................... 71
5.2 Special Case .................................................................................................... 72
5.3 The Ritz Method ............................................................................................. 73
5.3.1 General .................................................................................................... 73
5.3.2 Displacement Functions ......................................................................... 74
5.3.3 Closure ..................................................................................................... 77
5.4 Results and Discussion ................................................................................... 78
5.4.1 Sensitivity Analysis ............................................................................... 78
5.4.2 Undamaged Shallow Shells......................................................... 131
5.4.3 Damaged Shallow Shells........................................................... 133
5.5 Concluding Remarks................................................................................ 135
6. FINITE ELEMENT ANALYSIS........................................................................... 166
6.1 Introduction.................................................................................................. 166
6.2 Finite Element Formulation ........................................................................ 168
  6.2.1 Theoretical Development.................................................................. 168
  6.2.2 Finite Element Idealization ............................................................ 171
  6.2.3 Solution Procedure.......................................................................... 174
6.3 Results and Discussion ............................................................................... 176
  6.3.1 Sensitivity Analysis ...................................................................... 176
  6.3.2 Undamaged Plate Analysis ........................................................... 177
  6.3.3 Damaged Plate Analysis ............................................................... 178
6.4 Concluding Remarks .................................................................................. 181
7. CONCLUSIONS AND RECOMMENDATIONS................................................. 206
  7.1 Concluding Remarks................................................................................. 206
  7.2 Future Research ...................................................................................... 211
APPENDICES.................................................................................................................... 214
  Appendix A: Description of the Internal State Variables......................... 214
  Appendix B: Enforcing the Geometric Boundary Conditions.................... 221
  Appendix C: Evaluation of Element Matrices ............................................. 225
  Appendix D: Computer Programs structures ........................................... 233
BIBLIOGRAPHY................................................................................................................... 237
# LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Frequency coefficients for an isotropic simply supported plate, ([86]. a/b=1 , a/h=10 , v = 0.3)</td>
<td>59</td>
</tr>
<tr>
<td>2.2 Frequency coefficients for an isotropic simply supported plate, ([86]. a/b=2 , a/h=10 , v = 0.3)</td>
<td>60</td>
</tr>
<tr>
<td>2.3 Frequency coefficients for a simply supported cross-ply laminated plate, ([86], a/b=1, E_1/E_2 = 25, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, \nu_{12} = 0.5)</td>
<td>61</td>
</tr>
<tr>
<td>2.4 Possible combinations of boundary conditions on each edge, ([14])</td>
<td>62</td>
</tr>
<tr>
<td>3.1 Changes in (E_x) and (\nu_{xy}) due matrix cracking in the 90°-plies of ([0°/90°])s laminate</td>
<td>85</td>
</tr>
<tr>
<td>3.2 Comparison of normalized axial stiffness ((E_y/E_x^0)) due to matrix cracking in the 90°-plies</td>
<td>86</td>
</tr>
<tr>
<td>3.3 Comparison of normalized axial stiffness ((E_y/E_x^0)) due to matrix cracking in the ±45°-plies</td>
<td>86</td>
</tr>
<tr>
<td>5.1 Convergence of the frequency parameter (\lambda^*) for graphite-epoxy laminated composite simply supported plate, ([0°/90°])s, a/b=1, a/h=100</td>
<td>153</td>
</tr>
<tr>
<td>5.2 Convergence of the frequency parameter (\lambda^*) for graphite-epoxy laminated composite completely free shallow cylindrical shell, ([±30°])s, a/b=1, a/h=100, a/R=0.5</td>
<td>154</td>
</tr>
</tbody>
</table>
5.3 Convergence of the frequency parameter $\lambda^*$ for graphite-epoxy laminated composite completely free shallow cylindrical shell, $[\pm 45^\circ]_s$, $a/b=1$, $a/h=100$, $a/R=0.5$.

5.4 Frequency parameter $(\lambda)^*$ for a square specially orthotropic clamped plate.

5.5 Variation of frequency parameter $\lambda^*$ with $a/R$ for graphite-epoxy laminated composite simply supported cylindrical shell, $[0^\circ/90^\circ]_s$, $a/b=1$, $a/h=100$, $E_{11}/E_{22}=15$, $G_{12}/E_{22}=0.5$, $\nu_{12}=0.25$, $k^2=5/6$.

5.6 Variation of frequency parameter $\lambda^*$ with $a/R$ for graphite-epoxy laminated composite simply supported cylindrical shell, $[0^\circ/90^\circ]_s$, $a/b=1$, $a/h=20$, $E_{11}/E_{22}=15$, $G_{12}/E_{22}=0.5$, $\nu_{12}=0.25$, $k^2=5/6$.

5.7 Variation of frequency parameter $\lambda^*$ with $a/R$ for graphite-epoxy laminated composite simply supported cylindrical shell, $[0^\circ/90^\circ]_s$, $a/b=1$, $a/h=10$, $E_{11}/E_{22}=15$, $G_{12}/E_{22}=0.5$, $\nu_{12}=0.25$, $k^2=5/6$.

5.8 Variation of frequency parameter $\lambda^*$ with $a/h$ and $a/R$ for graphite-epoxy laminated composite simply supported cylindrical shell, $[0^\circ/90^\circ]_s$, $a/b=1$, $E_{11}/E_{22}=15$, $G_{12}/E_{22}=0.5$, $\nu_{12}=0.25$, $k^2=5/6$.

5.9 Variation of the percentage of error in the present shallow cylindrical shell solution from the exact one with $(a/R)$ and $(a/h)$, $[0^\circ/90^\circ]_s$, $E_{11}/E_{22}=15$, $G_{12}/E_{22}=0.5$, $\nu_{12}=0.25$, $k^2=5/6$.

5.10 Variation of frequency parameter $\lambda^*$ with orthotropy ratio $E_{11}/E_{22}$ for graphite-epoxy laminated composite simply supported cylindrical shell, $[0^\circ/90^\circ]_s$, $a/b=1$, $a/h=100$, $a/R=0.3$, $G_{12}/E_{22}=0.5$, $\nu_{12}=0.25$, $k^2=5/6$.

5.11 Effect of damage on the frequency parameter $\lambda^*$ for graphite-epoxy laminated composite completely free shallow cylindrical shell, $[\pm 45^\circ]_s$, $a/b=1$, $a/h=100$, $a/R=0.5$.

5.12 Effect of damage on the frequency parameter $\lambda^*$ for graphite-epoxy laminated composite completely free shallow cylindrical shell, $[\pm 30^\circ]_s$, $a/b=1$, $a/h=100$, $a/R=0.5$.

5.13 Effect of damage on the frequency parameter $\lambda^*$ with $a/h$ and $a/R$ for
graphite-epoxy laminated composite simply supported cylindrical shell, 
\([0^\circ/90^\circ]_s\), \(a/b=1\), \(E_{11}/E_{22}=15\), \(G_{12}/E_{22}=0.5\), \(v_{12}=0.25\), \(k^2=5/6\) ........................................................................................................ 165

6.1 Material properties ........................................................................................................ 196

6.2 Comparison of Finite Element results with experimental work, [20],
for the fundamental frequency ...................................................................................... 197

6.3 Comparison of Finite Element results with experimental work, [20],
for the second mode frequency .................................................................................... 197

6.4 Comparison of Finite Element results with Abaqus for the fundamental frequency................................................................................................................................. 198

6.5 Comparison of Finite Element results with Abaqus for the second mode frequency................................................................................................................................. 198

6.6 Frequency parameter \((\lambda)^*\) for a square specially orthotropic clamped plate ................................................................................................................................. 199

6.7 Effect of \((h/a)\) on the frequency parameter \((\lambda)^*\) for a square simply supported specially orthotropic plate ................................................................................................................................. 200

6.8 Effect of \((E_{11}/E_{22})\) on the frequency parameter \((\lambda)^*\) for a square simply supported specially orthotropic plate (First mode) ................................................................................................................................. 201

6.9 Effect of the damaged zone shape on the fundamental frequency of crossply laminate ................................................................................................................................. 202

6.10 Frequency parameter \((\lambda)^*\) for a square simply supported specially orthotropic plate ................................................................................................................................. 203

6.11 Frequency parameter \((\lambda)^*\) for a square specially orthotropic clamped plate ................................................................................................................................. 204

6.12 Frequencies (hz) of a cantilever 8-ply graphite-epoxy 6x3 in. 
\([0_2/\pm 30]_s\) plate ........................................................................................................... 205

xvi
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Laminated composite plate geometry and coordinate system</td>
</tr>
<tr>
<td>1.2</td>
<td>Damage Modes of Composite Laminate</td>
</tr>
<tr>
<td>1.3</td>
<td>The motivation of the present study</td>
</tr>
<tr>
<td>1.4</td>
<td>The outline for the present work</td>
</tr>
<tr>
<td>2.1</td>
<td>Different damage modes for laminated composite plates. (a) matrix cracks, (b) delamination, and (c) fiber fracture</td>
</tr>
<tr>
<td>2.2</td>
<td>Coordinate system for a damaged composite laminate</td>
</tr>
<tr>
<td>2.3</td>
<td>Coordinate system and geometry for composite laminated shell</td>
</tr>
<tr>
<td>2.4</td>
<td>Shallow shell geometry and coordinate system</td>
</tr>
<tr>
<td>2.5</td>
<td>A typical stacking of a laminated shell</td>
</tr>
<tr>
<td>2.6</td>
<td>Effects of shear deformations and rotary inertia on the natural frequency of laminated plates, [110]</td>
</tr>
<tr>
<td>2.7</td>
<td>Deformation geometry due to delamination, [4]</td>
</tr>
<tr>
<td>2.8</td>
<td>Experimental and theoretical damage shapes and sizes [33]</td>
</tr>
<tr>
<td>3.1</td>
<td>Description of the crack geometry at point A</td>
</tr>
<tr>
<td>3.2</td>
<td>Matrix cracking and local coordinates for a single ply</td>
</tr>
<tr>
<td>3.3</td>
<td>Composite laminate local (1-2) and global (X₁-X₂) coordinate systems</td>
</tr>
<tr>
<td>3.4</td>
<td>Variation of normalized stiffness with the number of cracks, [0/90]₅</td>
</tr>
</tbody>
</table>
3.5 Variation of normalized stiffness with the number of cracks, \([0/90_2]_s\).............83
3.6 Variation of normalized stiffness with the number of cracks, \([0/90_3]_s\).............84
5.1 Coordinate system and geometry for composite laminated shell .......................138
5.2 Planform for a shallow shell with centrally elliptical damaged zone.............139
5.3 A typical circular cylindrical shell with damage ............................................140
5.4 Variation of frequency parameter \(\lambda = \alpha a^2 \sqrt{\rho / E_{11} h^2}\) with \(a/R\) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_s\), \(a/b=1, a/h=100, E_{11}/E_{22}=15, G_{12}/E_{22}=0.5, \nu_{12}=0.25, k^2=5/6\)..................................................................................141
5.5 Variation of frequency parameter \(\lambda = \alpha a^2 \sqrt{\rho / E_{11} h^2}\) with \(a/R\) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_s\), \(a/b=1, a/h=100, E_{11}/E_{22}=15, G_{12}/E_{22}=0.5, \nu_{12}=0.25, k^2=5/6\)..................................................................................142
5.6 Variation of frequency parameter \(\lambda = \alpha a^2 \sqrt{\rho / E_{11} h^2}\) with \(a/R\) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_s\), \(a/b=1, a/h=20, E_{11}/E_{22}=15, G_{12}/E_{22}=0.5, \nu_{12}=0.25, k^2=5/6\)..................................................................................143
5.7 Variation of frequency parameter \(\lambda = \alpha a^2 \sqrt{\rho / E_{11} h^2}\) with \(a/R\) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_s\), \(a/b=1, a/h=10, E_{11}/E_{22}=15, G_{12}/E_{22}=0.5, \nu_{12}=0.25, k^2=5/6\)..................................................................................144
5.8 Variation of frequency parameter \(\lambda = \alpha a^2 \sqrt{\rho / E_{11} h^2}\) with \(a/R\) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_s\), \(a/b=1, a/h=10, E_{11}/E_{22}=15, G_{12}/E_{22}=0.5, \nu_{12}=0.25, k^2=5/6\)..................................................................................145
5.9 Variation of the percentage of error in the present shallow cylindrical shell solution from the exact one with \((a/R)\) and \((b/a)\), \([0^\circ/90^\circ]_s\), \(E_{11}/E_{22}=15, G_{12}/E_{22}=0.5, \nu_{12}=0.25, k^2=5/6\)..................................................................................146
5.10 Relative change of frequency parameter \(\lambda = \alpha a^2 \sqrt{\rho / E_{11} h^2}\) with reference to mode number for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_s\), \(a/b=1, a/h=100, a/R=0.3, G_{12}/E_{22}=0.5, \nu_{12}=0.25, k^2=5/6\)..................................................................................147
5.11 % deviation in the frequency parameter \(\lambda = \alpha a^2 \sqrt{\rho / E_{11} h^2}\)
5.12 Effect of the shape and size of damage on the frequency parameter
\( \lambda = \omega a^2 \sqrt{\frac{\rho}{E_{11}} h^2} \) with \( a/h \) and \( a/R \) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_s\), \( a/b=1, a/h=100, a/R=0.5, N=40 \) cracks/in. .......................................................... 149

5.13 Effect of the shape and size of damage on the frequency parameter
\( \lambda = \omega a^2 \sqrt{\frac{\rho}{E_{11}} h^2} \) for graphite-epoxy laminated composite completely free shallow cylindrical shell, \([\pm 45^\circ]_s\), \( a/b=1, a/h=100, a/R=0.5, N=20 \) cracks/in ............................................................ 150

5.14 Effect of damage on the fundamental frequency parameter
\( \lambda = \omega a^2 \sqrt{\frac{\rho}{E_{11}} h^2} \) for graphite-epoxy laminated composite simply supported shallow cylindrical shell, \([0^\circ/90^\circ]_s\), \( a/b=1, a/h=100, a/R=0.5, N=40 \) cracks/in., \( A_d=30\% \) .................................................. 151

5.15 Effect of damage on the fifth mode frequency parameter
\( \lambda = \omega a^2 \sqrt{\frac{\rho}{E_{11}} h^2} \) for graphite-epoxy laminated composite simply supported shallow cylindrical shell, \([0^\circ/90^\circ]_s\), \( a/b=1, a/h=100, a/R=0.5, N=40 \) cracks/in., \( A_d=30\% \) .................................................. 152

6.1 Local and global coordinate systems and degrees of freedom for "Heterosis" element .................................................. 183

6.2 Convergence of the finite element results with mesh refinement
\( \% \text{ error} = \left( \frac{\omega}{\omega_0} - 1 \right) \times 100, \omega_0 \text{ is the exact frequency}, [99] \) .............................................. 184

6.3 Effect of transverse shear deformation theory on the frequency response of crossply laminated composite \( (a/b=1, a/h=10) \) .......................................................... 185

6.4 Variation of error in the frequency response for various vibration modes (crossply laminated plate, \( a/b=1, a/h=10) \) .......................................................... 186

6.5 A representative local volume element in D-zone of a general laminate .............. 187

6.6 Plate geometry, coordinate systems and the location of damage ......................... 188

6.7 Effect of the damaged zone shape on the fundamental frequency response ............ 189
6.8 Effect of the damage size on the first five frequencies of crossply laminate
6.9 Variation of the normalized frequency for various damaged zone representation, [45/0/45]
6.10 Variation of the normalized frequency for various damaged zone representation, [90/0/90]
6.11 Variation of the normalized frequency for extended circular damaged zone, (simply supported [45/0/45] plate )
6.12 Variation of the normalized frequency for extended square damaged zone, (simply supported [45/0/45] plate )
6.13 Variation of the normalized frequency for extended square damaged zone, (clamped [45/0/45] plate )
A.1 Matrix cracking and delamination in a composite laminate
A.2 matrix cracking and local coordinates for a single ply
C.1 Local and global coordinate systems and degrees of freedom for "Heterosis" element
D.1 The finite element program structure
D.2 The Ritz method program structure
NOMENCLATURE

\( A_y \) = extensional stiffness;  
\( A_0 \) = area of plate or shell;  
\( A_d \) = area of damaged zone;  
\( a \) = dimension of laminate along \( x_1 \)-axis;  
\( a_c \) = matrix cracks spacing;  
\( \bar{a} \) = ellipse major axis;  
\( a_y, \bar{a}_y \) = coefficients for \( \text{Rtiz} \) displacement functions;  
\( B_y \) = bending-extensional stiffness;  
\( b \) = dimension of laminate along \( x_2 \)-axis;  
\( \bar{b} \) = ellipse minor axis;  
\( b_y, \bar{b}_y \) = coefficients for \( \text{Rtiz} \) displacement functions;  
\( c_0, c_1 \) = tracing factors for deep shell;  
\( c_y, \bar{c}_y \) = coefficients for \( \text{Rtiz} \) displacement functions;  
\( D_y \) = bending stiffnesses;  
\( d_y, \bar{d}_y \) = coefficients for \( \text{Rtiz} \) displacement functions;  
\( E_{11}, E_{22} \) = Young's moduli in 1 and 2 material principal directions;  
\( F_y \) = stiffness coefficients for deep shell;  
\( f_y, \bar{f}_y \) = coefficients for \( \text{Rtiz} \) displacement functions;  
\( G_{ij} \) = shear moduli in \( i-j \) surfaces, respectively;  
\( [G] \) = constraint matrix;  
\( h \) = thickness of shell;  
\( H_i \) = inertia terms;  
\( h_k \) = \( k \)-th ply thickness;  
\( k^2 \) = shear correction factor;  
\( [K] \) = system stiffness matrix;  
\( [\bar{K}] \) = reduced system stiffness matrix;  
\( M \) = total number of elements;
\[ M_{\alpha} = \text{moment resultants}; \]
\[ [M] = \text{system mass matrix}; \]
\[ \bar{M} = \text{reduced system mass matrix}; \]
\[ N = \text{number of cracks per inch}; \]
\[ N_x = \text{normal stress resultants}; \]
\[ Q_{ij} = \text{lamina stiffness coefficients}; \]
\[ \hat{Q}_{ij} = \text{lamina damage-reduced stiffness coefficients}; \]
\[ Q_{\alpha} = \text{shear stress resultants (\( \alpha = 1, 2 \))}; \]
\[ \{q\} = \text{vector of unknown coefficients}; \]
\[ \{\bar{q}\} = \text{reduced vector of unknown coefficients}; \]
\[ R = \text{defines a region (plate or shell)}; \]
\[ R_{12} = \text{radius of twist}; \]
\[ R_{\alpha} = \text{principal radii of curvature (\( \alpha = 1, 2 \))}; \]
\[ T = \text{total kinetic energy}; \]
\[ U_s = \text{total strain energy}; \]
\[ U_{mn} = \text{amplitudes of displacement, } u_{1}; \]
\[ u_{ij} = \text{displacements of mid-surface}; \]
\[ u_i = \text{displacements of point in shell}; \]
\[ V_{mn} = \text{amplitudes of displacement, } u_2; \]
\[ W_{mn} = \text{amplitudes of displacement, } u_3; \]
\[ \alpha^M = 2\text{nd order tensor for matrix crack parameters}; \]
\[ \sigma_{m} = m\pi/a; \]
\[ \beta_{n} = n\pi/b; \]
\[ \delta = \text{variational operator}; \]
\[ \phi_i = \text{rotations about normal to shell mid-surface}; \]
\[ \Phi_{1,mn} = \text{amplitudes of rotation, } \phi_1; \]
\[ \Phi_{2,mn} = \text{amplitudes of rotation, } \phi_2; \]
\[ \varepsilon_{ij} = \text{strain components}; \]
\[ \gamma_{ij} = \text{shear strain components}; \]
\[ \sigma_{ij} = \text{stress components}; \]
\[ \xi = \text{nondimensional coordinate in } x_1\text{-direction}; \]
\[ \eta = \text{nondimensional coordinate in } x_2\text{-direction}; \]
\[ \nu_y = \text{Poisson's ratios}; \]
\( \lambda \) = frequency parameter, and
\( \omega \) = natural circular frequency.
CHAPTER 1

INTRODUCTION

1.1 GENERAL

Continuous fiber reinforced polymer composite materials have been reliably used in many primary as well as secondary aerospace structural components. This class of materials is also widely used in the automobiles industry, sports facilities and equipments, and large span roof structures. Such a wide usage of these materials is due to their many unique characteristics such as the high strength/weight ratio, corrosion resistance, and the possibility for optimal design by tailoring the mechanical properties of structural components for specific applications. Moreover, the geometric designs with such materials are very flexible which makes it possible to improve the load carrying capacity of various structural components. Despite of these good features of the polymer composite materials, they are prone to various kinds of damage modes during their manufacturing phase and/or during normal service loading conditions. For example, delamination and debonding types
of damage may develop during manufacturing due to improper consolidation of plies. While in service loading conditions, such as fatigue and impact loadings, the structural components may exhibit fiber fracture, debonding, matrix cracking as well as delaminations.

Considerable efforts have been devoted to quantitatively determine stresses, strains, deformations, natural frequencies and buckling loads for different structural composites and under different loading conditions. However, there has been very little study of damage characterization of such structural components, especially in the area of dynamics. There is an important need to quantify the residual structural integrity of various components possessing such damage modes. In this regard, the dynamic response of composite plates and shells (the most common structural elements used in practice) containing damage is a serious concern.

1.2 MOTIVATION FOR THE PRESENT WORK

In the area of structural dynamics, the importance of delineating the vibration modes and frequencies of various structural components accurately enough is very well understood. Such modes and frequencies have now become an important element in the design process. This design process involving laminated polymer has become more complex due
to their several unique characteristics such as elastic anisotropy and inplane coupling as well as bending-stretching coupling phenomena.

Considerable research efforts have been devoted to quantitatively determine stresses, strains, deformations, natural frequencies, and buckling loads for a wide variety of laminated composite structural components. Efforts have also been made to experimentally characterize their static as well as dynamic loading conditions. Impact-induced damage has been recognized to be a serious problem in many practical situations. The induced damage tends to reduce the stiffness and strength properties of the laminated composite component. Therefore, static as well as dynamic characteristics of a structural element containing a damaged zone may be severely affected. The primary damage modes in laminated composites include: matrix cracks, delaminations, and fiber fracture. Both impact loading and environmental conditions may also induce or add some curvature to the composite panel. This curvature will change the original geometrical configuration used in the design process.

The primary observations made in the vast literature under various loading conditions include the following key features:

1. The damage, flaws, or defects in laminated polymer composite structural components (Figure 1.1) may exist due to the following loading conditions:

   (a) Mechanical loading.

   (b) Environmental conditions.
(c) Foreign object impact loading, or
(d) Combination of all the above loading conditions.

2. Fiber-fracture, debonding, matrix cracking or fiber-matrix interface debonding or delaminations are the primary types of damage modes, especially during low-velocity impact type of loading conditions.

3. Among the above types of damage modes, the transverse matrix cracking appears to be the precursor for fiber-debonding and inter-ply delaminations.

The above types of damage present in a structural component (shown in Figure (1.2)) tend to affect its further structural response and/or service life. It, therefore, appears very important to investigate the residual dynamic characteristics of such damaged components. A continuous monitoring of their residual response may eventually be necessary to assess their structural integrity and durability to ensure their safe usability. The present work is aimed at achieving a quantitative understanding of the residual dynamic response of laminated polymer composite plates and shells containing primarily the matrix crack type of damage. Such study should enable designers to make efficient use of such fast growing industrial materials. Also, the investigation of the effect the degree and size of damage on the global structural response can provide us with a better grip on the issue of the allowable damage tolerance limits which are currently used in the design process of composite structures on adhoc basis. This may also lay down a foundation for the development of a Non-Destructive Evaluation (NDE) technique that can be used to
monitor the structural integrity of a component (as shown in Figure (1.3)). It may be necessary to mention here that, to the best of our knowledge, there are no significant studies on the above issue existing in the open literature.

1.3 PRIMARY OBJECTIVES AND SCOPE

The primary objectives of the present work are:

- To review the existing damage models for selecting the appropriate one, which can then be simplified and developed further for its use in the structural dynamical analysis of laminated composite plates and shells. This study is restricted to consider the transverse matrix crack type of damage only.
- To develop a first order shear deformation theory for the dynamic structural analysis of laminated polymer composite plates and shallow shells containing damage regions with matrix crack type of damage in various plies.
- To develop appropriate approximate solution procedures that can be used to obtain the residual dynamic response of the above structural members. These solution procedures are: the Ritz method and the finite element with isoparametric element for laminated composite plates. The study is restricted to obtaining the frequency response only.
• To perform a parametric study to obtain the frequency response of shallow shells and plates by changing the size and extent of damage and the shape of the damage regions (which are centrally located within the structural configurations), curvature ratio, thickness ratio, degree of orthotropy as well as support conditions.

1.4 PRESENT WORK OUTLINES

To achieve the objectives discussed in the previous section, the present study is organized in the fashion shown in Fig.(1.4).
Fig.(1.1): Laminated composite plate geometry and coordinate system
Damage zone

Fig. (1.2) Damage Modes of Composite Laminate
Laminated Composite Structural Component

Static, Dynamic and Environmental Loading Conditions

Internal Strains, Stresses and Various Types of Damage Modes

Change in the Static and Dynamic Characteristics of the Structural Component

Reliable Tools for the Measurements of these Changes

Experimental Observations

Analytical Modeling (Present Study)

Build a Possible Correlation

Fig.(1.3): The motivation of the present study
Chapter (1) Introduction

Statement of the Problem
Motivation of the Present Work
Objectives and Scope

Chapter (2) Literature Review

Undamaged Zone analysis
Damaged Zone Modeling
Solution Procedures

Chapter (3) Modeling of Damage

Damage Mechanics modeling
Matrix Cracks Modeling
Elastic Constants Expressions

A

(continued)

Fig.(1.4): The outline for the present work
Energy Formulation

Chapter (4)

Composite Shallow Shell
Simplified Shallow Shell
Composite Flat Plate

Chapter (5)

Ritz Method Formulation

Displacement Functions
Continuity & Boundary Conditions
Results & discussion

Chapter (6)

Frequency Response by Finite Elements

Discrete Laminate
Finite Elements Modeling
Results & discussion

Chapter (7)

Conclusions
CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

The anisotropy, coupling phenomena and complexity of the failure modes observed in laminated polymer composite plate and shell type of structural components make the problem formulation cumbersome. The analytical as well as numerical determination of the distributions of stresses, strains, and deformation as well as the frequency response and buckling loads for a wide variety of material systems become difficult and challenging. Due to cost limitations and the difficulties involved in experimental investigations, only a limited set of good data is available for these of structural components. However, there exist a great deal of theoretical research output through the use of both the analytical and numerical approaches. For laminated composite plates and shells type of structural components, some of the analytical solution techniques include classical plate theory, higher order theories, boundary layer theory, and variational principles. While the numerical techniques are primarily confined to the finite element methods. the finite
difference methods, and Rayleigh-Ritz method. It has been cited in the literature that some of these solution techniques, either analytical or numerical, are only applicable under certain conditions. As an example, the displacement-based finite element models are incapable of accurately predicting the stress field in a free-edge delamination specimen, consequently the damage evolution and failure process of such specimens cannot be correctly predicted. This drawback was the motivation behind introducing some other theories such as the discrete laminated plate theory, and the global-local model so that such stress fields can be estimated more accurately.

As discussed earlier, the existence of various damage modes either during manufacturing processes and/or during the service loading conditions is quite natural in these types of laminated composite materials. Also, it should be understood that the existence of such damage modes will affect both the static and the dynamic characteristics of the structural components. Therefore, it becomes very important to address the problems of identifying potential correlations between the structural response and type, degree and size of these damage modes. A lot of research efforts, both theoretical as well as experimental, have been devoted to static loading situations. However, such efforts toward dynamic analysis such the frequency response are very limited.

The primary damage modes in laminated composites include; matrix cracks, delaminations, and fiber fracture, as shown in Fig.(2.1). However, the present work will focus only on the matrix crack type of damage mode and its effects on the frequency response of laminated plates and shells.
Before developing appropriate mathematical models for the problem under consideration, it is most appropriate to critically review first the relevant available literature.

2.2 THE UNDAMAGED ZONE

2.2.1 Introduction

In deriving the governing field equations of a laminated composite shell or plate type of structural component, there exist three main theoretical categories for the treatment of the transverse shear deformations. These categories are:

- The classical plate theory (CPT);
- The first order shear deformable theory (FSDT); and
- The higher order shear deformable theories (HSDT).

By using the basic configuration of a composite laminate shown in figure (2.2), the above three theories are briefly discussed in this section.
(i) The classical plate theory (CPT)

This theory is based on Kirchhoff hypothesis and it is well-documented in the literature (e.g. Whitney [110], Vinson and Sierakowski [106] and Reddy [82]). The displacement field can be written as:

$$u_\alpha (x_i) = u^0_\alpha (x_\beta) - x_3 u_{3,\alpha} (x_\beta)$$  \hspace{1cm} (2.1)

where, $u^0_\alpha$ are the inplane displacements of the midsurface of the plate, and the transverse displacement is given by,

$$u_3 (x_i) = u_3 (x_\beta)$$  \hspace{1cm} (2.2)

Hence, the strain-displacement relationships are given by,

$$\varepsilon_{\alpha\beta} = u^0_{\alpha,\beta} - x_3 u_{3,\alpha\beta}$$  \hspace{1cm} (2.3)

$$\varepsilon_{\alpha3} = \frac{1}{2} (u_{\alpha,3} + u_{3,\alpha}) = 0$$  \hspace{1cm} (2.4)

where, the strains $(\varepsilon_{\alpha\beta})$ are the components of the Euler symmetric strain tensor.

It is clear from the above equations that the transverse shear deformations are neglected, despite the fact that these deformations are known to play an important role in such laminated composite material systems. The importance of including the transverse shear deformation in these materials is due to their highly anisotropic nature which plays a significant role even if the thickness is relatively small. When these deformations are
neglected, which is the case for the (CPT), the resulting approximate solutions underestimate deflections and overestimate the natural frequencies especially for relatively thick composite plates or shells.

(ii) First order shear deformable theory (FSDT)

Yang [114], incorporated the effect of shear deformations for laminated composite plates in the way suggested by Mindlin [66] for homogeneous isotropic plates. The theory is based on linear inplane displacement field and constant transverse displacement field through the plate thickness, and which can be written as:

\[ u_\alpha(x_\beta) = u^0_\alpha(x_\beta) + x_3 \Phi_\alpha(x_\beta) \]  

(2.5)

and the transverse displacement field is given by Eqn.(2.2).

