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TEACHERS' WAYS OF KNOWING AND TEACHING: A CLASSROOM INVESTIGATION OF ONE EXPERIENCED TEACHER'S USE OF HIS KNOWLEDGE OF BOTH MATHEMATICAL AND PEDAGOGICAL REPRESENTATIONS ABOUT ALGEBRAIC MULTIPLICATION

DISSERTATION

Presented in Partial Fulfillment of the Requirement for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By
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The Ohio State University
1997

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Approved by

College of Education
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1997
Whoever you are, please manifest yourself in such a way that I can see you through my blind eyes and understand your meanings with my primitive language.
This study investigated one case of the phenomenon of teachers' knowledge of representations and use of that knowledge during classroom instruction in the context of algebraic multiplication (AM) in eighth-grade mathematics. The major objectives were (a) to examine in detail one teacher's knowledge of mathematical representations (symbolic representations and proofs) about AM, (b) to examine in detail the teacher's knowledge of pedagogical representations (pictures and story problems) about AM, and (c) to examine in detail the pedagogical events (explanations, representations, and questions) that the teacher constructed when teaching AM from the perspectives of use of both mathematical and pedagogical representations and conceptual and procedural knowledge.

Through a content analysis of 10 consecutive lessons of the textbook dealing with AM 41 main mathematical ideas called mathematical content curriculum events (CCEs) were identified. Using interviews and questionnaires, Mr. Kantor, the participant of the study, was asked to provide, when appropriate, a symbolic representation, a mathematical proof, a pictorial representation and a story-problem representation for each of these CCEs. Videotapes of his teaching were taken daily for 12 days to examine his use of his knowledge of representations during classroom instruction and to examine the pedagogical events from the perspective of conceptual and procedural knowledge.

It was found that Mr. Kantor's knowledge of story-problem representations, symbolic representations, pictorial representations, and mathematical proofs was very strong, strong, strong, and very weak, respectively. However, his use of these representations during
classroom instruction was very limited, ranging from 0% in the case of proofs to 29% in the case of story problems. Mr. Kantor's explanations and questions involved some degree of both conceptual and procedural knowledge but the degree of both conceptual and procedural knowledge in the explanations and questions was less than ideal. In addition, Mr. Kantor constructed explanations for only 18 of the 41 CCEs. The relative difficulty of the CCEs influenced the complexity of Mr. Kantor's explanations.

These findings suggest that the relationship between teachers' knowledge of representation and use of that knowledge during classroom instruction is far from being linear.
To my wife Soco

To my children, José Francisco and Marco Vinicio

To my parents

To my teachers

To my siblings

To my friends

To the rest of the family
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An intellectual odyssey has come to an end. During this journey, at several times, I thought I was trapped, as Bastian Baltasar Bux was, in A Neverending Story. But many people helped me to find my way home. I would like to express my appreciation to each of them.

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VITA

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FIELDS OF STUDY

Major Field: Education
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CHAPTER 1

INTRODUCTION: THE PROBLEM AND ITS SETTING

The single factor which seems to have the greatest power to carry forward our understanding of the teachers' role is the phenomenon of teachers' knowledge (Elbaz, 1981, p. 45).

Teaching involves using systems of knowledge, pedagogical skills, and making pedagogical and curricular decisions (Brown & Borko, 1992). The present study examines the phenomenon of teaching from the perspective of teachers' knowledge. The statement that teachers need knowledge to carry out most of their instructional activities is neither a new nor a controversial issue (Fennema & Franke, 1992). Less obvious is the answer to the question proposed by Wilson, Shulman, and Richert (1987): What do teachers need to know? Some scholars (e.g., Buchman, 1984) have strongly advocated that teachers should possess a deep and broad knowledge of their subject matter to be good teachers: "Teachers who never explain or demonstrate anything, who neither answer questions nor question answers, may be engaged in some useful activity, but they do not teach.... [These] activities of teaching presuppose subject matter knowledge on the part of teachers" (p. 31). Other scholars (e.g., Gage, 1985) argue that knowledge of effective teaching techniques is the most critical component of teachers' knowledge. Still other scholars have advocated that knowledge of how students learn should be useful for teachers to help their students to learn while others (e.g., Grant & Secada, 1990) suggest that knowledge of how to teach a culturally diverse population is a crucial element of effective teaching. Finally, Shulman and his associates (Shulman, 1986a; Wilson, Shulman, & Richert, 1987) argue that...
teachers need a special integrated combination of content and pedagogy that they have labeled pedagogical content knowledge.

The main goal of the present study is to provide some partial answers to two critical questions: (a) what do teachers know? And, perhaps more important, (b) how do teachers use their knowledge during their instruction? The search for answering these two questions is approached through the concept of subject matter content knowledge (mathematical representations) and the theoretical construct of pedagogical content knowledge (pedagogical representations).

The theoretical construct of pedagogical content knowledge is relatively new and has initially been described by Shulman (1986a) as:

The most useful forms of representations of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations, in a word, the ways of representing and formulating the subject matter that make it comprehensible to others ... [it] also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning. (p. 9)

This definition includes two major interrelated components. The first component, ways of representing subject matter to students, is termed pedagogical representations. As described by Shulman, knowledge of pedagogical representations includes, in addition to knowledge of representations, knowledge of analogies, illustrations, explanations, and demonstrations. A pedagogical representation has two features: (a) it is an authentic representation of a mathematical idea and (b) it helps students to construct genuine mathematical knowledge. The second component, knowledge of students' understanding of the subject matter concepts, is termed knowledge of students' specific cognitions. It includes knowledge of students' misconceptions, difficulties, stages of understanding, and strategies related to learning a specific content topic.

Another category of pedagogical content knowledge is that of how to assess students' prior knowledge and current understanding. This category is mentioned by Carpenter, Fennema, Peterson, and Carey (1988) and it came out empirically out of Marks' research
(1989, 1990). Figure 1.1 shows these three specific categories of pedagogical content knowledge with some specific components. The components pertaining to the specific research questions, as stated in a later section, are described in more detail in Chapter 2.

![Diagram of Pedagogical Content Knowledge]

**Figure 1.1 Categories and components of pedagogical content knowledge**

The knowledgeable mathematics teacher possesses a detailed and rich network of mathematical knowledge in the topics he or she teaches that somewhat parallels the knowledge held by a mathematician in those topics. In this sense, the knowledgeable
mathematics teacher and the mathematician hold similar subject matter knowledge about a particular mathematical topic. What might differentiate, in general terms, a good mathematics teacher from a mathematician is that the former possesses general pedagogical knowledge (e.g., knowledge of principles and theories of teaching and learning, knowledge of learners, knowledge of classroom organization and management, and knowledge of effective teaching principles) that the latter is not required to have. However, this general pedagogical knowledge is similar, in general terms, to that held by a pedagogue. From this perspective, it has been said that mathematics teachers, and other teachers, lack a specialized body of knowledge unique to their profession (Feiman-Nemser & Floden, 1986; Noddings, 1992). Shulman's (1986a) theoretical construct of pedagogical content knowledge has the potential of providing a unique kind of knowledge specific to the teaching profession. In the case of mathematics teachers, they need to possess mathematical pedagogical content knowledge, that is, knowledge of specific pedagogical representations and knowledge of students' cognitions about the specific mathematical ideas represented. This kind of knowledge would not be commonly held by either a mathematician or a pedagogue.

Research on teachers' knowledge is already a well-established area of educational research. For the most part, researchers have focused on general pedagogical knowledge, but teachers' subject-matter knowledge and pedagogical content knowledge with special emphasis on knowledge of representations and knowledge of students' cognitions are becoming fertile areas of research on teachers' knowledge (Ball & McDiarmid, 1990; Ball, 1991; Carter, 1990; Fennema & Franke, 1992).

The Problem

Shulman and his colleagues (Shulman, 1987; Wilson, Shulman, and Richert, 1987) conceptualize of teaching as a process of transforming content knowledge into representations that are understood by students. From this point of view, the fundamental
problem of teaching mathematics is finding the most powerful representations of each of
the mathematical topics of the curriculum. According to these scholars, during the process
of transformation teachers draw on two main types of knowledge: content knowledge and
pedagogical content knowledge. From this perspective, teachers need to have rich content
and pedagogical content knowledge. Rich content knowledge involves knowledge of
mathematical ideas (procedures, definitions, proofs, formulas, theorems, axioms, etc.)
supported by conceptual knowledge. However, "it remains unclear what teachers know
about their subject matter and how they choose to represent that subject matter during
instruction" (Wilson, Shulman, and Richert, 1987, p. 108). Similarly, Simon (1993) asks:
"What is the nature of practicing and prospective teachers' mathematical knowledge today?
He also makes the critical observation that teachers' knowledge needs to be examined in
social contexts in which the teachers use their knowledge" (p. 234). Finally, Fennema &
Franke (1992) ask: "Do teachers know the representations of the content they ordinarily
teach? Does knowing those representations make any difference on how teachers teach
...?" (p. 154). These questions and observations provide authoritative support for
investigating questions related to teachers' content and pedagogical content knowledge.

Some researchers studying mathematics teachers' knowledge of subject matter have
examined teachers' conceptions and beliefs about mathematics (e.g., Blaire, 1981; Ernest,
1989; Ferrini-Mundy, 1986; Kuhs, 1981; Lerman, 1983; Thompson, 1984) while others
have focused on teachers' understandings of specific mathematical concepts, procedures,
and principles (e.g., Ball 1990a, 1990b; Even, 1989, 1990, 1993; Graeber & Tirosh,
1988; Graeber, Tirosh, & Glover, 1986, 1989; Hadass & Bransky, 1991; Post, Harel,
1990a, 1990b; 1991). However, as we will see in the review of literature chapter,
teachers' knowledge about algebraic multiplication remains without research attention. The
authors of *The University of Chicago School Mathematics Project Algebra* (McConnell,
Brown, Eddins, Hackworth, Sachs, Woodward, Flanders, Hirschhorn, Hynes, Polonsky, & Usiskin, 1990) include the following ideas as central to algebraic multiplication in eighth-grade mathematics: area, arrays, multiplication of fractions and rates, solving $ax = b$ and $ax < b$, etc. Another gap in the research literature is the lack of research about the impact of teachers' content knowledge on their classroom instruction. In particular, teachers' knowledge of mathematical representations (e.g., symbolic representations and proofs) and the use of their knowledge during classroom instruction are underrepresented areas of research on teachers' mathematical content knowledge. The present study investigates the phenomenon of teachers' mathematical knowledge and knowledge use about symbolic representations and proofs (two types of mathematical representations) in the context of algebraic multiplication.

Most research on teachers' pedagogical content knowledge has been conducted mainly within the *Knowledge Growth in Teaching Project* at Stanford University by Shulman, his associates and students in several curricular areas such as social studies (Gudmundsdottir, 1987a, 1987b, 1988; Gudmundsdottir & Shulman, 1987, 1989), English (Grossman, 1989), science (Carlsen, 1988; Hashweh, 1987), and mathematics (Marks, 1989, 1990). Other researchers have also used Shulman's theoretical construct of pedagogical content knowledge as a conceptual framework for studying teachers' knowledge (Carpenter, Fennema, Peterson, and Carey, 1988; Even, 1993; Even & Markovits, 1993; Sánchez & Llinares, 1992). However, teachers' knowledge of pedagogical representations, a component of pedagogical content knowledge, is an underrepresented area of research on teachers' knowledge (Fennema & Franke, 1992; Simon, 1993; Wilson, Shulman, & Richert, 1987). Another gap in the literature is how teachers use their pedagogical content knowledge during classroom instruction. In particular, we do not know whether teachers know the story-problem representations and pictorial representations about mathematical ideas and whether knowing those representations makes any difference in how teachers
teach. The present study investigates the phenomenon of teachers' knowledge of pedagogical representations and knowledge use in the context of algebraic multiplication.

A more complete review of the research literature on teachers' content knowledge and pedagogical content knowledge is found in Chapter 3.

Purpose and Research Questions

The purpose of the present study is twofold. First, it examines, in detail, one experienced middle school teacher's knowledge of mathematical representations and pedagogical representations about algebraic multiplication. Second, it investigates the extent to which Mr. Kantor, the participant teacher, uses his content knowledge (mathematical representations) and pedagogical content knowledge (pedagogical representations) when constructing pedagogical events (explanations, representations, and questions) for teaching algebraic multiplication. To achieve these purposes, the following research questions are investigated:

I. What is Mr. Kantor's knowledge of mathematical representations about algebraic multiplication?
   1. What is Mr. Kantor's knowledge of symbolic representations?
   2. What is Mr. Kantor's knowledge of mathematical proofs?

II. What is Mr. Kantor's knowledge of pedagogical representations about algebraic multiplication?
   3. What is Mr. Kantor's knowledge of pictorial representations?
   4. What is Mr. Kantor's knowledge of story-problem representations?

III. How does Mr. Kantor use his knowledge of representations when teaching algebraic multiplication?
   5. What representations does Mr. Kantor use?
   6. What explanations does Mr. Kantor construct?
   7. What questions does Mr. Kantor pose?
Rationale and significance

Several researchers have noticed the lack of research on teachers' subject matter content knowledge and pedagogical content knowledge and have called for research in this area. For example, Shavelson and Stern (1981) in their review of teachers' pedagogical thoughts, judgments, decisions, and behavior noted:

Very little attention has been paid to how knowledge of a subject matter is integrated into teachers' instructional planning and the conduct of teaching ... Nevertheless, the structure of the subject matter and the manner in which is taught is extremely important to what students learn and their attitudes toward learning and the subject matter. (p. 491)

Similarly, Thompson (1982) pointed out that the question "of how teachers integrate their knowledge of mathematics into instructional practice ... ha[s] been largely ignored" (p. 1).

This lack of attention to the role of subject-matter knowledge in teachers' instructional behavior has been called by Shulman (1986b) the "missing paradigm" in research on teaching. He says:

In their necessary simplification of the complexities of classroom teaching, investigators ignored one central aspect of classroom life: the content of instruction, the subject matter .... No one asked how subject matter was transformed from the knowledge of the teacher into the content of instruction. (p. 8)

With particular reference to mathematics, Fennema and Franke (1992), as mentioned earlier, call for research on teachers' knowledge of representations and the influence of this knowledge on their classroom instruction. They are puzzled by the fact that we do not know whether teachers know the representations of the content they usually teach and the impact of that knowledge on their classroom instruction. Similarly, Simon (1993) noted that research is needed in the area of how teachers use their knowledge of mathematics in social situations such as classrooms. Geddis, Onslow, Beynon, and Oesch (1993) have also highlighted the importance of the role that pedagogical content knowledge plays "in transforming subject matter into forms that are more accessible to students" (p. 582). But, at the same time, "there have been relatively few studies that have provided detailed
articulation of the pedagogical content knowledge employed by teachers to transform particular content matter for specific students" (Geddis, Onslow, Beynon & Oesch, 1993, p. 582). The present study attempts to make some contribution to this area by focusing on some portions of the content and pedagogical content knowledge held by one teacher and how he uses those portions of knowledge when constructing pedagogical events (explanations, representations, and questions) when teaching algebraic multiplication.

Research on teachers' content knowledge and pedagogical content knowledge draws its importance from several sources. The first one is related to teaching and learning mathematics with understanding. Teaching and learning with conceptual understanding is a central issue to current reforms in mathematics education (e.g., National Council of Teachers of Mathematics [NCTM], 1989, 1991) as well as in current research on teaching and learning mathematics (e.g., Hiebert & Carpenter, 1992). One of the most important factors that influences the creation of opportunities that students have for learning mathematics with understanding is the teacher. It seems evident that if we want teachers to teach mathematics with understanding, then the teacher himself needs to have a deep conceptual understanding of the mathematical content to be taught. Then the answers to the questions posed above by Wilson, Shulman and Richert (1987), Fennema and Franke (1992), and Simon (1993) about the nature of practicing and prospective teachers' mathematical and pedagogical content knowledge are of primordial importance in research on teaching and teacher education. Not surprisingly, as described above, several researchers have examined teachers' knowledge of specific mathematical concepts such as function (e.g., Even, 1993; Wilson, 1994), division (Ball, 1990b; Graeber and Tirosh, 1988; Graeber, Tirosh, and Glover, 1989; Simon, 1993; Tirosh & Graeber, 1989), number theory (Zazkis & Campbell, 1994a, 1994b, 1996a, 1996b) and other mathematical topics. Teachers' knowledge about algebraic multiplication remains unexplored. In the
present study I examine two components of teachers' content knowledge: knowledge of symbolic representations and knowledge of mathematical proofs.

One type of symbolic representations are the mathematical definitions that can be represented symbolically. Teachers' knowledge of definitions and symbolic representations is an important area of research because of the relationship of definitions and symbolic representations to advanced mathematical thinking. As stated by Tall (1992) "advanced mathematical thinking ... is characterized by two important components: precise mathematical definitions (including the statement of axioms in axiomatic theories) and logical deductions of theorems based upon them" (p. 495). If we want to understand students' (including teachers) development of advanced mathematical thinking then we need to pay attention to teachers' own advanced mathematical thinking because of the potential impact it may have on their construction of pedagogical events to help their students develop that kind of thinking. Yet, a review of the literature suggested that teachers' knowledge of definitions or symbolic representations has been ignored in research on teachers' knowledge. An exception is the study by Even (1993). She asked 162 prospective secondary mathematics teachers, among other things, to give a definition of a function. Because her purpose was to examine prospective teachers' awareness of the arbitrariness and univalence nature of the function concept she did not categorize their responses based on their correctness and accuracy. Then it remains unknown the extent to which students knew the correct and precise mathematical definition of function. The present study describes in detail one teacher's knowledge about definitions or symbolic representations of a common curriculum topic: multiplication in algebra. We need to continue this area of research to have a complete picture about what teachers know about the definitions or symbolic representations of mathematical entities or ideas.

A second focus of the present study relates to teachers' proof frames. As stated by Tall (1992), definitions and proofs are the two main components of advanced mathematical
thinking. Yet, they remain pretty much without research attention. An exception is the study conducted by Martin and Harel (1989) who examined preservice elementary teachers' knowledge and conceptions about mathematical proofs. However, teachers' knowledge of most of the mathematical topics remains unexamined. The present study attempts to investigate one teacher's knowledge of mathematical proofs about topics related to algebraic multiplication.

Some researchers (e.g., Ball, 1988; Llinares & Sánchez, 1991) have investigated teachers' knowledge of pictorial or concrete representations. However, it still remains an underrepresented area of research on teachers' knowledge (Fennema & Franke, 1992). The present study investigates in depth one teacher's knowledge of pictorial representations of topics related to algebraic multiplication.

Even though some researchers (e.g., Azim, 1995, 1996; Ball, 1990b; Simon, 1993) have examined teachers' knowledge of story-problem representations, this area remains an underrepresented area of research on teachers' knowledge (Fennema & Franke, 1992). The present study documents in detail one teacher's knowledge of story problem representations about topics related to algebraic multiplication.

In mathematics, as well as in other subjects, it is likely that there is a strong, subtle, and complex relationship between teachers' knowledge and their instructional decisions, especially as it concerns content knowledge and pedagogical content knowledge. While most research on teachers' knowledge examines teachers' understanding of specific mathematical topics, few studies have examined the impact of teachers' knowledge on their classroom instruction. This claim is supported by Fennema and Franke (1992), Simon (1993), Wilson et al. (1987) who state that the extent to which teachers use their knowledge on their classroom instruction is in need of investigation. Examining teachers' use of their knowledge helps to understand the relationship between teachers' knowledge and knowledge use during classroom instruction. In addition, it is important to examine
teachers' use of their knowledge because the extent to which teachers use their knowledge contributes to the opportunities that students have for learning the intended mathematical content. The present study examines the extent to which one knowledgeable teacher uses his knowledge of symbolic representations, mathematical proofs, pictorial representations, and story-problem representations on his classroom instruction. Thus, the present study goes a step further in identifying the nature of the relationship between some components of teachers' content and pedagogical content knowledge and classroom instruction. This is an important consideration from a pedagogical perspective. Some studies have examined what teachers know about their subject matter, but we do not know how teachers actually use their knowledge for teaching purposes. Taking a step further, other studies have examined how teachers would represent subject matter knowledge to hypothetical students in hypothetical situations. Hence, from a pedagogical perspective, the present study goes a step further in examining how teachers actually use their knowledge for making pedagogical and curricular decisions (Brophy, 1991; Simon, 1993).

Another source of importance of teachers' knowledge and use of their knowledge is related to the role that teacher education programs have in helping teachers learn the representations of the material that they will likely to teach. We can examine some of the influence of teacher education programs by investigating the extent to which teachers use the knowledge that they learn in teacher education programs. The present study helps us to gain a deeper understanding of the influence that teacher education programs may have on teachers' knowledge of representations and the influence of that knowledge on their classroom instruction. It examines one teacher's knowledge and knowledge use of two types of knowledge that teachers are supposed to learn in teacher education programs: mathematical content knowledge (mathematical representations) and mathematical pedagogical content knowledge (pedagogical representations).
Further, the importance of examining teachers' knowledge comes from a student's perspective. One of the goals of current reforms in mathematics education is that all students become mathematically literate (NCTM, 1989). Mathematics teachers form a sample of a population whose mathematics education we want to improve. In particular, teachers are learners and once were students. Improving students' knowledge of mathematics implies improving teachers' knowledge of mathematics.

**Delimitations of the Study**

I have described several sources of importance of the present study. However, as does any other study in mathematics education, this study has some delimitations. One of the most important delimitations of the study relates to the size of the sample. I observed only one teacher's knowledge and how the teacher used his knowledge about a specific mathematical topic during classroom instruction. Therefore, generalizations about the phenomenon under investigations to a larger population are not warranted. Even though the sample size is consistent with the purpose of the study, we also need to consider larger samples or additional case studies to examine the phenomenon from a broader perspective.

A second delimitation of the present study deals with generalizability beyond the mathematical topics selected. A larger sample of mathematical topics would have added some generalizability to the findings about the phenomenon of teachers' knowledge and knowledge use. However, additional data would force the sacrifice of detail which is a major strength and purpose of the present study.

A third delimitation of the present study is that it focuses on the teaching phenomenon *per se* without addressing issues about the links between teaching and learning. We certainly need to investigate relationships between teaching and learning. However, we should not forget that studies about teaching *per se* are fulfilling their promise of understanding the complexity of the teaching phenomenon. In the present study, a case of
the teaching phenomenon is examined from the perspective of knowledge and knowledge use.

A fourth delimitation of the present study is that it is a descriptive study and, therefore, does not investigate reasons, attributions, or rationales. That is, I describe one teacher's knowledge and how he uses his knowledge when teaching algebraic multiplication. It is not the purpose to explain in depth why he has connected or disconnected knowledge about algebraic multiplication and why he uses or fails to use aspects of his knowledge when teaching algebraic multiplication. However, I will follow Wolcott's (1994) suggestion to use theory, expectations, and personal experience to provide or suggest possible interpretations, explanations, or even speculations about the findings.
CHAPTER 2

THEORETICAL FRAMEWORK

This chapter describes the theoretical framework that guided the statement of the research questions and that also guided the description, analysis, and interpretation of data collected to answer the following research questions described in Chapter 1:

I. What is Mr. Kantor's knowledge of mathematical representations about algebraic multiplication?
   1. What is Mr. Kantor's knowledge of symbolic representations?
   2. What is Mr. Kantor's knowledge of mathematical proofs?

II. What is Mr. Kantor's knowledge of pedagogical representations about algebraic multiplication?
   3. What is Mr. Kantor's knowledge of pictorial representations?
   4. What is Mr. Kantor's knowledge of story-problem representations?

III. How does Mr. Kantor use his knowledge of representations when teaching algebraic multiplication?
   5. What representations does Mr. Kantor use?
   6. What explanations does Mr. Kantor construct?
   7. What questions does Mr. Kantor pose?
Several overlapping conceptual perspectives about teaching mathematics contributed to the theoretical framework for this study. The first perspective is about the relationship between external representations and internal representations. This is discussed next.

**External and Internal Representations**

While for communicating mathematical ideas to others we need to use external representations, for thinking about mathematical ideas we need to represent them internally in our minds (Hiebert & Carpenter, 1992). Research in cognitive science has provided evidence that human beings translate their external experiences into internal representations which allow the human mind to think about and operate internally on those experiences (Wilson, Shulman, & Richert, 1987; Hiebert & Carpenter, 1992). In particular, to communicate mathematical ideas to students, we need to represent them externally through a variety of means such as spoken language, written symbols, pictures, physical objects, or a combination of these (cf. Hiebert & Carpenter, 1992; Lesh, Post, & Behr, 1987). As mathematics teachers, we want to choose those external representations of the mathematical ideas that are comprehensible to students. This remark leads us to the second perspective: Teaching is the process of constructing pedagogical, psychological or instructional representations of subject-matter knowledge (Ball, 1988a; Bruner, 1966; Byrne, 1983; Dewey, 1969; Shulman, 1986a, 1987; Wilson, Shulman, & Richert, 1987; Wineburg & Wilson, 1988).

**Teaching as the Construction of Instructional Representations**

One of the first scholars to talk about the process of constructing psychological representations of subject-matter was Dewey (1969). He said that a scientist's main concern is the creation of new knowledge. A teacher's main concern, on the other hand, is to psychologize the subject matter so that it can help children in their intellectual development. In his terms:

[teachers' concerns include] ways in which that subject may become a part of experience; what there is in the child's present that is usable with reference to it; how much elements are to be used; how his
own knowledge of the subject-matter may assist in interpreting the child's needs and doings, and determine the medium in which the child should be placed in order that his growth may be properly directed. He is concerned, not with the subject-matter as such, but with the subject matter as a related factor in a total and growing experience. (pp. 187-188)

Another scholar who was concerned with the problem of finding psychological representations of subject matter was Bruner (1966). According to Bruner (1966), the fundamental problem of the psychology of teaching and learning mathematics is to find various representational modes that embody abstract mathematical ideas and that can help children to learn the intended content. He contends that concepts can be represented using mainly three types of representations: enactive, iconic, and symbolic. He contends that all the three types of representations can help students to learn the intended subject matter.

Byrne's (1983) concept of teaching also involves finding ways to represent the subject matter to students. In his own words:

A teacher should certainly possess a certain minimum level of facility with, and understanding of, the subject to be taught. However, there is another aspect of the teacher's subject-knowledge which is also important. To appreciate this, it is necessary to withdraw from the commonsense view of knowledge usually adopted by researchers in this area and to adopt instead a more subtle perspective. This concerns a teacher's capacity for representing the knowledge to be taught (Emphasis added). (p. 18)

Shulman and his colleagues (Shulman, 1986a, 1987; Wilson, Shulman, and Richert, 1987) also conceive of teaching as a process of constructing pedagogical representations. They argue that teachers need not only content knowledge but also a blend of content knowledge and pedagogical knowledge that Shulman (1986a) has termed pedagogical content knowledge. Pedagogical content knowledge includes:

The most useful forms of representations of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations, in a word, the ways of representing and formulating the subject matter that make it comprehensible to others ... [it] also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning. (p. 9)

This definition includes two major interrelated components. The first component, ways of representing subject-matter to students, is termed pedagogical representations.
The second component, knowledge of students' understanding of the subject matter concepts, will be termed knowledge of students' specific cognitions.

Wineburg and Wilson (1988) also see instructional representations as central to the process of teaching subject matter. They state:

Creating a representation is an act of pedagogical reasoning. Teachers must first turn inward to comprehend and ponder the key ideas, events, concepts, and interpretations of their discipline. But in fashioning representations, teachers must also turn outward. They must try to think themselves into the minds of students who lack the depth of understanding that they, as teachers, possess. An instructional representation emerges as the product of teachers' comprehension of content and their understanding of the needs, motivations, and abilities of learners. (p. 57)

Ball and her colleagues (Ball, 1988a; McDiarmid, Ball, & Anderson, 1989) also acknowledge the importance of instructional representations in teaching. They argue that "whether or not they [teachers] are aware of it, teachers are constantly engaged in a process of constructing and using instructional representations" (McDiarmid, Ball, & Anderson, p. 194).

Fennema and Franke (1992) are also concerned with instructional representations. They argue that teachers need a type of knowledge about "how mathematics should be represented in instruction" (p. 153). In their terms:

This involves taking complex subject matter and translating into representations that can be understood by students.... Mathematics is composed of a large set of highly related abstractions, and if teachers do not know how to translate those abstractions into a form that enables learners to relate the mathematics to what they already know, they will not learn with understanding. (p. 153)

My conception of teaching also involves the construction or use of instructional representations. This conceptualization is described in the following sections.

**Mathematical and Pedagogical Representations**

My purpose in this section is to integrate several perspectives on the use of representations on teaching and learning mathematics and provide a classification of those from a pedagogical content knowledge perspective. Mathematical ideas or objects include concepts, definitions, axioms, theorems, formulas, algorithms, procedures and proofs. Each of these mathematical ideas will be termed a mathematical content curriculum event. These mathematical events can be represented using several types of
representations that I will call content representations (Fennema & Franke, 1992; Hiebert & Carpenter, 1992; Lesh, Post, & Behr, 1987). Lesh, Post and Behr (1987) identified five of these representations: (a) real scripts, (b) manipulative models, (c) static pictures, (d) written symbols, and (e) spoken language. Building on Lesh, Post, and Behr (1987), Hiebert and Carpenter (1992) contend that communicating mathematical ideas involves the use of spoken language, writing symbols, pictures, and physical objects. Fennema and Franke (1992), on the other hand, describe three contexts in which mathematical ideas can be represented: (a) real-world situations and problems, (b) concrete objects or manipulatives, and (c) pictorial representations.

I will categorize the representations in two broad categories; mathematical representations and pedagogical representations. Mathematical representations are the standard representations of the content. They are symbolic and abstract in nature. Mathematical representations include the symbolic representations of the mathematical content curriculum events. Pedagogical representations on the other hand, include (a) real-world situations and problems, (b) manipulatives, and (c) pictorial representations.

Regarding mathematical representations, I will distinguish between different levels of symbolic representations. I will use the terms numerical representations, numeric-symbolic representations and symbolic representations. I believe that this classification is useful when talking about teaching and learning algebra, in particular, algebraic multiplication. According to Usiskin (1988), school algebra includes three types of studies: (a) the study of the generalization of arithmetic relationships, (b) the study of procedures for solving certain kinds of problems, and (c) the study of relationships among quantities. The feature of algebra is the use of variables. A numerical representation of an algebraic object involves only the use of numbers. For example, 3 \times 5 = 5 \times 3 is a numerical representation of the commutative property of multiplication. A numeric-symbolic representation involves the use of both numbers and variables to stand
for numbers. For example, $3 \cdot x = x \cdot 3$ is an example of this type of symbolic representations. The final type of symbolic representations will be termed symbolic representations. It involves the use of variables to stand for numbers whenever is appropriate. For example, $xy = yx$ is the symbolic representation of the commutative property of multiplication. Pedagogical representations include mainly pictorial representations, story-problem representations, and physical objects or manipulatives. Because the subject matter of this dissertation is algebraic multiplication, numerical representations will be grouped under pedagogical representations. This type of representations are considered symbolic representations in arithmetic. The symbolic representation of algebraic objects will be termed mathematical representations.

I will also consider algebraic proofs as another type of mathematical representations because they involve the use of symbolic elements and they represent the truth of mathematical statements. The seven main types of representations (mathematical proofs, symbolic representations, numerical representations, numerical-symbolic representations, pictorial representations, story-problem representations, and physical representations or manipulatives) can be used to show the mathematical truth or meaning, or application of mathematical statements, concepts or procedures to students. Figure 2.1 summarizes the conceptual framework about mathematical and pedagogical representations.

Both mathematical and pedagogical representations embody content representations (i.e., mathematical knowledge). The difference is that the pedagogical representations are not necessarily the standard representations of the content knowledge. Mathematical representations are symbolic, abstract, and conceptual in nature. Pedagogical representations, on the other hand, can be pictorial, concrete, or physical.
Learning and Instructional Representations

Why is it important to use a variety of mathematical and pedagogical representations during teaching? All the scholars whose conceptualization of teaching includes the concept of representation contend that using a variety of representations helps students to learn subject-matter. For example, Bruner (1966) provided theoretical and empirical evidence that the use of multiple modes of representations helps students to understand subject-matter knowledge. McDiarmid, Ball, and Anderson (1989) argue that "the instructional representations that students encounter define their formal opportunities for learning about the subject matter" (p. 194). Hiebert and Carpenter (1992) suggest that:

the form of an external representation (physical materials, pictures, symbols, etc.) with which a student interacts makes a difference in the way the student represents the quantity or relationship internally. Conversely, the way in which a student deals with or generates an external representation reveals something of how the student has represented that information internally. (p. 66)

Finally, Fennema and Franke (1992) state that the "use of both real-world situations and concrete or pictorial representations help students learn the abstract ideas of mathematics with understanding" (p. 154).
Mathematical Knowledge

I have been talking about teachers representing and students learning mathematical knowledge. But what is mathematical knowledge? According to Schwab (1964, 1978), it encompasses two main interwoven components, namely, substantive knowledge of the subject and syntactic structure of the discipline. In Ball's (1991) terms, knowledge of the subject and knowledge about the subject. Substantive knowledge of mathematics or knowledge of mathematics includes knowledge of definitions, conventions, and concepts; axioms and theorems and their proofs; formulas and algorithms; conjectures and problems; as well as knowledge of the connections between and among those different components. Syntactic knowledge of mathematics, or knowledge about mathematics, includes knowledge of how claims are justified in the field, historical development of its concepts, applications of the subject within itself and to other fields, what doing mathematics entails, and epistemological issues of the domain, etc. Shulman (1986a) highlights the importance of these two components in the following excerpt:

Teachers must not only be capable of telling students what are the accepted truths in a domain. They must be able to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and without, both theoretical and practical. (p. 19)

In the present research project I am concerned with knowledge of mathematics (henceforth referred to as mathematical knowledge). Hiebert and Lefevre (1986) describe mathematical knowledge as consisting of both conceptual and procedural knowledge. These scholars define conceptual knowledge as "knowledge that is rich in relationships" (p. 3). Hiebert and Carpenter (1992) elaborate this description in the following terms: "A unit of conceptual knowledge is not stored as an isolated piece of information; it is conceptual knowledge only if it is part of a network" (p. 78). On the other hand, procedural knowledge includes knowledge of the symbols and their syntax as well as knowledge of rules, algorithms or procedures that are used to carry out mathematical
tasks (Hiebert & Lefevre, 1986). These two types of knowledge are not independent. As
stated by Hiebert and Carpenter (1992):

Conceptual knowledge contributes to mathematical expertise through its relationships with procedural
knowledge.... [These relationships] depend upon the connections learners construct between their
representations. From an expert's point of view, procedures in mathematics always depend upon
principles represented conceptually.... Relationships between procedural and conceptual knowledge
may range from no relationship to a relationship so close that they become difficult to distinguish. (p.
78)

Teaching as the Construction of Pedagogical Events

So far teaching has been characterized as the process of constructing content
representations. But content representations about what? Doyle (1992) argues that
teaching and curriculum are interwoven. He states "a curriculum is intended to frame or
guide teaching practice and cannot be achieved except during acts of teaching. Similarly,
teaching is always about something so it cannot escape curriculum" (p. 486). I conceive
of a curriculum as made up of content curriculum events. A *mathematical content*

*curriculum event* is each mathematical idea or object (e.g., concepts, formulas, theorems,
axioms, algorithms or procedures, etc.) identified in the curriculum or in a curriculum text
such as curriculum guides or textbooks. During a particular teaching episode a teacher
has as purpose to help students construct the mathematical knowledge represented by a
specific content curriculum event. To this end, teachers not only construct
representations but also explanations and ask questions as well. I will use the term
pedagogical events when referring to explanations, representations, and questions. An
explanation is described by Leinhardt, Putnam, Stein and Baxter (1991) as "an activity in
which teachers communicate subject-matter ... [it] is not only what a teacher says or
shows to the student, but also includes the systematic arrangement of experiences so that
the student can construct a meaningful understanding of a concept or procedure" (p. 89).
These scholars define representations as "physical or conceptual objects or systems of
objects that embody mathematical entities or ideas" (p. 89). A question is a formulation
that calls for an answer. The purpose of most teachers' questions is to monitor, check, or
assess students' learning. The purpose of these pedagogical events is to help or facilitate students' construction of mathematical knowledge (learning).

I conceptualize teaching as the process of constructing pedagogical events about content curriculum events with the purpose of helping students construct mathematical knowledge. Several factors influence curriculum, teaching, and learning. Those are described below.

An Integrated and General Theoretical Framework

Through reflection, initial analysis of the data, and review of the literature a theoretical framework has been created that can be helpful for examining the teaching phenomenon. While Shulman and his associates conceive of teaching as a process of transforming personal understanding of subject-matter knowledge into representations that are understandable to students (i.e., teaching is a pedagogical process of transformation), Doyle (1992) conceives of teaching as both a pedagogical and curricular process of transformation. It is a curricular process because a curriculum cannot be implemented without acts of teaching, and teaching is always about something. In this sense teaching and curriculum are inherently interwoven. But there is an explicit link missing in that process of transformation: students' learning. We cannot fully understand the teaching phenomenon without addressing its link with learning. The model illustrated in Figure 2.2 portrays the interrelationship of teaching to curriculum and learning. It also portrays my conceptualization of teaching as the construction of pedagogical events (explanations, representations, and questions) about content curriculum events with the purpose of helping students to construct mathematical knowledge.
As depicted in Figure 2.2, teaching as well as the curriculum affect the learning phenomenon. This, in turn, affects the teaching act. The nature of the content curriculum event (e.g., its difficulty, abstractness, novelty, etc.) is likely to affect the way students learn it (Kieran, 1992). In the same way, students' ways of learning and cognitions (e.g., difficulties, prior knowledge and misconceptions, strategies, mistakes, etc.) may have an impact on what pedagogical events the teacher constructs.

What are the factors that influence teachers' construction of pedagogical events about content curriculum events? The first factor that seems to be evident is teachers' content knowledge. The importance of this type of knowledge is supported by several research studies that have examined teachers' knowledge about specific mathematical content (e.g., Ball, 1990a, 1990b; Even, 1993; Fennema & Franke, 1992; Simon, 1993; Tirosh & Graeber, 1989; Wilson, 1994) or that have investigated how teachers use their mathematical knowledge in teaching situations (e.g., Ball, 1991; Fennema & Franke, 1992; Thompson & Thompson, 1994; Thompson & Thompson, 1996). Other factors are teachers' conceptions about the nature of mathematics (Dossey, 1992), teachers' beliefs
about teaching and learning (Thompson, 1992), and other affective factors such as
teachers' attitudes and emotions (McLeod, 1992). In addition to content knowledge,
Shulman and his colleagues (Shulman, 1986a; Wilson et al., 1987) argue that teachers
draw on pedagogical content knowledge and curricular content knowledge1 during the
process of transformation. Regarding learning, four factors seem to contribute to how
students construct mathematical knowledge: students' cognitions, teaching, and the
content itself (curriculum) (Kieran, 1992) and affective factors (McLeod, 1992). Several
other factors may affect the process of transforming curriculum events into pedagogical
events. Among those are: language, race, gender, socioeconomic status, etc. They are
indicated in the model under "other factors." Finally, the curriculum, teaching, and
learning are embedded in a particular context, culture, or situation that influences
teaching and particularly what students learn (Brown, Collins, & Duguid, 1989; Doyle,
1992). Figure 2.3 depicts the general theoretical framework that represents my
conceptualization of teaching and some of the factors that may have an influence on
teaching.

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1Curricular content knowledge includes knowledge of instructional materials such as manipulatives,
computer software, activities, etc., for teaching a specific mathematical content curriculum event.
Figure 2.3 A theoretical framework for examining the teaching phenomenon

In this particular study I am focusing on the process from curriculum to teaching and on two factors that influence that relationship: mathematical content knowledge and pedagogical content knowledge, as stated in the research questions. To reiterate, I examined one teacher's knowledge of mathematical representations (symbolic representations and mathematical proofs) and pedagogical representations (pictorial representations and story-problem representations) and the pedagogical events (explanations, representations, and questions) that he constructed for teaching the content curriculum events identified in the lessons taught. Figure 2.4 depicts the portions of the theoretical model that guided the present study.
As indicated in the model, I am interested in one component of mathematical content knowledge, mathematical representations, and in one component of pedagogical content knowledge, pedagogical representations. In the next chapter I provide a detailed review of the literature related to the fundamental components of the theoretical framework that have a bearing on this study: teachers' content knowledge (e.g., mathematical representations) and pedagogical content knowledge (e.g., pedagogical representations).
CHAPTER 3

REVIEW OF LITERATURE

Those who understand teach (Shulman, 1986a)

This chapter reviews the relevant literature related to the research questions and relevant portions of the theoretical framework presented in Chapter 2. As stated in the research questions, I examine the phenomena of teachers' knowledge and knowledge use of mathematical representations and pedagogical representations about topics related to algebraic multiplication. To give the reader a context for this study within research on teachers' knowledge, the first section deals with research on teachers' knowledge and how it has been approached in research on teaching and teacher education. The second section presents some relevant literature related to research on teachers' general pedagogical knowledge. The third section addresses research on teachers' mathematical content knowledge. The fourth section reviews research on teachers' pedagogical content knowledge in the case of mathematics. Finally, the fifth section reviews literature on teachers' use of mathematical and pedagogical representations during classroom instruction.

Teachers' Knowledge

Teachers' knowledge has been examined using approaches differing in assumptions, theoretical frameworks, and methodologies. Carter (1990) identified three such approaches: information-processing studies, studies of teachers' practical knowledge, and studies of teachers' pedagogical content knowledge (Figure 3.1).
Figure 3.1 Research on teachers' knowledge

**Information-Processing Studies**

Studies framed in the information-processing tradition have focused on describing the cognitive processes that teachers use in thinking prior to, during, and after classroom instruction. Generally, these studies have used technical language of psychology, have been conducted under controlled conditions, and have used standardized tasks in data collection procedures. Within the information-processing tradition, two lines of inquiry have emerged. The first line deals with teachers' generic pedagogical knowledge which includes knowledge of planning and decision making. The second line of inquiry has focused on finding, and accounting for, differences in thinking and instructional behavior between novice and expert teachers.
One of the most classical works within studies of planning and decision making is probably that of Jackson (1968) who identified three phases of teaching: pre-active or planning phase, interactive phase, and post-active or reflective phase. Results of this line of research document that teachers do make year plans, unit plans, and daily plans, which contain a diversity of topics such as objectives, what to teach, how to teach it, activities to engage students in, and how to adapt instruction to students. With respect to research on interactive decision making, this has provided us with information on the kind of decisions teachers make when they are actually teaching. Among the decisions teachers make during actual teaching are the following: modification of lesson's structure and content, specific ways of responding to a particular child, requesting particular students to answer a question, accepting or rejecting students' questions and answers, etc. (Fennema & Franke, 1992). Finally, this line of research has documented also that some teachers do reflect after their classroom instruction. This reflection includes things that went wrong and ways to improve the same material when it is presented to similar students. It also includes reflection on what students learned or did not learn during the class. All this information can be used by teachers to design the instruction of the following day (Fennema & Franke, 1992). A more detailed and complete review of this kind of research is provided elsewhere (Borko & Shavelson, 1990; Clark & Peterson, 1986; Shavelson, 1983; Shavelson & Stern, 1981).

Studies of teacher planning and decision making made a significant advancement in our understanding of teachers' knowledge and teaching. However, as noted by Shulman (1986b), most of this kind of research focused on a few narrow features of teachers' cognition without putting any attention to the knowledge behind that thinking.

Studies framed in the expert-novice research tradition have provided us with rich information about differences in knowledge, thinking, and action between expert and novice teachers. Those studies have cast some light on our understanding of how expert
knowledge is acquired. Expert-novice research has been conducted with an eye on
generic teaching (Berliner, 1987; Carter, Sabers, Cushing, Pinnegar, & Berliner, 1987)
using as context several curricular content areas such as physical education (Housner &
Griffey, 1985); social studies (Peterson & Comeaux, 1987); and mathematics (Borko &
Livingston, 1989; Leinhardt, 1989; Leinhardt & Greeno, 1986; Leinhardt, Putnam, Stein,
a specific subject matter as context, most of the research in expert-novice studies has
been conducted from a generic point of view and, therefore, those studies have focused
on generic differences between expert and novice teachers. Some generic patterns
provided by expert-novice studies are summarized below.

Berliner and his associates (Berliner, 1986, 1987; Carter, Sabers, Cushing, Pinnegar,
& Berliner, 1987) have used standardized tasks and simulated classroom situations to
compare ways of thinking about students and classrooms of three groups of teachers. The
first group included mathematics and science expert teachers with over five years of
experience who were nominated as excellent by their principals or school
superintendents. The second group consisted of student teachers and first-year teachers
of mathematics or science. They were classified as novices. The final group was formed
of mathematicians and scientists who were employed in business, industry, or research
organizations and who had expressed an interest in obtaining a teaching certificate
without taking any education course. They were considered postulants. Findings from
this research show that these three groups differed on their thinking across several tasks,
such as about students to be taught, about the validity of information used to describe
students by other teachers, etc. Teachers' thinking about students to be taught differed
across the three groups, but the most notable difference found was between experts and
postulants. For example, expert teachers were much less interested in specific
information about students (e.g., grades, demographic information, attendance) than were
postulants who used such information to categorize students (e.g., good students, bad students) or to rank students according to grades. Experts, in contrast, were critical of the information because it could not be used for making decisions about the content to be covered, for getting students involved in the work, etc. In addition, expert teachers stated that they would disregard the information provided by the other teacher because they like to get their own impressions by interacting with their students. Postulants, in contrast, attached a high degree of validity to students' information collected by other teachers. The general conclusion reached by Berliner (1987) is that "novices and postulants alike [as opposed to experts], rarely judged the relevance of one piece of information against another or generated instructional plans based on the information they had" (p. 68). For experts, on the other hand, "the only information that was deemed important was information that had instructional significance" (p. 68) such as number of students, age of students, grade level, etc. This group of participants was found to differ in other tasks such as thinking about taking over a class, about tests and homework, etc.

Studies of expert-novice teachers have provided rich examples of qualitative differences between novices and experts. Those studies, however, are faulty in at least two points: they do not illuminate the process of becoming an expert teacher and they tend, in general, to ignore teachers' knowledge behind their thinking and actions (Carter, 1990). Some studies under this tradition have been an important step toward examining and understanding teachers' knowledge and the path to expertise.

**Teachers' Practical Knowledge**

Researchers focusing on teachers' practical knowledge (Clandinin, 1985; Connelly & Clandinin, 1984, 1985, 1986; Elbaz, 1981, 1983) have examined the knowledge that teachers gain from practice and experience, the knowledge that they use to solve everyday problems, and the knowledge that guides their thinking and action in everyday interactions with their students.
To some extent, three aspects of teachers' personal practical knowledge were examined in Elbaz's (1981) research: content of practical knowledge, orientations of practical knowledge, and structure of practical knowledge. The first aspect refers to the content itself of practical knowledge categories which are "knowledge of subject matter, curriculum, instruction, self, and the milieu of schooling" (p. 48). Elbaz (1981) identified five orientations of practical knowledge: situational, social, personal, experiential, and theoretical. Situational orientation refers to the idea that teachers' practical knowledge is specific to a particular situation, oriented to a particular practical context or problem. Different contextual situations may call for different pieces of practical knowledge. Social orientation is reflected in the ways teachers adapt subject matter to the diversity of abilities, ethnic, social, and economic backgrounds held by their students and also on the ways teachers reflect students', parents', and principals' expectations of subject matter (e.g., as a set of skills to be mastered, as an integrated conceptual domain, etc.). Personal orientation, on the other hand, is reflected on the personal taste toward the subject matter and on the teachers' personal ways of viewing and presenting the subject matter to their students. Experiential orientation refers to the effect that experience has on teachers practical knowledge since experience enriches, extends, and shapes teachers' practical knowledge. Finally, theoretical orientation addresses the ways teachers view the role of theory in their teaching and their own theoretical position regarding the teaching and learning of the subject matter and how their experiences shape their theoretical views of students' learning of subject matter.

Regarding the structure of practical knowledge, Elbaz (1981) talks about three organizational levels, namely, rules of practice, practical principles, and images. The first level, that of rule of practice, "is simply a brief, clearly formulated statement of what to do or how to do it in a particular or specific situation frequently encountered in practice" (p. 61). The second level, that of practical principle, on the other hand, is "a broader,
more inclusive statement than the rule. Practical principles embody purpose in a
deliberate and reflective way ... the statement of a principle enunciates, or at least
implies, the rationale that emerges at the end of a process of deliberation on a problem”
(p. 61). The third level, that of images, is the most implicit of the three. At this level,
"the teacher's feelings, values, needs, and beliefs combine as she formulates brief
metaphoric statements of how teaching should be and marshals experience, theoretical
knowledge, and school folklore to give substance to these images” (p. 61). Images, in a
word, is a general orienting framework that guides teachers' actions.

Studies under this tradition represent a further step in examining and understanding
teachers' knowledge by focusing on the substance and organization of teachers' practical
knowledge. We notice, however, that studies of teachers' practical knowledge and
information-processing studies ignored the role that content knowledge plays in the
process of teaching. The pedagogical content knowledge tradition focuses on this critical
aspect and it is introduced next.

**Pedagogical Content Knowledge**

Another approach to studying teachers' knowledge deals with what teachers know
about their subject matter (content knowledge) and how they transform that knowledge
into representations (pedagogical content knowledge) that are intended to be meaningful
to students. According to Shulman and Sykes (1986) (cited in Carter, 1990), pedagogical
content knowledge includes:

> Understanding the central topics in each subject matter as it is generally taught to children of a
> particular grade level and being able to ask the following kinds of questions about each topic: what
> are core concepts, skills, and attitudes which this topic has the potential of conveying to students? ...
> What are the aspects of this topic that are most difficult to understand for students? What is the
> greatest intrinsic interest? What analogies, metaphors, examples, similes, demonstrations, simulations,
> manipulations, or the like, are most effective in communicating the appropriate understandings or
> attitudes of this topic to students of particular backgrounds and prerequisites? What students'
> preconceptions are likely to get in the way of learning? (p. 305)

This line of inquiry has been led by Shulman and his associates since 1985. It keeps
great promise for casting light in the search of what teachers need to know and in the
understanding of the process of teaching. It is with studies examining teachers' pedagogical content knowledge that research on teaching has taken into consideration the role of content knowledge in teaching. As described by Schulman's definition, pedagogical content knowledge is a combination of general pedagogical knowledge and content knowledge. I will review some literature dealing with general pedagogical knowledge, content knowledge and pedagogical content knowledge from a content knowledge perspective.

**Teachers' General Pedagogical Knowledge**

The construct of general pedagogical knowledge is a multidimensional one. The following components have been identified: knowledge of principles and theories of teaching and learning, knowledge of learners, knowledge of classroom organization and management, and knowledge of effective teaching principles. Since the present study deals with the integration of general pedagogical knowledge and content knowledge, that is, pedagogical content knowledge, I will only deal with research on teachers' general pedagogical knowledge that has been approached from a content point of view. From this perspective, it is with studies of effective teaching principles where subject matter knowledge has played a potential role in teaching and learning. I will briefly review this research next. Comprehensive reviews of some other components of teachers' general pedagogical knowledge have been done by Brophy and Good (1986) and Clark and Peterson (1986) and Gage (1985).

**Effective Teaching and Subject-matter Knowledge**

The common sense belief that subject matter knowledge plays a critical role in teachers' instructional practices which, in turn, affect what students learn in the classroom has been shared by some scholars through time (e.g., Ball, 1988a; Begle, 1979, Buchman, 1984; Conant, 1963; Post, Harel, Behr, & Lesh, 1991; Wilson, Shulman & Richert, 1987). Hence, it is not surprising that researchers have attempted to shed light on our
understanding of the relationship between teachers' subject matter knowledge and students' learning. Mathematics content knowledge has figured, faded, and reappeared as potential variable in our search for the critical variables in effective mathematics teaching. Ball (1991) identified three stages which effective teaching has gone through. The first stage, known as the stage of presage-product research, focused on teachers' characteristics and their relationship to students' learning. The second stage, known as the stage of process-product research, focused on the relationship between teacher behaviors and student achievement. Finally, the third stage, termed the stage of teachers' thinking and decision making, has been focusing on unraveling the knowledge, beliefs, and values that underlie teachers' thinking and decision making.

During the stage of presage-product research, researchers were interested in describing what effective teachers are like. Several presage characteristics were identified (e.g., good appearance, enthusiasm, helpfulness, knowledge of subject matter, etc.) based on students', supervisors', or principals' scale ratings. The first studies did not attempt to validate empirically the relationship between these variables and teachers' instructional practice or students' learning (for a complete review of this type of research see Barr, Worcester, Abell, Beecher, Jensen, Peronto, Ringness, Schmidt (1961) and Mitzel (1960)). Some later studies were designed to fill that gap. One of these studies used mathematics as a context variable and was conducted by the School Mathematics Study Group (Begle, 1972). One of the purposes of this study was to describe the relationship between teachers' mathematics knowledge as measured by the number of mathematics courses taken in college and students' learning as measured by achievement tests. Surprisingly, no significant correlations were found. A similar study conducted by Eisenberg (1977) produced similar findings. These discouraging results led Begle (1979) to conclude that "the effects of teachers' subject matter knowledge ... seem to be less powerful than most of us had realized.... Our attempts to improve mathematics education
would not profit from further studies of teachers" (pp. 54-55). A complete review of this type of presage-product research has been done by Medley (1979).

Recent research not only in mathematics but in other areas such as science (Carlsen, 1988; Hashweh, 1987), English (Grossman, 1989; Gudmundsdottir, 1988), history (Wineburg & Wilson, 1991), and social studies (Gudmundsdottir, 1987a, 1987b, 1988) is providing a great deal of evidence that teachers' knowledge plays a significant, although subtle, role on teachers' instructional practice which, in turn, might influence what students learn. So, a question remains as to why early research failed to identify teachers' subject matter knowledge as a significant variable in teaching or learning. Seen from a current perspective, Fennema and Franke (1992) identified three shortcomings of that research. The first one deals with the use of faulty operational definitions since teachers' subject matter knowledge was assessed by global measures such as the number or credits of college mathematics courses taken and scores obtained in standardized tests such as those of the *National Teachers Examination*. Similarly, students' knowledge was measured by standardized tests. The second shortcoming refers to the fact that there were no studies designed to investigate the relationship between what teachers knew and what they actually did in the classroom. The third one, and most important, the use of statistical techniques with which is difficult to capture the subtle, qualitative, and complex aspects of teaching.

Because studies of teachers' subject-matter background seemed to be unrelated to students' learning as operationalized in the presage-product research, researchers turned their attention to studying the relationship between what teachers actually did (teachers' behaviors) and students' achievement. The stage of process-product research had arrived. Much of this research focused on unraveling generic principles of effective teaching practices (general pedagogical knowledge). Because this research used the subject matter as context, not as the object of research, I will review some of the major results of this
line of investigation briefly. Complete reviews are those of Brophy and Good (1986), Dunkin and Bidle (1974), Gage (1985), Medley and Mitzel (1963), and Rosenshine and Furst (1973) from a generic perspective and those of Driscoll (n. d.) and Good, Grouws, and Ebmeier (1983) with a focus on effective mathematics teaching.

Two major programs of process-product research with a mathematics as context variable can be identified in the literature: (a) Evertson, Brophy and their colleagues' program and (b) Good, Grouws and their colleagues' program. Driscoll (n. d.) identified the following five teacher's behaviors associated with student achievement: (a) style of class organization, (b) questioning, (c) encouragement, (d) modeling, (e) and clarity. In addition, the developmental portion of a lesson was identified as another important variable in effective teaching in Good and Grouws' (1977) research. In their study of seventh- and eighth-grade mathematics, Evertson, Emmer, and Brophy (1980) found that more effective teachers devote most of the time of each period to lecture and discussion and less time for individual seat work than did less effective teachers. This allocation of time gives effective teachers more opportunities for additional examples, elaborated explanations of difficult topics, and assessing students' understanding of the content.

Regarding questions, Evertson, Emmer and Brophy found that effective teachers pose more questions asking for explanations (process questions) and also more questions asking for short answers (product questions) than did less effective teachers. In another study, Evertson, Anderson, Anderson, and Brophy (1980) found that encouraging and allowing students to ask questions and to make comments was another behavior of more effective teachers. It was also found that effective teachers model how to solve problems. Clarity of lessons was found to be an important component of effective teachers in Smith's (1977) study. He found that effective teachers tend to use fewer vague terms when teaching direct variation to first-year algebra students. Regarding the quality of the developmental portion of a lesson, Good and Grouws (1977) found, experimentally, that
this variable was an important component of effective teachers. They define the developmental portion of a lesson as that part of the lesson devoted to increase students' understanding of concepts, procedures and their relationships.

In trying to understand teacher behaviors some researchers began to examine teachers' thinking. It is with studies of teachers' thinking and decision making that teachers' subject matter, as well as teachers' beliefs and conceptions about it, began to reappear as a significant variable in studies of teaching. In one of the first studies of this kind, Schroyer (1982) studied some junior high mathematics teachers. She was interested in investigating how teachers with different mathematical backgrounds dealt with some critical moments in the teaching of the subject such as students' difficulties or unusual responses. She reported that teachers with less mathematical knowledge find it harder to provide alternative responses to those critical situations.

Research on teachers' subject matter knowledge is undergoing, metaphorically speaking, a renaissance. Some of this line of research is reviewed in the teachers' content knowledge section.

**Teachers' Content Knowledge**

The belief that subject-matter knowledge has a powerful impact on teachers' instructional behavior which, in turn, affects what students learn is shared by many scholars in the field. Consider, for example, the following assertions "It is obvious that knowledge of mathematics is basic [fundamental] to being able to help someone else learn it" (Ball, 1988b, p. 43). Conant (1963), when referring to teachers' subject matter writes "if a teacher is largely ignorant or uninformed he can do much harm" (p. 93). "A firm grasp of the underlying concepts is an important and necessary framework for the elementary teachers to possess" (Post, Harel, Behr, & Lesh, 1991, p. 210). "One would think that the effectiveness of any teaching is dependent, to some extent, on what the teacher knows about the subject matter to be taught" (Wilson, Shulman, & Richert, 1987.
p. 105). Given those assertions, it is not surprising that researchers have tried to examine the influence of teachers' subject-matter on classroom instruction and students' learning. As seen before, earlier research failed to provide insights on the impact of teachers' content knowledge on students' learning due, in part, to the research methodology used. The lack of correlation between teachers' subject matter and students' learning led researchers to consider teachers' content knowledge as unrelated to effective teaching.

Some studies of teachers' beliefs opened the doors to teachers' knowledge of and about mathematics. This new line of research has examined, among other things, the following questions: What do teachers know about subject-matter content? (Wilson, Shulman, and Richert, 1987). What is the role of teachers' subject-matter in instruction? How do teachers use their knowledge for purposes of teaching? What is the influence of teachers' understanding of their subject matter on their classroom discourse and practices? (Ball, 1991). In which ways does teachers' knowledge impact on student learning? How is subject matter knowledge integrated into teachers' instructional decisions during the pre-active and interactive phases of teaching? (Shavelson & Stern, 1981). The literature review in this section will provide partial answers to some of these questions. The focus will be on mathematics subject-matter, but I will draw on other curricular areas where there is relevant information available.

Until recently, most cognitive processes and content understandings studied by researchers were those related to students. Teachers' understandings of their subject matter were taken for granted in research on teaching and teacher education. But now this largely ignored area is gaining considerable attention and respect. A sampling of research will be reviewed to give the reader a taste of this promising area of research. The review starts with an eye toward providing some insights into the answer to the question of how teachers understand, or misunderstand, some of the content they ordinarily teach.
Teachers' understanding of the concept of division has been relatively well investigated. Part of this research has focused on the influence of primitive models of division on prospective elementary teachers' selection of operation in verbal problems involving division and multiplication. The concept of primitive model was first developed by Fischbein, Deri, Nello, and Marino (1985) as a result of their research with children in grades 5, 7 and 9. The researchers posed to those children a variety of division and multiplication word problems. As a result of their analysis, the researchers described two primitive models of division held by students that influence their actions. The first model, partitive division, interprets division as sharing: a collection of objects (the dividend) is divided into a number (the divisor) of sub-collections with the same number of elements each (the quotient). Several restrictions apply to this model: a) the dividend (the operand) must be greater than the divisor (the operator), b) the divisor must be a whole number, and c) the quotient must be less than the dividend. The second model, quotitive division or measurement division, is interpreted as how many times a given quantity is contained in a larger quantity. There is only one restriction in this model: the dividend must be larger than the divisor.

Graeber, Tirosh and Glover's (1989) study explored whether pre-service elementary teachers held some of the misconceptions identified among students (e.g., the divisor has to be a whole number, multiplication always makes bigger, and division always makes smaller). They posed verbal problems to the teachers to find out whether they would choose the correct operation when faced with multiplication and division problems conflicting with those beliefs. The sample involved in this study consisted of 129 female prospective elementary teachers enrolled in a large southeastern American university. They completed a 13 verbal problems test obtained from a modification of the Fischbein et al.'s (1985) test. In general, the prospective teachers did a much better job in problems matching the primitive models. About 39% of those teachers got four or more problems
wrong. Analysis of the responses showed two sources of difficulty: decimals as operators and decimal operators less than 1. To gain a deeper picture of participants' conceptions and reasoning, 33 subjects, who got wrong answers to one or more of the eight most commonly missed problems, were interviewed. The interviews revealed that most of those teachers held at least one of the misconceptions mentioned before. Some teachers faced tremendous difficulty in selecting the right operation to the verbal problems that did not conform to the primitive models of multiplication and division. Regarding multiplication problems, 8 out of the 33 prospective teachers interviewed selected a division expression (instead of multiplication) as appropriate probably because the context suggested that the answer was less than the given operand (as in the task, to get .75 of 15, the answer is less than 15, we solve $\frac{15}{.75}$. Six teachers gave the justification that division always makes bigger. As the children of the Fischbein at al.'s (1985) study, the prospective teachers in Graeber et al.'s study did not face difficulty with the size of the whole-number operator or presence of a decimal operand in problems involving multiplication. Regarding division, some teachers interchanged the role of divisor and dividend. Interviews revealed that the teachers did so because they believed that the large number is always divided by the smaller one. Others said that is not possible to divide a smaller number by a bigger one. Some teachers were confronted with the dilemma of dividing by non-whole numbers as in $\frac{5}{3.25}$ or dividing by a whole number larger than the dividend as in $\frac{3.25}{5}$. Those teachers choose to divide the decimal number by the larger one. This situation shows us that some situations are more conflictive than others and that some teachers choose the right operation, not because they understand the problem, but because they avoid dealing with more problematic or conflictive situations: in this case, to have a decimal number as divisor.
In a later study, Tirosh and Graeber (1990) interviewed pre-service teachers who explicitly said that in division problems the quotient is always less than the dividend. In addition, they examined the impact of cognitive conflict on those teachers when confronted with problems that did not match their conceptions. Three tests were designed for this study. The first one consisted of computations involving division and multiplication with decimals. The second one dealt with true-false statements about multiplication and division and justification of their responses. The third one was about writing expressions which would produce the solution to word problems. Those tests were administered to a sample of 58 preservice teachers (1 male and 57 female). Out of those 58, 21 agreed with the statement "the quotient must be less than the dividend" and got the correct answer to the computation $\frac{3.75}{.75}$. During the interview sections, those teachers were asked to verbalize their meaning attached to the concept of division, their ideas about the relative size of the dividend and quotient. They were also led to detect their inconsistencies and to reflect on the sources of those misconceptions. Regarding teachers' conceptions of division, 12 subjects interpreted division only as partitive, three gave both the partitive and measurement interpretations, three interpreted division as the inverse of multiplication, and the remaining three did not give a clear interpretation. As to explicit ideas about the quotient and the dividend, three teachers stated that the quotient can be either bigger or smaller than the dividend, three subjects stated this misconception, but said that they needed to check it. The remaining 15 agreed that the quotient is always smaller than the dividend based on their partitive interpretation of division, lack of counterexamples, algorithmic procedures, or because they conceived of division as the inverse operation of multiplication (and multiplication always makes bigger).

Regarding the detection of inconsistencies, three out of 18 prospective teachers immediately found a counterexample, three recognized the inconsistency when they were
asked to solve $\frac{4}{.5}$, four needed a second prompting question to recognize the inconsistency, and eight did not detect the inconsistency mostly due to the algorithmic procedures (e.g., $\frac{40}{5} = 8$ and $8 < 40$). These eight subjects were asked to compute $4 + \frac{1}{2}$. Three of these subjects recognized the contradiction and five argued that the answer was 2. Finally, after being asked to use the measurement interpretation of division, four realized the conflict while the remaining subject did not reach it because she computed $\frac{4}{.5}$ as .8 and $4 + \frac{1}{2}$ as 2. In addition, she was not aware of the new inconsistency. Finally, the researchers identified some sources of teachers' misconceptions. Those include experience only with whole numbers, overgeneralizations of statements about whole numbers to decimals, difficulty with decimals, and reliance on the standard algorithm.

In her study of 19 prospective teachers, Ball (1990b) examined their understanding of division in three contexts: division with fractions, division by zero, and division with algebraic equations. The sample consisted of 10 elementary teaching majors and nine secondary prospective teachers with either a major or a minor in mathematics. Regarding division by fractions, the participants were asked to solve $\frac{3}{4} + \frac{1}{2}$ and to provide a representation either in concrete form or as a story word problem. Nine elementary majors and eight secondary student teachers got the right answer to the computation but only five secondary, all math majors, were able to give an adequate representation; and no elementary major was able to provide an appropriate representation. (More details about this part of the study will be described in the section teachers' pedagogical content knowledge.)

