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TOPICS IN MEASURE-VALUED PROCESSES

Dissertation

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Lihua Yao, B. S., M. S.

* * * * *

The Ohio State University

1997

Dissertation Committee:

Professor Peter March, Adviser
Professor Neil Falkner
Professor Boris Pittel

Approved by

Department of Mathematics
ABSTRACT

A measure valued Markov process is a Markov process whose state space is $M(E)$, the space of Radon measures on $(E,\mathcal{E})$, where $E$ is a Polish space, and $\mathcal{E}=\mathcal{B}(E)$ is the $\sigma$-algebra of Borel subsets of $E$. Fleming - Viot processes form a class of probability measure valued processes, which arise as high density diffusion approximations to certain interacting particle systems, most notably the Ohta-Kimura stepwise mutation model in population genetics theory. Its generator is

$$(\mathcal{L}\varphi)(\mu) = \frac{1}{2} \int E \int E \mu(dx)((\delta_x(dy) - \mu(dy))) \frac{\delta^2 \varphi(\mu)}{\delta \mu(x) \delta \mu(y)} + \int E \mu(dx) A(\frac{\delta \varphi(\mu)}{\delta \mu(\cdot)})(x).$$

where $\delta \varphi(\mu)/\delta \mu(x) = \lim_{\epsilon \to 0^+} \varphi(\mu + \epsilon \delta_x) - \varphi(\mu)/\epsilon$, and $A$ is the generator for a Markov process in $E$. $\delta_x \in \mathcal{P}(E)$ denotes the unit mass at $x \in E$.

We assume that $A$ is the generator of a pure jump Markov process. It is known that if $A$ is the uniform mutation, i.e.

$$Af(x) = \frac{\theta}{2} \int (f(\zeta) - f(x))m_1(d\zeta),$$

where $m_1$ is a probability measure, $\theta$ is a positive constant, then the Fleming - Viot Process is reversible.

So there is the conjecture that, if the Fleming - Viot Process is reversible, can we say that $A$ is the uniform mutation?
When $E$ is finite, Shiga has shown that the Fleming-Viot process is reversible implies the mutation operator $A$ is of the form (2) \cite{EK4}. We proved that this result is true for any Polish space $E$ under the condition that the starting process is reversible, i.e., $A$ is reversible. Moreover, we also proved that this form of $A$ is also a necessary and sufficient condition for the appropriate $n$-particle systems (Moran process) to be reversible under the appropriate stationary distribution for any $n \in N$. We also give an explicit formula for $L_n^*$, the adjoint operator of $L_n$ with respect to the appropriate stationary distribution, for any $n \in N$, where $L_n$ is the generator of appropriate $n$-particle systems (Moran process).

In chapter 2, we discuss classes of measure-valued processes which preserve polynomials. We started by consider $\xi_n = \xi_0 Q_1 Q_2 \cdots Q_n$, where $\xi_0$ is a probability measure, and $Q_1, Q_2, \cdots, Q_n$ are independent random kernel. Observe that $\xi_n$ is a Markov chain, is like a random walk, and preserves polynomials. We consider the limit of $\xi(\cdot)$ by choosing appropriate distributions for $Q(\cdot)$. By using metric on $M_1(E)$, the space of probability measures on $E$, a compact space, we have the tightness result. We give a class of measure-valued process which preserves polynomials.
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VITA

Sep. 27, 1964........ Born in Hilongjiang, People's Republic of China

1985.....................B. S., Northeast Normal University, Changchun

1988.....................M. S., Wuhan University, Wuhan

1988-1990............Engineer, China Aerospace Standard Institute, Beijing

1991-1997.............Graduate Teaching Associate, Department of Mathematics, The Ohio State University

Columbus, OH 43210

Field of Study

Major Field: Mathematics

Probability
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INTRODUCTION

A measure-valued Markov process is a Markov process whose state space is $M(E)$, the space of Radon measures on $(E,\mathcal{E})$, where $E$ is a Polish space, and $\mathcal{E}=\mathcal{B}(E)$ is the $\sigma$-algebra of Borel subsets of $E$. For probabilists, the main interest is to study measures, laws of random variables, so it is natural to study laws of measure-valued random variables. So probabilists became interested in the study of measure-valued Markov process. A rapidly developing area of research in measure-valued processes is the study of Fleming–Viot processes, which form a class of probability measure-valued processes. Several authors, for example, Wright [1939, 1950], Moran [1958] studied this process when space $E$ is finite, then Ohta, Kimura [1973], Watterson [1976], extended the study to countably infinite space which is not much different from finite space. Fleming and Viot [1974] studied the process when the space $E$ is compact space by topologizing the space with the weak topology, so the process was named Fleming–Viot process. Fleming–Viot processes can be obtained as the limit of empirical measure processes associated with a suitably scaled sequence of Moran models.

Consider a system of $N$ $E$-valued processes $\{Z_i(t) : i = 1, \cdots, N, t \geq 0\}$. Roughly speaking, the $N$ particles move independently according to the same law, which has generator $A$, a linear operator describing the dynamic behavior of each particle. After an exponentially distributed amount of time of rate $\frac{1}{2}(n - 1)$, a particle jumps to the position of another particle which is randomly chosen, then moves on. The other particles move on as before. The associated empirical measure...
process is defined by

\[ X(t)^N = N^{-1} \sum_{i=1}^{N} \delta_{Z_i(t)}. \]

It is known[DD] that an empirical measure process of this type in which the \( \{Z_i\} \) form an exchangeable Markov system are measure-valued Markov processes. Exchangeable particle systems arise naturally in many fields including statistical physics, population biology and genetic algorithms. Of course in many applications it is the finite particle systems themselves which are of primary interest. However qualitative properties of the limiting process can often provide insight into the collective behavior of the former. And moreover these limiting processes possess rich mathematical structures which are of interest in their own right. It is shown that as \( N \to \infty \), the processes \( X(t)^N \) converge (in the sense of weak convergence of finite dimensional distributions) to \( \mu_* \), a measure-valued process introduced by Fleming and Viot [FV]. This process is called the Fleming-Viot process with mutation operator \( \Lambda \), where \( \Lambda \) is the generator of the starting particles. Let \( L^n \) be the generator of Moran process. The generator of Fleming-Viot process is

\[
(\mathcal{L}F_n)(\mu) = < L^n f_n, \mu^n > \\
= \frac{1}{2} \sum_{i \neq j} ( < \Phi_{ij}^n f_n, \mu^{n-1} > - < f_n, \mu^n > ) + < A^n f_n, \mu^n > ,
\]

for all \( f_n \in \mathcal{D}(A^n) \), where

\[
\Phi_{ij}^n : B(E^k) \to B(E^{k-1}), \\
(\Phi_{ij}^n f_n)(x_1, x_2, \ldots, x_{n-1}) = f_n(s_{ij}(x_1, \ldots, x_{n-1})) = f_n(x_1, x_2, x_{j-1}, x_i, x_j, \ldots, x_{n-1}), \\
F_n(\mu) = < f_n, \mu^n > = \int_{E^n} f_n(x_1, \ldots, x_n)\mu(dx_1)\ldots\mu(dx_n), \\
A^n f_n = \sum_{i=1}^{n} A_i f_n,
\]
and $A_i f_n$ is the generator $A$ acting on the $i$th variable only. $A$ generates a Feller semigroup $T(t)$ on $C_0(E)$, the space of real continuous functions on $E$ vanishing at infinity. We can see from (1) that the first term of the generator corresponding to the jump part, the second term corresponding to independent motion.

The relation (1) is called the duality property, in the second part of the thesis we call it polynomial preserving property, and it plays an important role in the study of the Fleming-Viot processes. For suitable $F \in B(M_1(E))$, the space of bounded functions on the space of probability measures on $E$,

$$F(\mu_t) - \int_0^t LF(\mu_s) ds,$$

is a martingale.

When $E$ is finite, i.e. $M_1(E) = \{(p_1, \cdots, p_k : p_1, \cdots, p_k \geq 0, p_1 + \cdots p_k = 1\}$, the generator of Fleming-Viot process is

$$LF(p) = \frac{1}{2} \sum_{i,j \in E} p_i (\delta_{ij} - p_j) \frac{\partial^2 F(p)}{\partial p_i \partial p_j} + \sum_{j \in E} (\sum_{i \in E} q_{ij} p_i) \frac{\partial F(p)}{\partial p_j},$$

where $(q_{ij})$ is the mutation matrix, $q_{ij}$ is the intensity of mutation from type $i$ to $j$. In the special case when $k = 3$, the space is a triangle in $R^3$. Particles move on the plane, never go off.

We assume in the thesis that $A$ is the generator of a pure jump Markov process, that is

$$Af(x) = \frac{\theta(x)}{2} \int (f(\zeta) - f(x)) p(x, d\zeta),$$

where $\theta(x)$ is a nonnegative continuous function on $E$, and $p(x, d\zeta)$ is a transition kernel. It is known that if $A$ is the uniform mutation, i.e

$$Af(x) = \frac{\theta}{2} \int (f(\zeta) - f(x)) m_1(d\zeta),$$

(2)
where \( m_1 \) is a probability measure, \( \theta \) is a positive constant, then the Fleming - Viot Process is reversible. Here are some known results

(1) This result is from [EK 4]:

The Fleming - Viot process with type space \( E \) and mutation operator \( A \) defined by

\[
(2) \text{has a unique stationary distribution } \prod_{\theta, m_1} \in \mathcal{M}(\mathcal{M}(E)), \text{ which is given by}
\]

\[
\prod_{\theta, m_1}(\cdot) = P\{\sum_{i=1}^{\infty} \rho_i \delta_{\xi_i} \in \cdot\},
\]

where \( (\rho_1, \rho_2, \cdots) \) has the Poisson - Dirichlet distribution with parameter \( \theta \), and \( \xi_1, \xi_2, \cdots \) are i.i.d - \( m_1 \), independent of \( (\rho_1, \rho_2, \cdots) \).

(2) This result is from [EK 3]:

The Fleming - Viot process as above is reversible with respect to \( \prod_{\theta, m_1} \). In other words, if \( \{\mu_t, -\infty < t < \infty\} \) is a stationary Fleming - Viot process with type space \( E \) and mutation operator \( A \) defined by (2), then

\[
\{\mu_t, -\infty < t < \infty\} \equiv_D \{\mu_{-t}, -\infty < t < \infty\}.
\]

Here \( \equiv_D \) means the same distribution.

So there is the conjecture that, are there any other Fleming - Viot Processes of having bounded mutation operator \( A \) which are reversible? ([EK 4])

When \( E \) is finite, Shiga has shown that the Fleming - Viot process is reversible implies the mutation operator \( A \) is of the form (2) [EK 4]. We proved that this result is true for any Polish space \( E \) (\( E \) is uncountable) under the condition that the starting process is reversible, i.e \( A \) is reversible. This is the main result given in Theorem 1.4, Chapter 1. Here is the result:

Let \( E \) be a Polish space. Let \( f \in B(E) \). Let

\[
Af(x) = \frac{\theta(x)}{2} \int_{\zeta \in E} (f(\zeta) - f(x))p(x, d\zeta),
\]
where \( \theta(x) > 0 \) is a bounded continuous function, and the starting process is reversible with a stationary probability measure \( m_1 \) which is nonatomic. Then:

The Fleming-Viot process is reversible iff \( p(x, d\zeta) = m_1(d\zeta), \theta(x) = \theta \). That is

\[
\mathcal{L} = \mathcal{L}^* \iff p(x, d\zeta) = m_1(d\zeta), \theta(x) = \theta.
\]

Here \( \mathcal{L}^* \) is the adjoint operator of \( \mathcal{L} \) with respect to the stationary distribution \( m \) of the Fleming-Viot process, and \( \theta \) is a positive constant.

Moreover, we also proved that this form of \( A \) is also a necessary and sufficient condition for the appropriate n-particle systems (Moran process) to be reversible under the appropriate stationary distribution for any \( n \in \mathcal{N} \). This result is given in Theorem 1.3, Chapter 1. Here is the statement of this result:

Let \( A \) be the generator of a jump process, ie

\[
Af(x) = \frac{\theta(x)}{2} \int (f(\zeta) - f(x))p(x, d\zeta).
\]

Suppose \( \theta(x) > 0 \) is a bounded continuous function, and \( m_1 \) is the stationary distribution which is nonatomic, satisfying

\[
< Af, m_1 > = 0.
\]

Let \( \overline{L}_n^* \) is the adjoint operator of \( \overline{L}_n \) in the space \( L^2(m_n) \), satisfying

\[
\int_{E^n} g_n(x)\overline{L}_n f_n(x)m_n(dx) = \int_{E^n} f_n(x)\overline{L}_n^* g_n(x)m_n(dx),
\]

for any \( g_n(x), f_n(x) \in B(E^n) \), where \( m_n \)‘s are stationary distribution of the n-particle system. Then the n particle process is reversible or

\[
\overline{L}_n = \overline{L}_n^*, \text{ for } n \in \mathcal{N}, \text{ iff } \theta(x) = \theta, p(x, d\zeta) = m_1(d\zeta).
\]

The main proof of the above result is given in Theorem 1.2, Chapter 1, which is the following:
Let $A$ be the generator of the pure jump process, $\theta(x) > 0$ is a bounded continuous function, and $m_1$ is the probability measure which is nonatomic such that $<Af, m_1> = 0$. Let

$$L_2f(x, y) = \frac{f(x, x) + f(y, y)}{2} - f(x, y) + A^2f(x, y).$$

Suppose $A = A^*$, also suppose $L_2 = L_2^*$. Then $\theta(x) = \theta$, a positive constant, and $p(x, d\zeta) = m_1(d\zeta)$.

Also in Chapter 1, we have Theorem 1.1, which gives an explicit formula for $L_n^*$, the adjoint operator of $L_n$ with respect to the appropriate stationary distribution, for any $n \in N$, where $L_n$'s are the generators of the appropriate $n$-particle systems (Moran process).

Note that a hypothesis in all the results of Chapter 1 is that the transition function $p(x, \cdot)$ of the generator $A$ or $m_1$ is nonatomic (i.e., have no atoms) for each $x \in E$. This model is called Infinitely-many-neutral-alleles model (Kimura and Crow 1964; Watterson 1976; Ethier and Kurtz 1981, 1986, 1987). The assumption of this model means that every mutant is of a new type, need $E$ uncountable). This requirement is biologically natural. Also note that in the case when $E$ is countable but not finite, there is no definite answer get to the reversibility conjecture.

In chapter 2, we discuss classes of measure-valued processes which preserve polynomials. We give new examples of measure-valued processes. Fleming-Viot processes which can be constructed as empirical limits of particle processes have the property of preserving polynomials, which describes the relations between $n$-th moment measures of the limiting measured-valued process and the semigroup of $n$-particle process. So it is natural to ask, besides Fleming-Viot processes, what other measured-valued processes have this preserving polynomials property.
We started by consider $\xi_n = \xi_0 Q_1 Q_2 \cdots Q_n$, where $\xi_0$ is a probability measure, and $Q_1, Q_2, \cdots, Q_n$ are independent random kernel. Observe that $\xi_n$ is a Markov chain (Theorem 2.1), is like a random walk, and preserves polynomials, lemma 2.5. We consider the limit of $\xi(t)$ by choosing appropriate distributions for $Q(t)$. By using Markov metric on $\mathcal{M}_1(E)$, the space of probability measures on $E$, a compact space, we have the tightness result, lemma 2.3. Theorem 2.4 gives the main result which states that any measured valued jump process which of generator

$$\mathcal{L} < f_n, \mu^n > = \int_Q ( < f_n, (\mu Q)^n > - < f_n, \mu^n > ) m(dQ),$$

and

$$\int ||Q - I|| m(dQ) < \infty.$$

preserves polynomials. Theorem 2.5 gives a class of measure-valued process which preserves polynomials based on Theorem 2.4. It says that Fleming-Viot processes plus the jump processes as above preserve polynomials. There are more questions need to be answered on this subject.
Chapter I

THE REVERSIBILITY OF

FLEMING-VIOT PROCESSES

First, we will discuss the definitions of adjoint operators and study the relations of stationary distributions of the particle-systems and the Fleming-Viot processes. We will also discuss some general results we will be using to prove the main result.

Let $E$ be a Polish space. Let $B(E^k)$ be the set of bounded Borel measurable functions on $E^k$. A probability measure-valued diffusion process called Fleming-Viot process, with type space $E$, and mutation operator $A$. Its generator is

$$
(\mathcal{L}f_n)(\mu) = \sum_{1 \leq i < j \leq n} (\langle \Phi^n_{ij} f_n, \mu^{n-1} \rangle - \langle f_n, \mu^n \rangle) + \langle A^n f_n, \mu^n \rangle
$$

for all $f_n \in \mathcal{D}(A^n)$, where

$$
\Phi^n_{ij} : B(E^k) \to B(E^{k-1}),
$$

$$(\Phi^n_{ij} f_n)(x_1, x_2, \cdots x_{n-1}) = f_n(s_{ij}(x_1, \cdots x_{n-1})) = f_n(x_1, x_2, x_{j-1}, x_i, x_j, \cdots x_{n-1}),$$

$$F_n(\mu) = \langle f_n, \mu^n \rangle = \int_{E^n} f_n(x_1, \cdots x_n) \mu(dx_1) \cdots \mu(dx_n).$$

$A$ generates a Feller semigroup $T(t)$ on $\hat{\mathcal{C}}(E)$, the space of real continuous functions on $E$ vanishing at infinity. $T(t)$ is given by a transition function $p(t, x, d\zeta)$. and
\[ T(t)f_1(x) = \int_E f_1(\xi)p(t,x,d\xi). \] Define the semigroup \( \{T_k(t)\} \) on \( B(E^k) \) by
\[ T_k(t)f_k(x_1, \ldots, x_k) = \int_E \cdots \int_E f_k(\xi_1, \cdots, \xi_k)p(t,x_1,d\xi_1) \cdots p(t,x_k,d\xi_k). \tag{4} \]

\( A^k \) denotes its generator, \( D(A^k) \) is the domain of \( A^k \), it is a subspace of \( B(E^k) \).

We can see that
\[ A^kf_k(x_1, \ldots, x_k) = \sum_{i=1}^k A_if_k(x_1, \ldots, x_k), \tag{5} \]
where \( A_if_k(x_1, \ldots, x_k) \) means \( A \) acting on the ith variable alone.

Let 
\[ \sigma_{ij}(x_1, \ldots, x_n) = (x_1, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n), \]
then
\[ (L_nf_n)(x_1, \ldots, x_n) = \sum_{1 \leq i < j \leq n} (f_n(\sigma_{ij}(x_1, \ldots, x_n)) - f_n(x_1, \ldots, x_n)) + A^n f_n(x_1, \ldots, x_n) \equiv L_n f_n, \tag{6} \]
for \( f_n \in B(E^n) \). We interpret \( L \) as an operator with domain \( B(E^\infty) \). \( L \) is a generator for an \( E^\infty \) valued process \( X = (X_t^1, X_t^2, \ldots) \). The jth component \( X_j \) of this process evolves as a Markov process with generator \( A \) independently of the other components (conditioned on its initial position) until after an exponentially waiting time with parameter \( j-1 \). It then jumps to some component \( i, i < j \), which is randomly chosen, then assumes the value of \( X_i \) at that time, then evolves independently for another exponentially waiting time. \( L_n \) is the generator of \( E^n \)-valued process \( (X_t^1, X_t^2, \ldots, X_t^n) \). See [DM], theorem 3.B.3.

Let \( f_n \in B(E^n) \). Define
\[ \overline{L_n} f_n(x) = \frac{1}{2} \sum_{i \neq j} (f_n(\sigma_{ij}(x)) - f_n(x)) + A^n f_n(x). \tag{7} \]

From [EK 3] Theorem 6.1, we know that the \( E^n \)-valued process \( (X_t^1, \ldots, X_t^n) \) (look down process) can be coupled to an \( E^n \)-valued process \( (Y_t^1, \ldots, Y_t^n) \) (Moran
process ) with generator $L_n$, in such a way that

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{Y^i} = \frac{1}{n} \sum_{i=1}^{n} \delta_{X^i}.$$

It is not hard to see that $(LF_n)(\mu) = \langle L_n f_n, \mu^n \rangle = \langle \overline{L}_n f_n, \mu^n \rangle$ for any $\mu \in \mathcal{M}_1(E)$.

The following lemma gives the relation between stationary distribution of a Fleming - Viot process and stationary distribution of appropriate $E^n$-valued process for any $n \in \mathcal{N}$.

A measure $m_n$ on $E^n$ is symmetric if for a permutation $\pi \in S_n$ and all Borel sets $A_1, \ldots, A_n \in E^n$,

$$m_n(A_1 \times \cdots \times A_n) = m_n(A_{\pi(1)} \times \cdots \times A_{\pi(n)}).$$

(8)

A sequence $(m_n)_{n=1}^{\infty}$ of measures $m_n$ on $E^n$ is consistent if for all Borel sets $A_1, \ldots, A_n \subset E^n$, and $n \in \mathcal{N}$, we have

$$m_{n+1}(A_1 \times \cdots \times A_n \times E) = m_n(A_1 \times \cdots \times A_n).$$

(9)

**Lemma 1.** If $m$ is a Borel measure on $\mathcal{M}_1(E)$, the set of probability measures on $E$, such that for every $n \geq 1$, $f_n \in \mathcal{D}(A^n)$, $\int_{\mathcal{M}_1(E)} < f_n, \mu^n > m(d\mu) < \infty$, then there is a unique consistent family of symmetric, regular, Borel measures $(m_n)_{n=1}^{\infty}$, such that

$$\int_{\mathcal{M}_1(E)} < f_n, \mu^n > m(d\mu) = \int_{E^n} f_n(x)m_n(dx).$$

(10)

Conversely, if $(m_n)_{n=1}^{\infty}$ is any consistent family of symmetric, regular Borel measures such that for every $n \geq 1$, $f_n \in \mathcal{D}(A^n)$, $\int_{E^n} f_n(x)m_n(dx) < \infty$, then there exists a unique Borel measure $m$ on $\mathcal{M}_1(E)$ such that the equation above holds.
Proof: The idea of the proof comes from Dawson and March Peter when $E$ is $R^d$. Now for any Polish space $E$ the proof is similar.

In the case $m$ is actually a probability measure and conversely if the $m_n$’s are probabilities on $E^n$, then the result is contained in Lemma 6.1 of [DH] which is in space $R^d$. Study the proof of it, we can see the use of Kolmogorov Consistency Theorem and Regularity of space $R^d$ will also hold for Polish space $E$. So Lemma 6.1 in [DH] will hold for $E$.

For the extension to the general case, the proof is outlined below.

If $m$ is as indicated then $f_n \mapsto \int_{\mathcal{M}_1(E)} F_{f_n} m(d\mu)$ is a positive linear functional on $\mathcal{D}(A^n)$ and so is represented by a unique, regular Borel measure $m_n$ [GF]

$$\int_{\mathcal{M}_1(E)} F_{f_n}(\mu) m(d\mu) = \int_{E^n} f_n(x) m_n(dx).$$

(11)

To check symmetry note that if $\pi \in S_n$ is any permutation and we define $\pi f_n \in \mathcal{D}(A^n)$ by $\pi f_n(x_1, \ldots, x_n) = f_n(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then $F_{\pi f_n}(\mu) = F_{f_n}(\pi \mu)$ for every $\mu \in \mathcal{M}_1(E)$. Consistency follows from the fact that for any $f_n \in \mathcal{D}(A^n)$, $F_{f_n} = F_{f_n \otimes 1}(\mu)$ and from an approximation of $f_n \otimes 1$ by functions in $\mathcal{D}(A^{n+1})$.

