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OPTIMAL ALLOCATION OF TOLERANCES AND CLEARANCES IN COMPLEX ASSEMBLIES TO MINIMIZE THE COST OF MANUFACTURING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

by

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* * * *

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ABSTRACT

A procedure has been developed for the optimal allocation of tolerances and clearances in complex assemblies for minimum manufacturing cost and sensitivity to variation keeping the deviations of the output from the ideal within designer specified limits. The assemblies that have been considered are the single loop planar and spatial mechanisms including analytically defined surface contact joints. The procedure and the program emanating from the work will serve as a design tool to be used in the early stages of design. The procedure allows the incorporation of manufacturing tolerances and clearances based on actual manufacturing process and cost information. This allows the designer's intent and the manufacturing and process capabilities to be tied together in the development of the product or assembly. The incorporation of tolerances in the case of analytically defined surface contacts like sphere to sphere contact, or sphere to cylinder contact etc. have also been studied in this work.

For each individual design parameter, the sensitivity of the output function with respect to the variations of these parameters are established. The sensitivities and the cost of incorporating individual tolerances and clearances are then studied together to see the overall effect on the assembly and also the cost of manufacturing. Sensitivities of the output with respect to the design parameters and also the link length and joint clearance parameters is often a very good information for the designer to have in studying the problem of variation of the output. This can be obtained as a part of the output generated in this work. It has been found that this gives a lot of insight on the actual behavior of the mechanism or assembly.

The allocation of tolerances are done with the idea of minimizing the overall "life cycle cost" of the assembly. The life cycle cost not only involves the actual manufacturing cost but also the cost involved in maintaining the design variables within their tolerable values. The second cost is implemented by minimizing the effects of the sensitivities of the individual variables to the output variation.
A multicriteria optimization procedure has been developed to handle the two aforementioned conflicting objectives. Use of the minimal $L_2$-norm along with the objective weighting method have been found to be quite effective in this work. The multicriteria optimization procedure has been incorporated in an interactive fashion where the designer is allowed to interpret the results and modify the weighting factors of the objectives based on the current results and the designer's overall intent of the design.

An equivalent "clearance link" model has been proposed for various joint types with clearance and they have been tested with example cases. It has been assumed in the research that the way the clearance links line up at a joint depend on the forces at the joint. But the forces at the joint will change from the no clearance situation simply because of the addition of the clearance links. To take care of this problem, an iterative procedure has been used to obtain the modified force solutions at the joint with clearance. This procedure has aided in the incorporation of the sensitivities of the clearances in the overall optimization process in a much more accurate and practical way. Clearance analysis procedure has been developed for the revolute, cylindrical, prismatic, and spherical joint types.

The multicriteria optimization based procedure makes the design "robust" against manufacturing variations and subsequent operating variations. This is also achieved by using the designer specified assembly tolerance constraints. Different cost vs. tolerance charts or functions can be used for estimating the cost of manufacture and the procedure can be used to evaluate the viability of alternative process and machine combinations to obtain the least manufacturing cost for the individual components of the designed mechanism.
To Boisali And Esha
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missed her Daddy during the long hours of work.

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FIELDS OF STUDY

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Planar and Spatial Kinematics; Design Optimization; CAD/CAM; Manufacturing Methods; Rapid Design Prototyping.
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CHAPTER 1

INTRODUCTION

1.0 Abstract

Tolerances on the dimensions of individual parts and clearances in joints are inevitable due
to variations in the manufacturing processes. Even though individual tolerances and
clearances are within small specified limits, the effect of accumulation of these tolerances
are often quite significant. The individual tolerances and clearances can thus have adverse
effects on the final assembly and its functional requirement. They tend to offset the output
of the assembly from the designer's original intended value. Tighter individual tolerances
would be preferred to keep the variation of the output from the ideal at a minimum.
However, this would increase the cost of manufacturing of the parts. Very loose tolerances
would be easy to manufacture but again might increase the cost in rework when the
assembly output is in error. The cost of manufacturing and the efficient operating cost of
an assembly is dependent on the tolerances and clearances associated with the parts and
joints. There is, therefore, a requirement of a value-based tolerance analysis methodology
as an aid to design for manufacturing.

The current research involves the development of procedures for optimally allocating the
tolerances and clearances in complex assemblies for minimum cost and sensitivity to
variation with deviations within tolerance limits from the ideal assemblies. The assemblies
considered are modeled as statically determinate structures and single loop planar and
spatial mechanisms. An emphasis is placed on mechanisms which have general contact
joints because in many instances these are more cost-effective than traditional pin and slider
joints.
joints. The cost of incorporating individual tolerances and clearances using specific manufacturing processes and their overall effects (sensitivities) on the whole assembly is established for each individual part. These are then incorporated into a multicriteria nonlinear optimization procedure to obtain the best tolerance specification for a part and the best manufacturing process.

The procedure for tolerance allocation developed is for use in the early design stages of an assembly to allow the design to accommodate dimensional variations occurring during manufacturing. Procedures are also developed to make the design as insensitive as possible to these variations, i.e. to make the design "robust" against the manufacturing operating variations.

1.1 Overview

There is a lack of readily available quality tools for aiding in efficient and systematic specification of dimensional tolerances in complex assemblies such as mechanisms. The availability of these tools to the designers could be an important aspect in increasing productivity. Significant improvements in product performance and cost reduction may be obtained by considering manufacturing tolerances from the beginning of the design stage. Design engineers are aware of the fact that tolerance stacking in assemblies has a critical effect on clearances and interferences in design and on the functional requirements of the part and its performance. But they cannot assign tight tolerances and ignore the manufacturing engineers' concerns for increasing cost. Tolerances also influence the selection of production processes and the final assemblablility of the finished product. Tolerance specification is therefore one of the most important aspects in product design and manufacturing and has been receiving more and more attention as the need to improve quality and reduce costs accelerates. It is much less expensive to make changes in the design stage than in the manufacturing or inspection stage. Once the choice of a specific manufacturing process is made, it is often impractical to transfer to a manufacturing process with reduced manufacturing costs.

The random nature of tolerances is a problem to account for when assigning design tolerances. Thus there is a need for a unified procedure for optimally allocating the tolerances. Optimum allocation of tolerances has been addressed to some extent when simple joints are involved, but very little has been done when the assembly involves point
contact between adjacent surfaces. Manufacturing of these is becoming increasingly important, and these are used in a wide variety of assemblies such as parts that are optically or magnetically read.

A very simple example involves an inexpensive mechanism for transmitting motion between two shafts which are non-parallel. One such mechanism is the standard universal joint but a simpler device to manufacture contains two directly contacting bent rods. Even though the three-dimensional cam contact joint is a five-degree-of-freedom joint, a three-link mechanism with two one-degree-of-freedom base joints is a one-degree-of-freedom mechanism. The bent shaft universal joint shown in Fig. 1.1 was analyzed by Beggs (1966) and Brat (1971). This would be very easy to manufacture compared to a traditional universal joint. Hence the use of this mechanism instead of a universal joint, wherever possible, would reduce the cost of the assembly by a significant amount. The effects of manufacturing tolerances on such assemblies have not been systematically studied, however, in part because of the difficulties in analyzing generally contacting surfaces.

Fig. 1.1: Bent Shaft Universal Joint
1.2 Current Work

The current work has resulted in a procedure and a computer program for systematic optimal allocation of link tolerances and joint clearances. The objectives of the optimization process are the minimum manufacturing cost of the mechanism or assembly and minimum sensitivity of variation of the output to the individual tolerances and clearances when the output assembly tolerance is within specified limits. Planar mechanisms and spatial mechanisms with general surface contact have been successfully optimized for tolerance and clearance. Three dimensional general cam contacts like the bent shaft universal joint of Fig. 1.1 have been modeled without any problem.

Clearance analysis models have been developed for revolute, cylindrical, prismatic, and spherical joints using the concept of equivalent clearance links. The clearance analysis model is an iterative procedure. A procedure has also been developed for estimating the sensitivity of the output to individual tolerance and clearance perturbation. A multicriteria optimization procedure has been developed for the actual optimization process. From the examples to follow it can be observed that the optimization procedure works very well.

1.3 Organization Of Dissertation

The dissertation is organized in the following manner. Chapter 1 gives a brief introduction to the work and a general outline of the problem. The nature of the problem and an overview of the approach taken to solve the problem is introduced in Chapter 2. Chapter 3 gives an outline of existing literature in the field and highlights the inadequacies of those works and how the current work addresses the problem more directly. Chapter 4 is an introduction to the generalized procedure used for the kinematic and frictionless force analysis of spatial mechanisms. An example mechanism analysis is also shown for ease of understanding. The procedure used for obtaining the assembly tolerance of a multidimensional assembly and the variation of the manufacturing cost for different operations with different tolerance and surface finish requirements is discussed in Chapter 5. A generalized cost tolerance function is also presented in this chapter. Chapter 6 gives an outline of the optimization process in design. It also highlights the specific optimization methods implemented in the research including several multicriteria optimization methods. Procedures for obtaining sensitivity factors and final results are also outlined in Chapter 6.
The details of the analysis procedure used when joint clearances are considered is given in Chapter 7. Equivalent joint clearance models have been discussed for a variety of joint types. The iterative force solution procedure is also explained in this chapter. An example is also provided for the process for ease of understanding. Chapter 8 gives a brief outline of the actual optimization procedure along with a flow chart. Chapter 9 provides a variety of examples from simple four-bar mechanism to complex surfaces as swash plate mechanism. Chapter 10 gives a summary of the work, research contributions, and scope for future development. Appendix A gives the shape matrix definition for a general link used in the kinematic modeling in the research. Appendices B and C give the pair matrix definition for various joints and floating coordinate definitions for surface contacts respectively. The frictionless force constraints for different joints are listed in Appendix D. Appendix E covers the formulas that can be used to estimate machining cost for simple machining operations. The derivation of the equation for tolerance of the assembly based on individual parameter tolerances is given in Appendix F. Appendix G gives a brief overview of the routines used in the program and their usage.
CHAPTER 2

DEFINITION OF PROBLEM

2.0 Introduction

Complex assemblies involving either mechanisms or statically determinate structures are difficult to evaluate for tolerance effects, and these are the assemblies that are addressed in this work. In the case of mechanisms, some clearances must be maintained if the assembly is to move and the joints are to be lubricated. The extent of clearance will be influenced by the tolerances in each "pin" and "hole". These clearances can have profound effect on the accuracy of the mechanism. Most often, the manner in which the clearances are taken up is a function of the way the mechanism is loaded. Thus for some loading conditions, the output tolerance could be within allowable limits, while for some other loading the output tolerance could be beyond acceptable limits. Since the presence, location, and nature of the loading are so important, any accurate clearance analysis must be preceded by a force analysis of the system. This makes the task of tolerance allocation a little more involved.

When a complex assembly can be modeled as a statically determinate structure or a mechanism, the effect of tolerances on either the location of critical points or on some functional output of the assembly can be considered together. It is very interesting to note that, the nominally zero-dof structure can be analyzed as a mechanism when the presence of clearances and tolerances are considered. Hence, the allocation of tolerances in mechanisms and statically determinate structures can be evaluated using the same procedure. Once the effect of individual tolerances on the output of the assembly is established, the tolerances may be allocated in an optimum fashion. The effect of
tolerances on the output of simple mechanisms has probably received more attention than
tolerances on any other type of assembly.

2.1 Outline Of Problem

The effect of clearances on several simple assemblies have been studied by various authors
and in most cases, the extremes of the error in the output has been considered. Also, as
discussed later, the works have been concerned mostly with planar mechanisms. The
effect of clearances in three dimensional assemblies can be much more significant than in
planar devices. This aspect has not received its due attention in the available literature.

Most of the work in the literature, involves systems assembled with pin and sliding joints.
The investigators that did consider spatial situations were usually interested in very special
cases. Lower pair joints characterized by surface contact between the mating parts have
been considered by a vast majority of studies. The kinematics of general higher pair joints
characterized by point or line contact (such as cam or gear contact) has not been studied
extensively. Unfortunately, when clearances are involved, all joints become higher pairs.
Also, the higher pairs are often less expensive to manufacture than lower pairs and this
makes them attractive for products in the consumer market. For this reason, one of the
major thrusts of this work is tolerance and clearance allocation for these kinds of
assemblies.

There are commercially available programs such as ADAMS, I-DEAS, Pro/MECHANICA
Motion etc. for the analysis of mechanisms and assemblies. But most of them do not
handle joints characterized by general surface contacts between the joint elements. These
software also do not have the direct capabilities to optimally allocate tolerances on
individual parts. There is therefore a need for the development of procedures for the
optimal allocation of tolerances and clearances for all kinds of assemblies especially for
those that have spatial surface contacts.

The modern challenge is to design products with nominal values and tolerances in such a
way that the overall design is "robust" against manufacturing and other natural variations.
This is one of the underlying ideas of Design for Manufacturability. In the tolerance
allocation situation, the designer strives to assign the tolerances for the parts in such a way
that the assembly is somewhat insensitive to the expected variations. In this process,
higher and consistent manufacturing yields are obtained. The other challenge associated with this is the requirement that the designer should also strive for an assembly that involves the least cost possible over the life of the part, which includes manufacturing cost and maintenance cost.

The need for developing and validating methodologies for improving the robustness of engineering designs for manufacturing variations has been identified by the report CRTD-15 by the Center for Research and Technology Development (1988). A probabilistic approach to engineering design has been recognized as a viable method when uncertainties in the system and manufacturing processes exist. Taguchi (1971) introduced the concept of robustness in design to reduce the sensitivity of the performance to variations in the environment as well as manufacturing processes. There is a need to study the relations of these and similar parameters to statistical tolerance analysis and synthesis. The concept of a value-based tolerance analysis methodology as a design-aid is an absolute necessity to survive in the global market.

With the above in mind, the tolerance situations for statically determinate structures and general one degree of freedom linkages has been addressed in this work. The specific aim of the research was to develop a procedure for optimally allocating the tolerances on the dimensions of individual parts and clearances in the joints of an assembly. The main objectives that are satisfied in the process are the cost of the manufactured assembly and the sensitivity of variation of the output at minimum possible values. The particular joint types of interest which could be the most important from a manufacturing cost standpoint are three dimensional general cam joints involving cylindrical, spherical and other analytically and non-analytically defined surfaces. There is no general procedure available in the literature that handles these. Also most of the works reviewed in the literature, treat clearances and tolerances separately. The estimation of sensitivity factors can be a formidable task even for the simplest assemblies.

The tolerance allocations in designs are most often based on experience and generally accepted local company standards. It is well known that manufacturing costs increase as tolerances become tighter and the relationship between them is highly non-linear. There are other indirect costs associated with tight tolerances - those associated with scrap, reworking, and selective assembly. No general procedure is available for the cost and performance analysis based on tolerance specifications even though a number of studies have been conducted. For example, Wade (1967) suggested a charting method for
determining the cost-tolerance relationship for individual parts of an assembly. This method could be used to determine the cost-tolerance relationship for various other production options. The cost of manufacturing and assembly are also influenced by the shape and size of the part as reflected in a classification scheme developed by Poli and Boothroyd (1981). Suh, et. al. (1981) discussed the improvements in manufacturing processes with regard to nominal dimensions of individual components.

To determine the impact on cost of an isolated part tolerance on the overall tolerance and cost of an assembly, it is necessary to know both the cost-tolerance relationships for each part and the effect of each part tolerance on the location of the critical points in an assembly. The first of these problems deals principally with manufacturing and the second with design. The cost-tolerance relationships will be functions of the manufacturing methods used for each part and this must be known or developed for each manufacturing operation. Trucks (1987) gives comparative lists for the costs associated with obtaining specific tolerances by different manufacturing methods. These tend to relate to specific parts and hence their effect on the assembly is independent of the assembly configuration. However, the effect of individual tolerances on an assembly depends on both the assembly configuration and the magnitude of the tolerances. Spotts (1983) has provided comprehensive guidelines for designers for optimally allocating tolerances for most efficient assemblies at minimum cost. The standard tolerances for various machining operations have also been listed as a quick reference.

Various other studies have taken place that investigate tolerance allocation from both a statistical and deterministic point of view. The most economical procedure is to use a statistical approach. This approach is appropriate as long as the tolerances are randomly distributed and the distribution can be defined either numerically or analytically. This is generally true if the parts are mass produced. With this assumption, Harry and Stewart (1988) presented a procedure for analyzing and allocating assembly tolerances. Their procedure resulted in a mechanical assembly which exhibits 6σ (assemblies insensitive to ±3 times standard deviation variations) performance. Process mean shifts and biases were considered in the final result to obtain design tolerances that are "robust" against such variations. These and other relevant literature will be discussed in Chapter 3.
2.2 Definition Of Parameters For The Research

The parameters used in the research are explained with a simple example here. A four-bar linkage is shown in Fig. 2.1. Fig. 2.2 shows a very bare detail of each link and the designer assigned link length tolerances for this linkage. A typical link length tolerance may be assigned in this fashion. Fig. 2.3 shows one of the revolute joints from Fig. 2.1

![Four-Bar Mechanism Used For Illustration](image)

(joint number 3) in a very exaggerated manner. The joint clearance is dependent on the hole and pin diameter and their tolerances. To take the example in this case, the extreme value of the diametral clearance can vary between \((D_h + \Delta h_u) - (D_p - \Delta p_1)\) and \((D_h - \Delta h_l) - (D_p + \Delta p_u)\) inches. Here \(D_h\) and \(D_p\) are the hole and the pin diameter at the joint respectively. \(\Delta h_u\), \(\Delta p_u\), \(\Delta h_l\), and \(\Delta p_l\) are the upper and lower tolerance limits for the hole and pin diameter respectively. The clearance considered in this research and shown in Fig. 2.3 is the radial clearance, which is the difference between the radius of the "pin" and the "hole". Even though the clearance is dependent on two variables' tolerances, its value is going to vary between zero and a very small positive number. In this research,
clearance variables are considered in a similar fashion as tolerance variables for optimization purposes. The interpretations of output results are slightly different. While the optimized value of a tolerance variable can be incorporated directly as the tolerance on that particular link "length" parameter, the optimized values of the clearance variables have to be incorporated indirectly into the "pin" and "hole" diameter tolerances. This can be done by using the Basic Hole system or the Basic Shaft system (Fortini, 1967) as the

![Diagram of the four-bar mechanism](image)

**Fig. 2.2: Basic Link Length And Tolerance Definitions For The Four-Bar Mechanism**
designer wishes so that the overall tolerance falls within the optimized range. This also leaves a flexibility for the manufacturing engineer to choose and mix and match individual links in an assembly as long as the overall link tolerances and the combination of "pin" and "hole" tolerances matched appropriately make the clearance within the optimized value.

Fig. 2.3: Clearance Definition For Revolute Joint 3 Of The Four-Bar Linkage

The output error and sensitivity of output error are used in the research for tolerance allocation. The output type for a mechanism is dependent on the mechanism itself. For the four-bar mechanism in Fig. 2.1, this could be the angular position of the output link 4 as shown in Fig. 2.4 or x-coordinate or y-coordinate of a point on the output link or the coupler (e.g. center of gravity of Link 4) as given in Fig. 2.5. For the mechanisms, it could be angular position, spatial position of any link and coordinates of points or even velocity of a point or link. With these optimization variables, the objectives are set up as the total cost of manufacturing and sensitivity of variation of output. The constraint is the tolerance of the assembly to be within limits set up by the designer. Fig. 2.6 gives an insight into the amount and type of variation of the output variable for a fixed variation or tolerance of a link parameter. The four-bar linkage in Fig. 2.1 is analyzed with a small perturbation (0.005") on link length L₃. The variation of the output from the ideal output is shown in Fig. 2.6.

The values used for the four-bar mechanism to get Fig. 2.6 are:
L₁ = 6 in.  L₂ = 1 in.  L₃ = 5 in.  L₄ = 4 in.

Fig. 2.4: Typical Output Crank Position Plot For The Four-Bar Mechanism Of Fig. 2.1

Fig. 2.5: Typical Output Crank Center Of Gravity Position Plot For The Four-Bar Mechanism Of Fig. 2.1

and tolerance on L₃ as 0.005 in.

The optimization procedure solves a multicriteria problem at all positions to obtain the "best" possible solution. The procedure will be discussed in detail in later chapters.
Fig. 2.6: Output Angle Error For Positive Tolerance (0.005") On Link Length $L_3$
CHAPTER 3

TOLERANCE AND JOINT CLEARANCE STUDIES

3.0 Introduction

In the studies of the real motion of a mechanism, it is important to consider clearances in joints. The clearances are required for motion and lubrication, and the size of the clearances are induced by manufacturing tolerances. The kinematic and dynamic performances of the mechanism are heavily influenced by the joint clearances. Normal operational wear can result in increased clearances causing impacts, vibrations, and noise. The performance and cost of the mechanism also depend greatly on the tolerance of the components. For precision mechanisms, the effect of clearances and tolerances should be considered at the design stages. Also the incorporation of these at the design stages is cost effective in the long run.

3.1 Studies On Clearances And/Or Tolerances In Mechanisms

Various investigators have studied the effects of clearances and/or tolerances in kinematic analysis and synthesis. Some of the investigators have focused on the resulting mechanism inaccuracies and some of the others have studied the effects on mechanism dynamics. The general details of each of the studies are given in this section.

The presence of clearances and tolerances produce errors in the ideal output of the assembled mechanism. However, both of them are somewhat inevitable. Joint clearances
are required for the satisfactory operation of the mechanism and tolerances are required to minimize manufacturing costs. Hence, neither of them can be eliminated completely. From a mathematical standpoint, tolerances can be considered as uncertainties associated with any parameters in the linkage, and clearances are associated with gaps in the linkage joints.

In the literature, only a few investigations have conducted combined studies of joint clearances and tolerance of link dimensions together. The study of the joint clearance problem, often referred to as "backlash" will be discussed first. The effects and characteristics of backlash have been studied by a number of investigators (Goodman, 1962, Fawcett & Burdess, 1971). Clearances have effects on the joint reaction forces, mechanism accuracy and repeatability (Vocaturo, 1983), noise levels of operation (Pereira and Dubowsky, 1979) and impacts in the joints (Dubowsky et al, 1984). Some investigators studied the dynamic effects of clearances on flexible or elastic mechanisms (Dubowsky and Gardner, 1977 and DeRung, 1986, and Dubowsky et al, 1987, to name a few). Link elasticity has not been considered in this research, and therefore these elasticity based studies will not be discussed here.

3.1.1 Studies On Mechanism Joint Clearances

Ideally a kinematic joint is characterized by a known physical contact of two bodies with predictable relative rolling and/or sliding interface motions. The physical interface between bodies making up the joints is either a surface, line, or point type. A lower pair joint is defined to be one that is characterized by surface contact between the joint parts. The higher pair joints have line and point contacts. The only lower pair joints commonly used are the revolute, prismatic, cylindrical, spherical, helical, and planar joints. Gear tooth contact, ball and roller bearings contacting the races and cam contact are some examples of higher pair joints. Due to the presence of clearances in the joints, ideal lower pair joints have the potential of becoming higher pairs. This is due to the fact that the ideal surface contacting interfaces really exhibit line or point contacts.

The studies on joint clearances in the literature can be classified into two broad categories. One of these is based on the idea that there is no loss of contact i.e. the bodies involved in the joint do not experience separation. The other type of study assumes that contact loss
actually occurs between the joint interfacing bodies and at some instant they are in a "floating" mode.

In the present work, it has been assumed that either structures or slowly moving mechanisms are involved so that joint contact is always maintained. The limitations of this assumption will be discussed later. The studies on contact loss models will be discussed in less detail for this reason.

3.1.1.1 Joint Clearance Studies: Contact Loss Models

A number of studies in joint clearance have considered situations where the bodies in contact in the joint can actually separate. For example, in a revolute joint with radial clearances, it is possible under some circumstances for the pin to "float" or reside anywhere in the mating hole without touching the inside surface of the hole. To take care of this, the contact loss type joint clearance models are always based on some sort of statistical studies.

Dubowsky and Freudenstein (1971 and 1971a) did pioneering work in contact loss joint clearance analyses. Joint dynamic equations were developed for one dimensional joints using Newton's laws for an "impact pair" model. The two joint types studied were the internal ball connection (one dimensional joint analysis) and revolute joint. During the contact period of the joints, contact compliances were calculated based on the configuration of the connection using classical Hertz surface contact analysis. Numerical techniques were employed to study the dynamic behaviors of the joints. This model was basic to the determination of the dynamic response of mechanical and electromechanical systems with clearances, including determination of the dynamic force amplification, frequency response, time-displacement characteristics, etc.

The impact pair model was used to study the clearance at the crank pin of a slider-crank mechanism (Dubowsky, 1974) that established correlations between the cyclically driven mechanism and the displacement forced impact pair dynamic response. The impact pair model was used further (Dubowsky, 1974a) to study the effects of clearances in complex multi-degree of freedom systems. The major aim of the study was to determine the dynamic force amplification due to impact between joint surfaces at clearance joint connections.
The impact pair model was modified to the "impact beam" model in the case where link flexibility was considered (Dubowsky & Gardner, 1975). It was observed that the introduction of the link elasticity substantially reduced connection forces in joints. However, they were still much higher than the ideal case of no-clearances. Clearance effects in a scotch yoke and a four-bar linkage were studied using the impact beam model. The results of the analysis compared very well with actual experimental values.

Deck and Dubowsky (1994) also used the impact beam model to study the limitations of the predictions of the dynamic response of machines with clearance studies. Analytical model of a spatial slider crank mechanism with flexible links and joint clearances was studied. The results suggested large amount of variability and high sensitivity to small parameter changes, and operating conditions were inherent properties of the dynamic response of the system. Experimental verifications of the simulation studies were performed. Both the simulations and the experimental verifications showed the nature of uncertainties involved in the prediction of the dynamic response and highlighted the need for running a large number of simulations with varying parameters to obtain a good response.

A joint model based on momentum exchange during impacts rather than the spring and dash pot of the impact pair was developed by Townsend and Mansour (1975). Two modes of motion were considered, the "free flight" mode and the "impact" mode. Contact was considered with various frictional conditions: smooth, rough and Coulomb type stick-slip. This study evaluated impulsive forces and torques due to clearance connections and obtained time history over a cycle of operation.

Another study (Wu and Earles, 1977) developed a method for analyzing the kinematic and dynamic response of a linkage mechanism in which one bearing had a known clearance. Assuming the clearance to be represented by a massless link, the analysis predicted the occurrence of contact-loss and studied the effects of contact loss on mechanism performance. The effects on performance were studied as a function of clearance size. The work was continued further (Earles and Wu, 1977) and a design criterion useful for evaluating and minimizing the effects of bearing clearances in a mechanism was developed.

Grant and Fawcett (1979) performed an analysis that predicted and verified experimentally contact loss based on a prior no-clearance analysis. They considered the clearance only at the coupler rocker bearing of a 4-bar linkage. The clearance connections were modeled using the concept of a massless "clearance link" (Earles and Wu, 1977). This transformed
the 4-bar linkage into a two-degree of freedom five-bar linkage, a common technique for no-contact loss models. A similar type of clearance link analysis was used by Osman, Bahgat, and Sankar (1980) to predict joint separation in high speed mechanisms. Multiple clearances in a slider-crank mechanism were considered where all revolutes had radial clearances and the prismatic joint had no clearances. Grant and Fawcett (1978) developed a theory to design against contact loss using the predicted contact loss information. It was ensured that joint forces were large enough and were in the right direction to prevent separation of surfaces of joints.

Dynamic effects of multiple clearance situations in planar mechanisms were studied by Shimojima, Ogawa and Matsumoto (1978) using Lagrange's equations. Friction in the joints was considered for predicting the conditions for joint separation, and the predicted results were compared with experimentally observed values. Osman, Bahgat, and Sankar (1983) examined the clearances in a planar quick return mechanism with the goal of computing the bearing reactions in terms of the input angle. The underlying assumption was that the variations in mechanism displacement and velocity due to clearance effects were negligible. Wilson and Fawcett (1972) considered the dynamics and experimental verification of piston slap which is the effect of clearance on a sliding joint.

The momentum exchange model (Townsend and Mansour, 1975) was extended to a three mode model (Miedema and Mansour, 1976) by adding a "following" mode to the "free-flight" and "impact" modes of joint motion. The joints experienced relative sliding while still in contact in this "following" mode. The slider-crank mechanism was taken as an example of study. Soong and Thompson (1990) modified this model to perform theoretical and experimental investigations of the dynamic response of planar mechanisms with radial clearances in revolute bearing connections. Besides the contact, free flight, and impact modes, a "transient mode" was considered. In this mode, that begins after the first impact, a sequence of several free-flight modes of progressively smaller time durations and several impact modes take place. This mode finally culminates in the contact mode. A slider crank mechanism with clearance in the pin joint between slider and connecting rod was taken as an example.

A vector network method (Rogers and Andrews, 1977) was used to simulate the dynamic behavior of planar clearance connections in revolute joints. A computer program called PLANET II created mathematical models of bearing elements which took into account the effects of clearance, surface compliance, and presence of lubricant. The joint model used
was similar to that of Dubowsky and Freudenstein (1971) in the use of spring and dash pot elements. Kinematic results and the results of simulation were compared. The displacement and angular response of the bearing connections of a mechanism were the main thrusts of this study with an offset slider-crank mechanism with radial clearances in both the crank pin and the wrist pin taken as an example.

The clearances in joints other than the often studied planar revolute and prismatic joints have been studied by a few investigators. The Geneva mechanism was the object of one study (Rao and Gavane, 1982a and 1982b). Tolerances were optimally distributed in the external and internal Geneva mechanism. Nonlinear programming techniques were employed in this study and the objective was to stay within specified acceleration and jerk errors for the mechanism. A stochastic analysis was performed based on the assumption of contact loss. The results presented a statistical band of operation for the mechanism based on the statistical distributions of tolerances and clearances. Besides considering clearances in revolute joints, this investigation also looked into the "pin in a slot" higher pair two-degree of freedom connection. The analysis of elastic machinery with clearances was performed earlier by Winfrey, Anderson and Gnilka (1973). The separation of cam and follower joints in an elastic cam mechanism were considered. The dynamic forces of impact on contact after separation were the main concerns in this study.

Bengisu et al (1986) made a theoretical and experimental investigation of contact loss for a four-bar mechanism. Friction in the bearings were neglected for the dynamic analysis and inelastic impact was assumed for the contact of journal and bearing at the end of the "free flight" stage. The numerical results showed that contact loss occurs when the magnitude of the bearing force is at minimum while the direction changes rapidly. The results compared well with earlier findings and experimental results. A non-dimensional parameter was offered that would predict the positions and number of separations per cycle.

Rhee and Akay (1996) examined the dynamic response of a four-bar mechanism with a clearance at one of its connections. The results showed nonlinear dependence on both the clearance size and the coefficient of friction between journal and bearing. The pin motion inside the bearing exhibited nonlinearities by doubling, tripling or quadrupling the period. In certain cases the pin motion even showed chaotic response. The studies were based on earlier three mode (sliding, free flight and impact) models. By performing the computations for a large number of crank periods, the existence of high, aperiodic impact velocities of the pin motion were determined.
3.1.1.2 Joint Clearance Studies: No Contact Loss Models

When a mechanism moves, there is always the presence of some kind of joint and link forces. These forces would tend to cause joint contact. That is, even in the presence of joint clearances the joint entities would not "float". In fact, they would assume a position dictated by these forces. In other words, due to the presence of static and inertial effects, it is reasonable to assume no contact loss in the joints in question, unless high speeds are involved. However, the contact positions are continuously changing and depend on external and friction forces among other things. In keeping with this idea, some investigators have developed clearance analyses in which the joint models are assumed to have no loss of contact between the joint surfaces.

Kolhatkar and Yajnik (1970) used the concept of equivalent links to model multiple clearances in revolute joints. Vector loop equations for a mechanism were partially differentiated with respect to joint angles of the equivalent clearance linkage, and joint angles corresponding to the extreme positions of the mechanism were found using min./max. theory. This study aimed at investigating the relationship between joint clearance and accuracy of mechanism output angular displacement. This investigation also attempted to develop an equivalent clearance link for sliding joints, but a tractable set of equivalent links was not obtained. The concept of equivalent links was also applied to multiple clearance situations by Bahgat, Osman, and Sankar (1981). However, this investigation was primarily concerned with dynamic forces rather than kinematic positions. The assumption used here was that the contact loss, if present, was for infinitesimally short period of time and was neglected in the analysis. Lagrangian mechanics was applied to obtain dynamic displacements, velocities, accelerations and bearing force responses.

Another study involved the analysis of slider-crank mechanism with clearances in three revolute joints and elastic links (Rao, 1978). No clearance in the sliding joint was considered. As a final result of this work an equation relating the slider position with respect to the crank angle and all the clearances was obtained. Further work with this no-contact-loss model (Rao, 1978a) included the effects of clearance, elasticity, and dimensional tolerances to develop an accurate mechanism synthesis. Joint reaction forces including the effect of Coulomb friction were used to determine how the clearances were taken up in the joints in question. This was a synthesis study and the deviation in the
desired output was used as a modification parameter iteratively to accurately synthesize the linkage parameters for the desired output.

Kinzel and Hall (1975) used a matrix transformation method to describe the effect of clearance connection between two mating links. The matrix technique modeled the joint clearance of a revolute joint as being made up of a diametral and axial clearance, a clearance rotation, and the plane in which the clearance rotation occurs. The method was used to develop an error "box" for the actual position of the end of a spatial six degree of freedom linkage based on estimated clearances and tolerances and the loading conditions. This was a general treatment of the situation, and the procedure could be applied theoretically to any joint type planar or spatial.

A dynamic based study of the clearance connections in a four-bar linkage was conducted (Furuhashi, Morita and Matsuura, 1978, 1978a, 1978b, 1978c) using a continuous contact model for the joint. The joint model used was similar to the equivalent clearance links of Kolhatkar and Yajnik (1970). The first of these papers developed the general theory for the study. The second and third investigated different cases of single clearance situations in the four-bar linkages. The fourth paper discussed and compared the results from the other two analyses. The dynamic response for displacements, velocities, accelerations and required torsional input were calculated. These studies involved clearances in all four revolute joints, and results were compared with experimental values for verification.

### 3.1.2 Studies On Mechanism Tolerance

The other aspect of mechanism error is a result of dimensional tolerances or uncertainties in the mechanism links. The tolerances are required by the manufacturing processes used to make the parts. Tolerances cannot be eliminated and to hold parts to unnecessarily tight tolerances increases the cost of the links. If the parts are mass produced, the tolerance generally follows some statistical distribution. The nominal length of the link varies due to the presence of tolerances. The nominal clearances specified for a linkage are also affected by the tolerances in the links.

Tolerance analysis and tolerance synthesis are the two major areas of studies on tolerances of linkages. The effect of known tolerances on the output of a mechanism are the principal concerns of the analysis studies. The synthesis studies concentrate on optimally allocating
tolerances to obtain a desired output behavior. Many of the tolerance investigations have only considered planar mechanisms.

An analysis accounting for the manufacturing tolerances in a double Hooke joint was performed by Austin, Denavit and Hartenberg (1965). The matrix approach was used to study the influence of known tolerances on the output displacement, velocity, and acceleration of the mechanism. Tolerances were assumed to be very small compared to the link dimensions and the principle of superposition was used.

The spherical four-bar linkage was considered for tolerance analysis by Tavkhelidze, Davitashvili and Demurishvili (1979). The position error of the driven link due to link length inaccuracies was determined. A Taylor series expansion was used to develop an equation for the error in the output due to link length inaccuracies. The resulting equation developed from the linear form of the series gave sensitivities of the output to the individual link tolerances.

Another investigation (Coit and Riley 1981) studied an inverted slider-crank straight line generator and its sensitivity to link length tolerances. Analytical results were presented on the screen of a computer terminal. The designer could make intelligent choices for the tolerances of the links based on the performance requirements and then observe the mechanism output. The work involved a kind of manual tolerance optimization.

Mechanical errors in the coupler curve of a four-bar linkage were estimated in terms of translational and rotational components of position error by Chatterjee and Mallik (1987). It was observed that cognate linkages having the same coupler curves had different sensitivity to the errors in the link lengths. Numerical results were used to identify the most critical link of a given mechanism as far as link tolerances are concerned. This study, though, applicable to four-bar linkages only, incorporated the basic idea of identifying which links needed tight tolerances and on which links the tolerances could be relaxed. Fenton, Cleghorn and Fu (1989) also presented a procedure to determine the tolerance bands for the dimensions of multiple loop planar mechanisms to keep the variation of the output within specified limits.

Lakshminarayana and Narayanamurthi (1971) used a vector loop closure equation to determine the influence of tolerances on mechanism output. This was a sensitivity based output, and it was easy to use since the development of an input-output function was not required. The individual loop closure equations for a mechanism were differentiated with
respect to the individual tolerance parameters for obtaining the results of the sensitivity analysis.

Yang and Blanche (1990) developed a computer-aided procedure for the analysis and synthesis of cycloid drives with tolerances. Two simple and practical formulas for design synthesis of cycloid drives were developed. One formula was for estimating the backlash of a given drive and the other was for estimating the torque variations. Chen and Freudenstein (1986) performed a dynamic analysis of a standard cardan universal joint with manufacturing tolerances. This was a continuation of the work by Fischer and Freudenstein (1984). The results provided the bearing forces under high speed operation of the joint. This information could be used in the design stages for sizing and dynamic balancing of the cardan joint.

General formulations were developed for the design of 4-bar linkages with tolerances for the crank-angle coupler-point coordination and rigid body guidance problems by Sutherland and Karwa (1978). The interaction of mechanical error with structural error was taken into account by noting the tolerance of the links being synthesized. The synthesis based on both types of errors were performed using least square techniques and also nonlinear optimization techniques. Tolerances were incorporated into the total error equation using results from an earlier Sutherland and Roth (1975) work. The final result of this work was again a sensitivity of the output to the variances in link lengths. The sensitivity relations of the output to the link tolerances were also studied by other investigators. One study (Knappe, 1963) determined the frequency distributions for the chance of randomly assembling a mechanism with dimensions at extreme tolerance limits. Lucifredi (1974) developed "macros" to calculate the tolerance sensitivities for a number of basic mechanism building blocks such as crank, dyad, slider etc.

Probability theory was also used to analyze 4-bar path generating linkages as a function of manufacturing tolerances (Baumgarten and Fixemer, 1976). The effect of random combinations of linkage tolerances could be determined by the developed theory. An error band was obtained for the generated path considering the link tolerances. Kim and Newcombe (1978) developed a probabilistic method for obtaining the variances in displacement, velocity and acceleration of followers in cam motions for any given manufacturing tolerance on the cam profile. The follower motion equation was obtained by expansion in a Taylor series considering both linear and second order terms. 3-sigma statistical bands on kinematic displacement, velocity, and acceleration of the follower were
analyzed using stochastic techniques. Rao and Gavane (1982) also used probability theory to evaluate the mechanical error in the kinematic and dynamic response of cam-follower systems. A 3-sigma band of confidence level was also obtained from the error. A nonlinear programming technique was used for optimally distributing tolerances to minimize manufacturing costs with constraints on the standard deviation of dynamic response at lift and dwell periods. The earlier work (Kim and Newcombe, 1978) did not involve dynamics of the system.

In another work, (Choubey and Rao, 1982) mechanical errors were added to the structural errors to synthesize a four-bar linkage to minimize the total error. Normal distributions of tolerances were assumed. The work showed that synthesizing based on structural error alone might effectively neglect a significant amount of error in the output. The mechanical error can be of the same order and in a direction opposite to the structural error. The maximum tolerance could be specified in order to control the mechanical error below a specified level. Sutherland and Roth (1975) also incorporated mechanical error into function generating synthesis. The design parameters of a system were calculable by considering both the structural error and tolerances due to manufacturing error. The optimum distribution of tolerances was determined to minimize manufacturing costs while satisfying a maximum allowable function generating constraint.

Cleghorn et al (1993) presented a procedure for determining the optimum set of tolerance bands for the dimensions of a planar mechanism. The optimum tolerance bands allow the largest values of dimensional tolerances while ensuring that the desired motion is kept within acceptable deviations from the ideal output in no tolerance case. The procedure also looked at the interaction of the various link parameters and identified the most sensitive link dimension, the most sensitive combination of input errors and the most sensitive period within the entire cycle of motion. While the study of the sensitivities of the output were performed using sensitivity factors, the optimal solution did not, however, consider the manufacturing cost or any other objectives.

3.1.3 Studies On Tolerances And Clearances In Mechanisms

Very few investigators have considered the effects of both tolerance and clearance effects in the analysis or synthesis of mechanisms. Considering both effects together and obtaining a
deterministic solution is somewhat complex. Hence, mostly probabilistic or statistical approaches have been incorporated in these available investigations.

Garrett and Hall (1969) developed a statistical technique for defining "mobility bands" for mechanisms based on tolerance and clearance specifications. Vector loop equations were written for a four-bar mechanism incorporating clearances. A large number of sample of mechanisms with a certain range and distribution of tolerances on the link lengths was studied by random variation of link lengths for the four-bar mechanism. The mobility band was determined from about 300 computer simulated mechanisms directly using the "delta method" rather than a large number of sampling.

Dhande and Chakraborty (1973) developed a stochastic model of the four-bar linkage considering tolerances and clearances and the probable location of the pins of revolute joints in their bushings. The "floating" revolute pins altered the effective lengths of links - dimensional tolerances were incorporated into the clearance analysis in this method. The analysis portion of the study provided statistical bands of operation for the analyzed mechanism with a 3-sigma confidence level. In the synthesis portion of the study, tolerances and clearances were optimally allocated to different members of the linkage for a specified maximum allowable mechanical error. This was accomplished by the use of dynamic programming methods. It was claimed that both mechanical error and structural error could be used in the optimization process. This work was extended to spatial linkages (Dhande and Chakraborty, 1978) where a no contact loss joint model was assumed. Probability density functions were used to model clearance errors. Revolute and prismatic joints were modeled with the constraint that revolute joints were not allowed to wobble in their sockets. The results included analysis and synthesis solutions for RRSS and RSSR mechanisms. Chakraborty (1975) also made use of parametric programming to perform mechanical error synthesis.

The principle of virtual work was used to calculate the sensitivity coefficients for clearances and tolerances in mechanisms (Lakshminarayana and Ramaiyan, 1976). This appeared to be a reasonable approach, since the manner in which the clearances are taken up in a mechanism depends on the loading. The results consisted of influence or sensitivity coefficients for various clearances. The contact forces for kinematic pairs were formulated for revolute, prismatic, cylindrical and spherical pairs. Truly 3-d clearances were allowed in the contact force developments. Examples of the links that were considered were planar four-bar, planar seven-bar, spatial CSSR and a grooved cam mechanism. A general design
rule that was inferred was that highly loaded links should be finely tolerated and highly loaded joints should have low clearances to reduce mechanical errors.

