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A FAMILY OF GENERAL RECORD MODELS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

Glenn Hofmann, M. S.

* * * * *

The Ohio State University

1997

Dissertation Committee:
H. N. Nagaraja, Adviser
Thomas J. Santner
Robert Bartoszynski

Approved by

H. N. Nagaraja
Adviser
Department of Statistics
ABSTRACT

First we introduce a general record model where the observations $X_i$ have distribution $F^\alpha_i$ with $F$ being a continuous c. d. f. and $\alpha_i$ a positive random variable. Given the $\alpha_i$'s the observations $X_i$ are assumed to be conditionally independent. As a result the observations need neither be independent nor identically distributed yielding a much less restrictive record model. This model includes the classical model as well as some models proposed by other authors as special cases. We discuss finite-sample and asymptotic properties of record values, record times and inter-record times for this model. We will also consider some variations of the secretary problem in the context of the random $F^\alpha$ model.

Next we consider an $F^\alpha$ model where the $\alpha_i$'s are fixed and a random number $N$ of observations arrive. For this model we obtain the joint distribution of record values and inter-record times. We investigate in detail the distribution of the number of records when the $\alpha$'s increase geometrically or linearly and $N$ has one of the common distributions. A particularly interesting case of the model arises when the observations arrive at time points paced by an independent point process. When the pacing process is Poisson, we obtain exact and asymptotic distributional results for the inter-record arrival times for a large class of combinations of $\alpha$-structures and intensity functions of the process.
To my parents
ACKNOWLEDGMENTS

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VITA

March 21, 1970  ......................... Born - Lichtenstein, Germany

June 1991  ............................... Vordiplom (B.S.) Mathematics, Technical University Chemnitz, Germany

June 1994  ............................... M.S. Statistics, The Ohio State University, Columbus, Ohio

PUBLICATIONS

Research Publications


FIELDS OF STUDY

Major Field: Statistics
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<tr>
<td>c. d. f.</td>
<td>cumulative distribution function</td>
</tr>
<tr>
<td>i. i. d.</td>
<td>independent and identically distributed</td>
</tr>
<tr>
<td>iff</td>
<td>if and only if</td>
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| $N \sim \text{Bern}(p)$ | $N$ has Bernoulli distribution with parameter $p$,
|                 | i. e., $P(N = 1) = p$, $P(N = 0) = 1 - p$                              |
| $N \sim \text{Bin}(n, p)$ | $N$ has Binomial distribution with parameters $n$, $p$,
|                 | i. e., $P(N = k) = \binom{n}{k} p^k (1 - p)^{n-k}$ for $0 \leq k \leq n$ |
| $N \sim \text{Poi}(\lambda)$ | $N$ has Poisson distribution with mean $\lambda$,
|                 | i. e., $P(N = k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $k = 0, 1, \ldots$ |
| $N \sim \text{Geom}(p)$ | $N$ has Geometric distribution with parameter $p$,
|                 | i. e., $P(N = k) = pq^k$, $k \geq 0$, $q = 1 - p$.                    |
| $N \sim \text{Negative}$ | $N$ has Negative Binomial distribution with parameters $r$, $p$
|                 | $P(N = k) = \binom{r+k-1}{k} p^r (1 - p)^k$, $k \geq 0$.                |
| $X \sim \mathcal{N}(\mu, \sigma^2)$ | $X$ is normally distributed with mean $\mu$ and variance $\sigma^2$ |
| $X \sim \text{Exp}(\lambda)$ | $X$ has exponential distribution with mean $\lambda$,
|                 | i. e., the c. d. f. of $X$ is $F(x) = 1 - e^{-\frac{x}{\lambda}}$.  |
| $X \sim \text{Gamma}(a, b)$ | $X$ has Gamma distribution with density function
|                 | $f(x|a, b) = \frac{1}{\Gamma(a)} x^{a-1} e^{-\frac{x}{b}}$, $0 < x < \infty$, $a, b > 0$.  |
| $X \sim \text{Beta}(a, b)$ | $X$ has Beta distribution with density function
<p>|                 | $f(x|a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$, $0 \leq x \leq 1$, $a, b &gt; 0$.  |</p>
<table>
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<tr>
<td>$X \overset{d}{=} Y$</td>
<td>The distribution functions of the random variables $X$ and $Y$ are equal.</td>
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<tr>
<td>$X_n \xrightarrow{c} N(0, 1)$</td>
<td>The c. d. f. of $X_n$ converges in distribution to a standard normal c. d. f.</td>
</tr>
<tr>
<td>$X_n \xrightarrow{p} c$</td>
<td>$X_n$ converges in probability to the constant $c$, i.e., $P(</td>
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CHAPTER 1

INTRODUCTION

A record observation of some phenomenon is the largest (or smallest) observation anyone has ever made. Because of this extreme character, records attract attention and draw public interest. For example, weather records and record performances in sports or the stock market are often discussed. A minor industry is devoted to the production of new records, such as the highest number of balls juggled simultaneously (10) or the record time to sit in a tree (25 years), which are listed in the Guinness Book of World Records (Young, 1997).

Statisticians and mathematicians have also taken an interest in records. The formal study of record value theory probably started with a paper by Chandler in 1952. Since then several models have been proposed to describe the stochastic behavior of occurrence times and values of new records in a sequence of observations. In the classical model, an infinite sequence of independent, identically distributed (i. i. d.) random variables is assumed. Rényi (1962, 1976) was the first to study the classical model in detail; he contributed many basic results. A large literature on the problem has subsequently been developed, and several reviews of classical record literature have appeared. Glick (1978) gives an informal introduction and reviews many basic results. Galambos (1987) includes a chapter on records and Resnick (1987) presents
a mathematical theory of record related topics; both books give an extensive bibliography. Nevzorov (1987) and Nagaraja (1988) provide short review articles on the classical record model and some of its generalizations.

While being very useful in some situations, the classical model does not always fit observed record breaking behavior; sports records serve as one example. In fact, athletic records are broken much more frequently than the classical model predicts. Acknowledging this fact, Yang (1975) was one of the first to suggest a more general model for Olympic records. Yang's model was later generalized by Ballerini and Resnick (1987b) and Nevzorov (1985, 1986a,b, 1990, 1995) to what is now known as the (fixed) $F^\alpha$ model. While this model can adjust for growing populations and (non-random) linear trends, dependent sequences are still problematic. Recently Ballerini (1994) and Nevzorova, Nevzorov and Balakrishnan (1997) have started to incorporate dependency structures in the underlying sequence of observations. A brief discussion of these models is given in the Appendix for ready reference.

In this thesis we study several particular extensions of the models discussed above. First, we allow the $\alpha$-parameters from the fixed $F^\alpha$ model to be random variables resulting in a "random $F^\alpha$ model". As a consequence we obtain a natural generalization of all previously mentioned models. The random $F^\alpha$ model is explored in Chapter 2. We give a hierarchy diagram that includes all these models and discuss how they relate to the new model. Then we present small sample and asymptotic properties of record times, record counts, inter-record times and record values.

One interesting application of record value theory is the so called secretary problem. The classical secretary problem is to select the best of $n$ candidates that arrive sequentially in random order for interviews without the possibility to recall candi-
dates interviewed in the past. It is also known as the dowry or the marriage problem. Freeman (1983) provides a very readable review of the large literature that exists on the many variations of this problem. For an entertaining account of some historical aspects see Ferguson (1989). The connection between the secretary problem and record value models is immediate, since the best of the $n$ candidates corresponds to the last record in this finite sequence and arrives at a record time. In Chapter 3 we give the optimal solution to two variations of the secretary problem when the underlying sequence is modeled in random $F^\alpha$ fashion.

Another interesting situation of record modeling arises when the observations arrive in continuous time according to an underlying point process $P$. In the i. i. d. case such a “point process record model” was first proposed by Pickands (1971) for whom $P$ was a homogeneous Poisson process. Gaver (1976) and Gaver and Jacobs (1978) considered several different $P$'s and proved various results for record and inter-record times as well as the number of records in a given time interval. Westcott (1977) clarified the link between the classical and point process record models and derived asymptotic properties of record statistics under various assumptions for $P$. Westcott (1979), Embrechts and Omey (1983) and Yakymiv (1986) studied the connection between the distributions of inter-arrival times of observations and records. Deheuvels (1982) and Bunge and Nagaraja (1992a) obtained a simple distributional representation of the inter-record times. Bruss (1988, for example) applied such models to variants of the secretary problem. Recent developments in i. i. d. point process record models are found in Bruss and Rogers (1991), Bunge and Nagaraja (1991, 1992a, 1992b) and Browne and Bunge (1995). A summary of these results as well as record value theory in general is given in the upcoming book by Arnold, Balakrishnan and
All the point process record models that have been considered in the literature outlined above assume an i. i. d. sequence of observations. In Chapter 4 we discuss point process record models for observations with a (fixed) $F^\alpha$ structure where $P$ is a Poisson process. For homogeneous Poisson $P$’s we obtain distributional representations for the inter-record times. Then we consider non-homogeneous Poisson pacing processes and present asymptotic results for inter-record times for a class determined by combinations of $\alpha$-structures and intensity functions of $P$. In Chapter 5 we give some concluding remarks and an outlook on possible future research.
CHAPTER 2

THE RANDOM $F^\alpha$ MODEL

2.1 Introduction

Let $X_1, X_2, \ldots$ be a sequence of random variables. An observation will be called an (upper) record if it exceeds all previous observations. We define the record times $T_n$, the record values $R_n$ and the inter-record times $\Delta_n$ as follows:

\[
T_1 = 1, \quad T_n = \min\left\{j > T_{n-1} : X_j > X_{T_{n-1}}\right\} \quad (n = 2, 3, \ldots)
\]

\[
R_n = X_{T_n} \quad (n = 1, 2, \ldots)
\]

\[
\Delta_n = T_n - T_{n-1} \quad (n = 2, 3, \ldots).
\]

Let $N_n$ be the number of records among $X_1, \ldots, X_n$. A convenient way of describing $N_n$ is by means of record indicators $I_n$, where

\[
I_n = 1\{x_n \text{ is a record}\}.
\]

Clearly, $N_n = \sum_{i=1}^{n} I_i$. Finally, we define the record rate to be $p_n = P(I_n = 1)$.

The origins of the record model considered here go back to Yang (1975). The results of his paper and some of the other papers on non-classical record models that
are relevant to our work are described in the Appendix. Yang looked at the breaking of Olympic records and took \( Y_i = \max \{ X_{i1}, \ldots, X_{i\alpha_i} \} \) where \( X_{ij} \) are i. i. d. random variables with c. d. f. \( F(x) \) for \( i = 1, 2, \ldots; j = 1, 2, \ldots, \alpha_i \). The measurement \( Y_i \) corresponds to the champion of the \( i \)th games. Yang assumed \( \alpha_i \) to be the world population at the time of the \( i \)th event that increases at a geometric rate. However, Olympic records are in fact broken faster than Yang's model predicts.

Nevzorov (1986a,b, 1990, 1995) and Ballerini and Resnick (1987b) generalized the model to include positive (not necessarily integer-valued) \( \alpha_i \)'s, i. e., the observations are independent and have c. d. f. \( F^{\alpha_i}(x) \), \( \alpha_i > 0 \). They proved that the record indicators are independent and developed various small sample and asymptotic properties. These "fixed \( F^\alpha \)" models include the classical model \( (\alpha_1 = \alpha_2 = \cdots = 1) \) as well as the linear trend models considered by Ballerini and Resnick (1985, 1987a,b). One assumption in all these models that is hardly realistic for most applications is the independence of the observations.

There is some work in the literature on record models where the \( \alpha \)'s are taken to be random. Alpuim (1985) considers the \( \alpha \)'s to be independent Poisson and Binomial random variables and Nevzorov (1987) briefly mentions the possibility of random \( \alpha \)'s. Ballerini (1994) and Nevzorova, Nevzorov and Balakrishnan (1997) consider record models with a limited dependence structure by using a copula approach.

We present a natural generalization of all these ideas that results in a broader model while retaining several attractive properties of the fixed \( F^\alpha \) model. It is introduced in Section 2.3. In Section 2.4 we show that all the aforementioned models are contained in our model and summarize their relations. In Section 2.5 we briefly discuss the properties of inter-record times. Section 2.6 is devoted to asymptotic
properties of record times and record counts. We will show central limit theorems and martingale properties under various sets of conditions. The asymptotic behavior of record values is investigated in Section 2.7. The main result there is a duality theorem that relates the possible limit distributions of records to the extreme value distributions for maxima from an associated c. d. f. We will see that the class of record limit distributions we obtain is actually more general than what one obtains for the classical model.

2.2 Definition

Let F be a continuous c. d. f. and let $\alpha_1, \alpha_2, \ldots$ be (a. s. finite) positive random variables. Given $\alpha_1, \ldots, \alpha_n$ the observations $X_1, \ldots, X_n$ are conditionally independent. The joint c. d. f. of $X_1, \ldots, X_n$ is given by

$$G_n(x_1, \ldots, x_n) = \mathbb{E}\left[ F^{\alpha_1}(x_1) \cdot \ldots \cdot F^{\alpha_n}(x_n) \right]$$

for all $x_i \in \mathbb{R}, i = 1, \ldots, n$ where the expectation is taken with respect to the $\alpha$’s. If (2.1) holds for all $n \geq 1$, we call the associated record model a random $F^\alpha$ model.

The marginal distribution of $X_k$ is $F_{X_k}(x) = EF^{\alpha_k}(x)$. If additionally, the $\alpha$’s are i. i. d. we recover the classical model.

We now introduce random variables

$$S(n) = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \pi_n = \frac{\alpha_n}{S(n)}, \quad n \geq 1.$$  

In the fixed $F^\alpha$ model ($\alpha$’s constant), the record indicators were independent and $P(I_n = 1) = \frac{\alpha_n}{S(n)}$ (see Nevzorov (1986b) or Ballerini and Resnick (1987b) - reviewed in appendix A). In our case $P(I_n = 1) = E\pi_n$.

One might wonder for what sequences of random variables $\{X_n\}_{n \geq 1}$ a random $F^\alpha$ model defined by (2.1) exists. For $x \in (0, 1)$ let $F^{-1}(x) = \min\{y : F(y) = x\}$. Since
$F$ is continuous, this minimum always exists and $F(F^{-1}(x)) = x$. Let us recall (2.1).
the definition of the joint distribution of $X_1, \ldots, X_n$ and replace $x_i$ by $F^{-1}(e^{-\lambda_i})$, $\lambda_i > 0$. We obtain

$$G_n \left( F^{-1}(e^{-\lambda_1}), \ldots, F^{-1}(e^{-\lambda_n}) \right) = E \left[ e^{-\lambda_1 \alpha_1 - \cdots - \lambda_n \alpha_n} \right]. \quad (2.2)$$

The right hand side of (2.2) can be viewed as the multivariate equivalent of a Laplace transform. We will make this precise using the following definition. Let $(Y_1, \ldots, Y_k)$ be a vector of nonnegative random variables with c. d. f. $F_i(x_1, \ldots, x_k)$. Then its Laplace transform is defined by

$$F_i(A_1, \ldots, A_k) = \int_{\mathbb{R}_+^k} e^{-\lambda_1 x_1 - \cdots - \lambda_k x_k} dF_i(x_1, \ldots, x_k), \lambda_i \geq 0, i = 1, \ldots, k$$

where $\mathbb{R}_+^k = (0, \infty)^k$. Clearly, $F_i(0, \ldots, 0) = 1$, $F_i(\infty, \ldots, \infty) = F_i(0, \ldots, 0)$. In direct analogy to the univariate case (see, for example, Theorems 6.6.2, 6.6.3 in Chung (1974)) the following results hold.

**Lemma 2.1** If for $j = 1, 2$, $\hat{F}_j$ is the Laplace transform of the c. d. f. $F_j$ with support in $\mathbb{R}_+^k$ then $\hat{F}_1 = \hat{F}_2$ iff $F_1 = F_2$.

**Lemma 2.2** Let $\{F_n\}_{n \geq 1}$ be a sequence of subdistribution functions (correspond to a measure $\mu$ with $\mu(\mathbb{R}_+^k) < 1$) with supports in $\mathbb{R}_+^k$ and $\hat{F}_n$ the corresponding Laplace transform. Then $F_n \xrightarrow{c} F_\infty$, where $F_\infty$ is a c. d. f., iff

(i) $\lim_{n \to \infty} \hat{F}_n(\lambda_1, \ldots, \lambda_k)$ exists for all $\lambda_i > 0$, $i = 1, \ldots, k$ and

(ii) $\lim_{\lambda_i \to 0, i = 1, \ldots, k} \lim_{n \to \infty} \hat{F}_n(\lambda_1, \ldots, \lambda_k) = 1$.

Then $\lim_{n \to \infty} \hat{F}_n$ is the Laplace transform of $F_\infty$. 
Let a function \( f \) on \( \mathbb{R}^k_+ = [0, \infty)^k \) be called **completely monotone** if its partial derivatives satisfy the condition
\[
(-1)^{j_1 + \cdots + j_k} \frac{\partial^{j_1 + \cdots + j_k}}{\partial x_1^{j_1} \cdots \partial x_k^{j_k}} f(x_1, \ldots, x_k) \geq 0
\]
for all \( j_i \geq 0 \) and all \( x_i > 0, \ i = 1, \ldots, k \). As is known in the univariate case, we give necessary and sufficient conditions for a function to be the Laplace transform of a c. d. f.

**Lemma 2.3** A function \( f \) on \( [0, \infty)^k \) is the Laplace transform of a c. d. f. \( F \) iff it is completely monotone and \( \lim_{\lambda_i \to 0+, \ i=1,\ldots,k} f(\lambda_1, \ldots, \lambda_k) = f(0, \ldots, 0) = 1 \).

**Proof:** The "only if" part follows immediately from
\[
\frac{\partial^{j_1 + \cdots + j_k}}{\partial \lambda_1^{j_1} \cdots \partial \lambda_k^{j_k}} f(\lambda_1, \ldots, \lambda_k) = (-1)^{j_1 + \cdots + j_k} \int_{\mathbb{R}^k_+} x_1^{j_1} \cdots x_k^{j_k} e^{-\lambda_1 x_1 - \cdots - \lambda_k x_k} dF(x_1, \ldots, x_k).
\]
For the "if" part let
\[
F_n(x_1, \ldots, x_k) = \sum_{j_1=0}^{[nx_1]} \cdots \sum_{j_k=0}^{[nx_k]} \frac{(-n)^{j_1 + \cdots + j_k}}{j_1! \cdots j_k!} \frac{\partial^{j_1 + \cdots + j_k}}{\partial \lambda_1^{j_1} \cdots \partial \lambda_k^{j_k}} f(n, \ldots, n)
\]
for \( x_i \geq 0, \ i = 1, \ldots, k \) and \( F_n(x_1, \ldots, x_k) = 0 \) if \( x_i < 0 \) for some \( i \). Clearly, \( F_n(-\infty, \ldots, -\infty) = 0 \) and for \( 0 < \epsilon < n \)
\[
1 = f(0, \ldots, 0) \geq f(\epsilon, \ldots, \epsilon) \\
\geq \sum_{j=0}^{m} \frac{1}{j!} (\epsilon - n)^j \left( \frac{\partial}{\partial \lambda_1} + \cdots + \frac{\partial}{\partial \lambda_k} \right)^j f(n, \ldots, n) \\
1 \geq \sum_{j_1 + \cdots + j_k \leq m} \frac{1}{j_1! \cdots j_k!} (\epsilon - n)^{j_1 + \cdots + j_k} \frac{\partial^{j_1 + \cdots + j_k}}{\partial \lambda_1^{j_1} \cdots \partial \lambda_k^{j_k}} f(n, \ldots, n).
\]
Letting $\epsilon \searrow 0$ and then $m \to \infty$ we see that $F_n(\infty, \ldots, \infty) \leq 1$. Clearly $F_n$ is right continuous and nondecreasing in all variables and the probability measure induced by $F_n$ is nonnegative on $k$-dimensional rectangles. Therefore $F_n$ is a subdistribution function. The Laplace transform of $F_n$ is

$$
\hat{F}_n = \int_{\mathbb{R}_+^k} e^{-\lambda_1 x_1 - \cdots - \lambda_k x_k} dF_n(x_1, \ldots, x_k)
$$

$$
= \sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} e^{-\lambda_1 \frac{j_1}{n} - \cdots - \lambda_k \frac{j_k}{n}} \frac{(-n)^{j_1+\cdots+j_k}}{j_1! \cdots j_k!} \frac{\partial^{j_1+\cdots+j_k}}{\partial \lambda_1^{j_1} \cdots \partial \lambda_k^{j_k}} f(n, \ldots, n).
$$

Summing over $j = j_1 + \cdots + j_k$ this simplifies to

$$
\hat{F}_n = \sum_{j=0}^{\infty} \frac{1}{j!} (-n)^j \left\{ e^{-n \lambda_1} \frac{\partial}{\partial \lambda_1} + \cdots + e^{-n \lambda_k} \frac{\partial}{\partial \lambda_k} \right\}^j f(n, \ldots, n)
$$

$$
= \sum_{j=0}^{\infty} \frac{1}{j!} \left\{ \left[ n \left( 1 - e^{-\lambda_1} \right) - n \right] \frac{\partial}{\partial \lambda_1} + \cdots + \left[ n \left( 1 - e^{-\lambda_k} \right) - n \right] \frac{\partial}{\partial \lambda_k} \right\}^j f(n, \ldots, n).
$$

However, this is nothing but the Taylor series expansion of

$$
f \left( n \left( 1 - e^{-\lambda_1} \right), \ldots, n \left( 1 - e^{-\lambda_k} \right) \right).$$

Since $f$ is continuous and $n \left( 1 - e^{-\lambda_i} \right) \to \lambda$ we obtain

$$
\hat{F}_n(\lambda_1, \ldots, \lambda_k) = f \left( n \left( 1 - e^{-\lambda_1} \right), \ldots, n \left( 1 - e^{-\lambda_k} \right) \right) \to f(\lambda_1, \ldots, \lambda_k)
$$

for all $\lambda_i \geq 0$, $i = 1, \ldots, k$. The result now follows from Lemma 2.2. □

We note that the right hand side of (2.2) is the Laplace transform of $(\alpha_1, \ldots, \alpha_n)$. Thus, from (2.1) and Lemma 2.3 we obtain the following result.

**Theorem 2.4** Let $\{X_n\}_{n \geq 1}$ be a sequence of continuous random variables with all $X_i$ having common support $S \subset \mathbb{R}$ and $G_k(x_1, \ldots, x_k)$ being the c. d. f. of $(X_1, \ldots, X_k)$, $k \geq 1$. Then $\{X_n\}$ can be modeled by a random $F^n$ model iff there exists a continuous c. d. f. $F$ (on $\mathbb{R}$) such that for all $k \geq 1$ and $\lambda_i > 0$ ($i = 1, \ldots, k$),

$$
G_k \left( F^{-1} \left( e^{-\lambda_1} \right), \ldots, F^{-1} \left( e^{-\lambda_k} \right) \right)
$$

is completely monotone as a function of $\lambda_1, \ldots, \lambda_k$. 10
Remark: It should be noted that there are sequences of random variables \( X_1, X_2, \ldots \) that cannot be modeled with a random \( F^\alpha \) model. For example, let \( 0 < x_1, x_2 < 1 \) and

\[
G_1(x_1) = \begin{cases} 
2x_1^2 & \text{for } x_1 \in (0, \frac{1}{2}] , \\
-1 + 4x_1 - 2x_1^2 & \text{for } x_1 \in \left[ \frac{1}{2}, 1 \right) 
\end{cases}, \quad G_2(x_1, x_2) = G_1(x_1)x_2.
\]

In order for \( G_2 \left( F^{-1} \left( e^{-\lambda_1} \right), F^{-1} \left( e^{-\lambda_2} \right) \right) = G_1 \left( F^{-1} \left( e^{-\lambda_1} \right) \right) F^{-1} \left( e^{-\lambda_2} \right) \) to be completely monotone, \( G_1(F^{-1}) \) and \( F^{-1} \) have to be differentiable infinitely often. This would also imply that \( G_1 \) is differentiable infinitely often which is clearly not the case. Hence, by Theorem 2.4, \( (X_1, X_2) \) with joint c. d. f. \( G_2 \) cannot be modeled with a random \( F^\alpha \) structure.

Theorem 2.4 could be used as an equivalent definition of the random \( F^\alpha \) model. However, note that different sets of \( \{ F, \alpha_i, i = 1, 2, \ldots \} \) can give the same \( G_k \)'s (i.e., the same model). As an example, consider \( F(x) = x, \tilde{F}(x) = \frac{1}{2}x(x + 1) \) for \( x \in [0, 1] \) and let \( \tilde{\alpha}_i \)'s be associated with \( \tilde{F} \). Suppose the distributions of \( \alpha_1, \alpha_2, \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) are as given below.

\[
P(\tilde{\alpha}_1 = 2, \tilde{\alpha}_2 = 1) = p, \quad P(\tilde{\alpha}_1 = 1, \tilde{\alpha}_2 = 2) = q \quad (p, q > 0; p + q = 1)
\]

\[
P(\alpha_1 = 4, \alpha_2 = 1) = P(\alpha_1 = 4, \alpha_2 = 2) = P(\alpha_1 = 2, \alpha_2 = 1) = \frac{p}{8},
\]

\[
P(\alpha_1 = 1, \alpha_2 = 4) = P(\alpha_1 = 2, \alpha_2 = 4) = P(\alpha_1 = 1, \alpha_2 = 2) = \frac{q}{8},
\]

\[
P(\alpha_1 = 3, \alpha_2 = 1) = P(\alpha_1 = 3, \alpha_2 = 2) = \frac{p}{4},
\]

\[
P(\alpha_1 = 1, \alpha_2 = 3) = P(\alpha_1 = 2, \alpha_2 = 3) = \frac{q}{4},
\]

\[
P(\alpha_1 = 2, \alpha_2 = 2) = \frac{1}{8}.
\]
It can easily be seen that $E \left[ F_{\alpha_1}(x_1)F_{\alpha_2}(x_2) \right] = E \left[ \tilde{F}_{\alpha_1}(x_1)\tilde{F}_{\alpha_2}(x_2) \right]$. Similar examples can be found for $k > 2$.

We see that a large class of sequences of random variables can be modeled by random $F^\alpha$ models. In particular, the $X_i$'s need neither be independent nor identically distributed. For a further treatment on what this class includes, see Section 2.4.

While the record count $N_n$ is always a proper (a. s. finite) random variable, the record time $T_n$ is proper only under certain conditions. They are given below.

**Theorem 2.5** For a random $F^\alpha$ model statements (i)-(viii) below are equivalent:

(i) $T_2$ is a proper random variable ($P(T_2 > n) \to 0$)

(ii) $E \left( \frac{1}{S(n)} \right) \to 0$

(iii) $S(n) \overset{p}{\to} \infty$

(iv) $S(n) \to \infty$ a. s.

(v) $\sum_{n=1}^{\infty} \pi_n = \infty$ a. s.

(vi) $T_n$ is proper for all $n$

(vii) $P(M_n \leq x) \overset{n \to \infty}{\to} 0$ for all $x \in \mathbb{R}$ with $F(x) \in (0,1)$

(or $M_n \overset{p}{\to} F^{-1}(1)$)

(viii) $P(M_n \leq x_0) \overset{n \to \infty}{\to} 0$ for some $x_0 \in \mathbb{R}$ with $F(x_0) \in (0,1)$

A necessary condition for (i)-(viii) is that $\sum_{n=1}^{\infty} E\pi_n = \infty$ or equivalently

$\prod_{n=1}^{\infty} (1 - E\pi_n) = 0$. 

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Proof: (i)⇒(ii): Note that
\[ P(T_2 > n) = E \left[ \frac{\alpha_1}{S(2)} \frac{S(2)}{S(3)} \cdots \frac{S(n-1)}{S(n)} \right] = E \left[ \frac{\alpha_1}{S(n)} \right] \]
and
\[ E \left( \frac{\alpha_1}{S(n)} \right) \geq E \left( \frac{\alpha_1}{S(n)} 1_{\{\alpha_i \geq \epsilon\}} \right) \quad \forall \epsilon > 0 \]
\[ \geq \epsilon E \left( \frac{1}{S(n)} 1_{\{\alpha_i \geq \epsilon\}} \right) \]
\[ \epsilon^{-1} E \left( \frac{\alpha_1}{S(n)} \right) \geq E \left( \frac{1}{S(n)} 1_{\{\alpha_i \geq \epsilon\}} \right) . \]

Thus (i) implies
\[ \lim_{n \to \infty} E \left( \frac{1}{S(n)} 1_{\{\alpha_i \geq \epsilon\}} \right) = 0 \quad \forall \epsilon > 0. \]

Now we let \( \epsilon \to 0 \):
\[ 0 = \lim_{\epsilon \to 0} \lim_{n \to \infty} E \left( \frac{1}{S(n)} 1_{\{\alpha_i \geq \epsilon\}} \right) = \lim_{n \to \infty} \lim_{\epsilon \to 0} E \left( \frac{1}{S(n)} 1_{\{\alpha_i \geq \epsilon\}} \right). \]

Note that \( \frac{1}{S(n)} 1_{\{\alpha_i \geq \epsilon\}} \) is non-decreasing when \( \epsilon \) decreases to zero. By the monotone convergence theorem:
\[ \lim_{\epsilon \to 0} E \left( \frac{1}{S(n)} 1_{\{\alpha_i \geq \epsilon\}} \right) = E \lim_{\epsilon \to 0} \left( \frac{1}{S(n)} 1_{\{\alpha_i \geq \epsilon\}} \right) = E \left( \frac{1}{S(n)} \right). \]

Combining the last two equations we get (ii).

(ii)⇒(iii): Using (ii) and the Chebyshev inequality we get \( \frac{1}{S(n)} \to 0 \) in probability which is the same as \( S(n) \to \infty \) in probability since \( S(n) \) is positive.