On the other hand, let $m_n(x_1; dx_2 \cdots dx_n)$ be the regular conditional distribution of $m_n$ relative to $m_1$. By consistency, for any Borel set $C \subset E$,

$$m_1(C) = m_n(C \times E^{n-1}) = \int_C m_n(x_1; E^{n-1}) m_1(dx_1).$$

(12)

Hence $m_n(x_1; dx_2 \cdots dx_n)$ is a probability measure on $E^{n-1}$ for $m_1 - a, e.x_1$. For such $x_1$,

$(m_n(x_1; \cdot))_{n=2}^\infty$ is a consistent family of symmetric, regular Borel measures. Once again by Lemma 6.1 of [DH], there is a unique Borel measure $m(x_1; d\mu)$ on $\mathcal{M}_1(E)$.
having \((m_n(x_1, \cdot))_{n=1}^{\infty}\) as its family of moment measures. Thus

\[
m(d\mu) = \int_E m(x_1; d\mu)m_1(dx_1)
\]
satisfies

\[
\int_{\mathcal{M}_1(E)} F_{f_n}(\mu)m(d\mu)
= \int_E \int_{\mathcal{M}_1(E)} \int_{E^{n-1}} f_n(x_1, x_2, \cdots, x_n)\mu(dx_2) \cdots \mu(dx_n)m(x_1; d\mu)m_1(dx_1)
= \int_E \int_{E^{n-1}} f_n(x_1, x_2, \cdots, x_n)m_n(x_1, dx_2 \cdots dx_n)m_1(dx_1)
= \int_{E^n} f_n(x)m_n(dx)
\]

(13)

To see that \((m_n)_{n=1}^{\infty}\) specifies \(m\) uniquely note that \(\cup_{n=1}^{\infty}\{F_{f_n} : f_n \in \mathcal{D}(A^n)\}\)
where when \(n = 0\), we use constant function, is dense in the space of continuous functions on \(\mathcal{M}_1(E)\) vanishing at \(\infty\).

The Fleming-Viot process \(\mu_t\) has a version

\[
\mu_t = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i}.
\]

(14)

Given that \((X_1(0), \cdots, X_n(0), \cdots)\) is an exchangeable sequence of \(E\)-valued random variables [EK 3] Theorem 6.1.

Suppose the Markov process \(X_t\) with generator \(A\) is stationary of stationary distribution \(m_1\). i.e

\[
\int_E p(t, x, C)m_1(dx) = m_1(C), C \in B(E),
\]

(15)

where \(p(t, x, dy)\) is the transition function of this process. If the closure of \(A\) generates the semigroup corresponding to \(p(t, x, dy)\) on a subspace of \(B(E)\) that is separating, then the above equality is equivalent to

\[
< Af, m_1 > = 0, f \in \mathcal{D}(A).
\]

(16)
The proof is as the following lemma 1.2. Then as a result, the appropriate Fleming-Viot process is also stationary of a unique stationary distribution.

Let $m$ be the stationary distribution of the Fleming-Viot process, with transition function $p(t, \mu, d\nu)$, then we have

$$\int_{\mathcal{M}_1(E)} p(t, \mu, d\nu)m(d\mu) = m(d\nu). \quad (17)$$

Let the semigroup of this process to be $\mathcal{P}_t$, then

$$\mathcal{P}_t F(\mu) = \int_{\mathcal{M}_1(E)} p(t, \mu, d\nu)F(\nu). \quad (18)$$

Let the generator to be $\mathcal{L}$, i.e.

$$\mathcal{L} F(\mu) = \lim_{t \to 0} \frac{\mathcal{P}_t F(\mu) - F(\mu)}{t}. \quad (19)$$

We have the following lemma

**Lemma 1.2** $m$ is stationary $\iff \int_{\mathcal{M}_1(E)} \mathcal{L} F(\mu)m(d\mu) = 0$, where $F \in \mathcal{D}(\mathcal{L})$

**Proof:** By Hille-Yosido theorem, we know

$$\int_{\mathcal{M}_1(E)} \mathcal{L} F(\mu)m(d\mu) = 0 \iff \int_{\mathcal{M}_1(E)} \mathcal{P}_t F(\mu)m(d\mu) = \int_{\mathcal{M}_1(E)} F(\mu)m(d\mu). \quad (20)$$

If $m$ is stationary, then

$$\int_{\mathcal{M}_1(E)} \mathcal{P}_t F(\mu)m(d\mu) = \int_{\mathcal{M}_1(E)} \int_{\mathcal{M}_1(E)} p(t, \mu, d\nu)F(\nu)m(d\mu)$$

$$= \int_{\mathcal{M}_1(E)} F(\nu)(\int_{\mathcal{M}_1(E)} p(t, \mu, d\nu)m(d\mu)) = \int_{\mathcal{M}_1(E)} F(\nu)m(d\nu). \quad (21)$$

On the other hand, if the above equality holds for any $F \in \mathcal{D}(\mathcal{L})$ then it also holds for any $F \in B(\mathcal{M}_1(E))$, then as a result $m$ is stationary. The lemma is proved. 

$\square$
Define

\[ q(t, \mu, d\nu) = \frac{p(t, \nu, d\mu)m(d\nu)}{m(d\mu)}. \]  

(22)

It is the transition function of the dual process, say \( \mu_{-t} \).

Let \( \mathcal{P}_{-t} \) be the semigroup of this dual process, i.e

\[ \mathcal{P}_{-t}G(\mu) = \int_{\mathcal{M}_1(E)} G(\nu)q(t, \mu, d\nu). \]  

(23)

Let \( \mathcal{L}^* \) be the generator of it, i.e

\[ \mathcal{L}^* G(\mu) = \lim_{t \to 0} \frac{\mathcal{P}_{-t} G(\mu) - G(\mu)}{t}. \]  

(24)

Then we have the following lemma:

**Lemma 1.3** Let \( G \in \mathcal{D}(\mathcal{L}^*), F \in \mathcal{D}(\mathcal{L}) \), Then

\[ \int_{\mathcal{M}_1(E)} G(\mu)\mathcal{L}F(\mu)m(d\mu) = \int_{\mathcal{M}_1(E)} F(\mu)\mathcal{L}^* G(\mu)m(d\mu). \]  

(25)

when the above integrals exist. We call \( \mathcal{L}^* \) the adjoint operator of \( \mathcal{L} \) relative to space \( L^2(m) \)

**Proof:**

\[
\int_{\mathcal{M}_1(E)} G(\mu)\mathcal{P}_t F(\mu)m(d\mu)
= \int_{\mathcal{M}_1(E)} G(\mu)(\int_{\mathcal{M}_1(E)} F(\nu)p(t, \mu, d\nu)m(d\mu))m(d\mu)
= \int_{\mathcal{M}_1(E)} \int_{\mathcal{M}_1(E)} G(\mu)F(\nu)p(t, \mu, d\nu)m(d\mu)
= \int_{\mathcal{M}_1(E)} \int_{\mathcal{M}_1(E)} G(\mu)q(t, \nu, d\mu)m(d\nu)
= \int_{\mathcal{M}_1(E)} F(\nu)(\int_{\mathcal{M}_1(E)} G(\mu)q(t, \nu, d\mu)m(d\nu))m(d\nu)
= \int_{\mathcal{M}_1(E)} F(\nu)\mathcal{P}_{-t} G(\nu)m(d\nu). \]  

(26)
Therefore
\[
\int_{\mathcal{M}_1(E)} G(\mu) \frac{P_1 F(\mu) - F(\mu)}{t} m(d\mu) = \int_{\mathcal{M}_1(E)} F(\mu) \frac{P_{-1} G(\mu) - G(\mu)}{t} m(d\mu).
\]  
(27)

Let \( t \to 0 \), we have the result of the proposition. \( \square \)

Next we will discuss the similar definition and result as above to \( m_n \), where \( m_n \) is as lemma 1.1.

**Lemma 1.4** Let \( m_1 \) be a measure such that \( < Af, m_1 > = 0 \). Let \( m \) be the stationary distribution of the Fleming-Viot process. Then \( m_n \) is a stationary distribution of process \((Y_t^1, \cdots, Y_t^n)\) for each \( n \). Moreover \((m_n)_{n=1}^\infty\) are given by the formulas
\[
m_n = (S_n^{n-1} m_{n-1}) R_n^n_{(n(n-1))},
\]
(28)

which is uniquely determined by \( m_1 \), and
\[
< S_n^{n-1} m_{n-1}, f_n > = < m_{n-1}, S_n^{n-1} f_n >,
\]
(29)
\[
S_n^{n-1} f_n(x_1, \cdots x_{n-1}) = \frac{1}{2} \sum_{1 \leq i \neq j \leq n} f_n(s_{ij}(x_1, \cdots x_{n-1})),
\]
(30)
\[
R_n^n_{(n(n-1))} = \left( \frac{n(n-1)}{2} - A_n^n \right)^{-1},
\]
(31)

**Proof**: Let \( f_n \in C(E^n) \). We have
\[
\int_{\mathcal{M}_1(E)} \mathcal{L} < f_n, \mu > m(d\mu) = \int_{E^n} \mathcal{L} f_n(x) m_n(dx) = 0.
\]
(32)

So \( m_n \) is a stationary distribution for \( n \)-particle system.
\[
\int_{\mathcal{M}_1(E)} \mathcal{L} < f_n, \mu > m(d\mu)
\]
\[
= \int_{E^{n-1}} S^{n-1} f_n(x_1, \cdots, x_{n-1}) m_{n-1}(dx_1, \cdots, dx_{n-1})
\]

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\[-\int_{\mathcal{E}^n} \left( \frac{n(n-1)}{2} f_n(x_1, \ldots, x_n) - A^n f_n(x_1, \ldots, x_n) \right) m_n(dx_1, \ldots, dx_n) = 0. \] 

(33)

So

\[ \int_{\mathcal{E}^n} f_n(x) S_{n-1}^n m_{n-1}(dx) = \int_{\mathcal{E}^n} \left( \frac{n(n-1)}{2} - A^n \right) f_n(x) m_n(dx). \] 

(34)

Therefore

\[ \int_{\mathcal{E}^n} R_{n(n-1)}^n f_n(x) S_{n-1}^n m_{n-1}(dx) = \int_{\mathcal{E}^n} f_n(x) m_n(dx). \] 

(35)

So $m_n = S_{n-1}^n m_{n-1} R_{n(n-1)}^n$.

\[ \square \]

Let $p^n(t, x, dy)$ be the transition function of the generator $\mathcal{L}_{n}$.

Define

\[ q^n(t, x, dy) = \frac{p^n(t, y, dx) m_n(dy)}{m_n(dx)}. \] 

(36)

It is the transition function of the dual process.

Let $\mathcal{P}^n_{-t}$ be the semigroup of it, i.e

\[ \mathcal{P}^n_{-t} f_n(x) = \int_{\mathcal{E}^n} f_n(y) q^n(t, x, dy). \] 

(37)

Let $\mathcal{L}^*_{n}$ be the generator of it, i.e

\[ \mathcal{L}^*_{n} g_n(x) = \lim_{t \to 0} \frac{\mathcal{P}^n_{-t} g_n(x) - g_n(x)}{t}. \] 

(38)

Then we have the following result:

**Lemma 1.5** Let $g_n \in \mathcal{D}(L^*_n)$, $f_n \in \mathcal{D}(L_n)$, then

\[ \int_{\mathcal{E}^n} g_n(x) \mathcal{L}^*_n f_n(x) m_n(dx) = \int_{\mathcal{E}^n} f_n(x) \mathcal{L}^*_n g_n(x) m_n(dx). \] 

(39)

when the above integrals exist. We call $\mathcal{L}^*_n$ the adjoint operator of $\mathcal{L}_n$ in space $L^2(m_n)$. 

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Proof: It is the same as lemma 1.3. □

Remark 1.1 We can see that Fleming-Viot process is reversible iff $q(t, v, d\mu) = p(t, v, d\mu)$ iff $P_t = P_{-t}$ iff $\mathcal{L} = \mathcal{L}^*.$

Remark 1.2 We can generalize the definitions of $\mathcal{L}^*, \mathcal{L}_n^*$ to bigger spaces, the corresponding Banach spaces. The equalities in lemma 1.3, 1.5 will still hold as long as the integrals exist.

Let's discuss some general results we will be using throughout this chapter.

Let $E$ be a Polish space, and $\mathcal{K}$ be a collection of open subset of $E$ such that $\mathcal{K}$ is countable and dense in $E$, and also $\mathcal{B}(E) = \sigma(\mathcal{K}).$

Lemma 1.5 (Lebesgue-Radon-Nikodym theorem) Let $\nu$ be a $\sigma$-finite signed measure and $\mu$ a $\sigma$-finite positive measure on $(E, \mu), ~ \nu \ll \mu,$ Then there exists an measurable function $f : E \mapsto \mathbb{R}^+$, unique up to a $\mu$-null set, such that $d\nu = f d\mu.$

denote $f = \frac{d\nu}{d\mu}.$

Lemma 1.7 Let $\mathcal{F}^n = \{ C_1 \times \cdots \times C_n : (C_i)_{i=1}^n \text{ are open sets in } E \}.$

Let $\mathcal{J}^n = \{ \text{ set of intersections of sets from } \mathcal{F}^n \}$

Then $\mathcal{J}^n$ is a ring in $(E)^n.$

Proof: If $C$ is an open set, then $C^c$ is a closed set and can be represented by intersections of open sets. Moreover, we can choose these open sets to be decreasing. Also unions of open sets is open. So the lemma is proved. □

Lemma 1.8 The extension theorem: Every $\sigma$-finite premeasure $\mu$ on a ring $\mathcal{R}$ in $\Omega$ can be extended to a measure $\tilde{\mu}$ on $\sigma(\mathcal{R})$ in exactly one way. [HB]. Theorem 1.5.7.
Lemma 1. 9

\[ \frac{1}{2} \int_{(E)} \sum_{1 \leq i\neq j \leq n} \delta_{\sigma_{ij}}(y)(A)m_n(dy) = S_{n-1}^{n-1}m_{n-1}(A). \]  

(40)

for \( A \in B((E)^n) \)

Proof : \(< S_{n-1}^{n-1}m_{n-1}, f > = < S^{n-1}f, m_{n-1} > . \)

Let \( f(x_1, \ldots, x_n) = 1_{A_1 \times A_2 \times \ldots \times A_n}(x_1, \ldots, x_n), A_i \in B(E) \)

Then

\[ S_{n-1}^{n-1}m_{n-1}(A_1 \times \cdots \times A_n) \]

\[ = \frac{1}{2} \sum_{1 \leq i\neq j \leq n} \int_{A_1 \times \cdots \times A_n} \sigma_{ij}(x_1, \ldots, x_{n-1})m_{n-1}(dx_1, \ldots, dx_{n-1}) \]

\[ = \frac{1}{2} \sum_{1 \leq i\neq j \leq n} m_{n-1}(A_1 \times A_2 \times \cdots \times A_i \cap A_j \times \cdots \times A_n). \]  

(41)

Now that

\[ \frac{1}{2} \int \sum_{1 \leq i\neq j \leq n} \delta_{\sigma_{ij}}(y)(A_1 \times \cdots \times A_n)m_n(dy) \]

\[ = \frac{1}{2} \sum_{1 \leq i\neq j \leq n} \int_{\{\sigma_{ij}(y) \in (A_1 \times \cdots \times A_n)\}} m_n(dy) \]

\[ = \frac{1}{2} \sum_{1 \leq i\neq j \leq n} \int_{\{y_i \in A_1, \ldots, y_i \in A_i, y_i \in A_1 \times \cdots \times A_n\}} m_n(dy) \]

\[ = \frac{1}{2} \sum_{1 \leq i\neq j \leq n} m_n(A_1 \cdots \times A_i \cap A_j \times \cdots \times A_{j-1} \times E \times \cdots \times A_n) \]

\[ = \frac{1}{2} \sum_{1 \leq i\neq j \leq n} m_{n-1}(A_1 \times A_2 \times \cdots \times A_i \cap A_j \times \cdots \times A_n). \]  

(42)

So

\[ \frac{1}{2} \int \sum_{1 \leq i\neq j \leq n} \delta_{\sigma_{ij}}(y)(A_1 \times \cdots \times A_n)m_n(dy) = S_{n-1}^{n-1}m_{n-1}(A_1 \times \cdots \times A_n). \]  

(43)

So the result was proved by \( \pi - \lambda \) system. \( \square \)
Lemma 1.10 For any $B \in B((E)^n)$

$$\frac{1}{2} \int_B \sum_{1 \leq i < j \leq n} \delta_{\sigma_{ij}(y)}(\cdot) m_n(dy) \ll S_{n-1}^{n-1}(\cdot).$$

Proof: If $S_{n-1}^{n-1}(A) = 0$, Then by lemma 1.9, we have

$$\frac{1}{2} \sum_{1 \leq i < j \leq n} \delta_{\sigma_{ij}(y)}(A) = 0, m_n.a.e.$$

So $\frac{1}{2} \int_B \sum_{1 \leq i < j \leq n} \delta_{\sigma_{ij}(y)}(A) m_n(dy) = 0$. Therefore the lemma was proved. □

Remark 1.3 So by lemma 1.10, there exists a measurable function

$$f^n_B(x) = \frac{\int_B \frac{1}{2} \sum_{1 \leq i < j \leq n} \delta_{\sigma_{ij}(y)}(dx) m_n(dy)}{S_{n-1}^{n-1}(dx)},$$

for $x \in H^n$, where $H^n = \bigcup_{1 \leq i < j \leq n} \{x_i = x_j\}$, such that

$$\frac{1}{2} \int_B \sum_{1 \leq i < j \leq n} \delta_{\sigma_{ij}(y)}(C) m_n(dy) = \int_C f^n_B(x) S_{n-1}^{n-1}(dx).$$

for any $C \in B((E)^n)$, and this $f^n_B(x)$ is unique, up to a $S_{n-1}^{n-1}$-null set.

Let $T^n = \{C_1 \times \cdots C_n : C_i \subset K \text{ for } i = 1, \ldots, n\}$

Now define a measure on $T^n$ as $\mu_n(x, B) = f^n_B(x)$ if $B \in T^n$

It is uniquely determined, only up to a $S_{n-1}^{n-1}$-null set. Since for each $B_k \in T^n$, \exists $F_k$, such that $S_{n-1}^{n-1}(F_k) = 0$ and $f^n_{B_k}(x)$ is uniquely defined for $x \in H^n \cap F_k$

So for each $B_k \in T^n$, $\mu_n(x, B_k) = f^n_{B_k}(x)$ is uniquely defined on $T^n$ for $x \in H^n \cap F^c$

where $F = \bigcup_{k=1}^\infty F_k, S_{n-1}^{n-1}(F) = 0$

Let $B \in F^n$, where $F^n = \{C_1 \times \cdots C_n : (C_i)_{i=1}^n \text{ are open sets in } E\}$, then we can find $(B_k)_{k=1}^\infty \in T^n$, and $B_k \not\subset B$, define

$$\mu_n(x, B) = \lim_{k \to \infty} \mu_n(x, B_k), x \in H^n \cap F^c.$$
It is well defined, since from (47), we know if \( B_2 \supseteq B_1, B_1, B_2 \in T^n \), then
\[
\int_C (f_{B_2}^n(x) - f_{B_1}^n(x))S_n^{n-1}m_{n-1}(dx) \geq 0. \quad (48)
\]
for any \( C \in B((E)^n) \), so \( f_{B_2}^n(x) \geq f_{B_1}^n(x), S_n^{n-1}m_{n-1} \) a.e
So \( \mu_n(x, B_2) \geq \mu_n(x, B_1) \) for \( x \in F^c \). So \( \lim_{k \to \infty} \mu_n(x, B_k) \) exists.

This method of proof will be used a few times later on, so we will give a general result.

**Lemma 1.11** Let \( E \) be a Polish space, \( K \) be a collection of open subsets of \( E \), such that \( K \) is countable, dense in \( E \), therefore \( B(E) = \sigma(K) \).

Let \( \mathcal{F} = \{ \text{open sets in } E \} \)
\[
\mathcal{J} = \{ \text{sets of intersections of sets from } \mathcal{F} \}.
\]
Let \( T_1(y, dx) \) be a kernel, such that
i) for any \( y, T_1(y, \cdot) \) is a measure.
ii) for any \( A \in B(E), T_1(y, A) \) is a measurable function of \( y \).

Let \( T_2(dx), T_3(dx) \) be two measures. Suppose
\[
\int_A T_1(y, \cdot)T_2(dy) \ll T_3(\cdot) \quad (49)
\]
for any \( A \in B(E) \). Then
\[
\mu(x, A) = \frac{\int_A T_1(y, dx)T_2(dy)}{T_3(dx)} \quad (50)
\]
is a measurable function of \( x \), for any \( A \in B(E) \), and \( \mu(x, \cdot) \) is a \( \sigma \)-finite premeasure on \( \mathcal{J} \) for \( T_3 \) a.e \( x \).
Proof: Let \((B_m)_{m=1}^\infty \in K\). Since
\[
\int_{B_i} T_1(y, \cdot)T_2(dy) \ll T_3(\cdot)
\]
for all \(B_i \in K\). Therefore, there exist measurable functions
\[
f_{B_i}(x) = \frac{\int_{B_i} T_1(y, dx)T_2(dy)}{T_3(dx)}
\]
which is unique for \(x \in F_1^c\), where \(T_3(F_i) = 0\).

Define \(\mu(x, B) = f_B(x)\) for any \(B \in K\). Then it is a measurable function of \(x\), uniquely determined for \(x \in F_1^c\), where \(F = \bigcup_{i=1}^\infty F_i\), and \(T_3(F) = 0\).

Let \(A \in \mathcal{F}\), then we can find \((B_m)_{m=1}^\infty \in K\), and \(B_m\) increasing to \(A\). Define
\[
\mu(x, A) = \lim_{m \to \infty} \mu(x, B_m).
\]
It is well-defined, since for any \(B_1, B_2 \in K\), if \(B_2 \supset B_1\), then \(f_{B_2}(x) \geq f_{B_1}(x)\).
because for any \(C \in \mathcal{B}(E)\)
\[
\int_C f_{B_1}(x)T_3(dx) = \int_{B_1} T_1(y, C)T_2(dy).
\]
\[
\int_C f_{B_2}(x)T_3(dx) = \int_{B_2} T_1(y, C)T_2(dy).
\]
Therefore
\[
\int_C (f_{B_2}(x) - f_{B_1}(x))T_3(dx) \geq 0.
\]
for any \(C \in \mathcal{B}(E)\), so \(f_{B_2}(x) \geq f_{B_1}(x)\), \(T_3\) a.e.

So \(\mu(x, B_2) \geq \mu(x, B_1)\) for \(x \in F_1^c\). Therefore \(\lim_{m \to \infty} \mu(x, B_m)\) exists.