Concerning division by zero, the teacher candidates were asked for the meaning of division. Some of them explained that, for example, 7 can not be divided by zero because there is nothing multiplied by zero to get 7. Others stated that the quotient increases
dramatically as the divisor approaches zero. Seven participants stated the rule that it is impossible to divide by zero but were not able to provide any reason of why is so. Five participants, all elementary teachers, stated an incorrect rule arguing that anything divided by zero is zero. Finally, two prospective elementary teachers did not remember what the answer of dividing 7 by zero was.

Regarding the context of algebraic equations, teachers were asked how they would explain to students how to solve the equation \( \frac{x}{0.2} = 5 \). Only one teacher, an elementary major, focused on the meaning of division, 14 teacher candidates, all the nine math majors, and five elementary majors, focused on the mechanics (e.g., get rid of 0.2, isolate \( x \), multiply both sides by 0.2). Four prospective teachers, all elementary majors, were not able to solve the problem.

The present study attempts to make a contribution to the area of teachers' content knowledge by examining teachers' knowledge of mathematical representations (symbolic representations and proofs) of content curriculum events related to algebraic multiplication. The literature on teachers' knowledge of mathematical representation, is scarce. For example, the chapter on Teachers Knowledge and its impact by Fennema and Franke (1992) in The Handbook of Research on Teaching and Learning Mathematics (Grouws, 1992) does not include any research related to teachers' knowledge of mathematical representations. Similarly, the chapter on Advanced Mathematical Thinking by Tall (1992) in the same handbook does not describe teachers' knowledge of mathematical representations.

One of the few studies dealing with teachers' knowledge of definitions, one type of symbolic representation, is that of Even (1993). It deals with teachers' knowledge of the function concept. She asked 162 prospective teachers to give a definition of a function. However, she did not focus on the correctness and accuracy of the definition. She was interested in the extent to which teachers' responses reflected a modern definition of
function (arbitrariness and univalence). Then, prospective teachers' knowledge of the
definition of a function remains unknown. In addition, teachers' knowledge of symbolic
representations of most mathematical entities has not been examined. Therefore, research
is needed to examine teachers' knowledge about definitions and other symbolic
representations of mathematical entities. In particular, we do not know the extent to
which teachers know the symbolic representations of topics related to algebraic
multiplication.

A few researchers (e.g., Martin & Harel, 1989; Movshovitz-Hadar, 1991) have
examined teachers' knowledge of mathematical proofs. The findings suggest that
teachers' knowledge of mathematical proofs is not well developed. For example, Martin
and Harel (1989) examined the proof frames of 101 preservice elementary teachers
enrolled in a required sophomore-level mathematics course. The participants had taken a
prerequisite high school geometry course and the idea of mathematical proof was
addressed explicitly throughout the university mathematics course. The prospective
teachers were asked to judge the mathematical correctness of inductive and deductive
verifications of mathematical statements in familiar and unfamiliar contexts. An example
of familiar setting was: "If the sum of the digits of a whole number is divisible by 3, then
the number is divisible by 3" (p. 43). An example of unfamiliar generalization was: "If a
divides b, and b divides c, then a divides c" (p. 43). They investigated six questions. The
first question was about the extent to which the prospective teachers accept inductive
arguments as proofs of familiar and unfamiliar mathematical statements. The researchers
found that for each statement, more than 50% of the prospective teachers accepted an
inductive argument as a proof of a mathematical statement. The familiarity of the
statement did not influence teachers' evaluations of inductive arguments. A second
question was whether prospective teachers accept deductive arguments as valid
mathematical proofs. The researchers found that more than 60% of the prospective
teachers accepted a correct deductive argument as a mathematical proof and that the familiarity of the statement did not influence their acceptance of the deductive arguments. A third question was whether prospective teachers were more convinced by some types of inductive arguments than others. Four types of inductive arguments were used in the instrument: (a) two particular instances of the generalization involving small numbers, (b) a pattern containing a sequence of instances of the generalization, (c) a particular instance of the generalization involving large numbers, and (d) an example and a nonexample. The researchers found that the prospective teachers were not more convinced by some types of inductive arguments than others. The fourth finding was that prospective teachers' judgments of arguments are influenced by the appearance of the argument rather than the correctness of the argument. As a result, many prospective teachers who accepted a correct general proof as a valid proof did not reject a fallacious proof of the mathematical statement. Also, more than half of the prospective teachers accepted an incorrect deductive argument as a correct mathematical proof for unfamiliar statements. The fifth finding was related to the role of a particular proof. A particular proof is "a correct proof of the generalization, including statements justifying each step, in which particular numbers were substituted for each of the variables" (p. 45). The researchers found that most prospective teachers who accepted a correct general proof also accepted a particular proof. Finally, the sixth finding is that over one third of prospective teachers accepted both inductive and correct deductive arguments as valid proofs of mathematical statements.

Although Martin and Harel's study provide some insights into the nature of teachers' proof frames, research is needed to examine teachers' knowledge of mathematical proofs of most of theorems. In particular, research is needed to understand teachers' knowledge of mathematical proofs of mathematical statements related to algebraic multiplication.
Studies of teachers' content knowledge have advanced our knowledge of what mathematics teachers know about the subject matter. If we know little about teachers' content knowledge, we know much less about what pedagogical representations teachers know about mathematical topics and which of these representations teachers use when teaching mathematical knowledge. The next two sections review some research related to these concerns.

**Teachers' Pedagogical Content Knowledge**

Not only what teachers know about their subject-matter is important, but also how such knowledge should be understood and used for teaching purposes. The process of transforming the subject matter for purposes of teaching was called by Dewey to "psychologize" the subject matter. As described in the theoretical framework chapter, Shulman's (1986a) concept of pedagogical content knowledge consists of two major components one of which is pedagogical representations of the subject matter. This observation led us to pose questions such as the following: What psychological or pedagogical representations do teachers know about the content to be represented? In this section, some research that provides some insights into the answer to this question will be reviewed.

**Pedagogical Representations**

In Chapter 2, I categorized pedagogical representations into four types: story-problem representations, pictorial representations, physical representations and manipulatives, and numerical representations. I will only review some research related to the first two types of representations because they are the central components of the research questions.

In their study, as part of a larger research project whose purpose was to examine teachers' pedagogical content knowledge of fractions, Sánchez and Llinares (1992) examined prospective elementary teachers' understanding of the connections between
symbolic and concrete representations of fractions such as chips and drawings and how the teachers use concrete referents to explain their procedural steps to generate equivalent fractions. The researchers conducted semi-structured clinical interviews with 26 prospective elementary teachers enrolled in a Spanish university. As a general picture, 21 prospective teachers relied on memorized procedural steps for generating equivalent fractions and the remaining five failed to find the answer to the tasks presented to them.

As a closer picture, when the teachers were asked how to find an equivalent fraction to $\frac{9}{12}$, one of the teachers said that "this could be done in two ways ... dividing the numerator by the denominator would give the same right?... multiply across ... can I divide these two [9 and 12]?" (p. 276). Teachers' thinking seemed to be directed at verifying whether two fractions were equivalent rather than searching for an equivalent fraction. In some moment the teacher realized that another fraction was needed: "I have to find ... wait ... four sevenths [$\frac{4}{7}$]" (p. 276), but he choose an additive strategy.

Similarly, other teachers used the erroneous additive strategy for finding out equivalent fractions of $\frac{9}{12}$. For example, one teacher provided the answer $\frac{6}{9}$ explaining that he had subtracted 3 from 9 and 12. When asked to explain the procedure followed using concrete referents, the teacher drew a rectangle and divided it into 12 parts and colored 9. After that, he said that he would divide the rectangle into 9 parts and color 6, but he was unable to go further. Other teachers who also used the additive strategy said, "But I don't know how to do it with playing chips.... What should I do? Should I take away the same number of playing chips from the two and I get the equivalent?" (p. 277). Other teachers used the strategy of multiplying (or dividing) numerator and denominator by the same number. For example, one of the teachers got $\frac{18}{24}$ and explained that he had multiplied the numerator and denominator by the same number. However, those teachers face much difficulty when asked to represent those steps using the concrete referents. For example.
one of the teachers said, "Honestly, I know how to do it ... but with playing chips, it's harder.... I know how to represent it, but as a separate fraction" (p. 278-279). Another teacher got $\frac{16}{24}$ as equivalent to $\frac{8}{12}$ using the multiplication strategy, but was unable to explain the procedure using rectangles.

Some teachers were able to find equivalent fractions and transfer the procedure somehow to a modeling process using rectangles. To illustrate, one prospective teacher generated the following explanation for the process of solving $\frac{9}{12} = \frac{6}{?}$ "I'm going to the rectangle ... I'm drawing that it's nine twelfths [he divides the rectangle into 12 parts and makes marks every three separations] ... three, six, nine ... this would be three fourths" (p. 279). At this point, the teacher realized that $\frac{9}{12}$ is equivalent to $\frac{3}{4}$. To make the connection to $\frac{6}{8}$, the teacher multiplied 3 and 4 by 2. At this moment, the teacher drew another rectangle intended to be congruent to the previous one. Then he divided it into 8 parts and concluded that $\frac{6}{8}$ was the same as $\frac{9}{12}$. However, this teacher failed to apply the same reasoning to the solution of the problem $\frac{9}{12} = \frac{15}{?}$.

Ball (1990b) asked 19 prospective teachers to provide a representation for either as a story-problem or any other kind of model. None of the 10 elementary majors and only five of nine secondary teachers constructed an appropriate representation.

Three elementary and two secondary teacher candidates gave representations that did not match the division $1\frac{3}{4} + \frac{1}{2}$. The most common error was to represent $1\frac{3}{4} - \frac{1}{2}$ instead.

This was illustrated by Barb, a mathematics major, who gave the following word story:

If we had one and three-quarter pizzas left and there were two of us dying to split it, then how would we be able to split that? (p. 135)

Eight teachers (six elementary and two secondary) were not able to generate a representation, correct or incorrect, for the situation proposed.
Orton (1988) (cited in Fennema & Franke, 1992), investigated also teachers' knowledge of representations of fraction concepts. The sample consisted of 29 practicing elementary teachers. He asked participants how they would teach a fraction concept to a student with a specific misconception. The researcher reported that most of the teachers provided symbolic representations as opposed to a concrete representation.

Post, Cramer, Behr, Lesh, and Harel (1993) asked preservice and practicing teachers to write a story-problem representation for \( \frac{3}{4} \times \frac{1}{2} \). They reported that most teachers did not construct a representation.

Simon's (1993) study included problems about story-problem representations to examine teachers' knowledge of division. He asked 33 prospective elementary teachers to write three realistic story problems that would be solved by dividing 51 by 4 and for which the answers would be, respectively, 12\( \frac{3}{4} \), 13, and 12. Simon found that about 76% of the prospective teachers provided a correct story problem for the case 12\( \frac{3}{4} \), about 36% of the prospective teachers constructed a correct story problem for the case 13, and about 61% of the prospective teachers provided a correct story problem for the case 12. Simon also asked the prospective teachers to provide a story problem that would be solved by \( \frac{3}{4} \) divided by \( \frac{1}{4} \). The found that only 30% of the prospective teachers were able to construct the story problem requested.

Azim (1995, 1996) investigated preservice elementary teachers' understanding of multiplication and division with fractions. Fifty prospective teachers enrolled in two sections of a methods course participated in the studies. In her study dealing with multiplication, Azim (1995) asked participants to create and solve a word problem for which the solution could be found by each of the following multiplication expressions: \( 24 \times 37, 7 \times \frac{1}{4}, \frac{1}{2} \times \frac{1}{3} \) and \( \frac{2}{3} \times \frac{3}{4} \). The researcher found that: (a) 16% of the preservice teachers created a word problem modeled by each of the four mathematical expressions.
(b) 36% of the prospective teachers did not provide a word problem for any of the three fraction multiplication expressions, and (c) an additional 36% of the preservice teachers were able to construct a word problem for only one of the three fraction multiplication expressions, the expression $7 \times \frac{1}{4}$. Azim also reported that four students were not able to create a word problem for $24 \times 37$.

In her study dealing with teachers' understanding of division, Azim (1996) asked the participants to construct and solve a story problem modeled by each of the following division expressions, $57 \div 8$, $34 \div 4$, $\frac{1}{2} \div 4$, $2 + \frac{1}{4}$, $\frac{3}{4} + \frac{1}{2}$. The percentages of prospective teachers constructing and solving the story problems were, respectively, 100%, 96%, 72%, 56%, and 4%. The preservice teachers were also given five story problems involving fractions modeled by addition, subtraction, multiplication, partitive division, and quotitive division. The participants were asked to identify and solve the division problems. Regarding the partitive story problem for $1 \frac{1}{2} + 3$, 92% of the prospective teachers identified the story problem as modeled by division and 83% of the participants were able to compute the answer to the division. With respect to the quotitive story problem for $2 \div \frac{4}{5}$, 82% of the preservice teachers identified the story problem as modeled by division, and 68% solved the division story problem.

Research reviewed in this section has enhanced our understanding of teachers' knowledge of pedagogical representations of some mathematical topics. However, research addressing teachers' knowledge of other mathematical topics is needed. In particular, teachers' knowledge of topics related to algebraic remains unexplored.

I have reviewed literature related to teachers' knowledge of mathematical and pedagogical representations. But as Simon (1993) stated, we need to know how teachers use that knowledge in social contexts, especially classrooms. This is reviewed in the next section.
As described in the theoretical framework, there are three main pedagogical events that teachers construct when engaged in the activity of teaching: explanations, representations, and questions. I will discuss teachers' use of both mathematical and pedagogical representations when constructing pedagogical events. Because of the limited existence of research on teachers' use of mathematical and pedagogical representations during classroom instruction, the research reviewed in this section includes also research about how teachers think they would use their knowledge of these type of representations when teaching hypothetical students in hypothetical situations.

In her study of 162 prospective secondary teachers, Even (1993) asked them to define a function and how they would define it for students. The responses showed that teachers would describe a function in other terms when asked to define a function for students. For example, of 78 prospective teachers who appealed to the modern concept of function, 51 gave a different definition for the case where students have difficulties with the modern concept. To illustrate, Valerie, one of the teachers interviewed, defined a function as "a 1-1 mapping of a set of points x onto y" (p. 104), but for the students, she provided the following explanation: "You take a group of numbers. You perform some operations on the numbers (such as multiplication). This gives you a second group of numbers. The operation you did is called a function" (p. 104). Valerie stated that she changed her definition because students would probably not understand what a mapping is. This case supports the claim that knowledge of students might have an impact on teachers' choices of explaining, representing and assessing mathematical content. In the same area, while just three subjects (out of 152) used "the vertical line test" in their definition of function, 26 used it when they were faced with an hypothetical classroom teaching situation. To illustrate, a student wrote "by graphing the function and doing the vertical line test, a line never crosses the graph more than once" (p. 108). Even reported
that the vertical line test was used as a rule without making the connection with the verbal
definition those teachers had provided. For example, one of the teachers said "If they're
told to figure out whether it's a function or not, using the definition, they probably
wouldn't be able to do it. If they know the vertical line test works, even if they do not
know why it works, they can see right away why this is a function, because they can go
through with a ruler or a straightedge and vertically go across the function, looking for
places where there are two points" (p. 108). From these observations we can see that
students' cognitions potentially influence teachers' use of alternative definitions and
explanations to students. A classroom investigation is needed to find out how real
teachers represent real mathematical content to real students in real teaching situations.

Baxter (1987) examined how two teachers' understanding of two computer topics,
looping and sorting, influenced their classroom instruction. As in this study, she used the
concepts of content knowledge and pedagogical content knowledge as her research focus.
Regarding teachers' understanding of the programming topics, Baxter found that one of
the teachers, Mr. Herron, tended to gave precise definitions of the topics but that the
teacher gave few relationships among the topics. In contrast, the other teacher, Mr.
Schelle, provided general concepts with many relationships among topics. Regarding
teachers' explanations, the researcher analyzed them in terms of four themes: use of
representations, treatment of difficult topics, degree of integration of concepts, and
overall structure of the unit. She found that while Mr. Herron used mainly one type of
representation for teaching loops, BASIC code, Mr. Schelle used a variety of
representations such as flows charts, summaries in English, BASIC code and other types
of diagrams. Regarding treatment of difficult topics, the researcher found that the
teachers' explanations incorporated different techniques when dealing this issue. She
reported that while Mr. Herron tended to provide specific algorithms, Mr. Schelle, tended
to provide more general heuristics. Regarding teachers' degree of integration of concepts
in their explanations, Baxter found that Mr. Herron focused on particular terms within programming and made no connections to previous lessons, while Mr. Schelle incorporated concepts from other domains and he often made connections to topics from previous lessons. Finally, the researcher found that there were differences regarding the overall structure of teachers’ lessons. She found that while Mr. Herron began with general concepts and then he constructed specific examples, Mr. Schelle move from simple to complex relationship among concepts.

In the area of mathematics education, researchers are beginning to examine the complexity of teachers' use of their content knowledge and pedagogical content knowledge for teaching purposes. For example, Borko, Eisenhardt, Brown, Underhill, Jones, and Agard (1992) examined one teaching episode in which a middle school student teacher failed to construct a conceptual explanations for the algorithm of division of fractions. The researchers reported that the student teacher attempted to incorporate in her explanations story-problem representations and pictorial representations but that she constructed incorrect representations. Borko et al. concluded that the student's failure was due, in part, to a poor conceptual understanding of the algorithm.

Thompson and Thompson (1994) examined the struggles that one teacher, Bill, had in helping a student to understand rates conceptually and why he failed to accomplish his goal. They concluded that the teacher had a deep understanding of rates but that the language that he used to communicate his conceptualization of rates was "encapsulated in the language of numbers and operations, and this undermined his effort to help the student understand rates conceptually" (p. 279). In their sequel to this study (Thompson & Thompson, 1996), they examined the instructional activities that one of the researchers, Pat, designed to help the student understand rates conceptually. They conclude that Pat was successful because his instruction was cognitively guided.
In conclusion, the pieces of research reviewed in this chapter have provided us with some understanding of some aspects of teachers' content knowledge and pedagogical content knowledge in the context of division, division of whole numbers and fractions, equivalence of fractions, functions, and rates. The present study provides further insights on teachers' content knowledge and pedagogical content knowledge by investigating the research questions proposed in Chapter 1.

The present study extends previous research on teachers' knowledge of mathematical and pedagogical representations in four major ways: (a) it examines one experienced and secondary-certified mathematics teacher as opposed to prospective teachers, (b) it examines the teacher's knowledge in the context of algebraic multiplication, as opposed to the other topics reported in the literature, (c) it provides a greater depth of information regarding the teacher's knowledge of mathematical and pedagogical representations that is currently available and (d) it examines the impact of the teacher's knowledge of these representations on his classroom instruction.
CHAPTER 4

METHODOLOGY

The denial of complexity is the beginning of tyranny (Anonymous)

This chapter provides a description of the research methodology that was used for this study. It consists of the following sections: design of the study, data collection and instrumentation, procedures, data analysis, trustworthiness, and data reporting. A rationale for some methodological decisions to be taken is provided when judged appropriate.

Design of the Study

Table 4.1 displays, in summary form, aspects related to the design of the study as described by Patton (1990). I will discuss each of them in a separate subsection.

Choosing a Study and a Problem

As a teacher, I have always been interested in knowing how other teachers engage in the complex activity of teaching. I was delighted to learn that examining the teaching phenomenon was a legitimate area of empirical inquiry. I read the literature on teachers' knowledge with the hope of finding a researchable topic that was considered important by the research community and interesting to me. I judged that a recommendation by very-well known researchers would provide part of the rationale for my study. I read Simon's (1993) article on prospective elementary teachers' knowledge of division. He made the critical point that we need to examine how teachers use their knowledge in social settings such as classrooms. At this point I formulated part of my research questions: to examine
the relationship between teachers' knowledge and knowledge use during classroom instruction.

During the review of the literature I came across Shulman's research project *Knowledge Growth in a Profession* (Shulman, 1986a) and his theoretical construct of pedagogical content knowledge. This concept was so appealing to me that I added another aspect to my study: teachers' pedagogical content knowledge and use of their pedagogical content knowledge during classroom instruction. This concept was appealing to me because it involves the specific knowledge that a teacher needs to know for teaching a specific mathematical topic. As a teacher trained in mathematics education, I tried to convince my colleagues to take inservice courses dealing with pedagogy when I was working as a high school mathematics teacher. One of them used to say "if you show me how to teach the quadratic formula so that students learn it well, I will take those courses." Experiencing difficulties myself in getting students to understand the quadratic formula, I gave up on trying to convince my colleague to take methods courses on teaching mathematics. As a graduate student, one day I came across an issue of a professional journal. It contained an invitation for attending a meeting and the advertisement was something like "if you are looking for ways to teach next Monday, this meeting is not for you, but if you are looking for ways to improve your teaching in general, then this meeting is for you." A posteriori, I found in this advertisement some support for my decision of not insisting on my colleague to take pedagogical courses years earlier. It supported the view that while mathematics education research can provide some global recommendations about how to teach mathematics in general, it can not provide specific recommendations for teaching a particular mathematical topic beyond the general suggestions. When looking for a dissertation topic, I learned about Shulman's construct of pedagogical content knowledge. This construct describes the
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<td>Ethical issues</td>
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Table 4.1 Components of the design of the study
name of the knowledge that teachers need to know for teaching the Monday topic "the quadratic formula." I then decided to incorporate it into my research questions. The second task was to decide the kind of content. Then I turned to the *Handbook of Mathematics Teaching and Learning* and read the chapter by Fennema and Franke (1992) entitled *Teachers' Knowledge and its Impact*. Everything sounded interesting, but the topic that definitely caught my attention was teachers' knowledge of story-problem representations and pictorial representations, two main categories of pedagogical content knowledge. I said "That is it. I will examine teachers pedagogical content knowledge of representations." Wanting also to know teachers' formal ways of knowing, I added another type of representation: mathematical proofs. After the first intense collection of data, I began to analyze the data to see if I would have enough and appropriate quality data to answer the research questions. One thing that surprised me was that Mr. Kantor was not teaching the definitions and symbolic representations of the concepts involved in the lesson. I wondered if he knew the definitions and symbolic representations of these concepts. As a result, I added another type of representation, and the last one, to my research study: symbolic representations. As a result of readings, searches and experiences, the research questions were formulated in terms of teachers' knowledge of mathematical representations (mathematical content knowledge), knowledge of pedagogical representations (pedagogical content knowledge), and which of these types of representations are used for the construction of three types of pedagogical events: explanations, representations, and questions.

**Type of Data Collected and Controls Exercised**

This is a naturalistic qualitative study with an observational case study format. It is a naturalistic study because it examines the pedagogical events that one teacher, Mr. Kantor, constructed when teaching some content curriculum events about algebraic multiplication in a natural classroom. As Lincoln and Guba (1985) say, in a natural...
setting the phenomena of study "take their meaning as much from their context as they do from themselves" (p. 189). A naturalistic study also allows us to better understand the nuances and complexities of the phenomenon under study. It is qualitative because the data collected are "qualitative," that is, they consist of "direct quotations from people about their experiences, opinions, feelings, and knowledge.... [or] detailed descriptions of people's activities, behaviors, actions, ... [or] excerpts, quotations, or entire passages [of documents]" (Patton, 1990, p.10). It is a case study because "it is a detailed examination of one setting, or a single subject, a single depository of documents, or one particular event" (Bogdan & Biklen, 1992, p. 62). It is observational because one of the major data gathering techniques is participant observation (Bogdan & Biklen, 1992).

What Is the Primary Purpose of the Study?

The research purpose (e.g., basic research, applied research, action research, etc.) determines decisions about design, measurement, analysis, and reporting (Patton, 1990). Therefore, it is appropriate to make clear the purpose of this study. Before doing that, I will discuss briefly the meaning of those terms. It is helpful to conceive of research along a continuum where on one end stands pure basic research and on the other stands applied research.

The main purpose of basic research is to develop a theory to understand and explain a particular phenomenon without any concern to solve human problems. Applied research, on the other hand, has as a main purpose to "generate potential solutions to human and societal problems" (Patton, 1990, p. 154). The extreme case of applied educational research is termed action research. Basic research is judged by its methodological rigor and its contribution to theory. Action research, on the other hand, is judged according to the extent to which it helps to understand a particular phenomenon in a specific place. I see the present study as situated somewhere in the middle of this continuum. On one hand, I constructed a theoretical framework that framed the research
questions and guided the analysis of the data. This theoretical framework has the potential of contributing toward a conceptualization of teaching and a theory of teachers' knowledge and its relationship to classroom practice. On the other hand, the research findings of the study might have some potential applications to the enterprise of teacher education, as described on the rationale and significance of the study in Chapter 1.

As a result of reflection, additional literature review, and initial analysis of the data, I developed a theoretical framework as described in Chapter 2. I believe that this theoretical framework has potential for examining and gaining further understanding of the phenomena of teaching from the perspective of knowledge and knowledge use.

Marshall and Rossman (1995) mention that there are four types of purposes within which qualitative research fall: exploratory, explanatory, descriptive, and predictive. Since the purpose of this study is to describe, in detail, one teacher's knowledge of both mathematical and pedagogical representations and his use of these representations when constructing pedagogical events for teaching algebraic multiplication, it can be categorized essentially as descriptive.

What Is the Focus of the Study?

An important design decision to be taken is that of breadth versus depth. Again, this is determined by the purpose and research questions of the study. I wanted to investigate the phenomena of teachers' knowledge of representations and knowledge use. As suggested by Patton (1990), I could approach this problem (a) with a focus on a diversity of teachers with mathematics in general as a context, or (b) with a number of middle school teachers and a specific mathematical topic such as multiplication of rational numbers, or (c) studying a single middle school teacher and the whole eighth-grade mathematics curriculum sampled at different points throughout the year, or (d) one teacher and one specific segment of mathematical content. The focus of this study is related to this last option. Specifically, I investigated in detail one teacher's knowledge of
mathematical representations, pedagogical representations, and his use of these representations when constructing pedagogical events for teaching algebraic multiplication.

The decision of choosing a specific piece of mathematical content is supported, in part, by current research conducted within the cognitive science tradition. Cognitive science research is focusing, among other things, on how students think about and learn specific mathematics concepts as opposed to how students learn mathematics or learning in general (Bromme, 1987; Fennema & Franke, 1992; Hiebert & Carpenter, 1992). For example, there is research data on how young children solve addition and subtraction problems (Carpenter, 1985; Carpenter & Moser, 1982, 1983, 1984; Riley, Greeno, & Heller, 1983), on how children think about place value (Fuson, 1990), and on how children think about rational numbers (Behr, Harel, Post, & Lesh, 1992). This research has provided a great deal of evidence that an analysis of learners' thinking and behaviors in a specific content domain provides better prediction and understanding of the learners' thinking and behavior than does learners' performance in a more general domain (Peterson, Fennema, Carpenter, & Loef, 1989; Romberg & Carpenter, 1986). Similarly, to better understand how teachers' knowledge and beliefs influence their thinking, decision making and teaching, researchers should analyze teachers' knowledge and beliefs within a specific content topic and grade level (Peterson, Fennema, Carpenter & Loef, 1989) such as multiplication or division in algebra. As an example, Wilson (1994) examined in detail the knowledge held by one prospective teacher about the function concept.

Another rationale for choosing a specific piece of mathematical content comes from the definition of pedagogical content knowledge itself. This kind of knowledge deals with content-specific knowledge that teachers need to meaningfully represent specific content knowledge (e.g., multiplication of fractions, decimal fractions, relationship
between division and multiplication, subtraction and addition, etc.) to a particular student population.

Units of Analysis

The unit of analysis was each of the four representations (symbolic representations, proofs, story-problem representations, and pictorial representations) that Mr. Kantor constructed during the interviews for almost each of 41 content curriculum events as well as each of the pedagogical events (explanations, representations, and questions) that Mr. Kantor constructed about each of the curriculum events during teaching. I will say more about this in the data analysis section.

What Were the Sampling Strategies?

A purposeful sampling strategy was followed. The intent of purposeful sampling is "to select information-rich cases whose study will illuminate the questions under study" (Patton, 1990, p. 169). With that intent in mind, an "intensity sampling" strategy was followed in choosing one experienced teacher. In intensity sampling, "information-rich cases that manifest the phenomenon intensely, but not extremely [above average]" (Patton, 1990, p. 182) are selected. Other purposeful sampling strategies were followed when analyzing the data. Those are discussed in the data analysis section.

Why a Sample Size of One?

I selected "only" one teacher for this study because, as stated before, my purpose was to describe and examine in detail the nuances and complexities of teachers' knowledge and knowledge use. This purpose and approach is consistent with current research in mathematics teaching and learning as well as in educational research. To illustrate, Baxter (1987) conducted two case studies of teacher explanations in computer programming. She examined two lessons for each teacher. As another example, Thompson and Thompson (1994) examined the struggles of one teacher when teaching rates for understanding to a middle school student. I agree with Baxter's observation that
a larger sample would add breath to the study but would limit severely its depth. In qualitative research the size of the sample is not the most important factor in the design of a study. It is the detailed understanding of complexity, as suggested in the title of Peshkin’s (1988) article, *Understanding complexity: A gift of qualitative inquiry* by which qualitative studies are judged and not by the "large" number of subjects.

**Why an Experienced Teacher?**

I selected an experienced teacher because I thought he was more likely to focus on teaching particular subject matter as opposed to focusing on the generic aspects of teaching such as classroom management. I also thought that experienced teachers are more likely to be on their way to expertise. Research described by Leinhardt, Putnam, Stein, and Baxter (1991) suggests that expert teachers hold knowledge that is more organized, and differentiated and "more accessible for use when needed and more adaptable to the particular circumstances that arise as lesson develops" (Brophy, 1991, p. xiv). I wanted two master teachers (as defined by having won national awards, recommendations of mathematics education faculty as being experts, and whose students were high achievers) because they were more likely to exhibit the phenomenon under examination intensely. I found it difficult to locate and obtain permission from two master teachers, but I did get cooperation of one master teacher and one other experienced teacher. I found that the pedagogical events that Mr. Kantor constructed were more conceptually oriented and more complex than the ones constructed by the master teacher. This dissertation reports on Mr. Kantor, the experienced teacher. As noted above, there is precedence for one teacher (Patton, 1990; Thompson & Thompson, 1994).

**Why Ludwig School?**

I wanted not only two master teachers but also a school district where students were known for high achievement and teachers were implementing an innovative conceptually-
based mathematics curriculum. The reason was that these teachers were more likely to teach for conceptual understanding and would have rich content and pedagogical content knowledge. Because it was difficult to get transportation, I also wanted a school district not too far away from the place I lived. Fortunately, I located these two teachers working under these conditions. The school district has two middle schools and Mr. Kantor is teaching at Ludwig. They are implementing *The University of Chicago School Mathematics Project* that is considered as a conceptually-based curriculum.

**What Analytical Approaches Were Used?**

Patton (1990) describes two approaches for the design of a study: inductive and deductive. In a pure deductive approach the components of the design of a study is determined before data collection begins: explicit theoretical framework, research questions, hypotheses, operationalized variables, instruments (e.g., closed-ended questionnaires), coding schemes, and detailed, specific and predetermined procedures and methods of analysis, etc. In an inductive approach, in contrast, theoretical framework, research questions, categories or constructs, and general patterns emerge from the data. In addition, open-ended interviews and an evolving and flexible design characterize the inductive approach to research. However, as Patton suggests, it is more helpful to see the dichotomy inductive or deductive as a continuum ranging from a pure inductive approach on one end and deductive approach on the other. Since I used an explicit theoretical framework, predetermined research questions, and some predetermined categories, my analytical approach to data collection and analysis has some elements of the deductive approach. On the other hand, because I designed open-ended interviews and some categories and patterns emerged from the data, this study has some elements of the inductive approach. Hence, this study is situated somewhere in the middle.
What Was the Time Frame for Data Collection and Analysis?

As many qualitative studies, I did not have a specific, predetermined, and precise time frame for data collection, data reduction, data display and verification of conclusions, the principal stages of a qualitative study (Miles & Huberman, 1994). These phases were interwoven. I cycled several times through the different stages of the research because I was modifying research techniques and refocusing questions as needed. I should say that they were not totally interwoven. For practical reasons, data collection, data reduction, data display, and verification of conclusions were not done simultaneously or concurrently as Miles and Huberman suggests. Then, we can identify time periods during which each of those activities was carried out with great intensity. This is displayed in Table 4.2.

This display illustrates the time frame for the larger project in the case of Mr. Kantor. Because this dissertation does not report the other teacher’s knowledge of representations and construction of pedagogical events, I have not indicated a time frame for the activities of data collection and data analysis in the case of the other teacher. This dissertation reports only on Mr. Kantor’s knowledge of representations and his use of that knowledge during classroom instruction. In the procedures section, I describe in more detail the procedures I followed for carrying out only the activities of data collection and data analysis pertaining to this dissertation.

Issues pertaining to logistic and practicalities as well as ethical issues will be described in the procedures section. Issues related to trustworthiness are described in a separate section after the data analysis section.

Data Collection and Instrumentation

In this section describe the instruments used for data collection as well as the data sources.
Table 4.2 Time frame for data collection, analysis, and write up

<table>
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<th>Activity</th>
<th>Time Frame</th>
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<td>Videotape classes</td>
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<tr>
<td>Preliminary interviews</td>
<td>October 6, 1994-October 14, 1994 (Interviews 1-3)</td>
</tr>
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<tr>
<td>Interviews</td>
<td>October 14, 1994-December 19, 1994 (Interviews 3-22)</td>
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<tr>
<td>Interviews &amp; Questionnaires</td>
<td>December 19, 1994-January 10, 1995 (Interviews 22-25)</td>
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<td></td>
<td>May 30, 1995-June 20, 1995 (interviews 26-32)</td>
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<td></td>
<td>August 25, 1995 (Interview 33)</td>
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<td>Content analysis of the textbook</td>
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<td>Data reduction</td>
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<td>Data display</td>
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<td>October 1995-June 1997</td>
</tr>
<tr>
<td>Writing up</td>
<td>December 1995-June 1997</td>
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The Instruments

In qualitative research, the researcher is the main instrument since "he or she observes, asks questions, and interacts with research participants" (Glesne & Peshkin, 1992, p. 6). Not all interview questions are ready ahead of time. Questions related to the teacher's knowledge and instruction were developed during the study. The researcher's story may impact various data collection decisions along the way because he or she is the main instrument. Therefore, I should tell my story next.
I was born in a small rural area in Mexico in 1962. I remember that I was always interested in becoming a teacher. At that time there were no middle schools in that area so I had to go to the nearest town. After graduating from middle school, I attended the high school located in the state capital, Guanajuato, because there were no high schools in the town where I graduated from middle school. I felt and I was told that I was good at mathematics so after graduating from high school I chose an engineering field: geology. I never felt at home in the school of mines, metallurgy and geology. It was at that institution that I met a mathematician. At that time I thought of doing graduate studies in mathematics and becoming a mathematics teacher, something that I always wanted to be. Fortunately, after two years of being a geology student, a school of mathematics was opened. I transferred to that school and got a Bachelors' Degree in mathematics with a minor in mathematics education. Almost at the same time that I began my undergraduate studies in mathematics I was offered and accepted a mathematics teacher position at the high school level. I taught there for about seven years. My mentor in mathematics education was Alfinio Flores, currently a professor of Mathematics Education at Arizona State University. He guided me through all the mathematics education courses and some mathematics courses. When I was a senior student, I learned through him that Ohio State was a good place for doctoral studies in mathematics education and I decided to come to Ohio State. As a graduate student at Ohio State, I took courses in statistical methods, qualitative methodology, quantitative research in education, teacher education, mathematics education, and mathematics.

**Data Collection Methods and Data Sources**

Three major data gathering techniques were used for the research reported here: participant observation (videotaping), interviewing, and analysis of documents (textbook).
I was a participant observer because I was present, seated in the back of the classroom, observing and videotaping Mr. Kantor's instruction. However, my role was limited to that of observer and handler of a camcorder. No interactions between me and the students took place beyond exchanging pleasantries.

The role of videotaping was as a means to replay or slow down Mr. Kantor's pedagogical events so I could be able to fully and richly describe those pedagogical events to examine his use of mathematical and pedagogical representations. Because the purpose was to provide a detailed description of the pedagogical events, particularly explanations, constructed by Mr. Kantor during classroom instruction, the videotaping was invaluable.

The second source of data was interviewing. The interview questions were aimed at gathering diverse kinds of information. First, some interview questions produced some background information such as years of teaching experience, mathematics courses taken in college, kinds of experiences, etc., about Mr. Kantor (See Interview Schedule 1, Appendix A). Second, other interview questions were aimed at producing some information related to why the teacher chose teaching as a profession in general and mathematics teaching in particular, as well as information about his or her conceptions of both mathematics teaching and learning (See Interview Schedule 2, Appendix A). Third, other interview questions were aimed at capturing Mr. Kantor's knowledge of representations in detail (See Interview Schedule 3 and 4, Appendix A). In all the cases, the interviews were recorded and fully transcribed for the purpose of analysis.

I also relied on the textbooks as an additional source of data and for purposes of question generation. The analysis of the textbook gave origin to interview questions such as those found in Interview schedules 3 and 4, Appendix A.
Procedures

Regarding research participants, the first thing I did was to ask mathematics education faculty members for the names of some potential research subjects. I described to them, in general terms, the purpose of the study and what kind of teachers I wanted and why. They suggested a master teacher, a middle school teacher who was a recipient of a presidential award for teaching. I phoned the teacher to give him a general overview of my research project and to arrange an appointment to further discuss some features of the project. During this meeting, I explained the purpose of the project along with the particular research questions that I planned to investigate, and I explained what his involvement would consist of. He agreed to participate. Since I did not find another master teacher, it was recommended that I might consider doing a pilot study with a student teacher who was going to do his student teaching under the direction of Mr. Kantor in Ludwig (same school district) and who was willing to volunteer in the pilot study. At that time I was getting close to the beginning of Spring quarter, 1994, during which I was hoping to collect data for the pilot study. I got the permission from The Human Subjects Review Committee on time but I was told that the permission from the school district was going to take longer than expected. Fortunately, I got the school district approval just in time and conducted the pilot study with Steve, the student teacher, to get a feeling of how to conduct a qualitative study and to foresee potential difficulties. During the pilot study I met Mr. Kantor. A faculty member said that he was a good teacher. Since I was still looking for another teacher for Autumn quarter, I told Mr. Kantor about the project and asked him if he would be interested in participating. He volunteered to participate. During a meeting Mr. Kantor gave informed consent as required by The Human Subjects Review Committee. The consent form made clear that his participation was totally voluntary and that he could drop at any time without any
penalty. In addition, "Kantor" and "Ludwig" are pseudonymous. No real names are used so we can maintain the anonymity of the participant and the school.

I began interviewing Mr. Kantor on October 6, 1994. The first two interviews and part of the third interview had as a purpose to get some information about Mr. Kantor's background, education, and conceptions and beliefs about mathematics, teaching, and learning. Interviews three through sixteen were mostly video stimulated in nature. I selected teaching episodes to ask Mr. Kantor's purpose and thinking in constructing specific pedagogical events during. My general purpose was to unravel Mr. Kantor's knowledge about students' specific cognitions— a research question in a larger project. In some cases it was not necessary to use the VCR to help Mr. Kantor remember a particular teaching episode. The stimulated recall interviews lasted from October 14 to November 23. The third type of interviews also had as a purpose to unravel Mr. Kantor's knowledge about students' cognitions as evidenced on students' tests. I conducted a content analysis of students' tests to find out their common difficulties, mistakes, and strategies. I then asked Mr. Kantor to describe students' common cognitions when solving selected problems and questions from the tests.

The fourth type of interviews were related to Mr. Kantor's knowledge of representations. It is important to note here that the interviews about this type of knowledge were conducted after Mr. Kantor finished teaching the lessons related to algebraic multiplication. In this way, the potential danger of influencing his use of representations during his classroom instruction was minimized. These interviews were conducted during three phases. During the first phase, from December 19, 1994 through January 10, 1995, I asked Mr. Kantor two types of questions: to provide a proof and to construct either a story-problem representation or a pictorial representation of selected concepts and theorems. At this time I had not yet coined the term content curriculum event. These concepts and theorems were identified through a content analysis of the
lessons related to multiplication in algebra—chapter four and first lesson of chapter five of the textbook. As mentioned before, after an initial analysis of the videotapes I realized that Mr. Kantor was not constructing representations about the definitions of important concepts during his classroom instruction. I wondered if he knew these kinds or representations. So I planned to ask him questions related to definitions during our next meeting. This meeting was delayed because I was analyzing the interview data to find out if the data was sufficient to answer the research questions. During that initial analysis I found that some of Mr. Kantor's responses were not very explicit and then I planned to probe to clarify meaning. During the first phase I only asked Mr. Kantor questions about a sample of curriculum events as planned and done in most, if not all, research in teachers' knowledge. I began to wonder if his knowledge about other related curriculum events was similar. I also thought that additional questions would provide additional evidence to my conclusions. The next meeting became the beginning of the second phase of this set of interviews. I could not meet Mr. Kantor until May 30 because from January through May I was transcribing and analyzing my interview data for completeness. To make sure that this time I would have all meanings clarified and questions about his knowledge answered, I conducted another content analysis of the textbook. This content analysis resulted in about 41 content curriculum events (See Table 5.1). For each of these curriculum events, I asked Mr. Kantor to provide, when appropriate, a symbolic representation, a mathematical proof, a story-problem representation, and a pictorial representation. If he had already provided one during the first phase of this type of interviews and it had been judged explicit I did not ask him again. As many qualitative researchers find, after transcribing and analyzing Mr. Kantor's knowledge about representations, for the second phase I found that some questions were still needing clarification. We arranged for the last interview on August 25, 1995. As I am writing the
final version of this dissertation, I still have some questions about Mr. Kantor's knowledge.

The selected class was videotaped on a daily basis for 12 days from October 13, 1994 to November 1, 1994. The camcorder was in the back of the classroom to minimize disruption.

During the months of January through May, I transcribed Mr. Kantor's interviews almost word for word. I wanted to ensure that I have all my data "there." During June and July I transcribed Mr. Kantor's videotapes the same way: word for word and without much editing.

Data Analysis

The term analysis may be used in a general sense to mean the transformation of data from field notes, audio and videotape transcriptions into research accounts. Wolcott (1994) discusses extensively this process of transformation. He identifies three components of this process: description, analysis, and interpretation. Miles and Huberman (1994), on the other hand, define analysis as "consisting of three concurrent flows of activity: data reduction, data display, and conclusion drawing/verification" (p. 10). I will combine both frameworks to describe the analysis pertaining to the present study.

Data Reduction and Integration

Miles and Huberman (1994) define data reduction as "the process of selecting, focusing, simplifying, abstracting, and transforming the data that appear in written-up field notes or transcriptions" (p. 10). I have added the word "integration" because the data needed to answer the research questions about each content curriculum event were spread throughout several interviews and sometimes throughout different periods of time. Regarding Mr. Kantor's knowledge of representations, I made a computer file for each lesson. Each lesson included several sections, each one corresponding to one content
curriculum event pertaining to that lesson. For each curriculum event I listed four subsections, each corresponding to one type of representation. Under each representation I entered raw data related to that representation and content curriculum event. Most of the time these data came from different interviews. For each excerpt, I indicated the number of the interview from where it was taken. I should mention that data reduction is not a one-time event. Data reduction occurred at different levels of specificity. During the first stage of reduction I only discarded the data that was not clearly related to the research questions. At that stage, I did not make any effort to summarize data.

Regarding the pedagogical events that Mr. Kantor constructed, I began by discarding data not related to a particular curriculum event. I also created a computer file for each lesson. Each file consisted of several excerpts describing a set of pedagogical events about a curriculum event. I also began editing and numbering the speeches made by Mr. Kantor and his students. (For an example, see Chapter 5, Excerpt 1). Again, at this stage I did not make any effort to summarize data.

Data Display and Reduction

The second phase was to display data in tables. Regarding Mr. Kantor's knowledge of representations, I made a computer file for each lesson. Each computer file contained a table with five columns. The first column was entitled mathematical concept [content curriculum event] and each of the remaining columns was named after each of the four types of representations. During this time I realized that I was entering too much data, especially in those cases where it was difficult for Mr. Kantor to construct a representation. I also realized that the representations needed editing to add clarity. So I began to reduce data by summarizing and editing them. After I had done that I began to write Chapter 5, the results of the study that includes the description, analysis, and interpretation of Mr. Kantor's knowledge and use of his knowledge.
A similar process of data reduction was needed when I began working with the pedagogical events. At that time I had already begun writing Chapter 5. So the remaining data reduction, analysis, and writing was carried out simultaneously and cyclically.

The next stage of reduction of data occurred when I was analyzing the data and writing Chapter 5 for the first time. Regarding Mr. Kantor's knowledge of representations, most came from interview data. I summarized again; this time discarding data that was irrelevant to answer the research questions. When a comment was pertinent, such as Mr. Kantor's struggles with the representation, this was made in the author's narrative. Regarding data reduction for the pedagogical events, I follow the same criteria: I discarded data and just displayed the pedagogical events constructed. I left out most of the students comments unless they were judged to be critical for the understanding of critical pedagogical events. In some cases I describe those students' comments in the author's narrative.

Data Display

Miles and Huberman (1994) define a display as an "organized, compressed assembly of information that permits conclusion drawing and action" (p. 11). They add "the creation and use of displays is not separate from analysis, it is part of analysis" (emphasis in the original) (p. 11). Designing a display—deciding on the rows and columns of a matrix for qualitative data and deciding what data, in which form, should be entered in the cells—are analytic activities" (p. 11). I used two display strategies, tables and excerpts, because I believe they are well suited for drawing conclusions. Using only tables would portray data as cool, without context. It would lose one of the rich and critical features of qualitative studies: the context. Miles and Huberman (1994) make the point that "It is important not to strip the data at hand from the context in which they occur" (p. 11).
Conclusions Drawing and Verification

This is essentially a descriptive case study. However, in many cases, interpretations beyond the "facts" may be needed. This situation led me to provide detailed descriptions and data segments so that the reader and I can have enough data for drawing conclusions and for verification. In addition, one of the purposes of this study was to examine in detail one teacher's knowledge of representations and the pedagogical events that he constructed when teaching algebraic multiplication in algebra from the perspective of knowledge use and, therefore, detailed descriptions are not only appropriate but essential.

Description, Analysis, and Interpretation

In addition to those activities, other strategies were used for presenting descriptions, approaching analysis, and approaching interpretation.

Description. Within the process of description, Wolcott (1994) suggests 10 different ways of organizing and presenting description. Out of those, the following three ways were used because they were judged to be appropriate for this study: (a) chronological order, (b) critical or key event, and (c) following an analytical framework. In the chronological order of presenting description, events are depicted in the order they occurred. This is illustrated by the way the data was organized especially the pedagogical events. Describing critical or key events means to describe those concrete examples of abstract analytical constructs that are related to the emergent categories or to the categories from the theoretical framework. This last example illustrates the case of using an analytical framework for descriptive purposes. I organized the material by research questions and by the categories of the theoretical framework related to the research questions.

Analysis. Regarding analysis, Wolcott (1994) suggests 10 ways of approaching it. The following six were judged to be appropriate in this research study: (a) to highlight the findings, (b) to display the findings, (c) to flesh out the analytical framework that guided
the data collection, (d) to identify patterned regularities in the data, (e) to compare with another case, and (f) to contextualize in a broader analytical framework. Within the approach of highlighting findings, I highlighted information already presented or described material at a finer level of detail. The data were presented in a way that highlights the underlying structural properties of the findings and relationships among them. Displaying findings consists of using other than a narrative structure to communicate findings such as tables, charts, diagrams, and figures. As indicated above, I used mainly tables. Fleshing out the analytical framework, as its name suggests, consists of collecting and reporting data that the framework calls for. The formulation of the research questions, the data collection, data analysis, and report of findings followed a theoretical framework. Another approach is that of carrying out a content analysis to identify patterned regularities in the data corpus. The content analysis was carried out to see the extent to which Mr. Kantor knew the representations, whether he constructed the pedagogical events for each curriculum event, the degree of conceptual and procedural knowledge in the pedagogical events, and whether he used his knowledge of representations for the construction of the pedagogical events. It is sometimes adequate to compare one case with another to highlight similarities and differences in the patterns identified. I compare Mr. Kantor's knowledge of one type of representation with other types to highlight similarities and differences. A case may be another individual or category. Finally, to contextualize your findings in a broader analytical framework means to draw connections between the research findings and external authority sources such as theory, expectations, own experience and conventional wisdom. I contextualized the findings using theory, expectation and personal experience when judged appropriate and relevant.
In the section below, I describe the coding schemes and definitions of terms that I created to categorize and organize Mr. Kantor's knowledge of representations and Mr. Kantor's use of his knowledge of representations.

**Coding scheme for Mr. Kantor's knowledge of symbolic representations.** Table C.2 (Appendix C) displays the categorization of each of the content curriculum events according to the coding scheme created by the researcher. The table also displays the categorization of Mr. Kantor's symbolic representations according to that coding scheme and according to the degree of correctness.

I classified the symbolic representations in three categories according to the degree of mathematical symbolism that can be used to represent them in writing: verbal representations, verbal-symbolic representations, and symbolic representations. A content curriculum event was judged to have a verbal representation if it can only be represented in writing involving almost no mathematical symbols. The following five content curriculum events (CCEs) were categorized as having a verbal representation: the rate model for multiplication (CCE 71); reciprocal of zero (CCE 16); the solution to $0x = b, b \neq 0$ (CCE 20); the solution to $0x = 0$ (CCE 21); and division by zero (CCE 37). A content curriculum event was judged to have a verbal-symbolic representation if it could only be represented using both verbal elements (nonmathematical symbols) and mathematical symbols. The following six content curriculum events were classified within that category: both area models for multiplication (CCEs 1 & 3), volume of a rectangular solid (CCE 4), the multiplication counting principle (CCE 28), the permutation theorem (CCE 34), and meaning of division (CCE 35). A content curriculum event was judged to have a symbolic representation if it could be represented mostly using mathematical symbols. For example, the commutative property (If $a, b \in R$ then $a \cdot b = b \cdot a$), associative property (For $a, b, c \in R (a \cdot b) \cdot c = a \cdot (b \cdot c)$), the rules of signs

\footnote{The content curriculum events are referred to by numbers according to Table 5.1.}
(e.g., if $a < 0$, $b > 0$ then $ab < 0$ or if $a > 0$, $b > 0$ then $-a(b) = -(ab)$) were categorized in this class. Most of the content curriculum events (30) were categorized as having a symbolic representation.

A symbolic representation constructed by Mr. Kantor was judged to be correct if it met two requirements: (a) was correct and (2) agreed with the researcher's categorization regarding its degree of symbolism. For example, Mr. Kantor's representation for the product of a negative number and a positive number is a negative number, "if $a$ is $-$ and $b$ is $+$ then $\frac{a}{b} = -$," although correct, it was judged partially correct because it can be represented using more mathematical symbols (e.g., if $a \in R^-$, and $b \in R^+$ then $\frac{a}{b} \in R^-$ or if $a < 0$ and $b > 0$ then $\frac{a}{b} < 0$). I asked Mr. Kantor to provide a symbolic representation for 38 of the 41 CCEs. Of the 38, I considered 32 of Mr. Kantor's representations as correct. A representation constructed by Mr. Kantor was considered partially correct if the representation had some flaws from a mathematical point of view or did not agree with the researcher's categorization regarding the degree of symbolism. I judged six representations as being partially correct. A representation was judged incorrect in the case where the representation was clearly incorrect or had major flaws from a mathematical point of view. I did not categorize any of Mr. Kantor's representations within this category.

As a measure of Mr. Kantor's knowledge of symbolic (including verbal and verbal-symbolic) representations, I obtained an index of degree of correctness. First, I counted the number of correct representations and partially correct representations. Second, each correct representation was worth 2 points, each partially correct representation was worth 1 point, and incorrect representations were worth 0 points. The index of degree of correctness was obtained by dividing the total number of points for correctness by 76, the total number of possible points, since I asked Mr. Kantor definitions or symbolic
representation for 38 content curriculum events. For the content curriculum events for which Mr. Kantor was asked to provide a symbolic representation more than once I averaged the correctness (0, 1, or 2) of the representations and I rounded the average.

Another measure of Mr. Kantor's knowledge of symbolic representations was obtained by considering only the content curriculum events for which a symbolic representation can be constructed (30) following a similar scheme. The total possible number of points was 58 because I asked Mr. Kantor to construct a symbolic representation for 29 of the 30 content curriculum events. A qualitative description is provided by giving examples of each the categories in which Mr. Kantor's representations were classified (i.e., verbal representations, verbal-symbolic representations, and symbolic representations; correct, partially correct).

**Coding scheme for Mr. Kantor's knowledge of proofs.** I categorized the proofs of the content curriculum events as axioms, definitions, proofs by contradiction, geometric proofs, algebraic proofs, and proofs by induction. Mr. Kantor's knowledge of proofs was categorized following that code, namely, knowledge of statements that are definitions, statements that can be accepted as axioms, statements that can be proved by contradiction, statements that can be proved geometrically, statements that can be proved by induction, and statements that can be proved algebraically. I considered that for 36 content curriculum events a mathematical proof was appropriate. Of those 36 content curriculum events I failed to ask Mr. Kantor a mathematical proof for only one content curriculum event. Table D.1 (Appendix D) displays Mr. Kantor's mathematical proofs for the 35 content curriculum events.

A proof was judged correct if every critical step was justified either (a) by mentioning explicitly previously proved content curriculum events, or (b) it was proved when needed (e.g., no previous curriculum event to rely on), or (c) it was mentioned by Mr. Kantor that a proof was needed. In the case when a mathematical statement was a
definition or could be accepted as an axiom, Mr. Kantor had to say so in order for his proof to be accepted as correct.

A proof was judged partially correct when the steps of the proof follow logically but Mr. Kantor did not provide all the mathematical justifications as required for a correct proof.

A proof was judged incorrect when Mr. Kantor either gave no proof at all, or the steps did not follow logically to create a correct proof, or the respondent’s arguments were based on physical or pictorial representations of the concepts involved.

As a measure of Mr. Kantor’s knowledge of mathematical proofs, I obtained an index of degree of correctness. First, I counted the number of content curriculum events for which he constructed a correct proof, a partially correct proof or an incorrect proof. Second, each correct proof was worth 2 points, each partially correct proof was worth 1 point, and incorrect proofs were worth 0 points. The index of degree of correctness was obtained by dividing the total number of points for correctness by 70, the total number of possible points since I asked Mr. Kantor to construct a proof for 35 content curriculum events. I also counted the number of curriculum events which he knew were accepted as definitions, axioms, or a proof was needed and was constructed. For the content curriculum events for which Mr. Kantor was asked to provide a mathematical proof more than once I averaged the correctness (0, 1, or 2) of the proofs and I rounded the average.

As a qualitative description of Mr. Kantor’s knowledge of mathematical proofs, I provide examples of each type of category (definitions, axioms, algebraic proofs, proofs by induction, proofs by contradiction and geometric proofs; correct proofs, and incorrect proofs).

Coding scheme for Mr. Kantor’s knowledge of pictorial representations. Mr. Kantor’s knowledge of representations was examined for 37 content curriculum events. I categorized Mr. Kantor’s pictorial representations according to two systems of
classification: degree of correctness (correct, partially correct, and incorrect) and level of explicitness (explicit, partially explicit, and implicit). A pictorial representation was judged *correct* if it involved pictorial elements and it could be interpreted as representing the intended curriculum event. A pictorial representation was judged *partially correct* if either involved few pictorial elements or the connection to the intended curriculum event was not clear. A pictorial representation was judged *incorrect* if either it did not involve pictorial elements or it could not be interpreted as representing the intended curriculum event. A pictorial representation was judged to be *explicit* if the connection to the intended curriculum event was explicit. A pictorial representation was judged *partially explicit* if the connection to the intended curriculum event needed some interpretation beyond that stated by Mr. Kantor. A pictorial representation was judged *implicit* if the connection to the intended curriculum event was not very clear. Table E.1 (Appendix E) displays Mr. Kantor's pictorial representations for the 37 content curriculum events.

As a measure of Mr. Kantor's knowledge of pictorial representations, I obtained an index of degree of correctness. First, I counted the number of content curriculum events for which he constructed a correct pictorial representation, a partially correct pictorial representation or an incorrect pictorial representation. Second, each correct pictorial representation was worth 2 points, each partially correct pictorial representation was worth 1 point, and incorrect pictorial representations were worth 0 points. The index of degree of correctness was obtained by dividing the total number of points for correctness by 76, the total number of possible points since I asked Mr. Kantor to construct 38 pictorial representations. For the content curriculum events for which Mr. Kantor was asked to provide a pictorial representation more than once I averaged the correctness (0, 1, or 2) of the representations and I rounded the average. As a qualitative description, I provide examples of each type of pictorial representations.

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2Since I was particularly interested in Mr. Kantor's knowledge of pictorial representations about conditional probability, I considered 29c as another separate CCE giving origin to 38 CCEs.
Coding scheme for Mr. Kantor's knowledge of story problem representations. I asked Mr. Kantor to construct a story-problem representation for 38 of 41 content curriculum events. For the content curriculum events, probability of an event, I asked Mr. Kantor to construct a story-problem representation for each of the three cases P(A), P(A and B), and P(A | B). For two content curriculum events, meaning of division and algebraic definition of division, I asked him to construct several story problems involving a variety of numbers (whole numbers, fractions, and combinations of whole numbers and fractions). I failed to ask Mr. Kantor a story-problem representation for the content curriculum event volume of a rectangular solid. I categorized Mr. Kantor's story-problem representations in three categories: correct, partially correct, and incorrect. A story problem was judged to be correct if it represented the intended content curriculum event and did not involve explicit reference to mathematical procedures (e.g., multiply both sides by five). The story-problem representations for 35 content curriculum events were judged as correct. A story-problem representation was judged as partially correct if the problem could be solved by using the intended content curriculum event but did not meet the criteria for correctness. The story-problem representations for the following three content curriculum events were judged as partially correct: the area model for multiplication in the continuous case (CCE 1), multiplication property of equality (CCE 18), and to solve \(-x = a\) (CCE 23). The word problem for the multiplication property of equality involved reference to "multiply both sides by five." The word problem for the solution to \(-x = a\) involved "the opposite of what you prefer ...." Finally, the word problem for the area model of multiplication in the continuous case involved whole numbers rather than decimals and thus I considered that the word problem illustrated the discrete version of the area model. Table F.1 (Appendix F) displays the story-problem representations constructed by Mr. Kantor.
As a measure of Mr. Kantor's knowledge of story-problem representations, I obtained an index of degree of correctness. First, I counted the number of content curriculum events for which he constructed a correct story-problem representation, a partially correct story-problem representation or an incorrect story-problem representation. Second, each correct story-problem representation was worth 2 points, each partially correct story-problem representation was worth 1 point, and incorrect story-problem representations were worth 0 points. The index of degree of correctness was obtained by dividing the total number of points for correctness by 76, the total number of possible points since I asked Mr. Kantor to construct story problems for 38 content curriculum events. For the content curriculum events for which Mr. Kantor was asked to provide a story-problem representation more than once I averaged the correctness (0, 1, or 2) of the representations and I rounded the average. As a qualitative description, I provide examples of story-problem representations constructed by Mr. Kantor.

Mr. Kantor's knowledge of each of the types of representations (symbolic, proofs, pictorial, and story problems) was labeled as very strong, strong, medium, weak, and very weak if the corresponding index of correctness fell, respectively within the following intervals: [.95, 1], [.85, .95), [.70, .85), [.60, .70), and [0, .6).

In the section above, I have explained coding schemes, measures and qualitative descriptions of Mr. Kantor's knowledge of symbolic representations, proofs, pictures and story problems. In the section below, I describe coding schemes and measures of Mr. Kantor's use of his knowledge of symbolic representations, proofs, pictures and story problems.

**Coding scheme for Mr. Kantor's use of his knowledge of representations.** To answer research question five, "What representations does Mr. Kantor use?," I analyzed Mr. Kantor's explanations to examine whether he used his knowledge of representations in constructing those explanations. His use of representations was examined in terms of
five types of representations: (a) numerical representations, (b) symbolic representations, (c) mathematical proofs, (d) pictorial representations, and (e) story-problem representations. The first type of representations, numerical representations, emerged from the data. I had not thought of analyzing Mr. Kantor's use of numerical representations. Because numerical representations involve the use of symbols (numerical symbols) and Mr. Kantor used, in some cases, numerical representations instead of symbolic representations, numerical representations will be described in the section dealing with symbolic representations.

Regarding Mr. Kantor's use of his knowledge of symbolic representations during classroom instruction, I considered that he used his knowledge of symbolic representations during classroom instruction when he explicitly stated the symbolic representation of the content curriculum events (CCEs) or asked students to explain the meaning of the symbols involved in the definitions, properties, etc. Informal definitions or numerical examples (numerical representations) were not categorized as symbolic representations. In the cases where Mr. Kantor did not provide the symbolic representations of the CCEs but provided numerical examples those are discussed in the section about numerical explanations. The coding scheme was applied to each of the 30 content curriculum events for which a symbolic representation can be constructed. Thus, to each content curriculum event was assigned the label "yes" or "no" to indicate whether Mr. Kantor constructed its symbolic representation. A measure of Mr. Kantor's use of his knowledge of symbolic representations was provided by counting the number of content curriculum events for which he provided a symbolic representation during classroom instruction. I compared this number with both the number of content curriculum events for which a symbolic representation can be provided and with the measure of his knowledge of this type of representations. A qualitative description of Mr. Kantor's use
of this type of knowledge is provided by attempting to provide a summary and examples of the symbolic representations he used during classroom instruction.

As to Mr. Kantor's use of his knowledge of mathematical proofs during classroom instruction, I considered that he used this type of knowledge when he constructed a mathematical proof of the content curriculum events. Construction of pictorial or story-problem representations were not considered as mathematical proofs. In the cases where Mr. Kantor provided other representations to justify why some content curriculum events (theorems) are true from a conceptual point of view, those are discussed in procedural and conceptual explanations. This coding scheme was applied to each of the 36 content curriculum events for which I considered that a mathematical proof or mathematical justification (e.g., axioms, definitions, etc.) is appropriate.

As a measure of Mr. Kantor's use of mathematical proofs, I counted the number of content curriculum events for which Mr. Kantor, during classroom instruction, constructed a mathematical proof or mathematical justification whichever was more appropriate from a mathematical point of view. I compared this number with both the number of CCEs for which a mathematical proof or justification is needed and with Mr. Kantor's measure of his knowledge of mathematical proofs. As a qualitative description of Mr. Kantor's use of mathematical proofs, I describe some examples, if any, for which he constructed a mathematical proof or justification.

Regarding Mr. Kantor's use of pictorial representations, I applied the following criteria to each of the 39 content curriculum events for which I considered a pictorial representation can be given. If Mr. Kantor constructed a pictorial representation to illustrate the meaning of a content curricular event during classroom instruction, I considered that he used his knowledge of pictorial representations for that particular event during classroom instruction. As a measure of Mr. Kantor's use of his knowledge during classroom instruction, I counted the number of content curriculum events for which a
pictorial representation was constructed by Mr. Kantor during classroom instruction. I compared this number with the number of content curriculum events for which a pictorial representation can be constructed and with the measure of his knowledge of pictorial representations.

As to Mr. Kantor's use of story-problem representations during classroom instruction, I considered that Mr. Kantor used this type of knowledge when he constructed a story-problem representation to illustrate the meaning of the content curriculum event in the cases when it was possible to do so. If not, then I considered that he used his knowledge of story-problem representations when he constructed a representation of this type to illustrate the application of the content curriculum event. As a measure of Mr. Kantor's use of his knowledge of story-problem representations, I counted the number of content curriculum events for which he used his knowledge of this type of representations and compared with both the number of content curriculum events for which a story-problem representation is possible to construct and with the measure of his knowledge of story-problem representations.

Tables G.1 through G.5 (Appendix G) display the different types of representations constructed by Mr. Kantor during classroom instruction.

To answer research question six, "What explanations does Mr. Kantor construct?," I used the following categories to analyze Mr. Kantor's explanations: (a) use of both mathematical and pedagogical representations (integration of representations), (b) relationship between students' difficulties and explanations (treatment of difficult topics), (c) operational and structural conceptions of algebraic objects, (d) integration of concepts, and (e) conceptual and procedural elements.

Regarding use and integration of representations, I categorized explanations by the types of representations involved: numerical explanations, symbolic explanations, symbolic-pictorial explanations, story-problem explanations, etc. Numerical
explanations are those explanations that involve only numerical representations of algebraic objects. Symbolic explanations are those explanations that involve symbolic representations of concepts and procedures. Pictorial explanations are those explanations that involve significant use of pictorial representations. Story-problem explanations are those explanations that involve significant use of story-problem representations. In the case where explanations involved significant use of several types of representations, they were classified accordingly (e.g., numerical-pictorial explanations, numerical-story problem explanations, etc.). Because almost all of the explanations involved verbal representations, this category was not used as the main element in the categorization.

With respect to treatment of difficult topics, I categorized Mr. Kantor's explanations in four categories (a) no explanations, (b) trivial or easy explanations, (c) middle range explanations, and (d) difficult explanations. No explanations are the explanations that are not constructed at all. Trivial or easy explanations are explanations that are constructed for topics that are considered "easy" from the teacher's perspective. They consist of the mere statement or telling of the content curriculum events without making any kind of connections between and among related content curriculum events. Middle range explanations involve some elements of procedural and conceptual knowledge. The teacher and students do not struggle to explain and to "understand" the mathematical idea. Difficult explanations are more elaborate explanations and they involve strong elements of conceptual knowledge. The teacher and students struggle to explain and understand, respectively, the connection between the procedure and the conceptual base.