(iii)⇒(iv): If \( S(n) \to \infty \) in probability, there exists a subsequence \( n_k \) such that 
\( S(n_k) \to \infty \) a. s. However, \( S(n_k) \leq S(m) \leq S(n_{k+1}) \) \( \forall n_k \leq m \leq n_{k+1} \), which means also \( S(n) \to \infty \) a. s.
(iv)⇒(v): follows directly from the Abel-Dini test (see for example Knopp (1931. Chapter IX)).

(v)⇒(iv):

\[ S(n) = \sum_{i=1}^{n} \alpha_i \geq \sum_{i=1}^{n} \frac{\alpha_i}{S(i)} \mu_i = \alpha_1 \sum_{i=1}^{n} \frac{\alpha_i}{S(i)} \rightarrow \infty \text{ a. s.} \]

(iv)⇒(vi):

\[ P(T_n > j) = P(N_j < n) \rightarrow P(N_\infty < n) = P(\{I_k = 1\} \text{ at most } n - 1 \text{ times}) \]

Hence \( P(T_n > j) \rightarrow 0 \iff P(I_n = 1 \text{ i. o.}) = 1 \). For any fixed \( F^\alpha \) model, we know that \( P(I_n = 1 \text{ i. o.}) = 1 \) as long as \( S(n) \rightarrow \infty \). Therefore since \( S(n) \rightarrow \infty \text{ a. s.} \), we have \( P(I_n = 1 \text{ i. o.} | \alpha_1, \alpha_2, \ldots) = 1 \text{ a. s.} \). It follows that

\[ P(I_n = 1 \text{ i. o.}) = EP(I_n = 1 \text{ i. o.} | \alpha_1, \alpha_2, \ldots) = 1 \]

(iv)⇒(vii): Let \( \lambda = -\log F(x) \). Since \( F(x) \in (0,1) \), \( \lambda > 0 \). From (2.2) it follows that

\[ P(M_n \leq x) = G_n(x, \ldots, x) = E\left[ \left. e^{-\lambda S(n)} \right| e^{-\lambda} \right]. \quad (2.3) \]

(iv) implies \( e^{-\lambda S(n)} \rightarrow 0 \text{ a. s.} \). Since \( |e^{-\lambda S(n)}| \leq 1 \) the bounded convergence theorem gives

\[ \lim_{n \rightarrow \infty} P(M_n \leq x) = \lim_{n \rightarrow \infty} E\left[ e^{-\lambda S(n)} \right] = E\left[ \lim_{n \rightarrow \infty} e^{-\lambda S(n)} \right] = 0. \]

(viii)⇒(iii): In view of (2.3), (vii) implies

\[ P(M_n \leq x_0) = G_n(x_0, \ldots, x_0) = E\left[ e^{-\lambda S(n)} \right] \rightarrow 0 \]
where \( \lambda = -\log F(x_0) > 0 \). However, by Chebyshev's inequality, for all \( c > 0 \)
\[
E \left[ e^{-\lambda S(n)} \right] \geq cP \left( e^{-\lambda S(n)} \geq c \right) = cP \left( S(n) \leq \frac{-\log c}{\lambda} \right).
\]
Hence \( P \left( S(n) \leq \frac{-\log c}{\lambda} \right) \to 0 \) for all \( c > 0 \) which is equivalent to (iii). Finally, the implications (vi)\( \Rightarrow \) (i) and (vii)\( \Rightarrow \) (viii) are trivial. Thus we have proved the equivalence of (i) - (viii). If any of these conditions hold, since \( P(I_n = 1) = E\pi_n \), the Borel-Cantelli lemma and \( P(I_n = 1 \ i.o.) = 1 \) imply \( \sum_{n=1}^{\infty} E\pi_n = \infty \). \( \square \)

Remark: Note that the necessary condition \( \sum_{n=1}^{\infty} E\pi_n = \infty \) is not sufficient. For example, let \( \alpha_1 \equiv 1 \),
\[
\alpha_2 = \begin{cases} 
1 & \text{with probability } \frac{1}{2} \\
\frac{1}{3} & \text{with probability } \frac{1}{2}
\end{cases}, \quad \alpha_n = \begin{cases} 
1 & \text{if } \alpha_2 = 1 \\
\frac{2}{n(n+1)} & \text{if } \alpha_2 = \frac{1}{3}
\end{cases}, \quad n \geq 3.
\]
Then
\[
\pi_2 = \begin{cases} 
\frac{1}{2} & \text{with probability } \frac{1}{2} \\
\frac{1}{4} & \text{with probability } \frac{1}{2}
\end{cases}, \quad \pi_n = \frac{\alpha_n}{\alpha_1 + \cdots + \alpha_n} = \begin{cases} 
\frac{1}{n} & \text{if } \pi_2 = \frac{1}{2} \\
\frac{1}{n^2} & \text{if } \pi_2 = \frac{1}{4}
\end{cases}, \quad n \geq 3.
\]
Hence, \( \sum_{n=1}^{\infty} E\pi_n \geq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty \), but with probability \( \frac{1}{2} \), \( \sum_{n=1}^{\infty} \pi_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \) and therefore (by (v)) the model is not proper.

Example 2.1 (Dirichlet distribution) Let \( (\alpha_1, \ldots, \alpha_n, 1 - \sum_{i=1}^{n} \alpha_i) \) have a
Dirichlet distribution with parameters \( \gamma_1, \ldots, \gamma_n, \delta_n \) where \( \gamma_i > 0, \delta_n = \sum_{i=n+1}^{\infty} \gamma_i \) and
\( \gamma = \sum_{i=1}^{\infty} \gamma_i < \infty \):
\[
f_{\alpha_1, \ldots, \alpha_n}(y_1, \ldots, y_n) = \frac{\Gamma(\gamma)}{\Gamma(\gamma_1) \cdots \Gamma(\gamma_n) \Gamma(\delta_n)} \prod_{i=1}^{n} y_i^{\gamma_i-1} \left( 1 - \sum_{i=1}^{n} y_i \right)^{\delta_n-1}
\]
where \( y_i \geq 0 \) and \( \sum_{i=1}^{n} y_i < 1 \). This sequence of \( \alpha \)'s taken together with any continuous
c. d. f. \( F \) gives a random \( F^\alpha \) model. However, \( S(n) = \alpha_1 + \cdots + \alpha_n \leq 1 \ a. s. \) for all
Therefore, by Theorem 2.5, $T_n (n \geq 2)$ is not a.s. finite. This implies that there is a positive probability of the records dying out.

A random $F^\alpha$ model is called proper if any of the conditions (i)-(viii) of Theorem 2.5 hold. In the following we will only consider proper random $F^\alpha$ models.

2.3 Small sample properties

The record indicators are (possibly dependent) Bernoulli random variables with $P(I_n = 1) = \pi_n$. However, conditioned on the $\alpha$'s they are independent. The record count is $N_n = \sum_{j=1}^n I_j$. Hence

$$E N_n = \sum_{j=1}^n E \pi_j,$$

and

$$\text{Var}(N_n) = E \left[ \text{Var}(N_n | \text{\alpha's}) \right] + \text{Var} \left( E[N_n | \text{\alpha's}] \right).$$

By conditioning on the $\alpha$'s, we get

$$\text{Var}(N_n) = \sum_{j=1}^n E \pi_j(1 - \pi_j) + \text{Var} \left( \sum_{j=1}^n \pi_j \right)$$

The record times $T_n$ form a Markov chain with transition probabilities

$$P(T_n = k | T_{n-1} = l) = P(I_{i+1} = 0, \ldots, I_{k-1} = 0, I_k = 1 | T_{n-1} = l)$$

and

$$= E^{\alpha'} P(I_{i+1} = 0, \ldots, I_{k-1} = 0, I_k = 1 | T_{n-1} = l, \alpha_1, \ldots, \alpha_k)$$

$$= E \left[ \frac{S(l)}{S(l + 1)} \ldots \frac{S(k - 2)}{S(k - 1)} \frac{\alpha_k}{S(k)} \right]$$

$$= E \left[ \frac{S(l) \alpha_k}{S(k) S(k - 1)} \right].$$

(2.5)
In proper random $F^\alpha$ models, the joint distribution of record times is given by
\[ P(T_2 = t_2, \ldots, T_n = t_n) \]
\[ = P(I_2 = 0, \ldots, I_{t_2-1} = 0, I_{t_2} = 1, I_{t_2+1} = 0, \ldots, I_{t_3} = 1, \ldots, I_{t_n} = 1) \]
\[ = E \left[ \frac{\alpha_1}{S(2)} \cdots \frac{S(t_2-2)}{S(t_2-1)} \frac{\alpha_{t_2}}{S(t_2)} \frac{S(t_2)}{S(t_2+1)} \cdots \frac{\alpha_{t_n}}{S(t_n)} \right]. \]

In other words,
\[ P(T_2 = t_2, \ldots, T_n = t_n) = E \left[ \frac{\alpha_1}{S(t_n)} \prod_{i=2}^{n} \frac{\alpha_{t_i}}{S(t_i-1)} \right]. \quad (2.6) \]

Traditionally, the independence of record indicators formed the basis for asymptotic considerations. Although we will prove our asymptotic results in a different way, record indicator independence is still an important property. For example, we will use it in the next section to relate our model to previously studied record models. In our case
\[ P(I_k = 1) = E \pi_k, \quad P(I_k = 1, I_l = 1) = E [\pi_k \pi_l], \ldots \]

Hence with the use of the following condition for random variables $Y_1, \ldots, Y_n$,
\[ E [Y_{i_1} \cdots Y_{i_k}] = E [Y_{i_1}] \cdots E [Y_{i_k}] \text{ for all } 1 \leq i_1 < \ldots < i_k \leq n, \quad k = 2, \ldots, n. \quad (2.7) \]
we can conclude:

**Theorem 2.6** In a random $F^\alpha$ model the record indicators are independent iff $\pi_1, \ldots, \pi_n$ satisfy (2.7) for all $n \geq 2$.

**Remark:** When the record indicators are independent, the necessary condition in Theorem 2.5 is also sufficient, since for independent events the converse of the Borel-Cantelli lemma holds. Further, in that case, $\text{Var}(N_n)$ in (2.4) reduces to $\sum_{j=1}^{n} E \pi_j (1 - E \pi_j)$. 

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There are two trivial members of the class of proper random $F^\alpha$ models, the classical model ($\alpha$'s i. i. d.) and the fixed $F^\alpha$ model ($\alpha$'s degenerate). For both it is known that the record indicators are independent. An interesting side product of this fact is that if $\alpha_1, \alpha_2, \ldots$ are positive i. i. d. random variables (2.7) holds for

\[
\frac{\alpha_n}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}, \ n \geq 3.
\]

One more simple conclusion can be drawn from Condition (2.7) in the case of independent $\alpha$'s.

**Theorem 2.7** In a random $F^\alpha$ model with independent $\alpha$'s, if the record indicators are independent either all $\alpha$'s are degenerate or all $\alpha$'s are non-degenerate.

**Proof:** Say $\alpha_1$ is degenerate and $n_0 = \min\{n: \alpha_n \text{ is non-degenerate}\}$.

Let $S = \alpha_1 + \alpha_2 + \cdots + \alpha_{n_0-1}$ (non-random). \(\frac{\alpha_{n_0}}{S + \alpha_{n_0}}\) is increasing and \(\frac{c}{S + c + \alpha_{n_0}}\) is decreasing in $\alpha_{n_0}$ for a fixed $c$. It follows that

\[
\text{cov}\left(\frac{\alpha_{n_0}}{S + \alpha_{n_0}}, \frac{c}{S + c + \alpha_{n_0}}\right) = E\left(\frac{\alpha_{n_0}}{S + \alpha_{n_0}} \cdot \frac{c}{S + c + \alpha_{n_0}}\right) - E\left(\frac{\alpha_{n_0}}{S + \alpha_{n_0}}\right) E\left(\frac{c}{S + c + \alpha_{n_0}}\right) < 0
\]

for all constant $c > 0$.

Since $\alpha_{n_0}$ and $\alpha_{n_0+1}$ are independent, conditioning on $\alpha_{n_0+1} = c$ shows that

\[
E\left(\frac{\alpha_{n_0}}{S + \alpha_{n_0}} \cdot \frac{c}{S + c + \alpha_{n_0}} | \alpha_{n_0+1} = c\right) - E\left(\frac{\alpha_{n_0}}{S + \alpha_{n_0}}\right) E\left(\frac{c}{S + c + \alpha_{n_0}} | \alpha_{n_0+1} = c\right) < 0 \forall c > 0.
\]

Hence by taking expectations with respect to $\alpha_{n_0+1}$

\[
\text{cov}\left(\frac{\alpha_{n_0}}{\alpha_1 + \cdots + \alpha_{n_0}}, \frac{\alpha_{n_0+1}}{\alpha_1 + \cdots + \alpha_{n_0+1}}\right) < 0.
\]
Similarly if $\alpha_1$ is non-degenerate and $n_0 = \min \{ n : \alpha_n \text{ is degenerate} \}$ then

$S(n_0 + 1) = \alpha_1 + \cdots + \alpha_{n_0+1}$ is random, $\alpha_{n_0}$ is fixed and the conditioning argument used above implies

$$\text{cov} \left( \frac{\alpha_{n_0}}{\alpha_1 + \cdots + \alpha_{n_0}}, \frac{\alpha_{n_0+1}}{\alpha_1 + \cdots + \alpha_{n_0+1}} \right) > 0. \quad \square$$

Returning to the dependent case, it is hard to see exactly what $\alpha$-structures satisfy Condition (2.7). However, if the $\alpha$'s are continuous random variables, we can relate them to the $\pi$'s using a transformation. Let $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\tilde{\pi} = (\pi_0, \pi_2, \ldots, \pi_n)$. $\pi_0 = \frac{1}{\alpha_1}$. Then

$$\left| \frac{\partial \tilde{\pi}}{\partial \tilde{\alpha}} \right| =$$

$$\begin{vmatrix}
-\frac{1}{\alpha_1^2} & 0 & 0 & \cdots & \cdots & 0 \\
-\frac{\alpha_2}{(\alpha_1+\alpha_2)^2} & \frac{\alpha_1}{(\alpha_1+\alpha_2)^2} & 0 & \cdots & \cdots & 0 \\
-\frac{\alpha_3}{(\alpha_1+\alpha_2+\alpha_3)^2} & -\frac{\alpha_2}{(\alpha_1+\alpha_2+\alpha_3)^2} & \frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2+\alpha_3)^2} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-\frac{\alpha_n}{(\alpha_1+\cdots+\alpha_n)^2} & \cdots & \cdots & \cdots & \frac{-\sum_{i=1}^n \alpha_i}{(\alpha_1+\cdots+\alpha_n)^2} & \frac{\alpha_1+\cdots+\alpha_n-1}{(\alpha_1+\cdots+\alpha_n)^2} \\
\end{vmatrix}$$

$$\left| \frac{\partial \tilde{\pi}}{\partial \tilde{\alpha}} \right| = \left( \prod_{i=1}^n \frac{1}{\alpha_1 + \cdots + \alpha_i} \right) \frac{1}{\alpha_1 + \cdots + \alpha_n}. \quad (2.8)$$

If we denote by $h$ the joint density of $\pi_0, \pi_2, \ldots, \pi_n$, the joint density of the $\alpha$'s can be written as

$$f_{\alpha_1, \ldots, \alpha_n}(a_1, \ldots, a_n) = \frac{h \left( \frac{1}{a_1}, \frac{a_1}{a_1+a_2}, \ldots, \frac{a_1+\cdots+a_n-1}{a_1+\cdots+a_n} \right)}{\left( \prod_{i=1}^n (a_1 + \cdots + a_i) \right) (a_1 + \cdots + a_n)}. \quad (2.9)$$

Such $\alpha$'s now give a random $F^\alpha$ model with independent record indicators whenever $h$ is such that the $\pi_k$'s for $k \geq 2$ satisfy (2.7).
Example 2.2 (Pareto distribution) Let $\alpha_1, \ldots, \alpha_n$ have the following joint density:

$$f_{\alpha_1, \ldots, \alpha_n}(y_1, \ldots, y_n) = \frac{\Gamma(\gamma)}{\Gamma(\gamma_1) \cdot \cdots \cdot \Gamma(\gamma_n) \Gamma(\delta_n)} \sigma^{\alpha(\gamma-1)} a^n (\sigma^{-a} - y_1^{-a})^{\gamma_1-1} \cdot \ldots \cdot (y_i^{1-a} - (y_i + y_{i+1})^{-a})^{\gamma_i-1} \cdot \ldots \cdot ((y_1 + \cdots + y_{n-1})^{-a} - (y_1 + \cdots + y_n)^{-a})^{\gamma_n-1} \cdot \frac{1}{\prod_{j=1}^{n} \sum_{j=1}^{i} y_j} \left(\sum_{j=1}^{n} y_j\right)^{-a-1}$$

where $\gamma = \sum_{i=1}^{\infty} \gamma_i < \infty$, $\delta_n = \sum_{i=n+1}^{\infty} \gamma_i$, $\gamma_i > 0 \forall i$, $a, \sigma > 0$, $y_1 > \sigma$, $y_i > 0 \forall i \geq 2$.

For positive integers $i_1 = \gamma_1, i_2 - i_1 = \gamma_2, \ldots, i_n - i_{n-1} = \gamma_n, m - i_n = \delta_n - 1$ $(\alpha_1, \ldots, \alpha_n)$ is equal in distribution to $(Y_{i_1:m}, Y_{i_2:m} - Y_{i_1:m}, \ldots, Y_{i_n:m} - Y_{i_{n-1}:m})$ where $Y_{k:m}$ is the $k$th order statistic of a random sample of size $m$ from a Pareto($\sigma, a$) distribution. It can be shown that $\pi_0 = \alpha_1, \pi_2, \ldots, \pi_n$ are independent. Let

$$Z_1 = 1 - \left(\frac{\alpha_1}{\sigma}\right)^{-a}$$

$$Z_i = \left(\frac{\alpha_1 + \cdots + \alpha_{i-1}}{\sigma}\right)^{-a} - \left(\frac{\alpha_1 + \cdots + \alpha_i}{\sigma}\right)^{-a} \text{ for } i = 2, \ldots, n$$

and

$$Y_n = \left(\frac{\alpha_1 + \cdots + \alpha_n}{\sigma}\right)^{-a}$$

then $(Z_1, \ldots, Z_n, Y_n)$ has a Dirichlet distribution with parameters $(\gamma_1, \ldots, \gamma_n, \delta_n)$. Therefore

$$Y_n \sim \text{Beta}(\delta_n, \gamma - \delta_n) \rightarrow \text{Beta}(0, \gamma) \equiv 0.$$

$$\Rightarrow Y_n \rightarrow 0 \text{ in probability } \Rightarrow S(n) = \sigma Y_n^{-\frac{1}{a}} \rightarrow \infty.$$

By Theorem 2.5 this random $F^\alpha$ model is proper.
Let $M_n = \max\{X_1, \ldots, X_n\}$. Another interesting property of most record models is the independence of $I_1, \ldots, I_n, M_n$. Ballerini and Resnick (1987b) prove it for their model and Nevzorov (1990, 1995) characterizes the fixed $F^\alpha$ model (among models with independent observations) using this independence. Let us see when it holds for the random $F^\alpha$ model.

**Theorem 2.8** In the random $F^\alpha$ model, $I_1, \ldots, I_n, M_n$ are independent iff $\pi_2, \ldots, \pi_n, x^{\alpha_1 + \cdots + \alpha_n}$ satisfy (2.7) for all $x \in (0, 1)$.

**Proof:** By conditioning on the $\alpha$'s and using the above mentioned result by Nevzorov (1990, 1995) for fixed $F^\alpha$ models we get

$$P(I_2 = i_2, \ldots, I_n = i_n, M_n \leq y | \alpha_1, \ldots, \alpha_n)$$

$$= P(I_2 = i_2 | \alpha_1, \ldots, \alpha_n) \cdot \ldots \cdot P(I_n = i_n | \alpha_1, \ldots, \alpha_n)$$

$$\cdot P(M_n \leq y | \alpha_1, \ldots, \alpha_n)$$

a. s. for $i_2, \ldots, i_n \in \{0, 1\}$.

$$P(I_2 = i_2, \ldots, I_n = i_n, M_n \leq y) = E \left[ \prod_{j=2}^{n} \pi_j^{i_j} (1 - \pi_j)^{1-i_j} F^{\alpha_1 + \cdots + \alpha_n}(y) \right]$$

and

$$P(M_n \leq y) \prod_{j=2}^{n} P(I_j = i_j) = EF^{\alpha_1 + \cdots + \alpha_n}(y) \cdot \prod_{j=2}^{n} (E\pi_j)^{i_j} (1 - E\pi_j)^{1-i_j},$$

Since $F$ is continuous and therefore assumes every value in $(0, 1)$, comparing the last two equations yields the result. □
Remark: Note that the condition that $\pi_2, \ldots, \pi_n, x^{\alpha_1 + \cdots + \alpha_n}$ satisfy (2.7) for all $x \in (0,1)$ is much weaker than the condition that (2.7) holds for $\pi_2, \ldots, \pi_n$ and $(\pi_2, \ldots, \pi_n)$ and $\alpha_1 + \cdots + \alpha_n$ are independent. For example, let $n = 2$, $Z$ be a uniform random variable on $(0,1) \cup (1,2)$,

$$\alpha_1 = \frac{1}{2} (Z - [Z]) \{1 + \text{sgn}(Z - 1) (Z - [Z])\}$$

$$\alpha_2 = \frac{1}{2} (Z - [Z]) \{1 - \text{sgn}(Z - 1) (Z - [Z])\}$$

where

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$[x] = \text{greatest integer } \leq x.$$ 

Then

$$\alpha_1 + \alpha_2 = Z - [Z], \quad \pi_2 = \frac{1}{2} \{1 - \text{sgn}(Z - 1) (Z - [Z])\}.$$ 

Note that $\pi_2$ and $\alpha_1 + \alpha_2$ are clearly dependent. Now

$$E\left\{ \pi_2 - \frac{1}{2} \right\} = 0 \quad \text{and} \quad E\left\{ \left( \pi_2 - \frac{1}{2} \right) x^{\alpha_1 + \alpha_2} \right\} = -\frac{1}{2} E \{\text{sgn}(Z - 1) \cdot (Z - [Z]) x^{z-[z]}\} = 0$$

for all $x \in (0,1)$.

Hence $\pi_2$ and $x^{\alpha_1 + \alpha_2}$ are uncorrelated for all $x \in (0,1)$.

Example 2.3 (Gamma distribution) Let $\alpha_1, \ldots, \alpha_n$ be independent and $\alpha_i$ have Gamma$(a_i, b)$ distribution with $a_1, \ldots, a_n, b > 0$. By transformation one can show that $\pi_2, \ldots, \pi_n, \alpha_1 + \cdots + \alpha_n$ are independent, $\pi_i$ $(i = 2, \ldots, n)$ has
Beta \( \left( a_i, \sum_{k=1}^{i-1} a_k \right) \) distribution and \( \alpha_1 + \cdots + \alpha_n \) has Gamma \( \left( \sum_{i=1}^{n} a_i, b \right) \) distribution. Note also that \( S(n) = \alpha_1 + \cdots + \alpha_n \to \infty \) a.s. iff \( a_1 + \cdots + a_n \to \infty \) and therefore by Theorem 2.5 the model is proper under this condition. Further, by Theorem 2.8, \( I_1, \ldots, I_n \) and \( M_n \) are mutually independent.

### 2.4 Relation to other record models

In Section 2.1 we mentioned a number of record models considered in the literature. The details of those models can be found in Appendix A. Here we will describe how they all relate to each other and to the random \( F^\alpha \) model.

The classical model assumed the observations to be i.i.d. Yang (1975) was the first to introduce an \( F^\alpha \)-type model, where his \( \alpha \)'s were positive, geometrically increasing integers of the form \( \alpha_n = \left[ n \alpha_{n-1} + \frac{1}{2} \right] \) \( (n \geq 2), \alpha_1 > 0, \lambda \geq 0 \). For \( \lambda = 1 \), the \( \alpha \)'s are all equal and the classical model is recovered. Nevzorov (1986a,b, 1990, 1995) introduced the fixed \( F^\alpha \) scheme, where the \( \alpha \)'s can be arbitrary positive constants. It clearly contains Yang's model as a special case. Ballerini and Resnick (1987b) introduced an embedding model (BR model) that is almost identical to Nevzorov's model. The authors introduce it by means of embedding the sequence of maxima \( M_1, M_2, \ldots \) into an extremal \( F \)-process. (The only difference from the fixed \( F^\alpha \) model is that they define it by giving the joint distribution of the sequence of successive maxima rather than the joint distribution of the sequence of observations. Technically, this makes it slightly more general since 2 different sequences of observations can give the same maxima sequence. We will ignore this nuance.) Ballerini (1994) took the fixed \( F^\alpha \) model for the marginal distributions of his observations and introduced a specific dependence structure using an Archimedean copula. For that dependent \( F^\alpha \)
model, the joint c. d. f. of observations $X_1, \ldots, X_n$ is given by

$$G_n(x_1, \ldots, x_n) = \exp \left\{ - \left( \sum_{i=1}^{n} [-a_i \log F_1(x_i)]^\gamma \right)^{\frac{1}{\gamma}} \right\}, \quad (2.10)$$

where $\gamma \geq 1, a > i > 0 \forall i$ and $F_1$ is an absolutely continuous c. d. f. For $\gamma = 1$, this model coincides with the fixed $F^\alpha$ model. For $\gamma > 1$, however, the observations are dependent. Nevzorova, Nevzorov and Balakrishnan (1997) (NNB) went one step further and noticed that the record indicators are still independent in certain Archimedean copula models without marginal $F^\alpha$ structure. The joint c. d. f. of $X_1, \ldots, X_n$ for the NNB copula model is given by

$$G_n(x_1, \ldots, x_n) = \eta \left( \sum_{i=1}^{n} c_i \nu \left(F_2(x_i)\right) \right)$$

where $F_2$ is a continuous c. d. f., the $c_i$ ($i = 1, \ldots, n$) are positive constants, $\eta$ is a positive, completely monotone function with $\eta(0) = 1$ and $\nu$ is the inverse of $\eta$. For $c_i = 1$ and $\eta(x) = \exp \left(-x^{\frac{1}{\gamma}}\right)$ this coincides with Ballerini's model given in (2.10).

We will now demonstrate that the random $F^\alpha$ model contains the NNB copula model as a special case. Let $F(x) = e^{-\nu(F_2(x))}$. Since $\nu(1) = 0$, $\nu(0+) = \infty$, $F$ is a continuous c. d. f. with the same support as $F_2$. From (2.11) we get

$$G_n \left(F^{-1} \left(e^{-\lambda_1}\right), \ldots, F^{-1} \left(e^{-\lambda_n}\right)\right) = \eta \left( \sum_{i=1}^{n} c_i \nu \left(F_2(F_2^{-1}(\eta(\lambda_i)))\right) \right)$$

$$= \eta \left( \sum_{i=1}^{n} c_i \lambda_i \right).$$

The right hand side is clearly completely monotone as a function of $\lambda_1, \ldots, \lambda_n$. Therefore it follows from Theorem 2.4 that this NNB copula model is also a random $F^\alpha$ model. Nevzorova et al. (1997) showed that in their model, $I_1, \ldots, I_n, M_n$ are independent. Therefore the NNB copula models are contained in the set of $F^\alpha$ models.
with independent $I_1, \ldots, I_n, M_n$ (as described by Theorem 2.8), which in turn is a sub-
set of the random $F^\alpha$ models with independent record indicators (see Theorem 2.7).
The hierarchy of the above mentioned models is described in Figure 2.1 given below.

Figure 2.1: Hierarchy of record models

We have demonstrated that all inclusions up to the NNB copula model (and BR
model) are strict. We will now show the same for the last 3 inclusions (involving random \(F^\alpha\) models) by giving the examples referred to in the figure.

**Example 2.4** (random \(F^\alpha\) model with dependent indicators)

Let \(P(\alpha_1 = 1) = P(\alpha_1 = 2) = \frac{1}{2}\) and let \(\alpha_2, \alpha_3, \ldots\) be i. i. d. with \(P(\alpha_2 = 1) = 1 - \gamma, \ P(\alpha_2 = 2) = \gamma, \ \gamma \in (0, 1)\). As \(S(n) = \alpha_1 + \cdots + \alpha_n \geq n \to \infty\) a.s., combined with any continuous c. d. f. \(F\) this \(\alpha\)-sequence forms a proper random \(F^\alpha\) model. However, simple calculations show that

\[
E[\pi_2 \pi_3] - E[\pi_2] \ E[\pi_3] = \frac{1}{720} (\gamma - \frac{1}{2}) (2\gamma^2 - 7\gamma - 5)
\]

which is nonzero for \(\gamma \neq \frac{1}{2}\). By Theorem 2.7, \(I_2\) and \(I_3\) are therefore dependent.

**Example 2.5** (random \(F^\alpha\) model with dependent \(I_1, \ldots, I_n, M_n\) but independent indicators)

Let us recall Example 2.2 from page 20. We have seen before that \(\pi_2, \ldots, \pi_n\) are independent for all \(n\) and therefore by Theorem 2.7 the record indicators \(I_1, \ldots, I_n\) are independent. Let us consider the special case where \(\gamma_1 = \gamma_2 = \delta_2 = a = \sigma = 1, \ \gamma = 3\).

The density of \((\alpha_1, \alpha_2)\) reduces to

\[
f_{\alpha_1, \alpha_2}(a_1, a_2) = \frac{2}{a_1^2(a_1 + a_2)^2} \text{ for } a_1 > 1, \ a_2 > 0.
\]

Let \(\pi_0 = S(2) = \alpha_1 + \alpha_2, \ \pi_2 = \frac{\alpha_2}{\alpha_1 + \alpha_2}\). Then we obtain the joint density of \(\pi_0, \pi_2\) to be:

\[
f_{\pi_0, \pi_2}(b_0, b_2) = \frac{2}{b_0^2(1-b_2)^2} \text{ for } b_0 > 1, \ b_2 > 1 - \frac{1}{b_0}.
\]

Simple calculations show

\[
E[\pi_2 \pi_0] - E[\pi_2] \ E[\pi_0] = \int_1^\infty \frac{x^b}{b^3} (b - 1 - 2 \log b) \, db
\]
which, for example, equals \(-.032\) for \(x = .5\). By Theorem 2.8, \(I_2\) and \(M_2\) are therefore dependent.