Let \(A \in \mathcal{J}\), then \(A = \bigcap_{i=1}^\infty B_i\), where \(B_i \in \mathcal{F}\), and \(B_1 \supset B_2 \supset \cdots\). So we can define \(\mu(x, A) = \lim_{i \to \infty} \mu(x, B_i)\). \(\mu(x, A)\) is a premeasure, because:

(i) \(\mu_n(x, \emptyset) = 0\)
(ii) \( \mu_n(x, \cdot) \geq 0. \)

(iii) For every sequence \( (F_m)_{m=1}^{\infty} \) of pairwise disjoint sets of \( \mathcal{J} \), with \( \bigcup_{m=1}^{\infty} F_m \in \mathcal{J} \), we have \( F_m = \bigcap_{l=1}^{\infty} B_l^{m} \), where \( (B_l^{m})_{l=1}^{\infty} \in \mathcal{F} \) are a decreasing sequence, for each \( m \), and \( (B_l^{m})_{m=1}^{\infty} \) are disjoint with each other for each \( l \). Since for any \( C \in \mathcal{B}(E) \), we have

\[
\int_C f_{\bigcup_{m=1}^{\infty} B_l^{m}}(x)T_3(dx) = \int_{\bigcup_{m=1}^{\infty} B_l^{m}} T_1(y, C)T_2(dy) = \sum_{m=1}^{\infty} \int_{B_l^{m}} T_1(y, C)T_2(dy) = \sum_{m=1}^{\infty} \int_C f_{B_l^{m}}(x)T_3(dx).
\]

(57)

So \( \int_{\bigcup_{m=1}^{\infty} B_l^{m}}(x) = \sum_{m=1}^{\infty} f_{B_l^{m}}(x) \), i.e.

\[
\mu(x, \bigcup_{m=1}^{\infty} B_l^{m}) = \sum_{m=1}^{\infty} \mu(x, B_l^{m}),
\]

(58)

for each \( l \). Therefore we have

\[
\mu(x, \bigcup_{m=1}^{\infty} F_m) = \mu(x, \bigcup_{m=1}^{\infty} \bigcap_{l=1}^{\infty} B_l^{m})
= \mu(x, \bigcap_{l=1}^{\infty} \bigcup_{m=1}^{\infty} B_l^{m})
= \lim_{l \to \infty} \mu(x, \bigcup_{m=1}^{\infty} B_l^{m})
= \sum_{m=1}^{\infty} \lim_{l \to \infty} \mu(x, B_l^{m}) = \sum_{m=1}^{\infty} \mu(x, F_m).
\]

(59)

\( \mu(x, \cdot) \) is \( \sigma \)-finite is clear. \( \Box \)

**Lemma 1.2** Suppose we have the same condition as lemma 1.11. Then there exists a measure \( \tilde{\mu}(x, \cdot) \) on \( \mathcal{B}(E) \), it is uniquely determined by \( \mu(x, \cdot) \) as in lemma 1.11, and for any \( A \in \mathcal{B}(E) \)

\[
\tilde{\mu}(x, A) = \frac{\int_A T_1(y, dx)T_2(dy)}{T_3(dx)},
\]

(60)
for $T_3$ a.e $x$. Also for any $A \in \mathcal{B}(E)$, $\tilde{\mu}(x, A)$ is a measurable function of $x$. We write as

$$\tilde{\mu}(x, dy) = \frac{T_1(y, dx)T_2(dy)}{T_3(dx)}.$$  \hfill (61)

**Proof**: This follows from lemma 1.11 and lemma 1.8, the extension theorem. □

**Lemma 1.13** There exists a measure $\tilde{\mu}_n(x, \cdot)$ on $\mathcal{B}((E)^n)$ for $x \in F^c \cap H^n$, it is uniquely determined up to null set $F$, where $H^n = \bigcup_{i \neq j=1}^{n} \{x_i = x_j\}$, and $S_n^{n-1}m_{n-1}(F) = 0$. Moreover, we also have

$$\tilde{\mu}_n(x, dy) = \frac{\frac{1}{2} \sum_{1 \leq i \neq j \leq n} \delta_{\sigma_{ij}(y)}(dx)m_n(dy)}{S_n^{n-1}m_{n-1}(dx)}.$$  \hfill (62)

**Proof**: By lemma 1.10 and lemma 1.11, we know

$$\mu_n(x, B) = \frac{1}{2} \int_B \sum_{1 \leq i \neq j \leq n} \delta_{\sigma_{ij}(y)}(dx)m_n(dy),$$  \hfill (63)

is a $\sigma$-finite premeasure on $J^n$, for $x \in H^n \cap F^c$, where $S_n^{n-1}m_{n-1}(F) = 0$. Therefore the result follows by lemma 1.8, the extension theorem. □

**Lemma 1.14** $\tilde{\mu}_n(x, (E)^n) = 1$ for $x \in H^n \cap F^c$, where $H^n = \bigcup_{i \neq j=1}^{n} \{x_i = x_j\}$, and $S_n^{n-1}m_{n-1}(F) = 0$.

**Proof**: Since

$$\frac{1}{2} \int_B \sum_{1 \leq i \neq j \leq n} \delta_{\sigma_{ij}(y)}(C)m_n(dy) = \int_C \tilde{\mu}_n(x, B)S_n^{n-1}m_{n-1}(dx),$$  \hfill (64)

for any $B \in \mathcal{B}((E)^n), C \in \mathcal{B}((E)^n)$.

Let $B \not\ni (E)^n$, LHS $\rightarrow \frac{1}{2} \int \sum_{1 \leq i \neq j \leq n} \delta_{\sigma_{ij}(y)}(C)m_n(dy) = S_n^{n-1}m_{n-1}(C)$. RHS $\rightarrow \int_C \tilde{\mu}_n(x, (E)^n)S_n^{n-1}m_{n-1}(dx)$, for any $C \in \mathcal{B}((E)^n)$.

So $\tilde{\mu}_n(x, (E)^n) = 1, S_n^{n-1}m_{n-1}$ a.e.

□
The above discussion is for any generator $A$. Now in the rest of this chapter, we will consider $A$ to be the generator of a jump process. We will derive a necessary and sufficient condition for the particle systems to be reversible. By the same idea of proof, we will derive a necessary condition for the Fleming-Viot process to be reversible.

Let $E$ be a Polish space, $B(E)$ be the space of real, bounded Borel functions on $E$. $C(E)$ be the space of continuous functions on $E$.

Let $f_n(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \psi_i(x_i)$, where $\psi_i \in B(E)$, for $i = 1, \ldots, n$.

\[
\mathcal{L} F_n(\mu) = \sum_{1 \leq i < j \leq n} (\langle \psi_i \psi_j, \mu \rangle - \langle \psi_i, \mu \rangle \langle \psi_j, \mu \rangle) \prod_{l \neq i, j} < \psi_l, \mu > + \sum_{i=1}^{n} < A \psi_i, \mu > \prod_{l \neq i} < \psi_l, \mu >, \tag{65}
\]

Where

\[
\mathcal{D}(\mathcal{L}) = \{ F_{\psi_1, \ldots, \psi_n} \in C(\mathcal{P}(E)) : \psi_1, \ldots, \psi_n \in \mathcal{D}(A), n \in \mathbb{N} \},
\]

\[
\mathcal{D}(A) = B(E),
\]

\[
F_{\psi_1, \ldots, \psi_n}(\mu) = \prod_{i=1}^{n} < \psi_i, \mu >,
\]

\[
A \psi(x) = \frac{\theta(x)}{2} \int_{E} (\psi(\zeta) - \psi(x)) p(x, d\zeta),
\]

\[
p(x, E) = 1.
\]

where $\theta(x)$ is a positive continuous function on $E$, and $p(x, d\zeta)$ is the transition kernel, satisfying:

i) $p(x, A)$ is a measurable function of $x$, for any $A \in B(E)$, the Borel measure sets of $E$.

ii) $p(x, \cdot)$ is a measure for any $x \in E$. 

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Let $m_1$ be a probability measure such that $< A\psi, m_1 > = 0$, for any $\psi \in B(E)$. Then there exists a stationary distribution $m$ for the Fleming-Viot process with type space $E$ and mutation operator $A$, [EK 3] Theorem 5.1. From lemma 1, we have a sequence of symmetric, consistent, Borel measures $(m_n)_{n=1}^{\infty}$, such that

$$\int_{\mathcal{M}_1(E)} < f_n, \mu_n > m(d\mu) = \int_{E^n} f_n(x)m_n(dx). \tag{66}$$

Except in the special case in which $p(x, d\zeta)$ doesn’t depend on $x$, the explicit form of $m$ seems to be unknown. But we have the following relations.

**Lemma 1.** 15:

$$\frac{n(n-1)}{2} + \sum_{i=1}^{n} \theta(x_i)m_n(dx_i) = S_{n-1} m_{n-1}(dx_i) + \frac{1}{2} K_n(dx). \tag{67}$$

Where

$$K_n(dx_1, \ldots dx_n) = \sum_{i=1}^{n} \int_{E} \theta(\zeta)p(\zeta, dx_i)m_n(dx_1, \ldots dx_{i-1}, d\zeta, dx_{i+1}, \ldots dx_n). \tag{68}$$

**Proof:** Let $\psi \in B(E)$, for $i = 1, \ldots n$. We have

$$\int_{\mathcal{M}_1(E)} \mathcal{L} F_{\psi_1, \ldots, \psi_n}(\mu)m(d\mu) = 0. \tag{69}$$

Since

$$\mathcal{L} F_{\psi_n}(\mu) = \sum_{1 \leq i < j \leq n} (< \psi_i \psi_j, \mu > - < \psi_i, \mu > < \psi_j, \mu >) \prod_{l \neq i, j} < \psi_l, \mu > + \sum_{i=1}^{n} < A\psi_i, \mu > \prod_{l \neq i} < f\psi_l, \mu >. \tag{70}$$

Therefore we have

$$\int_{E^n} \prod_{i=1}^{n} \psi_i(x_i)S_{n-1} m_{n-1}(dx_i) - \frac{n(n-1)}{2} \int_{E^n} \prod_{i=1}^{n} \psi_i(x_i)m_n(dx_i)$$
\[ + \sum_{i=1}^{n} \int_{\mathcal{M}_1(E)} < A\psi_i, \mu > \prod_{l \neq i} < \psi_l, \mu > m(d\mu) = 0. \tag{71} \]

Now the second term on RHS is

\[ \sum_{i=1}^{n} \int_{\mathcal{M}_1(E)} < A\psi_i, \mu > \prod_{l \neq i} < \psi_l, \mu > m(d\mu) = \sum_{i=1}^{n} \int_{E^n} \frac{\theta(x_i)}{2} \int_{E} \psi_i(\zeta)p(x_i, d\zeta) \prod_{l \neq i}^{n} \psi_l(x_l)m_n(dx) - \sum_{i=1}^{n} \int_{E^n} \frac{\theta(x_i)}{2} \prod_{i=1}^{n} \psi_i(x_i)m_n(dx). \tag{72} \]

By Changing the role of \( x \), and \( \zeta \) for the first term on RHS, we have

\[ \sum_{i=1}^{n} \int_{\mathcal{M}_1(E)} < A\psi_i, \mu > \prod_{l \neq i} < \psi_l, \mu > m(d\mu) = \sum_{i=1}^{n} \int_{E^n} \frac{\theta(x_i)}{2} \psi_i(\zeta)p(\zeta, dx_i) \prod_{l \neq i}^{n} \psi_l(x_l)m_n(dx_1, \ldots dx_{i-1}, d\zeta, dx_{i+1}, \ldots dx_n) - \sum_{i=1}^{n} \int_{E^n} \frac{\theta(x_i)}{2} \prod_{i=1}^{n} \psi_i(x_i)m_n(dx) = \frac{1}{2} \int_{E^n} \prod_{i=1}^{n} \psi_i(x_i)K_n(dx) - \int_{E^n} \frac{\sum_{i=1}^{n} \theta(x_i)}{2} \prod_{i=1}^{n} \psi_i(x_i)m_n(dx). \]

Therefore we will have

\[ \int_{E^n} \frac{n(n-1)}{2} + \sum_{i=1}^{n} \frac{\theta(x_i)}{2} \prod_{i=1}^{n} \psi_i(x_i)m_n(dx) = \int_{E^n} \prod_{i=1}^{n} \psi_i(x_i)m_{n-1}(dx) + \frac{1}{2} \int_{E^n} \prod_{i=1}^{n} \psi_i(x_i)K_n(dx). \tag{73} \]

Therefore for any \( f_n \in B(E^n) \), bounded Borel measurable functions on \( E^n \), we have

\[ \int_{E^n} \frac{n(n-1)}{2} + \sum_{i=1}^{n} \frac{\theta(x_i)}{2} f_n(x_1, \ldots, x_n)m_n(dx) \]
\[ = \int_{E^n} f_n(x_1, \ldots, x_n) S^{n-1}_* m_{n-1}(dx) \]
\[ + \frac{1}{2} \int_{E^n} f_n(x_1, \ldots, x_n) K_n(dx), \] (74)

because \( f_n(x_1, \ldots, x_n) \) can be approximated by linear combinations of form \( \prod_{i=1}^{n} \psi_i(x_i) \), where \( \psi_i \in B(E) \), for \( i = 1, \ldots, n \). So we have the equality of measures

\[
\frac{n(n-1) + \sum_{i=1}^{n} \theta(x_i)}{2} m_n(dx) = S^{n-1}_* m_{n-1}(dx) + \frac{1}{2} K_n(dx). \] (75)

\[ \square \]

**Lemma 1.16**: Let \( C_1, C_2 \in B(E) \). Then we have the following

\[
m_2(C_1, C_2) = m_1(C_1 \cap C_2) + \frac{1}{2} \left( \int_{\xi \in E} \theta(\xi) p(\xi, C_2) m_2(d\xi, C_1) - \int_{x_2 \in C_2} \theta(x_2) m_2(C_1, dx_2) \right) + \frac{1}{2} \int_{\xi \in E} \theta(\xi) p(\xi, C_1, m_2(d\xi, C_2) - \int_{x_2 \in C_1} \theta(x_2) m_2(C_2, dx_2)). \] (76)

**Proof**: Apply lemma 1.15 when \( n = 2 \), we have

\[
\frac{2 + \theta(x_1) + \theta(x_2)}{2} m_2(dx_1, dx_2)
\]
\[ = S_* m_1(dx_1) \]
\[ + \frac{1}{2} \left( \int_{\xi \in E} \theta(\xi) p(\xi, dx_1) m_2(d\xi, dx_2) \right) + \frac{1}{2} \int_{\xi \in E} \theta(\xi) p(\xi, dx_2) m_2(d\xi, dx_1). \] (77)

Therefore

\[
m_2(dx_1, dx_2)
\]

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\[ S \cdot m_1(dx_1) + \frac{1}{2} \left( \int_{\zeta \in E} \theta(\zeta)p(\zeta, dx_1)m_2(d\zeta, dx_2) - \theta(x_1)m_2(dx_1, dx_2) \right) + \frac{1}{2} \left( \int_{\zeta \in E} \theta(\zeta)p(\zeta, dx_2)m_2(d\zeta, dx_1) - \theta(x_2)m_2(dx_1, dx_2) \right). \]  

Integrate on both side over set \( C_1 \times C_2 \), we get the result.

Next, we are going to find the expression of \( L_n^- \) for \( n \in \mathcal{N} \). To do this, we need the following results

**Lemma 1.17**: Let \( \psi \in B(E) \). Let

\[ h_\psi ^i(dx_1, \ldots, dx_n) = \int_{\zeta \in E} \frac{\theta(\zeta)}{2} \psi(\zeta)p(\zeta, dx_1)m_n(dx_1 \cdots dx_{i-1}, d\zeta, dx_{i+1} \cdots dx_n). \]

Then \( h_\psi ^i \ll m_n \).

**Proof**: Let \( m_n(A_1 \times \cdots \times A_n) = 0 \), Then from lemma 1.15, we have

\[ K_n(A_1 \times \cdots \times A_n) = 0. \]

So

\[ \int_{\zeta \in E} \theta(\zeta)p(\zeta, A_i)m_n(A_1 \times \cdots \times A_{i-1}, d\zeta, A_{i+1} \cdots A_n) = 0, \forall i \in \{1, \cdots, n\}. \]

So \( \theta(\zeta)p(\zeta, A_i) = 0, m_1 \text{ a.e.} \).

Therefore

\[ \int_{\zeta \in E} \frac{\theta(\zeta)}{2} \psi(\zeta)p(\zeta, A_i)m_n(A_1 \times \cdots \times A_{i-1}, d\zeta, A_{i+1} \cdots A_n) = 0, \forall i \in \{1, \cdots, n\}. \]

So \( h_\psi ^i \ll m_n \) by \( \pi - \lambda \) Theorem. \( \square \)

From lemma 1.15, 1.17, we can see the existence of measurable functions

\[ \frac{S_n^{n-1}m_{n-1}(dx)K_n(dx)}{m_n(dx)} \]
and

\[ h_{\psi_1 \cdots \psi_n}(x_1, \cdots x_n) = \sum_{i=1}^{n} \prod_{k \neq i} \psi_k(x_k) \int_{E \in E} \frac{\theta(\psi_i(\xi)) p(\xi, dx_i) m_n(dx_1 \cdots dx_{i-1}, d\xi, dx_{i+1} \cdots dx_n)}{m_n(dx_1, \cdots dx_n)}, \]

for any \( \psi_i \in B(E) \), for \( i = 1, \cdots, n \).

**Theorem 1.1** Let \( \psi_1, \cdots, \psi_n \in B(E) \). Then

\[
\begin{align*}
L^n = \prod_{k=1}^{n} \psi_k(x_k) \\
= \int_{E^n} g_n(y) \tilde{\mu}_n(x, dy) \frac{S_{n-1}^{n-1} m_{n-1}(dx)}{m_n(dx)} \\
- \frac{n(n-1) + \sum_{i=1}^{n} \theta(x_i)}{2} \tilde{\mu}_n(x) \\
+ h_{\psi_1 \cdots \psi_n}(x_1, \cdots x_n) \\
= \int_{E^n} [g_n(y) - g_n(x)] \tilde{\mu}_n(x, dy) \frac{S_{n-1}^{n-1} m_{n-1}(dx)}{m_n(dx)} \\
+ B^n g_n(x),
\end{align*}
\]

where

\[
g_n(x) = \prod_{k=1}^{n} \psi_k(x_k),
\]

\[
\tilde{\mu}_n(x, dy) = \frac{1}{2} \sum_{1 \leq i \neq j \leq n} \delta_{\psi_i}(y)(dx) m_n(dy) \frac{S_{n-1}^{n-1} m_{n-1}(dx)}{m_n(dx)},
\]

\[
h_{\psi_1 \cdots \psi_n}(x_1, \cdots x_n),
\]

\[
= \sum_{i=1}^{n} \prod_{k \neq i} \psi_k(x_k) \int_{E \in E} \frac{\theta(\psi_i(\xi)) p(\xi, dx_i) m_n(dx_1 \cdots dx_{i-1}, d\xi, dx_{i+1} \cdots dx_n)}{m_n(dx_1, \cdots dx_n)},
\]

\[
B^n g_n(x) = h_{\psi_1 \cdots \psi_n}(x_1, \cdots x_n) - \frac{K_n(dx)}{2m_n(dx)} g_n(x),
\]

\[
= \sum_{i=1}^{n} \prod_{k \neq i} \psi_k(x_k) B \psi_i(x_i),
\]

\[
B \psi_i(x_i) = \int_{E \in E} \frac{\theta(\xi)}{2} (\psi_i(\xi) - \psi_i(x_i)) p(\xi, dx_i) m_n(dx_1 \cdots dx_{i-1}, d\xi, dx_{i+1} \cdots dx_n) \frac{m_n(dx)}{m_n(dx)},
\]

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The existence of $\mu_n(x, dy)$ is as in lemma 1.13, and the $m_n$'s, and $S_n^{n-1}m_{n-1}$ are as in lemma 1.4.

Proof: Let $f_n(x) = f_n(x_1, \cdots, x_n) = \prod_{i=1}^n \phi_i(x_i)$, where $\phi_i \in B(E)$, for $l = 1, 2, \cdots n$. Then

$$
\int_{E^n} g_n(x) L_n f_n(x) m_n(dx) = \frac{1}{2} \int_{E^n} g_n(x) \sum_{1 \leq i \neq j \leq n} f_n(\sigma_{ij}(x)) m_n(dx)
$$

$$
- \frac{n(n-1)}{2} \int_{E^n} g_n(x) f_n(x) m_n(dx) + \int_{E^n} g_n(x) A^n f_n(x) m_n(dx)
$$

$$
= \int_{E^n} f_n(x)(\int_{E^n} g_n(y) \mu_n(x, dy)) S_n^{n-1} m_{n-1}(dx)
$$

$$
- \frac{n(n-1)}{2} \int_{E^n} g_n(x) f_n(x) m_n(dx)
$$

$$
+ \sum_{i=1}^n \int_{E^n} \prod_{l=1}^n \psi_i(x_l) \prod_{k \neq i}^n \phi_k(x_k) \frac{\theta(x_i)}{2} \int_{E} \phi_i(\zeta)p(x_i, d\zeta) m_n(dx)
$$

$$
- \int_{E^n} \prod_{l=1}^n \psi_i(x_l) \frac{\sum_{i=1}^n \theta(x_i)}{2} m_n(dx)
$$

$$
= \int_{E^n} f_n(x)(\int_{E^n} g_n(y) \mu_n(x, dy)) S_n^{n-1} m_{n-1}(dx)
$$

$$
- \frac{n(n-1)}{2} \int_{E^n} g_n(x) f_n(x) m_n(dx)
$$

$$
+ \int_{E^n} \prod_{l=1}^n \phi_i(x_l) \sum_{i=1}^n \prod_{k \neq i}^n \psi_k(x_k) \int_{E} \frac{\theta(x)}{2} \psi_i(\zeta)p(\zeta, dx_i)m_n(dx_1, \cdots dx_{i-1}, dx_{i+1}, \cdots dx_n)
$$

$$
- \int_{E^n} f_n(x) g_n(x) \frac{\sum_{i=1}^n \theta(x_i)}{2} m_n(dx)
$$

$$
= \int_{E^n} f_n(x) L_n^* g_n(x) m_n(dx).
$$

So the theorem is proved.

□

In the formula of $L_n^*$, the first part is like some boundary condition, the second part is like a generator of n-dimensional Markov process acting on the
function, where the starting process is a jump process with kernel

$$k(x_{i}, d\zeta) = \frac{p(\zeta, dx_{i})m_{n}(dx_{1}, \cdots, dx_{i-1}, d\zeta, \cdots dx_{n})}{m_{n}(dx)}.$$ \hspace{1cm} (83)

which depends on $x$, the position of the $n$-particle.

From the formula of $L_{n}$, we can not find the condition for the process to be reversible. But in the rest of this section, we are able to find the condition directly without using the explicit formula of $L_{n}$.

**Lemma 1.18** Let $f \in B(E)$

$$Af(x) = \frac{\theta(x)}{2} \int_{\zeta \in E} (f(\zeta) - f(x)) p(x, d\zeta).$$ \hspace{1cm} (84)

where $\theta(x) \geq 0$ is a bounded continuous function, and $m_{1}\{x: \theta(x) = 0\} = 0$, and $m_{1}$ is the probability measure such that $<Af, m_{1}> = 0$, for any $f \in B(E)$. Then

i) $p(x, d\zeta) << m_{1}(d\zeta)$ for $m_{1}$ a.e. $x$.

ii) $p(x, d\zeta)$ has density $p(x, \zeta)$ relative to $m_{1}(d\zeta)$ and $p(x, \zeta)$ is a jointly measurable function of $(x, \zeta)$

iii) $\int_{\zeta \in E} \theta(x)p(x, \zeta)m_{1}(dx) = \theta(\zeta), \text{ for } m_{1} \text{ a.e } \zeta$.

**Proof:** Since

$$<Af, m_{1}> = 0, Af(x) = \frac{\theta(x)}{2} \int_{\zeta \in E} (f(\zeta) - f(x)) p(x, d\zeta).$$

Therefore $\int_{\zeta \in E} \theta(x) \int_{\zeta \in E} f(\zeta)p(x, d\zeta)m_{1}(dx) = \int_{\zeta \in E} \theta(x)f(x)m_{1}(dx)$.

So $\int_{\zeta \in E} f(\zeta)(\int_{\zeta \in E} \theta(x)p(x, d\zeta)m_{1}(dx) = \int_{\zeta \in E} \theta(\zeta)f(\zeta)m_{1}(d\zeta)$.

We know that if $f \in B(E)\nu_{1}(dx) = \int_{\zeta \in E} f(x)\nu_{2}(dx)$ for any $f \in B(E)$, then the two measures $\nu_{1}, \nu_{2}$ are equal.
So $\int_{x \in E} \theta(x)p(x,d\zeta)m_1(dx) = \theta(\zeta)m_1(d\zeta)$.