I also classified Mr. Kantor's explanations in terms of the conceptions of algebraic objects as operational and structural. The term operational after Sfard (1987, 1991) (Kieran (1992) uses procedural) refers to arithmetic computational operations carried out on numbers to yield numbers. The objects operated on are not algebraic expressions but their numerical instances. I will use the term structural as Kieran (1992) uses the term, a
"set of operations that are carried out, not on numbers, but on algebraic expressions."

That is, "the operations that are carried out are not computational" and "the results are yet
algebraic expressions" (p. 392).

Regarding the degree of integration of concepts, I classified the content curriculum
events for which Mr. Kantor constructed pedagogical events according to their
relationship to other concepts into the following categories: (a) concepts with zero degree
of integration, (b) concepts with one degree of integration, (c) concepts with two degrees
of integration, and (d) concepts with more than two degrees of integration. For example,
if I identified that a content curriculum event had a close relationship with only another
concept or procedure, I classified it as a content curriculum event with one degree of
integration. When I identified that a concept could be related to two concepts or
procedures, I categorized as a concept with two degrees of integration. Similarly, Mr.
Kantor's explanations were categorized according to their degree of integration of
concepts into the following categories: explanations with zero degree of integration,
explanations with one degree of integration, explanations with two degrees of integration,
and explanations with more than two degrees of integration using the same scheme.
Appendix H lists the concepts that I identified are related to each of the content
curriculum events for which Mr. Kantor constructed pedagogical events. It also lists the
connections that Mr. Kantor made and did not make when teaching these content
curriculum events. The list gives a rationale for the categorization of Mr. Kantor's
explanations according to the degree of integration of concepts.

From the procedural-conceptual perspective, Mr. Kantor's explanations were
categorized in four types: (a) procedural-instrumental, (b) procedural conceptual, (c)
conceptual procedural, and (d) conceptual-procedural rich explanations. Instrumental-
procedural explanations are those explanations that mostly involve procedures without
any kind of justification either mathematical, heuristic or pedagogical (using pictorial or
story-problem representations or manipulatives). *Procedural-conceptual* explanations are those explanations that involve procedures with verbal, numerical or symbolic representations and the justification is mostly symbolic or numerical. *Conceptual-procedural* explanations are those explanations that involve references to other concepts and the justification of the content curriculum event is connected to both mathematical and pedagogical representations. *Conceptual-procedural rich* explanations are those explanations that represent a content curriculum event using a variety of representations (integration of representations) and connection to other concepts (integration of concepts).

To answer research question 7, "What questions does Mr. Kantor's pose?," I categorize Mr. Kantor's questions into two major categories: *procedural questions* and *conceptual questions*. Questions asking for procedural knowledge were classified as procedural and questions asking for conceptual knowledge were classified as conceptual. The content curriculum events were classified according to the types of questions asked into three categories: content curriculum events with only procedural questions (CPQ), content curriculum events with only conceptual questions (CCQ), and content curriculum events with both procedural and conceptual questions (CPCQ).

**Interpretation.** Regarding interpretation, Wolcott (1994) discusses 11 ways of approaching it out of which the following six were appropriate and used for this study: (a) to extend the analysis, (b) to mark and then to make the leap, (c) to stop when one comes to the end, (d) to do as directed or suggested, (e) to turn to theory, and (f) to connect findings with personal experience. To extend the analysis means to note or suggest implications from the findings that one may conclude without actually concluding them. To mark and then make the leap means making inductive inferences when carrying out the analysis of the data. To stop when one comes to the end means to make interpretations as far as possible without losing confidence. One way of doing that is to
point out what is needed to make an inference and suggest steps to fill in that gap. To do what is directed or suggested means to follow indications of people with political power such as committee members or reviewers. A powerful way of approaching interpretation is to use a theory to link events or to discusses the meaning of categories, or to examine the cases in light of different theoretical stances. Finally, the connection of findings with personal experience helps to provide additional interpretation to our findings.

**Trustworthiness**

This concept refers to the validity of the inquiry regarding the internal (that of participants) and external judgments (that of fellow researchers) about the "consistency of its procedures and the neutrality of its findings or discussions" (Erlandson, Harris, Skipper, & Allen, 1993, p. 29). Trustworthiness can be built into a naturalistic inquiry through credibility, transferability, dependability and confirmability.

Credibility refers to the relationship or "compatibility of the constructed realities that exist in the minds of the inquiry's respondents with those that are attributed to them" (Erlandson, Harris, Skipper, & Allen, 1993, p. 30). To enhance the credibility of the research findings, the following strategies may be used: prolonged engagement, persistent observation, triangulation, referential adequacy materials, peer debriefing, and member checks. I used prolonged engagement, persistent observation, and triangulation to establish the credibility of this study. Regarding prolonged engagement, Mr. Kantor and I had many informal talks during the pilot study which lasted for about one month. During the time of the pilot study Mr. Kantor helped the student teacher when faced with a difficult topic to explain. I also visited Mr. Kantor's classroom for about three days before beginning the study in early October. I began interviewing Mr. Kantor on October 6, 1994. I videotaped his class from October 13 to November 1, all the time he spent teaching multiplication in algebra. All those visits provided a prolonged engagement long enough to establish trustworthiness of the study of Mr. Kantor's knowledge and use
of his knowledge. The persistent observation consisted of videotaping Mr. Kantor's teaching. The data are available when needed for verifying findings and conclusions. I can observe as persistently as needed. Triangulation refers to the use of multiple methods of data collection (interviews, observations, videotapes, documents), multiple sources of data (time, space, person), different questions, multiple investigators, and multiple theoretical perspectives (Denzin, 1970; Erlandson et al., 1993). I used two main sources of data: interviews and videotaping. I probed to "triangulate" conclusions. Most of the time I could judge whether he knew the representation based on how explicit he was in constructing it. Most, if not all of the time, my tentative hypotheses were checked at another time (multiple questions). I considered that interviews, videotaping, and rich description would give us enough context and data for interpreting and triangulating conclusions. To establish context, I added a section in Chapter 5 to describe some elements of the contextual nature of this study.

Transferability refers to "the extent to which ... findings can be applied in other contexts or with other respondents" (Erlandson, Harris, Skipper, & Allen, 1993, p. 31). This kind of generalization is to similar cases rather than to a population of cases. Stake (1978) calls them "naturalistic generalizations." Two strategies that were used to facilitate transferability were rich description and purposive sampling. Erlandson, Harris, Skipper, and Allen describe rich description as "sufficiently detailed descriptions of data in context and reports them with sufficient detail and precision to allow judgments about transferability" (p. 33). Regarding purposeful sampling, as mentioned before, I chose an experienced teacher with the potential of having above average knowledge of representations. I also sampled purposively for typical data and divergent data related to the purpose of the study (Erlandson, Harris, Skipper, & Allen, 1993).

Dependability refers to the consistency of the findings. If the inquiry "were replicated with the same or similar respondents (subjects) in the same (or a similar)
context, its findings would be repeated" (Erlandson, Harris, Skipper, & Allen, 1993, p. 33). Dependability can be facilitated by "audit trail" that "provides documentation (through critical incidents, documents, and interview notes) and a running account of the process (such as the investigator's daily journal) of the inquiry" (Erlandson et al., 1993, p. 34). I think that rich description also helps to add some dependability to the study. The readers can check the consistency of findings. Another thing that I did was to ask Mr. Kantor similar questions at different periods in time. I found consistency in his responses.

Confirmability refers to "the degree to which [research] findings are the product of the focus of the inquiry and not the biases of the researcher" (Erlandson et al., 1993, p.40). "This means that the data (constructions, assertions, facts, and so on) can be tracked to their sources, and that the logic used to assemble the interpretations into structurally and corroborating wholes is both explicit and implicit" (Guba & Lincoln, 1989, p. 243). Confirmability can also be established through an audit trail. I have indicated the number of the interview or classroom observation during which supporting data were collected so that readers and I can track the data to their original sources. I use a parenthesis with three entries to sources of the data. The first entry indicates the type of data source: I means interview and CO means classroom observation. The second entry indicates the number of the interview or classroom observation from which the excerpt comes. The third entry indicates the data the interview was conducted or the classroom observation made. Again, rich description and tracking enhance the confirmability of the findings.

Data Reporting

The findings of this study are reported as a narrative focusing on the research questions posed in Chapter 1. The narrative integrates description, analysis, and interpretation. The narrative follows Erickson's framework for reporting data. Erickson
(1986) mentions nine main elements of a report of field work research: (a) empirical assertions, (b) analytic narrative vignettes, (c) quotes from interviews, (d) synoptic data reports, (e) particular descriptions, (f) general description, (g) interpretative commentaries, (h) theoretical discussions, and (i) report of the natural history of inquiry in the study.

Empirical assertions are statements that are generated through a content analysis of the data and therefore are empirically tested against the data corpus. These empirical assertions vary in scope and in level of inference. To warrant those empirical assertions the researcher provides analytic narrative vignettes, quotes from interviews, synoptic data reports and particular description. Analytic narrative vignettes are "vivid portrayal of the conduct of an event of everyday life, in which sights and sounds of what was being said and done are described in the natural sequence of their occurrence in real time (Erickson, 1986, p. 149-150). The narrative vignette has three functions: rhetorical, analytic, and evidentiary. It has rhetorical functions because it purposes to convince the reader that things have happened in the setting as the researcher claim they happened. The vignette has analytical and evidentiary functions because it empirically grounds abstract analytical concepts or categories by providing concrete examples of their occurrence. This gives the reader empirical evidence to those analytical concepts. The description of those specific examples showing instantiation of analytical constructs is what Erickson terms particular description. I provide some narrative vignettes in the form of excerpts and in the form of extracts from interviews and videotapes.

The purpose of general description is to establish the generalizability of the patterns and events. The general description can be presented by citing analogous instances in the form of vignettes or providing a summary of those instances using synoptic reports such as charts or frequency tables to show the overall distribution of instances in the data corpus.
Interpretive commentaries fill in "the information beyond the story itself that is necessary for the reader to interpret the story in a way similar to that of the author" (Erickson, 1986, p. 152). Ways of approaching interpretive commentaries (inferences, theoretical discussions, etc.) are discussed throughout Chapter 5.

I have described the methodological decisions I made and the rationale behind them. In the next chapter I present the results of the data organized to respond to the research questions that were the focus of this study.
CHAPTER 5

RESULTS

In this chapter I present the results of the investigation of the phenomenon of teaching from the perspective of knowledge of both mathematical and pedagogical representations and use of that knowledge in the construction of pedagogical events (explanations, representations, and questions) when teaching some specific content curriculum events. This phenomenon was examined in the case of Mr. Kantor and in the context of algebraic multiplication as stated in the research questions:

I. What is Mr. Kantor's knowledge of mathematical representations about algebraic multiplication?

1. What is Mr. Kantor's knowledge of symbolic representations?

2. What is Mr. Kantor's knowledge of mathematical proofs?

II. What is Mr. Kantor's knowledge of pedagogical representations about algebraic multiplication?

3. What is Mr. Kantor's knowledge of pictorial representations?

4. What is Mr. Kantor's knowledge of story-problem representations?

III. How does Mr. Kantor use his knowledge of representations when teaching algebraic multiplication?

5. What representations does Mr. Kantor use?

6. What explanations does Mr. Kantor construct?
7. What questions does Mr. Kantor pose?

This chapter has been organized in the following sections: the context, mathematical content curriculum events, Mr. Kantor's knowledge of symbolic representations, Mr. Kantor's knowledge of proofs, Mr. Kantor's knowledge of pictorial representations, Mr. Kantor's knowledge of story-problem representations, Mr. Kantor's use of representations, Mr. Kantor's explanations, and Mr. Kantor's questions.

Multiplication in algebra is covered in 10 lessons in the textbook. Nine lessons in chapter four and one lesson in chapter five. The lessons of chapter four are the following: (a) areas, arrays, and volumes; (b) multiplying fractions and rates; (c) special numbers in multiplication; (d) solving $ax = b$; (e) special numbers in equations; (f) solving $ax < b$; (g) multiplication counting principle; (h) multiplying probabilities; and (i) the factorial symbol. The definition of division, first lesson of chapter five, division in algebra, was included because it is directly related to multiplication. As described in more detail later, Mr. Kantor followed the order of the textbook. I will begin by describing some components of the context in which Mr. Kantor was teaching to help us to better understand his construction of pedagogical events when teaching content curriculum events related to algebraic multiplication.

The Context

I divide this section into the following three parts: school setting, Mr. Kantor's background and conceptions, and classroom culture. In the first section, the school setting, I provide a description of the school setting where Mr. Kantor was teaching. In the second section, Mr. Kantor's background and conceptions, I describe some aspects of Mr. Kantor's background and some aspects of his conceptions about mathematics as well as about mathematics teaching and learning. In the third section, the classroom culture, I describe some rituals that Mr. Kantor and his students go through when they are engaged in the process of constructing mathematical knowledge.
School Setting

The school and its environment. The study took place in Ludwig, a three-year middle school located in a upper-middle class suburban area in a large Midwestern U.S. city. The grades are 6, 7, and 8.

Students' background. There were 678 students enrolled at Ludwig during the academic year, 1993-1994, of which 358 were boys and 320 girls. There were 21 minority students.

Teachers' background. There were about 44 teachers working in this school during the 1993-1994 academic year. The average experience for this group of teachers was 19.2 years. About 36% were male and 63.3% were female. About 2% of teachers belonged to minority ethnic groups.

Academic expectations. One of the objectives of the district in which Ludwig is located is that "100% of our students will achieve at their fullest individual potential" (School District Annual Report). This is reflected to some extent on student academic achievement as measured by standardized tests where students "achieve above expectations" (School District Annual Report).

Mr. Kantor's Background and Conceptions

Mr. Kantor's story. As an undergraduate, Mr. Kantor was in mathematics education in which he got a secondary teaching certificate. In the middle of student teaching he realized that teaching was not what he wanted to do at that time because of two different factors. The first one, that he calls immaturity, was the low social status of the teaching profession as reflected in salary payment. The second one was the difficulty of not much difference in ages between himself and his students. He transferred to business and got a degree in finance. He worked in business for fifteen years. He enjoyed his career a lot. However, two factors made him to decide to go back to teaching. For one, it had become time to decide whether to move up in a management position or stay at the
same level. However, management did not appeal to him, and because he is better at analytical thinking than at technical skills, he considered starting another career. Also during that time he coached children's teams, and people complimented him on his coaching skills. These events, in addition to witnessing some bad coaching, led Mr. Kantor to believe he could be a good teacher. Consequently, he returned to the teaching profession.

Mr. Kantor's conceptions about mathematics. Mr. Kantor conceives of mathematics mainly as a framework to put problems into, and as a valuable tool to use to solve problems. He said that there are some other interesting things in mathematics such as number theory, but he does not teach mathematics just for the sake of mathematics.

Mr. Kantor's beliefs about the role of the student as learner of mathematics. Mr. Kantor described the role of the student in the following words:

[Students] should stay focused, try to get forty two minutes to math, keep an open mind, try not to get frustrated ... trying and trying again. . . . the big step is to try something and read and understand the problem, reread and understand the problem, try something to see if that fits, [if not] try something else. I want them to come out being better problem solvers. (I, 1, 10-06-94)

Mr. Kantor's beliefs about his role in the mathematics teaching-learning process. According to Mr. Kantor, the ultimate goal of teaching mathematics is just to make students good problem solvers. To achieve this goal, Mr. Kantor stated:

I try to make sure that my students understand it [mathematics] and know where it might be applied, and know the relevance of it.... Try to get as many kids involved as possible. Try to go forwards and backwards as much as you can. Showing what we have already done, how that ties into this lesson. How that ties into future math and teaching for understanding rather than just content. (I, 1, 10-06-94)

He added that he should also:

Make the environment conducive to learning—... that when they fail, they don't feel bad about it. . . . It didn't work, so what? It's important that I keep everyone understanding that trying is important. Trying and trying and trying again—... I value any attempt that they make, try to be positive and encourage them. (I, 1, 10-06-94)

Geometry had a tremendous impact on his thinking of how students learn mathematics. Geometry helped him:
To realize that some people really like hands-on things. It showed me how to use as many different methods, try to use visual, try to use verbal, just try to use different techniques because some people—... one thing hits them easier than another thing does. (I, 1, 10-06-94)

Mr. Kantor attaches a lot of value to his 15-years experience in business and less value to most course work he took in college. That is because:

I don't know how I would honestly answer some of the questions kids have: Where am I going to use this? Why is this important? Paying attention to detail. I know first hand how important it is to pay attention to detail. I know, you know, making presentations to words and doing errors in presentations... it's all correct. I checked it. They didn't check it—there is a value to this—Showing work, checking work. (I, 1, 10-06-94)

Another influence on his beliefs about his role on the teaching-learning process was a high school teacher from whom he took calculus. The teacher:

... was very methodical. He explained calculus in a way that I understood it—to go back and review it. He was an exceptional teacher in that way. He spent a good deal of time at getting us to understand and really, and not just know how to do things but understand what you are doing. That had an impact on me, that method of teaching for understanding, not browsing through something with a very superficial understanding. (I, 1, 10-06-94)

Mr. Kantor considers that there are two main factors that help to be a good teacher. The first one is knowledge of the subject matter, and the second one is liking children. In his words:

I was very strong in math.... That's what makes a good teacher—real understanding of the subject.... Knowledge of math is one thing, but just knowledge of math doesn't make you a good teacher. But you have to have the basics, I think, to be successful in teaching a subject. You have to understand the subject but the thing that is important is liking children and to help everyone succeed.... I don't know if I just have it naturally or but I really like all the kids that I have—even the kids that test your patience—that try to upset you at times. They don't bother me. I see other teachers—that they don't like a certain kid. I think that makes a really tough job, when you have somebody you don't like that you try to teach. (I, 1, 10-06-94)

Classroom Culture

**Classroom routines.** Mr. Kantor follows some classroom routines: checking homework, answering questions, and then constructing some pedagogical events for discussing the new lesson and the homework for the following day. This observation agrees with what he said when I asked him about his classroom routine:

There are routine things that we do—... check homework every morning. I go around and make sure that they show work.... I really don't check at that point to see whether they understood it. I can't physically do that. As soon as I check their homework and give them a grade, ... we go over problems. [Sometimes students] present problems. Sometimes I just ask them if they have questions.
and go over questions.... Sometimes it's class work; sometimes it's group work. Sometimes it's somewhat lecture, but usually it's a big variety. I'm trying to have them discover some type of concept. I guide the discovery. For certain topics it's not much discovery other than leading into it, because I find a lot of times what I expect them to discover they don't come close to discovering it.... It's a timing thing. If we've done a quiz on a quiz day, there is no time to discuss the next day's lesson because I spent [some time on the] quiz. They're now responsible to read the lesson on their own and do the next homework.... The next day, the day after the quiz, I probably spend a little time reviewing the lesson they read themselves.... So it's a mixture because of time to get through everything is in the book. (f, 3, 10-14-94)

**The role of the teacher and students.** Mr. Kantor's instruction can be best characterized as direct instruction. He is the authority in the classroom and he constructs most of the explanations, representations, and questions for helping children to construct mathematical knowledge. The students feel quite free to ask questions. In general, he tends to emphasize both procedural and conceptual knowledge, especially when he believes that students do not know why a procedure works.

**The role of the textbook.** Mr. Kantor followed the order of the lessons in the textbook but the textbook is not a driving force in Mr. Kantor's construction of pedagogical events. The main influence of the textbook on Mr. Kantor's instruction, at least for the lessons examined, is as a curricular guide. However, he does not always construct pedagogical events about the main curriculum events. With difficult topics he tends to spend more time than with less difficult topics. His explanations, representations, and questions differ notably from those of the textbook.

In the section above, I have described the context in which Mr. Kantor was teaching including the school setting, his background and his conceptions about mathematics and about mathematics teaching and learning. Finally, I have described the classroom culture. In the remainder of this chapter I will address the research questions. First, the content curricular events dealt with in each of the lesson are mentioned. Second, Mr. Kantor's knowledge of each of the four representations related to each of the content curricular events is discussed. Third, Mr. Kantor's pedagogical events and knowledge use are examined for each of the curriculum events.
In the next section I mention the content curriculum events identified in the lessons of the textbook dealing with algebraic multiplication.

Mathematical Content Curriculum Events

Table 5.1 displays the content curriculum events for which Mr. Kantor's knowledge and knowledge use were examined.

For almost each of the content curriculum events displayed in Table 5.1, I asked Mr. Kantor, whenever appropriate, to provide two types of mathematical representations (a symbolic representation and a mathematical proof) and two types of pedagogical representations (a pictorial representation and a story-problem representation.) Next, I present the results corresponding to research question 1, Mr. Kantor's knowledge of symbolic representations.

Mr. Kantor's Knowledge of Symbolic Representations

Table C.1 in the Appendix C displays the symbolic representations constructed by Mr. Kantor for 38 of the content curriculum events displayed in Table 5.1. I did not ask Mr. Kantor the symbolic representation for the following three content curriculum events: volume of a rectangular solid (CCE 4), the product of a negative number and a positive number is a negative number (CCE 11), and the meaning of division (CCE 35) (the CCEs are numbered with reference to Table 5.1). I categorized the symbolic representations as verbal representations, verbal-symbolic representations and symbolic representations. Mr. Kantor provided a symbolic representation that I judged was correct for 32 of the 38 (84%) content curriculum events and a partially correct representation for six of the 38 (16%) content curriculum events. None of Mr. Kantor's symbolic representations was judged as incorrect. Some examples for each of the three types of symbolic representations follows. Table C.2 displays a summary of Mr. Kantor's categorization of the symbolic representations and their degree of correctness.
<table>
<thead>
<tr>
<th>Lesson</th>
<th>Content curriculum events</th>
</tr>
</thead>
</table>
| 1. Areas, arrays, and volumes | 1. Area model for multiplication (Continuous case)  
2. Commutative property of multiplication  
3. Area model for multiplication (Discrete version, array model)  
4. Volume of a rectangular solid  
5. Associative property of multiplication |
| 2. Multiplying fractions and rates | 6. Rule for multiplication of fractions  
7. Rate model for multiplication  
8. The product of two positive numbers is positive  
9. The product of two negative numbers is positive  
10. The product of a positive number and a negative number is negative  
11. The product of a negative number and a positive number is negative |
| 3. Special numbers in multiplication | 12. Multiplicative identity of 1 \((a\cdot 1 = 1\cdot a = a)\)  
13. Multiplicative property of \(-1\) \((-1\cdot a = -a)\)  
14. Definition of reciprocals  
15. The reciprocal of \(a\) is \(\frac{1}{a}\), \(a \neq 0\)  
16. Reciprocal of zero  
17. Multiplication property of zero |
| 4. Solving \(ax = b\) | 18. Multiplication property of equality  
19. Solving \(ax = b, a \neq 0\) |

(To be continued)

Table 5.1 Content curriculum events related to algebraic multiplication
<table>
<thead>
<tr>
<th>Table 5.1 (Continued)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. Special numbers in equations</td>
</tr>
<tr>
<td>20. Solve $0x = b$, $b \neq 0$</td>
</tr>
<tr>
<td>21. Solve $0x = 0$</td>
</tr>
<tr>
<td>22. Solve $ax = 0$, $a \neq 0$</td>
</tr>
<tr>
<td>23. Solve $-x = a$</td>
</tr>
<tr>
<td>24. Definition of $a &gt; b$.</td>
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<tr>
<td>6. Solving $ax &lt; b$</td>
</tr>
<tr>
<td>25. The multiplication property of inequality (part 1)</td>
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<tr>
<td>26. The multiplication property of inequality (part 2)</td>
</tr>
<tr>
<td>27. Solving $ax &lt; b$</td>
</tr>
<tr>
<td>28. Multiplication counting principle</td>
</tr>
<tr>
<td>29a. $P(A)$</td>
</tr>
<tr>
<td>29b. $P(A \cap B)$</td>
</tr>
<tr>
<td>29c. Classical definition of conditional probability</td>
</tr>
<tr>
<td>8. Multiplying probabilities</td>
</tr>
<tr>
<td>30. Second definition of conditional probability $[P(B \mid A) = P(A \cap B)/P(A)]$</td>
</tr>
<tr>
<td>31. Conditional probability formula $(P(A \cap B) = P(A)P(B \mid A)$</td>
</tr>
<tr>
<td>32. $n!$</td>
</tr>
<tr>
<td>9. The factorial symbol</td>
</tr>
<tr>
<td>33. $0!$</td>
</tr>
<tr>
<td>34. Permutation theorem</td>
</tr>
<tr>
<td>10. Algebraic definition of division</td>
</tr>
<tr>
<td>35. Meaning of division</td>
</tr>
<tr>
<td>36. Algebraic definition of division</td>
</tr>
<tr>
<td>37. Division by zero</td>
</tr>
</tbody>
</table>
Table 5.1 (Continued)

38. The quotient of two positive numbers is positive
39. The quotient of two negative numbers is positive
40. The quotient of a positive number and a positive number is negative
41. The quotient of a negative number and a positive number is negative

Table 5.1 Content curriculum events related to algebraic multiplication

**Verbal Representations**

I found five content curriculum events (CCEs) for which a verbal representation is appropriate: the rate model for multiplication (CCE 7); reciprocal of zero (CCE 16); solving \(0x = b, b \neq 0\) (CCE 20); solving \(0x = 0\) (CCE 21), and division by zero (CCE 37). Because it is not easy to find symbolic representations for these five content curriculum events and because the textbook gives verbal representations, I accepted all of Mr. Kantor's verbal representations as correct, since they were similar to the ones provided by the textbook. For example, his representation for the rate model was as follows:

> It's supposed to be the same thing (as multiplication of fractions), you just have units too, so that the units have to be consistent, and the units in your answer have to make sense [they have to have meaning to multiplying together].... You take a quantity and multiply it by a rate. (I, 26. 05-30-95)

Mr. Kantor's representation for the reciprocal of zero was "0 has no reciprocal" and his representation for division by zero was "[we] can't divide by zero."

**Verbal-Symbolic Representations**

Verbal-symbolic representations involve both verbal and symbolic elements. The following six content curriculum events were considered to have representations of this category: area model for multiplication (continuous case) (CCE 1), array model for multiplication (CCE 3), volume of a rectangular solid (CCE 4), multiplication counting principle (CCE 28), the permutation theorem (CCE 34), and the meaning of division (CCE 107).
35). I asked Mr. Kantor the representation for four of them. Three of the four representations constructed by Mr. Kantor were judged as correct (for CCEs 1, 3, and 34). For the discrete version of the area model (CCE 3), the array model, Mr. Kantor provided the following representation "You talk about an array of things ... If you have $n$ rows with $m$ elements in each row then there is $n$ times $m$ elements in the array." The representation for the multiplication counting principle was judged as partially correct. Mr. Kantor stated the representation for the multiplication counting principle in the following terms "If there is ... $x$ ways to make one choice [...] and $y$ ways to make a second choice. How many different pairs of choices could be made? Different pairs of choices would be $x$ times $y$.

**Symbolic Representations**

For 30 of the 41 curriculum events a symbolic representation can be given. Mr. Kantor was asked to provide a symbolic representation for 29 of these 30 content curriculum events. Kantor constructed a symbolic representation for 24 of the 29 content curriculum events. All of the 24 symbolic representations were judged as correct. To illustrate, Mr. Kantor provided the following symbolic representation for the associative property, "For $a, b, c \in R, (a-b)-c = a-(b-c)" and he represented one of the rules of signs as "Given $a > 0, b < 0$ then $a\cdot b < 0.$"

For the remaining five of the 29 content curriculum events having a symbolic representation (the definition of probability of an event (CCE 29) and the four rules of signs for division (CCEs 38-41)), Mr. Kantor constructed a verbal-symbolic representation. These representations were judged as partially correct because a symbolic representation can be constructed. To illustrate, he provided the following representation for one of the rules of signs for division: "If $a$ is + and $b$ is – then $\frac{a}{b} = -$." A correct symbolic representation would be "If $a > 0$ and $b < 0$ then $\frac{a}{b} < 0$" or "If $a \in R$ and $b \in R^-$ then $\frac{a}{b} \in R^-" or "let $a > 0$ and $b > 0$, then $\frac{a}{-b} = -\frac{a}{b}.$"
I have described Mr. Kantor's knowledge of symbolic representations from the perspectives of correctness and symbolism. I will describe another aspect of his knowledge of symbolic representations: the degree of development of this type of knowledge.

The Development of Mr. Kantor's Knowledge About Symbolic Representations

I should note here that Mr. Kantor did not construct the symbolic representations immediately. Rather, he tended to construct informal representations. For example, when I asked him to provide the definition of the commutative property he said "When you multiply quantities together the order doesn't make any difference." I had to constantly remind him that I would prefer the symbolic representation of the mathematical concept. He then provided a symbolic representation when he could do it.

Some readers may say that the definition of the rate model for multiplication is informal (CCE 7). I accepted it as correct because there is no obvious way to state a symbolic representation for that model. The textbook defines it in the following terms:

(When a rate is multiplied by another quantity, the unit of the product is the product of units. Units are multiplied as though they were fractions. The product has meaning when the units have meaning. (McConnel et al., 1990, p. 164)

When I asked Mr. Kantor a symbolic representations for each of the rules of signs for multiplication, he seemed to have trouble articulating them. I thought that he probably did not have a well developed knowledge about these representations. To gather more evidence about that tentative hypothesis, I asked him again, in later interviews, the symbolic representations of the rules of signs for multiplication. That time he provided the representations entered in cells 8-10, Table C.1, which I considered as correct. Some support to the statement that Mr. Kantor's knowledge of representations about the rules of signs is not well articulated comes from his reaction to a student's remark that "the number
of negatives is even, so the signs cancel out, and when the number of negative signs is odd, you put the negative sign." Mr. Kantor commented that:

In practice that's how everybody does it anyway. I mean, I want them to understand that can facilitate what you are doing, but it doesn't mean they are negatives. Just think of the opposite of the opposite is back to... I don't know even how to say it. I kind of wrap myself in a circle. The opposite of the opposite is the original, I mean.... I don't have a terminology to use. Negative times negative is positive. It's an easy terminology but it's not technically correct. (I, 6, 10-20-94)

However, it is interesting to notice that Mr. Kantor provided correct representations for the rules of signs. It is also worthwhile to notice that he did not provide the following representations for the rules of signs for multiplication: if \( a > 0 \) and \( b > 0 \) then \((-a)(-b) = (ab)\), \((-a)(b) = -(ab)\), \((a)(-b) = -(ab)\). When I asked Mr. Kantor to construct the symbolic representations for the rules of signs in the case of division, he did not construct correct symbolic representations as mentioned above.

As indicated in Table C.1, Mr. Kantor defined \( a > b \) if \( a - b > 0 \) which is an acceptable mathematical definition. We still need to define what \( c > 0 \) means where \( c \) is a real number; that is, we need to define our set of positive numbers. However, it was not the purpose of this study to examine Mr. Kantor's knowledge of the foundations of mathematics. That representation indicates that Mr. Kantor knows how to define algebraically that \( a \) is greater than \( b \). It is interesting to notice that Mr. Kantor did not define the related statement, \( a < b \), in terms of \( a > b \). In mathematics we tend to be economical. Those relations are connected and therefore we can define \( a < b \) as \( a < b \) if and only if \( b > a \). Regarding the symbolic representations of the multiplicative property of inequalities, Mr. Kantor did not construct the standard representations. For example, he defined the multiplication property of inequalities in the following terms: given \( ax < b \) when \( a, x, \) and \( b \) are in \( R \) then \( x < \frac{b}{a} \) when \( a > 0 \) [and] \( x > \frac{b}{a} \) when \( a < 0 \). The standard mathematical representation of the multiplication property of inequalities is: If \( a < b \) then \( ax < bx \) for \( x > 0 \) and \( ax > bx \) for \( x < 0 \).
Other content curriculum events for which Mr. Kantor's knowledge of definitions seems to be underdeveloped and tends to be intuitive and informal include the definition of probability. As we can see from Table C.1, Mr. Kantor's definitions of probability of an event, probability of the intersection of two events, and conditional probability, represent "common sense" definitions of those probabilities because he did not use appropriate language. For example, he did not use the word "number" for defining those probabilities and, as shown on those tables, he did not talk about experiments, outcomes, sample space, event, and his definitions were not rigorous, mathematically speaking. A more formal, yet intuitive, definition of probability of an event, $P(A)$, would be "the quotient of the number of successful outcomes and the total number of possible outcomes if all possible outcomes are equally likely." His responses suggest that he probably does not possess a well articulated body of knowledge about the mathematical definitions of those probabilities. Since some textbooks define $P(B \mid A)$ as $\frac{P(A \cap B)}{P(A)}$, I asked Mr. Kantor if there was a formula related to probability of $B$ given $A$; his response was such formula (Table C.1, CCE 30). I did not ask him explicitly the representation of the conditional probability formula ($P(A \cap B) = P(A) \cdot P(B \mid A)$) because he had already mentioned that relationship during classroom instruction and in other interviews.

We notice also that Mr. Kantor did not provide symbolic representations but verbal-symbolic representations for the rules of signs for division. These representations add some validity to my earlier statement that Mr. Kantor's knowledge about how to represent the rules of signs is not well developed. All this information helps to conclude that Mr. Kantor's knowledge of symbolic representations about some mathematical content curriculum events is not well articulated. Those content curriculum events include the rules of signs in the case of multiplication and division, the definition of $a < b$ and its relationship to $a > b$, the multiplicative properties of inequalities and the definition of probability.
As a summary, Mr. Kantor constructed a correct symbolic representation for 32 content curriculum events and a partially correct symbolic representations for 6 content curriculum events. Mr. Kantor's knowledge of symbolic representations was judged to be strong because the index of correctness of his symbolic representations (including verbal and verbal-symbolic representations) was .92. Using only the CCEs for which a symbolic representation can be constructed the index of correctness was .91. But a critical question remains without an answer: "What is the impact of Mr. Kantor's knowledge of symbolic representations on his constructing of pedagogical events?" Most of the literature on teachers' knowledge has ignored the role that teachers' knowledge plays in their instruction. This question will be addressed later.

Mr. Kantor's Knowledge of Formal Proofs

I will discuss Mr. Kantor's knowledge of formal proofs. This knowledge includes knowledge of statements that are definitions, statements that can be accepted as axioms, statements that can be proved by contradiction, statements that can be proved geometrically, statements that can be proved by induction, and statements that can be proved algebraically.

Table D.1 in Appendix D displays all the proofs constructed by Mr. Kantor. The reader may refer to either Table 5.1 for a fast identification of the content curriculum event or to Table D.1 in Appendix D for the content curriculum events and their corresponding proofs constructed by Mr. Kantor. For the following three content curriculum events I did not ask Mr. Kantor a proof because they were definitions clearly: probability of an event (CCE 29), the definition of greater than (CCE 24), and n! (CCE 32). For two content curriculum events, the rate model for multiplication and the meaning of division was not appropriate to construct a proof. I failed to ask Mr. Kantor proof for the content curriculum event the volume of a rectangular solid. Then, Mr. Kantor's knowledge of mathematical proofs was examined for 35 content curriculum events.
For some content curriculum events Mr. Kantor was asked to provide a proof more than once. In those cases he constructed different types of "proofs." For example, the first time Mr. Kantor was asked the mathematical proof of the rule of signs "the product of two positive numbers is a positive number" (CCE 8), he suggested that this rule could be a definition. On the other hand, the second time he justified the rule using the repeated addition model for multiplication. Then that curriculum event was entered under definitions and heuristic argument and was listed twice under the column *Mr. Kantor's incorrect proofs* in Table D.2. An acceptable proof would be that the statement "the product of two positive numbers is a positive number" is accepted as axiom in some developments of the real numbers. Table D.2 (Appendix D) displays Mr. Kantor's categorization of proofs and their degree of correctness.

**Mr. Kantor's Knowledge of Statements That Are Definitions**

I identified four statements that are definitions: a real number different from zero times its reciprocal equals 1¹ (CCE 14), second definition of conditional probability \[P(A \mid B) = \frac{P(A \cap B)}{P(B)}\] (CCE 30), \(0! = 1\) (CCE 33), and the definition of division, \(a + b = \frac{a}{b} = a \cdot \frac{1}{b}\) for \(b \neq 0\)² (CCE 36). Of those, Mr. Kantor justified all except the second definition of conditional probability as definitions. However, Mr. Kantor accepted also the following statements as definitions: the product of two positive numbers is a positive number (CCE 8), the product of two negative numbers is a positive number (CCE 9), the multiplicative

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¹Some mathematicians would argue that reciprocal is not an appropriate word and that multiplicative inverse would be a more appropriate word. However, both words are commonly used by textbook authors at both the college level (e.g., Long and DeTemple, 1996; Moise, 1990; Musser & Burger, 1997) and secondary level including the authors of the University of Chicago School Mathematics Project (McConnell et al., 1990).

²There is no general agreement among mathematicians about the definition of division. Some define \(a + b\) as \(a - \frac{1}{b}\), \(b \neq 0\) (e.g., Moise, 1990) while others (e.g., Long and DeTemple, 1996) define \(a + b = c\) if and only if \(a = b-c\) for a unique number \(c\). Since the textbook uses the first definition, I will refer to it as the definition of division.
property of \(-1 (-1a = -a)\) (CCE 13), multiplicative property of zero \((x \cdot 0 = 0)\) (CCE 17). In general, we can conclude that Mr. Kantor does not have a well developed knowledge of what mathematical statements are definitions.

**Mr. Kantor's Knowledge of Statements That Can Be Accepted as Axioms**

I identified that the following four statements can be accepted as axioms in some developments of the real numbers: multiplication is commutative (CCE 2), multiplication is associative (CCE 5), the product of two positive numbers is positive (CCE 8), and the multiplicative identity of \(1 \ (a \cdot 1 = a)\) (CCE 12). I was interested in examining whether Mr. Kantor knew the axiomatic status of these statements in some developments of real numbers. Mr. Kantor did not recognize any of these propositions as axioms nor did he mention that they can be proved if we accept other axioms in the construction of real numbers.

**Mr. Kantor's Knowledge of Proofs by Contradiction**

I identified that the following three curriculum events can be proved by contradiction: \(0\) has no reciprocal (CCE 16), the equation \(0x = b, b \neq 0\) has no solution (CCE 20), and division by zero is impossible (CCE 37). There is some evidence that Mr. Kantor attempted to prove these theorems by contradiction. Mr. Kantor's proof of the theorem "0 has no reciprocal" was "0 \(\cdot\) \(R = 1\), 0 has no reciprocal. There isn't something I multiply by zero to get one." I judged this proof as partially correct rather than correct because Mr. Kantor did not make explicit the connection to where the contradiction is. He started by assuming that the reciprocal of 0 is \(R\), and then \(0 \cdot R = 1\) by definition of reciprocal. His thinking was that there is no number that multiplied by zero gives 1 because 0 times any number gives 0; but we need to show that there is not a real number \(x\) such that \(x \cdot 0 = 1\). An acceptable proof would be like the following, assume that 0 has a reciprocal, \(R\). Then \(0 \cdot R = 1\) by definition of reciprocal. On the other hand, \(0 = 0 \cdot R\) by the multiplicative
property of zero. We then conclude that $0 = 1$, a contradiction. Therefore, $R$ can not exist.

(Of course, we have to prove that $0 \neq 1$, but that is beyond the purpose.)

Mr. Kantor also attempted to prove that $0x = b$, $b \neq 0$ has no solution by contradiction but he failed to do so. His thinking was as follows:

Assume there is a solution $[x]$. Again, $b$ times zero has to equal $x$; $b$ times $x$ has to equal zero. Let’s see $b$ times zero has to equal $x$. If $x$ is zero that would work. Then $x$ is not given. Let’s see, that will make, $x$ is zero, $b$ times zero equals zero, that works. $b$ divided by $x$ will have to equal zero. If there is a solution, then $b$ divided by $x$ will have to equal zero, correct? And then it follows that. [It's] just a circular argument, the next time zero equals $b$ [over $x$], that’s what I already said. If there is a solution then $b$ times zero equals $x$, $b$ times $x$ equals zero. That says those things are zero, but they can’t be zero. I can’t think right now. (I, 27, 06-02-95)

I judged Mr. Kantor’s proof by contradiction that division by zero is impossible as partially correct. His proof was stated in the following terms:

if there was an answer to that, then the related fact had to be true.... If this has an answer $\frac{a}{0}$ then $a$

divided by zero equals $b$ $\frac{a}{0} = b$. Then $b$ times 0 has to be equal $a$. The only way that can be is if $a$

were zero and in that case $b$ can be anything. So everything works for $b$. So that’s why 0 divided by 0

is kind of meaningless too. (I, 23, 01-03-95)

In general, Mr. Kantor’s knowledge of proofs by contradiction is not well developed. He needs to make explicit the nature of the contradiction of many proofs.

Mr. Kantor’s Knowledge of Proofs by Induction

I identified that the following three content curriculum events could be proved by induction: the number of elements in an array (the discrete model for multiplication) (CCE 3), the multiplication counting principle (CCE 28), and the permutation theorem (CCE 34). Because Mr. Kantor did not use, explicitly, the principle of mathematical induction to prove these theorems, I judged the three proofs as partially correct. To illustrate, his proof for the number of elements in an array with dimensions $r$ and $s$ was as follows:

$s$ seats per row then in every row you have that many seats, you have $r$ of those items and it’s repeated addition. (I, 31, 06-19-95)
\[ s + s + s + \ldots + s = r \cdot s \]

\[ r \]

As displayed in Table D.1, Mr. Kantor used similar arguments for proving the other two content curriculum events. Then, we can suggest that Mr. Kantor can give good arguments for theorems that require the use of mathematical induction but that he does not have a good understanding that mathematical induction is needed.

**Mr. Kantor's Knowledge of Geometric Proofs**

I considered that the area model for multiplication (CCE 1) can be proved using a geometric argument, if we accept as an axiom (or other axioms) that the area of a square with side \( a \) is \( a \cdot a \). However, Mr. Kantor did not provide a geometric proof nor any other kind of proof for the area model of multiplication.

**Mr. Kantor's Knowledge of Algebraic Proofs**

I grouped all the remaining 20 content curriculum events (CCEs) as having an algebraic proof. Some algebraic proofs required the use of theorems not mentioned previously and I asked Mr. Kantor to prove those lemmas. Other proofs could be deduced by making connections to previous theorems in a non-straightforward way. Still other proofs could be constructed by straightforward algebraic manipulation and using previous theorems. I identified the following eight curriculum events as having a proof that requires the use of theorems not mentioned previously (See Table 5.1 or Table D.1 in Appendix D): multiplication of fraction theorem (CCE 6), the product of two negative numbers is a positive number (CCE 9), the product of a positive number and a negative number is a negative number (CCE 10), the multiplicative property of \(-1\) (CCE 13), multiplicative property of zero (CCE 17), multiplication property of equality (CCE 18), first multiplicative property of inequalities (CCE 25), and the second multiplicative property of inequalities (CCE 26). Not surprisingly, the proofs of these theorems were the most
difficult for Mr. Kantor. He was not able to construct a correct mathematical proof for any of these eight content curriculum events. Mr. Kantor categorized two content curriculum events (the product of two negative numbers is a positive number, CCE 9, and the multiplicative property of -1, CCE 13) as definitions and provided an heuristic argument for some critical steps of some of the remaining proofs. To illustrate, Mr. Kantor constructed the following proof for the multiplication of fractions theorem,

\[
\frac{a}{b} \cdot \frac{c}{d} = \frac{(a + b)(c + d)}{(K: \text{This is the same as } a \div b \times c \div d.)} \\
= \frac{(a \cdot \frac{1}{b})(c \cdot \frac{1}{d})}{(K: \text{a times the reciprocal of } b, \text{c times the reciprocal of } d.)} \\
= \frac{a \cdot c \cdot \frac{1}{b} \cdot \frac{1}{d}}{(K: \text{Drop all parenthesis and regroup.)}} \\
= \frac{ac}{bd}
\]

A critical step is to prove that \( \frac{1}{b} \cdot \frac{1}{d} = \frac{1}{bd} \) (\( b, d \neq 0 \)). When Mr. Kantor attempted to prove it, he gave the following heuristic argument “let’s see, how do you go from there \( [a \cdot c \cdot \frac{1}{b} \cdot \frac{1}{d}] \) to there \( \frac{ac}{bd} \)? This \( \frac{1}{b} \) times this \( \frac{1}{d} \) is the same thing as \( \frac{1}{bd} \). You can look at it this way. You cut something into \( b \) parts and cut that into \( d \) parts. You cut it into \( b \) times \( d \) parts” (I, 23, 01-03-95). Although that argument can be used to justify intuitively that \( \frac{1}{b} \cdot \frac{1}{d} = \frac{1}{bd} \), cannot be accepted as a mathematical proof and, thus, the proof of the multiplication of fraction theorem was categorized as partially correct.

Regarding the proofs of making connections to previous curriculum events, I categorized the following five content curriculum events as falling within this category: The product of a negative number and a positive number is a negative number (CCE 11), and the four rules of signs for division (CCEs 38-41). Mr. Kantor was able to provide a correct proof of all of these content curriculum events. To illustrate, he proved that the product of a negative number and a positive number is a negative number by using the commutative property of multiplication in the case of a positive number multiplied by a negative number is a negative number. I consider this to be an elegant proof because it is a
simple deduction of a previous theorem. As a second example, he proved that the quotient of a negative number and a negative number is a positive number in the following terms:

If $a$ and $b$ are $-$, then $a + b = a \cdot \frac{1}{b}$ [is] $-$ then $\frac{1}{b}$ [is] $-$ [b is negative then one over $b$ is negative because $b$ times one over $b$ equals a positive $\frac{1}{b} \cdot \frac{1}{b} = +$. This $\frac{1}{b}$ is negative, that one $\frac{1}{b}$ has to be negative. And so then we know this is negative $[a]$ and that’s negative $[\frac{1}{b}]$, equals positive $[- \cdot +]$ [using multiplication rules]. (I, 30, 06-16-95)

Regarding the proofs using algebraic manipulation, I identified the following seven curriculum events as belonging to this category: the reciprocal of $a$ is $\frac{1}{a}$ ($a \neq 0$) (CCE 15), the solution of $ax = b$, $a \neq 0$ (CCE 19), the solution of $0x = 0$ (CCE 21), the solution to $ax = 0$, $a \neq 0$ (CCE 22), the solution to $-x = a$ (CCE 23), the solution to $ax < b$ (CCE 27), and the conditional probability formula (CCE 31). I judged Mr. Kantor’s proofs for solving $ax = b$, $a \neq 0$, the solution to $ax = 0$, $a \neq 0$, and the solution to $0x = 0$ as correct; and the proofs for the reciprocal of $a$ is $\frac{1}{a}$ ($a \neq 0$), the solution to $-x = a$, and the solution to $ax < b$ as partially correct. Mr. Kantor did not construct a proof for the conditional probability formula. To illustrate his knowledge, Mr. Kantor constructed the following proof for the solution of $ox = b$, $a \neq 0$ (CCE 19) "$\frac{1}{a}ox = b$ $\frac{1}{a} \neq 0$ (Multiply both sides by the reciprocal of $a$.) $x = \frac{b}{a}$, $a \in R$, except 0." We notice that this proof is somewhat sketchy. I would have preferred a proof in the following terms:

If $a \neq 0$, $a^{-1}$ exists. Then $ax = b$ implies $a^{-1}(ax) = a^{-1}b$ by multiplicative property of equality. From here it follows that $(a^{-1} \cdot a)x = a^{-1}b$ by the associative property. Then we have that $1 \cdot x = a^{-1}b$ by the property of inverses. By the identity property of 1 we have that $x = a^{-1}b$. Finally, we get that $x = \frac{b}{a}$ because $a^{-1} = \frac{1}{a}$ and $\frac{1}{a} \cdot b = \frac{b}{a}$. All the steps are reversible and therefore the solution of $ax = b$ is $\frac{b}{a}$.

In conclusion, Mr. Kantor’s knowledge of algebraic proofs is not well developed.
Mr. Kantor's Knowledge of Heuristic Arguments to Justify Definitions

Some statements accepted as definitions require of an heuristic argument to provide a rationale why the definition makes sense, at least from a pedagogical point of view. I identified the following definitions as requiring an heuristic argument to establish their reasonability: conditional probability defined as \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} \) (CCE 30), \( 0! = 1 \) (CCE 33), and the algebraic definition of division, \( a \div b = a \cdot \frac{1}{b} \) (CCE 36). He justified \( 0! = 1 \) with heuristic arguments using the fact that the number of combinations of seven objects taking seven at a time is 1 and it would also be computed using the expression \( \frac{7!}{0! \cdot 7!} \). I asked Mr. Kantor a justification about why \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} \) when \( P(B) \neq 0 \).

Some textbooks define \( P(A \mid B) \) as \( \frac{P(A \cap B)}{P(B)} \) and they provide the following heuristic argument, \( P(A \mid B) = \frac{n(A \cap B)}{n(B)} = \frac{n(A \cap B)}{n(B)} \cdot \frac{n(A \cap B) / n}{n(B) / n} = \frac{P(A \cap B)}{P(B)} \) where \( n \) denotes the cardinality of the event and \( N \) is the total number of possible equally likely outcomes.

Mr. Kantor did not provide this or any other similar argument to justify that \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} \) during the interviews. Once we know that formula then we can deduce that \( P(A \cap B) = P(B) \cdot P(A \mid B) \). We can also construct an heuristic argument to show that \( P(A \cap B) = P(B) \cdot P(A \mid B) \) and then deduce that \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} \) when \( P(B) \neq 0 \). For example, \( P(A \cap B) = \frac{n(A \cap B)}{N} = \frac{n(B)}{N} \cdot \frac{n(A \cap B)}{n(B)} = P(B) \cdot P(A \mid B) \). Mr. Kantor did not construct any argument to justify any of those formulas and then deduce the other.

Regarding the definition of division, Mr. Kantor knew that we can define \( a \div b \) as \( a \cdot \frac{1}{b} \) \((b \neq 0)\) and that in the case when \( b = 0 \), \( a \div b \) is impossible. During classroom instruction he justified the definition of division, \( a \div b = a \cdot \frac{1}{b} \), with story-problem representations. I asked him, during interview sessions, if there was another way to justify the definition besides a story problem. He said that he did not know. I was interested in finding out if
he knew other justifications such as \[ \frac{a}{b} = \frac{a}{b} \cdot \frac{1}{1} = a \cdot \frac{1}{b} \quad (b \neq 0) \] which is used

\[ \frac{a}{b} = \frac{ad}{bc} \] when \( b, c, \) and \( d \) are non zero. Another justification that

I was interested in finding if he knew was using the theorem of multiplication of fractions:

\[ a \cdot \frac{1}{b} = a \cdot \frac{1}{b} = a \cdot \frac{1}{b} \]. Still another way to justify the definition is to apply the repeated addition model of multiplication to \( a \cdot \frac{1}{b} \) when \( a \) is a whole number. In this case we get \( a \) terms of \( \frac{1}{b} \)

which equals to \( \frac{a}{b} \). That is, \( \frac{1}{b} + \frac{1}{b} + \cdots + \frac{1}{b} = \frac{a}{b} \). I judged his justification of \( a + b = a \cdot \frac{1}{b} \)

by definition as correct because the textbook and other mathematical texts define division in that way and the story-problem representations that he constructed show that he know why it is reasonable to define division in that way.

I have discussed Mr. Kantor's knowledge of each of the types of mathematical proofs. In the next section I will take another perspective and I will discuss some proofs that he had trouble with. This will give us a general feeling of his conceptions of what a proof involves and about his thinking when constructing mathematical proofs.

Mr. Kantor's Conceptions About Proofs

The proofs that Mr. Kantor provided revealed some of his conceptions about what a mathematical proofs entails. To illustrate, I will examine the proofs constructed by Mr. Kantor about the following content curriculum events: the multiplication of fractions theorem (CCE 6), rules of signs for multiplication and division (CCEs 8-11), multiplication property of \(-1\) (CCE 13), solving \( ax = b, a \neq 0 \) (CCE 19), and the second part of the multiplicative property of inequalities (CCE 26). As noted before, he almost constructed a formal proof for the theorem of multiplication of fractions, but he failed to construct a formal argument of the final step: why \( \frac{1}{b} \cdot \frac{1}{d} = \frac{1}{bd} \). Rather, he constructed a concrete and
informal justification as described above. Regarding the rules of signs, there is more than one entry in all cases except in content curriculum event ten (Table D.1). That is because I asked Mr. Kantor to give me the mathematical proof at different times as a way of giving him more opportunities to articulate his knowledge of mathematical proofs and test my emergent hypothesis that he did not have a well articulated knowledge of how to prove these curricular events. With the first representations (e.g., \((-a)(-b) = ab, (-a)(b) = -(ab)\). etc.), he attempted to construct a formal argument to prove those theorems (See Table D.1. cells 9-11). However, we notice that there are two steps that need to be justified formally: why \(-a = -1 \cdot a\) and why \(-(1) = 1\). I had asked him for the proof of \(-a = -1 \cdot a\) previously, but he was not able to construct a formal one. He tended to construct concrete examples using the number line and the repeated addition model for multiplication. For example, regarding the proof that \(-(1) = 1\) we held the following exchange:

J: Can we prove that?
K: Sure, on the number line. The opposite of something \([-1]\) is that [1]
J: Can we prove that the opposite of negative one has to be one, algebraically?
K: I don't remember how to do it. (I, 26, 05-30-95)

Even though he was working with formal proofs, Mr. Kantor was not bothered with his concrete justifications. I had to press him to construct formal proofs. That suggests that he does not have a well structured conception of what a formal proof means in mathematics. In that case, and in many other contexts, he tended to give me concrete, intuitive, or informal justifications. To further illustrate, the next excerpt helps us feel his struggles and his thinking in constructing proofs by examining his attempts to prove the multiplicative property of \(-1\):

If \(a \geq 0\), \(a(-1) = -a\)  If \(a \geq 0\), I guess you can take up the examples. If \(a\) is greater than or equal to zero \([a\geq0]\). I guess you can say that this \([-a]\) means the opposite of \(a\). There are a couple of ways I do it.
\(a-2 = a + a\)  If you look at it as repeated addition I get something like this, \(a\) times two equals \(a\) added twice \([a-2 = a + a]\)
\(a(-2) = -a - a = -2a\)  and \(a\) times a negative two \([a(-2)]\). You can really talk about subtracting a twice and that's how it looks before but you can do that. That's \([-2a]\) the opposite of this \([a-2]\)
\(a(1) = a\)  and \(a\) if this means one.
$a(-1) = -a$

We can talk about being the opposite.... This times a negative one you gonna get that. If $a$ is less than zero then the same thing is going to happen. Yeah (laughing) that doesn't work.... I just wonder whether we—if it's really by definition that we do it.... The opposite of $a$ is the same thing as I don't know if we do that by definition or ... If you look at the examples.... It didn't work this time but may be it'll work now. I don't know whether we know that by definition or not. I have to look it up. (I, 23, 01-03-95)

Regarding the case of solving $ax = b$, he showed a mathematical proof of the solution (cell 19, Table D.1). We notice that he did not justify every step of the algebraic manipulation. I accepted that proof as correct because other data suggested that he knew these justifications.

Regarding the proof when we multiply both sides of an inequality by a negative number, I asked Mr. Kantor several cases. The first case is a particular case (If $a < b$ then $-a > -b$) and a proof is given in the section Perspectives Chapter 4 of the teachers' edition of the textbook as follows: Given $a < b$. Add $-a$ to each side: $-a + a < -a + b$, $0 < -a + b$. Add $-b$ to each side: $0 < -a + b + -b$, $-b < -a$, $-a > -b$. With this theorem and with the part 1 of the multiplicative property of inequalities we can prove the second part in the way Mr. Kantor attempted to do it as shown below:

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; y$ then $-x &gt; -y$</td>
<td>$-ax &gt; -ay$</td>
</tr>
<tr>
<td>I don't know whether that's a definition. That this is the case. If that is a definition.</td>
<td>and multiply this $[-x &gt; -y]$ by positive $a$ then we maintain the relationship</td>
</tr>
<tr>
<td>$-ax &lt; -ay$</td>
<td>$a'x &gt; a'y$</td>
</tr>
<tr>
<td>$a'x &gt; a'y$</td>
<td>And since this is positive $a$ is the same thing as saying OK. that's $[a']$ a negative</td>
</tr>
</tbody>
</table>

We notice that he admitted that he did not know how to prove those properties ($a < b$ implies $-a > -b$ and $ac < bc$ if $c$ is positive). Because he failed to provide a mathematical justification for the critical steps, I judged Mr. Kantor's proof about the second part of the multiplicative property of inequalities as incorrect. When I asked him again, in a later interview, to construct a proof for the second part of the inequality he said that we can use a
number line to prove it. Again, his answer suggests that he did not have a well developed conception of what a formal mathematical proof entails.

We have seen that Mr. Kantor does not have a well developed knowledge about what mathematical statements are accepted as axioms and what statements are definitions and about what a proof involves as well as how to prove mathematical theorems. As a summary, Mr. Kantor provided 11 (31%) correct proofs, 10 (29%) partially correct proofs, and 14 (40%) incorrect proofs of 35 curriculum events. Mr. Kantor's knowledge of formal mathematical proofs was judged to be very weak because the index of correctness of his mathematical proofs was .46. Thus, we can conclude that, in general, Mr. Kantor's knowledge of mathematical proofs is underdeveloped.

**Mr. Kantor's Knowledge of Pictorial Representations**

I asked Mr. Kantor to construct pictorial representations for 35 of the 41 content curriculum events. Table E.1 in Appendix E displays all the pictorial representations constructed by Mr. Kantor and his thinking behind the representations. I did not ask Mr. Kantor the pictorial representation for the following six curriculum events: volume of a rectangular solid (CCE 4), reciprocal of zero (CCE 16), greater than (CCE 24), solving $ax < b$ (CCE 27), $n!$ (CCE 32), 0! (CCE 33). I did not ask Mr. Kantor to construct a pictorial representation about $a > b$ (greater than) because he had already mentioned in class that $a > b$ means that $a$ is to the right of $b$ on a number line and he said something similar for the multiplication property of inequality during interview sessions (See content curriculum event 25, the first part of the multiplicative property of inequalities in Table E.1). For content curriculum event solving $ax < b$, I just overlooked it in asking him, but he provided a pictorial representation in class. The pictorial representations for these two CCEs (definition of $a > b$, and solving $ax < b$) have been entered in the corresponding cells of Table E.1. Because I was particularly interested in Mr. Kantor's knowledge of pictorial representations about conditional probability (CCEs 29c, 30 and 31), I have counted CCE
29c, the classical definition of conditional probability, as another pictorial representation. Thus, Mr. Kantor's knowledge of pictorial representations was examined for 38 CCEs.

For the following three of the 35 content curriculum events I asked Mr. Kantor to construct more than one pictorial representation: definition of probability (CCE 29a, 29b, 29c), meaning of division (CCE 35) and definition of division (CCE 36). For the definition of probability, I asked Mr. Kantor to construct three pictorial representations each one illustrating a different case: the probability of an event, the probability of the intersection of two events, and the probability of an event given that another event has already happened (the classical definition of conditional probability). For each of the other two content curriculum events (CCEs 35 and 36), I asked Mr. Kantor to construct five pictorial representations. I wanted to explore Mr. Kantor's knowledge of pictorial representations for these two curriculum events using a variety of examples (e.g., whole numbers, whole numbers and fractions, and fractions).

I categorized each of Mr. Kantor's pictorial representations as correct, partially correct, and incorrect. For the CCEs for which I asked Mr. Kantor more than one pictorial representation I averaged and rounded the correctness. I also categorized them as explicit, partially explicit, and implicit. A pictorial representation, by its own nature, might not represent explicitly the content curriculum event. An explanation of why a picture represents an intended content curriculum event may be needed for teaching purposes. The criteria used to categorized Mr. Kantor's pictorial representations is described in Chapter 4, section "analysis," "Coding schema for Mr. Kantor's knowledge of pictorial representations." Table E.2 displays a summary of Mr. Kantor's categorization of pictorial representations according to the level of explicitness and degree of correctness. I will provide examples of each of these categories to illustrate Mr. Kantor's knowledge of pictorial representations.
Mr. Kantor's Correct Pictorial Representations

Mr. Kantor constructed 32 correct pictorial representations. To illustrate some correct pictorial representations constructed by Mr. Kantor, I will use several pictorial representation. Figure 5.1 displays the pictorial representation that he constructed for showing that the solution of $ax = b$ is $b \over a$ (CCE 19). His thinking was as follows, "$a$ times $x$—the area of that thing is $b$, and if we cut $b$ up into $a$ equal parts, that's what each of the parts is.... So that end [the block] is something that's one by $x$; so that's the same as $x$ times one which is $x$.

\[
\begin{align*}
\text{Area } b &= x \cdot 1 = x \\
\frac{b}{a} &= x \\
\end{align*}
\]

Figure 5.1 A pictorial representation to show how to solve $ax = b$

As another example, Mr. Kantor constructed the pictorial representations displayed in Figure 5.2 for illustrating $\frac{2}{3} + \frac{1}{2}$. His thinking was as follows,

That's two thirds and that's one half. Bring this down over here, so half goes into there once [and you have one left over]. So, once you do that, you try to see how many halves you have. You have one half there and you have a third of a half, so you have one and a third. One and a third halves.... That's how many times a half goes into two thirds. It's gonna take two thirds and make one and a third. You make one and a third halves out of it. (I, 29, 06-08-95)
Another example relates to represent why \(5 + \frac{1}{2} = 5(2)\) (CCE 36b). Mr. Kantor used the partitive model for division: we want to divide 5 objects into \(\frac{1}{2}\) groups and \(5 + \frac{1}{2}\) represents the number of objects per group (Figure 5.3). Figure 5.3b was meant initially to represent \(5 + \frac{1}{2}\) and Figure 5.3c was meant to represent \(5(2)\). Since those two figures have the same number of elements then \(5 + \frac{1}{2} = 5(2)\). I asked Mr. Kantor to make explicit the connection of why \(5 + \frac{1}{2} = 5(2)\). He then pointed to Figure 5.3b and said "if five is half of the group ... you double what you have \([5(2)]\) and get the whole group." We see that in this case Mr. Kantor used the same figure to represent both \(5 + \frac{1}{2}\) and \(5(2)\). Therefore, it is clear to me why \(5 + \frac{1}{2} = 5(2)\) using a pictorial representation without computing the answer for both \(5 + \frac{1}{2}\) and \(5(2)\).

One representation that I considered correct but partially explicit was the case of \(7 \div 2 = 7 \cdot \frac{1}{2}\) (CCE 36a) (Figure 5.4). He used the partitive model of division: a collection of seven objects is partitioned into two equal sized groups. Seven divided by two represents the number of objects per group. What is circled in Figure 5.4a represents the pictorial solution to \(7 \div 2\). Each group has three and a half objects. To represent \(7 \cdot \frac{1}{2}\), Mr. Kantor used the of model for multiplication in connection to division: Seven divided by two is one half of seven. That is, the number of elements in each group represents \(7 \div 2\) and also \(\frac{1}{2} \cdot 7\).
Figure 5.3 Pictorial representations about why $5 + \frac{1}{2} = 5(2)$ or $7 - \frac{1}{2}$.

Figure 5.4b represents $7 - \frac{1}{2}$ according to this interpretation. Figure 5.4c is an elaboration of 5.4b using the repeated addition model for multiplication. It represents $7 - \frac{1}{2}$ because it means taking "seven copies of a half and add them together [and] that's three and a half." Since each group in Figures 5.4a, 5.4b, and 5.4c has the same number of objects, three and a half, Mr. Kantor concluded that $7 + 2 = 7 - \frac{1}{2}$.

Figure 5.4 Pictorial representations for $\frac{7}{2} = 7 - \frac{1}{2}$
From my point of view, the pictorial representations are not making totally explicit why \(7 \div 2 = 7 \cdot \frac{1}{2}\) without counting the number of objects per group and because he used different interpretations of division and different pictures for the same situation. A representation such as the following would be more explicit as to why \(7 \div 2 = 7 \cdot \frac{1}{2}\):

We notice that those representations are essentially the same. That is the point of my representation: We can interpret the same figure as representing \(7 \div 2\) or \(7 \cdot \frac{1}{2}\) thus concluding that \(7 \div 2 = 7 \cdot \frac{1}{2}\). In the first representation I am dividing a number of objects into two groups. Either the upper part or lower part of the division represents \(7 \div 2\). Another way we can divide a number of objects into a certain number of groups is represented in the right figure: We divide each object into the same number of groups. Then half of each divided circle represents \(\frac{1}{2}\). Since we have 7 objects then \(7 \cdot \frac{1}{2}\) represents the number of objects per group. Notice that I am using essentially the same picture and the same interpretation or model for division: dividing a certain number of objects into a specific number of groups.