It has to be noted that in this theory the geometrical normal plane to the midsurface before deformations remains plane but not necessarily normal to the midsurface after deformations. Also, if the continuity of the interlaminar shear stresses at the inter-faces between the plies is enforced, the result is a constant shear stress over the plate thickness. In addition, the resulting transverse shear stresses on the top and bottom surfaces of the plate are non-zero. It means that the top and the bottom surfaces of the plate are not traction free. In order to eliminate such an anomalous situation, Srinivas [98,99] introduced a modified model based on the same assumption of linear inplane displacement field and constant transverse displacement across the thickness. It has been
also cited in the literature, e.g. [82,110], that the first order shear deformable theory (FSDT) is capable of predicting an improved displacement field and global response for relatively thick plates. On the other hand the stresses calculated from (FSDT) are identical to those of (CPT). This is due to the fact that the stresses are linear in $x_3$ within each ply for (FSDT). While, improvement in stresses requires higher order terms in the displacement field. This was the motivation behind introducing various higher order theories.

(iii) Higher order shear deformable theories (HSDT)

Whitney and Sun [101], assumed a linear inplane displacement field and quadratic transverse displacement field in terms of the transverse coordinate. In reference [111], they proposed another higher order theory into which the inplane displacements are also quadratic in $x_3$ but the transverse displacement is linear through the thickness. A more complicated displacement field was assumed by Lo et al. [61], into which the inplane displacements are cubic in $x_3$ and the transverse is quadratic over the thickness. Noting that all the previously stated displacement field do not satisfy the traction free top
and bottom surface of the plate, Bert [9] proposed a modified theory. In his theory the condition of free tractions of the plate top and bottom surfaces is explicitly imposed. There is no change in the transverse displacement field given by Eqn.(2.2), while the resulting inplane displacement field was given by.

\[ u_{\alpha}(x_r) = u_{\alpha}^0(x_p) + x_3\phi_{\alpha}(x_p) + \frac{4x_3^3}{3h^2} (\phi_{\alpha} + u_{3,\alpha}) \]  

(2.6)

It can easily be seen that the distribution of the transverse shear stresses is parabolic over the plate thickness with zero values at the top and bottom surfaces. Hence, there is no shear correction factor involved in the solution process.

Murthy [68] and Reddy [84], developed a very similar theory to that of Bert [9] with some differences in deriving the governing equations based on a variational principle. Other theories using different displacement fields were introduced e.g. Sun et al.[101], Seide [92], and Whitney [110].

Many investigations, e.g. [15,23,84-86], concluded that using higher order theory instead of (FSDT) results in very slight improvement in the natural frequency of the plate. Also, Simivas [98,99] mentioned that the application of Mindlin plate theory, in which the inplane displacements are assumed linear and the transverse displacement is constant across the thickness, and the effect of the transverse shear deformations is included, is accurate. Moreover, through three-dimensional analysis of simply supported laminates, even thick ones, with the outer plies having high elastic moduli as compared to those of the inner plies, the displacements \( u_\alpha \) are linear and \( u_3 \) is nearly constant.
across the thickness. This conclusion can be easily observed from Tables (2.1)-(2.3), which also indicate that the effect of rotary inertia is negligibly small. In summary, the inclusion of the transverse shear deformation is essential but the first order shear deformable theory is practically sufficient for vibration analysis. In addition, the effect of rotary inertia is very small and can be neglected without any significant error.

2.2.2 The governing equations

2.2.2.1 Laminated composite shallow shell

(a) Introduction

As it has been stated previously, some curvature might exist in a laminated composite plate. Such curvature could be the result of faulty manufacturing of the composite panel or impact damage or environmental effects. In general this type of curvature is very small compared with the panel dimensions. Thus, it is safe and reasonably accurate to assume that the panel will behave like a shallow shell. Hence, the major objective of this section is to discuss the basic governing equations for laminated composite shallow shells. Since all of the fundamental relationships are widely available in the open literature their derivations are not tackled in the present work. Interested readers
might consult the dedicated monographs of Leissa [54,55] as well as other materials on shells, e.g. [77-80].

The shell shown in Figure(2.3) is practically considered to be shallow if the central rise is one-fifth or less than the smaller dimension, [14,31,54], i.e.

\[
\frac{f}{a} \text{ or } \frac{f}{b} \leq \frac{1}{5}
\]

(2.7)

For such a case, in the subsequent mathematical analysis, terms such as \( (x_{3,\alpha})^2 \) and other higher order terms being small in comparison to unity can be neglected, Gould [31]. The practical interpretation of such assumption is that the curvilinear coordinates \( (\theta_1) \) and \( (\theta_2) \) may be selected as the Cartesian \( (x_\alpha) \) with the Lame' parameters \( A=B=1 \) (see Figure(2.4)). Additional approximations are introduced into the shallow shell theory at the level of equilibrium equations later in this section.

(b) Kinematic relations

In the Cartesian coordinates, the strains are related to the displacements through the following relations [14,31]:

\[
\varepsilon_{11} = \frac{1}{(1 + x_3/R_1)}(\varepsilon_{11}^0 + x_3\kappa_{11})
\]

(2.8)

\[
\varepsilon_{22} = \frac{1}{(1 + x_3/R_2)}(\varepsilon_{22}^0 + x_3\kappa_{22})
\]

(2.9)

\[
\varepsilon_{12} = \frac{1}{(1 + x_3/R_1)}(\varepsilon_{12}^0 + x_3\kappa_{12})
\]

(2.10)
\[ \varepsilon_{21} = \frac{1}{(1 + x_3 R_2)} (\varepsilon_{21}^0 + x_3 \kappa_{21}) \quad (2.11) \]

\[ \gamma_{12} = \varepsilon_{12} + \varepsilon_{21} \quad (2.12) \]

\[ \varepsilon_{\alpha 3} = \frac{\gamma_{\alpha 3}^0}{(1 + x_3 R_\alpha)} \quad (1.13) \]

where

\[ \varepsilon_{11}^0 = u_{1,1}^0 + \frac{u_3^0}{R_1} \quad (2.14) \]

\[ \varepsilon_{32}^0 = u_{2,2}^0 + \frac{u_3^0}{R_2} \quad (2.15) \]

\[ \gamma_{12}^0 = u_{2,1}^0 + \frac{u_3^0}{R_{12}} \quad (2.16) \]

\[ \gamma_{21}^0 = u_{1,2}^0 + \frac{u_3^0}{R_{12}} \quad (2.17) \]

\[ \gamma_{13}^0 = u_{3,1}^0 - \frac{u_1^0}{R_1} - \frac{u_2^0}{R_{12}} + \phi_1 \quad (2.18) \]

\[ \gamma_{23}^0 = u_{3,2}^0 - \frac{u_2^0}{R_2} - \frac{u_1^0}{R_{12}} + \phi_2 \quad (2.19) \]

and

\[ \kappa_{11} = \phi_{1,1} \quad \kappa_{22} = \phi_{2,2} \]

\[ \kappa_{12} = \phi_{2,1} \quad \kappa_{21} = \phi_{1,2} \quad (2.20) \]
(c) Moment and force resultants

The stresses in the k-th layer can be related to the strains through the conventional transformed stiffness coefficients which can be written in the following general form:

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{bmatrix}^k =
\begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} & 0 & 0 & \overline{Q}_{16} \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{23} & 0 & 0 & \overline{Q}_{26} \\
\overline{Q}_{13} & \overline{Q}_{23} & \overline{Q}_{33} & 0 & 0 & \overline{Q}_{36} \\
0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\
0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0 \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{36} & 0 & 0 & \overline{Q}_{66}
\end{bmatrix}^k
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{bmatrix}^k
\]  

(2.21)

In general, the normal strain component (\( \varepsilon_{33} \)) is assumed to be negligibly small \([105,106]\), i.e. \( \varepsilon_{33} = 0 \). In order to obtain the equations of moment and force resultants for a given laminated shell, Eqn.(2.21) can be integrated over the laminate thickness. The integral forms are given by:

\[
\begin{bmatrix}
N_{11} \\
N_{12} \\
Q_1
\end{bmatrix} = \frac{h}{2} \int_{-h/2}^{h/2} \begin{bmatrix}
\sigma_{11} \\
\sigma_{12} \\
\sigma_{13}
\end{bmatrix} (1 + \frac{x_3}{R_2}) dx_3
\]  

(2.22)

\[
\begin{bmatrix}
N_{22} \\
N_{21} \\
Q_2
\end{bmatrix} = \frac{h}{2} \int_{-h/2}^{h/2} \begin{bmatrix}
\sigma_{22} \\
\sigma_{21} \\
\sigma_{23}
\end{bmatrix} (1 + \frac{x_3}{R_1}) dx_3
\]  

(2.23)

22
\[
\begin{align*}
\begin{bmatrix}
  M_{11} \\
  M_{12}
\end{bmatrix} &= \int_{-h/2}^{h/2} \begin{bmatrix}
  \sigma_{11} \\
  \sigma_{12}
\end{bmatrix} \left(1 + \frac{x_3}{R_2}\right) x_3 dx_3 \\
\begin{bmatrix}
  M_{22} \\
  M_{21}
\end{bmatrix} &= \int_{-h/2}^{h/2} \begin{bmatrix}
  \sigma_{22} \\
  \sigma_{21}
\end{bmatrix} \left(1 + \frac{x_3}{R_2}\right) x_3 dx_3
\end{align*}
\] (2.24) (2.25)

In the above expressions one should note that despite the equality of the inplane shear stresses, i.e. \( \sigma_{12} = \sigma_{21} \), in general \( N_{12} \neq N_{21} \) and \( M_{12} \neq M_{21} \) except for the special case of a spherical shell where the principle radii of curvature \( (R_1 \text{ and } R_2) \) are equal.

The terms \( \left(x_3^2/R_\alpha R_\beta\right) \) are assumed to be small compared to unity and can be neglected.\([14]\). By using such approximation, the integral forms given by Eqns(2.22) through (2.25) can be evaluated to yield the following force and moment resultants:
\[
\begin{pmatrix}
N_{11} \\
N_{22} \\
N_{12} \\
N_{21}
\end{pmatrix}
= 
\begin{pmatrix}
A_{11} & A_{12} & A_{16} & A_{16} & | & B_{11} & B_{12} & B_{16} & B_{16} \\
A_{12} & A_{22} & A_{26} & A_{26} & | & B_{12} & B_{22} & B_{26} & B_{26} \\
A_{16} & A_{26} & A_{66} & A_{66} & | & B_{16} & B_{26} & B_{66} & B_{66} \\
A_{16} & A_{26} & A_{66} & A_{66} & | & B_{16} & B_{26} & B_{66} & B_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\gamma_{12} \\
\gamma_{21}
\end{pmatrix}
\]

\[
\begin{pmatrix}
M_{11} \\
M_{22} \\
M_{12} \\
M_{21}
\end{pmatrix}
= 
\begin{pmatrix}
B_{11} & B_{12} & B_{16} & B_{16} & | & D_{11} & D_{12} & D_{16} & D_{16} \\
B_{12} & B_{22} & B_{26} & B_{26} & | & D_{12} & D_{22} & D_{26} & D_{26} \\
B_{16} & B_{26} & B_{66} & B_{66} & | & D_{16} & D_{26} & D_{66} & D_{66} \\
B_{16} & B_{26} & B_{66} & B_{66} & | & D_{16} & D_{26} & D_{66} & D_{66}
\end{pmatrix}
\begin{pmatrix}
\kappa_{11} \\
\kappa_{22} \\
\kappa_{12} \\
\kappa_{21}
\end{pmatrix}
\]

\[
\begin{pmatrix}
-B_{11} & 0 & -B_{16} & 0 & | & -D_{11} & 0 & -D_{16} & 0 \\
0 & B_{22} & 0 & B_{26} & | & 0 & D_{22} & 0 & D_{26} \\
-B_{16} & 0 & -B_{66} & 0 & | & -D_{16} & 0 & -D_{66} & 0 \\
0 & B_{26} & 0 & B_{66} & | & 0 & D_{26} & 0 & D_{66}
\end{pmatrix} + C_0
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\gamma_{12} \\
\gamma_{21}
\end{pmatrix}
\]

(2.26)

and

\[
\begin{pmatrix}
Q_1 \\
Q_2
\end{pmatrix}
= 
\begin{pmatrix}
A_{55} & A_{45} \\
A_{45} & A_{44}
\end{pmatrix}
\begin{pmatrix}
\gamma_{13} \\
\gamma_{23}
\end{pmatrix}
+ C_0 
\begin{pmatrix}
-B_{55} & 0 \\
0 & B_{44}
\end{pmatrix}
\begin{pmatrix}
\gamma_{13} \\
\gamma_{23}
\end{pmatrix}
\]

(2.27)
where

\[ C_0 = \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \]  \hspace{1cm} (2.28)

and the stiffness coefficients \( A_{ij}, B_{ij}, D_{ij}, \) and \( F_{ij} \) (\( i,j=1,2,6 \)), \( A_{ij} \) and \( B_{ij} \) (\( i,j=4,5 \)) are defined as follows:

\[ A_{ij} = \sum_{k=1}^{N} \overline{Q}_{ij}^k (h_k - h_{k-1}) \quad i,j=1,2,6 \]  \hspace{1cm} (2.29)

\[ B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \overline{Q}_{ij}^k (h_k^2 - h_{k-1}^2) \quad i,j=1,2,6 \]  \hspace{1cm} (2.30)

\[ D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \overline{Q}_{ij}^k (h_k^3 - h_{k-1}^3) \quad i,j=1,2,6 \]  \hspace{1cm} (2.31)

\[ F_{ij} = \frac{1}{4} \sum_{k=1}^{N} \overline{Q}_{ij}^k (h_k^4 - h_{k-1}^4) \quad i,j=1,2,6 \]  \hspace{1cm} (2.32)

\[ A_{ij} = k^2 \sum_{k=1}^{N} \overline{Q}_{ij}^k (h_k - h_{k-1}) \quad i,j=4,5 \]  \hspace{1cm} (2.33)

\[ B_{ij} = \frac{k^2}{2} \sum_{k=1}^{N} \overline{Q}_{ij}^k (h_k^2 - h_{k-1}^2) \quad i,j=4,5 \]  \hspace{1cm} (2.34)

where \( h_k \) is the distance measured from the shell midsurface to the top of \( k \)-th layer in the \( x_3 \) direction (see Figure(2.5)) and \( k^2 \) is a shear correction factor, the discussion about which will follow in the transverse shear analysis section.
(d) Equations of motion in terms of stresses

For linear small deformation theory and in the absence of body forces, the equations of motion in terms of stresses are given by,

\[ \sigma_{ij,j} = \rho_0 \ddot{u}_i \]  \hspace{1cm} (2.35)

where, \( \rho_0 \) is the mass density and the superimposed dots indicate derivatives with respect to time (t).

(d) Equations of motion in terms of force and moment resultants

In the absence external forces and body forces and by using the previously discussed shallow shell simplified theory, the equations of motion in terms of the force and moment resultants can be obtained. By integrating Equation(2.35) and substituting for the integral form of the stress components using Equations (2.22-2.25), we get:

\[ N_{11,1} + N_{21,2} + \frac{Q_1}{R_1} + \frac{Q_2}{R_{12}} = I_1 \ddot{u}_1^0 + I_2 \ddot{\Phi}_1 \]  \hspace{1cm} (2.36)

\[ N_{12,1} + N_{22,2} + \frac{Q_2}{R_2} + \frac{Q_1}{R_{12}} = I_1 \ddot{u}_2^0 + I_2 \ddot{\Phi}_2 \]  \hspace{1cm} (2.37)
\[ M_{11,1} + M_{21,2} - Q_1 = I_2 \ddot{u}_1^0 + I_3 \ddot{\Phi}_1 \] (2.38)

\[ M_{12,1} + M_{22,2} - Q_2 = I_2 \ddot{u}_2^0 + I_3 \ddot{\Phi}_2 \] (2.39)

\[ Q_{1,1} + Q_{2,2} - \frac{N_{11}}{R_1} - \frac{N_{22}}{R_2} - \frac{(N_{12} + N_{21})}{R_{12}} = I_3 \dddot{u}_3^0 \] (2.40)

Where the superimposed dots indicate time derivatives and the terms \( I_1, I_2, \) and \( I_3 \) are defined as follows:

\[ I_1 = \int_{-h/2}^{h/2} \rho_0 dx_3 \] (2.41)

\[ I_2 = \int_{-h/2}^{h/2} \rho_0 x_3 (1 + \frac{x_3}{R_1})(1 + \frac{x_3}{R_2}) dx_3 \] (2.42)

\[ I_3 = \int_{-h/2}^{h/2} \rho_0 x_3^2 dx_3 \] (2.43)

The above equations of motion can be further simplified or specialized for a very shallow shell or a flat plate respectively by introducing some more assumptions. This issue is elaborated on in Chapter (4), Sections (4.4.1) and (4.4.4).

For the special case when the surface curvatures \( R_i \to \infty \) the shell becomes a flat plate with a more simplified governing equations. The reduced form of these equations for such a special case is the topic of discussion of the following section.
2.2.2.2 Laminated composite plates

Since part of the focus of the present work is on the vibration analysis of laminated composite, it is reasonable to derive and specialize the shell equations derived in the previous section for the case of a flat plate. As previously discussed, the governing equations are based on a first order transverse shear deformable theory (FSDT).

(i) The displacement field

As it has been previously discussed, the first order shear deformable theory is the basis for this work. For convenience, the displacement field given by Eqns.(2.2) and (2.5) can be recited here as.

\[ u_\alpha (x_i, t) = u_\alpha^0 (x_\beta, t) + x_3 \phi_\alpha (x_\beta, t) \]

and the transverse displacement is given by:

\[ u_3 (x_i, t) = u_3 (x_\beta, t) \]

(ii) The kinematic relations

The components of the Euler strain tensor are related to the plate displacements through.

\[ \varepsilon_{\alpha \beta} \equiv u_{(\alpha, \beta)} = \overline{u}_{(\alpha, \beta)} + x_3 \phi_{(\alpha, \beta)} \]  

(2.44)
\[ \varepsilon_{\alpha 3} \equiv \frac{1}{2} (u_{\alpha,3} + u_{3,\alpha}) = \frac{1}{2} (\phi_{\alpha} + u_{3,\alpha}) \quad (2.45) \]

\[ \varepsilon_{33} = u_{3,3} = 0 \quad (2.46) \]

(iii) The constitutive equations

The force resultants are related to the plate strains and curvatures through the constitutive equations given by.

\[
\begin{align*}
\begin{bmatrix}
N \\ M
\end{bmatrix} &= 
\begin{bmatrix}
A & B \\
B & D
\end{bmatrix}
\begin{bmatrix}
\varepsilon \\ \kappa
\end{bmatrix}
\end{align*}
\] (2.47)

And the transverse resultant shear forces constitutive relations are given by.

\[ Q_{\alpha} = A_{\alpha \beta} \varepsilon_{\beta 3} \quad (2.48) \]

\[ \{N\} = \{N_{11}, N_{22}, N_{12}\}^T \quad (2.49) \]

\[ \{M\} = \{M_{11}, M_{22}, M_{12}\}^T \quad (2.50) \]

\[ \{\varepsilon\} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}\}^T \quad (2.51) \]

\[ \{\kappa\} = \{\kappa_{11}, \kappa_{22}, \kappa_{12}\}^T \quad (2.52) \]
\[ [A] = k^2 \begin{bmatrix} A_{55} & A_{45} \\ A_{45} & A_{44} \end{bmatrix} \] (2.53)

where, \( k^2 \) is a shear correction factor.

Into the above expressions, the force and moment components can be defined as:

\[ \{N_{\alpha\beta}, M_{\alpha\beta}\} = \int_{-h/2}^{h/2} \{1, x_3\} \sigma_{\alpha\beta} \, dx_3 \] (2.54)

and

\[ \{A_{ij}, B_{ij}, D_{ij}\} = \int_{-h/2}^{h/2} \{1, x_3, x_3^2\} \overline{Q}_{ij} \, dx_3 \] (2.55)

where, \( \overline{Q}_{ij} \) are the conventional stiffness coefficients of the laminated plate \((i,j=1,2,6)\) and \( h \) is the total laminate thickness.

\[ A_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij} \, dx_3 \quad i,j=4,5 \] (2.56)

\[ \kappa_{\alpha\beta} = \frac{1}{2} (\Phi_{\alpha,\beta} + \Phi_{\beta,\alpha}) \] (2.57)

(iv) Equations of motion in terms of force resultants
By integrating Eqn.(2.35) and utilizing the above definitions of the force resultants with some algebraic manipulations and in the absence of transverse as well as the inplane loading effects, the plate equations of motion are given by:

\[ N_{\alpha\beta,\beta} = \rho \ddot{u}_\alpha^0 + H \dddot{\phi}_\alpha \]  
\[ M_{\alpha\beta,\beta} - Q_\alpha = H\ddot{u}_\alpha^0 + I \dddot{\phi}_\alpha \]  
\[ Q_{\alpha,\alpha} = \rho \dddot{u}_3 \]

where,

\[ \{\rho, H, I\} = \int_{-h/2}^{h/2} \left\{1, x_3, x_3^2\right\} \rho_0 dx_3 \]

The above expressions can easily be obtained using Eqns.(2.35)-(2.39) directly after setting \( R_{\alpha} \rightarrow \infty \) (\( \alpha = 1, 2 \)) and \( R_{12} \rightarrow \infty \), which is the case for a flat plate. The governing field equations, Eqns.(2.58)-(2.60), can be written explicitly in terms of the displacement components \( u_i \), using Hamilton's energy principle. Which is briefly outlined in the next section together with the resulting consistent set of boundary conditions.
2.3 SOLUTION PROCEDURE

2.3.1 General

Since our main focus is on the dynamical aspect, the energy formulation of the problem should take into account the time-dependent applied forces and geometrical constraints. The extension of the principles of virtual work, which are limited to static equilibrium, to the time domain yields the well-known Hamilton's Principle for dynamical problems. For conservative systems, i.e. the sum of the potential and kinetic energies in conserved, the Hamilton's Principle can be written [52,65,82,109] as follows.

\[
\delta \int_{t_1}^{t_2} L dt = 0
\]  

(2.62)

where, \( L \) is called the Lagrangian function and is given by,

\[
L = T - U_E
\]  

(2.63)

For a body with a volume \( V \) and if the mass density is independent of time, then the kinetic energy \( T \) is given by.

\[
T = \frac{1}{2} \int_V \rho \dot{u}_i u_i dV
\]  

(2.64)

And in the absence of body forces and surface tractions, the total potential energy, the same as the strain energy in this case and can be written as,
\[ U_E = \frac{1}{2} \int \sigma_{ij} \varepsilon_{ij} \ dV \quad (2.65) \]

It can be easily verified that the first variation of the Lagrangian (L) given by Eqn. (2.63) leads to the governing equations of motion for a laminated shallow shell given by Eqns. (2.35)-(2.39) or those for a laminated plate given by Eqns. (2.58)-(2.60). The topic of energy formulation and expanded expression in terms of midsurface displacements are the subjects of discussion in Chapter (4). Also, from this variational statement, a set of consistent boundary conditions will be presented in the next section.

2.3.2 The boundary conditions

A set of consistent boundary conditions results from the application of Hamilton's Principle. For the first order shear deformable theory (FSDT), the boundary conditions are given by [52,106,110],

\[
\begin{align*}
\hat{u}_n &= \hat{u}_n \quad \text{or,} \quad N_n = \hat{N}_n \\
\hat{u}_s &= \hat{u}_s \quad \text{or,} \quad N_{ns} = \hat{N}_{ns} \\
\hat{\phi}_n &= \hat{\phi}_n \quad \text{or,} \quad M_n = \hat{M}_n \\
\hat{\phi}_s &= \hat{\phi}_s \quad \text{or,} \quad M_{ns} = \hat{M}_{ns} \\
\hat{u}_3 &= \hat{u}_3 \quad \text{or,} \quad Q_n = \hat{Q}_3
\end{align*}
\quad (2.66)
\]

where, n and s refer to the normal and tangential directions at an edge respectively, and the quantities with "hat" represent a specified quantity at the boundary.
The expressions given by Eqn.(2.66) are expanded for various possible combinations of boundary conditions in Table (2.4), Chang [14] and Hong [42]. Some of the widely used edge conditions can be written explicitly as follows.

(i) Simply supported edge

\[ N_n = N_{ns} = M_n = \phi_3 = u_2 = 0 \]  \hspace{1cm} (2.67)

(ii) Clamped edge

\[ u_n = u_s = \phi_n = \phi_s = u_3 = 0 \]  \hspace{1cm} (2.68)

(iii) Free edge

\[ N_n = N_{ns} = M_n = M_{ns} = Q_n = 0 \]  \hspace{1cm} (2.69)

It should be noted that for (FSDT), five different boundary conditions, either essential or natural, can be specified on the boundary, while for the (CPT) only four can be specified, which results in a paradox at the free edge since five natural boundary conditions are needed.
2.3.3 Continuity conditions

The domain of the problem consists of two distinct regions namely; the undamaged and the damaged parts, or simply \( Z_1 \) and \( Z_2 \). However, the plate is assumed to be intact and its integrity is well-maintained. But, in the formulation process each part has its own field equations, hence it is essential to enforce the continuity conditions at the common boundary of the two zones. These continuity conditions can be written [2.20.43] as follows:

\[
\begin{align*}
\begin{bmatrix}
\hat{u}_i \\
\hat{u}_3' \\
\hat{\phi}_\alpha \\
\hat{M}_{\alpha\beta} \\
\hat{N}_\alpha \\
\hat{Q}_\alpha \\
\end{bmatrix} & \quad \text{on } \partial Z_2 \quad \text{(damaged)} \\
\begin{bmatrix}
\dot{u}_i \\
\dot{u}_3' \\
\dot{\phi}_\alpha \\
\dot{M}_{\alpha\beta} \\
\dot{N}_\alpha \\
\dot{Q}_\alpha \\
\end{bmatrix} & \quad \text{on } \partial Z_2 \quad \text{(undamaged)} \\
\end{align*}
\]

Where, \( Z_2 \) is an open connected region in the Euclidean space which represent the damaged zone and \( \partial Z_2 \) is its boundary. The variables with "hat" in Eqn.(2.70) indicate that they belong to the region \( R \), while the ones without "hat" are for the undamaged region. Also, the primed displacements represent their spatial derivatives.

When considering the continuity, it must be understood that in general not all the conditions given by Eqn.(2.70) can be satisfied simultaneously. This usually depends on the geometry and the approximate solution technique used. For instance, in a displacement-based finite element technique, only the continuity of displacements and
slopes can be enforced. While for stress-based technique, the continuity of the internal resultant forces can be achieved.

It must be mentioned that the above-stated displacement-based continuity conditions do not contradict the jump discontinuities at the plies interfaces that might result from delaminations. Because these discontinuities must be treated at each lamina level by careful investigation of the through-thickness direction of the laminate. While, the continuity conditions given by Eqn. (2.70) must be enforced on the level of the structural laminate.

2.4 THE DAMAGED-ZONE MODELING

2.4.1 General

It is a well-established fact that composite materials develop extensive patterns of microstructure damage as a result of mechanical and/or environmental load history. Many composite structures can be damage tolerant because if they can retain load carrying capacity and structural integrity after the development of microstructural damage. The mechanical as well as physical properties of the composite laminates are altered by the induced damage. Furthermore, the development and accumulation of such damage are known to be the precursor to structural failure. Such microstructural damage or cracks can result from fatigue, periodic overloading, or even impact during
normal use of the structures. They may also be inadvertently introduced during the fabrication process. The discussion in the following section is an overview on the different damage modes that have been observed in laminated composites under low-velocity impact loading. Since our focus is on post-damage characterization, hence the discussion will be extended to some existing damage evolution models that could be used to estimate the required mechanical properties of the damaged portions of the composite laminates.

2.4.2 Damage modes

The experimental investigations have indicated that the damage in laminated composites can be a combination of interply matrix cracking, interply delamination, and fiber fracture. These modes are shown schematically in Fig.(2.1).

Generally, the early stage of damage development is dominated by matrix cracking along the fiber directions. The number of these matrix cracks increases with the increase in the applied loading, the number of cycles in constant stress fatigue situations, or as a result of a foreign-object-impact. As these cracks reach the interply surfaces, interlaminar cracks are initiated. Subsequently, these interlaminar cracks merge leading to partial or total delamination which may result in loss of integrity of the laminate. The final stage of damage development may become highly localized with increased instability and large scale fiber breakage. From the above discussion, it is obvious that the influence of the
combined damage modes of matrix cracking and delamination are of serious concern to the study of post-damaged dynamic behavior of laminated composite plates and shells.

2.4.3 Modeling of damage

Approaches to the analysis of damage in composite laminates have been based on either micromechanics models or phenomenological ones. In the first category, generally the transverse matrix cracks are treated using the fracture mechanics approach. While in the second one evolution of damage can be treated and in principle the models in this category are experimentally oriented. In this section, some of the existing models are briefly outlined.

As it has been previously stated, the damage can be produced by both static and dynamic loadings. Great attention has been given to study the free-edge delamination due to inplane static and fatigue loadings. These investigations include both theoretical and experimental techniques to study the stress fields at the free-edge that might cause delamination, [25]. On the other hand, the research work that has been devoted to the evaluation of the mechanical properties of damaged laminated composite plates is very limited. And as cited by Sierakowski and Chaturvedi [93], efforts are needed in the area of damage modeling and the residual properties assessment. Apparently, the influence of damage on the dynamic characteristics of laminated composite plates is a relatively new
area that needs further attention. An overview of such areas and problems that need further investigations has recently been presented by Abrate [1], Bert [10], Cantwell and Morton [11] and Gibson [29,30]. As a matter of fact the motivation of this work was clearly stated in [29,30]. It was mentioned that damage in the form of local cracks or delaminations causes a reduction in the natural frequencies and an increase in vibration damping. It is believed that measuring the changes in the natural frequencies of a structure can be an effective tool to characterize the state of damage and material residual strength. It is beyond the scope of this work to review all the efforts that have been done in this area. However, the pioneer and benchmark investigations in the field are discussed below.

Some attempts were introduced to characterize the mechanical behavior under dynamic loading of sheet moulding compound (SMC), which is one of the important classes of short fiber composites. Sun et al. [107], studied the dynamic response of SMC clamped beam under the impact of steel ball. The beam was tested and the resulting strain history was compared to the finite element solution. The first attempt to characterize post-impact tensile stiffness and strength of SMC-R50 under static loading was due to Chaturvedi and Sierakowski [17,18]. Also, in their work [19], it was concluded that the tensile strength, plate stiffness, and the damage growth are dependent on the impactor size and mass. They also found that the damage-growth is nonuniform in the plane of the plate. Gu and Sun [35] used the thickness change of the impacted specimens as a measure of the reduced modulus and strength of SMC-R50 composites. Obviously, this
cannot be the only characterizing parameter for damage-growth for such class of materials.