Therefore, if $m_1(F) = 0$ for $F \in B(E)$, then

$$\int_{x \in E} \theta(x)p(x,F)m_1(dx) = 0.$$  \hspace{1cm} (85)

We would have $\theta(x)p(x,F) = 0$, for $m_1$ a.e $x$. Since $\theta(x) > 0$, for $m_1$ a.e $x$, therefore $p(x,F) = 0$, for $m_1$ a.e $x$. So for any $C \in B(E)$, we have

$$\int_{x \in C} p(x,F)m_1(dx) = 0.$$  \hspace{1cm} (86)

So $\int_{x \in C} p(x,\cdot)m_1(dx) \ll m_1(\cdot)$. Therefore by lemma 1.6, Radon-Nikodym Theorem, there exists a measurable function

$$f_{\cdot}(\zeta) = \frac{\int_{x \in C} p(x,d\zeta)m_1(dx)}{m_1(d\zeta)}.$$  \hspace{1cm} (87)

Now apply lemma 1.12, we define a kernel

$$\tilde{\mu}(\zeta,dx) = \frac{p(x,d\zeta)m_1(dx)}{m_1(d\zeta)}.$$  \hspace{1cm} (88)

$\tilde{\mu}(\zeta,A)$ is a measurable function of $\zeta$ for any $A \in B(E)$, and

$$\tilde{\mu}(\zeta,A) = \int_{x \in A} \frac{p(x,d\zeta)m_1(dx)}{m_1(d\zeta)}.$$  \hspace{1cm} (89)

$$\int_{\zeta \in C} \tilde{\mu}(\zeta,A)m_1(d\zeta) = \int_{\zeta \in C} \int_{x \in A} p(x,d\zeta)m_1(dx),$$  \hspace{1cm} (90)

for any $A,C \in B(E)$. Also clearly $\tilde{\mu}(\zeta,\cdot) \ll m_1(\cdot)$, so we can suppose

$$\frac{\tilde{\mu}(\zeta,dx)}{m_1(dx)} = p(x,\zeta),$$  \hspace{1cm} (91)

which is a jointly measurable function of $(x,\zeta)$ by [PM], page 154. Therefore we have
\[ p(x, d\zeta) = p(x, \zeta)m_1(d\zeta) \], where \( p(x, \zeta) \) is a jointly measurable function of \((x, \zeta)\).

That proved i) and ii).

As for iii), it is a result from \( \int_{x \in E} \theta(x)p(x, d\zeta)m_1(dx) = \theta(\zeta)m_1(d\zeta) \), and ii) which means

\[
\int_{\zeta \in A} \int_{x \in E} \theta(x)p(x, d\zeta)m_1(dx) = \int_{\zeta \in A} \int_{x \in E} \theta(x)p(x, \zeta)m_1(d\zeta)m_1(dx) \\
= \int_{\zeta \in A} \theta(\zeta)m_1(d\zeta),
\]

(92)

for any \( A \in B(E) \). Therefore \( \int_{x \in E} \theta(x)p(x, \zeta)m_1(dx) = \theta(\zeta) \), for \( m_1 \) a.e \( \zeta \).

\[ \square \]

**Lemma 1.19** If we suppose \( L_1 = L_1^\ast \), ie

\[
\int_{x \in E} g_1(x)L_1f_1(x)m_1(dx) = \int_{x \in E} f_1(x)L_1g_1(x)m_1(dx)
\]

, for any \( f_1, g_1 \in B(E) \). Then \( \theta(x)p(x, \zeta) \) is symmetric in \( x \) and \( \zeta \), for \( m_1 \times m_1 \) a.e \((x, \zeta)\), ie \( \theta(x)p(x, \zeta) = \theta(\zeta)p(\zeta, x) \), for \( m_1 \times m_1 \) a.e \((x, \zeta)\).

**Proof:** \( \int_{x \in E} g_1(x)L_1f_1(x)m_1(dx) = \int_{x \in E} f_1(x)L_1g_1(x)m_1(dx) \).

So \( \int_{x \in E} g_1(x)Af_1(x)m_1(dx) = \int_{x \in E} f_1(x)Ag_1(x)m_1(dx) \).

\[
\int_{x \in E} g_1(x)\theta(x)\int_{\zeta \in E} f_1(\zeta)p(x, d\zeta)m_1(dx) \\
- \int_{x \in E} g_1(x)\theta(x)f_1(x)m_1(dx) \\
= \int_{x \in E} f_1(x)\theta(x)\int_{\zeta \in E} g_1(\zeta)p(x, d\zeta)m_1(dx) \\
- \int_{x \in E} g_1(x)\theta(x)f_1(x)m_1(dx).
\]

(93)

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So

\[ \int_{x \in E} g_1(x) \theta(x) \int_{\zeta \in E} f_1(\zeta)p(x, \zeta)m_1(dx) = \int_{x \in E} f_1(x) \theta(x) \int_{\zeta \in E} g_1(\zeta)p(x, \zeta)m_1(dx). \quad (94) \]

Therefore

\[ \int_{E \times E} g_1(x)f_1(\zeta)\theta(x)p(x, \zeta)m_1(dx)m_1(d\zeta) \]
\[ = \int_{E \times E} g_1(\zeta)f_1(x)\theta(x)p(x, \zeta)m_1(dx)m_1(d\zeta) \]
\[ = \int_{E \times E} g_1(x)f_1(\zeta)\theta(\zeta)p(\zeta, x)m_1(dx)m_1(d\zeta). \quad (95) \]

Let \( f_2(x, \zeta) \) be any Borel measurable function of two variables, then it can be approximated by linear combinations of form \( g_1(x)f_1(\zeta) \), where \( g_1, f_1 \in B(E) \).

Therefore

\[ \int_{E \times E} f_2(x, \zeta)\theta(x)p(x, \zeta)m_1(dx)m_1(d\zeta) = \int_{E \times E} f_2(x, \zeta)\theta(\zeta)p(\zeta, x)m_1(dx)m_1(d\zeta), \quad (96) \]

for any function \( f_2 \in B(E \times E) \). So \( \theta(x)p(x, \zeta) = \theta(\zeta)p(\zeta, x) \), for \( m_1 \times m_1 \) a.e \((x, \zeta)\).  

\[ \square \]

**Lemma 1. 20** If \( \int_{E \times E} g_1(x)L_1f_1(y)m_2(dx, dy) = \int_{E \times E} f_1(x)L_1g_1(y)m_2(dx, dy) \), for any \( f_1, g_1 \in B(E) \), then

\[ \int_{E \times E} \theta(\zeta)p(\zeta, dy)m_2(dx, d\zeta) - \theta(y)m_2(dx, dy) \] is symmetric in \( dx, dy \), i.e

\[ \int_{E \times E} \theta(\zeta)p(\zeta, dy)m_2(dx, d\zeta) - \theta(y)m_2(dx, dy) \]
\[ = \int_{E \times E} \theta(\zeta)p(\zeta, dx)m_2(dy, d\zeta) - \theta(x)m_2(dy, dx). \quad (97) \]
Proof: \( \int_{E \times E} g_1(x) g_1(y) m_2(dx, dy) = \int_{E \times E} f_1(x) L_1 g_1(y) m_2(dx, dy) \) implies

\[
\int_{E \times E} g_1(x) \frac{\theta(y)}{2} \left( \int_{\zeta \in E} f_1(\zeta)p(y, d\zeta) \right) m_2(dx, dy) \\
- \frac{1}{2} \int_{E \times E} g_1(x) \theta(y) f_1(y) m_2(dx, dy) \\
= \int_{E \times E} f_1(y) \frac{\theta(x)}{2} \left( \int_{\zeta \in E} g_1(\zeta)p(x, d\zeta) \right) m_2(dx, dy) \\
- \frac{1}{2} \int_{E \times E} g_1(x) \theta(x) f_1(y) m_2(dx, dy). \tag{98}
\]

Here we use the fact that \( m_2(dx, dy) = m_2(dy, dx) \), and also we change the role of \( x \), and \( y \) on the RHS.

So by changing the order of integrals, and changing the role of \( y \) and \( \zeta \) for the first term on the LHS, and changing the role of \( x \) and \( \zeta \) for the first term on the RHS, we have the following:

\[
\int_{E \times E} g_1(x) f_1(y) \left( \int_{\zeta \in E} \frac{\theta(\zeta)}{2} p(\zeta, dy) m_2(dx, d\zeta) - \frac{\theta(y)}{2} m_2(dx, dy) \right) \\
= \int_{E \times E} f_1(y) g_1(x) \left( \int_{\zeta \in E} \frac{\theta(\zeta)}{2} p(\zeta, dx) m_2(dy, d\zeta) - \frac{\theta(x)}{2} m_2(dx, dy) \right). \tag{99}
\]

Therefore

\[
\int_{E \times E} f_2(x, y) \left( \int_{\zeta \in E} \frac{\theta(\zeta)}{2} p(\zeta, dy) m_2(dx, d\zeta) - \frac{\theta(y)}{2} m_2(dx, dy) \right) \\
= \int_{E \times E} f_2(x, y) \left( \int_{\zeta \in E} \frac{\theta(\zeta)}{2} p(\zeta, dx) m_2(dy, d\zeta) - \frac{\theta(x)}{2} m_2(dx, dy) \right). \tag{100}
\]

for any measurable function \( f_2 \in B(E \times E) \), because it can be approximated by linear combinations of form \( g_1(x) f_1(y) \), where \( g_1, f_1 \in B(E) \). Therefore the two measures on both sides are equal, i.e

\[
\int_{\zeta \in E} \frac{\theta(\zeta)}{2} p(\zeta, dy) m_2(dx, d\zeta) - \frac{\theta(y)}{2} m_2(dx, dy) \tag{101}
\]

is symmetric in \( dx, dy \). \( \Box \)
Lemma 1.21 If \( \int_{E \times E} g_1(x)L_1 f_1(y)m_2(dx, dy) = \int_{E \times E} f_1(x)L_1 g_1(y)m_2(dx, dy) \), for any \( f_1, g_1 \in B(E) \), then

\[
m_2(C_1, C_2) = m_1(C_1 \cap C_2) + \int_{\xi \in E} \theta(\xi)p(\xi, C_2)m_2(d\xi, C_1) - \int_{x_2 \in C_2} \theta(x_2)m_2(C_1, dx_2),
\]

for any \( C_1, C_2 \in B(E) \).

**Proof**: It is a direct result from lemma 1.16 and lemma 1.20.

Lemma 1.22 \( m_2(A, dx) = f_A(x)m_1(dx) \), for some measurable function \( f_A(x) \), for any \( A \in B(E) \), and this \( f_A(x) \) is uniquely determined, up to a \( m_1 \)-null set.

And \( f_A(x) = \mu(x, A) \), for some kernel \( \mu(x, dy) \).

**Proof**: Let \( m_1(B) = 0 \), then \( m_2(E, B) = 0 \). So \( m_2(A, B) \leq m_2(E, B) = 0 \), So \( m_2(A, \cdot) \ll m_1(\cdot) \) for any set \( A \in B(E) \).

We can apply lemma 1.12, the result follows. □

Lemma 1.23 Let \( m_2(C, dx) = f_c(x)m_1(dx) \), for any \( C \in B(E) \). Suppose

\[
\int_{\xi \times x \in D} (f_c(x) - f_c(\xi))m_1(d\xi)m_1(dx) = 0,
\]

for any \( D = A \times B \), where \( A, B \in B(E \setminus C) \).

Then \( f_c(x) = km_1(C) \), when \( x \in (C)^c \cap M^c \), where \( m_1(M) = 0 \).

**Proof**: line (105) is the same as

\[
\int_{\xi \times x \in D} f_c(x_2)m_1(d\xi)m_1(dx_2) = \int_{\xi \times x \in D} f_c(\xi)m_1(d\xi)m_1(dx_2).
\]

Therefore, we have

\[
\int_{\xi \times x \in D} m_2(C, dx_2)m_1(d\xi) = \int_{\xi \times x \in D} m_2(C, d\xi)m_1(dx_2).
\]
Therefore

\[ m_2(C, B)m_1(A) = m_2(C, A)m_1(B), \quad (106) \]

for any \( A, B \in B((E \setminus C)) \). We can see from (108) that \( m_2(C, B) / m_1(B) \) does not depend on \( B \). Now since that \( m_2(C, B) / m_1(B)m_1(C) \) is symmetric in \( B, C \), and does not depend on \( B \), so it does not depend on \( C \) either. Therefore, we have

\[ k = \frac{m_2(C, B)}{m_1(C)m_1(B)}, \quad (107) \]

a constant.

Now for any \( A, C \in B(E) \), we have

\[ \int_{A \cap C^c} m_2(C, dx) = \int_{A \cap C^c} f_c(x)m_1(dx) = \int_A f_c(x)1_{C^c}(x)m_1(dx). \quad (108) \]

But the right hand side of (110) is

\[ m_2(C, A \cap C^c) = km_1(C)m_1(A \cap C^c) = \int_A km_1(C)1_{C^c}(x)m_1(dx). \quad (109) \]

Therefore, we have

\[ \int_A f_c(x)1_{C^c}(x)m_1(dx) = \int_A km_1(C)1_{C^c}(x)m_1(dx). \quad (110) \]

Since \( A \) is arbitrary, so we have

\[ f_c(x)1_{C^c}(x) = km_1(C)1_{C^c}(x), \quad m_1 \text{ a.e } x. \]

Therefore \( f_C(x_2) = km_1(C) \), when \( x_2 \in (C)^c \cap M^c \), where \( m_1(M) = 0 \).

\[ \square \]

**Lemma 1.24** Let the generator \( A \) be the generator of a jump process as in (86). Assume \( \theta(x) = \theta \), a positive constant. \( m_1 \) is a probability measure satisfying
< A, m_1 > = 0, m_2 are as in lemma 1.4. If

$$\int_{\zeta \in E} p(\zeta, C_2)m_2(d\zeta, C_1) = m_1(C_1)m_1(C_2). \quad (111)$$

for any $C_1, C_2 \in B(E)$, satisfying $C_1 \cap C_2 = \emptyset$. Then

$$p(x, d\zeta) = m_1(d\zeta). \quad (112)$$

**Proof:** The condition of (113) implies

$$\int_{\zeta \in E} p(\zeta, C_2)m_2(d\zeta, E \setminus C_2) = m_1(C_2)m_1(E \setminus C_2),$$

for any $C_2 \in B(E)$.

So $\int_{\zeta \in E} p(\zeta, C_2)m_1(d\zeta) - \int_{\zeta \in E} p(\zeta, C_2)m_2(d\zeta, C_2) = m_1(C_2) - m_1(C_2)^2$.

Since $\int_{\zeta \in E} p(\zeta, C_2)m_1(d\zeta) = m_1(C_2)$ by lemma 1.18 iii). Therefore we have

$$\int_{\zeta \in E} p(\zeta, C_2)m_2(d\zeta, C_2) = m_1(C_2)m_1(C_2). \quad (113)$$

Now for any set $A, B$, let $A \cap B = C_1, A \setminus C_1 = C_2, B \setminus C_1 = C_3$.

Then $A = C_1 \cup C_2, B = C_1 \cup C_3$, and

$$\int_{\zeta \in E} p(\zeta, A)m_2(d\zeta, B) = \int_{\zeta \in E} (p(\zeta, C_2) + p(\zeta, C_1))(m_2(d\zeta, C_1) + m_2(d\zeta, C_3))$$

$$= \int_{\zeta \in E} p(\zeta, C_1)m_2(d\zeta, C_1) + \int_{\zeta \in E} p(\zeta, C_1)m_2(d\zeta, C_3)S_{n-1}m_{n-1}$$

$$+ \int_{\zeta \in E} p(\zeta, C_2)m_2(d\zeta, C_1) + \int_{\zeta \in E} p(\zeta, C_2)m_2(d\zeta, C_3)S_{n-1}m_{n-1}$$

$$= m_1(C_1)m_1(C_1) + m_1(C_1)m_1(C_3) + m_1(C_1)m_1(C_2) + m_1(C_2)m_1(C_3)$$

$$= m_1(A)m_1(B). \quad (114)$$

Therefore

$$\int_{\zeta \in E} p(\zeta, dx)m_2(d\zeta, dy) = m_1(dx)m_1(dy). \quad (115)$$
Now by lemma 1.21 and (117), we have

\[(1 + \theta)m_2(dx, dy) = S_m(dx, dy) + \theta m_1(dx)m_1(dy).\] (116)

Now we multiply both sides of (118) by \(p(x, z)\), and integrate with respect to \(x\) over \(E\), we have the following

\[
(1 + \theta) \int_{x \in E} p(x, z)m_2(dx, dy) = \int_{x \in E} p(x, z)S_m(dx, dy)
+ \theta \int_{x \in E} p(x, z)m_1(dx)m_1(dy).
\] (117)

Since \(\int_{y \in E} \int_{x \in E} g(y)f(x, y)S_m(dx, dy) = \int_{y \in E} g(y)f(y, y)m_1(dy)\).

So we have the following equality of measures

\[
\int_{x \in E} f(x, y)S_m(dx, dy) = f(y, y)m_1(dy).
\] (118)

Now choose \(f(x, y) = p(x, z)\) in line (120), we will have

\[
\int_{x \in E} p(x, z)S_m(dx, dy) = p(y, z)m_1(dy).
\] (119)

Now apply the equality in line (121) into (119), and also observe that

\[
\int_{x \in E} p(x, z)m_1(dx) = 1,
\] (120)

for \(m_1\) a.e \(z\), we will have

\[
(1 + \theta) \int_{x \in E} p(x, z)m_2(dx, dy) = p(y, z)m_1(dy) + \theta m_1(dy),
\] (121)

for \(m_1\) a.e \(z\). By multiply both sides of (123) by \(m_1(dz)\), we have

\[
(1 + \theta) \int_{x \in E} p(x, dz)m_2(dx, dy) = p(y, z)m_1(dz)m_1(dy) + \theta m_1(dy)m_1(dz).
\] (122)
That is

\[(1 + \theta)m_1(dy)m_1(dz) = p(y, dz)m_1(dy) + \theta m_1(dy)m_1(dz), \quad (123)\]

by using (117). and \(p(y, dz) = p(y, z)m(dz)\). Therefore \(p(y, dz) = m_1(dz)\). So we get the result of this lemma. □

**Theorem 1.2** Let \(A\) be the generator of the pure jump process as in (86). \(\theta(x) > 0\) is a bounded continuous function, and the transition kernel \(p(x, \cdot)\) is nonatomic for any \(x\), i.e. \(p(x, \{\zeta\}) = 0\) for any \(x, \zeta \in E\) (which means that every mutant is of a new type). \(m_1\) is the probability measure such that \(< A f, m_1 > = 0\). Let

\[\overline{L}_2f(x, y) = f(x, x) + f(y, y) - f(x, y) + A^2f(x, y). \quad (124)\]

Suppose \(L_1 = L_1^*\), also suppose \(\overline{L}_2 = \overline{L}_2^*\). Then \(\theta(x) = \theta\), a positive constant, and \(p(x, d\zeta) = m_1(d\zeta)\)

**Proof:** By lemma 18 iii), we have

\[\int \theta(x)p(x, \{\zeta\})m_1(dx) = \theta(\zeta)m_1(\{\zeta\}). \quad (125)\]

We can see from this equation that \(m_1(\{\zeta\}) = 0\) iff \(p(x, \{\zeta\}) = 0\) for \(m_1\) a.e \(x\). Therefore we have \(m_1(\{\zeta\}) = 0\) for any \(\zeta \in E\). Now \(\overline{L}_2 = \overline{L}_2^*\). So

\[\int_{E^2} g_1(x_1)g_2(x_2)\overline{L}_2 f_1(x_1)f_2(x_2)m_2(dx_1, dx_2) = \int_{E^2} f_1(x_1)f_2(x_2)\overline{L}_2 g_1(x_1)g_2(x_2)m_2(dx_1, dx_2). \quad (126)\]

for any \(g_1, g_2, f_1, f_2 \in B(E)\).

Let \(g_1(x_1) = 1_{B_1}(x_1), g_2(x_2) = 1_{B_2}(x_2), f_1(x_1) = 1_{A_1}(x_1), f_2(x_2) = 1_{A_2}(x_2)\). And suppose \(A_1 \cap A_2 = \emptyset, B_1 \cap B_2 = \emptyset\), then

\[\int_{E^2} g_1(x_1)g_2(x_2)A^2 f_1(x_1)f_2(x_2)m_2(dx_1, dx_2)\]

40
\[ = \int_{E^2} f_1(x_1)f_2(x_2)A^2g_1(x_1)g_2(x_2)m_2(dx_1, dx_2). \quad (127) \]

So we have

\[
\int_{E^2} 1_{B_1}(x_1)1_{B_2}(x_2)1_{A_1}(x_1)\theta p(x_2, A_2)m_2(dx_1, dx_2)
\]

\[- \int_{E^2} 1_{B_1}(x_1)1_{B_2}(x_2)1_{A_2}(x_2)\theta m_2(dx_1, dx_2)\]

\[+ \int_{E^2} 1_{B_1}(x_1)1_{B_2}(x_2)1_{A_2}(x_2)p(x_1, A_1)\theta m_2(dx_1, dx_2)\]

\[- \int_{E^2} 1_{B_1}(x_1)1_{B_2}(x_2)1_{A_1}(x_1)1_{A_2}(x_2)m_2(dx_1, dx_2)\]

\[= \int_{E^2} 1_{A_1}(x_1)1_{A_2}(x_2)1_{B_1}(x_1)\theta p(x_2, B_2)m_2(dx_1, dx_2)\]

\[- \int_{E^2} 1_{A_1}(x_1)1_{A_2}(x_2)1_{B_1}(x_1)1_{B_2}(x_2)\theta m_2(dx_1, dx_2)\]

\[+ \int_{E^2} 1_{A_1}(x_1)1_{A_2}(x_2)1_{B_2}(x_2)\theta(x_1)p(x_1, B_1)m_2(dx_1, dx_2)\]

\[- \int_{E^2} 1_{A_1}(x_1)1_{A_2}(x_2)1_{B_1}(x_1)1_{B_2}(x_2)\theta m_2(dx_1, dx_2). \quad (128)\]

Therefore, we have

\[
\int_{E^2} 1_{A_1}\cap B_1(x_1)1_{B_2}(x_2)p(x_2, A_2)m_2(dx_1, dx_2)
\]

\[- \int_{E^2} 1_{A_1}\cap B_1(x_1)1_{A_2}(x_2)p(x_2, B_2)m_2(dx_1, dx_2)\]

\[= \int_{E^2} 1_{A_2}\cap B_2(x_2)1_{A_1}(x_1)p(x_1, B_1)m_2(dx_1, dx_2)\]

\[- \int_{E^2} 1_{A_2}\cap B_2(x_2)1_{B_1}(x_1)p(x_1, A_1)m_2(dx_1, dx_2). \quad (129)\]

RHS is symmetric in \( A_2, B_2\), so LHS is also symmetric in \( A_2, B_2\). But LHS is also antisymmetric in \( A_2, B_2\). So these implies

\[
\int_{E^2} 1_{A_1}\cap B_1(x_1)1_{B_2}(x_2)p(x_2, A_2)m_2(dx_1, dx_2)
\]

\[= \int_{E^2} 1_{A_1}\cap B_1(x_1)1_{A_2}(x_2)p(x_2, B_2)m_2(dx_1, dx_2). \quad (130)\]
Now that

\[
\int_{E^2} 1_{A_1 \cap B_1}(x_1) 1_{B_2}(x_2)p(x_2, A_2)m_2(dx_1, dx_2)
= \int_{E} 1_{B_2}(x_2)p(x_2, A_2)m_2(A_1 \cap B_1, dx_2)
= \int_{\times x_2 \in A_2 \times B_2} p(x_2, \zeta)m_1(d\zeta)m_2(A_1 \cap B_1, dx_2)
= \int_{\times x_2 \in A_2 \times B_2} p(x_2, \zeta) \frac{m_2(A_1 \cap B_1, dx_2)}{m_1(dx_2)}m_1(d\zeta)m_1(dx_2)
= \int_{\times x_2 \in A_2 \times B_2} f_{A_1 \cap B_1}(x_2)p(x_2, \zeta)m_1(d\zeta)m_1(dx_2),
\]

where the last equality is by lemma 1.22. So by changing the roles of variables, we have

\[
\int_{\times x_2 \in A_2 \times B_2} f_{A_1 \cap B_1}(x_2)p(x_2, \zeta)m_1(d\zeta)m_1(dx_2)
= \int_{\times x_2 \in B_2 \times A_2} f_{A_1 \cap B_1}(x_2)p(x_2, \zeta)m_1(d\zeta)m_1(dx_2)
= \int_{\times x_2 \in A_2 \times B_2} f_{A_1 \cap B_1}(\zeta)p(\zeta, x_2)m_1(\zeta)m_1(dx_2).
\]

Let $A_1 = B_1 = C$, be arbitrarily fixed. Since $L_1 = L_1^*$, so by lemma 1.19, we have $p(x_2, \zeta) = p(\zeta, x_2)$. Therefore

\[
\int_{\times x_2 \in A_2 \times B_2} (f_C(x_2) - f_C(\zeta))p(x_2, \zeta)m_1(d\zeta)m_1(dx_2) = 0.
\]

It is true for any $A_2 \cap C = \emptyset, B_2 \cap C = \emptyset$.