Both clearances and tolerances were also considered by Agarwal (1981). This study included the use of a parametric programming technique and considered the total range of motion of the mechanism. It was not limited as was the earlier work (Dhande and Chakraborty, 1973) to mechanical error optimization for one position at a time. Multiple sets of acceptable clearances and tolerances for the mechanism were provided in the event that selective assembly was a possibility. The procedure was applied to various function generator mechanisms. The result was statistical confidence bands on the mechanism operation rather than the actual operation alone.

Another investigation developed design charts for specifying clearances and tolerances such that a given mechanism accuracy would be obtained (Corderman and Mabie, 1972). Joints were assumed not to separate and clearances were modeled using common equivalent clearance links concept. The charts were developed by a computer generation of a large population of mechanisms with varying clearances and tolerances. Trends of mechanism output behavior were recognized from the resulting outputs.

A stochastic model for the four-bar, path-generating linkage was made by Mallik and Dhande (1987). Tolerances and clearances were assumed to be random variables. A 3-sigma band of confidence level was obtained for the mechanical error in the required path of a coupler point. The error depended on whether the original mechanism or the cognate linkage was analyzed. A synthesis procedure for optimally distributing the tolerances and clearances so as to restrict the error in the path of the coupler point within specified limits was also developed.

Yin and Wu (1990) presented an optimal model of a function generator 4-bar linkage. Optimum dimensions were first obtained for a no-clearance mechanism. Dynamic analysis of the resulting mechanism was then performed to obtain the directions and magnitudes of joint forces. The concept of equivalent links was used to finally obtain synthesized values of link lengths to minimize overall structural error and clearances. It was claimed that the method could be generalized for most planar mechanisms.

Rhyu and Kwak (1988) presented a formulation of stochastic design of four-bar mechanisms considering tolerances on link lengths and clearances in joints. The mechanical error was analyzed for the 3σ band of confidence level through a new stochastic
model of the linkage. The objective included mechanical error and manufacturing cost reduction. Tolerance widths and nominal clearance sizes were taken as design variables. Trade-off curves were used to arrive at the optimal compromise solution. Problems from four-bar path generation and function generation were taken as examples.

Contrary to the common trends of considering mostly four-bar mechanisms, Sharfi and Smith (1983) considered a ten-bar linkage for tolerance and clearance synthesis. The variation of the sensitivity of the output of the mechanism to variation in mechanism parameters which arises from the change in geometry of the mechanism throughout its cycle of operation was accounted for. The effect of tolerance and clearance was essentially that of altering the nominal link lengths of the mechanism. All tolerances and clearances were assumed to have the same effect on output deviation.

Most of the investigations on mechanical error effects of mechanisms deal with planar mechanisms. However, Rao and Ambekar (1974) developed a probabilistic approach to the analysis of function generating spatial mechanisms. The special case considered was that of spherical mechanisms. The clearance in the joints was studied with the contact loss model. The final result was a statistical confidence band for mechanism operation.

The effect of tolerance and clearance in the Geneva mechanism was studied by Takanashi (1973) using a no contact-loss model. The effects were investigated during both the indexing periods and the dwell periods. The approach resulted in deterministic equations for analysis. There was no mention as to the range of mechanism accuracy due to a distribution of link tolerances and clearances. A qualitative discussion of the output sensitivity to mechanism design parameters was presented.

Another study (Lee et al, 1993) presented a systematic and general method for evaluating performances of dynamic planar mechanical systems whose dimensions are uncertain because of tolerances. The random uncertainties considered were pin locations, link length tolerances, and radial clearances. Probabilistic behavior of these uncertainties were studied by developing 'effective' link length models for the links with clearance joints. Sensitivity studies and standard deviations of the mechanical errors were performed. Optimal design problems were solved with the objective of minimizing representative manufacturing cost. A very simplistic model was used for cost analysis for demonstration purposes and no effort was made to minimize the life cycle variations. Examples were taken from planar
slider crank and planar two link robot manipulator with clearance connections and link tolerances.

3.2 Methods Of Tolerance Definitions On Surface Geometries

One of the main thrusts of the research proposed here is the tolerance allocation on joints where two surfaces come into contact. The available techniques of tolerance definitions on surface geometries are discussed in this section. The representation of 3-d curved surfaces require the development of a mathematical model. This model should be able to represent uniformly the geometry of a wide variety of objects. The bicubic B-spline formulation has been studied by various authors. The drawback of this is that it cannot interpolate a network of points. A B-spline formulation with interpolation capabilities has been studied by Barsky and Greenberg (1982). Accurate representations of curved surfaces using this procedure has been obtained for a surface initially defined to interpolate a specified network of points.

Parkinson (1985) developed a method to associate an overall risk of failure to assemble to the specification of an assembly given the component nominal dimensions and tolerances. Parkinson then adjusted dimensions and tolerances to increase the chances of acceptability. The algorithm developed could essentially do a cost-balance type optimization to obtain the values of the tolerances and minor changes in nominal dimensions that will minimize the cost of manufacture and assembly.

Requicha (1984) proposed the notion of 'offset surfaces' constructed from the nominal form of the surface. The volume of uncertainty enclosed by the two offset surfaces (one for the maximum tolerance limit, and the other for the minimum limit) forms the so-called 'tolerance zone'. In the above differential view, tolerance is represented as a single instance of a variation in geometry. This concept of analysis using offset surfaces is a very useful procedure for handling surface uncertainties.

Michael and Siddal (1981) proposed a 'vector-space' formulation for tolerance synthesis. In this formulation, a vector space was defined with its coordinates corresponding to the independent dimensions. 'Tolerance' and 'design' regions were then constructed in that vector space from identified tolerance variables and design variables with their limits given. The objective was to find the upper and lower tolerance limits for the dimensions that
minimize the overall manufacturing cost (also a function of tolerance limits), subject to the constraint that the ‘tolerance’ region be completely contained within the ‘design’ region. The idea was also extended to statistical tolerancing.

Hillyard and Braid (1978) developed a deterministic tolerance-analysis approach. The concept of variational geometry was used for analyzing the inconsistency in the specification of dimensions and tolerances in computer-aided mechanical design. The variational geometry method is a constraint-based dimension-driven technique. The geometry of an object is dictated by specified dimensions. The method regards dimensions as constraints between vertex geometries. The user defined part topology and set of dimensions are used to derive the exact geometry. The part geometry can be modified by changing the suitable dimension constraints.

Jayaraman and Srinivasan (1989 and 1989a) have developed the virtual boundary requirements (VBR) approach for tolerance synthesis. Requicha’s work (1984) defines composite features that cannot be uniformly handled, procedures for measured entities that do not work well with convex shapes, and incomplete conditions for position tolerance. For particular classes of requirements, the VBR approach defines necessary and sufficient conditions for acceptable tolerances that are quite useful for tolerance synthesis.

Chen and Ravani (1986) described a procedure for computationally obtaining offset surfaces based on a surface patch. The offset surface approximation method is based on normal distance least-squares fitting method. An algorithm for detection and removal of closed loop on offset surface due to its self intersection was also presented. The offset surface generation procedure was based on design and manufacturing considerations.

Lu and Wilhelm (1991) described a program CASCADE-T for tolerance synthesis and analysis. It is based on a detailed description of the part, its functional requirements, and associated tolerance relations. This program makes use of a variety of artificial intelligence rules like rule-based reasoning, constraint networks, and dependency tracking. Tolerance primitives based on virtual boundary requirements are provided to build tolerance specifications for complex cases. CASCADE-T can be used in determining the solutions consistent with the design problem. Performing manipulations of the constraint network, an optimal design within the original functional requirements can then be obtained.

Pandit and Starkey (1988) proposed a tolerance analysis procedure for the assembly of parts with prismatic and cylindrical surfaces. The nominal dimensions of the parts were
represented by boundary representations in the solid modeler and tolerances were connected to the models. By randomly perturbing the nominal geometries within the tolerance zones, instances of the parts were obtained. The assembled instances were checked for assembly clearances and tolerances and results of numerous cases were combined to get a statistical estimate of assemblability.

3.3 Research Objectives

In the literature review, it was observed that the clearance and tolerance analysis evaluations were most commonly developed for planar mechanisms and assemblies. Those studies that considered spatial situations were restricted to special cases only. The current research is intended for a more general class of spatial assemblies. Also since planar assemblies will exhibit spatial characteristics when clearances are considered, a spatial analysis is necessary to accurately analyze even planar assemblies. It is the intention that the results of this research would yield a method for evaluating the kinematic characteristics of spatial assemblies due to the effect of clearances and tolerances. The results also include a method for optimally allocating the clearances and the manufacturing tolerances to minimize the cost of manufacturing for an optimum mechanism output performance or assembly features. As discussed earlier, the main thrust was to study the tolerance and clearance effects on surfaces coming in contact e.g., 3-d cam contacts along with planar lower pair joints. Studies in this field are mainly concerned with the dynamic behavior of the joints. Kinematic performance analysis with the incorporation of joint clearances and manufacturing tolerances are normally not performed. This is one of the major thrusts in this work.

The joints that are considered in the analysis and the tolerance allocation procedure are the revolute, rectangular prismatic, cylindrical, spherical and three-dimensional general cam joints. Gutkowski (1990) developed a generalized method for the position and force analysis of closed single loop, one degree-of-freedom mechanisms. Singular value decomposition was used to solve under and over constrained systems. Various planar and spatial mechanisms have been analyzed successfully using these procedures and this has been used as the backbone kinematic analysis procedure in this work.

As a consequence of considering joint clearances, the behavior of the joints alter to some extent. They are modeled with that in mind. The clearance problem is treated first.
Analysis with clearances is performed using the "equivalent" links concept. The main idea behind that is the assumption of the existence of a massless fictitious link for the clearance and equal in length to the clearance at the joint. The whole kinematic definition of the linkage changes because of these links. The tolerance effects are incorporated next through the calculation of sensitivity factors. These are used to determine the tolerance on the mechanism output based on the individual dimensional tolerances in the mechanism. Surface contact joints are modeled using the concepts of "offset surfaces" to incorporate the tolerances on the surfaces. This is similar to the concept of surface offsets in ANSI Y14.5M GD&T standards. Optimization techniques are used to obtain the design tolerance specification to obtain a design for manufacturability. A "robust" design at the lowest possible cost of manufacturing and sensitivity of output variation is then obtained.

The general class of mechanisms that are analyzed include one-degree-of-freedom single loop spatial mechanisms. The clearance joint model used is of a continuous contact type. This assumption is realistic, even though contact loss is possible in some situations. This is a variation from the often used mode of contact loss where it is assumed that the "pin" floats in the hole during some time in the motion cycle. Stochastic and dynamic methods have generally been used in analyzing these. In this current work a more deterministic approach is taken and the "pin" location is known or can be obtained. A low speed, one-degree-of-freedom loaded mechanism would have its joints reoriented and aligned to maintain contact position to satisfy force equilibrium. For this reason, a force analysis of the ideal mechanism provides a starting point in an optimization procedure for determining the way the clearances are taken up in the actual mechanism.

Some studies have analyzed joint clearances and tolerances using the equivalent link concept for planar four-bar and other mechanisms. The massless clearance links in these cases are assumed to be aligned along force directions obtained from no clearance situation. The changes occurring in force balance because of these links and the effective reorientation of the links are normally not considered. Also, most of the work only considers planar revolute joints. The current work uses the concept of clearance link, but the kinematic and force solutions are done again with these links in an iterative fashion.

Statistical tolerance evaluations are often done by using the Monte Carlo Simulation techniques. There has been some work (Iannuzzi and Sandgren, 1994) that has been reported in this direction. Although the results are promising in many instances, the numerical overheads for these methods for a reasonably good estimate is quite expensive.
and often prohibitive. The current work with its model of statistical combination of individual tolerances and clearances along with the process mean shifts incorporated in the optimization procedure is a very good and viable alternative.

One work in the literature has tried to incorporate the manufacturing alternatives for assemblies from cost standpoint. Alternative manufacturing processes can be evaluated to find the best possible solution for an assembly. Very simple assemblies like tolerance stacks were considered in this case. The current work can incorporate a similar procedure for complex assemblies like mechanisms. The designer can setup various manufacturing alternatives for each part with the different cost-tolerance functions as explained in Chapter 5. These can then be executed in the optimization process to find out the globally best manufacturing process for each part from total cost and sensitivity point of view.

Also the concept of minimization of life-cycle cost (minimum manufacturing cost and minimum sensitivity of output to the variation in variables) is a relatively unique consideration for mechanism design.

### 3.4 Research Issues

The research program involved both theoretical work and the development of efficient computer programs. The main research issues are identified here:

- Need of producing assemblies with least life cycle cost and relatively insensitive to process deviations
- Need for an efficient procedure for incorporating optimum tolerance and clearance for assemblies and mechanisms during the design stages
- Need for development of a procedure for analyzing the sensitivities of the output to the individual tolerances and clearances in the tolerance allocation process
- Need for a procedure for incorporating cost of manufacturing using alternative manufacturing methods in the optimization process
• Need for a procedure to obtain the feasible optimum between the conflicting objectives of minimum manufacturing cost and minimum sensitivity of mechanical error using multicriteria optimization method

• Need for a study of the procedure of allocation of individual tolerances with the output error as a constraint

• Requirement of the development of a general procedure for position analysis of complex mechanisms and assemblies

• Need for development of a procedure for force analysis along with clearance allocation since the way the joint clearances are taken up depend on the joint forces

• Need for generalized analysis and tolerance allocation techniques for three-dimensional contacting surfaces including non-analytically defined surfaces through the use of surface patches
CHAPTER 4

SPATIAL MECHANISM KINEMATIC AND FORCE ANALYSIS

4.0 Introduction

The way in which the clearances are taken up in a mechanism is directly dependent on the mechanism configuration and the loading conditions. A closed-form solution to the general spatial mechanism problem is a formidable task, hence an iterative procedure is necessary to solve the position problem. The nature of the position displacement analysis is highly nonlinear and for this reason good starting values for all the variables are required for the initialization of the iterative procedure. Since the clearances will be small, results from a no-clearance configuration can serve as a reasonable starting point for the clearance analysis. This entails the requirement of a procedure for ideal position analysis of one-degree-of-freedom, single loop, spatial mechanisms. The force analysis must also be performed because the clearances are taken up in the direction of the internal joint reactions. Again, the force analysis of the no-clearance ideal mechanism would provide an initial guess for the points of contact in the joints. Outline of the procedure developed by Gutkowski (1990) will be discussed here.

4.1 Outline Of Generalized Spatial Mechanism Position and Force Analysis Procedure

General spatial mechanisms can be most easily analyzed using matrix methods and coordinate transformations. In this procedure, a coordinate system is attached to each link
of the mechanism (Fig. 4.1) so that the relationship between successive coordinate systems is defined in terms of the physical parameters such as link lengths and other link geometry features. Successive coordinate transformations are then made around the mechanism. If the transformations are made around a closed loop, then the resulting matrix equation is as follows:

$$T_{1,1} = T_{1,2}T_{2,3} \cdots T_{n-1,n}T_{n,1} = I_{4x4}$$  \hspace{1cm} (4.1)$$

where \(T_{i,j}\) = the 4x4 transformation from link i to link j

\(I_{4x4}\) = the 4x4 identity matrix

\[Fig. \ 4.1: \text{Coordinate Transformations For Spatial Mechanism Analysis}\]

The transformation matrices are functions of the link dimensions and the link pair variables. Expanding the matrix equations in terms of the pair variables provides equations for the variables as a function of the mechanism input variables. The matrix representation of many of the basic link and lower pair joint combinations are available in the literature
(Hartenberg and Denavit, 1964, Kinzel et al., 1972). The scalar equations represented by the closed-loop matrix formulation are highly nonlinear and can be solved using several methods. The methods developed by Uicker et al (1963) and Pieper and Roth (1969) are well known. The kinematic position, velocity, and acceleration of a general spatial mechanism can also be solved using the vector method proposed by Chace (1964).

In the position analysis of a direct contact mechanism, the procedure is to locate the contact points by defining the location of the two coincident points in the contacting surfaces. To do this, it is assumed that both surfaces are defined as mathematical functions. If the surfaces of the two bodies at the area of contact are defined by a table of numbers, a mathematical surface can be fitted through the data. The procedure used to determine the contact point depends on the types of surfaces involved although a nonlinear solution procedure such as Newton’s methods is required.

The position analysis is the most difficult part of the kinematic analysis. Once the positions of all the points of interest are found, the velocity and accelerations are obtainable by direct differentiation. This process gives a set of linear equations in the unknown velocity and acceleration terms so that the resulting equations can be easily solved. Additional matrix representations are required for higher pair joints.

Once the position, velocity, and acceleration analyses are performed, a quasi-static force analysis of a spatial mechanism is relatively straightforward. The inertial moments and forces are computed based on the acceleration, mass, and geometry information. The force analysis can again be approached from the matrix or the vector formulation. A friction-less analysis is performed which gives rise to a set of simultaneous linear matrix equations. The solutions can be easily obtained by linear matrix techniques.

4.2 Procedure For Clearance And Tolerance Analysis

The direction in which the backlash or the clearances are taken up in a loaded mechanism can be established once the force analysis solution is obtained. This must be done before the effects of tolerances can be determined. The analysis of clearances is a major concern in joints which are nominally designed to be revolute and sliding joints. Clearance errors can be easily characterized in these types of joint pairs. To represent the most common form of clearance existence, the clearance geometry of the standard lower pair joint is
considered as an example. A qualitative idea of the modeling can be obtained from Fig. 4.2.

Fig. 4.2: Revolute, Cylindrical, Prismatic, And Spherical Joints With Clearances (a) Revolute Joint With Radial And Limited Axial Clearance (b) Cylindrical Joint With Radial And Unlimited Axial Clearance (c) Prismatic Joint with Lateral and Unlimited Axial Clearance (d) Spherical Joint with Radial Shell Clearance

To consider the case of a revolute joint as an example, it can be divided into a male element (R+) and a female element (R-) as shown in Fig. 4.3. In a functional revolute joint, h+ must be larger than h- and D+ must be smaller than D- to avoid interference.
Fig. 4.3: Revolute Joint With Diametral And Axial Clearance $C_D$ And $C_A$

These will give rise to axial clearance ($C_A$) and diametral clearance ($C_D$). The effect of clearances in spatial mechanisms is more difficult to analyze than in the planar mechanisms. Unless the clearances are extremely small, the error potential becomes significant in spatial mechanisms. In a planar revolute joint, the element $R^+$ can move relative to $R^-$ in a radial direction only, and hence the axial clearance $C_A$ is unimportant. As shown in Fig. 4.4, the
position of link \( k \) relative to link \( j \) is restricted to a region defined by the diametral clearance \( C_D \) and the pair angle \( \theta_{kj} \). For a given angle \( \theta_{kj} \), the possible displacement error for point A of link \( k \) is a function of \( C_D \) only and is not affected by the link length or other geometry.
However, for the spatial mechanisms, the nominal one-dof revolute joint becomes a six-dof spatial joint, and both $C_A$ and $C_D$ along with much of the rest of the joint geometry are significant. Element $R^+$ can translate along as well as rotate about three independent axes.
relative to element R. As shown in Fig. 4.5, for two links at a nominal angle \( \theta_{kj} \), the displacement error for the point A of link k depends on both the clearances and geometry.

To determine the final configuration of a mechanism, it is necessary to determine the interaction of the joints' contacting surfaces. When the clearances are taken up, the contacting surfaces adhere to the following rules: they must be tangent to each other, must not pass through each other, and must have surface normals at the contact point directed along the force vector for that joint. To establish these requirements, the joint surfaces must be mathematically defined. For the simple joints that have surfaces constructed from planes, cylinders, and spheres, the surfaces would be represented exactly and surface tangents and normals would be established easily. Some sort of surface approximating techniques are required for describing the general three-dimensional cam joint. The surfaces can be described using Coon's surface patches (Rogers and Adams, 1976).

### 4.3 Analysis Procedure

To study the effect of clearances in spatial linkages, it is convenient to establish a coordinate system on link k and another on link j at the R+ and R- elements of the joint. The effect of clearances can then be considered via coordinate transformations. The coordinate transformation matrix used is based on work by Hall (1967) and Sheth and Uicker (1971) and is similar to that used by Brat (1971) for mechanisms with five-dof spatial joints.

The Denavit and Hartenberg (1955) approach is another very well known technique for embedding the coordinate systems in the linkage. This approach is very widely used especially for "axial" type of joints. The Sheth and Uicker's approach is more amenable to "non-axial" types of joints, is more "building block" oriented and is thus more easily programmed than the other. The four parameters of the Denavit and Hartenberg notation depend not only on the shape of the link under consideration, but also on the shape of the previous link in the kinematic loop being considered. Thus, it is not as suitable as the other approach for programmability.

The overall concept in the Sheth and Uicker's method is similar to the Denavit and Hartenberg technique. Local coordinate systems are attached to each link, and the transformation matrices between these systems are functions of the joint degrees-of
freedom. The equations that are obtained contain the constant link dimensions and variable joint degrees of freedom. The method however, clearly “decouples” the transforming effects of the constant link dimensions from the effects of the joint variables. This results in a procedure more amenable to a building block implementation.

Two links, \(i-1\) and \(i\) are shown in Fig. 4.6. For simplicity, the two links are joined by a revolute joint shown in an exploded view. Each link has two attached coordinate systems rather than one – these are labeled as systems \((uvw)\) and \((xyz)\). These systems are rigidly affixed to the links at their pairing element locations. The system’s orientations are not important. For the links in Fig. 4.6, the transformation matrix of interest would be \(T_{i-1,i}\) (i.e., the transformation matrix converting system \((uvw)_i\) coordinates to system \((uvw)_{i-1}\) coordinates). The method develops this in two stages. The first step is to determine the transformation from system \((uvw)_{i-1}\) to system \((xyz)_{i-1}\). This will be called \(S_{i-1}\) (shape matrix). \(S_{i-1}\) is solely a function of the shape of link \(i-1\) since both systems \((xyz)_{i-1}\) and \((uvw)_{i-1}\) are embedded in link \(i-1\). Next the transformation from system \((xyz)_{i-1}\) to system \((uvw)_i\) must be formed. This transformation will be labeled as \(P_{i-1,i}\) (the pair matrix). It is entirely a function of the degrees-of-freedom permitted by the joint pair. The transformation \(T_{i-1,i}\) may then be calculated from:

\[
T_{i-1,i} = S_{i-1}P_{i-1,i}
\]  

where

\[
T_{i-1,i} \text{ = the transformation matrix from system } (uvw)_{i-1} \text{ to system } (uvw)_i
\]

\[
S_{i-1} \text{ = the transformation matrix from system } (uvw)_{i-1} \text{ to system } (xyz)_{i-1} \text{ (the constant shape matrix)}
\]

\[
P_{i-1,i} \text{ = the transformation matrix from system } (xyz)_{i-1} \text{ to system } (uvw)_i \text{ (the variable pair matrix)}
\]

A single closed loop mechanism containing \(n\) links could then be described with transformation matrices as follows:

\[
I_{4x4} = S_1P_{12}S_2P_{23}S_3P_{34} \ldots S_{n-1}P_{n-1,n}S_nP_{n,1}
\]  

The \(S\) matrices are constant for a given mechanism, and the \(P\) matrices are functions of the degrees of freedom of the joints.
For the coordinate systems \((uvw)_i\) and \((xyz)_i\) attached to link \(i\) in Fig. 4.7, the transformation from system \((xyz)_i\) to \((uvw)_i\) can be formulated by considering the matrix product of six simple one-dof transformations. First the common normal to \(w_i\) and \(z_i\) is constructed. This normal intersects \(w_i\) at \(p\) and \(z_i\) at \(q\). A vector, \(t_i\), is directed from \(p\) to \(q\) along the common normal. Starting at system \((uvw)_i\), system \((xyz)_i\) is obtained by translation of \(c_i\) along \(w_i\), a rotation of \(\gamma_i\) about \(w_i\), a translation of \(a_i\) along \(t_i\), a rotation of \(\theta_i\) about \(w_i\), and a translation of \(b_i\) along \(t_i\).
\(\alpha_i\) about \(t_i\), a translation of \(b_i\) along \(z_i\) and a rotation of \(\beta_i\) about \(z_i\). The definition of the dimensions are as follows:

\[c_i = \text{the distance from the origin of } (uvw)_i, O_{uvw}, \text{ to the point of intersection, } p \text{ of } w_i \text{ with } t_i. \text{ It is positive in the direction of } w_i.\]
\( \gamma_i \) = the angle required to rotate \( u_i \) about \( w_i \) into \( t_i \). It is defined positive counterclockwise looking down \( w_i \).

\( a_i \) = the length of the common normal (i.e., the distance between \( p \) and \( q \)). It is positive in the direction of \( t_i \), by definition.

\( \alpha_i \) = the angle required to rotate \( w_i \) about \( t_i \) into \( z_i \). It is positive counterclockwise looking down \( t_i \).

\( b_i \) = the distance from the point of intersection, \( q \) of \( t_i \) with \( z_i \) to the origin of \((xyz)_i \), \( O_{xyz} \).

It is defined positive in the direction of \( z_i \).

\( \beta_i \) = the angle required to rotate \( t_i \) about \( z_i \) into \( x_i \). It is defined positive counterclockwise looking down \( z_i \).

These displacements and rotations can be mathematically described using rigid body translation and rotation matrices. In other words, system \((uvw)_i \) can be rotated and translated such that it is coincident and parallel to system \((xyz)_i \). The combination of these matrices will yield the transformation matrix from system \((xyz)_i \) to \((uvw)_i \). This is the shape matrix \( S_i \). The details of the procedure are given in Appendix A.

Using these general definitions for the translation and rotation matrices, the shape matrix for a given link can be obtained as function of the six parameters \( a, \alpha, b, \beta, c, \) and \( \gamma \):

\[
S(a, \alpha, b, \beta, c, \gamma) = 
\begin{bmatrix}
  c\beta_i \gamma_i - c\alpha_i s\beta_i s \gamma_i & -s\beta_i \gamma_i & s\alpha_i s \gamma_i & b_i s\alpha_i s \gamma_i + a_i c \gamma_i \\
  c\beta_i s \gamma_i + c\alpha_i s\beta_i c \gamma_i & -s\beta_i s \gamma_i + c\alpha_i c\beta_i c \gamma_i & -s\alpha_i c \gamma_i & -b_i s\alpha_i c \gamma_i + a_i s \gamma_i \\
  s\alpha_i s\beta_i & s\alpha_i c \beta_i & c \alpha_i & b_i c \alpha_i + c_i \\
  0 & 0 & 0 & 1
\end{bmatrix}
\] (4.4)

where \( s\alpha = \sin \alpha \), \( c\alpha = \cos \alpha \), etc. The six parameters \( a, \alpha, b, \beta, c, \) and \( \gamma \) will be constant for a given link.

**Pair Matrices**

After the pairing element coordinate systems \((uvw)\) and \((xyz)\) have been established for each link in a mechanism, the links may be joined together with pair matrices \( P \). These pair matrices are transformations between the \((xyz)\) and \((uvw)\) systems of adjacent links. A given pair matrix is a function of the degrees-of-freedom permitted by the joint being modeled. The pair matrices are unrelated to the shape of the link connected by them. A
standard library of pair matrices suitable for use with many link shapes have been
developed by Gutkowski (1990) and are summarized in Appendix B. As an example of the
nature of the pair matrices, the revolute pair matrix is shown in Fig. 4.8.

The joining of two links by a particular joint might force the alignment of certain element
pair axes on the links being joined. For the revolute joint, the following specifications are
required:

The origins of \( (xyz)_{i-1} \) and \( (uvw)_i \) are coincident.

\( z_{i-1} \) and \( w_i \) are collinear and in the same direction

\( \theta_i \) is measured about \( z_{i-1} \) from \( x_{i-1} \) to \( u_i \) and is positive counterclockwise looking
down \( z_{i-1} \)

The pair matrix in this case is given by:

\[
P(\theta_i) = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i & 0 & 0 \\
\sin \theta_i & \cos \theta_i & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(4.5)

The pair matrices for other commonly observed joints can be similarly obtained by a case
by case study as indicated in Appendix B.

Fig. 4.8: Description And Specifications For Revolute Pair Matrix
4.4 Procedure For Analytically Defined Surface Contact Joints

Using the Sheth and Uicker method, it is possible to develop pair matrices that can simulate the kinematic behavior of joints consisting of two contacting three-dimensional surfaces if the surfaces can be analytically defined and are convex (Gutkowski and Kinzel, 1995). Convex surfaces can have one and only one contact point. Consider the surface contact joint for two analytically defined surfaces shown in Fig. 4.9. The link i-1 is connected to link i via two surfaces in contact. Surface \( \psi_{i-1} \) is rigidly attached to link i-1 and is completely defined in terms of system \((xyz)_{i-1}\). Similarly, the surface \( \psi_i \) is rigidly attached to link i and defined in terms of system \((uvw)_i\). The pair matrix must be able to describe the relative motion permitted between links i-1 and i.

![Fig. 4.9: A 3-dimensional Analytical Surface Contact Joint](image.png)
Points A and B are the respective points of contact on the two surfaces \( \psi_{i-1} \) and \( \psi_i \). The transformation from system \((xyz)_{i-1}\) to \((uvw)_i\) is not obvious. “Floating” coordinate systems are placed at points A and B on the two surfaces. The reference to the term floating is made due to the fact that they are always located at the point of contact on each surface. The pair matrix between the two surfaces may then be obtained by a series of transformations:

\[
P_{i-1,i} = T_{i-1,A} T_{A,B} T_{B,i}
\]  

(4.6)

where

- \( P_{i-1,i} \) = the pair matrix for the surface contact joint between links \( i-1 \) and \( i \)
- \( T_{B,i} \) = transformation to obtain the system \((uvw)_i\) with respect to the intermediate floating system B
- \( T_{A,B} \) = transformation to obtain the intermediate floating system B with respect to the intermediate floating system A
- \( T_{i-1,A} \) = transformation to obtain the floating intermediate system A with respect to the system \((xyz)_{i-1}\)

For surface \( \psi_{i-1} \), the floating system at A is based on two mutually perpendicular vectors at A. The first one, \( \hat{n}_{i-1} \) is normal (assumed outward) to the surface at A. The other, \( \hat{t}_{i-1} \) is tangent to the surface at A. A third vector is obtained by taking the cross product of these two vectors. The set of three mutually perpendicular vectors is used to establish the floating coordinate system on \( \psi_{i-1} \) with origin at A. Similarly the floating coordinate system at B on \( \psi_i \) is established.

For simple surfaces in contacting motion, the contact must be at a point, and the surface normals at the point of contact must be collinear. This implies that the origins of systems A and B must be coincident, and the axis of system A selected to be along the surface normal of \( \psi_{i-1} \) must be collinear with the axis of system B selected to be along the surface normal of \( \psi_i \). Thus, the transformation from system B to A or from A to B can only involve a single degree of freedom rotation about the surfaces’ common normals. For a surface that can be defined analytically, any point on the surface can be described by means of a parametric representation in terms of two variables. The surface’s normal and tangent vectors can be easily obtained at the point defined by the two parameters. Using these two vectors, the transformation from the floating system to the surface definition system or vice
versa can be easily obtained. The transformations $T_{i-1,A}$ and $T_{B_i}$ for $P_{i-1,i}$ can thus be obtained according to the procedure outlined above.

The remaining transformation $T_{A,B}$ from one floating system to the other floating system at the point of contact is needed to completely implement the procedure. The transformations between the floating systems requires a simple rotation about the common normal at the point of contact. The transformation from floating system B to floating system A has been developed in two stages.

Fig. 4.10 shows the transformation procedure between the two floating systems. In the first stage, a transformation $T_{B:B'}$ is made from system B to an intermediate system $B'$. This is obtained by a rotation $\pi$ about the x-axis of system B. This aligns the $z'$-axis of system $B'$ with the $z$-axis of system A. Next a transformation $T_{A,B'}$ about the $z'$-axis of $B'$ is performed. This rotates the $x'$-axis of system $B'$ to the $x$-axis of system A. The angle $\phi$ required to do this is simply the angle between the $x$-axes of systems B and A measured about the $z$-axis of system A. The transformation from the floating system B to floating system A is expressed as:

$$T_{\text{floating, floating}} = T_{A,B'} T_{B:B'}$$

or,

$$T_{f,f} = \begin{bmatrix}
\cos \phi & -\sin \phi & 0 & 0 & 0 \\
\sin \phi & \cos \phi & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \cos \pi & -\sin \pi & 0 & 0 \\
0 & \sin \pi & \cos \pi & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
\cos \phi & \sin \phi & 0 & 0 \\
\sin \phi & -\cos \phi & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

It is clear that the above transformation is not a function of the contacting surfaces. However, the floating coordinate systems have to be defined as above and the surfaces should not contact at more than one point.

It is known from a geometric standpoint that a surface contact joint has five degrees of freedom. The only motion it can constrain is the displacements along the common surface normal at the point of contact. As shown above, the transformation from a surface definition system to the floating system (or vice versa) is a function of the surface's two defining parameters. There are two of these transformations involved providing a total of four degrees of freedom. The transformation between floating systems provides one more degree of freedom, thus making the total to five. Gutkowski (1990) has developed the "definition to floating" and "floating to definition" coordinate transformations for various
surfaces namely spherical, cylindrical, planar, ellipsoids, paraboloids, and cones. The details of these are given in Appendix C.

Fig. 4.10: Floating Coordinate Systems On Two Contacting Surfaces

4.5 Analysis Procedure For General Surface Contact Joints

General surfaces cannot be described accurately in terms of the analytical forms such as spheres, cones, ellipsoids etc. A common means for mathematically describing a general surface is through the use of surface patches (Rogers and Adams, 1976, and Farin, 1991). Very complex surfaces can be constructed using a collection of these relatively easy to
define patches. As with analytical surfaces, the only requirement is the ability to define a well behaved floating coordinate system at a point on the surface.

A patch is bounded by four curves in space that are defined by parameters ranging from 0 to 1. The surface described by the patch is a function of these boundary curves. The effects of the curves are blended together to form a smooth surface. The boundaries of a surface patch are shown in Fig. 4.11. The patch is defined by families of curves in two directions on the surface of the patch. The four boundary curves \( P(0,w) \), \( P(1,w) \), \( P(u,0) \), and \( P(u,1) \) are specified and the surface is obtained through a blending process. A possible implementation of a surface patch is the bicubic surface patch, where cubic splines are used as boundary curves. Given a set of points that lie on a curve, cubic splines are easily generated to pass through the points and approximate the curve. Cubics have the advantage that they satisfy piece wise zero and first order continuity. Also, given a set of points lying on a curve, it is relatively easy to generate the cubic spline passing through the points and approximating the curve. The cubic spline will also be approximately embedded in the same surface as the original curve. For this reason, surface patches approximating a surface are meshed with cubic splines.

\[
\begin{align*}
\text{Fig. 4.11: The Boundary Curves Of A Surface Patch}
\end{align*}
\]

The equation for a parametric cubic can be shown to be (Rogers and Adams, 1976):

52
\[ P(t) = F_1(t)P(0) + F_2(t)P(1) + F_3(t)P'(0) + F_4(t)P'(1) \]  

where \( P(0) \), \( P(1) \), \( P'(0) \), and \( P'(1) \) are the coordinates and slopes for the corner points in the patch. The \( F_i(t) \) are given by:

\[
\begin{bmatrix}
F_1(t) & F_2(t) & F_3(t) & F_4(t)
\end{bmatrix} = \begin{bmatrix}
2 & -2 & 1 & 1 \\
-3 & 3 & -2 & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \]

The \( F_i(t) \) are the blending functions in the above and are used to ensure zero and first order continuity between adjacent pieces. If two curves are known at the opposite sides of a patch, then a lofted surface can be easily established between them. This is given by:

\[ Q(u,w) = P(u,0)(1-w) + P(u,1)w \]

where \( P(u,0) \) and \( P(u,1) \) are the \( u \)-direction boundary curves at \( w = 0 \) and \( w = 1 \). Here \( Q(u,w) \) is a vector containing the components of a point on the lofted surface specified by the parameters \( (u,w) \). Using the blending functions, the equation for the lofted surface between the two \( u \)-direction curves is obtained as:

\[ Q(u,w) = P(0,w)(1 - 3u^2 + 2u^3) + P(1,w)(3u^2 - 2u^3) + P_u(0,w)(u - 2u^2 + u^3) + P_u(1,w)(-u^2 + u^3) \]

where \( P_u(u_i,w) = \frac{\partial P(u,w)}{\partial u} \bigg|_{u = u_i} \)

Another surface can be similarly lofted between the other two \( w \)-direction boundary curves. The two lofted surfaces can then be combined to obtain the final form of the approximated surface as:

\[ Q(u,w) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \mathbf{N}\mathbf{P}^T \begin{bmatrix} w^3 \\ w^2 \\ w \\ 1 \end{bmatrix} \]

Here \( \mathbf{N} = \mathbf{M}^{-1} \) and is given by:
Here, \[ P = \begin{bmatrix} P(0,0) & P(1,0) & P_w(0,0) & P_w(1,0) \\ P(1,0) & P(1,1) & P_w(1,0) & P_w(1,1) \\ P_u(0,0) & P_u(0,1) & P_u,w(0,0) & P_u,w(0,1) \\ P_u(1,0) & P_u(1,1) & P_u,w(1,0) & P_u,w(1,1) \end{bmatrix} \]

and \[ M^{-1} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \] (4.14)

Here,

\[ P_u(u_i, w_j) = \frac{\partial P(u, w)}{\partial u} \bigg|_{u=u_i, w=w_j} \] (4.16)

\[ P_w(u_i, w_j) = \frac{\partial P(u, w)}{\partial w} \bigg|_{u=u_i, w=w_j} \] (4.17)

\[ P_{u,w}(u_i, w_j) = \frac{\partial P(u, w)}{\partial u \partial w} \bigg|_{u=u_i, w=w_j} \] (4.18)

The twist vectors of Eq. (4.18) are often difficult to ascertain, and therefore, are sometimes assumed to be zero. This is called the F-patch. This might cause some flattening of the surface but the use of smaller patches will counteract this problem. The F-patch can also be studied for the surface contact modeling.

Having given the definition of a surface patch, the floating coordinate system may be easily constructed at any point on the surface. The following procedure will be used. If \( u \) and \( w \) are specified on the surface, the floating coordinate system will have the associated point as its origin. The surface normals required will be obtained via an indirect approach. If two non-collinear surface tangents can be determined at a point, then a surface normal at that point may be obtained easily. The cross product of the surface tangents will produce a mutually perpendicular vector normal to the surface. By taking the partial derivatives of the defining equations of the patches at any point, the tangent vectors may be obtained. The partial derivatives are to be taken once with respect to \( u \) and once with respect to \( w \). Thus, we have:
Next a rule can be defined stating that cross product of $u$ and $w$ produces an outward normal. The surface patches must be defined to adhere to this rule. That is, the cross product $t_u \times t_w$ will always produce an outward normal at the point $(u,w)$. The $z$ axis of the floating system will be given by:

$$\hat{z} = \frac{\vec{t}_u \times \vec{t}_w}{|\vec{t}_u \times \vec{t}_w|}$$ \hfill (4.21)

Now, $\hat{z}$ must be perpendicular to $\vec{t}_u$. So, let,

$$\hat{x} = \frac{\vec{t}_u}{|\vec{t}_u|}$$ \hfill (4.22)

The $y$ axis will then be given by:

$$\hat{y} = \hat{z} \times \hat{x}$$ \hfill (4.23)

The components of the unit vectors represent the direction cosines of the axes in the definition system of the surface, which is the system in which the corner point coordinates and tangents are defined. Thus, using the components of $\hat{x}$, $\hat{y}$, and $\hat{z}$, the rotation partition of the floating-to-definition transformation matrix can be obtained. The displacement partition can be determined from $Q(u,w)$ as this gives the coordinates of the floating system's origin relative to the definition system. Therefore, the floating-to-definition system is given as:

\[
\bar{t}_u = \frac{\partial Q(u,w)}{\partial u} = \begin{bmatrix} 3u^2 & 2u & 1 \end{bmatrix} NPN^T \begin{bmatrix} w^3 \\ w^2 \\ w \\ 1 \end{bmatrix} \tag{4.19}
\]

\[
\bar{t}_w = \frac{\partial Q(u,w)}{\partial w} = \begin{bmatrix} u^3 & u^2 & u \end{bmatrix} NPN^T \begin{bmatrix} 3w^2 \\ 2w \\ 1 \\ 0 \end{bmatrix} \tag{4.20}
\]
The definition-to-floating transformation is obtained by inverting the above matrix according to:

\[
T_{\text{definition, floating}} = \begin{bmatrix}
R_{\text{td}}^T & -R_{\text{td}}^T D_{\text{td}} \\
0^T & 1
\end{bmatrix}
\]  

(4.25)

where \( R_{\text{td}} \) is the (3x3) rotation partition and \( D_{\text{td}} \) is the displacement partition of the floating-to-definition transformation. Once the above are obtained, the analysis procedure can follow exactly the procedure developed by Gutkowski (1990).

The procedure considers the link and joint dimensions and characteristics as separate entities in the solution process. This can be implemented through the incorporation of shape and pair matrices as discussed earlier. Complex interactions in the joints like joint clearances can be considered using this. Joints with clearances have increased numbers of degrees-of-freedom. Ideally, single degree-of-freedom mechanisms and zero degree-of-freedom assemblies become multi degree-of-freedom assemblies. The solutions to these systems are dependent on the loading of the assemblies and mechanisms. For this reason, a single iterative process solving the position problem and satisfying the dynamic force equilibrium has been developed.

### 4.6 Summary Of The Procedure For Closed Loop Position, Velocity, And Acceleration Analysis

Suppose that links 1 to N are connected by joints to form a closed chain assembly. As discussed earlier, the relative motion between adjacent bodies in the chain will be functions of joint degrees-of-freedom. Let \( q_{ij} \) represent the joint degrees-of-freedom in the joint between links \( i \) and \( j \). Using the matrix product chain rule for the closed loop, we have

\[
T(Q)_{1,1} = T(q_{1,2})_{1,2} T(q_{2,3})_{2,3} \ldots T(q_{N-1,N})_{N-1,N} T(q_{N,1})_{N,1}
\]
\[ T(q_{1,2}, q_{2,3}, \ldots, q_{N-1,N}, q_{N,1})_{1,1} = I_4 \]  
\[ \text{where,} \quad Q = [q_{1,2} \quad q_{2,3} \quad \ldots \quad q_{N-1,N} \quad q_{N,1}]^T \]

The transformation matrix product around a closed loop of connected bodies is always equal to the identity matrix. This can be used to produce six independent equations to solve for six joint degree-of-freedoms. This will provide the position or displacement analysis solution. Three equations are selected from the rotation partition and three from the displacement partition. The equations from the rotation partition must not be from the same row or column.