**Example 2.6** (a general random \(F^\alpha\) model with independent \(I_1, \ldots, I_n, M_n\))

Before we give the example, we need to derive a property. Say, we have a random \(F^\alpha\) model which is also a NNB copula model. Then from (2.1) and (2.11) it follows that

\[
G_n(x_1, \ldots, x_n) = E[F^{\alpha_1}(x_1) \cdots F^{\alpha_n}(x_n)] = \eta \left( \sum_{i=1}^{n} c_i \nu \left( F_2(x_i) \right) \right)
\]  

(2.12)

for all \(n \geq 1\). Let \(n = 1\). Then

\[
EF^{\alpha_1}(x) = \eta \left( c_1 \nu \left( F_2(x) \right) \right)
\]

\[
\nu \left( F_2(x) \right) = \frac{1}{c_1} \nu \left( EF^{\alpha_1}(x) \right).
\]

Further let \(F(x_i) = e^{-\mu_i}\). We obtain

\[
\nu \left( F_2(x_i) \right) = \frac{1}{c_1} \nu \left( E e^{-\mu_i} \right).
\]

Therefore (2.12) reduces to

\[
EF \left[ e^{-\mu_1} \alpha_1 \cdots \mu_n \alpha_n \right] = \eta \left( \sum_{i=1}^{n} \frac{c_i}{c_1} \nu \left( E e^{-\mu_1} \right) \right)
\]

\[
\nu \left( \nu \left[ e^{-\mu_1} \alpha_1 \cdots \mu_n \alpha_n \right] \right) = \sum_{i=1}^{n} \frac{c_i}{c_1} \nu \left( E e^{-\mu_1} \right)
\]  

(2.13)

for all \(\mu_i > 0\), \(i = 1, \ldots, n\). Since \(\nu\) and \(\eta\) are differentiable any number of times, we can take the derivative with respect to \(\mu_i\) and \(\mu_j\) \((1 \leq i < j \leq n)\) on both sides.

\[
\nu'' \left( \nu \left[ e^{-\mu_1} \alpha_1 \cdots \mu_n \alpha_n \right] \right) E \left[ \alpha_i e^{-\mu_1} \alpha_1 \cdots \mu_n \alpha_n \right] E \left[ \alpha_j e^{-\mu_1} \alpha_1 \cdots \mu_n \alpha_n \right] +
\]

\[
+ \nu' \left( \nu \left[ e^{-\mu_1} \alpha_1 \cdots \mu_n \alpha_n \right] \right) E \left[ \alpha_i \alpha_j e^{-\mu_1} \alpha_1 \cdots \mu_n \alpha_n \right] = 0.
\]

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Let $\mu_l \to 0$ for all $l = 1, \ldots, n$. Then

$$\frac{E[\alpha_i \alpha_j]}{E \alpha_i E \alpha_j} = -\lim_{x \to 1^-} \frac{\nu''(x)}{\nu'(x)} \text{ for all } i \neq j. \tag{2.14}$$

The right hand side is a clearly defined constant in $(0, \infty)$ (since the left hand side is in $(0, \infty)$). Now consider the following example. Let $P(\alpha_1 = 1, \alpha_2 = 2) = P(\alpha_1 = 2, \alpha_2 = 1) = \frac{1}{2}$, $\alpha_n = 1$ for $n \geq 3$. With any continuous $F$ this is a proper random $F^\alpha$ model. Since $\pi_n$ and $S(n)$ are constants for all $n \geq 3$, $I_1, \ldots, I_n, M_n$ are trivially independent by Theorem 2.8. However,

$$\frac{E[\alpha_1 \alpha_2]}{E \alpha_1 E \alpha_2} = \frac{8}{9} \neq 1 = \frac{E[\alpha_1 \alpha_3]}{E \alpha_1 E \alpha_3},$$

which violates (2.14). Thus it cannot be a NNB copula model.

**Remark:** One could think of applying the random $\alpha$ idea to the Ballerini model (2.10) and therefore generalizing the random $F^\alpha$ model even further. However, it does not work. Letting the $\alpha$'s in (2.10) be random just yields a regular random $F^\alpha$ model. To see this, let $F(x) = \exp \{-(\log F_1(x))^\gamma\}$. Then $F^{-1}(e^{-\lambda}) = F_1^{-1}(e^{-(\lambda^{\frac{1}{\gamma}})})$ and by (2.10) (with random $\alpha$'s)

$$G_n \left( F^{-1}(e^{-\lambda_1}), \ldots, F^{-1}(e^{-\lambda_n}) \right) = E \exp \left\{ -\left[ \sum_{i=1}^n \lambda_i \alpha_i^\gamma \right]^{\frac{1}{\gamma}} \right\}.$$

Since $\gamma > 1$, it is easy to see that the right hand side is completely monotone as a function of $\lambda_1, \ldots, \lambda_n$. From Theorem 2.4 it therefore follows that it is indeed a random $F^\alpha$ model.

Also, recall that the independence of record indicators has been used to show asymptotic results in previous models. In the random $F^\alpha$ model, we still need it for some asymptotic properties, but we can prove some others without it (see Sections 2.6, 2.7).

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2.5 Inter-record times

Recall that the inter-record times were defined as $\Delta_n = T_n - T_{n-1}$, $n \geq 2$. From (2.6) we can derive the joint distribution of $\Delta_2, \ldots, \Delta_n$ as

$$P(\Delta_2 = k_2, \ldots, \Delta_n = k_n) = P(T_2 = 1 + k_2, T_3 = 1 + k_2 + k_3, \ldots, T_n = 1 + k_2 + \cdots + k_n).$$

Thus,

$$P(\Delta_2 = k_2, \ldots, \Delta_n = k_n) = E\left[\frac{\alpha_1}{S(1 + k_2 + \cdots + k_n)} \prod_{i=2}^{n} \frac{\alpha_{1+k_2+\cdots+k_i}}{S(k_2 + \cdots + k_i)}\right] \quad (2.15)$$

for $k_i = 1, 2, \ldots; i = 2, \ldots, n; n \geq 2$.

If $\pi_n \rightarrow p \in (0, 1)$ a.s., we can derive the limit distribution of $(\Delta_n, \ldots, \Delta_{n+l})$ using the bounded convergence theorem. Specifically,

$$\lim_{n \rightarrow \infty} P(\Delta_n = k_0, \ldots, \Delta_{n+l} = k_l | T_{n-1} = t)$$

$$= \lim_{n \rightarrow \infty} E\left[(1 - \pi_{t+1}) \cdots (1 - \pi_{t+k_{0-1}}) \pi_{t+k_0} (1 - \pi_{t+k_0+1}) \cdots \pi_{t+k_l} \cdots \pi_{t+k_l}\right]$$

$$= \prod_{i=0}^{l} p(1-p)^{k_{i-1}} \text{ for all } k_i \geq 1, i = 0, \ldots, l, l \geq 0.$$ 

Therefore $\{\Delta_k\}_{k \geq n}$ approaches an i.i.d. sequence with Geom($p$) distribution. If $\pi_n \rightarrow 0$ a.s. $\Delta_n \rightarrow \infty$. This has been observed for the fixed $F^\alpha$ model by Ballerini and Resnick (1987b).

Let us now consider the ratio of inter-record and record times. From (2.5) it follows that

$$P\left(\frac{\Delta_n}{T_n} > x | T_{n-1} = t\right) = P\left(T_n > \frac{t}{1-x} | T_{n-1} = t\right)$$

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\[
\begin{align*}
&= \sum_{j=\left[\frac{t}{1-x}\right]+1}^{\infty} \mathbb{E}\left[\frac{S(t)\alpha_j}{S(j-1)S(j)}\right] \\
&= \mathbb{E}\left[\sum_{j=\left[\frac{t}{1-x}\right]+1}^{\infty} \left(\frac{1}{S(j-1)} - \frac{1}{S(j)}\right)S(t)\right] \\
&= \mathbb{E}\left[\frac{S(t)}{S\left(\left[\frac{t}{1-x}\right]\right)}\right].
\end{align*}
\]

Hence, if \(\lim_{t\to\infty} \mathbb{E}\left[\frac{S(1-x)t}{S(t)}\right]\) exists for all \(x \in (0,1)\) then \(\frac{\Delta_n}{S_n}\) goes in law to a random variable with c. d. f. \(F(x) = 1 - \lim_{t\to\infty} \mathbb{E}\left[\frac{S(1-x)t}{S(t)}\right].\)

### 2.6 Asymptotic properties of record times and counts

In this section first we present a central limit theorem for the general random \(F^\alpha\) model. Next we give a number of results in the case where the record indicators are \(m\)-dependent. Lastly, we present some properties that hold only when the record indicators are independent. Some of these results parallel the known results from the fixed \(F^\alpha\) model obtained by Nevzorov (1985, 1986b, 1990, 1995) and Ballerini and Resnick (1987b).

**Theorem 2.9**  
(i) In a proper random \(F^\alpha\) model, if there exist sequences \(\gamma(n), \delta(n) \to \infty\) such that
\[
\frac{\sum_{i=1}^{n} \pi_i(1 - \pi_i)}{\delta^2(n)} \to 1 \quad \text{and} \\
\frac{\sum_{i=1}^{n} \pi_i - \gamma(n)}{\delta(n)} \to 0
\]
(2.16) (2.17)
then
\[
\frac{N_n - \gamma(n)}{\delta(n)} \to N(0,1).
\]

(ii) If further \(\gamma(n) \sim c\delta^2(n), c > 0\) and \(\gamma(n)\) nondecreasing, then
\[
\frac{\gamma(T_n) - n}{\sqrt{\delta(E)}} \to N(0,1).
\]

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Proof: Recall that in a proper random $F^a$ model $S(n) \to \infty$ a.s. It follows from a result in Nevzorov (1995) for fixed $F^a$ models that

$$P \left( \frac{N_n - \sum_{i=1}^{n} \pi_i}{\sqrt{\sum_{i=1}^{n} \pi_i(1 - \pi_i)}} \leq x | \alpha_1, \alpha_2, \ldots \right) \to \Phi(x) \quad \text{a.s.}$$

where $\Phi(x)$ is the standard normal c.d.f. Using the bounded convergence theorem it follows that

$$\lim_{n \to \infty} P \left( \frac{N_n - \sum_{i=1}^{n} \pi_i}{\sqrt{\sum_{i=1}^{n} \pi_i(1 - \pi_i)}} \leq x \right)$$

$$= \lim_{n \to \infty} E P \left( \frac{N_n - \sum_{i=1}^{n} \pi_i}{\sqrt{\sum_{i=1}^{n} \pi_i(1 - \pi_i)}} \leq x | \alpha_1, \alpha_2, \ldots \right)$$

$$= E \lim_{n \to \infty} P \left( \frac{N_n - \sum_{i=1}^{n} \pi_i}{\sqrt{\sum_{i=1}^{n} \pi_i(1 - \pi_i)}} \leq x | \alpha_1, \alpha_2, \ldots \right) = \Phi(x)$$

which means

$$\frac{N_n - \sum_{i=1}^{n} \pi_i}{\sqrt{\sum_{i=1}^{n} \pi_i(1 - \pi_i)}} \overset{d}{\to} N(0, 1). \quad (2.18)$$

Part (i) now follows from (2.16)-(2.18) and Slutzky's theorem.

For the second part, consider

$$P(N_n \leq k) \sim P \left( \frac{N_n - \gamma(n)}{\delta(n)} \leq x \right) \sim \Phi(x)$$

where $k = \gamma(n) + x\delta(n) \sim c\delta^2(n) + x\delta(n)$. Hence

$$\delta^2(n) = \frac{1}{4c^2} \left( -x + \sqrt{x^2 + 4ck} \right)^2$$

$$= \frac{k}{c} - \frac{x}{c\sqrt{c}} \sqrt{k} + o(\sqrt{k})$$

$$\gamma(n) = k - \frac{x}{\sqrt{c}} \sqrt{k} + o(\sqrt{k}).$$
Now

\[ \Phi(x) \sim P(N_n \leq k) = P(T_k \geq n) \]

\[ = P(\gamma(T_k) \geq \gamma(n)) \]

\[ = P \left( \gamma(T_k) \geq k - \frac{x}{\sqrt{c}} \sqrt{k} + o(\sqrt{k}) \right) \]

\[ \sim P \left( \frac{k - \gamma(T_k)}{\sqrt{\frac{k}{c}}} \leq x \right) \]

and part (ii) follows. □

**Remark:** The assumptions of the theorem are satisfied for example when

\[ \sqrt{n}(\pi_n - p) \xrightarrow{P} 0, \ p \in (0,1). \]

In this case we can choose \( \gamma(n) = np \) and \( \delta^2(n) = np(1-p) \).

Next we will consider the random \( F^\alpha \) model where the record indicators \( I_n \) are \( m \)-dependent. Let us start by examining for what \( \pi \)-sequences this is feasible.

**Theorem 2.10** In a random \( F^\alpha \) model the record indicators are \( m \)-dependent iff

\[ E \left[ \pi_{i_1} \cdots \pi_{i_k} \pi_{i_{k+1}} \cdots \pi_{i_{k+l}} \right] = E \left[ \pi_{i_1} \cdots \pi_{i_k} \right] E \left[ \pi_{i_{k+1}} \cdots \pi_{i_{k+l}} \right] \quad (2.19) \]

for all \( 1 \leq i_1 < \ldots < i_{k+l}, \ k, l \geq 1, \ i_{k+1} - i_k > m \).

**Proof:** Since \( P(I_k = 1) = E[\pi_k], \ P(I_k = 1, I_l = 1) = E[\pi_k \pi_l], \ldots \) the record indicators are \( m \)-dependent iff

\[ P(I_{i_1} = 1, \ldots, I_{i_k} = 1, I_{i_{k+1}} = 1, \ldots, I_{i_{k+l}} = 1) \]

\[ = P(I_{i_1} = 1, \ldots, I_{i_k} = 1)P(I_{i_{k+1}} = 1, \ldots, I_{i_{k+l}} = 1) \]

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Again it is hard to describe what (2.19) means for the $\alpha$’s, however, for continuous $\alpha$’s the transformation (2.9) applies and results in $m$-dependent indicators whenever $h$ is such that the $\pi_k$’s for $k \geq 2$ satisfy (2.19).

**Example 2.7** Let $\theta_1, \theta_2, \ldots$ be i. i. d. Exp(1). Given $\theta_k$, let $\pi_{km+1}(\pi_0$ for $k = 0), \pi_{km+2}, \ldots, \pi_{(k+1)m}$ be conditionally i. i. d. Beta$(1, \theta_k)$ for $k = 0, 1, \ldots$; $\pi_{km+1}, \ldots, \pi_{(k+1)m}$ be independent of $\pi_{lm+1}, \ldots, \pi_{(l+1)m}$ for $k \neq l$. This construction clearly gives $m$-dependent $\pi$’s (hence (2.19) holds) that do not satisfy (2.7). The joint density of these $\pi$’s turns out to be

$$f_{\pi_{0}, \pi_{1}, \ldots, \pi_{(k+1)m}}(b_1, \ldots, b_{(k+1)m}) = (m!)^{k+1} \prod_{j=1}^{(k+1)m} \frac{1}{1-b_j} \prod_{j=0}^{k} \frac{1}{1 - \log((1 - b_{jm+1}) \cdots (1 - b_{(j+1)m}))}.$$  

Using (2.9) the joint density of the corresponding $\alpha$’s is

$$f_{\alpha_1, \ldots, \alpha_{(k+1)m}}(a_1, \ldots, a_{(k+1)m}) = \frac{(m!)^{k+1}}{\prod_{i=1}^{(k+1)m} (a_1 + \cdots + a_i) \prod_{j=0}^{k} (1 - \log \left( \frac{a_1 + \cdots + a_{jm}}{a_1 + \cdots + a_{(j+1)m}} \right))}.$$  

Now, let us present the asymptotic results for random $F^\alpha$ models with $m$-dependent record indicators. Let $p_n = P(I_n = 1) = E\pi_n$,

$$A(n) = EN_n = \sum_{i=1}^{n} p_i,$$

$$B(n) = \sum_{i=1}^{n} p_i^2 - 2 \sum_{1 \leq i < j \leq n} \text{Cov}(I_i, I_j)$$

(2.20)

$$D^2(n) = \text{Var}(N_n) = A(n) - B(n), C(x) = D \left( A^{-1}(x) \right).$$
Theorem 2.11 In a random $F^n$ model with $m$-dependent record indicators the following statements hold.

(i) If $p_n \to p \in [0, 1]$ then $\frac{N_n}{n} \to p$, $\frac{T_n}{n} \to \frac{1}{p}$ a.s. (for $p = 0$, $\frac{T_n}{n} \to \infty$ a.s.)

(ii) If $D(n) \to \infty$ and $\limsup \frac{B(n)}{\lambda(n)} < 1$ then

$$\frac{N_n - A(n)}{D(n)} \overset{c}{\to} N(0, 1).$$

(2.21)

Proof: Recall that $N_n = I_1 + \cdots + I_n$, $N_{T_n} = n$. Part (i) is the strong law of large numbers for $m$-dependent sequences. See, for example, Billingsley (1986, Exercise 6.11). For part (ii) note that $\text{Cov}(I_i, I_j) \leq EI_iI_j \leq p_i$. From the $m$-dependence it follows that $2 \sum_{1 \leq i < j \leq n} \text{Cov}(I_i, I_j) \leq 2mA(n)$. Therefore $\lim \inf \frac{B(n)}{\lambda(n)} > -m$ and hence $\frac{A(n)}{D^2(n)} = O(1)$. The result now follows from the central limit theorem for partial sums of $m$-dependent sequences as proven, for example, in Orey (1958). $\square$

Theorem 2.12 Let $A(n)$, $B(n)$, $C(n)$ and $D(n)$ be as defined in (2.20) and the record indicators are $m$-dependent. Assume $D(n) \to \infty$ and $\limsup \frac{B(n)}{\lambda(n)} < 1$. Then the following statements hold.

(a) If $\frac{B(n)}{\lambda^2(n)} \to 0$ then

$$\frac{A(T_n) - n}{C(n)} \overset{c}{\to} N(0, 1).$$

(b) If $\frac{B(n)}{\lambda(n)} \to 0$ then

(i) $\frac{N_n - A(n)}{\sqrt{A(n)}} \overset{c}{\to} N(0, 1)$,

(ii) $\frac{A(T_n) - n}{\sqrt{n}} \overset{c}{\to} N(0, 1)$.

(c) If $\frac{1}{n} \sum_{i=1}^{n} p_i(1 - p_i) \to p(1 - p)$, $\frac{2}{n} \sum_{1 \leq i < j \leq n} \text{Cov}(I_i, I_j) \to c_1 \in \mathbb{R}$ with $p \in (0, 1)$, $\sigma^2_0 = p(1 - p) + c_1 > 0$, then

(i) $\frac{N_n - A(n)}{\sqrt{n}} \overset{c}{\to} N(0, \sigma^2_0)$,

(ii) $\frac{A(T_n) - n}{\sqrt{n}} \overset{c}{\to} N\left(0, \frac{\sigma^2_0}{p}\right)$. 

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(d) If $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (p_i - p) \to c_2 \in R$, $\frac{1}{n} \sum_{i=1}^{n} p_i^2 \to p^2$,

$\frac{2}{n} \sum_{1 \leq i < j \leq n} \text{Cov}(I_i, I_j) \to c_1 \in R$ with $p \in (0, 1)$, $p(1 - p) + c_1 > 0$. then

(i) $\frac{N_n - np}{\sqrt{n}} \xrightarrow{d} N(c_2, \sigma_0^2)$,

(ii) $\frac{T_n - \frac{n}{p}}{\sqrt{n}} \xrightarrow{d} N \left( \frac{c_2}{\sqrt{p}}, \frac{\sigma_0^2}{p} \right)$.

Proof: (a) Since $\frac{B(n)}{A(n)} \to 0$, $\frac{D(n)}{A(n)} = \sqrt{\frac{A(n) - B(n)}{A^2(n)}} \to 0$. Further as

$A(n), D(n) \to \infty$, it follows by L'Hospital's rule that $\frac{D'(x)}{A'(x)} \to 0$ ($A(x), D(x)$ are

made smooth and strictly increasing between integers such that $\lim_{x \to \infty} \frac{D'(x)}{A'(x)}$ exists).

Therefore

$$C'(x) = \frac{D'(A^{-1}(x))}{A'(A^{-1}(x))} \to 0.$$ 

The Taylor expansion of $C(n)$ can be written as

$$C(n) = C(A(T_n)) + (n - A(T_n))C'(\theta), \quad \theta \geq \min\{n, A(T_n)\}.$$ 

or

$$\frac{C(n)}{C(A(T_n))} = 1 + \frac{n - A(T_n)}{C(A(T_n))} C'(\theta), \quad \theta \geq A(n) \quad (2.22)$$

By replacing $n$ with $T_n$ in (2.21) we obtain

$$\frac{n - A(T_n)}{D(T_n)} = \frac{n - A(T_n)}{C(A(T_n))} \xrightarrow{d} N(0, 1). \quad (2.23)$$

Since $A(n) \to \infty$ and $C'(x) \to 0$, it follows from (2.22) that $\frac{C(n)}{C(A(T_n))} \xrightarrow{P} 1$. combining

this with (2.23) proves part (a).

(b) Here $\frac{D(n)}{\sqrt{A(n)}} = \sqrt{\frac{A(n) - B(n)}{A(n)}} \to 1$. Hence

$$\frac{N_n - A(n)}{\sqrt{A(n)}} = \frac{N_n - A(n)}{D(n)} \frac{D(n)}{\sqrt{A(n)}} \xrightarrow{d} N(0, 1).$$
The second part can be shown exactly like part (ii) of Theorem 2.9.

\[(c)\] Note that \( \frac{D^2(n)}{n} = \frac{1}{n} \sum_{i=1}^{n} p_i - \frac{1}{n} \sum_{i=1}^{n} p_i^2 + \frac{2}{n} \sum_{1 \leq i < j \leq n} \text{Cov}(I_i, I_j) \rightarrow \sigma_0^2 > 0. \)

Hence
\[
\frac{N_n - A(n)}{\sqrt{n}\sigma_0} = \frac{N_n - A(n)}{D(n)} \frac{D(n)}{\sqrt{n}\sigma_0} \xrightarrow{\mathcal{L}} N(0, 1).
\]
As \( \frac{N_n - A(n)}{\sqrt{A(n)}} \xrightarrow{\mathcal{L}} N\left(0, \frac{\sigma_0^2}{\rho}\right) \), the proof of (ii) follows along the lines of the proof of part (ii) of Theorem 2.9.

\[(d)\]
\[
\frac{N_n - np}{\sqrt{n}\sigma_0} = \frac{N_n - A(n)}{\sqrt{n}\sigma_0} + \frac{A(n) - np - c_2\sqrt{n}}{\sqrt{n}\sigma_0}
\]
\[= \frac{N_n - A(n)}{\sqrt{n}\sigma_0} + \frac{1}{\sigma_0} \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (p_i - p) - c_2 \right]
\]
\[\xrightarrow{\mathcal{L}} N(0, 1) \text{ by part (c)}.
\]

The proof of the second part is analogous to the proof of part (ii) of Theorem 2.9. \(\square\)

\textbf{Remark:} The condition about the sum of covariances in parts (c) and (d) holds if for example the record indicator sequence is asymptotically second order stationary. i.e., \( \text{Cov}(I_n, I_{n+k}) \rightarrow r_k \) for all \( 1 \leq k \leq m \) and \( p(1 - p) + 2(r_1 + \cdots + r_m) > 0 \). By use of strong mixing, similar asymptotic properties can be shown when the record indicator sequence is a stationary Markov chain. However, we consider the stationarity assumption to be too strong for our purpose of presenting general non-i.i.d. record models.

In the last part of this section we consider random \( F^\alpha \) models where the record indicators are independent (see Theorem 2.6 and Figure 2.1). The behavior of the
record times $T_n$ depends only on the record indicators. Therefore, since the indicators are independent, the properties of the record times are the same as in a fixed $F^\alpha$ model with $P(I_n = 1) = E \pi_n = p_n = \frac{\alpha_n}{S^*(n)}$ (observe the model non-identifiability from record times). Such a fixed $F^\alpha$ model can be achieved by setting $S^*(1) = \alpha_1^* = 1$, $S^*(n) = \prod_{i=2}^{n} \frac{1}{1-p_i}$ (and $\alpha_n^* = p_n S^*(n)$), $n \geq 2$. Note that now

$$A(n) = EN_n = \sum_{i=1}^{n} p_i, \quad B(n) = A(n) - \text{Var}(N_n) = \sum_{i=1}^{n} p_i^2.$$ 

We can now conclude the following theorem from known results for the fixed $F^\alpha$ model; part (a) from Nevzorov (1990) and part (b) from Nevzorov (1985).

**Theorem 2.13** In a random $F^\alpha$ model with independent record indicators the following properties hold.

(a) $n - A(T_n), (A(T_n) - n)^2 - A(T_n) + B(T_n)$ and $rac{(1+c)^n}{\prod_{k=1}^{n} (1+\frac{c}{\alpha_1^* + \ldots + \alpha_k^*})}$ (for any $c > 0$) are martingales.

(b) If $p_n \to 0$ then for any fixed $k \geq 2$ the random variables $\frac{S^*(T_n)}{S^*(T_{n+1})}, \ldots, \frac{S^*(T_{n+k-1})}{S^*(T_{n+k})}$ are asymptotically independent and $\lim_{n \to \infty} P\left(\frac{S^*(T_n)}{S^*(T_{n+1})} \leq x\right) = x$ for all $0 < x < 1$.

### 2.7 Asymptotic properties of record values

Here we will explore the possible limit distributions for record values from a random $F^\alpha$ model. We start by assuming $F$ to be standard exponential and then extend our results to an arbitrary continuous c. d. f. $F$. Let $M(n) = \max\{X_1, \ldots, X_n\}$ and

$$A_1(n) = \sum_{i=1}^{n} \pi_i, B_1(n) = \sum_{i=1}^{n} \pi_i^2, D_1^2(n) = A_1(n) - B_1(n), \quad (2.24)$$

**Lemma 2.14** In a proper random $F^\alpha$ model, if $F(x) = 1 - e^{-x}, x > 0$ then $P(R_n - \log S(T_n) \leq x) \to \exp\{-\exp\{-x\}\}$. 

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Proof:

\[ P(M(k) - \log S(k) \leq x | T_n = k) \]

\[ = E^{\alpha's} [P(M(k) - \log S(k) \leq x | T_n = k, \alpha_1, \alpha_2, \ldots)] \]

Since given the \( \alpha \)'s, \( M(k) \) and \( \{T_n = k\} = \{I_1 + \cdots + I_{k-1} = n - 1, I_k = n\} \) are independent:

\[ P(M(k) - \log S(k) \leq x | T_n = k) \]

\[ = E[P(M(k) - \log S(k) \leq x | \alpha_1, \alpha_2, \ldots)] \]

\[ = E \left[ (1 - e^{-x - \log S(k)})^{S(k)} \right] = E \left[ \left( 1 - \frac{e^{-x}}{S(k)} \right)^{S(k)} \right] \]

\[ P(M(k) - \log S(k) \leq x | T_n = k) = \exp\{-\exp\{-x\}\} E \left[ \exp\{O\left(\frac{1}{S(k)}\right)\} \right]. \quad (2.25) \]

Now

\[ P(M(k) - \log S(k) \leq x) \]

\[ = \sum_{k=n}^{\infty} P(R_n - \log S(T_n) \leq x | T_n = k) P(T_n = k) \]

\[ = \exp\{-\exp\{-x\}\} \sum_{k=n}^{\infty} E \left[ \exp\{O\left(\frac{1}{S(k)}\right)\} \right] P(T_n = k). \]

However, since \( S(k) \to \infty \) a. s., \( 1 \leq \exp\{O\left(\frac{1}{S(k)}\right)\} \leq \exp\{O\left(\frac{1}{S(n)}\right)\} \to 1 \) and the result follows. \( \Box \)

**Theorem 2.15** If \( \frac{B_1(n)}{\sqrt{A_1(n)}} \xrightarrow{P} c_0 \in [0, \infty) \) and \( \sum_{i=2}^{n} \frac{\pi_i^3}{(1-\pi_i^2)^2} \xrightarrow{P} 0 \), where \( A_1, B_1 \) are as in (2.24), then \( \frac{\log S(T_n) - n}{\sqrt{n}} - \frac{c_0}{2} \xrightarrow{L} N(0,1). \)
Proof: Using a Taylor expansion, we can derive that

\[ \log S(n) = -\log \left( \frac{\alpha_1}{S(n)} \right) \]

\[ = \sum_{i=2}^{n} -\log(1 - \pi_i) \]

\[ = \sum_{i=2}^{n} \pi_i + \frac{1}{2} \sum_{i=2}^{n} \pi_i^2 + \frac{1}{3} \sum_{i=2}^{n} \frac{\pi_i^3}{(1 - \theta_i)^3} \text{ with } \theta_i \in (0, \pi). \]

In view of the inequality \( 0 \leq \sum_{i=2}^{n} \frac{\pi_i^3}{(1 - \theta_i)^3} \leq \sum_{i=2}^{n} \frac{\pi_i^3}{(1 - \pi_i)^3} \), this implies that

\[ \log S(n) \sim A_1(n) + \frac{c_0}{2} \sqrt{A_1(n)}. \]

Hence

\[ \frac{D_1(n)}{\sqrt{\log S(n)}} \xrightarrow{P} 1, \quad \frac{A_1(n) - \log S(n)}{\sqrt{\log S(n)}} \xrightarrow{P} \frac{c_0}{2}. \]

Combining this with (2.18) from the proof of Theorem 2.9 gives

\[ \frac{N_n - \log S(n)}{\sqrt{\log S(n)}} - \frac{c_0}{2} \xrightarrow{L} N(0, 1). \]

The result now follows along the lines of the second part of the proof of Theorem 2.9.