For laminated composite plates Wang and Yew [108], introduced another damage evolution model into which Galerkins approximation of the potential energy functional together with Hashin’s failure criteria [40] were used. These criteria were used to reduce the material properties due to failure of its constituents under low-velocity impact. They developed a simple model for characterizing the loss in the material strength for the damaged zone depending on the type of failure. It was concluded that the majority of the induced damage in the plate is due to matrix cracking and delamination. It was also cited that the stacking sequence, i.e. fiber orientation, plays an important role in the damage process.

Jao et al. [46], applied the acoustic emission technique (AE) to assess damage in notched specimens made of SMC composites. The specimens were stretched and the accumulated damage was related to the acoustic emission counts. An empirical relation for the stress intensity factor $K_I$ was introduced.

Sun and Sierakowski [100], used two different analytical techniques to characterize the fracture toughness of SMC. The stresses in a compact tension specimen were calculated using a solid SAP finite element model. Also, the fracture toughness of the specimen was evaluated using the critical strain energy release rate.

Chaturvedi and Zayyad [20,115], were probably the first to study the post-impact dynamic behavior of SMC circular plates. The plate was treated as a linear elastic isotropic
material in both damaged and undamaged zones with symmetric damage growth due to impact. A quintic spline technique was used to solve the governing differential equation and the results were compared to SAP IV standard finite element solution. In their model the stiffness variation within the damaged zone was assumed linear, in the following form:

\[
E(r) = \begin{cases} 
(E_0 - E_d) \frac{r}{b} + E_d & 0 < r \leq b \\
E_0 & b \leq r \leq a 
\end{cases}
\]  

(2.71)

where, (a) and (b) are the radii of the plate and the damaged zone, respectively.

In a more representative form of the inflicted damage, Eqn.(2.71) was written as:

\[
E_d = \alpha E_0 \quad \text{and} \quad 0 < \alpha \leq 1
\]  

(2.72)

where, (\(\alpha\)) is the degree of stiffness loss, clearly, if (\(\alpha = 1\)) then, there is no damage.

It was concluded that decreasing the value of (\(\alpha\)) or increasing the percentage of the total damaged area results in a decrease in the plate frequency especially for the higher modes. One should note that despite the simplicity of the model it is a good starting point and it can be further extended to study the dynamic response of laminated composite plates. Reifsnider [87], developed a generic progressive damage model based on what has been called “Critical Element Concept” in a way similar to “Critical Path Concept” since the model can predict the residual life as well as the residual strength.

41
Poon et al. [75], investigated the impact resistance for different high strain-toughened Epoxy composites. The residual strength properties were determined by a post-impact compression test. It was concluded that the compression strength reduction ranged from 63% to 80% depending on the tested composite system. It was also demonstrated that the extensive delaminations resulting from impact were more severe in reducing the compression properties than a 2.54 cm. circular hole in the specimen.

Sun and Jen [81], conducted an experimental program to study the effect of the 90°-ply matrix cracks in a cross-ply laminates. A finite element model was used and the results were compared to the experimental values. It was concluded that 90°-ply matrix cracks could result in 30% reduction in cross-ply laminates strength loaded in the 0°-ply direction.

Wu and Springer [113], developed a model to evaluate the impact damage in cross-ply laminates. The displacements and stresses were calculated using a three-dimensional finite element model. The damage in the laminated plates was estimated using Tsai-Hill failure criterion. They claim that the model is in good agreement with the experimental results, however, no results were presented to confirm that claim.

Rajaiah and Rao [81], presented a good review of the different non-destructive evaluation (NDE) techniques used for damage assessment in laminated composite members.

Frantziskonis, Joshi, and Goode [26, 27, 47], developed a damage evolution model for laminated composite plates. They characterized the damage through an internal state
variable (ISV) tensor that represents the average measures of specific damage modes. The material was considered to be of two components, first one being intact (i.e. undamaged part) and the second one incorporates the effects of microcrack growth (i.e. damaged part). Then, theory of mixtures was utilized to provide an effective tool capable of describing the average influence of damage on the residual stiffness coefficients \((A_{ij}, B_{ij}, \text{and } D_{ij})\). It is interesting to note that the damage-dependent laminate stiffnesses were given by:

\[
A_{ij} = \sum_{k=1}^{N} \left\{ (1-r)Q_{ij} + r \overline{Q}_{ij} \right\}_k (x_3^k - x_3^{k-1})
\]

(2.73)

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \left\{ (1-r)Q_{ij} + r \overline{Q}_{ij} \right\}_k ((x_3^k)^2 - (x_3^{k-1})^2)
\]

(2.74)

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \left\{ (1-r)Q_{ij} + r \overline{Q}_{ij} \right\}_k ((x_3^k)^3 - (x_3^{k-1})^3)
\]

(2.75)

Into which, \((r_k)\) is the internal state variable representing the degree of damage and takes a critical value, \((r_{kcr})\) at the k-ply failure according to Tsai-Hill failure theory. And the stiffness \((\overline{Q}_{ij}^k)\) for the damaged and undamaged zones are defined in the regular way with the exception that the elastic moduli for the damaged zone are modified as presented in [47]. An iterative solution scheme was developed to obtain the strain field in each ply up to failure (i.e. \(r = r_{kcr}\)). The damage modes were identified by comparing the
calculated strains with the failure strains of each ply. Hence, degradation of the ply properties can be introduced based on the identified failure mode. The calculations were carried out using a 9-node, 45-DOF isoparametric finite element model. It is to be mentioned, here, that the above models include only two modes; matrix cracking and fiber breakage. And, the interply delamination mode was not discussed despite its important role in the reduction of the material strength.

Talerja [104], proposed that the damage can be modeled using a set of independent vector variables. For transverse matrix cracking in cross ply laminates, one parameter was used to represent this mode of damage. One should note that the model neither indicates how to measure the material damage parameter nor how to evaluate the stiffness loss due to damage.

Another important model is due to Allen, Harris, and Groves [2-6,36-39]. The model is based on the general continuum mechanics and thermodynamics formulation. This model assumes that the material is linear elastic and the damage is statistically homogeneous in a local volume element. In the absence of residual stresses and temperature effects, the modified constitutive equations are given by

\[
\begin{align*}
\{ N \} &= \begin{pmatrix} \bar{A} & \bar{B} \end{pmatrix} \{ \varepsilon \} \\
\{ M \} &= \begin{pmatrix} \bar{B} & \bar{D} \end{pmatrix} \{ \kappa \}
\end{align*}
\]

(2.76)
In the above expression, \( \tilde{A}, \tilde{B} \) and \( \tilde{D} \) define the modified stiffnesses due to the presence of damage. Which are functions of a second-order internal state variable tensor or a "Strain-like Tensor", \([\alpha]\) which depends on the local energy due to damage. The model is based on a modified Kirchhoff plate theory that allows for jump discontinuities at interlaminar planes that might result from delaminations. Referring to Fig.(2.7), the modified displacement field was given by [4]:

\[
\begin{align*}
    u_\alpha(x_i) &= \bar{u}_\alpha - x_3 \left( \beta^0_\alpha + H(x_3 - x_{3n}) \beta^d_\alpha \right) + H(x_3 - x_{3n}) u^d_{\alpha n} \quad (2.76) \\
    u_3(x_i) &= \bar{u}_3 + H(x_3 - x_{3n}) u^d_3 \quad (2.77)
\end{align*}
\]

where,

\[
\begin{align*}
    \beta^0_\alpha &= u_{3,\alpha} \text{ the undamaged ply rotation.} \\
    \beta^0_\alpha &= \text{ply jump rotation due to delamination at interface } n. \\
    u_{\alpha n} &= \text{ply jump displacement in the } i\text{-th direction at the interface } n, \text{ and} \\
    H(x_3 - x_{3n}) &= \text{heavy side step function.}
\end{align*}
\]

A detailed discussion has been presented in references [2,3] dealing with the specialization of the model to the matrix cracking of laminated composite plates. The extension of the model to study both matrix cracking and interply delaminations was discussed in [4,37,38]. However, it is important to mention that there exist a paradox in
the model because the undamaged ply rotations are the first derivative of the transverse deflection. Consequently, the resulting shear stresses are only due to the damage induced anisotropy. While, it is well-established fact that the shear deformations play a significant role in the damage process. In fact, the model might be improved if the modifications were introduced to Mindlin's plate theory instead of using Kirchhoff's.

Grady and Meyn [32], devoted their efforts to measure the change in the mechanical properties and vibration frequency of 1" X 7" X 0.1" cross-ply laminated specimens after impact by using a non-destructive evaluation technique. During fabrication an artificial delamination starter was embedded in the specimens and they were tested before and after the application of the impact loading. These artificial delaminations were used to simulate the ones that might result during fabrication or in-service loading. Also, they have developed a simple finite element model to evaluate approximately the vibration frequency of the laminated cantilever beam. In their finite element model a plane-strain rectangular element with uniform distributed damage zone was used. The elements along the midsurface of that area were eliminated as a tool to reduce the stiffness due to damage. One should note that this model does not address the issue of characterizing the damage zone itself. The model only considers total specimen-width delamination and neglects both partial delamination and matrix cracking. Also, the elimination of the whole element after local delamination is not appropriate since at least the inplane-stiffness is still effective. Moreover, the proposed model could be suitable only for
one-dimensional specimens, which can be described through the thickness, and as such can not be applied to deal with two- or three-dimensional laminated composites.

2.5 SUMMARY OF DAMAGE MODELING

Based on the literature review, some of which has been discussed above, the significant aspects of relevant damage modeling approaches can be summarized as follows:

- direct reduction of elastic constants
- fracture mechanics approach
- damage mechanics approach (ISV):

In the damage mechanics approach, there exist three different models which are:

1. Talreja model, [104].
2. Joshi et. al model, [26-27,47].
3. Allen et. al model, [2-6,36-39].

From the above discussion, it can be concluded that among various damage models, the third model that has been proposed by Allen et. al [2-6,36-39] is more effective to characterize the matrix cracks type of damage due to the following key features:

1. The ISV's are dependent on the surface area of damage, energy release rate and the damage growth direction.
2. The model reflects clearly the anisotropy resulting from the induced damage.
3. Using this model, with some modifications, one can write the reduced stiffness coefficients in terms of ISV's.

4. The above feature is important since it can facilitate the use of the existing solution algorithms.

5. The model is experimentally oriented, which requires some modifications for its extended use as an analytical approach.

2.6 CONCLUDING REMARKS

From the previous investigation and discussion of the different damage evolution and assessment models, it is obvious that the characterization of damage is an important research field. It was established that the different available models are focusing on the accumulation of damage due to different loading and environmental conditions. In addition, most of these research efforts were devoted to study both static and during-impact behavior of laminated composite members. On the other hand, very little efforts were focused on the post-damage behavior especially, the dynamic characterization of damaged laminated composite plates and shells. Since the vibration properties are critical in many applications, it would be reasonable and effective to investigate a correlation between the stiffness reduction due to damage and the dynamic characteristics.
The extensive studies, both experimental and theoretical, of the damage modes indicate that matrix cracks [16,24] and interply delaminations [32,71] can significantly reduce laminate stiffness. Hence, a model is to be established to study the effects of these types of damage modes on the dynamic response of laminated composite plates and shallow shells. In addition, both impact loadings and environmental conditions may induce a little curvature to the damaged plate. Such curvature will cause the composite panel to be like a shallow shell with one or more of the aforementioned damage modes. Therefore, we need to incorporate the influence of such factor on the frequency response.

In the analysis phase, many investigations neglect the transverse shear deformations and adopt Kirchhoff's plate theory. But, it is well-documented that these stresses play an important role in the damage process, especially the interply delamination mode. Therefore, a shear deformable theory should be utilized for accurate characterization of post-damage behavior.

The shape size and extent of damage growth due to low-velocity impact was found to be dependent on the stacking sequence and fiber orientation of the laminated composite plate. Despite of the usual irregularity of the damage shape, it is desirable to develop a model that has the capability of solving the commonly observed damage shapes.

The solution phase of the problem requires using an approximate analytical solution technique, since the exact solutions are generally absent for such class of problems. And from the discussion on the different solution methods, we recognize that the finite
element method, Rayleigh-Ritz method or splines technique can provide a reliable tool for approximate, but yet accurate solution.

In conclusion, to investigate the post-damage dynamic behavior one can propose the following research aspects:

- Utilize a shear deformable theory to derive the governing field equations for a laminated composite shells and plates.
- Characterize the damage zone, shape and size, due to low-velocity impact.
- Develop a model that has the capacity to evaluate different elastic constants due to various modes of damage by extending the concept of internal state variable (ISV) constitutive relationships.
- Investigate the combined effect of curvature and matrix cracks damage mode on the frequency response.
- Study the free vibration properties for different stacking sequence, boundary conditions and damage modes.
- Compare the model solutions with the experimental results or other models, when applicable, to check the reliability of the proposed model.
Figure 2.1: Different damage modes for laminated composite plates. (a) matrix cracks, (b) delamination, and (c) fiber fracture.
Figure (2.2): Coordinate system for a damaged composite laminate.
Figure (2.3): Coordinate system and geometry for composite laminated shell.
Figure (2.4): Shallow geometry and coordinate systems.
Fig. (2.5): A typical stacking of a laminated shell
Figure (2.6): Effects of shear deformations and rotary inertia on the natural frequency of laminated plates, [110]
Figure (2.7): Deformation geometry due to delamination, [4].
Figure (2.8): Experimental and theoretical damage shapes and sizes [33]
Table (2.1): Frequency coefficients for an isotropic simply supported plate, [86].

\[ \frac{a}{b} = 1, \frac{a}{h} = 10, v = 0.3 \]
Table (2.2): Frequency coefficients for an isotropic simply supported plate, [86].

\[ \frac{a}{b} = 2, \frac{a}{h} = 10, \nu = 0.3 \]
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Table (2.3): Frequency coefficients for a simply supported cross-ply laminated plate. [86]

\[ \frac{a}{b} = 1, \quad \frac{E_1}{E_2} = 25, G_{12} = G_{13} = 0.5E_2, \quad G_{23} = 0.2E_2, \quad v_{12} = 0.5 \]
Table (2.4): Possible combinations of boundary conditions on each edge,[14].
Table (2.4): Continued

(b) Simply supported edge

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<td>S8</td>
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(continued)
(c) Clamped edge

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CHAPTER 3

MODELING OF DAMAGE

3.1 GENERAL

Among various damage modeling methods, the damage mechanics approach can be considered an effective tool for predicting the overall effect of matrix crack type of damage modes on all the stiffness components of a composite laminate. In this type of approach the effect of damage can be accounted for through a set of second order tensor valued internal state variables (ISV’s) expressions for matrix cracking. The resulting reduced elastic parameters can be used to investigate the influence of the influence of transverse matrix cracking on the free vibration frequency response of various composite plates and shells.
3.2 MATRIX CRACKS MODELING

The internal state variables (\( \alpha_{ij}^\eta \)) as defined in references [2,3] and described in (Appendix- A) are derived by using a locally volume averaged diadic product of the crack opening displacement vector (\( u \)) and the corresponding unit normal vector (\( n \)) as shown in figure(3.1), which can be written as:

\[
\alpha_{ij}^\eta = \frac{1}{V} \int_{S^n} u_i n_j dS
\]  

(3.1)

where the superscript (\( \eta \)) denotes each specific damage mode.

For a laminate with statistically homogeneous matrix cracks shown in figure(3.2), the internal state variables that describe the damage state due to matrix cracks type of damage can be written as:

\[
\alpha_{ij}^M = \frac{1}{V} \int_S u_i n_j dS
\]  

(3.2)

where,

\[ u_i = \text{displacement components on the crack faces}, \]

\[ n_i = \text{components of a unit normal to the crack surface}, \]

\[ V = \text{local volume element over which cracks are distributed, and} \]

\[ S = \text{surface area of cracks in the volume element (V)}. \]

It is to be noted here that the Internal State Variables describe the kinematics of the cracking process and they are strain-like quantities. In Voigt notation as given in reference
the stress-strain relation for a composite lamina with matrix cracks can be written as:

\[
\sigma_i = C_{ij} \varepsilon_j + D_{ij} \alpha_j^M
\]  

(3.3)

where, \(D_{ij}\) are the damage moduli and \(\alpha_j^M\) are the ISV's.

For a typically cracked ply shown in figure(3.3), in terms of local ply coordinates, the ISV for the shown transverse matrix cracks are given by Eqn.(3.2). For cracks normal to the laminate midplane, the unit outer normal vector is:

\[
\{n\} = \{0e_1, n_2 e_2, 0e_3\}
\]  

(3.4)

By using Eqns.(3.2) and (3.4), the only nonzero components of the damage parameters \(\{\alpha_j^M\}\) are \(\alpha_{22}^M, \alpha_{12}^M, \alpha_{32}^M\). The parameter \(\alpha_{32}^M\) represents out-of-plane mixed mode cracking that develops due to curved matrix cracks which can be assumed to be negligible for the special case of vertical cracks as shown in reference [38]. Therefore, the only none-zero ISV's for vertical matrix cracks which are normal to laminate midsurface, in the cracked ply local coordinates, become:

\[
\{\alpha_j^M\} = \begin{bmatrix}
\alpha_{22}^M \\
\alpha_{12}^M \\
\alpha_{32}^M
\end{bmatrix}
\]  

(3.5)
The damage tensor \( \{ \alpha^M \} \), given by Eqn.(3.5) can be transformed to the global laminate coordinates through the following transformation [34].

\[
\{ \alpha_{ij}^M \} = [T_{ij}] \{ \alpha^M \}
\] (3.6)

where the transformation matrix \([T_{ij}]\) is given by, [33]:

\[
[T_{ij}] = \begin{bmatrix}
     s^2 & -c s \\
     c^2 & c s \\
    -c s & -c^2 \\
    -c s & -s^2
\end{bmatrix}
\] (3.7)

Into which, \( c = \cos \theta \), and \( s = \sin \theta \), where \( \theta \) is the rotation of the local coordinate system \((1,2)\) with respect to the laminate global system \((x_1,x_2)\) as shown in Figure (3.3).

### 3.3 Expressions for Elastic Constants

Based on the internal state variables (ISV’s), Allen et al. [2-6], Groves et al. [33,34] and Harris et al. [37-39] introduced general expressions to estimate the effective elastic constants. Their work was mainly devoted to evaluate the axial stiffness of crossply laminate [2,4] and angle-ply laminate [37] under vertical matrix cracks. However, Harris et al. [38] extended the case of vertical matrix cracks to study the effect of curved ones.
They also investigated the effect of matrix cracks on Poisson's ratio for a crossply laminate. After some algebraic manipulation, general expressions for elastic constants of N-layers crossply laminate can be written as.

\[
E_x = \frac{1}{N} \sum_{k=1}^{N} \frac{Q_{11}^k}{Q_{11}^k} \left\{ 1 - \frac{\partial \alpha_{x}^M}{\partial \varepsilon_{x}} \right\}^k
\]

(3.8)

\[
E_y = \frac{1}{N} \sum_{k=1}^{N} \frac{Q_{22}^k}{Q_{22}^k} \left\{ 1 - \frac{\partial \alpha_{y}^M}{\partial \varepsilon_{y}} \right\}^k
\]

(3.9)

\[
\nu_{xy} = \frac{1}{N} \sum_{k=1}^{N} \frac{Q_{12}^k}{Q_{12}^k} \left\{ 1 - \frac{\partial \alpha_{r}^M}{\partial \varepsilon_{r}} \right\}^k \frac{\partial \alpha_{x}^M}{\partial \varepsilon_{xy}} \left\{ 1 - \frac{\partial \alpha_{y}^M}{\partial \varepsilon_{y}} \right\}^k
\]

(3.10)

This expression for \( \nu_{xy} \) might be approximated as, [38]:

\[
\nu_{xy} = \frac{1}{N} \sum_{k=1}^{N} \frac{Q_{12}^k}{Q_{12}^k} \left\{ 1 - \frac{\partial \alpha_{x}^M}{\partial \varepsilon_{x}} \right\}^k \frac{\partial \alpha_{x}^M}{\partial \varepsilon_{x}} \left\{ 1 - \frac{\partial \alpha_{y}^M}{\partial \varepsilon_{y}} \right\}^k
\]

(3.11)

For the case of vertical cracks in the 90-degree plies, there is no coupling between shearing and normal components and the cracking mechanism is directly related to
Mode I crack-opening displacement. As a result the change in $\frac{\partial \alpha_y^M}{\partial \varepsilon_y}$ is negligibly small, then $(E_y)$ might remain unchanged for preliminary calculations and the expression for Poisson's ratio, Eqn.(3.11), reduces to:

$$v_{12} = \frac{1}{N} \sum_{k=1}^{N} \frac{Q_{12}^k}{Q_{22}^k} \left\{ 1 - \frac{Q_{12} \partial \alpha_y^M}{Q_{22} \partial \varepsilon_x} \right\}^k$$

(3.12)

Lee et al. [53] concluded that the effect of the adjacent layer constraint on the cracked one is negligibly small [53]. Therefore, the above expressions can be specified for the case of a single lamina. Under the assumption of matrix cracks normal to laminate midplane and for a single layer in the local coordinate system, we can get the following expressions:

$$\hat{Q}_{11} = Q_{11} \left\{ 1 - \left( \frac{Q_{12}}{Q_{11}} \right)^2 \frac{\partial \alpha_y^M}{\partial \varepsilon_x^{22}} \right\}$$

(3.13)

$$\hat{Q}_{22} = Q_{22} \left\{ 1 - \frac{\partial \alpha_y^M}{\partial \varepsilon_x^{22}} \right\}$$

(3.14)

$$\hat{Q}_{66} = Q_{66} \left\{ 1 - \frac{\partial \alpha_y^M}{\partial \varepsilon_x^{12}} \right\}$$

(3.15)

$$\hat{v}_{12} = v_{12} \left\{ 1 - v_{12} \frac{\partial \alpha_y^{M}}{\partial \varepsilon_x^{22}} \right\}$$

(3.16)
Where, \(( Q_{ij}^k)\) and \(( \tilde{v}_{12}^k)\) are the damaged lamina stiffness components and Poisson's ratio respectively and \(( Q_{ij}^k)\) are the undamaged initial conventional lamina stiffnesses.

On the basis of the standard mechanics of composites [106,110], the approximate expressions for the undamaged initial conventional lamina stiffness components can be related to the lamina elastic constants through the following relations.

\[
Q_{11}^k = \frac{E_{11}^k}{(1 - \nu_{12}^k \nu_{21}^k)}
\]

(3.17)

\[
Q_{22}^k = \frac{E_{22}^k}{(1 - \nu_{12}^k \nu_{21}^k)}
\]

(3.18)

\[
Q_{12}^k = \frac{\nu_{12}^k E_{22}^k}{(1 - \nu_{12}^k \nu_{21}^k)}
\]

(3.19)

\[
Q_{66}^k = G_{12}^k
\]

(3.20)

By using Eqns.(3.13-3.16) and Eqns.(3.17)-(3.20), the reduced elastic constants can be calculated provided that expressions for \(( \frac{\partial \alpha^M_j}{\partial \epsilon_{ij}})\) can be found. The discussion to follow will elaborate on this matter.

In order to evaluate the partial derivatives of the internal state variables with respect to the corresponding strain components, Allen et al. [2-4] used an experimentally-based
approach. The local energy loss due to matrix cracks was related to an experimentally determined value of the strain energy release rate for cracks developed in the 90° ply only.

An expression for the damage parameter \( \frac{\partial \alpha^{M}_{22}}{\partial \varepsilon_{22}} \) for a single 90° ply in a crossply laminate such as \([0^o, 90^o]_s\) can be written as:

\[
\frac{\partial \alpha^{M}_{22}}{\partial \varepsilon_{22}} = \frac{n(q + r)}{2r} \frac{E^0_x}{E_{22}} \left( \zeta(s) - 1 \right)
\]

where \( n \) is the number of 90-degree sub-layers in the 90-degree ply and \( \zeta(s) \) is an experimental expression which depends on the crack density and \( E^0_x \) is the initial undamaged laminate modulus. The total surface area of the matrix cracks \( s \) is taken as a measure for this crack density. From tests on Graphite/Epoxy of \([0^o, 90^o, 0^o]_s\) laminate, an expression of \( \zeta(s) \) was given by as Allen et al. [3]:

\[
\zeta(s) = 0.9969 - 0.061607(s) + 0.04623(s)^2
\]

The other derivatives of the internal state variables were neglected [2,3] as an approximation to evaluate the effective normal stiffness of a crossply laminate. Harris et
al. [38], extended the use of the model proposed by Allen et al. [2,3] to predict various components of the effective stiffness for quasi-isotropic laminates with matrix cracks. For angle-ply laminate a scaling factor \( K \) was introduced to account for the adjacent layer constraints, thus:

\[
\frac{\partial \alpha_{22}^M}{\partial \varepsilon_{22}} \bigg|_{\theta_1, 90/\theta_2} = K(\theta_1, \theta_2) \frac{\partial \alpha_{22}^M}{\partial \varepsilon_{22}} \bigg|_{(0, 90/0)_5} \tag{3.23}
\]

where \( K \) is approximately equal unity for \( 0^\circ \leq \theta_1 = \theta_2 = \theta \leq 60^\circ \) as it was revealed by the finite element investigation [39]. The approximation given by Eqn. (3.23) was generalized to evaluate the other derivatives of the internal state variables. In a similar fashion, for vertical cracks the other component can be written as:

\[
\frac{\partial \alpha_{22}^M}{\partial \varepsilon_{11}} \bigg|_{\theta_1, 90/\theta_2} = \frac{Q_{12}}{Q_{11}} K(\theta_1, \theta_2) \frac{\partial \alpha_{22}^M}{\partial \varepsilon_{22}} \bigg|_{(0, 90/0)_5} \tag{3.24}
\]

Harris et al. [39] used the results of tensile tests of Graphite/Epoxy \([\pm 45^\circ]_{25}\) which are essentially in a state of pure shear, and assumed a linear relationship between \( G_{12} \) and the surface area of the matrix cracks to derive an expression for \( \frac{\partial \alpha_{12}^M}{\partial \varepsilon_{12}} \), which can be written as:
where \( G_{12}^e \) is the experimental damaged shear modulus and \( G_{12}^0 \) is the initial undamaged shear modulus and \( S_m \) is the matrix crack surface area in a local volume element for a single ply and \( S_e \) is the experimental surface area of cracks at which \( G_{12}^e \) is measured. The experimental investigations [33] show that for Graphite-Epoxy laminate \( G_{12}^e \) = 0.822 measured at \( S_e=21 \) cracks per inch in each ply of \([\pm45^\circ]_{2s}\) laminate. It was concluded, [33] that the expression given by Eqn.(3.25) may be used to determine the damage parameter for any ply with matrix crack damage. The orientation of the cracked ply is accounted for by the coordinate transformation as given in Appendix A.

It must be noted that the evaluation of the above expressions for the elastic constants depend on estimating the internal state variables \( \alpha_{12}^M \). The previously presented model of Allen et al. [2-6] can evaluate the reduced axial stiffness of crossply laminates based on an experimental relations for the degraded laminate modulus corresponding to different crack damage states. It is obvious that these experimental expressions are not always easy to obtain because of financial and time limitations. Therefore, analytical expressions, though might be approximate, are more desirable. Such approach was introduced by Lee et al. [53] for analytical approximation of the internal state variable expressions. The procedure are based on the assumption that the

\[
\frac{\partial \alpha_{12}^M}{\partial \varepsilon_{12}} = 2 \left(1 - \frac{G_{12}^e}{G_{12}^0}\right) \frac{S_m}{S_e}
\]  

(3.25)
crack plane remains plane throughout the deformation process. Which simply means that the transverse shear deformations due to matrix cracks to be neglected. Such approximation might be accepted as a penalty to get the required expressions. Accordingly, the displacement field was assumed [53] to be linear in the plane of the crack while in the normal direction, i.e., the opening direction of the crack, the displacement field was assumed a trigonometric polynomial, which can be written as:

\[ u^c = \frac{v_0}{a_c} x_2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \alpha x_2 \cos \beta x_3 \]  

(3.26)

where:

\[ \alpha = \frac{(2m-1)\pi}{2a_c} ; \]

\[ \alpha = \frac{(2n-1)\pi}{2t} ; \]

\( a_c \) and \( t \) are defined in Figure (3.2).

By assuming that the cracks are statistically arranged and by using a local volume element \( (a_c x 2t x 2b_c) \), as shown in Figure (3.2), a fictitious boundary value problem is obtained. By minimizing the total potential energy in the local volume element, an approximate analytical expression for the partial derivative of the damage parameter \( \frac{\partial \alpha^{M}_{22}}{\partial \varepsilon_{22}} \) can be written as:
where, \( \mu \) is a function of the lamina Poisson's ratios, and can be written as:

\[
\mu = \left(1 - \nu_{12}^2\right)/(1 - \nu_{21} - 2\nu_{12}^2)(1 + \nu_{21})
\]  

(3.28)

and,

\[
\beta_c = \sum_{m,n=1}^{\infty} \frac{1}{\{2(m-1)^2(2n-1)^2 + (2n-1)^4 \frac{G_{12}(a_c/t)^2}{\mu E_{22}}\}}
\]  

(3.29)

One should note here that \((2a_c)\) is the crack spacing, and \((2t)\) is the crack length. In this way, the damage parameter is related to the material parameters as well as the matrix crack density \((N = a_c/t)\).

In fact, Eqns.(3.25) through (3.27) represent the simplest analytical for the required derivatives of the damage parameters with respect to the corresponding strain components. By substituting the results of Eqns.(3.25) and (3.27) into Eqns.(3.13) through (3.16), we can obtain the damaged stiffness components for a cracked ply in its local coordinate system. For the expressions of the damaged elastic parameters in the global coordinate system, Eqns.(3.8)-(3.10) and Eqn.(3.12) can be used. For such a case the derivatives of the damage parameters given by Eqns.(3.25) and (3.27) should be
transformed to the global coordinate system using the transformation given in the Appendix (A).

Now, before concluding our discussion on deriving expression for the elastic constants due to matrix cracks, we present some numerical results. Table (3.1) shows the material properties of Graphite/Epoxy Laminate used in this numerical example. Two different crossply laminates are investigated as shown in Table (3.2). From the comparison of the effective normal stiffness, there is good agreement between the two analytical models and the experimental investigation. Assuming the validity of the approximation proposed by Equations (3.23) and (3.24) for angle-ply laminates, another two laminates are analyzed using the same material properties given in Table (3.1). The two laminates are \([0/\pm 45\degree]_s\) and \([\pm 45\degree]_s\) and the normalized stiffness is evaluated using 21 cracks per inch in each 45\degree-plies. The results are compared with those of the experimental investigation [33] in Table (3.3). The depicted results show that the margin of error increases from less than 1\% for crossply laminate \([0/90]_s\).