Let the elements of the closed loop matrix product be represented by \( t(Q)^{i,j} \) where \( i \) is the row number and \( j \) the column number. Sandor and Erdman (1984) used the following equations for nonlinear displacement analysis:

\[ t(Q)^{1,3} + t(Q)^{1,1} + t(Q)^{3,3} - 2 = 0 \]  
\[ t(Q)^{2,1} + t(Q)^{1,1} + t(Q)^{2,2} - 2 = 0 \]  
\[ t(Q)^{3,2} + t(Q)^{2,2} + t(Q)^{3,3} - 2 = 0 \]  
\[ t(Q)^{1,4} = 0 \]  
\[ t(Q)^{2,4} = 0 \]  
\[ t(Q)^{3,4} = 0 \]

The Eqs. (4.28) - (4.33) work fairly well for the nonlinear solutions.

The closed loop velocity analysis is performed by differentiating the displacement analysis equation with respect to time:

\[ \frac{d}{dt} T(Q)_{1,1} = \frac{d}{dt} I_4 \]

or

\[ \sum_{i=1}^{F} \frac{\partial T(Q)_{1,1}}{\partial q_i} \dot{q}_i = 0_4 \]

where the \( q_i \)'s are the components of \( Q \) and \( F \) is the total number of degrees-of-freedom in the loop. The above equation provides six independent, linear equations in terms of the
unknown $q_i$. In order to obtain a unique solution for the system, there should not be more than six unknown $q_i$'s. The independent set of velocity equations are often written as:

$$J\dot{Q} = 0$$

(4.35)

where $J$ is called the Jacobian matrix and contains the partial derivative terms. This is a straightforward linear system of equations, and it can be solved for the unknowns unless $J$ is singular.

Similarly, the equations for acceleration analysis are obtained from differentiating with respect to time, the velocity equations:

$$\sum_{i=1}^{F} \frac{\partial T(Q)}{\partial q_i} \ddot{q}_i = -(A_n + A_c) = B$$

(4.36)

where $A_n$ and $A_c$ are the normal and Coriolis acceleration terms (known). As with the velocity analysis, six independent equations may be chosen and solved for using the matrix equation $J\dot{Q} = b$.

### 4.7 Summary Of The Procedure For Closed Loop Dynamic Force Analysis

Consider the links shown in Fig. 4.12. System $G_i$ is assumed aligned with the principal axes and placed at the link center of mass. Systems $u_i$ and $x_i$ represent the pairing element coordinate systems. The affixed coordinate system at $G_i$ is called the kinetic reference frame (Waldron, 1992) for the link $i$. If $^i\omega_i$ and $^i\alpha_i$ are the angular velocity and angular acceleration of link $i$ with respect to ground resolved in a system embedded in ground, then the dynamic force equilibrium equations may be written down as:

$$^if_{ext} = m^iA_{\alpha_i}$$

(4.37)

and

$$\sum^if_{ext} + \sum^i r_i \times f_{ext} = I_{Gl}^i\alpha_i + ^i\omega_i \times I_{Gl}^i\omega_i$$

(4.38)

In order to sum the forces and moments, the force state at system $u_{i+1}$ of link $i+1$ must be replaced by an equivalent force state at the point of contact with link $i$. The links in Fig. 4.12 are in contact at A and B. Given a force state at $u_{i+1}$, an equivalent state at B on the
Fig. 4.12: Force Distribution Between Link $i$ And Link $i+1$

same link $i+1$ can be found from:

$$1_f_B = 1_f_{u_{i+1}}$$  \hspace{1cm} (4.39)

and

$$1_m_B = 1_m_{u_{i+1}} - 1_r_{B/u_{i+1}} \times 1_f_{u_{i+1}}$$  \hspace{1cm} (4.40)
From which, the equivalent force system at A on link i can be obtained as:

\[ 1f_A = 1f_{u,i} \]  
\[ (4.41) \]

and

\[ 1m_A = 1m_{u,i} + 1r_{B/u,i} \times 1f_{u,i} \]  
\[ (4.42) \]

Here,

\[ 1f_{ext} = \text{ External applied forces resolved in system 1 (ground link)} \]
\[ m = \text{ mass of the body} \]
\[ 1m_{ext} = \text{ externally applied moments resolved in system 1} \]
\[ 1r_i = \text{ position vector, resolved in system 1, from center of mass of link i to external force} \]
\[ I_{gi} = \text{ diagonal matrix of principal moments of inertia for link i} \]
\[ 1\omega_i = \text{ angular velocity of link i relative to ground} \]
\[ 1\alpha_i = \text{ angular acceleration of link i relative to ground} \]
\[ 1f_v = \text{ force at v resolved in ground coordinate system (system 1)} \]
\[ 1m_v = \text{ moment at v resolved in system 1} \]

Summing forces and moments about \( G_i \) (the center of mass) on link i yields:

\[ 1f_{u,i} - 1f_{u,i} + \sum 1f_{ext} = m_i 1A_{G_i} \]  
\[ (4.43) \]

and

\[ 1m_{u,i} - 1m_{u,i} + 1r_{B/u,i} \times 1f_{u,i} + 1r_{u/G_i} \times 1f_{u,i} + 1r_{A/G_i} \times (-1f_{u,i}) \]
\[ + \sum 1m_{ext} + \sum 1r_{ext} \times 1f_{ext} = I_{gi} 1\alpha_i + 1\omega_i \times I_{gi} 1\omega_i \]  
\[ (4.44) \]

Now, the terms \( 1r_{B/u,i} \times 1f_{u,i} \) and \( 1r_{A/G_i} \times (-1f_{u,i}) \) can be combined as follows:

\[ 1r_{B/u,i} \times 1f_{u,i} + 1r_{A/G_i} \times (-1f_{u,i}) = -(1r_{A/G_i} + 1r_{B/O_i}) \times 1f_{u,i} = -1r_{u/Gi} \times 1f_{u,i} \]

Hence, the final form of the force system about \( G_i \) on link i, is given by:

\[ 1f_{u,i} - 1f_{u,i} + \sum 1f_{ext} = m_i 1A_{G_i} \]  
\[ (4.45) \]
The unknown joint forces and moments are easily dealt with when they are resolved in the $u_i$ coordinate system. The forces and moments may be resolved using:

$$f_{u_i} = C_{1,u_i}^u f_{u_i}$$

(4.47)

$$f_{u_{i+1}} = C_{1,u_{i+1}}^{u_{i+1}} f_{u_{i+1}}$$

(4.48)

$$m_{u_i} = C_{1,u_i}^{u_{i+1}} m_{u_i}$$

(4.49)

$$m_{u_{i+1}} = C_{1,u_{i+1}}^{u_{i+1}} m_{u_{i+1}}$$

(4.50)

where the $C$'s are the change of basis transformations, available from the rotation partitions of transformation matrices developed during displacement analysis. $C_{1,u_{i+1}}$ is the change of basis transformation from system $i+1$ to the ground (link 1).

The system of force equations then become:

$$C_{1,u_i}^u f_{u_i} - C_{1,u_{i+1}}^{u_{i+1}} f_{u_{i+1}} + \sum f_{\text{ext}} = m_i^1 A_{G_i}$$

(4.51)

and

$$C_{1,u_i}^u m_{u_i} - C_{1,u_{i+1}}^{u_{i+1}} m_{u_{i+1}} + C_{1,u_i}^{u_{i+1}} (u_i r_{u_i/G_i} \times u_i f_{u_i})$$

$$- C_{1,u_{i+1}}^{u_{i+1}} (u_{i+1} r_{u_{i+1}/G_i} \times u_{i+1} f_{u_{i+1}}) + \sum m_{\text{ext}} + \sum r_{\text{ext}} \times f_{\text{ext}}$$

$$= I_{G_i}^1 \alpha_i + I_{G_i}^1 \omega_i$$

(4.52)

The unknowns involved in the Eqs. (4.51) and (4.52) are the joint forces and moments. The angular velocity and acceleration of the link are known from the kinematic analysis. The change of basis transformations are rotation transformations available from kinematic analysis. The external forces and moments and the position vectors for the joint locations from center of mass and the link inertia and mass are known properties. Putting all the known terms of the above equations to the right hand side yields the following equations:

$$C_{1,u_i}^u f_{u_i} - C_{1,u_{i+1}}^{u_{i+1}} f_{u_{i+1}} = m_i^1 A_{G_i} - \sum f_{\text{ext}}$$

(4.53)
The Eqs. (4.53) and (4.54) are a linear system of equations for the frictionless case. To solve this system of equations, however, additional joint force constraint equations are required along with the known values and kinematic solutions. Given a joint described by \( n_f \) degrees of freedom, it is possible to write \( n_f \) joint bearing force equations related to those degrees of freedom. A typical ideal model for the revolute joint has one degree of freedom in the form of a rotation about the revolute's z-axis. In the frictionless joint case, the revolute can transfer no torque about its rotation axis. Hence, for this case, if \( \mathbf{m}_{revolute}^z \) is the moment transferred about the revolute joint about its z axis, then the constraint equation may be written as:

\[
\mathbf{m}_{\text{revolute}}^z = 0
\]

The joint bearing forces/moments used in the dynamic force equilibrium equations are always resolved in the link's u coordinate system. Joint force constraint equations may be obtained for all types of joints. Appendix D contains a table of the joint force constraint equations for the frictionless case.

4.8 Numerical Solution For Kinematic And Force Analyses

Due to the complexity of the problems, the above systems of equations for kinematic and force analyses must be solved numerically. Uicker et al (1964) developed a unique matrix oriented displacement analysis procedure for spatial mechanisms. Denavit et al (1965) extended the procedure to analysis of velocities, accelerations, and static forces in single dof spatial mechanisms having revolute and prismatic pairs. Method of virtual work was combined with the loop equations to obtain the static forces. Hall, Root, and Sandgren (1977) presented a generalized method for solving the position equations in three dimensional mechanisms. It was based on optimization methods used to close the loop equations iteratively to obtain starting values for Newton's method. Once relatively close
starting values were provided, Newton's method can obtain the solutions for normal positions quickly.

Gutkowski (1990) presented a procedure for solving the displacement, velocity, acceleration, and dynamic force analysis problems for general spatial mechanisms. For the displacement analysis, a two stage solution process was used. An optimization procedure based on the Davidon Fletcher Powell (DFP) method (Arora, 1989) was used to obtain the solution for the unknowns in the first position. The subsequent positions were solved using Newton's method using the solutions from the preceding position as the starting values for the approximation. The procedure has proved to be quite robust. The force analysis procedure solves for both the linear frictionless situation and the case of Coulomb friction in the joints. The Singular Value Decomposition (SVD) method was applied in the position analysis of mechanisms. For overconstrained systems, the SVD procedure yields a consistent least square solution which is the exact solution. For an underconstrained system of equations, SVD provides the infinite number of solutions possible by obtaining a particular solution and the system's nullspace basis vectors. SVD method was also used in the solution of the dynamic force analysis problem. This eliminated the possibility of a numerical problem in the solutions emanating from under or overconstrained systems.

4.9 Example Mechanism Analysis

A spatial slider crank (RSSP) mechanism has been analyzed by Sandor and Erdman (1984). Fig. 4.13 shows a layout drawing for the mechanism. The link 3 has two spherical joints at the two joints 2 and 3. The analysis of this mechanism using the procedure is presented here.

Based on the geometry and axes definitions, the link shape matrices may be defined indirectly by specifying two points on the "z-axis" and one on the "x-axis" for both the uvw and the xyz coordinate systems based on the link definition coordinate system. The link definition coordinates for each link are marked with the subscripts L_i for link i. Fig. 4.14 also shows the center of gravity (c.g.) locations for each link along with the c.g. coordinate systems for each link. The rotation angle of the crank link about the z1 axis measured from x1 gives the input link angular position and the location of joint 3 on the slider link 4 defines the output position. The indirect specifications of the link shape matrices are given in Table 4.1.
Based on the geometry definitions and the coordinate system definitions, the shape matrices for the links are obtained for all the links in this example. As discussed earlier and in Appendix B, there are many pair matrices applicable to modeling spherical joints. The particular implementation used here is the "x-Rotation/z-Rotation/x-Rotation" formulation.

The shape matrices obtained in this case are given by:

\[
S_1 = \begin{bmatrix}
0 & -1 & 0 & 4 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Fig. 4.14: Axes Definitions For The Spatial Slider Crank Mechanism

\[
S_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Table 4.1: Definition of uvw and xyz coordinate systems based on link definition coordinate system for RSSP linkage

<table>
<thead>
<tr>
<th>Link</th>
<th>System</th>
<th>Point 1</th>
<th>Point 2</th>
<th>Point 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(u,v,w)</td>
<td>(0,0,0)</td>
<td>(0,0,1)</td>
<td>(1,0,0)</td>
</tr>
<tr>
<td></td>
<td>(x,y,z)</td>
<td>(4,0,2)</td>
<td>(4,0,3)</td>
<td>(4,1,2)</td>
</tr>
<tr>
<td>2</td>
<td>(u,v,w)</td>
<td>(0,1.5,0)</td>
<td>(1,1.5,0)</td>
<td>(0,1.5,1)</td>
</tr>
<tr>
<td></td>
<td>(x,y,z)</td>
<td>(0,-1.5,0)</td>
<td>(0,-1.5,1)</td>
<td>(0,-2.5,0)</td>
</tr>
<tr>
<td>3</td>
<td>(u,v,w)</td>
<td>(-5,0,0)</td>
<td>(-5,0,1)</td>
<td>(-6,0,0)</td>
</tr>
<tr>
<td></td>
<td>(x,y,z)</td>
<td>(5,0,0)</td>
<td>(5,0,1)</td>
<td>(6,0,0)</td>
</tr>
<tr>
<td>4</td>
<td>(u,v,w)</td>
<td>(1,0,0)</td>
<td>(1,1,0)</td>
<td>(2,0,0)</td>
</tr>
<tr>
<td></td>
<td>(x,y,z)</td>
<td>(0,0,0)</td>
<td>(0,0,1)</td>
<td>(1,0,0)</td>
</tr>
</tbody>
</table>

The kinematic parameters used in the analysis are:

\[ S_3 = \begin{bmatrix} -1 & 0 & 0 & -10 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ S_4 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

The kinematic parameters used in the analysis are:

\[ ^1\omega_2 = 6 \text{ rad/sec} \quad ^1\alpha_2 = 0 \]

There is an external force of 40 lbs along the z axis of link 4.

The existence of an idle degree of freedom about the axis of link 3 has to be handled in this case. The idle degree of freedom indicates a mathematically underconstrained system of equations that describe the mechanism. No special suppression of the idle degree of freedom is required to handle this situation. Since this is an underconstrained mechanism, the mathematical concept of nullspace solution has to be followed. Using the technique of
Singular Value Decomposition, the nullspace basis vectors are automatically determined during the solution process if any exist. The RSSP mechanism is underconstrained by one equation, hence, one nullspace basis vector exist. For linear problems, the nullspace basis vectors may be added in any linear combination to the "particular" solution to yield another valid solution.

During the solution process, the unknown joint variables are placed in an "unknown" vector. For the RSSP mechanism, this is given by:

\[
Q = \begin{bmatrix}
\alpha_2 \\
\beta_2 \\
\gamma_2 \\
\alpha_3 \\
\beta_3 \\
\gamma_3 \\
\beta_4 \\
\end{bmatrix} = \begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4 \\
q_5 \\
q_6 \\
q_7 \\
\end{bmatrix}
\]

Hence, it is evident that the system will be underconstrained. There are seven unknowns for which there are only six (at most) equations to solve. The SVD approach for this mechanism yields a nullspace basis vector that is identical for each position analyzed. The nullspace basis vector, \( Q^* \) is always found to be:

\[
Q^* = \begin{bmatrix}
0 & 0 & 0.7071 & 0.7071 & 0 & 0 & 0
\end{bmatrix}^T
\]

It is clear that the axes about which \( \gamma_z \) and \( \alpha_z \) rotate are collinear. Thus this nullspace vector may in fact be multiplied by any constant and added to the "particular" solution obtained. The SVD method solves this system automatically detecting the idle degree of freedom without any special treatment. In a more traditional displacement analysis procedure, the user has to realize the existence of the idle degree of freedom and suppress or set to zero the particular variable involved. But the use of the SVD method takes that overhead away from the user definition making it simple to define the mechanism, but at the same time highlighting the existence of it in the results.

A nullspace vector is also obtained when performing the velocity and acceleration analyses on the RSSP mechanism. This stands to reason as there will be seven unknown time derivatives in both cases. The nullspace vectors obtained are identical to the one produced
by the displacement analysis. Link 3 can spin about its axis as a function of time at any rate without affecting the rest of the mechanism.

Figs. 4.15 - 4.18 show the plots of output slider position, velocity, and acceleration and the input torque required for the ideal spatial slider crank mechanism and the results match very well with the analysis done by Sandor and Erdman (1984).

![Graph of Output Slider Position](image1)

**Fig. 4.15:** Slider Displacement For The Spatial Slider Crank Example

![Graph of Output Slider Velocity](image2)

**Fig. 4.16:** Output Slider Velocity For The Spatial Slider Crank Example
Fig. 4.17: Output Slider Acceleration For The Spatial Slider Crank Example

Fig. 4.18: Input Torque Requirement For Spatial Slider Crank Example

Brief descriptions for the routines and the programs used for this work are given in Appendix G.
5.0 Introduction

One of the objectives of the optimization process as described in this work is to minimize the manufacturing cost when the tolerances of the assembly are held within constraints. This entails the study of the assembly tolerance techniques and the generation of cost-tolerance functions to read cost values from. This chapter briefly gives the overview of the assembly tolerance model and the nature of manufacturing cost and tolerance/surface finish variation.

5.1 General Tolerance Analysis And Synthesis Studies

Tolerance analysis in design is an analytical model for the accumulation of tolerances in a mechanical assembly of components. The worst case analysis and statistical analysis are two widely used tolerance analysis techniques. In the worst case analysis, each component dimension is assumed to be at its maximum or minimum limit, resulting in the worst possible assembly limits. In the statistical analysis method, the effect of component tolerances are estimated as root sum squared. The details of these are given in the next section.

Of the well known tolerance allocation techniques, the allocation by proportional scaling and allocation by constant precision factor are quite commonly used. These are given in
Tolerance allocation by optimization techniques is also a very practical method. The component tolerances are allocated in the design such that the projected cost of production of the assembly is minimized. The optimization process considers the tolerance for each component and determines the combination of individual tolerances that minimizes the production cost. For the optimum tolerance allocation to work, a definition of cost vs. tolerance function is necessary. Several algebraic functions have been proposed and the details of some promising ones are discussed in a later section.

Several investigators have given tolerancing procedures which aim towards minimizing cost for assemblies. Peters (1970) studied the different possibilities of distributing the tolerances between the components of an assembly in order to obtain a minimum cost assembly. Smathers and Ostward (1972) developed a process of optimization of dimensions and tolerances where the objective was to maintain overall assembly tolerance at minimum cost. Bellman’s Principle of Optimality was used to optimize the sum of specified tolerances with respect to cost while maintaining the constraints of design. Discrete variable linear programming techniques were used for minimum-cost tolerance allocation by Ostward and Huang (1977). Michael and Siddal (1981) developed a procedure for optimal assignment of manufacturing tolerances. This was achieved by treating the tolerances also as design variables. The work was based on a worst case analysis and was dependent on the use of a good cost model. General tolerancing procedures which aim toward minimum cost for assemblies are also discussed by Pike and Silverberg (1953), Burr and James (1974), Ermer and Wu (1977) and many others. Chen et al (1984) demonstrated a linear programming based design algorithm for the tolerance design with an example of printer actuator. Some investigators (Iannuzzi and Sandgren, 1994) have used genetic algorithms and Monte-Carlo simulation to handle tolerance allocation problems. These methods are usually plagued with a huge computational burden.

Chase and Greenwood (1987) have given a comprehensive discussion of the available tolerance design tools and their limitations. A simple new model has also been presented that can give a more realistic representation of the manufactured parts. The model proposed is called the Estimated Mean Shift model. For each component, a midpoint tolerance zone was defined and a mean shift factor was associated with each process. This is a combination of the two tolerance allocation methods with the mean shift model.
Cagan and Kurfess (1992) presented a procedure for optimally allocating tolerances over multiple manufacturing alternatives. The manufacturing cost of the parts was the objective that was minimized. The stochastic optimization of simulated annealing was used for the development. This work emanated from the idea that merely allocating tolerances are not enough but the machining processes are also to be assigned to minimize the cost of a part. This is one of the very few studies to realize that. However, the procedure was applied to very simple cases of tolerance stack-up. The idea of reducing the overall variation along with cost was not looked into.

5.2 Assembly Tolerance Models

Tolerance analysis is necessary in the design of assemblies where the tolerances of individual parts interact. Typically, one or more critical dimensions of the assembly must be maintained to assure its proper function. If the manufacturing process for a part is known, reasonably attainable tolerances may be selected by following tolerance guidelines for the process. Useful data may also be obtained from company or industry standards or plant process capability charts. The tolerance of the assembly may then be predicted from tolerance analysis models. The tolerance analysis models are analytical tools for the estimation of tolerance accumulation in mechanical assemblies of component parts. The common models used in engineering design are described briefly here. More details are given in Chase and Greenwood (1988), Fortini (1967), and Krishnaswami and Mayne (1994).

5.2.1 Worst Limit Model

In the worst limit model, the assembly tolerance is determined by summing the component tolerances linearly. Each component dimension is assumed to be at its maximum or minimum limit resulting in the worst possible assembly limits.

For one dimensional assemblies, this is given by:

\[ T_{\text{assy}} = \sum T_i \]  

(5.1)

For multi-dimensional assemblies such as a mechanism chain, this is given by:
\[ T_{\text{assy}} = \sum \left( \left| \frac{\partial f}{\partial x_i} \right| T_i \right) \]  

(5.2)

where \( T_{\text{assy}} \) is the resulting tolerance of the assembly

\( x_i \) are the individual component dimensions

\( T_i \) are the tolerances on the individual design variables

\( f(x_i) \) is the assembly function describing the effects of the individual tolerances and \( \frac{\partial f}{\partial x_i} \) represent the sensitivity of the assembly tolerance to the variation of individual component dimensions.

This model is valid only if the linearizations implied by the summation of the sensitivities is sufficiently accurate for the range of tolerances and geometric dependence considered. Balling et al (1986) demonstrated this method for tolerance allocation and tolerance synthesis problems. Emch and Parkinson (1994) developed a procedure for robust optimal design based on worst case tolerance model.

### 5.2.2 Statistical Tolerance Model

In this model, the component tolerances are combined as Root Sum Squared (RSS). The low probability of the occurrence of worst case combination is accounted for by assuming a Gaussian or Normal distribution of individual component variations. For the variation of individual components within six standard deviations (± 3\( \sigma \)), the tolerance in the assembly can be obtained (Evans, 1974, 1975) as follows:

For one dimensional assemblies, this is given by:

\[ T_{\text{assy}} = \left[ \sum T_i^2 \right]^{1/2} \]  

(5.3)

For multi-dimensional assemblies, this is given by:

\[ T_{\text{assy}} = \left[ \sum \left( \frac{\partial f}{\partial x_i} \right)^2 T_i^2 \right]^{1/2} \]  

(5.4)

This model is valid only if the conditions of linearity are valid and the variables are independent in the true sense. The details of the derivation are given in Appendix F.
For multi-dimensional assemblies, for a generalized case (other than ±3σ case), this is given by:

\[ T_{asy} = Z \left[ \sum \left( \frac{\partial f}{\partial x_i} \right)^2 \left( \frac{T_i}{Z_i} \right)^2 \right]^{1/2} \]  

(5.5)

where \( Z \) is the number of standard deviations desired for the assembly tolerance and \( Z_i \) describes the expected number of standard deviations for each component tolerance.

### 5.2.3 Limitations Of The Worst Case And Statistical Tolerance Models

Both the worst limit model and the statistical model for tolerance accumulation have distinct limitations in tolerance allocation cases. Some of these are listed here:

a) Since the worst limit model assumes the worst possible combination of individual component tolerances, it results in component tolerances which are tight and costly to manufacture.

b) Statistical models result in looser tolerances, but often predict higher assembly yields than is actually achievable in production.

c) For some extreme cases, the statistical tolerance model results are worse than the worst limit model. This is true when the number of components is low (less than three or four), or when one component tolerance is much greater than the rest of the tolerances. Examples of this are given in Greenwood and Chase (1987).

d) Most manufactured parts have a bias or skewness in their variation. The statistical model assumes that the manufacturing variations follow a normal distribution symmetrically positioned at the midpoint of the tolerance limits and fail to account for the bias. Fig. 5.1 shows the unexpected rejects resulting from a simple one dimensional statistical tolerance allocation case.

The effect of the bias is to shift the nominal dimension. In an assembly, it can accumulate and result in unexpectedly high rejection rates. Bias results from a variety of parameters.
such as, tooling, fixture or setup errors or tool wear. Sometimes bias is deliberately introduced to compensate for tool wear or to reduce the cost of rework.

Fig. 5.1: Unexpected Rejects In RSS Model Because Of Bias (Chase & Greenwood, 1988)

5.2.4 Estimated Mean Shift Model

In the estimated mean shift model of tolerance accumulation, the designer estimates the bias for individual component in the assembly. A zone about the midpoint of the tolerance range is defined for the probable location of the mean of a typical batch of parts (Fig. 5.2). The midpoint tolerance zone is expressed as fraction (a number between 0 and 1) of the specified tolerance range for the part dimension. If the process used to produce the part is closely controlled, then a low value for a mean shift factor (signifying the bias) is assumed typically in the range of 0.1 to 0.2. For less confidence in process parameters, a large factor (0.7 - 0.8) is assumed. For common processes, the factor can be selected from the
interpretations of available process control history data. The assembly tolerance can be calculated once the individual mean shift factors are estimated.

For one dimensional assemblies, this is given by:

\[
T_{\text{asy}} = \sum m_i T_i + \left[ \sum (1-m_i)^2 T_i^2 \right]^{1/2}
\]  

(5.6)

For multi-dimensional assemblies this is given by:

\[
T_{\text{asy}} = \sum m_i \frac{\partial f}{\partial x_i} T_i + \left[ \sum (1-m_i)^2 \left( \frac{\partial f}{\partial x_i} \right)^2 T_i^2 \right]^{1/2}
\]  

(5.7)

where \( m_i \) is the mean shift factor for the \( i \)th component.

Fig. 5.2: Mean Shift Factor (Chase & Greenwood, 1988)

The first summation in the model is the worst limit summation and the second is the statistical sum of the component tolerances and mean shifts along with the mean shifts. Hence this model is a combination of both the models. When all the mean shifts are zero, the resulting assembly tolerance reduces to the statistical model; and the assembly tolerance reduces to the worst limit model when all the mean shift factors are 1. The estimated mean shift model thus covers the entire range between the two extremes.
The estimated mean shift model provides interaction between the design engineers and the manufacturing engineers and has advantages over the other models. The mean shift factors may be mixed in the assembly. In the assembly, some parts may be nearly worst case and some parts may be nearly statistical. There is no need to penalize the entire assembly with a worst limit analysis because of one poorly controlled component.

5.3 Factors Contributing To Manufacturing Cost Increase

To some extent, the price of a product or component is related to the manufacturing cost. The higher the manufacturing cost, the higher the component price. If market pressures and some other non-engineering factors are ignored, the price of a product may be described as a function of the manufacturing cost. This implies that if the manufacturing cost is affected by the tolerances, so is the price of the product. A variety of factors contribute to the increase in production cost of a particular part or assembly. An effort will not be made to discuss all of these factors, but a representative few of the machining processes and costs will be briefly discussed. The main idea behind the discussions in this section is to highlight the nature of variations of manufacturing cost with tolerance and surface finish and essentially get a feel for the cost-tolerance functions developed in the next section.

5.3.1 Dimensional Tolerances And Surface Roughnesses

Fig. 5.3 gives a general relationship of actual dimensional tolerance to surface roughness for machining processes. The dimensional tolerances and surface roughnesses on a part should not be specified to limits beyond the requirement of design. This will ensure lowest possible cost and fastest possible production. From Fig. 5.3, it can be inferred that it would not be possible to hold a tolerance of 0.0001 in. on a part which is to be machined to an average roughness of 125 \( \mu \text{in. rms} \). Fig. 5.3 also shows the relative cost increase as tolerances and surface roughnesses become finer.

Fig. 5.4 shows the tolerance range of general machining processes. It is clear from the figure that various processes overlap in the ranges and sizes that they can handle. Each of the processes have their limitations and driving them towards the limits result in a very high
increase in cost. For example, it is not practical to build a 16 in. diameter drill to machine a
hole to high precision. It would need a boring tool to get close tolerances. Fig. 5.4 also
shows the tolerances ranges that can be held by the processes. Trucks (1987) has given
some examples to show that the cost increase due to fabrication requirements is related to
the tolerances and surface roughness on similar parts.

![Graph showing relationship between dimensional tolerance and surface finish]

**Fig. 5.3: Relationship Between Dimensional Tolerance And Surface Finish**
*(Trucks, 1987)*

<table>
<thead>
<tr>
<th>RANGE OF SIZES FROM</th>
<th>THROUGH</th>
<th>TOLERANCES ±</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.599</td>
<td>0.00016 0.0002 0.0003 0.0005 0.0008 0.0012 0.002 0.003 0.005</td>
</tr>
<tr>
<td>0.800</td>
<td>0.999</td>
<td>0.00016 0.00025 0.0004 0.0006 0.0008 0.001 0.0016 0.0025 0.004 0.006</td>
</tr>
<tr>
<td>1.000</td>
<td>1.489</td>
<td>0.0002 0.0003 0.0006 0.0008 0.001 0.0016 0.0025 0.004 0.006 0.01</td>
</tr>
<tr>
<td>1.500</td>
<td>2.799</td>
<td>0.00025 0.0004 0.0008 0.001 0.0016 0.0025 0.004 0.006 0.01 0.015</td>
</tr>
<tr>
<td>2.500</td>
<td>4.499</td>
<td>0.0003 0.0006 0.0008 0.0012 0.002 0.003 0.005 0.008 0.012 0.01</td>
</tr>
<tr>
<td>4.500</td>
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</tr>
<tr>
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<td>13.899</td>
<td>0.0005 0.0008 0.0012 0.002 0.003 0.006 0.008 0.012 0.02 0.02</td>
</tr>
<tr>
<td>13.500</td>
<td>20.999</td>
<td>0.0006 0.001 0.0015 0.0025 0.004 0.006 0.01 0.015 0.025</td>
</tr>
</tbody>
</table>

**Fig. 5.4: Tolerance Ranges Of Machining Processes** *(Trucks, 1987)*
5.3.2 Tolerances And Costs For Drilling Operations

Even though drilling is the simplest machining operation, the cost can be high if the tolerances and depth of holes are not designed properly. Charts are available for standard drill hole tolerances and should be used wherever possible to keep the manufacturing cost the minimum. For closer tolerances beyond the limits of drill tolerances, additional operations like boring, grinding or reaming are necessary. Each of these additional operations contribute to higher costs.

The depth of the hole also contributes very much in drilling operation cost. Fig. 5.5 shows the effect of hole depth on drilling cost. The curve is based on standard equipment and shows that the cost of drilling a hole increases proportionally to its depth for some depth. Beyond a depth of three times the diameter, the cost increases at a much higher rate.

![Graph showing the effect of drill hole depth on machining cost](image)

**Fig. 5.5: Effect Of Drill Hole Depth To Machining Cost (Trucks, 1987)**

Another factor that increases the cost of drilled holes is the location tolerance of holes. The method always needs to be very carefully analyzed for drilling operations. Fig. 5.6 shows the dramatic effect the true position tolerances have on drilling cost. As the true position tolerances become tighter, the cost of production increases and often entails the requirement of special equipment to even hold the tolerances between the hole positions.
5.3.3 Tolerances And Costs For Turning Operations

Turning operations constitute one of the primary machining operations employed in the manufacturing of parts. With today's modern CNC lathes, tolerances below ±0.001 in. can be maintained easily. Surface finishes can be held to 10 μ in. rms, but 32 μ in. rms is a more economical range. Concentricity of circular features can be held to 0.0005 in. in these machines. Fig. 5.7 shows the effect of surface roughness requirement on cost increase of turning operations. For very fine surface roughness requirements, the cost increase is very high.

5.3.4 Tolerances And Costs For Milling Operations

After turning and drilling, milling is the most widely used machining process. It can be adapted very well to economical production of parts and has almost unlimited versatility as a production process. There are economical tolerance ranges for the milling process too. Again for parts that have very fine tolerances, additional operations might be necessary after the milling operation. Tolerances in the range of 0.001 in. can be met by conventional milling machines. The tolerance range of CNC mills are finer than these, but in many cases the overall cost of producing a part is higher than conventional milling.
Fig. 5.7: Effect Of Tolerance And Surface Finish On Turning Cost (Trucks, 1987)

Fig. 5.8: Effect Of Tolerance And Surface Finish On Face Milling Cost (Trucks, 1987)
Fig. 5.8 illustrates the effect of dimensional tolerances and surface finish on face milling costs. As expected, finer tolerances and surface finishes mean a dramatic increase in machining costs. With a tolerance under ±0.002 in., there is an approximate 40% difference in cost between machining to 63 μ in. and 125 μ in surface roughness.

5.3.5 Tolerances And Costs For Grinding Operations

Often additional operations are required beyond milling, drilling, or turning operations to obtain the designer's specified tolerances and surface finishes. Production grinding is one of the most widely used methods of finishing parts to extremely close tolerances and fine surface finishes. There are a variety of grinders available for different types of grinding operations. Classes of grinding machines include: cylindrical grinders, centerless grinders, internal grinders, surface grinders, rotary surface grinders. Particular design features of a part dictate to a large degree the type of grinding machine required for the job. Many types of forgings and other parts are finished completely with the grinding wheel at appreciable savings of time and machining cost. In this case, the grinding operation is used not only as a finishing operation but also as a machining operation. The grinding operations can hold tolerances in the range of ±0.0001 in. in general. As expected, the cost increases as finer tolerances and finishes are required from the grinding processes.

Fig. 5.9 shows the effect of tolerances and surface finishes on cylindrical grinding cost. This is a nonuniform cost increase pattern. For 40 μ in. surface roughness, there is a minor cost rise between 0.001 in. and 0.0005 in. tolerance range, followed by an abrupt rise in cost for tolerances between 0.0005 in. and 0.00025 in. The cost seems to level off for tolerances below 0.0025 in. A different cost increase pattern is obtained for 25 μ in. surface roughness case.

Fig. 5.10 shows the effect of surface roughness and tolerance on reciprocating grinding cost. In this case, the dimensional tolerance range has greater effect on cost than does the surface quality range. Below 0.001 in. tolerance range the cost rises sharply for all three different surface finish requirements.
Fig. 5.9: Effect Of Tolerance And Surface Finish On Cylindrical Grinding Cost (Trucks, 1987)

Fig. 5.10: Effect Of Tolerance And Surface Finish On Reciprocating Grinding Cost (Trucks, 1987)
Figs. 5.11 and 5.12 similarly show the effect of tolerance and surface finish on rotary grinding and centerless grinding costs. In all the grinding processes, about half of the cost increase occurs between 0.001 in. and 0.0005 in. For rotary surface grinding however, the cost increase is not as dramatic as with the rest of the grinding processes.

![Graph showing the effect of tolerance and surface finish on rotary grinding cost.](image)

**Fig. 5.11: Effect Of Tolerance And Surface Finish On Rotary Grinding Cost**

![Graph showing the effect of tolerance and surface finish on centerless grinding cost.](image)

**Fig. 5.12: Effect Of Tolerance And Surface Finish On Centerless Grinding Cost**
5.4 Cost vs. Tolerance Functions

One of the key factors in optimum tolerance allocation for minimizing the manufacturing cost is the specification of cost vs. tolerance functions for manufacturing processes. In many cases, this data may be available in the form of manufacturing cost tolerance charts for a company or a manufacturing unit for a particular material. The cost of machining is affected by the surface finish requirements. As discussed in the previous section, the machining cost increases dramatically for all the processes when finer tolerances or surface finishes are required. The nature of the cost vs. tolerance curves are similar for all processes. A typical chart for the variation of relative cost in machining steel with type of operation and required finish is shown in Fig. 5.13. The increase in cost is associated with the need for using additional finishing operations such as grinding etc. It is to be noted that the nature of the variation in Fig. 5.13 is the same as the one presented in the last section. It is, however, a mirror image of those curves because of the x-axis values being in reverse order.

The overall machining costs are highly dependent on the dimensional tolerances requirements. In most cases, close tolerance control also dictates finer surface finishes. The relationships of dimensional tolerance, surface finish, and their overall effect on cost of machining were discussed in the previous section.

As discussed earlier, often cost tolerance charts may be available at a particular manufacturing site for a particular type of parts. Machining cost may also be estimated by using formulas incorporating all the factors from tool cost to machining time to tool replacement costs etc. and overheads. Some of the typical formulas are given in Appendix E based on recommendations of Machinability Data Center (1980). In the case, where no such data is available an algebraic function may be used to estimate the manufacturing cost for obtaining a given tolerance. These functions essentially try to simulate the effects of tolerance and surface finish on manufacturing cost as discussed in the previous section for a generic situation. Several of these functions have been proposed. Three of these are worth mentioning. In each of the models, there is a constant coefficient $A_i$ that may include setup, tooling, material etc. cost and $B_i$ determines the cost of producing a single component dimension to a specified tolerance $T_i$. 

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Fig. 5.13: Relative Machining Costs And Surface Finish For Steel

Reciprocal Squared Function was proposed by Spotts (1973):

\[ C_i = A_i + \frac{B_i}{T_i^2} \]  \hfill (5.8)

Reciprocal Function was proposed by Chase and Greenwood (1986):

\[ C_i = A_i + \frac{B_i}{T_i} \]  \hfill (5.9)
**Exponential Function** was proposed by Speckhart (1972):

\[ C_i = A_i e^{-B_i T_i} \]  

(5.10)

Trucks (1987) and Jamieson (1982) have given the machining costs for a variety of actual parts for several metal-removal processes. Process/cost results were correlated to the tolerance vs. size charts available and the data were curve fit by regression analysis for each cost model. Fig. 5.14 shows a typical result for a particular material and process combinations. The reciprocal tolerance curve appears to fit the machining process cost data the best.

![Graph showing cost vs. tolerance models](image)

**Fig. 5.14: Comparison Of Cost vs. Tolerance Models (Adapted From Chase & Greenwood, 1988)**

Many of these functions might be obtained for a variety of material at different machining conditions. Normally, the machining time also varies with the material, and this causes the machining cost to go up accordingly. Fig. 5.15 shows the relative cost of machining for different metals. For Fig 5.15, the relative costs are based on a part for which the
machining costs $10 for 7075-T6 Aluminum bar stock. The costs are approximate and do not reflect the costs of material itself, cutting tool, or heat treatment etc. But the cost tolerance functions are essentially of the same nature and the only differences are the actual values of the factors $A_i$ and $B_i$ for the cost estimation.

![Machining Cost Comparison](image)

**Fig. 5.15: Machining Cost Comparison For Different Material**

5.5 Assembly Tolerance Models And Cost-Tolerance Functions Used In This Research

As discussed earlier, the estimated mean shift method seems to be a good model to use for mass-produced multi-dimensional assemblies. This is a combination of the worst case model and the statistical model.

The expression for the tolerance on an output variable at the $j$th position of the mechanism is given by:
where $S_j(x_j)$ is the sensitivity of the output with respect to the variation in variable $x_j$, $x_j$ is the current value of tolerance/clearance variable $i$, $m_i$ is the estimated mean shift factor for variable $x_j$, and $(T_{asy})_j$ is the tolerance of the assembly at the $j$th position. For most of the cases, a value of 0.1 to 0.2 has been used for $m_i$ by default, unless the user defines a different value. A value of 0.1 to 0.2 is what seems to be the consensus for $m_i$ in the literature.

The computer program developed as a part of this research can read user-defined cost vs. tolerance lookup tables for estimating manufacturing cost or can even calculate using Eqs. from Appendix E. But in the absence of any such data readily available from the user, generic cost-tolerance functions may be used. The reciprocal function as defined in the previous section has been used in most cases. The reciprocal function is given by:

$$\text{Cost}_i = A_i + \frac{B_i}{\text{Tol}_i}$$

A typical function used has values of:

$A_i = 0.75$

and $B_i = 0.0025$

and Tol$_i$ is in inches

These are just representative values and have been interpolated from the function shown in Fig. 5.14. The use of these functions and assembly tolerance models will be explained with examples in Chapter 9.
CHAPTER 6

DESIGN OPTIMIZATION STRATEGIES

6.0 Introduction

The main thrust of the research is the optimal allocation of tolerances and clearances in
kinematic assemblies to obtain minimum manufacturing cost and minimum sensitivity of
variation combination when the assembly tolerance is within user defined constraint
limits. This requires the ability to handle multiple nonlinear objectives in the presence of
design constraints, which is the crux of the generalized multicriteria optimization process.
This chapter studies the design optimization methods and their use and relevance in the
present work. Details of constrained nonlinear optimization methods and multicriteria
optimization are discussed briefly in this chapter. The specific multicriteria optimization
methods used here are the weighting objective method and the norm method or a
combination of both. For constrained optimization performed inside the multicriteria
optimization, the Generalized Reduced Gradient (GRG) method is used. Also an
unconstrained optimization method (Davidon, Fletcher, Powell or DFP method) is used
for the numerical solution of the kinematic position problem (only as an aid to solve the
first position to obtain good starting values for the Newton's method). A very brief
description for these methods is also given in this chapter. The end of the chapter
introduces the optimization formulation for this work along with the sensitivity
evaluation and other factors that need to be considered for formulating the optimal
tolerance and clearance allocation problem.
6.1 Design Optimization Problem

Engineering design is often a trial-and-error process in arriving at the design parameters. Optimization techniques can be used to automate the trial-and-error aspects of the engineering design process. The optimization techniques essentially set up the problem in the form of minimizing one or more objectives in the presence or absence of design constraints. The algorithms for solving the problems can be formulated as:

\[
\text{Find } x = x^* \quad (6.1)
\]

such that

\[
f(x) \text{ is a minimum}
\]

subject to

\[
g_i(x) \leq 0 \quad i = 1, 2, \ldots, m \quad (6.2)
\]

\[
h_i(x) = 0; \quad i = 1, \ldots, p \quad (6.3)
\]

\[
x_l \leq x \leq x_u \quad (6.4)
\]

where

\[
x^* \in \mathbb{R}^n \quad : \text{optimal values of design variables}
\]

\[
f(x) \quad : \text{given objective function}
\]

\[
g_i(x) \leq 0 \quad i = 1, 2, \ldots, m \quad : \text{given constraint functions}
\]

\[
x_l \in \mathbb{R}^n, \quad x_u \in \mathbb{R}^n \quad : \text{given lower and upper limits of design variables}
\]

Most engineering problems are nonlinear in nature. This results in some or all of the objective and/or constraint functions being nonlinear. These problems are termed as nonlinear programming (NLP) problems and iterative numerical methods are used for solving them. Even though analytical solutions to engineering optimization problems are preferred, most of the times, the analytical methods are insufficient or inefficient in solving the NLP problems. There could be a variety of reasons for this. The number of
design variables and constraints can be large and the nonlinear nature of them often prohibits solution in closed-form. Sometimes the dimension of the problem is small, but the necessary conditions for analytical methods could be highly nonlinear and intractable. In many cases, the cost or constraint functions are not available in explicit form for use in the numerical methods.