□

Remark: Loosely speaking, the theorem is intended for the case where \( \pi_n \) converges to zero or, in other words, where the record rate approaches zero. Of course, it cannot go to zero too fast, then the model would not be proper; nor too slow, then \( \frac{B_1(n)}{\sqrt{A_1(n)}} \) would diverge, and there can be a (sparse) subsequence \( n_k \) such that \( \pi_{n_k} \) does not approach zero. However, if \( \pi_n \to 0 \) a. s., \( \frac{B_1(n)}{\sqrt{A_1(n)}} \xrightarrow{P} c_0 \in [0, \infty) \) trivially implies

\[ \sum_{i=2}^{n} \frac{\pi_i^2}{(1 - \pi_i)^3} \xrightarrow{P} 0. \]

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Corollary 2.16 When \( F(x) = 1 - e^{-x}, x > 0 \), under the assumptions of Theorem 2.15,
\[
\frac{R_n - n}{\sqrt{n}} - \frac{c_0}{2} \overset{\mathcal{L}}{\to} N(0, 1).
\]

Proof:
\[
\frac{R_n - n}{\sqrt{n}} - \frac{c_0}{2} = \frac{R_n - \log S(T_n)}{\sqrt{n}} + \left( \frac{\log S(T_n) - n - c_0}{2} \right).
\]

By Lemma 2.14 the first part goes to zero in probability. □

The next theorem applies if, loosely speaking, \( \pi_n \) converges to \( p \in (0, 1) \).

Theorem 2.17 If \( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\pi_i - p) \overset{P}{\to} c_1 \in \mathbb{R}, \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{\pi_i - p}{1 - \pi_i} \right)^2 \overset{P}{\to} 0 \) with \( p \in (0, 1) \) then
\[
\log S(T_n) - \frac{np(-\log(1-p))}{\sqrt{n}(1-p(-\log(1-p)))} = \frac{c_1}{\sqrt{p(1-p)}} \left( 1 - \frac{p}{-(1-p)\log(1-p)} \right) \overset{\mathcal{L}}{\to} N(0, 1).
\]

Proof: Using a Taylor expansion we can conclude that
\[
\log S(n) \sim -\log \left( \frac{\alpha_1}{S(n)} \right)
\]
\[
= \sum_{i=2}^{n} -\log(1 - \pi_i)
\]
\[
= -(n-1)\log(1-p) + \sum_{i=2}^{n} -\log \left( 1 - \frac{\pi_i - p}{1 - p} \right)
\]
\[
= -(n-1)\log(1-p) + \frac{1}{1-p} \sum_{i=2}^{n} (\pi_i - p) + \frac{1}{(1-p)^2} \sum_{i=2}^{n} \frac{(\pi_i - p)^2}{(1-\theta_i)^2}
\]
(2.26)

with \( \theta_i \in \left( 0, \frac{\pi_i - p}{1-p} \right) \cup \left( \frac{\pi_i - p}{1-p}, 0 \right) \). However,
\[
\left( \frac{\pi_i - p}{1 - \theta_i} \right)^2 \leq \max \left\{ (\pi_i - p)^2, (1-p)^2 \left( \frac{\pi_i - p}{1 - \pi_i} \right)^2 \right\} \leq \left( \frac{\pi_i - p}{1 - \pi_i} \right)^2.
\]

In view of (2.26) this implies
\[
\log S(n) \sim -n\log(1-p) + \frac{c_1}{1-p} \sqrt{n} + o(\sqrt{n}).
\]
(2.27)
Also,

\[ D_1^2(n) = \sum_{i=1}^{n} \pi_i - \sum_{i=1}^{n} \pi_i^2 \]

\[ = np(1-p) - \sum_{i=1}^{n}(\pi_i - p)^2 + (2p - 1) \sum_{i=1}^{n}(p - \pi_i) \]

\[ D_1^2(n) \sim np(1-p) + O(\sqrt{n}). \quad (2.28) \]

Hence, from (2.27) and (2.28) we obtain

\[ \frac{D_1(n)}{\sqrt{\log S(n)}\sqrt{\frac{p(1-p)}{-\log(1-p)}}} \xrightarrow{p} 1 \text{ and } \quad (2.29) \]

\[ \frac{A_1(n) - \frac{p}{-\log(1-p)} \log S(n)}{\sqrt{\log S(n)}\sqrt{\frac{p(1-p)}{-\log(1-p)}}} \sim \sum_{i=1}^{n}(\pi_i - p) - \frac{c_1 \sqrt{n}}{-(1-p)\log(1-p)} + o(\sqrt{n}) \]

\[ \sim \frac{c_1}{\sqrt{p(1-p)}} \left( 1 - \frac{p}{-(1-p)\log(1-p)} \right). \quad (2.30) \]

Combining (2.29) and (2.30) with (2.18) gives

\[ \frac{N_n - \frac{p}{-\log(1-p)} \log S(n)}{\sqrt{\log S(n)}\sqrt{\frac{p(1-p)}{-\log(1-p)}}} - \frac{c_1}{\sqrt{p(1-p)}} \left( 1 - \frac{p}{-(1-p)\log(1-p)} \right) \xrightarrow{c} N(0,1). \]

The result follows from this along the lines of the second part of the proof of Theorem 2.9. □

We can easily conclude the following.

**Corollary 2.18** When \( F(x) = 1 - e^{-x}, x > 0, \) under the assumptions of Theorem 2.17,

\[ \frac{R_n - \frac{p}{-\log(1-p)}}{\sqrt{n}\sqrt{1-p}\frac{-\log(1-p)}{p}} - \frac{c_1}{\sqrt{p(1-p)}} \left( 1 - \frac{p}{-(1-p)\log(1-p)} \right) \xrightarrow{c} N(0,1). \]
What limit distributions are possible if $F$ is not exponential? For fixed $\alpha$'s, let 
\[ \{Y_n\}_{n \geq 1} \] be random variables with c. d. f. $G_n(x) = (1 - e^{-x})^{\alpha_n} (x > 0)$ and 
\[ \{X_n\}_{n \geq 1} \] be random variables with c. d. f. $F_n(x) = F^{\alpha_n}(x)$. ($F$ continuous), $n \geq 1$. It follows that

\[ X_n \overset{d}{=} F_n^{-1}(G_n(Y_n)) = F^{-1}(1 - e^{-Y_n}) \]

and 
\[ R_n \overset{d}{=} F^{-1}(1 - e^{-R_n^*}) \]

where $R_n^*$ is the $n$th record value of the $Y$ sequence.

Returning to random $\alpha$'s this means:

\[ P(R_n \leq x | \alpha_1, \alpha_2, \ldots) = P\left(F^{-1}(1 - e^{-R_n}) \leq x | \alpha_1, \alpha_2, \ldots\right) \text{ a.s. } \alpha. \]

It remains to take expectations with respect to the $\alpha$'s and we get

\[ R_n \overset{d}{=} F^{-1}(1 - e^{-R_n^*}). \tag{2.31} \]

Let $\gamma(x) = -\log(1 - F(x))$; then (2.31) becomes

\[ R_n^* \overset{d}{=} \gamma(R_n). \tag{2.32} \]

Once we have (2.32) and the convergence results of corollaries 2.16, 2.18 for the exponential case, we can discuss the possible limit distributions for records from a random $F^\alpha$ model. Let us begin with the following lemma.

**Lemma 2.19** Suppose either (2.33) or (2.34) below holds.

\[ \frac{B_1(n)}{\sqrt{A_1(n)}} = \frac{\sum_{i=1}^n \pi_i^2}{\sqrt{\sum_{i=1}^n \pi_i}} \overset{P}{\to} c_0 \in [0, \infty), \quad \frac{\sum_{i=1}^n \pi_i^3}{\sqrt{\sum_{i=1}^n \pi_i}} \overset{P}{\to} 0 \text{ or} \tag{2.33} \]

\[ \frac{1}{\sqrt{n}} \sum_{i=1}^n (\pi_i - p) \overset{P}{\to} c_1 \in \mathbb{R}, \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{\pi_i - p}{1 - \pi_i}\right)^2 \overset{P}{\to} 0, \quad p \in (0, 1). \tag{2.34} \]

Then there exist $a(n)$ and $b(n) > 0$ such that

\[ \frac{R_n - a(n)}{b(n)} \overset{c}{\to} H \tag{2.35} \]
where $H$ is a nondegenerate c. d. f. iff there exists a non-decreasing function $g(x)$ with more than one point of increase such that

$$
\frac{\gamma(a(n) + b(n)x) - n - \frac{c_0}{2} \sqrt{n}}{\sqrt{n}} \rightarrow g(x) \text{ or }
$$

$$
\frac{\gamma(a(n) + b(n)x) - \frac{a}{p}(-\log(1-p))}{\sqrt{n} / 1 - p \log(1-p) \over p} - \frac{c_1}{\sqrt{p(1-p)}} \left(1 - \frac{p}{-(1-p) \log(1-p)}\right) \rightarrow g(x),
$$

respectively at all continuity points $x$ of $g(x)$. In this case $H(\cdot)$ is of the form

$$
H(x) = \Phi(g(x)).
$$

**Proof:** In view of Corollary 2.16 and

$$
P(R_n \leq a(n) + b(n)x) = P \left( \frac{R_n^* - n - \frac{c_0}{2} \sqrt{n}}{\sqrt{n}} \leq \frac{\gamma(a(n) + b(n)x) - n - \frac{c_0}{2} \sqrt{n}}{\sqrt{n}} \right)
$$

we see that (2.35) holds if (2.36) and (2.38) are valid. The second part is analogous.

\[ \Box \]

Let $M_n$ be the maximum of $n$ i. i. d. random variables with c. d. f. $F$. We write

$F \in D_M(G, a(n), b(n))$ and say that $F$ is in the domain of maximal attraction of the extreme value c. d. f. $G$ if

$$
P \left( \frac{M_n - \hat{a}(n)}{\hat{b}(n)} \right) \leq x = F^n(\hat{a}(n) + \hat{b}(n)x) \rightarrow G(x)
$$

\[ \forall \hat{a}(n), \hat{b}(n) > 0 : \frac{\hat{b}(n)}{\hat{b}(n)} \rightarrow 1, \frac{\hat{a}(n) - a(n)}{\hat{b}(n)} \rightarrow 0. \]

Similarly, we write $F \in D_R(H, a(n), b(n))$ and say that $F$ is in the domain of record attraction of c. d. f. $H$ if

$$
P \left( \frac{R_n - \hat{a}(n)}{\hat{b}(n)} \right) \leq x \rightarrow H(x) \text{ for } \hat{a}(n), \hat{b}(n) \text{ as above.}$$
Further let $\delta > 0$ and

$$
\Lambda(x) = \exp(-\exp(-x)), \ x \in \mathbb{R}
$$

$$
\Phi_{\delta}(x) = \exp(-x^{-\delta}), \ x > 0
$$

$$
\Psi_{\delta}(x) = \exp(-(x)^{-\delta}), \ x < 0
$$

$$
N_{1,\delta,c}(x) = \Phi(\delta \log x - c), \ x > 0, \ c \in \mathbb{R}
$$

$$
N_{2,\delta,c}(x) = \Phi(-\delta \log(-x) - c), \ x < 0, \ c \in \mathbb{R}
$$

$$
\hat{F}(x) = 1 - \exp\left\{-(-\log(1 - F(x)))^{\frac{1}{2}}\right\} = 1 - \exp\left\{-\gamma^{\frac{1}{2}}(x)\right\}. \quad (2.39)
$$

$\hat{F}(x)$ is called the associated c. d. f. (Resnick, 1973b). Now we are ready to state the duality theorem. The proof is similar to the one in Resnick (1973b), but we will present it here in full detail for completeness.

**Theorem 2.20** If conditions in (2.33) or (2.34) hold the following dualities are true:

$$
F \in DR(N, a(n), b(n)) \iff \hat{F} \in DM(\Lambda, a(\theta/\log(n)^2) - c_2 b(\theta/\log(n)^2), c_3^{-1} b(\theta/\log(n)^2))
$$

$$
F \in DR(N_{1,\delta,c_2}, a(n), b(n)) \iff \hat{F} \in DM(\Phi_{\delta}, a(\theta/\log(n)^2), b(\theta/\log(n)^2))
$$

$$
F \in DR(N_{2,\delta,c_2}, a(n), b(n)) \iff \hat{F} \in DM(\Psi_{\delta}, a(\theta/\log(n)^2), b(\theta/\log(n)^2))
$$

where if (2.33) holds, $\theta = 1$, $c_2 = \frac{a}{2}$ and $c_3 = \frac{1}{2}$ and if (2.34) holds, $\theta = -\frac{p}{\log(1-p)}$,

$c_2 = \frac{c_1}{\sqrt{p(1-p)}} \left(1 - \frac{p}{-(1-p)\log(1-p)}\right)$ and $c_3 = \sqrt{\frac{-(1-p)\log(1-p)}{4p}}$.

**Proof:** Let us first consider the conditions in (2.33). From (2.36) we have

$$
\gamma(a_{n+1} + b_{n+1} x) - n - \frac{a}{2}\sqrt{n}
$$
\[ \gamma(a_{n+1} + b_{n+1}x) - (n + 1) - \frac{\alpha n}{2} \sqrt{n + 1} \left( \frac{n + 1}{n} \right)^{\frac{1}{2}} + \frac{1}{\sqrt{n}} + \frac{c_0}{2} \left[ \left( \frac{n + 1}{n} \right)^{\frac{1}{2}} - 1 \right] \]

\[ \rightarrow g(x). \]

Hence \( P(R_n \leq a_{n+1} + b_{n+1}x) \rightarrow \Phi(g(x)) \) and this coupled with (2.35) and the convergence-to-types theorem (see for example Billingsley (1986, Theorem 14.2)) gives \( b(n) \sim b_{n+1}, \frac{a_{n+1} - a_n}{b(n)} \rightarrow 0 \). Now we can switch from the discrete parameter \( n \) to a continuous parameter \( s \) and rewrite (2.36) as

\[ \frac{\gamma(a(s) + b(s)x) - s}{\sqrt{s}} \rightarrow g(x) + \frac{c_0}{2} \tag{2.40} \]

at all continuity points of \( g(x) \). From (2.40) clearly

\[ \gamma(a(s) + b(s)x) \sim s. \tag{2.41} \]

Now

\[ \frac{\gamma(a(s) + b(s)x) - s}{\sqrt{s}} \]

\[ = \left( \gamma^{\frac{1}{2}}(a(s) + b(s)x) - \sqrt{s} \right) \frac{\gamma^{\frac{1}{2}}(a(s) + b(s)x) + \sqrt{s}}{\sqrt{s}} \]

Consequently, (2.40) and (2.41) together imply

\[ \left( \gamma^{\frac{1}{2}}(a(s) + b(s)x) - \sqrt{s} \right) \rightarrow \frac{1}{2} \left( g(x) + \frac{c_0}{2} \right). \tag{2.42} \]

Following similar steps under Conditions (2.34) (using (2.37) instead of (2.36)) we obtain

\[ \sqrt{s} \] 

\[ \gamma^{\frac{1}{2}}(a(s) + b(s)x) - \frac{\sqrt{s}}{\sqrt{p}} \sqrt{-\log(1 - p)} \]

\[ \rightarrow \sqrt{\frac{(1 - p)(-\log(1 - p))}{4p}} \left( g(x) + \frac{c_1}{\sqrt{p(1 - p)}} \left[ 1 - \frac{p}{(1 - p) \log(1 - p)} \right] \right). \tag{2.43} \]
From (2.42) and (2.43) it follows that

\[ \gamma^{\frac{1}{2}}(a(s) + b(s)x) - \frac{\sqrt{s}}{\sqrt{\theta}} \rightarrow c_3(g(x) + c_2), \]

or equivalently

\[ \exp\left(\frac{\sqrt{s}}{\sqrt{\theta}}\right) \exp\left(-\gamma^{\frac{1}{2}}(a(s) + b(s)x)\right) \rightarrow \exp(-c_3(g(x) + c_2)), \quad (2.44) \]

where \( \theta, c_2, c_3 \) are given above in the statement of the theorem. By substituting \( y = \exp\left(\frac{\sqrt{s}}{\sqrt{\theta}}\right) \) this becomes

\[ y \left(1 - \hat{F} \left(\theta (\log y)^2\right) + b \left(\theta (\log y)^2\right) x\right) \rightarrow \exp(-c_3(g(x) + c_2)) \quad (2.45) \]

or since \( 1 - \hat{F}(z) \sim -\log \hat{F}(z) \) for \( z \to \infty \), this is equivalent to

\[ \hat{F}' \left(\theta (\log y)^2\right) + b \left(\theta (\log y)^2\right) x \rightarrow \exp(-\exp(-c_3(g(x) + c_2))) \quad (2.46) \]

(or \( P(M_n \leq c(n) + d(n)x) \rightarrow \exp(-\exp(-c_3(g(x) + c_2))) \) where \( c(n) = a(\theta (\log n)^2) \), \( d(n) = b(\theta (\log n)^2) \) and \( M_n \) is the maximum of \( n \) variables from an i. i. d. sequence with c. d. f. \( \hat{F} \) if \( y = n \) is an integer.)

By Lemma 2.19, (2.46) is equivalent to \( F \in D_R(H, a(n), b(n)) \) where \( H(x) = \Phi(g(x)) \).

However, (2.46) means that \( \hat{F} \) is in the domain of attraction of an extreme value distribution \( G(\cdot) \) (see Galambos (1987), section 2.4) and

\[ G(x) = \exp(-\exp(-c_3(g(x) + c_2))). \quad (2.47) \]

The only possibilities for \( G \) are \( \Lambda, \Phi_\delta, \Psi_\delta \).

For \( G = \Lambda, g(x) \) is a linear function and the constants \( c_2, c_3 \) in (2.47) cause location and scale changes, respectively. In this case we can rewrite (2.46) as (technically by changing \( g \) to say \( g_1 \))

\[ \hat{F} \left(a \left(\theta (\log y)^2\right) + c_2b \left(\theta (\log y)^2\right) + c_3^{-1}b \left(\theta (\log y)^2\right) x\right) \rightarrow \exp(-\exp(-g(x))) = \Lambda(x) \]
and the limit distribution of $\frac{R_n - a(n)}{b(n)}$ is $H(x) = \Phi(g(x)) = \Phi(x)$.

For $G(x) = \Phi_\delta(x) = \exp(-x^{-\delta})$, $x > 0$, we get from (2.47) that

$$-\exp(-c_3(g(x) + c_2)) = -x^{-\delta}$$

$$g(x) = c_3^{-1}\delta \log x - c_2.$$

We will substitute $\delta$ for $c_3^{-1}\delta$ and hence rewrite (2.46) as

$$\tilde{F}(a(\theta \log(y)) + b(\theta \log(y)^2) x) \to \exp(-x^{-\frac{\delta}{2}}) = \Phi_{c_3\delta}(x).$$

Then $H(x) = \Phi(g(x)) = \Phi(\delta \log x - c_2)$. The case $G = \Psi_\delta$ is equivalent to $G = \Phi_\delta$.

\[\Box\]

**Remark:** The parameter $c_2$ in the limit distributions $N_{1,\delta,c_2}$ and $N_{2,\delta,c_2}$ expands the class of possible limit distributions for $R_n$. Note that for the classical model, $c_2 \equiv 0$.

**Theorem 2.21** Under the assumptions of Theorem 2.20 the norming constants $a(n), b(n)$ for $R_n$ can be taken to be:

1. $F \in D_R(\Phi, a(n), b(n)) : a(n) = \gamma^{-1}\left(\frac{n}{\theta}\right) + c_2 b(n),$

   $$b(n) = c_3 \left(\gamma^{-1}\left(1 + \sqrt{\frac{n}{\theta}}\right)^2 - \gamma^{-1}\left(\frac{n}{\theta}\right)\right)$$

2. $F \in D_R(N_{1,\delta,c_2}, a(n), b(n)) : a(n) = 0, b(n) = \gamma^{-1}\left(\frac{n}{\theta}\right)$

3. $F \in D_R(N_{2,\delta,c_2}, a(n), b(n)) : a(n) = F^{-1}(1), b(n) = F^{-1}(1) - \gamma^{-1}\left(\frac{n}{\theta}\right)$

where

$$\gamma^{-1}\left(\frac{n}{\theta}\right) = F^{-1}\left(1 - \exp\left(-\frac{n}{\theta}\right)\right),$$

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\[ \gamma^{-1} \left( \left( 1 + \sqrt{\frac{n}{\theta}} \right)^2 \right) = F^{-1} \left( 1 - \exp \left( - \left( 1 + \sqrt{\frac{n}{\theta}} \right)^2 \right) \right). \]

**Proof:** From (2.39) it follows that

\[ \hat{F}^{-1}(x) = \gamma^{-1} \left( ( - \log(1 - x))^2 \right) = F^{-1} \left( 1 - \exp \left( - ( - \log(1 - x))^2 \right) \right). \quad (2.48) \]

Possible forms of the norming constants that ensure convergence of the maximum to one of the extreme value distributions are known (see for example Galambos (1987) or Arnold, Balakrishnan and Nagaraja (1992), Theorem 8.3.4). In view of the latter reference and Theorem 2.20 we can take for \( F \in D_R(\Phi, a(n), b(n)) \):

\[
\begin{align*}
& a \left( \theta \left( \log n \right)^2 \right) - c_2b \left( \theta \left( \log n \right)^2 \right) = \hat{F}^{-1} \left( 1 - \frac{1}{n} \right), \\
& c_3^{-1}b \left( \theta \left( \log n \right)^2 \right) = \hat{F}^{-1} \left( 1 - \frac{1}{ne} \right) - \hat{F}^{-1} \left( 1 - \frac{1}{n} \right),
\end{align*}
\]

for \( F \in D_R(N_1, \delta, c_2, a(n), b(n)) \):

\[
\begin{align*}
& a \left( \theta \left( \log n \right)^2 \right) = 0, \ b \left( \theta \left( \log n \right)^2 \right) = \hat{F}^{-1} \left( 1 - \frac{1}{n} \right),
\end{align*}
\]

for \( F \in D_R(N_2, \delta, c_2, a(n), b(n)) \):

\[
\begin{align*}
& a \left( \theta \left( \log n \right)^2 \right) = \hat{F}^{-1}(1), \ b \left( \theta \left( \log n \right)^2 \right) = \hat{F}^{-1}(1) - \hat{F}^{-1} \left( 1 - \frac{1}{n} \right).
\end{align*}
\]

It is easy to see that the result follows from this by (2.48). □

The following three theorems give necessary and sufficient conditions for \( F \) to be in the domain of attraction of one of the three possible limit distributions. Their proofs are equivalent to those in Resnick (1973b) and will not be given here.
Theorem 2.22 If (2.33) or (2.34) holds, the following statements are equivalent:

(i) \( F(x) \in D_R(N_{1,\delta,c}) \) (with some \( a(n), b(n) \))

(ii) \( 1 - \hat{F}(x) = \exp \left\{ -\gamma^{\frac{1}{2}}(x) \right\} = x^{-c_3\delta}L(x), \) \( L \) a slowly varying function.

(iii) \( \gamma^{\frac{1}{2}}(x) = c(x) + \int_1^x a(t)\frac{dt}{t} \) where \( c(x) \to c, |c| < \infty \) and

\[
a(x) \to c_3 \delta \text{ as } x \to \infty
\]

(iv) \( \lim_{x \to \infty} \frac{\gamma(x) - \gamma(x)}{\gamma(x)} = 2c_3\delta \log t. \)

Theorem 2.23 Let the right end of \( F, F^{-1}(1) < \infty. \) If (2.33) or (2.34) holds, the following statements are equivalent:

(i) \( F(x) \in D_R(N_{2,\delta,c}) \)

(ii) \( 1 - \hat{F}\left(F^{-1}(1) - \frac{1}{2}\right) = \exp \left\{ -\gamma^{\frac{1}{2}}\left(F^{-1}(1) - \frac{1}{2}\right) \right\} = x^{-c_3\delta}L(x), \)

where \( L(x) \) is a slowly varying function.

(iii) \( \gamma^{\frac{1}{2}}\left(F^{-1}(1) - \frac{1}{2}\right) = c(x) = \int_1^x a(t)\frac{dt}{t}, \) where \( c(x) \to c, |c| < \infty. \)

\[
a(x) \to c_3 \delta \text{ as } x \to \infty.
\]

(iv) \( \lim_{x \to 0^+} \frac{\gamma\left(F^{-1}(1) - x\right) - \gamma\left(F^{-1}(1) - \frac{1}{2}\right)}{\gamma\left(F^{-1}(1) - x\right)} = -2c_3\delta \log t \) for all \( t > 0. \)

Theorem 2.24 If (2.33) or (2.34) holds, the following statements are equivalent:

(i) \( F(x) \in D_R(\Phi) \)

(ii) For all \( x > 0, y > 0, \)

\[
\lim_{s \to \infty} \frac{\gamma^{-1}(\log(sz)^2) - \gamma^{-1}(\log sz)^2}{\gamma^{-1}(\log sz)^2} = \frac{\log x}{\log y}
\]

(iii) For all \( x, \) \( \lim_{s \to \infty} \frac{\gamma^{-1}(s+x\sqrt{2}) - \gamma^{-1}(s)}{\gamma^{-1}(s+x\sqrt{2})} = x \)

Remark: Resnick (1973a) investigated the possibility of both the maximum and the record sequence having limit distribution. His results (in particular Theorems 8-10) also hold in our case.
Example 2.8 (Weibull distribution) If $F(x) = 1 - e^{-x^c}$, $x > 0$, $c > 0$, then

$$\gamma(x) = x^c, \quad \hat{F}(x) = 1 - e^{-x^c}.$$ 

$$\lim_{s \to \infty} \gamma^{-1}(s + x\sqrt{s}) - \gamma^{-1}(s) = \lim_{s \to \infty} \frac{(s + x\sqrt{s})^{\frac{1}{c}} - s^{\frac{1}{c}}}{(s + \sqrt{s})^{\frac{1}{c}} - s^{\frac{1}{c}}}$$

$$= \lim_{s \to \infty} \frac{(1 + \frac{x}{\sqrt{s}})^{\frac{1}{c}} - 1}{(1 + \frac{1}{\sqrt{s}})^{\frac{1}{c}} - 1} = x.$$ 

Consequently, by Theorem 2.24 the records are asymptotically normal.

Example 2.9 (logistic distribution) If $F(x) = \frac{e^x}{1 + e^x}$, $x \in \mathbb{R}$ then

$$\gamma(x) = -\log \left( \frac{1}{1 + e^x} \right) = \log(1 + e^x), \quad \gamma^{-1}(x) = \log(e^x - 1)$$

$$\lim_{s \to \infty} \gamma^{-1}(s + x\sqrt{s}) - \gamma^{-1}(s) = \lim_{s \to \infty} \frac{\log \left( \frac{e^{x\sqrt{s} - 1}}{e^x - 1} \right)}{\log \left( \frac{e^{x\sqrt{s} - 1}}{e^x - 1} \right)} = x.$$ 

Again, the records are asymptotically normal by Theorem 2.24.

Example 2.10 (Pareto distribution) If $F(x) = 1 - \left( \frac{x}{c} \right)^d$, $x > c; \ c, d > 0$ then

$$\gamma(x) = d \log \left( \frac{x}{c} \right), \quad \gamma^{-1}(x) = ce^\frac{x}{d}$$

$$\lim_{x \to \infty} \frac{\gamma(tx) - \gamma(x)}{\sqrt{\gamma(x)}} = \lim_{x \to \infty} \frac{\log t}{\sqrt{d \log \left( \frac{t}{c} \right)}} = 0.$$ 

Therefore by Theorem 2.22 $F \not\in D_R(\mathcal{N}_1, \delta_c)$. Now,

$$\lim_{s \to \infty} \frac{\gamma^{-1}(s + x\sqrt{s}) - \gamma^{-1}(s)}{\gamma^{-1}(s + \sqrt{s} - \gamma^{-1}(s))} = \lim_{s \to \infty} \frac{e^{\frac{e^{x\sqrt{s}} - 1}{d}} - e^{\frac{1}{d}}}{e^{\frac{e^{x\sqrt{s}} - 1}{d}} - e^{\frac{x}{d}}}$$

$$= \lim_{s \to \infty} e^{\frac{d}{x}\left(x - 1\right)} = \begin{cases} \infty & x > 1 \\ 0 & x < 1 \end{cases}.$$
Hence, $F \not\in D(\Phi)$. Neither is it in the domain of attraction of $N_{2,\delta,c}$ since $F^{-1}(1) = \infty$.

It follows that there do not exist norming constants $a(n)$, $b(n)$ such that $\frac{R_n - a(n)}{b(n)}$ has a nondegenerate limit.

**Example 2.11** $F(x) = 1 - e^{-a^2[\log x]^2}$, $x > 1$, $a > 0$.

Then $\gamma(x) = a^2[\log x]^2$ and

$$\lim_{x \to \infty} \frac{\gamma(tx) - \gamma(x)}{\sqrt{\gamma(x)}} = \lim_{x \to \infty} \frac{a^2[\log tx]^2 - a^2[\log x]^2}{a \log x}$$

$$= \lim_{x \to \infty} \frac{2 \log t \log x + [\log t]^2}{\log x}$$

$$= 2a \log t \text{ for all } t > 0.$$

Hence it follows from Theorems 2.21 and 2.22 that normed by $a(n) = 0$, $b(n) = e^{-\sqrt{\gamma}}$ the record value distribution converges to $N_{1, c_1^{-1}a, c_2}$.