The above discussion was developed for the case of matrix cracks in a plane normal to the laminate mid-surface. However, experimental investigations show that matrix cracks are often curved with respect to the normal to laminate midplane. This type of crack develops for laminate with three or more 90-degree plies are stacked in succession. This result is reflected in Figures (3.4) through (3.6) which compare the present results with experimental investigation [33] for three crossply laminates. They have the stacking sequence of \([0^\circ / 90^\circ]_s\), \([0^\circ / 90_2^\circ]_s\) and \([0^\circ / 90_3^\circ]_s\). One can see that the predicted
results departure from the experimental one increases for thicker 90-degree plies as the number of cracks per inch increases. Then, one can conclude that the developed expressions are reasonably accurate and that the assumption of vertical matrix cracks which are normal to the laminate midsurface can be justified for thin to moderately-thick laminates.

The expressions for the reduced stiffness components developed in this chapter are used in the next two chapters to study and quantify the changes in the frequency response of laminated composite plates and shallow cylindrical shells containing matrix crack type of damage mode.
Fig.(3.1): Description of the crack geometry at point A.
Fig.(3.2): Matrix cracking and local coordinates for a single ply.
Fig. 3.3: Composite laminate local (1-2) and global (X₁-X₂) coordinate systems.
Figure (3.4) Variation of normalized stiffness with number of cracks, $[0^\circ/90^\circ]_8$.
Figure (3.5) Variation of normalized stiffness with number of cracks, [0/90₂]₅
Figure (3.6): Variation of normalized stiffness with number of cracks, [0/90]s
<table>
<thead>
<tr>
<th>N*</th>
<th>$E_x/E_x^0$</th>
<th>$\nu_{xy}/\nu_{xy}^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>10</td>
<td>0.9979</td>
<td>0.9440</td>
</tr>
<tr>
<td>20</td>
<td>0.99377</td>
<td>0.8969</td>
</tr>
<tr>
<td>30</td>
<td>0.98923</td>
<td>0.8595</td>
</tr>
<tr>
<td>40</td>
<td>0.98463</td>
<td>0.8510</td>
</tr>
</tbody>
</table>

*N=number of cracks per in.

Table (3.1): Changes in $E_x$ and $\nu_{xy}$ due matrix cracking in the $90^\circ$-plies of $[0^\circ/90^\circ]_s$ laminate.
### Table 3.2: Comparison of normalized axial stiffness ($E_x/E_x^0$) due to matrix cracking in the 90°-plies.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0°/90°]_s$</td>
<td>46</td>
<td>0.969</td>
<td>0.962</td>
<td>0.975</td>
</tr>
<tr>
<td>$[0_{2}°/90_{2}°]_s$</td>
<td>46</td>
<td>0.958</td>
<td>0.953</td>
<td>0.960</td>
</tr>
</tbody>
</table>

*N=number of cracks per in.

---

### Table 3.3: Comparison of normalized axial stiffness ($E_x/E_x^0$) due to matrix cracking in the ±45°-plies.

<table>
<thead>
<tr>
<th>Laminate</th>
<th>$N^*$</th>
<th>Experimental [33]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0°±45°]_s$</td>
<td>21</td>
<td>0.94</td>
<td>0.974</td>
</tr>
<tr>
<td>$[±45°]_s$</td>
<td>21</td>
<td>0.822</td>
<td>0.822</td>
</tr>
</tbody>
</table>

*N=number of cracks per in.

---

Table 3.2: Comparison of normalized axial stiffness ($E_x/E_x^0$) due to matrix cracking in the 90°-plies.

86
CHAPTER 4

ENERGY FORMULATION FOR LAMINATED COMPOSITE SHALLOW SHELLS

4.1 INTRODUCTION

In area of structural mechanics, an energy formulation is often viewed as an alternative to the differential formulation of the initial as well as boundary value problems. In this section, we introduce the energy formulation which is also considered a good basis for many powerful numerical methods that can be used to obtain approximate solutions for many problems.

In this section, we highlight the energy formulation as well as the governing equations of motion for a general laminated composite shallow shell as discussed in Chapter (2), Section (2.2.2.1). Then some specializations as well as simplifications are introduced for special shell geometry. Also, the reduced energy expressions for such special cases are compared. when possible, with similar expressions in the open literature.
4.2 STRAIN ENERGY

The strain energy for an elastic body \( (U_E) \) may be written as [109]:

\[
U_E' = \frac{1}{2} \int \int \int_V \sigma_{ij} \epsilon_{ij} dV
\]  

(4.1)

where \( V \) represent the volume of the continuum.

By using the Cartesian coordinate system discussed and shown in Figure (2.4) and by assuming that the normal strain \( \epsilon_{33} = 0 \), in, the above expression for the strain energy for an N-layer laminated composite shallow shell may be written as:

\[
U_E = \frac{1}{2} \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \int_A \left\{ \sigma_{11}\epsilon_{11} + \sigma_{22}\epsilon_{22} + \sigma_{12}\epsilon_{12} + \sigma_{13}\epsilon_{13} + \sigma_{23}\epsilon_{23} \right\} (1 + \frac{x_3}{R_1})(1 + \frac{x_3}{R_2}) dx_1 dx_2 dx_3
\]  

(4.2)

Substituting Eqn.(2.21) into the above expression, Eqn. (4.2), for the k-th lamina we can write:
\[ U_E = \frac{1}{2} \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \int_{A} \{ \overline{Q}^{(k)}_{11} \varepsilon_{11}^2 + \overline{Q}^{(k)}_{22} \varepsilon_{22}^2 + \overline{Q}^{(k)}_{66} \varepsilon_{12}^2 + \overline{Q}^{(k)}_{55} \varepsilon_{13}^2 + \overline{Q}^{(k)}_{44} \varepsilon_{23}^2 \\
+ 2 \overline{Q}^{(k)}_{12} \varepsilon_{11} \varepsilon_{22} + 2 \overline{Q}^{(k)}_{16} \varepsilon_{11} \varepsilon_{12} + 2 \overline{Q}^{(k)}_{26} \varepsilon_{22} \varepsilon_{12} \\
+ 2 \overline{Q}^{(k)}_{45} \varepsilon_{13} \varepsilon_{23} \} \left(1 + \frac{x_3}{R_1}\right) \left(1 + \frac{x_3}{R_2}\right) dx_1 dx_2 dx_3 \] (4.3)

Eqn.(4.3) is the basis for the finite element solution developed and discussed in Chapter(6) which uses a discrete laminate theory for the special case of laminated composite flat plate.

In the above expression, Eqns.(2.14)-(2.20) can be used to substitute for the strains in terms of the midsurface displacements. Also, the definitions of the stiffness coefficients given by Eqns.(2.29)-(2.34) can be employed to rewrite Eqn.(4.3) explicitly in terms of generalized displacements; \( u \) and \( \phi_a \) after dropping the superscript \( (0) \) for simplicity.

By using the assumption introduced in Chapter (2) for \( h^2/R_\alpha R_\beta << 1 \) and integration over the laminate thickness, Eqn.(4.3) may be written as:

\[ U_E = U_s + U_b + U_u + U_t \] (4.4)

where

- \( U_s \) = the energy terms due to stretching,
- \( U_b \) = the energy terms due to bending,
- \( U_u \) = the energy terms due to unsymmetrical coupling coefficients, and
$U'$ are the energy terms due to transverse shear stresses.

The above terms can be written explicitly as follows [14]:

$$U' = \frac{1}{2} \int_A \left\{ A_{11} \left( u_{1,1} + \frac{u_3}{R_1} \right)^2 + A_{22} \left( u_{2,2} + \frac{u_3}{R_2} \right)^2 + A_{66} \left( u_{1,2} + u_{2,1} + \frac{2u_3}{R_{12}} \right)^2 
+ 2A_{12} \left( u_{1,1} + \frac{u_3}{R_1} \right) (u_{2,2} + \frac{u_3}{R_2}) 
+ 2A_{16} (u_{1,1} + \frac{u_3}{R_1})(u_{1,2} + u_{2,1} - \frac{2u_3}{R_{12}}) 
+ 2A_{26} (u_{2,2} + \frac{u_3}{R_2})(u_{1,2} + u_{2,1} - \frac{2u_3}{R_{12}}) \right\} dA$$

(4.5)

$$U_b = \frac{1}{2} \int_A \left\{ D_{11} \phi_{1,1}^2 + D_{22} \phi_{2,2}^2 + D_{66} (\phi_{1,2} + \phi_{2,1})^2 + 2D_{12} \phi_{1,1} \phi_{2,2} 
+ 2D_{16} (\phi_{1,1} \phi_{2,1} + \phi_{1,1} \phi_{1,2}) + 2D_{26} (\phi_{2,1} \phi_{2,2} + \phi_{1,2} \phi_{2,2}) 
- 2C_0 \left[ D_{11} \left( u_{1,1} + \frac{u_3}{R_1} \right) \phi_{1,1} 
- D_{22} \left( u_{2,2} + \frac{u_3}{R_2} \right) \phi_{2,2} 
+ D_{66} \left( u_{2,1} + \frac{u_3}{R_{12}} \right) \phi_{2,1} - D_{66} \left( u_{1,2} + \frac{u_3}{R_{12}} \right) \phi_{1,2} 
+ D_{16} \left( u_{1,1} + \frac{u_3}{R_1} \right) \phi_{2,1} + D_{16} \left( u_{2,1} + \frac{u_3}{R_{12}} \right) \phi_{1,1} 
- D_{26} \left( u_{2,2} + \frac{u_3}{R_2} \right) \phi_{1,2} - D_{26} \left( u_{1,2} + \frac{u_3}{R_{12}} \right) \phi_{2,2} \right] \right\} dA$$

(4.6)
\[ U_\alpha = \int_A \left\{ B_{11}(u_{1,1} + \frac{u_3}{R_1})\phi_{1,1} + B_{22}(u_{2,2} + \frac{u_3}{R_2})\phi_{2,2} \right. \]
\[ + B_{66}(u_{1,2} + u_{2,1} + \frac{2u_3}{R_{12}})(\phi_{1,2} + \phi_{2,1}) \]
\[ + B_{12}(u_{1,1} + \frac{u_3}{R_1})\phi_{2,2} + B_{12}(u_{2,2} + \frac{u_3}{R_2})\phi_{1,1} \]
\[ + B_{16}(u_{1,1} + \frac{u_3}{R_1})(\phi_{1,2} + \phi_{2,1}) + B_{16}(u_{1,2} + u_{2,1} + \frac{2u_3}{R_{12}})\phi_{1,1} \]
\[ + B_{26}(u_{2,2} + \frac{u_3}{R_2})(\phi_{1,2} + \phi_{2,1}) + B_{26}(u_{1,2} + u_{2,1} + \frac{2u_3}{R_{12}})\phi_{2,2} \]
\[ - \frac{1}{2} C_0 \left[ B_{11}(u_{1,1} + \frac{u_3}{R_1})^2 - B_{22}(u_{2,2} + \frac{u_3}{R_2})^2 \right. \]
\[ + B_{66}(u_{2,1} + \frac{u_3}{R_{12}})^2 - B_{66}(u_{1,2} + \frac{u_3}{R_{12}})^2 \]
\[ + 2B_{16}(u_{1,1} + \frac{u_3}{R_1})(u_{2,1} + \frac{u_3}{R_{12}}) \]
\[ + 2B_{26}(u_{2,2} + \frac{u_3}{R_2})(u_{1,2} + \frac{u_3}{R_{12}}) \]
\[ - F_{11}\phi_{1,1}^2 - F_{22}\phi_{2,2}^2 + F_{66}(\phi_{2,1}^2 - \phi_{1,2}^2) \]
\[ + 2F_{16}\phi_{1,1}\phi_{2,1} - 2F_{26}\phi_{2,2}\phi_{1,2} \]
\[ + B_{55}(u_{3,1} - \frac{u_1}{R_1} + \phi_{1})^2 \]
\[ + B_{44}(u_{3,2} - \frac{u_2}{R_2} - \frac{u_1}{R_{12}} + \phi_{2})^2 \left. \right\} dA \]

(4.7)
\[ U_\ell = \frac{1}{2} \int \left\{ A_{3\ell}(u_{3,1} - \frac{u_1}{R_1} - \frac{u_2}{R_{12}} + \Phi_1)^2 + A_{4\ell}(u_{3,2} - \frac{u_2}{R_2} - \frac{u_1}{R_{12}} + \Phi_2)^2 \right. \\
\left. + 2A_{45}(u_{3,1} - \frac{u_1}{R_1} - \frac{u_2}{R_{12}} + \Phi_1)(u_{3,2} - \frac{u_2}{R_2} - \frac{u_1}{R_{12}} + \Phi_2) \right\} dA \]

(4.8)

where,

\[ C_0 = r_0 \left\{ \frac{1}{R_1} - \frac{1}{R_2} \right\} \]

(4.9)

\[ C_1 = r_0 \left\{ \frac{1}{R_1} + \frac{1}{R_2} \right\} \]

(4.10)

In the expressions for \( C_0 \) and \( C_1 \) given by Eqns. (4.9) and (4.10), the symbol \( r_0 \) is used as a tracer for including or excluding the terms with \( (1 \pm \frac{x_3}{R_1}) \). Such terms are excluded from the analysis if \( r_0 = 0 \) and are included if \( r_0 = 1 \).

It is important to note that the stiffness components (\( A_{ij}, B_{ij}, D_{ij} \) and \( F_{ij} \)) in the above are defined by Eqns. (2.29)-(2.34) and are constructed separately for each zone, i.e. damaged and undamaged zones using the appropriate stiffness coefficients.

### 4.3 KINETIC ENERGY

For a body with a volume \( V \) with mass density independent of time, the kinetic energy \( (T) \) can be written as:

92
The kinetic energy can be written explicitly as:

\[
T = \frac{1}{2} \int_V \rho \left( \dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2 \right) + 2H_2 (\dot{\phi}_1 + \dot{\phi}_2) \\
+ H_3 (\dot{\phi}_1 + \dot{\phi}_2) \\
+ C_1 \left[ H_2 (\dot{u}_1 + \dot{u}_2 + \dot{u}_3) + 2H_3 (\dot{\phi}_1 + \dot{\phi}_2) \\
+ H_4 (\dot{\phi}_1 + \dot{\phi}_2) \right] dA
\]  

where

\[
H_i = \sum_{k=1}^{N} h_i \int_{h_{i-1}}^{h_i} \rho^{(k)}(x_3)^{i-1} dx_3 \quad (i=1,2,3,4)
\]

4.4 SPECIALIZATIONS

4.4.1 Simplified Shallow Shell

As it has been already mentioned in Section (2.2.2.1), some assumptions may be introduced to reduce the degree of complexity. Before integrating over the laminate thickness and by assuming that \( (1 \pm \frac{x_3}{R_i}) = 1 \) and neglecting the effect of the inplane
displacement components on the transverse shear strains, the strain energy functionals
given by Eqns. (4.5) through (4.8) reduce to:

\[
U' = \frac{1}{2} \int_A \left\{ A_{11}(u_{1,1} + \frac{u_3}{R_1})^2 + A_{22}(u_{2,2} + \frac{u_3}{R_2})^2 + A_{66}(u_{1,2} + u_{2,1} + \frac{2u_3}{R_{12}})\right) \\
+ 2A_{12}(u_{1,1} + \frac{u_3}{R_1})(u_{2,2} + \frac{u_3}{R_2}) + 2A_{16}(u_{1,1} + \frac{u_3}{R_1})(u_{1,2} + u_{2,1} - \frac{2u_3}{R_{12}}) \\
+ 2A_{26}(u_{2,2} + \frac{u_3}{R_2})(u_{1,2} + u_{2,1} - \frac{2u_3}{R_{12}})\} \, dA
\]  

(4.14)

\[
U_b = \frac{1}{2} \int_A \left\{ D_{11}\phi_{1,1}^2 + D_{22}\phi_{2,2}^2 + D_{66}(\phi_{1,2} + \phi_{2,1})^2 + 2D_{12}\phi_{1,1}\phi_{2,2} \\
+ 2D_{16}(\phi_{1,1}\phi_{2,1} + \phi_{1,1}\phi_{1,2}) + 2D_{26}(\phi_{2,1}\phi_{2,2} + \phi_{1,2}\phi_{2,2})\} \, dA
\]  

(4.15)
\[
U'_u = \int_A \left\{ B_{11} \left( u_{1,1} + \frac{u_3}{R_1}\right) \phi_{1,1} + B_{22} \left( u_{2,2} + \frac{u_3}{R_2}\right) \phi_{2,2} + B_{66} \left( u_{1,2} + u_{2,1} + \frac{2u_3}{R_{12}}\right) \left( \phi_{1,2} + \phi_{2,1}\right) + B_{12} \left( u_{1,1} + \frac{u_3}{R_1}\right) \phi_{2,2} + B_{12} \left( u_{2,2} + \frac{u_3}{R_2}\right) \phi_{1,1} + B_{16} \left( u_{1,1} + \frac{u_3}{R_1}\right) \left( \phi_{1,2} + \phi_{2,1}\right) + B_{16} \left( u_{1,2} + u_{2,1} + \frac{2u_3}{R_{12}}\right) \phi_{1,1} + B_{26} \left( u_{2,2} + \frac{u_3}{R_2}\right) \left( \phi_{1,2} + \phi_{2,1}\right) + B_{26} \left( u_{1,2} + u_{2,1} + \frac{2u_3}{R_{12}}\right) \phi_{2,2} \right\} dA
\]

(4.16)

\[
U'_1 = \frac{1}{2} \int \left\{ A_{55} \left( u_{3,1} + \phi_1\right)^2 + A_{44} \left( u_{3,2} + \phi_2\right)^2 + 2A_{45} \left( u_{3,1} + \phi_1\right)(u_{3,2} + \phi_2) \right\} dA
\]

(4.17)

and the kinetic energy given by Eqn.(4.12) becomes:

\[
T = \frac{1}{2} \int_A \left\{ H_1 \left( \ddot{u}_1^2 + \ddot{u}_2^2 + \ddot{u}_3^2\right) + 2H_2 \left( \dot{u}_1 \dot{\phi}_1 + \dot{u}_2 \dot{\phi}_2\right) + H_3 \left( \dot{\phi}_1^2 + \dot{\phi}_2^2\right) \right\} dA
\]

(4.18)
**4.4.2 Spherical Shell**

Additional specializations can be made for some known shell geometry. For a spherical shell, for example the radii of curvatures $R_{12} \rightarrow \infty$ and $R_1=R_2=R$. Hence, the strain energy expressions for the simplified shallow shell given by Eqns.(4.14) and (4.16) can be rewritten as:

$$U_s = \frac{1}{2} \int_A \left\{ A_{11}(u_{1,1} + \frac{u_3}{R})^2 + A_{22}(u_{2,2} + \frac{u_3}{R})^2 + A_{66}(u_{1,2} + u_{2,1})^2 \\
+ 2A_{12}(u_{1,1} + \frac{u_3}{R})(u_{2,2} + \frac{u_3}{R}) + 2A_{16}(u_{1,1} + \frac{u_3}{R})(u_{1,2} + u_{2,1}) \\
+ 2A_{26}(u_{2,2} + \frac{u_3}{R})(u_{1,2} + u_{2,1}) \right\} dA$$

(4.19)

$$U_u = \int_A \left\{ B_{11}(u_{1,1} + \frac{u_3}{R})\phi_{1,1} + B_{22}(u_{2,2} + \frac{u_3}{R})\phi_{2,2} \\
+ B_{66}(u_{1,2} + u_{2,1})(\phi_{1,2} + \phi_{2,1}) \\
+ B_{12}(u_{1,1} + \frac{u_3}{R})\phi_{2,2} + B_{12}(u_{2,2} + \frac{u_3}{R})\phi_{1,1} \\
+ B_{16}(u_{1,1} + \frac{u_3}{R})(\phi_{1,2} + \phi_{2,1}) + B_{16}(u_{1,2} + u_{2,1})\phi_{1,1} \\
+ B_{26}(u_{2,2} + \frac{u_3}{R})(\phi_{1,2} + \phi_{2,1}) + B_{26}(u_{1,2} + u_{2,1})\phi_{2,2} \right\} dA$$

(4.20)
The other parts of the strain energy and the kinetic energy will stay the same as given by Eqns.(4.15) and (4.17-4.18).

**4.4.3 Cylindrical Shell**

For a shallow circular cylindrical shell, the radii of curvatures are reduced to: $R_{12} \to \infty$, $R_1 \to \infty$ and $R_2 = R$. Then, the energy expressions given by Eqns.(4.14) and (4.16) take the following simple form:

\[
U' = \frac{1}{2} \int_A \left\{ A_{11}(u_{1,1})^2 + A_{22}(u_{2,2} + \frac{u_3}{R})^2 + A_{66}(u_{1,2} + u_{2,1})^2 \\
+ 2A_{12}(u_{1,1})(u_{2,2} + \frac{u_3}{R}) + 2A_{16}(u_{1,1})(u_{1,2} + u_{2,1}) \\
+ 2A_{26}(u_{2,2} + \frac{u_3}{R})(u_{1,2} + u_{2,1}) \right\} dA
\]  

(4.21)
\[
U_u = \int_A \left\{ B_{11}(u_{1,1})\phi_{1,1} + B_{22}(u_{2,2} + \frac{u_3}{R})\phi_{2,2} \\
+ B_{66}(u_{1,2} + u_{2,1})(\phi_{1,2} + \phi_{2,1}) \\
+ B_{12}(u_{1,1})\phi_{2,2} + B_{12}(u_{2,2} + \frac{u_3}{R})\phi_{1,1} \\
+ B_{16}(u_{1,1})(\phi_{1,2} + \phi_{2,1}) + B_{16}(u_{1,2} + u_{2,1})\phi_{1,1} \\
+ B_{26}(u_{2,2} + \frac{u_3}{R})(\phi_{1,2} + \phi_{2,1}) \\
+ B_{26}(u_{1,2} + u_{2,1})\phi_{2,2} \right\} dA
\]

(4.22)

The above expressions are the same as those given by researches in the references [7, 14, 77, 106].

4.4.4 Laminated Composite Plate

For a laminated composite plate, with radii of curvatures \( R_i \to \infty \) (i=1,2,1-2) and the effect of curvature will vanish. Hence, Eqns.(4.21) and (4.22) are reduced to the following simple form:
\[ U_s = \frac{1}{2} \int A \left\{ A_{11}(u_{1,1})^2 + A_{22}(u_{2,2})^2 + A_{66}(u_{1,2} + u_{2,1})^2 \\
+ 2A_{12}(u_{1,1})(u_{2,2}) + 2A_{16}(u_{1,1})(u_{1,2} + u_{2,1}) \\
+ 2A_{26}(u_{2,2})(u_{1,2} + u_{2,1}) \right\} dA \]  

(4.23)

\[ U_u = \int A \left\{ B_{11}(u_{1,1})\phi_{1,1} + B_{22}(u_{2,2})\phi_{2,2} \\
+ B_{66}(u_{1,2} + u_{2,1})(\phi_{1,2} + \phi_{2,1}) \\
+ B_{12}(u_{1,1})\phi_{2,2} + B_{12}(u_{2,2})\phi_{1,1} \\
+ B_{16}(u_{1,1})(\phi_{1,2} + \phi_{2,1}) + B_{16}(u_{1,2} + u_{2,1})\phi_{1,1} \\
+ B_{26}(u_{2,2})(\phi_{1,2} + \phi_{2,1}) \\
+ B_{26}(u_{1,2} + u_{2,1})\phi_{2,2} \right\} dA \]  

(4.24)

For convenience, the different components of the strain energy for such a special case of a laminated composite plate are added together. This is to facilitate comparison and use in the finite element analysis and to accommodate within the discrete laminate model. Therefore, adding Eqns (4.15), (4.17), (4.23) and (4.24) the strain energy for a laminated composite plate with first order shear deformations can be written as:
4.5 THE GOVERNING EQUATIONS OF MOTION

4.5.1 Equations of Motion for Laminated Shallow Shell

The equations of motion (2.56)-(2.58) for a laminated shallow shell, in terms of the generalized displacements \((u_i)\) and \((\phi_a)\) after dropping the subscripts ‘0’, can be written using the following differential operators form:
\[ L_{ij} u_i + M_{ij} \ddot{u}_i = 0 \quad (i,j=1, \ldots, 5) \]

where, \( L_{ij} \) are differential operators and \( M_{ij} \) are the components of mass matrix.

The above expression can be written in an expanded matrix form as:

\[
\begin{bmatrix}
L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\
L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\
L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\
L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\
L_{51} & L_{52} & L_{53} & L_{54} & L_{55}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\phi_1 \\
\phi_2
\end{bmatrix}
+ \begin{bmatrix}
M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\
M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \\
M_{31} & M_{32} & M_{33} & M_{34} & M_{35} \\
M_{41} & M_{42} & M_{43} & M_{44} & M_{45} \\
M_{51} & M_{52} & M_{53} & M_{54} & M_{55}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\phi_1 \\
\phi_2
\end{bmatrix}
= \{0\}
\]

(4.25)

where the displacement vector is given by:

\[ u_i = \{u_1, u_2, u_3, \phi_1, \phi_2\}^T. \]

and the mass matrix \( M_i = M_j \), where the components are given by:

\[ M_{11} = M_{22} = M_{33} = (H_1+C_1H_2) \]

\[ M_{14} = M_{25} = (H_2+C_1H_3) \]
\[ M_{44} = M_{55} = (H_3 + C_1 H_4) \]

all other \( M_{ij} = 0 \) \hspace{1cm} (4.26)

where the curvature coefficient \( C_1 \) is given by Eqn.(4.10) and the inertia terms \( (H_i) \) \((i=1,2,3,4)\) are given by Equation(4.13).

The differential operators, \( L_{ii} = L_{ji} \), become:

\[
L_{11} = A_{11} \frac{\partial^2}{\partial x_1^2} + 2A_{16} \frac{\partial^2}{\partial x_1 \partial x_2} + A_{66} \frac{\partial^2}{\partial x_2^2} - \frac{A_{55}}{R_1} - \frac{1}{R_{12}} \left( \frac{2A_{45}}{R_1} + \frac{A_{44}}{R_{12}} \right) \\
+ C_0 \left( -B_{11} \frac{\partial^2}{\partial x_1^2} + B_{66} \frac{\partial^2}{\partial x_2^2} + \frac{B_{55}}{R_1^2} - \frac{B_{44}}{R_{12}^2} \right)
\]

\[
L_{12} = A_{16} \frac{\partial^2}{\partial x_1^2} + (A_{12} + A_{16}) \frac{\partial^2}{\partial x_1 \partial x_2} + A_{26} \frac{\partial^2}{\partial x_2^2} \\
- \frac{A_{45}}{R_1 R_2} - \frac{1}{R_{12}} \left( \frac{A_{55}}{R_1} + \frac{A_{44}}{R_2} + \frac{A_{45}}{R_{12}} \right) \\
+ C_0 \left( -B_{16} \frac{\partial^2}{\partial x_1^2} + B_{26} \frac{\partial^2}{\partial x_2^2} + \frac{B_{55}}{R_1 R_{12}} - \frac{B_{44}}{R_2 R_{12}} \right)
\]

102
\[ L_{11} = \left[ \frac{(A_{11} + A_{55})}{R_1} + \frac{A_{12}}{R_2} + \frac{(2A_{16} + A_{45})}{R_{12}} \right] \frac{\partial}{\partial x_1} + \]

\[ \left[ \frac{(A_{16} + A_{45})}{R_1} + \frac{A_{26}}{R_2} + \frac{(2A_{66} + A_{44})}{R_{12}} \right] \frac{\partial}{\partial x_1} + C_0 \left[ -\left( \frac{B_{11} + B_{55}}{R_1} - \frac{B_{16}}{R_{12}} \right) \frac{\partial}{\partial x_1} + \left( \frac{B_{26}}{R_2} + \frac{B_{66} + B_{44}}{R_{12}} \right) \right] \frac{\partial}{\partial x_2} \]

\[ L_{14} = B_{11} \frac{\partial^2}{\partial x_1^2} + 2B_{16} \frac{\partial^2}{\partial x_1 \partial x_2} + B_{66} \frac{\partial^2}{\partial x_2^2} + \frac{A_{55}}{R_1} + \frac{A_{45}}{R_{12}} + C_0 \left( -D_{11} \frac{\partial^2}{\partial x_1^2} + D_{66} \frac{\partial^2}{\partial x_2^2} - \frac{B_{55}}{R_1} \right) \]

\[ L_{15} = B_{16} \frac{\partial^2}{\partial x_1^2} + (B_{12} + B_{66}) \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{A_{45}}{R_1} + \frac{A_{44}}{R_{12}} + C_0 \left( -D_{16} \frac{\partial^2}{\partial x_1^2} + D_{26} \frac{\partial^2}{\partial x_2^2} - \frac{B_{44}}{R_{12}} \right) \]

\[ L_{22} = A_{56} \frac{\partial^2}{\partial x_1^2} + 2A_{26} \frac{\partial^2}{\partial x_1 \partial x_2} + A_{44} \frac{\partial^2}{\partial x_2^2} + \frac{A_{44}}{R_1} - \frac{1}{R_{12}} \left( \frac{2A_{45} + A_{44}}{R_2} \right) + C_0 \left( -B_{66} \frac{\partial^2}{\partial x_1^2} + B_{22} \frac{\partial^2}{\partial x_2^2} - \frac{B_{44}}{R_2} + \frac{B_{55}}{R_{12}} \right) \]

103
\[ L_{26} = \frac{A_{16}}{R_1} + \frac{(A_{26} + A_{45})}{R_2} + \frac{(2A_{66} + A_{55})}{R_{12}} \frac{\partial}{\partial x_2} + \frac{A_{12}}{R_1} + \frac{(A_{22} + A_{44})}{R_2} + \frac{(2A_{26} + A_{45})}{R_{12}} \frac{\partial}{\partial x_1} \]

\[ L_{24} = B_{16} \frac{\partial^2}{\partial x_2^2} + (B_{12} + B_{66}) \frac{\partial^2}{\partial x_1 \partial x_2} + B_{26} \frac{\partial^2}{\partial x_2^2} + \frac{A_{45} + A_{55}}{R_{12}} \]

\[ + C_0 \left(-D_{16} \frac{\partial^2}{\partial x_1^2} + D_{26} \frac{\partial^2}{\partial x_2^2} - \frac{B_{55}}{R_{12}}\right) \]

\[ L_{25} = B_{66} \frac{\partial^2}{\partial x_1^2} + 2B_{26} \frac{\partial^2}{\partial x_1 \partial x_2} + B_{22} \frac{\partial^2}{\partial x_2^2} + \frac{A_{44} + A_{45}}{R_{12}} \]

\[ + C_0 \left(-D_{66} \frac{\partial^2}{\partial x_1^2} + D_{22} \frac{\partial^2}{\partial x_2^2} + \frac{B_{44}}{R_2}\right) \]

\[ L_{11} = A_{45} \frac{\partial^2}{\partial x_1^2} - 2A_{44} \frac{\partial^2}{\partial x_1 \partial x_2} - A_{44} \frac{\partial^2}{\partial x_2^2} + \frac{A_{11}}{R_1} + \frac{2A_{12}}{R_1 R_2} \]

\[ + \frac{A_{22}}{R_2} + \frac{4}{R_{12}} \left( \frac{A_{16}}{R_1} + \frac{A_{26}}{R_2} + \frac{A_{66}}{R_{12}} \right) \]

\[ + C_0 \left[B_{55} \frac{\partial^2}{\partial x_1^2} - B_{44} \frac{\partial^2}{\partial x_2^2} - \frac{B_{11}}{R_1} + \frac{B_{22}}{R_2} - \frac{2}{R_{12}} \left( \frac{B_{16}}{R_1} + \frac{B_{26}}{R_2} \right) \right] \]
\[ L_{34} = \left( -A_{55} + \frac{B_{11}}{R_1} + \frac{B_{12}}{R_2} + \frac{2B_{16}}{R_{12}} \right) \frac{\partial}{\partial x_1} \\
+ \left( -A_{45} + \frac{B_{16}}{R_1} + \frac{B_{26}}{R_2} + \frac{2B_{66}}{R_{12}} \right) \frac{\partial}{\partial x_2} \\
+ C \left( B_{55} - \frac{D_{11}}{R_1} + \frac{D_{16}}{R_{12}} - \frac{D_{26}}{R_{12}} \right) \frac{\partial}{\partial x_1} + \left( \frac{D_{26}}{R_2} + \frac{D_{66}}{R_{12}} \right) \frac{\partial}{\partial x_2} \]

\[ L_{35} = \left( -A_{45} + \frac{B_{16}}{R_1} + \frac{B_{26}}{R_2} + \frac{2B_{66}}{R_{12}} \right) \frac{\partial}{\partial x_1} \\
+ \left( -A_{44} + \frac{B_{12}}{R_1} + \frac{B_{22}}{R_2} + \frac{2B_{26}}{R_{12}} \right) \frac{\partial}{\partial x_2} \\
+ C \left( -\frac{D_{16}}{R_1} + \frac{D_{66}}{R_{12}} - \frac{D_{26}}{R_{12}} \right) \frac{\partial}{\partial x_1} + \left( -B_{44} + \frac{D_{22}}{R_2} + \frac{D_{26}}{R_{12}} \right) \frac{\partial}{\partial x_2} \]

\[ L_{44} = -A_{55} + D_{11} \frac{\partial^2}{\partial x_1^2} + 2D_{16} \frac{\partial^2}{\partial x_1 \partial x_2} + D_{66} \frac{\partial^2}{\partial x_2^2} \\
+ C \left( B_{55} - F_{11} \frac{\partial^2}{\partial x_1^2} + F_{66} \frac{\partial^2}{\partial x_2^2} \right) \]

\[ L_{45} = -A_{45} + D_{16} \frac{\partial^2}{\partial x_1^2} + (D_{12} + D_{66}) \frac{\partial^2}{\partial x_1 \partial x_2} + D_{26} \frac{\partial^2}{\partial x_2^2} \\
+ C \left( -F_{16} \frac{\partial^2}{\partial x_1^2} + F_{26} \frac{\partial^2}{\partial x_2^2} \right) \]
\[ L_{55} = -A_{44} + D_{66} \frac{\partial^2}{\partial x_1^2} + 2D_{26} \frac{\partial^2}{\partial x_1 \partial x_2} + D_{22} \frac{\partial^2}{\partial x_2^2} + C_0 \left( -B_{44} - F_{66} \frac{\partial^2}{\partial x_1^2} + F_{22} \frac{\partial^2}{\partial x_2^2} \right) \]

(4.27)

where, \( C_0 \) is given by Equation (4.9) and the other stiffness coefficients, \( A_{ij}, B_{ij}, D_{ij}, \) and \( F_{ij} \) are defined by Equations (2.29) through (2.34).