6.2 Unconstrained Design Optimization

The unconstrained optimum design problems are the simplest of the nonlinear programming problems. There are a host of numerical methods available to get the optimum set for these problems (Arora, 1989). Research in this area is still continuing to develop more efficient and robust techniques. The unconstrained optimization problem can be stated as:

Find \( \mathbf{x}^* = (x_1^*, ..., x_n^*) \), a design variable vector of dimension \( n \), to minimize a cost function given by:

\[
\mathbf{f} = f(\mathbf{x}) \quad (6.5)
\]

In the direct numerical search methods to solve unconstrained optimization problems, an initial design is selected for the iterative process. A move is made in the design space to another "point" such that the objective function value is lower. The iterative process is continued until no further moves are possible and the optimality conditions are satisfied. The iteration step at each stage can be represented as:

\[
\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \Delta \mathbf{x}^{(k)}; \quad k = 0, 1, 2, \ldots \quad (6.6)
\]

The superscript \( k \) represents the iteration number, \( \mathbf{x}^{(k)} \) is the current design estimate, and \( \Delta \mathbf{x}^{(k)} \) represents a small change in the current design. The changes in design \( \Delta \mathbf{x}^{(k)} \) is decomposed as:

\[
\Delta \mathbf{x}^{(k)} = \alpha_k \mathbf{d}^{(k)} \quad (6.7)
\]

where \( \alpha_k \) is a positive scalar step size in the desirable search direction \( \mathbf{d}^{(k)} \). Thus, every step of the design improvement involves the solution of the search direction and step size
determination subproblems. Solution of both the subproblems involve values of the cost function. Hence at each step, the new design point is obtained from:

$$x^{(k+1)} = x^{(k)} + \alpha_k d^{(k)}$$  (6.8)

Once the search direction is determined in any problem, the step size determination is a one-dimensional minimization problem. Often one of the three well known methods namely Equal Interval Search, Golden Section Search, or Polynomial Interpolation is used to solve the step size determination.

There are, again, a number of methods for the determination of search direction. The methods differ in the way the search direction is determined and whether just the first order (gradient) or second order (Hessian) derivative information is used at each stage. The Steepest Descent Method, the Conjugate Gradient Method (Fletcher and Reeves, 1964) and Newton's Method are well known methods for solving unconstrained optimization problems. Newton's method uses the Hessian of the objective function to determine the search direction and has a quadratic rate of convergence. However, Newton's method also has its own drawbacks and might not be the best method for some problems.

To overcome some of the problems associated with the Newton's method, the Quasi-Newton methods were developed. These methods use first order derivatives to generate approximations for the Hessian matrix. Also the convergence to the minimum is speeded up by using information from the previous iteration at each step. Some of these methods estimate the Hessian itself and some of them estimate the inverse of the Hessian at each direction search stage. One of the very powerful methods in this category is the DFP (Davidon-Fletcher-Powell) method. In this method only the first derivative is used to build the approximate inverse of the Hessian (Davidon, 1959, Fletcher and Powell, 1963). Another of these methods is the BFGS (Broyden-Fletcher-Goldfarb-Shanno) method which is also very effective (Gill et al, 1981).

As stated before, the DFP method has been used in the optimization procedure in the numerical solution of the kinematic displacement equations in the first position. It has been noted to be quite stable and simple to implement for this situation.
6.3 Constrained Design Optimization

Most of the engineering problems are not only nonlinear, they also have nonlinear constraints. Similar to the unconstrained optimization problems, several methods are available for the general constrained optimization models (Arora, 1989). The general procedure followed by most methods is two phase as in the unconstrained cases: a \textit{search direction determination} phase and a \textit{step length determination} phase.

Thus, the design improvement involves the solution of the search direction and step size determination subproblems. Solution of both the subproblems can involve values of cost and constraint functions as well as their gradients at each design point. The constraints have to be considered at every stage including the search direction and the step size determination for the constrained problem.

All constrained optimization algorithms need a design estimate to initiate the iterative process. The iterative process depends on whether the starting design is feasible or infeasible. If it is inside the feasible region (Fig. 6.1), then we can reduce the cost function by moving along a descent direction. The example for steepest descent direction (-c) is shown in Fig. 6.1. Such iterations are continued until either a constraint boundary is reached or unconstrained minimum point is reached.

![Fig. 6.1: Steps Of Constrained Optimization Algorithms Starting From A Feasible Point (Arora, 1989)](image-url)
Fig. 6.2: Steps Of Constrained Optimization Algorithms Starting From An Infeasible Point (Arora, 1989)

Fig. 6.2 shows the steps of constrained optimization algorithm based on an infeasible start point. At all stages of the constrained optimization process, the constraints are to be checked and correcting steps to be taken.

Several algorithms based on these strategies have been developed and evaluated. Some algorithms are better for a certain class of problems than others. A few algorithms work well if the problem has only inequality constraints whereas others can handle both equality and inequality constraints simultaneously.

Fig. 6.3: Status Of A Constraint At Design Points A, B, C And D (Arora, 1989)
The status of a constraint at a design point is an important factor for constrained optimum design. An inequality constraint can be either active, \( \varepsilon \)-active, violated or inactive at a design point (\( \varepsilon \) is a very small number). On the other hand, an equality constraint is either active or violated at a design point. The development of the numerical methods need precise definitions of the status of a constraint at a design point. The details of the status are shown in Fig. 6.3. Point A is in the feasible region for the inequality constraint \( g_i(x) < 0 \), B is in the \( \varepsilon \)-active region, at C \( g_i(x) \) is active and at D, it is violated.

### 6.4 Solution Of Constrained Optimization Problem

For the unconstrained optimization methods, the objective function is used as the descent function to monitor the progress of the algorithm to the optimum. For constrained problems, the descent function is usually constructed by adding a penalty for constraint violations to the current value of the cost function. The value of the cost function has to be same as the value of the objective function at that point.

One of the common methods of solving constrained optimization problems is by linearizing the problem. Design change is computed by solving an approximate subproblem which is obtained by expanding the cost and constraint function in a linear Taylor series. Linear Programming (LP) methods are then used at each iteration. This process is called *Sequential Linear Programming* method or SLP. The method, however, has some drawbacks, the principal one being the lack of robustness.

Many methods and their variations for constrained optimization have been developed. The *Gradient Projection Method* and the *Generalized Reduced Gradient Method* are two of the well known ones. Rosen (1961) developed the gradient projection method. The method uses the first order information about the problem at the current point. The direction vector in this method can be calculated easily but it is not as good as the feasible direction method. However, this method has sufficient computational advantages over the feasible direction method.

The gradient projection method starts with an initial point. When the start point is inside the feasible region, the steepest descent direction of the objective function is used until a constraint boundary is reached. If the starting point is infeasible, then a series of
constraint correction steps are taken to reach the constraint boundary. If the starting point is on the constraint boundary, a direction tangent to the constraint surface is used. The direction is determined by projecting the steepest descent direction for the cost function on to the tangent plane. A step is taken in the negative projected gradient direction. The new point will be infeasible and needs a series of correction steps to reach the feasible region again (Fig. 6.4).

6.4.1 Generalized Reduced Gradient Method

The Generalized Reduced Gradient Method (GRG) is a very versatile algorithm and works very well for engineering optimization problems. GRG method has been used in this work for handling constrained optimization stages of the multicriteria optimization. Wolfe developed the reduced gradient method in 1967 based on a simple variable elimination technique for equality constrained problems (Abadie, 1970). The Generalized Reduced Gradient (GRG) method is an extension of the reduced gradient method to accommodate nonlinear inequality constraints. In the GRG method, a search direction is found, such that for any small move, the current active constraints remain active. If some active constraints are not satisfied due to nonlinearity of constraint functions, the Newton-Raphson method is used to return to the constraint boundary. In this respect, the GRG method is somewhat similar to the gradient projection method.

Since, inequality constraints can always be converted to equalities by adding slack variables, an equality constrained Non Linear Programming model can be formed. Also, potential constraint energy can be employed to treat all the constraints in the subproblem as equalities. The subproblem for finding the direction in the GRG method can be defined as given by Arora (1989). The design variable vector $x$ is partitioned as: $[y^T, z^T]^T$ where $y_{(a-p)}$ and $z_{(p)}$ are vectors of independent and dependent design variables, respectively. First order changes in the cost and constraint functions (treated as equalities) are given as:

$$\Delta f = \frac{\partial f^T}{\partial y} \Delta y + \frac{\partial f^T}{\partial z} \Delta z \quad (6.9)$$

$$\Delta h_i = \frac{\partial h_i^T}{\partial y} \Delta y + \frac{\partial h_i^T}{\partial z} \Delta z \quad (6.10)$$
Since, the process was started with a feasible design, any change in the variables keeps the current equalities satisfied at least to first order, i.e. \( \Delta h_i = 0 \). Therefore, the above equation can be written in the matrix form as:

\[
A^T \Delta y + B^T \Delta z = 0
\]

or,

\[
\Delta z = -(B^{-T}A^T)\Delta y
\]

where columns of matrices \( A_{(n-p)x_p} \) and \( B_{(p)x_p} \) contain gradients of equality constraints with respect to \( y \) and \( z \) respectively. Eq. (6.12) can be looked at as the one to use to determine the change in the dependent variable \( \Delta z \) when the change in the independent variable \( \Delta y \) is specified. Substituting \( \Delta z \) from Eq. (6.12) into Eq. (6.9), we get:

\[
\Delta f = \left( \frac{\partial f^T}{\partial y} - \frac{\partial f^T}{\partial z} B^{-T}A^T \right) \Delta y
\]

or,

\[
\frac{df}{dy} = \frac{\partial f}{\partial y} - AB^{-1} \frac{\partial f}{\partial z}
\]

This is known as the generalized reduced gradient and can be viewed as the gradient of an unconstrained function.
In line search, the objective function is treated as the descent function. For a trial value of $\alpha$, the design variables are updated using $\Delta y = -\alpha \frac{df}{dy}$ and $\Delta z$ from Eq. (6.12). If the trial design is found to be infeasible, then independent design variables are considered to be fixed and dependent variables are changed iteratively until a feasible design point is obtained. Newton-Raphson method is used in the above iterative processes. If the new feasible design obtained thus satisfies the descent condition, then the line search is terminated; otherwise, the procedure is repeated with a reduced step size. It may be noted that when $\frac{df}{dy} = 0$ in Eq. (6.14), the Kuhn-Tucker conditions of optimality are satisfied for the original Non Linear Programming problem.

The main computational burden associated with the GRG method arises from the use of the Newton-Raphson iterations during line search. The gradients of the constraints need to be recalculated and the Jacobian matrix $B$ needs to be inverted at every Newton-Raphson iteration during line search. This could become very expensive for problems with large number of variables and functions. Many efficient numerical schemes have been suggested to take care of this problem. One such method uses a quasi-Newton formula to update $B^{-1}$ without recomputing gradients but requiring only constraint function values. However, this can cause problems if the set of independent variables changes every iteration. Also, if the starting points are not "good", then special algorithms have to be developed to handle arbitrary starting points.

6.5 Multicriteria Optimization

Engineering design problems including the current work most often involve multiple objectives which are conflicting in nature. These problems form a special class of optimization problems called the Multicriteria Optimization Problem. This involves multiple conflicting objectives like the manufacturing cost minimization and error sensitivity minimization of this research. This section introduces the multicriteria optimization problem and briefly describes some of the popular approaches that have been found to be useful in finding a compromise solution in the presence of conflicting objectives. There are a host of other methods and strategies that have been used in solving engineering problems (Jayaraman, 1993).
6.5.1 Multicriteria Optimization Literature

Pareto (1848-1923), the French-Italian economist and sociologist, established an optimality concept in the field of economics based on a multitude of objectives, i.e., on the permanent conflict of interests and antagonisms in social life (Pareto 1927). Hence this field of optimization is also called Pareto Optimization.

Multicriteria optimization methods have been used mainly in the field of social sciences (Cohon, 1978), (Zeleny, 1982). In production management, some analyses using multicriteria optimization methods have been published (Huckert et al., 1980), (Hitomi, Nakamura and Hatakeyama, 1980), (Takakuwa and Hitomi, 1981), (Hamada and Hitomi, 1982). The application of multicriteria optimization to problems in technology in general took quite a long time, and the number of applications is still limited.

Stadler (1975, 1978) referred to the scientific application of Pareto's optimality concept for the first time, and also published several papers especially on natural shapes.

Over the last two decades, several different approaches and strategies have been proposed to handle optimization problems with multiple criteria. Each of these techniques is effectively applicable only to a few categories of problems. These can be characterized by their mathematical behavior, complexity, and computational aspects. Consequently, the approaches preferred in different fields, and sometimes even between applications within a specific area, say machine design, vary. The application of multicriteria optimization strategies to machine design and structural design began in the seventies.

Bartel and Marks (1974) explained trade-off relationships between conflicting evaluative characteristics of hydrodynamic journal bearings. Two bearing design problems were considered — a full journal bearing operating at a constant speed under a constant load, and a full journal bearing operating under a dynamic load.

Osyczka (1978) developed a method for obtaining a compromise solution for a lathe gearbox design. The objective was to simultaneously minimize: (1) the volume of material used for the elements mounted on the shafts, (2) maximal peripheral velocity between the gears (reflects the dynamic behavior of the gearbox in terms of vibration and noise), (3) width of the gearbox, and (4) the distance between the axes of the input and output shafts.
Carmichael (1980) applied the constraint method for multiobjective optimization to a simple truss design.

Vincent (1983) examined the role of game theory in the engineering design process, and demonstrated its utility as a design tool applied to scalar as well as multicriteria optimization problems. He also examined the multicriteria optimization task from the perspective of a single designer, and from the perspective of team design where the design problem is assigned to several designers each responsible for one aspect of the total design.

Metwalli et al. (1984) adopted a method of multiplying competing objectives and raising them to different exponents, and applied it to optimize the finite journal bearing performance under steady loading conditions. The basic competing objectives were minimum power loss, maximum minimum film thickness, minimum side leakage, minimum temperature rise, and minimum bearing size.

Yoshimura et al. (1984) presented a method of multiobjective optimization to handle conflicting objectives in designing machine-tool spindles. They derived a Pareto optimum set showing a trade-off relationship between minimizing the total weight of the spindle and the static torsional or bending compliance using the Kuhn-Tucker necessary conditions for optimality, and through analyses of the objective and constraint functions.

Rao and Eslampour (1986) presented a methodology for multistage multiobjective optimization of gearboxes using a goal programming approach. The goal programming approach, in which the ideal feasible solutions of the individual objectives are treated as their respective goals, was suggested as a competitive alternative to the traditional method of linear combination of conflicting objectives. This approach was applied to the multiobjective optimization problems of kinematic and strength designs of multispeed gearboxes. In the kinematic design stage, the objectives were to minimize the deviation of output speeds from specified values and the overall center distance of the gearbox. In the strength design stage, the objectives were to minimize the volume of the material of the gears and to maximize the power transmitted by the gearbox.

Koski and Silvennoinen (1987) discussed a family of scalarization techniques based on the $l_p$-norm in a unified form and suggested a parametrization suitable for different interactive design systems. They also proposed an alternative approach which, instead of
scalarization, reduces the dimension of the multicriteria problem. This *partial weighting method* can be interpreted as a generalization of the traditional scalarization technique where the weighted sum of the criteria is used as the objective function. Several illustrative truss examples were presented.

Tseng and Lu (1990) proposed a *minimax* multiobjective optimization model for structural optimization, and solved such a problem using three typical techniques — goal programming, compromise programming and the *surrogate worth trade-off method*. Their paper focused on the application of multiobjective optimization techniques to the selection of system parameters and large scale structural design optimization problems.

Montusiewicz and Osyczka (1990) presented a heuristic four-stage decomposition strategy for multicriteria optimization of large-scale systems. The first stage optimizes the overall system with respect to basic decision variables. The second stage optimizes all subsystems which are considered separately. Interaction between subsystems and between the first and second stages are treated as coordination variables. The third stage optimizes the overall system with respect to coordination variables. The final stage selects the Pareto optimal set of solutions and makes the final decision. This strategy was applied in designing machine tool spindle systems with hydrostatic bearings.

Rao and Freiheit (1991) introduced a *modified game theory* approach to multiobjective optimization. The cooperative game theory method is hard to automate since it involves a two step optimization process — minimization of the scalarized criterion, and maximization of a numerical measure of compromise. In the modified approach, the generation of a compromise solution and the maximization of a numerical measure of compromise (supercriterion) are performed simultaneously. The proposed algorithm was tested on a numerical example dealing with the probabilistic design of an eighteen speed machine tool gear train.

### 6.5.2 Interactive Multicriteria Optimization

Interactive procedures that integrate the decision making process into multicriteria optimization algorithms began to emerge as multicriteria optimization strategies found more applications. As with the multicriteria optimization applications, a great number of publications in the field of interactive multicriteria optimization first appeared in the areas
of economics and operations research in the 1970's and 1980's. A large number of books and monographs on the subject Multicriteria Decision Making or Multiobjective Decision Making were published. Conference papers on this topic have regularly appeared since 1975 but without dealing with mechanical engineering problems.

For the design process, interactive procedures have been mentioned or applied in only a few cases (Carmichael, 1980), (Osyczka, 1985), (Diaz, 1987), (Koski, 1988). This is basically due to the immense non-linearity of the occurring problems. Furthermore, in an area of application such as structural design optimization, finite element/boundary element analyses must be carried out numerically, and this is very time-consuming.

Diaz (1987) presented sensitivity analysis as a natural addition to interactive multiobjective optimization methods based on compromise programming to reduce the number of interactions needed before a satisfactory solution is found. He demonstrated that useful information regarding trade-offs in the objectives can be generated effectively by means of an analysis of the sensitivity of solutions to variations in preference structures. An implementation based on sequential quadratic programming was provided.

6.6 The Multicriteria Optimization Problem

A feasible set in the design space \( \mathbb{R}^n \), for the design variable vector \( x \), is defined by

\[
\Omega = \left\{ x \in \mathbb{R}^n \mid g(x) \leq 0, \ h(x) = 0, \ x^{(l)} \leq x \leq x^{(u)} \right\} \Rightarrow \Omega \subseteq \mathbb{R}^n \quad (6.15)
\]

where the functional constraints are

\[
g : \mathbb{R}^n \rightarrow \mathbb{R}^r, \quad h : \mathbb{R}^n \rightarrow \mathbb{R}^s \quad (6.16)
\]

A single criterion (scalar) optimization problem is defined as

\[
\min_{x \in \Omega} f(x) \quad (6.17)
\]

with the scalar objective function

\[
f : \Omega \rightarrow \mathbb{R} \quad (6.18)
\]

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An extension of the dimension of the objective function yields the multicriteria (vector) optimization problem

$$\min_{x \in \Omega} f(x)$$  \hspace{2cm} (6.19)

with the vector objective function

$$f(x) = \{ f_j(x) \mid j=1,2,\ldots,m \}^T, \quad f : \Omega \to \mathbb{R}^m$$  \hspace{2cm} (6.20)

The image of the feasible set $\Omega$ in the criterion space $\mathbb{R}^m$, known as the *attainable set*, consists of all the feasible criterion values, and is defined by

$$\Lambda = \{ y \in \mathbb{R}^m \mid y = f(x), \ x \in \Omega \} \quad \Rightarrow \quad \Lambda \subseteq \mathbb{R}^m$$  \hspace{2cm} (6.21)

A characteristic feature of such optimization problems with multiple criteria is the appearance of an 'objective conflict', i.e., none of the feasible solutions allows the simultaneous minimization of all objectives. Consequently, the subject of multicriteria optimization deals with all kinds of conflicting problems.

### 6.6.1 Convexity Of A Multicriteria Optimization Problem

A multicriteria optimization problem on $\mathbb{R}^m$ is convex if and only if

(a) the components of the objective function vector $f(x)$ are convex

(b) the components of the vector of inequality constraints $g(x)$ are convex

(c) the components of the vector of equality constraints $h(x)$ are linear functions of $x$

### 6.6.2 Pareto-Optimal Set And Minimal Set

A vector $x^* \in \Omega$ is *Pareto-optimal* for the problem in Eq. (6.19), if and only if there exists no vector $x \in \Omega$ such that

$$f_j(x) \leq f_j(x^*) \quad \forall \ j = 1,2,\ldots,m$$
and
\[ f_j(x) < f_j(x^*) \quad \text{for at least one } j \in \{1, 2, \ldots, m\} \]  \hspace{1cm} (6.22)

With the above definition, the Pareto-optimal set in the design space \( \mathbb{R}^n \) is given by
\[ \Omega^* = \{ x^* \in \Omega \} \]  \hspace{1cm} (6.23)

The image of the Pareto-optimal set in the criterion space \( \mathbb{R}^m \), known as the efficient set or minimal set or non-dominated set, is defined by
\[ \Lambda^* = \{ y^* \in \mathbb{R}^m \mid y^* = f(x^*), \ x^* \in \Omega^* \} \]  \hspace{1cm} (6.24)

For all non-Pareto-optimal vectors, the value of at least one objective function \( f_j \) can be reduced without increasing the function values of the other components. Fig. 6.5 shows a mapping of the feasible design space into the attainable criterion space, \( \Omega \subset \mathbb{R}^2 \rightarrow \Lambda \subset \mathbb{R}^2 \), and the mapping of the Pareto-optimal set into the minimal set, \( \Omega^* \rightarrow \Lambda^* \). The Pareto-optimal / minimal sets can be obtained by using any of the techniques described in Section 6.7. The characteristics of the different regions in the bounded criterion space illustrated in Fig. 6.5 are presented in Table 6.1.

### 6.6.3 Weak Pareto-Optimal Set

A vector \( x^* \in \Omega \) is weakly Pareto-optimal for the problem in Eq. (6.19), if and only if there exists no vector \( x \in \Omega \) such that
\[ f_j(x) < f_j(x^*) \quad \forall \ j = 1, 2, \ldots, m \]  \hspace{1cm} (6.25)

Obviously, every Pareto-optimum is also weakly Pareto-optimal, but the converse is not true.

### 6.6.4 Substitute Problem And Preference Function

Although a vector optimization problem is regarded as solved when the Pareto-optimal set has been determined, practical applications usually need a single 'preferred' solution from this set. Solutions of non-linear vector optimization problems can be found in
Fig. 6.5: Mapping Of The Feasible Design Space Into The Attainable Criterion Space (Eschenauer et al., 1990)
Table 6.1: Characteristics Of The Regions In Criterion Space Of Fig. 6.5  
(Eschenauer et al., 1990)

<table>
<thead>
<tr>
<th>Region</th>
<th>Feasibility</th>
<th>Optimality</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Infeasible</td>
<td>Super-minimal</td>
</tr>
<tr>
<td>(\Lambda^*)</td>
<td>Feasible</td>
<td>Minimal</td>
</tr>
<tr>
<td>II</td>
<td>Feasible</td>
<td>Sub-minimal</td>
</tr>
<tr>
<td>III</td>
<td>Infeasible</td>
<td>Sub-minimal</td>
</tr>
</tbody>
</table>

different ways. By defining substitute problems, these are normally reduced to scalar optimization problems.

The problem

\[
\min_{x \in \Omega} P[f(x)] = \min_{f \in \Lambda} P[f(x)]
\]

(6.26)

is a substitute problem if there exists an \(\tilde{x} \in \Omega^*\) such that

\[
P[f(\tilde{x})] = \min_{x \in \Omega} P[f(x)] = \min_{f \in \Lambda} P[f(x)]
\]

(6.27)

The function \(P\) is called preference function or substitute objective function. It is important to study whether the solutions \(\tilde{x}\) of the substitute problems are Pareto-optimal with respect to the feasible set, and to the attainable set, i.e.,

\[
\tilde{x} \in \Omega^* \quad \Leftrightarrow \quad f(\tilde{x}) \in \Lambda^*
\]

(6.28)

6.6.5 Normalization

Scalarization of multicriteria problems by defining substitute problems introduces another important aspect for consideration, which is relatively unimportant for regular scalar problems — normalization of the objectives. Although normalization of the objective
functions is done sometimes for scalar problems from a numerical standpoint or so, it is absolutely essential for scalarized substitute problems since the preference function is a combination of all the objectives. There are quite a few different ways of normalizing the objectives in order to scale them to the same relative order of magnitude. For instance, a useful non-dimensional normalization is

$$z_j(x) = \frac{f_j(x)}{f_j^{\text{min}}}$$

where $f_j(x)$ is the value of the $j$th objective function and $f_j^{\text{min}}$ is the utopian minimum of the $j$th objective function minimized as a single criterion.

In this report, all functions are assumed to be already in some normalized form unless a specific normalization scheme is explicitly presented.

### 6.6.6 Parametrization

Formulation of the substitute problem as in Section 6.6.4 results in a single solution, which may not be satisfactory. Parametrization of the substitute problem for the original multicriteria problem is achieved by introducing more degrees of freedom into the preference function $P[f(x)]$. Usually, the additional parameters are weighting factors $w_j$, which help generate a range of solutions with the preference function $P[f(x), w]$. However, it is important that any such parametrization yields only minimal solutions; conversely, it is desirable to have the parametrization span all minimal solutions so that a designer can have the maximum choice possible.

### 6.7 Overview Of Multicriteria Optimization Strategies

A number of publications have dealt with various methods for transforming vector optimization problems into substitute problems. In the following sections, some of the popular multicriteria optimization 'strategies' are presented. However, it is to be emphasized that most of these methods are problem-dependent.
6.7.1 Linear Combination Of Objectives

Objective weighting is one of the most common substitute models for vector optimization problems. This scheme permits a preference formulation that is independent of the individual objective minima for positive weights, $w_j$. The preference function or utility function is determined by a linear combination of the criteria \( f = \{ f_j \mid j=1,2,\ldots,m \}^T \) together with the corresponding weighting factors \( w = \{ w_j \mid j=1,2,\ldots,m \}^T \):

\[
P[f(x),w] = \sum_{j=1}^{m} [w_j f_j(x)] = w^T f(x)
\]

(6.30)

\( w \) is usually chosen such that

\[
\sum_{j=1}^{m} w_j = 1, \quad 0 \leq w_j \leq 1 \quad (j=1,2,\ldots,m)
\]

(6.31)

As stated before, the components of \( f(x) \) need to be in normalized form.

**Parametrization:**

Parametrization is automatic as the Pareto-optima for the original problem in Eq. (6.19) can be generated by varying the weights \( w_j \) in the preference function. For a given \( w \), this represents a solution in the criterion space, at the point(s) of tangency of the hyper-plane, \( \{ w^T f(x) = c \} \), to the minimal set \( A^* \), such that

\[
c = \min_{f \in A} \{ P[f(x),w] \} = P[f(x^*),w]
\]

(6.32)

Linear combination guarantees the minimality of all such generated solutions for convex problems. However, for non-convex problems, the converse is not true, i.e., parametrization does not yield the entire Pareto-optimal set. This is illustrated geometrically in Fig. 6.6.
Fig. 6.6: Parametrization By Linear Combination Of Objectives (Osyczka, 1984)
6.7.2 Compromise Programming Or Norm Method

Norms or distance functions are frequently applied leading to a scalarization of the vector optimization problem. A designer specifies ‘demand levels’, \( \bar{f} = \{ \bar{f}_j \mid j=1,2,...,m \}^T \), a set of objective function values to be achieved in the best possible way. However, the selection of an inappropriate demand level vector, \( \bar{f} \), will lead to non-minimal solutions.

The preference function in general is

\[
P[f(x),p] = \left[ \sum_{j=1}^{m} \left| f_j(x) - \bar{f}_j \right|^p \right]^{1/p} = \| f(x) - \bar{f} \|_p,
\]

\( 1 \leq p < \infty, \quad x \in \Omega \) \hfill (6.33)

\[
= \max_{j=1,2,...,m} \left| f_j(x) - \bar{f}_j \right| = \| f(x) - \bar{f} \|_{\infty},
\]

\( p \to \infty, \quad x \in \Omega \) \hfill (6.34)

where \( \|\|_p \) represents the \( l_p \)-norm, and \( p \) defines a family of 'hyper-spheres' in the criterion space \( \mathbb{R}^m \) with center \( \bar{f} \) and radius \( R \):

\[
S_p(R) = \{ y \in \mathbb{R}^m \mid y = f(x), \ R = \| f(x) - \bar{f} \|_p \} \hfill (6.35)
\]

Geometrically, the variation of \( p \) from 1 through \( \infty \) represents a corresponding transformation in the radius of curvature of the hyper-spheres from \( \infty \) through 0. This can be observed from the family of hyper-spheres illustrated in Fig. 6.7 for the compromise solution at the point(s) of tangency of the hyper-spheres to the minimal set \( \Lambda^* \), with \( f(x) = f(x^*) \).

It is worth noting that the minimization of \( L_2(f) \) leads to the minimization of the Euclidean distance between \( \bar{x} \in \mathbb{X} \) and the ideal solution. For most practical problems, the demand levels are the individual minima of the \( m \) objective functions, \( f_j^{\text{min}} \). The selection of such an ideal vector or utopia point, \( f^{\text{min}} \), is appropriate but requires \textit{a priori} knowledge of the individual minima which is not easy to determine for non-convex problems. The most frequently used norms are summarized in Table 6.2.
Koski (1981) has suggested $L_p$-metrics with a normalized vector objective function of the form:

\[ f_j(x) = \frac{f_j(x) - f_j^{\text{min}}}{f_j^{\text{max}} - f_j^{\text{min}}} \quad (6.36) \]

where

\[ f_j^{\text{max}} = \text{Max}[f_j(x)] \quad \text{in } x \in X \]

and

\[ f_j^{\text{min}} = \text{Min}[f_j(x)] \quad \text{in } x \in X \quad \text{for the } j\text{th objective function} \]

In this case, the values of every normalized function are limited to the range $[0, 1]$.

**Parametrization:**

For the preference formulation defined in Eqs. (6.33) and (6.34), parametrization is achieved by introducing more parameters in the form of weighting factors $w$:  

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<table>
<thead>
<tr>
<th>Norm</th>
<th>Preference Function — $P[f(x),p]$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 1$</td>
<td>$\sum_{j=1}^{m}</td>
<td>f_j(x) - f_j^{\text{min}}</td>
</tr>
<tr>
<td>$p = 2$</td>
<td>(Euclidean)</td>
<td></td>
</tr>
<tr>
<td>$p \to \infty$ (Chebyshev)</td>
<td>$\max_{j=1,2,...,m}</td>
<td>f_j(x) - f_j^{\text{min}}</td>
</tr>
</tbody>
</table>

Table 6.2: Some Popular Norms

$$P[f(x),p,w] = \left[ \sum_{j=1}^{m} \left( w_j \left| f_j(x) - \bar{f}_j \right| \right)^p \right]^{1/p}.$$}

$$1 \leq p < \infty, \quad x \in \Omega \quad (6.37)$$

$$= \max_{j=1,2,...,m} \left( w_j \left| f_j(x) - \bar{f}_j \right| \right), \quad p \to \infty, \quad x \in \Omega \quad (6.38)$$

For $w$ that satisfies Eq. (6.33), this represents a solution in the criterion space at the point(s) of tangency of the hyper-ellipsoid, $\{ \sum_{j=1}^{m} \left( w_j \left| f_j(x) - \bar{f}_j \right| \right)^p = c; p \neq \infty \}$, to the minimal set $\Lambda^*$, such that:

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\[ c^{1/p} = \min_{f \in \Lambda} \{ P[f(x), p, w] \} = P[f(x^*), p, w], \quad p \neq \infty \]  

The above parametrization guarantees the minimality of all such generated solutions, and the range of the parameter, \( p \in [1, \infty) \), ensures the generation of all Pareto-optima for both convex as well as non-convex problems. This is geometrically illustrated in Fig. 6.8.

### 6.7.3 Min-Max Formulation

This formulation is based on the minimization of the relative deviations of the objective functions from the respective demand levels. Clearly, this suggests a close relationship to the \( l_\infty \) norm defined in Eq. (6.34), and its parametrized form in Eq. (6.38). At the same time, this formulation has a property that is distinct from the other members of the family of \( l_p \)-norms, and is widely used in several forms. With the demand levels \( \bar{f}_j \) specified for the objective functions, a general parametrized form of the scalar substitute problem for the min-max formulation can be defined as:

\[ P[f(x), w] = \max_{j=1,2,\ldots,m} \left[ w_j z_j(x) \right] \]  

with

\[ z_j(x) = \frac{f_j(x) - \bar{f}_j}{\bar{f}_j}, \quad \bar{f}_j > 0 \quad (j = 1,2,\ldots,m) \]

\[ = f_j(x), \quad \bar{f}_j = 0 \quad (j = 1,2,\ldots,m) \]  

At the compromise solution, all the weighted relative deviations of the objectives from their demand levels are equal:

\[ w_j z_j(x^*) = \min_{f \in \Lambda} \{ P[f(x), w] \} = P[f(x^*), w] \quad (j = 1,2,\ldots,m) \]  

Also, as in the norm method, the individual minima of the objectives, \( f_j^{\text{min}} \), are often chosen as the demand levels, \( \bar{f}_j \).
Fig. 6.8: Parametrization By Compromise Programming ($L_p$-Norm) (Eschenauer et al., 1990)
For \( \mathbf{w} \) that satisfies Eq. (6.33), the parametrization in Eq. (6.42) represents a solution in the criterion space at the point of tangency of the hyper-cuboid. \( \{ c_j = z_j(x) \mid j=1,2,...,m \} \), to the minimal set \( \Lambda^* \), such that

\[
 w_j c_j = \min_{f \in \Lambda} \left\{ P[f(x),\mathbf{w}] \right\} = P[f(x^*),\mathbf{w}] \quad (j = 1,2,...,m) \tag{6.43}
\]

The above parametrization guarantees the minimality of all such generated solutions and ensures the generation of all Pareto-optima for both convex as well as non-convex problems. The geometrical illustration is included in Fig. 6.8.

### 6.8 Optimization Problem Defined

The definition of the main optimization problem depend on the output sensitivity analysis and the setup of the multicriteria optimization objectives and their handling. This section gives the details of the optimization problem formulation.

#### 6.8.1 Sensitivity Analysis

As discussed earlier, one of the objective functions and the constraint both depend on the sensitivity analysis results of the output with respect to each variable. The output variable of interest could be various things. It will depend on what the designer considers to be the most critical output parameter(s) in the control and the intended use of the mechanism or assembly. If the design requirement is to make sure that an output link is within some tolerance of given angles at every position, then the angle of the output link would be the \( f \) parameter used. In the optimization process, constraints would also be defined on this variable. Similarly, if the intent is to control the position of some link in space within given tolerance, then the \( x, y, z \) coordinates of that link could be the output variables of interest. If \( f \) is an output variable of interest, then we define:

\[
 \frac{\partial f}{\partial x_i} = \frac{f(x_1, x_2, ..., x_i + \delta x_i, ..., x_n) - f(x_1, x_2, ..., x_i - \delta x_i, ..., x_n)}{2\delta x_i} \tag{6.44}
\]

\[
 \frac{\partial f}{\partial x_i} = \frac{f(x_1, x_2, ..., x_i + \delta x_i, ..., x_n) - f(x_1, x_2, ..., x_i, ..., x_n)}{\delta x_i} \tag{6.45}
\]
\( \delta x_i \) is a very small value used for the perturbation of the variable \( x_i \). \( x_i \)'s are the variables that are being optimized e.g. tolerance ranges on individual link parameters and joint clearances. Eqs. (6.44) and (6.45) can be evaluated for every driver position of the mechanism and constitute the central difference method and the positive difference method, respectively. \( x_i \) is the current value of the \( i \)th variable (tolerance or clearance). Eq. (6.44) is supposed to be used for calculating sensitivities of tolerance variables and Eq. (6.45) for sensitivities of clearance variables. For the clearance case, a negative perturbation below nominal zero is not meaningful hence a one-sided perturbation is done.

In the case when a single output variable is tracked (e.g. the angle of the output link in planar case), the Sensitivity \( S_j(x_i) \) of variable \( x_i \) at position \( j \) is defined as:

\[
S_j(x_i) = \left( \frac{\partial f}{\partial x_i} \right)_j
\]  

(6.46)

where \( j \) is the position number.

For multi level output variables (e.g. \( x, y, z \) coordinates of a point on a general output link), the following is used for \( S_j(x_i) \) of variable \( x_i \) at position \( j \):

\[
S_j(x_i) = \left[ \sum_{i=1}^{m} \left( \frac{\partial f}{\partial x_i} \right)_j^2 \right]^{1/2}
\]  

(6.47)

where \( m \) is the total number of degrees of the output variable and \( j \) is the position number. The signs of the sensitivities are lost in the above Eq. (6.47), but wherever the sensitivity information is used, its absolute value is used. Thus no generality is lost by using the Eq. (6.47) in the above form.

\textit{Sensitivity Factor} \( \text{SF}(x_i) \) for the variable \( x_i \) over the whole motion range is defined as:

\[
\text{SF}(x_i) = \left[ \sum_{j=1}^{k} S_j(x_i)^2 \right]^{1/2}
\]  

(6.48)

where \( k \) is the total number of input positions defined by the designer.

\textit{Sensitivity Index} \( \text{SI}(x_i) \) for the variable \( x_i \) is defined as:
where $n$ is the total number of active variables consisting of tolerances and clearances. Sensitivity Index defined here is used in formulating the second objective function of the multicriteria optimization procedure.

### 6.8.2 Cost Objective Function

The primary objective of the optimal tolerance allocation in this research is minimization of manufacturing cost. A representative estimate of the manufacturing cost model is therefore required. The Cost Objective Function is defined as:

$$ C = \sum_{i=1}^{n} C_i(x_i) $$ (6.50)

where $C_i(x_i)$ is the estimated total cost of manufacturing associated with maintaining a variable $x_i$ at the current value.

The total manufacturing cost may be estimated from any function or table defined for the purpose for the particular variable in the particular mechanism. Cost estimate tables or charts may be obtained by looking at the manufacturing processes required for the design. Estimations can also be done by using the methods outlined in Appendix E for manufacturing processes. In the absence of real tangible data on machining cost, a representative cost vs. tolerance chart may be used. A generic function may be used for all such cases. It may be given as:

$$ C_i(x_i) = A_i + \frac{B_i}{x_i} $$ (6.51)

$A_i$ is the factor for overhead costs etc. and $B_i$ is the cost associated with producing a part with a fixed tolerance. Each variable may have a different cost-tolerance function defined at different ranges of the tolerances of manufacturing. The cost objective function is dependent on the current values of the variables and has the same value for every position of the mechanism.
6.8.3 Sensitivity Objective Function

As stated earlier, the two main objectives to be considered in the multicriteria optimization problem are the overall cost of the assembly (C) and the variation of the output (δ) from the desired value. The vector objective function \( f(x) = [C \ \delta]^T \) can then be minimized simultaneously. On close scrutiny, it seems that the two objectives are in very strong conflict. To reduce the δ, the individual tolerances are to be maintained tightly, thus incurring more cost. Likewise to reduce the cost of the assembly, the variation has to be increased. It also does not make much sense in trying to reduce the overall variation of the output of the mechanism below the level that is already acceptable to the designer and increasing the cost in the process.

Even though, it would be nice to have the second objective as the variation of the output from the ideal, there are potential problems with this approach. As discussed earlier, the objective is in very strong conflict with the cost objective. Also, if the designer is able to assign a limit on the variation and is able to accommodate that, there is no sense in trying to reduce that variation below what is required. It will definitely increase cost unnecessarily.

One of the prime concerns when employing the multicriteria optimization methods is the nature of conflicts between the separate objectives. However the existence of the conflicts is inherent in the system, since the multicriteria formulation is used only when there exists several competing design objectives. Thus, it is useful to analyze the conflict of the candidate criteria before forming the vector objective function. The local and global conflicts are considered separately. Functions \( f_i \) and \( f_j \) are said to be locally conflicting at the point \( x \) if there exists no \( c > 0 \) such that

\[
\nabla f_i(x) = c \nabla f_j(x)
\]

(6.52)

The angle \( \phi \) between the two gradients is a measure for the degree of local conflict. If \( \phi = \pi \), then there is complete conflict and if \( \phi = 0 \) then the criteria are collinear and there is no conflict. Normally \( 0 < \phi < \pi \) and the case \( \phi > \pi/2 \) is called a strong conflict. Similarly, the objectives \( f_i \) and \( f_j \) are said to be globally conflicting in \( \Omega \) if the scalar problems:

\[
\min_{x \in \Omega} f_i(x) \text{ and } \min_{x \in \Omega} f_j(x)
\]
have different solutions. Since the global conflict between the criteria cannot be proven generally, the degree of the global conflict can be assessed by minimizing every criteria separately in Ω. This is usually done during some sort of normalization of the criteria at the beginning of the multicriteria optimization process. The above normalization also brings the various objectives to the same scale or dimension and provides uniformity to the problem.

Generally the criteria are not similar and their numerical values can differ greatly. Many normalization procedures are developed in the available literature. Two of them are considered in this work. The first one is given by Eq. (6.29) and is repeated here again:

\[ f_j(x) = \frac{f(x)}{f_{j_{\text{max}}}}, \quad j = 1, 2, ..., n \]  

(6.53)

which is suitable when every criteria achieves only strictly positive values. \( f_{j_{\text{min}}} \) and \( f_{j_{\text{max}}} \) are used for the minimum and maximum values of the criterion \( f_j \) in \( \Omega \). If non dimensional criteria with equal variation ranges are wanted, an alternative form can be used as given in Eq. (6.36) and repeated here:

\[ f_j(x) = \frac{f_j(x) - f_{j_{\text{min}}}}{f_{j_{\text{max}}} - f_{j_{\text{min}}}}, \quad j = 1, 2, ..., n \]  

(6.54)

The values of each normalized objective \( f_j(x) \) are limited between zero and unity; i.e., \( f_j(x) \in [0, 1] \) for \( j = 1, 2, ..., m \).