**Example 2.12** (standard normal distribution)

When $F(x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$, for $x \to \infty$

$$1 - \Phi(x) \sim \frac{\phi(x)}{x} = \frac{1}{\sqrt{2\pi x}} e^{-\frac{x^2}{2}}$$

$$\gamma(x) = -\log(1 - \Phi(x)) \sim \frac{x^2}{2} + \log x$$

$$\gamma^{-1}(x) \sim \sqrt{2x}.$$  

Thus,

$$\lim_{s \to \infty} \frac{\gamma^{-1}(s + x\sqrt{s}) - \gamma^{-1}(s)}{\gamma^{-1}(s + \sqrt{s}) - \gamma^{-1}(s)} = \lim_{s \to \infty} \frac{\sqrt{s + x\sqrt{s} - \sqrt{s}}}{\sqrt{s + \sqrt{s} - \sqrt{s}}}$$

$$= \frac{\sqrt{x}}{1} = x.$$  

Hence, the records are asymptotically normal.
CHAPTER 3

THE SECRETARY PROBLEM FOR RANDOM $F^\alpha$ MODELS

3.1 The classical secretary problem

The classical (no-information) secretary problem is as follows. A known number of items (candidates for a secretary position, girl/boy-friends, dowries,...) is to be presented one by one in random order, all $n!$ possible orders being equally likely. The observer is able at any time to rank the items that have so far been presented in order of desirability. As each item is presented he must either accept it, that stops the process, or reject it, in which case the next item in the sequence is presented and the observer faces the same choice as before. If the last item is presented, he must accept it. The observer aims to maximize the probability that the chosen item is indeed the best. The problem is also known as the marriage problem, the dowry problem, the candidate problem and the apartment problem. There are many variations of the classical secretary problem.

Only this secretary problem has been considered for the $F^\alpha$ setup. In the fixed $F^\alpha$ context Pfeifer (1989) has given the optimal as well as an asymptotically optimal solution that is easy to obtain. For a no-information secretary problem the optimal
solution depends only on record times and counts. Therefore it follows from an earlier conclusion that finding solutions in fixed $F^a$ and random $F^a$ cases are equivalent.

Let $S_{n,c}$ be the number of records between observation $c$ and $n$, i.e., $S_{n,c} = \sum_{i=c}^{n} I_i$. Let

$$a_k = E\pi_k \prod_{i=k+1}^{n} (1 - E\pi_i), \quad k = 1, \ldots, n - 1, \quad a_n = E\pi_n;$$

$$A(k) = \prod_{i=k+1}^{n} (1 - E\pi_i), \quad k = 1, \ldots, n - 1; \quad A(n) = 1.$$  

Following Pfeifer (1989) the optimal strategy is to take the first record after $c^* - 1$ observations where $c^*$ is the number which maximizes

$$P(S_{n,c} = 1) = \left(\prod_{k=c}^{n} (1 - E\pi_k)\right) \sum_{k=c}^{n} \frac{E\pi_k}{1 - E\pi_k}.$$  

When $p_n = E\pi_n \to 0$ an easier asymptotic solution uses $c^* = A^{-1}(e^{-1})$. The asymptotic winning probability is $e^{-1}$. When $p_n \to p \in (0, 1)$:

$$P(S_{n,c} = 1) = \prod_{i=c}^{n} (1 - p_i) \sum_{i=c}^{n} \frac{p_i}{1 - p_i} 
\approx (1 - p)^{n-c+1} \cdot (n - c + 1) \frac{p}{1 - p}.$$  

This is maximized by $c^* = n - 1 - \frac{1}{\log(1 - p)}$. The asymptotic winning probability is

$$P(S_{n,c} = 1) = \frac{p}{e(1 - p) \log(1 - p)}$$  

which is greater than $e^{-1}$.

### 3.2 Arrivals according to a Poisson process with known intensity

The goal is the same as in the classical problem. Only now we do not have a fixed number of candidates, instead we have a fixed time $T$ during which candidates arrive
according to a Poisson process with intensity $\lambda$. Cowan and Zabczyk (1978) solved this problem for the i. i. d. case. I rely on their ideas for the following derivation in the random $F^\alpha$ case.

Naturally all decisions will take place only at the epochs (jumps) of the Poisson process. Indeed all non-trivial decisions will occur when the interviewed candidate is the best of those to date. We shall call these epochs of the Poisson process *serious epochs*. Consider the situation when the $m$th epoch is serious and a time $t$ remains (i.e. $T - t$ has elapsed). We define a stochastic process $\{X_n, n \geq 0\}$, connected with these serious epochs. We set $X_0$ equal to a state labeled $\delta$ with probability one. For $n = 1, 2, \ldots$ we define $X_n$ by

\begin{align*}
\{X_n = (m, t)\} &\iff \\
\{X_n = \partial\} &\iff \{\text{there are less than } n \text{ serious epochs in } [0, T]\}.
\end{align*}

Here $(m, t) \in \{n, n + 1, \ldots\} \times [0, T]$ while $\partial$ is a label for an absorbing state. In fact, $\{X_n\}$ is a homogeneous Markov process with parameter space $\{n : n = 0, 1, 2, \ldots\}$ and state space $X = \{\delta\} \cup \{\partial\} \cup (\{1, 2, 3, \ldots\} \times [0, T])$. It is a discrete process embedded in continuous time. The process changes state whenever a serious epoch occurs and remains in that state until the next serious epoch or until time $T$. At time $T$, it enters the absorbing state $\partial$. If no candidates arrive in $[0, T]$, the process goes directly from state $\delta$ to state $\partial$. We are able to find the densities of the transition probabilities of this Markov process. Using the notation $p[x, \{y\}] = P(X_{n+1} = y | X_n = x)$, we have

\begin{align*}
p[\delta, \{\partial\}] &= e^{-\lambda T}, \quad p[\partial, \{\partial\}] = 1,
\end{align*}

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\[ p(\delta, \{(m, t)\}) = \begin{cases} \lambda e^{-\lambda(T-t)} & \text{for } m = 1 \\ 0 & \text{for } m > 1 \end{cases} \]

\[ p((m, t), \{\emptyset\}) = \sum_{j=0}^{\infty} \left( \frac{(\lambda t)^j}{j!} e^{-\lambda t} \prod_{k=m+1}^{m+j} (1 - E\pi_k) \right) \]

\[ p((m, t), \{(m + k, t - u)\}) = \left( \prod_{j=m+1}^{m+k} (1 - E\pi_j) \right) E\pi_{m+k} \frac{\lambda^k u^{k-1} e^{-\lambda u}}{(k-1)!} . \]

Let \( \phi(x) \) be the winning probability if we stop the process \( \{X_n\}_{n=0}^{\infty} \) in state \( x \) and \( P\phi(x) = \int_X \phi(y) p[x, \{dy\}] \) the winning probability if we stop at the next record (includes the probability that it exists). If \( \Theta \) is the space of all stopping times, then our objective is to find the \( \theta \in \Theta \) which maximizes \( E(\phi(X_{\theta})) \). Cowan and Zabczyk (1978) show that \( \theta \) is given by \( \theta = \min \{ n : X_n \in G \} \) where \( G = \{ x \in X : P\phi(x) \leq \phi(x) \} \).

\[ \phi(m, t) = \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} \prod_{k=m+1}^{m+j} (1 - E\pi_k) \]

\[ = e^{-\lambda t} \left( 1 + \sum_{j=1}^{\infty} \frac{(\lambda t)^j}{j!} \prod_{k=m+1}^{m+j} (1 - E\pi_k) \right) \]

\[ P\phi(m, t) = e^{-\lambda t} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{(\lambda t)^i}{i!} \frac{E\pi_{m+i}}{1 - E\pi_{m+k}} \frac{(\lambda t)^j}{(j+k)!} \]

\[ = e^{-\lambda t} \sum_{i=1}^{\infty} \frac{(\lambda t)^i}{i!} \left( \prod_{j=m+1}^{m+i} (1 - \pi_j) \right) \sum_{k=1}^{i} \frac{E\pi_{m+k}}{1 - E\pi_{m+k}} \]

Let

\[ a_{im} = \frac{1}{i!} \prod_{j=m+1}^{m+i} (1 - E\pi_j), \quad b_{im} = \sum_{k=1}^{i} \frac{E\pi_{m+k}}{1 - E\pi_{m+k}} \]

then

\[ P\phi(m, t) \leq \phi(m, t) \Leftrightarrow \sum_{i=1}^{\infty} (\lambda t)^i a_{im} b_{im} \leq 1 + \sum_{i=1}^{\infty} (\lambda t)^i a_{im} \]

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\[ h(x) = \phi(m, t) - P\phi(m, t) = 1 + \sum_{n=1}^{\infty} (1 - b_{nm})a_{nm}x^n > 0 \]

where \( x = \lambda t \).

If \( \sum_{k=2}^{\infty} \frac{E\pi_k}{1 - E\pi_k} = \infty \) (which always holds for proper models since \( \sum_{k=2}^{\infty} \frac{E\pi_k}{1 - E\pi_k} \geq \sum_{k=2}^{\infty} E\pi_k = \infty \)) then \( b_{nm} \to \infty \forall m \) as \( n \to \infty \). Since \( b_{nm} \) is also monotone increasing in \( n \), there exists an \( r \) such that:

\[ h(x) = 1 + \sum_{n=1}^{r} (1 - b_{nm})a_{nm}x^n + \sum_{n=r+1}^{\infty} (b_{nm} - 1)a_{nm}x^n \]

with both summations containing positive terms. Note that

\[ h^{(k)}(x) = -\sum_{n=k}^{\infty} (b_{nm} - 1)a'_{nmk}x^{n-k} \text{ for } k > r \]

\[ h^{(k)}(x) = \sum_{n=k}^{r} (1 - b_{nm})a'_{nmk}x^{n-k} - \sum_{n=r+1}^{\infty} (b_{nm} - 1)a'_{nmk}x^{n-k} \]

for \( k \leq r \)

where \( a'_{nmk} = a_{nm} \prod_{i=n-k+1}^{n} i > 0 \).

For each \( k \geq 0 \), \( h^{(k)}(x) \) is negative for sufficiently large \( x \). That is, there exists a sequence of non-negative numbers \( \{y_k\} \) such that \( h^{(k)}(x) < 0 \) for \( x \geq y_k \). Naturally \( y_k = 0 \) for \( k > r \). Thus for \( 0 \leq k \leq r \), \( h^{(k)}(0) > 0 \) and \( h^{(k)}(y_k) < 0 \) and therefore from continuity, there exists at least one root of the equation \( h^{(k)}(x) = 0 \). We now show that there is only one root. Since \( h^{(r+1)}(x) < 0 \) for \( x \geq 0 \), \( h^{(r)}(x) \) is decreasing and \( h^{(r)}(x) = 0 \) has a unique root \( \hat{x}_r \). Also, \( h^{(r)}(x) > 0 \) for \( x < \hat{x}_r \) and \( h^{(r)}(x) < 0 \) for \( x > \hat{x}_r \).

Now \( h^{(r)}(\hat{x}_r) = 0 \) and \( h^{(r+1)}(\hat{x}_r) < 0 \) implies that \( h^{(r-1)} \) achieves its maximum in the interval \([0, y_{r-1}]\) at \( \hat{x}_r \). Since \( h^{(r-1)}(0) > 0 \) and \( h^{(r-1)}(\hat{x}_r) > 0 \) and \( h^{(r-1)} \) has its
unique root, say \( x_{r-1} \), in the interval \([x_r, y_{r-1}]\). This again implies that \( h^{(r-2)} \) has its positive unique maximum in \( x_{r-1} \). Continuing this argument shows that \( h(x) \) has a unique positive root \( x_0 = x_m \) and \( h(x) > 0 \) for \( x < x_m \) and \( h(x) < 0 \) for \( x > x_m \). Therefore

\[
P(\phi(m, t) \leq \phi(m, t)) \Leftrightarrow \lambda t \leq x_m
\]

and the optimal solution is to stop in state \((m, t)\) if \( \lambda t \leq x_m \).

### 3.3 Arrivals according to a Poisson process with unknown intensity

We consider the same problem as in the previous section except that the intensity \( \lambda \) is now unknown in the sense of an improper uniform prior distribution \( \pi(\lambda) = 1 \) for \( \lambda \in (0, \infty) \). Let \( N = N(T) \) be the total number of arrivals in time \( T \) and \( \tau_j \) the time of the \( j \)th arrival \((j \leq N) \). This problem has been considered for i.i.d. observations by Bruss (1987). He derived the posterior distribution on \( N \) given the first \( j \) arrival times to be:

\[
P(N = n | \tau_1, \ldots, \tau_{j-1}, \tau_j = t) = \binom{n}{j} \left( \frac{t}{T} \right)^{j+1} \left( 1 - \frac{t}{T} \right)^{n-j}.
\]

If the \( j \)th observation arrives at time \( t \) and is a record then the probability that it is the best candidate is:

\[
p_j(t) = \sum_{n=j}^{\infty} \binom{n}{j} \left( \frac{t}{T} \right)^{j+1} \left( 1 - \frac{t}{T} \right)^{n-j} \prod_{k=j+1}^{n} (1 - E\pi_k).
\]

The probability that the next record arrives and is the best candidate is:

\[
q_j(t) = \sum_{n=j+1}^{\infty} \binom{n}{j} \left( \frac{t}{T} \right)^{j+1} \left( 1 - \frac{t}{T} \right)^{n-j} \left( \prod_{k=j+1}^{n} (1 - E\pi_k) \right) \sum_{k=j+1}^{n} \frac{E\pi_k}{1 - E\pi_k}.
\]
Let
\[ a_{nj} = \prod_{k=j+1}^{n} (1 - E\pi_k) > 0 \] and
\[ b_{nj} = \sum_{k=j+1}^{n} \frac{E\pi_k}{1 - E\pi_k} \]
then
\[ h_j\left(\frac{t}{T}\right) = p_j(t) - q_j(t) = \left(\frac{t}{T}\right)^{j+1} + \sum_{n=j+1}^{\infty} \binom{n}{j} \left(\frac{t}{T}\right)^{j+1} (1 - \frac{t}{T})^{n-j} a_{nj}(1 - b_{nj}) \]
\[ h_j(x) = x^{j+1} \left[ 1 + \sum_{k=1}^{\infty} \binom{k+j}{j} a_{k+j,j}(1 - b_{k+j,j})(1 - x)^k \right]. \]

The optimal strategy is again to select the first record (= jth observation, at time t) with \( h_j\left(\frac{t}{T}\right) > 0 \).

\[ h_j(0) = 0 \] and \( h_j(x) = 0 \) for \( x \in (0, 1) \)
\[ \Leftrightarrow g_j(y) = 1 + \sum_{k=1}^{\infty} \binom{k+j}{j} a_{k+j,j}(1 - b_{k+j,j})y^k = 0 \]
where \( y = 1 - x \).

In the same manner as in the previous section we can conclude that \( g_j \) has a unique root in \((0, \infty)\), say \( y_j \), and \( g_j(y) > 0 \) for \( y < y_j \), \( g_j(y) < 0 \) for \( y > y_j \). The optimal strategy therefore is to accept the first record with \( \frac{t}{T} \geq 1 - y_j \).
CHAPTER 4

RANDOM AND POINT PROCESS RECORD MODELS IN THE $F^\alpha$ SETUP

4.1 Introduction

So far we considered an infinite sequence of observations. In this chapter we will assume the number of available observations to be a random variable. This is called the random record model. For the classical (i. i. d.) case, this and also the point process record model have been discussed in the literature. See for example Arnold, Balakrishnan and Nagaraja (1998) for a summary. We will consider the random record model for a fixed $F^\alpha$ setup. It will be introduced in the next section. A random record model situation arises naturally when the observations arrive at time points determined by an independent point process $P$. We call this scenario a point process record model. A basic introduction is provided in Section 4.3.

Section 4.4 concentrates on the case where $P$ is homogeneous Poisson. We provide distributional representations for the (continuous) inter-record times when the $\alpha_n$'s are geometrically increasing, linearly increasing or have a special polynomial form. We look at non-homogeneous Poisson pacing processes in Section 4.5. First we give a characterization for the process to be homogeneous, using the first two inter-record
times. Next we explore the limit behavior of the inter-record times for a class of combinations of \(\alpha\)-structures of the \(F^\alpha\) model and intensity functions of \(P\).

Section 4.6 briefly discusses a case where the observations are dependent and their distributions depend on their arrival times.

4.2 Random record model

We consider a sequence \(\{X_i\}_{i \geq 1}\) of observations according to a (fixed) \(F^\alpha\) model. However, only \(N + 1\) such values are observed where \(N\) is a non-negative, integer-valued random variable with \(P(N = k) = t_k, k \geq 0\). The sequence observed is \(X_1, \ldots, X_{N+1}\) where \(X_1\) is always observed and is trivially a record. There is a positive probability that this record will never be broken:

\[
P(X_1 = \max \{X_1, \ldots, X_{N+1}\}) = E \left[ \frac{\alpha_1}{S(N + 1)} \right]
\]

where the expectation is with respect to \(N\). The distribution theory for records will therefore have to include the event that insures their presence.

For \(n \geq 1\) let \(D_n\) be the event that more than \(n\) records occur. Then for \(k \geq n, r_1 < \ldots < r_{n+1}:

\[
P(D_n \cap R_1 \in dr_1, \ldots, R_{n+1} \in dr_{n+1} | N = k) =
\]

\[
= \sum_{1=1, i_1 < i_2 < \ldots < i_{n+1} \leq k+1} d[F^{\alpha_{i_1}}(r_1)] \cdots d[F^{\alpha_{i_{n+1}}}(r_{n+1})] F^{S(i_{2+1})-S(i_1)}(r_1) \cdots 
\]

\[
\cdot \cdot F^{S(i_{n+1}+1)-S(i_n)}(r_n)
\]

where \(\{R_i \in dr_i\} = \{r_i < R_i \leq r_i + dr_i\}\)

\[
= \sum_{1=1, i_1 < i_2 < \ldots < i_{n+1} \leq k+1} \prod_{j=1}^{n} \alpha_j F^{S(i_{j+1}+1)-S(i_{j-1})}(r_j) dF(r_j)
\]
\[ \alpha_{n+1} F_{\alpha_{n+1} - 1}(r_{n+1})dF(r_{n+1}), \]

where \( S(0) = 0 \). Hence the distribution of the first \( n + 1 \) records is

\[
p_{n,N} := P(D_n \cap R_1 \in dr_1, \ldots, R_{n+1} \in dr_{n+1})
\]

\[
= \sum_{k=n}^{\infty} t_k \sum_{1 = i_1 < i_2 < \ldots < i_{n+1} \leq k+1} \prod_{i=1}^{n} \alpha_{i_j} F^{S(i_{j+1} - 1) - S(i_j - 1)}(r_j)dF(r_j) \quad \text{(4.1)}
\]

Note that in the classical record model, given the records, the inter-record times are conditionally independent and have a geometric distribution. For the random record model in the case of i.i.d. observations Bunge and Nagaraja (1991) showed that the first \( n \) inter-record times are conditionally independent if \( N \) has a geometric tail beyond \( n \). Recall that the inter-record times were defined as \( \Delta_n = T_n - T_{n-1}, \quad n \geq 2 \).

In our case, we have

\[
P(\Delta_2 = j_2, \ldots, \Delta_{n+1} = j_{n+1} \mid D_n \cap R_1 \in dr_1, \ldots, R_{n+1} \in dr_{n+1})
\]

\[
= \prod_{i=1}^{n+1} dF(r_i) \alpha_1 \alpha_1 + j_2 \cdots \alpha_1 + j_2 + \cdots + j_{n+1} F^{S(j_2 - 1)}(r_1) \cdot F^{S(j_2 + j_3) - S(j_2 - 1)}(r_2) \cdots F^{S(j_2 + \cdots + j_{n+1}) - S(j_2 + \cdots + j_1) - 1}(r_n) \quad \text{(4.2)}
\]

\[
\cdot F_{\alpha_1 + j_2 + \cdots + j_{n+1} - 1}(r_{n+1}) P(N \geq j_2 + \cdots + j_{n+1})
\]

and

\[
P(\Delta_2 = j_2, \ldots, \Delta_{n+1} = j_{n+1} \mid D_n \cap R_1 \in dr_1, \ldots, R_{n+1} \in dr_{n+1})
\]

\[
= \frac{1}{p_{n,N}} \cdot (\text{right hand side of (4.2)}). \quad \text{(4.3)}
\]
Theorem 4.1 In the random record model under (fixed) $F^a$ setup, if given the records $R_1, R_2, R_3$, the inter-record times $\Delta_2, \Delta_3$ are conditionally independent, then $\alpha_3 = \alpha_4 = \ldots = \alpha$ and $N$ has a geometric tail beyond 2, i.e., $P(N \geq k) = e^{k-2}P(N \geq 2)\ (0 < c < 1)$ for all $k \geq 2$. Conversely, if $\alpha_3 = \alpha_4 = \ldots = \alpha$ and $N$ has a geometric tail beyond $n\ (n \geq 2)$, then for all $k \geq n+1$ given the records $R_1, \ldots, R_k$ the inter-record times $\Delta_2, \ldots, \Delta_k$ are conditionally independent. While $\Delta_k, k > 2$ are (conditionally) geometric, $\Delta_2$ has a geometric tail after 1.

Proof: $\Delta_2$ and $\Delta_3$ are conditionally independent iff (4.3) factors into functions of $j_2$ and $j_3$. Note that $p_{n,N}$ does not depend on the $j_i$'s. By ignoring the factors that only depend on $j_2$, it follows that

$$\alpha_1+j_2+j_3 P(N \geq j_2 + j_3) F^{S(j_2+j_3)}(r_2) F^{\alpha_1+j_2+j_3}(r_3)$$

has to factor into individual functions of $j_2, j_3$. By letting $r_3 \to \infty; r_2, r_3 \to \infty$ we obtain that each of the expressions

$$\alpha_1+j_2+j_3 P(N \geq j_2 + j_3), F^{S(j_2+j_3)}(r_2)$$

factors into functions of $j_2, j_3$. Let us consider the second expression. There must exist a function $g$ such that

$$S(j_2 + j_3) = g(j_2) + g(j_3) \text{ for all } j_2, j_3 \geq 1.$$

For $n \geq 3$, $S(n) = g(n-2)+g(2), S(n-1) = g(n-2)+g(1)$. Hence, $\alpha_n = S(n)-S(n-1) = g(2) - g(1) = \alpha$ for all $n \geq 3$. Let us now look at the first expression in (4.4). Since $\alpha_1+j_2+j_3 = \alpha$, there exists a function $h$ such that $P(N \geq j_2 + j_3) = h(j_2)h(j_3)$ for all $j_2, j_3 \geq 1$. Let $c = \frac{h(2)}{h(1)}\ (< 1)$. For $n \geq 3, P(N \geq n) = h(n-2)h(2) = h(n-1)h(1).$
Hence, \( h(n - 1) = ch(n - 2) \) for \( n \geq 3 \) which implies \( P(N \geq k) = c^{k-2}h^2(1) = c^{k-2}P(N \geq 2) \) for all \( k \geq 2 \).

To prove the second part of the theorem, first note that (4.3) simplifies to

\[
P(\Delta_2 = j_2, \ldots, \Delta_{n+1} = j_{n+1} | D_n \cap R_i \in dr_1, \ldots, R_{n+1} \in dr_{n+1})
\]

\[
= c^a_i \alpha_{1+j_2} F^{S(j_2)-1}(r_1) F^{a_{j_2+1}+(j_3-1)\alpha_r-1}(r_2) F^{a_{j_3+1}-1}(r_3) \ldots F^{a_{j_{n+1}+1}-1}(r_n).
\]

\[
\cdot F^{a_{n+1}}(r_{n+1}) c^{j_2+\ldots+j_{n+1}-n} P(N \geq n)
\]

\[
= c^a_i \alpha_{1+j_2} F^{S(j_2)-1}(r_1) F^{a_{j_2+1}+(j_3-1)\alpha_r-1}(r_2) c^{j_2-1} P(N \geq n) n+1 \prod_{i=3}^{n+1} [F^{a}(r_{i-1})] c^{j_{i-1}}^{-1} (4.5)
\]

where \( c^* \), \( c^{**} \) are constants. Hence, conditionally, the \( \Delta_i \)'s are independent and have geometric distributions with parameters \( 1-cF^{a}(r_{i-1}) \) for \( i = 3, \ldots, n+1 \). From (4.5), \( P(\Delta_2 = j | D_1 \cap R_1 \in dr_1, R_2 \in dr_2) \propto \alpha_{1+j_2} F^{S(j)-1}(r_1) F^{a_{j_2+1}}(r_2) c^{j_2-1} \). Hence for \( j > 1 \), 

\[
P(\Delta_2 = j | D_1 \cap R_1 \in dr_1, R_2 \in dr_2) \propto [cF^{a}(r_1)]^{j_2-1} \square
\]

In the random record model since the number of observations is a.s. finite, so is the number of records. In the following we will derive the distribution of the number \( M \) of non-trivial records (excluding \( R_1 \)).

\[
P(M = m | D_1 \cap R_1 \in dr_1, \ldots, R_{m+1} \in dr_{m+1})
\]

\[
= \sum_{k=m}^{\infty} t_k \left( \prod_{i=1}^{m+1} dF(r_i) \right) \sum_{1=i_1<\ldots<i_{m+1} \leq k+1} \alpha_{i_1} \cdots \alpha_{i_{m+1}} F^{S(i_2-1)-1}(r_1) \cdot F^{S(i_3-1)-S(i_2-1)-1}(r_2) \cdot \ldots F^{S(k+1)-S(i_{m+1}-1)-1}(r_{m+1})
\]

Setting \( y_i = F(r_i) \) and integrating gives

\[
P(M = m) = \sum_{k=m}^{\infty} t_k \sum_{1=i_1<\ldots<i_{m+1} \leq k+1} \alpha_{i_1} \cdots \alpha_{i_{m+1}}.
\]

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\[
\int_{0 \leq y_1 < \ldots < y_{m+1} \leq 1} y_1^{S(i_2 - 1) - 1} y_2^{S(i_3 - 1) - 1} \ldots y_{m+1}^{S(k+1) - S(i_{m+1} - 1) - 1} dy_1 \ldots dy_{m+1}
\]

\[
= \sum_{k=m}^{\infty} t_k \sum_{1 \leq i_1 < \ldots < i_{m+1} \leq k+1} \frac{\alpha_{i_1} \ldots \alpha_{i_{m+1}}}{S(i_2 - 1) \ldots S(i_{m+1} - 1) S(k + 1)}.
\]

Let \( j_k = i_{k+1} - 1 \). Then

\[
P(M = m) = \begin{cases} 
\sum_{k=m}^{\infty} t_k \sum_{1 \leq j_1 < \ldots < j_m \leq k} \frac{\alpha_{j_1} \ldots \alpha_{j_m}}{S(j_1) \ldots S(j_m)} & \text{for } m \geq 1 \\
\sum_{k=0}^{\infty} \frac{t_k}{S(k+1)} & \text{for } m = 0
\end{cases}.
\]

Let us first consider the i.i.d. case, i.e., \( \alpha_i = 1 \forall i \). Then (4.6) reduces to

\[
P(M = m) = \begin{cases} 
\sum_{k=0}^{\infty} \frac{t_k}{S(k+1)} & \text{for } m \geq 1 \\
\sum_{k=0}^{\infty} \frac{t_k}{S(k+1)} & \text{for } m = 0
\end{cases}.
\]

Bunge and Nagaraja (1991) treated this and found that the distribution has a nice form if \( N + 1 \) is a geometric random variable. I will give a simpler proof of their result.

**Theorem 4.2** If in the i.i.d. random record model \( N + 1 \) is geometric, in the sense that

\[
P(N = k) = pq^k, \quad k \geq 0, \quad q = 1 - p,
\]

then

\[
P(M = m) = \frac{p (-\log p)^{m+1}}{q (m+1)!}, \quad m \geq 0.
\]

**Proof:** Step 1: \( m = 0 \):

\[
P(M = 0) = \sum_{k=0}^{\infty} \frac{pq^k}{k+1} = \sum_{k=0}^{\infty} \frac{pq^{k+1}}{qk+1}
\]

\[
= \frac{p}{q} \int_0^q \frac{1}{1-x} dx = \frac{p}{q} \int_0^q \frac{1}{1-x} dx
\]

\[
= \frac{p}{q} (-\log p).
\]
Step 2: $m > 0$:

$$P(M = m) = \sum_{k=m}^{\infty} \frac{pq^k}{k+1} A_m(k) \text{ for } m \geq 0$$

where

$$A_m(k) = \sum_{1 \leq j_1 < \cdots < j_m \leq k} \frac{1}{j_1 \cdots j_m} \text{ and } A_0(k) \equiv 1.$$ 

Let

$$S_m = \sum_{k=m}^{\infty} \frac{q^{k+1}}{k+1} A_m(k) = \sum_{k=m+1}^{\infty} \frac{q^k}{k} A_m(k-1) \text{ and } T_m = \frac{(-\log p)^{m+1}}{(m+1)!}.$$ 

We have to show $S_m = T_m$ for all $m \geq 1$. The proof is by induction, let $m \geq 1$ and assume $S_{m-1} = T_{m-1}$.

$$T_m = \frac{(-\log(1-q))^{m+1}}{(m+1)!} = \frac{1}{(m+1)!} \left( \sum_{k=1}^{\infty} \frac{q^k}{k} \right)^{m+1}.$$ 

$S_m$ and $T_m$ are both polynomials in $q$ with $m+1$ being the smallest power. Two such polynomials are equal iff all the coefficients are equal, but since $m+1 \geq 1$ this holds iff the first derivatives are equal. Hence we are done if we show that $S'_m(q) = T'_m(q)$.

$$T'_m(q) = \frac{1}{m!} \left( \sum_{k=1}^{\infty} q^k \right)^m \sum_{k=1}^{\infty} q^{k-1} = \frac{T_{m-1}(q)}{1-q} = \frac{S_{m-1}(q)}{1-q}.$$ 

It remains to show that $(1-q)S'_m(q) = S_{m-1}(q)$. First note that for $m \geq 1, k \geq m+1$

$$A_m(k-1) = \sum_{1 \leq j_1 < \cdots < j_m \leq k-1} \frac{1}{j_1 \cdots j_m}$$

$$= \sum_{j_m=m}^{k-1} \frac{1}{j_m} \sum_{1 \leq j_1 < \cdots < j_{m-1} \leq j_m-1} \frac{1}{j_1 \cdots j_{m-1}}$$

$$= \sum_{n=m-1}^{k-2} \frac{1}{n+1} A_{m-1}(n).$$

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Now

\[(1 - q)S^r_m(q) = (1 - q) \sum_{k=m+1}^{\infty} q^{k-1} A_m(k - 1) \]

\[= \sum_{k=m+1}^{\infty} [A_m(k) - A_m(k - 1)] q^k + q^m A_m(m) \]

\[= \sum_{k=m+1}^{\infty} \left[ \sum_{n=m-1}^{k-1} \frac{1}{n+1} A_{m-1}(n) - \sum_{n=m-1}^{k-2} \frac{1}{n+1} A_{m-1}(n) \right] q^k \]

\[+ q^m \frac{1}{m} A_{m-1}(m - 1) \]

\[= \sum_{k=m+1}^{\infty} \frac{A_{m-1}(k - 1)}{k} q^k + \frac{A_{m-1}(m - 1)}{m} q^m \]

\[= \sum_{k=m}^{\infty} \frac{q^k}{k} A_{m-1}(k - 1) = S_{m-1}(q). \Box \]

Next let us consider geometrically increasing α's. Let \(\alpha_1 = c, \alpha_k = (c - 1)c^{k-1}\) for \(k \geq 2\), \(S(k) = \alpha_1 + \cdots + \alpha_k = c^k, c > 1\). Note that Yang (1975) used similar α's for his original model for sports records. Then (4.6) simplifies to

\[P(M = m) = \sum_{k=m}^{\infty} \frac{t_k}{c^k+1} \sum_{1 \leq j_1 < \cdots < j_m \leq k} (c - 1)^m \]

\[= (c - 1)^m \sum_{k=m}^{\infty} t_k c^{-k} \binom{k}{m} \] for all \(m \geq 0\). \hspace{1cm} (4.7)

**Example 4.1** Suppose \(N\) has a negative binomial distribution with parameters \((r, p)\), i.e.,

\[t_k = \binom{r + k - 1}{k} p^r q^k, q = 1 - p.\]

Then

\[P(M = m) = (c - 1)^m \sum_{k=m}^{\infty} \binom{r + k - 1}{k} p^r q^k c^{-k} \binom{k}{m} \]

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The sum at the end is nothing but the Taylor series of \((r + m - 1)!(1 - x)^{-(r+m)}\).