Another correct and partially explicit pictorial representation constructed by Mr. Kantor was for the associative property of multiplication (CCE 5). This representation is displayed in Figure 5.5. Mr. Kantor's thinking was as follows:

Each of these is a row.... Call this length, the width, the height. You think as if we put crates in, crates across the bottom of it. You have so many in a length, you have so many widths so you have \(l\) times \(w\) crates on the bottom and they are stacked \(h\) high so \(l\) times \(w\) times \(h\); that gives you the volume of the room. For this one you can say, OK, let's make a wall of crates and you know it's \(h\) high; let's do it this way, the bottom has, the width of the room has this crates and you have \(h\) of those rows high, now you just take all those rows down the length of the room so you have \(l\) of them. You have the same volume, it's the same room. (I, 33, 08-25-95)
I judged this representation partially explicit because Mr. Kantor did not make the connection as to why the number of crates (volume of the box) is \((lw)h\) or \(l(wh)\) by using repeated addition. He knew why that is as evidenced in the proof he constructed about the number of elements in an array (see curriculum event 3, Table D.1).

\[
\text{lw crates bottom} \\
\text{stacked } h \text{ height} \\
\]

\[(lw)h = l(wh)\]

Figure 5.5 The associative property of multiplication

Regarding the pictorial representation of why \(\frac{2}{3} + \frac{1}{2} = \frac{2}{3} \cdot 2\), Mr. Kantor was not very explicit initially about why Figure 4 in cell 36d, Table E.1 (Appendix E) also represented \(\frac{2}{3} \cdot 2\). After some probing he finally said "this is half of a group, you double to get the whole group" and he pointed to \(\frac{2}{3} \cdot 2\).

Making Mr. Kantor be explicit about why Figure 5c (cell 36e, Table E.1) also represented \(\frac{1}{2} \cdot \frac{3}{2}\) was troublesome as we can see from the speeches entered in that cell. After several attempts he finally said something similar to the fact that the number of \(\frac{1}{2}\)'s illustrated in Figure 5c is \(\frac{3}{2}\) and therefore \(\frac{1}{2} + \frac{2}{3} = \frac{1}{2} \cdot \frac{3}{2}\).
I would like to illustrate Mr. Kantor’s knowledge of pictorial representations with another example, division by zero. Because of the complexity of this pictorial representation I will discuss it to give the trial audit an opportunity to see my interpretation.

This pictorial representation is illustrated in Figure 5.6. His thinking was as follows,

So you divide by a tenth to get a big number. You divide the same thing by a hundredth to get a bigger number. If this keeps getting smaller, eventually you get bigger and bigger, [no matter] how close you get to zero you still [keep getting] bigger so … infinity, divided by zero equals infinity, but I can’t represent any bigger than that. I guess, eventually you start, you start getting down to where these things, these things are rectangles … lines, they have no thickness, they have no thickness, and they can’t fill the space. The rectangles, they can fill the space, [but once they are not] they cannot fill the space. (I, 33, 08-25-95)

---

Mr. Kantor’s pictorial representation corresponding to division by zero suggests the mathematical fact that \( \lim_{b \to 0} \frac{a}{b} = \infty \) when \( b \) approaches zero and \( a \neq 0 \). As long as \( b \neq 0 \) we can divide \( a \), represented as a rectangle with sides 1 and \( a \), by \( b \), represented by a rectangle with sides 1 and \( b \). \( \frac{a}{b} \) represents the number of smaller rectangles needed to fill the larger rectangle. This process can be done as long as \( b \neq 0 \). As \( b \) approaches 0 we need a larger and larger number of such rectangles to fill the larger rectangle. Once \( b = 0 \) the rectangle with sides 1 and \( b \) becomes a line which does not have thickness and therefore cannot fill
the larger rectangle. (Actually, we need an infinite number of lines to fill the rectangle.) Therefore we cannot divide a by zero, that is, division by zero is impossible. Then the pictorial representation suggests why we cannot divide by zero.

The pictorial representation that Mr. Kantor constructed to illustrate why \( 21 + \frac{3}{4} = 21 \cdot \frac{4}{3} \) is particularly illuminating because it shows the complexities of attempting research participants to be explicit about their knowledge. In this case Mr. Kantor was using the same representation (Figure 5.7b) when talking about \( 21 + \frac{3}{4} \) and \( 21 \cdot \frac{4}{3} \) but I wanted him to tell me explicitly why he was thinking that Figure 5.7b also represented \( 21 \cdot \frac{4}{3} \). I requested Mr. Kantor three times to be explicit as to why Figure 5.7b represented \( 21 \cdot \frac{4}{3} \).

The first time his thinking was as follows:

Twenty one is three fourths of a group. Well, it's seven, seven. It's twenty one, and it's three fourths of a group [Figure 5.7a]. Add another row of sevens in. That's 28. [The first interpretation will be] If twenty one is three fourths of a group, the whole group ends up [having] to add another fourth of that group to it [Figure 5.7b]. This is three fourths of a group and this is gonna be a fourth of a group. [So in this case you multiply 21 times by] Fourth thirds. (I, 29. 06-08-95)

![Figure 5.7 A pictorial representation for \( 21 + \frac{3}{4} = 21 \cdot \frac{4}{3} \)](image)
I did not consider this explanation as totally explicit about why figure 5.7b represents also 21\(\cdot\frac{4}{3}\). So I asked him again and his response was:

Twenty one is three fourths of a group. Find out what the group is \([21 + \frac{3}{4}]\). The way to do it \\
\([21 \cdot \frac{4}{3}]\) is, you cut what you have into thirds ... three equal parts. What you have is three fourths of a 
group, that's three equal parts \([\frac{3}{3}]\). Take one more equal part \([\frac{1}{3}]\), so you have four equal parts and 
that's what the whole group is [that's what four thirds is] \([21 \cdot \frac{4}{3}]\). [It's 21 times four thirds] because 
the three divides it into three parts, divides what you have into three parts, and the four for the full 
collection I guess you can say. (I, 30, 06-16-95)

I wanted him to be more explicit and I asked him once more to explain why Figure 
5.7b represents 21\(\cdot\frac{4}{3}\). He said:

Well, if this is, if twenty one is three out of four parts of a group, then the twenty one, you have 
three groups, three groups of threes \([\frac{3}{3}]\) out of twenty one and we need one more group \([\frac{1}{3}]\) to get a 
whole, so it's four, four thirds times twenty one. (I, 31, 06-19-95)

I think that Mr. Kantor's thinking in this speech is more explicit about that connection.

Figure 5.7b represents 21\(\cdot\frac{3}{4}\) because it represents the number of objects per group when 
we want to divide 21 objects in \(\frac{3}{4}\) groups. This figure also represents 21\(\cdot\frac{4}{3}\) because it can 
be read as the number of 21s that we have: the number of 21s that we have was \(\frac{3}{3}\) (or 1) 
and the number of 21s that we add was \(\frac{1}{3}\). Thus the total number of 21s is \(\frac{4}{3}\), that is, the 
number of objects per group is \(\frac{4}{3}\cdot21\) or 21\(\cdot\frac{4}{3}\).

The pictorial representation for \(P(A \cap B) = P(A)P(B | A)^3\) (CCE 31) is another 
example that illustrates the complexity of teachers' knowledge and some difficulties that I 
faced in trying to ask Mr. Kantor to explain explicitly his pictorial representations. This is 
described below.

\(^3\)When Mr. Kantor and I were discussing events related to conditional probability we used both formulas 
\(P(A \cap B) = P(A)-P(B \mid A)\) and \(P(A \cap B) = P(B)-P(A \mid B)\).
Mr. Kantor's pictorial representation for $P(A \cap B) = P(A)P(B \mid A)$ was judged as correct but partially explicit. Figure 5.8 displays the pictorial representations constructed by Mr. Kantor, during the interviews, to illustrate the formula $P(A \cap B) = P(B)P(A \mid B)$. These data complement the data from class collected on videotapes. From an initial analysis of his videotaped teaching, it was clear to me that Mr. Kantor knew some pictorial representations to illustrate that formula but I wanted to know if he knew more related representations. He constructed the representation displayed in Figure 5.8a. This representation was constructed when I asked him to construct a representation to illustrate $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$. He said that he could represent $P(A \cap B) = P(B)P(A \mid B)$ instead.

His thinking was as follows:

If you do it the other way, where it is probability of A given B times probability of B gives you this [$P(A \cap B)$], then I can represent that with a rectangle. Over here is the probability of not B [Area AGHD] and this is the probability of B happening [Area GBCH]. This is the probability of B happening; so over here, that's the probability of A given that B has already happened [BF]. Let's see. Probability of A given B. This inside is the probability of the intersection [Area GBFE = $P(A \cap B)$]. That's the probability of not A given that B has happened [CF]. If you take that [$P(A \cap B)$] and divide it by this [$P(B)$]... let's see, A and B, that will give you this [$P(A \mid B)$]. That's the probability of B [GB], right? You take this area [GBFE] and you divide it by probability of B [BG], and you get the probability of A given B [BF]. (1, 25, 01-10-95)

Figure 5.8 Pictorial representations for $P(A \cap B) = P(B)P(A \mid B)$
The representation constructed in Figure 5.8a is particularly interesting because it indeed represents $P(A \cap B) = P(B)P(A \mid B)$ if several assumptions hold (e.g., $AD = 1$, $AB = 1$). Under these assumptions, it is clear to me that the areas associated with the events $B$, not $B$, and $A \cap B$ represent, respectively, $P(B)$, $P(\text{not } B)$ and $P(A \cap B)$ (The total area is 1). However, it is not very clear why $CF$ represents $P(A' \mid B)$, $FB$ represents $P(A \mid B)$ and $BG$ represents $P(B)$ and whether Mr. Kantor knew, explicitly, the assumptions he made implicitly and whether he knew why the segments represent the probabilities. We can use two arguments about why $GB = P(B)$. The first argument is as follows: $P(B) = \frac{\text{Area} \ (GBCH)}{1} = \text{Area} \ (GBCH) = GB \cdot BC = GB \cdot 1 = GB$. The second argument is as follows. Since $AB = CD = 1$, we have that $P(B) = \frac{\text{Area} \ (GBCH)}{\text{Area} \ (ABCD)} = \frac{GB \cdot BC}{AB \cdot BC} = \frac{GB}{AB} = \frac{GB}{1} = GB$. However, only the following argument, or an equivalent, is valid to show why $BF$ represents $P(A \mid B)$: $P(A \mid B) = \frac{\text{Area} \ (GBFE)}{\text{Area} \ (GBCH)} = \frac{GB \cdot BF}{GB \cdot BC} = \frac{BF}{BC} = \frac{BF}{1} = BF$.

Another acceptable argument is the following: since we are assuming that the event $B$ happened and $BC = 1$, then $BF$ represents the "proportion" of the event $B$ associated with event $A$, that is, $P(A \mid B) = \frac{BF}{BC} = BF$. Neither of these or another equivalent argument was provided by Mr. Kantor to justify why $P(A \mid B) = BF$. Once we know that $P(A \cap B) = P(B)P(A \mid B)$ then it follows, algebraically, that $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$. This is probably what Mr. Kantor had in mind when he said "You take this area [GBFE or $P(A \cap B)$] and you divide it by probability of $B$ [BG], and you get the probability of $A$ given $B$ [BF]." Since during his instruction Mr. Kantor was more explicit about the meaning of the representations used, I thought that he probably knew why $P(A \cap B) = P(B)P(A \mid B)$. That is, why $P(A \mid B) = BF$. To check my hypothesis, I asked him, in a later interview, to construct a pictorial representation that illustrates that formula and to explain why the representation illustrated the formula. Figure 5.8b illustrates the second representation that
he provided. We changed the notation of the events. Figure 5.8a represents \( P(A \cap B) = P(B) - P(A \mid B) \) and Figure 5.8b represents \( P(A \cap B) = P(A) \cdot P(B \mid A) \). His thinking when constructing the representation depicted in Figure 5.8b was as follows:

This will be probability of \( A \) [ABCD]. This is the probability of not \( A \) [BGHC] ... and these are not necessarily the same areas. So this is the probability of \( A \) and \( B \) happen [EFCD].... This is probability of \( B \) given \( A \) [DE].... The probability of \( A \) and \( B \) equals that [EFCD]. This is the probability of \( A \) but not \( B \). This is the probability of \( B \) but not \( A \). This is the probability of not \( A \) not \( B \). (I, 28, 06-06-95)

At this point I asked him why \( DE \) represents \( P(B \mid A) \) and he said:

because if you just isolate this, if I just look at, this represents that \( A \) has happened [ABCD] ... and some of the time there is a probability that \( B \) will happen [EFCD] and that's the probability of \( B \) [DE] there but it's given that \( A \) has happened. (I, 28, 06-06-95)

We notice that some understanding about why \( P(B \mid A) \) can be represented by \( DE \) is being explicit by Mr. Kantor's explanation. However, because Mr. Kantor did not provide a more explicit explanation about why the pictorial representations illustrated the conditional probability formula, \( P(A \cap B) = P(A) \cdot P(B \mid A) \), I categorized the representation as partially explicit.

**Mr. Kantor's Partially Correct Pictorial Representations**

I judged Mr. Kantor's pictorial representations for the following four content curriculum events as partially correct: area model for multiplication (continuous case) (CCE 1), the product of a negative number and a positive number is a negative number (CCE 11), probability of an event (\( P(A) \), \( P(A \text{ and } B) \)) (CCE 29a and 29b), and the second definition of conditional probability (CCE 30). I will begin to illustrate Mr. Kantor's partially correct representations with probability of an event. I judged these pictorial representations as partially correct because Mr. Kantor did not make explicit the difference between an event and its probability. This is discussed below.

**Probability of an event \( A \), \( P(A) \).** Mr. Kantor constructed the pictorial representations displayed in Figure 5.9 for illustrating the meaning of probability of an event. For Figure 5.9a his thinking was as follows,
If these are all the things that are possible; and then you have some section, let's say here. That it's gonna be the probability of A, all of those things that are possible where A occurs and, so this is success and this is possible. (I, 28, 06-06-95)

\[ P(A) = \frac{\text{Area A}}{\text{Total area}} \]

(a) (b)

Figure 5.9 Pictorial representations for \( P(A) \)

It is not clear to me whether Mr. Kantor is attempting to construct the so called classical definition of probability, \( P(A) = \frac{n(A)}{N} \), where \( n(A) \) represents the number of outcomes in A and \( N \) is the number of equally likely outcomes of the sample space or whether he is using the definition of probability in the case of geometric regions, \( P(A) = \frac{m(A)}{m(S)} \), where \( m(A) \) is the measure of the region representing event A and \( m(S) \) is the measure of the region representing the sample space. The measure can be area, length, etc.

I think that he is mixing up those two types of probabilities. Another thing that I notice is that he is not using the appropriate representation of the probability of the event A. The probability of the event A is the ratio of the area representing the event A (or the number of successes) and the total area (or the number of possible equal likely outcomes). Strictly speaking, he is taking the area representing the event A as the probability of A in Figure 5.9a and that is not correct, mathematically speaking. This was in contrast with his use of
the term when explaining probability. This is a good time to mention a similar phenomenon that occurred when he attempted to construct a story problem for the second multiplicative property of inequalities (see Appendix I). In both of these cases Mr. Kantor was not very precise about the vocabulary used to describe the representations. I wanted to explore this phenomenon more, and I asked him again, in a later interview, to construct another pictorial representation. He constructed the representation illustrated in Figure 5.9b. The thinking behind that representation was as follows:

Divide it into as many things as possible.... Let's say all those, all those things are possible and if you are making the same size, which I try to do, then they will be equally likely.... So let's say two of them are A. [The event A consists of the regions labeled A]. These are all the things that are possible ... then the probability of A is the area of A over the total area. (I, 33, 08-25-95)

In this case we notice that this pictorial representation and the corresponding verbal or symbolic representation match much better but there are still some elements of vagueness. He is using the same label, A, to denote the event, the area of each of the five regions, and the area of two of the regions. That is somewhat confusing. I think that we need to be careful about the language that we use to represent mathematical objects. I am not advocating using a very precise mathematical language but using appropriate notation to represent mathematical knowledge. Those intuitive definitions could be clearer if Mr. Kantor would use more appropriate notation. As a tentative conclusion, I would say that Mr. Kantor's explicit pictorial representations about probability of an event are not totally correct. The next curriculum event, P(A ∩ B), will provide more opportunities to examine his knowledge of pictorial representations about probability of an event.

**Probability of the intersection of two events, P(A ∩ B).** Figures 5.10a and 5.10b display the pictorial representations constructed by Mr. Kantor about P(A ∩ B). The thinking behind the pictorial representation in Figure 5.10a was as follows:

A Venn diagram. The probability that A happens is this area [pointing to region A], the probability that B happens is that area [pointing to region B].... The total area are all the things that are possible so the shaded area compared to the total area [pointing to both regions] is the probability that A and B happen. (I, 31, 06-19-95)
In this case we notice that his representations are not totally appropriate. His is taking both $P(A)$ and $P(B)$ as the corresponding areas, which can be done if we take the total area as 1. (This can be done by renaming the unit used for area.) However, when he is talking about $P(A \cap B)$, he is not taking the region representing $A \cap B$ as $P(A \cap B)$. I asked him for another representation to see what his thinking would be. The representation is illustrated in Figure 5.10b and his thinking was as follows:

Two events, A and B. That's all the things possible [pointing to the rectangle], so the probability of A and B is all this area [pointing to the intersection of A and B]. (I, 33, 08-25-95)

Strictly speaking, $P(A \cap B) = \frac{\text{Area } (A \cap B)}{\text{Total area}}$. We notice again that Mr. Kantor is not being explicit about the fact that the region representing the event $A \cap B$ can be taken as $P(A \cap B)$ by changing the units used to measure area. Again, these representations support the earlier conclusion that Mr. Kantor's knowledge of pictorial representations about probability of an event is not very explicit about their appropriateness since he did not mention that by making the total area 1, $P(A \cap B) = \text{Area } (A \cap B)$. 

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It is also interesting to describe and analyze the difficulties that I had when asking Mr. Kantor to be explicit about some pictorial representations or the difficulties that he faced when constructing partially correct representations. There is only one case in which he struggled to construct a pictorial representation and was partially successful after several attempts: the case of the second definition of conditional probability \( P(A | B) = \frac{P(A \cap B)}{P(B)} \). The description and analysis is related in Appendix J.

**Mr. Kantor's Incorrect Pictorial Representations**

A pictorial representation that I judged incorrect was for the content curriculum event that the quotient of a positive number and a negative number is a negative number (QPNN) (CCE 41). Mr. Kantor constructed the pictorial representation depicted in Figure 5.11 to illustrate QPNN. His thinking was as follows:

I can do the same thing. If you are walking, if I walk 3 hours, I walk a hundred and fifty steps positive, where was I three hours ago? Here is where I am now, present time, where was I three hours ago? ... Three hours ago. So I lost fifty paces per hour. (L. 33, 08-25-95)

![Figure 5.11 Pictorial representation for QPNN](image)

I must mention that I do not know a good representation that illustrates the rule of signs pictorially. Mr. Kantor's pictorial representation involves the concept of time. Time cannot be represented pictorially. However, some representations involving time can represent the intended mathematical idea indirectly. Mathematical ideas are abstract and creating good pictures to represent them may be a difficult, if not an impossible, task. I
will discuss Mr. Kantor's representations about the rule of signs in the case of division so readers can assess my judgment of Mr. Kantor's pictorial representation for QPNN. While I consider Mr. Kantor's pictorial representation for "the quotient of a positive number and a positive number is positive" (Appendix E, CCE 38) to be straightforward because it is the one we encounter commonly in real life, the representation for "the quotient of a negative number and a negative number is a positive number" needs a little more interpretation: some people are walking and we are filming, then we back up the film two hours \([-2]\) and they are now 100 steps behind \([-100]\) with respect to where they are now \([0]\). Then when we divide the distance \(-100\) by the time \(-2\) we get that they are walking 50 steps per hour which is positive. The picture representing a negative number divided by a positive number is also easy to interpret: a distance in the negative direction \([-100\text{ miles}]\) divided by the time it took to travel it \([5\text{ days}]\) gives us the rate of traveling, \(-20\text{ miles per day}\). The representation for the content curriculum event "the quotient of a positive number and a negative number is negative" is the most problematic to me to interpret. One interpretation seems to be that at \(t = 0, d = 150\) and when \(t = -3, d = 0\). In this case we have that we have been walking 50 steps per hour because when we go back three hours \([-3]\) we lost 150 \([-150]\) then \(-\frac{150}{-3} = 50\) or we walked 150 steps \([d = 150]\) in three hours \([t = 3]\) and so we have \(-\frac{150}{3} = 50\). This interpretation is not illustrating that a positive number divided by a negative number is negative. Another possible interpretation to me is that when \(t = 0, d = 0\) and when \(t = -3, d = 150\). In this case we can have \(\frac{150}{-3}\) which is equivalent, pictorially, to \(-\frac{150}{3}\). I am not sure which interpretation Mr. Kantor was making explicit.

Based on the analysis of the pictorial representations constructed by Mr. Kantor and on the previous discussions, I can conclude that Mr. Kantor's knowledge of pictorial representations about curriculum events related to algebraic multiplication is strong. Even though he did not know the pictorial representations for \(P(A \cap B) = \frac{P(A \cap B)}{P(B)}\) he knew
most of the pictorial representations of the main curriculum events. The passages illustrate that he learned that representation during our exchanges. As a summary, I found that Mr. Kantor was able to construct 32 (about 84%) correct representations, four (about 11%) partially correct representations, and two (about 5%) incorrect representations. Mr. Kantor's knowledge of pictorial representations was judged to be strong because the index of correctness of his pictorial representations was .89.

**Mr. Kantor's Knowledge of Story-Problem Representations**

Table F.1 of Appendix F displays all the story-problem representations constructed by Mr. Kantor for 38 of 41 content curriculum events. As was the case for pictorial representations, for some curriculum events I asked Mr. Kantor several story-problem representations. The representations for the following three content curriculum events were judged partially correct: area model for multiplication (continuous case) (CCE 1), multiplicative property of equality (CCE 18), and solving $-x = a$ (CCE 23). The story-problem representations for the remaining 35 CCEs were judged correct. Thus, Mr. Kantor's knowledge of story-problem representations can be described as very strong since the index of correctness of his story-problem representations was .96.

To illustrate in detail his knowledge of story-problem representations, the following four content curriculum events were selected for description: solving $ax = b$, solving $ax < b$, the multiplication counting principle, and the algebraic definition of division.

Mr. Kantor constructed, among others, the following story problem to represent solving $ax = b$: "Forty bucks a radio. How many can you buy for six hundred dollars? [40R = 600]." I see this representation as pedagogically powerful for helping students to construct the connection between the symbolic representation of solving $40R = 600$, and equations of the form $ax = b$ in general, and the meaning of division. The answer to the problem provided by Mr. Kantor can be represented by the equation $40R = 600$ and the division $600 ÷ 40$. Therefore the solution to that equation is $600 ÷ 40$. While the equation
40R = 600 represents the missing-factor model of division, the quotient 600 ÷ 40 models the repeated-subtraction model of division. Mr. Kantor constructed several story problems whose solution can be represented by an inequality of the form \( ax < b \). One of them was as follows "Let's say that you have three crates, of equal size and content and they hold less than a thousand baseballs. How many are in each crate? \([3x < 1000]\)." Mr. Kantor illustrated the multiplication counting principle with a story problem in following terms

"You can set up things ... a hundred and something [items] in a box, and you put them in a van. You have 20 boxes in a van and you got 15 vans, how many of those items do you have?" Mr. Kantor constructed several story problems to represent why \( a + b \) is equal to \( \frac{a}{b} \). He constructed the following problem to illustrate why \( 5 + \frac{1}{2} = 5\frac{1}{2} \): "You have five dollars, how many bags of fifty cents potato chips could you buy? That \([5 + \frac{1}{2}]\) tells you ten bags. [This means] you can buy two bags per dollar, so two times five.... Each dollar gives you two bags, [and you have] five dollars, [so it is] five times two."

In general, Mr. Kantor was explicit about the story-problem representations. An exception is the case of showing why \( \frac{1}{2} + \frac{2}{3} = \frac{1}{2} \cdot \frac{3}{2} \) (See content curriculum event 36e, Table F.1 Appendix F). The first time I asked Mr. Kantor this representation he was explicit but he said two instead of one (situation i). The following day I asked him about why it was two but he was not able to articulate why. Rather, he represented why \( \frac{1}{2} + \frac{2}{3} = 3 \cdot \frac{1}{4} = \frac{3}{4} \) (situation ii). I asked him to be explicit about the meaning of \( \frac{1}{2} : \frac{3}{2} \) using the same problem but he preferred to change the situation as illustrated in problem iii.

However, he represented why \( \frac{1}{2} + \frac{2}{3} = 3 \cdot \frac{1}{4} = \frac{3}{4} \). Situation iv almost represented why \( \frac{1}{2} + \frac{2}{3} = \frac{1}{2} \cdot \frac{3}{2} \) but Mr. Kantor focused on the computational equivalence \( \frac{3}{4} \) rather than on the meaning of \( \frac{1}{2} \cdot \frac{3}{2} \).
As a general conclusion to Mr. Kantor's knowledge of representations, I can say that his knowledge of story-problem representations was very strong, his knowledge of pictorial representations and his knowledge of symbolic representations was strong, and his knowledge of proofs was very weak. In the next section I will present the results about Mr. Kantor's use of his knowledge of representations when constructing pedagogical events during classroom instruction.

Mr. Kantor's Use of His Knowledge of Representations

The purpose of this section is to provide answers to research question 5: What representations does Mr. Kantor use? This question is answered using five types of representations (categories). Four types of representations were determined from the theoretical framework: (a) symbolic representations, (b) mathematical proofs, (c) pictorial representations, and (d) story-problem representations. The other type of representation emerged from the data: numerical representations. Because Mr. Kantor used numerical representations instead of symbolic representations in most of the cases, I will discuss them under the section numerical and symbolic representations. Tables G.1 through G.5 contain representations constructed by Mr. Kantor when teaching each of the lessons related to algebraic multiplication. In this section I analyze Mr. Kantor's explanations in terms of the representations he used when teaching algebraic multiplication.

Mr. Kantor's Use of His Knowledge of Numerical and Symbolic Representations

Table 5.2 displays the content curriculum events for which Mr. Kantor constructed numerical or symbolic representations in three contexts: (a) when teaching the meaning of the content curriculum events, (b) when using the content curriculum event for solving problems and (c) during review sessions. The representations are displayed in tables in
Appendix G. The numbers entered in Table 5.2 indicate the number of the content curriculum event as displayed in Table 5.1.

<table>
<thead>
<tr>
<th>No representations constructed at all</th>
<th>Content curriculum events for which meaning was taught</th>
<th>Content curriculum events used for solving problems</th>
<th>Content curriculum events used during review</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1, 2, 4, 5, 8, 9, 10, 11, 13, 17, 18, 20, 21, 22, 23, 30, 33, 37, 38, 39, 40, 41 (22)</td>
<td>1, 2, 5, 8, 9, 10, 11, 12, 13, 14, 18, 20, 21, 22, 23, 24, 27, 29, 30, 33, 37, 38, 39, 40, 41 (25)</td>
<td>3, 4, 8, 9, 10, 11, 12, 13, 16, 17, 18, 24, 25, 26, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41 (27)</td>
</tr>
<tr>
<td>Numerical representations</td>
<td>3, 6, 7, 14, 15, 19, 24, 25, 26, 27, 28, 29c, 31, 32, 34, 35, 36 (17)</td>
<td>3, 4, 7, 17, 19, 28, 31, 32, 34, 35 (10)</td>
<td>6, 7, 14, 19, 20, 21, 22, 23, 27, 31 (10)</td>
</tr>
<tr>
<td>Numerical-symbolic representations</td>
<td>7, 14, 16 (3)</td>
<td>4, 6, 15, 16, 25, 26 (6)</td>
<td>1, 2, 6, 15 (4)</td>
</tr>
<tr>
<td>Symbolic representations</td>
<td>12, 15, 36 (3)</td>
<td>36 (1)</td>
<td>5, 14 (2)</td>
</tr>
</tbody>
</table>

Table 5.2 Content curriculum events and types of representations constructed by Mr. Kantor during classroom instruction

I will divide this section into three parts. The first part deals with the representations that Mr. Kantor constructed for teaching the meaning of specific content curriculum events. This is the part of the lesson when the mathematical content curriculum events are introduced and explanations, representations, and questions are construct for teaching the meaning of the CCEs. The second part deals with the representations that Mr. Kantor constructed when solving problems using some of the content curriculum events displayed in Table 5.1. The objective of this part of the lesson is to use the CCEs for solving
problems or exercises. The third section deals with content curriculum events for which representations were constructed during review sessions.

**Teaching the meaning of the content curriculum events.** As indicated on Table 5.2, Mr. Kantor constructed numerical representations for the following 17 content curriculum events during regular teaching sessions (when the content curriculum events are introduced or the conceptual base is taught): area model for multiplication (discrete version) (CCE 3), rate model for multiplication (CCE 7), rule for multiplication of fractions (CCE 6), a number times its reciprocal equals 1 (CCE 14), the reciprocal of \( x \) is \( \frac{1}{x} \), \( x \neq 0 \) (CCE 15), solving \( ax = b \) (CCE 19), informal definition of \( a > b \) (CCE 24), multiplicative properties of inequalities (CCEs 25 & 26), solving \( ax < b \) (CCE 27), multiplication counting principle (CCE 28), classical definition of conditional probability (CCE 29c), conditional probability formula (CCE 31), \( n! \) (CCE 32), permutation theorem (CCE 34), meaning of division (CCE 35), and algebraic definition of division (CCE 36). (The numbers in parenthesis represent the number associated with the content curriculum event as referred to in Table 5.1.) To illustrate Mr. Kantor's use of numerical representations when teaching the meaning of the content curriculum events, he constructed the following two representations for solving \( ax = b \), \( 40x=600 \), and \( \frac{3}{4}x=15 \). As a second example, he constructed the following numerical representations for solving \( ax < b \), \( -5x < 10 \), \(-12 < 48-X \), \(-10u < 0 \), and \( 10-5a < 10 \). Table G.1 displays the other numerical representations constructed by Mr. Kantor when explaining the meaning of the content curriculum events. Mr. Kantor constructed numerical representations for 17 (41%) of 41 content curriculum events during regular teaching sessions.

Mr. Kantor constructed the symbolic representations for the following three (of 30) when teaching the meaning of the content curriculum events: multiplicative identity of 1 (CCE 12), the reciprocal of a real number \( x \) is \( \frac{1}{x} \), \( x \neq 0 \) (CCE 15), and the algebraic
definition of division (CCE 36). To illustrate, Mr. Kantor provided the following symbolic representation for the multiplicative identity of 1, "Multiplication has an identity also. If I start with this [A], what do I multiply by [A • = A ] [to end up with the same thing I started with?].... OK [A • 1 = A]." As another example, Mr. Kantor provided the following symbolic representation for the definition of division: \( a + b = a \cdot \frac{1}{b} \), \((b \neq 0)\).

Table G.1 in Appendix G also displays the symbolic representations constructed by Mr. Kantor for teaching the meaning of the content curriculum event. As mentioned in Chapter 4, I considered that for some content curriculum event a symbolic representation can not be constructed easily. In some cases a verbal representation or verbal-symbolic representation is a formal representation. For example, the representation "division by zero is impossible" is a verbal representation. In this case a verbal-symbolic representation can be given (there does not exist a real number \( q \) such that \( x + 0 = q \) for every number real \( x \neq 0 \). In the case of \( 0 + 0 \), the real number \( q \) is not unique and therefore \( 0 + 0 \) is not defined in mathematics either), but I accepted the verbal representation as appropriate. For 30 content curriculum events a symbolic representation can be constructed, for six content curriculum events a verbal-symbolic representation is appropriate, and for five content curriculum events a verbal representation is appropriate.

Since Mr. Kantor constructed only a symbolic representation for three out of 29 content curriculum events for which I asked him a symbolic representation, the relationship between Mr. Kantor's knowledge of symbolic representation and use of his knowledge during classroom instruction can be categorized as very weak. The index of relationship was .10 (3/29). Mr. Kantor did not construct any type of symbolic or numerical representation for 22 CCEs. Thus, the index of relationship between Mr. Kantor's knowledge of numerical or symbolic representations was .46 (19/41).

**Using the content curriculum events for solving problems.** Another context in which we can examine teachers' use of symbolic representations is when
teachers use the content curriculum events for solving problems. While Mr. Kantor did not construct numerical or symbolic representations for teaching the meaning of the following two content curriculum, he used them for solving numerical or numerical-symbolic problems: volume of a rectangular solid (CCE 4), and division by zero (CCE 37). In addition, Mr. Kantor used the following 15 content curriculum events for solving problems: area model for multiplication (discrete version) (CCE 3), theorem of multiplication of fractions (CCE 6), rate model for multiplication (CCE 7), the reciprocal of $x$ is $1/x$ ($x \neq 0$) (CCE 15), reciprocal of zero (CCE 16), multiplication property of zero (CCE 17), solving $ax = b$ (CCE 19), multiplicative properties of inequalities (CCEs 25 & 26), multiplication counting principle (CCE 28), conditional probability formula (CCE 31), the factorial symbol (CCE 32), permutation theorem (CCE 34), the meaning of division (CCE 35), and algebraic definition of division (CCE 36). To illustrate Mr. Kantor's use of numerical and numerical-symbolic representations when solving problems I will use three examples. He used the formula of a rectangular solid when solving the problem "The volume of a box needs to be 500 cubic centimeters. The base of the box has dimensions 12.5 cm and 5 cm. How high must the box be?" As a second example, he used and illustrated the multiplicative property of inequalities for solving the inequality $-5X > 10$. He said "I multiply both sides by negative, that switches. You saw that that happens, $[\left(-\frac{1}{5}\right)\cdot 5\cdot X > 10\left(-\frac{1}{5}\right)]$... As soon as you multiply both sides by a negative, change the sense." As third example, he used and illustrated the permutation theorem to solve the problem "In softball, there are 10 people who can bat. In how many ways can the manager of a softball team arrange the batting order?"

Table G.2 displays all the content curriculum events that Mr. Kantor used for solving numerical, numerical-symbolic, or symbolic problems. He used 16 (39%) of 41 content curriculum events in this context.
Content curriculum events covered during review. While Mr. Kantor did not construct representations for six content curriculum events during regular teaching sessions nor during problem solving sessions, he constructed representations for these CCEs during review sessions. For the commutative property of multiplication (CCE 2) he constructed a numerical-symbolic representation ($3x = x3$) and for the associative property (CCE 5), a symbolic representation $[(ab)c = a(bc)]$. For four content curriculum events (solving $0x = b, b \neq 0$; solving $0x = 0$, solving $ax = 0, a \neq 0$, solving $-x = a$) he constructed, respectively, the following numerical representations $0 \cdot w = 14, 0 = a \cdot 0, 7y = 0, -l = 3 - 5$ (Table G.3, Appendix G).

Mr. Kantor's Use of His Knowledge of Mathematical Proofs

I judged that for six of 41 content curriculum events it was not appropriate to construct a mathematical proof. Then, I asked Mr. Kantor to construct a mathematical proof for 35 content curriculum events. As mentioned previously, of the 35 proofs, Mr. Kantor constructed 11 correct proofs, 10 partially correct proofs, and 14 incorrect proofs. For none of the 35 content curriculum events did Mr. Kantor provide the proof during classroom instruction. Since Mr. Kantor did not know the proof of 14 content curriculum events, we cannot expect him to construct proofs during classroom instruction. In addition, middle school students are not expected to learn the proofs of some theorems related to algebraic multiplication. However, I believe that Mr. Kantor could have constructed mathematical justifications of some theorems related to algebraic multiplication. Then, I will describe some opportunities that Mr. Kantor missed for constructing proofs during classroom instruction. Mr. Kantor could have used the repeated addition model for multiplication to justify the following three content curriculum events: (a) array model for multiplication (CCE 3), the multiplication counting principle (CCE 28), and the permutation theorem (CCE 34). He also could have proved, by contradiction, that division by zero is
impossible (CCE 37). Mr. Kantor could have also proved the four rules of signs in the case of division based on the corresponding properties of multiplication (CCEs 38-41).

Mr. Kantor provided heuristic arguments for two content curriculum events multiplication of fractions theorem and zero has no reciprocal. For the multiplication of fractions rule he said,

\[
\frac{3}{4} \times \frac{3}{5} = \frac{9}{20} \]

That's three times a fourth, three times a fifth—change the order.

Three times three times a fourth times a fifth \([=3 \cdot \text{fourth} \cdot \text{fifth}]\).... using the commutative and associative properties.... If you cut something into fourths ... and then cut those into fifths. So a fourth times a fifth is a twentieth \(\frac{9}{20}\). (CO, 1, 10-13-94)

Regarding the fact that zero has no reciprocal, Mr. Kantor provided the following argument "If I multiply B by anything I can go back to where I started but unless I multiply by zero. Once I multiply by zero I wipe out. I can't multiply that by something to get back to \(B\) \([B \cdot 0 \cdot = B]\). . . . There isn't something I multiply by zero to get one."

Mr. Kantor did not take advantage of some proofs provided by the textbook. For example, the textbook has the following example (p. 169).

Simplify \(-x \cdot -y\).

Solution Since \(-x = -1 \cdot x\) and \(-y = -1 \cdot y\)

Then \(-x \cdot -y\)  
\[= (-1 \cdot -1)xy\]
\[= xy\]

If \(x\) and \(y\) are positive numbers then \(-x\) and \(-y\) are negative numbers. This example illustrates the fact that the product of two negative numbers is positive. Another example can be found in the teachers' textbook edition. This example shows that if \(a < b\) then \(-a > -b\). The argument is as follows:

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If $a < b$ then $a + -a < b + -a$
\[0 < b + (-a)\]
\[0 + -b < b + -a + -b\]
\[-b < -a\]
\[-a > -b\]

This illustrates a special case of the multiplicative property of inequalities: When we multiply both sides of an inequality by negative 1, the sense of the inequality changes.

In summary, Mr. Kantor's use of mathematical proofs was nonexistent. He did not construct the proof of any of the 31 content curriculum events for which he knew a proof (correct or partially correct) during classroom instruction.

**Mr. Kantor's Use of His Knowledge of Pictorial Representations**

Table G.4 displays the pictorial representations that Mr. Kantor constructed for teaching the meaning of the content curriculum events. Mr. Kantor only constructed pictorial representations for the following eight of the 32 (25%) possible content curriculum events for which he constructed a correct pictorial representation during the interviews: rule for multiplication of fractions (CCE 6), $a > b$ (CCE 24), multiplicative properties of inequalities (first and second part) (CCEs 25 & 26), solving $ax < b$ (CCE 27), multiplication counting principle (CCE 28), classical definition of conditional probability (CCE 29c), conditional probability formula (CCE 31). To illustrate, Mr. Kantor constructed the pictorial representation depicted in Figure 5.12 to explain why the multiplication of fractions rule holds. We notice that he used the area model for multiplication. The area of the rectangle with vertices B, E and H, is, by construction $\frac{9}{20}$.

On the other hand, that area can also be represented by $\frac{3}{4} \cdot \frac{3}{5}$. Therefore, $\frac{3}{4} \cdot \frac{3}{5} = \frac{9}{20}$. I will further illustrate Mr. Kantor's use of his knowledge of pictorial representations when describing his explanations.
Mr. Kantor's Use of His Knowledge of Story-Problem Representations

Table G.5 displays the story-problem representations constructed by Mr. Kantor for teaching the meaning of the content curriculum events. He only constructed story-problem representations for the following 10 of the 35 (29%) content curriculum events for which he knew a story-problem representation: rate model for multiplication (CCE 7), the multiplicative properties of inequalities (CCEs 25 & 26), multiplication counting principle (CCE 28), classical definition of conditional probability (CCE 29c), conditional probability formula (CCE 31), n! (CCE 32), permutation theorem (CCE 34), meaning of division (CCE 35), algebraic definition of division (CCE 36). To illustrate, Mr. Kantor constructed
several story-problem representations to explain the meaning of the rate model for multiplication. He began his construction of pedagogical events with an example to illustrate when it is appropriate to use the rate model. The example was the following: $1.55 \frac{\text{mi}}{\text{hr}} \cdot 12 \text{ hr}$ has meaning if $55 \frac{\text{mi}}{\text{hr}}$ and $12 \text{ hr}$ are related, belong to the same trip. Another example was the following "Use the reciprocal of the rates to do the multiplication $6 \frac{\text{dollars}}{\text{lb}} \cdot 30 \frac{\text{shrimp}}{\text{lb}} = \frac{\text{shrimp}}{\text{dollar}}$." A third example was "Suppose a laser printer prints 5 pages per minute. How long will it take to print 2400 documents with 3 pages per document?" Still another example was to convert $\frac{4\text{ft}}{\text{sec}^2}$ to $\frac{\text{in}}{\text{hr}^2}$. I will further illustrate Mr. Kantor's use of his knowledge of story-problem representations when describing Mr. Kantor's explanations.

Mr. Kantor's Explanations

The explanations constructed by Mr. Kantor during classroom instruction were analyzed in terms of five themes: (a) use of both mathematical and pedagogical representations (integration of representations), (b) relationship between students' difficulties and explanations (treatment of difficult topics), (c) operational and structural conceptions of algebraic objects, (d) integration of concepts, and (e) conceptual and procedural elements.

Explanations and Use and Integration of Representations

I categorized explanations also by the types of representations involved: verbal explanations, numerical explanations, symbolic explanations, symbolic-pictorial explanations, story-problem explanations, etc. Verbal explanations are those explanations that involve the use of spoken language to represent mathematical knowledge. Numerical explanations are those explanations that involve only numerical representations of algebraic objects. Symbolic explanations are those explanations that involve symbolic representations of concepts and procedures. Pictorial explanations are those explanations...
that involve significant use of pictorial representations. Story-problem explanations are those explanations that involve significant use of story-problem representations. Table 5.3 shows the categorization of the Mr. Kantor’s explanations according to the use of representations. As we can see from the table, Mr. Kantor only constructed explanations for 18 of the possible 41 (44%) content curriculum events.

We see from Table 5.3. that Mr. Kantor used a variety of representations for constructing explanations about the 18 content curriculum events and then the explanations are categorized in multiple categories. The use of spoken language plays a predominant role not only as a means of communication in general but also to communicate mathematical ideas in particular. Not surprisingly, Mr. Kantor used verbal representations for 17 content curriculum events.

**Numerical explanations.** We see that Mr. Kantor constructed representations involving numerical elements for 17 explanations. Explanations for solving equations of the form \( ax = b \) and inequalities of the type \( ax < b \) were dominated by numerical representations. To illustrate, Mr. Kantor used the following numerical representations: \( 40R = 600, \frac{3}{4}B = 15, \frac{7}{9}q = 140, -4p = 12, 12r = -4, 6j = 11 \) and \( -20k = \frac{2}{5} \) and the story problem "The volume of a box needs to be 500 cubic centimeters. The base of the box has dimensions 12.5 cm and 5 cm. How high must the box be?" when explaining how to solve equations of the form \( ax = b \). We can see that the numerical representations dominated over symbolic, pictorial, and story-problem representations.

**Verbal explanations.** Mr. Kantor constructed representations involving verbal elements for 17 of the 18 explanations.
<table>
<thead>
<tr>
<th>Content curriculum event</th>
<th>Verbal (17)</th>
<th>Numerical (17)</th>
<th>Symbolic (3)</th>
<th>Proof (0)</th>
<th>Pictorial (8)</th>
<th>Story-problem (10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. Rule for multiplying fractions (Pictorial)</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>7. Rate model for multiplication (Story-problem)</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>12. The Multiplicative identity of 1 (Symbolic)</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14. Definition of reciprocal (Numerical-symbolic)</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15. The reciprocal of a is 1/a (a ≠ 0) (Numerical-symbolic)</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16. Reciprocal of zero (Verbal-Numerical)</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19. Solving ax = b (Numerical)</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24. Definition of a &gt; b (Verbal-numerical)</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>25. The multiplicative property of inequalities (first part) (Numerical-Pictorial)</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>26. The multiplicative property of inequalities (Second part) (Numerical-pictorial)</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>27. Solving ax &lt; b (Numerical)</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
</tbody>
</table>

(To be continued)

Table 5.3 Categorization of Mr. Kantor's explanations according to the use of representations
<table>
<thead>
<tr>
<th>Table 5.3 (Continued)</th>
</tr>
</thead>
<tbody>
<tr>
<td>28. Multiplication counting principle (Story-problem)</td>
</tr>
<tr>
<td>29c. Classical definition of conditional probability (Pictorial and Story-problem)</td>
</tr>
<tr>
<td>31. Conditional probability formula (Pictorial and story-problem)</td>
</tr>
<tr>
<td>32. $n!$ (Story-problem)</td>
</tr>
<tr>
<td>34. Permutation theorem (Story-problem)</td>
</tr>
<tr>
<td>35. Meaning of division (Story-problem)</td>
</tr>
<tr>
<td>36. Algebraic definition of division (Story-problem)</td>
</tr>
</tbody>
</table>

Table 5.3 Categorization of Mr. Kantor's explanations according to the use of representations

**Symbolic explanations.** Mr. Kantor constructed representations involving symbolic elements for three of the 18 explanations. However, only the explanation for the multiplicative identity of 1 was characterized as mainly symbolic. The explanation was as follows:

When you look at the operation of addition, what's the identity? ... What an identity means is this, if I start with something [$A$], in that operation, what do I add to that to end up with the same thing I started with [$A + = A$]? ... Zero [$A + 0 = A$]. Now, multiplication has an identity also. If I start with this [$A$], what do I multiply by [$A \cdot = A$] to end up with the same thing I started with? ... OK [$A \cdot 1 = A$] (CO, 2, 10-14-94)

**Explanations involving proofs.** Mr. Kantor did not construct for any content curriculum event an explanation involving the use of proof.

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**Pictorial explanations.** Mr. Kantor constructed explanations involving a pictorial representation for 8 of the 18 curriculum events. Only the explanation for the multiplication of fractions theorem was characterized as mainly pictorial. This explanation is described below.

Mr. Kantor gave students a hand-out activity (Appendix B). His purpose was to use the activity to guide students to discover the multiplication of fractions theorem using a pictorial representation for the operational example of $\frac{3}{4} \times \frac{3}{5} = \frac{9}{20}$. As he stated during an interview, his purpose was that students "can visualize multiplication of fractions [with] something concrete." (I, 4, 10-17-94)

The excerpt in Appendix K relates the beginning of the explanation (Refer to Figure 5.12 of Mr. Kantor’s representation about why $\frac{3}{4} \times \frac{3}{5} = \frac{9}{20}$). Mr. Kantor began his explanation by considering a square unit and constructing a representation of $\frac{3}{4}$ as a length (BE = $\frac{3}{4}$) (speeches 1-7) and dividing the square into four equal rectangles each of them representing $\frac{1}{4}$ (speeches 1-4). Then Mr. Kantor constructed a representation of $\frac{3}{5}$ and divides each of those four rectangles into five smaller parts (rectangles) which will represent 4 times 5 (the multiplication of the denominators of the fractions $\frac{3}{4}$ and $\frac{3}{5}$) (speeches 8-13). Then he asks students what the area of each of those 20 parts is and explains that its area is $\frac{1}{20}$ (speech 10). Mr. Kantor's next purpose is to make explicit the representation of $\frac{3}{4} \cdot \frac{3}{5} = \frac{9}{20}$ (Figure 5.11). This is related in Excerpt 1.

**Excerpt 1**

1. K: Now, carefully outline the rectangle that I will put Xs through the quarters, BE and EH... It's comprised of nine smaller rectangles. Each of those has an area of what?
2. S: Twenty five.
4. K: Each of those rectangles is a twentieth.
5. S: Is nine twentieths?
6. K: The area of this rectangle is nine twentieths. One way to think of fractions is, the numerator tells me how many I have and the denominator tells me, in effect, the size of the pieces. I have nine twentieths. Yes?

7. S: Why is that a twentieth?

8. K: The whole thing is one. I cut it into twenty equal parts so each of these is a twentieth. Is that true? ... So, by a construction we come up with the area of nine twentieths, correct? ... We know that another way to find the area of a rectangle is what?


10. K: Length times width. Its length is three fourths, its width is three fifths. Those together should be equal the area which we have constructed to be nine twentieths. Our construction verifies that to multiply fractions you multiply the numerators together, multiply the denominators together

\[
\frac{3}{4} \times \frac{3}{5} = \frac{9}{20}. \quad \text{(CO, 1, 10-13-94)}
\]

At the beginning of Excerpt 1, speech 1, Mr. Kantor is constructing a pictorial representation of \( \frac{3}{4} \times \frac{3}{5} \) using the area model for multiplication (At that moment Mr. Kantor does not focus on that fact, however). Then Mr. Kantor and the students discuss that the area of the rectangle with sides \( \frac{3}{4} \) and \( \frac{3}{5} \) is \( \frac{9}{20} \) because it is composed of nine parts [the product of 3 and 3, the numerators of the fractions], each one having an area of \( \frac{1}{20} \). Next, he uses the fact that the area of that rectangle can be expressed as \( \frac{3}{4} \times \frac{3}{5} \) (width times length) (speeches 8-10). He then concludes that those areas should be equal. Therefore, \( \frac{3}{4} \times \frac{3}{5} = \frac{9}{20} \) (speech 10). He then notes the pattern, to multiply fractions you multiply the numerators together (3x3 = 9) and the denominators together (4x5) (speech 10).

We notice that Mr. Kantor used the pattern of a specific example to generalize to the general pattern. He has shown a specific pattern. However, he has not explained explicitly why that particular pattern holds. That is, he has shown that \( \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \) using the specific example \( \frac{3}{4} \times \frac{3}{5} = \frac{9}{20} \) but he has not made explicit why that is. He attempts to do that in Excerpt 2.

Excerpt 2

1. K: [He begins to read his handout] Therefore the area of the larger rectangle equals three fourths times three fifths. I've rewritten those as three fourths times three fifths \( \left( \frac{3}{4} \times \frac{3}{5} = \frac{9}{20} \right) \).
Then I changed the order of these. That's three times a fourth, three times a fifth—change the order. Three times three times a fourth times a fifth [=3·3·fourth·fifth]... using the commutative and associative properties.... Now, it says we saw that if you cut something into fourths—that's what we did with the horizontal lines—and then cut those into fifths, what do you end up with?

2. S: Twentieths.
3. K: Twentieths. So a fourth times a fifth is a twentieth. The same thing you get when you multiply the denominators together. That's why it works. To check, cut something into fourths and each of those cut into five parts then you get twentieths and this gets me that I have nine of them [9 twentieths]. So we've shown, using this construction, that that's the way you multiply fractions. That's why the algorithm works.... Does this make sense that this works? The construction showed you that that's the case. Any questions on that? (CO, 1, 10-13-94)

We notice, however, that Mr. Kantor has not been very explicit in showing why $\frac{3}{4} \times \frac{3}{5} = \frac{9}{20}$, that is, why the particular pattern for multiplying fractions holds. An explicit explanation would say that he has divided the unit square into four smaller rectangles and each of these four rectangles into five smaller rectangles having 4x5 smaller rectangles as a result and that the area representing $\frac{3}{4} \times \frac{3}{5}$ consists of 3x3 rectangles out of 4x5. An explicit explanation would include saying why it is 3x3 and 4x5. Some students might think that the 9 (representing the 9 rectangles) is 3x3 and the 20 (representing the total number of rectangles) is 4x5 rather than thinking 4x5 as representing having divided the square unit into four smaller rectangles and then each of those rectangles into 5 rectangles making a total of 4x5 rectangles and 3x3 representing the total number of small rectangles making up the rectangle whose sides are $\frac{3}{4}$ and $\frac{3}{5}$ (see Figure 5.12).

**Story-problem explanations.** Mr. Kantor constructed explanations involving story-problem representations for 10 of the 18 curriculum events (see Table 5.3). In the following five content curriculum events the story-problem representations dominated over the other types of representations: rate model for multiplication (CCE 7), multiplication counting principle (CCE 28), $n!$ (CCE 32), permutation theorem (CCE 34), meaning of division (CCE 35), and algebraic definition of division (CCE 36). From the perspective of using story-problem representations, Mr. Kantor was at his best when explaining the
algebraic definition of division. Mr. Kantor began his explanations about why \( a + b = a \cdot \frac{1}{b} \)
as related in Excerpt 3.

**Excerpt 3: Why \( 7 + 2 = 7 \cdot \frac{1}{2} \)?**

1. **K:** What's the algebraic definition of division?
2. **S:** ... for all real numbers if \( b \) doesn't equal zero \( a \) divided by \( b \) equals \( a \) times \( a \) \( \frac{1}{b} \) over \( b \).
3. **K:** [Mr. Kantor writes \( a + b = a \cdot \frac{1}{b} \) on the board]. Now, if \( a \) is seven and \( b \) is two. Seven divided by two is the same thing as what?
4. **S:** Seven times one over two.
5. **K:** Seven times one over two \( 7 + 2 = 7 \cdot \frac{1}{2} \). Can someone give me a concrete example of seven divided by two, a situation in which seven divided by two gives you the answer?
6. **S:** Seven pieces of pizza divided by two people.
7. **K:** You have seven pieces of pizza and you want to divide those among two people. Now, that gives you three and a half. Seven pieces of pizza divided among two people you get three and a half. Jeff, can you get the same answer taking seven times one half?
8. **J:** Yes.
9. **K:** If you do seven times one half, what happens? You get three point five. Now, using that example, explain why this works. Using that example, you said take seven pieces of pizza, divide them in half, you get three and a half.
10. **S:** Because that's the reciprocal.
11. **K:** Show me in the context of the problem why the same situation can be described as seven times one half.
12. **S:** Because multiplication is the opposite of division. One half is the opposite of two.
13. **K:** Again, you are getting too abstract. You are right but I want to express it in a way that someone who doesn't know anything about the inverses or reciprocals can understand. When you were at elementary school you understood that dividing by two is the same as taking half of something, right?
14. **E:** OK. If you have seven pieces of pizza and each person gets a half of that, that's seven times one half.
15. **K:** OK. Everybody gets half of that, all right? So you understand, you conceptualize, you internalize that that divided by two, taking something and cutting in half is the same thing as taking half of it. Half of it, one half times, correct? Same thing as taking fifty percent of it, you know that. So there is an example where you know that the algebraic definition of division works. Dividing by this is the same thing as multiplying by the reciprocal. (CO, 12, 11-01-94)

In Excerpt 3, Mr. Kantor's purpose is that students understand why \( a + b = a \cdot \frac{1}{b} \), \( b \neq 0 \) using a story problem whose solution can be represented by \( 7 + 2 \) and \( 7 \cdot \frac{1}{2} \) and thus concluding that \( 7 + 2 = 7 \cdot \frac{1}{2} \). While students seem to understand that the solution to the story problem can be represented by \( 7 + 2 \), some of them do not understand why the solution can also be represented by \( 7 \cdot \frac{1}{2} \) (speeches 10 and 12). Then, Ethel, one of his best students, gives an explanation with some degree of conceptual understanding using the
of model for multiplication (speech 14). Mr. Kantor elaborates on Ethel's explanations in speech 15. We notice that Mr. Kantor's explanation is also in terms of the of model for multiplication: seven divided by two \((7 \div 2)\) is the same as taking half of seven \(\left(\frac{1}{2} \cdot 7\right)\) (speech 15). However, a question arises: why is it that \(\frac{1}{2}\) models taking half of seven?

Students may be just memorizing that taking half of something can be modeled by multiplying by one half. The interviews showed that this is not the case for Mr. Kantor. He knows why that is as shown on the following excerpt:

[Int you have seven pieces of pizza and each person gets a half of that]. Half of each pizza, that's what each person gets, each person gets three and a half [pizzas]. That's ... a half seven times. Seven dollars, it costs two dollars for a pizza, you can buy three and half pizzas. Looking it differently, a dollar will pay for half a pizza... Seven times a half. Well, a half a dollar is gonna. No, a dollar can buy a half a pizza. So each dollar buys a half a pizza. You have seven, your seven dollars times, it's a rate, seven times half pizza per dollar. (I, 29, 06-08-95)

It is interesting to notice that Mr. Kantor gives the explanation using the of model when he knew others that have more potential to produce more powerful conceptual-based understanding. In addition, he also mentioned during the interviews that only some students know that taking half of something means to multiply by one half but that they do not know why that is. He did not elaborate further on his explanation of the connection between dividing by two and taking half of it. He then constructed an example involving dividing by a fraction. His goal was, again, to help students construct the connection between division and multiplication. The pedagogical events constructed for this example are described in Excerpt 4.

Excerpt 4: Why \(5 + \frac{1}{2} = 5 \cdot 2\)?

1. K: Give me problem, make up a problem where five divided by one half represents that problem. Five divided by one half.
2. S: What do you mean?
3. K: Give me a story problem where I can express that, directly as five divided by one half.
4. S: ah, I don't know.
5. K: Let me give you a start. Let's say that one half represents a half a dollar. Give me a story problem with five divided by one half represents that problem.
6. S: Five people have to divide a half a dollar among them.
7. K: ... That will be a half divided by five. I want five divided by a half. I want a situation where I'm trying to figure out how many times one half goes into five.
8. S: It goes into five ten times.
9. K: It goes into five ten times \[10 = 5 + \frac{1}{2}\]. What's a problem that talks about a half going into five.

Jacinta? What would be a problem? [No answer] ... A problem?

10. C: You have five dollars. You wanna know how many times ... how many pencils you can get for five dollars?

11. K: How much the pencils cost?

12. C: Fifty cents.

13. K: Right. So, you have five dollars. You wanna find out how many pencils you can buy that cost fifty cents a piece. You're trying to find out ... how many times a half goes into five. That makes sense? You have five dollars, pencils is fifty cents a piece, how many times does fifty cents go into five dollars?


17. K: Now, in the context of that problem why does five times two give me the answer?

18. S: Because it equals ten.

19. K: Because it equals ten.... six plus four equals ten, too. I can use this to get the answer but that doesn't mean it's the mechanism to employ this in other problems. What?

20. S: Because when you divide a fraction you can do the same thing by multiplying by the reciprocal.

21. K: Tell me in a concrete way why that gives you the same thing ... I can figure out how many half dollars are in five bucks, right?

22. A: Because it is.

23. K: I can find out how many times it goes into five. The process of doing that, how can I do it?

24. A: Because it is.... Because it's.

25 K: Let me change the problem just a little bit. The same type of thing. Let's say that I have candy for Halloween, but a lot of times people don't come to my house, so I don't buy a lot. So what I wanna do is I wanna make sure I got a bag of money in case I run out of candy. I wanna make sure I have money to give you. I wanna give each kid fifty cents. I have five dollar bills, all right? Now, I figure out first of all with five dollar bills I can give ten kids fifty cents, right? Now, I have five dollar bills. I've gotta to convert that. I have to go to the bank. I have to go the bank and get fifty cent pieces for five dollar bills. For every dollar bill they give what?


27. K: Two fifty cent pieces, right? and so that's five times two is how many fifty cent pieces I get. The same thing. How often does a half go into five, how many times a half goes into five is the same as five times two. (CO, 12, 11-01-94)

One of the most critical events in Excerpt 4 is the fact that Mr. Kantor is attempting students to explain why a story problem for the division \[5 \div \frac{1}{2}\] can also be solved by the multiplication \(5 \cdot 2\) to conclude that \[5 + \frac{1}{2} = 5\cdot2\] (speeches 13 and 15). Again, students do not seem to understand what the question is asking. For example, one of them says the opposite (speech 16). Mr. Kantor asks them why five divided by one half gives the same answer as five times a half (speech 17). The student gives the inadequate explanation that because it equals ten (speech 18). Mr. Kantor then makes the point that six plus four equals ten, too, but that it is an incorrect procedure. Another student says "because when
you divide … you can do the same thing by multiplying by the reciprocal” (speech 20). We notice that the student is justifying the procedure using the procedure itself. Mr. Kantor does not give up and paraphrases the question in speeches 21 and 23. A student answers “because it is” (speeches 22 and 24). Mr. Kantor does not continue asking students for an explanation but rather, in speeches 25 and 27, he constructs an explanation to help students construct the connection of why $5 \times \frac{1}{2} = 5 \cdot 2$. We notice that his explanation is conceptually based. Mr. Kantor then constructs another example, $21 \div \frac{3}{4} = 21 \cdot \frac{4}{3}$, to help students understand that $a \div b = a \cdot \frac{a}{b}$. This is described in Appendix L. It is worth to mention here that his explanation illustrates directly why $21 \div \frac{3}{4} = 21 \cdot \frac{4}{3}$, not why $21 \div \frac{3}{4} = 21 \cdot \frac{4}{3}$. His explanation of the second example and responses to interviews show that he knows why $21 \div \frac{3}{4} = 21 \cdot \frac{4}{3}$.

**Explanations and Treatment of Difficult Topics**

A pattern of Mr. Kantor’s explanations emerged regarding the degree of difficulty of the content curriculum events from both Mr. Kantor’s and students’ point of view. From the perspective of difficulty associated with the curriculum events, I categorized Mr. Kantor’s explanations in four categories (a) no explanations, (b) trivial or easy explanations, (c) middle range explanations, and (d) difficult explanations. No explanations are the explanations that are not constructed at all. Trivial or easy explanations are explanations that are constructed for topics that are considered “easy.” They consist of the mere statement or telling of the content curriculum events without making any kind of connections between and among related content curriculum events. The third level of explanations are called middle range explanations. They involve some elements of procedural and conceptual knowledge. The teacher and students do not struggle to explain and to “understand” the mathematical idea. Difficult explanations are more elaborate explanations and they involve strong elements of conceptual knowledge. The teacher and
students struggle to explain and understand, respectively, the connection between the procedure and the conceptual base.

Table 5.4 shows the categorization of Mr. Kantor's explanations according to the difficulty of the content curriculum events. As before, the numbers in parenthesis indicate the number associated to the content curriculum events as displayed in Table 5.1.

<table>
<thead>
<tr>
<th>No explanations (23)</th>
<th>Trivial explanations (0)</th>
<th>Middle range explanations (12)</th>
<th>Difficult explanations (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2, 3, 4, 5, 8, 9, 10, 11, 13, 17, 18, 20, 21, 22, 23, 30, 33, 37, 38, 39, 40, 41.</td>
<td>6, 12, 14, 15, 16, 19, 24, 25, 28, 32, 34, 35.</td>
<td>7, 26, 27, 29c, 31, 36.</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.4 Categorization of Mr. Kantor's explanations according to treatment of difficult topics

No explanations and trivial explanations. Mr. Kantor did not construct explanations for 23 of 41 content curriculum events. Some of these curriculum events are: the associative and commutative properties (CCEs 2 & 5), the array model for multiplication (CCE 3), multiplicative properties of -1 and 0 (CCEs 13 & 17), the rules of signs for multiplication and division (CCEs 8-11 & 38-41). I did not categorize any of Mr. Kantor's explanations within the trivial-explanations category.

Middle range explanations. I categorized the explanations for the following 12 content curriculum events as middle range explanations: the multiplication of fractions theorem (CCE 6), multiplicative property of 1 (CCE 12), the definition of reciprocal (CCE 14), the reciprocal of \( x \) is \( \frac{1}{x} \), \( x \neq 0 \) (15), reciprocal of zero (CCE 16), solving \( ax = b \) (CCE 19), definition of \( a > b \) (CCE 24), first part of the multiplicative property of inequalities.
(CCE 25), multiplication counting principle (CCE 28), the factorial symbol (CCE 32), the permutation theorem (CCE 34), and meaning of division (CCE 35). His explanations for solving equations of the form \( ax = b, b \neq 0 \) are as described and analyzed below.

Mr. Kantor began teaching how to solve \( ax = b \) by connecting it to how to solve a numerical example in the addition case, \( x + 3.1 = 5.2 \). He first asked students procedural questions about what the identity, inverse, and related operation for addition are. He then explained the rationale for the method of solving \( x + 3.1 = 5.2 \):

I'm adding the inverse of three point one. I'm whipping out the effect of adding three point one. To go backwards into the problem... I want to reverse this process by using, by applying the inverse. OK? Now, ... what's another way I can do, almost the same thing as adding negative three point one? ... Subtracting three point one... So I can reverse on a process by applying the inverse ... or using the related operation... Either way works. You can use either method you want to solve these equations. (CO, 3, 10-17-94)

It seems that his purpose is to connect the way of solving \( x + a = b \), using a numerical example, to the procedure for solving \( ax = b \). Excerpt 5 describes the first explanation constructed by Mr. Kantor to help students construct procedural and conceptual knowledge about solving equations of the type \( ax = b \) as well as the interchange between Mr. Kantor and his students.

**Excerpt 5: Solving \( 40R = 600 \)**

1. K: Now, if you look at an equation that involves what operation? \( [40R = 600] \)
2. S: Multiplication.
3. K: Multiplication. You got the same thing, what's the identity in multiplication?
4. S: One.
5. K: What's the inverse?
7. K: The reciprocal. What's the related operation?
8. S: Division.
9. K: Division. Now, if you apply the inverse to solve this, what do you use? What do you multiply by? Inverse is multiplying by?
10. S: One over forty.
11. K: Multiply both sides by one over forty \( \left[ \frac{1}{40} \right] 40R = 600 \left( \frac{1}{40} \right) \), and that will get the answer. I get R on this side and fifteen on the other \( \left[ \frac{1}{40} \right] 40R = 600 \left( \frac{1}{40} \right) = 15 \). So I can undo the multiplication. I can reverse the process by applying the inverse or I can reverse the process by
12. S: Or by applying the reciprocal.
13. K: That's the same thing. The inverse is the reciprocal.
17. K: Dividing, that's the related operation. Divide by forty $\left(\frac{1}{40}\right)40R = 600\left(\frac{1}{40}\right) = 15$. Here, [He wrote this information on a transparency] this is the way you solve equations. Reverse the process. You can either apply the inverse or do the opposite—do the related operation. (CO, 3, 10-17-94)

We notice in Excerpt 5 that Mr. Kantor asked procedural questions about the identity, inverse, and related operation in the case of multiplication. His goal is, at least implicitly, to make a comparison between the procedure for solving equations of the type $x + a = b$ and $ax = b$, and to conclude that both cases can be solved by applying the inverse or the related operation. We notice that he did not explicitly justify (or asked students to justify) every step of the procedure. It seems that he was implicitly assuming that students already knew and understood some aspects of the mathematical theory behind the procedure. He is probably correct about some students. However, there may be some other students that are constructing or reconstructing the mathematical justification of the procedure of solving $ax = b$. Mr. Kantor provided other examples. The next one was to solve an equation involving fractions: $\frac{3}{4}B = 15$. The explanation is described in Appendix M. It is worth to notice that the approach was structural in nature: working on algebraic objects to produce equivalent algebraic objects. However, we should notice that he did not construct, or ask students to construct, explanations about every step that needs to be justified mathematically using the associative property, property of reciprocals, identity, and multiplication of fractions. Then, his explanations tend to be semiconceptual and semistructural as well as the representations. They have some structural elements because the procedure involves applying algebraic properties to create algebraic objects (e.g., multiplying both sides by the reciprocal).