### 4.5.2 Equations of Motion for Simplified Shallow Shells

For a simplified shallow shells one might use the assumptions developed in section (4.4.1) as well as the discussion of Section (2.2.2.1). Therefore, Equations (4.26) and (4.27) may be greatly simplified. Then, the original coefficients in Equation (4.25) can be given in a reduced form as follows:

The inertia coefficients (4.26) when simplified by setting \( C_1 = 0 \), will yield:

\[
\begin{align*}
M_{ij} & = M_{ji} \\
M_{11} & = M_{22} = -M_{33} = -H_1 \\
M_{14} & = M_{25} = -H_2 \\
M_{44} & = M_{55} = -H_3 \\
\text{all other } M_{ij} & = 0
\end{align*}
\]

(4.28)
where, again the $H_i$ $(i=1,2,3)$ are given in (4.13).

and the differential operators $L_i = L_j$ are given by

\[
L_{11} = A_{11} \frac{\partial^2}{\partial x_1^2} + 2A_{16} \frac{\partial^2}{\partial x_1 \partial x_2} + A_{66} \frac{\partial^2}{\partial x_2^2}
\]

\[
L_{12} = A_{16} \frac{\partial^2}{\partial x_2^2} + (A_{12} + A_{66}) \frac{\partial^2}{\partial x_1 \partial x_2} + A_{26} \frac{\partial^2}{\partial x_2^2}
\]

\[
L_{13} = \left[ \frac{A_{11}}{R_1} + \frac{A_{12}}{R_2} + \frac{2A_{16}}{R_{12}} \right] \frac{\partial}{\partial x_1} \left[ \frac{A_{16}}{R_1} + \frac{A_{26}}{R_2} + \frac{2A_{66}}{R_{12}} \right] \frac{\partial}{\partial x_2}
\]

\[
L_{14} = B_{11} \frac{\partial^2}{\partial x_1^2} + 2B_{16} \frac{\partial^2}{\partial x_1 \partial x_2} + B_{66} \frac{\partial^2}{\partial x_2^2}
\]

\[
L_{15} = B_{16} \frac{\partial^2}{\partial x_1^2} + (B_{12} + B_{66}) \frac{\partial^2}{\partial x_1 \partial x_2} + B_{26} \frac{\partial^2}{\partial x_2^2}
\]

\[
L_{22} = A_{66} \frac{\partial^2}{\partial x_1^2} + 2A_{26} \frac{\partial^2}{\partial x_1 \partial x_2} + A_{22} \frac{\partial^2}{\partial x_2^2}
\]

\[
L_{23} = \left[ \frac{A_{16}}{R_1} + \frac{A_{26}}{R_2} + \frac{2A_{16}}{R_{12}} \right] \frac{\partial}{\partial x_1} \left[ \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} + \frac{2A_{26}}{R_{12}} \right] \frac{\partial}{\partial x_2}
\]

\[
L_{24} = B_{16} \frac{\partial^2}{\partial x_1^2} + (B_{12} + B_{66}) \frac{\partial^2}{\partial x_1 \partial x_2} + B_{26} \frac{\partial^2}{\partial x_2^2}
\]

\[
L_{25} = B_{66} \frac{\partial^2}{\partial x_1^2} + 2B_{26} \frac{\partial^2}{\partial x_1 \partial x_2} + B_{22} \frac{\partial^2}{\partial x_2^2}
\]

\[
L_{13} = -A_{55} \frac{\partial^2}{\partial x_1^2} - 2A_{45} \frac{\partial^2}{\partial x_1 \partial x_2} - A_{44} \frac{\partial^2}{\partial x_2^2} + \frac{A_{11}}{R_1^2} + \frac{2A_{12}}{R_1R_2} + \frac{A_{22}}{R_2^2}
\]
It should be mentioned here that the expressions given by Eqn.(4.29) can be obtained directly from Eqn.(4.27) by setting \( C_0 = 0 \) and adjusting the terms corresponding to the transverse shear deformation. i.e. \( A_{ij}/R_i R_j \) and \( B_{ij}/R_i R_j \) (\( i,j = 4,5 \)) are all set equal to zero.
4.5.3 Equations of Motion for flat plates

The governing equations can be further simplified for the special case of a flat plate. In such a case all the curvature vanishes, the Eqn. (4.28) remains the same while Eqn.(4.29) reduces to the following form:

\[ L_{11} = A_{11} \frac{\partial^2}{\partial x_1^2} + 2A_{16} \frac{\partial^2}{\partial x_1 \partial x_2} + A_{66} \frac{\partial^2}{\partial x_2^2} \]

\[ L_{12} = A_{16} \frac{\partial^2}{\partial x_1^2} + (A_{12} + A_{66}) \frac{\partial^2}{\partial x_1 \partial x_2} + A_{26} \frac{\partial^2}{\partial x_2^2} \]

\[ L_{13} = 0 \]

\[ L_{14} = B_{11} \frac{\partial^2}{\partial x_1^2} + 2B_{16} \frac{\partial^2}{\partial x_1 \partial x_2} + B_{66} \frac{\partial^2}{\partial x_2^2} \]

\[ L_{15} = B_{16} \frac{\partial^2}{\partial x_1^2} + (B_{12} + B_{66}) \frac{\partial^2}{\partial x_1 \partial x_2} + B_{26} \frac{\partial^2}{\partial x_2^2} \]

\[ L_{22} = A_{66} \frac{\partial^2}{\partial x_1^2} + 2A_{26} \frac{\partial^2}{\partial x_1 \partial x_2} + A_{22} \frac{\partial^2}{\partial x_2^2} \]

\[ L_{23} = 0 \]

\[ L_{24} = B_{16} \frac{\partial^2}{\partial x_1^2} + (B_{12} + B_{66}) \frac{\partial^2}{\partial x_1 \partial x_2} + B_{26} \frac{\partial^2}{\partial x_2^2} \]

\[ L_{25} = B_{66} \frac{\partial^2}{\partial x_1^2} + 2B_{26} \frac{\partial^2}{\partial x_1 \partial x_2} + B_{22} \frac{\partial^2}{\partial x_2^2} \]
\[\begin{align*}
L_{11} &= -A_{55} \frac{\partial^2}{\partial x_1^2} - 2A_{45} \frac{\partial^2}{\partial x_1 \partial x_2} - A_{44} \frac{\partial^2}{\partial x_2^2} \\
L_{34} &= -A_{55} \frac{\partial}{\partial x_1} - A_{45} \frac{\partial}{\partial x_2} \\
L_{35} &= -A_{45} \frac{\partial}{\partial x_1} - A_{44} \frac{\partial}{\partial x_2} \\
L_{44} &= A_{55} + D_{11} \frac{\partial^2}{\partial x_1^2} + 2D_{16} \frac{\partial^2}{\partial x_1 \partial x_2} + D_{66} \frac{\partial^2}{\partial x_2^2} \\
L_{45} &= A_{45} + D_{16} \frac{\partial^2}{\partial x_1^2} + (D_{12} + D_{66}) \frac{\partial^2}{\partial x_1 \partial x_2} + D_{26} \frac{\partial^2}{\partial x_2^2} \\
L_{55} &= A_{44} + D_{16} \frac{\partial^2}{\partial x_1^2} + 2D_{26} \frac{\partial^2}{\partial x_1 \partial x_2} + D_{22} \frac{\partial^2}{\partial x_2^2}
\end{align*}\]

(4.30)
4.6 CLOSURE

In this chapter the variational formulation with various specializations to laminated composite plates and shallow shells have been presented. Such variational statements form a powerful basis for obtaining solutions for various problems in structural mechanics, many of which can be intractable otherwise. The single most important step for generating appropriate solutions using the presented variational formulation is the selection of the coordinate functions (the displacement functions) which will minimize the total energy functional and satisfy a prescribed set of boundary conditions as well as the initial conditions (as given in Chapter (2), Table (2.4)). The development and discussion of solution procedures for shallow cylindrical shells as well as flat plates containing matrix crack types of damage zone with various degrees of damage, size and shape of damage and support conditions are the topics of the next two chapters, by using the Ritz method of solution and the finite element method of solution. The development of these solution procedures require the use of appropriate energy equations developed in this chapter along with the boundary conditions and the continuity conditions between the damaged and the undamaged regions discussed in Chapter (2), (Eqn.(2.70)).
CHAPTER 5

RITZ METHOD FOR LAMINATED COMPOSITE SHALLOW SHELLS AND PLATES WITH MATRIX CRACKS

5.1 INTRODUCTION

Quite often the formulation of the eigenvalue problem for most practical cases does not have a closed form solution. This is frequently the case when the stiffness or mass distribution of a given structural system is nonuniform resulting into energy expressions which may be too complicated to handle. This is also the case when the geometric boundary conditions and/or the continuity equations cannot be adequately described by known functions. In such cases, one may be content with approximate solutions of the boundary-value or the eigenvalue problem. From the previous formulations of damage in Chapter(3) and the shallow shell geometry and energy expressions presented in Chapter(4), one can immediately realize that the problem in hand does not have an exact solution due to the existence of many complicating factors. Therefore, the main objective of this chapter is to seek an 'approximate' solution procedure to the problems of laminated composite shallow shells and plates that contains a centralized damaged zone. The two
simplest and widely used approximate solutions in the field of structural mechanics especially in the areas of vibration and buckling are the Ritz method and the Finite Element method. The solution using the last approach is presented in the next chapter while in the present chapter we consider an approximate solution procedure using Ritz method. However, before proceeding we first introduce a known closed-form solution for a special case without any type of damage so that it can be used as a basis for comparison with the approximate solution.

5.2 SPECIAL CASE

In this section we consider a closed-form solution of a crossply laminated composite shallow shell with shear diaphragm boundaries without damage. This type of boundary is often referred to as simply supported or freely supported boundary. This specific problem is widely used in the open literature since it exhibits a closed-form solution, which provides a basis for comparison with approximate or numerical analyses. In this regard one can refer to the solutions given by Leissa [55] for isotropic shallow shell, Qatu [77] for orthotropic shallow shells without shear deformations, and Chang [14] for laminated shallow shell with shear deformations. Here we present the closed-form solution of Chang [14] since it is more inclusive and the other solutions may be derived as special cases from this general solution.
Consider the laminated composite shell shown in Figure (5.1). The shell boundary conditions are shear diaphragms on all four edges, i.e. S2 type boundary conditions as described in Section (2.5.3) and given in Table (2.4), and which can be written as:

\[
\begin{align*}
&u_1^0 = M_{11} = N_{11} = u_2^0 = \phi_2 = 0 \quad \text{on} \quad x_1 = 0, \ a \\
&u_1^0 = M_{22} = N_{22} = u_2^0 = \phi_1 = 0 \quad \text{on} \quad x_2 = 0, \ b 
\end{align*}
\]  

(5.1)

The displacement functions that can satisfy the above boundary conditions exactly for a cross-ply laminated shallow shells having principal curvatures in the \( x_1 \) and \( x_2 \) directions, i.e. \( R_{12} = \infty \), can be chosen as [14]:

\[
\begin{align*}
&u_1^0(x_a, t) = \sum_{m,n=1}^\infty U_{mn} \cos(\alpha_m x_1) \sin(\beta_n x_2) \sin \omega t \\
&u_2^0(x_a, t) = \sum_{m,n=1}^\infty V_{mn} \sin(\alpha_m x_1) \cos(\beta_n x_2) \sin \omega t \\
&u_3^0(x_a, t) = \sum_{m,n=1}^\infty W_{mn} \sin(\alpha_m x_1) \sin(\beta_n x_2) \sin \omega t \\
&\phi_1(x_a, t) = \sum_{m,n=1}^\infty \Phi_{1mn} \cos(\alpha_m x_1) \sin(\beta_n x_2) \sin \omega t \\
&\phi_2(x_a, t) = \sum_{m,n=1}^\infty \Phi_{2mn} \sin(\alpha_m x_1) \cos(\beta_n x_2) \sin \omega t
\end{align*}
\]  

(5.2)

where

\[
\alpha_m = m\pi / a, \quad \beta_n = n\pi / b
\]
and \( \omega \) is the free vibration frequency. For the analysis of static deflections, one may discard \( \sin \omega t \) and for dynamic analysis the summation can be omitted, since each term of the series in (5.2) represents a mode shape by itself.

In the absence of body and external forces and couples, the substitution of (5.2) into (4.26) and (4.27) in Section (4.5.1) will yield a set of five linear algebraic equations in terms of the unknown amplitudes \( U_{mn}, V_{mn}, W_{mn}, \Phi_{1mn} \) and \( \Phi_{2mn} \) for the free vibration of laminated composite shallow shell. In a matrix form these equations can be written as:

\[
[K] \{q\} + \omega^2 [M] \{q\} = \{0\}
\]

where

\([K]\) is the stiffness matrix;

\([M]\) is the mass matrix;

\(\{q\} = \{U_{mn}, V_{mn}, W_{mn}, \Phi_{1mn}, \Phi_{2mn}\}^T\) is the amplitude vector;

The elements of the mass matrix are given by:

\[
M_{11} = M_{22} = M_{33} = H_1 + C_1 H_2
\]

\[
M_{14} = M_{25} = H_2 + C_1 H_3
\]

\[
M_{44} = M_{55} = H_3 + C_1 H_4
\]

\[
M_{ij} = M_{ji}, \text{ and all other elements } = 0;
\]

and the elements of the stiffness matrix are given by:
\[
\begin{align*}
K_{11} &= -A_1 \alpha_m - A_{66} \beta_n - \frac{A_{45}}{R_1} + c_0 \left( \frac{B_{11} \alpha_m - B_{66} \beta_n}{R_1} + \frac{B_{55}}{R_1} \right) \\
K_{12} &= -(A_{12} - A_{66}) \alpha_m \beta_n \\
K_{13} &= \left( \frac{A_{11} + A_{55}}{R_1} + \frac{A_{12}}{R_2} \right) \alpha_m - c_0 \left( \frac{B_{11} + B_{55}}{R_1} \right) \\
K_{14} &= -B_{11} \alpha_m - B_{66} \beta_n + \frac{A_{55}}{R_1} + c_0 \left( \frac{D_{11} \alpha_m - D_{66} \beta_n}{R_1} + \frac{B_{55}}{R_1} \right) \\
K_{15} &= -(B_{12} + B_{66}) \alpha_m \beta_n \\
K_{22} &= -A_{66} \alpha_m - A_{22} \beta_n - \frac{A_{44}}{R_2} + c_0 \left( \frac{B_{66} \alpha_m - B_{22} \beta_n}{R_2} + \frac{B_{44}}{R_2} \right) \\
K_{23} &= \left( \frac{A_{12}}{R_1} + \frac{A_{22} + A_{44}}{R_2} \right) \beta_n + c_0 \left( \frac{B_{22} + B_{44}}{R_2} \right) \beta_n \\
K_{24} &= -(B_{12} + B_{66}) \alpha_m \beta_n \\
K_{25} &= -B_{66} \alpha_m - B_{22} \beta_n + \frac{A_{44}}{R_2} + c_0 \left( D_{66} \alpha_m - D_{22} \beta_n + B_{44} \right) \\
K_{44} &= -\frac{A_{11}}{R_1} - \frac{A_{22}}{R_2} - \frac{2A_{12}}{R_1 R_2} - (A_{44} \alpha_m + A_{44} \beta_n) \\
&\quad + c_0 \left( \frac{B_{22}}{R_2} - \frac{B_{55}}{R_2} + B_{55} \alpha_m - B_{44} \beta_n \right) \\
K_{54} &= \left( \frac{B_{11}}{R_1} + \frac{B_{12}}{R_2} - A_{55} \right) \alpha_m - c_0 \left( \frac{D_{11}}{R_1} - B_{55} \right) \alpha_m
\end{align*}
\]
In a similar manner, the above closed-form solutions can also be used for the simplified shallow theory in Section 4.5.2. By substituting Equations (5.2) into Equations (4.28) and (4.29), and by restricting the problem to cross-ply laminated shells having principal curvatures in the $x_1$ and $x_2$ directions, the expressions of the mass matrix as given by Equation (5.4) are reduced to:

\[
M_{11} = M_{22} = M_{33} = H_1
\]
\[
M_{14} = M_{25} = H_2
\]
\[
M_{44} = M_{55} = H_3
\]
\[
M_{ij} = M_{ji}, \text{ and all other elements } = 0; \quad (5.6)
\]

and the elements of the stiffness matrix reduce to:

\[
K_{11} = -A_{11} \alpha_m \beta_n^2 - A_{66} \beta_n^2
\]
\[
K_{12} = -(A_{12} + A_{66}) \alpha_m \beta_n^2
\]
\[
K_{13} = \left(\frac{A_{11}}{R_1} + \frac{A_{12}}{R_2}\right) \alpha_m
\]
\[ \begin{align*}
K_{14} &= -B_{11} \alpha_m^2 - B_{66} \beta_n^2 \\
K_{15} &= -(B_{12} + B_{66}) \alpha_m \beta_n \\
K_{22} &= -A_{66} \alpha_m^2 - A_{44} \beta_n^2 \\
K_{23} &= \left( \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} \right) \beta_n \\
K_{24} &= -(B_{12} + B_{66}) \alpha_m \beta_n \\
K_{25} &= -B_{66} \alpha_m^2 - B_{44} \beta_n^2 \\
K_{33} &= -\frac{A_{11}}{R_1^2} - \frac{A_{32}}{R_2^2} - \frac{2A_{12}}{R_1 R_2} - \left( A_{55} \alpha_m^2 + A_{44} \beta_n^2 \right) \\
K_{34} &= \left( \frac{B_{11}}{R_1} + \frac{B_{12}}{R_2} - A_{55} \right) \alpha_m \\
K_{35} &= \left( \frac{B_{12}}{R_1} + \frac{B_{22}}{R_2} - A_{44} \right) \beta_n \\
K_{44} &= -A_{55} - D_{11} \alpha_m^2 - D_{66} \beta_n^2 \\
K_{45} &= -(D_{12} + D_{66}) \alpha_m \beta_n \\
K_{55} &= -A_{44} - D_{66} \alpha_m^2 - D_{22} \beta_n^2 \\
K_{ij} &= K_{ji} \quad (5.7)
\end{align*} \]
5.3 THE RITZ METHOD OF SOLUTION

5.3.1 General

In general closed-form solutions of the differential equations of motion are not easily attainable for many problems in structural mechanics. In such cases, one should seek approximate solutions for the initial or boundary value problems. As it has been discussed earlier in Section (4.1), the energy formulations provide an approximate way of solutions to the boundary value problem without solving the differential equations. There are various schemes for obtaining approximate solutions for the eigenvalue problem using energy formulations. However, all such solution schemes have one factor in common: they all approximate a continuous system by an n-degrees of freedom system. Quite often these type of approximate solutions are sufficient for most physical applications where a limited number of lowest frequencies are of practical importance. Of the several possibilities available, the Rayleigh-Ritz method is perhaps the simplest and widely used technique in the area of vibration [54-59]. The method is a direct application of the calculus of variations for the minimization of the energy expression for achieving a true solution. One should mention that the finite element method, which is very similar to the Ritz method, is used in Chapter (6).
5.3.2 Displacement functions

Consider the laminated composite shallow shell with a planform (axb) with a centrally located elliptical damaged zone with $\alpha$ and $\beta$ as the major and minor axes respectively. (Figure (5.2)). We shall designate the undamaged region of the shell as zone-1 and the damaged region with matrix cracks type of damage as zone-2 or simply Z1 and Z2 respectively. Despite that the domain of the problem consists of two regions, the damaged and undamaged zones, the shallow shell is assumed to be intact at the interface between the damaged and the undamaged zones and its integrity is well-maintained. Hence, it is essential to enforce the continuity conditions at the common boundary of the two zones Z1 and Z2. The general form of these continuity conditions can be written as (Chapter(2)):

$$\begin{bmatrix}
\hat{u}_t \\
\phi_\alpha \\
\hat{u}_3' \\
\hat{M}_{\alpha\beta} \\
\hat{N}_\alpha \\
\hat{Q}_\alpha
\end{bmatrix} = \begin{bmatrix}
\bar{u}_t \\
\phi_\alpha \\
\bar{u}_3' \\
\bar{M}_{\alpha\beta} \\
\bar{N}_\alpha \\
\bar{Q}_\alpha
\end{bmatrix}$$

(5.8)

(on the outer boundary of Z1 ) (on the inner boundary of Z1 )
The variables with a hat on the left side of Eqn.(5.8) indicate that they belong to the region Z2, while the variables without a hat to the left are for the undamaged region Z1. Also, \( u' \) and \( \hat{u}' \) represent the first derivatives of the transverse deflections. One should note that higher and other derivatives might be included depending on the type of approximate solution procedure used and the desired accuracy.

In general not all the conditions given by Eqn.(5.8) can be satisfied simultaneously. This usually depends on the geometry and the approximate solution technique used. For instance, using Ritz method will only require the satisfactions of the geometric continuity conditions. Leissa [54,55], Gould [31], and Meirovitch [65]. Also, using a displacement-based finite element solutions only the continuity of displacements and slopes can be enforced. While, for stress-based solutions, only the continuity of the internal force resultants can be achieved [25,42].

The task now is to seek a displacement field that will:

1. Ensure that the functional of the total potential energy, Equation (4.4), is stationary with respect to the displacement field.
2. Satisfy at least all the geometric boundary conditions.
3. Satisfy the geometric continuity conditions at the interface between the damaged and the undamaged regions.
One should mention here that a solution based on the above conditions will satisfy the equilibrium equations and the strain compatibility conditions. Which illustrates one of the advantages of using a reasonably assumed displacement field in Ritz method.

Hence, the Ritz solution technique consists of assuming the displacement field as a linear combination of a family of trial functions which can be written as:

$$u(x) = \sum_{i=0}^{N} a_i \varphi_i(x)$$

(5.9)

where, \((\varphi_i)\) are the known functions of the spatial coordinates and they are linearly independent over the domain. The \((a_i)\) are the unknown coefficients to be determined through the minimization of the total potential energy expression. By doing so we approximate a continuous solid having infinite number of degrees of freedom by only an \(N\)-degrees of freedom. This essentially has the effect of raising the stiffness of the system, which is the shallow shell or plate for our case. Consequently, the estimated frequencies are expected to be higher than the true ones. They can be considered the upper bounds. On the other hand, the estimated static deflections are lower than the true ones, and are considered to be the lower bounds. In practice, improved results are obtained by increasing the number of terms used in the sequence. Clearly, selecting the displacement functions \((\varphi_i)\) that can closely resemble the actual displacement field is the second important factor, which can accelerate convergence to the exact solution. The value of engineering intuition in such selection is obvious. A third factor that might be considered to accelerate convergence is the use of orthogonal functions which will eliminate many
computational errors. Therefore, a wide variety of possibilities is available for the choice
of \((\varphi,\iota)\) according to the problem. Commonly, algebraic polynomials or elementary
trigonometric functions are used. For the present work, the displacement functions are
selected to be simple algebraic polynomials. The reasons for such selections can be stated
as follows:

a) They are capable of handling arbitrary boundary conditions.

b) They form a mathematically complete set of functions, thereby permitting
   convergence to the exact solution when a sufficient number of terms is used.

c) They are simpler to manage in algebraic operations as well as in computational
   programming.

Based on the above discussion, the displacement field for the undamaged zone can be
assumed as:

\[
\begin{align*}
  u_1 &= \sum_{i=0}^{I} \sum_{j=0}^{J} a_{ij} \xi^i \eta^j \\
  u_2 &= \sum_{i=0}^{K} \sum_{j=0}^{L} b_{ij} \xi^i \eta^j \\
  u_3 &= \sum_{i=0}^{M} \sum_{j=0}^{N} c_{ij} \xi^i \eta^j
\end{align*}
\]
For the damaged zone, we need to ensure that the geometric continuity conditions given by Equation (5.8) are satisfied. The basic idea is to add a continuous function to the displacement field described by Equations (5.10) which should vanish on the boundary of Z2, i.e. the boundary of the ellipse (Figure 5.2). This extension of Ritz method is known as Kantorovich method, [31]. This approach was used by Singh [95] to study the buckling of delaminated composite plates. In his work similar continuity conditions were enforced at the boundary of the delaminated regions. In a similar fashion, for an elliptical damage zone centrally located in the shell planform, one can write the displacement field as:

$$\bar{u}_1 = (1 - a_1^2 \xi^2 - b_1^2 \eta^2) \sum_{i=0}^{L} \sum_{j=0}^{J} \bar{a}_{ij} \xi^i \eta^j + \sum_{i=0}^{L} \sum_{j=0}^{J} a_{ij} \xi^i \eta^j$$

$$\bar{u}_2 = (1 - a_2^2 \xi^2 - b_2^2 \eta^2) \sum_{i=0}^{K} \sum_{j=0}^{L} \bar{b}_{ij} \xi^i \eta^j + \sum_{i=0}^{K} \sum_{j=0}^{L} b_{ij} \xi^i \eta^j$$

$$\bar{u}_3 = (1 - a_3^2 \xi^2 - b_3^2 \eta^2) \sum_{i=0}^{M} \sum_{j=0}^{N} \bar{c}_{ij} \xi^i \eta^j + \sum_{i=0}^{M} \sum_{j=0}^{N} c_{ij} \xi^i \eta^j$$

$$\bar{\phi}_1 = (1 - a_4^2 \xi^2 - b_4^2 \eta^2) \sum_{i=0}^{P} \sum_{j=0}^{R} \bar{d}_{ij} \xi^i \eta^j + \sum_{i=0}^{P} \sum_{j=0}^{R} d_{ij} \xi^i \eta^j$$

124
In the above expressions (Equations (5.10) and (5.11)), the following nondimensional coordinates are defined:

\[ \xi = \frac{x_1}{a}; \eta = \frac{x_2}{b}; a_1 = \frac{a}{\bar{a}}; \text{ and } b_1 = \frac{b}{\bar{b}} \]

and \( a_1, b_1, \ldots, f_1 \) and \( \bar{b}, \bar{b}^\prime, \ldots, \bar{f}_{ij} \) are the amplitudes or the unknown displacement coefficients. Also, the superscript \( (k) \) in Equation (5.11) is an integer used to insure the continuity conditions. By choosing \( k=1 \) in the expressions of \( \bar{u}_1, \bar{u}_2, \bar{\phi}_1, \) and \( \bar{\phi}_2 \) will insure the continuity of the inplane displacement components and rotations on the boundary between the damaged and the undamaged zones. And by setting \( k=2 \) in the expression of \( \bar{u}_3 \), the transverse displacement components and their slopes are continuous at the common boundary of the two regions.

Clearly, the solution format suggested by Eqns.(5.10) and (5.11) is applicable to any of the general shells or the various specializations presented previously in Chapter (4) for both static and dynamic analysis. However, our discussions and solutions form this point onward are limited to the free vibration of circular cylindrical simplified shallow shells and of course, the special case of a flat plate.