The other alternative is to consider the sensitivity of the variation \( \delta \) with respect to the individual tolerances and the overall cost \( C \) as the design criteria. The vector objective function will look like:

\[ f(x) = \left[ C \frac{\partial \delta}{\partial x_1} \frac{\partial \delta}{\partial x_2} ... \frac{\partial \delta}{\partial x_n} \right]^T \]  

(6.55)

where \( x_i \) \( i = 1, 2, ..., n \) are the design variables. These objectives do not pose such a great conflict between themselves. However the number of criteria increases by a large number. A prime concern when formulating optimization problems is a way where both computing costs and the efforts in handling the results are kept reasonably small. This can be achieved in multicriteria problems by restricting the number of the criteria to be as small as possible. When engineering
judgment cannot reduce the number of criteria, a purely mathematical basis with certain parameters may be used to do so. With this in mind, it might be advisable to combine all the sensitivities into a single criterion. A combination of the n normalized sensitivities can be used as another criterion in addition to the cost of manufacture.

The second objective function is selected which is connected to the variations of output in a more "life cycle cost" sense. This objective function is the sensitivity objective function and minimizes the sensitivity of the output variation to variation in individual variables. The Sensitivity Objective Function is also dependent on the current values of the variables. It may be defined as:

\[ \Delta = \sum_{i=1}^{n} [SI(x_i)x_i]^2 \]  \hspace{1cm} (6.56)

where SI(x_i) is the Sensitivity Index defined in Section 6.8.1.

In Eq. (6.53), the sensitivity indices are combined in this fashion to make their effects be felt in the statistical way.

The vector objective problem then becomes

\[ \min_{x \in \Omega} [C \Delta]^T \]  \hspace{1cm} (6.57)

The basic constraints within which the multi-objective optimization problem operates are the tolerance limits on each of the design dimensions \( x_{i1} \leq x_i \leq x_{i2} \) \( i = 1, ..., m \) and the limit on the variation of the output tolerance \( T_{asy} \leq T_{allow} \) for each position.

### 6.8.4 Constraint Function

As discussed earlier, the variation of the output from the ideal is used as a constraint in the optimal allocation of tolerances and clearances in this work. The user is able to define the maximum allowable variation in assembly tolerance that can be accommodated without sacrificing the mechanism performance. The designer also has the flexibility to define different values of allowable assembly tolerances based on the design requirements at different driver positions. The optimization procedure has the flexibility to handle that based on the fact that an independent optimization procedure is performed.
at each driver position. The global optimum is obtained which satisfies all the constraints and minimization requirements at all positions. This ensures that the optimal tolerances and clearances obtained are minimum in the global sense. The history of the optimization process for each driver position is also given to the designer. Based on that the designer might be able to modify some constraints or design variables and rerun the optimization process to remove any inefficacy due to some stray situations in some positions.

The constraints used in the optimization case is given by:

\[
T_{\text{assy}} \leq T_{\text{max}}
\]  

(6.58)

where \(T_{\text{assy}}\) is the tolerance of the assembly and \(T_{\text{max}}\) is the maximum allowed from the design standpoint.

The normalized constraint is given by:

\[
\frac{T_{\text{assy}}}{T_{\text{max}}} - 1 \leq 0
\]

(6.59)

As discussed in Chapter 5, the assembly tolerance is modeled by using the mean shift method which is a combination of the worst case combination and the statistical method by using an estimate of the mean shift in manufacturing processes. The expression for the assembly tolerance at driver position \(j\) is given by:

\[
(T_{\text{assy}})_j = \sum_{i=1}^{n} m_i |S_j(x_i)|x_i + \left[ \sum_{i=1}^{n} (1 - m_i)^2 (S_j(x_i))^2 x_i^2 \right]^{1/2}
\]

(6.60)

where \(S_j(x_i)\) is the sensitivity of the output with respect to the variation in variable \(x_i\), \(x_i\) is the current value of tolerance/clearance variable \(i\) and \(m_i\) is the estimated mean shift factor for variable \(x_i\). For most of the cases, a value of 0.1 to 0.2 has been used for \(m_i\) by default unless the user defines a different value. A value of 0.1 to 0.2 is what seems to be the consensus for \(m_i\) in literature for closely controlled processes.

### 6.8.5 Note On Objective Weighting Factors

Pareto-optima for the original problem is obtained by varying the weights \(w_i\) in the preference function. Let \(C\) be the objective function for manufacturing cost minimization.
and $\Delta$ be the objective function for mechanical error sensitivity minimization and $x$ be the vector of design variables (the clearances and tolerances). Then the optimization problem may be formulated as:

$$\min_{x \in \Omega} [C \quad \Delta]^T$$

subject to

$$\frac{T_{asys}}{T_{max}} - 1 \leq 0$$

and $x_i \leq x_i \leq x_i$ $i = 1, ..., m$

where the objective functions $C$ and $\Delta$ are given by Eqs. (6.50) and (6.56) and the assembly tolerance is given by Eq. (6.60). $x_{i_l}$ and $x_{i_u}$ are the lower and upper limits on the $i$th design variable $x_i$.

The norm method is used to get a preference function for minimizing. The $L_2$-norm as discussed in Section 6.7.2 has been used with success in this work. $L_2$-norm minimizes the Euclidean distance from the ideal solution. In this case, the preference function is defined as:

$$p[f(x)] = \left[ \left( \frac{C - C_{min}}{C_{min}} \right)^2 + \left( \frac{\Delta - \Delta_{min}}{\Delta_{min}} \right)^2 \right]^{1/2}$$

(6.61)

or in most cases:

$$p[f(x)] = \left[ w_1^2 \left( \frac{C - C_{min}}{C_{min}} \right)^2 + w_2^2 \left( \frac{\Delta - \Delta_{min}}{\Delta_{min}} \right)^2 \right]^{1/2}$$

(6.62)

where $C_{min}$ is the ideal minimum optimized value of the cost objective function optimized alone, and $\Delta_{min}$ is the ideal minimum optimized value of the sensitivity objective function optimized alone, and $w_1$ and $w_2$ are the preferred weighting factors. The weighting factors are interactively input by the user in the program developed as a part of this research.
CHAPTER 7

KINEMATIC AND FORCE SOLUTIONS FOR JOINTS WITH CLEARANCES

7.0 Introduction

This chapter outlines the procedure for modeling the joints with clearances and solving for position, velocity, acceleration, and force in a mechanism with clearance joints. The results from the force analysis is used in an iterative fashion along with the position, velocity, and acceleration analysis to finally obtain the kinematic solution for the mechanism with clearances at the joints. The iterative procedure is general and essentially remains the same for all joint types. A general outline of the procedure is given first. The modeling of the other joints are introduced next. A flowchart for the process is also given.

7.1 Iterative Kinematic And Force Solution For Mechanisms With Joint Clearances

As discussed earlier, it is assumed in this work that even with the presence of clearances, there is no contact loss at the joints. The presence of forces in the joints always ensures that the "pin" at a joint is always in contact with the "bearing" surface. The words "pin" and "bearing" are used in a general sense here. They can mean, for example, the internal spherical surface and the external spherical surface respectively for a spherical pair. The clearances are assumed to be taken up along the force directions at a joint. Due to the presence of the joint clearances, the centers of the "pin" and the "hole" do not coincide.

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There is a gap between them which is essentially the joint clearance value. Fig. 7.1 shows the details of the situation at a joint with clearance. The revolute joint is shown for illustration purposes, but the procedure is same for all the joint types considered. For each joint type, the changes or variations are only due to the differences in the individual joints’ parameters and pair variables. For analysis purposes, it is assumed that there is a fictitious link between the pin center and the journal center. This link is assumed to be massless. Each joint with clearance is assumed to have a fictitious link associated with it. The length of this massless link is the actual clearance at the joint. This is the concept of the equivalent clearance link model. This is the model used for the clearance studies in this work. An iterative procedure is developed for obtaining the solution of a mechanism with a number of clearance links. It is assumed that the clearance links always align along the force directions in the joint. To begin, the force solutions for the no clearances case is initially used to estimate approximate directions of the clearance links. With these in place, the kinematic problem is solved again using the modified pair variables at the joints to take care of the clearances. New force solutions and force directions are obtained after that which are then used again for obtaining kinematic solutions. The process is repeated until the difference between the old solution and new solution falls within specified iteration tolerances.

7.1.1 Joint Clearance Model And Position Solution

Fig. 7.1 gives the details of a joint with clearance. It is assumed without any loss of generality, that the pin is fixed to link i+1. $F_{i,i+1}$ is the force from link i to link i+1 without clearance. $F_{i+1,i}$ is the force from link i+1 to link i without clearance. The massless clearance link aligns along the direction of the force at the joint at all time. Since the Sheth-Uicker's method of matrix definition is used for the kinematic modeling, the joint pair matrix can be decoupled from the link shape matrix. This allows the incorporation of the joint clearances into a modified pair matrix without making any changes to the link shape matrices. The clearance link at this joint is represented by $C_{i,i+1}$. The pair matrix ($P_{i,i+1}$) at a clearance joint connection for a revolute joint is given by:

$$P_{i,i+1} = [P_{i,C_{i+1}}(q_{i,C_{i+1}})] [S_{c_{i+1}}(L_{C_{i+1}})] [P_{C_{i+1},i+1}(q_{C_{i+1},i+1})]$$

(7.1)

where
Fig. 7.1: Details Of A Joint With Clearance

\[
\begin{bmatrix}
P_{i,c,i+1}(q_{i,c,i+1})
\end{bmatrix}
\text{is the pair matrix between link } i \text{ and the clearance link } C_{i,i+1}
\]

\[
\begin{bmatrix}
S_{c,i+1}(L_{C,i+1})
\end{bmatrix}
\text{is the shape matrix for the clearance link}
\]

and \[
\begin{bmatrix}
P_{c,i+1,i+1}(q_{c,i+1,i+1})
\end{bmatrix}
\text{is the pair matrix between the clearance link } C_{i,i+1} \text{ and link } i+1
\]

\[q_{i,c,i+1}\] is the angle between the link \(i\) and the massless clearance link \((C_{i,i+1})\) between links \(i\) and \(i+1\)

\[q_{c,i+1,i+1}\] is the angle between the massless clearance link \((C_{i,i+1})\) between links \(i\) and \(i+1\) and the link \(i+1\)

For the case of a revolute joint with clearance, the pair matrix \((P_{i,i+1})\) is given by:

\[
P_{i,i+1} = \begin{bmatrix}
P_{i,c,i+1}(\theta_{i,c,i+1})
S_{c,i+1}(L_{C,i+1})
P_{c,i+1,i+1}(\theta_{c,i+1,i+1})
\end{bmatrix}
\]  

(7.2)
where

\[ \theta_{i,C_{i+1}} \] is the angle between the link i and the massless clearance link \((C_{i,i+1})\) between links \(i\) and \(i+1\)

\[ \theta_{C_{i+1},i+1} \] is the angle between the massless clearance link \((C_{i,i+1})\) between links \(i\) and \(i+1\), and the link \(i+1\)

\[ P_{i,C_{i+1}}(\theta_{i,C_{i+1}}) \] is the pair matrix between the link i, and \((C_{i,i+1})\) and is a function of the known angle \(\theta_{i,C_{i+1}}\) only

\[ P_{C_{i+1},i+1}(\theta_{C_{i+1},i+1}) \] is the pair matrix between \((C_{i,i+1})\), and link \(i+1\) and is a function of angle \(\theta_{C_{i+1},i+1}\) only. This angle is unknown in general, but will be known for the driver link as detailed later

\[ S_{C_{i+1}}(L_{C_{i+1}}) \] is the shape matrix for the clearance link and is a function of the length of the clearance link, \(L_{C_{i+1}}\) only

The length of the clearance link for the revolute joint is given by: \(L_{C_{i+1}} = R_{\text{hole}} - R_{\text{pin}}\). It is assumed for this development that the mechanism remains a planar mechanism even with the presence of the clearance.

If link \(i\) is the ground link and link \(i+1\) is the driver in a mechanism, then \(\theta_{i,i+1}\) will be considered the apparent driver angle. \(\theta_{C_{i+1},i+1} = \pi + \theta_{i,i+1} - \theta_{i,C_{i+1}}\) is chosen from this relationship. Fig. 7.2 shows the details of the pair matrix definition for this joint.

If \(\theta_i\) is the pair variable for a revolute joint, then the pair matrix is given by:

\[
P_{i,i+1} = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

(7.3)

where \(c\theta_i = \cos(\theta_i)\) and \(s\theta_i = \sin(\theta_i)\).

The general expression for the Shape Matrix in terms of its parameters is given by (Appendix A):

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For the shape matrix for $C_{i,i+1}$, the following parameters are identified:

$$c = 0, \gamma = 180^\circ, \alpha = 0, b = 0, \beta = 0$$

Substituting these values in Eq. (7.3) we get the Shape Matrix $S_{C_{i,i+1}}$ as:

$$S_{C_{i,i+1}} = \begin{bmatrix}
-1 & 0 & 0 & -L_{C_{i,i+1}} \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad (7.5)$$

The Pair Matrix at the joint with clearance is then:

$$P_{i,i+1} = P_{i,C_{i,i+1}}(\theta_{i,C_{i,i+1}})S_{C_{i,i+1}}(L_{C_{i,i+1}})P_{C_{i+1,i}}(\theta_{C_{i+1,i+1}}) \quad (7.6)$$
\[
\begin{bmatrix}
 c\theta_{i,C_{i-1}} & -s\theta_{i,C_{i-1}} & 0 & 0 & -L_{C_{i-1}} & c\theta_{i,C_{i-1},i-1} & -s\theta_{i,C_{i-1},i-1} & 0 & 0 \\
 s\theta_{i,C_{i-1}} & c\theta_{i,C_{i-1}} & 0 & 0 & -L_{C_{i-1}} & s\theta_{i,C_{i-1},i-1} & c\theta_{i,C_{i-1},i-1} & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}
\]

\( P_{i,i+1} = \)

\[
\begin{bmatrix}
 -c\theta_{i,C_{i-1}} & c\theta_{i,C_{i-1},i-1} & s\theta_{i,C_{i-1}} & s\theta_{i,C_{i-1},i-1} & 0 & -L_{C_{i-1}} & c\theta_{i,C_{i-1},i-1} & 0 & -L_{C_{i-1}} & s\theta_{i,C_{i-1},i-1} \\
 -s\theta_{i,C_{i-1}} & c\theta_{i,C_{i-1},i-1} & c\theta_{i,C_{i-1}} & s\theta_{i,C_{i-1},i-1} & -c\theta_{i,C_{i-1}} & c\theta_{i,C_{i-1},i-1} & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 
\end{bmatrix}
\]

(7.7)

or, \( P_{i,i+1} = \)

\[
\begin{bmatrix}
 -c(\theta_{i,C_{i-1}} + \theta_{C_{i-1},i-1}) & s(\theta_{i,C_{i-1}} + \theta_{C_{i-1},i-1}) & 0 & -L_{C_{i-1}} & c\theta_{i,C_{i-1}} \\
 -s(\theta_{i,C_{i-1}} + \theta_{C_{i-1},i-1}) & -c(\theta_{i,C_{i-1}} + \theta_{C_{i-1},i-1}) & 0 & -L_{C_{i-1}} & s\theta_{i,C_{i-1}} \\
 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 
\end{bmatrix}
\]

(7.8)

Now from Fig. 7.2, it is clear that \( \theta_{i,C_{i-1}} + \theta_{C_{i-1},i-1} = \pi + \theta_{i,i+1} \)

i.e. \( \cos(\theta_{i,C_{i-1}} + \theta_{C_{i-1},i-1}) = \cos(\pi + \theta_{i,i+1}) = -\cos \theta_{i,i+1} \)

(7.9)

and \( \sin(\theta_{i,C_{i-1}} + \theta_{C_{i-1},i-1}) = \sin(\pi + \theta_{i,i+1}) = -\sin \theta_{i,i+1} \)

(7.10)

Substituting Eqs. (7.9) and (7.10) in Eq. (7.8), we get the pair matrix as:

\[
\begin{bmatrix}
 c\theta_{i,i+1} & -s\theta_{i,i+1} & 0 & -L_{C_{i-1}} & c\theta_{i,C_{i-1}} \\
 s\theta_{i,i+1} & c\theta_{i,i+1} & 0 & -L_{C_{i-1}} & s\theta_{i,C_{i-1}} \\
 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 
\end{bmatrix}
\]

or, \( P_{i,i+1} = \)

(7.11)

The form of the pair matrix in Eq. (7.11) is similar to the regular revolute pair matrix given in Eq. (7.3) except for the displacement terms from the clearance link length. For a general case,
As an example, Fig. 7.3 shows the exaggerated view of an RRRR four-bar mechanism with clearances at all the revolute joints. The transformation matrix product for a four-bar mechanism position without clearances is given by:

\[ P_{i,i+1} = P_{i,C_{i+1}}(q_{i,C_{i+1}}) \begin{bmatrix} -1 & 0 & 0 & -L_{C_{i+1}} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P_{C_{i+1},i+1}(q_{C_{i+1},i+1}) \]  

(7.12)

where

\[ S_i \] is the Shape Matrix for link \( i \)

and \( P_{i,i+1} \) is the Pair Matrix between the links \( i \) and \( i+1 \) at the joint between links \( i \) and \( i+1 \).

Fig. 7.3: Four-Bar Mechanism With Clearances At All Joints

Small clearance links are assumed at the joints which (to begin with) are assumed to be aligned according to the force direction at the joint. For the four-bar mechanism with clearances at the joints, using the model from before, Eq. (7.13) changes to:

\[ I_4 = T_{1,2}T_{2,3}T_{3,4}T_{4,1} = S_1P_{1,2}S_2P_{2,3}S_3P_{3,4}S_4P_{4,1} \]  

(7.13)
\[ I_4 = S_{i_1}P_{i_1}S_{i_2}P_{i_2}S_{i_3}P_{i_3}S_{i_4}P_{i_4}S_{i_5}P_{i_5}S_{i_6}P_{i_6}S_{i_7}P_{i_7}S_{i_8}P_{i_8}S_{i_9}P_{i_9}S_{i_{10}}P_{i_{10}} \]  

As discussed before, \( P_{i_{10}} \) is a function of the driver angle \( \theta_{i_{10}} \). \( P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}, P_{i_5}, P_{i_6}, P_{i_7}, P_{i_8}, P_{i_9} \) are functions of known angles of forces at these joints. \( S_{i_1}, S_{i_2}, S_{i_3}, S_{i_4}, S_{i_5} \) are functions of known clearances at the respective joints. \( P_{i_{11}}, P_{i_{12}}, P_{i_{13}}, P_{i_{14}}, P_{i_{15}} \) are functions of unknown angles to be solved for.

### 7.1.2 Clearance Joint Models

The clearance joint models of some of the common joint types are detailed in this section. The general kinematic and force solutions have already been discussed in the previous section. The revolute, cylindrical and prismatic joints in this section are still assumed to remain planar.

#### 7.1.2.1 Revolute Joint Clearance Model

The model for a revolute joint with clearance has already been discussed, and the equations are summarized here only. Fig. 7.1 gives the detailed view of the joint. The modified pair matrix for the revolute joint is given by Eq. (7.8) and is repeated here again:

\[
P_{i,i+1} = \begin{bmatrix}
-c(\theta_{i,C_{i,i+1}} + \theta_{C_{i,i+1},i+1}) & s(\theta_{i,C_{i,i+1}} + \theta_{C_{i,i+1},i+1}) & 0 & -L_{C_{i,i+1}}c\theta_{i,C_{i,i+1}} \\
-s(\theta_{i,C_{i,i+1}} + \theta_{C_{i,i+1},i+1}) & -c(\theta_{i,C_{i,i+1}} + \theta_{C_{i,i+1},i+1}) & 0 & -L_{C_{i,i+1}}s\theta_{i,C_{i,i+1}} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(7.15)

where

\( \theta_{i,C_{i,i+1}} \) is the angle between the link i and the massless clearance link \( (C_{i,i+1}) \) between links i and i+1

\( \theta_{C_{i,i+1},i+1} \) is the angle between the massless clearance link \( (C_{i,i+1}) \) between links i and i+1, and the link i+1

and \( L_{C_{i,i+1}} = R_{\text{hole}} - R_{\text{pin}} \) is the length of the equivalent clearance link.
7.1.2.2 Cylindrical Joint Clearance Model

Fig. 7.4 shows the details of the cylindrical joint with clearance. The "pin" is fixed to link i+1. \( F_{i,i+1} \) is the force from link i to link i+1 without clearance. \( F_{i+1,i} \) is the force from link i+1 to link i without clearance. \( z_i \) and \( w_{i+1} \) (not shown) are in same direction at the joint. But they are not collinear.

![Fig. 7.4: Details Of A Cylindrical Joint With Clearance](image)

The massless clearance link aligns along the direction of the force at the joint. The pair matrix \( (P_{i,i+1}) \) at a clearance joint connection for a cylindrical joint is given by:

\[
P_{i,i+1} = P_{i,C_{i,i+1}}(\theta_{i,C_{i,i+1}})S_{C_{i,i+1}}(L_{C_{i,i+1}})P_{C_{i,i+1},i+1}(\theta_{C_{i,i+1},i+1},s_{i+1})
\]

(7.16)
where
\[ \theta_{i,C_{i+1}} \] is the angle between the link \( i \) and the massless clearance link \( (C_{i,i+1}) \) between links \( i \) and \( i+1 \)

\[ \theta_{c_{i,i+1}} \] is the angle between the massless clearance link \( (C_{i,i+1}) \) between links \( i \) and \( i+1 \), and the link \( i+1 \)

\[ P_{c_{i,i+1}}(\theta_{i,C_{i,i+1}}) \] is the pair matrix between the link \( i \) and \( (C_{i,i+1}) \) and is a function of known angle \( \theta_{i,C_{i,i+1}} \) only

\[ P_{c_{i,i+1}}(\theta_{c_{i,i+1}},s_{i+1}) \] is the pair matrix between \( (C_{i,i+1}) \) and the link \( i+1 \) and is a function of angle \( \theta_{c_{i,i+1},s_{i+1}} \) and \( s_{i+1} \)

\[ S_{c_{i,i+1}}(L_{c_{i,i+1}}) \] is the shape matrix for the clearance link and is a function of the length of the clearance link, \( L_{c_{i,i+1}} \) only

The length of the clearance link is given by: \( L_{c_{i,i+1}} = R_o - R_i \) where \( R_o \) is the radius of outside cylinder and \( R_i \) is the radius of the inside cylinder.

The general expression for the Shape Matrix in terms of its parameters is given by (Appendix A):

\[
S = \begin{bmatrix}
  c\beta \gamma - \cos \beta \gamma & -s \beta \gamma - \cos \beta \gamma & \sin \gamma & b \sin \gamma + a \gamma \\
  c\beta \gamma + \cos \beta \gamma & -s \beta \gamma + \cos \beta \gamma & -\sin \gamma & -b \cos \gamma + a \gamma \\
  s \alpha \beta & s \alpha \beta & c \alpha & b c \alpha + c \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]  

(7.17)

For the shape matrix for \( C_{i,i+1} \), the following parameters are identified:

\( c = 0, \, \gamma = 180^\circ, \, a = L_{c_{i,i+1}}, \, \alpha = 0, \, b = 0, \, \beta = 0 \)

Hence, the Shape Matrix \( S_{c_{i,i+1}} \) is given by:

\[
S_{c_{i,i+1}} = \begin{bmatrix}
  -1 & 0 & 0 & -L_{c_{i,i+1}} \\
  0 & -1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]  

(7.18)

The Pair Matrix at the joint with clearance is then:
Simplifying Eq. (7.20) we get:

\[ P_{i,i+1} = \begin{bmatrix}
-c(\theta_{i,c_{iunt}} + \theta_{c_{iunt},i+1}) & s(\theta_{i,c_{iunt}} + \theta_{c_{iunt},i+1}) & 0 & -L_{c_{iunt}} c\theta_{i,c_{iunt}} \\
-s(\theta_{i,c_{iunt}} + \theta_{c_{iunt},i+1}) & -c(\theta_{i,c_{iunt}} + \theta_{c_{iunt},i+1}) & 0 & -L_{c_{iunt}} s\theta_{i,c_{iunt}} \\
0 & 0 & 1 & s_{i+1}
\end{bmatrix} \]  

(7.21)

\[ P_{i,i+1} = P_{c_{iunt},i} S_{c_{iunt}} (L_{c_{iunt}}) P_{c_{iunt},i+1}(\theta_{c_{iunt},i+1}, s_{i+1}) \]

(7.19)

\[ P_{i,i+1} = \begin{bmatrix}
c\theta_{i,c_{iunt}} & -s\theta_{i,c_{iunt}} & 0 & 0 \-L_{c_{iunt}} c\theta_{i,c_{iunt}} & -s\theta_{i,c_{iunt},i+1} & 0 & 0 \\
s\theta_{i,c_{iunt}} & c\theta_{i,c_{iunt}} & 0 & 0 \0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \]

7.1.2.3 Prismatic Joint Clearance Model

Figs. 7.5 and 7.6 show the details of a prismatic pair with clearance. \( F_{i,i+1} \) is the force from link \( i \) to link \( i+1 \) without clearance. \( F_{i+1,i} \) is the force from link \( i+1 \) to link \( i \) without clearance. \( z_i \) and \( w_{i+1} \) are in same direction at the joint. But they are not collinear. The massless clearance link aligns along the direction of the force at the joint. The pair matrix \( (P_{i,i+1}) \) at a clearance joint connection for a prismatic joint is given by:

\[ P_{i,i+1} = \begin{bmatrix}
c\theta_{i,c_{iunt}} + \theta_{c_{iunt},i+1} & s\theta_{i,c_{iunt}} + \theta_{c_{iunt},i+1} & 0 & -L_{c_{iunt}} c\theta_{i,c_{iunt}} \\
-s\theta_{i,c_{iunt}} + \theta_{c_{iunt},i+1} & -c\theta_{i,c_{iunt}} + \theta_{c_{iunt},i+1} & 0 & -L_{c_{iunt}} s\theta_{i,c_{iunt}} \\
0 & 0 & 1 & s_{i+1}
\end{bmatrix} \]  

(7.22)

where

\( \theta_{i,c_{iunt}} \) is the angle between the link \( i \) and the massless clearance link \( (C_{i,i+1}) \) between links \( i \) and \( i+1 \)
$P_{i, C_{i+1}}(\theta_{i, C_{i+1}})$ is the pair matrix between the link $i$ and $(C_{i+1})$ and is a function of known angle $\theta_{i, C_{i+1}}$ only.

$P_{C_{i+1}, i+1}(s_{i+1})$ is the pair matrix between $(C_{i+1})$ and the link $i+1$ and is a function of distance $s_{i+1}$.

$S_{C_{i+1}}(L_{C_{i+1}})$ is the shape matrix for the clearance link and is a function of the length of the clearance link, $L_{C_{i+1}}$ only.

Fig. 7.5: Details Of A Prismatic Joint With Clearance
Fig. 7.6 gives the details of the calculation of the length of the equivalent clearance link. \( \alpha \) is the angle of the clearance link from the nominal 45 degrees angular position. From Fig. 7.6, we have

\[
\sin \alpha = \frac{O_{i+1}A}{O_iO_{i+1}} \tag{7.23}
\]

Hence, we get:

\[
O_{i+1}A = L_{c_{i+1}} \sin \alpha \tag{7.24}
\]

In \( \Delta PQR \),

\[
PQ = PR = O_{i+1}A = L_{c_{i+1}} \sin \alpha
\]

and

\[
O_iA = O_iR - AR = O_iR - (AP + PR) = L_{c_{i+1}} \cos \alpha
\]

or,

\[
\frac{1}{\sqrt{2}} (b - a) - L_{c_{i+1}} \sin \alpha = L_{c_{i+1}} \cos \alpha
\]

or,

\[
L_{c_{i+1}} = \frac{(b - a)}{\sqrt{2} (\sin \alpha + \cos \alpha)} \tag{7.25}
\]

![Fig. 7.6: Calculation Of The Length Of Clearance Link In A General Case](image-url)
In ΔPQR, \( PQ = PR = O_{i+1}A = L_{c_{i+1}} \sin \alpha \)

and \( O_iA = O_iR - AR = O_iR - (AP + PR) = L_{c_{i+1}} \cos \alpha \)

or, \( \frac{1}{\sqrt{2}}(b-a) - L_{c_{i+1}} \sin \alpha = L_{c_{i+1}} \cos \alpha \)

or, \( L_{c_{i+1}} = \frac{(b-a)}{\sqrt{2}(\sin \alpha + \cos \alpha)} \) (7.26)

The length of the clearance link is given by the above equation when both the shapes are considered as squares with sides equal to \( a \) and \( b \).

The Shape Matrix \( S_{c_{i+1}} \) is given by:

\[
S_{c_{i+1}} = \begin{bmatrix}
-1 & 0 & 0 & -L_{c_{i+1}} \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (7.27)

The Pair Matrix at the joint with clearance for this case is then:

\[
P_{i,i+1} = P_{i,c_{i+1}} (\theta_{i,c_{i+1}}) S_{c_{i+1}} (L_{c_{i+1}}) P_{c_{i+1},i+1}(s_{i+1})
\] (7.28)

\[
= \begin{bmatrix}
c\theta_{i,c_{i+1}} & -s\theta_{i,c_{i+1}} & 0 & 0 \\
s\theta_{i,c_{i+1}} & c\theta_{i,c_{i+1}} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 & -L_{c_{i+1}} \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

or,

\[
P_{i,i+1} = \begin{bmatrix}
-c\theta_{i,c_{i+1}} & s\theta_{i,c_{i+1}} & 0 & -L_{c_{i+1}} c\theta_{i,c_{i+1}} \\
s\theta_{i,c_{i+1}} & -c\theta_{i,c_{i+1}} & 0 & -L_{c_{i+1}} s\theta_{i,c_{i+1}} \\
0 & 0 & 1 & s_{i+1} \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (7.29)

where

\[
L_{c_{i+1}} = \frac{(b-a)}{\sqrt{2}(\sin \alpha + \cos \alpha)}
\] (7.30)

and \( \alpha \) is the angular deviation from the nominal 45 degrees.
7.1.2.4 Spherical Joint Clearance Model

The development is done for the spherical pair defined by Euler angle transformations of the X-Z-X type (Spherical pair Type II, Appendix B). Similar models can be developed for the other spherical joint types. Fig. 7.7 shows the situation of a spherical pair with clearance. The massless clearance link aligns along the direction of the force at the joint.

The pair matrix \( P_{i,i+1} \) at a clearance joint connection for a spherical joint is given by:

\[
P_{i,i+1} = [P_{i,C_{i+1}}(\theta_i, \phi_i, \psi_i)] S_{C_{i+1}}(L_{C_{i+1}}) [P_{C_{i+1},i+1}(\theta_{i+1}, \phi_{i+1}, \psi_{i+1})]
\]

where \( \theta_i, \phi_i, \psi_i \) and \( \theta_{i+1}, \phi_{i+1}, \psi_{i+1} \) are two sets of Euler angles between the link \( i \) and the massless clearance link \( (C_{i,i+1}) \) between links \( i \) and \( i+1 \), and between \( C_{i,i+1} \) and link \( i+1 \) respectively. Fig. 7.8 shows the details of the joint and angle transformations.

![Fig. 7.7: Details Of A Spherical Joint With Clearance](image)
\( P_{i,C_{i+1}}(\theta_i, \phi_i, \psi_i) \) is the pair matrix between the link \( i \) and \( (C_{i+1}) \) and is a function of the angles \( \theta_i, \phi_i, \psi_i \) only.

\( P_{C_{i+1},i+1}(\theta_{i+1}, \phi_{i+1}, \psi_{i+1}) \) is the pair matrix between \( (C_{i+1}) \) and the link \( i+1 \) and is a function of the angles \( \theta_{i+1}, \phi_{i+1}, \psi_{i+1} \) only.

\( S_{C_{i+1}}(L_{C_{i+1}}) \) is the shape matrix for the clearance link and is a function of the length of the clearance link, \( L_{C_{i+1}} \) only.

![Fig. 7.8: Modified Pair Matrix Definition For A Spherical Joint With Clearance](image)

The Shape Matrix \( S_{C_{i+1}} \) is given by:

\[
S_{C_{i+1}} = \begin{bmatrix}
-1 & 0 & 0 & -L_{C_{i+1}} \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(7.32)

The Pair Matrix at the joint with clearance is then:
The complete pair matrix is a product of a series of transformations. There is a $\theta_i$ rotation of $z_i$ about $x_i$ to define a "floating" axis $w_i'$, followed by a $\phi_i$ rotation of $x_i$ about $w_i'$ into $u_i$. Subsequently, there is a $\psi_i$ rotation of $w_i'$ about $u_i$ into $w_i$. This is at the one end of the clearance link. A similar series of transformations follow at the other end of the clearance link to link $i+1$. There is a $\theta_{i+1}$ rotation of $z_{i+1}$ about $x_{i+1}$ to define a "floating" axis $w_{i+1}'$, followed by a $\phi_{i+1}$ rotation of $x_{i+1}$ about $w_{i+1}'$ into $u_{i+1}$. Subsequently, there is a $\psi_{i+1}$ rotation of $w_{i+1}'$ about $u_{i+1}$ into $w_{i+1}$. The details of the angle notations for these transformations are given in Fig. 7.8.

\[
P_{i,i+1} = \begin{bmatrix}
    c\phi_i & -s\phi_i c\psi_i & s\phi_i s\psi_i & 0 \\
    c\theta_i s\phi_i & -s\theta_i s\psi_i + c\theta_i c\phi_i c\psi_i & -s\theta_i c\psi_i - c\theta_i c\phi_i s\psi_i & 0 \\
    s\theta_i s\phi_i & c\theta_i s\psi_i + s\theta_i c\phi_i c\psi_i & c\theta_i c\psi_i - s\theta_i c\phi_i s\psi_i & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix} \times \begin{bmatrix}
    -1 & 0 & 0 & -L_{C_{i,i+1}} \\
    0 & -1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

Expanding the above matrix product, the following elements are obtained for the pair matrix:

\[
P_{i,i+1}(1,1) = -c\phi_i c\phi_{i+1} + s\phi_i c\psi_i c\theta_{i+1} s\phi_{i+1} + s\phi_i s\psi_i s\theta_{i+1} s\phi_{i+1} \tag{7.35}
\]

\[
P_{i,i+1}(1,2) = c\phi_i s\phi_{i+1} c\psi_{i+1} + s\phi_i c\psi_i (-s\theta_i s\psi_{i+1} + c\theta_i c\phi_{i+1} c\psi_{i+1}) + s\phi_i s\psi_i (c\theta_{i+1} s\psi_{i+1} + s\theta_{i+1} c\phi_{i+1} c\psi_{i+1}) \tag{7.36}
\]

\[
P_{i,i+1}(1,3) = -c\phi_i s\phi_{i+1} s\psi_{i+1} + s\phi_i c\psi_i (-s\theta_{i+1} c\psi_{i+1} - c\theta_{i+1} c\phi_{i+1} s\psi_{i+1}) + s\phi_i s\psi_i (c\theta_{i+1} c\phi_{i+1} s\psi_{i+1} - s\theta_{i+1} c\phi_{i+1} s\psi_{i+1}) \tag{7.37}
\]
\[ P_{i+1}(1,4) = -L_{c_{i+1},s}c_{\phi_i} \] (7.38)

\[ P_{i+1}(2,1) = -c_{\theta_i}s_{\phi_i}c_{\phi_{i+1}} + (s_{\theta_i}s_{\psi_i} - c_{\theta_i}c_{\phi_i}c_{\psi_i})c_{\theta_{i+1},s}c_{\phi_{i+1}} \\
+ (-s_{\theta_i}c_{\psi_i} - c_{\theta_i}c_{\phi_i}c_{\psi_i})s_{\theta_{i+1},s}c_{\phi_{i+1}} \] (7.39)

\[ P_{i+1}(2,2) = c_{\theta_i}s_{\phi_i}c_{\phi_{i+1}}c_{\psi_{i+1}} + (s_{\theta_i}s_{\psi_i} - c_{\theta_i}c_{\phi_i}c_{\psi_i})(-s_{\theta_{i+1},s}c_{\phi_{i+1},c_{\psi_{i+1}}} + c_{\theta_{i+1},c_{\phi_{i+1},c_{\psi_{i+1}}}}) + (-s_{\theta_{i+1}}c_{\psi_{i+1}} - c_{\theta_{i+1}}c_{\phi_{i+1}}c_{\psi_{i+1}})(c_{\theta_{i+1},s}c_{\psi_{i+1}} + s_{\theta_{i+1},c_{\phi_{i+1},c_{\psi_{i+1}}}}) \] (7.40)

\[ P_{i+1}(2,3) = -c_{\theta_i}s_{\phi_i}c_{\phi_{i+1}} + (s_{\theta_i}s_{\psi_i} - c_{\theta_i}c_{\phi_i}c_{\psi_i}) \\
(-s_{\theta_{i+1},c_{\psi_{i+1}}} - c_{\theta_{i+1}}c_{\phi_{i+1},c_{\psi_{i+1}}}) + (-s_{\theta_{i+1}}c_{\psi_{i+1}} - c_{\theta_{i+1}}c_{\phi_{i+1}}c_{\psi_{i+1}}) \\
(c_{\theta_{i+1},c_{\psi_{i+1}}} - s_{\theta_{i+1}}c_{\phi_{i+1},c_{\psi_{i+1}}}) \] (7.41)

\[ P_{i+1}(2,4) = -L_{c_{i+1},s}c_{\theta_i}c_{\phi_i} \] (7.42)

\[ P_{i+1}(3,1) = -s_{\theta_i}s_{\phi_i}c_{\phi_{i+1}} + (-c_{\theta_i}s_{\psi_i} - s_{\theta_i}c_{\phi_i}c_{\psi_i})c_{\theta_{i+1},s}c_{\phi_{i+1}} \\
+ (c_{\theta_i}c_{\psi_i} - s_{\theta_i}c_{\phi_i}c_{\psi_i})s_{\theta_{i+1},s}c_{\phi_{i+1}} \] (7.43)

\[ P_{i+1}(3,2) = s_{\theta_i}s_{\phi_i}c_{\phi_{i+1}}c_{\psi_{i+1}} + (-c_{\theta_i}s_{\psi_i} - s_{\theta_i}c_{\phi_i}c_{\psi_i}) \\
(-s_{\theta_{i+1},c_{\psi_{i+1}}} + c_{\theta_{i+1}}c_{\phi_{i+1},c_{\psi_{i+1}}}) + (c_{\theta_{i+1},c_{\psi_{i+1}}}) + (-s_{\theta_{i+1}}c_{\psi_{i+1}} - c_{\theta_{i+1}}c_{\phi_{i+1}}c_{\psi_{i+1}}) \\
(c_{\theta_{i+1},c_{\psi_{i+1}}} + s_{\theta_{i+1}}c_{\phi_{i+1},c_{\psi_{i+1}}}) \] (7.44)

\[ P_{i+1}(3,3) = -s_{\theta_i}s_{\phi_i}c_{\phi_{i+1}} + (-c_{\theta_i}s_{\psi_i} - s_{\theta_i}c_{\phi_i}c_{\psi_i}) \\
(-s_{\theta_{i+1},c_{\psi_{i+1}}} - c_{\theta_{i+1}}c_{\phi_{i+1},c_{\psi_{i+1}}}) + (c_{\theta_{i+1},c_{\psi_{i+1}}}) \\
(c_{\theta_{i+1},c_{\psi_{i+1}}} + s_{\theta_{i+1}}c_{\phi_{i+1},c_{\psi_{i+1}}}) \] (7.45)

\[ P_{i+1}(3,4) = -L_{c_{i+1},s}c_{\theta_i}s_{\phi_i} \] (7.46)

\[ P_{i+1}(4,1) = 0 \] (7.47)

\[ P_{i+1}(4,2) = 0 \] (7.48)

\[ P_{i+1}(4,3) = 0 \] (7.49)

\[ P_{i+1}(4,4) = 1 \] (7.50)
7.1.3 Clearance Joint Velocity And Acceleration Analysis

Let \( q_i \) be the generalized coordinate representation of the individual joint variables. The completed displacement analysis of a given mechanism produces values of \( q_i \) for the elements of \( Q \) – vector of joint variables for the joints in the mechanism. Let \( F \) be the total number of joint degrees of freedom. The velocity analysis results provide the time rate of change of the elements of \( Q \) symbolized as \( \dot{Q} \).

The position solution of joint variables are obtained from:

\[
T(Q)_{i,1} = I_4
\]  

(7.51)

The velocity formulation for a closed loop mechanism is obtained by differentiating the above equation with respect to time:

\[
\frac{d}{dt}[T(Q)_{i,1}] = \frac{d}{dt}[I_4] = 0_4
\]  

(7.52)

i.e.

\[
\sum_{i=1}^{F} \frac{\partial T(Q)_{i,1}}{\partial q_i} \dot{q}_i = 0_4
\]  

(7.53)

Six independent equations in general in the joint variables can be obtained from the above equation. Input \( \dot{q}_i \)'s are the independent joint degrees of freedom velocities and the dependent \( \dot{q}_i \)'s are the resulting velocities of the remaining joint degrees of freedom. Let,

\[
\frac{\partial T(Q)_{i,1}}{\partial q_i} = \frac{\partial}{\partial q_i} T_{i,1} = \begin{bmatrix}
    t_{q_i}^{1,1} & t_{q_i}^{1,2} & t_{q_i}^{1,3} & t_{q_i}^{1,4} \\
    t_{q_i}^{2,1} & t_{q_i}^{2,2} & t_{q_i}^{2,3} & t_{q_i}^{2,4} \\
    t_{q_i}^{3,1} & t_{q_i}^{3,2} & t_{q_i}^{3,3} & t_{q_i}^{3,4} \\
    0 & 0 & 0 & 0
\end{bmatrix}
\]  

(7.54)

Here \( t_{i,j}^{k} \) is the \((i,j)\)th element in the matrix \( T_{i,1} \) and \( t_{q_i}^{k} = \frac{\partial t_{i,j}^{k}}{\partial q_i} \). The last row of \( \frac{\partial T(Q)_{i,1}}{\partial q_i} \) is always zero. Hence 12 nontrivial equations are obtained from the above. They can be represented in the matrix form as:
Six independent equations are obtained from this matrix. For solving, three equations are taken from the rotation partition and three from the displacement partition. The one used from the rotation partition cannot be all from the same row or column. The actual equations used are the ones as described in Chapter 4. The Jacobian or the resulting matrix of the first derivatives is signified as $J$. The resulting set of linear equations is represented as:

$$J \dot{Q} = 0 \quad (7.56)$$

The solution for the acceleration of the joint parameters follow a similar procedure. Differentiating the velocity equations with respect to time, the following equations are obtained:

$$\sum_{i=1}^{n} \frac{\partial T(Q)_{i1}}{\partial \dot{q}_i} \ddot{q}_i + \sum_{i=1}^{n} \frac{\partial^2 T(Q)_{i1}}{\partial q_i^2} \ddot{q}_i^2 + 2 \sum_{i=1}^{n} \left( \sum_{j=i+1}^{n} \frac{\partial^2 T(Q)_{i1}}{\partial q_i \partial \dot{q}_j} \dot{q}_i \dot{q}_j \right) = 0 \quad (7.57)$$

The first summation is the "tangential" component of the force, the second one is the "normal" component and the third the "coriolis" component of the force. Again, this can be solved for up to six independent variables and the Jacobian formulation is:

$$J \ddot{Q} = b \quad (7.58)$$

where $b$ consists of known terms.