The next problem for which Mr. Kantor constructed pedagogical events was asked during the homework session. It was finding the height of a box with a volume of 500 cubic centimeters and whose base has dimensions 12.5 cm and 5 cm. The pedagogical
events constructed by Mr. Kantor are described in Appendix M. Briefly, he represented the solution of the problem with an equation, $500 = 62.5h$. He then asked students how to solve the equation. He mentioned several equivalent ways. His approach to solving this problem was structural (representing it with an equation, solving the equation algebraically, and carrying the units through the problem). However, the questions asked tended to be procedural in nature.

**Difficult explanations.** The following content curriculum events were the most difficult for Mr. Kantor to explain: the rate model for multiplication (CCE 7), the second part of the multiplicative property of inequalities (CCE 26), solving inequalities of the form $ax < b$ (CCE 27), the classical definition of conditional probability (CCE 29c), the conditional probability formula (CCE 31), and the algebraic definition of division (CCE 36). The most difficult content curriculum events to teach were the classical definition of conditional probability and the conditional probability formula $[P(A \cap B) = P(A) P(B \mid A)]$. The explanations are described and analyzed below.

Mr. Kantor began teaching this lesson on October 25, 1994. The day before the students had a quiz that included three lessons, solving $ax = b$, special numbers in equations and solving $ax < b$. The following lesson of the textbook deals with the multiplication counting principle: "if one choice can be made in $m$ ways and a second choice can be made in $n$ ways, then there are $mn$ ways of making the first choice followed by the second choice" (McConnell et al., 1990, p. 187). The day of the quiz Mr. Kantor asked all his students (about 76, four periods) to write down on the top of the test page the global language and music option they were taking. The following day Mr. Kantor began teaching the lesson about conditional probability by asking students how many different master schedules are possible just looking at global language and music options. A student said two and another four times four. Mr. Kantor said that there are sixteen. Then he
passed a sheet with the data he collected from students organized in a table similar to Table 5.5 (See Appendix B for the complete handout).

<table>
<thead>
<tr>
<th>Language</th>
<th>Choir</th>
<th>Music</th>
<th>Band</th>
<th>Orchestra</th>
</tr>
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<tbody>
<tr>
<td>Spanish</td>
<td>25</td>
<td>15</td>
<td>11</td>
<td>4</td>
</tr>
<tr>
<td>French</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>German</td>
<td>4</td>
<td></td>
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<td>2</td>
</tr>
<tr>
<td>Reading</td>
<td>1</td>
<td>1</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5.5 Distribution of students according to language and music options

The sheet had also a tree diagram like the one shown in Figure 5.13 (without the percentages). Mr. Kantor and his students got the percentage of students that take each language. This percentages are displayed in that figure. Then Mr. Kantor asked how we check that answer and why the sum of percentages add up to 100 percent. After that, Mr. Kantor asked students to look just at the branch for Spanish. His general purpose was to lead students to discover that \( P(A \cap B) = P(A) P(B \mid A) \). The remaining of the teaching episode is related in Excerpt 6.

**Excerpt 6: An attempt to discover \( P(A \cap B) = P(A) P(B \mid A) \)**

1. K: Just look at the branch now for Spanish. At this point there are four options, and again, we follow the way the chart is, choir, music, band, and orchestra. This is important to understand. The percentages for this, presumed that you are at this point here. You are looking at only those people that take Spanish and the percentage of time those people branch up, so what do these percentages add up to? [Seventy two point thirty seven, one hundred, and forty five point six were the answers] One hundred percent because there are 55 people that take Spanish.... Take choir divided by 55.... Do you understand that? There are only 55 people that get to this point. How do those people branch up? So you are dividing by 55.

2. S: Why are you dividing by 55?
3. K: It's important that you pay attention because this gets very complicated. What this means, the branching here, these percentages. Those 55 people, how many even go in each of these paths? ... So come up with those and we'll talk about that. And, again, they have to add up to a hundred percent, right? [Some students begin to pack]... I need to finish up. Don't pack up yet, don't pack up yet. Now the percentage of people taking choir. Hurry up.... Forty five point forty five. Music? Hurry up with that.... Jeff, you got music for us? ... [The student got twenty seven point twenty seven] Let me stop here. I get one thing I need to cover. Sarah, I need to cover this. What I want you to do is to figure this out. I wanna know what percentage of people takes Spanish and Choir. How do you figure that out?

4. A: I don't know

5. K: How do you figure it out Emma?

6. E: You take the number of people that are in Spanish and Choir, ... twenty five and you divide it by seventy six.

7 K: Number of people that are in Spanish and Choir divided by the total number of people. Now, there is another way to do it. Let me show you because we don't have time for you to discover it. Look, seventy two point three seven percent of people get down to this point. Forty five point four five percent of those. Multiply those together. Just keep working on this and do your homework. [The bell rings and students begin to leave] (CO, 8, 10-25-94)

Figure 5.13 A pictorial representation for $P(A \cap B) = P(A)P(B \mid A)$
I will paraphrase and discuss these teaching events in terms of Mr. Kantor's purpose and using the language of probability. He wants students to construct the meaning of the conditional probability formula: \( P(A \cap B) = P(A) \cdot P(B \mid A) \) that in this example translates into \( P(S \cap C) = P(S) \cdot P(C \mid S) \). The situation is the following: we have 76 students in Mr. Kantor's periods and the experiment consists of choosing one student at random. \( S \) represents the event "the student is taking Spanish," \( C \) is the event "the student is taking choir," \( S \cap C \) is the event "the student is taking Spanish and Choir," \( C \mid S \) is the event "a Spanish student who is taking choir." There were four critical events in that teaching episode. The first one is computing the percentage of students who take Spanish (\( \frac{55}{76} \times 100 \) or 72.37 %). Since \( 0 \leq P(S) \leq 1 \), 72.37 represents \( P(S) \) in percent form. The second critical event is the computing of the percentage of students taking Spanish that are also taking Choir (\( \frac{25}{55} \times 100 \) or 45.45%) (speech 1). This represents \( P(C \mid S) \) in percent form. They also compute \( P(M \mid S) \), the percentage of students taking Spanish that are also taking music. The third critical event is describing the way of computing \( P(S \cap C) \) using the definition of probability of an event, in this case \( \frac{n(S \cap C)}{76} \). The fourth critical event is computing \( P(S \cap C) \) following an alternative reasoning, namely, multiplying \( P(S) \) and \( P(C \mid S) \) in that order with reference to the meaning of that multiplication. \( P(A \cap B) = P(A) \cdot P(B \mid A) \) is an important formula that represents a connection between \( P(A \cap B) \) and \( P(B \mid A) \). The equivalent representation \( P(B \mid A) = \frac{P(A \cap B)}{P(A)} \), if \( P(A) \neq 0 \), is the formula commonly used to represent the connection between \( P(A \cap B) \) and \( P(B \mid A) \). Theoretical arguments suggest that using multiple representation of concepts promotes conceptual understanding. As a teacher, I would use both formulas to represent the connection between \( P(A \cap B) \) and \( P(B \mid A) \) as well as to explain why those representations hold. To discover the formula \( P(S \cap C) = P(S) \cdot P(C \mid S) \) is the most critical event, and paradoxically, the one to which Mr. Kantor devoted relatively less time. I understand that
his reasoning was as follows: the percentage of students who are taking Spanish is 72.37%; of those students, 45.45% are taking choir. Therefore, the percentage that are taking choir and Spanish is (.4545)(72.37). To get the percentage of an amount we multiply the percentage by the amount. In this case we want the percentage of students who are taking Spanish and Choir. We can get this percentage using the standard meaning of percentage (part/total). Another way is to take the percentage of Spanish students who are also taking Choir (P(C | S) times the percentage of students who taking Spanish (P(S)). That is, percentage of students taking Spanish and choir equals the percentage of Spanish students who are also taking choir times the percentage of students who are taking Spanish.

We notice that the model of multiplication that has some potential to help us understand the meaning of that equivalence is the of model, taking a fractions of a fraction, or a percentage of a percentage. An alternative argument can be stated in the following terms: Let \( n(S \cap C) \), \( n(C \mid S) \), represent, respectively, the number of students taking both Spanish and choir and the number of Spanish students taking choir. Let \( p(S) \), \( p(C \mid S) \) represent, the percent, in decimal form, of students who are taking Spanish and Spanish students taking choir, respectively. Let \( N \) represent the total number of students. We have \( n(S \cap C) = n(C \mid S)n(S) = p(C \mid S)n(S) = p(C \mid S)(p(S)\cdot N) = (p(C \mid S)p(S))\cdot N. \) From here we have that \( \frac{n(S \cap C)}{N} = p(C \mid S)p(S) \). That is, the percent, in decimal form, of students who are taking Spanish and choir equals the percent, in decimal form, of Spanish students taking choir times the percent of students who are taking Spanish. I have been somewhat explicit about what I think Mr. Kantor had in mind. He was less explicit about the connection between

\[
\frac{n(S \cap C)}{N} = p(C \mid S)p(S)
\]

This can be used to deduce the conditional probability formula. Thus, \( P(S \cap C) = \frac{n(S \cap C)}{N} = p(C \mid S)p(S) = \frac{n(C \mid S)}{n(S)} \cdot \frac{n(S)}{N} = p(C \mid S)p(S). \) Another way to deduce the conditional probability formula is as follows: \( P(S \cap C) = \frac{n(S \cap C)}{N} = \frac{n(C \mid S)}{n(S)} \cdot \frac{n(S)}{N} = p(C \mid S)p(S). \)
probability of the intersection of two events, \( P(S \cap C) \), and conditional probability, \( P(C \mid S) \), as described in speech 7, Excerpt 6:

Number of people that are in Spanish and Choir divided by the total number of people. Now, there is another way to do it.... Seventy two point three seven percent of people get down to this point. Forty five point four five percent of those. Multiply those together.

In an informal talk Mr. Kantor told me that he hates to rush through the material but that he needed to finish that lesson and that some students from the previous period had actually discovered that connection. He knew that his explanations did not have the expected impact on students' learning, at least on students from the second period. In the following teaching episodes we will see that some students, if not all, had a hard time with the concept of conditional probability. We also will have an opportunity to analyze Mr. Kantor's degree of explicitness when talking about the equivalence between \( P(A \cap B) \) and \( P(A)P(B \mid A) \). That will also help to add more validity to my interpretation about Mr. Kantor's thinking when explaining that \( P(S \cap C) = P(S)P(C \mid S) \).

The next day, October 26, 1994, Mr. Kantor began his instruction by going over problem six of the questions at the end of the lesson. Problem 6 makes reference to Example 2 of the textbook. The combined problems read as follows, "suppose that two cards are drawn from a well-shuffled deck. The first card is not put back before the second is drawn ... " (Example 2). "Copy and compute the probabilities" (Problem 6) (McConnell et al., pp. 192 & 194). The accompanying diagram for Problem 6 was similar to the one depicted in Figure 5.14. This was brought by Mr. Kantor on a transparency. Mr. Kantor explained why the probability of drawing a heart in the first draw and the probability of not getting a heart on the first draw were \( \frac{1}{4} \) and \( \frac{3}{4} \), respectively. Then he constructed pedagogical events about getting a heart on the second draw given that a heart was drawn in the first draw. This is related in Excerpt 7.
Figure 5.14 A pictorial and story-problem representation
for \( P(A \cap B) = P(A)P(B \mid A) \)

Excerpt 7: Splitting the probability of \( P(A) \) as \( P(A \cap B) + P(A \cap B') \)

1. K: Now, the point of understanding in this diagram. What happens is, I take this and I branch this out [pointing to A, the event getting a heart in the first draw]. Twenty five percent chances I go down this and then I split out the twenty five percent. Split out that twenty five percent. Some of the time on the second draw, you get another heart [pointing B], sometimes you don’t [pointing C]. But I’m splitting out twenty five percent when I do that. When I find out the percent of time you get a heart and then another heart. It’s simply this multiplied by that. What I am doing is, I’m splitting out this twenty five percent. Yes?

2. S: So basically you just multiply the fractions, the first chance and the second chance together. \( \frac{1}{4} \times \frac{12}{51} \)


4. S: How do you figure out the chance of not getting a heart in the second draw? \( \frac{39}{51} \)

5. K: There are 51 cards left once I draw one. Thirty nine of those are not hearts. Twelve are hearts and 39 are not. So there are 39 out of 51 chances that it won’t be a heart. Now, look here. These percentages add up to twenty five percent because I split out the twenty five percent. Twenty five percent of the time I go down this path and almost six percent of that I draw another heart, nineteen percent of that I don’t draw a heart. That’s important that you look at that conditional probability, what is really happening, what’s really happening. Does everybody understand that? We are splitting out those probabilities. Does that make sense? Let’s see if you understand this situation. How much of the time you draw two hearts, if you look at that overhead? … Almost six percent of the time if you draw two cards they both gonna be hearts. (CO, 9, 10-26-94)

During this excerpt Mr. Kantor constructed more pedagogical events for teaching a case of the conditional probability formula, \( P(A \cap B) = P(A) P(B \mid A) \), in this case

\[
P(H_1 \cap H_2) = P(H_1) \cdot P(H_2 \mid H_1)
\]

where \( H_1 \) is the event "obtaining a heart on the first draw" and \( H_2 \) is the event "getting a heart on the second draw." It seems to me that his thinking was similar to the following: 25% (or \( \frac{1}{4} \)) of the time we get a heart on the first draw.
\[ P(H_1) = \frac{1}{4}. \] Then we partition this \( \frac{1}{4} [P(H_1)] \) into two probabilities, \( P(H_1 \cap H_2) \) (about 6% out of 25% we obtain a heart on the second draw and a heart on the first draw) and \( P(H_1 \cap H_1') \) (about 19% out of 25% we do not get a heart on the second draw and a heart on the first draw). That is, he is representing that \( P(H_1) = P(H_1 \cap H_2) + P(H_1 \cap H_1') \). We notice, however, that he is not totally explicit about the connection of those probabilities and it is not clear why the "splitting" is relevant in getting to the final result. I do not mean to suggest that the representation just mentioned is important. What it is interesting to notice is that Mr. Kantor is not very explicit about the connections that he attempts to represent. Another thing worth noting is the representation that he is using a tree diagram. He attempts to use different types of representations so that students construct conceptual knowledge about the conditional probability formula. Another important phenomenon to notice is that while he used the definition of probability of an event to compute \( P(S \cap C) \) \( \frac{n(S \cap C)}{76} \), he did not use that definition to compute \( P(H_1 \cap H_2) \) \( \frac{13 \cdot 12}{52 \cdot 51} \). Puzzled by this I asked him, during the interview sessions, another way to find \( P(H_1 \cap H_2) \). He said that \( \frac{13 \cdot 12}{52 \cdot 51} \) was another way. We can see that this is a segment of mathematical knowledge that he did not use. This way of computing \( P(H_1 \cap H_2) \) provides students with another context in which to apply the multiplication counting principle and see the result as a product of probabilities \( \left[ \frac{13 \cdot 12}{52 \cdot 51} = \frac{13}{52} \cdot \frac{12}{51} \right] \). We notice also that he is assuming that students know, at least procedurally, how to get \( P(H_2 | H_1) \), that is \( P(B | A) \). He is not attempting to explain the meaning of that probability and the meaning of \( P(H_1 \cap H_2) = P(H_1) \cdot P(H_2 | H_1) \), specially from a conceptual point of view. As stated before, we will see later that some students still were having difficulties with the concept of conditional

\[ 5 \text{From a mathematical point of view, } P(H_1) = P(H_1 \cap H_2) + P(H_1 \cap H_1') \text{ is important because it is a representation of the total probability theorem (Bain & Engelhardt, 1992). However, it is not relevant here because it does not lead to the conditional probability formula.} \]
probability. After asking how much of the time we get two hearts, Mr. Kantor asked students other probabilities that can be computed from the diagram. After that a student asked Mr. Kantor a question about one example from the textbook. She was having difficulty with the concept of conditional probability. The pedagogical events constructed by Mr. Kantor are related in Excerpt 8.

Excerpt 8. Some students' struggles with conditional probability

1. K: The probability of drawing a heart as the first card and as the second card.... They use this 
\[
\frac{13}{52} \cdot \frac{12}{51}
\]. That's confusing? You take a deck, there are fifty two cards in there. This is the probability of drawing a heart in the first draw because there are 13 cards out of 52 that are hearts. Thirteen successes out of the fifty two possible. And they are saying this, don't put the card back. In the second draw you have already taken a card out and it's already a heart. This probability 
\[
\frac{12}{51}
\]. This is the probability you draw a heart on the second given that you drew a heart on the first \(P(H_2 \text{ given } H_1)\).

2. A: I don't understand that probability.

3. K: It means it has already happened.

4. H: What does it mean, has already happened?

5. A: Explain that probability, Mr. Kantor.

6. K: The probability a heart on the first and a heart on the second is the probability that you get a heart on the first one times the probability that you have a heart on the second given that this has already happened \(P(H_1 \text{ and } H_2) = P(H_1) \cdot P(H_2 \text{ given } H_1)\).

7. S: That's too confusing.

8. K: The point is this.... Twenty five percent of the time you draw a heart in the first one. Some of the time you draw a heart on the second and some of the time you don't. You're splitting up this probability by multiplying it by another percentage.... Let me go on, once you have already drawn a heart, there are only 51 cards left ... and you only have 12 hearts left because you pull one out of it. So now the chances of drawing a heart on the second one is 12 hearts out of 51 cards. That's why it is twelve over fifty one.... Yes?

9. H: What does given mean?

10. K: Given means it has already happened.

11. H: How can ... already happened ... how ... [Another student tries to explain this concept to Hanna].

(CO, 9, 10-26-94)

Excerpt 8 gives an opportunity to examine Mr. Kantor's pedagogical knowledge (knowledge of pictorial and story-problem representations) in action. Mr. Kantor is faced with the teaching problem of helping some confused student to understand the concept of conditional probability (speeches 2-11). There are several speeches where we can see that some students have some difficulties with the concept of probability (speeches 2, 4, 5, 7, 9, 11). I would say that those students are having major difficulties with the concept of
conditional probability. If there are some places where Mr. Kantor is going to use his knowledge of representations, this is probably one. I will stress the pedagogical events that he constructed for helping students construct the meaning of conditional probability, in this case $P(H_2 \mid H_1)$ when computing $P(H_1 \cap H_2) = P(H_1)P(H_2 \mid H_1) = \frac{13}{52} \cdot \frac{12}{51}$. Mr. Kantor explained that $\frac{13}{52}$ is the probability of drawing a heart on the first draw using the definition successes/possible (speech 1). Then he says: Assuming that we do not put the first card back, $\frac{12}{51}$ is "the probability you draw a heart on the second given that you draw a heart on the first" (speech 1) and points to this expression $P(H_2 \mid H_1)$. The students are confused by the word given and then he states the verbal representation of the formula $P(H_1 \text{ and } H_2) = P(H_1) \cdot P(H_2 \mid H_1)$ (speech 6). We notice that this representation does not help us to understand what $P(H_2 \mid H_1)$ means. One student says that it is confusing and then Mr. Kantor constructs the following explanation: "Twenty five percent of the time you draw a heart in the first one $[P(H_1) = \frac{1}{4}]$. Some of the time you draw a heart on the second and some of the time you don't $[P(H_1) = P(H_1 \cap H_2) + P(H_1 \cap H_2')]$. You're splitting up this probability by multiplying it by another percentage $[P(H_1) = P(H_1 \cap H_2) + P(H_1 \cap H_2') = P(H_1)P(H_2 \mid H_1) + P(H_1)P(H_2' \mid H_1)]$" (speech 8). Again, that explanation is not explicitly addressing the concept of conditional probability and therefore it is not surprising that the student is still confused. Then Mr. Kantor explains "once you have already drawn a heart $[H_1 \text{ has happened}]$, there are only 51 cards left ... And you only have 12 hearts left because you pull one out of it. So now the chances of drawing a heart on the second one is 12 hearts out of 51 cards. That's why it is twelve over fifty one" (speech 8). We notice, again, that this explanation is not totally addressing the concept of $H_2 \mid H_1$. As a consequence, the student has not understood that concept because she asks "What does given mean?" (speech 9). Mr. Kantor replies "Given means it has already happened" (speech 10). The student is still confused. At this point
we have a situation where Mr. Kantor has not been very successful in helping students
construct the concept of conditional probability and the students are insisting on
understanding that concept. A question emerges: "What pedagogical events is Mr. Kantor
going to construct to explain that concept?" Those pedagogical events will provide
additional opportunities to examine Mr. Kantor's knowledge of representations. The next
one is related in Excerpt 9.

Excerpt 9

1. K: Look, if you want to find out this probability [He writes P(Blonde and Run Track) on the board].
   Probability that someone is blonde and they run track. Take the probability that they're blonde and
   we can figure that out... Let's say there is 25% blondes in the school, and let's say that 40% of
   people in the school run track. So the way to find this out [P(Blonde and run track)], take the
   percentages of blondes times the percentage of people that run track [P(Blonde) • P(Run Track)].
   That does not give you the answer. You have to take the percentage of people who run track that
   are blonde [P(Blonde) • P(Run Track given Blonde)].
2. S: Ahh.
4. K: That's how you do it. These percentages. There is overlap there. You have to deal with it. If I say
   percentage of people who run track given that you are looking at blondes. [Students are struggling
to understand that]. (CO, 9, 10-26-94)

In Excerpt 9 Mr. Kantor uses another example to illustrate the concept of conditional
probability. The story-problem representation is the following: 25% of students are
blonde, 40% of students run track, find P(Blonde and runs track), that is, the probability
that a randomly selected student is blonde and runs track [P(B ∩ R)]. It seems to me that
he represented P(B ∩ R) as P(B) • P(R) but that he realized immediately that that
representation was incorrect. This immediate realization provides additional evidence that
his knowledge is well connected. We notice, however, that that problem did not represent
P(B ∩ R) = P(B) • P(R | B) because P(R | B) is not known. Nevertheless, it is
pedagogically appropriate that Mr. Kantor constructs an example to illustrate an incorrect
formula that some students use to find P(B ∩ R). During interviews he stated that some
students think that P(A ∩ B) = P(A) • P(B). My teaching experience agrees with that.
After stating that P(B ∩ R) cannot be found by P(B) • P(R) he wrote the right formula on
the board, P(Blonde) • P(Run Track given Blonde). We notice also that the example did
not accomplish its purpose of representing conditional probability with a story-problem representation because \( P(R \mid B) \) is unknown. The next excerpt describes another attempt to construct a problem to illustrate the conditional probability formula.

Excerpt 10

Let's do it this way. How many people who are blonde are here? ... one, two, three, four, five, six. Six out of 24 are blonde. Now, how many people run track? ... one, two, three, four, five, six, seven.

Look, I don't multiply those percentages together to get the percentage of people \( \frac{6}{24} \cdot \frac{7}{24} \). Listen, listen. If I multiply those together ... What I should do is this.... I should say, OK. Now all the blondes, how many of you run track because that's what I'm trying to find. One, two, three, four.

Four out of six \( \frac{4}{6} \). Now, when I multiply that out six cancels out. \( \frac{6}{24} \cdot \frac{4}{6} = \frac{4}{24} \). Four out of twenty four students are blonde and go run track ... one given the other. (CO, 9, 10-26-94)

We notice that the story-problem representation is a more realistic problem. He is using data from his students to contextualize the problem of finding \( P(B \cap R) \). There are several things worth noticing here. The first has just been mentioned. It is the use of real data to represent the probabilities, \( P(B) \), \( P(R) \), \( P(R \mid B) \). There were 6 students who were blonde, 7 students who run track, and 4 blonde students that run track. The problem is to find \( P(B \cap R) \), the probability that a student is blonde and runs track and the probabilities are \( P(B) = \frac{6}{24} \), \( P(R) = \frac{7}{24} \), \( P(R \mid B) = \frac{4}{6} \). Mr. Kantor computed \( P(B) \cdot P(R) = \frac{6}{24} \cdot \frac{7}{24} \) and \( P(B \cap R) = P(B) \cdot P(R \mid B) = \frac{6}{24} \cdot \frac{4}{6} = \frac{4}{24} \). A second important thing is that while he computed both \( P(B) \cdot P(R) = \frac{6}{24} \cdot \frac{7}{24} \) and \( P(B \cap R) = P(B) \cdot P(R \mid B) = \frac{6}{24} \cdot \frac{4}{6} = \frac{4}{24} \) he did not stress that \( P(B) \cdot P(R) = \frac{6}{24} \cdot \frac{7}{24} \) does not give \( P(B \cap R) \) because there are 4 students out of 24 who are blonde and run track. He did not stress either that \( P(B) \cdot P(R \mid B) = \frac{6}{24} \cdot \frac{4}{6} = \frac{4}{24} \) agrees with finding \( P(B \cap R) \) using the definition, \( \frac{n(B \cap R)}{N} = \frac{4}{24} \). A third interesting thing is that he did not take advantage of this realistic and concrete example to explain the meaning of \( P(R \mid B) \). It seems to me that students were taking too literally the fact that the verbal representation of \( P(R \mid B) \) is the probability of R given that B has already
happened. Another equivalent verbal representation for \( P(R \mid B) \) is the probability of \( R \) with respect to the sample space \( B \). Still another verbal representation is the probability of choosing a blonde person that runs track, that is, \( P(R \mid B) = \frac{n(R \cap B)}{n(B)} \).

During the review session for a quiz Mr. Kantor used the following example:

Six students in a class of 25 have the flu. Two of these six are girls. Thirteen of the 25 students in the class are boys. Draw a tree diagram. a) What is the probability that a randomly chosen student is a girl? b) What is the probability that a randomly chosen student with the flu is a girl? c) What is the probability that a randomly chosen student is girl with the flu? (McConnell et al., 1990, p. 200)

Mr. Kantor brought a transparency with a display similar to the one depicted in Figure 5.15. The pedagogical events are related in Excerpt 11.

Excerpt 11

I started with breaking the class up by boy/girl. Thirteen boys which means there must be twelve girls. So thirteen twenty-fifths of the class is boys [pointing out to the diagram]. Twelve twenty-fifths of the class is girls.... Next, it tells you that six people have the flu. Two of those are girls. What does that mean? ... So that means, four out of the 13 boys have the flu, so the other nine out of 13 do not, and two of the 12 girls have the flu and 10 out of 12 do not.... Let me explain this again because it's important you understand this. When you talk about conditional probability, because one thing that matters in this question is, if you pick up a person at random, what are the chances that it's a boy that has the flu. Now, they are saying the way to get those probabilities you multiply this probability \( \frac{13}{25} \) times that \( \frac{4}{13} \). It's important that you understand why that is. Literally one half a class is boys and four thirteenths of that, four thirteenths of that are those who have the flu [pointing out to the corresponding numbers], so you are breaking it up, you are taking a fraction of it [pointing out both fractions, referring to taking \( \frac{4}{13} \) of \( \frac{13}{25} \)]. You are multiplying to get this \( \frac{4}{25} \). You multiply this \( \frac{9}{13} \) times that \( \frac{13}{25} \). This fraction \( \frac{9}{13} \) of that \( \frac{13}{25} \) are the boys that do not have the flu. This fraction \( \frac{2}{12} \) of this \( \frac{12}{25} \) are those girls that have the flu \( \frac{2}{25} \). ... It's important that you understand that's what conditional probability really is, that you are taking a probability and splitting it apart. (CO, 10, 10-27-94)
I will use the following notation when discussing this excerpt: the experiment is to select a student. Let $B$ be the event "the student is a boy," $G$ the event "the student is a girl," $F$ the event "the student has the flu," $NF$ the event "the student does not have the flu," $F \mid B$ the event "the student is a boy that has the flu," $NF \mid B$ the event "the student is a boy that does not have the flu," $F \mid G$ the event "the student is a girl that has the flu," $NF \mid G$ the event "the student is a girl that does not have the flu," $B \cap F$ the event "the student is a boy and has the flu," $B \cap NF$ the event "the student is a boy and does not have the flu," $G \cap F$ the event "the student is a girl and has the flu," $G \cap NF$ the event "the student is a girl and does not have the flu." In Excerpt 1, Mr. Kantor computes $P(B)$ and $P(G)$ without making, as expected, many comments. It is when he begins talking about conditional probability that he stresses the probabilities. Mr. Kantor is explaining that to find the probability of selecting a student that is a boy and has the flu $[P(B \cap F)]$, we multiply $\frac{13}{25} [P(B)]$ by $\frac{4}{13} [P(F \mid B)]$. The reason we multiply those probabilities is that "Literally one half a class is boys $[\frac{13}{25}]$ and four thirteenths of that, four thirteenths of that are those who have the flu, so you are breaking it up, you are taking a fraction of it."
\[ \frac{4}{13} \text{ of } \frac{13}{25} \text{ or } \frac{4}{13} \cdot \frac{13}{25} \] is the fraction of students who are boys and have the flu, \( \frac{4}{25} \). We notice that he is explaining why \( P(B \cap F) = P(B) \cdot P(F \mid B) \). However, Mr. Kantor does not justify why \( \frac{4}{13} \cdot \frac{13}{25} \) represents the fraction of students who are boys and have the flu. Probably he does not realize that the associative property of multiplication is hidden here.

He then explains with less explicitness that \( \frac{9}{13} \) of \( \frac{13}{25} \) is the fraction of students who are boys and do not have the flu, "this fraction \( \frac{9}{13} \) of that \( \frac{13}{25} \) are the boys that do not have the flu" and similarly that \( \frac{2}{12} \) of \( \frac{12}{25} \) is the fraction of girls with flu, "this fraction \( \frac{2}{12} \) of this \( \frac{12}{25} \) are those girls that have the flu \( \frac{2}{25} \)." Those are potentially powerful representations since they show why \( P(B \cap F) = P(B) \cdot P(F \mid B) \) if teachers explain why they are true. However, we notice also that Mr. Kantor missed this opportunity for constructing explanations about the meaning of conditional probability, \( P(F \mid B) \). Excerpt 12 relates the next teaching episode in which Mr. Kantor constructed another pictorial representation for illustrating \( P(A \cap B) = P(A) \cdot P(B \mid A) \) similar to the one depicted in Figure 5.16.

**Excerpt 12**

That line is a hundred percent of the length. If you have these two lines together, their lengths add up to that. Obviously, split out boys and girls, that's a hundred percent.... All I'm doing is splitting the class up: boy, girl, flu, no flu, in each of these categories. This line here is thirteen twenty-fifths of this line because that's the proportion of the class that is boys. It's important that you understand here. I take four thirteenths of this length, four thirteenths of this length is the boys that have the flu. I end up with this. This is four twenty-fifths of the total class. It's a way of seeing in a picture what actually is the case.... You see how by splitting that out I can kind of see that's what's happening. Because that's what's happening in conditional probability. I take a portion of the class and split that into flu/no flu. Take this portion and split it out. And I do that by multiplying probabilities. This \[ \frac{13}{25} \] times this \[ \frac{4}{13} \] gives me four twenty-fifths of the total. This \[ \frac{13}{25} \] times this \[ \frac{9}{13} \] gives me nine twenty-fifths of the total. I multiply this \[ \frac{2}{12} \] times this portion \[ \frac{12}{25} \] and I end up with two twenty-fifths of the total class, does that make sense? That there are twelve girls in class, two of those twelve have the flu. That means two out the twenty five have the flu. Look here. Thirteen boys in the class. Nine of the thirteen do not have the flu. That means nine of the twenty five of the class are boys who don't have the flu. Make sense? (CO, 10, 10-27-94)
We notice that he is illustrating each of the probabilities with a graphical representation:

\[
P(B) = \frac{13}{25} \text{ (AB and CE)}, \quad P(G) = \frac{12}{25} \text{ (BJ and KM)}, \quad P(F \mid B) = \frac{4}{13} \text{ (CD)}, \quad P(NF \mid B) = \frac{9}{13} \text{ (DE)},
\]

\[
P(F \mid G) = \frac{2}{12} \text{ (KL)}, \quad P(NF \mid G) = \frac{10}{12} \text{ (LM)}, \quad P(B \cap F) = \frac{4}{25} \text{ (FG)}, \quad P(B \cap NF) = \frac{9}{25} \text{ (HI)}, \quad P(G \cap F) = \frac{2}{25} \text{ (NO)}, \quad P(G \cap NF) = \frac{10}{25} \text{ (PQ)}.
\]

He is also representing pictorially that \( P(B \cap F) = P(B) \cdot P(F \mid B). \) However, he is more explicit in some cases than in others. For example, he is more explicit representing verbally \( P(B) \) than \( P(F \mid B) \) and \( P(B \cap F) = P(B) \cdot P(F \mid B). \) To illustrate, when Mr. Kantor is talking about \( P(B) \) he says 13 out of 25 students are boys and therefore \( P(B) = \frac{13}{25}. \) On the other hand, when he is talking about \( P(F \mid B) \) and \( P(B \cap F) \) he says "I take four thirteenths of this length, four thirteenths of this length is the boys that have the flu. I end up with this. This is four twenty fifths of the total class." It is not clear for me in what moment he is talking about \( P(F \mid B) \) and \( P(B \cap F). \) It seems to be that he is saying only that for finding \( P(B \cap F) \) we take \( \frac{4}{13} \) of \( \frac{13}{25} \). When he says 4/13 of this length that may mean \( \frac{4}{13}, \frac{13}{25} \), which is \( P(B \cap\)
Or it may mean \( \frac{4}{13} \) with respect to that length and in this case that means \( P(F \mid B) \). In both cases it is the same length. While CD with respect to CE represents \( P(F \mid B) \), it represents \( P(B \cap F) \) with respect to AJ. Mr. Kantor is not explicit about that critical distinction. That may be due to the fact that his knowledge is not totally connected. As we saw before, when he was constructing pictorial representations about conditional probability \( [P(B \mid A)] \) and \( P(A \cap B) \) he had some trouble constructing a pictorial representation for \( P(B \mid A) \). He is constructing here, at least implicitly, four of them: CD with respect to CE represents \( P(F \mid B) \), DE with respect to CE represents \( P(NF \mid B) \), KL with respect to KM represents \( P(F \mid G) \), and LM with respect to KM represents \( P(NF \mid G) \). In those cases we take CE and KM as having one unit. Probably he was not totally explicit about conditional probability because he himself did not have a totally explicit knowledge about the fact that \( n(B \mid A) = n(A \cap B) \) but that \( P(B \mid A) \) and \( P(A \cap B) \) refer to different sample spaces. In that teaching episode he was more explicit about the fact that \( P(B) - P(F \mid B) \) gives \( P(A \cap B) \), "I multiply this \( \left[ \frac{2}{12} \right] \) times this portion \( \left[ \frac{12}{25} \right] \) and I end up with two twenty fifths of the total class, does that make sense? That there are twelve girls in class, two of those twelve have the flu, that means two out the twenty five have the flu. Look here. Thirteen boys in the class. Nine of the thirteen do not have the flu. That means nine of the twenty five of the class are boys who don't have the flu, \( \left[ \frac{13}{25}, \frac{9}{25} \right] \). Then Mr. Kantor and his students compute other probabilities, \( P(NF) = \frac{n(NF)}{25} = \frac{19}{25} \) and \( P(F) = P(B \cap F) + P(G \cap F) = \frac{4}{25} + \frac{2}{25} = \frac{6}{25} \). After this Mr. Kantor said that some students still were having trouble with the concept given. It was not surprising to me because Mr. Kantor had not constructed explanations and representations addressing the concept of conditional probability very explicitly.
The next example that Mr. Kantor constructed to illustrate the meaning of "given" was in the context of cards. This teaching episode is described in Appendix N. Briefly, the situation was to draw two cards out of a deck of 52 and not putting back the first card before drawing the second card. The problem consisted of finding the probability of obtaining two diamonds using the formula $P(D_1 \cap D_2) = P(D_1) \cdot P(D_2 \mid D_1)$ where $D_1$ represents the event "to get a diamond on the first draw," and $D_2$ represents the event "to obtain a diamond on the second draw." Mr. Kantor focused on the meaning of $P(D_2 \mid D_1)$ in terms of an event that has already happened. As we saw earlier, the concept of "has already happened" did not help students much to understand what "given means."

The use of deck of cards is a common context for teaching the concept of conditional probability. However, I think that $P(D_2 \mid D_1)$ is more difficult to conceptualize than either $P(R \mid B)$ (the example of blondes who run track) or $P(F \mid B)$ (the example of boys with flu). Even though students need to have a variety of experiences to construct mathematical knowledge, in this case, the concept of conditional probability, I believe that we need to choose contexts that are easier to conceptualize and then move to more abstract situations. After that teaching episode (Appendix N), Mr. Kantor talked about the different representations of conditional probability that he had constructed. This is related in Excerpt 13.

Excerpt 13

1. K: Look at this again. Probability it's a boy and the person has the flu $[P(B \cap F)]$. Probability it's a boy $[P(B)]$, there are thirteen boys out of the twenty five $\frac{13}{25}$, times the probability somebody has the flu given a boy $[P(F \mid B)]$. That means, when you do this just look at the boys; to get this problem you just look at the boys. There are only thirteen boys and four of those have the flu $\frac{4}{13}$. This one $[P(B)]$ looks at the whole class. This one $[P(F \mid B)]$ just looks at the boys. This one $[P(D_1)]$ looks at the whole deck. This one $[P(D_2 \mid D_1)]$ just looks at fifty one cards.... They look at different things. It has to be that way, otherwise, if I get this, I do the probability of you pick up a boy times the probability somebody has the flu. That is mixing things up ...
2. S: The way I figured it out ...
3. K: ... Does that make sense, what given means? given something has already happened. These percentages depend on what happened on the first draw. The probability of drawing a diamond on the second draw if you didn't draw one on the first is different. You have more chances on the second draw of getting a diamond if you didn't get one on the first than if you did because there is one less card. It's like, you know, if you are working on a straight, the probability of getting a straight, thirteen diamonds on the first fifty two times twelve out of the next fifty one times eleven out of forty nine times nine out of forty eight
\[
\begin{array}{cccc}
13 & 12 & 11 & 10 & 9 \\
52 & 51 & 50 & 49 & 48 \\
\end{array}
\]
That's really a small probability. (CO, 10, 10-27-94)

In speech 1, Mr. Kantor explains what conditional probability means in the context of the probability that a selected boy has the flu. He says "that means, when you do this just look at the boys; to get this problem you just look at the boys. There are only thirteen boys and four of those have the flu \[ \frac{4}{13} \]." That example is more concrete than the one referring to the deck of cards. Then he says "This one \([P(B)]\) looks at the whole class. This one \([P(F given B)]\) just looks at the boys. This one \([P(D_1)]\) looks at the whole deck. This one \([P(D_2 given D_1)]\) just looks at fifty one cards.... They look at different things." We notice that in that speech Mr. Kantor is making reference to the fact that \(P(A)\) and \(P(B | A)\) are computed with respect to different sample spaces. That is the critical property of conditional probability. From my point of view, all probabilities are conditional because they refer to a certain sample space. There is no universal sample space. Each experiment gives origin to the "standard sample space." After those explanations Mr. Kantor computed the probability of getting a straight of diamonds when drawing 5 cards from a deck (speech 3). That example provides a generalized representation of the conditional probability formula. After that representation Mr. Kantor and his students talked about the probabilities related to the lottery. But this is another story and should be told in another time.

Mr. Kantor constructed pedagogical events for the curriculum events of classical definition of conditional probability \((P(B | A) = \frac{n(B | A)}{n(A)}\) and the conditional probability formula \([P(A \cap B) = P(A)P(B | A)]\). The explanations and representations involved strong
components of both procedural and conceptual knowledge with the conceptual dominating the procedural ones. Mr. Kantor not only wanted students to use the formulas of these two curriculum events but he also wanted them to have an understanding of why the formulas work. He did not construct pedagogical events for the second definition of conditional probability \[ P(B \mid A) = \frac{P(A \cap B)}{P(A)} \], probably due to his lack of well-articulated knowledge about that curriculum event.

**Explanations and Operational-Structural Conceptions of Algebraic Objects**

The categorization of explanations as operational or structural did not emerge from the analysis of the data. They were brought to the data. As noted in Chapter 4, the term operational refers to computational operations carried out on numerical examples and which yield a numerical result. On the other hand, the term structural refers to "a set of operations that are carried out, not on numbers, but on algebraic expressions." That is, "the operations that are carried out are not computational" and "the results are yet algebraic expressions" (Kieran, 1992, p. 392).

**Operational explanations.** Mr. Kantor constructed explanations involving operational elements for the following 11 curriculum events: rule for multiplying fractions (CCE 6), definition of greater than (CCE 24), the multiplicative properties of inequalities (CCEs 25 & 26), the multiplication counting principle (CCE 28), classical definition of probability (CCE 29), conditional probability formula (CCE 31), the factorial symbol (CCE 32), the permutation theorem (CCE 34), meaning of division (CCE 35), and algebraic definition of division (CCE 36). I will illustrate Mr. Kantor's operational explanations with the multiplicative property of inequalities. This is related below.

Mr. Kantor began the teaching of the lesson solving \( ax < b \) with constructing some numerical (i.e., operational) examples of the first part and second part of the multiplicative property of inequalities. This teaching episode is described in Excerpt 14.
Excerpt 14: Operational examples of the multiplicative property of inequality:
If $5 < 7$ then $10 < 14$ and $-10 > -14$

1. K: OK. On your paper, write down those two numbers $[5 \quad 7]$. Five and seven, what's the relationship between those two numbers? ... Are they equal, less than, greater than?
2. S: Seven is greater than five.
3. K: So I can point to five, correct? $[5 < 7]$. Frank, If I double five and get ten, double seven and get fourteen,—double five and double seven $[10 \quad 14]$, what's the relationship then?
4. S: Ten is still less than fourteen.
5. K: The relationship stays $[10 < 14]$ and write all this down. Write everything down that I do. I want you to follow some logically.... Now, if I take and multiply each of these by negative one, negative ten, negative fourteen $[-10 \quad -14]$, we get negative ten and negative fourteen, what do I put in between those?
6. S: Greater [Students talking about that].
7. K: Negative ten is greater than negative fourteen $[-10 > -14]$. (CO, 5, 10-19-94)

As Excerpt 14 shows, Mr. Kantor's first representations of the multiplicative property of inequalities were the two numerical examples: if $5 < 7$ then $10 < 14$ and $-10 > -14$. The teaching-learning event in which M. Kantor multiplies both sides of the inequality $5 < 7$ by 2 to obtain $10 < 14$ is critical (speech 3). He has constructed an operational example of the multiplication property of inequalities whose purpose is that students generalize and begin to construct this portion of mathematical knowledge: When we multiply both sides of an inequality by a positive number the sign of the inequality stays the same. After that, Mr. Kantor multiplied both sides of the inequality $10 < 14$ by $-1$ to obtain $-10 > -14$. This is another critical event. Mr. Kantor has constructed an operational example of the second part of the multiplicative property of inequalities: When we multiply both sides of an inequality by a negative number the sense of the inequality changes. His purpose is that students generalize and construct that segment of mathematical knowledge. But probably he believes, as I do based on my experience as teacher and mathematics educator, that those representations are not enough for some students to construct powerful representations of this segment of mathematical knowledge. They need to have some experience with more and other kinds of representations. Probably that was Mr. Kantor's conscious or unconscious purpose when he constructed some pictorial representations as described in Excerpt 15.

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Excerpt 15: Pictorial representations of If $5 < 7$ then $10 < 14$ and $-10 > -14$

1. K: I want you to draw some number line like this,... and lets say zero is about, in the middle. Here is five and here is seven, that's the order. Now, when I multiply by two, this one right here, at ten, this one right here at fourteen.... Those lines didn't cross, five is less than seven, ten is less than fourteen. They stay in the same order. Now, when I went down to multiply by negative, all of sudden, they switch the order. Ten became negative ten, and fourteen crossed over this and became negative fourteen. So the order switches, ten and fourteen. Now we switch, negative fourteen and negative ten.

2. S: That's weird, though.

3. K: Look, if you look at it on a number line, ... this is to the left of this $5 < 7$. This is to the left of this $10 < 14$. It's a bigger path. Lines didn't cross. But when we multiply by negative, now all of sudden, this is to the right of that $-10 > -14$ [See figure below]. (CO, 5, 10-19-94)

In this short episode Mr. Kantor constructed some pictorial representations to illustrate geometrically that when we multiply both sides of the inequality $5 < 7$ by a positive number we get an inequality with the same sense, $10 < 14$, but that when we multiply both sides by a negative number, the sense changes. Although those pictorial representations only reflect number facts, they may help some students to construct the meaning of the multiplicative property of inequalities. At this point in time a student asked Mr. Kantor his definition of greater than. Mr. Kantor said that:

Greater than on a number line ... is to the right of something else.... We will use these numbers, negative three and a positive four. We draw a line.... Here is zero, negative three is here, positive four is here [What's the relationship between those two?].... Negative three is less than four [$-3 < 4$], it's to the left of four. Negative three is to the left of four and so it's less than four. (CO, 5, 10-19-94)

In this teaching episode Mr. Kantor provided an informal definition of what it means that a number being greater than another one and illustrated that definition with $-3$ and $4$. 187
The number -3 is less than 4 because -3 is to the left of 4. He did not use his algebraic definition that he had in the interview to say that -3 is less than 4 because \(-3 - 4 = -7\) is less than zero. After that Mr. Kantor constructed another numerical representation of the multiplicative property of inequalities. This teaching event is described in the following excerpt.

Excerpt 16: Pictorial representations for if \(-3 < 4\) then \(-6 < 8\) and \(6 > -8\)

1. K: When I double that \([-3 < 4]\). I double that. I get negative six and I get eight. And now, double this one, get negative six, double this one, get eight, and still this side is to the left of that one \([-6 < 8]\). It's still less than. And now, when I multiply both sides by negative... This one goes over here. This one goes over there and now, what's the relationship [see figure below]. Six is greater than negative eight \([6 > -8]\). Now the question is this, you've seen this happen. If I have an inequality and I multiply both sides by a negative number, what happens?

\[
\begin{array}{c|c}
-3 & 4 \\
-6 & 8 \\
-8 & 6 \\
\end{array}
\]

2. S: It switches
3. K: The sense switches. This is how the sense's switching. The sense switches. The sense switches. Multiply both sides by a negative number and the sense switches. (CO, 5, 10-19-94)

As we notice in Excerpt 16, Mr. Kantor uses an example in which one of the numbers is negative to illustrate another particular case of the multiplicative properties of inequalities. Again, those critical incidents have as their purpose to help students construct the meaning of that mathematical curriculum event. After these and the earlier examples Mr. Kantor probably feels that students are somewhat ready for stating the second part of the "great theorem." He does so in the following terms: "multiply both sides by a negative number and the sense switches" (speech 3). Once students have had some experience with that property, some of them may have the cognitive structures to construct the next segment of mathematical knowledge: how to solve inequalities of the form \(ax < b\) with understanding. After this teaching episode, Mr. Kantor constructed some examples of how to solve inequalities of the form \(ax < b\) (\(-5x < 10, -12 < 48x, -10u < 0, 10-5a < 10\)). Since
students were having difficulties in understanding the application of the multiplicative property of inequalities for solving those types of inequalities, Mr. Kantor attempted to construct a story-problem representation of the multiplicative property of inequalities after teaching how to solve $10 - 5a < 10$. The story problem is described in Appendix I.

**Structural explanations.** Mr. Kantor constructed explanations involving structural elements for the following seven content curriculum events: rate model for multiplication, the multiplicative identity of 1, definition of reciprocals, the reciprocal of $x$ is $1/x$, reciprocal of zero, solving $ax = b$, and solving $ax < b$. I will illustrate Mr. Kantor's structural explanations with the rate model for multiplication.

I will begin with the description of the rate model of multiplication provided by the textbook,

> When a rate is multiplied by another quantity, the unit of the product is the product of the units. Units are multiplied as though they were fractions. The product has meaning when the units have meaning (McConnell et al., 1990, p. 164).

Mathematically speaking, units can not be multiplied because they are not numbers. However, the point of the rate model of multiplication is not to state that units represent numbers and as a consequence they can be multiplied but to point out that "units are treated as if they were factors [variables or numbers] in a multiplication" (McConnell et al., 1990, p. 163). The rate model for multiplication is one of the bases for multiplying physical quantities, as is done in dimensional analysis.

Mr. Kantor began his teaching of the rate model for multiplication by asking a student to read a relevant passage from the textbook. Then he constructed some pedagogical events about the meaning of what the textbook said. This is related in Excerpt 17.
Excerpt 17

1. S: Rates can be multiplied by other quantities. The units are treated as if they were factors in a multiplication.

2. K: Rates can be multiplied by other quantities. Here is a rate $\frac{55 \text{ mi}}{\text{hr}}$, and I'm going to multiply it by another quantity [12 hr].... What does that mean? $\frac{55 \text{ mi}}{\text{hr}} \cdot 12 \text{hr}$ ...

3. S: Like, for example, if you go 55 miles per hour for 12 hours, how far do you go?

4. K: But that's not the example.... The problem is this. I go 55 miles per hour and you go on a trip that takes 12 hours. So, I say OK. I'll take 55 miles per hour times 12 hours [pointing $\frac{55 \text{ mi}}{\text{hr}} \cdot 12 \text{hr}$], ... what's the problem with my answer? [inaudible] Right. My speed has nothing to do with her trip.... I can multiply a rate by a quantity if the quantity is related to the rate. The book doesn't tell you that. If Emily goes in a twelve-hour trip that averages 55 miles per hour. this $\frac{55 \text{ mi}}{\text{hr}} \cdot 12 \text{hr}$ has meaning. Rate times time. The reason it has meaning. We talked about this last year. That twelve-hours trip [12 hr], when I multiply it by this $\frac{55 \text{ mi}}{\text{hr}}$ it's not getting any bigger, what's fifty five times 12? [a student answers six hundred and sixty miles].... Now, does that [660 mi] look bigger than this? [12 hr] [a students says yeah] But it's not. It's exactly the same thing as this [12hr].

5. S: How?

6. K: Our 12 hour trip is 660 miles long.... Twelve hours is the same trip as 660 miles in her case. That's a trip [12 hr], that's a trip [660 mi]. When I multiply it by a rate $\frac{55 \text{ mi}}{\text{hr}}$, it's not getting any bigger, it's just changing the way it looks. Remember, a rate has a value of one $\frac{55 \text{ mi}}{\text{hr}}$, 55 mi = 1 hr. Fifty five miles and an hour is the same thing. It's how far she goes in one hour. They're different units but they are talking about the same thing. That's why I multiply a rate times a quantity. I'm not getting bigger. It's multiplication where I don't get bigger. It's multiplication by one. $\frac{55 \text{ mi}}{\text{hr}} \cdot 12 \text{hr}$ [pointing to $\frac{55 \text{ mi}}{\text{hr}}$] OK? (CO, 1, 10-13-94)

The main idea of the rate model is that a rate can be multiplied by other quantities but only if they are related. In this excerpt, Mr. Kantor makes that point by giving an example in which the rate, $\frac{55 \text{ mi}}{\text{hr}}$, is not related to the quantity, 12 hr, and therefore the multiplication of them has no meaning. He goes on to emphasize that in the case where the quantity and the rate are related, and therefore the quantity 12 hr can be multiplied by the rate $\frac{55 \text{ mi}}{\text{hr}}$, multiplying by a rate does not change the quantity but that only the quantity is changed to different units because the rate has a value of one. His comments can be
interpreted that he is thinking of 12 hr as equal to 660 mi. Mathematically speaking, however, that is not the case because 12 hr is a time quantity and 660 mi is a distance quantity.

I have transcribed Excerpt 17 in detail to give the reader a sense of the complexity of Mr. Kantor's pedagogical events and the elements involved in constructing them. The remaining explanations constructed by Mr. Kantor when teaching the rate model for multiplication are described in Appendix O.

Explanations 2 and 3 (Appendix O) were provided as a reaction to the way students solved the corresponding problems. As we noticed before, the students used operational approaches instead of using the rate model of multiplication for solving the problems 2 and 3 described in Appendix O. Mr. Kantor, on the other hand, constructed structural explanations because his goal is that students learn to use algebraic approaches. It was difficult for students to think of units as variables. As a consequence, Mr. Kantor constructed more examples to help students learn this main idea of the rate model. The corresponding explanations are 4 and 5 and are described in Appendix O. In explanation 4, Mr. Kantor provided several examples in which the units were a fundamental component of a physical situation. After that he constructed the example $\frac{3x \cdot 6y}{l \cdot 2x}$ to illustrate the property of cancellation of variables and made the remark that the units are like variables and then they can be canceled out, too. A student was perplexed by this and asked him why that was possible. Andrea, on the other hand, made the remark that cancellation was possible for variables. She was probably thinking that while cancellation made sense for variables it was not possible for units. Mr. Kantor then asked students if three feet was the same as three times a foot $[3ft = 3ft]$. A student said yes and then Mr. Kantor asked "how can you just get rid of it? (the units) probably hoping that students would answer by carrying them through the problem. However, a student said that he would put them at the end, to which Mr. Kantor replied that there were problems where it is not easy to figure
that out and he posed the problem of converting $\frac{4 \text{ ft}}{\text{sec}^2}$ to $\frac{\text{in}}{\text{hr}^2}$. Some students proposed to do the conversion by multiplying and dividing (without using the rate model). Explanation 5 continues Mr. Kantor's efforts to make students see the point of the rate model. In this explanation Mr. Kantor poses a conversion problem which is not easy to do without the rate model of multiplication. When the students attempted to do the conversion they did have some trouble in figuring what to multiply by. That trouble was not enough to convince Andrea of the effectiveness of this model after Mr. Kantor used it to do the conversion.

Although the rate model of multiplication allows us to treat units as variables or numbers, it is not correct to state that reciprocal rates are the same or that $3 \text{ ft} = 3 \cdot \text{ft}$ or that "three pages is the same thing as a document" as Mr. Kantor did in all these three cases. We need to emphasize this observation and the fact that the rate model of multiplication justifies to treat units as variables or numbers but that the units are not variables nor numbers.

In summary, Mr. Kantor's explanations about the rate model involve strong elements of structural elements in the sense that he is emphasizing the use of the rate model of multiplication. The operations that Mr. Kantor carried out were not only on numbers but they were carried out in "algebraic expressions" to produce "algebraic expressions" by using the rate model for multiplication. (Mathematically speaking, of course, expressions like $\frac{\text{mi}}{\text{hr}}$ are not algebraic expressions, but they can be treated as if they were algebraic expressions.)

Explanations and Integration of Concepts

I categorized explanations also by their degree of integration of concepts into four types: (a) explanations with zero degree of integration, (b) explanations with one degree of integration, (c) explanations with two degrees of integration, and (d) explanations with
more than two degrees of integration. I define explanations with zero degree of integration as those explanations that do not involve connections to any other concept. The concepts stand in isolation from one another. Explanations with one degree of integration involve some connections to another concept. Explanations with two degrees of integration involve some connections to other two concepts. Explanations with more than two degrees of integration involve connections to more than two concepts. Table 5.6 displays researcher’s classification of the content curriculum events for which Mr. Kantor constructed explanations during regular teaching sessions. It also displays the categorization of the content curriculum events according to Mr. Kantor’s degree of integration during classroom instruction. A more complete description of why these content curriculum events were classified as shown in the table can be found in Appendix H.

<table>
<thead>
<tr>
<th>Researcher’s classification of content curriculum events according to their degree of integration</th>
<th>Classification of Mr. Kantor’s explanations according to their degree of integration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Content curriculum events with one degree of integration</td>
<td>14, 15, 24, 28, 32 (5)</td>
</tr>
<tr>
<td>Content curriculum events with two degrees of integration</td>
<td>12, 14, 15, 16, 19, 28, 29c, 31, 32, 34, 35, 36 (12)</td>
</tr>
<tr>
<td>Content curriculum events with more than two degrees of integration</td>
<td>6, 7, 25, 26, 27 (5)</td>
</tr>
</tbody>
</table>

Table 5.6 Classification of content curriculum events according to their relationship to other concepts
All but one of the explanations constructed by Mr. Kantor involved some integration of concepts (definition of $a > b$). Mr. Kantor constructed explanations with one degree of integration for the following 12 content curriculum events: the multiplicative identity of one (CCE 12), definition of reciprocal (CCE 14), the reciprocal of $a$ is $1/a$ (CCE 15), reciprocal of zero (CCE 16), solving $ax = b$ (CCE 19), multiplication counting principle (CCE 28), classical definition of conditional probability (CCE 29c), conditional probability formula (CCE 31), $n!$ (CCE 32), permutation theorem (CCE 34), meaning of division (CCE 35), and algebraic definition of division (CCE 36). Of those content curriculum events, I categorized the following seven as having at least two degrees of integration: the multiplicative identity of one (CCE 12), solving $ax = b$ (CCE 19), classical definition of conditional probability (CCE 29c), conditional probability formula (CCE 31), permutation theorem (CCE 34), meaning of division (CCE 35) and the definition of division (CCE 36). To illustrate Mr. Kantor's explanations with one degree of integration, he constructed the explanation displayed in Excerpt 18 for the reciprocal of zero.

**Excerpt 18:** 0 has no reciprocal

1. K: Now, every number has an opposite, everybody listen, every number has an opposite but not every number has a reciprocal. What number does not have a reciprocal?

2. S: Zero

3. K: Look, one way of looking at this is this. If I multiply B by anything I can go back to where I started but unless I multiply by zero. Once I multiply by zero I wipe out. I can't multiply that by something to get back to B [$B \cdot 0 \cdot = B$]. Remember I talked about this being called in Europe the eliminator? If you multiply by that it eliminates.... The eliminator doesn't have a reciprocal. listen, the eliminator doesn't have a reciprocal because once you multiply by it you can't go back to where you started. (CO, 2, 10-14-94)

We notice that Mr. Kantor went beyond stating that zero has no reciprocal. He provided a more elaborated explanation by making a connection to the definition of reciprocal he provided. Later, during the summary of the lesson *special numbers in multiplication*, a student asked Mr. Kantor the reciprocal of zero. He said that zero has no reciprocal because "there isn't something I multiply by zero to get one."
Mr. Kantor constructed explanations with two degrees of integration for the following five content curriculum events: rule for multiplying fractions (CCE 6), the rate model for multiplication (CCE 7), the two multiplicative properties of inequalities (CCEs 25 and 26), and solving $ax < b$ (CCE 27). Of these five content curriculum events, I categorized the following two as having more than two degrees of integration: rule for multiplying fractions (CCE 6) and the rate model for multiplication (CCE 7).

In total, Mr. Kantor downgraded the degree of integration of his explanations for 10 of the 18 content curriculum events. For example, Mr. Kantor constructed a first degree explanation for the algebraic definition of division when this content curriculum event has connections to at least the multiplication of fractions theorem and to the equal fractions property. As another example, Mr. Kantor constructed an explanation with one degree of integration for solving $ax = b$. He did not make the connection to the concept of division.

**Explanations and Procedural and Conceptual Knowledge**

From procedural-conceptual perspective, Mr. Kantor's explanations were categorized in four types: instrumental-procedural, procedural-conceptual, conceptual-procedural, and conceptual-procedural rich explanations. Instrumental-procedural explanations are those explanations that mostly involve procedures without any kind of justification either mathematical, heuristic or pedagogical (using pictorial or story-problem representations or manipulatives). Procedural-conceptual explanations are those explanations that involve procedures with verbal, numerical or symbolic representations and the justification is mostly symbolic or numerical. Conceptual-procedural explanations are those explanations that involve reference to other concepts and the justification of the content curriculum event is connected to both mathematical and pedagogical representations. Conceptual-procedural rich explanations are those explanations that represent a content curriculum event using a variety of representations (integration of representations) and connection to other concepts (integration of concepts).
**Instrumental-procedural explanations.** I did not categorize any of Mr. Kantor's explanation as instrumental procedural.

**Procedural-conceptual explanations.** I categorized the explanations for the following ten content curriculum events as procedural-conceptual explanations: the multiplicative identity of one (CCE 12), definition of reciprocals (CCE 14), the reciprocal of $x$ is $\frac{1}{x}$ (CCE 15), reciprocal of zero (CCE 16), solving $ax = b$ (CCE 19), definition of $a > b$ (CCE 24), solving $ax < b$ (CCE 27), multiplication counting principle (CCE 28), $n!$ (CCE 32), and the permutation theorem (CCE 34). I will illustrate Mr. Kantor's procedural-conceptual explanations with solving $ax < b$. This is described below.

After having constructed the pictorial representation for "If $-3 < 4$, then $-6 < 8$ and $6 > -8,"$ Mr. Kantor illustrated how to solve inequalities of the form $ax < b$ using the following examples $-5x < 10, -12 < 48X, -10u < 0, 10-5a < 10$. The pedagogical events that Mr. Kantor constructed for the first example are described in Excerpt 19.

**Excerpt 19: Solving $-5x < 10$**

1. **K:** Now, an example of that $[-5x < 10]...$ Now, how do I solve that problem?
2. **S:** Add the reciprocal of that, of five to both sides.
3. **K:** How do I undo multiplication?
4. **S:** Division.
5. **K:** Division or multiplying by
6. **S:** The reciprocal.
7. **K:** Now. So I'm gonna multiply by the reciprocal of the coefficient. This is the coefficient. Multiply both sides by negative one fifth, negative one fifth, and as soon as I do that what happens to it?

\[
\begin{bmatrix}
-5x < 10 \\
\left(-\frac{1}{5}\right) \\
\left(-\frac{1}{5}\right)
\end{bmatrix}.
\]

8. **S:** Switches.
9. **K:** It switches. I multiply both sides by negative, that switches. You saw that that happens,

\[
\left(\frac{-1}{5}\right) -5 \cdot x > 10\left(\frac{-1}{5}\right).\]

That is what happens.... On the left side, this times its reciprocal is obviously one. $X$ is greater than, this times this is negative two. Here is my answer $[x > -2].$

[The] number one problem with this is people will get this right $[-2]$, but they will have the wrong sense.... As soon as you multiply both sides by a negative, change the sense. The set up in this way it's easy to see when I multiply by a negative that the sense changes.... Now, how would you check this?

10. **S:** Pick a number greater than negative two.
11. K: What's the easiest one to pick, greater than negative two?
12. S: Negative three.

Negative five times zero is zero. It is less than ten. It does check. \(\frac{-5 \cdot 0}{0} < \frac{10}{0}\). I got the right sense. If I had it the other way around I would check negative there and that doesn't work. The book talks about there being two ways of doing problems that are like this. Depending on whether \(a\) is positive or negative and that's true. If \(a\) is positive you don't change the sense. That's what happens here. If I multiply both sides by something positive it stays the same, but if I multiply by negative it switches. It wipes over.... Let me show you another problem. (CO, 5, 10-19-94)

As we notice in Excerpt 19, Mr. Kantor uses explicitly the multiplicative property of inequalities to solve \(-5x < 10\). He said "[The sign] switches. I multiply both sides by negative, that switches" and wrote on the board \(\left(-\frac{1}{5}\right) - 5 \cdot x > 10\left(-\frac{1}{5}\right)\) (speech 9). It is worth noting that Mr. Kantor changed the sense of the inequality when he multiplied both sides of the inequality. As a mathematics teacher, I have seen that most of my students do not change the sense of the inequality when they multiply by a negative number and some of them usually change it in the next step. This last strategy would be \(\left(-\frac{1}{5}\right) - 5 \cdot x < 10\left(-\frac{1}{5}\right)\) and then change the sign as follows \(x > -2\). Not surprisingly, a content analysis of students' answers to a similar problem on the quiz revealed that most of the students did not change the sign at all, some changed in the next step and only a few changed it when they multiplied by a negative number. That is another incident that suggests that Mr. Kantor's knowledge tends to be structural. After this teaching event, Mr. Kantor said that the solution to the problem was \(x > -2\) and stressed that structural aspect of this problem and also the common wrong answer: "[The number one problem with this is people will get this right \([-2]\), but they will have the wrong sense ... as soon as you multiply both sides by a negative change the sense. Set up in this way, it's easy to see when I multiply by a negative that sense changes" (speech 9). After this remark Mr. Kantor stressed the idea of how to check the solution to an inequality (speeches 9-15).

Finally, Mr. Kantor stated in a structural way the way to solve inequalities of the form \(ax < \)
b: "Depending on whether a is positive or negative and that's true. If a is positive you
don't change the sense. That's what happens here. If I multiply both sides by something
positive it stays the same, but if I multiply by negative it switches" (speech 15). A thing
worth noting about the pedagogical events constructed by Mr. Kantor is that he did not use
his knowledge of the associative property to justify multiplying $\frac{1}{5}$ and 5. He did not
either mention the commutative property to justify multiplying one side of the inequality by
the right and the other by the left, and he did not ask students the justifications to those
steps. Probably he believes that students already have conceptual and procedural
knowledge about those steps. His focus in this lesson is probably on students solving
inequalities of the form $ax < b$ and understanding the foundation of the procedure: the
multiplicative property of inequalities. Mr. Kantor probably believes, as I do, that students
need more experience in constructing the procedural and conceptual knowledge about
solving inequalities of the form $ax < b$. Mr. Kantor provided more examples of how to
solve inequalities. Tables 5.7, and Appendix P display other representations and
explanations, respectively, constructed by Mr. Kantor in teaching how to solve inequalities
of the form $ax < b$.