Let $F = \{D \subseteq (E \setminus C) \times (E \setminus C) : \int_{\times x_2 \in D} (f_C(x_2) - f_C(\zeta))p(x_2, \zeta)m_1(d\zeta)m_1(dx_2) = 0\}$.

Then it is clear that $F$ is a Borel system.

\{ $A_2 \times B_2 : A_2 \in E \setminus C, B_2 \in E \setminus C$ \} is a $\pi$-system, and satisfies conditions of $F$. So by $\pi - \lambda$ system, $F$ is the $\sigma$-algebra on $(E \setminus C) \times (E \setminus C)$.

Let $D^+ = \{ \zeta \times x_2 \in (E \setminus C) \times (E \setminus C) : f_C(x_2) - f_C(\zeta) > 0\}$.
and $D^- = \{ \zeta \times x_2 \in (E \setminus C) \times (E \setminus C) : f_C(x_2) - f_C(\zeta) < 0 \}$.

Then $D^+, D^- \in B((E \setminus C) \times (E \setminus C)$ So $D^+, D^- \in \mathcal{F}$, which implies

$$m_1 \times m_1(D^+) = 0, m_1 \times m_1(D^-) = 0. \quad (134)$$

Therefore

$$\int_{\zeta \times x_2 \in D} (f_C(x_2) - f_C(\zeta))m_1(d\zeta)m_1(dx_2) = 0, \quad (135)$$

for any $D = A \times B$, where $A, B \in B(E \setminus C)$. So by lemma 1.23, we have

$f_C(x_2) = km_1(C)$, when $x_2 \in (C)^c \cap M^c$, where $m_1(M) = 0$.

Observe that $L_2 = L_2^*$ implies the conditions of lemma 1.21, therefore we can apply lemma 1.21, which is

$m_2(C_1, C_2) = m_1(C_1 \cap C_2) + \int_{\zeta \in E} \theta(\zeta)p(\zeta, C_2)m_2(d\zeta, C_1) - \int_{x_2 \in C_2} \theta(x_2)m_2(C_1, dx_2), \quad (136)$

for any $C_1, C_2 \in B(E)$. Let $C_2 \in B(E)$ be arbitrary. So

$$f_C(x_2) = \int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_2(d\zeta, C_1) - \theta(x_2)f_{C_1}(x_2), \quad (137)$$

when $x_2 \in (C_1)^c$.

Therefore

$$\int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_2(d\zeta, C_1) = (\theta(x_2) + 1)f_{C_1}(x_2). \quad (138)$$

Therefore we have

$$\int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_2(d\zeta, C_1) = (\theta(x_2) + 1)km_1(C_1). \quad (139)$$

when $x_2 \in (C_1)^c \cap M^c$, for some set $M$, such that $m_1(M) = 0$, and $k$ does not depend on $C_1$. Now that

$$\int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_2(d\zeta, C_1 \cup M) = \int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_2(d\zeta, C_1)$$
\[ + \int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_2(d\zeta, M) \]
\[ = \int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_2(d\zeta, C_1), \quad (140) \]

since \( \int_{\zeta \in E} m_2(d\zeta, M) = m_1(M) = 0 \), which implies that \( \int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_2(d\zeta, M) = 0 \). But also we have

\[ (\theta(x_2) + 1)km_1(C_1 \cup M) = (\theta(x_2) + 1)km_1(C_1). \quad (141) \]

Therefore we can define a new set \( C_1 \) to be \( C_1 \cup M \), and still have

\[ \int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_2(d\zeta, C_1) = (\theta(x_2) + 1)km_1(C_1), \quad (142) \]

when \( x_2 \in (C_1)^c \), and \( k \) does not depend on \( C_1 \).

Therefore we have

\[ \frac{1}{\theta(x_2)} \int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_2(d\zeta, C_1) = \frac{(\theta(x_2) + 1)}{\theta(x_2)}km_1(C_1). \quad (143) \]

when \( x_2 \in (C_1)^c \). We will show

\[ \frac{(\theta(x_2) + 1)}{\theta(x_2)}k = 1. \quad (144) \]

Since

\[ \frac{1}{\theta(x_2)} \int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_2(d\zeta, E) \]

\[ = \frac{1}{\theta(x_2)} \int_{\zeta \in E} \theta(\zeta)p(\zeta, x_2)m_1(d\zeta) = 1, \quad (145) \]

for \( m_1 \) a.e \( x_2 \). So

\[ \lim_{C \rightarrow E} \int_{\zeta \in E} \frac{1}{\theta(x_2)}\theta(\zeta)p(\zeta, x_2)m_2(d\zeta, C) = 1, \quad (146) \]
for \( m_1 \) a.e. \( x_2 \)

Let

\[
A = \sup_{x_2 \in B} \frac{1 + \theta(x_2)}{\theta(x_2)} k.
\]  

(147)

\[
B = \min_{x_2 \in B} \frac{1 + \theta(x_2)}{\theta(x_2)} k.
\]  

(148)

We will show \( A = B = 1 \).

For any \( \epsilon > 0 \), there exists \( C_1 \), such that \( m_1(C_1) < 1 \), and \( x_2 \in C_1^c \)

( if \( x_2 \in C_1 \), then we take \( C_1 \setminus \{x_2\} \), such that

\[
1 - \int_{C_1} \frac{1}{\theta(x_2)} \theta(\zeta) p(\zeta, x_2) m_2(d\zeta, C_1) < \epsilon.
\]  

(149)

Therefore we have

\[
1 - \frac{1 + \theta(x_2)}{\theta(x_2)} km_1(C_1) < \epsilon.
\]  

(150)

\[
\frac{1 + \theta(x_2)}{\theta(x_2)} k > \frac{1 - \epsilon}{m_1(C_1)} > 1 - \epsilon.
\]  

(151)

So

\[
A > 1 - \epsilon.
\]  

(152)

Since \( \epsilon > 0 \) was arbitrary, so \( A \geq 1 \).

If \( A > 1 \), then we can find \( \epsilon \), such that \( A - \epsilon > 1 \).

\[
\frac{1 + \theta(x_2)}{\theta(x_2)} k
\]  

(153)

is a continuous function of \( x_2 \). So there exists open set \( F \) such that for any \( x \in F \), we have

\[
\frac{1 + \theta(x)}{\theta(x)} k > A - \epsilon.
\]  

(154)

We can find \( C_1 \), such that

\[
1 > m_1(C_1) > \frac{1}{A - \epsilon},
\]  

(155)
and also \( m_1(C_i^c \cap F) \neq 0 \). Therefore

\[
\int_{z \in \mathcal{E}} \frac{1}{\theta(x_2)} \theta(z) p(z, x_2) m_2(dz, C_i) = \frac{1 + \theta(x_2)}{\theta(x_2)} k m_1(C_i) > (A - \epsilon)m_1(C_i) > 1 \quad (156)
\]

for

\[
x_2 \in \{ x : \frac{1 + \theta(x)}{\theta(x)} k > A - \epsilon \} \cap C_i^c = F \cap C_i^c. \quad (157)
\]

Which is clearly impossible.

So \( A = 1 \).

Therefore \( B < 1 \).

If we suppose \( B < 1 \). Then there exists \( \epsilon \), such that \( B + \epsilon < 1 \). But we can choose \( C_1 \) large such that \( m_1(C_1) = 1 \), and also we can find

\[
x_2 \in C_i^c \cap \{ x : \frac{1 + \theta(x)}{\theta(x)} k < B + \epsilon \}, \quad (158)
\]

such that

\[
\int_{z \in \mathcal{E}} \frac{1}{\theta(x_2)} \theta(z) p(z, x_2) m_2(dz, C_i) = \frac{1 + \theta(x_2)}{\theta(x_2)} k m_1(C_i) < (B + \epsilon)m_1(C_i) < 1. \quad (159)
\]

But on the other hand, we would also have

\[
\int_{z \in \mathcal{E}} \frac{1}{\theta(x_2)} \theta(z) p(z, x_2) m_2(dz, C_i) = 1. \quad (160)
\]

We have a contraction.

So \( A = B = 1 \). Therefore \( \theta(x) = \theta \), a constant. And

\[
\int_{z \in \mathcal{E}} \frac{1}{\theta(x_2)} \theta(z) p(z, x_2) m_2(dz, C_i) = \int_{z \in \mathcal{E}} p(z, x_2) m_2(dz, C_i) = m_1(C_i). \quad (161)
\]
for $m_1$ a.e $x_2$. Then we will get

$$\int_{\zeta \in E} p(\zeta, C_2) m_2(d\zeta, C_1) = m_1(C_1)m_1(C_2). \quad (162)$$

when $C_1 \cap C_2 = \emptyset$.

So now we can apply lemma 1.24. Therefore the theorem was proved. □

A little computation shows that $p(x, d\zeta) = m_1(d\zeta), \theta(x) = \theta$ is not only necessary for $A = A^*$, and $\mathcal{L}_2 = \mathcal{L}_2^*$ but also sufficient. As the following lemma shows.

**Lemma 1.25** Conditions are the same as in Theorem 1.2, suppose $A = A^*$

$$\mathcal{L}_2 = \mathcal{L}_2^* \iff p(x, d\zeta) = m_1(d\zeta), \theta(x) = \theta \quad (163)$$

**Proof** : $\implies$ follows from theorem 1.2.

$\iff$ Now suppose $p(x, d\zeta) = m_1(d\zeta), \theta(x) = \theta$.

Then we will have

$$m_2(dx, dy) = \frac{S_m(dx, dy)}{1 + \theta} + \frac{\theta}{1 + \theta} m_1(dx)m_1(dy). \quad (164)$$

And

$$\mathcal{L}_2 f_1(x_1) f_2(x_2) = \frac{1}{2} (f_1(x_1) f_2(x_1) + f_1(x_2) f_2(x_2)) - f_1(x_1) f_2(x_2)$$

$$+ \frac{\theta}{2} < f_1, m_1 > f_2(x_2) + \frac{\theta}{2} < f_2, m_1 > f_1(x_1) - \theta f_1(x_1) f_2(x_2). \quad (165)$$

Therefore

$$\int_{E^2} g_1(x_1) g_2(x_2) \mathcal{L}_2 f_1(x_1) f_2(x_2) m_2(dx_1, dx_2)$$

$$= \frac{1}{2} \int_{E^2} g_1(x_1) g_2(x_2) (f_1(x_1) f_2(x_1) + f_1(x_2) f_2(x_2))$$

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\[
\begin{align*}
\left[ \frac{S_\theta m_1(dx_1, dx_2)}{1 + \theta} + \frac{\theta}{1 + \theta} m_1(dx_1) m_1(dx_2) \right] \\
- (\theta + 1) \int_{E^2} g_1(x_1) g_2(x_2) f_1(x_1) f_2(x_2) m_2(dx_1, dx_2) \\
+ \frac{\theta}{2} < f_1, m_1 > \int_{E^2} g_1(x_1) g_2(x_2) f_2(x_2) \left( \frac{S_\theta m_1(dx_1, dx_2)}{1 + \theta} + \frac{\theta}{1 + \theta} m_1(dx_1) m_1(dx_2) \right) \\
+ \frac{\theta}{2} < f_2, m_1 > \int_{E^2} g_1(x_1) g_2(x_2) f_1(x_1) \left( \frac{S_\theta m_1(dx_1, dx_2)}{1 + \theta} + \frac{\theta}{1 + \theta} m_1(dx_1) m_1(dx_2) \right) \\
= \frac{1}{1 + \theta} < g_1 g_2 f_1 f_2, m_1 > + \frac{\theta}{2(1 + \theta)} < g_2, m_1 > < g_1 f_1 f_2, m_1 > \\
+ \frac{\theta}{2(1 + \theta)} < g_1, m_1 > < g_2 f_1 f_2, m_1 > \\
- (\theta + 1) \int_{E^2} g_1(x_1) g_2(x_2) f_1(x_1) f_2(x_2) m_2(dx_1, dx_2) \\
+ \frac{\theta}{2} < f_1, m_1 > \left( \frac{1}{1 + \theta} < g_1 g_2 f_2, m_1 > + \frac{\theta}{1 + \theta} < g_1, m_1 > < g_2 f_2, m_1 > \right) \\
+ \frac{\theta}{2} < f_2, m_1 > \left( \frac{1}{1 + \theta} < g_1 g_2 f_1, m_1 > + \frac{\theta}{1 + \theta} < g_2, m_1 > < g_1 f_1, m_1 > \right) \\
= \int_{E^2} f_1(x_1)) f_2(x_2)) \overline{L}_2 g_1(x_1)) g_2(x_2)) m_2(dx_1, dx_2). \quad (166)
\end{align*}
\]

Therefore \( \overline{L}_2 = \overline{L}_2^* \).

\[\square\]

Because of the above result, we would think of a more general result as the following theorem shows:

**Theorem 1.3** Let \( A \) be the generator of a jump process as (86). Suppose \( \theta(x) > 0 \) continuous, and the transition kernel \( p(x, \cdot) \) is nonatomic for any \( x \in E \).

\( m_1 \) is the stationary distribution satisfying

\[< Af, m_1 > = 0. \ m_n \text{'s are as in lemma 1.4.} \ Let

\[\overline{L}_n f_n(x) = \frac{1}{2} \sum_{1 \leq i \neq j \leq n} (f_n(\sigma_{ij}(x)) - f_n(x)) + A^n f_n(x). \quad (167)\]

\( \overline{L}_n^* \) is the adjoint operator of \( \overline{L}_n \) in the space \( L^2(m_n) \), satisfying

\[\int_{E^n} g_n(x) \overline{L}_n f_n(x) m_n(dx) = \int_{E^n} f_n(x) \overline{L}_n^* g_n(x) m_n(dx), \quad (168)\]

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for any \( g_n(x), f_n(x) \in B(E^n) \). Then

\[ \bar{L}_n = L^n, \text{ for } n \in \mathcal{N}, \text{ iff } \theta(x) = \theta, p(x, d\zeta) = m_1(d\zeta). \]

**Proof:** \( \Rightarrow \) follows from theorem 1.2.

\( \Leftarrow \) Now suppose \( \theta(x) = \theta, p(x, d\zeta) = m_1(d\zeta) \).

\[
\int_{E^n} \prod_{l=1}^{n} g_l(x_l) \bar{L}_n \prod_{l=1}^{n} f_l(x_l) m_n(dx)
\]

\[
= \frac{1}{2} \sum_{1 \leq i < j \leq n} \int_{E^n} \prod_{l=1, l \neq i,j}^{n} g_l(x_l)f_i(x_i)f_j(x_j) \prod_{l \neq i,j}^{n} f_l(x_l) m_n(dx)
\]

\[
- \frac{n(n-1+\theta)}{2} \int_{E^n} \prod_{l=1}^{n} g_l(x_l)f_l(x_l) m_n(dx)
\]

\[
+ \frac{\theta}{2} \sum_{i=1}^{n} <f_i, m_1> \int_{E^n} \prod_{l=1, l \neq i}^{n} g_l(x_l)f_l(x_l) m_n(dx). \quad (169)
\]

So to show

\[
\int_{E^n} \prod_{l=1}^{n} g_l(x_l) \bar{L}_n \prod_{l=1}^{n} f_l(x_l) m_n(dx) = \int_{E^n} \prod_{l=1}^{n} f_l(x_l) \bar{L}_n \prod_{l=1}^{n} g_l(x_l) m_n(dx). \quad (170)
\]

We need to show

\[
\frac{1}{2} \sum_{j \neq i, j=1}^{n} \int_{E^n} \prod_{l \neq i,j}^{n} g_l f_i(x_i)(f_i f_j g_j)(x_j) g_i(x_i) m_n(dx)
\]

\[
+ \frac{\theta}{2} \sum_{i=1}^{n} <f_i, m_1> \int_{E^n} \prod_{l=1, l \neq i}^{n} g_l f_i(x_i) g_i(x_i) m_n(dx), \quad (171)
\]

is symmetric in \( f_i, g_i \).

Let \( h_l = f_l g_l, l \neq i; h_i = f_i \).

So we need to show

\[
\frac{1}{2} \sum_{j \neq i, j=1}^{n} \int_{E^n} \prod_{l \neq i,j}^{n} h_l(x_l)(h_l h_j)(x_j) g_i(x_i) m_n(dx)
\]
From \([E]\), we know
\[
\int_{\mathbb{E}^n} \prod_{i=1}^{n} f_i(x_i) m_n(dx) = \sum_{d=1}^{n} \sum_{\pi(n,d)} p(\beta) \prod_{k=1}^{d} f_{i_k} m_1 > .
\] (173)
where \(\pi(n,d)\) is the collection of partitions \(\beta\) of \(\{1, \ldots, n\}\) into \(d\) unordered non-empty subsets \(\beta_1, \ldots, \beta_d\). and
\[
p(\beta) = (|\beta_1| - 1)! \cdots (|\beta_d| - 1)! \frac{\theta^{d-1}}{(1 + \theta) \cdots (n - 1 + \theta)}. \] (174)
Now let \(d \in \{1, \ldots, n\}\), and let partitions \(\beta\) of \(\{1, \ldots, n\}\) be \(\beta_1, \ldots, \beta_d\).
The coefficient of \(g_i, m_1 > \prod_{k=1}^{d} f_{i_k} m_1 >\) on RHS is \(\frac{\theta}{2} p(\beta)\).
To find the coefficient of it on LHS, suppose \(i \in \beta_i\). The coefficient on LHS is
\[
\frac{1}{2} 1_{\{\beta_i \geq 2\}} \frac{\theta^d}{(1 + \theta) \cdots (n - 1 + \theta)} (|\beta_1| - 1)! \cdots (|\beta_d| - 1)! + \frac{\theta}{2} 1_{\{|\beta_i| = 1\}} \frac{\theta^{d-1}}{(1 + \theta) \cdots (n - 1 + \theta)} (|\beta_1| - 1)! \cdots (|\beta_d| - 1)! = \frac{\theta}{2} 1_{\{\beta_i \geq 2\}} p(\beta) + \frac{\theta}{2} 1_{\{|\beta_i| = 1\}} p(\beta)
\] (175)
So the coefficient of \(g_i, m_1 > \prod_{k=1}^{d} f_{i_k} m_1 >\) on both sides are equal.
Now let \(d \in \{1, \ldots, n, n + 1\}\), let \(\beta = \{\beta_1, \ldots, \beta_d\}\) be partitions of \(\{1, 2, \ldots, n + 1\}\), let
\[
p(\beta) = (|\beta_1| - 1)! \cdots (|\beta_d| - 1)! \frac{\theta^{d-1}}{(1 + \theta) \cdots (n - 1 + \theta)}. \] (176)
We have \( h_1, h_2, \cdots, h_n, h_{n+1} = g_i \).

Suppose \( n+1 \in \beta_i, i \in \beta_i \). Then the coefficient of \( \prod_{i=1}^d \prod_{j \in \beta_i} h_j, m_1 > \), when \( |\beta_k| \geq 2 \) is

\[
RHS = \frac{1}{2} \frac{\theta^{d-1}}{(1+\theta) \cdots (n-1+\theta)}(|\beta_1|-1)! \cdots (|\beta_d|-1)! = \frac{1}{2^d} p(\beta).
\]

\[
LHS = \frac{1}{2} 1_{\{\beta_i \geq 2\}} \frac{\theta^{d-1}}{(1+\theta) \cdots (n-1+\theta)}(|\beta_1|-1)! \cdots (|\beta_d|-1)! \\
+ \frac{\theta}{2} 1_{\{\beta_i = 1\}} \frac{\theta^{d-2}}{(1+\theta) \cdots (n-1+\theta)}(|\beta_1|-1)! \cdots (|\beta_d|-1)! \\
= \frac{1}{2} 1_{\{\beta_i \geq 1\}} p(\beta) + \frac{1}{2} 1_{\{\beta_i \geq 2\}} p(\beta) = \frac{1}{2} p(\beta).
\]

So \( RHS = LHS \), therefore \( L_n = \bar{L}_n \) for any \( n \in N \).

The theorem was proved. \( \square \)

Remark 1. 4 Let \( \theta = \sup_{x \in E} \theta(x) \), and to avoid trivialities, assume \( \theta > 0 \). Define transition function \( q(x, dy) \) by

\[
q(x, dy) = (1 - \frac{\theta(x)}{\theta}) \delta_x(dy) + \frac{\theta(x)}{\theta} p(x, dy).
\]

Then

\[
Af(x) = \theta \int_E (f(y) - f(x)) q(x, dy).
\]

Also observe that if \( q(x, dy) = m_1(dy) \), then \( p(x, dy) = m_1(dy) \), and \( \theta(x) = \theta \), a constant. Because:

\[
m_1(C) = \int_C q(x, dy) = \frac{\theta(x)}{\theta} p(x, C),
\]

for any \( x \in C^c \), while \( C \in B(E) \). So for any \( x \), we can choose \( C \) large such that \( m_1(C) = 1 \), and also \( x \in C^c \). This would imply \( \theta(x) = \theta \), for any \( x \), and as a result, \( m_1(C) = p(x, C) \), for any \( C \in B(E) \).
So, without loose of generality, we can suppose $\theta(x) = \theta$, a positive constant.

It is not hard to see that $\mathcal{L}_n = \mathcal{L}_n^*$ for any $n \in \mathcal{N}$ implies $\mathcal{L} = \mathcal{L}^*$. It would be reasonable to guess that the other-way around is also true, or equivalently $\mathcal{L} = \mathcal{L}^*$ implies $\theta(x) = \theta, p(x, \mathcal{d}\zeta) = m_1(d\zeta)$. Shiga showed it right when $E$ is a finite set. In the next Theorem, we will show that this conjecture is true for any Polish space $E$ if $A$ is reversible.

**Theorem 1.4** Let $E$ be a Polish space. Let $f \in \mathcal{B}(E)$. Let

$$Af(x) = \frac{\theta(x)}{2} \int_{\zeta \in E} (f(\zeta) - f(x))p(x, \mathcal{d}\zeta),$$

where $\theta(x) > 0$ is a bounded continuous function, and the transition kernel $p(x, \cdot)$ is nonatomic for any $x \in E$, which means that every mutant is of a new type. $m_1$ is the probability measure satisfying:

$<Af, m_1> = 0$, for $f \in \mathcal{B}(E)$. Suppose $A = A^*$, then:

The Fleming-Viot process is reversible iff $p(x, d\zeta) = m_1(d\zeta), \theta(x) = \theta$. That is

$$\mathcal{L} = \mathcal{L}^* \iff p(x, d\zeta) = m_1(d\zeta), \theta(x) = \theta.$$  

$\theta$ is a positive constant.