Therefore

\[
P(M = m) = (c - 1)^m p^r \left( \frac{r + m - 1}{m} \right) x^m (1 - x)^{-(r+m)}
\]

\[
= (c - 1)^m p^r \left( \frac{r + m - 1}{m} \right) \frac{q^m}{(c - q)^{r+m}}
\]

\[
= \left( \frac{r + m - 1}{m} \right) \left( \frac{pc}{c - p} \right)^r \left( \frac{c - 1}{c - q} \right)^m.
\]

Hence, \(M\) is negative binomial \((r, \frac{pc}{c - q})\).

**Example 4.2** Let \(N \sim \text{Poi}(\lambda)\). Then

\[
P(M = m) = (c - 1)^m \sum_{k=m}^{\infty} \binom{k}{m} c^{-k} e^{-\lambda} \frac{\lambda^k}{k!}
\]

\[
= (c - 1)^m \frac{1}{m!} e^{-\lambda} \sum_{k=m}^{\infty} \frac{1}{(k - m)!} \left( \frac{\lambda}{c} \right)^k
\]

\[
= (c - 1)^m \frac{1}{m!} e^{-\lambda} \left( \frac{\lambda}{c} \right)^m \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{\lambda}{c} \right)^k
\]

\[
= \left( \frac{c - 1}{c} \right)^m \frac{1}{m!} \lambda^m e^{-\lambda (1 - c^{-1})}.
\]

Hence, \(M \sim \text{Poi}(\lambda (1 - c^{-1}))\).

**Example 4.3** Let \(N \sim \text{Bin}(n, p)\). Then

\[
P(M = m) = (c - 1)^m \sum_{k=m}^{n} \binom{k}{m} e^{-k} \binom{n}{k} p^k q^{n-k}, \quad 0 \leq m \leq n
\]

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\[
= (c - 1)^m \frac{1}{m!} \sum_{k=m}^{n} \frac{n!}{(n-k)!(k-m)!} \left( \frac{p}{c} \right)^k q^{n-k}
\]

\[
= \frac{(c - 1)^m}{m!} \left( \frac{p}{c} \right)^m \frac{n!}{(n-m)!} \sum_{k=0}^{n-m} \binom{n-m}{k} \left( \frac{p}{c} \right)^k q^{n-k-m}
\]

\[
= \binom{n}{m} (c - 1)^m c^{-n} p^m (p + cq)^{n-m}
\]

\[
= \binom{n}{m} \left( \frac{(c - 1)p}{c} \right)^m \left[ \frac{p + cq}{c} \right]^{n-m}.
\]

Hence, \( M \sim \text{Bin}(n, \frac{c-1}{c} p) \).

As another manageable \( \alpha \) structure, let us consider \( \alpha_n = n \) for all \( n \). Then (4.6) reduces to

\[
P(M = m) = \sum_{k=m}^{\infty} \frac{2}{k+1}(k+2) \sum_{1 \leq j_1 < \cdots < j_m \leq k} \frac{2}{j_1} \cdots \frac{2}{j_m}
\]

\[
= 2^{m+1} \sum_{k=m}^{\infty} \frac{t_k}{(k+1)(k+2)} A_m(k).
\]

**Example 4.4** Let \( \alpha_n = n \), \( N + 1 \sim \text{Geom}(p) \), \( t_k = pq^k \). Then

\[
P(M = m) = 2^{m+1} \frac{p}{q^2} \sum_{k=m}^{\infty} x^{k+1} A_m(k)
\]

\[
P(M = m) = 2^{m+1} \frac{p}{q^2} \int_0^q \sum_{k=m}^{\infty} \frac{x^{k+1}}{k+1} A_m(k) dx.
\]

Notice that the integrand is equal to \( S_k \) from the proof of Theorem 4.2. Therefore

\[
P(M = m) = 2^{m+1} \frac{p}{q^2} \frac{1}{(m+1)!} \int_0^{\log(1-x)} \frac{(-\log(1-x))^{m+1}}{(m+1)!} dx.
\]

By substituting \( y = \log(1-x) \), \( 1-x = e^{-y} \), we obtain

\[
P(M = m) = 2^{m+1} \frac{p}{q^2} \frac{1}{(m+1)!} \int_0^{\log p} \frac{y^{m+1}}{y} e^{-y} dy
\]

\[
= 2^{m+1} \frac{p}{q^2} \left( 1 - \frac{1}{(m+1)!} \int_{\log p}^{\infty} y^{m+1} e^{-y} dy \right).
\]
The integral is an incomplete Gamma integral. It is well known that (can, for example, be derived by setting equal the probabilities of less than \( m+2 \) arrivals until time \( t \) and the waiting time until the \((m+2)\)nd arrival being greater than \( t \) in a homogeneous Poisson process)

\[
\sum_{k=0}^{m+1} \frac{t^k}{k!} e^{-t} = \frac{1}{(m+1)!} \int_t^\infty y^{m+1} e^{-y} dy.
\]

Hence,

\[
P(M = m) = \frac{2^{m+1} p}{q^2} \left( 1 - p \sum_{k=0}^{m+1} \frac{(-\log p)^k}{k!} \right)
\]

\[
= \frac{2^{m+1} p}{q^2 (m+1)!} \int_0^{-\log p} y^{m+1} e^{-y} dy, \quad m \geq 0.
\]

Similar formulas can be obtained when \( N + 1 \) is geometric and \( \alpha_n = n^2, n^3, \ldots \). Derivations, however, become increasingly complicated.

### 4.3 Point process models in the \( F^\alpha \) setup

So far we assumed that the \( X_n \)'s are observed at a random number of equally spaced time points. Now suppose \( X_n \) is observed at the time of occurrence of the \( n \)th event in a simple point process \( P \) defined on \([0, \infty)\) where \( P \) is independent of the sequence of \( X_n \)'s and has a point at 0 with probability 1. Let \( X_1 \) be observed at time 0. Let \( \{Z_n, \ n \geq 0\} \) be the sequence of arrival times characterizing \( P \), i.e., \( X_n \) arrives at time \( Z_{n-1} \) \((Z_0 = 0)\). Let \( \{V_n, \ n \geq 1\} \) be the sequence of inter-arrival times of the observations, meaning

\[
V_1 = Z_1, \quad V_n = Z_n - Z_{n-1} \text{ for } n \geq 2.
\]
Define $T_n, N_n, R_n, \Delta_n, I_n$ as in the regular (fixed) $F^\alpha$ model (see page 5). Now $T_n$ represents the $n$th record index and $\Delta_n$ the $n$th inter-record count. Of special interest in this model are the (continuous) record arrival times

$$W_0 = 0, W_n = \sum_{i=1}^{T_{n+1} - 1} V_i, n \geq 1$$

and record inter-arrival times $U_n = \sum_{i=I_n}^{T_{n+1} - 1} V_i, n \geq 1$. Notice that $W_n = \sum_{i=1}^{n} U_i$.

Finally, we introduce $N(t) = \max \{n : Z_n < t\}$ and $M(t) = \max \{n : W_n \leq t\}$ representing the number of observations and the number of records in $(0, t]$, respectively.

When the $\alpha$’s are fixed, $T_n, N_n, R_n, \Delta_n$ and $I_n$ behave just as in the regular (fixed) $F^\alpha$ model and therefore require no special treatment. The random variables $U_n, W_n$ and $M(t)$, however, are of independent interest. We now investigate their properties for various pacing processes $P$ in the following sections. Such point process models have not yet been considered in the $F^\alpha$ scheme, although literature exists for the classical (i. i. d.) setup. In the i. i. d. case, such a model was first formulated by Pickands (1971) with $P$ being a homogeneous Poisson process with unit intensity. Gaver (1976), Gaver and Jacobs (1978) and Westcott (1977) considered several commonly used $P$’s and obtained various results for $U_n, W_n$ and $M(t)$. Assuming $P$ is a renewal process with inter-arrival c. d. f. $F_V$, Westcott (1977, 1979), Embrechts and Omey (1983) and Yakymiv (1986) studied the connection between the tail behavior of $F_V$ and the c. d. f.’s of $U_n, W_n$. Deheuvels (1982) and Bunge and Nagaraja (1992a) obtained a simple distributional representation for the $U_n$’s. Bruss (1988, for example) applied such models to variants of the secretary problem. Recent developments in i. i. d. point process record models are found in Bruss and Rogers (1991), Bunge and Nagaraja (1991, 1992a, 1992b) and Browne and Bunge (1995). Our treatment of
the $F^a$ point process record model will include generalizations of results in the above mentioned paper when possible.

4.4 Homogeneous Poisson pacing process

We now assume $\mathbf{P}$ to be a homogeneous Poisson process with unit intensity. Let us look at the joint distribution of the first $n$ record inter-arrival times $U_1, \ldots, U_n$. Since $U_n = \sum_{i=1}^{T_{n+1} - 1} V_i \ (n \geq 1)$, given $T_1, \ldots, T_{n+1}$ the record inter-arrival times $U_1, \ldots, U_n$ are conditionally independent. Further, $U_k$ has $\text{Gamma}(T_{k+1} - T_k, 1)$ distribution $(1 \leq k \leq n)$. Hence,

$$f_{\tilde{U}}(u_1, \ldots, u_n) = \sum_{1=t_1<t_2<\ldots<t_{n+1}} P(T_2 = t_2, \ldots, T_{n+1} = t_{n+1}) \cdot \prod_{i=1}^{n} \frac{1}{\Gamma (t_{i+1} - t_i)} u_i^{t_{i+1} - t_i - 1} e^{-u_i}$$

where $\tilde{U} = (U_1, \ldots, U_n)$. Recall that

$$P(T_2 = t_2, \ldots, T_{n+1} = t_{n+1})$$

$$= P \left( I_2 = 0, \ldots, I_{t_2 - 1} = 0, I_{t_2} = 1, I_{t_2 + 1} = 0, \ldots, I_{t_3} = 1, \ldots, I_{t_{n+1}} = 1 \right)$$

$$= \frac{\alpha_1}{\alpha_1 + \alpha_2} \cdot \cdots \cdot \frac{\alpha_1 + \cdots + \alpha_{t_2 - 2}}{\alpha_1 + \cdots + \alpha_{t_2 - 1}} \cdot \frac{\alpha_1 + \cdots + \alpha_{t_2}}{\alpha_1 + \cdots + \alpha_{t_2} + 1} \cdot \frac{\alpha_{t_{n+1}}}{\alpha_1 + \cdots + \alpha_{t_{n+1}}}$$

$$= \frac{\alpha_1}{\alpha_1 + \cdots + \alpha_{t_2 - 1}} \cdot \frac{\alpha_{t_2}}{\alpha_1 + \cdots + \alpha_{t_3 - 1}} \cdot \cdots \cdot \frac{\alpha_{t_{n+1}}}{\alpha_1 + \cdots + \alpha_{t_{n+1}}}$$

$$= \frac{\alpha_1}{S(t_{n+1})} \prod_{i=2}^{n+1} \frac{\alpha_i}{S(t_i - 1)}.$$

$$\text{(4.8)}$$
Substituting this in the above formula and setting \( k_i = t_{i+1} - t_i, i \geq 1 \), yields

\[
f_\theta(u_1, \ldots, u_n) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{\alpha_i}{S(1 + k_1 + \cdots + k_n)} \cdot 
 \prod_{i=1}^{n} \frac{\alpha_1 + \cdots + \alpha_i}{S(1 + k_1 + \cdots + k_i)} \prod_{i=1}^{n} \frac{1}{\Gamma(k_i)} u_i^{k_i-1} e^{-u_i}.
\] (4.9)

We first consider geometrically increasing \( \alpha \)'s, i.e., as done in Section 4.2 we let \( \alpha_1 = c, \alpha_n = (c - 1)c^{n-1}, n \geq 2 \). Equation (4.9) then reduces to

\[
f_\theta(u_1, \ldots, u_n) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} c^{-(k_1 + \cdots + k_n)}(c - 1)^n \prod_{i=1}^{n} \frac{1}{\Gamma(k_i)} u_i^{k_i-1} e^{-u_i}.
\]

\[
= \prod_{i=1}^{n} (c - 1)e^{-u_i} \sum_{k_i=1}^{\infty} \frac{1}{(k_i - 1)!} c^{-k_i} u_i^{k_i-1}
\]

\[
= \prod_{i=1}^{n} \frac{c - 1}{c} e^{-u_i} e^{(\frac{u_i}{c})}
\]

\[
= \prod_{i=1}^{n} \left(1 - c^{-1}\right) e^{-u_i} \left(1 - c^{-1}\right).
\] (4.10)

This implies that \( U_1, U_2, \ldots \) are i.i.d. with an \( \text{Exp}\left(\frac{c}{c-1}\right) \) distribution. In other words we have the following theorem which was already proven by Bunge and Nagaraja (1992a) in a slightly different context.

**Theorem 4.3** In an \( F^\alpha \) point process record model with homogeneous Poisson pacing process with unit intensity and geometrically increasing \( \alpha \)'s as defined above, the record arrival process (after the first observation) is again a homogeneous Poisson process with intensity \( (1 - c^{-1}) \).

For finding the distribution of \( M(t) \), note that

\[
P(M(t) \leq k) = P(U_1 + \cdots + U_k \geq t)
\]

\[
= 1 - G_k(t)
\]

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where $G_k$ is the c. d. f. of a Gamma$\left( k, \frac{e}{z-1} \right)$ distribution. Hence

$$P(M(t) = k) = P(M(t) \leq k) - P(M(t) \leq k-1)$$

$$= G_{k-1}(t) - G_k(t).$$

Next let us consider linearly increasing $\alpha$'s, in particular let $\alpha_n = n$. To derive a distributional representation for the record inter-arrival times, we need the following two results.

**Lemma 4.4** (see Abramowitz and Stegun (1977, p. 556, formula 15.1.20) or Bateman (1953, p. 101))

If $x$, $y$, $z$ are complex numbers such that $Re(y + 1) > 0$, $Re(z - y) > 0$ and $Re(z + x + 2) > 0$ then

$$\eta(x, y, z) = \sum_{k=1}^{\infty} \frac{\Gamma(x + k)\Gamma(y + k)}{\Gamma(z + x + 1 + k)\Gamma(k)} = \frac{\Gamma(x + 1)\Gamma(y + 1)\Gamma(z - y)}{\Gamma(z + 1)\Gamma(z + x - y + 1)}.$$

**Lemma 4.5** Let $A_i$ be i. i. d. Exp(1), $B_i$ be i. i. d. $Exp\left( \frac{1}{2} \right)$, $1 \leq i \leq n$; $A_i$'s and $B_i$'s independent and $C_j = \sum_{i=1}^{j} B_i$ for $1 \leq j \leq n$.

Then $(C_1 + \log A_1, C_2 + \log A_2, \ldots, C_n + \log A_n)$ has characteristic function

$$\phi(\bar{z}) = \prod_{j=1}^{n} 2\Gamma(z_j + 1) \left( 2 - \sum_{i=j}^{n} z_i \right)^{-1},$$

where $\bar{z} = (z_1, \ldots, z_n)$ and the $z_i$'s are imaginary numbers.

**Proof:**

$$\phi(\bar{z}) = E \left\{ \exp \left[ \sum_{j=1}^{n} z_j (C_j + \log A_j) \right] \right\}$$

$$= \int_{0}^{\infty} \cdots \int_{0}^{\infty} e^{z_1(y_1 + \log x_1)} \cdots e^{z_n(y_n + \cdots + y_n + \log x_n)}.$$
\[-e^{-x_1} \cdots e^{-x_n} 2e^{-2y_1} \cdots 2e^{-2y_n} dx_1 \cdots dx_n dy_1 \cdots dy_n \]

\[= \prod_{j=1}^{n} \left( \int_{0}^{\infty} e^{z_j \log x_j e^{-x_j}} dx_j \right) \left( \int_{0}^{\infty} e^{y_j \left( \sum_{i=j}^{n} z_i \right)} 2e^{-2y_j} dy_j \right) \]

\[= \prod_{j=1}^{n} 2\Gamma(z_j + 1) \left( 2 - \sum_{i=j}^{n} z_i \right)^{-1}. \square \]

**Theorem 4.6** Let \( \alpha_n = cn \) \((c > 0)\), \( P \) be a homogeneous Poisson process with unit intensity and \( A_i, C_i \) be as in Lemma 4.5. Then

\[(\log U_1, \ldots, \log U_n) \overset{d}{=} (C_1 + \log A_1, \ldots, C_n + \log A_n).\]

**Proof:** Let \( \phi(\mathbf{z}) = \phi(z_1, \ldots, z_n) \) be the characteristic function of \((\log U_1, \ldots, \log U_n)\) where \( z_1, \ldots, z_n \) are imaginary numbers. Given \( T_1, \ldots, T_{n+1} \) the record inter-arrival times \( U_1, \ldots, U_n \) are conditionally independent and have Gamma distributions. Using (4.8) we therefore obtain

\[\phi(\mathbf{z}) = E \left\{ \exp \left[ \sum_{j=1}^{n} z_j \log U_j \right] \right\} = E \left[ \prod_{j=1}^{n} U_j^{z_j} \right] = \sum_{1=t_1 < t_2 < \cdots < t_{n+1}} P(T_2 = t_2, \ldots, T_{n+1} = t_{n+1}) \prod_{i=1}^{n} EU_i^{z_i} = \sum_{1=t_1 < t_2 < \cdots < t_{n+1}} \frac{\alpha_1}{S(t_{n+1})} \prod_{i=2}^{n+1} \frac{\alpha_{t_i}}{S(t_i - 1)} \prod_{i=1}^{n} \frac{\Gamma(t_{i+1} - t_i + z_i)}{\Gamma(t_{i+1} - t_i)}.\]

Setting \( k_i = t_{i+1} - t_i \) and using \( \alpha_k = ck, S(k) = c \frac{k(k+1)}{2} \) \((c > 0)\) gives

\[\phi(\mathbf{z}) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{2}{(1 + k_1 + \cdots + k_n)(2 + k_1 + \cdots + k_n)} \cdot \prod_{i=1}^{n} \frac{2}{(k_1 + \cdots + k_i)} \frac{\Gamma(k_i + z_i)}{\Gamma(k_i)}.\]

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\[
= 2^{n+1} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{1}{(1 + k_1 + \cdots + k_n)(2 + k_1 + \cdots + k_n)} \cdot \prod_{i=1}^{n} \frac{\Gamma(k_i + z_i)}{\Gamma(k_i)}.
\]

Now we will sum over \( k_n \), then over \( k_{n-1}, \ldots \) and repeatedly use Lemma 4.4 and the relation \( t_{i+1} = t_i + k_i \). Note that \((t_{n+1} - 1)t_{n+1}(t_{n+1} + 1)^{-1}\) can be expressed as \(\Gamma(t_n - 1 + k_n)/\Gamma(t_n + 2 + k_n)\). Thus, on summing over \( k_n \), we obtain \(\eta(t_n - 1, z_n, 2)\) of Lemma 4.4. On using \(\eta\)'s second form, the sum over \( k_n \) can be expressed as

\[
\frac{1}{2} \Gamma(z_n + 1) \frac{\Gamma(2 - z_n)\Gamma(t_{n-1} + k_{n-1})}{\Gamma(t_{n-1} + 2 + k_{n-1} - z_n)}
\]

Next, summing over \( k_{n-1} \), we get

\[
\frac{1}{2} \Gamma(z_n + 1) \Gamma(t_n - 2 + 2 + k_{n-2} - z_n).
\]

Continuing in this fashion, the final summation (over \( k_1 \)) yields

\[
\phi(\mathbf{z}) = \prod_{j=1}^{n} 2\Gamma(z_j + 1) \left(2 - \sum_{i=j}^{n} z_i\right)^{-1}.
\]

The result now follows from Lemma 4.5. □

Using Theorem 4.6, we can say something about the distribution of \(M(t)\). Note that

\[
\{M(t) \geq m\} = \{U_1 + \cdots + U_m \leq t\}, \quad U_1 + \cdots + U_m = \sum_{i=1}^{m} e^{\log U_i}.
\]

Therefore

\[
P(M(t) \geq m) = \mathcal{P} \left( \sum_{i=1}^{m} e^{C_i + \log A_i} \leq t \right) = \mathcal{P} \left( \sum_{i=1}^{m} A_ie^{C_i} \leq t \right).
\]
It might not be possible to obtain this expression in closed form, but it could certainly be obtained numerically.

Note that Bunge and Nagaraja (1992a) have shown a similar theorem for the classical model, i.e., $\alpha_n \equiv 1$, $p_n = \frac{1}{n}$. In our case, $\alpha_n = cn$, $p_n = \frac{2}{n + 1}$. This multiplicative factor of 2 in the $p_n$'s seems to change the distribution of the $B_i$'s from Exp(1) to Exp($\frac{1}{2}$). We can make this observation precise and show a similar result for $\alpha_n$'s having a special polynomial form. Let

$$Q_0(n) \equiv 1, \quad Q_k(n) = \sum_{i=1}^{n} Q_{k-1}(i), \quad k \geq 1, \quad n \geq 1.$$

By induction on $n + k$ we can show that

$$Q_k(n) = \frac{n(n + 1) \ldots (n + k - 1)}{k!} = \binom{n + k - 1}{k}. \quad (4.11)$$

Clearly, $Q_0(n) = 1$ for all $n \geq 1$ and $Q_k(1) = 1$ for all $k \geq 0$. Let (4.11) hold for $n + k \leq m$. Then

$$Q_l(m + 1 - l) = Q_l(m - l) + Q_{l-1}(m + 1 - l)$$

$$= \binom{m - 1}{l} + \binom{m - 1}{l - 1}$$

$$= \binom{m}{l} \quad \text{for} \quad 1 \leq l \leq m.$$

**Theorem 4.7** Let $\alpha_n = cQ_k(n) = c\binom{n + k - 1}{k}$, for fixed $k \geq 0$, $c > 0$ and $P$ be a homogeneous Poisson process with unit intensity. Let $A_i$ be i.i.d. Exp(1), $B_i$ be i.i.d. Exp($\frac{1}{k+1}$), $1 \leq i \leq n$; $A_i$'s and $B_i$'s be independent and $C_j = \sum_{i=1}^{j} B_i$ for $1 \leq j \leq n$. Then

$$(\log U_1, \ldots, \log U_n) \overset{d}{=} (C_1 + \log A_1, \ldots, C_n + \log A_n).$$
Proof: The case $k = 0$ (classical model, $\alpha_n = 1$) was proven by Bunge and Nagaraja (1992a). For $k = 1$ we recover Theorem 4.6. For $k > 1$,

$$\frac{\alpha_n}{S(n-1)} = \frac{cQ_k(n)}{cQ_{k+1}(n-1)} = \left(\frac{n+k-1}{k}\right) = \frac{k+1}{n-1}, \quad n \geq 2.$$  

Using this identity, a proof equivalent to those of Lemma 4.5 and Theorem 4.6 can be constructed. □

4.5 Non-homogeneous Poisson pacing process

Now suppose $P$ is a Poisson process with positive rate function $\lambda(t)$ such that $
abla_0^\infty \lambda(x)dx = \infty$, $\int_0^\infty \lambda(x)dx < \infty \forall \epsilon > 0$. Let

$$\Lambda(t) = \int_0^t \lambda(x)dx$$

and

$$\Psi(t) = \Lambda^{-1}(t) = \inf \{s : \Lambda(s) > t\}, \quad t > 0.$$  

Then $Z_1^* = \Lambda(Z_1)$, $Z_2^* = \Lambda(Z_2)$, ... are the arrival times of a homogeneous Poisson process with unit intensity and $W_1^* = \Lambda(W_1)$, $W_2^* = \Lambda(W_2)$, ..., $U_1^* = \Lambda(U_1)$, $U_2^* = \Lambda(U_1 + U_2) - \Lambda(U_1)$, ... the associated record arrival and inter-arrival times, respectively. The distribution of $(U_1, \ldots, U_n)$ can therefore be obtained by a density transformation from (4.9).

$$\begin{vmatrix} \partial (U_1^*, \ldots, U_n^*) \\ \partial (U_1, \ldots, U_n) \end{vmatrix} = \begin{vmatrix} \lambda(U_1) & 0 & \cdots & 0 \\ \ast & \lambda(U_1 + U_2) & \cdots & 0 \\ \ast & \ast & \ddots & \vdots \\ \vdots & \ddots & \ast & 0 \\ \ast & \cdots & \cdots & \lambda(U_1 + \cdots + U_n) \end{vmatrix}$$
\[
\prod_{i=1}^{n} \lambda(U_1 + \cdots + U_i).
\]
Hence,
\[
f_U(u_1, \ldots, u_n) = \exp\left(-\Lambda(u_1 + \cdots + u_n)\right) \prod_{i=1}^{n} \lambda(u_1 + \cdots + u_i) \cdot \\
\sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{\alpha_1}{S(1 + k_1 + \cdots + k_n)} \prod_{i=1}^{n} \frac{\alpha_1 + k_1 + \cdots + k_i}{S(k_1 + \cdots + k_i)} \cdot \\
\frac{1}{\Gamma(k_i)} \left(\Lambda(u_1 + \cdots + u_i) - \Lambda(u_1 + \cdots + u_{i-1})\right)^{k_i-1}. (4.12)
\]
Again we first consider geometrically increasing \(\alpha\)'s, i.e., \(\alpha_1 = c\), \(\alpha_n = (c - 1)c^{n-1}\).
\(n \geq 2\). For this case we can also obtain the distribution of \((U_1, \ldots, U_n)\) through the same density transformation from (4.10):
\[
f_U(u_1, \ldots, u_n) = (1 - c^{-1})^{n} \exp\left(-\Lambda(u_1 + \cdots + u_n)\right) \left(1 - c^{-1}\right) \cdot (4.13)
\]
\[
\prod_{i=1}^{n} \lambda(u_1 + \cdots + u_i).
\]
One could ask under what conditions the above \(U_n\)'s are independent.

**Theorem 4.8** If the \(\alpha\)'s are geometrically increasing as defined above and (i) \(U_1, U_2\) are independent or (ii) \(U_1, U_2\) are identically distributed, then \(\lambda(t)\) is constant, i.e., the pacing process is in fact homogeneous Poisson and the \(U_n\)'s are i.i.d.

**Proof:** Let \(\theta = 1 - c^{-1}\).

(i) From (4.13) with \(n = 1\) it follows that the density of \(U_1\) is
\[
f_{U_1}(u) = \theta \lambda(u) e^{-\theta \Lambda(u)} \tag{4.14}
\]
and with \(n = 2\) in (4.13) we obtain
\[
P(U_2 > u_2, U_1 \in du) = \int_{u_2}^{\infty} \theta^2 e^{-\theta \Lambda(u + x)} \lambda(u + x)dx \lambda(u)du
\]
\[
= \theta \lambda(u) e^{-\theta \Lambda(u + u_2)} du. \tag{4.15}
\]
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From (4.14) and (4.15) we get

\[ P(U_2 > u_2 | U_1 = u) = e^{-\theta(\Lambda(u+u_2)-\Lambda(u))} \forall u, u_2 > 0. \]

Since \( U_1, U_2 \) are independent, this does not depend on \( u \). Therefore \( \Lambda(u + u_2) - \Lambda(u) \) does not depend on \( u \) either. By using \( u = 0 \), \( \Lambda(0) = 0 \) we obtain the well-known Cauchy functional equation \( \Lambda(u + u_2) = \Lambda(u) + \Lambda(u_2) \) whose only continuous solutions are linear. Hence, \( \lambda(u) = \Lambda'(u) \) is constant.