For the free vibration analysis, the displacement functions given by Eqns. (5.10) and (5.11) are substituted into the maximum strain energy expression \( U_E \), the different components

\[
\bar{\phi}_2 = (1 - a_1^2 \xi^2 - b_1^2 \eta^2)^k \sum_{i=0}^{\bar{s}} \sum_{j=0}^{\bar{r}} \bar{f}_{ij} \xi^i \eta^j + \sum_{i=0}^{\bar{s}} \sum_{j=0}^{\bar{r}} f_{ij} \xi^i \eta^j \quad (5.10)
\]
are given by Eqns. (4.15), (4.17) and (4.21)-(4.22), and into the maximum kinetic energy (T) as given by Eqn. (4.18). For obtaining a solution, the Ritz method then requires the minimization of the energy functional \((U_E - T)\) with respect to the unknown displacement coefficients. This can be accomplished as follows:

\[
\begin{align*}
\frac{\partial (U_E - T)}{\partial a_{ij}} &= 0 \\
\vdots & \\
\frac{\partial (U_E - T)}{\partial \tilde{f}_{ij}} &= 0
\end{align*}
\]

Which yields a set of simultaneous, linear, homogeneous equations in equal number of the unknown coefficients. This set of equations can be written as:

\[
[K - \lambda^2 M] \{q\} = 0
\]

where:

- \([K]\) = the system stiffness matrix;
- \([M]\) = the system mass matrix;
- \(\{q\}\) = the vector of the unknown coefficients; and
- \(\lambda\) = the vibration frequency parameter.

126
For obtaining a solution to Eqn.(5.13), the determinant of the coefficients matrix is set equal to zero, which is a generalized eigenvalue problem. Clearly, substituting each of the resulting eigenvalues back into Eqn.(5.13) yields the corresponding eigenvector or amplitudes. Then, by using Eqns.(5.10) and (5.11), the mode shapes can be determined. However, our solution will be limited only to the frequency response. This is not to say that the mode shapes are not important but they need a separate investigation for any shifts or changes in the mode shapes due to damage especially if it is not centrally located.

In the above discussion of the solution procedure no word has been mentioned concerning the boundary conditions. This is one of the advantages of using algebraic polynomials. Following the procedures given in references [7], [49], and [95], Eqn.(5.13) is modified in accordance to the approach given in the Appendix (B). Hence, the eigenvalue system given by Eqn.(5.13) can be rewritten as:

\[
\begin{bmatrix}
\bar{K} - \lambda^2 \bar{M}
\end{bmatrix} \{q\} = 0
\]  

(5.14)

Into which:

\[
\begin{align*}
[\bar{K}] &= [G]^T [K] [G] \\
[\bar{M}] &= [G]^T [M] [G] \\
\{q\} &= [G] \{\bar{q}\}
\end{align*}
\]

where:
Therefore, by forming the constraints matrix \([G]\) for any given set of boundary conditions the displacement functions are made to satisfy the geometric boundary conditions. Then, solving the reduced eigenvalue system given by Eqn.(5.14) yields the required frequency parameters.

5.3.3 Closure

The above solution procedures are implemented into a FORTRAN computer program that has been designed specifically to solve the present problem. The program can handle any laminated composite plate or shallow cylindrical shell with arbitrary support conditions. The plate or the shallow shell may contain a centrally located damaged region which can be either a circle or an ellipse. The structure of the computer program and an outline of the main modules are given in Appendix (D), (Figure (D.2). The generated numerical results using the program are tested and discussed in the following section.
5.4 RESULTS AND DISCUSSION

5.4.1 Sensitivity analysis

As it has been mentioned earlier, the use of algebraic polynomials which form a complete set should converge to the exact solution as the number of terms is increased. As a compromise, one should search for sufficient number of terms that will give a reasonable accuracy of the numerical results. For such purpose, a convergence study is conducted using some representative cases of laminated composite plates and shallow cylindrical shells. The selected cases are without damage so that the results can be compared with the exact and/or numerical solutions in the open literature. The material used in the analysis is graphite-epoxy [AS/3501] with the following properties as given by Vinson and Sierakowski [106]:

\[
\begin{align*}
E_{11} & = 20.02 \times 10^9 \text{ psi} \\
E_{22} & = 1.3 \times 10^9 \text{ psi} \\
G_{12} & = 1.03 \times 10^6 \text{ psi} \\
\nu_{12} & = 0.3
\end{align*}
\]

The results for a simply supported crossply laminated plate with known exact solution is shown in Table (5.1). In the presented Ritz solution, the same number of terms is used for all the field variables \((u_i)\) and \((\phi_a)\) as given by Eqn.(5.10). The frequency parameter, \(\lambda = \omega a^2 \sqrt{\rho / E_{11} h^2}\), for the first mode is accurate up to five significant digits using 80 terms, i.e. 4x4 for each variable. Of course, the rate of convergence to the exact solution...
is slower for higher modes as it can be observed from Table (5.1). In the range of the lowest five frequencies, the third frequency is the slowest to converge. For such mode, using 80, 125, and 180 terms the errors are 25%, 1.45%, and 1.2% respectively.

Tables (5.2) and (5.3) show the convergence studies for two completely free symmetric angle-ply cylindrical shallow shells. The two shallow shells are graphite-epoxy with stacking sequence of \([\pm 30^\circ]_s\) and \([\pm 45^\circ]_s\). By using the aspect ratio \(a/b=1\), curvature ratio \(a/R=0.5\), and the thickness ratio \(a/h=100\), the present results are compared with thin shell numerical solutions, Qatu [77]. It is observed that using 125 terms; i.e. 5x5 for each field variable the accuracy is very reasonable. For the two cases, the results also show very slight improvement over the thin shell results. This result is expected since \(a/h=100\) for both cases represents a very thin shell.

The lowest five frequencies for a clamped crossply laminated graphite-epoxy plate are shown in Table (5.4). The results are generated using 180 terms and they are compared with the results of the finite element method, as discussed in Chapter (6). The results show very good agreement.

From the above results, it has been concluded that the solution is monolithically converging to the exact solution. One also might add that using 6x6 terms; i.e. 180 total number of terms, will insure a very reasonable accuracy for the range of the problems to be investigated in this study.
5.4.2 Undamaged shallow shell

A study of the effects of some important parameters on the frequency response of circular cylindrical laminated shallow shells is presented. By using some classical edge conditions and typical symmetric stacking sequences, the effects of the following parameters are investigated:

(i) The degree of curvature; \( a/R \).

(ii) The ratio of orthotropy; \( E_{11}/E_{22} \).

(iii) The thickness aspect ratio; \( a/h \).

Figure (5.4) and Tables (5.5) through (5.8) show the effect of the variation of \( (a/h) \) on the frequency parameter, \( \lambda = \omega a^2 \sqrt{\rho / E_{11}h^2} \), for various \( (a/R) \) values. The present results, using first order shear deformable theory (FSDT), are compared with the closed-form solution presented in the previous section (5.2) and generated by Chang [14]. The results are also compared with the thin shell theory solutions, Qatu [77]. For \( a/h=100 \) and \( a/R=0.5 \), the errors of the present solution and the thin shell solution with respect to the exact solution are 0.23\% and 0.26\%, respectively. When the shell is thicker, e.g. \( a/h=10 \), these errors become 2\% and 13\%. This result emphasizes the necessity of using at least a first order shear theory especially for moderately thick shells.
The effect of the curvature as measured by \((a/R)\) is depicted in Figures (5.5)-(5.8) and also in Tables (5.5)-(5.8). The ratio \((a/R)\) varies between \(10^{-10}\), i.e. a plate, to 0.5. As this ratio increases the frequency parameter increases which is an expected result since the added curvature will increase the shell stiffness. This change is very significant for higher values of \((a/h)\). For \(a/h=10\), the frequency parameter of a shell with \(a/R=0.5\) is only 3\% higher than that of a flat plate, while it is increased by 186\% for \(a/h=100\). In other words, the effect of changing the curvature is felt more as the shell gets thinner. This might be due to the fact that the added stiffness for a thin shell is a major component in comparison to the original stiffness. The results also show that the errors in the thin shell results are slightly affected by the changes in the curvatures, as shown in Figures (5.5)-(5.8). One might also add that the errors in the present solution increases with increasing \((a/R)\). Such result is expected since the present formulation is based on simplified shallow shell assumptions, as discussed in Section (4.4.1). In such case, as the curvature increases for the same planform the shell becomes deeper, then the approximate solution deviates from the exact one. Figure (5.9) and Table (5.9) show the deviation of the present simplified shallow shell solution from the exact solution, [14], for graphite-epoxy simply supported crossply laminated cylindrical shell. For the shell rise to side ratio \(f/a=0.2\), which is the limit defining a shallow shell, the total error is about 3\%. By noting that the present approximate solution has an error of about 1\%, it can be concluded that the simplified shallow shell theory has an error of about 2\% from the unsimplified shallow shell theory at the limit defining a shallow shell (i.e. \(f/a=0.2\)).
Table (5.10) and Figure (5.10) show the effect of changing the orthotropy ratio \( E_{11}/E_{22} \) on the frequency parameter \( \lambda \). Using \( a/h=100 \), \( a/R=0.3 \), the frequency parameter decreases as the orthotropy ratio increases. The rate of decrease is more rapid for higher modes. In this case the first and fifth modes are compared. The results also indicate that even for a very thin shell \( (a/h=100) \), the error in the fundamental frequency increases with increasing the orthotropy ratio. The variation of the error in the frequency parameter \( \lambda \) using thin shell theory for various \( (E_{11}/E_{22}) \) is shown in Figure (5.11). The depicted errors are in reference to the present (FSDT) solution. As it can be seen from that figure, the error in the first mode is about 0.25\% for \( E_{11}/E_{22}=15 \) and this error goes up to about 2\% for \( E_{11}/E_{22}=40 \). These errors are magnified for higher modes; for instance the fifth mode the corresponding errors are 0.75\% and 3\%. The results reflect the importance of including the transverse shear deformation in the analysis of laminated composites which are orthotropic by nature, especially when the higher modes are required.

5.4.3 Damaged shallow shell

In this section an investigation of the effect of the damage shape, size, and matrix cracks density along with the material and geometrical parameters is presented. The discussion here-in-after is limited to graphite-epoxy symmetric laminated cylindrical shells. The planform area of the shell is \( (A_0) \) and the area of a centrally located elliptical or circular
The damaged region is \( (A_d) \). The material properties are the same as those used in the previous section unless otherwise specified.

The effect of the shape and size or extent of damage is shown in Figures (5.12) and (5.13). Two symmetric stacking sequences are used; namely: \([0°/90°]\) and \([±45°]\) the first shell is simply supported while the other is completely free. The damaged region is either an ellipse or a circle and the size as represented by \% \( A_d/A_0 \) varies from zero, i.e. no damage, to 50\%. By using a crack density \( N=40 \) cracks per inch, the results show that the fundamental frequency is reduced by about 5.5\%. The results also indicate that the shape of the damaged zone has a very little influence on the reduction of the normalized frequency as depicted in Figures (5.12) and (5.13). Therefore, we decided to use a circular area of damage to investigate the influence of the other factors.

Figures (5.14) and (5.15) show the variation of the frequency parameter \( (\lambda) \) for various \( (a/R) \) ratios for damaged and undamaged crossply shell. By using \( N=40 \) cracks per inch in the 90°-ply and \( A_d/A_0 =30\% \), the fundamental frequency is reduced by 3\% for a flat plate \( (a/R=0) \). This reduction slightly increases to 4\% as the curvature ratio is increased to 0.3. This effect becomes more pronounced for higher modes as shown in Figure (5.15) for the fifth mode. For such case, the reduction in the frequency parameter is about 8\% for a flat plate \( (a/R=0) \) and the reduction goes up to 10\% for \( a/R=0.3 \).

Tables (5.11) and (5.12) show the damaged and undamaged lowest frequency parameters for completely free cylindrical shells using stacking sequences of \([±45°]\) and \([±30°]\). In
the analysis 20 cracks per inch in all plies and $A_d/A_0 = 30\%$ are used. For the two shells, the reduction in the frequency parameter is about 12\% and 11\%, respectively.

Table (5.13) is a study of the effect of the damage on the fundamental frequency using various $(a/h)$ ratios. For $a/h=100$ and $a/R=0.5$, the reduction in the fundamental frequency parameter is 4.25\%. This reduction slightly increases as $(a/h)$ decreases. For instance, the reduction becomes 5\% and 5.7\% for $a/h=20$ and 10 respectively. One explanation of such slight change might be due to the fact that by using the same planform, the shell gets thicker as $a/h$ decreases. Hence, the thickness of the cracked plies is increased and as discussed in Chapter (3) the damage will cause more reduction in the laminate stiffness. Consequently, the resulting frequency parameter is reduced as the thickness aspect ratio $(a/h)$ is decreased.

5.5 CONCLUDING REMARKS

From the above results and discussion, one can conclude the following:

1. The transverse shear deformations have a significant effect (error of up to 24\%) on the frequency response which should not be ignored especially for thick or highly orthotropic laminates. The use of a first order shear deformable theory (FSDT) yields
a very reasonable accuracy for global dynamic analysis of laminated composite plates or shallow shells.

2. The analysis of laminated composite plates and shallow cylindrical shells with matrix cracks depicts a typical trend of reduction in the vibration frequency due to this type of damage. The reduction is found to be mainly dependent on the size of the damage region and density of cracks. The typical change for the cases studied range from 3% to 5%. This change in the frequency, especially for the fundamental mode, appears to be less sensitive to the shell boundary conditions as well as small values of curvature. On the other hand, the reduction in the vibration frequency is more pronounced for higher modes (8%-12% for the fifth mode). This happens probably due to the fact that the presence of a small damage region within a plate or a shell is reflected more critically on the higher modes that exhibit more localized patterns of motions.

3. The study also shows that the degree of curvature may significantly affect the frequency response of undamaged laminated composite shallow shells. In general, the frequency parameters increase with increasing the degree of curvature (a/R) which is expected, since the added curvature will increase the shell stiffness. One might also add that this change in the frequency is very significant for thin shells which could be due to the fact that the added stiffness due to the contribution of the terms related to the curvature in thin shells becomes comparable with that of corresponding flat plate stiffness, which is not the case for thick shells.
4. The damage region was assumed to be either an ellipse or a circle. These uniform shapes are used to reasonably approximate the actual damage shapes for the laminate configurations considered here. These shapes in general, however, can be non-uniform and very much dependent on the stacking sequence and the source of damage. In the range of the studied cases, the frequency response has a very little sensitivity to the shape of the damaged region for a given a total area of damage zone provided that the damaged zone aspect ratio is of order unity. On the other hand, the extent of damage and the density of cracks are the major contributing factors to the reductions in the vibration frequencies. This result is an important contribution since it can facilitate the investigation of many similar problems by using any of the available numerical solution techniques. For instance, the Ritz method might be used assuming that the damage region is either an ellipse or a circle to avoid the difficulty of enforcing the continuity conditions for shapes with sharp edges that might result in some singularities.

5. The use of a simplified shallow shell theory yields very reasonable accuracy of the frequency response for the cases studied here including the damaged and the undamaged laminates. In comparison with the closed-form solutions (based on the unsimplified theory), the resulting error is less than 2% at the approximate limiting boundary for the shallow shell; i.e. \( f/a=0.2 \) which corresponds to \( a/R=0.69 \) for a circular cylindrical shell.
Figure (5.1): Coordinate system and geometry for composite laminated shell.
Figure (5.2): Planform for a shallow shell with centrally elliptical damaged zone
Figure (5.3): A typical circular cylindrical shell with damage.
Figure (5.4) Variation of frequency parameter \( \lambda = \alpha a^2 \sqrt{\frac{\rho}{E_{11}h^2}} \) with \( a/R \) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_s\), \( a/b = 1 \).

\[ E_{11}/E_{22} = 15, \ G_{12}/E_{22} = 0.5, \ \nu_{12} = 0.25, \ k^2 = 5/6 \]
Figure (5.5): Variation of frequency parameter \( \lambda = \omega a^2 \sqrt{\rho / E_{11} h^2} \) with \( a/R \) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_s\), \( a/b=1 \), \( a/h=1.00 \), \( E_{11}/E_{22}=15 \), \( G_{12}/E_{22}=0.5 \), \( v_{12}=0.25 \), \( k^2=5/6 \).
Figure (5 6): Variation of frequency parameter \( \lambda = \omega a^2 \sqrt{\rho / E_{11} h^2} \) with \( a/R \) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_n\), \( a/b = 1 \).

\( a/h = 0.20, \ E_{11}/E_{22} = 15, \ G_{12}/E_{22} = 0.5, \ \nu_{12} = 0.25, \ k^2 = 5/6 \)
Figure (5.7) Variation of frequency parameter \( \lambda \) with \( a/R \) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_8\), \( a/b=1 \).

\[
\lambda = \omega a^2 \sqrt{\rho / E_{11} h^2}
\]

\( a/h=10 \), \( E_{11}/E_{22}=15 \), \( G_{12}/E_{22}=0.5 \), \( v_{12}=0.25 \), \( k'=5/6 \)

144
Figure (5.8): Variation of frequency parameter \( \lambda = \omega a^2 \sqrt{\rho / E_{11} h^2} \) with \( a/R \) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_5\), \( a/b = 1 \).

\[
a/h=10, \ E_{11}/E_{22}=15, \ G_{12}/E_{22}=0.5, \ v_{12}=0.25, \ k^2=5/6
\]
Figure (5.9) Variation of the percentage of error in the present shallow cylindrical shell solution from the exact one with \((a/R)\) and \((f/a)\), \([0^\circ/90^\circ]_3\), 

\[
a/b=1, \ E_{11}/E_{22}=15, \ G_{12}/E_{22}=0.5, \ \nu_{12}=0.25, \ k^2=5/6
\]
Figure (5.10): Relative change of frequency parameter \( \lambda = \omega \sqrt{\frac{\rho}{E_{11} h^2}} \) with reference to mode number for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_s\), a/b=1, a/h=100, a/R=0.3, G_{12}/E_{22}=0.5, \nu_{12}=0.25, k^2=5/6
Figure (5.11): % deviation in the frequency parameter $\lambda = \omega a^2 \sqrt{\rho / E_{11} h^2}$ in reference to mode number for graphite-epoxy laminated composite simply supported cylindrical shell, $[0^\circ/90^\circ]_s$, $a/b=1$, $a/h=100$, $G_{12}/E_{22}=0.5$, $v_{12}=0.25$, $k^2=5/6$
Figure (5.12): Effect of the shape and size of damage on the frequency parameter

\[ \lambda = \omega a^2 \sqrt{\frac{\rho}{E_{11} h^2}} \]

with \( a/h \) and \( a/R \) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_s\), \( a/b=1 \), \( a/h=100 \), \( a/R=0.5 \), \( N=40 \) cracks /in.
Figure (5.13) Effect of the shape and size of damage on the frequency parameter

\[ \lambda = \omega a^2 \sqrt{\rho / E_{11} h^2} \]

for graphite-epoxy laminated composite completely free shallow cylindrical shell, [±45°]s, a/b=1, a/h=100, a/R=0.5, N=20 cracks/in.
Figure (5.14): Effect of damage on the fundamental frequency parameter

\[ \lambda = \alpha a^2 \sqrt{\rho / E_{11} h^2} \]

for graphite-epoxy laminated composite simply supported shallow cylindrical shell, $[0^\circ/90^\circ]_s$, $a/b=1$, $a/h=100$, $a/R=0.5$, $N=40$ cracks /in., $A_d=30\%$
Figure (5.15): Effect of damage on the fifth mode frequency parameter

\( \lambda = \omega a^2 \sqrt{\rho / E_{11} h^2} \) for graphite-epoxy laminated composite simply supported shallow cylindrical shell, \([0^\circ/90^\circ]_s\), \(a/b=1\), \(a/h=100\), \(a/R=0.5\), \(N=40\) cracks/in., \(A_d=30\%\)
<table>
<thead>
<tr>
<th>No. of Terms</th>
<th>Mode Number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>45</td>
<td>1.66051</td>
</tr>
<tr>
<td>80</td>
<td>1.66037</td>
</tr>
<tr>
<td>125</td>
<td>1.66037</td>
</tr>
<tr>
<td>180</td>
<td>1.66037</td>
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<tr>
<td>245</td>
<td>1.66037</td>
</tr>
<tr>
<td>Exact [107]</td>
<td>1.66037</td>
</tr>
</tbody>
</table>

\[ \lambda = \omega a^2 \sqrt{\rho / E_{11}h^2} \]

Table (5.1): Convergence of the frequency parameter \( \lambda^* \) for graphite-epoxy laminated composite simply supported plate, \([0^\circ/90^\circ]_s\), \(a/b=1\), \(a/h=100\).
<table>
<thead>
<tr>
<th>No. of Terms</th>
<th>Mode Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td></td>
<td>2.2951</td>
<td>6.2261</td>
<td>7.4851</td>
<td>10.7428</td>
<td>12.3608</td>
</tr>
<tr>
<td>80</td>
<td></td>
<td>2.2532</td>
<td>5.5915</td>
<td>5.7746</td>
<td>8.3628</td>
<td>8.6096</td>
</tr>
<tr>
<td>125</td>
<td></td>
<td>2.2164</td>
<td>5.1883</td>
<td>5.6064</td>
<td>7.7994</td>
<td>8.1248</td>
</tr>
<tr>
<td>180</td>
<td></td>
<td>2.2124</td>
<td>5.1183</td>
<td>5.5416</td>
<td>7.6560</td>
<td>7.9082</td>
</tr>
<tr>
<td>245</td>
<td></td>
<td>2.2112</td>
<td>5.1087</td>
<td>5.5344</td>
<td>7.6218</td>
<td>7.8815</td>
</tr>
<tr>
<td>Thin Shell**</td>
<td></td>
<td>2.2156</td>
<td>5.1241</td>
<td>5.5678</td>
<td>7.6824</td>
<td>7.9914</td>
</tr>
</tbody>
</table>

* \( \lambda = \omega a^2 \sqrt{\rho / E_{11} h^2} \)

** Qatu [77], using 192 terms.

Table (5.2): Convergence of the frequency parameter \( \lambda^* \) for graphite-epoxy laminated composite completely free shallow cylindrical shell. \([\pm 30^\circ]_s\), \(a/b=1\), \(a/h=100\), \(a/R=0.5\)
<table>
<thead>
<tr>
<th>Terms</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>2.3250</td>
<td>4.8689</td>
<td>7.5120</td>
<td>11.2042</td>
<td>13.2634</td>
</tr>
<tr>
<td>80</td>
<td>2.2821</td>
<td>4.3722</td>
<td>5.7990</td>
<td>8.7221</td>
<td>9.2389</td>
</tr>
<tr>
<td>125</td>
<td>2.2462</td>
<td>4.0569</td>
<td>5.6246</td>
<td>8.1351</td>
<td>8.7185</td>
</tr>
<tr>
<td>180</td>
<td>2.2422</td>
<td>4.0023</td>
<td>5.5595</td>
<td>7.9845</td>
<td>8.4860</td>
</tr>
<tr>
<td>245</td>
<td>2.2415</td>
<td>3.9948</td>
<td>5.5523</td>
<td>7.9488</td>
<td>8.4574</td>
</tr>
<tr>
<td>Thin Shell**</td>
<td>2.2423</td>
<td>4.0069</td>
<td>5.5858</td>
<td>8.0129</td>
<td>8.5775</td>
</tr>
</tbody>
</table>

* $\lambda = \omega a^2 \sqrt{\rho / E_{11} h^2}$
** Qatu [77], using 192 terms.

Table(5.3): Convergence of the frequency parameter $\lambda^*$ for graphite-epoxy laminated composite completely free shallow cylindrical shell, $[\pm 45^\circ]_s$, $a/b=1$, $a/h=100$, $a/R=0.5$
<table>
<thead>
<tr>
<th>Mode</th>
<th>Ritz Method</th>
<th>FEM**</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.719</td>
<td>7.712</td>
</tr>
<tr>
<td>2</td>
<td>10.100</td>
<td>10.095</td>
</tr>
<tr>
<td>3</td>
<td>15.060</td>
<td>14.873</td>
</tr>
<tr>
<td>4</td>
<td>20.180</td>
<td>20.117</td>
</tr>
<tr>
<td>5</td>
<td>21.740</td>
<td>21.675</td>
</tr>
</tbody>
</table>

* \[ \lambda = \left( \omega a^2 \sqrt{\rho / D_{22}} \right) / \pi^2 \]

** Chapter (6)

Table(5.4): Frequency parameter (\(\lambda\))* for a square specially orthotropic clamped plate

156
<table>
<thead>
<tr>
<th>$a/R$</th>
<th>Exact [14]</th>
<th>Thin Shell [77]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-10}$</td>
<td>3.17</td>
<td>3.1700</td>
<td>3.1700</td>
</tr>
<tr>
<td>0.001</td>
<td>-</td>
<td>3.1700</td>
<td>3.1700</td>
</tr>
<tr>
<td>0.01</td>
<td>-</td>
<td>3.1747</td>
<td>3.1745</td>
</tr>
<tr>
<td>0.02</td>
<td>-</td>
<td>3.1887</td>
<td>3.1883</td>
</tr>
<tr>
<td>0.05</td>
<td>-</td>
<td>3.2848</td>
<td>3.2838</td>
</tr>
<tr>
<td>0.1</td>
<td>3.6012</td>
<td>3.6071</td>
<td>3.6035</td>
</tr>
<tr>
<td>0.2</td>
<td>4.6663</td>
<td>4.6762</td>
<td>4.6732</td>
</tr>
<tr>
<td>0.3</td>
<td>6.0286</td>
<td>6.0431</td>
<td>6.0406</td>
</tr>
<tr>
<td>0.4</td>
<td>7.5182</td>
<td>7.5372</td>
<td>7.5351</td>
</tr>
<tr>
<td>0.5</td>
<td>9.0608</td>
<td>9.0844</td>
<td>9.0824</td>
</tr>
</tbody>
</table>

$\lambda^* = \omega a^2 \sqrt{\rho / E_{11} h^2}$

Table (5.5): Variation of frequency parameter $\lambda^*$ with $a/R$ for graphite-epoxy laminated composite simply supported cylindrical shell, $[0^\circ/90^\circ]_s$, $a/b=1$, $a/h=100$, $E_{11}/E_{22}=15$, $G_{12}/E_{22}=0.5$, $v_{12}=0.25$, $k^2=5/6$
<table>
<thead>
<tr>
<th>a/R</th>
<th>Exact [14]</th>
<th>Thin Shell [77]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{10}</td>
<td>3.0728</td>
<td>3.1700</td>
<td>3.0728</td>
</tr>
<tr>
<td>0.001</td>
<td>-</td>
<td>3.1700</td>
<td>3.0698</td>
</tr>
<tr>
<td>0.01</td>
<td>-</td>
<td>3.1702</td>
<td>3.0700</td>
</tr>
<tr>
<td>0.02</td>
<td>-</td>
<td>3.1707</td>
<td>3.0707</td>
</tr>
<tr>
<td>0.05</td>
<td>-</td>
<td>3.1743</td>
<td>3.0744</td>
</tr>
<tr>
<td>0.1</td>
<td>3.0882</td>
<td>3.1873</td>
<td>3.0906</td>
</tr>
<tr>
<td>0.2</td>
<td>3.1337</td>
<td>3.2383</td>
<td>3.1433</td>
</tr>
<tr>
<td>0.3</td>
<td>3.2077</td>
<td>3.3210</td>
<td>3.2285</td>
</tr>
<tr>
<td>0.4</td>
<td>3.3075</td>
<td>3.4322</td>
<td>3.3430</td>
</tr>
<tr>
<td>0.5</td>
<td>3.4300</td>
<td>3.5682</td>
<td>3.4827</td>
</tr>
</tbody>
</table>

\[ *\lambda = \omega a^2 \sqrt{\frac{\rho}{E_{11} h^2}} \]

Table (5.6): Variation of frequency parameter \( \lambda^* \) with \( a/R \) for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]\)_5, \( a/b=1 \), \( a/h=20 \), \( E_{11}/E_{22}=15 \), \( G_{12}/E_{22}=0.5 \), \( v_{12}=0.25 \), \( k^2=5/6 \)
<table>
<thead>
<tr>
<th>a/R</th>
<th>Exact [14]</th>
<th>Thin Shell</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-10}$</td>
<td>2.8329</td>
<td>3.1700</td>
<td>2.8442</td>
</tr>
<tr>
<td>0.001</td>
<td>-</td>
<td>3.1700</td>
<td>2.8442</td>
</tr>
<tr>
<td>0.01</td>
<td>-</td>
<td>3.1708</td>
<td>2.8448</td>
</tr>
<tr>
<td>0.02</td>
<td>-</td>
<td>3.1720</td>
<td>2.8457</td>
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<tr>
<td>0.05</td>
<td>-</td>
<td>3.1724</td>
<td>2.879</td>
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<td>2.8346</td>
<td>3.1732</td>
<td>2.8516</td>
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<td>2.8397</td>
<td>3.1828</td>
<td>2.8624</td>
</tr>
<tr>
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<td>2.8481</td>
<td>3.1986</td>
<td>2.8823</td>
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<tr>
<td>0.4</td>
<td>2.8599</td>
<td>3.2204</td>
<td>2.8942</td>
</tr>
<tr>
<td>0.5</td>
<td>2.8749</td>
<td>3.2477</td>
<td>2.9323</td>
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</tbody>
</table>

\[ \lambda^* = \omega a^2 \sqrt{\rho / E_{11} h^2} \]

Table (5.7): Variation of frequency parameter $\lambda^*$ with a/R for graphite-epoxy laminated composite simply supported cylindrical shell, $[0^\circ/90^\circ]_S$, $a/b=1$, $a/h=10$, $E_{11}/E_{22}=15$, $G_{12}/E_{22}=0.5$, $\nu_{12}=0.25$, $k^2=5/6$
\[ *\lambda = \omega a^2 \sqrt{\frac{\rho}{E_{11}}} h^2 \]

Table (5.8): Variation of frequency parameter \( \lambda^* \) with a/h and a/R for graphite-epoxy laminated composite simply supported cylindrical shell, \([0^\circ/90^\circ]_5\), a/b=1, \( E_{11}/E_{22} = 15 \), 
\( G_{12}/E_{22} = 0.5 \), \( v_{12} = 0.25 \), \( k^2 = 5/6 \)
<table>
<thead>
<tr>
<th>$\frac{a}{R}$</th>
<th>$\frac{f}{a}$</th>
<th>$%$ errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.05</td>
<td>0.30551</td>
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<td>0.4</td>
<td>0.10</td>
<td>1.179094</td>
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<tr>
<td>0.6</td>
<td>0.167</td>
<td>2.504351</td>
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<tr>
<td>0.69</td>
<td>0.2</td>
<td>3.03562</td>
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<tr>
<td>0.8</td>
<td>0.25</td>
<td>4.116174</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>5.838428</td>
</tr>
</tbody>
</table>

Table (5.9): Variation of the percentage of error in the present shallow cylindrical shell solution from the exact one with $(\frac{a}{R})$ and $(\frac{f}{a})$, $[0^\circ:90^\circ]_s$.