**Proof**: There is a proof in [E]. It is also not hard to prove it from Theorem 3, and the fact that $\mathcal{L}_n = \mathcal{L}_n^*$ implies $\mathcal{L} = \mathcal{L}^*$, because:

$$\{\frac{1}{n} \sum_{i=1}^{n} \delta_\nu, -\infty < t < \infty\} \equiv_{\mathcal{D}} \{\frac{1}{n} \sum_{i=1}^{n} \delta_\nu|t, -\infty < t < \infty\},$$

implying

$$\{\mu_t, -\infty < t < \infty\} \equiv_{\mathcal{D}} \{\mu_{-t}, -\infty < t < \infty\}.$$  

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Now suppose $L = L^*$. By remark 1.4, we can suppose $\theta(x) = \theta$, a constant.

Since $A = A^*$, therefore by lemma 1.19, we know $p(x, \xi)$ is symmetric in $x, \xi$, i.e. $p(x, \xi) = p(\xi, x)$.

Let $g_2 \in B(E^2)$, $f_1 \in B(E)$, then we will have
\[
\int_{A_1(E)} < g_2, \mu^2 > \mathcal{L} < f_1, \mu > m(d\mu) = \int_{A_1(E)} < f_1, \mu > \mathcal{L} < g_2, \mu^2 > m(d\mu).
\] (185)

Let $g_2(x_1, x_2) = 1_{A_1}(x_1)1_{A_2}(x_2)$, $f_1(x_3) = 1_{A_3}(x_3)$. Then line (187) is the following
\[
\int_{E^3} 1_{A_1}(x_1)1_{A_2}(x_2)[\frac{\theta}{2}p(x_3, A_3) - \frac{\theta}{2}1_{A_3}(x_3)]m_3(dx_1, dx_2, dx_3)
= \int_{E^3} 1_{A_3}(x_3)[\frac{1}{2}1_{A_1} \cap A_2(x_1) + 1_{A_1} \cap A_2(x_2) - 1_{A_1}(x_1)1_{A_2}(x_2)
+ \frac{\theta}{2}1_{A_1}(x_1)p(x_2, A_2) - \frac{\theta}{2}1_{A_1}(x_1)1_{A_2}(x_2)
+ \frac{\theta}{2}1_{A_3}(x_3)p(x_1, A_1) - \frac{\theta}{2}1_{A_1}(x_1)1_{A_2}(x_2)]m_3(dx_1, dx_2, dx_3).
\] (186)

Therefore
\[
\frac{\theta}{2} \int_{E} p(x_3, A_3)m_3(A_1, A_2, dx_3) - \frac{\theta}{2}m_3(A_1, A_2, A_3)
= m_2(A_1 \cap A_2, A_3) - m_3(A_1, A_2, A_3) + \frac{\theta}{2} \int_{E} p(x_2, A_2)m_3(A_1, dx_2, A_3)
+ \frac{\theta}{2} \int_{E} p(x_1, A_1)m_3(dx_1, A_2, A_3) - \theta m_3(A_1, A_2, A_3).
\] (187)

So we have the following equality:
\[
(1 + \frac{\theta}{2})m_3(A_1, A_2, A_3) = m_2(A_1 \cap A_2, A_3) + \frac{\theta}{2} \int_{E} p(x, A_1)m_3(dx, A_2, A_3)
+ \frac{\theta}{2} \int_{E} p(x, A_2)m_3(A_1, dx, A_3) - \frac{\theta}{2} \int_{E} p(x, A_3)m_3(A_1, A_2, dx).
\] (188)

Similarly, if we change the role of $A_1$ and $A_3$, we will have
\[
(1 + \frac{\theta}{2})m_3(A_1, A_2, A_3) = m_2(A_3 \cap A_2, A_1) + \frac{\theta}{2} \int_{E} p(x, A_3)m_3(dx, A_2, A_1)
\]
\[ +\frac{\theta}{2} \int_E p(x, A_2) m_3(A_3, dx, A_1) - \frac{\theta}{2} \int_E p(x, A_1) m_3(A_3, A_2, dx). \] (189)

Take the sum of the above two equalities, i.e. (190) + (191), and use the fact that \( m_3 \) is symmetric, we have

\[ 2(1 + \frac{\theta}{2}) m_3(A_1, A_2, A_3) = m_2(A_3 \cap A_2, A_1) + m_2(A_1 \cap A_2, A_3) + \theta \int_E p(x, A_2) m_3(A_1, A_3, dx). \] (190)

Let \( A_3 \cap A_2 = \emptyset \), then line (192) is the following:

\[ 2(1 + \frac{\theta}{2}) m_3(A_1, A_2, A_3) = m_2(A_1 \cap A_2, A_3) + \theta \int_E p(x, A_2) m_3(A_1, A_3, dx). \] (191)

So by changing the order of integrals for the second term on RHS of (193), we have

\[ 2(1 + \frac{\theta}{2}) \int_{z \in A_1} m_3(dz, A_2, A_3) = \int_{z \in A_1} 1_{A_2}(z) m_2(dz, A_3) + \theta \int_E p(x, A_2) m_3(dz, A_3, dx). \] (192)

for any \( A_1 \in B(E) \). Therefore we have the following equalities of measures:

\[ 2(1 + \frac{\theta}{2}) m_3(dz, A_2, A_3) = 1_{A_2}(z) m_2(dz, A_3) + \theta \int_E p(x, A_2) m_3(dz, A_3, dx). \] (193)

Now for any set \( C \in B(E) \), we multiply both sides of (195) by \( p(z, C) \), and integrate both sides with respect to \( z \) over \( E \), we have

\[ 2(1 + \frac{\theta}{2}) \int_{z \in E} p(z, C) m_3(dz, A_2, A_3) = \int_{z \in E} p(z, C) 1_{A_2}(z) m_2(dz, A_3) + \int_{z \in E, x \in E} p(z, C) p(x, A_2) m_3(dz, A_3, dx). \] (194)

Next we are going to show \( \int_{z \in E} p(z, C) 1_{A_2}(z) m_2(dz, A_3) \) is symmetric in \( C, A_2 \), i.e.

\[ \int_{z \in E} p(z, C) 1_{A_2}(z) m_2(dz, A_3) = \int_{z \in E} p(z, A_2) 1_C(z) m_2(dz, A_3). \] (195)
when $C \cap A_3 = \emptyset$, and $A_2 \cap A_3 = \emptyset$.

Now we can see clearly that

$$\int_{z \in E, x \in E} p(z, C)p(x, A_2)m_3(dz, dx, A_3)$$

is symmetric in $C, A_2$.

Also we will show that

$$\int_{z \in E} p(z, C)m_3(dx, A_2, A_3).$$

is symmetric in $C, A_2$. Here are the argument:

In line (190), we put $C$ in place of $A_1$, we will have:

$$(1 + \frac{\theta}{2})m_3(C, A_2, A_3) = m_2(A_2 \cap A_3, C)$$
$$+ \frac{\theta}{2} \int_E p(x, A_2)m_3(A_3, C, dx)$$
$$+ \frac{\theta}{2} \int_E p(x, A_3)m_3(A_2, C, dx) - \frac{\theta}{2} \int_E p(x, C)m_3(A_2, A_3, dx)$$
$$= \frac{\theta}{2} \int_E p(x, A_2)m_3(A_3, C, dx)$$
$$+ \frac{\theta}{2} \int_E p(x, A_3)m_3(A_2, C, dx) - \frac{\theta}{2} \int_E p(x, C)m_3(A_2, A_3, dx).$$

(198)

Therefore we have the following:

$$(1 + \frac{\theta}{2})m_3(C, A_2, A_3) - \frac{\theta}{2} \int_E p(x, A_3)m_3(A_2, C, dx)$$
$$= \frac{\theta}{2} \int_E p(x, A_2)m_3(A_3, C, dx) - \frac{\theta}{2} \int_E p(x, C)m_3(A_2, A_3, dx).$$

(199)

Since the LHS in (201) is symmetric in $C, A_2$, therefore the RHS

$$\frac{\theta}{2} \int_E p(x, A_2)m_3(A_3, C, dx) - \frac{\theta}{2} \int_E p(x, C)m_3(A_2, A_3, dx),$$

(200)

is also symmetric in $C, A_2$, but it is also antisymmetric in $C, A_2$. So we have

$$\frac{\theta}{2} \int_E p(x, A_2)m_3(A_3, C, dx) = \frac{\theta}{2} \int_E p(x, C)m_3(A_3, A_2, dx).$$

(201)
Therefore we have the desired result that

$$\int_{z \in E} p(z, C) 1_{A_2}(z)m_2(dz, A_3), \tag{202}$$

is symmetric in $C, A_2$, from (196), when $C \cap A_3 = \emptyset, A_2 \cap A_3 = \emptyset$. That is

$$\int_{z \in E} p(z, C) 1_{A_2}(z)m_2(dz, A_3) = \int_{z \in E} p(z, A_2) 1_{C}(z)m_2(dz, A_3). \tag{203}$$

Now that

$$\int_{z \in E} p(z, C) 1_{A_2}(z)m_2(dz, A_3)$$

$$= \int_{z \in E} p(z, C) 1_{A_2}(z)f_{A_3}(z)m_1(dz)$$

$$= \int_{z \in E, z \in C} p(z, \zeta)f_{A_3}(z)m_1(dz)m_1(d\zeta) \tag{204}$$

And also we have

$$\int_{z \in E} p(z, A_2) 1_{C}(z)m_2(dz, A_3)$$

$$= \int_{z \in E} p(z, A_2) 1_{C}(z)f_{A_3}(z)m_1(dz)$$

$$= \int_{z \in E, z \in C} p(z, \zeta)f_{A_3}(z)m_1(dz)m_1(d\zeta)$$

$$= \int_{z \in E, z \in C} p(\zeta, z)f_{A_3}(\zeta)m_1(d\zeta)m_1(dz). \tag{205}$$

Therefore by (205), (206), (207) and the fact that $p(z, \zeta) = p(\zeta, z)$, we have the following:

$$\int_{z \in E, z \in C} p(z, \zeta)(f_{A_3}(\zeta) - f_{A_3}(z))m_1(dz)m_1(d\zeta) = 0, \tag{206}$$

for any $C, A_2 \in B(E \cap A_3^c)$, for any fixed $A_3$. Now for arbitrary fixed $A_3 \in B(E)$, let

$$F = \{D \subseteq (E \setminus A_3) \times (E \setminus A_3) : \int_{x \times \zeta \in D}(f_{A_3}(x) - f_{A_3}(\zeta))p(x, \zeta)m_1(d\zeta)m_1(dx_2) = 0\}.$$
Then it is clear that $\mathcal{F}$ is a Borel system, since it satisfies the following conditions of being a Borel system:

i) increasing countable union of set in $\mathcal{F}$ still in $\mathcal{F}$.

ii) for $A, B \in \mathcal{F}$, and $B \subseteq A$, then $A \setminus B \in \mathcal{F}$

iii) $(E \setminus A_3) \times (E \setminus A_3) \in \mathcal{F}$.

Now let

$$G = \{A_2 \times B_2 : A_2 \in E \setminus C, B_2 \in E \setminus C\}.$$  

Clearly $G$ is a $\pi$-system, and is a subset of $\mathcal{F}$. So by $\pi - \lambda$ system, $\mathcal{F}$ is the $\sigma$-algebra on $(E \setminus C) \times (E \setminus C)$, i.e $\mathcal{F} = B((E \setminus C) \times (E \setminus C))$

Let

$$D^+ = \{\zeta \times x_2 \in (E \setminus C) \times (E \setminus C) : f_C(x_2) - f_C(\zeta) > 0\}.$$

and

$$D^- = \{\zeta \times x_2 \in (E \setminus C) \times (E \setminus C) : f_C(x_2) - f_C(\zeta) < 0\}.$$

Then $D^+, D^- \in B((E \setminus C) \times (E \setminus C)$, So $D^+, D^- \in \mathcal{F}$, which implies

$$m_1 \times m_1(D^+) = 0, m_1 \times m_1(D^-) = 0. \quad (207)$$

Therefore

$$\int_{\zeta \times x_2 \in D} (f_C(x_2) - f_C(\zeta)) m_1(d\zeta) m_1(dx_2) = 0, \quad (208)$$

for any $D = A \times B$, where $A, B \in B(E \setminus C)$. Therefore, by lemma 1.23, we have

$$f_C(x_2) = km_1(C), \quad (209)$$

when $x_2 \in (C)^c \cap M$, where $m_1(M) = 0$.

Observe that $\mathcal{L} = \mathcal{L}^*$ implies the conditions of lemma 1.21, therefore we can apply lemma 1.21, which is

$$m_2(C_1, C_2) = m_1(C_1 \setminus C_2) + \int_{\zeta \in E} \theta p(\zeta, C_2)m_2(d\zeta, C_1) - \int_{x_2 \in C_2} \theta m_2(C_1, dx_2), \quad (210)$$
for any $C_1, C_2 \in B(E)$. Let $C_2 \in B(E)$ be arbitrary. We have

\[
\int_{x_2 \in C_2} f_{C_1}(x_2) m_1(dx_2) \\
= \int_{x_2 \in C_2} m_2(C_1, dx_2) \\
= \int_{x_2 \in C_2} 1_{C_1}(x_2) m_1(dx_2) \\
+ \int_{x_2 \in C_2} \int_{\xi \in E} \theta p(\zeta, x_2) m_2(d\zeta, C_1) m_1(dx_2) - \int_{x_2 \in C_2} \theta f_{C_1}(x_2) m_1(dx_2) \] (211)

Therefore

\[
f_{C_1}(x_2) = \int_{\xi \in E} \theta p(\zeta, x_2) m_2(d\zeta, C_1) - \theta f_{C_1}(x_2),
\] (212)

when $x_2 \in (C_1)^c$.

Therefore

\[
\int_{\xi \in E} \theta p(\zeta, x_2) m_2(d\zeta, C_1) = (\theta + 1) f_{C_1}(x_2).
\] (213)

when $x_2 \in (C_1)^c$.

Therefore by (211), line (214) will be

\[
\int_{\xi \in E} \theta p(\zeta, x_2) m_2(d\zeta, C_1) = (\theta + 1) km_1(C_1).
\] (214)

when $x_2 \in (C_1)^c \cap M^c$, for some set $M$, such that $m_1(M) = 0$, and $k$ does not depend on $C_1$. Now that

\[
\int_{\xi \in E} \theta p(\zeta, x_2) m_2(d\zeta, C_1 \cup M) \\
= \int_{\xi \in E} \theta p(\zeta, x_2) m_2(d\zeta, C_1) \\
+ \int_{\xi \in E} \theta p(\zeta, x_2) m_2(d\zeta, M) \\
= \int_{\xi \in E} \theta p(\zeta, x_2) m_2(d\zeta, C_1),
\] (215)

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since \( \int_{\zeta \in E} m_2(d\zeta, M) = m_1(M) = 0 \), which implies that \( \int_{\zeta \in E} \theta p(\zeta, x_2) m_2(d\zeta, M) = 0 \). But also we have

\[
(\theta + 1)k m_1(C_1 \cup M) = (\theta + 1)k m_1(C_1). \tag{216}
\]

Therefore we can define a new set \( C_1 \) to be \( C_1 \cup M \), and still have

\[
\int_{\zeta \in E} \theta p(\zeta, x_2) m_2(d\zeta, C_1) = (\theta + 1)k m_1(C_1), \tag{217}
\]

when \( x_2 \in (C_1)^c \), and \( k \) does not depend on \( C_1 \).

Therefore we have

\[
\int_{\zeta \in E} p(\zeta, x_2) m_2(d\zeta, C_1) = \frac{(\theta + 1)}{\theta}k m_1(C_1). \tag{218}
\]

when \( x_2 \in (C_1)^c \). We will show

\[
\frac{(\theta + 1)}{\theta}k = 1. \tag{219}
\]

Since

\[
\int_{\zeta \in E} p(\zeta, x_2) m_2(d\zeta, E) = \int_{\zeta \in E} p(\zeta, x_2) m_1(d\zeta) = 1, \tag{220}
\]

for \( m_1 \) a.e. \( x_2 \)

And clearly we have

\[
\int_{\zeta \in E} p(\zeta, x_2) m_2(d\zeta, E \setminus \{x_2\})
= \int_{\zeta \in E} p(\zeta, x_2) m_2(d\zeta, E)
- \int_{\zeta \in E} p(\zeta, x_2) m_2(d\zeta, \{x_2\}). \tag{221}
\]

On the other hand, we also have

\[
\int_{\zeta \in E} p(\zeta, x_2) m_2(d\zeta, E \setminus \{x_2\})
= \frac{(\theta + 1)}{\theta}k m_1(E \setminus \{x_2\})
= \frac{(\theta + 1)}{\theta}k(1 - m_1 \{x_2\}). \tag{222}
\]
Observe that \( \int_{\mathcal{E}} m_2(d\zeta, \{x\}) = m_1\{x\} \). Therefore
\[
m_2(d\zeta, \{x\}) = 0 \text{ iff } m_1\{x\} = 0, \text{ for any } x \in E.
\]
Now by lemma 18 iii), we have
\[
\int \theta(x)p(x, \{x_0\})m_1(dx) = \theta(x_0)m_1\{x_0\}.
\]
Therefore \( m_1\{x_0\} = 0 \text{ iff } p(x, \{x_0\}) = 0 \text{ for } m_1 \text{ a.e } x \).

So we will have \( m_1\{x\} = 0 \text{ for any } x \in E \) by the condition that \( p(x, \cdot) \) is nonatomic for any \( x \in E \). And as a result \( m_2(d\zeta, \{x\}) = 0 \text{ for any } x \in E \).

So by (222), (223), (224), we have the result:
\[
\frac{(\theta + 1)}{\theta} k = 1.
\]
Therefore
\[
\int_{\zeta \in \mathcal{E}} p(\zeta, x_2)m_2(d\zeta, C_1) = m_1(C_1),
\]
for \( x_2 \in C_1^c \). Then we will get
\[
\int_{\zeta \in \mathcal{E}} p(\zeta, C_2)m_2(d\zeta, C_1) = m_1(C_1)m_1(C_2).
\]
when \( C_1 \cap C_2 = \emptyset \).

Therefore we can apply lemma 1.24 to get the result of this theorem. \( \square \)

Remark 1.5 In the proof of Theorem 1.4, we need to use the fact that
\( p(\zeta, x) = p(x, \zeta) \), which is equivalent to \( A = A^* \). We don’t have a proof without this condition or we don’t know if \( \mathcal{L} = \mathcal{L}^* \) would imply \( A = A^* \). The Fleming-Viot process is the empirical limit of the particle systems, so the Fleming-Viot process has some information about the particles, but will not necessarily preserve all the information. Since we have proved that the generator \( A \) is the uniform mutation is a necessary and sufficient condition for the particle systems to be reversible, so we guess that \( A = A^* \) is a necessary condition for the theorem.
Chapter 2

NEW EXAMPLES OF

MEASURE-VALUED PROCESSES

Let $E$ be a compact metric space with metric $\bar{d}$, and $\mathcal{E}$ be the Borel $\sigma$-algebra. Let $\mathcal{M}_F(E)$ be the space of finite measures on $E$. Let $\mathcal{M}_1(E)$ be the space of probability measures on $E$. The weak topology on $\mathcal{M}_F(E)$ is defined by

$$\mu_n \Rightarrow \mu \iff \langle \mu_n, f \rangle \to \langle \mu, f \rangle \text{ for any } f \in bC(E),$$

the space of bounded continuous functions on $E$. Let $d$ be the Prohorov metric, which is defined as the following:

$$d(\mu_1, \mu_2) = \inf\{\epsilon > 0 : \mu_1(F) \leq \mu_2(F^\epsilon) + \epsilon, \forall F \in \mathcal{N}\},$$  \hspace{1cm} \text{(229)}

where $\mathcal{N}$ is the collection of closed subsets of $E$ and

$$F^\epsilon = \{x \in E : \inf_{y \in F} \bar{d}(x, y) < \epsilon\}.$$  \hspace{1cm} \text{(230)}

The Prohorov metric and the weak topology are equivalent. Therefore $(\mathcal{M}_1(E), d)$ is a compact space of probability measures on $E$ with metric $d$ [EK 2, Chap 3]. Let $\mathcal{B}(\mathcal{M}_1(E))$ be the Borel $\sigma$-algebra, and $P$ is the probability distribution on $\mathcal{M}_1(E)$. So we have space $(\mathcal{M}_1(E), \mathcal{B}(\mathcal{M}_1(E)), P)$.

Let $\Omega$ be the sets of transition kernels. That is: for any $Q \in \Omega$, we have

1) for any $x \in E$, $Q(x, \cdot)$ is a measure on $E$.

2) for any $A \in \mathcal{B}(E)$, $Q(x, A)$ is a measurable function of $x$.
Let $\Omega_1$ be the sets of probability transition kernels, that is they are transition kernels of Markov chains.

Later, we will use the following notation: for any $Q_1, Q_2 \in \Omega_1$,

$$
\|Q_1 - Q_2\| = \max_{x \in E} d(Q_1(x, \cdot), Q_2(x, \cdot)),
$$

since for any $x$, $Q_1(x, \cdot), Q_2(x, \cdot) \in M_1(E)$.

$d(Q_1, Q_2) = \|Q_1 - Q_2\|$ is a metric on $\Omega_1$. Let $B(\Omega_1)$ be the Borel $\sigma$-algebra on $\Omega_1$, and $P$ be the distribution on it. So we have space $(\Omega_1, B(\Omega_1), P)$.

Let $D = D([0, \infty), M_1(E))$ with Skorohod topology.

$$
X_t : D \to M_1(E), X_t(\omega) := \omega(t), \omega \in D
$$

Let $\tilde{P}$ be the distribution on it.

Let $\xi_0 \in M_1(E)$ be random, and $Q_1, Q_2, \ldots, Q_n$ be independent random kernels from $\Omega_1$. Denote

$$
\mu Q(dy) = \int_{x \in E} \mu(dx)Q(x, dy), \text{ for } \mu \in M_F(E), Q \in \Omega.
$$

Now define $\xi_n = \xi_0 Q_1 Q_2 \cdots Q_n$.

**Definition 2.1** Let $\mathcal{P} = \{\Lambda(f_n)\mu | \Lambda(f_n)\mu = < f_n, \mu^n >, f_n \in bC(E^n), \forall n\}$. The semigroup of a measure valued process $T_t$ preserves Polynomial if for any $\Lambda(f_n)\mu \in \mathcal{P}$, we have the semigroup $T_t^n$ for a n-dimensional variable such that

$$
T_t \Lambda(f_n)\mu = \Lambda(T_t^n f_n)\mu.
$$

In this chapter, we will show that (Theorem 2.4) any measured valued jump process which of generator

$$
\mathcal{L} < f_n, \mu^n > = \int_Q (< f_n, (\mu Q)^n > - < f_n, \mu^n >) m(dQ),
$$
and
\[ \int ||Q - I||m(dQ) < \infty. \] (234)

preserves polynomial. To achieve this goal, we check the following:

i) The jump process can be approximated by rescale \( \xi_n = \xi_0 Q_1 Q_2 \cdots Q_n \), or more precisely, subordinate \( \xi_n \) by Poisson process to have \( \xi_{N(t)} \). In the spirit of observing that \( \xi_n = \xi_0 Q_1 Q_2 \cdots Q_n \) is like random walk, so we can consider similar things to random walk. We will obtain some "weak law of small number" [1], i.e. a sum of random variables that are positive with small probabilities approaches Poisson. We will consider more jumps per unit but each jump is small.

ii) The process \( \xi_{N(t)} \), and \( \xi_{[n]} \) preserves polynomial.

The main result of this chapter (Theorem 2.5, Theorem 2.6) gives classes of measure valued process which preserves polynomial.

**Theorem 2.1** \( \xi_n \)'s are measure valued Markov chain.

**Proof:** Let \( \mu_0 \) be the distribution of \( \xi_0 \), and \( m \) be the distribution of \( Q_i \), \( i = 1, \cdots, n \).