(ii) From (4.13) it follows that the density of \( U_2 \) is

\[ f_{U_2}(u) = \int_0^\infty \theta e^{-\theta \Lambda(u+x)} \lambda(u+x) \lambda(x) \, dx. \quad (4.16) \]

Let \( g(x) = \theta \lambda(x) e^{-\theta \Lambda(x)} \). Then since \( U_1, U_2 \) are identically distributed, it follows from (4.14) and (4.16) that

\[ g(u) = \int_0^\infty g(u + x) d\Lambda(x). \]

By the Lau-Rao theorem about this integrated Cauchy functional equation (see for example Ramachandran and Lau (1991, Chapter 2) or Rao and Shanbhag (1994, Chapter 2)) this implies

\[ g(x) = ae^{-bx} \text{ for some constants } a > 0, b. \]

On integrating both sides over \((x, \infty)\) and rearranging we obtain

\[ e^{\theta \Lambda(x) - bx} = \frac{b}{a}. \]

From \( \Lambda(0) = 0 \), it follows that \( e^{\theta \Lambda(x) - bx} = 1 \) or \( \Lambda(x) = \frac{b}{\theta}x \). Hence \( \lambda(x) \) is a constant.

\[ \square \]
As in the previous section, let us look at the case where $\alpha_n = c^{\left(n + k - 1\right)}$. $k \geq 0, c > 0$. Using the time axis transformation $A(t)$ we obtain the joint distribution of $(U_1, \ldots, U_n)$ from Theorem 4.7:

$$
(U_1, \ldots, U_n) \overset{d}{=} \left(\Lambda^{-1}\left(A_1e^{C_1}\right), \Lambda^{-1}\left(A_1e^{C_1} + A_2e^{C_2}\right) - \Lambda^{-1}\left(A_1e^{C_1}\right), \ldots \right.
$$

$$
\left.\ldots, \Lambda^{-1}\left(A_1e^{C_1} + \cdots + A_ne^{C_n}\right) - \Lambda^{-1}\left(A_1e^{C_1} + \cdots + A_{n-1}e^{C_{n-1}}\right)\right).
$$

Let us now look at the limit distribution of $(U_n, U_{n+1}, \ldots)$ for $n \to \infty$. It turns out that we can obtain it for a variety of combinations of $\alpha$-structures and mean intensities $\Lambda(t)$ of the pacing process. For pacing processes where $\Lambda(t)$ approaches zero, for example, when $\Lambda(t) = \log t$ or $\Lambda(t) = t^l, l < 1$, the inter-arrival time $V_n \overset{p}{\to} \infty$ indicating that $U_n \overset{p}{\to} \infty$. Hence we will restrict our attention to processes where $\Lambda(t)$ does not go to zero. For the i. i. d. model, assuming $\lim_{t \to \infty} \frac{\Lambda(t)}{\Lambda(t)} = c \in (0, \infty)$, Bunge and Nagaraja (1992b) showed that $U_n$ has an exponential limit. In our considerations for the $F^\alpha$ model (which includes the i.i.d. model as a special case) we only need that $\frac{\Lambda(t)}{\Lambda(t)}$ is bounded away from zero in the limit. This condition is used in the following lemma which plays an essential role in later results.

**Lemma 4.9** Let

$$
\liminf_{t \to \infty} \frac{\Lambda(t)}{\Lambda(t)} > 0. \quad (4.17)
$$

Then $|\Psi(W_n) - \Psi(T_{n+1})| \to 0$ a. s. and $|U_n - (\Psi(T_{n+1}) - \Psi(T_n))| \to 0$ a. s. where $\Psi = \Lambda^{-1}$.

**Proof:** Let $H(x) = \Psi(e^x)$. Then

$$
H'(x) = e^x\Psi'(e^x) = \frac{e^x}{\lambda(\Psi(e^x))}.
$$

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The Condition (4.17) means that $\frac{\lambda(t)}{\lambda(t)}$ is bounded in the limit. By replacing $t$ with $\Psi(e^x)$, it follows that $H'(x)$ is also bounded in the limit, i.e., there exist $c > 0$, $x_0 > 0$ such that $H'(x) \leq c$ for all $x > x_0$. Hence

$$|H(x) - H(y)| \leq |x - y|H'(\xi) \quad \text{where } \xi \in [x, y] \cup [y, x]$$

$$\leq c|x - y| \text{ for all } x, y > x_0.$$ (4.18)

Recall that $W^*_n = \sum_{i=1}^{T_{n+1}-1} V^*_i$, where the $V_i$'s are i.i.d. Exp(1) variables. From the strong law of large numbers, it follows that

$$\frac{W^*_n}{T_{n+1}} \to 1 \text{ a.s. or } \log W^*_n - \log T_{n+1} \to 0 \text{ a.s.}$$

Using (4.18) we get

$$|\Psi(W^*_n) - \Psi(T_{n+1})| = |H(\log W^*_n) - H(\log T_{n+1})|$$

$$\leq c|\log W^*_n - \log T_{n+1}|$$

for all $\log W^*_n$, $\log T_{n+1} > x_0$. Clearly $W^*_n$, $T_n \to \infty$ a.s., thus

$$|\Psi(W^*_n) - \Psi(T_{n+1})| \to 0 \text{ a.s.}$$

Since $U_n = \Psi(W^*_n) - \Psi(W^*_{n-1})$,

$$|U_n - (\Psi(T_{n+1}) - \Psi(T_n))| = |\Psi(W^*_n) - \Psi(T_{n+1}) + \Psi(T_n) - \Psi(W^*_{n-1})|$$

$$\leq |\Psi(W^*_n) - \Psi(T_{n+1})| + |\Psi(W^*_{n-1}) - \Psi(T_n)| \to 0 \text{ a.s.} \Box$$

We need one more lemma before showing the main results.

**Lemma 4.10** Let $D \subset (0, \infty)$ be countable. If $L(a) = \lim_{t \to \infty} \frac{S(t)}{S(t+a)}$ exists for all $a \in [0, \infty) \setminus D$ then either $L(a) \equiv 0$, $L(a) \equiv 1$ or $L(a) = e^{-ca}$ ($c > 0$) for all $a > 0$. 81
**Proof:** Clearly, \( L(a) \) is nonincreasing, \( L(0) = 1 \), \( L(a) \in [0, 1] \) for all \( a \in (0, \infty) \setminus D \) and

\[
L(a + b) = \lim_{t \to \infty} \frac{S(\Lambda(t))}{S(\Lambda(a + b + t))} = \lim_{t \to \infty} \frac{S(\Lambda(t))}{S(\Lambda(a + t))} \cdot \lim_{t \to \infty} \frac{S(\Lambda(a + t))}{S(\Lambda(a + b + t))}
\]

\( L(a + b) = L(a)L(b) \) for all \( a, b \in [0, \infty) \setminus D \). (4.19)

It is well known that the only nonincreasing solutions of this Cauchy type functional equation are (see, for example, Aczel (1966))

\[
L(a) \equiv 0, \; L(a) \equiv 1, \; L(a) = e^{-ca}, \; c > 0.
\]

Hence, if (4.19) held for all \( a, b \geq 0 \), the result would follow. For this, it remains to show that \( L(a) \) is continuous for all \( a \in [0, \infty) \) (where then \( L(a) := \lim_{n \to a} L(a_n) \) for \( a \in D \)). Say \( L \) is not continuous in \( a \in (0, \infty) \). Note that this implies \( L(a-) > 0 \).

Then there exist \( \varepsilon_n \to 0 \) with \( a - \varepsilon_n, 2\varepsilon_n \in (0, \infty) \setminus D \) for all \( n \) such that

\[
\frac{L(a + \varepsilon_n)}{L(a - \varepsilon_n)} = L(2\varepsilon_n) \leq \delta < 1 = L(0)
\]

for sufficiently large \( n \). Hence \( L \) is discontinuous at 0 with a jump of at least \((1 - \delta)\). In view of (4.20), however, this in turn implies that \( L \) has a jump of at least \((1 - \delta)\) \( L(a) \) in all \( x \in (0, a) \setminus D \) which contradicts the boundedness of \( L \). \( \Box \)

We will now look at some classes of combinations of \( \alpha \)-structures and mean intensities of the Poisson pacing process that lead to a limit for \( U_n \).

**Theorem 4.11** Let \( \liminf_{t \to \infty} \frac{\Lambda(t)}{\Lambda_\tau(t)} > 0 \) and \( \tau(t) = \log S(\Lambda(t)) \). If \( U_n \xrightarrow{\mathcal{L}} U \), then the c. d. f. of \( U \) is given by \( F_U(a) = 1 - \lim_{t \to \infty} \frac{S(\Lambda(t))}{S(\Lambda(t + a))} \) (\( a > 0 \)) and one of the
following holds:

\[ U \equiv 0 \quad \text{and} \quad \frac{\tau(t)}{t} \to \infty \ (t \to \infty) \quad (4.21) \]

\[ U \equiv \infty \quad \text{and} \quad \tau(t) = o(t) \quad (4.22) \]

\[ U \sim \text{Exp} \left( \frac{1}{\epsilon} \right) \quad \text{and} \quad \tau(t) \sim ct. \quad (4.23) \]

In the last case, the sequence of processes \( \{cU_{n+k}\}_{k \geq 1} \) converges in law to the i. i. d. process \( \{E_k\}_{k \geq 1} \), where \( E_k \) has Exp(1) distribution.

**Proof:** Let \( a > 0 \).

\[
P(\Psi(T_{n+1}) - \Psi(T_n) > a | T_n = i) = P(\Psi(T_{n+1}) > a + \Psi(T_n) | T_n = i) \]

\[ = P(T_{n+1} > \Lambda(a + \Psi(i)) | T_n = i) \]

\[ = \frac{S(i)}{S(\Lambda(a + \Psi(i)))}. \]

Since \( T_n \to \infty \) a. s. as \( n \to \infty \),

\[
\lim_{n \to \infty} P(\Psi(T_{n+1}) - \Psi(T_n) > a) = \lim_{x \to \infty} \frac{S(x)}{S(\Lambda(a + \Psi(x)))}
\]

if these limits exist. From Lemma 4.9 we conclude

\[
\lim_{n \to \infty} P(U_n > a) = \lim_{x \to \infty} \frac{S(x)}{S(\Lambda(a + \Psi(x)))} = \lim_{t \to \infty} \frac{S(\Lambda(t))}{S(\Lambda(a + t))}
\]

if these limits exist, i. e., if one limit exists the other one also exists and they are equal. The c. d. f. \( F_U \) can only be discontinuous on a countable set, say \( D \). Therefore, \( \lim_{n \to \infty} P(U_n > a) = \lim_{t \to \infty} \frac{S(\Lambda(t))}{S(\Lambda(a + t))} \) exists for all \( a \in (0, \infty) \setminus D \) and Lemma 4.10 yields the 3 possibilities for \( U \) given by (4.21) - (4.23). We only show the functional form of \( \tau \) in the last case. The first 2 cases can be proven on similar lines. We have

\[ e^{-ca} = 1 - F_U(a) = \lim_{t \to \infty} \frac{S(\Lambda(t))}{S(\Lambda(a + t))} = e^{\lim_{t \to \infty} [\tau(t) - \tau(a + t)]}. \]
Hence,
\[ \lim_{t \to \infty} [\tau(a + t) - \tau(t)] = ca \text{ for all } a > 0. \]

Consider \( a = 1, \epsilon > 0 \). There exists an integer \( t_0 > 0 \) such that
\[ c - \epsilon < \tau(t + 1) - \tau(t) < c + \epsilon \text{ for all } t \geq t_0. \]

On applying the first part of this inequality \([t]\) times, the second part \([t] + 1\) times and using the fact that \( \tau \) is strictly increasing we obtain
\[ ([t] - t_0)(c - \epsilon) < \tau(t) - \tau(t_0) < ([t] + 1 - t_0)(c + \epsilon) \]
for all \( t \geq t_0 \), where \([x]\) is the greatest integer that is not greater than \( x \). On dividing by \( t \) and letting \( t \to \infty \) we get
\[ c - \epsilon \leq \liminf_{t \to \infty} \frac{\tau(t)}{t} \leq \limsup_{t \to \infty} \frac{\tau(t)}{t} \leq c + \epsilon. \]

Now we let \( \epsilon \) go to zero and we see that \( \tau(t) \sim ct \).

It remains to show the process convergence in the exponential case. Recalling Lemma 4.9, it is clear that \( cU_n \xrightarrow{d} \text{Exp}(1) \) is equivalent to
\[ \lim_{n \to \infty} P(\Psi(T_{n+1}) - \Psi(T_n) > a) = e^{-ca}. \tag{4.24} \]

To show convergence in law of the process \((U_n, U_{n+1}, \ldots)\) it is sufficient to show convergence of finite-dimensional distributions (see, e.g., Pollard (1984), Chapter 5). We will show
\[ \lim_{n \to \infty} P(\Psi(T_{n+k+1}) - \Psi(T_{n+k}) > a_k, 1 \leq k \leq p) = e^{-c(a_1 + \cdots + a_p)} \tag{4.25} \]
for any integer \( p \geq 1 \) and any sequence \( \{a_k\}_{k \geq 1} \) with \( a_k > 0 \). The last part of the theorem follows from this by again applying Lemma 4.9. The proof of (4.25) is by
induction. The case \( p = 1 \) follows from (4.24). Assume (4.25) has been shown for \( p - 1 \). Let

\[
O_n = \{ \Psi(T_{n+k+1}) - \Psi(T_{n+k}) > a_k, 1 \leq k \leq p - 1 \}
\]

\[
Q_n = \{ \Psi(T_{n+p+1}) - \Psi(T_{n+p}) > a_p \}, \quad \sigma_n = \sigma(T_1, \ldots, T_{n+p-1})
\]

where \( \sigma(Y_1, \ldots, Y_n) \) is the smallest \( \sigma \)-algebra generated by \( (Y_1, \ldots, Y_n) \). Recalling that \( T_n \) is a Markov chain, we get from (4.24)

\[
P(Q_n|\sigma_n) \to e^{-c a_p}
\]

and from (4.25)

\[
E1_{O_n} = P(O_n) \to e^{-c(a_1 + \cdots + a_{p-1})}.
\]

Now

\[
P(O_n Q_n) = E[P(O_n Q_n|\sigma_n)] = E[1_{O_n} P(Q_n|\sigma_n)]
\]

and by the bounded convergence theorem

\[
\lim_{n \to \infty} P(O_n Q_n) = \lim_{n \to \infty} E[1_{O_n} P(Q_n|\sigma_n)]
\]

\[
= E \left[ \lim_{n \to \infty} 1_{O_n} P(Q_n|\sigma_n) \right] = e^{-c(a_1 + \cdots + a_p)}. \quad \square
\]

**Theorem 4.12** (converse of Theorem 4.11)

If \( \lim_{t \to \infty} \frac{S(\lambda(t))}{S(\lambda(a+t))} \) exists for all \( a > 0 \) (or equivalently for all \( a \) in some interval
$(0, \epsilon), \epsilon > 0$, we get

\[
\tau(t) \sim ct \implies U_n \xrightarrow{c} \text{Exp} \left( \frac{1}{c} \right)
\]

\[
\frac{\tau(t)}{t} \to \infty \implies U_n \xrightarrow{c} U \equiv 0
\]

\[
\tau(t) = o(t) \implies U_n \xrightarrow{c} U \equiv \infty.
\]

**Proof:** We just consider $\tau(t) \sim ct$. The other two cases can be handled similarly.

Since $\lim_{t \to \infty} \frac{S(\tau(t))}{S(\tau(a+t))} \in [0,1]$ exists for all $a > 0$, we have one of the following 3 possibilities:

- $U_n \xrightarrow{c} \text{Exp} \left( \frac{1}{c_0} \right)$ and $\lim_{t \to \infty} [\tau(a + t) - \tau(t) - \tau(a + t)] = 0$
- $U_n \xrightarrow{c} U \equiv 0$ and $\lim_{t \to \infty} [\tau(a + t) - \tau(t)] = 0$
- $U_n \xrightarrow{c} U \equiv \infty$ and $\lim_{t \to \infty} [\tau(a + t) - \tau(t)] = \infty$.

We will now show that the last two are not possible and the first can only hold with $c = c_0$. Suppose either the last case or the first case with $c_0 > c$, holds. (The second case and $c_0 < c$ can be shown to lead to a contradiction in the same manner.)

Let $0 < \epsilon < c_0 - c (\epsilon > 0$ for $U \equiv \infty), a = 1$. There exists a $t_0 > 0$ such that $\tau(t + 1) - \tau(t) > c + \epsilon$ for all $t \geq t_0$. Applying this inequality $n$ times, we obtain $\tau(n + t_0) - \tau(t_0) > n(c + \epsilon)$ for all positive integers $n$. On dividing by $n + t_0$ and letting $n \to \infty$ we get

\[
\lim_{n \to \infty} \inf \frac{\tau(n + t_0)}{n + t_0} \geq c + \epsilon.
\]

Hence,

\[
\lim_{t \to \infty} \sup \frac{\tau(t)}{t} \geq c + \epsilon
\]

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which contradicts $\tau(t) \sim ct$. □

**Example 4.5** Let $r \geq 1$, $c > 0$, $\Lambda(t) = e^{[(ct)^r]} - 1$ and

$$S(x) = e^{\left(\log(x+1)^{1/r}\right)} - 1 = \exp\left(\frac{1}{r} \log \log(x + 1)\right) - 1.$$ 

It follows that $\lambda(t) = r e^{t r^{-1} e^{[(ct)^r]}}$ and

$$\lim_{t \to \infty} \frac{\lambda(t)}{\Lambda(t)} = r e^{t r^{-1}} \lim_{t \to \infty} t r^{-1} = \begin{cases} \infty & \text{if } r = 1 \\ c & \text{if } r > 1 \end{cases} > 0.$$ 

Further, $S(\Lambda(t)) = e^{ct} - 1$, $\frac{S(\Lambda(t))}{S(\Lambda(a + t))} \to e^{-ca}$ and $\tau(t) = ct$. Therefore, by Theorem 4.12, $\{c U_{n+k}\}_{k \geq 1} \overset{C}{\to} \{E_k\}_{k \geq 1}$. Note that for $r = 1$, $S(x) = x$ and we have the classical model.

**Example 4.6** Let $S(x) = \log(x+e)$, $\Lambda(t) = \exp\left(\exp\left(t^r\right)\right) - e$, $r > 0$. Then $\tau(t) = t^r$ and

$$\lim_{t \to \infty} \frac{\lambda(t)}{\Lambda(t)} = \lim_{t \to \infty} r t^{r-1} e^{(r)} = \infty > 0,$$

$$\lim_{t \to \infty} \frac{S(\Lambda(t))}{S(\Lambda(a + t))} = \lim_{t \to \infty} \frac{e^{(r)}}{e^{(a+1)^r}} = \begin{cases} 1 & \text{if } r < 1 \\ e^{-a} & \text{if } r = 1 \\ 0 & \text{if } r > 1 \end{cases}.$$ 

By Theorem 4.12,

$$U_n \overset{C}{\to} \begin{cases} \infty & \text{if } r < 1 \\ \text{Exp}(1) & \text{if } r = 1 \\ 0 & \text{if } r > 1 \end{cases}.$$ 

In the cases where $U_n \to 0$ and $U_n \to \infty$ we would like to know how fast this convergence occurs, in particular, is it possible to find norming constants $a_n$, $b_n$ such

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that \( \frac{U_n - \alpha}{b_n} \) goes in law to a nondegenerate distribution? It turns out that we can. However, we need some conditions on the \( \alpha \)-structure itself (no such condition was required in Theorem 4.12). First we consider the case where \( p_n = \frac{\alpha_n}{S(n)} \to 0 \). Let 
\( A(n) = \sum_{i=1}^{n} a_i, B(n) = \sum_{i=1}^{n} p_i^2 \).

**Lemma 4.13** In a Poisson paced \( F^\alpha \) model, let \( p_n = \frac{\alpha_n}{S(n)} \to 0 \), \( \frac{B(n)}{\sqrt{A(n)}} \to 0 \). Let 
\( \tau(t) = \log S(A(t)) \) be differentiable and \( b_n = \frac{1}{\tau^{-1}(n)} \). If there exists a function \( h \) with \( h(n) \to \infty \) as \( n \to \infty \) and

\[
\gamma_n = 2\sqrt{2S^{-1}(e^{n+\sqrt{nh(n)}})\log \log (e^{n+\sqrt{nh(n)}})}\tau'(\tau^{-1}(n)) \sup_{\xi \in K_n} \Psi'(\xi) \to 0 \quad (4.26)
\]

where

\[
K_n = \left[ S^{-1}(e^{n-\sqrt{nh(n)}}) - 2\sqrt{2S^{-1}(e^{n-\sqrt{nh(n)}})\log \log (e^{n-\sqrt{nh(n)}})} \right] \\
S^{-1}(e^{n+\sqrt{nh(n)}}) + 2\sqrt{2S^{-1}(e^{n+\sqrt{nh(n)}})\log \log (e^{n+\sqrt{nh(n)}})}
\]

then

\[
\frac{|U_n - \frac{\Psi(T_{n+1}) - \Psi(T_n)}{b_n}|}{b_n} \overset{P}{\to} 0.
\]

**Proof:** Since \( W_n^* = V_1 + \cdots + V_{T_{n+1}-1} \) where the \( V_i \)'s are i. i. d. \( \text{Exp}(1) \) variables and \( T_n \to \infty \) a. s., it follows from the law of the iterated logarithm (see for example Chung (1974, section 7.5)) that

\[
\limsup_{n \to \infty} \frac{|W_n^* - (T_{n+1} - 1)|}{\sqrt{2(T_{n+1} - 1)\log \log(T_{n+1} - 1)}} \leq 1 \text{ a. s. or}
\]

\[
\limsup_{n \to \infty} \frac{|W_n^* - T_{n+1}|}{\sqrt{2T_{n+1} \log \log T_{n+1}}} \leq 1 \text{ a. s.}
\]

Hence, there exists an \( n_0 \) such that

\[
|W_n^* - T_{n+1}| \leq 2\sqrt{2T_{n+1} \log \log T_{n+1}} \text{ a. s. for all } (T_n \geq) n > n_0.
\]

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Thus,

\[ |\Psi(W_n^*) - \Psi(T_{n+1})| \leq 2\sqrt{2T_{n+1} \log \log T_{n+1}} \sup_{\xi \in J_n} \Psi'(\xi) \text{ a.s.} \]

where

\[ J_n = \left[ T_{n+1} - 2\sqrt{2T_{n+1} \log \log T_{n+1}}, T_{n+1} + 2\sqrt{2T_{n+1} \log \log T_{n+1}} \right]. \]

Hence,

\[ \frac{|\Psi(W_n^*) - \Psi(T_{n+1})|}{b_n} \leq 2\sqrt{2T_{n+1} \log \log T_{n+1}} \tau'(n^{-1}) \sup_{\xi \in J_n} \Psi'(\xi) \text{ a.s.} \quad (4.27) \]

and

\[ \frac{|\Psi(W_{n-1}^*) - \Psi(T_n)|}{b_n} \leq 2\sqrt{2T_n \log \log T_n} \tau'(n^{-1}) \sup_{\xi \in J_{n-1}} \Psi'(\xi) \text{ a.s.} \quad (4.28) \]

If the right hand side goes to zero in probability, so does the left hand side. However, from Nevzorov (1995),

\[ \frac{\log S(T_n) - n}{\sqrt{n}} \frac{P}{\sqrt{h(n)}} \rightarrow N(0, 1). \]

Therefore \( \frac{\log S(T_n) - n}{\sqrt{h(n)}} \rightarrow 0 \) for any \( h(n) \rightarrow \infty (n \rightarrow \infty) \), and also \( \frac{\log S(T_{n+1}) - n}{\sqrt{h(n)}} \rightarrow 0 \), which implies that

\[ P \left( S^{-1} \left( e^n \sqrt{h(n)} \right) \leq T_{n+1} \leq S^{-1} \left( e^n \sqrt{h(n)} \right) \right) \rightarrow 1 \]

and

\[ P \left( S^{-1} \left( e^{-n} \sqrt{h(n)} \right) \leq T_n \leq S^{-1} \left( e^{-n} \sqrt{h(n)} \right) \right) \rightarrow 1. \]

Hence, the right hand sides of (4.27) and (4.28) go in probability to zero if \( \gamma_n \rightarrow 0 \).

Recall that \( U_n = \Psi(W_n^*) - \Psi(W_{n-1}^*) \). From (4.27) and (4.28) it now follows that

\[ \frac{|U_n|}{b_n} = \frac{|\Psi(W_n^*) - \Psi(T_{n+1})|}{b_n} - \frac{|\Psi(W_{n-1}^*) - \Psi(T_n)|}{b_n} \leq \frac{|\Psi(W_n^*) - \Psi(T_{n+1})|}{b_n} + \frac{|\Psi(W_{n-1}^*) - \Psi(T_n)|}{b_n} \rightarrow 0. \]

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Theorem 4.14 In a Poisson paced $F^\alpha$ model, let the conditions of Lemma 4.13 hold and let $p_n = \frac{a_n}{s(n)} \to 0$, $\frac{B(n)}{\sqrt{A(n)}} \to 0$, where $A(n) = \sum_{i=1}^{n} p_i$, $B(n) = \sum_{i=1}^{n} p_i^2$. Suppose for all $x > 0$ there exists a function $g_x(n)$ with $g_x(n) \to \infty (n \to \infty)$ such that

$$b_n \inf_{t \in I_n} \tau'(t) \to 1 \quad \text{and} \quad b_n \sup_{t \in I_n} \tau'(t) \to 1$$

(4.29)

where

$$b_n = \frac{1}{\tau'(\tau^{-1}(n))}, \quad I_n = [\tau^{-1}(n - \sqrt{n} g_x(n)), \tau^{-1}(n + \sqrt{n} g_x(n)) + b_n x].$$

Then

$$\frac{U_n}{b_n} \xrightarrow{c} \exp(1).$$

Proof: Let $x > 0$. When $p_n \to 0$, $\frac{B(n)}{\sqrt{A(n)}} \to 0$, Nevzorov (1986b, 1995) shows that

$$\log S(T_n) - n \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1).$$

Therefore

$$\frac{\log S(T_n) - n}{\sqrt{n} g_x(n)} = \frac{\tau(\Psi(T_n)) - n}{\sqrt{n} g_x(n)} \xrightarrow{p} 0.$$ 

Hence

$$P \left( -1 \leq \frac{\tau(\Psi(T_n)) - n}{\sqrt{n} g_x(n)} \leq 1 \right) \to 1$$

$$P \left( n - \sqrt{n} g_x(n) \leq \tau(\Psi(T_n)) \leq n + \sqrt{n} g_x(n) \right) \to 1$$

$$P \left( \tau^{-1}(n - \sqrt{n} g_x(n)) \leq \Psi(T_n) \leq \tau^{-1}(n + \sqrt{n} g_x(n)) \right) \to 1$$

$$P \left( b_n \inf_{t \in I_n} \tau'(t) \leq b_n \tau'(\Psi(T_n) + \xi_n x b_n) \leq b_n \sup_{t \in I_n} \tau'(t) \right) \to 1 \quad \forall \xi_n \in [0, 1].$$

Since we assumed that both sides go to one,

$$b_n \tau'(\Psi(T_n) + \xi_n x b_n) \xrightarrow{p} 1 \quad \forall \xi_n \in [0, 1].$$

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However,

\[ Y_n = \tau(\Psi(T_n) + b_n x) - \tau(\Psi(T_n)) \]

\[ = xb_n \tau'(\Psi(T_n) + \xi_n xb_n) \text{ for some } \xi_n \in [0, 1] \]

which means \( Y_n \xrightarrow{p} x \). Therefore \( e^{-Y_n} \xrightarrow{p} e^{-x} \). Since \( \tau \) is strictly increasing, \( Y_n \geq 0 \). Hence \( |e^{-Y_n}| \leq 1 \) and \( e^{-Y_n} \) also converges in \( L^1 \) (Chung, 1974, Theorem 4.1.4); that is

\[ \lim_{n \to \infty} Ee^{-Y_n} = e^{-x} \forall x > 0. \quad (4.30) \]

Now let us look at \( \frac{U_n}{b_n} \). From Lemma 4.13 it follows that

\[ \lim_{n \to \infty} P \left( \frac{U_n}{b_n} > x \right) = \lim_{n \to \infty} P \left( \Psi(T_{n+1}) - \Psi(T_n) > b_n x \right) \]

\[ = \lim_{n \to \infty} P \left( T_{n+1} > \Lambda(\Psi(T_n) + b_n x) \right) \]

\[ = \lim_{n \to \infty} \sum_{i=n}^{\infty} P \left( T_{n+1} > \Lambda(\Psi(T_n) + b_n x) \mid T_n = i \right) \cdot P(T_n = i) \]

\[ = \lim_{n \to \infty} \sum_{i=n}^{\infty} \frac{S(i)}{S(\Lambda(\Psi(i) + b_n x))} P(T_n = i) \]

\[ = \lim_{n \to \infty} E \left[ \frac{S(T_n)}{S(\Lambda(\Psi(T_n) + b_n x))} \right] \]

\[ \lim_{n \to \infty} P \left( \frac{U_n}{b_n} > x \right) = \lim_{n \to \infty} Ee^{-\tau(\Psi(T_n)) - \tau(\Psi(T_n) + b_n x)} \]

\[ = \lim_{n \to \infty} Ee^{-Y_n} = e^{-x}. \]
Remark: The Conditions (4.26) in Lemma 4.13 and (4.29) in Theorem 4.14 look cumbersome, but are usually not very hard to check. In all the situations we have considered, the choice of $n^\epsilon$, $\epsilon \in (0, \frac{1}{2})$ or $\log n$ for $g(n)$ and $h(n)$ work. Below are a few examples.