We notice from Table 5.7 that Mr. Kantor used a variety of representations for
illustrating how to solve inequalities of the form $ax < b$: inequalities involving negative
numbers, fractions, decimals, etc. He used somewhat structural approaches for solving
inequalities (e.g., multiplying both sides by the reciprocal of the coefficient of $x$ and
changing the sense in the correct place when appropriate.
2) $-12 < 48X$ (CO, 5, 10-19-94)  
3) $-10u < 0$ (CO, 5, 10-19-94)  
4) $10 - 5a < 10$ (CO, 5, 10-19-94)  
5) $5X \geq 10$ (CO, 6, 10-20-94)  
6) $-3y < 300$ (CO, 6, 10-20-94)  
7) $-4A < -124$ (CO, 6, 10-20-94)  
8) $13 > 2Z$ (CO, 6, 10-20-94)  
9) $\frac{2}{3}P \leq \frac{1}{4}$ (CO, 6, 10-20-94)  
10) $.09 > -9C$ (CO, 6, 10-20-94)  
11) $-m < 8$ (CO, 6, 10-20-94)  
12) $-T < 0$ (CO, 6, 10-20-94)  
13) $-2 \leq -n$ (CO, 6, 10-20-94)

Table 5.7 Representations for solving inequalities of the form $ax < b$

Figure 5.17 displays one pictorial representation that Mr. Kantor constructed for solving $-2 \leq -n$. The corresponding explanations add more validity to these interpretations. They are described in Appendix P.

![Figure 5.17](image)

Figure 5.17 A pictorial representation for solving $-2 \leq -n$.

An analysis of the explanations described in Appendix P and the representations displayed in Table 5.7 suggests the following pattern: Mr. Kantor's explanations involve both procedural and conceptual knowledge. They involve procedural knowledge because they illustrate how to solve inequalities. They involve conceptual knowledge because Mr. Kantor constructed pictorial representations to illustrate that when we multiply both sides of an inequality by a negative number the sense of the inequality changes. We notice however
that the procedural elements are not as strong as they can be because Mr. Kantor did not use his knowledge of other segments of mathematical content to justify the steps of the procedure (e.g., associative property, property of reciprocals, and multiplicative identity of 1). For the same reason the conceptual elements are not very strong either. In addition, he only constructed a pictorial representation for one problem (solving \(-2 \leq -n\)), and his use of story-problem representations was very limited.

**Conceptual-procedural explanations.** I categorized the explanations for the following eight content curriculum events as conceptual-procedural explanations: rule for multiplying fractions, rate model for multiplication, the multiplicative properties of inequality, classical definition of probability, conditional probability formula, meaning of division, and the algebraic definition of division.

**Conceptual-procedural rich explanations.** I did not categorize any of Mr. Kantor's explanations as conceptual-procedural rich explanations.

**Mr. Kantor's Questions**

I categorized Mr. Kantor's questions into two major categories: procedural and conceptual. The coding scheme is described in Appendix Q. Table Q.1 (Appendix Q) displays most of the questions asked by Mr. Kantor during his teaching of algebraic multiplication. I have classified the content curriculum events according to the type of questions asked in three categories: content curriculum events with only procedural questions (CPQ), content curriculum events with only conceptual questions (CCQ), and content curriculum events with both procedural and conceptual questions (CPCQ).

Mr. Kantor asked only procedural questions for the following 13 content curriculum events: rule for multiplying fractions (CCE 6), the multiplicative identity of 1 (CCE 12), definition of reciprocal (CCE 14), the reciprocal of a is \(\frac{1}{a}\) (CCE 15), the reciprocal of zero (CCE 16), solving \(ax = b\) (CCE 19), definition of \(a > b\) (CCE 24), the
<table>
<thead>
<tr>
<th>Content curriculum event</th>
<th>Procedural questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. Rule for multiplying fractions</td>
<td>The length of the side of the square is one unit. Therefore the area of the square is We know that another way to find the area of a rectangle is what? If you cut something into fourths ... and then cut those into fifths, what do you end up with?</td>
</tr>
<tr>
<td>12. The multiplicative identity of 1</td>
<td>Multiplication has an identity also. If I start with this [A], what do I multiply by [A - = A ] [to get the same thing I started with]?</td>
</tr>
<tr>
<td>14a. Definition of reciprocal</td>
<td>If I take something [B] and make it twice as big [B-2] what do I multiply that by to get back to where I started from?</td>
</tr>
<tr>
<td>15. The reciprocal of a is ( \frac{1}{a} ) ( (a \neq 0) )</td>
<td>1. If that's the case, how do I solve for R? ( \frac{3}{5} - R = 1 )</td>
</tr>
<tr>
<td>16. Reciprocal of zero</td>
<td>Every number has an opposite but not every number has a reciprocal. What number does not have a reciprocal?</td>
</tr>
<tr>
<td>19. Solving ( ax = b )</td>
<td>Now, if you look at an equation that involves what operation? ([40R = 600]) If you apply the inverse to solve this, what do you use? What do you multiply by? Inverse is multiplying by? So I can undo the multiplication. I can reverse the process by applying the inverse or I can reverse the process by</td>
</tr>
<tr>
<td>24. ( a &gt; b )</td>
<td>Here is zero, negative three is here, positive four is here, [what's the relationship between those two?]</td>
</tr>
<tr>
<td>25. The multiplicative property of inequalities (first part)</td>
<td>Five and seven, what's the relationship between those two numbers? ... Are they equal, less than, greater than? If I double five and get ten, double seven and get fourteen ... what's the relationship then?</td>
</tr>
</tbody>
</table>

(To be continued)

Table 5.8 A sample of procedural questions

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Table 5.8 (Continued)

27. Solving $ax < b$
1. How do I solve that problem? $[-5x < 10]$
2. How do I undo multiplication?
3. Division or multiplying by
4. As soon as I do that what happens to it? [to the sign]
5. How would you check this?
6. What's the easiest one to pick, greater than negative two?

28. Multiplication counting principle
To figure out how many different schedules are possible.... For example, if you simply look at English and music, English and music, how many different schedules are possible?

And next we can look at global language \[
\begin{align*}
\frac{1}{English} \cdot \frac{4}{music} \cdot \frac{3}{math} \cdot \frac{1}{GL}
\end{align*}
\]

how many branches?

29. Classical definition of conditional probability
You are looking at only those people that take Spanish and the percentage of time those people branch up, so what do these percentages add up to?

Those 55 people, how many even go in each of these paths?

Now the percentage of people taking choir.

Now all the blondes, how many of you run track? because that's what I'm trying to find

31. Conditional probability formula
I wanna know what percentage of people takes Spanish and Choir. How do you figure that out?

32. $n!$
It gives you a hand. A hearts, two diamonds, five spades, three diamonds and $J$ hearts. In how many different ways can that hand be arranged in a row from left to right?

Table 5.8 A sample of procedural questions

multiplicative property of inequalities (first part) (CCE 25), solving $ax < b$ (CCE 27), the counting multiplication principle (CCE 28), classical definition of conditional probability (CCE 29c), conditional probability formula (CCE 31), $n!$ (CCE 32). Table 5.8 displays some questions asked by Mr. Kantor for each of the 13 content curriculum events for
which only procedural questions were asked. We notice that most of the procedural questions were related to carry out some sort of procedure (e.g., the length of the side of the square is one unit. Therefore the area of the square is; if that's the case, how do I solve for $R$?; how do I solve that problem? [$-5x < 10$]) or to remember a mathematical content curriculum event (e.g., We know that another way to find the area of a rectangle is what?: every number has an opposite but not every number has a reciprocal. What number does not have a reciprocal?; I can reverse the process by applying the inverse or I can reverse the process by?).

There were no content curriculum events for which Mr. Kantor constructed only conceptual explanations. For the following five content curriculum events Mr. Kantor asked both procedural and conceptual questions: rate model for multiplication (CCE 7), the second part of the multiplicative property of inequalities (26), permutation theorem (CCE 34), meaning of division (CCE 35), and algebraic definition of division (CCE 36). Table 5.9 displays some procedural and conceptual questions.

The most common procedural questions were about procedures (e.g., what's fifty five times 12?, If you do seven times one half, what happens?, etc.). Mr. Kantor only asked one question about symbolic representations of mathematical objects (what's the algebraic definition of division?). Regarding conceptual questions, most of them were of the type why (e.g., why is that ten factorial represents the number of different [batting orders of 10 people in a softball game]?; now, in the context of that problem, why does five times two give me the answer?). Mr. Kantor only asked students to construct a story-problem representation (Give me a problem, make up a problem where five divided by one half represents that problem. Five divided by one half.)

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Table 5.9 A sample of procedural and conceptual questions

<table>
<thead>
<tr>
<th>Content curriculum event</th>
<th>Procedural questions</th>
<th>Conceptual questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. Rate model for multiplication (Story-problem)</td>
<td>What's fifty five times 12?</td>
<td>What does that mean, rates can be multiplied by other quantities?</td>
</tr>
<tr>
<td></td>
<td>If I multiply that by three pages per document what happens?</td>
<td>I went on a trip at an average of 55 miles per hour. You went on a trip that took 12 hours... I multiply those together. What's the problem with my answer?</td>
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<td></td>
<td>A person is accelerating at four feet per second square and the person says what is the acceleration in inches per hour square?</td>
<td></td>
</tr>
<tr>
<td>26. The multiplicative property of inequalities (Second part) (Numerical-pictorial)</td>
<td>([-6 &lt; 8]) When I multiply both sides by negative, multiply both sides by negative... what's the relationship?</td>
<td>Negative three (W) is less than negative three (L) ([-3W &lt; -3L]). What does that mean? Why is this person now in a better situation? What does that mean? ... what does negative three mean?</td>
</tr>
<tr>
<td></td>
<td>What happens when we multiply both sides of this ([W &gt; L]) by a negative number?</td>
<td></td>
</tr>
<tr>
<td>34. Permutation theorem (Story-problem)</td>
<td>Problem 12: In softball, there are 10 people who can bat. In how many ways can the manager of a softball team arrange the batting order?</td>
<td>Why is that ten factorial represents the number of different [batting orders of 10 people in a softball game]?</td>
</tr>
<tr>
<td>35. Meaning of division [Story-problem]</td>
<td>You have five dollars, pencils is fifty cents a piece, how many times does fifty cents go into five dollars?</td>
<td>Give me a problem, make up a problem where five divided by one half represents that problem. Five divided by one half.</td>
</tr>
<tr>
<td>36. Algebraic definition of division (Story-problem)</td>
<td>What's the algebraic definition of division?</td>
<td>Show me in the context of the problem why the same situation can be described as seven times one half.</td>
</tr>
<tr>
<td></td>
<td>If you do seven times one half, what happens?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>How many times do three quarters go into twenty one quarters?</td>
<td>You know in this case that this (\left[ \frac{21 \cdot 4}{3} \right]) will give you the same answer, why?</td>
</tr>
</tbody>
</table>
In this chapter I have presented the results of the investigation to answer the research questions that guided this study. However, some questions such as the following remain to be discussed: What do the results mean? What are some implications for teacher education? What are some implications for further research? These are some of the questions that will be pursued in the final chapter.
Two critical questions in research on teaching were the focus of the present study: What is teachers' knowledge of representations? How do teachers use their knowledge of representations during classroom instruction? The purpose of this study was to shed some light on those issues by examining the case of Mr. Kantor, the participant of the study. I examined his knowledge of representations and the pedagogical events that he constructed for teaching content curriculum events about algebraic multiplication. As defined in Chapter 2, a content curriculum event is any specific mathematical object (definition, formula, axiom, theorem, algorithm, proof) that makes up a curriculum. It is the specific element that a teacher teaches in a particular period of time. Forty one content curriculum events were identified as related to algebraic multiplication. The pedagogical events examined for each curriculum event were: explanations, representations, and questions. Specifically, I investigated the following research questions:

I. What is Mr. Kantor's knowledge of mathematical representations about algebraic multiplication?
   1. What is Mr. Kantor's knowledge of definitions or symbolic representations?
   2. What is Mr. Kantor's knowledge of mathematical proofs?
II. What is Mr. Kantor's knowledge of pedagogical representations about algebraic multiplication?

3. What is Mr. Kantor's knowledge of pictorial representations?

4. What is Mr. Kantor's knowledge of story-problem representations?

III. How does Mr. Kantor use his knowledge of representations when teaching algebraic multiplication?

5. What representations does Mr. Kantor use?

6. What explanations does Mr. Kantor construct?

7. What questions does Mr. Kantor pose?

This chapter consists of six sections. In the first section I provide a summary of the dissertation. In the second section I discuss the significance of the findings. In the third section I discuss some other emerging contributions of this study. In the fourth section I summarize some recommendations for further research. In the fifth section I discuss and summarize some recommendations for teacher education. With the sixth section I conclude this dissertation.

Summary

This section is divided into two main parts. In the first part I describe the background of the study. In the second part I summarize the main results of the study.

Background of the Study

In this section I will provide a summary of the rationale and empirical background, methodology and procedures, and data analysis.

Rationale and Empirical Background. Some researchers (e.g., Fennema & Franke, 1992; Simon, 1993; Wilson, Shulman, and Richert, 1987) consider teachers' knowledge of representations to be a critical component of teachers' knowledge for effective teaching. Yet, it remains unknown the extent to which mathematics teachers know the representations of the content they teach and the impact of that knowledge

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during their classroom instruction (Fennema & Franke, 1992; Simon, 1993; Wilson, Shulman, and Richert, 1987). For example, Fennema and Franke (1992) ask "Do teachers know the representations of the content they ordinarily teach? Does knowing these representations make any difference in how teachers teach?" (p. 154). The importance of examining teachers' knowledge and use of that knowledge during classroom instruction is supported from three main sources. The first one is that teachers' knowledge may play a critical role in determining or creating the formal opportunities and environments that students encounter for learning subject matter (McDiarmid, Ball, & Anderson, 1989; Simon, 1993). The second one is related to a belief shared by some prominent mathematics education researchers (e.g., Fennema & Franke, 1992; Hiebert & Carpenter, 1992) that the use of both mathematical and pedagogical representations helps students construct both procedural and conceptual knowledge. The third one is related to gaining some additional understanding about the complexities of teaching from the perspective of use of knowledge. In particular, what is the relationship between teachers' knowledge and use of their knowledge during classroom instruction? The main purpose of this study was to provide some partial answers to the last questions in the case of algebraic multiplication in eighth grade by reporting the case of Mr. Kantor, the participant of the study.

**Methodology and Procedures.** This was a qualitative naturalistic study with a case-study format.

**Participant and setting.** At the time of the study, Mr. Kantor, the participant of the study, was teaching eighth-grade algebra and had been teaching mathematics for about five years in an suburban school district known for high student achievement. The school district is located in a upper-middle class suburban area in a large midwestern U.S. city. Mr. Kantor was selected purposefully based on three criteria: (a) the school district was implementing *The University of Chicago Mathematics Project* (McConnell et al., 1990).
a mathematics curriculum with emphasis on skills, properties, uses, and representations; 
(b) he was an experienced middle school teacher with secondary mathematics 
certification, (c) his knowledge of representations was strong in most of the cases. I 
thought that these factors would support Mr. Kantor's use of representations in the 
classroom.

**Data collection and procedures.** There were four major sources of data: (a) 
videotapes of lessons taught by Mr. Kantor, (b) interviews, (c) questionnaires, and (d) the 
textbook. I carried out a content analysis of the 9 lessons of chapter four, *Multiplication 
in Algebra,* and one lesson of chapter five, *Division in Algebra,* of the textbook to 
identify the main content curriculum events. A total of 41 content curriculum events 
were the base for analyzing Mr. Kantor's knowledge of representations and use of his 
knowledge. Mr. Kantor's teaching of the ten lessons was videotaped to examine whether 
he used the four types of representations (symbolic representations, proofs, pictures, and 
story problems) for each of the 41 content curriculum events. After the videotaping of 
the lessons, I interviewed Mr. Kantor to examine his knowledge of representations. For 
each content curriculum event, Mr. Kantor was asked, when appropriate, to provide a 
symbolic representation, a mathematical proof, a pictorial representation, and a story-
problem representation. In some cases questionnaires were also used, especially in those 
cases where I wanted Mr. Kantor to be explicit about his knowledge.

**Data Analysis.** Each type of representation constructed by Mr. Kantor for each 
content curriculum event was judged as correct, partially correct or incorrect. In addition, 
pictorial representations were judged as implicit, partially explicit and explicit. Mr. 
Kantor's knowledge of mathematical proofs was also examined to find out whether he 
could distinguish between mathematical statements accepted as definitions and axioms.
Results

I will divide this section into two main parts. In the first part I will describe the findings of the study in the case of Mr. Kantor to answer the question of what is teachers' knowledge of representations. In the second part I will describe some findings to answer for the case of Mr. Kantor the question of how teachers use their knowledge of representations during classroom instruction.

Mr. Kantor's Knowledge of Representations. As stated in the research questions above, the phenomenon of teachers' knowledge was examined in the case of Mr. Kantor using as a context algebraic multiplication and four types of representations: (a) symbolic representations, (b) mathematical proofs, (c) pictorial representations, and (d) story-problem representations. Mr. Kantor's knowledge of each of the four types of representations will be summarized below.

Mr. Kantor's knowledge of symbolic representations. This type of knowledge was examined for 38 of the 41 content curriculum events. I found that Mr. Kantor constructed 32 correct symbolic representations and 6 partially correct symbolic representations (including verbal and verbal-symbolic representations where a symbolic representation can not be constructed). His knowledge of symbolic representations was judged as strong. To illustrate Mr. Kantor's correct symbolic representations, he represented the associative property as "For a, b, c ∈ R, (a•b)•c = a•(b•c)." As another example, he represented the algebraic definition of division as \[ \frac{a}{b} = a \cdot \frac{1}{b}, \quad b \neq 0. \] As an example of partially correct representations, Mr. Kantor provided the following symbolic representation for the rule of signs the quotient of two negative numbers is a positive number "If a and b are negative then \( \frac{a}{b} \) is positive."

Mr. Kantor's knowledge of mathematical proofs. Mr. Kantor was asked in interviews to construct a mathematical proof or a mathematical justification for 35 of the 41 content curriculum events. I found that Mr. Kantor constructed a correct proof for 11
content curriculum events, a partially correct proof for ten content curriculum events, and an incorrect proof for 14 content curriculum events. I also found that Mr. Kantor did not have a well articulated knowledge about what statements are accepted as definitions or axioms. His knowledge of proofs was judged as very weak. To illustrate, Mr. Kantor categorized as definitions, among others, the product of two negative numbers is a positive number, multiplicative property of negative 1, the multiplicative property of zero. As one example of incorrect proofs, Mr. Kantor constructed the following proof for the multiplication property of zero:

\[ a \times 0 = 0. \]  
By definition of zero! ... I don’t know the formal proof but I can use repeated addition ... zero terms that you are adding together. (I, 23, 01-03-95)

As an example of partially correct proofs, Mr. Kantor constructed the following proof for the multiplication of fractions theorem:

\[
\frac{a}{b} \cdot \frac{c}{d} = (a + b) \cdot (c + d)  
\]  
(This is the same as \(a\) divided by \(b\) times \(c\) divided by \(d\).)

\[ = (a \cdot \frac{1}{b}) \cdot (c \cdot \frac{1}{d}) \]  
\(\) (a times the reciprocal \([\text{of} \ b]\), \(c\) times the reciprocal \([\text{of} \ d]\).)

\[ = a \cdot \frac{c \cdot \frac{1}{b} \cdot \frac{1}{d}}{bd} \]  
\(\) (Drop all parenthesis and regroup.)

\[ = \frac{ac}{bd} \]  
(I, 23, 01-03-95)

In some instances, Mr. Kantor was able to construct a proof using previous theorems. For example, for proving that the product of a negative number and a positive number he said "Multiplication is commutative. [If we already proved the other then we can prove that]." Similarly, he used the four rules of signs for multiplication for proving the corresponding rules of signs for division.

**Mr. Kantor's knowledge of pictorial representations.** Mr. Kantor's knowledge of pictorial representations was examined for 38 content curriculum events. Mr. Kantor constructed correct pictorial representations for 32 content curriculum events, partially correct representations for four content curriculum events, and incorrect representations
for two content curriculum events. His knowledge of pictorial representations was judged as strong. I will illustrate Mr. Kantor's knowledge of pictorial representations with two correct pictorial representations, one partially correct pictorial representation and one incorrect pictorial representations. Table E.1 in Appendix E displays all the pictorial representations constructed by Mr. Kantor during the interview sessions. Mr. Kantor constructed the pictorial representation displayed in Figure 6.1 to illustrate how to solve equations of the form $ax = b$. His thinking when constructing the representation was as follows, $a$ times $x$—the area of that thing is $b$, and if we cut $b$ up into $a$ equal parts, that's what each of the parts is.... So that end [the block] is something that's one by $x$; so that's the same as $x$ times one which is $x$.

![Figure 6.1 A pictorial representation for solving equations of the form $ax = b$](image)

Another example of a correct pictorial representations constructed by Mr. Kantor was for the associative property of multiplication. Figure 6.2 displays Mr. Kantor's pictorial representation:

Each of these is a row.... Call this length, the width, the height. You think as if we put crates in, crates across the bottom of it. You have so many in a length, you have so many widths so you have $l$ times $w$ crates on the bottom and they are stacked $h$ high so $l$ times $w$ times $h$; that gives you the volume of the room. For this one you can say, OK, let's make a wall of crates and you know it's $h$ high; let's do it this way, the bottom has, the width of the room has this crates and you have $h$ of those rows high, now you just take all those rows down the length of the room so you have $l$ of them. You have the same volume, it's the same room. (I, 33, 08-25-95)
One example of a partially correct representation constructed by Mr. Kantor was for the product of a negative number and a positive number is a negative number (PNPN). He constructed the representation displayed in Figure 6.3. His thinking was as follows: "Here is our goal, and we start in the wrong direction five miles per day \((-5 \frac{\text{miles}}{\text{day}})\). In ten days we are 50 miles from our goal."

One possible interpretation for this pictorial representation is \((10 \text{ days})(-5 \frac{\text{miles}}{\text{day}}) = -50 \text{ miles}\). This is the interpretation that comes naturally using the model of repeated addition for multiplication \((-5 + -5 + \ldots + -5 = 10(-5))\) and it is clear, given the physical meaning of the quantities, that the answer should be \(-50 \text{ miles}\). However, that pictorial representation would illustrate the product of a positive number and a negative number is a negative number instead. The other possible interpretation is related to the rate model for multiplication, that is, \((-5 \frac{\text{miles}}{\text{day}})(10 \text{ days}) = (-5)10 \text{ miles}\), but in this case it is not
very clear as to why the pictorial representation illustrates the intended mathematical and physical situation.

Figure 6.3 Pictorial representation for PNPN

Mr. Kantor's knowledge of story-problem representations. Mr. Kantor's knowledge of story-problem representations was examined for 38 content curriculum events. I found that Mr. Kantor constructed correct representations for 35 content curriculum events and partially correct story-problem representations for three content curriculum events. Table F.1 in Appendix F displays all the story-problem representations constructed by Mr. Kantor during the interview sessions. His knowledge of story-problem representations was judged as very strong. To illustrate his knowledge of representations, I will describe three. Mr. Kantor constructed the following story-problem representation to illustrate the meaning of the commutative property of multiplication:

The example of looking at an auditorium. How many seats are in a row times how many rows are tells you how many seats [there are] in the auditorium. You do the same thing with columns. Count how many seats in a column [times] how many columns are there [gives you] how many seats are in the auditorium. You get the same thing. (1, 26, 05-30-95)

Mr. Kantor constructed the following story-problem representation for illustrating the meaning of solving $ax = b$ using the particular case $40R = 600$, "Forty bucks a radio. How many can you buy for six hundred dollars?" As another example, he constructed the following story-problem representation to illustrate why $5 + \frac{1}{2} = 5\cdot2$:  

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You have five dollars, how many bags of fifty cents potato chips could you buy? That \(5 + \frac{1}{2}\) tells you ten bags. [This will be] you can buy two bags per dollar, so two times five... Each dollar gives you two bags. [and you have] five dollars. Five times two. (I, 29, 06-08-95) & (I, 30, 06-16-95)

Mr. Kantor's Use of His Knowledge of Representations During Classroom Instruction. In this section I will describe some findings for the case of Mr. Kantor to answer the question: How do teachers use their knowledge of representations during classroom instruction using as a context algebraic multiplication, the four types of representations described above and the three pedagogical events (representations, explanations, and questions?)

Mr. Kantor's use of his knowledge of symbolic representations during classroom instruction. While Mr. Kantor provided a correct symbolic representation for 24 of 29 possible content curriculum events for which a symbolic representation can be constructed, he only constructed symbolic representations for the following three content curriculum events during classroom instruction: multiplicative identity of 1, the reciprocal of \(x\) is \(\frac{1}{x}\) if \(x \neq 0\), and the algebraic definition of division. To illustrate, he represented the algebraic definition of division as \(a + b = a \cdot \frac{1}{b}\) for \(b \neq 0\). Thus, Mr. Kantor constructed a symbolic representation for about 12.5% of the content curriculum events he knew. Thus, we can observe that the relationship between Mr. Kantor's knowledge of symbolic representations and use of that knowledge during classroom instruction was very weak.

Mr. Kantor's use of his knowledge of mathematical proofs during classroom instruction. I found that Mr. Kantor did not construct a mathematical proof for any of the 11 content curriculum events for which he knew a mathematical proof. Then, Mr. Kantor's use of his knowledge of mathematical proofs was nonexistent.

Mr. Kantor's use of his knowledge of pictorial representations during classroom instruction. While Mr. Kantor knew the pictorial representations for 32 of 38 content
curriculum events, in class he only constructed a pictorial representation for eight content curriculum events. For example, he constructed a rectangle with sides \( \frac{3}{5} \) and \( \frac{3}{4} \) to show that the area is \( \frac{9}{20} \). Because the area can be represented as \( \frac{3 \cdot 3}{4 \cdot 5} \) the picture shows that \( \frac{3 \cdot 3}{4 \cdot 5} = \frac{9}{20} \). The relationship between Mr. Kantor's knowledge of pictorial representations and use of that knowledge during classroom instruction was very weak because the index of use was .25.

**Mr. Kantor's use of his knowledge of story-problem representations during classroom instruction.** While Mr. Kantor knew a story-problem representation for 35 of 38 content curriculum events, in class he constructed a story-problem representation for only 10 content curriculum events. To illustrate, he used the following story situation, among others, to illustrate the meaning of conditional probability: "six out of 24 people are blonde.... seven out of 24 people run track. Out of six blonde people, four run track. \( P(\text{Run track given blonde}) = \frac{4}{6} \)." The relationship between Mr. Kantor's knowledge of story-problem representations and use of that knowledge during classroom instruction was very weak because the index of use was .29.

**Mr. Kantor's explanations.** I found that Mr. Kantor constructed explanations for only 18 content curriculum events. I categorized each of Mr. Kantor's explanations according to five themes: (a) integration of representations, (b) treatment of difficult topics, (c) operational and structural conceptions of algebraic objects, (d) integration of concepts, and (e) conceptual and procedural elements.

Regarding integration of representations, some of Mr. Kantor's explanations integrated some types of representations. He used numerical representations for 17 content curriculum events. In addition to these representations, Mr. Kantor constructed a symbolic representation for three content curriculum events. For eight content curriculum events, Mr. Kantor constructed pictorial representations in addition to
numerical representations. For 10 content curriculum events, Mr. Kantor constructed story-problem representations in addition to numerical representations. For only five content curriculum events, he constructed both pictorial and story representations in addition to numerical representations. Mr. Kantor did not construct all three types of symbolic, pictorial and story-problem representations for any of the 18 content curriculum events.

With respect to treatment of difficult topics, I found that Mr. Kantor did not construct explanations for 23 content curriculum events, that he constructed middle-range explanations for 12 content curriculum events, and difficult explanations for six content curriculum events.

As to operational and structural explanations, I found that Mr. Kantor constructed explanations involving operational elements for 11 content curriculum events and structural elements for seven content curriculum events.

Regarding integration of concepts, I found that all the explanations constructed by Mr. Kantor involved some integration of concepts. However, for 10 content curriculum events Mr. Kantor failed to make additional connections to related concepts.

All of Mr. Kantor's explanations involved conceptual and procedural knowledge. Mr. Kantor emphasized procedural knowledge for ten content curriculum events and conceptual knowledge for eight content curriculum events. However, in both cases, Mr. Kantor's use of procedural and conceptual knowledge could have been much stronger.

Mr. Kantor's questions. Mr. Kantor asked questions for only 18 content curriculum events. I found that Mr. Kantor asked only procedural questions for 13 curriculum events (e.g., multiplication of fractions theorem, solving $ax = b$, etc.). Most of those questions were related to carrying out some sort of procedure (e.g., the area of the square is: how do I solve for $R$? etc.). There were no content curriculum events for which Mr. Kantor constructed only conceptual explanations. For five content curriculum events (e.g., meaning of division, definition of division, etc.), Mr. Kantor asked both procedural and
conceptual questions (e.g., what is the algebraic definition of division? Show me in the context of the problem why the same situation can be described as seven times one half, etc.).

Discussion

In this section I will discuss the importance, meaning and some implications of the findings. I will divide this section into three parts corresponding to each of the three main questions of the study: Mr. Kantor's knowledge of mathematical representations, Mr. Kantor's knowledge of pedagogical representations, and Mr. Kantor's pedagogical events.

Mr. Kantor's Knowledge of Mathematical Representations

Mr. Kantor's knowledge of mathematical representations was examined for two types of representations: symbolic representations and mathematical proofs. The discussion of the findings with respect to each of these types of representations follows.

**Mr. Kantor's knowledge of symbolic representations.** For 30 content curriculum events a symbolic representation can be constructed. Mr. Kantor's knowledge of symbolic representations was examined for 29 of the 30 content curriculum events. I found that Mr. Kantor knew the symbolic representations for 24 (83%) content curriculum events. The representations for the other five content curriculum events were judged as partially correct. In summary, Mr. Kantor’s knowledge of symbolic representations was judged as strong because the index of correctness was .91.

A first contribution of this study to the field of mathematics education is that the study examines a teacher's knowledge of symbolic representations (including symbolic definitions). Definitions and symbolic representations are areas of genuine research because of their relationship to advanced mathematical thinking. As stated by Tall (1992) "advanced mathematical thinking ... is characterized by two important components: precise mathematical definitions (including the statement of axioms in
axiomatic theories) and logical deductions of theorems based upon them" (p. 495). If we want to understand students' (including teachers) development of advanced mathematical thinking we need to pay attention to teachers' own advanced mathematical thinking because of the potential impact it may have on their construction of pedagogical events to help their students develop that kind of thinking and because teachers are a population whose mathematics education needs to be developed to its maximum potential. Yet, a review of the literature suggested that teachers' knowledge of definitions or symbolic representations has been ignored in research on teachers' knowledge and learning to teach. An exception is the study by Even (1993). She asked 162 prospective secondary mathematics teachers, among other things, to give a definition of a function. Because her purpose was to examine students' awareness of the arbitrariness and univalence nature of the function concept she did not categorize students' responses based on their correctness and accuracy. Then it remains unknown the extent to which the prospective teachers knew the correct and precise mathematical definition of function. The present study moves the mathematics education field further by documenting in detail one teacher's knowledge of symbolic representations of topics related to a common curriculum topic: multiplication in algebra. We need to continue this area of research to have a more complete picture about what teachers know about the symbolic representations of mathematical curriculum events. While this study provides an in-depth look at this phenomenon, we also need to consider larger samples to see the phenomenon from a broader perspective. These concerns give rise to questions in need of an answer regarding other teachers' or teacher population's knowledge of definitions or symbolic representations of the mathematical objects they ordinarily teach. Those populations should include prospective teachers (elementary teachers and secondary teachers) and practicing teachers (elementary and secondary teachers). Both in-depth case studies and large-sample studies of teachers need to be conducted to better understand the following
question about teachers' knowledge of symbolic representations (including symbolic
definitions) about content curriculum events: To what extent do teachers know the
symbolic representations of the content they ordinarily teach?

As described in the literature, most researchers, if not all, have reported that teachers' content knowledge tends to be very much underdeveloped. They argue that we cannot expect teachers to teach for rich procedural and conceptual understanding if they themselves lack those types of knowledge. The case of Mr. Kantor is worth examining because, contrary to findings reported in the literature dealing with teachers' content knowledge, his knowledge of symbolic representations (one type of content knowledge) was strong. This dissertation reports one of the first cases about a teacher with a strong knowledge of symbolic representations (including symbolic definitions). From a teacher education perspective, many questions arise: (a) Where did Mr. Kantor learn the symbolic representations of the content curriculum events related to algebraic multiplication? (b) What was the role of teacher education programs in helping Mr. Kantor develop his knowledge of symbolic representations? What has been the role of experience in shaping Mr. Kantor's knowledge of symbolic representations? What has been the role of the textbook? From a teaching perspective, Mr. Kantor is an excellent case to examine the influence of teachers' knowledge of representations on their classroom instruction. While teachers with poor knowledge of symbolic representations are unlikely to teach students symbolic representations of mathematical content curriculum events, teachers who know the symbolic representations are good cases for examining the phenomenon of the relationship between teachers' knowledge of symbolic representations and their classroom instruction. This dissertation contributes to that area and the findings will be discussed later.

**Mr. Kantor's knowledge of mathematical proofs.** Mr. Kantor's knowledge of mathematical proofs was examined for 35 content curriculum events. Mr. Kantor
constructed 11 correct proofs, 10 partially correct proofs, and 14 incorrect proofs. I also found that Mr. Kantor did not have a well developed knowledge about what mathematical statements are accepted as axioms or definitions. Even when we were working with formal proofs, Mr. Kantor was not bothered in providing informal justifications to many content curriculum events. In summary, I found that Mr. Kantor's knowledge of mathematical proofs tended to be weak.

A second contribution of this study relates to teachers' proof frames. As stated by Tall (1992), precise definitions and proofs are the two main components of advanced mathematical thinking. Yet, they remain pretty much without research attention. An earlier study that examined teachers' conceptions and knowledge about mathematical proofs was conducted by Martin and Harel (1989). They asked 101 preservice teachers the mathematical correctness of inductive and deductive verification of mathematical statements. The researchers found that many prospective teachers accepted inductive arguments as correct mathematical proofs. Martin and Harel's study and this dissertation suggest that teachers may lack this kind of knowledge. From a mathematics education perspective, those findings are puzzling. They reveal that many teachers lack a well developed knowledge of mathematical proof. Yet, knowledge of mathematical proofs is considered by many as a central component of a mathematical education, especially teachers' mathematical education. From this perspective, teachers with poor knowledge of mathematical proofs can be considered as lacking a complete mathematics education. From a teaching perspective, we cannot expect teachers to use mathematical proofs as justifications for establishing the truth of mathematical ideas during classroom instruction if they themselves lack that type of knowledge. From a teacher education perspective, we need to examine what teachers learn about mathematical proofs in the content courses that they take and how we can help teachers to develop their knowledge of mathematical proofs. The main contribution of this dissertation in the area of teachers' proof frames is
that it reports, in detail, one teacher's knowledge of mathematical proofs about content curriculum events related to algebraic multiplication. Of course, further research is needed, specially with larger samples, to back up that claim and to make, to the extent possible, generalizations to other mathematical topics. The question of as to what extent do teachers know the mathematical proofs of the content they ordinarily teach needs to be further addressed using both in-depth case studies and large-sample studies of prospective and practicing teachers.

Mr. Kantor's Knowledge of Pedagogical Representations

Mr. Kantor's knowledge of pedagogical representations was examined for two types of representations: pictorial representations and story-problem representations. Mr. Kantor's knowledge of each of these types of representations is discussed below.

Mr. Kantor's knowledge of pictorial representations. Mr. Kantor's knowledge of pictorial representations was examined for 38 content curriculum events. He constructed a correct pictorial representation for 32 content curriculum events, a partially correct pictorial representation for four content curriculum events and an incorrect pictorial representation for two content curriculum events. In summary, Mr. Kantor's knowledge of pictorial representations about content curriculum events related to algebraic multiplication was judged to be strong.

Some researchers (e.g., Ball, 1988a; Llinares & Sánchez, 1991) have investigated teachers' knowledge of pictorial or concrete representations. However, it still remains an underrepresented area of research on teachers' knowledge (Fennema & Franke, 1992). The third contribution of this study is related to this area. I examined in depth Mr. Kantor's knowledge of pictorial representations. Contrary to the findings of Ball (1988a) and Llinares and Sánchez (1991), I found Mr. Kantor's knowledge of pictorial representations to be strong. In addition, Ball (1988a) and Llinares and Sánchez's (1991) participant subjects were prospective teachers while this dissertation reports a practicing
teacher's knowledge of pictorial representations. Grade and topic are also different from those reported in the literature. The case of Mr. Kantor is worth reporting because he is a teacher with a strong knowledge of pictorial representations. Then, from a teacher education perspective, several questions arise about Mr. Kantor's case, (a) What are the sources of Mr. Kantor's knowledge of pictorial representations? (b) Where or when did he learn those pictorial representations? (c) What was the role of teacher education programs in helping Mr. Kantor construct a strong knowledge of pictorial representations? What has been the role of experience and textbooks in shaping Mr. Kantor's knowledge of pictorial representations? While I am not providing answers to those questions, they certainly suggest topics for further investigation in the area of teacher's knowledge of pictorial representations. From a teaching perspective, we need to examine the relationship between teachers' knowledge of pictorial representations and their classroom instruction. Scholars who find teachers' knowledge of pictorial representations to be weak argue that those teachers can not be expected to construct representations and explanations that would promote student's learning with understanding. Mr. Kantor, on the other hand, holds strong bodies of knowledge about pictorial representations and therefore he is a good case for examining in detail the relationship between teachers' knowledge of pictorial representations and their classroom instruction. Issues related to that relationship will be discussed later. From a research perspective, we still need to examine the phenomenon of teachers' knowledge of pictorial representations from a variety of perspectives (case studies, large sample, prospective teachers, practicing teachers, elementary teachers, secondary teachers, etc.). We need a more complete picture of the extent of teachers' knowledge of the pictorial representations of the content they ordinarily teach.

Mr. Kantor's knowledge of story-problem representations. Mr. Kantor's knowledge of story-problem representations was examined for 38 content curriculum events. Mr.
Kantor constructed correct story-problem representations for 35 content curriculum events and partially correct story-problem representations for three content curriculum events. In summary, Mr. Kantor's knowledge of story-problem representations was judges as very strong.

Even though some researchers (e.g., Ball, 1990b; Simon, 1993) have examined teachers' knowledge of story-problem representations, this area remains an underrepresented area of research on teachers' knowledge (Fennema & Franke, 1992). This study moves the field of mathematics education further by documenting in detail a teacher's knowledge of story-problem representations. Contrary to the finding of other researchers (e.g., Ball, 1990b; Post, Cramer, Behr, Lesh, and Harel, 1993; Simon, 1993), I found Mr. Kantor's knowledge of story-problem representations to be very strong. This finding makes it appropriate to report and describe his knowledge in detail as it was done in Chapter 5. From a teacher education perspective, the findings of the study raise the following questions: (a) What are the sources of Mr. Kantor's rich knowledge of story problem representations? (b) What was the role of teacher education programs in helping Mr. Kantor construct a rich knowledge of story-problem representations? (c) What was the role of experience and textbooks in shaping Mr. Kantor's knowledge of story-problem representations? These questions raise issues about sources of Mr. Kantor's knowledge of story-problem representations and provide suggestions for further research on teachers' knowledge of story-problem representations. From a teaching perspective, we need to examine the impact of teachers' knowledge of story-problem representations on their classroom instruction. As in the case of scholars who find teachers' knowledge of pictorial representations to be weak, researchers who find teachers' knowledge of story-problem representations to be poor argue that teachers are not likely to construct story-problem representations that have the potential to promote students' conceptual understanding of mathematical ideas. Since Mr. Kantor's knowledge of story-problem
representations is very strong, he is a good case for studying the phenomenon of the impact of teachers' knowledge of story-problem representations on their classroom instruction. Discussion of findings about that phenomenon is done later. From a research perspective, teachers' knowledge of story-problem representations is still an underrepresented area of research on teachers' knowledge. The question of to what extent teachers know the story-problem representations of the content they ordinarily teach remains without a complete answer. We need to examine a variety of teacher populations (e.g., prospective teachers, practicing teachers, novice teachers, experienced teachers, elementary teachers, secondary teachers, etc.) using a variety of methodologies (e.g., in-depth case studies, large-sample studies) in several contexts (addition, subtraction, multiplication, division, rational numbers, etc.), to have a better picture of the phenomenon of teachers' knowledge of story-problem representations.

In the next section I will discuss findings pertaining to how Mr. Kantor uses his knowledge of mathematical and pedagogical representations to construct pedagogical events (representations, explanations, and questions) when teaching algebraic multiplication.

**Mr. Kantor's Use of His Knowledge of Representations for Constructing Pedagogical Events**

In this section I will discuss the findings as pertaining to how Mr. Kantor used his knowledge of representations to construct three types of pedagogical events (representations, explanations, and questions) for teaching each of the content curriculum events related to algebraic multiplication.

**Mr. Kantor's use of representations for teaching algebraic multiplication.** In this section I will summarize the instructional representations that Mr. Kantor constructed when teaching each of the curriculum events related to algebraic multiplication. I will examine Mr. Kantor's use of numerical, symbolic representations, mathematical proofs.
pictorial representations, and story-problem representations constructed during classroom instruction.

With respect to numerical representations, Mr. Kantor constructed numerical representations for 17 of 41 content curriculum events for teaching their meaning. However, he only constructed symbolic representations for about three of 30 content curriculum events for which a symbolic representation can be constructed. This is a somewhat surprising result because Mr. Kantor knew the symbolic representations for 24 of 29 content curriculum events for which I asked him a symbolic representation. In addition, as stated before, *The University of Chicago School Mathematics Project* curriculum has as one of its main objectives to develop students' mathematical knowledge in four areas: skills, properties, uses, and representations (SPUR). Then, Mr. Kantor was expected to construct symbolic representations (properties) of the content curriculum events. From a teaching perspective this is a puzzling finding: the relationship between teachers' knowledge of symbolic representations and their use of that knowledge during classroom instruction may not be a linear but a complex one. From a learning perspective, students missed opportunities to strengthen their knowledge of symbolic representations as well as constructing connections between symbolic representations and other representations such as numerical, pictorial, and story-problem representations. This finding also leads us to rethink the role of teacher education programs about educating teachers regarding the importance of using symbolic representations for helping students to construct that type of mathematical knowledge. From a research perspective, this finding suggests two lines of research: Do other knowledgeable teachers use their knowledge of symbolic representations during classroom instruction? What prevented Mr. Kantor or what prevents other knowledgeable teachers from constructing symbolic representations during classroom instruction?
Regarding mathematical proofs, Mr. Kantor did not construct proofs for any of the 35 possible content curriculum events during classroom instruction. He knew the mathematical proof for 11 content curriculum events. Mr. Kantor missed many opportunities to engage students in justifying procedures using mathematical properties (definitions or symbolic representations). This finding leads us to rethink the role of mathematical proof in teaching middle grade mathematics: To what extent do teachers construct mathematical proofs when teaching eighth-grade algebra? To what extent should teachers construct mathematical proofs in eighth-grade algebra? From a teacher education perspective, the findings suggest that we need to reflect and gather empirical evidence to answer questions such as the following: What do teachers need to learn about the role of proof in school mathematics, especially in the middle grades?

Regarding pedagogical representations, I found that while Mr. Kantor was able to construct a correct pictorial representation for 32 of 38 content curriculum events and a correct story problem for 35 of 38 content curriculum events, he only constructed pictorial representations for eight content curriculum events and story-problem representations for 10 content curriculum events when teaching the meaning of the content curriculum events. Again, these findings are puzzling. From a teaching perspective, they suggest that the relationship between teachers' knowledge of pictorial and story-problem representations and its relationship to classroom practice is far from being linear. From a learning perspective, students missed some opportunities to construct connections between and among mathematical representations and pedagogical representations to strength their connections between procedural and conceptual knowledge. From a research point of view, we need to further examine the relationship between teachers' knowledge of pedagogical representations and their classroom instruction. Is Mr. Kantor a unique case? To what extent do other knowledgeable teachers use their knowledge of pedagogical representations during classroom
instruction? What factors help or prevent some teachers from using their knowledge of pictorial and story-problem representations on their classroom instruction? A variety of teacher populations (e.g., preservice, practicing, elementary, secondary, novice, experienced, etc.) needs to be examined using a diverse range of methodologies (in-depth case studies, large-sample studies) in a variety of contexts (addition, subtraction, multiplication, division, rational numbers, algebra, etc.) to have a complete picture about teachers' knowledge of representations and its impact on classroom instruction. Those research settings will also enhance our understanding of the factors that help or prevent teachers from using pedagogical representations for teaching mathematical ideas.

The fifth contribution of this study to the area of mathematics education has been in providing some partial answers to the question posed by Fennema & Franke (1992) in their chapter of the *Handbook for Research on Mathematics Teaching and Learning*, namely, "Does knowing [the] representations make any difference on how teachers teach?" (p. 154).

**Mr. Kantor's explanations.** In this section I will discuss the findings regarding Mr. Kantor's explanations. What kinds of explanations did Mr. Kantor construct during classroom instruction? Mr. Kantor only constructed explanations for 18 of 41 possible content curriculum events. Mr. Kantor's explanations were analyzed in terms of five themes: (a) use of both mathematical and pedagogical representations (integration of representations), (b) relationship between students' difficulties and explanations (treatment of difficult topics), (c) operational and structural conceptions of algebraic objects, (d) integration of concepts, and (e) conceptual and procedural elements. With respect to integration of representations I found two general patterns. First for explanations for which Mr. Kantor constructed a symbolic representation, he also constructed numerical and verbal representations. Second, explanations that did not involve the use of symbolic representations involved pictorial or story-problem
representations with story-problem representations dominating. These explanations also involved the use of numerical representations. Thus, Mr. Kantor did not tend to integrate both mathematical representations (symbolic representations) and pedagogical representations (pictorial or story-problem representations) in his explanations. The discussion of this finding is similar to teachers' use of representations: what prevented Mr. Kantor from using both mathematical and pedagogical representations in his explanations? How were students' constructions of internal connections between mathematical and pedagogical representations affected?

Regarding treatment of difficult topics, Mr. Kantor's explanations depended on the relative difficulty of the mathematical idea from the students' perspective: Mr. Kantor's explanations tended to be more elaborate for difficult topics than for less difficult or "easy" topics. This analysis and finding help to explain the lack of a strong relationship between Mr. Kantor's knowledge of representations and his classroom practices: What sense does it make to construct explanations using a variety of representations if the material is "easy" for students? Mr. Kantor stated in formal interviews and informal talks that most students have a fair understanding of the mathematical topics included in the first part of the textbook and that, therefore, he did not have to go over it in detail. From his perspective, many topics of the first part of the book were easy and much less difficult than the topics corresponding to the second part of the textbook and, as a consequence, he was going to spend more time on the second part of the textbook. We see that Mr. Kantor's beliefs and knowledge about the relative difficulty of mathematical topics had an impact on his constructing of explanations.

Regarding Mr. Kantor's explanations from the perspective of operational-structural conceptions, Mr. Kantor constructed operational explanations for 11 content curriculum events and structural explanations for seven content curriculum events. From a teaching perspective this is an interesting phenomenon. Because many students conceive of
algebraic objects from an operational point of view (Kieran, 1992), it is more likely that they will better understand operational explanations. However, the students also need experiences in dealing with algebraic objects from a structural perspective to construct a stronger conception of algebraic objects (Kieran, 1992). Kieran recommends the use of both types of representations (operational and structural) to help students develop operational and structural knowledge of algebraic objects.

Regarding the integration of concepts, I found that all the explanations constructed by Mr. Kantor involved some degree of integration of concepts. He made some connections to related concepts. However, Mr. Kantor did not construct many important connections between and among related concepts. He downgraded the degree of integration of his explanations for 10 of the 18 content curriculum events. From a teaching perspective, this is a puzzling finding. On one hand, teaching for conceptual understanding means constructing a variety of instructional representations. Some mathematics educators believe that the more experiences students have with external connections between and among related concepts, the richer is the knowledge that they construct (Fennema & Franke, 1992; Hiebert & Carpenter, 1992). On the other hand, Mr. Kantor knew the connections but he decided (consciously or unconsciously) not to construct pedagogical events about these connections. Again, this suggests that the relationship between teachers' knowledge and classroom instruction is far from being linear. From a practical point of view, to what extent should teachers construct pedagogical events to help students construct all "possible" relationships among concepts when teachers have to teach a large curriculum? If it were a few content curriculum events to be taught then the answer could be to the larger extent possible. But the fact that teachers are expected to "cover" a whole curriculum makes the answer to the question or the solution to the problem more difficult to find. This is a practical issue that mathematics educators need to reflect on and give a practical answer or solution.

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Finally, I categorized Mr. Kantor’s explanations according to their degree of procedural and conceptual elements based on the use of multiple representations and degree of integration of concepts on the following categories: instrumental-procedural, procedural-conceptual, conceptual-procedural, and conceptual procedural rich explanations. All of Mr. Kantor’s explanations were categorized either as procedural-conceptual (for 10 content curriculum events) or conceptual-procedural (for 8 content curriculum events). None of the explanations were categorized as conceptual procedural rich explanations because Mr. Kantor did not use the two types of representations (mathematical and pedagogical) and he did not make connections between and among all "possible" related concepts. That is an interesting and encouraging finding. Contrary to the findings and reports of several studies (e.g., Eisenhardt, Borko, Underhill, Brown, Jones, Agard, 1993; McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers, & Cooney, 1987; Mullis, Dossey, Owen, & Phillips, 1991; Porter, 1989) that indicate that only knowledge of rules and algorithms is emphasized in most schools, I found Mr. Kantor’s explanations involving both conceptual and procedural knowledge and making connections between those types of knowledge. I did not categorize any of Mr. Kantor’s explanations as stressing only rules and rote algorithms (instrumental-procedural explanations). From another point of view, Mr. Kantor’s explanations did not involve conceptual knowledge to its fullest extent. I did not categorize any of Mr. Kantor’s explanations as conceptual and procedural rich. This raises an issue: to what extent should teachers use as many representations as possible and make connections between and among all related concepts given that they have to teach a large curriculum? When we talk about teaching a specific topic for conceptual and procedural knowledge it makes clear sense that we need to use a variety of multiple representations and constructing connections between and among many related concepts. However, when we talk about teaching the large mathematics curriculum we are faced with the dilemma of to what
extent we should construct rich conceptual and procedural explanations for some topics at the expense of not constructing any representation or connection for other topics. Simon and Blume (Simon, 1995; Simon & Blume, 1994) report a teacher researcher (Simon) who spent several sessions helping prospective elementary teachers to construct the concept of the area of a rectangular region as a multiplicative relationship between the lengths of the sides. We need to reflect on what it means to teach for conceptual knowledge a large mathematics curriculum.

Teachers' explanations have been largely ignored in research on teaching. Yet, they are the heart of what we call teaching. Some exceptions are the studies by Borko, Eisenhardt, Brown, Underhill, Jones and Agard (1992), Kinach (1996), Thompson and Thompson (1994), and Thompson and Thompson (1996). We need to examine in detail teachers' explanations to better understand the phenomenon of teaching, especially from the perspective of knowledge. From the point of view of explanations, this study moves the field of mathematics education forward by examining and describing in detail the explanations that a knowledgeable teacher constructed about content curriculum events related to algebraic multiplication. In contrast to the study reported by Simon and Blume, this dissertation deals with a regular teacher as opposed to a teacher researcher. In contrast to the study of Borko et al., this study focuses on a teacher who knew the representations. From a research perspective, we need to examine and describe the explanations that knowledgeable teachers construct for teaching specific content curriculum events (e.g., multiplication of fractions theorem, meaning of conditional probability, solving equations of the form $ax = b$, etc.). Constructing theories of how knowledgeable teachers use their knowledge or what explanations knowledgeable teachers construct for teaching has potential to further enhance our understanding of the teaching phenomenon. As implied by Marton's (1989) pedagogical theory of content, while we cannot answer questions of what kinds of explanations teachers construct
without consideration to the specific content, we can construct theories of what explanations teachers construct for teaching specifically those pieces of mathematical content. Knowing the explanations that teachers construct may be very helpful for understanding the teaching phenomenon and suggesting specific recommendations for teacher education programs about their role in helping teachers to learn what explanations to construct for teaching specific mathematical content curriculum events.

Borko, Eisenhardt, Brown, Underhill, Jones, and Agard (1992) report the case of a student teacher who was not successful in providing a meaningful explanation for the algorithm of division of fractions. Their analysis revealed that the failure was due to the fact that the student teacher lacked knowledge about representations, especially story-problem representations. Their discussion and recommendations focus on the need of teacher education programs in developing student teachers' knowledge of representations as well as on the critical role that knowledge of representations plays in learning to construct conceptually based explanations. This study moves the field of mathematics education further because it examines the case of a teacher who has the knowledge that Borko et al. believe is critical to provide meaningful explanations but that he did not use it to its fullest extent. From a teacher education perspective, we need to develop teachers' knowledge and beliefs about using both mathematical and pedagogical representations that are likely to promote students' understanding.

Mr. Kantor's questions. I found that for 13 content curriculum events, he only asked procedural questions and that for five content curriculum events he asked both procedural and conceptual questions. However, in most of the content curriculum events the procedural questions dominated the conceptual questions. This finding has two sides. On one hand, it is not very practical to ask students the conceptual foundation of every step of each procedure, but it is very practical to ask students the steps of the procedures. That may be one reason why the procedural questions outnumber the conceptual
questions. The interesting side of the finding was that Mr. Kantor missed several opportunities to ask conceptual questions in addition to the procedural questions he asked.

Another interesting feature of Mr. Kantor’s questions was that only questions asked about why \( a + b = a \cdot \frac{1}{b} \) involved the use of story-problem representations. Most of the other questions did not involve asking students to construct any type of representation about the content curriculum events. This finding adds evidence to the claim that the relationship between teachers’ knowledge of representations and its use during classroom instruction is not linear; he did not construct any type of pedagogical event (representations, explanations, and questions) for many content curriculum events, and for some content curriculum events he only constructed some types of representations.

These findings are not only interesting but useful. From a research perspective, we need to examine other teachers’ questions to analyze them from the perspective of procedural and conceptual knowledge and the extent to which they involve asking for both mathematical and pedagogical representations. We also need to understand the reasons, conscious and unconscious, that had an influence on Mr. Kantor’s questions. From a teacher education perspective, we also need to develop teachers’ questions so they ask questions involving both procedural and conceptual knowledge as well as multiple representations.

As an additional discussion to this section and even though the purpose of the present study is descriptive, I will propose some tentative explanations and speculations to better understand two phenomena about Mr. Kantor’s pedagogical events, namely, his limited use of representations during classroom instruction and his lack of explanations for 23 of the 41 content curriculum events. One possible explanation is the difficulty that symbolic representations (including symbolic definitions) put on the learner. Mr. Kantor stated that students had a hard time with definitions and that, as a consequence, he did not
"emphasize" [formal] definitions (symbolic definitions). Other reasons include the relative difficulty of topics, the lack of time, and the pressure to "cover" the curriculum. Mr. Kantor told his students that the second part of the book was more difficult and therefore, he would devote more time to it. During informal interviews, Mr. Kantor commented that he could not "go over everything" that was in the book because of the lack of time. Additionally, he mentioned during formal interviews that students already knew some topics (e.g., commutative property, associative property, etc.) and that some topics were easy and, consequently, the students could understand them on their own. He also mentioned that creating good and real applications as well as using applications during classroom instruction took time. Finally, I speculate that he did not have a well-articulated knowledge about the potential usefulness of using both mathematical and pedagogical representations for promoting students' conceptual understanding.

Other Contribution of this Study

In this section I will discuss other contributions of the study that somehow emerged from it. Those contributions are: (a) I propose a distinction between content knowledge and pedagogical content knowledge, (b) I propose a theoretical framework for examining the phenomenon of teaching, and (c) I provide some data to construct theories of content-specific teaching. I will elaborate each of these themes below.

A Distinction Between Content Knowledge and Pedagogical Content Knowledge

The literature on research on content knowledge and pedagogical content knowledge makes no clear and consistent distinction between these two types of knowledge especially regarding representations. One of the main issues related to content knowledge is: What are mathematical representations? The corresponding issue related to pedagogical content knowledge is: What are pedagogical representations? An issue that links the two types of knowledge is: What is the relationship between subject-matter knowledge [mathematical representations] and pedagogical content knowledge
[pedagogical representations]? (Even, 1993). While some researchers (e.g., Borko et. al., 1992; Even, 1993) have used the construct of pedagogical content knowledge as a framework for examining teachers' knowledge, McEwan and Bull (1991) have challenged the distinction between content knowledge and pedagogical content knowledge on epistemological grounds. Some mathematics education researchers seem to agree, at least implicitly, with McEwan and Bull. They argue that all knowledge is pedagogical in nature. Even (1993), for example, uses the term pedagogical content knowledge as the knowledge that teachers use to represent the content knowledge to students. She asked prospective teachers to give a definition of function (content knowledge). Then she posed the following scenario and question: "A student says that he/she does not understand this definition. Give an alternate version that might help the student understand." She considered their responses as reflecting teachers' pedagogical content knowledge about the definition of function. I find a serious problem with the way Even used the two terms. In her sense all content knowledge is pedagogical in nature: teachers hold content knowledge but in the moment they represent it to students it becomes pedagogical content knowledge. In the same way, an author possesses content knowledge but when he/she represents it in the textbook it becomes pedagogical content knowledge. On the other hand, Even (1993) mentions that "teachers' subject-matter knowledge and its interrelationship with pedagogical content knowledge are still very much unknown" (p. 94). I agree with McEwan and Bull in the sense that we can not make an epistemological distinction between content knowledge [mathematical representations] and pedagogical content knowledge [pedagogical representations]. All pedagogical representations embody content representations. But these pedagogical representations are not necessarily the standard representations of the content knowledge. For example, the pictorial representations constructed by Mr. Kantor for illustrating the multiplication of fractions theorem and the conditional probability formula are
pedagogical in nature because they represent content knowledge, but these representations are not part of the standard mathematical content knowledge. Then it makes sense to talk about content representations and pedagogical content representations. In the context of mathematics, I propose to divide representations in two types: mathematical representations and pedagogical representations. Mathematical representations, in turn, can be divided in several types: definitions, formulas, axioms, theorems, algorithms, and proofs. Pedagogical representations can be divided in three groups: pictorial representations, story-problem representations, and physical representations or manipulatives (Figure 6.4).

<table>
<thead>
<tr>
<th>Representations</th>
<th>Mathematical Representations</th>
<th>Pedagogical Representations</th>
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<tr>
<td></td>
<td>Definitions</td>
<td>Pictorial representations</td>
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<tr>
<td></td>
<td>Concepts</td>
<td>Story - problem representations</td>
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<td></td>
<td>Formulas</td>
<td>Physical representations</td>
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<td>Theorems</td>
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<td></td>
<td>Proofs</td>
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</tbody>
</table>

Figure 6.4 A categorization of representations

The proposed distinction might be helpful to stimulate further research in the area of teachers' knowledge. I believe that teachers need more than content knowledge for
helping students construct subject-matter knowledge. Empirical research is needed to test or construct powerful pedagogical representations (pedagogical content knowledge) of content-specific representations.

A Theoretical Framework for Examining the Phenomenon of Teaching

Understanding the process of teaching mathematics is a fundamental problem of the psychology of mathematics teacher education just as understanding the process of learning mathematics is a fundamental problem of the psychology of mathematics education. We need to understand that process in order to help teachers construct environments pedagogically rich so that students learn the intended mathematical knowledge. Research can enhance our understanding of that process. However, a review of the literature revealed that relatively little attention has been paid to understanding that process. For example, Brown, Cooney, and Jones (1990) focused on philosophical issues in mathematics teacher education due to the very limited research available at that time. Most of the research on becoming a teacher reviewed by Brown and Borko (1992) has a generic focus rather than a specific mathematics focus. This situation is perceived by other mathematics education researchers. Sowder (1989), for example, states that research on mathematics teacher education is an underrepresented area. Why is that? Cruickshank (1990) suggests that one reason may be the lack of models to guide inquiry in teaching. We have some models that can help us to guide inquiry in teaching (e.g., Wilson, Shulman, & Richert, 1987) but we do not have models to guide inquiry in teaching that integrate curriculum, teaching and learning. Curriculum and teaching are inherently interwoven. As stated by Doyle (1992), "A curriculum is intended to frame or guide teaching and cannot be achieved except during acts of teaching. Similarly, teaching is always about something so it cannot escape curriculum" (p. 486). Teaching tends to be investigated in isolation from both curriculum and learning. Doyle (1992), for example, remarks that "when investigators try to capture the curriculum, pedagogy
[teaching] slips into the background, and when their attention turns to pedagogy, curriculum easily becomes invisible" (p. 507). That research needs to pay attention to curriculum and teaching is recommended by Kieran (1992) when she states, "there is still an enormous gap in the existing research literature on teaching regarding how algebra teachers interpret ... [the algebraic] content ... prescribed in the textbooks" (p. 413).

Wilson, Shulman, and Richert (1987) focus on the teaching act when they state that little research attention has been paid to how teachers represent the subject matter content knowledge to students during instruction. Fennema and Franke (1992) go beyond that by asking what the influence of teachers' use of representations on students' learning is.

I define the process of teaching as the process of constructing powerful pedagogical events about curriculum events to help students construct mathematical knowledge. According to Shulman and his colleagues (Shulman, 1986a; Wilson, Shulman, & Richert. 1987), teachers draw on three main types of knowledge to represent the subject-matter content to students: subject-matter content knowledge, pedagogical content knowledge, and curriculum knowledge. Other factors influencing teachers' representation of subject-matter knowledge are teachers' conceptions of mathematics (Dossey, 1992), teachers' beliefs (Thompson, 1992), students' cognitions (Kieran, 1992), and students' affective factors (McLeod, 1992). Figure 6.5 shows the proposed theoretical framework. As shown there, learning is not only influenced by teaching but also by the curriculum (the content) and students' cognitions and affective factors. In Chapter 2, I describe each of the components of this framework and provide a rationale for their inclusion in the model.

As stated in Chapter 2, the proposed theoretical framework emerged from literature, reflection, and some results of the present study. From an empirical perspective, one contribution of the present study to the theoretical framework is that some teachers focus only on some selected content curriculum events and that teachers only use a limited type
of representations. Also, teachers' explanations are complex differing in the integration of both mathematical and pedagogical representations, integration of concepts, involvement of conceptual and procedural knowledge, involvement of operational and structural conceptions of mathematical objects as well as in treatment of difficult topics. Because the focus of the present study was not to validate the model, research is needed to add empirical validation to the model as well as to provide examples of the connections among its diverse components. As stated by Strauss and Corbin (1990), "Any categories, hypothesis, and so forth generated by the literature have to be checked out against real (primary data) ... to enhance the conceptual richness of the theory" (p. 55).

One of the goals of the research generated by this model is to create theories to better understand the interrelationship between curriculum, teaching, and learning in the context of specific content knowledge. This information might be useful for redesigning our teacher education programs.
Constructing a Theory of How Teachers Use their Knowledge for Instruction

Current research in cognitive science and research on teachers' knowledge deals with how people handle particular contents. In the context of teaching and my theoretical framework, the ultimate goal of this type of research is to construct a theory of how teachers use their knowledge for constructing pedagogical events for teaching specific content knowledge (Doyle, 1992; Marton, 1989). In the same way that understanding students' cognitions in a specific content domain has potential implications for the psychology of mathematics education, understanding how teachers use their knowledge for constructing pedagogical events about specific content domains has potential...
implications for the psychology of teacher education. This study described the pedagogical events that one teacher, Mr. Kantor, constructed about the following curriculum events: multiplication of fractions theorem, rate model for multiplication, multiplicative identity of 1, definition of reciprocal, the reciprocal of $x$ is $\frac{1}{x}$ ($x \neq 0$), reciprocal of zero, solving $ax = b$, definition of greater than, multiplicative properties of inequalities, solving $ax < b$, multiplication counting principle, classical definition of conditional probability, conditional probability formula, $n!$, permutation theorem, meaning of division, and algebraic definition of division. Further research is needed to elaborate a complete theory of how teachers use their knowledge to construct pedagogical events to teach these and other mathematical content curriculum events.

**Recommendations for Further Research**

In this section I will summarize some of the identified research that I believe is needed in the area of teachers' knowledge of representations and its relationship to classroom instruction and that I have discussed earlier. The next 11 questions refer to empirical research.

1) What is teachers' knowledge of mathematical representations (definitions or symbolic representations and mathematical proof) of the content they ordinarily teach?

2) What is teachers' knowledge of pedagogical representations (pictorial and story-problem representations) of the content they normally teach?

3) What is the relationship between teachers' knowledge of each of these types of representations and their classroom instruction?

4) What pedagogical events (representations, explanations, and questions) do teachers construct for teaching other topics?

5) To what extent do the pedagogical events that knowledgeable teachers construct involve conceptual and procedural knowledge?
6) To what extent do the pedagogical events that knowledgeable teachers construct involve the use of both mathematical and pedagogical representations?

7) What are the factors that influence teachers' use of mathematical and pedagogical representations when teaching mathematical ideas?

8) What are the sources (experience, textbook, teacher education programs) of teachers' knowledge of representations?

9) What are the sources (experience, textbook, teacher education programs) of teachers' explanations?