The unknown joint rates are solved from the above in the usual way (as discussed in Chapter 4) by separating the independent and dependent set of equations. Once the joint variable velocities and accelerations are obtained, the values are used to obtain the velocity and acceleration values for the links. These are then used in the dynamic force analysis. The only difference in solving the systems proposed here is in the use of the angle $\theta_{i,i+1}$, the apparent joint angle in the revolute joint instead of the regular joint angle between links $i$ and $i+1$. The rest of the procedure after that for velocity and acceleration analysis is the
same as usual (Chapter 4). The identification of the joint variables for the other joints depend on the joint type and can be easily obtained for a joint.

![Diagram of joint parameter definition](image)

**Fig. 7.9: Definition Of Joint Parameter For Velocity And Acceleration Analyses**

For the revolute joint, once the position solution is obtained with clearances, the angle $\theta_{i,i+1}$ subtended at the instant center $O'_{i,i+1}$ between the links $i$ and $i+1$ is considered as the joint variable for obtaining joint rates for velocity and acceleration analyses (Fig. 7.9). $\theta_{i,i+1}$ is a function of known angles $\theta_{i,C_{i+1}}$ and $\theta_{C_{i+1},i+1}$ as given earlier.

### 7.1.4 Clearance Joint Force Analysis

Fig. 7.10 shows the details of the force analysis for a joint with clearance. In order to sum the forces and moments, the force state at system $u_{i+1}$ at $O_{i+1}$ of link $i+1$ must be replaced by an equivalent force state at point $O_i$. Given a force state at $u_{i+1}$, an equivalent state at $O_i$ on the same link $i+1$ can be found from:

$$ f_{O_i}^{(i+1)} = f_{u_{i+1}} $$

(7.59)
and \[ {\bf m}_{o_i}^{(i+1)} = {\bf m}_{u_{n+1}}^{(i+1)} - r_{o_i/u_{n+1}}^{(i)} \times f_{u_{n+1}}^{(i)} \] (7.60)

The directions of these forces are obvious and are not shown in Fig. 7.10 for the sake of clarity. Here, \( f_{u_{n+1}}^{(i)} \), \( m_{u_{n+1}}^{(i)} \) stand for the force and moments respectively with respect to link 1 (ground) coordinate systems at the origin of coordinate system \( u \) of link \( i+1 \). When a point designation such as \( O_j \) is used instead of the coordinate systems (e.g. \( f_{o_j}^{(i+1)} \), \( m_{o_j}^{(i+1)} \), the force and moments with respect to link 1 (ground) is referred to at that point \( O_i \) defined in link \( i+1 \). From which, the equivalent force system at \( O_i \) on link \( i \) can be obtained as:

\[ f_{o_i}^{(i)} = -f_{u_{n+1}}^{(i)} \] (7.61)

and \[ m_{o_i}^{(i)} = -m_{u_{n+1}}^{(i)} + r_{o_i/u_{n+1}}^{(i)} \times f_{u_{n+1}}^{(i)} \] (7.62)

where \( |r_{o_i/u_{n+1}}| = L_{c_{i+1}} \) is the clearance link length at the joint (7.63)

Summing forces and moments about the center of gravity, \( G_i \), on link \( i \) yields:

\[ f_{u}^{(i)} - f_{u_{n+1}}^{(i)} + \sum f_{ext} = m_{i}^{(i)} A_{G_i} \] (7.64)

and \[ m_{u}^{(i)} - m_{u_{n+1}}^{(i)} + r_{o_i/u_{n+1}}^{(i)} \times f_{u_{n+1}}^{(i)} + r_{u_{i}/G_i}^{(i)} \times f_{u}^{(i)} + r_{o_i/G_i}^{(i)} \times (-f_{u_{n+1}}^{(i)}) \]

\[ + \sum m_{ext} + \sum r_{ext} \times f_{ext} = I_{G_i}^{(i)} \omega_i + \omega_i^{(i)} \times I_{G_i}^{(i)} \omega_i \] (7.65)

Now, the terms \( r_{o_i/u_{n+1}} \times f_{u_{n+1}} \) and \( r_{o_i/G_i} \times (-f_{u_{n+1}}) \) can be combined as follows:

\[ r_{o_i/u_{n+1}} \times f_{u_{n+1}}^{(i)} + r_{o_i/G_i}^{(i)} \times (-f_{u_{n+1}}^{(i)}) = -r_{o_i/G_i}^{(i)} + r_{u_{n+1}/G_i}^{(i)} \times f_{u}^{(i)} = -r_{u_{n+1}/G_i}^{(i)} \times \] (7.66)

Hence, the final form of the force system about \( G_i \) on link \( i \), is given by:

\[ f_{u_{n+1}}^{(i)} + \sum f_{ext} = m_{i}^{(i)} A_{G_i} \] (7.67)

and \[ m_{u}^{(i)} - m_{u_{n+1}}^{(i)} + r_{u_{i}/G_i}^{(i)} \times f_{u}^{(i)} - r_{u_{n+1}/G_i}^{(i)} \times f_{u_{n+1}}^{(i)} + \sum m_{ext} + \sum r_{ext} \times f_{ext} \]

\[ = I_{G_i}^{(i)} \omega_i + \omega_i^{(i)} \times I_{G_i}^{(i)} \omega_i \] (7.68)

where the terms have the same meaning as in Chapter 4.
The unknown joint forces are easily dealt with when they are resolved into the \( u_i \) coordinate system. The forces and moments may be resolved using:
\begin{align*}
\mathbf{f}_{u_i} &= \mathbf{C}_{l,u_i} \mathbf{u}_i \mathbf{f}_{u_i} \\
\mathbf{f}_{a_{i+1}} &= \mathbf{C}_{l,u_{i+1}} \mathbf{u}_{i+1} \mathbf{f}_{a_{i+1}} \\
\mathbf{m}_{u_i} &= \mathbf{C}_{l,u_i} \mathbf{u}_i \mathbf{m}_{u_i} \\
\mathbf{m}_{a_{i+1}} &= \mathbf{C}_{l,a_{i+1}} \mathbf{u}_{i+1} \mathbf{m}_{a_{i+1}}
\end{align*}

(7.69) (7.70) (7.71) (7.72)

where the $C$'s are the change of basis transformation, available from the rotation partitions of transformation matrices developed during the displacement analysis.

The system of force equations then become:

\begin{align*}
\mathbf{C}_{l,u_i} \mathbf{u}_i \mathbf{f}_{u_i} - \mathbf{C}_{l,a_{i+1}} \mathbf{u}_{i+1} \mathbf{f}_{a_{i+1}} + \sum \mathbf{f}_{\text{ext}} &= m_i \mathbf{A}_{g_i} \\
\mathbf{C}_{l,u_i} \mathbf{u}_i \mathbf{m}_{u_i} - \mathbf{C}_{l,a_{i+1}} \mathbf{u}_{i+1} \mathbf{m}_{a_{i+1}} + \sum \mathbf{r}_{\text{ext}} \times \mathbf{f}_{\text{ext}} &= I_i \mathbf{r}_i + I_i \mathbf{r}_i \times \mathbf{\omega}_i
\end{align*}

(7.73) (7.74)

The frictionless force equilibrium yields a set of linear equations which can be solved directly. Dynamic force equilibrium equations similar to Eqs. (7.73) and (7.74) may be written for every link in a mechanism except for the ground link. These can be solved in the matrix form after separating into known and unknown terms.

### 7.1.5 Steps Of The Iterative Clearance Analysis Procedure

The steps of the iterative clearance analysis procedure can be listed as follows:

1. Obtain and store position, velocity, acceleration, and frictionless force solution for the mechanism using procedure outlined in Chapter 4.

2. Obtain joint force direction from current force solution.

3. Setup clearance pair variables and modeling with clearance for each joint using Eq. (7.12) or equivalent for the joint.
4. Assume clearance link along current force direction and solve for position, velocity, acceleration, and new forces at the joint using Eqs. (7.14), (7.56), (7.58), and (7.73), (7.74) or equivalent.

5. Compare force direction from current and previous solution.

6. If the comparison is satisfied within tolerance limits, save and go to step 8.

7. If tolerance limits are not satisfied then go to step 2.

8. Store current kinematic and frictionless force solution as the solution at this position.

9. Check to see if all positions are done. If yes, stop, else, go to step 1.

7.1.6 Flow Chart Of The Iterative Kinematic And Force Analysis Process

As indicated before, the whole process of solution for the position for a mechanism with clearances at the joints is an iterative procedure using position, velocity, acceleration, and dynamic force solution. The flow chart for the process is given here. The details of each of the steps are not described in Fig. 7.11.

7.2 Clearance Analysis Example

A very simple example of a four-bar linkage is shown in Fig. 7.12. In this example, a clearance of 0.002 in. was assumed at joint number 1. Fig. 7.12 shows the layout drawing for the planar four-bar linkage. Pair angle \( \theta_i \) is the input angle and \( \theta_o \) is the output angle in this case. Example output position, velocity, acceleration, and input torque are plotted for reference in the absence of clearance (Figs. 7.13 - 7.16).

When the output crank angle is checked for the effects of clearance at joint 1, the nature of the curve is the same as the ideal. The difference between the output with the clearance link and the ideal link is very small and are plotted in Fig. 7.17 for all positions. The iterative force and position solution is then performed as outlined in this chapter for every position.
Obtain Complete Kinematic Position, Velocity, Acceleration For The Mechanism With No Clearances For This Position

Obtain The Direction Of Forces At The Joints

Setup Clearance Pair Variables At The Joints With Clearances

Solve Position, Velocity, Acceleration And Dynamic Force With The Modified Mechanism Model (Clearance Along The Force Direction)

Save Solution Set

Difference Between Force Direction From Last Set and New Set Within Tolerance?

Yes

Store Final Solution For Mechanism With Clearances

No

All Positions Done?

Yes

End

No

Fig. 7.11: Flow Chart Of The Iterative Kinematic And Force Analysis Procedure For Mechanisms With Clearance Joints
For the example being considered, the iteration steps (the force direction as a representative output) at position number 25 is plotted in Figs. 7.18 and 7.19. It is observed that the iteration converges quite fast in this case. Other examples of the process are given in Chapter 9.
Fig. 7.14: Output Crank Angular Velocity For The Ideal Four-Bar Linkage

Fig. 7.15: Output Crank Angular Acceleration For The Ideal Four-Bar Linkage

Fig. 7.16: Plot Of Input Torque For The Four-Bar Linkage
Fig. 7.17: Variation Of Output Error For The Four-Bar Linkage With Clearance At Joint 1

Fig. 7.18: Force Iterations For RRRR At Joint 1 (Driver) When Driver Is At 125 Degrees
Fig. 7.19: Force Iterations For RRRR At Joint 1 (Driver) When Driver Is At 125 Degrees
CHAPTER 8

PROCEDURE FOR OPTIMAL TOLERANCE AND CLEARANCE ALLOCATION

8.0 Introduction

This chapter gives the outline of the procedure for optimal tolerance and clearance allocation. The main issue involved in the tolerance allocation process is the life cycle cost of an assembly. This includes the manufacturing and assembly cost and the maintenance cost to the end user. The allocation of tolerances affect each of these costs. The tolerances should be assigned in such a way that the overall manufacturing cost as well as the sensitivity of the mechanical error to the individual tolerances are at their minimum. Both of these contribute to the life cycle cost of the assembly. There is no sense in trying to reduce the mechanical error below designer-accepted limits in this process as that is invariably going to increase the cost. On the other hand, the allowable mechanical error can be used as a constraint that should be satisfied in the optimization process.

Because of the above, it was decided to keep the mechanical error at maximum allowable limits (used as a design constraint) and allocate machining tolerances to obtain the minimum manufacturing cost. Considering that, a multicriteria optimization problem with manufacturing cost and error sensitivities as the objectives is set up to be solved for every position of the mechanism. The details of the main stages of the procedure have been discussed previously. The following sections in this chapter give a brief outline of each stage of the procedure along with a flowchart of the procedure at the end.
8.1 Input

The procedure and program developed as a part of this research is mostly automatic once the user defines the required inputs. The inputs are constituted of not only the geometry and kinematic definitions of the mechanism but also input velocity, acceleration and forces. It also needs identification of the designer’s choice of variables to be considered for tolerance allocation. For the clearance analysis and allocation, information about the joints where clearance is to be considered has also to be input. The actual optimization process, needs some more inputs. Each variable needs an upper and lower limit and also the constraint for the output variation has to be defined. The output variation constraint could be the same at all positions of the mechanism or the user could define different values of constraints at different driver positions. This second option makes the optimization process more versatile from the fact that if the designer thinks that some of the positions are to be controlled more precisely than others, he or she can go through an "what-if" scenario and check the optimal tolerance results for selective tighter tolerances on the output. Another very important input is the cost-tolerance values for each variable. Depending on the complexity of a part or link, the user can define different cost tables or lookup charts or functions for each variable. This might incorporate the various manufacturing processes and costs involved in producing individual parts. The user also has to input relative weighting preference factors for each objective function in the optimization process. In some cases, further input might be required for offsetting some of these preferences based on the obtained pareto optimums.

8.2 Generalized Kinematic And Force Analysis

Once all the user input has been obtained, the first thing that is performed is the kinematic and frictionless force analysis for the ideal mechanism when tolerances and clearances are not considered. The results of this are stored to be used in the iterative kinematic analysis for clearance connections. The procedure followed for ideal mechanism kinematic and dynamic force solution is based on Gutkowski (1990). A procedure for solving the displacement, velocity, acceleration, and dynamic force analysis problems for general spatial mechanisms has been used. For the displacement analysis, a two stage solution process is used. An optimization procedure based on the DFP method (Arora, 1989) is used to obtain the solution for the unknowns in the first position. The subsequent
positions are solved using Newton's method using the solutions from the preceding position as the starting values for the approximation. The procedure has proved to be quite robust. The force analysis procedure solves for the linear frictionless situation. The Singular Value Decomposition (SVD) method is applied in the position analysis of mechanisms. For overconstrained systems, the SVD procedure yields a consistent least square solution which is the exact solution. For an underconstrained system of equations, SVD provides the infinite number of solutions possible by obtaining a particular solution and the system's nullspace basis vectors. SVD is also used in the solution of the dynamic force analysis problem. This eliminates the possibility of a numerical problem in the solutions emanating from under or overconstrained systems. The use of Sheth and Uicker's approach for mechanism modeling has been advantageous from the point of view of mechanism and joint pair definitions in the presence of clearances. This has been highlighted in the earlier chapters. The decoupling of the joint pair matrix and the link shape matrix has been found to be very advantageous.

8.3 Clearance Analysis

The kinematic and dynamic force analysis in the presence of clearances has been set up to follow an iterative procedure. This has been discussed with a flowchart in Chapter 7. The underlying assumption used here is that there is no contact loss in the joints and the way the "pins" are aligned in a joint is dependent on the direction of the forces acting at the joint. In other words, the forces at the joints always force the "pins" to align along a particular direction depending on force direction. Initially, the direction of the joint forces in the no clearance case is used to obtain the starting direction for joint force. The kinematics and dynamic forces are then solved using the modified joint model with clearance in question in an iterative fashion. A solution is obtained when subsequent joint force solutions are within a specified tolerance limit. This method has to be followed for every step of the input position when evaluating the sensitivity of the clearance joint in the mechanism. It is to be noted that the usual values of joint clearances are very small and often very close to the optimization tolerance limits used in the research. The results of clearance sensitivities are therefore required to be scrutinized very carefully to make sure that numerical fallacies are not encountered.
8.4 Sensitivity Analysis

As discussed earlier, the error sensitivity objective function and the assembly tolerance constraint both depend on the sensitivity analysis results of the output with respect to each variable. The sensitivity analysis is performed with respect to one optimization variable at a time. The variable is perturbed by a small value (0.005 in. for dimension variables and 0.002 in. for clearance variables in general). Output results for the entire range of motion are obtained by kinematic solution of the mechanism using the perturbed value for the variable. Sensitivity of each variable is estimated along with Sensitivity Factor and Sensitivity Index as given by Eqs. (6.47) to (6.49) for the entire range of motion. These are later used in setting up and solving the optimization problem.

8.5 Formulating The Cost Objective Function

The primary objective of the optimal tolerance allocation in this research is minimization of manufacturing cost. The Cost Objective Function is given by the summation of the costs of producing individual links and joints with the current values of tolerance and clearance variables. This is given by Eq. (6.50). The optimization procedure can read cost vs. tolerance values from user defined lookup tables. The designer might be able to input cost tolerance values based on machining, overhead, and other costs etc. Plant to plant variations within the same company may also be incorporated to arrive at the best possible production alternative. In the absence of any such table, a representative model of the cost tolerance function is required. Eq. (6.51) gives the equation of a typical function that has been used. The cost objective function is dependent on the current values of the variable but has the same value for every position of the mechanism.

8.6 Formulating The Sensitivity Objective Function

The second objective is the minimization of the sensitivity of the output error to individual variables. Output sensitivities to each variable from all the positions have been combined into a Sensitivity Index. The sensitivity objective function is based on the sensitivity index. Eq. (6.56) gives the sensitivity objective function. The sensitivity indices are combined in a fashion so that their effects are felt in a statistical way. The sensitivity
objective function is also dependent on current values of variable and is a combination of the sensitivity at all positions.

8.7 Formulating The Constraint Function

The variation of the output from the ideal is used as a constraint in the optimal allocation of tolerances and clearances in this work. The user is able to define the maximum allowable variation in assembly tolerance that can be accommodated without sacrificing the mechanism performance. The designer also has the flexibility to define different values of allowable assembly tolerances based on the design requirements at different driver positions. The optimization procedure has the flexibility to handle that based on the fact that an independent optimization procedure is performed at each driver position before arriving at a global optimum. The global optimum is obtained which satisfies all the constraints and minimization requirements at all positions.

As discussed in Chapter 5, the assembly tolerance is modeled by using the mean shift method which is a combination of the worst case combination and the statistical method by using an estimate of the mean shift in manufacturing processes. The expression for the assembly tolerance is given by Eq. (6.60). For most of the cases, a value of 0.1 to 0.2 has been used for $m_i$ by default unless the user defines a different value. This can be easily modified to the statistical method by defining an $m_i$ value of 0.

8.8 Objective Weighting Factors

The multicriteria optimization problems are solved by defining a linear substitute problem with the objective functions. The procedure is set up to use the $L_2$-norm minimization of the multiple objectives. The objectives are normalized at all stages for the optimization to work well. The objectives are also assigned individual weight factors as specified in Chapter 6 to add more flexibility to the optimization process. The designer has more control on the optimization results by the way of changes to the objective weight factors. Using these factors, a spectrum of solutions may be obtained, that the designer would be able to look into and make a judicious decision on the relative weights of the objectives and a final optimization solution.
8.9 Flowchart Of The Process

Fig. 8.1 shows the details of the flowchart for the process of optimal tolerance allocation procedure as developed in this research. The process is mostly automatic once the designer defines all the input. But in some cases, user interaction may be required in fine tuning some of the objective weighting factors for the multicriteria optimization process. The main steps of the process are:

1) Get input information
2) Verify and validate input information
3) Perform kinematic analysis and frictionless force analysis for ideal mechanism
4) Store solution from above for later use
5) Setup kinematic and force solution for clearance joints
6) Identify user defined optimization variables
7) Perturb one variable's base value (positive and negative perturbation for dimension variables and positive perturbations for clearance variables)
8) Obtain kinematic solution for step (7) and obtain expected output error sensitivity for variable
9) Repeat steps (7) and (8) for all optimization variables
10) Obtain "Sensitivity Index" for all variables
11) Setup tolerance cost lookup table for each variable
12) Perform multicriteria optimization for minimum manufacturing cost and minimum error sensitivity at one position
13) Repeat step (12) for all positions
14) Store solution if it satisfies all positions
15) Repeat steps (12) to (14) for different sets of weight factors
16) Obtain the best solution from all the sets satisfying constraints at all positions
Fig. 8.1: Flowchart For Optimal Tolerance Allocation
CHAPTER 9

EXAMPLES: OPTIMAL ALLOCATION OF TOLERANCES AND CLEARANCES

9.0 Introduction

The procedure developed as a part of this work has been tested with a variety of mechanisms. In some simple cases, only link length tolerances are considered, and in the later cases both link length tolerances and joint clearances are considered. The optimization schemes considered, reduce the manufacturing cost and output variation sensitivity.

9.1 Cost vs. Tolerance Functions

As has been discussed earlier, the optimization procedure needs a good cost tolerance chart or look-up table over the range of each variable. The procedure developed can accept any user-supplied look-up table or chart as cost representation. However, for the examples in this chapter, a representative cost tolerance function is used for most of the design variables (unless otherwise stated). The equation of the cost-tolerance function is given by:

\[
\text{Cost} = 0.75 + \frac{0.0025}{\text{Tol.}}
\]

(9.1)

where
Cost is the manufacturing cost in representative dollars

Tol. is the tolerance in inches

This function gives a cost of 1 when the tolerance is 0.01 in. and a cost of 5.75 when the tolerance is at 0.0005 in. The plot of this function is shown in Fig. 9.1. This is the cost-tolerance function 1 and is used for most variables. In some examples, where higher cost is expected for a variable, a different function is used. This also demonstrates the option of using different functions for different variables.

![Cost-tolerance function graph](image)

**Fig. 9.1: Typical Cost-Tolerance Functions For Examples**

For the cases, where a variable is associated with a higher cost function, the cost-tolerance function 2 is used:

\[
\text{Cost} = 1.5 + \frac{0.005}{\text{Tol.}}
\]  

(9.2)

Fig. 9.1 also shows a plot of this cost-tolerance function.

### 9.2 Link Length Tolerance Studies

In the set of examples in this section, the optimization of tolerances for minimum cost and output variation are performed for the simple cases when only link length tolerances are
considered. Three examples are executed here to obtain the tolerances in link lengths for minimum manufacturing cost. These are a planar four-bar linkage, a spatial bent shaft universal joint, and a four-bar linkage with spherical joints on the coupler (RSSP).

### 9.2.1 Planar Four-Bar Mechanism

The schematics for the planar four-bar linkage are shown in Fig. 9.2. The link lengths for this mechanism seen earlier, are given by:

\[
\begin{align*}
L_1 &= 6 \text{ in.} & L_2 &= 1 \text{ in.} & L_3 &= 5 \text{ in.} & L_4 &= 4 \text{ in.}
\end{align*}
\]

The motion specifications are:

\[
\begin{align*}
^{1}\omega_2 &= 10 \text{ rad/sec} & ^{1}\alpha_2 &= 0
\end{align*}
\]

Here \(\theta_1\) is the input angle and \(\theta_4\) is the output angle. Fig. 9.3 shows the axes specifications for the pairing element definition. Based on the axes and geometry definitions for the ideal mechanism, the shape matrices for the links are given by:

---

Fig. 9.2: Layout Of The Planar Four-Bar Linkage

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Fig. 9.3: Axes Definitions For The Planar Four-Bar Linkage

\[
S_1 = \begin{bmatrix}
-1 & 0 & 0 & -6 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \quad S_2 = \begin{bmatrix}
-1 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
S_3 = \begin{bmatrix}
-1 & 0 & 0 & -5 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \quad S_4 = \begin{bmatrix}
-1 & 0 & 0 & -4 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Figs. 9.4 - 9.6 show the output position, angular velocity and angular acceleration for the ideal linkage. Link 4 is the output link in this case. The output variation that is used in the optimization process is the variation of the angle of the output link for all positions. The tolerances on link lengths \( L_1, L_2, L_3, \) and \( L_4 \) are the optimization variables.
Fig. 9.4: Output Link Angular Position For Ideal Four-Bar Linkage

Fig. 9.5: Angular Velocity Of Output Link For Ideal Four-Bar Linkage

Fig. 9.6: Angular Acceleration Of Output Link For Ideal Four-Bar Linkage
Figs. 9.7 and 9.8 show the variation of the output from the ideal over the range of motion for positive and negative perturbation respectively in link lengths. The perturbation values used in each case is 0.005 in. The nature of variation for each variable in both the cases are the same, except the signs are opposite. Fig. 9.9 shows the sensitivity of the output link angular positions to the variations in link lengths.

Fig. 9.7: Output Variation For Positive Perturbation Of Four-Bar Link Lengths

Fig. 9.8: Output Variation For Negative Perturbation Of Four-Bar Link Lengths
Two optimization schemes were tried on this example. In the first case, both the objectives of minimum total manufacturing cost and sensitivity of variation were given equal weights. In the second case, the manufacturing cost objective was given higher preference (as might be a logical intent of the designer). In each of the cases, the optimization was performed at each angular position of the input crank. Figs. 9.10 and 9.11 show the minimized cost objective function for each case over the entire range of input crank motion. Similarly, Figs. 9.12 and 9.13 show the minimized sensitivity variation objective function for each case over the entire range of input crank motion. The values of the tolerance variables that
have to be chosen must satisfy all the constraints at all positions the crank needs to go through. With this idea, the whole range of motion as defined by the user and not just a few positions is studied. Since the entire range has to be satisfied, the absolute worst case of the optimized output has to be selected as the optimized output in each case. This guarantees that the tolerance values selected will satisfy all constraints at all user defined positions. Lower limit for each variable was set at 0.001 in. and the maximum allowable variation for the output angle was set at 0.01 radians (0.57 deg., constraint) for each case.

![Graph](image)

**Fig. 9.11:** Variation Of The Four-Bar Linkage Total Manufacturing Cost Objective (Cost Objective Has Higher Weight)

![Graph](image)

**Fig. 9.12:** Variation Of The Sensitivity Of Four-Bar Link Error Objective (Equal Weights Of The Two Objectives)
The optimized values of the tolerances in the first case where both the objectives have same weights are:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Tolerance on L_i</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tolerance on L_1</td>
<td>3.781E-3 in.</td>
</tr>
<tr>
<td>2</td>
<td>Tolerance on L_2</td>
<td>4.539E-2 in.</td>
</tr>
<tr>
<td>3</td>
<td>Tolerance on L_3</td>
<td>1.000E-3 in.</td>
</tr>
<tr>
<td>4</td>
<td>Tolerance on L_4</td>
<td>8.290E-2 in.</td>
</tr>
</tbody>
</table>

Total Manufacturing Cost: $ 6.2467 (representative)

![Fig. 9.13: Variation Of The Sensitivity Of Four-Bar Link Error Objective](image)

The design tolerance specification for the link lengths based on this and practical considerations would then be:

- L_1: 6.000 ± 0.004 in.
- L_2: 1.000 ± 0.045 in.
- L_3: 5.000 ± 0.001 in.
- L_4: 4.000 ± 0.082 in.

The optimized values of the tolerances in the second case when the manufacturing cost objective is given a preference (relative weights of the objectives 0.9 and 0.1 respectively) are as:

Variable 1: Tolerance on L_1: 2.1644E-2 in.
Variable 2: Tolerance on $L_2$: 5.9544E-3 in.
Variable 3: Tolerance on $L_3$: 1.4985E-3 in.
Variable 4: Tolerance on $L_4$: 8.2081E-2 in.
Total Manufacturing Cost: $ 5.2341 (representative)

The design tolerance specification for the link lengths based on this and practical considerations would be:

\[
\begin{align*}
L_1: & \quad 6.000 \pm 0.022 \text{ in.} \\
L_2: & \quad 1.000 \pm 0.006 \text{ in.} \\
L_3: & \quad 5.000 \pm 0.001 \text{ in.} \\
L_4: & \quad 4.000 \pm 0.082 \text{ in.}
\end{align*}
\]

The constraints are satisfied in both the cases, but the design variables have sufficiently looser tolerances in the second case and the representative cost is also less. It is thus observed that putting equal weights on the two objectives puts an unnecessary burden on the design process and results in higher cost. However, this could well be the designer's choice for optimization. It will, thus, have to be the designer's decision as to how much emphasis to put on the life cycle variation objective. But, whatever, the details of the design intent, optimized values for the variables may be obtained accordingly after the user specifies the relative weights interactively.

Looking at the plots for the optimized values of the objective functions, the reason to look at the entire range of motion can be easily justified. It is clear from Figs. 9.10 - 9.13, that the objective function values do not vary much for most of the range, but there are some sudden changes of values. These are due to the complex interactions of the sensitivities in the optimization process. In this example, the sensitivity values of length $L_2$ and $L_4$ reach peaks in between 190 and 230 degrees. These seem to affect the optimization results very much. The effect also seems to be more pronounced when the objectives are given equal importance. The constraints and the second objective function are both highly dependent on the sensitivities and they seem to adversely affect the system at some points where the sensitivities change values rapidly. For a different value of constraint, the effect is not that pronounced. In both the objective functions, effectively the squared values of the sensitivities are used, and hence, their signs do not have much effect on the optimization process as does their values.
9.2.2 Bent Shaft Universal Joint

A bent shaft linkage which is a low cost way of transmitting rotary motion between two non-parallel or skew shafts can be used as a universal joint in some cases. The schematics for the bent shaft linkage (Gutkowski and Kinzel, 1995) are shown in Fig. 9.14. Angular position of the output link 3 is the optimization parameter. The variables that are identified to be optimized are the tolerances on the parameters given by:

\[ L_2 = 1.5 \text{ in.} \quad L_3 = 1 \text{ in.} \quad R_1 = 0.25 \text{ in.} \quad R_2 = 0.25 \text{ in.} \]

Fig. 9.15 shows the axes specifications for the pairing element definition. The input angle is the pair variable angle $\theta_1$ at joint 1 and the output variable is pair angle $\theta_3$ at joint 3. Fig. 9.16 shows the output position for the ideal linkage. Based on the axes and geometry definitions, the shape matrices for the ideal links are given by:

![Diagram of Bent Shaft Universal Joint](image)

**Fig. 9.14: Bent Shaft Universal Joint** (Gutkowski & Kinzel, 1995)
Fig. 9.15: Pairing Element Coordinate System For Bent Shaft Universal Joint
Figs. 9.17 and 9.18 show the variation of the output from the ideal over the range of motion for positive and negative perturbations respectively in the link parameters. The nature of variation for each variable in both the cases are same, except the signs are opposite. Fig. 9.19 shows the sensitivity of the output link angular position to the variations in link parameters.
As before, two optimization schemes were tried on this example. In the first case, both the objectives of minimum total manufacturing cost and sensitivity of variation were given equal weights or preferences. In the second case, the manufacturing cost objective was given higher preference. In each of the cases, the optimization was performed at each angular position of the input crank. Figs. 9.20 and 9.21 show the minimized cost objective function for each case over the entire range of input crank motion. Similarly, Figs. 9.22 and 9.23 show the minimized sensitivity variation objective function for each case over the entire range of input crank motion. Lower limit for each variable was again set at 0.001 in. and the maximum allowable variation for the output angle was set at 0.01 radians (0.57 deg., constraint) for each case.

Fig. 9.18: Output Variation For Variable Negative Perturbation For Bent Shaft

Fig. 9.19: Sensitivity Of The Output To Variables For Bent Shaft
The optimized values of the tolerances in the first case where both objectives have same weights are as:

Variable 1: Tolerance on $L_2$: 3.1592E-2 in.
Variable 3: Tolerance on $R_1$: 1.0000E-3 in.
Variable 4: Tolerance on $R_2$: 1.0000E-3 in.
Total Manufacturing Cost: $8.2893 (representative)

Fig. 9.20: Variation Of The Total Manufacturing Cost Objective For Bent Shaft (Equal Weights Of The Two Objectives)

Fig. 9.21: Variation Of The Total Manufacturing Cost Objective For Bent Shaft (Cost Objective Has Higher Weight)
The design tolerance specification for the link variables based on this and practical considerations would be:

\[
\begin{align*}
L_2 & : \quad 1.500 \pm 0.032 \text{ in.} \\
L_3 & : \quad 1.000 \pm 0.012 \text{ in.} \\
R_1 & : \quad 0.0250 \pm 0.001 \text{ in.} \\
R_2 & : \quad 0.0250 \pm 0.001 \text{ in.}
\end{align*}
\]

**Fig. 9.22: Variation Of The Sensitivity Of Error Objective For Bent Shaft**
*(Equal Weights Of The Two Objectives)*

**Fig. 9.23: Variation Of The Sensitivity Of Error Objective For Bent Shaft**
*(Cost Objective Has Higher Weight)*
The optimized values of the tolerances in the second case when the manufacturing cost objective is given a higher preference (weight factors are 0.9 and 0.1 respectively) are as:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Tolerance on L_2</th>
<th>Tolerance on L_3</th>
<th>Tolerance on R_1</th>
<th>Tolerance on R_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable 1:</td>
<td>8.9039E-3 in.</td>
<td>7.9400E-3 in.</td>
<td>4.6025E-3 in.</td>
<td>4.6025E-3 in.</td>
</tr>
<tr>
<td>Variable 2:</td>
<td>8.9039E-3 in.</td>
<td>7.9400E-3 in.</td>
<td>4.6025E-3 in.</td>
<td>4.6025E-3 in.</td>
</tr>
<tr>
<td>Variable 3:</td>
<td>8.9039E-3 in.</td>
<td>7.9400E-3 in.</td>
<td>4.6025E-3 in.</td>
<td>4.6025E-3 in.</td>
</tr>
<tr>
<td>Variable 4:</td>
<td>8.9039E-3 in.</td>
<td>7.9400E-3 in.</td>
<td>4.6025E-3 in.</td>
<td>4.6025E-3 in.</td>
</tr>
<tr>
<td>Total Manufacturing Cost:</td>
<td>$ 4.6819 (representative)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 9.24: Variation Of Output At Positive And Negative Worst Case Combinations Of Optimized Variables For Bent Shaft

The design tolerance specification for the link lengths based on this and practical considerations would be:

- L_1: 1.500 ± 0.009 in.
- L_2: 1.000 ± 0.008 in.
- L_3: 0.0250 ± 0.005 in.
- L_4: 0.0250 ± 0.005 in.

Fig. 9.24 is an effort to estimate the effect of the tolerances obtained from the second case on the actual design performance. The worst case combination of the tolerances are checked for variation in output angular position. Both the cases are checked when all the link parameters are at their smallest values and when they are at their largest values as specified by the second optimization case. This is not a statistical combination that is being checked, but the worst case combination. The variations are still found to be within
reasonable limits, signifying that the optimized results are good designs from this standpoint. Another interesting aspect of the second optimization scheme in this case is that all variables are tolerated to the thousandths of an inch, but the overall manufacturing cost is lower than the first case.

The jumps in the objective function values observed, are again due to the adverse combination of the sensitivities, resulting in the optimization variables being forced to go to the allowable limits within the constraint range defined.

### 9.2.3 Four-Bar RSSR Linkage

The example linkage in this case is a modification of the RRRR linkage from Example 9.1.1. The joints on link 3 are changed to spherical joints. The variables that are identified to be optimized are the tolerances on the following:

- \( L_1 = 6 \text{ in.} \)
- \( L_2 = 1 \text{ in.} \)
- \( L_3 = 5 \text{ in.} \)
- \( L_4 = 4 \text{ in.} \)

The motion specifications are:

\[ \omega_2 = 10 \text{ rad/sec} \quad \alpha_2 = 0 \]

A schematic drawing of the mechanism is shown in Fig. 9.25. The axes definitions remain the same as the ones for the planar four-bar linkage (Fig. 9.3). \( \theta_i \) and \( \theta_4 \) are the input and output pair variable angles of interest. Based on the axes and geometry definitions for the ideal mechanism, the shape matrices for the links are given by:

\[
S_1 = \begin{bmatrix}
-1 & 0 & 0 & -6 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
S_2 = \begin{bmatrix}
-1 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
S_3 = \begin{bmatrix}
-1 & 0 & 0 & -5 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
S_4 = \begin{bmatrix}
-1 & 0 & 0 & -4 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Fig. 9.25: Schematic Drawing Of The RSSR Linkage

Fig. 9.26 shows a plot of the output crank position for the ideal linkage. Link 4 is the output link in this case. Output angular position of link 4 is chosen as the optimization parameter in this case.

Fig. 9.26: Output Link Angular Position For Ideal RSSR Linkage
The cost of producing the link 3 that has two spherical joints at the ends will be sufficiently higher than the other links. To simulate this and also to show that each variable may be associated with a different cost-tolerance function, the cost function 2 is used for the 3rd link. Figs. 9.27 and 9.28 show the variation of the output from the ideal over the range of motion for positive and negative perturbations respectively in link lengths. The absolute value of the perturbation in each case is 0.005 inches. The nature of variation for each variable in both the cases are the same, except the signs are opposite. Fig. 9.29 shows the sensitivity of the output link angular position to the variations in link lengths. These plots are same as the plots for the RRRR case.

In this example, the manufacturing cost objective was given higher preference (relative weights 0.9 and 0.1 respectively). The optimization was performed at each angular position of the input crank. Fig. 9.30 shows the minimized cost objective function for this case over the entire range of input crank motion. Fig. 9.31 shows the minimized sensitivity variation objective function over the entire range of input crank motion. Lower limit for each variable was again set at 0.001 in. and the maximum allowable variation for the output angle was set at 0.01 radians (constraint).

![Graph showing output variation for positive perturbation of RSSR link lengths](image)

**Fig. 9.27: Output Variation For Positive Perturbation Of RSSR Link Lengths**

The optimized values of the tolerances in this case when the manufacturing cost objective is given a preference are as:
Variable 1: Tolerance on $L_1$ 2.3147E-2 in.
Variable 2: Tolerance on $L_2$ 4.8071E-3 in.
Variable 3: Tolerance on $L_3$ 2.2171E-3 in.
Variable 4: Tolerance on $L_4$ 8.3191E-2 in.
Total Cost: $6.6631$ (representative)

Fig. 9.28: Output Variation For Negative Perturbation Of RSSR Link Lengths

Fig. 9.29: Sensitivity Of The Output To Variations In RSSR Link Lengths
The design tolerance specification for the link lengths based on this and practical considerations would be:

\[ L_1: \quad 6.000 \pm 0.024 \text{ in.} \]
\[ L_2: \quad 1.000 \pm 0.048 \text{ in.} \]
\[ L_3: \quad 5.000 \pm 0.002 \text{ in.} \]
\[ L_4: \quad 4.000 \pm 0.084 \text{ in.} \]
Fig. 9.32: Variation Of Output At Positive & Negative Worst Case Combinations Of Optimized Variables For RSSR Mechanism

Fig. 9.32 shows the worst case combination of the tolerances for variation in output angular position. Both the cases are checked when all the link parameters are at their smallest values and when they are at their largest values as specified by the optimization output. The variations are still found to be within reasonable limits, signifying that the optimized results are good designs from this standpoint.

9.3 Clearance And Tolerance Studies

In the set of examples in this section, the optimization for minimum cost and output variation are performed for the cases when both link length tolerances and joint clearances are considered. Four examples are considered here to obtain the tolerances in link lengths and clearances for minimum manufacturing cost. The examples considered are a swash plate mechanism with a spherical follower, a bent shaft universal joint, a planar four-bar mechanism where all the joint clearances and link length tolerances are considered and a spatial RSSP slider crank mechanism. A representative cost tolerance function as in Section 9.1 is used for most of the design variables (unless otherwise stated).
9.3.1 Swash Plate Mechanism With Spherical Follower

Swash plate mechanisms are found in hydraulic pumps and motors. The schematic drawing for a swash plate mechanism with spherical follower is shown in Fig. 9.33. The displacement of the follower s is the output variable, \( \theta_2 \) the pair variable angle about \( w_2 \) axis is the input variable in this case. The linkage parameters for this example are given by:

\[
L_1 = 1.45 \text{ in.} \quad L_2 = 2.60 \text{ in.} \quad L_3 = 2.35 \text{ in.}
\]

\[
R_1 = 0.0625 \text{ in.}
\]

The motion specifications for the swash plate are:

\[
^1\omega_2 = 6 \text{ rad/sec} \quad ^1\alpha_2 = 0
\]

An external force of 10 lbs upward is applied on link 3.

![Fig. 9.33: Layout Of The Swash Plate Mechanism With Spherical Follower](image-url)
Fig. 9.34: Axes Definitions For The Swash Plate Mechanism

Fig. 9.35: Follower Link Position For Ideal Swash Plate Mechanism

Fig. 9.34 shows the axes specifications for the pairing element definition. Figs. 9.35 - 9.38 show the output position, angular velocity and angular acceleration and input torque.
for the ideal mechanism. Link 3 is the output follower in this case. Based on the axes and geometry definitions, the shape matrices for the ideal links are given by:

\[
S_1 = \begin{bmatrix}
-1 & 0 & 0 & 1.45 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 2.6 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
S_2 = \begin{bmatrix}
0.9659 & 0 & 0.2588 & 0 \\
0 & 1 & 0 & 0 \\
-0.2588 & 0 & -0.9659 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
S_3 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -2.35 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

The variables identified for optimization are the tolerances on link parameters \(L_1, L_2, L_3, R_1\) and clearance \(C_1\) at revolute joint 1. Figs. 9.39 and 9.40 show the variation of the output from the ideal over the range of motion for positive and negative perturbations respectively in link parameters. The nature of variation for each variable in both the cases are the same, except the signs are opposite. Fig. 9.41 shows the variation of the output from the ideal for a 0.002 perturbation of clearance variable at revolute joint 1. Fig. 9.42 shows the sensitivity of the follower position to the variations in swash plate mechanism optimization parameters.

![Follower Velocity For Ideal Swash Plate Mechanism](image)

**Fig. 9.36: Follower Velocity For Ideal Swash Plate Mechanism**
Again, two optimization schemes (one with equal weights of both the objectives of minimum total manufacturing cost and sensitivity of variation and the other with higher preference for the manufacturing cost objective) were tried for this example. Figs. 9.43 and 9.44 show the minimized cost objective function and sensitivity of error for the first case over the entire range of swash plate motion. Similarly, Figs. 9.45 and 9.46 show the same for the second case over the entire range of swash plate motion. Since the entire range of motion has to be satisfied, the absolute worst case of the optimized output has to be selected as the optimized output in each case. Lower limit for the tolerance variables were set at 0.001 in. and the maximum allowable variation for the output position was set
at 0.005 inches (constraint) for each case. The maximum allowable value for clearance variable was set at 0.003 inches.

![Graph](image)

**Fig. 9.39:** Output Variation For Positive Perturbation Of Swash Plate Mechanism Link Lengths And Sphere Radius

![Graph](image)

**Fig. 9.40:** Output Variation For Negative Perturbation Of Swash Plate Mechanism Link Lengths And Sphere Radius

The optimized values of the tolerances in the first case where both objectives have same weights are as:

- **Variable 1:** Tolerance on $L_1$: 2.2980E-2 in.
- **Variable 2:** Tolerance on $L_2$: 1.0000E-3 in.
Variable 3: Tolerance on $L_3$: $1.0000E-3$ in.
Variable 4: Tolerance on $R_1$: $1.0000E-3$ in.
Variable 5: Clearance $C_1$ at Joint 1: $3.0000E-3$ in.