Let \( A = \{ \xi_0 \in B_0, \xi_1 \in B_1, \cdots, \xi_n \in B_n \} \in \sigma(\xi_0, \xi_1, \cdots, \xi_n) \). Then we have

\[
\int_A \mathbb{1}_{\xi_{n+1} \in B_{n+1}} dP
\]
\[
= P(\xi_0 \in B_0, \xi_1 \in B_1, \cdots, \xi_n \in B_n, \xi_{n+1} \in B_{n+1})
\]
\[
= \int \mathbb{1}_{y_0 \in B_0} \mu_0(dy_0) \int \mathbb{1}_{y_0 \in B_1} m(dQ_1) \cdots \int \mathbb{1}_{y_0 \cdots y_{n+1} \in B_{n+1}} m(dQ_{n+1})
\]
\[
= \int_A (\int_1 \mathbb{1}_{\xi_n Q_{n+1} \in B_{n+1}} m(dQ_{n+1}) dP \tag{235}
\]

The collection of sets for which (235) holds is a \( \lambda \)-system, and the collection for
which it has been proved is a $\pi$-system, so it follows from the $\pi - \lambda$ theorem that

$$P(\xi_{n+1} \in B_{n+1}/\sigma(\xi_0, \xi_1, \cdots, \xi_n)) = \int 1_{\xi_n \in B_n} m(dQ_{n+1}).$$

But since for any $C = \{\xi_n \in B\}$, we have

$$\int_C 1_{\xi_{n+1} \in B_{n+1}} dP = P(\xi_n \in B, \xi_{n+1} \in B_{n+1}) = \int_C \int 1_{\xi_n \in B_n} m(dQ_{n+1}) dP.$$

Therefore by $\pi - \lambda$ system argument as above, we have

$$\int 1_{\xi_n \in B_n} m(dQ_{n+1}) = E(1_{\xi_{n+1} \in B_{n+1}}/\sigma(\xi_n)).$$

So

$$P(\xi_{n+1} \in B_{n+1}/\sigma(\xi_0, \xi_1, \cdots, \xi_n)) = P(\xi_{n+1} \in B_{n+1}/\sigma(\xi_n)).$$

This proves that $\xi_n$ are Markov chain. □

Before we start measure-valued process, we first have some warm up: we will discuss some "weak law of small numbers" for measure-valued random variable.

**Definition 2.2** A measure valued random variable $\xi$ is called of measured-valued Poisson distribution of rate $\lambda$ of kernel $P_1$ if it has the following:

$$P(\xi = \xi_0 P_1^i) = \frac{e^{-\lambda} \lambda^i}{i!}.$$  \hspace{1cm} (240)

for $i = 0, 1, \cdots$, where $\xi_0$ is a probability measure, and $P_1$ is a transition kernel in $\Omega_1$.

Let $bpC(E)$ be the set of bounded non-negative continuous functions on $E$. A sequence of bounded non-negative Borel measurable functions $\{f_n\}$ is said to converge $bp$ to $f$ if $f_n(x) \to f(x)\forall x$, and there exist $M < \infty$ such that $\sup_n, x f_n(x) \leq M$.
**Theorem 2.2** Let $Q_{n,1}, \ldots, Q_{n,n}$ be independent from $\Omega$. Let $\xi_n = \xi_0 Q_{n,1} Q_{n,2} \cdots Q_{n,n}$.

Suppose
\[ P(Q_{n,m} = \delta_z(dy)) = 1 - p_{n,m}, P(Q_{n,m} = P_1) = p_{n,m}, \] (241)
where $P_1$ is a transition kernel in $\Omega$. Suppose further we have the following conditions:

i) $\sum_{m=1}^{n} p_{n,m} \to \lambda$, where $\lambda$ is a positive real number.

ii) $\max_{1 \leq m \leq n} p_{n,m} \to 0$.

Then $\xi_n \Rightarrow \xi$, as $n \to \infty$, where $\xi$ is the measured-valued Poisson distribution of rate $\lambda$ of kernel $P_1$, that is
\[ P(\xi = \xi_0 P_1^i) = \frac{e^{-\lambda} \lambda^i}{i!}. \] (242)

for $i = 0, 1, \ldots$.

**Proof:** Let $P_n \in M_1(M_1(E))$ be the distribution of $\xi_n$. To prove $P_n \Rightarrow P \in M_1(M_1(E))$ as $n \to \infty$ it suffices to show that [DD Corollary 3.2.7]

i) $\{P_n\}$ are tight.

ii) $L_n(f) = \int e^{-\langle \mu, f \rangle} P_n(d\mu)$ converges as $n \to \infty$ for each $f \in V$, where $V$ is a countable set: $\{f_n\} \subseteq bpC(E)$ such that $1 \in V$, and $V$ is closed with respect to addition operation, and the bp closure of $V$ is a bounded non-negative Borel measurable function on $E$.

The first condition i) is clearly satisfied since $M_1(E)$ is compact. Now
\[ L_n(f) = \int e^{-\langle \mu, f \rangle} P_n(d\mu) \]
where \( P(S_n = k) \) is as in Appendix 1, therefore
\[
P(S_n = k) \to \frac{e^{-\lambda} \lambda^k}{k!}
\]
as \( n \to \infty \). So
\[
L_n(f) \to \sum_{k=0}^{\infty} e^{-<f, P_k^* f>} \frac{e^{-\lambda} \lambda^k}{k!} = \int e^{-\mu(f)} P(\mu)
\]
(245)
Therefore \( \xi_n \to \xi \) as \( n \to \infty \), where \( \xi \) is the measured-valued Poisson distribution of rate \( \lambda \) of kernel \( P_1 \). □

Without too much trouble, we can generalize theorem 2.2 to the following theorem 3

**Theorem 2.3** Let \( Q_n, 1, \ldots, Q_n, n \) be independent from \( \Omega_1 \). Let \( \xi_n = \xi_0 Q_n, 1 Q_n, 2 \ldots Q_n, n \).

Suppose
\[
P(Q_n, m = \delta_x(dy)) = 1 - p_{n, m} - \epsilon_{n, m}, P(Q_n, m = P_1) = p_{n, m}, P(Q_n, m = \text{ others}) = \epsilon_{n, m}.
\]
(246)
where \( P_1 \) is a transition kernel in \( \Omega_1 \). Suppose further we have the following conditions:

i) \( \sum_{m=1}^{n} p_{n, m} \to \lambda \), where \( \lambda \) is a positive real number.

ii) \( \max_{1 \leq m \leq n} p_{n, m} \to 0 \).

iii) \( \sum_{m=1}^{n} \epsilon_{n, m} \to 0 \).

Then \( \xi_n \to \xi \), as \( n \to \infty \), where \( \xi \) is the measured-valued Poisson distribution of rate \( \lambda \) of kernel \( P_1 \), that is
\[
P(\xi = \xi_0 P_i) = \frac{e^{-\lambda} \lambda^i}{i!}.
\]
(247)
for \( i = 0, 1, \cdots \).

**Proof:** The proof is similar to the proof of Appendix 2. □

We have the following corollary

**Corollary 2.1** Let \( \xi_n = \xi_0 Q_1 Q_2 \cdots Q_n \), where \( Q_1, Q_2, \cdots Q_n \) are iid from \( \Omega_1 \).

Suppose

\[
\mathcal{P}(Q_i = \delta_x(dy)) = 1 - \frac{\lambda}{n}, \quad \mathcal{P}(Q_i = P_i) = \frac{\lambda}{n},
\]

where \( P_i \) is a transition kernel in \( \Omega \), and \( \lambda \) is a positive real number.

Then \( \xi_n \Rightarrow \xi \), as \( n \to \infty \), where \( \xi \) is the measured-valued Poisson distribution of rate \( \lambda \) of kernel \( P_1 \), that is

\[
P(\xi = \xi_0 P_1^i) = \frac{e^{-\lambda} \lambda^i}{i!}.
\]

for \( i = 0, 1, \cdots \).

□

**Corollary 2.2** Let \( \xi_n = \xi_0 Q_1 Q_2 \cdots Q_n \), where \( Q_1, Q_2, \cdots Q_n \) are iid from \( \Omega_1 \).

Suppose

\[
\mathcal{P}(Q_i = \delta_x(dy)) = 1 - \frac{1}{n^2}, \quad \mathcal{P}(Q_i = P_1) = \frac{1}{n^2},
\]

where \( P_1 \) is a transition kernel in \( \Omega_1 \), and \( \lambda \) is a positive real number.

Then \( \xi_n \Rightarrow \xi_0 \), as \( n \to \infty \).

□

Naturally, we extend measure-valued Poisson random variable to measure-valued Poisson process.
Definition 2.3 A Process \( \xi_t \) is called a measure-valued Poisson Process of rate \( \lambda \) of kernel \( P_1 \) if it has the following:

i) 

\[
P(\xi_t = \xi_0 P_1^i) = \frac{e^{-\lambda t} (\lambda t)^i}{i!},
\]

for \( i = 0, 1, \cdots \), where \( P_1 \) is a transition kernel from \( \Omega \).

ii) 

\[
P(\xi_t = \xi_0 P_1^{k+i}, \xi_s = \xi_0 P_1^k) = \frac{e^{-\lambda t} (\lambda t)^k e^{-\lambda (t-s)} (\lambda (t-s))^i}{k! i!},
\]

for \( i = 0, 1, \cdots \), and \( k = 0, 1, \cdots \).

Condition ii) in the definition 2 is equivalent to

\[
P(\xi_t - \xi_s = \xi_0 P_1^{k+i} - \xi_0 P_1^k, \xi_s = \xi_0 P_1^k) = \frac{e^{-\lambda t} (\lambda t)^k e^{-\lambda (t-s)} (\lambda (t-s))^i}{k! i!},
\]

which tells us that \( \xi_t - \xi_s \) and \( \xi_s \) are independent.

Our next goal is to show that measure-valued jump process preserves polynomial.

We will use the following lemma from [JM]

Lemma 2.1 Let \( (X^n)_{n=1}^{\infty} \) be a sequence of processes defined on their respective probability space \( (\Omega^n, \mathcal{A}, P_n) \) with values in the complete separable metric space \( \mathcal{H} \).

Then the sequence \( \{P_n\} \) of laws of the processes \( (X^n) \) form a tight sequence if and only if [T1] and [T2] hold, where

[T1] for any \( t \) in some dense subset \( T \) of \( R^+ \), the laws of the random variables \( (X^n_t) \) form a tight sequence of laws in \( \mathcal{H} \).

[T2] for any \( \epsilon > 0 \), \( T > 0 \), \( \beta > 0 \), there exist \( \delta > 0 \), such that

\[
\lim_{n \to \infty} \sup_n P_n \{ \omega : \omega \in \Omega^n, w^T (X^n(\cdot, \omega), \delta) > \beta \} \leq \epsilon,
\]
where

\[ w^n(X^n(\cdot, \omega), \delta) = \inf_{\min|t_{i+1}-t_i| \geq \delta} \max_{t_i \leq t < t_{i+1}} \sup_{u_i \leq t < t_{i+1}} d(X_u, X_t), \quad (255) \]

for \( X \in D(R^+; \mathcal{H}) \), and \( t_0 < t_1 < \cdots < t_n < T \) in \( R^+ \).

Lemma 2.2 Let \( \xi \) be a probability measure, and \( P_1, P_2, \cdots, P_n \) are transition kernel from \( \Omega_1 \). We have

\[ d(\xi P_1 P_2 \cdots P_n, \xi) \leq \sum_{i=1}^{n} ||P_i - I||. \quad (256) \]

where \( d \) is the Prohorov metric, and

\[ ||P_i - I|| = \max_{x \in E} d(P_i(x, \cdot), \delta_x(\cdot)). \quad (257) \]

Proof: First, we will show that

\[ d(\xi P_1, \xi) \leq ||P_1 - I||. \quad (258) \]

Let \( \varepsilon_y = d(P_1(y, \cdot), \delta_y(\cdot)) \). Let \( \varepsilon = \sup_y \varepsilon_y \). Then by the definition of Prohorov metric, we will have

\[ P_1(y, F) \leq \delta_y(F^c) + \varepsilon, \quad (259) \]

for any \( F \), a closed subset of \( E \), and for any \( y \in E \). Therefore

\[ \xi P_1(F) = \int_{y \in E} \xi(dy) P_1(y, F) \leq \int_{y \in E} \xi(dy) (\delta_y(F^c) + \varepsilon) = \xi(F^c) + \varepsilon. \quad (260) \]
Since (260) is true for any $F$, a closed subset of $E$, therefore

$$d(\xi P_1, \xi) \leq \epsilon.$$  \hspace{1cm} (261)

That is

$$d(\xi P_1, \xi) \leq ||P_1 - I||.$$  \hspace{1cm} (262)

Now by triangle inequality and line (262), we have

$$d(\xi P_1 \cdots P_n, \xi) = d(\xi P_1 \cdots P_n, \xi P_1 \cdots P_{n-1}) + d(\xi P_1 \cdots P_{n-1}, \xi) \leq ||P_n - I|| + d(\xi P_1 \cdots P_{n-1}, \xi).$$  \hspace{1cm} (263)

Therefore we have the result of this lemma by induction. □.

**Lemma 2.3** Let $\xi_n = \xi_0 Q_1 Q_2 \cdots Q_n$, where $Q_1, Q_2, \cdots Q_n$ are iid from $\Omega_1$, and of probability distribution $P_c$, which has $c > 0$ as a parameter.

Let $N(t)$ be the Poisson Process of rate $c$, that is

$$P(N(t) = k) = \frac{e^{-ct}(ct)^k}{k!},$$  \hspace{1cm} (264)

for $k = 0, 1, \cdots$.

Let $X^c_t = \xi_{N(t)}$. Suppose

1) $\int_{Q \in \Omega} F(Q) c P_c(dQ) \to \int_{Q \in \Omega} F(Q) m(dQ)$, as $c \to \infty$, where $F$ is a bounded function of $Q$.

2) $\int ||Q - I|| m(dQ) < \infty$.

Then $X^c_t \Rightarrow X_t$, as $c \to \infty$, where $X_t$ is a measure-valued Process.

**Proof:** We have a sequence of measure valued process $X^c_t$ defined on $(D^c, P^c)$ with values in $\mathcal{M}_1(E)$. The sequence $\{P^c\}$ are the laws of $X^c_t$. We will show $\{P^c\}$ are tight.
i) The first condition [T1] is easily seen from the compactness of $\mathcal{M}_1(E)$.

ii) Now prove [T2]. For any $\epsilon > 0$, $T > 0$, $\beta > 0$, choose $\delta > 0$, such that

$$\delta < \frac{\beta}{\int ||Q-I||m(dQ)},$$

and $\delta l = T$ for some integer $l$.

Let a partition be $t_0 = 0 < t_1 = \delta < t_2 = 2\delta < \cdots < t_l = T$.

$$d(X_t, X'_t) = d(\xi_0 Q_1 \cdots Q_{N(s)}, \xi_0 Q_1 \cdots Q_{N(t)})$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} P(N(s) = k, N(t) - N(s) = n) d(\xi_k Q_1 \cdots Q_n, \xi_k)$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} P(N(s) = k) P(N(t) - N(s) = n)$$

$$= \int_{P_1} \cdots \int_{P_n} d(\xi_k P_1 \cdots P_n, \xi_k) P_c(dP_1) \cdots P_c(dP_n),$$

where we use the fact that $N(t) - N(s)$, and $N(s)$ are independent, and $Q_1, \cdots, Q_n$ are iid of distribution $P_c$.

Now by lemma 2.2, we have

$$\int_{P_1} \cdots \int_{P_n} d(\xi_k P_1 \cdots P_n, \xi_k) P_c(dP_1) \cdots P_c(dP_n)$$

$$\leq \int_{P_1} \cdots \int_{P_n} \sum_{i=1}^{n} ||P_i - I|| P_c(dP_1) \cdots P_c(dP_n)$$

$$\leq \sum_{i=1}^{n} \int ||P_i - I|| P_c(dP_i)$$

$$= n \int ||P - I|| P_c(dP).$$

Therefore, by the fact that

$$P(N(t) - N(s) = n) = \frac{e^{-c(t-s)}(c(t-s))^n}{n!}.$$
and line (266), (267), we have
\[
d(X_s, X_t) \leq \sum_{n=0}^{\infty} \frac{e^{-c(t-s)}(c(t-s))^n}{n!} \int \|P - I\|P_c(dP) \\
= \int \|P - I\|P_c(dP)e^{-c(t-s)}(c(t-s))\sum_{n=1}^{\infty} \frac{(c(t-s))^{n-1}}{(n-1)!} \\
= \int \|P - I\|P_c(dP)(c(t-s)). \tag{268}
\]
Therefore we will have
\[
w^T(X^n(\cdot, \omega), \delta) \leq c \int \|P - I\|P_c(dP)\delta \tag{269}
\]
Then will have
\[
P^c\{\omega : \omega \in D, w^T(X^n(\cdot, \omega), \delta) > \beta\} \leq P^c\{\omega : \omega \in D, c \int \|P - I\|P_c(dP)\delta > \beta\} \tag{270}
\]
Since \(c \int \|P - I\|P_c(dP) \to \int \|P - I\|m(dP)\), therefore [T2] is satisfied, that is
\[
\lim_{c \to \infty} \sup_{\omega \in D} P^c\{w^T(X^n(\cdot, \omega), \delta) > \beta\} = 0 \leq \epsilon, \tag{271}
\]
\[
\square
\]
**Remark 2.1** \(X_t^c\) is a jump process. The process waited for an exponentially distributed amount of time of rate \(c\), then randomly choose a transition kernel which of distribution \(P_c\) and jump there. Let \(f_n \in B(E^n)\), the set of bounded functions on \(E^n\). The generator of \(X_t\) is
\[
\mathcal{L}c < f_n, \mu^n > = c \int_Q (\langle f_n, (\mu Q)^n > - \langle f_n, \mu^n >) P_c(dQ). \tag{272}
\]
The limiting process \(X_t\) is also a jump process. The generator of \(X_t\) is
\[
\mathcal{L} < f_n, \mu^n > = \int_Q (\langle f_n, (\mu Q)^n > - \langle f_n, \mu^n >) m(dQ), \tag{273}
\]
and \(\mathcal{L}c < f_n, \mu^n > \to \mathcal{L} < f_n, \mu^n >\) as \(c \to \infty\).
Remark 2.2 \(m\) can have a total mass of infinity, but we need \(\int |Q - I||m(dQ) < \infty\).

The following Corollary are easily derived from lemma 2.3.

Corollary 2.3 Let \(\xi_n = \xi_0 Q_1 Q_2 \cdots Q_n\), where \(Q_1, Q_2, \cdots Q_n\) are iid from \(\Omega_1\).

Suppose
\[
P(Q_i = \delta_x(dy)) = 1 - \frac{\lambda}{c}, P(Q_i = P_1) = \frac{\lambda}{c},
\]
for any \(i\), where \(P_1\) is a transition kernel in \(\Omega_1\), and \(\lambda\) and \(c\) are positive real numbers. Let \(N(t)\) be the Poisson Process of rate \(c\), that is
\[
P(N(t) = k) = \frac{e^{-ct}(ct)^k}{k!},
\]
for \(k = 0, 1, \cdots\)

Then \(\xi_{N(t)} \Rightarrow X_t\), as \(c \to \infty\), where \(X_t\) is the measure-valued Poisson Process of rate \(\lambda\) of kernel \(P_1\), that is
\[
P(X_t = \xi_0 P_1^i) = \frac{e^{-\lambda t}(\lambda t)^i}{i!},
\]
for \(i = 0, 1, \cdots\).

\Box

Corollary 2.4 Let \(\xi_n = \xi_0 Q_1 Q_2 \cdots Q_n\), where \(Q_1, Q_2, \cdots Q_n\) are iid from \(\Omega_1\).

Suppose
\[
P(Q_i = \delta_x(dy)) = 1 - \frac{\lambda}{c}, P(Q_i = P_k) = \frac{\lambda}{nc},
\]
for any \(i\), and \(k = 1, 2, \cdots, n\), where \(P_k\)'s are transition kernels in \(\Omega_1\), and \(\lambda\) and \(c\) are positive real numbers. Let \(N(t)\) be the Poisson Process of rate \(c\), that is
\[
P(N(t) = k) = \frac{e^{-ct}(ct)^k}{k!},
\]
for \(k = 0, 1, \cdots\).
for $k = 0, 1, \ldots$. 

Then $\xi_{\mathcal{N}(t)} \Rightarrow X_t$ as $c \to \infty$, where $\xi_t$ is the measure-valued Poisson Process of rate $\lambda$ of kernel $P_1, P_2, \ldots, P_n$, that is

$$X_t = \xi_0 R_1 R_2 \cdots R_{\mathcal{N}(t)},$$

(279)

where $\mathcal{N}(t)$ is of Poisson distribution of rate $\lambda$, and $R_1, R_2, \ldots$ are iid from $\Omega_1$ which of the following distribution

$$\mathcal{P}(R_i = P_k) = \frac{1}{n},$$

(280)

for any $i$, and $k = 1, 2, \ldots, n$.

Remark 2.3 Collary 2.3, and collary 2.4 are some special case where $m\{I^c\} = \lambda m_1$, where $\lambda$ is a real number, and $m_1$ is a probability measure.

Definition 2.4 Let $R(x, dy) \in \Omega_1$. Define $R^2$ by the following:

$$R^2(x, dy) = \int_{E} R(x, dz) R(z, dy).$$

(281)

Therefore, we have the definition of $R^n$ for any integer $n$.

Definition 2.5 For any $R \in \Omega_1$, define

$$e^{tR} = \sum_{k=0}^{\infty} \frac{t^k}{k!} R^k.$$  

(282)

OR more precisely

$$< f, e^{tR} > = \sum_{k=0}^{\infty} \frac{t^k}{k!} < f, R^k >,$$

(283)

for any $f \in \mathcal{BC}(E)$. 

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If we choose \( R(x, dy) = \delta_x(dy) \), then we have

\[
e^{t} = e^{td} = \sum_{k=0}^{\infty} \frac{t^k}{k!}.
\]

(284)

**Lemma 2.4** Let \( R_1, R_2 \in \Omega_1 \), then

\[
|| (R_1 - R_2)^k || \leq 2^{k-1} ||R_1 - R_2||,
\]

(285)

for any integer \( k \).

**Proof:** Since

\[
(R_1 - R_2)^2(x, dy) = \int_z (R_1 - R_2)(x, dz)(R_1 - R_2)(z, dy).
\]

(286)

Therefore

\[
|| (R_1 - R_2)^2 || \leq ||(R_1 - R_2)|| \sup_x \int_z (R_1 + R_2)(x, dz) = 2|| (R_1 - R_2)||.
\]

(287)

So the lemma was proved by induction.

The following lemma says that \( \xi_k \) has the property of preserving polynomial.

**Lemma 2.5** Let \( \xi_k = \xi_0 Q_1 \cdots Q_k \), where \( \xi_0 \) is a random probability measure, and \( Q_1, \cdots \), are iid of distribution \( P_c \). For any \( f_n \in bC(\mathbb{E}^n) \), let

\[
< f_n, (\xi_k)^n > = \int f_n(x_1, \cdots, x_n)\xi_k(dx_1) \cdots \xi_k(dx_k).
\]

(288)

Let

\[
R_n(x_1, dy_1) \cdots R_n(x_n, dy_n) = \int_Q Q(x_1, dy_1) \cdots Q(x_n, dy_n) P_c(dQ).
\]

(289)

Then we have

\[
E_\mu < f_n, (\xi_k)^n > = < f_n, (\mu R_n^k)^n >.
\]

(290)
Proof:

\[ E_\mu < f_n, (\xi_k)^n > = E_\mu (E_\mu ( < f_n, (\xi_k)^n > /\sigma(\xi_{k-1}))) = E_\mu ( < f_n, (\xi_{k-1}R_n)^n > ) = E_\mu ( < f_n, (\xi_{k-2}R_n^2)^n > ) = \cdots = < f_n, (\mu R_n^c)^n >. \] (291)

□

In the following, we will use notations like \((\cdot)_c\), or \((\cdot)^c\) to mean that they have \(c\) as a parameter.