10) What kinds of representations do teachers learn in teacher education programs?

11) What is the impact of teachers using both mathematical and pedagogical representations on students' learning?

Philosophical research is needed to clarify the following questions:

12) What does it mean to teach a content curriculum event for conceptual understanding?

13) What does it mean to teach the mathematics curriculum for conceptual understanding?

We need to address the empirical questions using a variety of samples (prospective teachers, practicing teachers, novice teachers, experienced teachers) using both in-depth case studies and large-sample studies for a diverse range of mathematical topics (addition and subtraction, multiplication and division, rational numbers, etc.) to have a better understanding of the complexity of teachers' knowledge, knowledge use during classroom instruction, as well as a better understanding of the pedagogical events that they construct. To have a more complete understanding of teachers' construction of pedagogical events, we need to understand teachers' motives, intentions, and thinking when engaged in the activity of teaching: Why do teachers do what they do? Both types
of research, philosophical and empirical, are needed to enhance our understanding of those phenomena.

**Recommendations for Teacher Education**

I have already mentioned some recommendations for teacher education. In this section I will elaborate on some of them. The main purposes of teacher education programs is to prepare teachers so they can help students learn subject-matter knowledge. As teacher educators, we need to know what knowledge base teachers need to hold to carry out that instructional activity. Most of the studies conducted in the area of teachers knowledge have reported that participant teachers lack strong components of conceptual knowledge. Consequently, the researchers tend to recommend that teacher education programs need to equip teachers with conceptually rich subject matter. These researchers seem to assume, at least implicitly, that there is a direct relationship between teachers' knowledge and the use of that knowledge in the construction of pedagogical events to help students learn subject-matter knowledge. This study suggests that this is not necessarily the case. While Mr. Kantor held a strong knowledge of symbolic, pictorial, and story-problem representations, his use of that knowledge for the construction of pedagogical events was very limited.

Mr. Kantor mentioned one reason why he did not use his knowledge of definitions or symbolic representations: students have a hard time with definitions. As a mathematics teacher, I have encountered many students who do not provide correct definitions about mathematical entities when asked to do so. That does not mean that we should not construct definitions of mathematical objects.

I found that Mr. Kantor did not have a well-articulated knowledge of mathematical proofs. From a teacher education perspective, this is somewhat puzzling since I agree that teachers should know and appreciate how the mathematical truth of theorems is established. If we want students to know and appreciate these components of
mathematics then teachers as well as teacher educators need to know about mathematical proofs. From a student perspective, I am less concerned with the fact that Mr. Kantor did not have a well articulated knowledge of mathematical proofs. At the middle school level, students, and teachers, are not expected to construct mathematical proofs of theorems.

It is more puzzling that Mr. Kantor did not construct pictorial and story-problem representations as much as he could have. Researchers in mathematics education believe that the use of a variety of external representations, especially pictorial and story-problem representations, influence to a large extent the internal representations that students construct about mathematical objects. The use of those representations also helps students to see connections between mathematics and the physical world. A tentative hypothesis of why Mr. Kantor did not use his knowledge of representations as much as he could have is that he did not see it as important. Probably he was not exposed to this type of instruction. Then, I would agree with McDiarmid, Ball and Anderson's (1989) recommendation that we, teacher educators, need to use these types of representations in our own teaching. We need to practice what we preach.

One problem with story-problem representations is that some of them do not represent a genuine physical situation but rather an artificial one. One type of story-problem representation that may have a greater impact on teachers' instruction is applications. This type of story-problem representation can be seen by teachers as more relevant and reflecting better the connections between mathematics and the physical world. Then another fertile area of research is to examine the impact of teachers' knowledge of applications on their instruction.

In summary, this study makes us reflect on the role of teacher education programs. We tend to assume that teachers will use what they learn in these programs. This study shows that we should not take this assertion for granted. We need to understand the
factors that influence teachers' use of knowledge during classroom instruction. The purpose of the pedagogical events is to help students learn content curriculum events with understanding. I would like to close this section with two questions: What can we do to help teachers use their knowledge of representations for constructing pedagogical events? How does knowledge of representations fit into the structure of teachers' knowledge? (Fennema & Franke, 1992).

In Conclusion

Research literature on teachers' knowledge (e.g., Ball, 1990b; Borko et al., 1992; Post, Cramer, Behr, Lesh, & Harel, 1993; Sánchez & Llinares, 1992; Simon, 1993) tends to portray teachers' knowledge of representations as weak. In contrast, I found that Mr. Kantor's knowledge of representations was strong except in the case of mathematical proofs. Therefore, Mr. Kantor was a good case for examining the phenomena of teachers' use of representations and teachers' explanations during classroom instruction. However, the analysis of the pedagogical events that he constructed for teaching the content curriculum events related to algebraic multiplication revealed that he did not put to use his strong knowledge of representations for teaching those content curriculum events. At the very least this finding reveals that the relationship between teachers' knowledge and their use of that knowledge during classroom instruction is not a linear but a complex relationship. In addition, it was found that Mr. Kantor only constructed explanations for 18 of 41 possible content curriculum events. Furthermore, Mr. Kantor's explanations varied in terms of integration of both mathematical and pedagogical representations, treatment of difficult topics, operational and structural conceptions of algebraic objects, integration of concepts, and conceptual and procedural elements.

We need additional case studies to examine the phenomenon of use of both mathematical and pedagogical representations by knowledgeable teachers to get a better picture of that phenomenon. This is a promising area of research to better understand the
relationship between teachers' knowledge of representations and their classroom instruction. Also, we need to investigate, in depth, the reasons or rationales about why teachers use or fail to use mathematical and pedagogical representations during classroom instruction.

As teacher educators, we need to reflect on the role that teacher education programs can play in helping teachers not only to develop their knowledge of representations but also to develop their knowledge of the importance of using multiple representations that have the potential to promote students' procedural and conceptual knowledge of mathematics.
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APPENDIX A: Interview questions
Interview Schedule 1: Teacher's Background

1. How many years have you been teaching mathematics?
2. How did you get your certification?
3. What math courses did you take [as an undergraduate]?
4. Could you remember some experiences that you liked most about those courses?
5. Did you get an elementary certification or secondary certification?
6. Do you have any other minors, other specialties, other degrees?
7. Tell me about that experience [working in business for 20 years]. Do you think it was valuable? were there any connections to teaching mathematics?
8. Is there any teacher that you remember you had like that [making sure that students understand]?
Interview Schedule 2: Teacher’s conceptions about mathematics and mathematics teaching and learning

9. Why did you choose teaching as a profession?

10. Tell me about your conceptions of mathematics. How do you define mathematics for yourself?

11. What are the aspects that you emphasize in your teaching?

12. What do you think that a teacher should know for teaching middle grade mathematics successfully?

13. What do you think the teacher’s role should be in teaching mathematics?

14. What about students, what should they do when they are learning mathematics?

15. What do you expect kids to know about mathematics?

16. What are your goals for teaching mathematics, in general?

17. What do you think about representing a mathematical idea using manipulatives?

18. What about using pictorial, graphical, and geometrical representations, not just using symbolic representations but also those kinds of representations?

19. How would you describe to an outsider how you go about for teaching a specific topic?
Interview Schedule 3: A sample of questions designed to examine Mr. Kantor's knowledge of representations during the first round of interviews

Illustrate the commutative property of multiplication with pictorial representations.

Make up some story problems that represent the commutative property of multiplication.

Illustrate the associative property of multiplication using a pictorial representation.

Illustrate with story problems the associative property of multiplication.

Use algebraic tools to proof the multiplication of fractions theorem.

Illustrate with a story problem that the multiplication of a positive number and a negative number give us a negative number.

Algebraic proof that $a(-1) = -a$.

Proof that $(-a)b = -(ab)$.

Define reciprocal.

How would you prove algebraically that $a \cdot 0 = 0$?

Illustrate some problems that can be solved by the equation $ax = b$.

How do you define algebraically $a > b$?

Illustrate with a pictorial representation that $x < y$ implies $-x > -y$.

Illustrate with a story problem that $x < y$ implies $-x > -y$.

Prove algebraically, that if $x < y$ and $a$ is negative then $ax > ay$. 

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Illustrate with a pictorial representation that $P(A/B) = P(\cap B)/P(B)$.

Why is that $a \neq 0$ is impossible?

Illustrate with a story-problem that division by zero is impossible.

Illustrate with a pictorial representation or a story problem that $a ÷ b$ is negative if $a$ is positive and $b$ is negative.
Interview Schedule 4: A second sample of questions designed to assess Mr. Kantor's knowledge of representations in depth (Second and third run of interviews)

Associative property of multiplication.

Definition / mathematical representation:

Pictorial representation:

Story-problem representation:

Mathematical proof:
Multiplicative property of inequalities (part 2).

Definition / mathematical representation:

Pictorial representation:

Story-problem representation:

Mathematical proof:
The quotient of two negative numbers is positive.

Mathematical representation:

Pictorial representation:

Story-problem representation:

Mathematical proof:
APPENDIX B: Handouts given by Mr. Kantor
36.1 Multiplication of Fractions

Name ______________

Period __

The length of the side of the square above is one unit. Therefore, the area of the square is __ square units.

Divide side AE into four equal lengths. Label the ends of each length B, C, D. Draw horizontal lines through B, C, and D and vertically through the square. This will cut the square into four rectangles of the same size.

The area of each rectangle = __ units$^2$

The length of BE = __ units

Divide side EJ into five equal lengths. Label the ends of each length F, G, H, I. Draw vertical lines through F, G, H, and I and horizontally through the square. This will create __ rectangles of the same size.

The area of each rectangle = __ units$^2$ (OVER)

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The length of \( \overline{EH} \) = _____ units

Carefully outline the rectangle with sides \( \overline{BE} \) and \( \overline{EH} \). It is comprised of _____ smaller rectangles, each with area _____ unit\(^2\). Therefore the area of the larger rectangle is _____ unit\(^2\).

The area of the larger rectangle also equals _____ \( \times \) _____ by the area model for multiplication.

The lengths of \( \overline{BE} \) and \( \overline{EH} \) are \( \frac{3}{4} \) and \( \frac{3}{5} \) respectively.

Therefore the area of the larger rectangle equals

\[
\frac{3}{4} \times \frac{3}{5} = 3 \times \frac{3}{4} \times \frac{3}{5} = 3 \times \frac{3}{20}
\]

By using the area of a rectangle we have shown how to multiply fractions.
15.2 Conditional Probability

<table>
<thead>
<tr>
<th></th>
<th>Choir</th>
<th>Music</th>
<th>Band</th>
<th>Orchestra</th>
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<tbody>
<tr>
<td>Spanish</td>
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<td>French</td>
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<td>Reading</td>
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</tbody>
</table>
APPENDIX C: Mr. Kantor's symbolic representations
<table>
<thead>
<tr>
<th>Content curriculum event*</th>
<th>Mathematical definition or symbolic representation**</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Area Model for Multiplication (Continuous case) (VS)</td>
<td>If you have a rectangle of length ( l ) and width ( w ) its area is ( l ) times ( w ). (VS, C) (I, 26, 05-30-95)</td>
</tr>
<tr>
<td>2. Commutative property of multiplication (S)</td>
<td>If ( a, b \in R ), then ( a \cdot b = b \cdot a ). (S, C) (I, 31, 06-19-95)</td>
</tr>
<tr>
<td>3. Area Model for Multiplication (Discrete version) (VS)</td>
<td>You talk about an array of things.... If you have ( n ) rows with ( m ) elements in each row then there is ( n ) times ( m ) elements in the array. (VS, C) (I, 26, 05-30-95)</td>
</tr>
<tr>
<td>4. Volume of a rectangular solid (VS)</td>
<td>Not asked</td>
</tr>
<tr>
<td>5. Associative property of multiplication (S)</td>
<td>For ( a, b, c \in R ), ((a \cdot b) \cdot c = a \cdot (b \cdot c)). (S, C) (I, 31, 06-19-95)</td>
</tr>
<tr>
<td>6. Multiplication of fractions theorem (S)</td>
<td>( a, b, c, d \in R, b, d \neq 0 ) then ( \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} ). (S, C) (I, 31, 06-19-95)</td>
</tr>
<tr>
<td>7. Rate model for multiplication (V)</td>
<td>It's supposed to be the same thing [as multiplication of fractions], you just have units too, so that the units have to be consistent, and the units in your answer have to make sense [they have to have meaning to multiplying together].... You take a quantity and multiply it by a rate. (V, C) (I, 26, 05-30-95)</td>
</tr>
<tr>
<td>8. The product of two positive numbers is a positive number (S)</td>
<td>If ( a &gt; 0 ) and ( b &gt; 0 ), then ( ab &gt; 0 ). (S, C) (I, 33, 08-25-95)</td>
</tr>
<tr>
<td>9. The product of two negative numbers is a positive number (S)</td>
<td>If ( a, b \in R ) and ( a &lt; 0 ) &amp; ( b &lt; 0 ) then ( a \cdot b &gt; 0 ). (S, C) (I, 31, 06-19-95)</td>
</tr>
<tr>
<td>10. The product of a positive number and a negative number is a negative number (S)</td>
<td>Given ( a &gt; 0 ), ( b &lt; 0 ) then ( a \cdot b &lt; 0 ) (Given ( a ) is greater than zero. ( b ) is less than zero then ( a ) times ( b ) is less than zero.) (S, C) (I, 31, 06-19-95)</td>
</tr>
</tbody>
</table>

(To be continued)

Table C.1 Mr. Kantor's symbolic representations

275
11. The product of a negative number and a positive number is a negative number (S) Not asked

12. Multiplicative identity of 1 (S) Given any real number \( x \), \( x \cdot 1 = x \). (S, C) (I, 27, 06-02-95)

13. Multiplicative property of \(-1\) (S) Given any real number \( x \), negative one times \( x \) equals the opposite of \( x \): \( -1 \cdot x = -x \). (S, C) (I, 27, 06-02-95)

14. Definition of reciprocals (S) It's the multiplicative inverse. You have something \([x]\) multiplied by its reciprocal equals one, \( x \cdot \frac{1}{x} = 1 \). (S, C) (I, 27, 06-02-95)

15. The reciprocal of a real number (S) The reciprocal of \( a \) is \( \frac{1}{a} \), \( a \neq 0 \). (S, C) (I, 23, 01-03-95)

16. Reciprocal of zero (V) 0 has no reciprocal (V, C) (I, 23, 01-03-95) & (CO, 3, 10-17-94)

17. Multiplication property of zero (S) \( x \times 0 = 0 \) where \( x \) is any real number, \( x \cdot 0 = 0 \) where \( x \in \mathbb{R} \). (S, C) (I, 27, 06-02-95)

18. Multiplication property of equality (S) If \( x = y \), then \( ax = ay \) [and that's true for] \( a, x, y \in \mathbb{R} \). (S, C) (I, 27, 06-02-95)

19. Solving \( ax = b, a \neq 0 \) (S) \[ x = \frac{b}{a}, a \in \mathbb{R}, \text{ except } 0. \] (S, C) (I, 27, 06-02-95)

20. Solve \( 0x = b, b \neq 0 \) (V) If \( b \) is not equal zero there is no solution. (V, C) (I, 27, 06-02-95)

21. Solve \( 0x = 0 \) (V) All real numbers. (V, C) (I, 27, 06-02-95)

22. Solve \( ax = 0, a \neq 0 \) (S) \( x = 0 \) (So \( x \) has to be equal to zero.) (S, C) (I, 27, 06-02-95)

23. Solve \( -x = a \) (S) \( x = -a \). (S, C) (I, 27, 06-02-95)

(To be continued)
### Table C.1 (Continued)

<table>
<thead>
<tr>
<th>Table C.1 (Continued)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>24. ( a &gt; b, a &lt; b ) (S)</strong></td>
<td>a) ( a ) greater than ( b ) is defined algebraically as ( a ) minus ( b ) greater than zero, ( a - b &gt; 0 ). (I, 24, 01-05-95)</td>
</tr>
<tr>
<td></td>
<td>b) ( a ) less than ( b ) is defined as ( a ) minus ( b ) less than zero ( a - b &lt; 0 ). (S, C) (I, 24, 01-05-95)</td>
</tr>
<tr>
<td><strong>25. The multiplication property of inequality (part 1) (S)</strong></td>
<td>Given ( ax &lt; b ) where ( a, x, ) and ( b ) are in ( R ) then ( x ) less than ( \frac{b}{a} ) when ( a &gt; 0 ). (S, C) (I, 27, 06-02-95)</td>
</tr>
<tr>
<td><strong>26. The multiplication property of inequality (part 2) (S)</strong></td>
<td>Given ( ax &lt; b ) where ( a, x, ) and ( b ) are in ( R ) then ( x &gt; \frac{b}{a} ) when ( a &lt; 0 ). (S, C) (I, 27, 06-02-95)</td>
</tr>
<tr>
<td><strong>27. Solving ( ax &lt; b ) (S)</strong></td>
<td>Given ( ax &lt; b ) where ( a, x, ) and ( b ) are in ( R ) then ( x &lt; \frac{b}{a} ) when ( a &gt; 0 ).</td>
</tr>
<tr>
<td></td>
<td>[and] ( x &gt; -\frac{b}{a} ) when ( a &lt; 0 ). (S, C) (I, 27, 06-02-95)</td>
</tr>
<tr>
<td><strong>28. Multiplication counting principle (VS)</strong></td>
<td>If there is ... ( x ) ways to make one choice [...] and ( y ) ways to make a second choice. How many different pairs of choices could be made? Different pairs of choices would be ( x ) times ( y ). (VS, PC) (I, 28, 06-06-95)</td>
</tr>
<tr>
<td><strong>29a. Probability of an event. P(A) (S)</strong></td>
<td>Assuming possible equally likely outcomes ... ( P(A) ) equals successful outcomes ( A ) occurs over possible equally likely outcomes ... of an event, I guess.</td>
</tr>
<tr>
<td></td>
<td>( P(A) = \frac{\text{Successful outcomes (A occurs)}}{\text{Possible equally likely outcomes of event}} ) (VS, PC)</td>
</tr>
<tr>
<td><strong>29b. Probability of the intersection of two events, P(A ( \cap ) B) (S)</strong></td>
<td>It's the probability that you look at successes meaning that ( A ) and ( B ) both happen, and that's over possible, possible events.</td>
</tr>
<tr>
<td></td>
<td>( P(A \cap B) = \frac{\text{Successes (A and B both happen)}}{\text{Possible events}} ) (VS, PC)</td>
</tr>
<tr>
<td><strong>(To be continued)</strong></td>
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<tr>
<td><strong>277</strong></td>
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<tr>
<td>Table C.1 (Continued)</td>
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<tr>
<td>-----------------------</td>
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<tr>
<td>29c. Classical definition of conditional probability,</td>
<td></td>
</tr>
<tr>
<td>$P(A</td>
<td>B) = \frac{n(A \cap B)}{n(B)}, n(B) \neq 0$</td>
</tr>
<tr>
<td>(S)</td>
<td></td>
</tr>
<tr>
<td>You look at possible outcomes ... given that B occurred and then successful given B occurred.</td>
<td></td>
</tr>
<tr>
<td>$P(A \text{ given } B) = \frac{\text{Successful given } B \text{ occurs}}{\text{Possible outcomes given that } B \text{ occurs}}$</td>
<td></td>
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<tr>
<td>(VS, PC)</td>
<td></td>
</tr>
<tr>
<td>(I, 28, 06-06-95)</td>
<td></td>
</tr>
<tr>
<td>30. Second definition of conditional probability</td>
<td></td>
</tr>
<tr>
<td>$P(B</td>
<td>A) = \frac{P(A \cap B)}{P(A)}, P(A) \neq 0$</td>
</tr>
<tr>
<td>(S)</td>
<td></td>
</tr>
<tr>
<td>31. Conditional probability formula,</td>
<td></td>
</tr>
<tr>
<td>$P(A \cap B) = P(A)P(B</td>
<td>A)$</td>
</tr>
<tr>
<td>(S)</td>
<td></td>
</tr>
<tr>
<td>32. $n!$</td>
<td></td>
</tr>
<tr>
<td>(S)</td>
<td></td>
</tr>
<tr>
<td>$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$</td>
<td></td>
</tr>
<tr>
<td>(S, C) (I, 33, 08-25-95)</td>
<td></td>
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<tr>
<td>33. $0!$</td>
<td></td>
</tr>
<tr>
<td>(S)</td>
<td></td>
</tr>
<tr>
<td>$0! = 1$</td>
<td></td>
</tr>
<tr>
<td>(S, C) (I, 33, 08-25-95)</td>
<td></td>
</tr>
<tr>
<td>34. Permutation theorem (VS)</td>
<td></td>
</tr>
<tr>
<td>You have n items, taking n at a time. There are n factorial possible arrangements.</td>
<td></td>
</tr>
<tr>
<td>(VS, C) (I, 28, 06-06-95)</td>
<td></td>
</tr>
<tr>
<td>35. Meaning of division (VS)</td>
<td></td>
</tr>
<tr>
<td>Not asked</td>
<td></td>
</tr>
<tr>
<td>36. Algebraic definition of division (S)</td>
<td></td>
</tr>
<tr>
<td>$\frac{a}{b} = a \cdot \frac{1}{b}$, $b \neq 0$</td>
<td></td>
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<tr>
<td>(S, C) (I, 31, 06-19-95)</td>
<td></td>
</tr>
<tr>
<td>37. Division by zero (V)</td>
<td></td>
</tr>
<tr>
<td>[We] can't divide by zero.</td>
<td></td>
</tr>
<tr>
<td>(CO, 2, 10-14-94)</td>
<td></td>
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<tr>
<td>Division by zero is impossible.</td>
<td></td>
</tr>
<tr>
<td>(V, C) (I, 33, 08-25-95)</td>
<td></td>
</tr>
<tr>
<td>38. The quotient of a positive number and a positive number is positive (S)</td>
<td></td>
</tr>
<tr>
<td>If $a$ and $b$ are real numbers that are $&gt; 0$ then $\frac{a}{b}$ is positive.</td>
<td></td>
</tr>
<tr>
<td>(VS, PC) (I, 29, 06-08-95)</td>
<td></td>
</tr>
<tr>
<td>(To be continued)</td>
<td></td>
</tr>
</tbody>
</table>
39. The quotient of a negative number and a negative number is positive (S)
   If $a$ and $b$ are negative then $\frac{a}{b}$ is positive. (VS, PC)
   (I, 30, 06-16-95)

40. The quotient of a negative number and a positive number is negative (S)
   If $a$ is $-$ and $b$ is $+$ then $\frac{a}{b} = -$. (VS, PC) (I, 30, 06-16-95)

41. The quotient of a positive number and a negative number is negative (S)
   If $a$ is $+$ and $b$ is $-$ then $\frac{a}{b} = -$. (VS, PC) (I, 30, 06-16-95)

* I Each content curriculum event was categorized according to the highest degree of symbolism that could be used to represent it in three categories: verbal representations (V), verbal-symbolic representations (VS), and symbolic representations (S).

** Mr. Kantor's definitions or symbolic representations were categorized according to the degree of symbolism involved in the three categories: verbal representations (V), verbal-symbolic representations (VS), and symbolic representations (S). I used three levels to judge the correctness of Mr. Kantor's symbolic representations: Correct (C), partially correct (PC) and incorrect (I). For the description of the criteria used for judging the degree of correctness of Mr. Kantor's definitions or symbolic representations see Chapter 4, section "Analysis," "Coding scheme for Mr. Kantor's knowledge of definitions or symbolic representations."

Table C.1 Mr. Kantor's symbolic representations
<table>
<thead>
<tr>
<th></th>
<th>Verbal representations</th>
<th>Verbal-symbolic representations</th>
<th>Symbolic representations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Researcher's categorization*</td>
<td>7**, 16, 20, 21, 37 (5)</td>
<td>1, 3, 4***, 28, 34, 35*** (6)</td>
<td>2, 5, 6, 8, 9, 10, 11***, 12, 13, 14, 15, 17, 18, 19, 22, 23, 24, 25, 26, 27, 29 (29a, 29b, 29c), 30, 31, 32, 33, 36, 38, 39, 40, 41 (30)</td>
</tr>
<tr>
<td>Categorization of Mr. Kantor's symbolic representations (38)</td>
<td>7, 16, 20, 21, 37 (5)</td>
<td>1, 3, 28, 29 (29a, 29b, 29c), 34, 38, 39, 40, 41 (9)</td>
<td>2, 5, 6, 8, 9, 10, 12, 13, 14, 15, 17, 18, 19, 22, 23, 24, 25, 26, 27, 30, 31, 32, 33, 36 (24)</td>
</tr>
<tr>
<td>Mr. Kantor's correct symbolic representations (32)</td>
<td>7, 16, 20, 21, 37 (5)</td>
<td>1, 3, 34 (3)</td>
<td>2, 5, 6, 8, 9, 10, 12, 13, 14, 15, 17, 18, 19, 22, 23, 24, 25, 26, 27, 30, 31, 32, 33, 36 (24)</td>
</tr>
<tr>
<td>Mr. Kantor's partially correct symbolic representations (6)</td>
<td></td>
<td>28, 29 (29a, 29b, 29c), 38, 39, 40, 41 (6)</td>
<td></td>
</tr>
</tbody>
</table>

*Each content curriculum event was categorized according to the highest degree of symbolism that could be used to represent it in three categories: verbal representations (V), verbal-symbolic representations (VS), and symbolic representations (S).

**Content curriculum events are referred to by numbers according to Table C.1. Thus, 7 refers to content curriculum event 7, rate model for multiplication.

*** Symbolic representation not asked.

Table C.2 Categorization of symbolic representations
APPENDIX D: Mr. Kantor's mathematical proofs
<table>
<thead>
<tr>
<th>Content curriculum event*</th>
<th>Mathematical proof**</th>
</tr>
</thead>
</table>
| 1. Area Model for Multiplication (Continuous case) (Geometric proof, axiom) | a) I don't know exactly if I prove it or if I just make them feel comfortable with it. (I) (I, 33, 08-25-95)  
b) I don't know. If you take something like ... a rows with b elements in each row, just by definition of multiplication you are talking about multiplication replacing repeated addition. (I) (I, 33, 08-25-93) |
| 2. Commutative property of multiplication (Axiom) | a) I can't recall if I have proved it.... I can't recall. (I) (I, 26, 05-30-95)  
b) Commutative is just, again, it's a point of view type of thing. I don't know if that's a proof but you have to look at the commutative rows, m elements in a row and n elements in a column, it's just, it's the same situation, it has been rotated ... just from a different point of view. (I) (I, 26, 05-30-95) |
| 3. Area Model for Multiplication (Discrete version) (Induction) | s seats per row then in every row you have that many seats, you have r of those items and it's repeated addition. (PC) (I, 31, 06-19-95)  
\[
s + s + s + \ldots + s = r \cdot s
\]
| 4. Volume of a rectangular solid | NA*** |
| 5. Associative property of multiplication (Axiom) | I imagine we could [prove it]. I'd go back and do the same thing [I do] with the commutative property (I) (I, 26, 05-30-95) |
| 6. Multiplication of fractions theorem (Algebraic proof) | \[
\frac{a}{b} \cdot \frac{c}{d} = (a + b)(c + d) \]  
(K: This is the same as a divided by b times c divided by d.)  
\[
= (a \cdot \frac{1}{b}) \cdot (c \cdot \frac{1}{d}) \]  
(K: a times the reciprocal [of b], c times the reciprocal [of d].) |

(To be continued)

Table D.1 Mr. Kantor's mathematical proofs
Table D.1 (Continued)

\[ a \cdot c \cdot \frac{I}{b} \cdot \frac{I}{d} \]  
(K: Drop all parenthesis and regroup.)

\[ \frac{ac}{bd} \]

J: Let's see how do you go from there \( [a \cdot c \cdot \frac{I}{b} \cdot \frac{I}{d}] \) to there \( \frac{ac}{bd} \).

K: This \( \frac{I}{b} \) times this \( \frac{I}{d} \) is the same thing as \( \frac{I}{bd} \). You can look it this way. You cut something into \( b \) parts and cut that into \( d \) parts. You cut it into \( b \) times \( d \) parts. (PC)

(I, 23, 01-03-95)

7. Rate model for multiplication

NA

8. The product of two positive numbers is a positive number (Axiom)

a) Is that a definition or? I don't know how to try to prove that. (I)

(b) You go back to repeated addition, \( a \) times \( b \) is \( a \) copies of \( b \) added together, so a positive plus a positive is a positive.... The book will take something as given without proving it which may be doing ... [in] this case. I don't know.... This is the algebraic definition of multiplication. It still goes back to repeated addition to show that that's the case.... I really don't [know how to prove it formally.] (I)

(I, 26, 05-30-95)

(I, 33, 08-25-95)

9. The product of two negative numbers is a positive number (Algebraic proof)

a) \(-a \cdot -b\)  
\(-1 \cdot a \cdot -1 \cdot b\)  
(The opposite of \( a \) is the same thing as negative one times \( a \).)

\(-1 \cdot -1 \cdot a \cdot b\)  
(You do that and then you rearrange things.)

\(-(-1) \cdot a \cdot b\)

\(1 \cdot a \cdot b\)  
(You have to be able to say negative one times negative one is positive one.... That means the opposite of this. The opposite of negative one is positive one)  
(PC)  
(I, 26, 05-30-95)

b) I'd go back to addition again.... How do you do repeated addition? ... It's reasonable that that should be a definition. (I)

(I, 33, 08-25-95)

(To be continued)

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10. The product of a positive number and a negative number is a negative number
   (Algebraic proof)
   \[ -a \cdot b \]
   ([Let] \( a \) be a positive [number]. So the opposite of \( a \) is a negative number.)
   \[ = -1 \cdot a \cdot b \]
   (The opposite of \( a \) is the same thing as negative one times \( a \).)
   \[ = -(a \cdot b) \]
   (Then group that.)
   \[ = -(a - b) \]
   (If you group that, a positive \( a \) times a positive \( b \) is a positive \( ab \) and the opposite of that has to be negative \( -(ab) \).) [Negative one times that \( -1(ab) \) is the same as the opposite of \( ab \) \(-a b\).] (PC)
   (I, 26, 05-30-95)

11. The product of a negative number and a positive number is a negative number
   (Algebraic proof)
   a) [An algebraic proof of \((-a)b = -(ab))\]
   \[ (-a)b \]
   \[ = (-1)a \cdot b \]
   \[ = -(a \cdot b) \]
   (This \((-a)\) is the same as that \(-ab\).) Once I have that \(-a = -1 \cdot a\) (C) (I, 23, 01-03-95)
   b) Proof? Multiplication is commutative. [If we already proved the other then we can prove that] Have we already proved the other? (C)
   (I, 30, 06-16-95)

12. Multiplicative identity of 1
   (Axiom)
   Well, if you look at the definition of multiplication, it's repeated addition, then you have one \( x \). (I) (I, 27, 06-02-95)

13. Multiplicative property of -1
   \[ [-1 \cdot a = -a]\]
   (Algebraic proof)
   I don't know whether we know that by definition or not. I have to look it up. (I) (I, 23, 01-03-95)

14. The definition of reciprocals
   \[ x \cdot R_x = 1\] [By definition] (C) (I, 23, 01-03-95)

15. The reciprocal of \( a \) is \( \frac{1}{a} \)
   \( (a \neq 0)\)
   \[ \frac{a \cdot 1}{a} = \frac{a}{a} = 1 \]
   And then by division of fractions [that] equals one. (PC) (I, 23, 01-03-95)

16. Reciprocal of zero (Proof by contradiction)
   \[ 0 \cdot R = 1 \]
   0 has no reciprocal. There isn't something I multiply by zero to get one (PC) (I, 33, 08-25-95)

17. Multiplication property of zero
   (Algebraic proof)
   \[ a \times 0 = 0 \]
   times 0 equals zero. By definition of zero! ... I don't know the formal proof but I can use repeated addition ... zero terms that you are adding together. (I) (I, 23, 01-03-95)

(To be continued)
<table>
<thead>
<tr>
<th>Table D.1 (Continued)</th>
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<tbody>
<tr>
<td>18. Multiplication property of equality (Algebraic proof)</td>
</tr>
<tr>
<td>19. Solving $ax = b$ ($a \neq 0$) (Algebraic proof)</td>
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<tr>
<td></td>
</tr>
<tr>
<td>20. Solve $0x = b$, $b \neq 0$ (Proof by contradiction)</td>
</tr>
<tr>
<td>21. Solve $0x = 0$ (Algebraic proof)</td>
</tr>
<tr>
<td>22. Solve $ax = 0, a \neq 0$ (Algebraic proof)</td>
</tr>
<tr>
<td>23. Solve $-x = a$ (Algebraic proof)</td>
</tr>
<tr>
<td>24. $a &gt; b, a &lt; b$ (Definition)</td>
</tr>
<tr>
<td>25. The multiplication property of inequality (part 1) (Algebraic proof)</td>
</tr>
</tbody>
</table>

(To be continued)
K: ... If \( m \) is less than \( n \), I don't know how rigorous this would be, but this implies that, first given \( m \) and \( n \) greater than zero, if \( m \) is less than \( n \) implies \( m \) is closer to 0. [And that] implies the absolute value of \( m \) is less than the absolute value of \( n \). It's the same thing. If I multiply both sides by \( K \), \( K \) is less than zero. I'm trying to think how do I prove. If I can prove this, that the opposite of \( m \) is bigger than the opposite of \( n \). I can show that. I know we don't prove that in the book, we just take it as [given] I mean, I showed it, I can show it by showing here is \( m \) and here is \( n \). This is positive, this is negative, so that's a theorem. It's gotta be the situation, it's gotta be the left of that, when I take the opposite using a compass, it's here, and using the compass that's here, now this one is less than that.... I don't do anything analytically. We just take it as true.

J: Do you want to think more about that?

K: No. [we laughed] [it's getting, that's too much of abstract algebra.]

26. The multiplication property of inequality (part 2)

(Algebraic proof)

a) If \( x < y \) then \( -x > -y \) (I don't know whether that's a definition. That this is the case. If that is a definition...)

If \( x < y \) then \( ax < ay \) (If we multiply both sides of this by positive \( a \))

\[-ax > -ay\] (and multiply this \([-x > -y]\) by positive \( a \) then we maintain the relationship.)

\[a'x > a'y\] (And since this is positive \( a \) is the same thing as saying OK. That's \([a']\) a negative)

J: Do you have any algebraic proof to these? [If \( x < y \) then \(-x > -y\) and if \( x < y \) then \( ax < ay \)]

K: (laughs). I don't know. (I) (I, 24, 01-05-95)

b) You could do it in that way, too. On a number line... I don't know how mathematically we can prove it, but we can show that, if this is true, \( ax \) is less than \([b]\) on the number line. This is positive, this is negative, \( ax \) has to be to the left of \( b \). \( x \) less than zero.... We just don't do that many proofs that proves is the same thing. I am trying to remember what we have already proven. Because you can look at different cases. Give every case that is possible, if you start with \( a \) less than zero, then look at the possibilities, \( x \) is less than zero. \( a \) times \( x \) has to be positive. This implies \( b \) is positive since \( a \) times \( x \) is less than \( b \). From that, \( b \) divided by \( a \) has to be less than zero, but that doesn't prove that \( x \) is bigger than that. Yeah, I'm sure I can do it but it may take some time. (I) (I, 28, 06-06-95)

27. Solving \( ax < b \) (Algebraic proof)

Given \( ax < b \) where \( a, x, \) and \( b \) are in \( R \) then \( x < \frac{b}{a} \) where \( a > 0 \).

\[x > \frac{b}{a}\] when \( a < 0 \) (PC) (I, 27, 06-02-95) & (I, 33, 08-25-95)

(To be continued)
Table D.1 (Continued)

28. Multiplication counting principle (Proof by induction)

The way to talk about it is that, if you're taking one choice at a time and there is, let's say four ways of making that first choice. For each of those choices there are may be three ways of making the second one. So then it's, if you have four choices, that doesn't have to be four, it can be $x$ choices, for each of those there are $y$ ways to make another choice. So there is $y$ plus $y$ plus $y$ plus $y$ and there are $x$ of those so it's $x$ times $y$. (PC) (I, 28, 06-06-95)

29a. Probability of an event, $P(A)$. (Definition) NA

29b. Probability of the intersection of two events, $P(A \cap B)$. (Definition) NA

29c. Classical definition of conditional probability

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)} , n(B) \neq 0$$

(Definition)

30. Second definition of conditional probability, $P(B \mid A)$ Since I can't do this one (the conditional probability formula), I don't think I can give you this one [I asked him first the conditional probability formula]. (I) (I, 31, 06-19-95)

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} , P(A) \neq 0$$

(Definition, heuristic argument)

31. Conditional probability formula, $P(A \cap B) = P(A)P(B \mid A)$ (Algebraic proof) [I could not show a proof at this time] (I). (I, 31, 06-19-95)

(To be continued)
<table>
<thead>
<tr>
<th>32.</th>
<th>( n! ) (Definition)</th>
<th>NA</th>
</tr>
</thead>
<tbody>
<tr>
<td>33.</td>
<td>( 0! ) (Definition, heuristic argument)</td>
<td>a) You need it to be able to do this ( \frac{7!}{(7-7)!7!} ) or this ( \frac{7!}{0!7!} ). You can by logic, let's see. I have to come up with the answer. I know that you need it for that, for that formula to work. Zero factorial has to be one ... divided by zero. I mean, ... if you had it undefined that wouldn't help either because you have to ... with zero for this formula to work... [you have seven] and you are taking seven at a time. You end up with this seven factorial [divided by 0 factorial] times [seven factorial] and I know the answer is one ( \frac{7!}{0!7!} = 1 ). I know I can't divide by zero and I don't want the formula to fall apart. So by definition I make this equal one... There may be properties of factorials that I just have never seen [to justify ( 0! = 1 ) in another way]. (C) (I, 25, 01-10-95)</td>
</tr>
<tr>
<td>b) That's by definition, I think. (C) (I, 33, 08-25-95)</td>
<td></td>
<td></td>
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<tr>
<td>34.</td>
<td>Permutation theorem (Proof by induction)</td>
<td>Just the same way, if I have ( n ) items to put here, I have ( n ) items to put here but I already use one, and the next one, I know I have used two ( {n 	imes (n-1) 	imes (n-2) \ldots } ) (PC) (I, 28, 06-06-95)</td>
</tr>
<tr>
<td>35.</td>
<td>Meaning of division</td>
<td>NA</td>
</tr>
<tr>
<td>36.</td>
<td>Algebraic definition of division (Definition, heuristic argument)</td>
<td>Dividing by a fraction is the same thing as multiplying by the reciprocal (by definition). (C) (CO, 3, 10-17-94) &amp; (I, 31, 06-19-95)</td>
</tr>
<tr>
<td>( a + b = a - \frac{1}{b} ) ( b \neq 0 ) by definition. (C) (I, 33, 08-25-95)</td>
<td></td>
<td></td>
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<tr>
<td>37.</td>
<td>Division by zero (Proof by contradiction)</td>
<td>( \frac{a}{0} ) is impossible] Well, if you look at different meanings of division... Talking about zero going into something a certain number of times.... It doesn't ... going into is, it is just an inconceivable question I guess. The other is, if there was an answer to that, then the related fact had to be true.... If this has an answer ( \frac{a}{0} ) (To be continued)</td>
</tr>
</tbody>
</table>
then a divided by zero equals b \( \frac{a}{0} = b \). Then b times 0 has to be equal a. The only way that can be is if a were zero and in that case b can be anything. So everything works for b. So that's why 0 divided by 0 is kind of meaningless too. (PC) (I, 23, 01-03-95)

38. The quotient of a positive number and a positive number is positive (Algebraic proof)

\[ a + b = a \cdot \frac{1}{b}, \ a \text{ and } b \text{ are } + \ (\text{We can go back to multiplication. } a \text{ divided by } b \text{ is the same thing as } a \times \text{ one over } b) \]

b is +, \( \frac{1}{b} \) is +, (and you know that b is positive, its reciprocal is positive, so you know, and if a and b are positive then, that's a positive \( [a] \) times a positive \( \frac{1}{b} \) which is always positive... We proved that before [if a number is positive then the reciprocal is positive], I think we did. We can prove it because a times its [reciprocal] has to equal one. b is positive times its reciprocal has to equal one. So positive times a positive equals one so the reciprocal has to be positive] (C) (I, 30, 06-16-95)

39. The quotient of a negative number and a negative number is positive (Algebraic proof)

If a and b are -, then \( a + b = a \cdot \frac{1}{b} \).

b [is] – then \( \frac{1}{b} \) [is] – [b is negative then one over b is negative]

because b times one over b equals a positive \( b \cdot \frac{1}{b} = + \). This [b] is negative, that one \( \frac{1}{b} \) has to be negative. And so then we know this is negative [a] and that's negative \( \frac{1}{b} \), equals positive \( \cdot + \) [using multiplication rules] (C) (I, 30, 06-16-95)

40. The quotient of a negative number and a positive number is negative (Algebraic proof)

If a [is] – and b [is] +. \( a + b = a \cdot \frac{1}{b} \) [a divided by b equals a times one over b]. If a is negative and b is positive, negative [a] times a positive \( \frac{1}{b} \) what you get is negative \( - + = - \) (C) (I, 30, 06-16-95)

(To be continued)
41. The quotient of a positive number and a negative number is negative (Algebraic proof)

Given $a$ is positive, $b$ is negative, $a$ divided by $b$ equals $a$ times $1$ over $b$ [$a + b = a \cdot \frac{1}{b}$]. We already showed [that if] $b$ is negative so it's this [$\frac{1}{b}$]. [positive $a$] times a negative [$\frac{1}{b}$] is negative [$\cdot - = -$]. (C) (I, 30, 06-16-95)

* Each content curriculum event was categorized according to the type of proof that we can use to establish its truth in the following categories: definitions, axioms, geometric proofs, algebraic proofs, proofs by contradiction, and proofs by induction.

** Each of Mr. Kantor's proofs was categorized according to their degree of correctness in the following categories: Correct (C), Partially correct (PC), Incorrect (I). The criteria used for judging the correctness of Mr. Kantor's mathematical proofs are described in Chapter 4, section "Analysis," "Coding schema for Mr. Kantor's knowledge of proofs."

*** Proof not asked or not applicable.

Table D.1 Mr. Kantor's mathematical proofs
Each content curriculum event was categorized according to the type of proof that we can use to establish its truth in the following categories: definitions, axioms, geometric proofs, algebraic proofs, proofs by contradiction, and proofs by induction.

For some content curriculum events I asked Mr. Kantor a proof more than once. In some of these cases he provided different arguments and hence the CCEs may appear in more than one cell. An average was taken for correctness. The criteria used for judging the correctness of Mr. Kantor's mathematical proofs are described in Chapter 4, section "Analysis," "Coding schema for Mr. Kantor's knowledge of proofs."

Table D.2 Categorization of Mr. Kantor's proofs
APPENDIX E: Mr. Kantor's pictorial representations
### Area Model for Multiplication

**1. Continuous case**

There is \( I \) squares this way, \( w \) rows of those. That's what \( I \) times \( w \) will give you. (PE, PC) (I, 26, 05-30-95)

![Area Model](image)

\[
lw = A
\]

**2. Commutative property of multiplication**

You can show it with rectangles. Just in different orientations, same sides. One standing on its width, [the other on] its length. (E, C) (I, 26, 05-30-95)

![Rectangle](image)

\[
A = lw
\]

**3. Discrete version**

You can do seats in an auditorium. You have so many seats in a row and so many rows, so seats in a row (s) times number of rows (r) gives you the capacity, let's say, of the auditorium.... [It is \( s \cdot r \) because you have] \( s \) seats per row, then in every row you have that \( r \) many seats. You have \( r \) of those items and it's repeated addition. (E, C) (I, 31, 06-19-95)

\[
s \cdot r = \text{capacity}
\]

(To be continued)

---

**Table E.1 Mr. Kantor's pictorial representations**

293
Table E.1 (Continued)

4. Volume of a rectangular solid  

Each of these is a row... Call this length, the width, the height. You think as if we put crates in, crates across the bottom of it. You have so many in a length, you have so many widths so you have \( l \) times \( w \) crates on the bottom and they are stacked \( h \) high so \( l \) times \( w \) times \( h \); that gives you the volume of the room. For this one you can say, OK, let's make a wall of crates and you know it's \( h \) high; let's do it this way, the bottom has, the width of the room has this crates and you have \( h \) of those rows high, now you just take all those rows down the length of the room so you have \( l \) of them. You have the same volume, it's the same room. [PE, C] (1, 33, 08-25-95)

\[
lw \text{ crates bottom} \\
(\text{stacked } h \text{ height}) \\
(lw)h = (wh)l
\]

5. Associative property of multiplication

6. Multiplication of fractions theorem

\[
\left[ \left( \frac{3}{5} \right) \left( \frac{4}{7} \right) \right]. \text{ A baseball team has played three fifths of their season and they've been winning four out of every seven games. What fraction of the total season have they already won? Show them with area. To show that's the entire season. So far we've played three fifths of the season and then you cut that into sevenths—won four of every seven [And the solution will be] what's double shaded. [It is thirty five because] you are cutting up the rectangle into thirty five equal parts. Seven this way and five that one. (E, C) (l, 31, 06-19-95)
\]

(To be continued)
Table E.1 (Continued)

7. Rate model for multiplication

Let's say you have something that's three and a half yards long and you want to find out how many feet that is—three feet per yard—that's half of it. (E, C) (I, 31, 06-19-95)

\[
\frac{3\text{ yd}}{2} \times \frac{3\text{ ft}}{\text{yd}}
\]

8. The product of two positive numbers is a positive number

Let's say you take a certain number of steps in a certain direction. Let's say you take ten steps per hour, how far are you in five hours? Fifty steps in that particular direction. Call that direction positive. Time in the future is positive. (E, C) (I, 26, 05-30-95)

Forward (+)

10 steps

50 steps

9. The product of two negative numbers is a positive number

We are saying this is positive here. You start walking [and filming]. Five steps an hour; that's five steps; that's five steps... that's five steps away from positive [−5 steps/hour], film this and all of sudden, [you start backing up film]... now you are going back in time [−2 hours]... and you end up positive ten away from the starting point, [from where you] started backing up film. (PE, C) (I, 33, 08-25-95)

(To be continued)
10. The product of a positive number and a negative number is a negative number.

You can do it with vectors again. If this is positive, you have a vector that is in the negative direction. So that will be $b$... Let's say we increase it $a$ times, so we expand it. So $a$ times $b$ will be in the negative direction. (E, C) (I, 31, 06-19-95)

11. The product of a negative number and a positive number is a negative number.

Here is our goal, and we start in the wrong direction five miles per day [−5 miles/day]. In ten days we are 50 miles [farther] from our goal. (E, PC) (I, 30, 06-16-95)

12. Multiplicative identity of 1

It seems kind of trivial but you have something that's one by $x$. There is $x$ squares that go into that. So its area would be $x$. (E, C) (I, 27, 06-02-95)
<table>
<thead>
<tr>
<th>Table E.1 (Continued)</th>
</tr>
</thead>
<tbody>
<tr>
<td>13. Multiplicative</td>
</tr>
<tr>
<td>property of $-1$</td>
</tr>
<tr>
<td>I can do the same thing, when you take a hundred steps, a hundred steps an hour. So here we are now, one hour ago we're a hundred steps, we're negative a hundred, we were negative hundred from where we are now. This is positive. (E, C) (I, 33, 08-25-95)</td>
</tr>
<tr>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>14 &amp; 15. Definition of</td>
</tr>
<tr>
<td>reciprocals. The</td>
</tr>
<tr>
<td>reciprocal of $a$ is $\frac{1}{a}$</td>
</tr>
<tr>
<td>You can take like a square ... one by one. Now each of these is a third. If you move these up here, now you've gonna have three times one third equals one. (E, C) (I, 27, 06-02-95)</td>
</tr>
<tr>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>16. Reciprocal of zero</td>
</tr>
<tr>
<td>NA</td>
</tr>
<tr>
<td>17. Multiplication</td>
</tr>
<tr>
<td>property of zero</td>
</tr>
<tr>
<td>You can use area, length $a$, 0 wide, what is its area? (E, C) (I, 23, 01-03-95)</td>
</tr>
<tr>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td>18. Multiplication</td>
</tr>
<tr>
<td>property of equality</td>
</tr>
<tr>
<td>There is the $x$ and here is the $y$ and if those are the same length and they have the same width they're gonna have the same area. It's easy to see that. (E, C) (I, 27, 06-02-95)</td>
</tr>
<tr>
<td><em>(To be continued)</em></td>
</tr>
</tbody>
</table>
19. Solving $ax = b$

$a$ times $x$—the area of that thing is $b$, and if we cut $b$ up into $a$ equal parts, that’s what each of the parts is. So that end [the block] is something that’s one by $x$; so that’s the same as $x$ times one which is $x$. (E, C) (I, 31, 06-19-95)

\[
\frac{b}{a} = x \cdot 1 = x
\]

20. Solve $0x = b$, $b \neq 0$

You have something $x$ long and there is no width, what’s the area? The area has to be zero. . . . If there is an area, that’s not true. (PE, C) (I, 27, 06-02-95)

21. Solve $0x = 0$

It doesn’t matter how long the side is. If it doesn’t have any width, it doesn’t have an area. For any length it’s not gonna have an area. (E, C) (I, 27, 06-02-95)

(To be continued)
22. Solve $ax = 0, a \neq 0$

If you say OK, $a$ times $x$ can represent the area of a rectangle and now we are saying that area is zero. Well, if this $a$ is non zero, to have an area of zero then the $x$ has to be zero. (E, C) (I, 27, 06-02-95)

23. Solve $-x = a$

You could have a vector come up here and let's say that vector is $x$, the opposite of that—we are gonna call it $a$—the opposite of that, back to $x$ again. (E, C) (I, 27, 06-02-95)

24. $a > b, a < b$ (T***)

Greater than on a number line ... is to the right of something else.... We will use these numbers, negative three and a positive four. We draw a line.... Here is zero, negative three is here, positive four is here, [what's the relationship between those two?] ... Negative three is less than four $[-3 < 4]$, it's to the left of four. Negative three is to the left of four and so it's less than four. (E, C) (CO, 5, 10-19-94)

25. The multiplication property of inequality (part 1)

In this case you choose $x$ to the left of $y$ ... $x$ is positive. They maintain that relationship.... What you can show is two $x$ is less than two $y$. Here is $x$. Here is two $x$.... Zero is here.... You use a compass, that's $2x$, that's $2y$.... The same thing works [if it's a negative]. Here is $2z$. So $z$ is less than $x$ and two $z$ is less than two $x$. (E, C) (I, 24, 01-05-95)

(To be continued)
Table E.1 (Continued)

26. The multiplication property of inequality (part 2)

a) \(x < y\) implies \(-x > -y\). Here is \(x\) and here is \(y\). \(x\) is less than \(y\) and then I find the opposite of \(y\) using a compass, the opposite of \(x\) using a compass. The opposite of \(x\) is bigger than that \([-y]\). (E, C) (I, 24, 01-05-95)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-y)</th>
<th>(0)</th>
<th>(y)</th>
<th>(-x)</th>
</tr>
</thead>
</table>

b) \([x < y\) and \(a\) is negative then \(ax > ay\)] Let's say \(a\) is negative two. Here is \(x\). If you measure it up that's the opposite of \(x\), and over here is \([-\text{negative}]\) two \(x\). Negative \(y\) is about here and \([-\text{negative}]\) two \(y\) over here, so it shows that negative two \(y\) is less than negative two \(x\). (E, C) (I, 24, 01-05-95)

| \(-2y\) | \(-y\) | \(-2x\) | \(-x\) | \(0\) | \(x\) | \(y\) |

27. Solving \(ax < b\) (T***)

What do I do to solve this one \([-2 \leq -n]\)? … Another way you can look at this, if this makes it less confusing. If I graph the opposite of \(n\) Graph all things that make that true. Everything that's greater than or equal to negative two. Now, the opposite of that is what \(n\) is. Here is zero. The opposite switches everything over. Now I have positive two. Everything changes to this way (See figure). What's the opposite of this? [how is this related to that?] (E, C) (CO, 6, 10-20-94)

| \(-n\) | \(-2\) | \(0\) | \(2\) |

(To be continued)
[A student asked Mr. Kantor to explain the process again] The opposite of $n$ is greater than or equal to negative two, correct? Now, when I graph what $n$ is, $n$ is the opposite of this $-n$, correct? $n$ is the opposite of that. Now, how do I find what the opposite of this number $[-2]$ is? How do I find it using a compass? ... It's on the other side of zero. I just draw an arc [he draws an arc with center at zero and radius 2] and that's where the opposite of this number is. That's where two is. It's on the opposite side. OK? Here is negative one. I wanna find the opposite of that. Same thing. I draw an arc from zero to there. It's on the other side. It's where one is. So that's the opposite of this one [one is the opposite of $-1$]. The opposite of this one $[-2]$ goes over here. So all these numbers. I put this one over. I put this one over (see figure). Well, take something that's over here, let's say this is number five, where is the opposite of that? ... On the other side. Draw an arc to the other way. You flip it over, OK? you flip it over. Now, it's just like this. What did I start with? What I started with was this. Everything... When I flip this, what happens? ... they all flip like this ... around zero and all flips over. So now I have $n$ being less than or equal to two. $[2 \geq n]$ you see that? (E, C) (CO, 6, 10-20-94)

28. Multiplication counting principle

If there is two choices what you can do [is] like a factor tree, this many things [3] for the first choice, and then out of each of those branches the second choice can be made, that type of thing... [we] just go on. Maybe there is two possibilities here and then let's see there. Usually I wouldn't show all, I just kind of keep one branch ... so they can see that. So that's a little more abstract than just showing every single one of them. (E, C) (I, 28, 06-06-95)
Table E.1 (Continued)

You can also look at it [in another way] I have three choices here, two choices there, those spaces are the choices, three dimensions by two. (E, C) (I, 28, 06-06-95)

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</table>

29a. Probability of an event, \( P(A) \)

If these are all the things that are possible; and then you have some section, let's say here. That it's gonna be the probability of \( A \), all of those [are the] things that are possible where \( A \) occurs. So this is success and this is possible. (E, PC) (I, 28, 06-06-95)

<table>
<thead>
<tr>
<th>Successes</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>( P(A) )</td>
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</table>

\[ Possible \]

Figure 1a

Divide it into as many things as possible.... Let's say all those, all those things are possible and if you are making them the same size, which I try to do, then they will be equally likely.... So let's say two of them are \( A \) [The event \( A \) consists of the regions labeled \( A \)]. These are all the things that are possible, ... then the probability of \( A \) is the area of \( A \) over the total area. (E, PC) (I, 33, 08-25-95)

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\[ P(A) = \frac{\text{Area } A}{\text{Total area}} \]

Figure 1b

(To be continued)
Table E.1 (Continued)

<table>
<thead>
<tr>
<th>29b. Probability of the intersection of two events, $P(A \cap B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Venn diagram. The probability that A happens is this area [pointing to region A], the probability that B happens is that area [pointing to region B]... The total area are all the things that are possible so the shaded area compared to the total area [pointing to both regions] is the probability that A and B happen. (E, PC) (I, 31, 06-19-95)</td>
</tr>
</tbody>
</table>

![Figure 2a](image)

<table>
<thead>
<tr>
<th>29c. Conditional probability, $P(A \mid B) = \frac{n(A \cap B)}{n(B)}$, $n(B) \neq 0$ or $P(A \mid B) = \frac{\text{Area} \ (A \cap B)}{\text{Area} \ (B)}$ (Classical definition)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Here are other things. Let's say, you take part of it and might say if that's the probability, this is the probability that B occurred [pointing to region GHFE] ... and right in here. Ah, let's see, you are gonna call that the probability of A [given B]. That probability is that area; that area compared to this area, I mean. If I call these the successes for A, then this is the probability of A given B, I mean, that area [region CDPE] compared to this total area [region GHFE] is probability of A given B. (E, C) (I, 28, 06-06-95)</td>
</tr>
</tbody>
</table>

![Figure 2b](image)

The definition of $P(A \mid B)$

(To be continued)
Table E.1 (Continued)

30. Second definition of conditional probability \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} \), \( P(B) \neq 0 \)

i) See 31 i) (IM, PC) (I, 25, 01, 10, 95)

ii) [That's the probability of A, area of ABCD]. That's the probability of not A [Area of BGHC]. [That's the probability of B] given A [Area of ABFE]. Why is that the probability of B given A? [The exchange continues through figure 6] (PE, I) (I, 31, 06-09-95)

If I call this the probability that A occurred [ABCD] and this is the probability that B occurred [EIHD]

(To be continued)
Table E.1 (Continued)

If we go back to this one [Figure 1].
That's the probability of A [ABCD].
That's the probability of B given A
[AE] [Mr. Kantor relabels some
probabilities]. When you multiply
them together you get this [ABFE or
P(A \cap B)]

Figure 3

[Can we also represent P(B/A) with
area?] You can do it also with area. If
I call this a unit square, the
probability of A is this length [DC].
Then this area is also the probability
of A [ABCD] ... and then we can call
this the probability of B [the length
DE], and this is the probability of B
given A [EFCD].... This area [EFCD]
compared to this area [ABCD] is the
probability of B given A. [Mr. Kantor
relabels EFCD as P(A \cap B)]. [PE,
P(C]

Figure 4

[EFCD represents P(A \cap B) because I
just multiplied probability of A times
probability of B. We don't care about
this [He erases FI].

Figure 5

(To be continued)
Table E.1 (Continued)

[The] probability of B given that A has happened, that's what this line [DE] is... Because we are looking at the probability of B. We are only looking at this section [EFCD? or AD?]. It's the probability of B given that A has already occurred.

![Figure 6](image)

iii) That area is the probability of B [ABCD]... [The] probability of B I shade it this way. The probability of A [Area EIH] I shade it this way.... This has a total area of one.... So this probability of B is a length [AB] and it is an area since it has a unit width.... This is the probability of A [DE] as well as that area; so then A given B is double crossed [Area of EFCD], and it is this, its length times its width. Wait a minute, that's not right. That's right. Mmh, OK. What are we doing wrong? Oh, I see. Yeah, that's, this [Area of EFCD] is the probability of A and B. That's not the probability of A given B... [P(A) and P(B) are the areas and the lengths because the figure is a square of area 1]... I don't think that works. (E, I) (I, 33, 08-25-95)

![Figure 7](image)

(To be continued)
iv) When working on a story problem for \( P(J \text{ and } H) = \frac{P(J \text{ and } H)}{P(H)} \) Mr. Kantor had an insight about this pictorial representation. If I change it, probability of a jack given that you have a heart... Yeah, this is where I got mixed up. This, that area [EFCDF] is the probability of A given B when compared to this [area ABCD]... Yes. So this is the probability of A given B when compared to anything that's shaded [that way]... and it is this \([P(A \cap B)]\) when compared to total area. Total area is 1. (E, PC) (I, 33, 08-25-95)

31. Conditional probability formula, \( P(A \cap B) = \frac{P(A \cap B)}{P(B)} \) (PE, C)

i) If you do it the other way, where it is probability of A given B times probability of B gives you this \([P(A \cap B)]\), then I can represent that with a rectangle. Over here is the probability of not B [Area AGHD] and this is the probability of B happening [Area GBCH]. This is the probability of B happening; so over here, that's the probability of A given that B has already happened [BF]... Let's see. Probability of A given B. This inside is the probability of the intersection [Area GBFE = P(A \cap B)]... That's the probability of not A given that B has happened [CF]. If you take that \([P(A \cap B)]\) and divide it by this \([P(B)]\) let's see, A and B, that will give you this \([P(A \mid B)]\). That's the probability of B [GB], right? You take this area [GBFE] and you divide it by probability of B [BG], and you get the probability of A given B [BF]. (PE, C) (I, 25, 01-10-95)

(To be continued)
Table E.1 (Continued)

ii) This will be probability of A [ABCD]. This is the probability of not A [BGHC] ... and these are not necessarily the same areas. So this is the probability of A and B happen [EFCD]... This is probability of B given A [DE]... The probability of A and B equals that [EFCD]. This is the probability of A but not B. This is the probability of B but not A. This is the probability of not A not B. [DE represents P(B | A)] because if you just isolate this, if I just look at, this represents that A has happened [ABCD] ... and some of the time there is a probability that B will happen [EFCD] and that's the probability of B [DE] there but it's given that A has happened. (PE, C) (I, 28, 06-06-95)

iii) Another representation for \( P(A \cap B) = P(A)P(B | A) \) was constructed when working on CCE 30, second definition of conditional probability. See in particular Figure 3 in cell 30. (PE, C) (I, 31, 06-09-95)

32. \( n! \)  
   NA

33. \( 0! \)  
   NA

34. Permutation theorem  
The type of same thing we did over here ... \( n \) things here and then there is \( n \) minus one left to put there, \( n \) minus two left to put there, eventually you have the last here. (E, I) (I, 28, 06-06-95)

35. Meaning of division. (E, C)  
   1) 7 + 2. Take seven and put them in groups of two, and see how many groupings you have. You have three and half a grouping. (E, C) (I, 29, 06-08-95)

To be continued)
Table E.1 (Continued)

35b.  2) \( \frac{5}{2} \). You have five things and how many halves are in each of those, cut each of those in halves, and count the number of halves that you have and [that number is] ten. (E, C) (I, 29, 06-08-95)

35c.  3) \( \frac{3}{4} \). It's kind of cumbersome, though, taking, cutting in quarters and how many three quarters you have. Probably a better way to do that is to, if you look at 21 dollars and that. In three dollars, we have four groups of three quarters. You can see these, your actual dollars. You end up doing it for seven groups. Seven groups of three dollars but each of these is really four groups of three quarters. I can show that if you want to draw each one of them here is one group, two groups, three groups, [and these] form the fourth group. Four times seven, altogether twenty eight groups of three quarters. (E, C) (I, 29, 06-08-95)

35d.  4) \( \frac{2}{3} + \frac{1}{2} \). That's two thirds and that's one half. Bring this down over here, so half goes into there once [and you have one left over]. So, once you do that, you try to see how many halves you have. You have one half there and you have a third of a half, so you have one and a third. One and a third halves... That's how many times a half goes into two thirds. It's gonna take two thirds and make one and a third. You make one and a third halves out of it. (E, C) (I, 29, 06-08-95)

(To be continued)
35e. (E, C)

5) \( \frac{1}{2} + \frac{2}{3} \)

i) Well, you can talk about this as cutting things up into pieces. If you take a half divided by two, you cut one half into two pieces, and a half is two thirds of what part and then what happens there is what you are doing here. You cut a half up into two parts and you add another part because now you got three. Then that half here is two thirds of what you created \([3/4]\). And that's taking this and showing it in this way. That's too indirect, though... I don't know if it's that much indirect because you're cutting one half into two thirds of a part, and that's two thirds of a part, and that's two thirds of a part [the 1/2]. That's two thirds of what.... Here's your half and normally when we divide we cut into parts, well, this is two thirds of a part. So, if that's two thirds of a part then that's two parts and other part. Now that's two thirds of this. That's the answer \([3/4]\).

(In, 22, 12-19-94)

ii) It doesn't go into a half [evenly]. Here is the half. [Here are the] two thirds. This is what you start with. This is two thirds. So that's three out of four parts of two thirds \([3/4]\).

(In, 29, 06-08-95)

36. Algebraic definition of division

\[
\frac{a}{b} = a \cdot \frac{1}{b}, \quad b \neq 0
\]

1) \( \frac{7}{2} = \frac{1}{2} \). Another way you can look at division, you're cutting this into two parts and that's what's in a part. So, if you are cutting into two parts, this is what a part is \([\text{Figure 1a}]\), and this is something you are doing in there. You are cutting in half, a half of a group \([\text{Figure 1b}]\). Here we cut the group into two equal parts \([\text{Figure 1a}]\). So it's the same

(To be continued)
Table E.1 (Continued)

thing. [The connection with seven
times one half is that] we were saying
we were taking half of a group there, I
mean, we are cutting this into two
equal groups and what's circled is what
one group is [Figure 1a]; and this one
you can just use repeating addition to.

You have seven halves \[7(\frac{1}{2})\]. That's
two, four, six, seven halves
there which is what one of those
groups, same as one of those
groups.... You cut seven into two
equal groups. The answer to that is
what's in a group.... That's the same
thing. You take seven copies of a half
and add them together. That's three and
half (Fig. Ic).... [This one half
represents] taking half of it [7].