$a/b=1$, $E_{11}/E_{22}=15$, $G_{12}/E_{22}=0.5$, $\nu_{12}=0.25$, $k^2=5/6$
<table>
<thead>
<tr>
<th>$E_{11}/E_{22}$</th>
<th>Mode</th>
<th>Thin Shell</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>6.0431</td>
<td>6.0497</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.24%)</td>
<td>(0.24%)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>11.3040</td>
<td>11.2633</td>
</tr>
<tr>
<td>15</td>
<td>3</td>
<td>11.4420</td>
<td>11.3802</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>13.6810</td>
<td>13.5825</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>17.661</td>
<td>17.5221</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>5.2064</td>
<td>5.1861</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.4%)</td>
<td>(1.4%)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>9.7090</td>
<td>9.6234</td>
</tr>
<tr>
<td>25</td>
<td>3</td>
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<td>10.9351</td>
</tr>
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<td>4</td>
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<td>12.7209</td>
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<td>5</td>
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<td>15.4334</td>
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<td></td>
<td>1</td>
<td>4.4052</td>
<td>4.3789</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.9%)</td>
<td>(1.9%)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8.2798</td>
<td>8.1804</td>
</tr>
<tr>
<td>40</td>
<td>3</td>
<td>10.9420</td>
<td>10.7450</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>12.3240</td>
<td>12.0311</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>13.7900</td>
<td>13.3804</td>
</tr>
</tbody>
</table>

$* \lambda = \omega a^2 \sqrt{\frac{\rho}{E_{11}h^2}}$

1. Values in parentheses are the exact values for the fundamental frequency parameter, Qatu [77] and Reddy [87].
2. Error in the fundamental frequency for thin shell.

Table (5.10): Variation of frequency parameter $\lambda^*$ with orthotropy ratio $E_{11}/E_{22}$ for graphite-epoxy laminated composite simply supported cylindrical shell,

$\{0^\circ/90^\circ\}_s$, $a/b=1$, $a/h=100$, $a/R=0.3$, $G_{12}/E_{22}=0.5$, $v_{12}=0.25$, $k'=5/6$

162
<table>
<thead>
<tr>
<th>Mode</th>
<th>Undamaged</th>
<th>Damaged**</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.2415</td>
<td>2.1974</td>
</tr>
<tr>
<td>2</td>
<td>3.9948</td>
<td>3.8494</td>
</tr>
<tr>
<td>3</td>
<td>5.5523</td>
<td>5.1737</td>
</tr>
<tr>
<td>4</td>
<td>7.9488</td>
<td>7.1875</td>
</tr>
<tr>
<td>5</td>
<td>8.4574</td>
<td>7.3855</td>
</tr>
</tbody>
</table>

* \( \lambda = \omega a^2 \sqrt{\rho / E_{11} h^2} \)

** \( N = 20 \) cracks 'in. \( A_e = 30\% \)

Table(5.11): Effect of damage on the frequency parameter \( \lambda^* \) for graphite-epoxy laminated composite completely free shallow cylindrical shell,

\([\pm 45^\circ]_s , a/b=1, a/h=100, a/R=0.5\)
<table>
<thead>
<tr>
<th>Mode</th>
<th>Undamaged</th>
<th>Damaged**</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.2112</td>
<td>2.1717</td>
</tr>
<tr>
<td>2</td>
<td>5.1087</td>
<td>4.9631</td>
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<tr>
<td>3</td>
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<td>5.2090</td>
</tr>
<tr>
<td>4</td>
<td>7.6218</td>
<td>6.9625</td>
</tr>
<tr>
<td>5</td>
<td>7.8815</td>
<td>6.9919</td>
</tr>
</tbody>
</table>

\[ \lambda = \omega ^2 \sqrt{\rho \cdot E_{11} h^2} \]

** N= 20 cracks /in. \( A_d = 30\% \)

Table (5.12): Effect of damage on the frequency parameter \( \lambda \) for graphite-epoxy laminated composite completely free shallow cylindrical shell.

\([\pm 30^\circ]_s, a/b=1, a/h=100, a/R=0.5\)
<table>
<thead>
<tr>
<th>a/h</th>
<th>a/R</th>
<th>Undamaged</th>
<th>Damaged**</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>3.1700</td>
<td>3.0518</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>3.6035</td>
<td>3.4624</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>4.6732</td>
<td>4.4868</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3</td>
<td>6.0406</td>
<td>5.7936</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>7.5351</td>
<td>7.2208</td>
</tr>
<tr>
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<td>0.5</td>
<td>9.0824</td>
<td>8.6960</td>
</tr>
<tr>
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<td>100</td>
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<td>2.9475</td>
</tr>
<tr>
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<td>0.1</td>
<td>3.0906</td>
<td>2.9578</td>
</tr>
<tr>
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<td>0.2</td>
<td>3.1433</td>
<td>3.0022</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3</td>
<td>3.2285</td>
<td>3.0790</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4</td>
<td>3.3430</td>
<td>3.1685</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>3.4827</td>
<td>3.3058</td>
</tr>
<tr>
<td>0.0</td>
<td>20</td>
<td>2.8442</td>
<td>2.6952</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>2.8516</td>
<td>2.6967</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>2.8624</td>
<td>2.7030</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3</td>
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<tr>
<td>0.4</td>
<td>0.4</td>
<td>2.8942</td>
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<tr>
<td>0.5</td>
<td>0.5</td>
<td>2.9323</td>
<td>2.7629</td>
</tr>
</tbody>
</table>

\[ \lambda = \omega a^2 \sqrt{\rho / E_{11} h^2} \]

** N= 40 cracks/in, A_0 = 30%

Table (5.13): Effect of damage on the frequency parameter \( \lambda \) with a/h and a/R for graphite-epoxy laminated composite simply supported cylindrical shell.

\([0^\circ/90^\circ]_s, a/b=1, E_{11}/E_{22}=15, G_{12}/E_{22}=0.5, v_{12}=0.25, k^2=5/6\]

165
6.1 INTRODUCTION

In the previous chapter, the Ritz method was used to generate approximate solutions for laminated composite shallow shells with matrix cracks. In a final conclusion, the method has been successfully utilized to investigate the effects of curvature, side to thickness ratio, the aspect ratio of the projected shell planform and the extent of damage. The damage mode is limited to matrix cracks which are assumed to be statistically homogenous within a centrally located elliptical or circular region. Clearly, such solution procedure in the format presented in Chapter (5) is incapable of handling any other shapes for the damaged zone such as square or rectangle. Such shapes will require major changes in the trial displacement functions to ensure the continuity if Ritz method is to be utilized. Even then, if the displacement field is modified, there might be some singularities associated with the existence of sharp edges within the domain of application. To complete the present work
and to make our investigation thorough, the work has been extended to use the finite element method as an alternative numerical approximate technique to handle such problems. Another advantage of generating solutions by the finite element method is its flexibility in discretizing the domain layer-wise, i.e. using discrete laminate theory rather than the smeared laminate theory. Such advantage is important in two aspects; first, incorporating the effect of damage on a layer-wise strategy facilitates the numerical calculations; second, it makes it possible to use a more accurate transverse shear deformation theory as proposed by Hong [42] and Reddy [83]. One should also add that the displacement continuity conditions are automatically satisfied on the inter-elements boundaries without any special treatment which is not the case for Ritz method.

In this chapter a brief outline of the implementation of the finite element solution for the analysis of laminated composite plates containing matrix crack type of damage mode is discussed. Also, a parametric study including several material and geometrical parameters that may affect the frequency response of damaged and undamaged laminated composite plates is presented.
6.2 FINITE ELEMENT FORMULATION

6.2.1 Theoretical Development

The focus of the current research is on the global dynamic behavior of composite laminates. Therefore, it is reasonable to seek a solution scheme which is based on first-order-shear deformable theory (FSDT). This conclusion is a consequent result of the discussion of this aspect in Chapter (2). The theory is well-documented in literature and was discussed in the previous chapter. Into that discussion it was concluded that using (FSDT) requires a shear correction factor \( k' \). Different methods were introduced to evaluate this factor. For instance, Mindlin [66] proposed a value of \( \pi^2/12 \) for the factor \( k' \) while Reissner [88] used a value of \( 5/6 \). These different values are the result of the different approaches adopted to derive the shear factor. Despite the fact that the values of \( k' \) were derived for the analysis of homogeneous isotropic plates, they were used without rigorous investigation for the analysis of laminated composite plates. But recently, Hong [42] derived what he called "a consistent deformable shear theory" or (CDST) for the analysis of laminated composite plates. These features of (CDST) provide a reasonable choice for our investigation. Although, it was concluded earlier that the use of smeared laminate theory has its justification, Hong’s approach is based on describe laminate theory. Which adds another advantage, since matrix cracks are generally developed in some of the layers and not all of them.
By separating the inplane equilibrium equations from the transverse ones, and ignoring the inertia terms, for the $k^{th}$ layer, Eqns. (2.35) can be expanded as:

\[
\sigma^{(k)}_{\alpha\beta,\beta} + \sigma^{(k)}_{\beta,3} = 0
\]  

\[
\sigma^{(k)}_{3,\alpha} + \sigma^{(k)}_{33,3} = 0
\]  

The inplane stresses which satisfy the static equilibrium equations are assumed to be linear function of the normal coordinate ($x_i^k$), then:

\[
\sigma^{k}_{\alpha\beta} = A^k_{\alpha\beta} + B^k_{\alpha\beta} x_3^k
\]  

where, $A^k_{\alpha\beta}$ and $B^k_{\alpha\beta}$ are independent of ($x_i^k$).

By using the definition of the force resultants and neglecting the body forces and surface tractions, with some calculus and algebraic manipulation, the transverse shear constitutive relations are written as[42]:

\[
Q^k_\alpha = \sum_{i=1}^{n} C^k_{\alpha\beta} (\Phi^i_\beta + \omega^i_{3,\beta})
\]
where \( C_{ab} \) are the coefficients of a symmetric matrix defined by the material properties, thickness of layers and the stacking sequence of a laminate which are given in [25.42].

Equations (6.4) represent the shear constitutive equations without the need for a shear correction factor. It also indicates that the shear force in a layer is a linear combination of the transverse shear strains from all the other layers for a given laminate.

Despite the advantages and flexibility achieved through this model, it puts some limitations on the size of problems that can be handled. This is due to the fact that this type of discretization requires more computer storage and consequently, more computational time.

So far, the above discussion concluded two important components of the finite element process; namely: the type of kinematic relations and the laminate discretization model. Folding this discussion requires the investigation of the type of element to be used in the finite element formulation. In this regard, there are a variety of elements that are developed for plate analysis. Among these elements, the isoparametric type elements which utilize Lagrange or Serendipity interpolation functions are considered superior.

However, Huges et al. [44] developed what is called "Heterosis Element" which is 9-node quadrilateral that uses Serendipity shape functions for the transverse displacements and Lagrange shape function for the rotational degrees of freedom. Also, a selective/ reduced integration scheme is used to evaluate the element stiffnesses. The results presented in [44] show that the element performs consistently well for different numerical tests even the cases that Serendipity and Lagrange elements have poor performance.
The same element was tested, [25], for the static case against quadrilateral 4-node element (Q4) and the results were far much better for the Heterosis Element. Therefore, the element is a good candidate for our finite element procedure. The element degrees of freedom are shown in Fig.(6.1). Also, the shape functions and their derivatives are presented in Appendix (C).

6.2.2 Finite Element Idealization

The finite element method is a general numerical technique for the construction of an approximate solution to many boundary value problems. The method is well-documented and we do not intend to elaborate on its basics but only some aspects directly related to the formulation of our problem will be briefly discussed.

The basic concept is to interpolate the solution over a small element of the domain and using a variational formulation technique to assemble the solution for the collection of elements of the whole domain.

The displacement field for an element can be interpolated using the nodal points displacements as follows:
\begin{align*}
\mathbf{u}_1 &= \sum_{i=1}^{n} u_{i} \Phi_i^1 \\
\mathbf{u}_2 &= \sum_{i=1}^{n} u_{i} \Phi_i^2 \\
\mathbf{u}_3 &= \sum_{i=1}^{n} u_{i} \Phi_i^3 \\
\Phi_1 &= \sum_{i=1}^{n} \psi_i^1 \Phi_i^1 \\
\Phi_2 &= \sum_{i=1}^{n} \psi_i^2 \Phi_i^2
\end{align*}
\tag{6.5}

where,

\(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\) is the displacement vector (\(j=1,2,3\) and \(\alpha=1,2\)).

\(\mathbf{u}_j^{\alpha}, \psi^{\alpha}_j\) is the displacement vector at the element nodal points.

\(\Phi^i\) are the interpolation functions, and \((n)\) is the number of the nodal points of the element.

For the element shown in Figure (6.1) and using the interpolation functions and transformations given in Appendix (C), Equations (6.5) can be written as:

\begin{align*}
\mathbf{u}_1^{(1)}(x,t) &= H_u^T(x)\mathbf{U}(t) \\
\Phi_1^{(1)}(x,t) &= H_\Phi^T(x)\Phi(t) \\
\mathbf{u}_i(x,t) &= H_\mathbf{u}^T(x)\mathbf{W}(t)
\end{align*}
\tag{6.6}
where

\[ u^{i\varepsilon} = \{u^{i\varepsilon}_1, u^{i\varepsilon}_2\}^T \]

\[ \phi^{i\varepsilon} = \{\phi^{i\varepsilon}_1, \phi^{i\varepsilon}_2\}^T \]

and \( U(t), \Phi(t) \) and \( W(t) \) are vector functions of time defined at the nodal points and \( H_u, H_\phi \) and \( H_\omega \) are respectively, the matrices of spatial interpolation functions for the field variables indicated by their subscripts.

In the same fashion, the generalized strains may be expressed as:

\[
\begin{align*}
\epsilon^{i\varepsilon}(x_i, t) &= T_u^T(x_i)U(t) \\
\kappa^{i\varepsilon}(x_i, t) &= T_\Phi^T(x_i)\Phi(t) \\
\gamma^{i\varepsilon}(x_i, t) &= T_\omega^T(x_i)W(t) + H_\omega^T(x_i)\Phi(t)
\end{align*}
\]

where,

\[
\begin{align*}
\epsilon^{i\varepsilon} &= \{\epsilon^{i\varepsilon}_{11}, \epsilon^{i\varepsilon}_{22}, 2\gamma^{i\varepsilon}_{12}\}^T \\
\gamma^{i\varepsilon} &= 2\{\epsilon^{i\varepsilon}_{11}, \epsilon^{i\varepsilon}_{22}\}^T \\
\kappa^{i\varepsilon} &= \{\kappa^{i\varepsilon}_{11}, \kappa^{i\varepsilon}_{22}, 2\kappa^{i\varepsilon}_{12}\}^T
\end{align*}
\]

and \( T_u, T_\Phi \) and \( T_\omega \) are the transformation matrices derived from the interpolation functions by suitable differentiation as shown in Appendix (C).
6.2.3 Solution procedure

For initial value problems, the first variation of the lagrangian, i.e. Hamilton's principle, of the energy expression gives the variational form of the equations of motion of the laminated plate. The present finite element formulation is based on the energy functional given by Equations (4.18) for the kinetic energy expression and (4.24) for the total potential energy after the necessary modifications for discrete laminate theory and the transverse shear deformation described earlier in this section.

For free vibration analysis, the potential energy due to applied loads is set to zero and neglecting the effect of damping, we get the well-known form:

\[ M\ddot{x} + Kx = 0 \]  

(6.8)

where:

\[ [M] \] is the system mass matrix, and

\[ [K] \] is the system stiffness matrix.

Assuming harmonic motion of the system, then all the time functions in Eqns. (6.6) can all be taken as \( e^{i\omega t} \) where \( \omega \) is the natural frequency and \( i = \sqrt{-1} \). Therefore, the solution can be written as:
\[ X = q(x_t) e^{i\omega t} \]  \hspace{1cm} (6.9)

where,

\{q\} is the amplitude vector, which are the unknown nodal points displacements.

By substituting Eqn.(6.9) into Eqn.(6.8), we get the generalized eigenvalue problem:

\[ \{K - \omega^2 M\} \{q\} = 0 \]  \hspace{1cm} (6.10)

The solution of Eqn.(6.10) is a standard mathematical problem which yields the vibration frequencies and optionally the mode shapes. For this purpose a finite element program was developed using the 'heterosis' plate bending element previously discussed. A compacted flow chart for the FORTRAN program showing main components and different routines is given in Appendix D (Figure (D.1)). One should mention that in the present formulation the geometric continuity conditions are automatically satisfied whereas the specified boundary conditions are enforced by modifying the stiffness and mass matrices as given in Appendix (C).
6.3 RESULTS AND DISCUSSION

6.3.1 Sensitivity analysis

The accuracy of the present formulation and the finite element code developed in this work has to be checked before investigating the effect of damage on the frequency response. For this purpose, an undamaged sandwich plate [90°/0°/90°] whose 3-dimensional elasticity solution is known, Srinivas[98], is considered. The plate is simply supported with a/b=1, a/h=10 and is made from graphite-epoxy composite having the material properties shown in Table (6.1). Figure (6.2) illustrates the convergence of the numerical solutions with the finite element mesh refinement. It is worth mentioning that the rate of convergence is very fast between (4) and (16) elements mesh and is slower for any refinement beyond the (16) elements mesh. By using only (16) element mesh, the deviation in the present formulation from the exact solution based on 3-dimensional elasticity formulation is only 0.5 % and it hardly improves with further refinement because of approximations involved in the first order shear deformation theory as well as in the numerical computation.

Tables (6.2) and (6.3) compare the present finite element results for the first and second frequencies for [±18°/90°]s and [±18°/90°10]s graphite-epoxy, [AS4/3501-6]. laminates with the experimental investigations [24]. The results are also compared with those generated by using one of the known commercial packages, ABAQUS in Tables (6.4) and (6.5). Table (6.6) is a comparison between the finite element results and the Ritz method results of the previous chapter.
All of the cases compared above reflect excellent agreement between the present finite element solution and both the analytical and experimental investigations. Therefore, one can easily conclude that the numerical results of the present finite element formulation and the corresponding FORTRAN code are reasonably accurate. Therefore, from this point inward all the results are generated using our own program which is designed specifically for the present work.

6.3.2 Undamaged plate analysis

In the numerical analysis to follow, a typical graphite/epoxy laminated composite plate with the material properties given in Table (6.1) is used. Unless otherwise specified, the plate is square (a/b=1) with a moderate thickness aspect ratio, a/h=10. The effect of transverse shear deformation on the frequency response is investigated for sandwich square plate. The effect of the thickness aspect ratio (h/a) on the first and second frequency modes is depicted in Table(6.7). Figure (6.3) is a comparison for the first four frequency computed using the present finite element results noted as (FSDT), higher order transverse shear deformation theory (HSDT),[86], classical plate theory (CPT),[99], and the exact solution,[99], for a crossply simply supported plate. The chart clearly shows that the classical plate theory overestimates the frequency and the error is bigger for higher frequency modes. The percentage of error in estimating the frequencies using the classical

177
plate theory is increasing nonlinearly for higher modes as shown in Figure (6.4). The maximum error for the fourth mode, for example, can be as high as 14%. The results, therefore reflect the importance of including the effect of transverse shear deformation by using at least a first order shear theory for an accurate prediction of the frequency response of such orthotropic materials.

The variation of the fundamental frequency mode with the degree of orthotropy of the composite laminate represented by the ratio \( \frac{E_{11}}{E_{22}} \) is shown in Table (6.8). These finite element results have been generated using only 16 element mesh. The deviation from the exact value of the fundamental frequency is 2.51% for the present finite element analysis compared to 24.06% for the thin plate theory for a ratio of \( \frac{E_{11}}{E_{22}} = 15 \).

Once again these results show clearly the fact that the thin plate theory assumption overestimates the vibration frequencies and that the error increases rapidly for larger values of the thickness aspect ratio as well as for larger degree of orthotropy.

### 6.3.3 Damaged plate analysis

A typical laminated composite plate of area \( A_0 \) with a centrally located damaged region of area \( A_d \) is shown in Figure (6.5). The damage is limited to matrix cracks mode only and the damaged region is saturated with cracks usually denoted in terms of the crack density defined as \( N \) cracks per inch. The shape of the damaged region may be either a rectangle, a circle or an ellipse. The plate geometry, coordinates system and location of
damage is shown in Figure (6.6). For such a composite plate with classical boundary conditions, we present some key results for the effect of damage on the frequency response. The following geometrical and material parameters are used to generate the parametric results:

- Shape of damage zone;
- Size and extent of damage;
- Boundary conditions; and
- Cut-out as the total damage.

The effect of the size and shape of damage on the frequency response for a simply supported crossply laminated square plates was investigated. The damage zone with three different shapes having the same damage area were considered. These shapes are rectangular, elliptical and circular. For a damage level of 40 cracks per in. in the 90°-plies, there is no significant difference in the vibration frequency between the three shapes as shown in Table (6.9) and graphically in Figure (6.7). Hence, the results presented hereafter are for rectangular damage zones because it is easier to handle the data input and mesh generation in the computer code. The size of damage is measured as the percentage of the D-zone with respect to the total area of the plate. Figure (6.8) shows the variation of the first five frequency modes of a simply supported crossply laminate with the percentage of damage. The graph reflects a decrease in the fundamental frequency of about 4% when 50% of 90°-plies are saturated with 40 cracks per inch. It also shows that this effect becomes more pronounced for higher (about 8% for the fifth mode) vibration modes.
probably due to the fact that the presence of damage within the plate is reflected more critically on the higher modes exhibiting more localized patterns of motion. It is very interesting to note that although this small-looking change in the frequency (4%-8%) is indeed very significant even from practical standpoint. For a crossply carbon-fiber-reinforced plastic plate, the change of the fundamental frequency within < 1% has been used to detect the size and location of damage by Cawley and Adams [22].

The frequency response for three composite laminates with various boundary conditions was investigated. The results shown in Tables (6.10) and (6.11) compare the first five frequency parameters for specially orthotropic simply supported and clamped plates respectively. The results for the undamaged plate show a very good agreement with the exact and other numerical solutions. Table (6.12) presents the frequencies for graphite-epoxy 8-angle-ply \([0^\circ/\pm30^\circ]\) cantilever plate. The predicted frequencies for the undamaged plate compare reasonably well with the experimental investigation [21]. For centrally located damage area of about 30% of the total plate area, there is a decrease in the fundamental frequency of about 4% irrespective of the plate boundary conditions. The reduction in the fifth mode frequency goes up to 10% for both the simply supported and the clamped plate boundary conditions.

In some real life situations, the damaged region may be cut out or repaired using some isotropic material, e.g. matrix type of filling. For such practical case, we investigated two simply supported laminated plates with stacking sequences of \([90^\circ/0^\circ/90^\circ]\) and \([45^\circ/0^\circ/45^\circ]\). The damaged zone is taken either a circular hole, a square cut out, an
isotropic matrix filling or matrix cracks damage mode with crack density of 20 cracks per inch. The results shown in Figures (6.9) and (6.10) show that the cut-outs either circular or square can significantly reduce the frequency. For the cases studied, the reductions are about 20% and 25% for [90°/0°/90°] and [45°/0°/45°] laminates, respectively. When the cut-outs are patched with an isotropic material, like the pure polymer matrix, the reduction in the fundamental frequency is only about 6%. The effects of extending the area of partial or total damage (cut-out) to about 81% for simply supported and clamped plates are shown in Figures (6.11)-(6.13). It is interesting to note that for large cut-outs (c/a=0.9) the fundamental frequencies are increased by a factor of 2.5 for a simply supported case and by a factor of 5 for the clamped case. This behavior results probably from the fact that plates with such large cut-outs may no longer behave as monolithic plates. The coupling between the boundary conditions and the conditions at the free edges of the cut-outs may be very strong. Therefore, further studies are required to fully understand this phenomenon, and these studies must include the mode shape behavior.

6.4 CONCLUDING REMARKS

The current investigation is the first attempt to quantify the change in the vibration response of moderately thick laminated composite plates due to matrix-cracks type of damage. The model used has incorporated the effect of shear deformation in a consistent manner leading to a finite element solution algorithm. The damage mechanics approach was used for the modeling of these matrix cracks which could then be coupled with the global macro-level of analysis. The analysis shows that this approach can be a very
effective tool to analyze the global dynamic response of damaged plates. The finite element results for various plate boundary conditions show that the matrix cracks type of damage reduces the vibration frequencies and the pattern of results is identical to that of the previous chapter. The change in the frequency for a given size and degree of damage appear to have a little sensitivity toward the plate support conditions (a reduction of about 3-4% for the first mode. This reduction is more pronounced for higher frequency modes (about 7%-8% for the fifth mode). This happens probably due to the fact that the presence of damage within the plate is reflected more critically on the higher modes exhibiting more localized patterns of motion. Moreover, the extent of damage and the crack density are the major contributing factors to the reductions in the vibration frequencies. On the other hand, the study shows that for a given damage level, the shape of the damaged zone has a very little influence on the frequency response, provided that the damaged zone-aspect ratio is of the order of unity. The damage region was assumed to be either a rectangle, a square, an ellipse or a circle. By recalling similar results from the previous chapter (using Ritz method), they can be an important contribution since they can facilitate the investigation of many similar problems by using any of the available numerical solution techniques. For instance, the Ritz method might be used assuming that the damage region is either an ellipse or a circle to avoid the difficulty of enforcing the continuity conditions for shapes with sharp edges that might result in some singularities. On the other hand, if the finite element is to be used, a square or a rectangular damage zone is easier to work with in terms of domain discretization and mesh generation.
Figure (6.1): Local and global coordinate systems and degrees of freedom for "Heterosis" element
Figure (6.2): Convergence of the finite element results with mesh refinement.

\[
\% \text{ error} = \left( \frac{\omega}{\omega_0} - 1 \right) \times 100, \quad \omega_0 \text{ is the exact frequency.} \quad [99]
\]
Figure 6.3. Effect of transverse shear deformation theory on the frequency response of a crossply laminated composite ($a/b=1, a/h=10$).
Figure (6.4): Variation of error in the frequency response for various vibration modes (crossply laminated plate, a/b=1, a/h=10).
Figure (6.5): A representative local volume element in D-zone of a general laminate.
Figure (6.6): Plate geometry, coordinate systems and the location of damage
Figure (6.7): Effect of the damaged zone shape on the fundamental frequency of crossply laminate
Figure (6.8): Effect of the damage size on the first five frequencies of crossply laminate
Figure (6.9): Variation of the normalized frequency for various damaged zone representations, [90°/0°/90°].
Figure (6.10): Variation of the normalized frequency for various damaged zone representation, $[45^\circ/0^\circ/45^\circ]$. 

192
Figure (6.11): Variation of the normalized frequency for extended circular damaged zone, (simply supported [45°/0°/45°] plate).
Figure (6.12): Variation of the normalized frequency for extended square damaged zone (simply supported $[45^\circ/0^\circ/45^\circ]$ plate).
Figure (6.13): Variation of the normalized frequency for extended square damaged zone (clamped $[45^\circ/0^\circ/45^\circ]$ plate).
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<tbody>
<tr>
<td>$E_{11}$</td>
<td>$18.5 \times 10^6$ psi</td>
</tr>
<tr>
<td>$E_{22}$</td>
<td>$1.6 \times 10^6$ psi</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.25</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>$0.65 \times 10^6$ psi</td>
</tr>
<tr>
<td>$\rho_k$</td>
<td>0.055 lb/in$^3$</td>
</tr>
<tr>
<td>$t_k$</td>
<td>0.0052 in</td>
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</table>

Table(6.1): Material properties.
<table>
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<th>Case</th>
<th>Stacking Seqn.</th>
<th>Dimensions (in)</th>
<th>FE result ($\omega_1$) (hz)</th>
<th>Experimental($\omega_1$) (hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$[\pm 18$ $^\circ$ $90$ $^\circ]$ $s$</td>
<td>6x6</td>
<td>589.657</td>
<td>588</td>
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<tr>
<td>2</td>
<td>$[\pm 18$ $^\circ$ $90$ $^\circ]$ $l$</td>
<td>10x10</td>
<td>353.789</td>
<td>343</td>
</tr>
</tbody>
</table>

Table(6.2): Comparison of Finite Element results with experimental work[24] for the fundamental frequency.

<table>
<thead>
<tr>
<th>Case</th>
<th>Stacking Seqn.</th>
<th>Dimensions (in)</th>
<th>FE result ($\omega_2$) (hz)</th>
<th>Experimental($\omega_2$) (hz)</th>
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<tbody>
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<td>6x6</td>
<td>1168.576</td>
<td>1180</td>
</tr>
<tr>
<td>2</td>
<td>$[\pm 18$ $^\circ$ $90$ $^\circ]$ $l$</td>
<td>10x10</td>
<td>700.983</td>
<td>790</td>
</tr>
</tbody>
</table>

Table(6.3): Comparison of Finite Element results with experimental work[24] for the second mode frequency.

197
### Table (6.4): Comparison of Finite Element results with Abaqus for the fundamental frequency.