Now by rescale \(\xi\) to be \(\xi_{N(t)}\), we will prove in the following that \(\xi_{N(t)}\) preserves polynomial.

**Lemma 2.6** Let \(\xi_n = \xi_0Q_1Q_2\cdots Q_n\), where \(\xi_0\) is a random probability measure, and \(Q_1, Q_2, \cdots Q_n\) are iid from \(\Omega_1\), and of probability distribution \(P_c\), which has \(c > 0\) as a parameter. Let \(X_t^n = \xi_{N(t)}\), where \(N(t)\) is a Poisson process of rate \(c\). Let \(T_t^n\) be the semigroup of \(X_t^n\). Then

\[ T_t^n \Lambda (f_n)\mu = \Lambda ((T_t^n)_c f_n)\mu, \] (292)

where \((T_t^n)_c\) is the semigroup of a \(n\) dimensional particles.

**Proof:**

\[ T_t^n < f_n, \mu^n > = \int_{x_1, \cdots, x_n} f_n(x_1, \cdots, x_n) \nu^n(dx_1, \cdots, dx_n) p^e(t, \mu, d\nu) = \int_{x_1, \cdots, x_n} f_n(x_1, \cdots, x_n) \int_\nu \nu^n(dx_1, \cdots, dx_n) \sum_{k=0}^{\infty} \frac{(ct)^k e^{-ct}}{k!} p^c(k, \mu, d\nu). \] (293)
where \( p^c(k, \mu, d\nu) = P(\xi_k \in d\nu/\xi_0 = \mu) \). Now that
\[
E_\mu < f_n, \xi^n_k > = \int f_n(x_1, \cdots, x_n) \xi_k(dx_1, \cdots, dx_n) = \int f_n(x_1, \cdots, x_n) \int \nu^n(dx_1, \cdots, dx_n) p^c(k, \mu, d\nu).
\]
(294)

Therefore by lemma 2.5, we have
\[
\int \nu^n(dx_1, \cdots, dx_n) p^c(k, \mu, d\nu) = \mu(R_n)^k_c(dx_1) \cdots \mu(R_n)^k_c(dx_n) = \int (R_n)^k_c(y_1, dx_1) \cdots (R_n)^k_c(y_n, dx_n) \mu^n(dy_1, \cdots, dy_n).
\]
(295)

So define transition function of \( n \) particles as
\[
p^n_c(t, y_1, \cdots, y_n; dx_1, \cdots, dx_n) = \sum_{k=0}^{\infty} \frac{(ct)^k e^{-ct}}{k!} (R_n)^k_c(y_1, dx_1) \cdots (R_n)^k_c(y_n, dx_n).
\]
(296)

We will have
\[
T^n_c < f_n, \mu^n > = \int f_n(x_1, \cdots, x_n) \int_{y_1, \cdots, y_n} p^n_c(t, y_1, \cdots, y_n; dx_1, \cdots, dx_n) \mu^n(dy_1 \cdots dy_n) = \int (T^n_c)_c f_n(y_1, \cdots, y_n) \mu^n(dy_1, \cdots, dy_n) = < (T^n_c)_c f_n, \mu^n >.
\]
(297)

□

Remark 2.4 Let \((Z^n_k)_{k=1}^{\infty}\) be a Markov chain of transition kernel \((R_n)_c\) and initial distribution \(\mu\). Then \(\mu(R_n)^k_c\) is the distribution of \((Z^n_k)_c\). Let \((Y_n)_c(t) = (Z^n_k)_c^{N(t)}\), where \(N(t)\) is the Poisson Process of rate \(c\). Let \((Y^n_k)_c^{(1)}, \cdots, (Y^n_k)_c^{(n)}\) be iid of same distribution as \((Y_n)_c\), then \((Y^n_k)_c^{(1)}, \cdots, (Y^n_k)_c^{(n)})\) is the \(n\) particles which of transition function \(p^n_c(t, y_1, \cdots, y_n; dx_1, \cdots, dx_n)\).

We know from lemma 2.3 that \(\xi_{N(t)}\) has limit \(X_t\), and from lemma 2.6, we know that \(\xi_{N(t)}\) preserves polynomial. Next, we will show that the limiting process \(X_t\)
also preserves polynomial. The following lemma 2.7, and 2.8 will lead to the proof.

**Lemma 2.7** Let $f_1, \cdots, f_n \in bC(E)$, $k$ is an integer, and $Q \in \Omega$ of distribution $P_c$. Let $||f_1(x_1) \cdots f_n(x_n)|| \leq M$. Then we have the following:

i) $$
\int_{x_1, \cdots, x_n} f_1(x_1) \cdots f_n(x_n) \left( \int (Q(y_1, dx_1) \cdots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \cdots \delta_{y_n}(dx_n)) cP_c(dQ) \right)^k
\to \int_{x_1, \cdots, x_n} f_1(x_1) \cdots f_n(x_n) \\
\left( \int (Q(y_1, dx_1) \cdots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \cdots \delta_{y_n}(dx_n)) \right)^k,
$$  \tag{298}

as $c \to \infty$.

ii) $$
< f_1 \cdots f_n, e^{ct} \int (Q(y_1, dx_1) \cdots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \cdots \delta_{y_n}(dx_n)) P_c(dQ) >
\to < f_1 \cdots f_n, e^{ct} \int (Q(y_1, dx_1) \cdots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \cdots \delta_{y_n}(dx_n)) m(dQ) >,
$$  \tag{299}

as $c \to \infty$.

**Proof:** To prove i) observe that

$$
\int_{x_1, \cdots, x_n} f_1(x_1) \cdots f_n(x_n) \left( \int (Q(y_1, dx_1) \cdots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \cdots \delta_{y_n}(dx_n)) cP_c(dQ) \right)^k
= \int_{Q_1 \in \mathcal{E}, \cdots, Q_k \in \mathcal{E}} \int_{x_1, \cdots, x_n} f_1(x_1) \cdots f_n(x_n)
\left( \sum_{i=1}^{2^k} A_i(y_1, dx_1) \cdots A_i(y_n, dx_n) \right) cP_c(dQ_1) \cdots cP_c(dQ_k),
$$  \tag{300}

where $A_i = B_{i_1} \cdots B_{i_k}$, and $B_{ij} = Q_j$ or $\delta$. Now let

$$
\int_{Q_1 \in \mathcal{E}, \cdots, Q_k \in \mathcal{E}} F_1(Q_1) \cdots F_k(Q_k) cP_c(dQ_1) \cdots cP_c(dQ_k)
= \int_{x_1, \cdots, x_n} f_1(x_1) \cdots f_n(x_n)
$$

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\[
\left( \int (Q(y_1, dx_1) \ldots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \ldots \delta_{y_n}(dx_n))c\pi_c(dQ) \right)^k.
\] (301)

Then we can see that \( F_1(Q_1) \ldots F_k(Q_k) \) is a continuous function of \( Q_1 \ldots Q_k \), and 
\[
||F_1(Q_1) \ldots F_k(Q_k)|| \leq M2^k.
\]
So \( cP_c \Rightarrow m \) imply that
\[
\int_{Q_1 \in \mathcal{F}, \ldots, Q_k \in \mathcal{F}} F_1(Q_1) \ldots F_k(Q_k)c\pi_c(dQ_1) \ldots c\pi_c(dQ_k)
\]
\[
\rightarrow \int_{Q_1 \in \mathcal{F}, \ldots, Q_k \in \mathcal{F}} F_1(Q_1) \ldots F_k(Q_k)m(dQ_1) \ldots m(dQ_k).
\] (302)

So we have result i).
To get ii), we know that 
\[
\left< f_1 \ldots f_n, e^{ct} \int (Q(y_1, dx_1) \ldots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \ldots \delta_{y_n}(dx_n))c\pi_c(dQ) \right>
\]
\[
= \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{Q_1 \in \mathcal{F}, \ldots, Q_k \in \mathcal{F}} F_1(Q_1) \ldots F_k(Q_k)c\pi_c(dQ_1) \ldots c\pi_c(dQ_k).
\] (303)

Now we can see that
\[
||\int_{Q} (Q(y_1, dx_1) \ldots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \ldots \delta_{y_n}(dx_n))m(dQ)||
\]
\[
||\int_{Q} Q(y_1, dx_1) \ldots Q(y_n, dx_n) - \delta_{y_1}(dx_1)(Q(y_2, dx_2) \ldots Q(y_n, dx_n)
\]
\[
+ \delta_{y_1}(dx_1)(Q(y_2, dx_2) \ldots Q(y_n, dx_n) + \cdots - \delta_{y_1}(dx_1) \ldots \delta_{y_n}(dx_n))m(dQ)||
\]
\[
\leq n \int_{Q} ||Q - I||m(dQ).
\] (304)

So by (304), and lemma 2.4, we have
\[
\sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{x_1, \ldots, x_n} f_1(x_1) \ldots f_n(x_n)
\]
\[
(\int (Q(y_1, dx_1) \ldots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \ldots \delta_{y_n}(dx_n))m(dQ))^k
\]
\[
\leq \sum_{k=0}^{\infty} \frac{t^k}{k!} M2^{k-1}n \int ||Q - I||m(dQ)
\]
\[
= Mn \int ||Q - I||m(dQ)e^{2t} < \infty.
\] (305)

Therefore by LDCT and i), we have result ii). □
Lemma 2.8 The semigroup \((T^n_t)_c\) in lemma 2.6 approaches \(T^n_t\), the semigroup of a \(n\) variable, as \(c\) goes to infinity.

Proof: Now that

\[
(p^n)_c(t, y_1, \cdots, y_n; dx_1, \cdots, dx_n) = \sum_{k=0}^{\infty} \frac{(ct)^k e^{-ct}}{k!} (R_n)_c(y_1, dx_1) \cdots (R_n)_c(y_n, dx_n)
\]

\[
= \sum_{k=0}^{\infty} \frac{(ct)^k e^{-ct}}{k!} ((R_n)_c(y_1, dx_1) \cdots (R_n)_c(y_n, dx_n))^k
\]

\[
= e^{-ct} e^{t(R_n)_c(y_1, dx_1) \cdots (R_n)_c(y_n, dx_n)}
\]

\[
= e^{ct} \int (Q(y_1, dx_1) \cdots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \cdots \delta_{y_n}(dx_n)) P_c(dQ)
\]

\[
(306)
\]

By lemma 2.7, we know

\[
< f_1 \cdots f_n, e^{ct} \int (Q(y_1, dx_1) \cdots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \cdots \delta_{y_n}(dx_n)) P_c(dQ) >
\]

\[
\rightarrow < f_1 \cdots f_n, e^t \int (Q(y_1, dx_1) \cdots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \cdots \delta_{y_n}(dx_n)) m(dQ) > .
\]

(307)

Now let

\[
p^n(t, y_1, \cdots, y_n; dx_1, \cdots, dx_n) = e^t \int (Q(y_1, dx_1) \cdots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \cdots \delta_{y_n}(dx_n)) m(dQ).
\]

(308)

Let

\[
T^n_t f_1(x_1) \cdots f_n(x_n) = \int_{y_1, \cdots, y_n} f_1(y_1) \cdots f_n(y_n) p^n(t, x_1, \cdots, x_n; dy_1, \cdots, dy_n),
\]

(309)

which is the semigroup of a \(n\) variable. Therefore the corresponding semigroup approaches, since \(f_1(x_1) \cdots f_n(x_n)\) is dense in \(bC(E^n)\).

Next theorem is a main result of this chapter, which gives a class of measure-valued process which preserves polynomial.
Theorem 2.4 Let $X_t$ be a measure valued jump process, with jump kernel having distribution $m$. Suppose $\int ||Q - I||m(dQ) < \infty$. The generator is

$$\mathcal{L}\Lambda(f)(\mu) = \int_Q (\Lambda(f)(\mu Q) - \Lambda(f)(\mu))m(dQ), \quad (310)$$

for $\Lambda(f)(\mu) \in \mathcal{P}$. Let $T_t$ be the semigroup. Then this jump process preserves Polynomial, that is

$$T_t\Lambda(f_n)\mu = \Lambda(T^n_t f_n)\mu, \quad \text{for } \Lambda(f_n)\mu \in \mathcal{P}, \quad \text{where } T^n_t \text{ is the semigroup of } n \text{ particles which of generator }$$

$$L_n f_n(x_1, \ldots, x_n) = \int_{y_1, \ldots, y_n} (f_n(y_1, \ldots, y_n) - f_n(x_1, \ldots, x_n)) R_n(x_1, dy_1) \cdots R_n(x_n, dy_n), \quad (311)$$

where

$$R_n(x_1, dy_1) \cdots R_n(x_n, dy_n) = \delta_{x_1}(dy_1) \cdots \delta_{x_n}(dy_n)$$

$$= \int_Q (Q(x_1, dy_1) \cdots Q(x_n, dy_n) - \delta_{x_1}(dy_1) \cdots \delta_{x_n}(dy_n))m(dQ). \quad (312)$$

So for any $\Lambda(f_n)\mu \in \mathcal{P}$, we have

$$\mathcal{L}\Lambda(f_n)\mu = \Lambda(L_n f_n)\mu. \quad (313)$$

Proof: The semigroup of $X_t$ preserves Polynomial is a result of lemma 2.3, lemma 2.6, and lemma 2.8. Let $f_1, \ldots, f_n \in bC(E)$. The semigroup of the $n$ particles is

$$T^n_t f_1(x_1) \cdots f_n(x_n)$$

$$= < f_1 \cdots f_n, e^{t\int (Q(y_1, dx_1) \cdots Q(y_n, dx_n) - \delta_{y_1}(dx_1) \cdots \delta_{y_n}(dx_n))m(dQ)} >$$

$$= < f_1 \cdots f_n, e^{t(R_n(y_1, dx_1) \cdots R_n(y_n, dx_n) - \delta_{y_1}(dx_1) \cdots \delta_{y_n}(dx_n))} > \quad (314)$$

Therefore

$$L_n f_1(x_1) \cdots f_n(x_n) = \lim_{t \to 0} \frac{T^n_t f_1(x_1) \cdots f_n(x_n) - f_1(x_1) \cdots f_n(x_n)}{t}$$

81
\[ = \int_{y_1, \ldots, y_n \in E} (f_1(y_1) \cdots f_n(y_n) - f_1(x_1) \cdots f_n(x_n))R_n(x_1, dy_1) \cdots R_n(x_n, dy_n) \]

Since \( f_1 \cdots f_n \) is dense in \( bC(E^n) \), therefore the generator of the \( n \) particles is \( L_n \).

\[ \square \]

We will have the following special case of Theorem 2.4.

**Corollary 2.5** Measure-valued Poisson process preserves polynomial.

We will prove another theorem which will use the following result from [EK2] page 261, problem 3:

**Lemma 2.9** Suppose \( \{T(t)\} \) is a semigroup on \( B(E) \) given by a transition function and has full generator \( \hat{A} \). Let

\[ Bf(x) = \lambda(x) \int (f(y) - f(x)) \mu(x, dy) \quad (316) \]

where \( \lambda \in B(E) \) is a nonnegative and \( \mu(x, \Gamma) \) is a transition function. Then \( \hat{A} + B \) is the full generator of a semigroup on \( B(E) \) given by a transition function.

The following result gives a class of measure-valued process which preserves polynomial.

**Theorem 2.5** Let \( L_1 \) be the generator of Fleming-Viot process, and \( L_2 \) be the generator of measured valued jump process. Then

i) \( L_1 + L_2 \) is the generator of a semigroup.

ii) The semigroup in i) preserves Polynomial.

**Proof:** i) is a result from lemma 2.9. As for ii), knowing that the semigroups of \( L_1 \) preserves Polynomials, and by theorem 2.4 and lemma 2.9, we have ii). \( \square \)
Our next goal is to achieve more classes of measure-valued process. We will rescale \( \xi_n \) to \( \xi_{[nt]} \). Let \( \xi_{[nt]} = Q_0 Q_{n,1} Q_{n,2} \cdots Q_{[n,nt]} \), for any \( t > 0 \), where \( Q_{n,1}, Q_{n,2}, \cdots \) are iid from \( \Omega_1 \), and of distribution \( m_n \). In the case of
\[
\int ||Q - I|| m_n(dQ) \to \int ||Q - I|| m(dQ) < \infty
\]
we will have the same result as lemma 2.3, so we are not getting more classes. So now we will consider the case that
\[
\int ||Q - I|| m_n(dQ) \to \infty
\]

**Definition 2.6** A process \( X_t \) is a measure-valued Brownian process of initial distribution \( \xi_0 \) if it has the following:

for any \( A \in \mathcal{B}(E) \),
\[
X_t(A) = \int_{x \in A} \int_y \frac{1}{(2\pi t)^{1/2}} \exp\left\{ -\frac{(y-x)^2}{2} \right\} dy \xi_0(dx) . \tag{317}
\]

Our next result will describe a class of measure-valued process which preserves polynomial, specially the measure-valued Brownian process preserves polynomial.

**Theorem 2.6** Let \( X_t^n = \xi_{[nt]} = Q_0 Q_{n,1} Q_{n,2} \cdots Q_{[n,nt]} \), for any \( t > 0 \), where \( Q_{n,1}, Q_{n,2}, \cdots \) are iid from \( \Omega_1 \), and of distribution \( m_n \). Suppose
\[
m_n(dQ) = (1 - a_n) \delta_{P_n}(dQ) + a_n \bar{m}(dQ), \tag{318}
\]
where \( a_n > 0 \) is a real number, and satisfies

i) \( a_n = O\left(\frac{1}{\sqrt{n}}\right) \),

ii) and for \( f_k \in \mathcal{B}(E^k) \), any integer \( k \), we have
\[
na_n \int < f_k, (\mu Q)^k > - < f_k, \mu^k > \bar{m}(dQ) \to \int < f_k, (\mu Q)^k > - < f_k, \mu^k > m(dQ), \tag{319}
\]
as \( n \to \infty \) for some measure \( m \). We choose \( P_n \) to be the following:
\[
P_n(x, dy) = \frac{1}{2} \delta_{x+\frac{1}{\sqrt{n}}}(dy) + \frac{1}{2} \delta_{x-\frac{1}{\sqrt{n}}}(dy). \tag{320}
\]
Then $X^n_t \Rightarrow X_t$, a measure-valued process which of generator $L$, as $n \rightarrow \infty$,
where

$$
\mathcal{L} < f_k, \mu^k > = < \tilde{L}_k f_k, \mu^k > = < \Delta f_k, \mu^k > + < L_k f_k, \mu^k > ,
$$

(321)

here $\Delta f_k = \sum_{i=1}^k \Delta_i f_k$, and $\Delta_i$ is the generator of Brownian motion acting on the $i$th variable only. $L_k$ is the generator of a jump process with jumping kernel having distribution $m$.

**Proof:** The generator of $X^n_t$ is

$$
\mathcal{L}_n < f_k, \mu^k > = n(1 - a_n) < f_k, (\mu P_n)^k > - < f_k, \mu^k > \\
+ n a_n \int < f_k, (\mu \mathbb{Q})^k > - < f_k, \mu^k > m(d\mathbb{Q}).
$$

(322)

Since

$$
\int f_1(x)n(P_n(y, dx) - \delta_y(dx)) \rightarrow \Delta f_1(y),
$$

(323)

for $f_1 \in \mathcal{D}(\Delta)$.

Now that

$$
n < f_k, (\mu P_n)^k > - < f_k, \mu^k > \\
= n \int f_k(x_1, \ldots, x_k) \int [\mu(dy_1)P_n(y_1, dx_1) \ldots \mu(dy_k)P_n(y_k, dx_k) \\
- \mu(dy_1) \ldots \mu(dy_k) \delta_{y_1}(dx_1) \ldots \delta_{y_k}(dx_k)] \\
= \int \mu(dy_1) \ldots \mu(dy_k) (\int f_k(x_1, \ldots, x_k) \\
- \delta_{y_1}(dx_1) \ldots \delta_{y_k}(dx_k)) \\
\{P_n(y_1, dx_1) \ldots P_n(y_k, dx_k) - \delta_{y_1}(dx_1) \ldots \delta_{y_k}(dx_k)\},
$$

(324)

and

$$
n [P_n(y_1, dx_1) \ldots P_n(y_k, dx_k) - \delta_{y_1}(dx_1) \ldots \delta_{y_k}(dx_k)]
$$
\[ n[P_n(y_1, dx_1) \cdots P_n(y_k, dx_k) - \delta_{y_1}(dx_1)P_n(y_2, dx_2) \cdots P_n(y_k, dx_k) \\
+ \delta_{y_1}(dx_1)P_n(y_2, dx_2) \cdots P_n(y_k, dx_k) - \cdots - \delta_{y_1}(dx_1) \cdots \delta_{y_k}(dx_k)] \] (325)

Therefore, as \( n \to \infty \), we have

\[ n(< f_k, (\mu P_n)^k > - < f_k, \mu^k >) \to < \Delta f_k, \mu^k > . \] (326)

And also it is not hard to see that

\[ na_n(< f_k, (\mu P_n)^k > - < f_k, \mu^k >) \to 0. \] (327)

Therefore

\[ \mathcal{L}_n < f_k, \mu^k > \to < \Delta f_k, \mu^k > + < L_k f_k, \mu^k > . \] (328)

Now by lemma 2.9, we have \( \tilde{L}_k = \Delta + L_k \), which is the generator of a uniquely determined semigroup.

Let \( \mathcal{L} < f_k, \mu^k > =< \tilde{L}_k f_k, \mu^k > . \) Then \( \mathcal{L} \) is the generator of a semigroup of a measure-valued process, which clearly preserves polynomial. □

**Remark 2.5** Since \( d(P_n, I) = \frac{1}{\sqrt{n}} \). Therefore the condition of the tightness proof of lemma 2.3 is not satisfied, because

\[ n \int ||Q - I||m_n(dQ) = n(1 - a_n)||P_n - I|| + na_n||P - I|| \] (329)

approaches \( \infty \) as \( n \) goes to \( \infty \).

**Remark 2.6** The limiting process in theorem 2.6 can be considered has \( \Delta \) as the diffusion part, and has \( L_k \) as the drift part.

**Appendix 1** The following result is from [DR] Theorem 6.1.
For each \( n \), let \( X_{n,m} \), \( 1 \leq m \leq n \), be independent random variables with

\[
P(X_{n,m} = 0) = 1 - p_{n,m}, P(X_{n,m} = 1) = p_{n,m},
\]

\[
\max_{1 \leq m \leq n} p_{n,m} \to 0,
\]

\[
\sum_{m=1}^{n} p_{n,m} \to \lambda.
\]

where \( \lambda \) is a positive real number. If \( S_n = X_{n,1} + \cdots + X_{n,n} \), then \( S_n \Rightarrow Z \) where \( Z \) is a Poisson of rate \( \lambda \).

Now

\[
E e^{itS_n} = \sum_{k=0}^{\infty} P(S_n = k) e^{itk}
\]

\[
\to E e^{itZ} = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} e^{itk}.
\]

Therefore \( P(S_n = k) \to \frac{e^{-\lambda} \lambda^k}{k!} \).

Appendix 2 This result is from [DR] Theorem 6.7. For each \( n \), let \( X_{n,m} \), \( 1 \leq m \leq n \), be independent nonnegative integer-valued random variables with

\[
P(X_{n,m} = 1) = p_{n,m}, P(X_{n,m} \geq 2) = \epsilon_{n,m},
\]

\[
\max_{1 \leq m \leq n} p_{n,m} \to 0,
\]

\[
\sum_{m=1}^{n} \epsilon_{n,m} \to 0,
\]

\[
\sum_{m=1}^{n} p_{n,m} \to \lambda
\]

where \( \lambda \) is a positive real number. If \( S_n = X_{n,1} + \cdots + X_{n,n} \), then \( S_n \Rightarrow Z \) where \( Z \) is Poisson of rate \( \lambda \).
References


[KC] M. Kimura and J.F. Crow (1964), The number of alleles that can be maintained in a finite population, Genetics, 49, pp.725-738.