Total Manufacturing Cost: $18.6921$ (representative)

Fig. 9.41: Output Variation For Perturbation Of Joint 1 Revolute Clearance For Swash Plate Mechanism

Fig. 9.42: Sensitivity Of The Output To Variations In Swash Plate Mechanism Parameters

The optimized values of the tolerances in the second case when the manufacturing cost objective is given a preference are as (weight factors 0.9 and 0.1):

Variable 1: Tolerance on $L_1$: $4.065E-3$ in.
Variable 2: Tolerance on L₂: 2.108E-3 in.
Variable 3: Tolerance on L₃: 2.974E-3 in.
Variable 4: Tolerance on Rₙ: 2.927E-3 in.
Variable 5: Clearance C₁ at Joint ₁: 3.000E-3 in.
Total Manufacturing Cost: $11.2727 (representative)

---

**Fig. 9.43:** Variation Of The Swash Plate Mechanism Total Manufacturing Cost Objective (Equal Weights Of The Two Objectives)

---

**Fig. 9.44:** Variation Of The Sensitivity Of Swash Plate Mechanism Error Objective (Equal Weights Of The Two Objectives)
Fig. 9.45: Variation Of The Swash Plate Mechanism Total Manufacturing Cost Objective (Weights Of The Objectives: 0.9 And 0.1)

Fig. 9.46: Variation Of The Sensitivity Of Swash Plate Mechanism Error Objective (Weights Of The Objectives: 0.9 And 0.1)

The design tolerance specification for the link parameters based on the second optimization results and practical considerations would be:

\[
\begin{align*}
L_1 &: 1.450 \pm 0.004 \text{ in.} \\
L_2 &: 2.600 \pm 0.002 \text{ in.} \\
L_3 &: 2.350 \pm 0.003 \text{ in.} \\
R_1 &: 0.625 \pm 0.003 \text{ in.} \\
C_1 &: 0.003 \text{ in.}
\end{align*}
\]
Fig. 9.47 estimates the effect of the tolerance and clearance obtained from these design specifications on the actual performance of the mechanism. The worst case combination of all the variables at their maximum is checked. The difference obtained from the ideal solution for this case is plotted in Fig. 9.47. Even though, this is a worst case combination and not a statistical combination, the resulting variations are still within the constraints (< 0.005 inches) in this case at all positions.

![Fig. 9.47: Variation Of Output At Positive Worst Case Combinations Of Optimized Variables For Swash Plate Mechanism](image)

**9.3.2 Bent Shaft Universal Joint**

The bent shaft linkage from 9.2.2 is used here for tolerance and clearance studies. The schematic diagram for the bent shaft linkage is shown in Fig. 9.14. The variables that are identified to be optimized are the tolerance on link parameters used earlier and the clearances. They are given by:

\[
L_2 = 1.5 \text{ in.} \quad L_3 = 1 \text{ in.} \quad R_1 = 0.25 \text{ in.}
\]

and \( C_1 = \text{Clearance at revolute joint #1} \quad C_3 = \text{Clearance at revolute joint #3} \)

An external torque of 100 in-lb is applied on link 3.
Fig. 9.48: Output Variation For Positive Perturbation Of Bent Shaft Link Lengths And Joint Clearances $C_1$ And $C_2$

Fig. 9.49: Output Variation For Negative Perturbation Of Link Parameters For Bent Shaft

Fig. 9.48 shows the variations in output angular positions for positive perturbations of link parameters $L_2$, $L_3$, $R_1$ and perturbations of clearance values $C_1$ and $C_3$ at joints 1 and 3 respectively. Fig. 9.49 shows the variation of the output from ideal over the range of motion for negative perturbation in the link parameters. The perturbation values used for Fig. 9.48 and Fig. 9.49 are 0.005 in. for link lengths and radius and 0.002 in. for clearances. The nature of variation for each link length or radius variable in both the cases are the same, except the signs are opposite. Fig. 9.50 shows the sensitivity of the output link angular position to the variations in link parameters and joint clearances.
Results for equal weights of the manufacturing cost objective and sensitivity of variation objective, and higher preference for cost objective are given below. In each of the cases, the optimization was performed at each angular position of the input crank. Figs. 9.51 and 9.52 show the minimized cost objective function for each case over the entire range of input crank motion. Similarly, Figs. 9.53 and 9.54 show the minimized sensitivity variation objective function for each case over the entire range of input crank motion. Lower limit for each length and radius variable was again set at 0.001 in. and the maximum allowable variation for the output angle was set at 0.001 radians (constraint) for each case. The lower and upper limits for the clearance variables were set at 0.0005 in. and 0.002 in. respectively.

The optimized values of the tolerances in the first case where both objectives have same weights are as:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Tolerance on L₂</th>
<th>Tolerance on L₃</th>
<th>Tolerance on R₁</th>
<th>Clearance C₁ at Joint 1</th>
<th>Clearance C₃ at Joint 3</th>
<th>Total Manufacturing Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable 1</td>
<td>3.1657E-2 in.</td>
<td>1.1981E-2 in.</td>
<td>1.0000E-3 in.</td>
<td>5.0000E-4 in.</td>
<td>5.0000E-4 in.</td>
<td>$16.5376 (representative)</td>
</tr>
</tbody>
</table>

Fig. 9.50: Sensitivity Of The Output To Link Lengths And Joint Clearances For Bent Shaft
The design tolerance specification for the link variables based on this and practical considerations would be:

- \( L_2 \): 1.500 ± 0.032 in.
- \( L_3 \): 1.000 ± 0.012 in.
- \( R_1 \): 0.0250 ± 0.001 in.
- \( C_1 \): 0.0005 in.
- \( C_3 \): 0.0005 in.
The optimized values of the tolerances in the second case when the manufacturing cost objective is given a preference (weight factors 0.9 and 0.1 respectively) are as:

Variable 1: Tolerance on $L_2$: $9.7417E-3$ in.
Variable 2: Tolerance on $L_3$: $8.5990E-3$ in.
Variable 3: Tolerance on $R_1$: $4.9496E-3$ in.
Variable 4: Clearance $C_1$ at Joint 1: $2.0000E-3$ in.
Variable 5: Clearance $C_3$ at Joint 3: $2.0000E-3$ in.
Total Manufacturing Cost: $7.3020$ (representative)

The design tolerance specification for the link lengths based on this and practical considerations would be:

$L_2$: $1.500 \pm 0.010$ in.
$L_3$: $1.000 \pm 0.009$ in.
$R_1$: $0.0250 \pm 0.005$ in.
$C_1$: $0.002$ in.
$C_3$: $0.002$ in.

The variation from the ideal obtained from the worst case combination of the tolerances and clearances were checked. This is the case when all the tolerances and clearances are at their largest values as specified by the second optimization case. The variations were still found...
to be within constraint limits, signifying that the optimized results are good designs from this standpoint. Another interesting aspect of the results is the fact that when sensitivity objective is given equal weight, the clearance values are driven down to lower values as expected. In fact their optimum values turn out to be the lower limits imposed on them. For the (0.9, 0.1) weights case, the clearance values end up at the largest allowed value to obtain the lowest cost.

Fig. 9.54: Variation Of The Sensitivity Of Error Objective For Bent Shaft Clearance And Tolerance (Weights Of The Two Objectives: 0.9 & 0.1)

9.3.3 Planar Four-Bar Mechanism

The planar four-bar mechanism as analyzed in 9.2.1 is studied again for link length tolerances on all four links and clearances at all four joints. The schematics for the planar four-bar Unkage are shown in Fig. 9.2. Additionally there is a 10 lb-in. torque about the z-axis on link 4.

Figs. 9.7 and 9.8 can be referred for the variation of the output from the ideal over the range of motion for positive and negative perturbations respectively in link lengths. The perturbations used for these variables were 0.005 inches. Fig. 9.9 shows the sensitivity of the output link angular position to the variations in link lengths.
Fig. 9.55 shows the variation of the output from ideal when the joint clearance variables are each perturbed independently with values of 0.002 inches. Fig. 9.56 shows the sensitivity of these variations over the entire range of motion.

Fig. 9.55: Output Variation For Perturbation Of Four-Bar Clearance Parameters C_1 through C_4

Fig. 9.56: Sensitivity Of The Output To Variations In Four-Bar Link Clearance Parameters C_1 Through C_4

In the optimization scheme used, the manufacturing cost objective was given higher preference. Weight factors of 0.88 and 0.12 respectively were used for the cost and sensitivity objective functions. The multicriteria optimization was performed at each
angular position of the input crank. Fig. 9.57 shows the minimized cost objective function for this case over the entire range of input crank motion. Fig. 9.58 shows the minimized sensitivity variation objective function for this case over the entire range of input crank motion. The values of the tolerance variables that have to be chosen must satisfy all the constraints at all positions the crank needs to go through. Lower limit for each tolerance variable was set at 0.001 in. and the maximum allowable variation for the output angle was set at 0.01 radians (constraint) for each case. The clearance variables were allowed a range between 0.0005 in. and 0.003 in.

![Graph showing variation of the four-bar linkage total manufacturing cost objective](image)

**Fig. 9.57: Variation Of The Four-Bar Linkage Total Manufacturing Cost Objective (Objective Weights 0.88 & 0.12 Respectively)**

Values obtained at optimum is (for relative weights at 0.88 and 0.12 respectively):

- **Variable 1**: Tolerance on $L_1$: 9.1799E-3 in.
- **Variable 2**: Tolerance on $L_2$: 8.0609E-3 in.
- **Variable 3**: Tolerance on $L_3$: 7.9851E-3 in.
- **Variable 4**: Tolerance on $L_4$: 1.3317E-2 in.
- **Variable 5**: Clearance $C_1$ at Joint 1: 3.0000E-3 in.
- **Variable 6**: Clearance $C_2$ at Joint 2: 3.0000E-3 in.
- **Variable 7**: Clearance $C_3$ at Joint 3: 3.0000E-3 in.
- **Variable 8**: Clearance $C_4$ at Joint 4: 3.0000E-3 in.
- **Total Manufacturing Cost**: $10.4166 (representative)
Fig. 9.58: Variation Of The Sensitivity Of Four-Bar Link Error Objective (Objective Weights 0.88 & 0.12 Respectively)

From practical considerations the optimized values for the variables are:

- \( L_1: \) 6.000 ± 0.009 in.
- \( L_2: \) 1.000 ± 0.008 in.
- \( L_3: \) 5.000 ± 0.008 in.
- \( L_4: \) 4.000 ± 0.013 in.
- \( C_1: \) 0.003 in.
- \( C_2: \) 0.003 in.
- \( C_3: \) 0.003 in.
- \( C_4: \) 0.003 in.

9.3.4 Spatial Slider Crank Mechanism (RSSP)

A spatial slider crank mechanism (Sandor and Erdman, 1984) with two spherical joints has been used for this example. The schematic drawing for the spatial slider crank mechanism is shown in Fig. 9.59. Angle \( \theta_2 \) is the input variable and the displacement of the slider is the output variable. The axes definitions and the link length parameters are shown in Fig. 9.60. The chosen link length parameters are:

\[
L_1 = 2.000 \quad L_2 = 3.000 \quad L_3 = 10.000 \quad L_4 = 1.000
\]
Based on the axes and geometry definitions, the shape matrices for the ideal links are given by:

\[ S_1 = \begin{bmatrix} 0 & -1 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

Fig. 9.59: Layout Of The Spatial Slider Crank (RSSP) Mechanism (Sandor & Erdman, 1984)
Fig. 9.60: Axes Definitions For Spatial Slider Crank Mechanism

\[
S_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
S_3 = \begin{bmatrix}
-1 & 0 & 0 & -1 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
S_4 = \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Figs. 9.61 - 9.64 show the plots of output slider position, velocity, and acceleration and the input torque required for the ideal spatial slider crank mechanism. These plots were given earlier in Chapter 4, but they are given here again for the sake of completeness. There is an external force of 40 lbs along the z axis of link 4.

Fig. 9.61: Position Of The Output Slider For The Spatial Slider Crank Mechanism

The optimization variables chosen for this example are link length parameters \(L_1, L_2, L_3, L_4\), and clearance variables \(C_1\) (at revolute joint 1), \(C_3\) (at spherical joint 3) and \(C_4\) (at prismatic joint 4). Fig. 9.65 shows the variation of the output from the ideal over the range of motion for perturbations in clearance parameters. The perturbations used for these variables were 0.002 inches. Fig. 9.69 shows the sensitivity of these variations over the entire range of motion.
Fig. 9.62: Velocity Of The Output Slider For The Spatial Slider Crank Mechanism

Fig. 9.63: Acceleration Of The Output Slider For The Spatial Slider Crank Mechanism

Fig. 9.64: Input Torque Required For The Spatial Slider Crank Mechanism
Fig. 9.65: Output Variation For Perturbation Of Revolute Clearance C1, Spherical Clearance C3, And Prismatic Clearance C4 For The Spatial Slider Crank Mechanism

Figs. 9.66 and 9.67 show the variation of the output from ideal when the link parameters are each perturbed independently with positive and negative perturbation values of 0.005 inches. The sensitivity of the variation of the output to the link parameters are shown in Fig. 9.68.

Fig. 9.66: Output Variation For Negative Perturbation Of The Spatial Slider Crank Mechanism Link Lengths (L1 Through L4)
In the optimization scheme used, the manufacturing cost objective was given higher preference. Weight factors of 0.88 and 0.12 respectively were used for the cost and sensitivity objective functions. The multicriteria optimization was performed at each angular position of the input crank. Fig. 9.70 shows the minimized cost objective function for this case over the entire range of input crank motion. Fig. 9.71 shows the minimized sensitivity variation objective function for this case. Lower limit for each tolerance variable was set at 0.001 in. and the maximum allowable variation for the output position was set at 0.005 inches (constraint) for each case. The clearance variables were allowed a range between 0.0005 in. and 0.001 inches. Increased costs were assigned for links 3 and 4.

Fig. 9.67: Output Variation For Positive Perturbation Of The Spatial Slider Crank Mechanism Link Lengths (L₁ Through L₄)

Fig. 9.68: Sensitivity Of The Output To Variations In Spatial Slider Crank Mechanism Length Parameters (L₁ Through L₄)
Fig. 9.69: Sensitivity Of The Output To Variations In Spatial Slider Crank Mechanism Joint Clearance Parameters ($C_1$, $C_3$, And $C_4$)

Fig. 9.70: Variation Of The Total Manufacturing Cost Objective For Spatial Slider Crank Mechanism (Weights Of The Objectives: 0.88 And 0.12)

The results of the optimization are given by:

- Variable 1: Tolerance on $L_1$: 1.355E-3 in.
- Variable 2: Tolerance on $L_2$: 1.557E-3 in.
- Variable 3: Tolerance on $L_3$: 1.706E-3 in.
- Variable 4: Tolerance on $L_4$: 2.209E-3 in.
- Variable 5: Clearance $C_1$ at Joint 1: 1.564E-3 in.
- Variable 6: Clearance $C_3$ at Joint 3: 1.354E-3 in.
Variable 7: Clearance $C_4$ at Joint 4: $1.565 \times 10^{-3}$ in.
Total Manufacturing Cost: $20.436$ (representative)

Fig. 9.71: Variation Of The Sensitivity Of Spatial Slider Crank Mechanism
Error Objective (Weights Of The Objectives: 0.88 And 0.12)

The values and allowable limits for variables and constraints chosen here for this problem seems to be a good combination. The sudden jumps in objective function values are not observed here. This emphasizes the need for the interactive and iterative process for the tolerance optimization to get the right set of input parameters. There is not "a" good set of parameters for all problems.
CHAPTER 10

CONCLUSIONS AND RECOMMENDATIONS

10.1 Conclusions

A procedure has been developed for the optimal allocation of tolerances and clearances in complex assemblies for minimum manufacturing cost and sensitivity to variation keeping the deviations of the output from the ideal within designer specified limits. The procedure works very well for single loop planar and spatial mechanisms including analytically defined surface contact joints. The results of the numerous example cases run with this procedure confirms the viability of using this as a design tool in the early stages of design. The procedure allows the incorporation of manufacturing tolerances and clearances based on actual manufacturing process and cost information. This allows the designer's intent and the manufacturing and process capabilities to be tied together in the development of the product or the assembly. Comparison of process and plant capabilities can also be performed during the optimization process to match the best manufacturing process for a component from the bigger life cycle cost perspective. The optimization procedure, thus, can evaluate alternative manufacturing processes for a particular component or assembly to obtain the best possible manufacturing plan.

The intent of the work was to strive for designs that are robust against variations. The variations that have been considered are the variations arising out of mass production manufacturing processes and also the variations due to wear and tear from the use of the mechanism or assembly. In this sense, the allocation of tolerances are done with the idea of minimizing the overall "life cycle cost" of the assembly. The life cycle cost not only
involves the actual manufacturing cost but also the cost involved in maintaining the design variables within their tolerable values. The second cost is implemented by minimizing the effects of the sensitivities of the individual variables to the output variation. This is based on the fact that over the life cycle of the assembly in use, some of the clearances and individual link tolerances are going to change causing alterations in the output function. The tolerances and clearances are chosen such that the variation in the output for the changes in these variables are also minimized. In this respect, the research has developed a unified procedure for the optimal allocation of tolerances from a life cycle cost point of view.

The examples presented here also show that the multicriteria optimization procedure handles the two conflicting objectives fairly well. Not too many engineering design problems have been solved using the multicriteria optimization techniques, even though the engineering design objectives are conflicting in nature. Use of the minimal $L_1$-norm along with the objective weighting method is quite effective. The interactive way in which the multicriteria optimization is implemented gives an extra latitude in the optimal tolerance allocation process. The designer can vary the importance of the objectives and do a what-if analysis to arrive at the best optimal solution for the particular case in hand. The examples also show how the output is affected by this. For some particular situation, one objective might take precedence over the other and this can be easily handled both in the pre-analysis or a post-analysis stages of design. Sensitivities of the output with respect to the parameters and also the link length and joint clearance parameters is often a very good information for the designer to have in studying the problem of variation of the output. This can be obtained as a part of the output generated in this work. It has been found that this gives a lot of insight on the actual behavior of the mechanism or assembly.

The use of clearances in the optimization process poses a problem not only from the point of view of more design variables, but also from the lack of effective ways to handle the clearances from the kinematics perspective. The equivalent "clearance link" modeling has been proposed for various joint types and they have been tested with example cases. The iterative procedure used in this research to obtain the modified force solutions at the joint with clearance has been found to yield valid, realistic results. The Sheth-Uicker's method of modeling has helped in the development of the procedure for clearance joint analysis. As has been observed from the examples, this procedure has aided in the incorporation of the sensitivities of the clearances in the overall optimization process in a much more accurate and practical way. Clearance analysis procedure has been developed for the
revolute, cylindrical, prismatic and spherical joint types. In the case of the spatial slider crank mechanism (RSSP), for example, clearances in revolute, prismatic, and spherical joints have been considered along with link length tolerance parameters. The procedure can handle complex interactions of variables in situations like this without any problem. The results obtained are very promising. The incorporation of tolerances on analytically defined surface contacts like sphere to plane surface contact, or sphere to cylinder contact etc. have also been studied in this work.

Complex joints and surface contacts and their combinations have been considered for tolerance and clearance allocation examples and they show that the procedure works very well for all of these cases. The equivalent clearance link model has also been found to work very well and has proved to be a versatile method for clearance analysis.

The overall research contributions are summarized in the next section.

10.2 Summary Of Research Contributions

The contributions of the research can be summarized briefly as:

- Development of a unified procedure for incorporating optimum tolerance and clearance for assemblies and mechanisms during the design stages
- Development of a procedure for producing designs with least expected life cycle cost and relatively insensitive to process deviations
- Development of a procedure for incorporating and minimizing cost of manufacturing using alternative manufacturing processes in the optimization stages
- Development of an efficient procedure to obtain the feasible optimum between the conflicting objectives of minimum manufacturing cost and minimum sensitivity of mechanical error using multicriteria optimization methods
- Development of a procedure for the allocation of tolerances and clearances with the output error as a constraint
- Development of an efficient procedure for force analysis along with clearance allocation for different kind of joints as the clearances align based on the force directions
10.3 Recommendations For Future Work

The procedure and the resulting program developed as a part of this research works very well for the general kind of problems as defined in this work. However, there is scope for further work and some of the directions that may be pursued will be discussed here.

It might be proper in certain situations to model the joints with clearance to allow for a spatial "tilt" or misalignment of joint axes. In the case of the revolute joint, for example, the pin might be tilted out of plane. The resulting mechanism and joint configuration will depend on the geometry of the pin and the bearing themselves along with the joint forces as has been considered here. The modeling of the joint pair itself has to be looked into for this case and an efficient procedure need to be developed. New modeling technique has to be developed, one that can incorporate the geometry constraints of the joint itself (like the "journal" and "bearing" possible contact points along with the pin "Shoulders" etc.) in the generalized pair definition. The friction forces at the joints might also be an important part of this kind of modeling. There will thus be a need for the development of the clearance joint models in the presence of friction forces. Even though, the kinematic analysis program used in this work can handle friction forces, the clearance joints have been modeled without friction. This was done with the idea of keeping things simple to get the other modeling and optimization procedures to work in a stable manner. The friction forces can be implemented in the iterative solution of clearance link position and forces as developed in this work.

The development of the procedure for the kinematic solution and optimal allocation of tolerances on general cam contacts has to be looked into in more detail. The representation of surfaces using surface patches has been studied to some extent, but a more detailed study can be undertaken. This is important from two point of views. For some cases of complex surface contact definitions, this might be one of the only way of modeling the contact points. Also, for cases where profile tolerances are allowed on even spherical or cylindrical surfaces, it might be worthwhile to model them as surface patches and look into the optimization problem based on that model instead of the offset surface model of this.
work. There is, thus, a need for studying the use of surface patches specially from optimization standpoint. Kinematic and dynamic force solution of mechanisms when joints are defined as general surface patches might also be a challenge.

In the procedure developed in this research, it is always assumed that the designer already has the individual dimensions of link lengths and other geometry parameters defined from other kinematic design procedure, and is trying to optimize the allocation of tolerances. It would be very useful to have a procedure that can look at the optimization procedure from a broader perspective and have these individual dimensions and their tolerances be part of the optimization variable list. The optimization procedure will then become part of the kinematic design problem and the resulting mechanism can be expected to be very robust. The number of variables required to be considered will probably make the optimization problem really complex. But it might still be worthwhile to look into this as this procedure will really be a unified design tool.
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APPENDICES
DEFINITION OF SHAPE MATRIX

For the coordinate systems (uvw)\textsubscript{i} and (xyz)\textsubscript{i} attached to link i in Fig. A.1, the transformation from system (xyz)\textsubscript{i} to (uvw)\textsubscript{i} can be formulated by considering the matrix product of six simple one-dof transformations. First the common normal to w\textsubscript{i} and z\textsubscript{i} is constructed. This normal intersects w\textsubscript{i} at p and z\textsubscript{i} at q. A vector, t\textsubscript{i} is directed from p to q along the common normal. Starting at system (uvw)\textsubscript{i}, system (xyz)\textsubscript{i} is obtained by translation of c\textsubscript{i} along w\textsubscript{i}, a rotation of \gamma\textsubscript{i} about w\textsubscript{i}, a translation of a\textsubscript{i} along t\textsubscript{i}, a rotation of \alpha\textsubscript{i} about t\textsubscript{i}, a translation of b\textsubscript{i} along z\textsubscript{i} and a rotation of \beta\textsubscript{i} about z\textsubscript{i}. The definition of the dimensions are as follows:

\begin{itemize}
  \item[c\textsubscript{i} = ] the distance from the origin of (uvw)\textsubscript{i}, O\textsubscript{uvw}, to the point of intersection, p of w\textsubscript{i} with \textsubscript{t}. It is positive in the direction of w\textsubscript{i}.
  \item[\gamma\textsubscript{i} = ] the angle required to rotate u\textsubscript{i} about w\textsubscript{i} into t\textsubscript{i}. It is defined positive counterclockwise looking down w\textsubscript{i}.
  \item[a\textsubscript{i} = ] the length of the common normal (i.e., the distance between p and q). It is positive in the direction of t\textsubscript{i}, by definition.
  \item[\alpha\textsubscript{i} = ] the angle required to rotate w\textsubscript{i} about t\textsubscript{i} into z\textsubscript{i}. It is positive counterclockwise looking down t\textsubscript{i}.
  \item[b\textsubscript{i} = ] the distance from the point of intersection, q of t\textsubscript{i} with z\textsubscript{i} to the origin of (xyz)\textsubscript{i}, O\textsubscript{xyz}. It is defined positive in the direction of z\textsubscript{i}.
\end{itemize}
\( \beta_i \) = the angle required to rotate \( t_i \) about \( z_i \) into \( x_i \). It is defined positive counterclockwise looking down \( z_i \).

Fig. A.1: Definition of Shape Matrix
These displacements and rotations can be mathematically described using rigid body translation and rotation matrices. In other words, system (uvw)\textsubscript{i} can be rotated and translated such that it is coincident and parallel to system (xyz)\textsubscript{i}. The combination of these matrices will yield the transformation matrix from system (xyz)\textsubscript{i} to (uvw)\textsubscript{i}. This is the shape matrix S\textsubscript{i}. The shape matrix is given by:

\[ S_i = D_x(c_i)R_z(\gamma_i)D_x(a_i)R_y(\alpha_i)D_z(b_i)R_z(\beta_i) \]  \hspace{1cm} (A.1)

Here, \[ D_x(c_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (A.2)

\[ D_x(a_i) = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (A.3)

\[ D_z(b_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (A.4)

\[ R_z(\gamma_i) = \begin{bmatrix} c\gamma_i & -s\gamma_i & 0 & 0 \\ s\gamma_i & c\gamma_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (A.5)

\[ R_y(\alpha_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (A.6)

\[ R_z(\beta_i) = \begin{bmatrix} c\beta_i & -s\beta_i & 0 & 0 \\ s\beta_i & c\beta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (A.7)

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\[ S_i = \begin{bmatrix} 1 & 0 & 0 & 0 & c\gamma_i & -s\gamma_i & 0 & 0 & 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 & s\gamma_i & c\gamma_i & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c_i & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \]

Expanding the matrices, we get:

\[ S(a_i, \alpha_i, b_i, \beta_i, c_i, \gamma_i) = \begin{bmatrix} c\beta_i c\gamma_i - \alpha_i s\beta_i s\gamma_i & -\beta_i c\gamma_i & -\alpha_i c\beta_i s\gamma_i & \alpha_i s\gamma_i & b_i s\alpha_i s\gamma_i + a_i c\gamma_i \\ c\beta_i s\gamma_i + \alpha_i s\beta_i c\gamma_i & -\beta_i s\gamma_i + \alpha_i c\beta_i c\gamma_i & -\alpha_i c\gamma_i & -b_i s\alpha_i c\gamma_i + a_i s\gamma_i \\ \alpha_i s\beta_i & \alpha_i c\beta_i & c\alpha_i & b_i c\alpha_i + c_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
APPENDIX B

DEFINITION OF PAIR MATRIX

B.1 Revolute Pair

Fig. B.1: Coordinate Systems For Revolute Pair

Pair Variable: \( \theta_i \)

Specifications:
- The origin of \((xyz)_{i-1}\) and \((uvw)_i\) are coincident
- \(z_{i-1}\) and \(w_i\) are collinear and in the same direction
\[ \theta_i \text{ is measured about } z_{i-1} \text{ from } x_{i-1} \text{ to } u_i \text{ positive counterclockwise looking down } z_{i-1} \]

Pair Matrix

\[ P(\theta_i) = \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 & 0 \\ \sin \theta_i & \cos \theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (B.1) \]

B.2 Prismatic Pair

![Coordinate Systems For Revolute Pair](image)

Fig. B.2: Coordinate Systems For Revolute Pair

Pair Variable: \( s_i \)

Specifications:
- \( z_{i-1} \) and \( w_i \) are collinear and in the same direction
- \( x_{i-1} \) and \( u_i \) are parallel and in the same direction
- \( s_i \) is measured along \( z_{i-1} \) from \( x_{i-1} \) to \( u_i \) positive in the direction of positive \( z_{i-1} \)

Pair Matrix
B.3 Helical Pair: Type I

Fig. B.3: Coordinate Systems For Type I Helical Pair

Pair Variable: $\theta_i$

Specifications:

- Behavior described in terms of joint rotation
- $z_{i-1}$ and $w_i$ are collinear and in the same direction
- $\theta_i$ is measured (in radians) about $z_{i-1}$ from $x_{i-1}$ to $u_i$ positive counterclockwise looking down $z_{i-1}$
- $s_i$ (function of $\theta_i$) is distance along $z_{i-1}$ from $x_{i-1}$ to $u_i$ positive in the direction of positive $z_{i-1}$
- Lead of helix is $L$
- Translation of joint, $s_i$, along $z_{i-1}$ is $\frac{L\theta_i}{2\pi}$

Pair Matrix
\[ P(\theta_i) = \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 & 0 \\ \sin \theta_i & \cos \theta_i & 0 & 0 \\ 0 & 0 & 1 & \frac{L \theta_i}{2\pi} \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  

(B.3)

B.4 Helical Pair : Type II

![Fig. B.4: Coordinate Systems For Type II Helical Pair](image)

Pair Variable: \( s_i \)

Specifications:

- Behavior described in terms of joint translation
- \( z_{i-1} \) and \( w_i \) are collinear and in the same direction
- \( s_i \) is measured along \( z_{i-1} \) from \( x_{i-1} \) to \( u_i \) positive in the direction of positive \( z_{i-1} \)
- \( \theta_i \) (function of \( s_i \)) is the angle about \( z_{i-1} \) from \( x_{i-1} \) to \( u_i \) positive counterclockwise looking down \( z_{i-1} \)
- Lead of helix is \( L \)
- Rotation of joint, \( \theta_i \) in radians, about \( z_{i-1} \) is \( \frac{2\pi s_i}{L} \)

Pair Matrix

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\[
P(s_i) = \begin{bmatrix} \cos \frac{2\pi s_i}{L} & -\sin \frac{2\pi s_i}{L} & 0 & 0 \\ \sin \frac{2\pi s_i}{L} & \cos \frac{2\pi s_i}{L} & 0 & 0 \\ 0 & 0 & 1 & s_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

(B.4)

B.5 Cylindrical Pair

Fig. B.5: Coordinate Systems For Cylindrical Pair

Pair Variable: \( \theta_i, s_i \)

Specifications:

- \( z_{i-1} \) and \( w_i \) are collinear and in the same direction
- \( \theta_i \) is measured about \( z_{i-1} \) from \( x_{i-1} \) to \( u_i \) positive counterclockwise looking down \( z_{i-1} \)
- \( s_i \) is measured along \( z_{i-1} \) from \( x_{i-1} \) to \( u_i \) positive in the direction of positive \( z_{i-1} \)

Pair Matrix
\[
P(\theta_i, s_i) = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i & 0 & 0 \\
\sin \theta_i & \cos \theta_i & 0 & 0 \\
0 & 0 & 1 & s_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (B.5)

### B.6 Planar Pair

**Pair Variable:** \(r_i, s_i, \theta_i\)

**Specifications:**

- \((xy)_{i-1}\) and \((uv)_i\) are coplanar
- \(z_{i-1}\) and \(w_i\) are parallel and point in the same direction
- The origin of \((uvw)_i\) is displaced from the origin of \((xyz)_{i-1}\) by \(r_i\) units along \(x_{i-1}\) and \(s_i\) units along \(y_{i-1}\)
- The angle from \(x_{i-1}\) to \(u_i\) is \(\theta_i\) and is measured about \(w_i\) positive counterclockwise looking down \(w_i\)

Fig. B.6: Coordinate Systems For Planar Pair
B.7 Spherical Pair: Type I

Pair Variable: $\theta_i$, $\phi_i$, $\psi_i$

Specifications:

- Euler type transformation matrix
- Transformation is product of three simple rotation matrices: $R_z(\theta_i)R_x(\phi_i)R_y(\psi_i)$
- The origins of $(xyz)_{i-1}$ and $(uvw)_i$ are coincident

$$P(r_i,s_i,\theta_i) = \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 & r_i \cos \theta_i + s_i \sin \theta_i \\ \sin \theta_i & \cos \theta_i & 0 & r_i \sin \theta_i - s_i \cos \theta_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ (B.6)

![Diagram of Coordinate Systems For Type I Spherical Pair](image_url)

Fig. B.7: Coordinate Systems For Type I Spherical Pair

- $R_z(\theta_i)$ rotates $x_{i-1}$ about $z_{i-1}$ to define a "floating" axis $u'_i$
- $R_x(\phi_i)$ rotates $z_{i-1}$ about $u'_i$ into $w_i$
- $R_y(\psi_i)$ rotates $u'_i$ about $w_i$ into $u_i$
• Angles are measured positive counterclockwise looking down the axes about which the rotation is described

Pair Matrix

\[
P(\theta_i, \phi_i, \psi_i) = \begin{bmatrix}
  c\theta_i c\psi_i - s\theta_i c\phi_i s\psi_i & -c\theta_i s\psi_i - s\theta_i c\phi_i c\psi_i & s\theta_i s\phi_i \\
n\theta_i c\psi_i + c\theta_i c\phi_i s\psi_i & -s\theta_i s\psi_i + c\theta_i c\phi_i c\psi_i & -c\theta_i s\phi_i \\
s\phi_i s\psi_i & s\phi_i c\psi_i & c\phi_i \\
0 & 0 & 0
\end{bmatrix}
\]  

(B.7)

B.8 Spherical Pair : Type II

Pair Variable: \( \theta_i, \phi_i, \psi_i \)

Specifications:
• Euler type transformation matrix

![Coordinate Systems For Type II Spherical Pair](image)

Fig. B.8: Coordinate Systems For Type II Spherical Pair

• Transformation is product of three simple rotation matrices: \( R_z(\theta_i)R_y(\phi_i)R_z(\psi_i) \)
• The origins of \((xyz)_{i-1}\) and \((uvw)_i\) are coincident

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• $R_z(\theta_i)$ rotates $z_{i-1}$ about $x_{i-1}$ to define a “floating” axis $w_i'$
• $R_x(\phi_i)$ rotates $x_{i-1}$ about $w_i'$ into $u_i$
• $R_z(\psi_i)$ rotates $w_i'$ about $u_i$ into $w_i$
• Angles are measured positive counterclockwise looking down the axes about which the rotation is described

Pair Matrix

$$P(\theta_i, \phi_i, \psi_i) = \begin{bmatrix} c\phi_i & -s\phi_i c\psi_i & s\phi_i s\psi_i & 0 \\ c\theta_i s\phi_i & -s\theta_i s\psi_i + c\theta_i c\phi_i c\psi_i & -s\theta_i c\psi_i - c\theta_i c\phi_i s\psi_i & 0 \\ s\theta_i s\phi_i & c\theta_i s\psi_i + s\theta_i c\phi_i c\psi_i & c\theta_i c\psi_i - s\theta_i c\phi_i s\psi_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (B.8)$$

B.9 Spherical Pair : Type III

Pair Variable: $\theta_i$, $\phi_i$, $\psi_i$

![Coordinate Systems For Type III Spherical Pair](image)

Fig. B.9: Coordinate Systems For Type III Spherical Pair

Specifications:
• Screw axis type transformation matrix based on rotation $\psi_i$ about an arbitrary axis $A_i$
  through origin
• $\theta_i$ and $\phi_i$ give the orientation of screw axis $A_i$ relative to system $(xyz)_{i-1}$
• With $z_{i-1}$ as the "North Pole", $\theta_i$ and $\phi_i$ give the longitude and latitude, respectively,
  of a point through which $A_i$ is directed from origin
• The origins of $(xyz)_{i-1}$ and $(uvw)_i$ are coincident
• $\theta_i$ is the angle measured about $z_{i-1}$ through which $A_i$ is rotated from the $x_{i-1}z_{i-1}$ plane
• $\phi_i$ is the angle between $z_{i-1}$ and $A_i$
• Angles are measured positive counterclockwise looking down the axes about which the
  rotation is described
• $(a_x, a_y, a_z)$ are the direction cosines of $A_i$, i.e. $a_x = \cos\theta_i \sin\phi_i$, $a_y = \sin\theta_i \sin\phi_i$, $a_z = \cos\phi_i$
• $\cos\psi = \cos\psi$, $\sin\psi = \cos\psi$, $\sin\psi = \vert \sin\psi \vert = 1 - \cos\psi$

Pair Matrix

\[
P(\theta_i, \phi_i, \psi_i) = \begin{bmatrix}
  a_x^2 \psi_i + c\psi_i & a_x a_y \psi_i - a_x s \psi_i & a_x a_z \psi_i + a_y s \psi_i & 0 \\
  a_x a_y \psi_i + a_z s \psi_i & a_y^2 \psi_i + c\psi_i & a_y a_z \psi_i - a_x s \psi_i & 0 \\
  a_x a_z \psi_i - a_y s \psi_i & a_y a_z \psi_i + a_x s \psi_i & a_z^2 \psi_i + c\psi_i & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\] (B.9)

B.10 Spherical Pair : Type IV

Pair Variable: $\theta_i$, $\phi_i$, $\psi_i$

Specifications:

• Transformation matrix formed by rotations about three mutually perpendicular z-axes
  (spherical joint representation of Sheth and Uicker)
• The origins of $(xyz)_{i-1}$ and $(uvw)_i$ are coincident
• Matrix constructed using $R_z(\theta_i)R_x(\pi/2)R_z(\phi_i)R_x(\pi/2)R_z(\psi_i)$
• Constant $R_z(\pi/2)$ matrices ensure the mutual perpendicularity of the z-axes about
  which the joint degrees-of-freedom are defined
• Angles are measured positive counterclockwise looking down the axes about which the rotation is described

Pair Matrix

\[
P(\theta_i, \phi_i, \psi_i) = \begin{bmatrix}
  c\theta_i c\phi_i c\psi_i + s\theta_i s\psi_i & -c\theta_i c\phi_i s\psi_i + s\theta_i c\psi_i & c\theta_i s\phi_i & 0 \\
  s\theta_i c\phi_i c\psi_i - c\theta_i s\psi_i & -s\theta_i c\phi_i s\psi_i - c\theta_i c\psi_i & s\theta_i s\phi_i & 0 \\
  s\phi_i c\psi_i & -s\phi_i s\psi_i & -c\phi_i & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]  
(B.10)

B.11 Spherical Pair : Type V

Pair Variable: \( \theta_i, \phi_i, \psi_i \)

Specifications:

• Transformation matrix formed by rotations about a z-axis, a y-axis, and an x-axis in that order ("Roll, Pitch, and Yaw" transformation)
• The origins of (xyz) \( _j \) and (uvw) \( _i \) are coincident
• Matrix constructed using \( R_z(\theta_i) R_y(\phi_i) R_x(\psi_i) \)
• Angles are measured positive counterclockwise looking down the axes about which the rotation is described

Pair Matrix

\[
P(\theta_i, \phi_i, \psi_i) = \begin{bmatrix}
  c\theta_i c\phi_i & c\theta_i s\phi_i s\psi_i - s\theta_i c\psi_i & c\theta_i s\phi_i c\psi_i + s\theta_i s\psi_i & 0 \\
  s\theta_i c\phi_i & s\theta_i s\phi_i s\psi_i + c\theta_i c\psi_i & s\theta_i s\phi_i c\psi_i - c\theta_i s\psi_i & 0 \\
  -s\phi_i & c\phi_i s\psi_i & c\phi_i c\psi_i & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]  
(B.11)
APPENDIX C

FLOATING COORDINATE SYSTEM AND TRANSFORMATIONS FOR SURFACE CONTACTS

C.1 Spherical Surface

Equation of sphere of radius r in cartesian form is given as:

\[ F(x, y, z) = x^2 + y^2 + z^2 = r^2 \]  \hspace{1cm} (C.1)

Points in the surface defined in parametric form is given by:

\[ P = r(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \]

where, \( 0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \pi \)

Normal for a point on the surface is given by:

\[ n = \nabla F = (2x, 2y, 2z) = 2r(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \]

The unit normal is given by:

\[ \hat{n} = \frac{2r(\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)}{2r\sqrt{(\cos \theta \sin \phi)^2 + (\sin \theta \sin \phi)^2 + (\cos \phi)^2}} = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \]  \hspace{1cm} (C.2)

The unit tangent vector is given by:

\[ \hat{t} = (\cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi) \]  \hspace{1cm} (C.3)
$$z\text{-axis direction cosine} = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$$

$$x\text{-axis direction cosine} = (\cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi)$$

$$y\text{-axis direction cosine} = \hat{n} \times \hat{i} = \begin{vmatrix} i & j & k \\ \cos \theta \sin \phi & \sin \theta \sin \phi & \cos \phi \\ \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi \end{vmatrix} = (-\sin \theta, \cos \theta, 0)$$

The transformation from floating system at a point on the sphere's surface to the sphere's definition system is then given by:

$$T_{d, f} = \begin{bmatrix} \cos \theta \cos \phi & -\sin \theta & \cos \theta \sin \phi & r \cos \theta \sin \phi \\ \sin \theta \cos \phi & \cos \theta & \sin \theta \sin \phi & r \sin \theta \sin \phi \\ -\sin \phi & 0 & \cos \phi & r \cos \phi \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ (C.4)
and \[ T_{r,d} = \begin{bmatrix} \cos \theta \cos \phi & \sin \theta \cos \phi & -\sin \phi & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ \cos \theta \sin \phi & \sin \theta \sin \phi & \cos \phi & -r \\ 0 & 0 & 0 & 1 \end{bmatrix} \] (C.5)

C.2 Cylindrical Surface

Equation of cylindrical surface of radius \( r \) in cartesian form is given as:

\[ F(x,y,z) = x^2 + y^2 = r^2 \] (C.6)

Points in the surface defined in parametric form is given by:

\[ P = (r \cos \theta, r \sin \theta, z) \quad \text{where,} \quad -\infty < z < \infty \]

Fig. C.2: Definition Of Cylindrical Surface

Normal for a point on the surface is given by:

\[ \mathbf{n} = \nabla F = (2x, 2y, 0) = 2(r \cos \theta, r \sin \theta, 0) \]

The unit normal is given by:

\[ \hat{\mathbf{n}} = (\cos \theta, \sin \theta, 0) \] (C.7)
The unit tangent vector is given by:  \[ \hat{t} = (-\sin \theta, \cos \theta, 0) \]  \[ \text{(C.8)} \]

z-axis direction cosine = \((\cos \theta, \sin \theta, 0)\)

x-axis direction cosine = \((-\sin \theta, \cos \theta, 0)\)

y-axis direction cosine = \(\hat{n} \times \hat{t} = (0, 0, 1)\)