**Example 4.7** Let $S(x) = x$ (classical model), $\Lambda(t) = e^{(r)} - 1$, $r > 0$.

It follows that $\tau(t) = \log (e^{(r)} - 1) \sim t^r$, $b_n = \frac{1}{\tau^{-1}(n)} = \frac{e^{n+1}}{e^n - 1} \left[ \log (e^n + 1) \right]^{\frac{1}{r} - 1} \sim \frac{1}{r} n^{\frac{1}{r} - 1}$, $\tau'(t) = r t^{r-1} \frac{e^{(r)}}{e^n - 1} \sim rt^{r-1}$. Note that $\tau'(t)$ is strictly increasing for large $t$ ($r > 1$), constant ($r = 1$) or strictly decreasing ($r < 1$). Therefore, to check (4.29), we only have to check whether $b_n \tau'(\tau^{-1}(n - \sqrt{n}g_x(n)))$ and $b_n \tau'(\tau^{-1}(n + \sqrt{n}g_x(n))) + b_n x$ go to one.

$$b_n \tau'(\tau^{-1}(n - \sqrt{n}g_x(n))) \sim \frac{1}{r} n^{\frac{1}{r} - 1} . \left\{ \left[ \log (e^{n-\sqrt{n}g_x(n)} + 1) \right]^{\frac{1}{r} - 1} \right\}^{\frac{1}{r} - 1}$$

$$\sim n^{\frac{1}{r} - 1} \left[ \log (e^{n-\sqrt{n}g_x(n)} + 1) \right]^{\frac{1}{r} - 1}$$

$$\rightarrow 1 \text{ for all } g_x(n) = n^\epsilon, \epsilon \in (0, \frac{1}{2}).$$

$$b_n \tau'(\tau^{-1}(n + \sqrt{n}g_x(n))) + b_n x \sim \frac{1}{r} n^{\frac{1}{r} - 1} \left\{ \left[ \log (e^{n+\sqrt{n}g_x(n)} + 1) \right]^{\frac{1}{r} + x} + \frac{x}{r} n^{\frac{1}{r} - 1} \right\}^{\frac{1}{r} - 1} \rightarrow 1$$

for $g_x(n) = n^\epsilon, \epsilon \in (0, \frac{1}{2})$. To check (4.26), note that

$$\Psi(t) = \left( \log(t + 1) \right)^{\frac{1}{r}}, S^{-1}(x) = x, \Psi'(t) = \frac{\left( \log(t + 1) \right)^{\frac{1}{r} - 1}}{r(t + 1)}.$$

Since $\Psi'(t)$ is strictly decreasing,

$$\sup_{\xi \in K_n} \Psi' (\xi) = \Psi' \left( e^{n-\sqrt{n}h(n)} - 2 \sqrt{2e^n-\sqrt{n}h(n)} \log(n - \sqrt{n}h(n)) \right).$$

Hence,

$$\gamma_n \sim 2 \sqrt{2e^n-\sqrt{n}h(n)} \log(n - \sqrt{n}h(n)) \sqrt{n^{\frac{1}{r} - 1}} \left[ n - \sqrt{n}h(n) \right]^{\frac{1}{r} - 1} \frac{1}{re^{n-\sqrt{n}h(n)}} \rightarrow 0$$
for \( h(n) = n^\epsilon, \epsilon \in (0, \frac{1}{2}) \).

It follows that \( \frac{\bar{U}_{n}}{n} \xrightarrow{L} \text{Exp}(1) \). Notice that \( b_n \) (and \( U_n \)) go to zero for \( r > 1 \) and to \( \infty \) for \( r < 1 \).

**Example 4.8** Let \( S(x) = x^r \), \( \Lambda(t) = e^{\frac{1}{2}t^r} - e^t \), \( r > 0 \). It follows that

\[
\tau(t) = r \log \left( e^{\frac{1}{2}t^r} - e^t \right), \quad \tau'(t) = \frac{e^{\frac{1}{2}t^r} - e^t}{e^{\frac{1}{2}t^r} - e^t} \sim e^t, \quad b_n \sim \frac{1}{n}, \quad \frac{p_n}{\bar{U}_{n}} = 1 - \frac{S(n-1)}{S(n)} = 1 - \frac{(n-1)^r}{n^r} \to 0. \quad \frac{\bar{U}_{n}}{\sqrt{A(n)}} \to 0 \text{ also holds. Since } \tau'(t) \text{ is strictly increasing for large } t,
\]

\[
b_n \inf_{\xi \in I_n} \tau'(\xi) \sim \frac{1}{n} \log(n - \sqrt{n} g_x(n)) \to 1 \text{ for } g_x(n) = n^\epsilon, \epsilon \in (0, \frac{1}{2}).
\]

Similarly, \( b_n \sup_{\xi \in I_n} \tau'(\xi) \to 1 \) for \( g_x(n) \) as above.

\[
\Psi(t) = \log \left( r \log \left( t + e^t \right) \right), \quad \Psi'(t) = \frac{1}{t + e^t} \log \left( t + e^t \right), \quad S^{-1}(x) = x^{\frac{1}{t}}.
\]

\[
\sup_{\xi \in I_n} \Psi'(\xi) \sim e^{-\frac{n - n^{\sqrt{h(n)}}}{t}} \frac{1}{\log \left( \frac{n - n^{\sqrt{h(n)}}}{t} \right)},
\]

\[
\gamma_n \sim 2 \sqrt{2e^{-\frac{n - n^{\sqrt{h(n)}}}{t}} \log \left( \frac{n + n^{\sqrt{h(n)}}}{t} \right)} n e^{-\frac{\sqrt{n^{\sqrt{h(n)}}}}{t}} \frac{1}{\log \left( \frac{n - n^{\sqrt{h(n)}}}{t} \right)} \to 0
\]

for \( h(n) = n^\epsilon, \epsilon \in (0, \frac{1}{2}) \).

Therefore \( nU_n \xrightarrow{L} \text{Exp}(1) \).

**Example 4.9** When \( \tau(t) = \log t \), it follows that \( \tau'(t) = \frac{1}{t} \) is strictly decreasing, \( \tau^{-1}(t) = e^t, b_n = e^n, \) and

\[
b_n \sup_{\xi \in I_n} \tau'(\xi) = e^n e^{-\left( n - \sqrt{n} g_x(n) \right)} = e^{\sqrt{n} g_x(n)} \to \infty \text{ for all } g_x \text{ with } g_x(n) \to \infty.
\]

Hence, (4.29) fails to hold and Theorem 4.14 cannot be applied. This happens for the classical record model \( S(x) = x \) associated with the homogeneous Poisson pacing.
process \((\Lambda(t) = t)\). In this case, recall from Theorem 4.7 that \(U_n \overset{d}{=} A_n e^{C_n}\), where \(A_n \sim \text{Exp}(1), C_n \sim \text{Gamma}(n, 1)\) and \(A_n, C_n\) independent. Hence \(EU_n = \infty\) for all \(n\) and indeed there cannot exist sequences \(a_n, b_n\) such that \(\frac{U_n - a_n}{b_n}\) has an exponential limit.

**Remark:** When \(p_n \to p \in (0, 1)\), techniques used in proving Lemma 4.13 and Theorem 4.14 do not work because \(\frac{\psi(W_n) - \psi(T_{n+1})}{b_n} \not\overset{P}{\to} 0\). However, if we let \(S(n) = c^n (c > 1)\), then \(p_n \equiv \frac{c-1}{c} \in (0, 1)\) and it follows from Bunge and Nagaraja (1992a) that \((1 - c^{-1})U_n \overset{\mathcal{L}}{\to} \text{Exp}(1)\). We conjecture therefore that there exist \(b_n\)'s such that \(\frac{U_n}{b_n}\) has an exponential limit in a much larger class defined by combinations of \(\alpha\)-structures and intensities of the Poisson process than given here.

### 4.6 Arrival time dependent observations

An interesting consideration would be to let the \(\alpha\)'s be time dependent instead of being fixed. To give an example where this arises naturally, consider the original model for sports records from Yang (1975). The \(\alpha\)'s there represented the world or athlete population at the time of the event. Since this population is growing more or less continuously, time dependent \(\alpha\)'s would be ideal. Under this assumption, we have \(\alpha_k = \alpha(Z_k) = \alpha(V_1 + \cdots + V_k)\) where \(\alpha\) is a function: \((0, \infty) \to (0, \infty)\). (Here we also let \(X_1\) occur at a random time \(Z_1 > 0\) determined by \(P\) to have all \(\alpha\)'s random.) Note that now the \(X_i\)'s are no longer independent. Further, the observations and the pacing process are dependent. See also Westcott (1977, sec. 7) for other models where the observations and the pacing process are dependent.
This is a special case of the random $F^a$ model and all the results from Chapter 2 for $T_n, N_n, R_n, \Delta_n$ hold. Let us look at the process related record statistics $M(t), U_n, W_n$.

From (4.6) it follows that for $m \geq 1, k \geq m$ and $0 < s_1 < \ldots < s_k < t < s_{k+1}$

$$P(M(t) = m | Z_1 = s_1, \ldots, Z_k = s_k \cap N(t) = k)$$

$$= \frac{\alpha(s_1)}{S(k) \prod_{i=1}^{k+1} S(s_i)} \sum_{1 \leq j_1 < \ldots < j_{m-1} \leq k-1} \frac{\alpha(s_{j_1+1}) \ldots \alpha(s_{j_{m-1}+1})}{S(j_1) \ldots S(j_{m-1})}$$

where $S(n) = \alpha(s_1) + \ldots + \alpha(s_n)$.

On integrating over the arrival times we obtain

$$P(M(t) = m | N(t) = k)$$

$$= \frac{1}{P(Z_k \leq t < Z_{k+1})} \int_{0 < s_1 < \ldots < s_k < t < s_{k+1}} \frac{\alpha(s_1)}{S(k)}$$

$$\cdot \sum_{1 \leq j_1 < \ldots < j_{m-1} \leq k-1} \frac{\alpha(s_{j_1+1}) \ldots \alpha(s_{j_{m-1}+1})}{S(j_1) \ldots S(j_{m-1})} f_Z(s_1, \ldots, s_{k+1}) \prod_{i=1}^{k+1} ds_i,$$

where $f_Z(s_1, \ldots, s_{k+1})$ is the joint p. d. f. of $Z_1, \ldots, Z_{k+1}$. Hence,

$$P(M(t) = m)$$

$$= \sum_{k=m}^{\infty} P(Z_k \leq t < Z_{k+1}) \int_{0 < s_1 < \ldots < s_k < t < s_{k+1}} \alpha(s_1) S(k)$$

$$\cdot \sum_{1 \leq j_1 < \ldots < j_{m-1} \leq k-1} \frac{\alpha(s_{j_1+1}) \ldots \alpha(s_{j_{m-1}+1})}{S(j_1) \ldots S(j_{m-1})} f_Z(s_1, \ldots, s_{k+1}) \prod_{i=1}^{k+1} ds_i,$$

where $t_k = P(N(t) = k)$. Unfortunately, even for "nice" processes and $\alpha$-functions this seems unmanageable analytically. The same integral-sum combination also appears in the distribution of the $U_n$'s.
CHAPTER 5

CONCLUSIONS AND FUTURE WORK

In this dissertation, we have examined two general record models. First we introduced the random $F^a$ model and discovered that it includes many previously considered record models as special cases. The hierarchy of these models was illustrated in Figure 2.1 on page 25. We discussed some small sample properties for various record statistics and then showed asymptotic results for record counts and record times under several different assumptions on the model. Especially noteworthy were the central limit theorems we obtained under the condition of $m$-dependent record indicators.

Towards the end of Chapter 2, we gave all possible limit distributions for record values. We noted that the set of these limit distributions is more general than encountered in a classical record model. In fact, two of the three limit laws have an extra parameter. From the perspective of Yang (1975) one can identify the (in his case fixed) $\alpha_i$'s with the size of the athletic population. Here, for each sports event only the largest observation is considered as a "candidate" for a record, where in the classical model since the population size is constant, each observation has to be considered. It appears that in some cases this difference cannot be accounted for in the limit distribution by a location and/or scale change.
The random $F^\alpha$ model is very flexible since the $\alpha_i$'s are arbitrary positive random variables and $F$ need only be a continuous c. d. f. One direction for my future research will be to apply this model to real data by putting a more specific structure on the $\alpha$'s and on $F$. Besides the traditional athletic records example, insurance claim data has been suggested to me as a possible application.

In Chapter 3 we gave the optimal solution to two variations of the secretary problem when the underlying sequence is modeled in random $F^\alpha$ fashion. In the future I plan to look at a number of other record related optimal stopping problems in the more general $F^\alpha$ context.

In Chapter 4 we considered the situation when the observations arrive in continuous time according to an underlying point process $P$. For homogeneous Poisson $P$'s we gave distributional representations for the (continuous) inter-record times. In the general situation of non-homogeneous Poisson pacing processes, we first classified the limit of the inter-record times $U_n$ themselves (without normalizing) in three cases that depend on the limiting properties of the composite function $S(\Lambda(t))$. Then we derived the limit distribution of (suitable normalized) $U_n$ for a class of combinations of $\alpha$-structures and intensity functions of $P$.

Several interesting questions remain in this area. At the end of Section 4.5 we gave an example where $U_n$ has an exponential limit for a combination of $\alpha$-structure and intensity function that does not fall in the above mentioned class. We therefore conjectured that a similar result holds for a larger class. Moreover, in the i. i. d. case point process record models have also been considered for pacing processes other than Poisson, e. g., birth and renewal processes (see Section 4.3 for some references). An exploration of such pacing processes in $F^\alpha$ point process record models promises to
be an exciting task. I plan to follow up on these ideas in future research.
APPENDIX A

SOME PREVIOUSLY STUDIED NON-CLASSICAL RECORD MODELS

A.1 Yang (1975)

The joint c. d. f. of $X_1, \ldots, X_n$ is

$$G_n(x_1, \ldots, x_n) = F_1(x_1) \cdot \ldots \cdot F_n(x_n)$$

where $F$ is a c. d. f. and $\alpha_i = \left[\lambda^{i-1} \alpha_1 + \frac{1}{2}\right]$ for $i > 1$, $\lambda \geq 1$. Yang's idea was to let $\alpha_i$ be the population size (world population or athlete population for a sport) which is geometrically increasing. He considered Olympic records and showed the following properties for inter-record times.

$$P(\Delta_2 > j) = \frac{\alpha_1}{S(j+1)}$$ (A.1)

$$P(\Delta_{n+1} > j)$$

$$= \sum_{k_1=2}^{\infty} \sum_{k_2=k_1+1}^{\infty} \ldots \sum_{k_{n-1}=k_{n-2}+1}^{\infty} \frac{\alpha_1 \alpha_{k_1} \cdots \alpha_{k_{n-1}}}{S(k_1-1) \cdots S(k_{n-1}-1) S(k_{n-1}+j)}$$ (A.2)

$$\lim_{n \to \infty} P(\Delta_n = j) = (\lambda - 1) \lambda^{-j}, \quad j = 1, 2, \ldots$$ (A.3)
A.2 Fixed $F^\alpha$

Let the joint c. d. f. of $X_1, \ldots, X_n$ ($n = 1, 2, \ldots$) be

$$G_n(x_1, \ldots, x_n) = F^{\alpha_1}(x_1) \cdots F^{\alpha_n}(x_n)$$

where $F$ is a continuous c. d. f. and the $\alpha_i$'s are positive constants. Nevzorov treated this model in a series of papers. Basic properties of the model and asymptotic results for record times and counts are discussed in Nevzorov (1986b) and Nevzorov (1985). The characterizations are taken from Nevzorov (1986a, 1990). All the martingale properties are considered in Nevzorov (1990) and the record value asymptotics in Nevzorov (1995). Nevzorov (1995) also summarizes most previous results. Let $S(n) = \alpha_1 + \cdots + \alpha_n$.

**Basic Properties**

$$p_n = P(I_n = 1) = \frac{\alpha_n}{S(n)}$$

(A.4)

**Lemma A.1** $T_2$ is a proper (a. s. finite) random variable if and only if $S(n) \to \infty$. If $T_2$ is proper, so is $T_n$ for all $n \geq 2$.

**Lemma A.2** (Ballerini and Resnick, 1987b) $I_1, \ldots, I_n, M_n$ are (mutually) independent for all $n$.

$$P(T_2 = t_2, \ldots, T_n = t_n) = \frac{\alpha_1}{S(t_n)} \prod_{i=2}^{n} \frac{\alpha_{t_i}}{S(t_i - 1)}$$

(A.5)

**Characterizations**

**Lemma A.3** Let $X_1, \ldots, X_n$ be independent continuous random variables with c. d. f.'s $F_1, \ldots, F_n$ whose supports intersect in more than one point. If $I_1, \ldots, I_n, M_n$ are independent, then there exist constants $\alpha_i$, $i = 2, \ldots, n$ such that $F_i(x) = F_1^{\alpha_i}(x)$ ($i = 2, \ldots, n$).
Lemma A.4 Let \( F(0) = 0, F(x) > 0 \) for \( x > 0 \). \( R_1, R_2 - R_1 \) are independent if and only if \( \alpha_3 = \alpha_4 = \ldots = 1 \) and \( F(x) = 1 - \exp \left( -\frac{x}{\sigma} \right), \sigma > 0. \)

**Asymptotics**

Let

\[
\begin{align*}
    p_n &= \frac{\alpha_n}{S(n)}, \quad A(n) = EN_n = \sum_{i=1}^{n} \frac{\alpha_i}{S(i)}, \\
    B(n) &= \sum_{i=1}^{n} \left( \frac{\alpha_i}{S(i)} \right)^2, \quad D^2(n) = \text{Var}(N_n) = A(n) - B(n), \\
    C(x) &= D(A^{-1}(x)).
\end{align*}
\]

If \( D^2(n) \to \infty \) then
\[
\frac{N_n - A(n)}{D(n)} \overset{\mathcal{L}}{\to} N(0, 1). \tag{A.6}
\]

If \( \frac{B(n)}{A(n)} \to 0 \) (especially true if \( p_n \to 0 \)) then
\[
\frac{N_n - A(n)}{\sqrt{A(n)}} \overset{\mathcal{L}}{\to} N(0, 1)
\]
or analogously in general for \( p_n \to p \in [0, 1) \)
\[
\frac{N_n - A(n)}{\sqrt{A(n)}\sqrt{1 - p}} \overset{\mathcal{L}}{\to} N(0, 1) \tag{A.7}
\]
and
\[
\frac{A(T_n) - n}{\sqrt{n\sqrt{1 - p}}} \overset{\mathcal{L}}{\to} N(0, 1).
\]

Let \( C(x) = D \left( A^{-1}(x) \right) \). If \( D(x) \to \infty \) and
\[
\frac{C(x)}{C_1(x + z\sqrt{x})} \to 1 \text{ for any } z,
\]
then
\[
\frac{A(T_n) - n}{C(n)} \overset{\mathcal{L}}{\to} N(0, 1); \text{ if also } \frac{B(n)}{A(n)} \to 0 \text{ then } \frac{A(T_n) - n}{\sqrt{n}} \overset{\mathcal{L}}{\to} N(0, 1). \tag{A.8}
\]

If \( p_n \to 0 \) and \( \frac{B(n)}{\sqrt{A(n)}} \to 0 \),
\[
\frac{N_n - \log S(n)}{\sqrt{\log S(n)}} \overset{\mathcal{L}}{\to} N(0, 1), \quad \frac{\log S(T_n) - n}{\sqrt{n}} \overset{\mathcal{L}}{\to} N(0, 1). \tag{A.9}
\]

If \( p_n \to p \in (0, 1), \quad \frac{\sum_{i=1}^{n} (p_i - p)}{\sqrt{n}} \to 0 \text{ and } \frac{\sum_{i=1}^{n} (p_i - p)^2}{\sqrt{n}} \to 0. \)
then \( \frac{N_n - p \log S(n)}{p(1-p) \log(S(n))} \to N(0,1) \) and \( \frac{\log S(T_n) + n \log(1-p)}{-p(1-p) \log(1-p)} \sim N(0,1). \) (A.10)

If \( \sum_{i=1}^{n} \frac{p_i - p}{\sqrt{n}} \to 0 \) and \( \sum_{i=1}^{n} \frac{|p_i - p|}{n} \to 0, \)

then \( \frac{N_n - np}{\sqrt{np(1-p)}} \to N(0,1) \) and \( \frac{T_n - \frac{n}{p}}{\sqrt{\frac{n(1-p)}{p^2}}} \sim N(0,1). \) (A.11)

The last result was shown by Ballerini and Resnick (1987b) and corrected by Nevzorov (1995). If \( p_n \to 0 \) then for any fixed \( k \geq 2 \) the random variables

\[ \frac{S(T_n)}{S(T_{n+1})}, \ldots, \frac{S(T_{n+k-1})}{S(T_{n+k})} \]

are asymptotically independent and

\[ \lim_{n \to \infty} \frac{\lim_{n \to \infty} P \left( \frac{S(T_n)}{S(T_{n+1})} < x \right)}{x} = 1 \text{ for all } 0 < x < 1. \] (A.13)

If \( F(x) = 1 - e^{-x} \) then \( P \left( R_n - \log S(T_n) \leq x \right) \to \exp \left( -e^{-x} \right). \) (A.14)

Nevzorov (1995) also gives some bounds on the rate of this convergence.

If \( F(x) = 1 - e^{-x}, p_n \to 0 \) a. s., \( B(n) = \frac{\log(n)}{A(n)} \to 0 \) a. s. then \( \frac{R_n - n}{\sqrt{n}} \to N(0,1). \) (A.15)

If \( F(x) = 1 - e^{-x}, p_n \to p \in (0,1) \) a. s., \( \sum_{i=1}^{n} \frac{p_i - p}{\sqrt{n}} \to 0 \) a. s.,

\[ \sum_{i=1}^{n} \frac{(p_i - p)^2}{\sqrt{n}} \to 0 \text{ a. s. then } \frac{pR_n + n \log(1-p)}{-p(1-p) \log(1-p)} \to N(0,1). \] (A.16)

Let \( F \in D_M(G, a_n, b_n) \) mean that \( F \) is in the domain of maximal attraction of the extreme value distribution \( G \), i.e.,

\[ P \left( \frac{M_n - \hat{a}_n}{\hat{b}_n} < x \right) = F^n(\hat{a}_n + \hat{b}_n x) \to G(x) \]

\[ \forall \hat{a}_n, \hat{b}_n > 0: \frac{\hat{a}_n - a_n}{\hat{b}_n} \to 1, \frac{\hat{a}_n - a_n}{\hat{b}_n} \to 0 \]

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and \( F \in D_R(H, a_n, b_n) \) that \( F \) is in the domain of record attraction of \( H \), i.e.,

\[
P \left( \frac{R_n - \hat{a}_n}{b_n} < x \right) \rightarrow H(x) \text{ for } \hat{a}_n, \hat{b}_n \text{ as above.}
\]

Further let

\[
\Lambda(x) = \exp(-\exp(-x)), \ x \in \mathbb{R}
\]

\[
\Phi_\gamma(x) = \exp(-x^\gamma), \ x > 0, \ \gamma > 0
\]

\[
\Psi_\gamma(x) = \exp(-(x^\gamma)), \ x < 0, \ \gamma > 0
\]

\[
N(x) = \Phi(x)
\]

\[
N_{1\gamma}(x) = \begin{cases} 
\Phi(\gamma \log x) & x > 0 \\
0 & x \leq 0 
\end{cases} \quad \gamma > 0
\]

\[
N_{2\gamma}(x) = \begin{cases} 
1 & x > 0 \\
\Phi(-\gamma \log(-x)) & x \leq 0 
\end{cases} \quad \gamma > 0
\]

\[
R(x) = -\log(1 - F(x)), \ R^{-1}(x) = F^{-1} \left( 1 - e^{-x} \right)
\]

\[
\hat{F}(x) = 1 - \exp \left\{ -\left( -\log(1 - F(x)) \right)^{\frac{1}{\gamma}} \right\} = 1 - \exp \left\{ -R \left( x \right)^{\frac{1}{\gamma}} \right\}
\]

\( \hat{F}(x) \) is called the associated c. d. f. (Resnick, 1973b).

**Lemma A.5** If \( p_n \rightarrow p = 0, \ \frac{B(n)}{\sqrt{\Lambda(n)}} \rightarrow 0 \) or \( p_n \rightarrow p \in (0, 1), \ \sum_{i=1}^{n} \frac{B_i - p_i}{\sqrt{n}} \rightarrow 0, \sum_{i=1}^{n} \frac{(p_i - p)^2}{\sqrt{n}} \rightarrow 0 \) then the following dualities holds.

\[
F \in D_R(N, a_n, b_n) \iff \hat{F} \in D_M \left( \Lambda, a_{[\hat{p} \log^2 n]}, \delta^{-1} b_{[\hat{p} \log^2 n]} \right)
\]

\[
F \in D_R(N_{1\gamma}, a_n, b_n) \iff \hat{F} \in D_M \left( \Phi_\gamma a_{[\hat{p} \log^2 n]}, b_{[\hat{p} \log^2 n]} \right)
\]

\[
F \in D_R(N_{2\gamma}, a_n, b_n) \iff \hat{F} \in D_M \left( \Psi_\gamma a_{[\hat{p} \log^2 n]}, b_{[\hat{p} \log^2 n]} \right)
\]

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where
\[ \tilde{p} = \begin{cases} -\frac{p}{\log(1-p)} & \text{for } p \in (0, 1) \\ 1 & \text{for } p = 0 \end{cases}, \quad \delta = \begin{cases} \sqrt{-\frac{(1-p)\log(1-p)}{4p}} & \text{for } p \in (0, 1) \\ \frac{1}{2} & \text{for } p = 0 \end{cases} \]

**Martingale Properties**

**Lemma A.6** \( n - A(T_n), (A(T_n) - n)^2 - A(T_n) + B(T_n) \) and \( \frac{(1+c)^n}{\prod_{k=1}^{n} \left(1 + \frac{c}{a_1 + \ldots + a_k}\right)} \) (for any \( c > 0 \)) are martingales.

**A.3 Ballerini and Resnick (1987b)**

Let \( F \) be a continuous c. d. f. A stochastic process \( \{Y(t), t > 0\} \) with finite-dimensional distributions

\[ F_{t_1, \ldots, t_n}(x_1, \ldots, x_n) = F_{t_1}^{x_1} \left( \bigwedge_{i=1}^{n} x_i \right) F_{t_2-t_1}^{x_2} \left( \bigwedge_{i=2}^{n} x_i \right) \ldots F_{t_n-t_{n-1}}^{x_n}(x_n) \]

for \( 0 < t_1 < \ldots < t_n \) is called an extremal-\( F \) process, where \( \bigwedge_{i=j}^{k} x_i = \min\{x_j, \ldots, x_k\} \).

Ballerini and Resnick (1987b) introduced their model by embedding the maxima sequence \( \{M_n\}_{n \geq 1} \) in such a process, i.e., \( \{M_n\}_{n \geq 1} \overset{d}{=} \{Y(a_n)\}_{n \geq 1} \) where \( 0 < a_1 < a_2 < \ldots \to \infty \). It follows that

\[ P \left( X_1 \leq \bigwedge_{i=1}^{n} x_i, X_2 \leq \bigwedge_{i=2}^{n} x_i, \ldots, X_n \leq x_n \right) \]

\[ = P(M_1 \leq x_1, \ldots, M_n \leq x_n) \]

\[ = F^{a_1} \left( \bigwedge_{i=1}^{n} x_i \right) F^{a_2-a_1} \left( \bigwedge_{i=2}^{n} x_i \right) \ldots F^{a_n-a_{n-1}}(x_n). \]

By taking \( \alpha_1 = a_1, \alpha_i = a_i - a_{i-1} (i \geq 2) \) and \( y_k = \bigwedge_{i=k}^{n} x_i \) we see that it is almost equivalent to the fixed \( F^a \) model

\[ P(X_1 \leq y_1, \ldots, X_n \leq y_n) = F^{\alpha_1}(y_1) \ldots F^{\alpha_n}(y_n). \]  

(A.17)
The only slight difference is that Ballerini and Resnick define their model in terms of successive maxima. Technically, this is somewhat more general since it only requires the probability of having a particular sequence of successive maxima to be equal to the same probability as obtained from (A.17).

Lemma A.7 \( I_1, \ldots, I_n, M_n \) are independent for all \( n \).

Lemma A.8 If \( p_n = \frac{\alpha}{S(n)} \to p \in [0,1] \) then \( \frac{N_n}{n} \to p \) a. s., \( \frac{T_n}{n} \to \frac{1}{p} \) a. s. If \( p \in (0,1) \) then

\[
\frac{N(n) - np}{\sqrt{np(1-p)}} \to N(0,1) \quad \text{and} \quad \frac{T_n - \frac{n}{p}}{\sqrt{\frac{n(1-p)}{p^2}}} \to N(0,1).
\]

Lemma A.9 Let \( W_k = \{\Delta_{n+k}\}_{n \geq 0}, W = \{\Gamma_n\}_{n \geq 0} \). \( \Gamma_n \)'s are i. i. d. Geom(\( p \)) and \( p_n \to p \in (0,1) \). Then \( W_k \overset{d}{\to} W \).

A.4 Special Archimedean copula (Ballerini, 1994)

Let the joint c. d. f. of \( X_1, \ldots, X_n \) be

\[
G_n(x_1, \ldots, x_n) = \exp \left\{ - \left( \sum_{i=1}^{n} [-\alpha_i \log F_i(x_i)]^\gamma \right) \right\} \quad (A.18)
\]

\( \gamma \geq 1, \alpha_i > 0 \ \forall i, \)

where \( F_i \) is an absolutely continuous c. d. f. Note that marginally \( P(X_i \leq x) = F_i^{\alpha_i}(x) \) as in the fixed \( F^\alpha \) model but the observations are now dependent (for \( \gamma > 1 \)).

Lemma A.10 \( I_1, \ldots, I_n, M_n \) are independent for all \( n \).
A.5 General copula model (Nevzorov, Nevzorova and Balkrishnan, 1997)

Let the joint c. d. f. of $X_1, \ldots, X_n$ be

$$G_n(x_1, \ldots, x_n) = \eta \left( \sum_{i=1}^{n} c_i \nu (F_2(x_i)) \right)$$  \hspace{1cm} (A.19)

where $F_2$ is a continuous c. d. f., $c_i (i = 1, \ldots, n)$ positive constants, $\eta$ is a positive, completely monotone function with $\eta(0) = 1$ and $\nu$ is the inverse of $\eta$.

Lemma A.11 $I_1, \ldots, I_n, M_n$ are independent for all $n$. 

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