<table>
<thead>
<tr>
<th>Case</th>
<th>Stacking Seqn.</th>
<th>Dimensions (in)</th>
<th>FE result ($\omega_1$) (hz)</th>
<th>Abaqus res. ($\omega_1$) (hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$[\pm18_{/90_0}]$</td>
<td>6x6</td>
<td>589.657</td>
<td>588.345</td>
</tr>
<tr>
<td>2</td>
<td>$[\pm18_{/90_{10}}]$</td>
<td>10x10</td>
<td>353.789</td>
<td>372.015</td>
</tr>
</tbody>
</table>

### Table (6.5): Comparison of Finite Element results with Abaqus for the second mode frequency.

<table>
<thead>
<tr>
<th>Case</th>
<th>Stacking Seqn.</th>
<th>Dimensions (in)</th>
<th>FE result ($\omega_2$) (hz)</th>
<th>Abaqus ($\omega_2$) (hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$[\pm18_{/90_0}]$</td>
<td>6x6</td>
<td>1168.576</td>
<td>1167.974</td>
</tr>
<tr>
<td>2</td>
<td>$[\pm18_{/90_{10}}]$</td>
<td>10x10</td>
<td>700.983</td>
<td>722.732</td>
</tr>
<tr>
<td>Mode</td>
<td>Frequency parameter by:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>------</td>
<td>-------------------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ritz Method**</td>
<td>FEM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>7.719</td>
<td>7.712</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10.100</td>
<td>10.095</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>15.060</td>
<td>14.873</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>20.180</td>
<td>20.117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>21.740</td>
<td>21.675</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* $\lambda = (\omega a^2 \sqrt{\frac{\rho}{D_{22}}}) \cdot \pi^2$

** Chapter (5).

Table (6.6): Frequency parameter ($\lambda$)* for a square specially orthotropic clamped plate
Table (6.7): Effect of (h/a) on the frequency parameter ($\lambda$)* for a square simply supported specially orthotropic plate

\[
\lambda = (\omega \sqrt{\frac{\rho h^2}{E_{11}^{(2)}}})
\]

* % error from the exact values

(a) First mode

<table>
<thead>
<tr>
<th>h/a</th>
<th>Exact [99]</th>
<th>Thin Plate [99]</th>
<th>FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.04742</td>
<td>0.04967</td>
<td>0.047656</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(4.74)</td>
<td>(0.5)</td>
</tr>
<tr>
<td>0.3</td>
<td>0.33200</td>
<td>0.44699</td>
<td>0.352584</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(34.64)</td>
<td>(6.2)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.70338</td>
<td>1.24160</td>
<td>0.771831</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(76.52)</td>
<td>(8.31)</td>
</tr>
</tbody>
</table>

(b) Second mode

<table>
<thead>
<tr>
<th>h/a</th>
<th>Exact [99]</th>
<th>FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.21697</td>
<td>0.22244</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2.52)</td>
</tr>
<tr>
<td>0.3</td>
<td>0.65043</td>
<td>0.71053</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(9.24)</td>
</tr>
<tr>
<td>0.5</td>
<td>1.08240</td>
<td>1.21683</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(12.42)</td>
</tr>
</tbody>
</table>
\begin{table}
ext \begin{tabular}{|c|c|c|c|}
\hline
$E_{11}/E_{22}$ & Exact [99] & Thin Plate [99] & FEM \\
\hline
1 & 0.04742 & 0.04967 \(4.74\) & 0.047656 \(0.5\) \\
\hline
5 & 0.077148 & 0.08533 \(10.61\) & 0.078267 \(1.45\) \\
\hline
15 & 0.70338 & 0.13899 \(24.06\) & 0.114835 \(2.51\) \\
\hline
\end{tabular}
\end{table}

\* $\lambda = (\omega \sqrt[2]{\rho h / E_{11}^{(2)}})$
\* \% error from the exact values

Table (6.8): Effect of \((E_{11}, E_{22})\) on the frequency parameter \((\lambda)\)* for a square simply supported specially orthotropic plate (First mode)
Normalized frequencies for damage at the center of the plate (area $A_0$), the damage area ($A_d$) with the shape of:

<table>
<thead>
<tr>
<th>$c_C \frac{A_d}{A_0}$</th>
<th>Rectangular</th>
<th>Circular</th>
<th>Elliptical</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>0.9948</td>
<td>0.9932</td>
<td>0.994</td>
</tr>
<tr>
<td>20</td>
<td>0.988</td>
<td>0.9851</td>
<td>0.9869</td>
</tr>
<tr>
<td>30</td>
<td>0.9772</td>
<td>0.9735</td>
<td>0.9753</td>
</tr>
<tr>
<td>40</td>
<td>0.964</td>
<td>0.9609</td>
<td>0.9628</td>
</tr>
<tr>
<td>50</td>
<td>0.9512</td>
<td>0.946</td>
<td>0.9491</td>
</tr>
</tbody>
</table>

Table(6.9): Effect of the damaged zone shape on the fundamental frequency of crossply laminate
<table>
<thead>
<tr>
<th>Mode</th>
<th>Exact</th>
<th>FEM undamaged</th>
<th>FEM damaged**</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.60555</td>
<td>3.60555</td>
<td>3.54782</td>
</tr>
<tr>
<td>2</td>
<td>5.83095</td>
<td>5.83104</td>
<td>5.69226</td>
</tr>
<tr>
<td>3</td>
<td>10.4403</td>
<td>10.4416</td>
<td>10.0667</td>
</tr>
<tr>
<td>4</td>
<td>13.0000</td>
<td>13.0024</td>
<td>12.2352</td>
</tr>
<tr>
<td>5</td>
<td>14.4220</td>
<td>14.4229</td>
<td>13.1825</td>
</tr>
</tbody>
</table>

\[ \bar{\lambda} = \left( \omega a^2 \sqrt{\rho / D_{22}} \right) \pi^2 \]

** 30% at the center of the plate

Table (6.10): Frequency parameter (\( \bar{\lambda} \)) for a square simply supported specially orthotropic plate

203
<table>
<thead>
<tr>
<th>Mode</th>
<th>Ritz Method***</th>
<th>FEM undamaged</th>
<th>FEM damaged**</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.719</td>
<td>7.712</td>
<td>7.433</td>
</tr>
<tr>
<td>2</td>
<td>10.100</td>
<td>10.095</td>
<td>9.584</td>
</tr>
<tr>
<td>3</td>
<td>15.060</td>
<td>14.873</td>
<td>14.091</td>
</tr>
<tr>
<td>4</td>
<td>20.180</td>
<td>20.117</td>
<td>18.951</td>
</tr>
<tr>
<td>5</td>
<td>21.740</td>
<td>21.675</td>
<td>20.165</td>
</tr>
</tbody>
</table>

\[
\lambda = \left( \omega a^2 \frac{1}{\sqrt{\rho \cdot D_{22}}} \right) \pi^2
\]

** 30% at the center of the plate

*** Chapter (5)

Table (6.11): Frequency parameter (\(\lambda\))* for a square specially orthotropic clamped plate
<table>
<thead>
<tr>
<th>Mode</th>
<th>Experiment[24]</th>
<th>FEM undamaged</th>
<th>FEM damaged*</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>58.3</td>
<td>64.235</td>
<td>62.326</td>
</tr>
<tr>
<td>2</td>
<td>148</td>
<td>136.546</td>
<td>131.443</td>
</tr>
<tr>
<td>3</td>
<td>362.7</td>
<td>407.204</td>
<td>392.065</td>
</tr>
<tr>
<td>4</td>
<td>508</td>
<td>525.557</td>
<td>502.128</td>
</tr>
<tr>
<td>5</td>
<td>546</td>
<td>587.025</td>
<td>567.982</td>
</tr>
</tbody>
</table>

* 30% at the center of the plate

Table(6.12): Frequencies (hz) of a cantilever 8-ply graphite-epoxy 6x3 in. [0_2/±30]_3 plate
CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 CONCLUDING REMARKS

The present study is very unique and can be considered an important building block in the field of structural dynamics. It has successfully tackled and provided solutions for a new, practical and very much needed in the field. In the absence of closed-form solutions, approximate ones have been presented for laminated polymer composite shallow shells and plates containing matrix crack type of damage with arbitrary support conditions. These solution are based on a first order transverse shear deformable theory which has been proven to be essential for the analysis of composite laminates. Another very important contribution of the present study is the fact that it has bridged the gap, probably for the first time, between the structural dynamics field and continuum damage mechanics field. We presented a simple analytical formulation for matrix cracking type of damage in continuous fiber laminated composites by using a damage mechanics approach. The damage is reflected in the laminate equations by using a second order tensor internal state
variables (ISV) which are strain-like quantities. These ISV’s are introduced at the ply level through the alteration of the lamina constitutive equations since the matrix cracks are fully contained within the cracked plies. In this manner, the lamina stiffness is directly related to the cracking density. Such approach has been found to be an effective tool to predict the global dynamic behavior of laminated composites.

The developed damage model was then utilized to study the frequency response of laminated composite plates and shallow cylindrical shells. Due to the absence of closed-form solutions because of several complicated factors, the Ritz method and the finite element solution have been used as approximate techniques. Using those two solution techniques together, made it possible to investigate the effect of almost all of the significant parameters on the frequency response of plates and shallow cylindrical shells. Two lengthy FORTRAN computer programs have been written to generate numerical results and to investigate various aspects of the problem. The focus of the investigation was to quantify the reduction in the free vibration frequencies due to matrix cracking type of damage mode. Such analysis is a very valuable contribution since it can facilitate any future research in this field by separating the effects of various damage modes. The study might also help the researchers to understand the interaction between these damage modes. More importantly, with more investigations and refinements, the numerical results can form a very useful database to develop a none-destructive evaluation technique (NDE). Such technique can be used to predict the damage status by reading the changes in the frequency response as it has been reported by Cawley and Adams [22]. By tackling
this new problem of the dynamics of a laminated composite shell with centrally located
damaged zone, the author hopes to open the door for further investigations. Which can be
easily done since the present formulation and solution procedures can be modified and or
expanded to study other damage modes as well as various laminate geometries.

More specific conclusions based upon the parametric analytical/ numerical results include
the following:

1. The accuracy of the predicted stiffness reductions are in a very good agreement with
both analytical and experimental results for thin to moderately thick crossply
laminates. The thickness of the cracked ply plays a very important role. As this ply
gets thicker, the results of the reduced stiffness deviates more from the experimental
investigations. One possible explanation for such error is the use of the assumption
that the plane of the crack is normal to the midsurface which might not be adequate for
thick laminates. For such case, the matrix cracks may start to branch, join and curve.
Obviously, these complicating factors need to be carefully modeled and incorporated
into the damage mechanism.

2. The model is shown to be effective for the analysis of crossply laminates. However, for
angle-ply laminates, it can be used with less degree of accuracy. The reason is the
coupling that do exist in the angle-ply laminates which may maximize the confinement
effect of the adjacent plies.
3. The transverse shear deformations have a significant effect (error of up to 24%) on the frequency response which should not be ignored especially for thick or highly orthotropic laminates. The use of a first order shear deformable theory (FSDT) yields a very reasonable accuracy for global dynamic analysis of laminated composite plates or shallow shells.

4. The analysis of various plates and shells with matrix cracks depicts a typical trend of reduction in the vibration frequency due to this type of damage. The reduction is found to be mainly dependent on the size of the damage region and density of cracks. The typical reduction in the fundamental frequency parameter for the studied cases ranges from 3.5%. The changes in the frequency, especially for the fundamental mode, appear to be less sensitive to the shell boundary conditions as well as small values of curvature. On the other hand, the reduction in the vibration frequency is more pronounced for higher modes (8-12% for the fifth mode). This happens probably due to the fact that the presence of a small damage region within a plate or a shell is reflected more critically on the higher modes that exhibit more localized patterns of motions.

5. The study also shows that the degree of curvature may significantly affect the frequency response of laminated composite shells. In general, the frequency parameters increase with increasing the degree of curvature (a/R) which is expected, since the added curvature will increase the shell stiffness. One might also add that this
change in the frequency is very significant for thin shells which could be due to the fact that the added stiffness due to contribution of terms related to the curvature in thin shells becomes comparable with that of corresponding flat plate stiffness, which is not the case for thick shells.

6. The damage region was assumed to be either a rectangle, a square, an ellipse or a circle. These uniform shapes are used to reasonably approximate the actual damage shapes for laminate configurations considered here. These shapes in general however, can be non-uniform and very much dependent on the stacking sequence and the source of damage. In the range of the studied cases, the frequency response has a very little sensitivity to the shape of the damaged region for a given total area of damage, provided that the damaged zone-aspect ratio is of the order of unity. On the other hand, the extent of damage and the cracking density are the major contributing factors to the reductions in the vibration frequencies. This result is an important contribution since it can facilitate the investigation of many similar problems by using any of the available numerical solution techniques. For instance, the Ritz method might be used assuming that the damage region is either an ellipse or a circle to avoid the difficulty of enforcing the continuity conditions for shapes with sharp edges that might result in some singularities. On the other hand, if the finite element is to be used, a square or a rectangular damage zone is easier to work with in terms of domain discretization and mesh generation.

210
7 The present treatment of the effect of matrix cracks is more realistic and more useful than the ply discount strategy which underestimates the residual stiffness of damaged plates and shells.

8. Depending on the use of the structural component and the damage tolerance limits, the numerical results may provide rough guidelines for the repair of damaged laminated composite panels.

9. The use of a simplified shallow shell theory yields very reasonable accuracy of the frequency response for the cases studied here including both damaged as well as undamaged laminated composites. In comparison with the closed-form solutions (based on the unsimplified theory), the resulting error is less than 2% at the approximate limiting boundary for the shallow shell; i.e. \( \ell_0/a = 0.2 \) which corresponds to \( a/R = 0.69 \) for a circular cylindrical shell.

7.2 FUTURE RESEARCH

Based on the results of the present study, one might recommend the following aspects for future research work and investigations in this area:

1. There is an urgent need of extensive experimental investigations to measure the changes in the material elastic constants due to matrix cracking. Such experimental work should include various stacking sequences for both plates and shells. Another
experimental program is needed also to study the frequency response of laminated composite plates and shells with various damage modes. Linking the two experimental programs will be very helpful in building a possible correlation between the experimental and numerical results.

2. The damage mechanics model can be extended to include some complicated factors such as:

- Curved matrix cracks;
- Cracks joining and branching;
- Adjacent ply confinement effect;
- Inter-ply delaminations.

3. The present free vibration analysis for shallow shells can be extended to deep shell analysis with matrix cracking type of damage or delaminations. This could be very helpful for investigating some special types of structural components; e.g. laminated composite pipes. Also, the present formulation can be used with appropriate modifications to study other types of shell geometries; e.g. spherical and hyperbolic paraboloidal shells.

4. An extensive study is required for any possible shifts in the shapes of the vibration modes, especially for damage zones which are displaced from the center of the structural components.
5. The present solution procedure can be extended to study forced vibration response of laminated composite plates and shells with matrix cracks and/or delaminations.
In thermodynamic systems, internal state variables (ISV) have been used to describe the microstructural changes or damage. Their use have been very effective in continuum damage mechanics to supplement the set of observable parameters of the undamaged state. To properly estimate the effect of damage, it is necessary to formulate the damage phenomenon in terms of mechanics. Then, it is possible to analyze various engineering problems using analytical and computational techniques. The internal state variables concept is one of such proper representation of damage for some complicated cases such as laminated composites into which the crack shape, size and direction are to be considered. In this section we briefly outline their definition and description which are adopted from references [2,3] and [33].

Using a second order tensor representation, the internal state variables are defined from a locally volume averaged diadic product of the crack opening displacement vector \((u^c)\) and the corresponding outward unit normal \((n^c)\). This results in:
where the superscript \( (\eta) \) denotes a specific damage mode, \( S^n \) is the damaged surface area, and \( V_L \) is the local volume. The local volume \( V_L \) is chosen such that the material constitutive response is statistically homogeneous. In the development to follow, the subscript \( (L) \) is dropped since it is understood that the description is for a local volume with a certain mode of damage. The ISV's describe the kinematics of the cracking process and are strain-like quantities.

As shown in Figure (A.1), the matrix cracks are assumed to be fully contained in each ply, hence, the local volume element for matrix cracks damage mode is a single ply. Therefore, all terms associated with matrix cracking are averaged each ply then incorporated into the laminate analysis. Thus, for a typical cracked ply shown in Figure (A.2) is used to develop the governing response. In terms of the local ply coordinates, i.e. (1-2) system, the ISV's for transverse matrix cracks can be written as (from Eqn.(A.1)):

\[
\alpha_{\eta}^M = \frac{1}{V_L} \int_{S^M} u^\varepsilon_i n^\varepsilon_j dS
\]  

(A.2)

where the crack opening displacement vector is given by:

\[
u^\varepsilon = u_1^\varepsilon e_1 + u_2^\varepsilon e_2 + 0e_3
\]  

(A.3)
and the unit normal vector is:

\[ n^i = 0e_1 + n_2^i e_2 + 0e_3 \]  \hspace{1cm} (A.4)

Then Eqn.(A.2) can be reduced to:

\[ \left[ \alpha^M \right] = \begin{bmatrix} \alpha_{22}^M \\ \alpha_{12}^M \end{bmatrix} \]  \hspace{1cm} (A.5)

One should note that \((\alpha^M_{ij})\) is generally an asymmetric tensor. One should also note that in the above reduced expression given by Eqn.(A.5), \(\alpha_{i3}^M\) (i=1,2,3) are all neglected for this special case. They represent out-of-plane mixed mode cracking that develop due to curved matrix cracks which is not the focus of the present work, therefore they are neglected. Hence, in the local coordinates, the only nonezero terms are \(\alpha_{22}^M\) and \(\alpha_{12}^M\) which represent crack modes I and II, respectively, as defined in fracture mechanics.

For an arbitrary orientation the transformation of the derivatives of the damage parameters follow the second order tensor transformation law given as [38]:

216
where, the unbarred quantities are in the local coordinate system and barred quantities are in the laminate coordinate system. And \( a_{ip} \) are the direction cosines relating the local ply coordinates to the laminate coordinates given as:

\[
a_{ip} = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The locally averaged thermo-mechanical constitutive equations for this damage mode were derived [3] by expanding the Helmholtz free energy in terms of strains, \( e_{ij} \), temperature, \( T \), and the ISV, \( \alpha_{ij}^M \), to obtain the following form:

\[
\sigma_{ij} = \sigma_{ij}^R + E_{ij} \Delta T + C_{ijkl} \varepsilon_{kl} + I_{ijkl} \alpha_{kl}^M
\]  

(A.7)

where \( \sigma_{ij}^R \) are the residual stresses and the tensors \( E, C, \) and \( I \) are material constants. It should be noted that all terms in the series expansion which are higher than the second order have been neglected. It was shown, [2], that \( I_{ijkl} = -C_{ijkl} \) if the stresses in the homogeneous damaged region are negligible. Assuming that the residual stresses are negligible and that all test are conducted at constant temperature, then Eqn.(A.7) can be written as:
Equation (A.8) can be simplified by using single index notation. This is done by incorporating the symmetry of the stress and strains, and the quadratic dependence of the Helmholtz free energy on strain. Using Voigt index notation, the resulting equation is:

\[ \sigma_{ij} = C_{ijkl} (\varepsilon_{kl} - \alpha^M_{kl}) \quad (A.8) \]

The above local constitutive equations are simplified further by applying the material symmetry. For laminated composites, the material is assumed to be initially transversely isotropic in the undamaged state. It is also assumed, [33], that transverse matrix cracks introduce orthotropy in the coordinates of the cracks.

\[ \sigma_i = C_{ij} (\varepsilon_j - \alpha^M_j) \quad i,j=1,2,...,6 \quad (A.9) \]
Figure(A.1): Matrix cracking and delaminations in a composite laminate
Figure (A.2): Matrix cracking and local coordinates for a single ply.
APPENDIX B

ENFORCING THE GEOMETRIC BOUNDARY CONDITIONS

In this section, enforcing the geometric boundary conditions is briefly outlined. The procedure presented herein after are adopted from references [7], [49] and [95]. It should be noted that the discussion is limited to displacement vector \((u_i)\), i.e. the cross-section rotations \((\phi_o)\) are not considered, since the goal is only to set an example of the applicability of the formulation.

The displacement functions may be written as:

\[
\begin{align*}
    u_1 &= \sum_{i=1}^{M} u_i \phi_i(x, y) \\
    u_2 &= \sum_{i=1}^{N} v_i \theta_i(x, y) \\
    u_3 &= \sum_{i=1}^{P} w_i \psi_i(x, y)
\end{align*}
\]

(B.1)
The Rayleigh-Ritz method requires that the functions are chosen such that they satisfy at least the geometric boundary conditions. However, any one of these functions may be written in terms of a linear combination of a complete set of functions, such as

\[
\phi_i = \sum_m a_m \Phi_m
\]

\[
\theta_i = \sum_n b_n \Theta_n
\]

\[
\psi_i = \sum_p c_p \Psi_p
\]

(B.2)

Then the equations (B.1) can be written

\[
u_1 = \sum_{i=1}^\infty u_i \left( \sum_m a_m \Phi_m \right)
\]

\[
u_2 = \sum_{i=1}^\infty v_i \left( \sum_n b_n \Theta_n \right)
\]

\[
u_3 = \sum_{i=1}^p w_i \left( \sum_p c_p \Psi_p \right)
\]

(B.3)

or

\[
u_1 = \sum_m u_m \Phi_m
\]

\[
u_2 = \sum_n v_n \Theta_n
\]

\[
u_3 = \sum_p w_p \Psi_p
\]

(B.4)
where \( u_m, v_n, \) and \( w_p \) are constrained and boundary conditions are satisfied.

Depending on boundary conditions for each case, the constraint equations are known; therefore, it is straightforward to find the elements of the matrices of transformation relating \( u_i \) to \( u_m \), \( v_i \) to \( v_n \), and \( w_i \) to \( w_p \).

\[
q_i = u_i, \quad 1 \leq i < M \\
q_j = v_j, \quad M < j \leq M+N \\
q_k = w_k, \quad M+N < k \leq M+N+P \tag{B.6}
\]

Similarly, vector \( \{ \bar{q} \} \) can be introduced in the following form:

\[
\bar{q}_i = u_i, \quad 1 \leq i < m \\
\bar{q}_j = v_j, \quad m < j \leq m+n \\
\bar{q}_k = w_k, \quad n+m < k \leq m+n+p \tag{B.6}
\]

The relation between \( \{ \bar{q} \} \) and \( \{q\} \) may be found from constraint equations, i.e.,

\[
\{q\} = [G] \{ \bar{q}_i \}
\]

and

\[
\{q_i\}^T = \{ \bar{q}_i \}^T [G]^T \tag{B.7}
\]

The energy functionals in the matrix form are written as:

\[
\bar{E} = 1/2 \{ \bar{q}_i \}^T [E] \{q_i\} \tag{B.8}
\]
Then, using equations (B.7), the functionals may be written in the following form:

\[
\bar{E} = 1/2 \{\bar{q}^T_i\} [G^T] [E] [G] \{\bar{q}^-_i\}
\]  \quad \text{(B.9)}

The advantage of equation (B.9) is that the matrix E may be evaluated from unconstrained displacement functions. By using the proper constraint equations for any given boundaries, the matrices G and \( G^T \) can then be found.
APPENDIX C

EVALUATION OF ELEMENT MATRICES

C.1 INTERPOLATION FUNCTIONS

The computer program developed incorporated the ‘Heterosis’ plate bending element [Hughes 1978] along with reduced/selective integration technique. The element matrices can be formed following the usual procedure of isoparametric element formulation. However, the ‘Heterosis’ element differs from other isoparametric elements in using different interpolation scheme for lateral displacement on the one hand and in-plane displacement and rotations of the cross section on the other. In-plane displacements $\mathbf{u}_a^{(1)}$ and rotations of cross-section $\mathbf{\phi}_a^{(k)}$ are approximated by quadratic functions for 8-node isoparametric element while the lateral displacement $w$ is approximated by 9-node Lagrange interpolation functions. Consequently, the number of degrees of freedom at the center node is less than that at other nodes by one. Interpolation scheme of the ‘Heterosis’ element is shown in Figure C.1. Using such interpolation scheme, Hughes [42] and Hong[44] were able to avoid spurious zero energy mode of stiffness matrix which can be caused by use of reduced integration.
Interpolations functions of 8-node isoparametric element and 9-node Lagrange element in terms of coordinates \((s-t)\) and their derivatives with respect to \((s)\) and \((t)\) are given below.

\[
N = \frac{1}{4} \begin{pmatrix}
(1-s)(1-t)(-1-s-t) \\
(1+s)(1-t)(-1+s-t) \\
(1+s)(1+t)(-1+s+t) \\
(1-s)(1+t)(-1-s+t)
\end{pmatrix}, \quad \frac{\partial N}{\partial s} = \frac{1}{4} \begin{pmatrix}
(1-t)(2s+t) \\
(1-t)(2s-t) \\
(1+t)(2s+t) \\
(1+t)(2s-t)
\end{pmatrix}, \quad \frac{\partial N}{\partial t} = \frac{1}{4} \begin{pmatrix}
(l-s)(2t+s) \\
(l+s)(2t-s) \\
(l+s)(2t+s) \\
(l-s)(2t-s)
\end{pmatrix}
\]

(C.1)
Here, \( N \) and \( L \) denote interpolation functions for 8-node isoparametric and 9-node Lagrange elements, respectively.

### C.2 EVALUATION OF STIFFNESS AND MASS MATRICES

Since the field variables are interpolated over an element in natural coordinates \((s-t)\), it is necessary to set up the relation of the global coordinates and natural (local) coordinates for evaluation of the element matrices defined in Chapter (6). We consider a mapping of global coordinate system \((x_1-x_2)\) to local coordinate system \((s-t)\). We assume that this mapping is one-to-one and onto. By chain rule, the derivative in each coordinate system is related by

\[
L = \frac{1}{4} \begin{pmatrix}
st(1-s)(1-t) \\
st(1+s)(1-t) \\
st(1+s)(1+t) \\
st(s-1)(t+1) \\
2t(1-s^2)(t-1) \\
2s(s+1)(1-t^2) \\
2t(1-s^2)(t+1) \\
2s(s-1)(1-t^2) \\
4(1-s^2)(1-t^2)
\end{pmatrix},
\]

\[
\frac{\partial L}{\partial s} = \frac{1}{4} \begin{pmatrix}
t(2s-1)(t-1) \\
t(2s+1)(t-1) \\
t(2s+1)(t+1) \\
t(2s-1)(t+1) \\
4st(1-t) \\
2(2s+1)(1-t^2) \\
-4st(t+1) \\
(2s-1)(1-t^2) \\
8st^2 - 1
\end{pmatrix},
\]

\[
\frac{\partial L}{\partial t} = \frac{1}{4} \begin{pmatrix}
s(2t-1)(s-1) \\
s(2t-1)(s+1) \\
s(2t+1)(s+1) \\
s(2t+1)(s-1) \\
2(2t+1)(1-s^2) \\
-4st(s+1) \\
2(2t+1)(1-s^2) \\
4st(1-s) \\
8t(s^2 - 1)
\end{pmatrix}
\]

(C.2)
\[
\left(\frac{\partial}{\partial s}\right) = J \left(\frac{\partial}{\partial x}\right) \quad \text{or} \quad \left(\frac{\partial}{\partial t}\right) = J^{-1} \left(\frac{\partial}{\partial s}\right)
\]

where the Jacobian matrix $J$ and its inverse is defined as

\[
J = \begin{pmatrix}
\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\
\frac{\partial x}{\partial x} & \frac{\partial y}{\partial y} \\
\frac{\partial t}{\partial t} & \frac{\partial x}{\partial t}
\end{pmatrix} \quad \text{and} \quad J^{-1} = \frac{1}{|J|} \begin{pmatrix}
\frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \\
\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s}
\end{pmatrix}
\]

Here, $|J|$ is the determinant of the Jacobian matrix. Using (C.3) and (C.4), one can obtain the expression of the matrices $T_e$, $T_o$ and $T_i$ defined in Eqns. (6.6) and (6.7) in natural coordinates. Following the concept of isoparametric formulation, global coordinates are interpolated over an element as

\[
x = \psi^T \bar{x}
\]

where $\psi$ is the vector of interpolation functions used for field variable, $\bar{x}$ is the vector of global coordinate values at nodal points. Since in the 'Heterosis' element different interpolation functions $N$ and $L$ are used for $\Phi^{(k)}_i$ and $w$, $\psi$ in (C.5) must be $L$ for evaluating $T_e$ and $T_o$ while $N$ is used for $T_i$.

228
Using (C.1)-(C.5), interpolation function defined in Eqns. (6.6)-(6.7). Explicitly.

\[ H_u = H_o = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \quad (C.6) \]

\[ H_w = N \quad (C.7) \]

\[ T_u^T = T_o^T = \frac{1}{|J_u|} \begin{pmatrix} y^T & 0 \\ 0 & -x^T \end{pmatrix} P \begin{pmatrix} -x^T & y^T \end{pmatrix} \quad (C.8) \]

\[ T_i^T = \frac{1}{|J_x|} \begin{pmatrix} y^T \\ -x^T \end{pmatrix} R \quad (C.9) \]

where

\[ |J_u| = -x^T P y \]

\[ |J_x| = -x^T R y \]

\[ P = L_{1,1} L_{1,2}^T - L_{1,2} L_{1,1}^T \]

\[ R = N_{1,1} N_{1,2}^T - N_{1,2} N_{1,1}^T \]

229
Here, a subscripted comma denotes partial differentiation with respect to the variables following the comma.

In element matrices integrands are functions of natural coordinates \((s-t)\). Therefore, the surface integration extends over the natural coordinate surface. Since, in general,

\[
dR = |J| dsdt
\]

integration in each coordinate system is related by

\[
\iint F(x, y) dxdy = \int \int F(s, t)|J| dsdt
\]

Using Gaussian quadrature

\[
\int F(x, y) dR = \sum_{i=1}^{m} \sum_{j=1}^{m} F(s_{ij}, t_{ij}) |J_{ij}| W_{ij}
\]

where \(m\) is the number of Gaussian quadrature points and \(W_{ij}\) are weighing values. Here, it should be mentioned that the 'Heterosis' element numerical integration was performed by selective/reduced integration technique, viz., two-point Gaussian quadrature for evaluation of transverse strain energy term while three-point quadrature is used for other quantities. Therefore, the elements of the transverse shear stiffness were evaluated by two-point quadrature and the remaining quantities were evaluated by three-point quadrature.
C.3 ENFORCING THE BOUNDARY CONDITIONS

The constraints equations describing various boundary conditions can be specified either at the element level or at the assembled system matrices level. In the present formulation it is easier to do so at the element level. For such case, the system matrices do not have to reside in-core.

In general, the element equations can be stated as:

\[
\begin{bmatrix}
  K_{11} & K_{12} \\
  K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
  q_1 \\
  q_2
\end{bmatrix} =
\begin{bmatrix}
  f_1 \\
  f_2
\end{bmatrix}
\]  \hspace{1cm} (C.13)

where \( \{q_1\} \) and \( \{q_2\} \) indicate the known and the unknown nodal point displacement vectors, respectively.

By adding dummy equations such that the solution will yield the known specified displacements, then Eqn.(C.13) can be written as:

\[
\begin{bmatrix}
  K_{11} & 0 \\
  0 & K_{22}
\end{bmatrix}
\begin{bmatrix}
  q_1 \\
  q_2
\end{bmatrix} =
\begin{bmatrix}
  f_1 - K_{12}q_2 \\
  f_2
\end{bmatrix}
\]  \hspace{1cm} (C.14)

The application of the specified boundary conditions then consists of specifying zeros on the rows and columns corresponding to the specified nodal degrees of freedom with identity on the diagonal entries. Of course, the right hand side is accordingly modified.
Figure (C.1): Local and global coordinate systems and degrees of freedom for "heterosis" element
Figure(D.1): The finite element program structure.
Print the resulting eigenvalues and optionally the corresponding eigenvectors.

Figure(D 1): The finite element program structure (continued).
Figure(A.1): The Ritz method program structure.
BIBLIOGRAPHY


[29] Gibson, R.F. Recent research on dynamic mechanical properties of fiber reinforced composite materials and structures, The Shock and Vibration Digest, Vol. 15, No.2,


[77] Qatu, M.S. *“Free vibration and static analysis of laminated composite shallow shells“* PHD Disseration, The Ohio State University, 1989.


246