The transformation from floating system at a point on the cylinder's surface to the cylinder's definition system is then given by:

\[
T_{d,r} = \begin{bmatrix}
-sin \theta & 0 & cos \theta & rcos \theta \\
0 & 0 & sin \theta & rsin \theta \\
0 & 1 & 0 & z \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[ \text{(C.9)} \]

and

\[
T_{r,d} = \begin{bmatrix}
-sin \theta & cos \theta & 0 & 0 \\
0 & 0 & 1 & -z \\
cos \theta & sin \theta & 0 & -r \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[ \text{(C.10)} \]

C.3 Plane Surface

![Diagram of Plane Surface](image-url)

Fig. C.3: Definition For Plane Surface

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Equation of plane surface in cartesian form is given as:

\[ F(x, y, z) = z = 0 \]  \hspace{1cm} (C.11)

Points in the surface defined in parametric form is given by:

\[ P = (x, y, 0) \quad \text{where,} \quad -\infty < x < \infty, \quad -\infty < y < \infty \]

The unit normal at any point on the surface is given by: \( \hat{n} = (0, 0, 1) \)  \hspace{1cm} (C.12)

The unit tangent vector is given by: \( \hat{t} = (1, 0, 0) \)  \hspace{1cm} (C.13)

z-axis direction cosine = (0,0,1)

x-axis direction cosine = (1,0,0)

y-axis direction cosine = \( \hat{n} \times \hat{t} = (0,1,0) \)

The transformation from the floating system to the definition system is then given by:

\[
T_{d.f} = \begin{bmatrix}
1 & 0 & 0 & x \\
0 & 1 & 0 & y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

and \[
T_{f.d} = \begin{bmatrix}
1 & 0 & 0 & -x \\
0 & 1 & 0 & -y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

(C.14) \hspace{1cm} (C.15)

C.4 Conical Surface

Equation of conical surface of radius \( r \) when \( z = c \) is given as:

\[
F(x, y, z) = \frac{x^2 + y^2}{r^2} - \frac{z^2}{c^2} = 0 
\]

(C.16)

Points in the surface defined in parametric form is given by:
\[ \mathbf{P} = \left( \frac{z}{c}, \frac{r \cos \theta}{c}, \frac{z}{c} \sin \theta, z \right) \quad \text{where}, \quad 0 < z < \infty \]

Normal for a point on the surface is given by:

\[ \mathbf{n} = \nabla \mathbf{F} = \left( \frac{2x}{r^2}, \frac{2y}{r^2}, -\frac{2z}{c^2} \right) = \frac{2z}{c} \left( \frac{\cos \theta}{r}, \frac{\sin \theta}{r}, -\frac{1}{c} \right) \]

\[ \text{Fig. C.4: Definition Of Conical Surface} \]

The unit normal is given by:

\[ \hat{n} = \left( \frac{c \cos \theta}{\sqrt{c^2 + r^2}}, \frac{c \sin \theta}{\sqrt{c^2 + r^2}}, -\frac{1}{\sqrt{c^2 + r^2}} \right) \]

\[ \text{(C.17)} \]

The unit tangent vector is given by:

\[ \hat{t} = (-\sin \theta, \cos \theta, 0) \]

\[ \text{(C.18)} \]
z-axis direction cosine = \left( \frac{c \cos \theta}{\sqrt{c^2 + r^2}}, \frac{c \sin \theta}{\sqrt{c^2 + r^2}}, -\frac{r}{\sqrt{c^2 + r^2}} \right)

x-axis direction cosine = (-\sin \theta, \cos \theta, 0)

y-axis direction cosine = \hat{n} \times \hat{t} = \left( \frac{r \cos \theta}{\sqrt{c^2 + r^2}}, \frac{r \sin \theta}{\sqrt{c^2 + r^2}}, -\frac{c}{\sqrt{c^2 + r^2}} \right)

The transformation from floating system at a point on the cone's surface to the cone's definition system is then given by:

\[
T_{d,f} = \begin{bmatrix}
-\sin \theta & \frac{r \cos \theta}{\sqrt{c^2 + r^2}} & \frac{c \cos \theta}{\sqrt{c^2 + r^2}} & \frac{r z \cos \theta}{c} \\
\frac{\cos \theta}{\sqrt{c^2 + r^2}} & \frac{r \sin \theta}{\sqrt{c^2 + r^2}} & \frac{c \sin \theta}{\sqrt{c^2 + r^2}} & \frac{r z \sin \theta}{c} \\
0 & 0 & 1 & 0
\end{bmatrix}
\] (C.19)

and

\[
T_{f,d} = \begin{bmatrix}
-\sin \theta & \frac{\cos \theta}{\sqrt{c^2 + r^2}} & 0 & 0 \\
\frac{r \cos \theta}{\sqrt{c^2 + r^2}} & \frac{r \sin \theta}{\sqrt{c^2 + r^2}} & \frac{c}{\sqrt{c^2 + r^2}} & -\frac{z \sqrt{c^2 + r^2}}{c} \\
\frac{c \cos \theta}{\sqrt{c^2 + r^2}} & \frac{c \sin \theta}{\sqrt{c^2 + r^2}} & 0 & 1
\end{bmatrix}
\] (C.20)

C.5 Paraboloid Surface

Equation of paraboloid surface of "radius" r when z = c is given as:

\[
F(x, y, z) = \frac{x^2 + y^2}{r^2} - \frac{z}{c} = 0
\] (C.21)

Points in the surface defined in parametric form is given by:

\[
P = \left( \frac{z c}{r}, \frac{z c \cos \theta}{r}, \frac{z c \sin \theta}{r} \right)
\]

where, \( 0 < z < \infty \)
Normal for a point on the surface is given by:

$$\mathbf{n} = \nabla \mathbf{F} = \left( \frac{2x}{r^2}, \frac{2y}{r^2}, -\frac{1}{c} \right) = \left\{ \frac{z}{c} \left( \frac{2 \cos \theta}{r} \right), \frac{z}{c} \left( \frac{2 \sin \theta}{r} \right), -\frac{1}{c} \right\}$$

The unit normal is given by:

$$\mathbf{n} = \left( \frac{2 \sqrt{zc} \cos \theta}{\sqrt{4zc + r^2}}, \frac{2 \sqrt{zc} \sin \theta}{\sqrt{4zc + r^2}}, -\frac{r}{\sqrt{4zc + r^2}} \right) \quad (C.22)$$

Fig. C.5: Paraboloid Surface
The unit tangent vector is given by: 
\[ \hat{t} = (-\sin \theta, \cos \theta, 0) \]  
(C.23)

\[ \text{z-axis direction cosine} = \left(\frac{2\sqrt{zc} \cos \theta}{\sqrt{4zc + r^2}}, \frac{2\sqrt{zc} \sin \theta}{\sqrt{4zc + r^2}}, -\frac{r}{\sqrt{4zc + r^2}}\right) \]

\[ \text{x-axis direction cosine} = (-\sin \theta, \cos \theta, 0) \]

\[ \text{y-axis direction cosine} = \hat{n} \times \hat{t} = \left(\frac{r \cos \theta}{\sqrt{4zc + r^2}}, \frac{r \sin \theta}{\sqrt{4zc + r^2}}, \frac{2\sqrt{zc}}{\sqrt{4zc + r^2}}\right) \]

The transformation from floating system at a point on the paraboloid's surface to the paraboloid's definition system is then given by:

\[ T_{d,f} = \begin{bmatrix} -\sin \theta & \frac{r \cos \theta}{\sqrt{4zc + r^2}} & -\frac{z \cos \theta}{\sqrt{c}} \\ \cos \theta & \frac{r \sin \theta}{\sqrt{4zc + r^2}} & \frac{z \sin \theta}{\sqrt{c}} \\ 0 & \frac{2\sqrt{zc}}{\sqrt{4zc + r^2}} & z \end{bmatrix} \]  
(C.24)

\[ T_{r,d} = \begin{bmatrix} -\sin \theta & \frac{\cos \theta}{\sqrt{4zc + r^2}} & 0 \frac{0}{\sqrt{c}} \\ \frac{r \cos \theta}{\sqrt{4zc + r^2}} & \frac{2\sqrt{zc}}{\sqrt{4zc + r^2}} & 0 \frac{-z}{\sqrt{4zc + r^2}} \\ \frac{0}{\sqrt{4zc + r^2}} & \frac{0}{\sqrt{4zc + r^2}} & \frac{-z \sqrt{c}}{\sqrt{4zc + r^2}} \end{bmatrix} \]  
(C.25)
APPENDIX D

FRICTIONLESS JOINT FORCE CONSTRAINTS

D.1 Revolute Joint

\[ m^z = 0 \]  \hspace{1cm} (D.1)

D.2 Prismatic Joint

\[ f^z = 0 \]  \hspace{1cm} (D.2)

D.3 Helical Joint

\[ f^z - \frac{m^z}{r_i \tan \alpha} = 0 \]  \hspace{1cm} (D.3)

where \( \alpha \) is the helix angle and \( r_i \) is the pitch radius of the thread.

D.4 Cylindrical Joint

\[ f^z = 0 \]  \hspace{1cm} (D.4)

\[ m^z = 0 \]  \hspace{1cm} (D.5)
D.5 Planar Joint

\[ f^x = 0 \] (D.6)

\[ f^y = 0 \] (D.7)

\[ m^z = 0 \] (D.8)

D.6 Spherical Joint

\[ m^t = 0 \] (D.9)

\[ m^l = 0 \] (D.10)

\[ m^r = 0 \] (D.11)

D.7 Surface Contact Joint

\[ t^{1,1}_{B,u} f^x + t^{1,2}_{B,u} f^y + t^{1,3}_{B,u} f^z = 0 \] (D.12)

\[ t^{2,1}_{B,u} f^x + t^{2,2}_{B,u} f^y + t^{2,3}_{B,u} f^z = 0 \] (D.13)

\[ m^t - r^y f^z + r^z f^y = 0 \] (D.14)

\[ m^l - r^x f^z + r^z f^x = 0 \] (D.15)

\[ m^r - r^x f^y + r^y f^x = 0 \] (D.16)

where \( t_{B,u}^{ij} \) are the \( ij \)th elements of \( T_{B,u} \) and \( (r^x, r^y, r^z) \) represent the displacement partition elements of \( T_{B,u}^{-1} \).
APPENDIX E

GENERALIZED MACHINING COST EQUATIONS

The following data is adapted from Machining Data Handbook by Machinability Data Center (Metcut Research Associates, Inc., Cincinnati)

For turning and milling operations, the cost of machining one workpiece ($/workpiece), C is given by:

\[ C = M \times (FT + RTT + CIT + TRT) + F2 \times (TDC + TSC + RRC + IC + GWC + TPC) \]

where,

- \( C \) = cost in $ per workpiece
- \( M \) = labor + overhead cost per min.
- \( FT \) = feeding time
- \( RTT \) = rapid traverse time
- \( CIT \) = cutter index time
- \( TRT \) = tool replacement time
- \( F2 \) = factor for other costs
- \( TDC \) = tool depreciation cost
- \( TSC \) = tool sharpening cost
- \( RRC \) = rebrazing or blade reset cost
- \( IC \) = insert or blade cost
- \( GWC \) = grinding wheel cost
- \( TPC \) = tool presetting cost

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For drilling and reaming the cost $C$ is given by:

$$C = M \times (FT + RTT + CIT + TRT) + F2 \times (TDC + TSC + TPC)$$

where the symbols have the same meaning.

Each of the variables or factors are dependent on a variety of parameters. The equations can be obtained by using the correct factors. The terms are given in Table E.1.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Turning</th>
<th>Milling</th>
<th>Drilling/Reaming</th>
</tr>
</thead>
<tbody>
<tr>
<td>FT</td>
<td>$D(L + e)$</td>
<td>$D(L + e)$</td>
<td>$D(L + e)$</td>
</tr>
<tr>
<td></td>
<td>$3.82f_v$</td>
<td>$3.82Zf_v$</td>
<td>$3.82f_v$</td>
</tr>
<tr>
<td>RTT</td>
<td>$\frac{R}{r}$</td>
<td>$\frac{R}{r}$</td>
<td>$\frac{R}{r}$</td>
</tr>
<tr>
<td>CIT</td>
<td>$t_i$</td>
<td>$t_i$</td>
<td>$t_i$</td>
</tr>
<tr>
<td>TRT</td>
<td>$\frac{DLt_d}{3.82f_vT}$</td>
<td>$\frac{Lt_d}{ZT_i}$</td>
<td>$\frac{Lt_d}{T_i}$</td>
</tr>
<tr>
<td>F2</td>
<td>$\frac{DL}{3.82f_vT}$</td>
<td>$\frac{L}{ZT_i}$</td>
<td>$\frac{L}{T_i}$</td>
</tr>
<tr>
<td>TDC</td>
<td>$\frac{C_p}{(k_i + 1)}$</td>
<td>$\frac{C_p}{(k_i + 1)}$</td>
<td>$\frac{C_p}{(k_i + 1)}$</td>
</tr>
<tr>
<td>TSC</td>
<td>$Gt_s$</td>
<td>$Gt_s$</td>
<td>$Gt_s$</td>
</tr>
<tr>
<td>RRC</td>
<td>$\frac{Gt_b}{k_2}$</td>
<td>$Gt_b$</td>
<td>$Gt_b$</td>
</tr>
<tr>
<td>IC</td>
<td>$\frac{C_s}{k_3}$</td>
<td>$\frac{C_s}{k_3}$</td>
<td>$\frac{C_s}{k_3}$</td>
</tr>
<tr>
<td>GWC</td>
<td>$C_w$</td>
<td>$C_w$</td>
<td>$C_w$</td>
</tr>
<tr>
<td>TPC</td>
<td>$Gt_p$</td>
<td>$Gt_p$</td>
<td>$Gt_p$</td>
</tr>
</tbody>
</table>

Table E.1: Factors For Estimating Machining Cost
Turning Operations:

\[ C = M \left[ \frac{D(L + e)}{3.82f_v r} + R + t_i + \frac{DLt_d}{3.82f_v T} \right] + \frac{DL}{3.82f_v r T} \left[ \frac{C_p}{(k_1 + 1)} + Gt_i + \frac{Gt_b}{k_2} + \frac{C_c}{k_3} + C_w + Gt_p \right] \]

Milling Operations:

\[ C = M \left[ \frac{D(L + e)}{3.82Zf_v r} + R + t_i + \frac{Lt_d}{ZT_i} \right] + \frac{L}{ZT_i} \left[ \frac{C_p}{(k_1 + 1)} + Gt_i + \frac{Gt_b}{k_2} + \frac{C_c}{k_3} + C_w + Gt_p \right] \]

Drilling and Reaming Operations:

\[ C = M \left[ \frac{D(L + e)}{3.82f_v r} + R + t_i + \frac{Lt_d}{T_i} \right] + \frac{L}{T_i} \left[ \frac{C_p}{(k_1 + 1)} + Gt_i + Gt_p \right] \]

where,

- \( C \) = cost for machining one workpiece; $/workpiece
- \( C_i \) = cost of each insert or inserted blade; $/blade
- \( C_p \) = purchase cost of tool or cutter; $/cutter
- \( C_w \) = cost of grinding wheel for resharpening tool or cutter; $/cutter
- \( D \) = diameter of work in turning, of tool in milling, drilling, reaming, tapping; in.
- \( e \) = extra travel at feed rate including approach, overtravel and all positioning moves; in.
- \( f_v \) = feed per revolution; in./rev.
- \( f_t \) = feed per tooth; in./tooth
- \( G \) = labor + overhead in tool reconditioning department
- \( k_1 \) = number of times tool is resharpened before being discarded
- \( k_2 \) = number of times lathe tool or milling cutter is resharpened before inserts or blades are rebrazed or reset
- \( k_3 \) = number of times inserts or tools are indexed before they are discarded
- \( L \) = length of workpiece in turning and milling or sum length of all holes of same diameter in drilling, reaming etc.; in.
- \( M \) = labor + overhead cost on lathe, milling machine or drilling machine; $/min.
- \( r \) = rapid traverse rate; in./min.
- \( R \) = total rapid traverse distance for a tool or cutter on one part; in.
- \( t_i \) = time to rebraze lathe tool or cutter teeth or reset blades; min.
- \( t_f \) = time to index from one type cutter to another between operations; min.
\( t_p \) = time to preset tools away from machine (in toolroom); min.
\( t_s \) = time to resharpen lathe tool, milling cutter, drill, reamer, etc.; min./tool
\( T \) = tool life measured in minutes to dull a lathe tool; min.
\( T_i \) = tool life measured in inches travel of work or tool to dull a drill, reamer or one milling cutter tooth; in.
\( v \) = cutting speed; ft./min.
\( Z \) = number of teeth in milling cutter
APPENDIX F

DEVELOPMENT OF ASSEMBLY TOLERANCE EQUATIONS

For a system dependent on a combination of individual component variables, the expected values of the system performance and the variance of the system may be obtained from the expected values and variances of the component variables. The method used to obtain this is called the generation of system moments, or statistical error propagation or the delta method.

Let the relationship between system performance $z$ and the component variables $x_1, x_2, ..., x_n$ be given by the function $z = f(x_1, x_2, ..., x_n)$. Let $E(x_i)$ be the mean or expected value and $\mu_k(x_i)$ be the $k$th central moment for the $i$th component variable. Similarly $E(z)$ and $\mu_k(z)$ denote the expected value and the $k$th moment about the mean for system performance. The method of obtaining $E(z)$ and $\mu_k(z)$ consists of expanding $z = f(x_1, x_2, ..., x_n)$ in a multivariable Taylor series about $[E(x_1), E(x_2), ..., E(x_n)]$. Assume that the component variables are uncorrelated.

Let

- $E(x_i) = \text{expected value for the } i\text{th component variable (}i = 1, 2, ..., n\text{)}$
- $E[x_i - E(x_i)]^r = \text{rth moment about mean for } i\text{th component variable (}i = 1, 2, ..., n; r = 2, 3, 4\text{)}$
- $E(z) = \text{expected value for system performance}$
- $E[z - E(z)]^r = \text{rth moment about mean for system performance (}r = 2, 3, 4\text{)}$

If the component variables are not correlated, then we can write (Hahn & Shapiro, 1967)

\[
E\left[\left(x_i - E(x_i)\right)\left(x_j - E(x_j)\right)\right] = E\left[x_i - E(x_i)\right] E\left[x_j - E(x_j)\right] \quad i \neq j \quad (F.1)
\]

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Expanding the output function, \( z = f(x_1, x_2, \ldots, x_n) \), we get:

\[
    f(x_1, x_2, \ldots, x_n) = f[E(x_1), E(x_2), \ldots, E(x_n)] + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}[x_i - E(x_i)] + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}[x_i - E(x_i)]^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}[x_i - E(x_i)][x_j - E(x_j)]
\]  

(F.2)

where all derivatives are evaluated at their expected values. Taking expected values of both sides of the above equation, we get:

\[
    E[f(x_1, x_2, \ldots, x_n)] = E[f[E(x_1), E(x_2), \ldots, E(x_n)]] + E\left\{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}[x_i - E(x_i)]\right\} + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}[x_i - E(x_i)]^2 + E\left\{\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}[x_i - E(x_i)][x_j - E(x_j)]\right\}
\]  

(F.3)

The following properties of the expectation operator are known:

\[
    E(y_1 + y_2 + \ldots + y_n) = E(y_1) + E(y_2) + \ldots + E(y_n)
\]

\[
    E(c) = c \quad E(cx) = cE(x) \text{ for a constant } c \text{ and random variable } x.
\]

Hence

\[
    E[f[E(x_1), E(x_2), \ldots, E(x_n)]] = f[E(x_1), E(x_2), \ldots, E(x_n)]
\]

and

\[
    E\left\{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}[x_i - E(x_i)]\right\} = \sum_{i=1}^{n} E\left\{\frac{\partial f}{\partial x_i}[x_i - E(x_i)]\right\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}E[x_i - E(x_i)] = 0 \quad (F.4)
\]

and

\[
    E\left\{\frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}[x_i - E(x_i)]^2\right\} = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}E[x_i - E(x_i)]^2 = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \text{Var}(x_i) \quad (F.5)
\]

since by definition:

\[
    E[x_i - E(x_i)]^2 = \text{Var}(x_i)
\]

Also,

\[
    E\left\{\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}[x_i - E(x_i)][x_j - E(x_j)]\right\} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}E[x_i - E(x_i)][x_j - E(x_j)] = 0 \quad (F.6)
\]

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since \( E(x_i - E(x_i)) = E(x_i - E(x_i))E(x_j - E(x_j)) = 0 \) \( \text{(F.7)} \)

Thus, we get:

\[
E(z) = f[E(x_1), E(x_2), ..., E(x_n)] + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \text{Var}(x_i) \quad \text{(F.8)}
\]

Now, \( \text{Var}(z) = E(z^2) - [E(z)]^2 \)

where \( \text{Var}(z) \) is the variance of the variable \( z \).

Substituting, \( z = f(x_1, x_2, ..., x_n) \) yields:

\[
\text{Var}[f(x_1, x_2, ..., x_n)] = E[f(x_1, x_2, ..., x_n)^2] - \{E[f(x_1, x_2, ..., x_n)]\}^2 \quad \text{(F.9)}
\]

To obtain an approximation to the first term of the right hand side of the above equation, we take expected values on a term by term basis and retain terms up to second order. This gives:

\[
E[f(x_1, x_2, ..., x_n)^2] = \{E[f(x_1, x_2, ..., x_n)]\}^2 + \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right)^2 E[x_i - E(x_i)]^2
\]

\[
+ f[E(x_1), E(x_2), ..., E(x_n)] \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} E[x_i - E(x_i)]^2 \quad \text{(F.10)}
\]

And,

\[
\{E[f(x_1, x_2, ..., x_n)]\}^2 = \left\{f[E(x_1), E(x_2), ..., E(x_n)] + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \text{Var}(x_i)\right\}^2
\]

\[
\{f[E(x_1), E(x_2), ..., E(x_n)]\}^2 + f[E(x_1), E(x_2), ..., E(x_n)] \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} E[x_i - E(x_i)]^2 \quad \text{(F.11)}
\]

if higher order terms are neglected.

Substituting the above expressions for the right hand sides we get:

\[
\text{Var}[f(x_1, x_2, ..., x_n)] = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right)^2 E[x_i - E(x_i)]^2
\]
i.e., \[ \text{Var}(z) = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right)^2 \text{Var}(x_i) \] (F.12)

which is the same as:

\[ T_{\text{asy}} = \left[ \sum \left( \frac{\partial f}{\partial x_i} \right)^2 T_i^2 \right]^{1/2} \] (F.13)
APPENDIX G

DESCRIPTION OF SUBROUTINES

KINFOR.F
Program for kinematic force analysis for spatial mechanisms.

SUBROUTINE FORCEI ( ISAMPL )
This subroutine performs the ideal (no friction) force analysis.

SUBROUTINE INERTA ( ISAMPL, FINERT )
This subroutine calculates inertia loading for each link (excluding ground).

SUBROUTINE EXTFRC ( FEXTRN )
This routine calculates external loading on each link.

SUBROUTINE CONFOR ( ISAMPL )
This routine calculates contact forces in the contact systems if there are any surface contact joints.

SUBROUTINE FORCEF ( ISAMPL )
This subroutine performs the friction force analysis.

SUBROUTINE FFEQU ( ILINK, IROW, FF, F, FEXTRN, R1G, R2G )
This routine calculates the residuals of the 6 force equilibrium equations for link "ilink" and the force state given in "f". "ff" is filled with the residuals accordingly starting at row "irow".

SUBROUTINE DFFEQU ( ILINK, IROW, DFF, R1G, R2G )
This routine calculates the portion of the jacobian of the force system of equations relating to the equilibrium equations of link "ilink". It fills matrix "dff" accordingly. The link equilibrium equations are completely general and are linear in the unknown forces. Therefore, for a given position, the portions of the Jacobian formed by this routine will not be functions of the unknown forces and hence will be constant. This routine essentially fills in a 6 x 12 partition of the Jacobian.
This routine retrieves the residual of the joint force constraint equations for joint "ijnt" due to the force state given in "f". Friction force case.

This subroutine retrieves residual of joint force constraint equation for revolute joint. Friction force case.

This subroutine retrieves residual of joint force constraint equation for prismatic joint. Friction force case.

This subroutine retrieves residual of joint force constraint equation for cylindrical joint. Friction force case.

This subroutine retrieves residual of joint force constraint equation for type i spherical joint. Friction force case.

This subroutine retrieves residual of joint force constraint equation for surface contact joint. Friction force case.

This routine determines the partial derivatives of the joint force constraint equation with respect to the force variables in "f" for joint "ijnt".

This routine obtains partial derivative for friction force analysis of revolute joints.

This routine obtains partial derivative for friction force analysis of prismatic joints.

This routine obtains partial derivative for friction force analysis of cylindrical joints.

This routine obtains partial derivative for friction force analysis of spherical joints of type i.

This routine obtains partial derivative for friction force analysis of surface contact joints.

This routine assembles the equilibrium equations for the linear (no friction) force analysis.

This routine will fill \( S \) and \( B \) with the coefficients for the joint force constraint equations for each of the degrees of freedom in joint "ijnt".
SUBROUTINE FEREVO ( JNT, NDOF, CF, CI, BF )
This routine returns the joint force constraint equation for an ideal frictionless revolute joint. The equation is returned in the form of 6 coefficients and a right hand side value. The equation is not evaluated.

SUBROUTINE FEPRIS ( JNT, NDOF, CF, CI, BF )
This routine returns the joint force constraint equation for an ideal frictionless prismatic joint. The equation is returned in the form of 6 coefficients and a right hand side value. The equation is not evaluated.

SUBROUTINE FECYLI ( JNT, NDOF, CF, CI, BF )
This routine returns the joint force constraint equations for an ideal frictionless cylindrical joint. The equations are returned in the form of 6 coefficients and a right hand side value. The equation is not evaluated.

SUBROUTINE FESPH0 ( JNT, NDOF, CF, CI, BF )
This routine returns the joint force constraint equations for an ideal frictionless spherical joint modeled by a [rz][rx][rz] transformation scheme. The equations are returned in the form of 6 coefficients and a right hand side value. The equations are not evaluated. The equations are developed by resolving the torques about the 3 moving axes of rotation used to model the joint into the joint X,y,z system.

SUBROUTINE FESPH1 ( JNT, NDOF, CF, CI, BF )
This routine returns the joint force constraint equations for an ideal frictionless spherical joint modeled by a [rx][rz][rx] transformation scheme. The equations are returned in the form of 6 coefficients and a right hand side value. The equations are not evaluated. The equations are developed by resolving the torques about the 3 moving axes of rotation used to model the joint into the joint X,y,z system.

SUBROUTINE FESPH2 ( JNT, NDOF, CF, CI, BF )
This routine returns the force equation representing the ideal (linear) force constraint on variable "iprm" of spherical joint "ijnt". The equation is returned in the form of 6 coefficients and a right hand side value. The equation is not evaluated.

SUBROUTINE FESPH3 ( JNT, NDOF, CF, CI, BF )
This routine returns the force equation representing the ideal (linear) force constraint on variable "iprm" of spherical joint "ijnt". The equation is returned in the form of 6 coefficients and a right hand side value. The equation is not evaluated.

SUBROUTINE FESURF ( JNT, NDOF, CF, CI, BF )
This routine returns the joint force constraint equations for an ideal frictionless surface contact. The equations are returned in the form of coefficients and a right hand side value. The equations are not evaluated. The equations are developed by resolving the forces and torques into a system having its xy plane as the point of contact's tangent plane and its z axis along the point of contact's surface normal.

KININP.F
Program for getting the input from the user.
FUNCTION XINPUT ( MD, TM)
This function defines the input position, velocity, acceleration equations for the driver link

SUBROUTINE INIDEF
Set up initial and default parameters

SUBROUTINE INPUT ( INFILE, OUFILE, WARN, ISTAT )
Input mechanism definition

SUBROUTINE IN000 ( LINPUT, FRMT, WARN )
Input analysis general info

SUBROUTINE IN100 ( LINPUT, FRMT, WARN )
Input joint descriptions

SUBROUTINE IN200 ( LINPUT, FRMT, WARN )
Input joint tolerance and friction specifications

SUBROUTINE IN300 ( LINPUT, FRMT, WARN )
Input link coordinate system definitions

SUBROUTINE IN400 ( LINPUT, FRMT, WARN )
Input link mass properties

SUBROUTINE IN500 ( LINPUT, FRMT, WARN )
Input auxiliary nodal points

SUBROUTINE IN600 ( LINPUT, FRMT, WARN )
Input applied forces

SUBROUTINE IN700 ( LINPUT, FRMT, WARN )
Input function and analysis time intervals

SUBROUTINE IN800 ( LINPUT, FRMT, WARN )
Input solution parameters

SUBROUTINE IN900 ( LINPUT, FRMT, WARN )
Input clearance/tolerance parameters

INTEGER FUNCTION LTRIM ( STRING )
Find the trimmed length of a string

KINKIN.F
Program comprising of the kinematic analysis routines.

SUBROUTINE KINMTC ( ISAMPL )
Calculate the kinematic properties of the links and node points relative to the ground frame.

SUBROUTINE CONPOS ( ILINK, IEND, ISURFC, TIPDIS )
This subroutine performs transformation for general surface contact joint.
SUBROUTINE DISPL ( IERROR )
This subroutine performs the displacement analysis.

SUBROUTINE VELOC
This subroutine performs the velocity analysis.

SUBROUTINE ACCEL
This subroutine performs the acceleration analysis.

SUBROUTINE CNACC ( CNROT, CNDIS )
This routine forms the matrix representing the coriolis and normal components of acceleration for a given position and velocity state. This matrix is known once the displacement and velocity analyses have been completed. Hence, the acceleration analysis is concerned with finding the tangential components of acceleration.

SUBROUTINE CNADD ( ITRFRM, IPARAM, ITYPE, INDX, JTRFRM, JPARAM, JTYPE, JNDX, CNROT, CNDIS )
Subroutine for Coriolis component of force calculations.

SUBROUTINE DENVTT ( ITRFRM, IDELTA, JDELTA, DELX )
Form a transformation matrix using the denavit/hartenberg approach. If 'idelta' or 'jdeleta' are nonzero, they correspond to parameter numbers for which a perturbation is desired. This is useful for partial derivative calculations.
ITRFRM = The transformation number
IDELTA = Parameter no. For which a perturbation is desired
JDELTA = Parameter no. For which a perturbation is desired
DELX = Perturbation value

SUBROUTINE JTREVO ( ITRFRM, IDELTA, DELI, JDELTA, DELJ, ROT, DIS )
This subroutine defines the pair matrix for revolute joint with or without clearances.

SUBROUTINE JTPRIS ( ITRFRM, IDELTA, DELI, JDELTA, DELJ, ROT, DIS )
This subroutine defines the pair matrix for prismatic joint with or without clearances.

SUBROUTINE JTCYLI ( ITRFRM, IDELTA, DELI, JDELTA, DELJ, ROT, DIS )
This subroutine defines the pair matrix for cylindrical joint with or without clearances.

SUBROUTINE JTSPHO ( ITRFRM, IDELTA, DELI, JDELTA, DELJ, ROT, DIS )
This subroutine defines the pair matrix for spherical joint type I.
Euler type transformations
Spherical joint - [rz][rx][rz]

SUBROUTINE JTSPH1 ( ITRFRM, IDELTA, DELI, JDELTA, DELJ, ROT, DIS )
This subroutine defines the pair matrix for spherical joint type II with or without clearances.
Spherical joint - [rx][rz][rx]

SUBROUTINE JTSPH2 ( ITRFRM, IDELTA, DELI, JDELTA, DELJ, ROT, DIS )
This subroutine defines the pair matrix for spherical joint.
Spherical joint - screw axis formulation : theta, phi, psi
SUBROUTINE JTSPH3 ( ITRFRM, IDELTA, DELI, JDELTA, DELJ, ROT, DIS )
This subroutine defines the pair matrix for spherical joint.
Spherical joint - sheth: [rz][rx(90)][rz][rx(90)][rz]
Theta   phi   psi

SUBROUTINE JTPLAN ( ITRFRM, IDELTA, DELI, JDELTA, DELJ, ROT, DIS )
This subroutine defines the pair matrix for planar joint.
r, s, theta

SUBROUTINE JTHELO ( ITRFRM, IDELTA, DELI, JDELTA, DEU, ROT, DIS )
Helical joint with translation as the independent parameter.

SUBROUTINE JTHELI ( ITRFRM, IDELTA, DELI, JDELTA, DELJ, ROT, DIS )
Helical joint with rotation as the independent parameter.

SUBROUTINE JTSURF ( ITRFRM, IDELTA, DELI, JDELTA, DELJ, T21ROT, T21DIS )
This subroutine generates pair matrix for 3d general surface contact joint

SUBROUTINE TSCYLI ( ISURF, THETA, Z, MODE, ROT, DIS )
This routine defines definition to floating and floating to definition transformation for cylindrical contact.

SUBROUTINE TSSPHE ( ISURF, THETA, PHI, MODE, ROT, DIS )
This routine defines definition to floating and floating to definition transformation for spherical contact.

SUBROUTINE TSPLNR ( ISURF, X, Y, MODE, ROT, DIS )
This routine defines definition to floating and floating to definition transformation for planar contact.

SUBROUTINE TSPLNP ( ISURF, R, T, MODE, ROT, DIS )
This routine defines definition to floating and floating to definition transformation for planar contact.

SUBROUTINE DFORM
Forms the [DU] matrices for all of the links. [DU] is a function of the link's constant dimensions. [DU] transforms from 'first' joint to 'second' joint on a link (Sheth's technique)

SUBROUTINE GFORM
Forms the [GIG] and [GGL] matrices for all of the links. [GIG] and [GGL] are functions of the link's constant dimensions. [GIG] transforms from the 'first' joint to that link's principal axis system

SUBROUTINE LTRFRM ( COORDS, ROT, DIS )
Subroutine for calculation involved for indirect specification of shape matrix.

KINMOM.F
Program for multicriteria optimization using the GRGSUB.F program.
SUBROUTINE OPTGRG
Performs Multicriteria Optimization using GRG2

SUBROUTINE SETINI(ISET,IVAL,NNVARS,XSET,XVAL)
Sets initial design

SUBROUTINE PREMFO (NVART, NFUNT, NNOBJT, NCONT, BIG, NUMOBJ, FMIN, FMAX, BUCON,CPAYOF,W,SMALL)
Finds objective extrema, performs Normalization/Parametrization

SUBROUTINE SETPAY(ID,NNOBJT,NUMOBJ,F,PAYOFF)
Add column to pay-off table

SUBROUTINE GCOMP(G,X)
Functions module for single objective/multicriteria norm

SUBROUTINE GCOMP2(F,X)
Subroutine to define objective function and constraints.

KINOPT.F
Program for performing optimization as a part of the spatial kinematic solution.

SUBROUTINE BRAKET (OBJF, FX, FY, DX, DY, AMBDA, ALFA, EST, DELTAX, DIR, HNRM, IPRINT, KOUNTB, IER)
This subroutine searches along hessian for positive directional derivative to search for a minimum

SUBROUTINE CALCNF (F, ROT, DIS)
Calculate newton's method objective functions

SUBROUTINE CALCOF (F, ROT, DIS)
Calculate objective function for transformation matrix

SUBROUTINE FMFP (OBJF, LIMIT, DELTAX, EPS, EST, IPRINT, IER, KOUNT)
This subroutine defines objective function.

SUBROUTINE INTERP (OBJF, FX, FY, DX, DY, DELTAX, AMBDA, DIR, KOUNTI)
Interpolate to find minimum within braketed minimum

SUBROUTINE LINMIN (OBJF, DY, EST, DELTAX, DIR, HNRM, IPRINT, KOUNTB, KOUNTI, IER)
This subroutine performs line minimization.

SUBROUTINE NEWF (DELTAX)
This subroutine defines objective function for Newton's method.

SUBROUTINE NLSYST (NEWF, MAXIT, DELTA, XTOL, FTOL, IPRINT, NSING, IER, KOUNT)
This subroutine solves a system of n non-linear equations by newton's method. The partial derivatives of the functions are estimated by difference quotients when a variable

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is perturbed by an amount equal to delta (delta is added). This is done for each variable in each function. Increments to improve the estimates for the x-values are computed from a system of equations using subroutine elim.

SUBROUTINE OBJF (DELTAX)
This subroutine computes objective function value and gradient vector

SUBROUTINE SVBKSB (U, W, V, M, N, MP, NP, B, X)
Solves \([a]{x} = {b}\) for a vector \(x\), where \(a\) is specified by the arrays \(u\), \(w\), and \(v\) as returned by svdcmp. \(M\) and \(n\) are the logical dimensions of \(a\), and will be equal for square matrices. \(M_p\) and \(n_p\) are the physical dimensions of \(a\). \(B\) is the input right-hand side. \(X\) is the output solution vector. No input quantities are destroyed, so the routine may be called sequentially with different \(b\)'s.

SUBROUTINE SVDCMP (A, M, N, MP, NP, W, V)
Given a matrix \(a\), with logical dimensions \(m\) by \(n\) and physical dimensions \(m_p\) by \(n_p\), this routine computes its singular value decomposition, \(a = [u][w][v]'\) (\('\) denotes transpose). The matrix \(u\) replaces \(a\) on output. The diagonal matrix of singular values \(w\) is output as a vector \(w\). The matrix \(v\) (not the transpose \(v'\)) is output as \(v\). \(M\) must be greater or equal to \(n\); if it is smaller, then \(a\) should be filled up to square with zero rows.

KINOUT.F
Program for output control and input data and results.

SUBROUTINE OUTFCN (TYPE)
Subroutine for optimization and Newton's method iteration results.

SUBROUTINE OUTVAR (TYPE, ISAMPL, KOUNT)

SUBROUTINE OUTPST (INFO, ISAMPL)
Produce required output info in output file

SUBROUTINE OUT001 (INFO, ISAMPL)
Echo input information to the post file

SUBROUTINE OUT002 (INFO, ISAMPL)
Write constant link transformations to the post file

SUBROUTINE OUT003 (INFO, ISAMPL)
Write displacement analysis result to the post file

SUBROUTINE OUT004 (INFO, ISAMPL)
Output velocity analysis results

SUBROUTINE OUT005 (INFO, ISAMPL)
Output acceleration analysis results

SUBROUTINE OUT006 (INFO, ISAMPL)
Output kinematic properties of links and nodes of mechanism
SUBROUTINE OUT007 ( INFO, ISAMPL )
Output frictionless force analysis results

SUBROUTINE OUT008 ( INFO, ISAMPL )
Output results from friction force analysis

SUBROUTINE OUT009 ( INFO, ISAMPL )
Output results from tolerance/clearance analysis

KINSPL.F
Program for the kinematic analysis of spatial mechanisms using Sheth and Uicker's method and Singular Value decomposition.

PROGRAM KINSPL
The actual main program

SUBROUTINE TIJD ( ILINK, TDROT, TDDIS )
This routine calculates the first time derivative of [tij]

SUBROUTINE TIJDD ( ILINK, TDDROT, TDDDIS )
This routine calculates the second time derivative of [tij]

SUBROUTINE NEXTX ( FORWRD, NREVRS, DONE )
This subroutine obtains the next value of independent driver position.

SUBROUTINE HALFX ( FACTOR, LIMCNT, FORWRD, NREVRS )
This subroutine does interval halving for singularity cases.

SUBROUTINE EXTREM ( XMIN, XMAX, YMIN, YMAX, ZMIN, ZMAX, NSAMPL )
This subroutine finds extreme values of a driver parameter from kinematic standpoint.

SUBROUTINE PRNCPL

SUBROUTINE DFWDPR ( ITRFRM, JTRFRM )
Calculate the 'forward' products when one of the transformations is a derivative matrix.

SUBROUTINE D2TPRD ( ITRFRM, IPARAM, JTRFRM, JPARAM )
This subroutine forms second partial derivative of (t) w.r.t.variables i,j

SUBROUTINE BWDPRD
Calculate the 'backward' products of transformations

SUBROUTINE DTPROD ( IDEP, DELX, TOOROT, TOODIS )
This subroutine calculates transformation product around loop with perturbation

SUBROUTINE FWDPDRD
Calculate the 'forward' products

SUBROUTINE OPNFLS ( IFILE, OFILE, PFILE, IERR )
Open input and output files
SUBROUTINE TFORM ( ITRFRM, IDEL, DELI, JDEL, DELJ )
This subroutine forms the transformation matrix at a particular position.

SUBROUTINE TINIT
Subroutine for initialization of the transformation matrix product.

SUBROUTINE TPROD ( T00ROT, T00DIS )
Calculate transformation product around loop

SUBROUTINE TUPDAT
Subroutine for updating transformation matrices.

SUBROUTINE TINVRS ( ROT, DIS )
Subroutine for inverting the transformation matrices.

KINTOL.F
Program for tolerance and clearance studies.

SUBROUTINE TOLCLR
Subroutine that controls tolerance and clearance analysis

SUBROUTINE INICLR
Subroutine to initialize the clearance solution process. Start by saving all non-clearance solutions and parameters.

SUBROUTINE FORDIR (ISAMPL,INDX,JINT)
Subroutine to calculate force directions at each joint for use in the clearance solution process.

SUBROUTINE CLDISP ( ISAMPL, IERROR )
Does displacement analysis with clearance joints.

SUBROUTINE CLVELO ( ISAMPL )
Does velocity analysis with clearance joints.

SUBROUTINE CLACCL ( ISAMPL )
Does acceleration analysis with clearance joints.

SUBROUTINE CHKCON (ISAMPL,ICONV,JINT)
Subroutine to check convergence of force directions at each joint in the iterative clearance solution process.

SUBROUTINE OPTVAR
Program for evaluation of output variation for all variable
At optimized value . Output values are read from files. Variation values at each position are stored in output file. Variation for positive values stored in rrrrvarp.Dat & rrrrvarn.Dat for negative case

SUBROUTINE OUTVAR
Subroutine for evaluation of output variation for variable i at perturbed value . Output values are read from files. Variation values at each position are stored in output file. Variation for positive values stored in rrrrvarp.Dat & rrrrvarn.Dat for negative case
SUBROUTINE SENCIJ
Subroutine for evaluation of sensitivity of output to variable i at position j for clearance variables. Output values are read from file. Sensitivity values at each position are stored in output file

SUBROUTINE SENFAC
Subroutine to calculate the sensitivity factors for all variables. Input is output from sensij.F, sensitivity at each position for each variable. Calculates sensitivity factor for each variable: combination of sensitivities at all positions. Also calculates sensitivity index for each variable

SUBROUTINE SENSIIJ
Subroutine for evaluation of sensitivity of output to variable i at position j. Output values are read from two files. Sensitivity values at each position are stored in output file