(P.E, C) (I, 29, 06-08-95)

\[36b. \text{(E, C) } 2) \quad 5 + \frac{1}{2} = 5 \cdot 2.\]

i) Again, you can look at that's half of
a group [Figure 2a]. If it's half of a
group, you double to get the whole
group [Figure 2b]. Five is half a
group. There is the other half of
group, so the whole group is ten.
Same thing as if you take five groups
of two, you get ten [Figure 2c] (I, 29,
06-08-95)

ii) [The connection to 2 is] I guess
you'd say if five is half of the group,
you double ... you double what you
have and get the whole group [5:2]
[Figure 2b] (I, 30, 06-16-95)

iii) [5:2 represents] two groups of
five. (I, 31, 06-19-95)

(To be continued)
Table E.1 (Continued)

<table>
<thead>
<tr>
<th>36c. (E, C)</th>
<th>3) $21 + \frac{3}{4} = 21 \cdot \frac{4}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>i) Twenty one is three fourths of a group. Well, it's seven, seven. It's twenty one, and it's three fourths of a group [Figure 3a]. Add another row of sevens in. That's 28. [The first interpretation will be] If twenty one is three fourths of a group, the whole group ends up [having] to add another fourth of that group to it [Figure 3b]. This is three fourths of a group and this is gonna be a fourth of a group. [So in this case you multiply 21 times by] Four thirds. (I, 29, 06-08-95)</td>
<td></td>
</tr>
<tr>
<td>ii) Twenty one is three fourths of a group. Find out what the group is $[21 + \left( -\frac{3}{4} \right)]$. The way to do it $[21 \left( -\frac{3}{4} \right)]$ is, you cut what you have into thirds ... three equal parts. What you have is three fourths of a group, that's three equal parts $[3/3]$. Take one more equal part $[1/3]$, so you have four equal parts and that's what the whole group is [that's what four thirds is] $\frac{4}{3} [21 \left( -\frac{3}{4} \right)]$. [It's 21 times four thirds] because the three divides it into three parts, divides what you have into three parts, and the four for the full collection I guess you can say. (I, 30, 06-16-95)</td>
<td></td>
</tr>
<tr>
<td>iii) Well, if this is, if twenty one is three out of four parts of a group, then the twenty one, you have three groups, three groups of threes $[3/3]$ out of twenty one and we need one more group $[1/3]$ to get a whole, so it's four, four thirds times twenty one. (I, 31, 06-19-95)</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3a

Figure 3b

(To be continued)
### Table E.1 (Continued)

<table>
<thead>
<tr>
<th>36d. (E, C)</th>
<th>4) $\frac{2}{3} + \frac{1}{2} = \frac{2}{3} \cdot \frac{2}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>i) Two thirds is half of a group. So add this, and now that's half of group, add another one to get the whole group. (I, 29, 06-08-95)</td>
</tr>
<tr>
<td></td>
<td>ii) This is half of a group so—that's half a group—so add two more... This is half of a group, you double to get the whole group [$\frac{2}{3} \cdot 2$].</td>
</tr>
<tr>
<td></td>
<td>Figure 4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>36e. (E, C)</th>
<th>5) $\frac{1}{2} + \frac{2}{3} = \frac{1}{2} \cdot \frac{2}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>i) Here is your half [Figure 5a], and normally when we divide we cut into parts. Well, this is two parts of a part.... [$\frac{1}{2} + \frac{2}{3}$] That's two parts and another part [Figure 5b]. Now, that's $\left[\frac{1}{2}\right]$ two thirds of this $\left[\frac{3}{4}\right]$. That is the answer $\left[\frac{3}{4}\right]$. And that's the same thing as doing it this way $\left[\frac{1}{2} \cdot \frac{3}{2}\right]...$</td>
</tr>
<tr>
<td></td>
<td>Figure 5a 1/2</td>
</tr>
<tr>
<td></td>
<td>This part is one half [the left half circle of Figure 5c]. Break it up this way [He redraws the horizontal diameter of the circle in figure 5c]. Cut that half in half $\left[\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}\right]$ and then take three of them $\left[3\left(\frac{1}{4}\right)\right]$. You have three fourths (Figure 5d). (I, 22, 12-19-94)</td>
</tr>
<tr>
<td></td>
<td>Figure 5b $\frac{1}{2} + \frac{2}{3}$</td>
</tr>
</tbody>
</table>

(To be continued)
Table E.1 (Continued)

ii) \( \frac{1}{2} + \frac{2}{3} \). A half is two thirds of a group. If this is two thirds of a group and then add another one (another third) to get the whole group (Figure 5d). That's two thirds and that, those are equal and each of those is a third, add another one, you have three thirds of a group. You have a whole group \( \frac{3}{4} \) (Figure 5d).

\[ \text{Figure 5c} \quad \frac{1}{2}, \quad \frac{3}{2} \]

\[ \text{Figure 5d} \quad \frac{1}{2} + \frac{2}{3} \]. Well, you take that half and cut it into two parts \( \frac{1}{2} \) and \( \frac{1}{4} \) and you need three of those \( \frac{3}{4} \). If that half represents two of the three parts of the group, you find what each part is \( \frac{1}{4} \) and then you add them together, in this case repeating add is just multiplying by three \( 3(\cdot) \).

(Figure 5c) (I, 29, 06-08-95)

iii) One half is two thirds of a group. So that means it's two parts. Add another part to get the whole group. Half a group, so you need, half a group is two of three parts so then you triple \( \text{[You triple]} \) each, triple the part \( 3\left(\frac{1}{4}\right) \). If a half is two of three \( \frac{1}{2} \) parts, so the whole, triple one part of what you have \( \text{[to get the whole]} \). OK. Here is a half \( \frac{1}{2} \), and then three halves is one and a half times that \( \frac{3}{2} \). You take, take your half of cake \( \frac{1}{2} \), multiply by one half \( \frac{1}{2} \). You need one

(To be continued)
and a half times that to get the whole
\[
\frac{1}{2} \times \frac{3}{2} = \frac{3}{2}.
\]
(I, 31, 06-19-95)

37. Division by zero

So you divide by a tenth to get a big number. You divide the same thing by a hundredth to get a bigger number. If this keeps getting smaller, eventually you get bigger and bigger, [no matter] how close you get to zero you still [keep getting] bigger so ... infinity, divided by zero equals infinity, but I can't represent any bigger than that. I guess, eventually you start, you start getting down to where these things, these things are rectangles ... lines, they have no thickness, they have no thickness, and they can't fill the space. The rectangles, they can fill the space, [but once they are not] they cannot fill the space. (E, C)
(I, 33, 08-25-95)

<table>
<thead>
<tr>
<th>/ \</th>
<th>\ 1/10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>/ \</td>
<td>1/100</td>
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<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

38. The quotient of a positive number and a positive number is positive

You split up 20 kids into five teams. Or, you can show that happens; that's one, two, three, four, five teams.... Four kids per team. (E, C)
(I, 30, 06-16-95)

(To be continued)
Table E.1 (Continued)

39. The quotient of a negative number and a negative number is positive

I guess you go back to the motion picture type of thing. Okay, if I run a film backwards two hours \([-2]\), ... if they're walking, because we call this positive, they're walking that way ... backwards. Now, they are backing up. This is where I start \([-100]\), and I'm going backwards they're going to end up a hundred from the positive direction where I start. [The connection to division is ] each hour is fifty. Each hour [is going forward] fifty steps, retracing fifty steps in the positive direction [negative one hundred divided by negative two] (IM, C) (I, 33, 08-25-95)

---

40. The quotient of a negative number and a positive number is negative

You could say that we have a goal in mind. This is our goal here, and you start walking in the wrong direction, so you got a hundred miles away from our goal \([-100]\). It took us five days \([5]\) to accomplish that. So we lost twenty miles per day toward our goal \([-100 + 5 = -20]\). That's a positive vector. That's a negative [vector]. (E, C) (I, 30, 06-16-95)
41. The quotient of a positive number and a negative number is negative.

I can do the same thing. If you are walking, if I walk 3 hours, I walk a hundred and fifty steps positive, where was I three hours ago? Here is where I am now, present time, where was I three hours ago? ... Three hours ago. So I lost fifty paces per hour. (IM, I) (I, 33, 08-25-95)

\[ \begin{array}{c}
3 \text{ hr} \\
\rightarrow \\
150 \\
\rightarrow \\
\leftarrow \\
-50 \\
\rightarrow \text{ Present time}
\end{array} \]

* Mr. Kantor's pictorial representations were categorized according to their degree of explicitness and to their degree of correctness in the following categories: Explicit (E), partially explicit (PE), and implicit (IM); and correct (C), partially correct (PC), and incorrect (I). The criteria used to categorized Mr. Kantor's pictorial representations is described in Chapter 4, section “analysis,” “Coding schema for Mr. Kantor's knowledge of pictorial representations.”

** NA: Not asked or not applicable.

*** T: Pictorial representation taken from Mr. Kantor's classroom instruction.

Table E.1 Mr. Kantor's pictorial representations
<table>
<thead>
<tr>
<th>Category</th>
<th>Correct (32)*</th>
<th>Partially correct (4)</th>
<th>Incorrect (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit (30*)</td>
<td>2**, 3, 6, 7, 8, 10, 12, 13, 14, 15, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29c, 35, 36, 37, 38, 40 (27)*</td>
<td>11, 29 (29a, 29b) (2)</td>
<td>34 (1)</td>
</tr>
<tr>
<td>Partially explicit (6)</td>
<td>5, 9, 20, 31 (4)</td>
<td>1, 30 (2)</td>
<td></td>
</tr>
<tr>
<td>Implicit (2)</td>
<td>39 (1)</td>
<td></td>
<td>41 (1)</td>
</tr>
</tbody>
</table>

*The number in parenthesis indicate the number of CCEs within the category.

**The content curriculum events as referred by numbers according to Table 5.1 or E.1.

Table E.2 Quantitative summary of Mr. Kantor's knowledge of pictorial representations
APPENDIX F: Mr. Kantor's story-problem representations
<table>
<thead>
<tr>
<th>Content curriculum event</th>
<th>Story-problem Representation*</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Area Model for</td>
<td>You have a room that's eleven by ten and you want the carpet</td>
</tr>
<tr>
<td>Multiplication (Continuous case)</td>
<td>squares—square feet. How many carpet squares do you need to cover the floor? (PC) (I, 26, 05-30-95)</td>
</tr>
<tr>
<td>2. Commutative property of multiplication</td>
<td>The example of looking at an auditorium. How many seats are in a row times how many rows are tells you how many seats [there are] in the auditorium. You do the same thing with columns. Count how many seats in a column [times] how many columns are there [gives you] how many seats are in the auditorium. You get the same thing. (C) (I, 26, 05-30-95)</td>
</tr>
<tr>
<td>3. Area Model for</td>
<td>You can do seats in an auditorium. You have so many seats in a row and so many rows, so seats in a row (s) times number of rows (r) that gives you the capacity, let's say, of the auditorium. (C) (I, 31, 06-19-95)</td>
</tr>
<tr>
<td>Multiplication (Discrete version, array model)</td>
<td></td>
</tr>
<tr>
<td>4. Volume of a rectangular solid</td>
<td>NA</td>
</tr>
<tr>
<td>5. Associative property of multiplication</td>
<td>You have a hundred trucks, fifty crates in each truck, [and some] boxes in each crate. What you could look at is, how many crates do you have? Trucks times crates gives you how many crates, times what's in each crate gives how many boxes. OK? Or you can look at the boxes in a, let's see, the boxes in a truck. Crates times boxes. ... that's how many boxes are in a truck. Multiply that by how many trucks you have. It still gives the same answer. (C) (I, 26, 05-30-95)</td>
</tr>
</tbody>
</table>
| 6. Multiplication of fractions theorem | a) \( \frac{3}{4} \cdot \frac{1}{2} \) You can do it with money. Somebody has three fourths of a dollar [75 cents] and wants to split it between two people. How much does each person get? (C) (I, 23, 01-03-95)  

b) \( \frac{3}{4} \cdot \frac{5}{6} \) Let's say you are coloring eggs, three of every four you color them a certain color and then five of every six [colored ones] you put some type of sparkling thing on it. So [how many do] you color that particular color and sparkling? (C) (I, 26, 05-30-95) |

(To be continued)

Table F.1 Mr. Kantor's story-problem representations
Table F.1 (Continued)

<table>
<thead>
<tr>
<th>7. Rate model for multiplication</th>
<th>a) I took a trip and I go fifty miles and I multiply that by times the mileage [hrs/miles]. Then it gives me how much time it took me. (C) (I, 26, 05-30-95)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>b) Let's say you have something that's three and a half yards long [and you want to] find out how many feet that is—three feet per yard. (C) (I, 31, 06-19-95)</td>
</tr>
<tr>
<td></td>
<td>c) You have a piece of wood that is three and a half yards long, and you want to find out how many one-yard sections you can get out of it. (C) (I, 31, 06-19-95)</td>
</tr>
<tr>
<td>8. The product of two positive numbers is a positive number</td>
<td>Let's say you take a certain number of steps in a certain direction. Let's say you take ten steps per hour. How far are you in five hours? Fifty steps in that particular direction. Call that direction positive. Time in the future is positive. (C) (I, 26, 05-30-95)</td>
</tr>
<tr>
<td>9. The product of two negative numbers is a positive number</td>
<td>a) How much did you weigh six days ago if you have been losing three ounces a day? Eighteen ounces more. (C) (I, 23, 01-03-95)</td>
</tr>
<tr>
<td></td>
<td>b) We can use the same example, walking. In this case you are walking in the opposite direction. You've been walking. If you are filming this and [all of a sudden you say] this is time zero and you keep filming. Let's go back two hours in time [−2 hrs]. If you are taking five steps an hour [−5 steps/hr] that way. Two hours before, ten steps, [in that direction] which is positive. (C) (I, 26, 05-30-95)</td>
</tr>
<tr>
<td>10. The product of a positive number and a negative number is a negative number</td>
<td>If you diet for five days and you lost two ounces per day. How much do you lose over that period of time? ... You lost ten ounces. (C) (I, 23, 01-03-95)</td>
</tr>
<tr>
<td>11. The product of a negative number and a positive number is a negative number</td>
<td>a) How much more or less did you weigh five days ago if you have been gaining two ounces a day? You weighed ten ounces less. (C) (I, 23, 01-03-95)</td>
</tr>
<tr>
<td></td>
<td>b) I lose five pounds a month. In three months I am 15 pounds lighter than today. (C) (I, 30, 06-16-95)</td>
</tr>
<tr>
<td>12. Multiplicative identity of 1</td>
<td>I guess you can say, you pay thirty-five dollars an hour, how much do you pay for one-hour's work? (C) (I, 31, 06-19-95)</td>
</tr>
<tr>
<td>13. Multiplicative property of −1</td>
<td>a) I guess we can talk about that a represents weight gain [a = weight gain = 2 oz]. Then one day ago, what was her weight? It was two ounces less than it is today. (C) (I, 23, 01-03-95)</td>
</tr>
</tbody>
</table>

(To be continued)
Table F.1 (Continued)

<table>
<thead>
<tr>
<th>14 &amp; 15. Definition of reciprocals. The reciprocal of $a$ is $1/a$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>b) If you have lost weight $[a = -2 \text{ oz}]$ then a day ago you weighed two ounces more than you weigh today. (C) (I, 23, 01-03-95)</td>
</tr>
<tr>
<td>c) You can talk about if you are in Las Vegas and you are gambling and you are winning one hundred dollars an hour—one hundred dollars per hour—one hour ago you were one hundred dollars poorer than now. (C) (I, 27, 06-02-95)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>16. Reciprocal of zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not asked</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>17. Multiplication property of zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) If a salesman will make zero dollars a day. How much does he make in $a$ days? (C) (I, 23, 01-03-95)</td>
</tr>
<tr>
<td>b) You gonna start saving five bucks a week and before you even start, how much money you can save? (C) (I, 24, 01-05-95)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>18. Multiplication property of equality</th>
</tr>
</thead>
<tbody>
<tr>
<td>You have a dozen and three eggs in a box, ... and a dozen and three is fifteen. Multiply both sides by five. That's seventy five eggs.... Yeah, so you have five dozens and fifteen which ends up being the same thing, 75. (PC) (I, 24, 01-05-95)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>19. Solving $ax = b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) $[40R = 600]$ Forty bucks a radio. How many can you buy for six hundred dollars? (C) (I, 7, 10-24-94)</td>
</tr>
<tr>
<td>b) $[\left(\frac{3}{4}\right)b = 15]$ Your area is fifteen units, and the box is three fourths of a unit by $b$ units. So how long is the box? (C) (I, 7, 10-24-94)</td>
</tr>
<tr>
<td>c) [We go out] and buy five cars. Each costs the same amount of money. We spend one hundred thousand dollars. How much does each car cost? (C) (I, 24, 01-05-95)</td>
</tr>
<tr>
<td>d) You travel for five hours and end up going one hundred and twenty miles. What was your rate? (C) (I, 24, 01-05-95)</td>
</tr>
<tr>
<td>e) You travel for three hours, let's say at some unknown speed, and you go one hundred and sixty five miles $[3hr S = 165\text{mi}]$. What is $S$? Well, $S$ is a hundred and sixty five miles divided by three hours. Time times speed equals distance. (C) (I, 27, 06-02-95)</td>
</tr>
</tbody>
</table>

(To be continued)
<p>| | |</p>
<table>
<thead>
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<tbody>
<tr>
<td>20. <strong>Solve</strong> $0x = b$, $b \neq 0$</td>
<td>If $b$ is not zero…. You save 0 dollars every day, how long does it take to save one hundred dollars? It can't be done. If $b$ is positive you can't do that problem. There is no solution. (C) (I, 24, 01-05-95)</td>
</tr>
<tr>
<td>21. <strong>Solve</strong> $0x = 0$</td>
<td>You can talk about if you don't gain weight … how long can you maintain the present weight? A day, three days from now, a year from now. Anything will work. (C) (I, 24, 01-05-95)</td>
</tr>
<tr>
<td>22. <strong>Solve</strong> $ax = 0$, $a \neq 0$</td>
<td>We can go back to the weight gain. You gain two ounces a day. How long can you maintain your present weight? You can't. It can't pass today. $t$ will be zero [$a = 0$]. (C) (I, 24, 01-05-95)</td>
</tr>
<tr>
<td>23. <strong>Solve</strong> $-x = a$</td>
<td>The opposite of what you prefer is gaining ten pounds. what do you prefer? That's losing ten pounds. (PC) (I, 27, 06-02-95)</td>
</tr>
<tr>
<td>24. $a &gt; b$, $a &lt; b$</td>
<td>Your wealth, temperature. (I, 24, 01-05-95)</td>
</tr>
<tr>
<td>25. <strong>The multiplication property of inequality</strong> (part 1)</td>
<td>I win two dollars a day. You win three dollars a day. As we look into the future, you will be winning more money than I will [$a$ will be the number of days] (I, 24, 01-05-95)</td>
</tr>
<tr>
<td>26. <strong>The multiplication property of inequality</strong> (part 2)</td>
<td>a) Negative is back in time…. [$a$] situation where you are winning more money than I am. You don't want to go back in time. If you go back in time, then I'm in a better situation…. [$y = 5$] and [$x = 7$] seven…. If you go back two days, then that's, I'm losing, giving up ten dollars that I already had…. So financially I'm 10 dollars poorer but you are fourteen dollars poorer. (C) (I, 24, 01-05-95)</td>
</tr>
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<tr>
<td>b) Let's say $x$ represents person's … change in wealth each day…. You look at $a$ [would be] a negative number, you look back in time so that a certain number of days ago, …..I can use negative two for this…. Two days ago I had two x less money than I do now so to speak, that's less than … a hundred dollars [$-2x &lt; -100$]. So in that case what you solve is—person makes more than—increases her wealth more than fifty dollars a day. So two days ago they were more than hundred dollars worse off than they are now. (C) (I, 28, 06-06-95)</td>
<td></td>
</tr>
<tr>
<td>27. <strong>Solving</strong> $ax &lt; b$</td>
<td>a) You gain eight ounces a day…. How long can you stay below a certain level of weight gain? [$8x &lt; b$]. And the other, when would you exceed that level of weight gain? (C) (I, 24, 01-05-95)</td>
</tr>
</tbody>
</table>

(To be continued)
Table F.1 (Continued)

b) You have a number of different people sat on cars, and they are all gonna travel for three hours. Then, what's the speed they have to stay under to avoid exceeding certain distance? \( 3s < d \); ... and the other one will be what speed they have to go to exceed that distance? (C) (I, 24, 01-05-95)

c) Let's say that you have three crates, of equal size and content and they hold less than a thousand baseballs. How many are in each crate? \( 3x < 1000 \). (C) (I, 27, 06-02-95)

d) You're gonna buy boxes of paper that are twelve bucks a box. That's what \( a \) is. You have a budget of a hundred and fifty dollars. How many boxes can you buy? That type of thing \( 12x < 150 \). (C) (I, 9, 10-26-94)

28. Multiplication counting principle

a) You can set up things ... a hundred and something [items] in a box, and you put them in a van. You have 20 boxes in a van and you got 15 vans, how many of those items do you have? (C) (I, 24, 01-05-95)

b) If you go on a trip and you take two pairs of shoes, three pairs of shorts, four shirts and two ties, how many different outfit combinations are those, assuming that they're all [used]? (C) (I, 28, 06-06-95)

29a. Probability of an event, \( P(A) \)

What's the probability of pulling a heart out of a deck of a standard deck of playing cards? ... If there are not jokers, [the probability will be] 13 out of 52 which is one over four. (C) (I, 28, 06-06-95)

29b. Probability of the intersection of two events, \( P(A \cap B) \)

a) What's the probability to pull two cards out that will be hearts? (C) (I, 28, 06-06-95)

b) Let's say you look at our house of kids. If you have a hundred and some students, ... find the probability of, if you pick up somebody at random, that person takes global language and algebra. Everybody takes math but some don't take algebra. Everybody takes a global language or reading. There are some who don't take global language.... Divide the number of students that are taking both by the total number of students. (C) (I, 31, 06-19-95)

29c. Classical definition of conditional probability, \( P(A/B) \)

What's the probability of pulling a heart out of a deck given you've already pulled one out, that you don't replace? (C) (I, 28, 06-06-95)

30. Second definition of conditional probability

That's using cards. Probability of getting a jack of hearts, a jack given that you already drew a heart. Probability you get a jack and a heart is one out of fifty two. Probability you draw a heart is thirteen out of fifty two. (C)

(To be continued)
Table F.1 (Continued)

<p>| | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>P( J of H) = P(J and H)</td>
<td>1</td>
</tr>
<tr>
<td>P(H)</td>
<td>52</td>
</tr>
<tr>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>52</td>
<td></td>
</tr>
</tbody>
</table>

(I, 33, 08-25-95)

31. **Conditional probability formula**, \( P(A \cap B) = \frac{P(A)P(B/A)}{P(A)} \)

Suppose that two cards are drawn from a well-shuffled deck. The first card is not put back before the second is drawn. What is the probability that both cards are hearts? (Example taken from his teaching). (CO, 9, 10-26-94)

32. \( n! \)

It's used to do arrangements. We can do that. If you draw five cards and you don't look at them, you put them in order, how many different arrangements are there with those five cards? ... If you have the cards, let's say they are these five then there is five things I can put here, four here, three here, two here, one here. (C) (I, 28, 06-06-95)

33. \( 0! \)

Not asked

34. Permutation theorem

Any time you use arrangements of things. Let's say, I have a collection of [items]... How many different ways can I arrange them in a straight line? (C) (I, 25, 01-10-95)

If you have five cards and put them in order, how many different arrangement are there with those five cards? (C) (I, 28, 06-06-95)

35. Meaning of division (C)

35a. (C)

[7 + 2 ] You have seven donuts. You wanna give two per person. How many people can you feed? (I, 29, 06-08-95)

35b. (C)

\[7 + \frac{1}{2}\] [If you have seven pieces of pizza, and each person gets a half of a pizza]... You have seven pizzas, you cut each of those in half then you get 14 pizzas [pieces]. So you can feed 14 people. (I, 15, 11-18-94)

35c. (C)

\[6 \frac{2}{3}\] A person can paint three yards of a fence in an hour. That's a third of an hour to do every yard. How long does it take to paint six and two thirds yards? (I, 15, 11-18-94)

(To be continued)
Table F.1 (Continued)

35d. (C) \[ \frac{5}{2} \] Five divided by a half. Same type of problem. You have five loaves of bread and give half of loaf to each family. How many families can you feed? (I, 29, 06-08-95)

35e. (C) \[ n \text{ dozens} + 3 \] You have a recipe that calls for using three eggs and you have a certain number of dozen eggs. How many times can you complete that recipe? Division gives you that answer, repeating subtracting three from how many dozens you have. (I, 29, 06-08-95)

35f. (C) \[ \frac{200}{4} \] As far as a rate goes, talk about taking a trip, you travel two hundred miles in four hours, express that situation as a rate: miles per hour. (I, 29, 06-08-95)

35g. (C) \[ \frac{21}{4} \] You have twenty one dollars. Let's see, a bag of chips is 75 cents, three quarters of a dollar. How many bags can you buy? (I, 29, 06-08-95)

35h. (C) \[ \frac{2}{3} + \frac{1}{2} \] Let's say you have a two thirds of a paper left to type. You can type half a page in an hour. How long will it take you to type what's left? (I, 29, 06-08-95)

35i. (C) \[ \frac{1}{2} + \frac{2}{3} \]

i) You can say I took two thirds of somebody's money, and I end up with a half a dollar. How much money did they have? (I, 22, 12-19-94)

ii) If the area of [a rectangle] is 1/2, and one of its dimensions is 2/3, what's the other dimension? (I, 22, 12-19-94)

iii) One half divided by two thirds... You have one half page left to type. You type two thirds of a page in an hour. How long will it take you to type a half page? (I, 29, 06-08-95)

36a. Algebraic definition of division

\[ \frac{a}{b} = a \cdot \frac{1}{b}, \quad b \neq 0 (C) \]

\[ \frac{7}{2} = \frac{7 \cdot 1}{2} \] [If you have seven pieces of pizza and each person gets a half of that]. Half of each pizza, that's what each person gets, each person gets three and a half [pizzas]. That's ... a half seven times. (I, 15, 11-18-94)

(To be continued)
Table F.1 (Continued)

Seven dollars, it costs two dollars for a pizza, you can buy three and a half pizzas. Looking at it differently, a dollar will pay for half a pizza. Seven times a half. Well, a half a dollar is gonna — No, a dollar can buy a half a pizza. So each dollar buys a half a pizza. You have seven, your seven dollars times — it's a rate, seven times half pizza per dollar. (I, 29, 06-08-95)

36b. (C) $5 + \frac{1}{2} = 5 \cdot 2$. You have five dollars, how many bags of fifty cents potato chips could you buy? That $[5 + \frac{1}{2}]$ tells you ten bags. [This means] you can buy two bags per dollar, so two times five. (I, 29, 06-08-95)

Each dollar gives you two bags, [and you have] five dollars, [so it is] five times two. (I, 30. 06-16-95)

36c. (C) $21 + \frac{3}{4} = 21 \cdot \frac{4}{3}$. [A two-liter bottle costs three fourths of a buck] There's twenty one bucks and let's see, two liter-bottle of pop. How many bottles of pop can you buy? That's not apparent that your dollar can buy four thirds bottles of pop (laughs), but it's apparent that you can buy four bottles with three dollars. So you could calculate the rate, four bottles for three dollars. So that is. The rate times the twenty one dollars gives you how many bottles you can buy. (I, 29, 06-08-95)

36d. (C) $\frac{2}{3} + \frac{1}{2} = \frac{2}{3} \cdot 2$

i) A team will eat half a [pot of] soup, how many teams could you feed if you have two thirds of this big party pizza [soup]. That $[\frac{2}{3} + \frac{1}{2}]$ tells you how many and so you know the party soup feeds two teams. You have two thirds of a soup. How many teams could you feed? Two thirds times two. [So this two will be] the number of teams that this party soup could feed. (I, 29, 06-08-95)

ii) [You have two thirds of a paper left to type. That's what you have. You can type half a page in an hour, ... how long will it take to type what's left?].... This is the paper, this is how much you have left to do. Two thirds of it. You have a half here. You can do that much in an hour so you can do [type] the whole paper in two hours.... [You can type] half a page in an hour. So I can type the whole page in two

(To be continued)
hours. So that paper I can also consider is two hours of typing. I have two thirds of it. So I have four thirds of hours left to do. [Two represents] I really did a conversion. Instead of talking about half a page an hour I am saying the whole page is two hours. It's a rate. [this will represent two hours and this] I have two thirds of a two-hour job to do. (I, 33, 08-25-95)

\[
\frac{1}{2} + \frac{2}{3} = \frac{1}{2} \cdot \frac{3}{2}.
\]

i) You have a half a [pot of] soup, and it takes two thirds of a [pot of] soup to feed a team. So how much of a team could you feed? You have a half a [pot of] soup, a [pot of] soup will feed, the whole [pot of] soup will feed two [one?] and a half [teams]. So how many teams could you feed with half a soup?

(I, 29, 06-08-95)

\[
\frac{1}{2} + \frac{2}{3} = \frac{1}{2} \cdot \frac{3}{2}.
\]

ii) \( \frac{1}{2} + \frac{2}{3} \) Well, I guess it's the same way. If this is two thirds of a team and that's two equal parts of a team, each part needs a fourth of this. Is that right? Yeah, ... a third of the team needs a fourth of the [pot of] soup. For the whole team, that's three thirds, ... It's three fourths for the whole team. (I, 30, 06-16-95)

\[
\frac{1}{2} + \frac{2}{3} = \frac{1}{2} \cdot \frac{3}{2}.
\]

iii) \( \frac{1}{2} + \frac{2}{3} \) Let's say I have 50 cents and that will pay for two thirds of a team to take the bus ride, how much do I need for the whole team? [Now one half times the reciprocal of this which is three halves] That's the point. That two thirds is two equal groups, two of three equal groups that form the team. And, so this part gives you ... the fourth for each group and there are three groups altogether. So three times this \( \frac{3}{2} \). (I, 30, 06-16-95)

\[
\frac{1}{2} + \frac{2}{3} = \frac{1}{2} \cdot \frac{3}{2}.
\]

iv) \( \frac{1}{2} + \frac{2}{3} \) There is a [pot of ] soup. You need two thirds for one [team], So two parts will feed one team. So I have another part. So now I will be able to feed three halves or one and a half team [with one pot of soup], but you only have half a [pot of] soup, and so I have one, two, three out of the four parts you need for a whole team. So it's three fourths. (C) (I, 31, 06-19-95)

37. Division by zero

So you can say, you know, there are one hundred dollars and you want to split it out and give everybody zero dollars. How many people could you split it out? I mean ... that's a stupid problem. If you can cut something into zero parts, what does that mean? (C) (I, 23, 01-03-95)

(To be continued)
Table F.1 (Continued)

<table>
<thead>
<tr>
<th>38. The quotient of a positive number and a positive number is positive</th>
<th>You have a certain amount of money that's left to six children equally, and so you take that amount of money and divide it by six. Each one will get a positive amount of money. (C) (I, 30, 06-16-95)</th>
</tr>
</thead>
<tbody>
<tr>
<td>39. The quotient of a negative number and a negative number is positive</td>
<td>a) Well, you can talk about going back in time. You are in the hole, let's say, twenty dollars, four days ago. He was [you were] in gambling. In the present day you break even. So what's happening? Have you been winning money? Have you been losing money during those four days? On average, you win five dollars a day. (C) (I, 23, 01-03-95)</td>
</tr>
<tr>
<td></td>
<td>b) You could talk about if you lost a thousand dollars and you've been losing two hundred dollars per day, how long did it take to loose the thousand dollars? ... Negative a thousand divided by negative two hundred per day. (C) (I, 30, 06-16-95)</td>
</tr>
<tr>
<td>40. The quotient of a negative number and a positive number is negative</td>
<td>a) Let's say you lost 30 dollars over six days. On average, how much did you lose? (C) (I, 23, 01-03-95)</td>
</tr>
<tr>
<td></td>
<td>b) You have been gambling. You lost a thousand dollars over seven days, how much did you lose per day? (C) (I, 30, 06-16-95)</td>
</tr>
<tr>
<td>41. The quotient of a positive number and a negative number is negative</td>
<td>You can talk about someone who has been losing two dollars a day.... They started with six dollars and they do not have any left. How long they have been losing two dollars a day? The answer would be negative three, three days ago they had six dollars. (C) (I, 23, 01-03-95)</td>
</tr>
</tbody>
</table>

*A description of the criteria used to judge the correctness of Mr. Kantor's representations is given in Chapter 4, section "Analysis," part "Coding scheme for Mr. Kantor's knowledge of story-problem representations.*

Table F.1 Mr. Kantor's story-problem representations
APPENDIX G: Mr. Kantor's use of representations
<table>
<thead>
<tr>
<th>Content curriculum event*</th>
<th>Representations**</th>
</tr>
</thead>
</table>
| 3. Area model for multiplication (Discrete version) | If I have an array like this, 
\[
\begin{array}{c|c|c|c}
\cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot \\
\end{array}
\], how do I find out how many elements are in that array? (N) (CO, 1, 10-13-94) |
| 6. Multiplication of fractions theorem | A pictorial representation for why \[ \frac{3}{4} \cdot \frac{3}{5} = \frac{9}{20} \] (N) (CO, 1, 10-13-94) |
| 7. Rate model for multiplication | Rates can be multiplied by other quantities. The units are treated as if they were factors in a multiplication. (V) (CO, 1, 10-13-94) |
| | \[ \frac{55 \text{ mi}}{\text{hr}} \cdot 12 \text{ hr} \] has meaning if \[ \frac{55 \text{ mi}}{\text{hr}} \] and 12 hr are related. (N) (CO, 1, 10-13-94) |
| | Use the reciprocal of [some of] the rates to do the multiplication |
| | \[ \frac{\text{dollars}}{\text{lb}} \cdot 30 \frac{\text{shrimp}}{\text{lb}} = \frac{\text{shrimp}}{\text{dollar}} \] (N) (CO, 2, 10-14-94) |
| | Suppose a laser printer prints 5 pages per minute. How long will it take to print 2400 documents with 3 pages per document? |
| | \[ \frac{2400 \text{ doc}}{1 \text{ doc}} \times \frac{3 \text{ pg}}{1 \text{ pg}} \times \frac{1 \text{ min}}{5 \text{ min}} \] (N) (CO, 2, 10-14-94) |
| | While Phyllis exercises, her rate is 150 beats per minute. If she exercises for \( m \) minutes at this rate, how many times her heart beat? (NS) (CO, 2, 10-14-94) |

(To be continued)

Table G.1 Representations constructed by Mr. Kantor when explaining the content curriculum events
Comparing a giraffe 18 feet to a person six feet tall \[ \frac{18 \text{ ft}}{6 \text{ ft}} = 3 \] [the units cancel out]. (N) (CO, 2, 10-14-94)

Convert \( \frac{4 \text{ ft}^2}{\text{sec}^2} \) into \( \frac{\text{in}^2}{\text{hr}^2} \):

\[
\frac{4 \text{ ft}^2 \cdot 12 \text{ in} \cdot 3600 \text{ sec} \cdot 3600 \text{ sec}}{\text{sec}^2 \cdot 1 \text{ ft} \cdot \text{hr} \cdot \text{hr}} = \frac{\text{in}^2}{\text{hr}^2}. \] (N) (CO, 2, 10-14-94)

12. Multiplicative identity of 1

Multiplication has an identity also. If I start with this \([A]\), what do I multiply by \([A \cdot 1 = A]\) to end up with the same thing I started with? ... OK \([A \cdot 1 = A]\) (S) (CO, 2, 10-14-94)

14. Definition of reciprocal

An inverse of an operation. If I apply the operation, what I then apply so I be back to where I started from? If I take something \((B)\) and make it twice as big \((B \cdot 2)\) what do I multiply that by to get back to where I started from? ... by multiplying by two I multiply by a half to go back to where I started \(B \cdot 2 \cdot \frac{1}{2}\) (NS) (CO, 2, 10-14-94)

\[
4 \cdot \frac{3}{5} = 1, \quad R = \frac{1}{4} \cdot \frac{3}{5}. \] (N) (CO, 3, 10-17-94)

15. The reciprocal of \(x\) is \(1/x\) \((x \neq 0)\)

The reciprocal of \(\pi\) is \(1/\pi\) (N) (CO, 3, 10-17-94)

The reciprocal of \(x^2\) is \(\frac{1}{x^2}\). (S) (CO, 3, 10-17-94)

16. The reciprocal of zero

1) Zero has no reciprocal \((V)\) (CO, 2, 10-14-94) & (CO, 3, 10-17-94)

2) If I multiply \(B\) by anything I can go back to where I started but unless I multiply by zero. Once I multiply by zero I wipe it out. I can't multiply that by something to get back to \(B\) \([B \cdot 0 = B]\) (NS) (CO, 2, 10-14-94)

3) There isn't something I multiply by zero to get one. \((V)\) (CO, 2, 10-14-94)

(To be continued)

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### Table G.1 (Continued)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$40R = 600$</td>
<td>$\frac{3}{4}B = 15$</td>
<td>(N) CO, 3, 10-17-94</td>
</tr>
<tr>
<td>$-3 &lt; 4$</td>
<td>$-3 &lt; 4$</td>
<td>(N) CO, 5, 10-19-94</td>
</tr>
<tr>
<td>$10 &lt; 14$</td>
<td>$-6 &lt; 8$</td>
<td>(N) CO, 5, 10-19-94</td>
</tr>
<tr>
<td>$-10 &gt; -14$</td>
<td>$6 &gt; -8$</td>
<td>(N) CO, 5, 10-19-94</td>
</tr>
<tr>
<td>$-5x &lt; 10$</td>
<td>$-12 &lt; 48 - x$, $-10u &lt; 0$, $10 - 5a &lt; 10$</td>
<td>(N) CO, 5, 10-19-94</td>
</tr>
<tr>
<td>$\frac{1}{4} \cdot \frac{3}{4} \cdot \frac{2}{4} = \frac{12}{51} = \frac{4}{3}$</td>
<td>( \text{I get ninety six, ninety six branches down there, ninety six different master schedules possible.} )</td>
<td>( \text{(N) CO, 8, 10-25-94} )</td>
</tr>
<tr>
<td>$P(H_1 \text{ and } H_2) = P(H_1) \cdot P(H_2 \text{ given } H_1)$</td>
<td>( \frac{13}{52} \cdot \frac{12}{51} = \frac{13 \cdot 12}{52 \cdot 51} )</td>
<td>( \text{(N) CO, 9, 10-26-94} )</td>
</tr>
<tr>
<td>$P(\text{Blonde and Run Track}) = \frac{6 \cdot 4}{24 \cdot 6} = \frac{4}{24}$</td>
<td>( \text{Six out of 24 are blonde, 7 run track, 4 out of 6 who run track are blonde.} )</td>
<td>( \text{(N) CO, 9, 10-26-94} )</td>
</tr>
</tbody>
</table>

(To be continued)
Six students in a class of 25 have the flu. Two of these six are girls.
Thirteen of the 25 students in the class are boys. Draw a tree diagram.
a) What is the probability that a randomly chosen student is a girl? b) What is the probability that a randomly chosen student with the flu is a girl? c) What is the probability that a randomly chosen student is girl with the flu? (N) (CO, 10, 10-27-94)

Probability that you pull a diamond out of the first draw and a diamond out of the second draw.

\[ P(D_1 \text{ and } D_2) = P(D_1) \cdot P(D_2 \text{ given } D_1) \]
\[ = \frac{13}{52} \cdot \frac{12}{51} \] (N) (CO, 10, 10-27-94)

If you are working on a straight, the probability of getting a straight, thirteen diamonds on the first fifty two times twelve out of the next fifty one times eleven out of fifty times ten out of forty nine times nine out of forty eight

\[ \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} \] (N)

(CO, 10, 10-27-94)

31. Conditional probability formula

<table>
<thead>
<tr>
<th>Choir</th>
<th>Music</th>
<th>Band</th>
<th>Orchestra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spanish</td>
<td>25</td>
<td>15</td>
<td>11</td>
</tr>
<tr>
<td>French</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>German</td>
<td>4</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Reading</td>
<td>1</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

K: What percentage of people takes Spanish and Choir? \[ P(S \text{ and } C) \]
… Number of people that are in Spanish and Choir divided by the total number of people. Now, there is another way to do it. Let me show you because we don’t have time for you to discover it. Look, seventy two point three seven percent of people get down to this point [students who take Spanish, P(S)]. Forty five point four five percent of those [take choir, P(C/S)]. Multiply those together. (N) (CO, 8, 10-25-94)
Six students in a class of 25 have the flu. Two of these six are girls. Thirteen of the 25 students in the class are boys. Draw a tree diagram. a) What is the probability that a randomly chosen student is a girl? b) What is the probability that a randomly chosen student with the flu is a girl? c) What is the probability that a randomly chosen student is girl with the flu? (N) (CO, 10, 10-27-94)

32. Factorial symbol (CO, 9, 10-26-94) A hearts, two diamonds, five spades, three diamonds and J hearts. In how many different ways can that hand be arranged in a row from left to right? $\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{121!}{119!} \cdot 120! \cdot 119! \cdot 120! \cdot 119!$ (N)

10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 (N)

34. Permutation theorem A hearts, two diamonds, five spades, three diamonds and J hearts. In how many different ways can that hand be arranged in a row from left to right? $\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{121!}{119!} \cdot 120! \cdot 119! \cdot 120! \cdot 119!$ (N) (CO, 9, 10-26-94)

35. Meaning of division $7 + 2, 5 + \frac{1}{2}, 21 + \frac{3}{4}$ (N) (CO, 12, 11-01-94)

36. Algebraic definition of division $a + b = a \cdot \frac{1}{b}$ (S) (CO, 12, 11-01-94)

(To be continued)
Table G.1 (Continued)

\[
7 + \frac{1}{2} = 7.5, \quad 5 + \frac{1}{2} = 5.5, \quad 21 + \frac{3}{4} = \frac{21 \cdot 4}{3} \quad (N)
\]

(Co, 12, 11-01-94)

* This table only contains the content curriculum events for which Mr. Kantor constructed explanations about their meaning. They are not numbered in consecutive order (1, 2, 3, etc.) but according to Table 5.1 so we have consistency across content curriculum events. In this way, CCE 5 refers to the same content curriculum event regardless to what table I refer to.

** The representations constructed by Mr. Kantor during classroom instruction were categorized as numerical (N), numerical-symbolic (NS), verbal (V), and symbolic (S) representations.

Table G.1 Representations constructed by Mr. Kantor when explaining the content curriculum events
3. Area model for multiplication (Discrete case) [Problem from the textbook]. In the figure at right, all angles are right angles. Find its area. (N) (CO, 1, 10-13-94)

![Area model for multiplication diagram]

4. Volume of a rectangular solid

The volume of a box needs to be 500 cubic centimeters. The base of the box has dimensions 12.5 cm and 5 cm. How high must the box be? (N) (CO, 4, 10-18-94)

Find the volume of the rectangular solid in which the area of the base is $9p^2$ and the height is $12p$. (NS) (CO, 1, 10-13-94)

6. Multiplication of fractions theorem

Find the volume of the rectangular solid in which the area of the base is $9p^2$ and the height is $12p$. (NS) (CO, 1, 10-13-94)

7. Rate model for multiplication

If Irma dribbles a basketball two times per second, and moves 4.5 ft per second, how many dribbles will she make moving 60 ft down court? (N) (CO, 4, 10-18-94)

\[
\frac{60 \text{ ft}}{4.5 \text{ ft/sec}} \cdot \frac{2 \text{ dribbles}}{\text{sec}} = 27 \text{ dribbles}
\]

15 & 16. Definition of reciprocal and the reciprocal of $x$ is $1/x$

Multiple choice. Which pairs of numbers are reciprocals?

(a) $2y$ and $\frac{1}{2y}$ (b) 0.4 and $\frac{2}{5}$ (c) $-6.4$ and $6.4$ (d) 0.8 and 0.2

(NS) (CO, 4, 10-18-94)

(To be continued)

Table G.2 Representations constructed when applying the content curriculum events to solve problems
## Table G.2 (Continued)

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>17. Multiplication property of zero</td>
<td>Suppose ( D = (w + 2)(w - 3)(w + 6) ). For what three values of ( w ) will ( D ) have a value of 0? ( D ) is three quantities multiplied together. Make any of those three quantities zero and ( D ) is zero. (N) (CO, 3, 10-17-94)</td>
</tr>
<tr>
<td>19. Solving ( ax = b )</td>
<td>The volume of a box needs to be 500 cubic centimeters. The base of the box has dimensions 12.5 cm and 5 cm. How high must the box be? (N) (CO, 4, 10-18-94)</td>
</tr>
<tr>
<td>25. Multiplicative property of inequalities (First part)</td>
<td>If ( a ) is positive you don't change the sense.... If I multiply both sides by something positive it stays the same, but if I multiply by negative it switches. (V) (CO, 5, 10-19-94) I multiplied both sides by a positive number so the sense does not change, the sense does not change, the sense does not change. ([-12 &lt; 48X] ) (NS) (CO, 5, 10-19-94)</td>
</tr>
<tr>
<td>26. Multiplicative property of inequalities (Second part)</td>
<td>I multiply both sides by negative, that switches. You saw that that happens, ([\left(\frac{1}{5}\right) - 5 \cdot X &gt; 10\left(\frac{1}{5}\right)].... As soon as you multiply both sides by a negative, change the sense. (V) (NS) (CO, 5, 10-19-94)</td>
</tr>
<tr>
<td>28. Multiplication counting principle</td>
<td>Out of a standard deck of playing cards, how many different poker hands are possible? ( \left[ \frac{52}{51} \cdot \frac{50}{49} \cdot \frac{48}{47} \right] = ) (N) (10, 9, 10-2694)</td>
</tr>
<tr>
<td>31. Conditional probability formula</td>
<td>Six students in a class of 25 have the flu. Two of these six are girls. Thirteen of the 25 students in the class are boys. Draw a tree diagram. a) What is the probability that a randomly chosen student is a girl? b) What is the probability that a randomly chosen student with the flu is a girl? c) What is the probability that a randomly chosen student is girl with the flu? (N) (CO, 10, 10-27-94)</td>
</tr>
<tr>
<td>32. The factorial symbol</td>
<td>(31)! (N) (CO, 10, 10-27-94)</td>
</tr>
<tr>
<td>34. Permutation theorem</td>
<td>In softball, there are 10 people who can bat. In how many ways can the manager of a softball team arrange the batting order? (N) (CO, 10, 10-27-94)</td>
</tr>
</tbody>
</table>
### Table G.2 (Continued)

<table>
<thead>
<tr>
<th>35. Meaning of division</th>
<th>Congruent figures are figures with the same size and shape. Split this region into 6 congruent pieces. (N) (CO, 12, 11-01-94)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><img src="image" alt="Diagram" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>36. Algebraic definition of division</th>
<th>Fill in the blanks (S) (CO, 12, 11-01-94)</th>
</tr>
</thead>
</table>
| \[
| \frac{m}{n} = m + \_
| \]                                                                                                  |

<table>
<thead>
<tr>
<th>37. Division by zero</th>
<th>I can't divide by zero. (V) (CO, 2, 10-14-94)</th>
</tr>
</thead>
</table>
| \[
| \frac{m}{n} = m \cdot ?
| \]                                                                                                  |

Table G.2 Representations constructed when applying the content curriculum events to solve problems
### Content curriculum event

<table>
<thead>
<tr>
<th>Event</th>
<th>Representations</th>
</tr>
</thead>
</table>
| 1. Area model for multiplication (Continuous case)     | Find the area of the figure (all angles are right angles) (NS) (CO, 7, 10-24-94) *
|                                                         | ![Area Model Diagram](#)                                                                                                                      |
| 2. Commutative property of multiplication               | $3x = x3$ (NS) (CO, 3, 10-17-94)                                                                                                           |
|                                                         | $8b = b8$ (NS) (CO, 3, 10-17-94)                                                                                                           |
| 5. Associative property of multiplication               | $(a\cdot b)c = a(bc)$ (S) (CO, 3, 10-17-94)                                                                                                  |
| 6. Multiplication of fractions theorem                  | $\frac{ax}{3} = \frac{6}{a}$ (N) (CO, 11, 10-28-94)                                                                                          |
|                                                         | $\frac{3R^2S}{7T} \cdot \frac{14T^2}{9RS^2}$ (NS, CO, 11, 10-28-94)                                                                      |
|                                                         | The large square at the right has length 1. What multiplication of fractions does the drawing represent? (N) (CO, 11, 10-28, 94) |
|                                                         | ![Fractional Multiplication Diagram](#)                                                                                                     |

(To be continued)

---

Table G.3 Representations constructed during review sessions

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340
<table>
<thead>
<tr>
<th>Table G.3 (Continued)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. Rate model for multiplication</td>
</tr>
</tbody>
</table>
| 14. Definition of reciprocal | N \cdot R = 1 (If you take a number times its reciprocal you get the identity) (S) (CO, 3, 10-17-94)  
\[ \frac{3}{4} \cdot \frac{4}{3} = 1 \] (Three fourths times its reciprocal gives you the multiplicative identity) (N) (CO, 3, 10-17-94) |
| 15. The reciprocal of \( x \) is \( \frac{1}{x} \) (\( x \neq 0 \)) | The reciprocal of \( \frac{3}{m} \) is \( \frac{1}{3m} \) (NS) (CO, 11, 10-28-94) |
| 19. Solving \( ax = b, b \neq 0 \) | \( \frac{4}{25}A = 6/5 \), \( 7/9)p = 140 \), \(-4p = 12 \), \( 12r = -4 \) (N)  
\( 6j = 11 \), \(-20k = 2/5 \) (N) (CO, 7, 10-24-94) |
| 20. Solving \( 0x = b, b \neq 0 \) | \( 0 \cdot w = 14 \) (N) (CO, 6, 10-20-94) |
| 21. Solving \( 0x = 0 \) | \( 0 = a \cdot 0 \) (N) (CO, 6, 10-20-94) |
| 22. Solving \( ax = 0, a \neq 0 \) | \( 7y = 0 \) (N) (CO, 6, 10-20-94) |
| 23. Solving \( -x = a \) | \(-l = 3 - 5 \) (N) (CO, 7, 10-24-94) |
| 27. Solving \( ax < b \) | \( 5x \geq 10 \), \(-3y < 300 \), \(-4A < -124 \), \( 13 > 2Z \), \( (2/3)p \leq (1/4) \), \( 0.09 > -9C \), \(-m < 8 \), \(-T < 0 \), \(-2 \leq -n \) (N) (CO, 6, 10-20-94)  
\( 4m < (1/10) \), \(-90 \geq -6n \), \( d + 2d < 39 \) (N) (CO, 7, 10-24-94)  
\(-y \leq -2 \), \(-3x < 6 \) (N) (CO, 11, 10-28-94) |
| 31. Conditional probability formula | Bill is a streak hitter in baseball. He gets hits 25% of the time he is at bat. But when he gets a hit his first time up, the probability he will get a hit the next time up is 32%. What is the probability Bill will get hits twice in a row at the beginning of a game? (N) (CO, 11, 10-28-94) |

Table G.3 Representations constructed during review sessions
Table G.4 Pictorial representations constructed for teaching the content curriculum events

<table>
<thead>
<tr>
<th>-14</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>4</td>
</tr>
<tr>
<td>-6</td>
<td>8</td>
</tr>
<tr>
<td>-8</td>
<td>6</td>
</tr>
</tbody>
</table>

27. Solving $ax < b$

(See Appendix P for Mr. Kantor's explanation)

(To be continued)
Table G.4 (Continued)

28. Multiplication counting principle
(CO, 8, 10-25-94)

<table>
<thead>
<tr>
<th>Subject</th>
<th>Options</th>
</tr>
</thead>
<tbody>
<tr>
<td>English</td>
<td>1 option</td>
</tr>
<tr>
<td>Music</td>
<td>4 options</td>
</tr>
<tr>
<td>Math</td>
<td>3 options</td>
</tr>
<tr>
<td>Global language</td>
<td>4 options</td>
</tr>
<tr>
<td>Social studies</td>
<td>2 options</td>
</tr>
</tbody>
</table>

29c. Classical definition of conditional probability

See the pictures for conditional probability formula.

31. Conditional probability formula

Figure 5.13 displays the first pictorial representation constructed by Mr. Kantor for teaching the conditional probability formula. (CO, 8, 10-25-94)

---

```
Figure 5.13

Draw

\[ \frac{1}{4} \text{ heart} \quad \frac{3}{4} \text{ no heart} \]

\[ \frac{12}{51} \text{ heart} \quad \frac{39}{51} \text{ no heart} \]

\[ \frac{13}{51} \text{ heart} \quad \frac{38}{51} \text{ no heart} \]

5.88% 19.12% 19.12% 55.88%

(See Excerpt 7, Chapter 5, for Mr. Kantor's explanation and Table G.1. cell 29c, for the word problem associated with this diagram) (CO, 9, 10-26-94)
```

(To be continued)
Table G.4 (Continued)

Table G.4 Pictorial representations constructed for teaching the content curriculum events

(See Excerpt 11, Chapter 5, for Mr. Kantor's explanation and Table G.1, cell 29c, for the word problem associated with this diagram)

(See Excerpt 12, Chapter 5, for Mr. Kantor's explanation and Table G.1, cell 29c, for the word problem associated with this diagram) (CO, 10-10-27-94)
7. Rate model for multiplication

55 \text{ mi} \cdot 12 \text{ hr} \text{ has meaning if } \frac{55 \text{ mi}}{\text{hr}} \text{ and } 12 \text{ hr} \text{ are related.} \quad (CO, 1, 10-13-94)

Use the reciprocal of the rates to do the multiplication

\[ \frac{6 \text{ dollars}}{\text{lb}} \cdot \frac{30 \text{ shrimp}}{\text{lb}} = \frac{\text{shrimp}}{\text{dollar}}. \]

The rate six dollars per pound is the same as one pound per six dollars

\[ \frac{\text{lb}}{6 \text{ dollars}} \cdot \frac{30 \text{ shrimp}}{\text{lb}} = \frac{\text{shrimp}}{\text{dollar}}. \quad (CO, 2, 10-14-94) \]

Suppose a laser printer prints 5 pages per minute. How long will it take to print 2400 documents with 3 pages per document?

\[ 2400 \frac{\text{doc}}{1} \times \frac{3 \text{ pg}}{1 \text{ doc}} \times \frac{1 \text{ min}}{5 \text{ pg}} \]

\[ 7200 \text{ pg} \quad 1440 \text{ min} \quad (CO, 2, 10-14-94) \]

While Phyllis exercises, her rate is 150 beats per minute. If she exercises for \( m \) minutes at this rate, how many times her heart beat?

\( (CO, 2, 10-14-94) \)

In 1983, the U.S birth rate was 15.5 babies per 1000 population. The population was 226,000,000. Use this multiplication,

\[ \frac{15.5 \text{ babies}}{1000 \text{ people}} \cdot 226,000,000, \]

\[ \text{to determine how many babies were born in 1983.} \quad (CO, 2, 10-14-94) \]

(To be continued)

Table G.5 Story-problem representations constructed for teaching the content curriculum events

346
Comparing a giraffe 18 feet to a person six feet tall \[
\frac{18 \text{ ft}}{6 \text{ ft}} = 3
\] [the units cancel out]. (CO, 2, 10-14-94)

Convert \[\frac{4 \text{ ft}}{\text{sec}^2}\] into \[\frac{\text{in}}{\text{hr}^2}\].

\[
\frac{4 \text{ ft}}{\text{sec}^2} \cdot \frac{12 \text{ in}}{3600 \text{ sec}} = \frac{\text{in}}{\text{hr}^2}
\] (CO, 2, 10-14-94)

25 & 26. The multiplicative property of inequalities

Let's say two people go to Las Vegas, they are gambling. Call one person W, one person L, winner and loser. Mr. W always wins the same amount every single day. Mr. L always loses the same amount every single day \([W > L]\). Now, this W is in a better situation, he is winning. The other guy is losing, correct? As I go into the future, this person \([W]\) is always in a better position that this person: two days \([2W > 2L]\), ten days in the future, \([10W > 10L]\)… Correct? This guy keeps winning the same amount every day, [and] this guy keeps losing the same amount every day. Now, look, what happens when we multiply both sides of this \([W > L]\) by a negative number? … Negative three W is less than negative three L \([-3W < -3L]\). What does that mean? Why is this person now in a better situation? What does that mean? … Look, the first day it wins \([W > L]\), after two days is in a better situation, ten days is much better. So what does this mean? … Three days ago. Three days ago he hadn't won all the money \([-3W]\). … He hadn't lost three days work \([-3L]\). This guy \([-3L]\) would like go back in time. [Students are discussing] Look, look. Let's say that they … [win or lose] exactly the same amount of money [every day]… Day zero. They each have a thousand dollars. Look, this guy W wins two hundred dollars a day. L loses a hundred dollars a day. W three days ago had how much money? … Pick time zero. He's been gambling before time zero and gambles that [200]. Three days before time zero he had six hundred dollars less. L three days before he hadn't lost three hundred dollars. He had thirteen hundred dollars \(\begin{bmatrix} W \\ \$400 \\ \$1300 \end{bmatrix}\). He [pointing out \(-3L\)] is in a better situation than he is [pointing out \(-3W\)]. True? … Look, look, from point zero into the future this is always better than \[
\begin{bmatrix}
W > L \\
2W > 2L \\
10W > 10L
\end{bmatrix}
\]

(To be continued)
If you go backwards, though. The loser going back in time [pointing out $-3L$]. He likes to do that…. The winner wouldn’t. Why? [surprise]…. If you went to Las Vegas for three days—for three days and you’re losing money [pointing out at $-3L$], wouldn’t you like to never have gone? (CO, 5, 10-19-94)

The loser is better off if they go back in time. The loser is better off than the winner. It reverses the process. The winner is always better off than the loser when you go forth in time $W > L$…. When you go backwards in time the opposite is true $-3W < -3L$. The loser is better off if they go backwards in time. He hasn’t lost that money and the winner hasn’t won that money. If you go back three days the loser hasn’t lost for three days and the winner hasn’t won for three days. (CO, 6, 10-20-94)

### 28. Multiplication counting principle

A master schedule is, just think of the courses, the classes that you take. I am not concerned about when you take them. I want to figure out how many different schedules are possible without considering what period of the day you take them. For example, everyone of you takes English. It has to be on every one’s master schedule. Everyone of you has to take some type of a math. All of you are obviously on algebra but some of your classmates don’t take algebra … Think about just eighth grade master schedules. So I want you to do is figure out how many different master schedules are possible. (CO, 8, 10-25-94)

Out of a standard deck of playing cards, how many different poker hands are possible? [Assuming that the order is important] (CO, 9, 10-26-94)

### 29c. Classical definition of conditional probability

<table>
<thead>
<tr>
<th></th>
<th>Choir</th>
<th>Music</th>
<th>Band</th>
<th>Orchestra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spanish</td>
<td>25</td>
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<td>4</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>German</td>
<td>4</td>
<td></td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Reading</td>
<td>1</td>
<td>1</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

a) Determine the percentage of people who take each of the four languages.

(To be continued)
b) Look at only those people who take Spanish. Determine the percentage of people who take each of the four music options [presuming they are taking Spanish]. (CO, 8, 10-25-94)

Six out of 24 people are blonde.... Seven out of 24 people run track.
Out of six blonde people, 4 run track.

\[
P(\text{Run track given blonde}) = \frac{4}{6} \text{ (CO, 9, 10-26-94)}
\]

The probability of getting a straight of diamonds

\[
\begin{array}{cccc}
13 & 12 & 11 & 10 \\
\_52 & 51 & 50 & 49 & 48
\end{array}
\]  

(CO, 10, 10-27-94)

31. Conditional probability formula

Figure out the percentage of people who take Spanish and Choir [using \(P(A \cap B) = \frac{25}{76}\) and taking a percentage of a percentage: 72.37\% of students take Spanish \((P(A) = .7237)\), and 45.45\% of those students take Choir \((P(B/A) = .4545)\). Then \((.7237)(.4545)100\) represents the percentage of students taking Spanish and Choir] (CO, 8, 10-25-94)

Suppose that two cards are drawn from a well-shuffled deck. The first card is not put back before the second is drawn.... Copy and compute the probabilities. (CO, 9, 10-26-94)

\[
\text{drawing}
\]

\[
\begin{array}{c}
1\text{st card} \\
\text{heart} \\
\text{no heart}
\end{array}
\]

\[
\begin{array}{c}
a \\
b \\
c
\end{array}
\]

\[
\begin{array}{c}
2\text{nd card} \\
\text{heart} \\
\text{no heart} \\
\text{heart} \\
\text{no heart}
\end{array}
\]

\[
P(\text{Blonde and Run track}) = P(\text{Blonde}) \cdot P(\text{Run track given blonde})
\]

(To be continued)
Six students in a class of 25 have the flu. Two of these six are girls. Thirteen of the 25 students in the class are boys. Draw a tree diagram. a) What is the probability that a randomly chosen student is a girl? b) What is the probability that a randomly chosen student with the flu is a girl? c) What is the probability that a randomly chosen student is girl with the flu? (CO, 10, 10-27-94)

What is the probability of pulling a diamond out of the first draw and a diamond out of the second draw? (CO, 10, 10-27-94)

32. n!

Out of a standard deck of 52 cards we got the following hand: A heart, 2 diamonds, 5 spades, 3 diamonds, and J hearts. In how many different ways can that hand be arranged in a row from left to right? (CO, 9, 10-26-94)

34. Permutation theorem

Out of a standard deck of 52 cards we got the following hand: A heart, 2 diamonds, 5 spades, 3 diamonds, and J hearts. In how many different ways can that hand be arranged in a row from left to right? (CO, 9, 10-26-94)

In softball, there are 10 people who can bat. In how many ways can the manager of a softball team arrange the batting order? (CO, 10, 10-27-94)

35. Meaning of division (CO, 12, 11-01-94)

Can someone give me a concrete example of seven divided by two, a situation in which seven divided by two gives you the answer?

Give me problem, make up a problem where five divided by one half represents that problem.... I want a situation where I'm trying to figure out how many times one half goes into five.... So, you have five dollars. You wanna find out how many pencils you can buy that cost fifty cents a piece. You're trying to find out ... how many times a half goes into five.

Twenty one divided by three fourths $\left[21 + \frac{3}{4}\right]$, and let's see that this is the situation ... I wanna take some of my students out to lunch and I know that I can buy a pop per seventy five cents per person, seventy five cents a can. I've got twenty one bucks so I have to figure out how many people I can buy pop. That gives me the answer, doesn't it? Tells me how often seventy five cents goes into twenty one dollars, right?

(To be continued)
Can someone give me a concrete example of seven divided by two, a situation in which seven divided by two gives you the answer? ... You have seven pieces of pizza and you want to divide those among two people.... If you do seven times one half, what happens?

You get three point five \[ \frac{7 + 2}{2} \]. Now, using that example, explain why this works.... Show me in the context of the problem why the same situation can be described as seven times one half.... When you were at elementary school you understood that dividing by two is the same as taking half of something, right? ... So you understand, you conceptualize, you internalize that that divided by two, taking something and cutting in half is the same thing as taking half of it. Half of it, one half times, correct?

Give me a story problem with five divided by one half represents that problem.... I want five divided by a half. I want a situation where I'm trying to figure out how many times one half goes into five. So, you have five dollars. You wanna find out how many pencils you can buy that cost fifty cents a piece. You're trying to find out ... how many times a half goes into five.... how many times does fifty cents go into five dollars? ... Ten times. That makes sense? Doesn't it make sense that you get the same answer as five times two? ... Why? Using the same situation explain why.... Let me change the problem just a little bit. The same type of thing. Let's say that I have candy for Halloween, but a lot of times people don't come to my house, so I don't buy a lot. So what I wanna do is I wanna make sure I got a bag of money in case I run out of candy. I wanna make sure I have money to give you. I wanna give each kid fifty cents. I have five dollar bills, all right? Now, I figure out first of all with five dollar bills I can give ten kids fifty cents , right? Now, I have five dollar bills. I've gotta to convert that. I have to go to the bank. I have to go the bank and get fifty cent pieces for five dollar bills. For every dollar bill they give what? ... Two fifty cent pieces, right? and so that's five times two is how many fifty cent pieces I get. The same thing. How often does a half go into five, how many times a half goes into five is the same as five times two.

Now, let me give you another example. Twenty one divided by three fourths \[ \left( \frac{21}{3} \right) \], and let's see that this is the situation .... I wanna take some of my students out to lunch and I know that I can buy a pop per seventy five cents per person, seventy five cents a can. I've got twenty one bucks so I have to figure out how many people I can buy pop. That gives me the answer, doesn't it? Tells me how often seventy five cents
goes into twenty one dollars, right? ... But now tell me this. We know by the algebraic definition of division that that's the same thing \[ 21 \div \frac{3}{4} = 21 \cdot \frac{4}{3} \]. How do we know that from a concrete standpoint that that's the same thing? ... To me the easiest way to do this problem ... I take twenty one dollars, I go to the bank and I say give me twenty one dollars worth in quarters. That's what that is, eighty four quarters and then what's this \[ \frac{84}{3} \]? I put it into groups of three and my first graders can do that. I end up with twenty eight groups of three \[ \frac{84}{3} = 28 \], all right?
APPENDIX H: Analysis of Mr. Kantor's explanations with respect to the degree of integration of concepts
In this appendix I describe the connections that Mr. Kantor made and did not make when teaching the content curriculum events. Only the content curriculum events for which Mr. Kantor constructed explanations are analyzed to assess Mr. Kantor's explanations in terms of the degree of integration of concepts.

6. Multiplication of fractions theorem
   (1) Relationship of a fraction to the repeated model for multiplication \( \frac{3}{4} \times \frac{3}{5} = \frac{9}{20} \) (3-fourths-three-fifths)
   (2) Relationship of multiplication of fraction to the area model for multiplication (the pictorial representation)
   (3) No relationship of a fraction to the area model of multiplication (\( \frac{3}{4} \) of \( \frac{3}{5} \))
   (4) No relationship of multiplication of fractions to the model array

Explanation with two degrees of integration

7. Rate model for multiplication

The explanation about the rate model of multiplication involved explicit connections to two concepts:
   (1) The value of some rates is equal to 1
   (2) Two reciprocal rates represent the same amount
   (3) No connection to the repeated-addition model of multiplication was made explicit.

Explanation with two degrees of integration

12. The multiplicative identity of 1
   (1) Connection to additive identity
   (2) No connection to the commutative property
   (3) No connection to the meaning of multiplication as repeated addition
   (4) No connection to the rate model of multiplication

Explanation with one degree of integration:
14. Definition of reciprocal

(1) A connection between a number times its reciprocal equals 1 and the reciprocal of \( x \) is \( \frac{1}{x} \).

Explanations with one degree of integration

15. The reciprocal of \( x \) is \( \frac{1}{x} \).

(1) Connection to the definition of reciprocal

Explanations with one degree of integration

16. Reciprocal of zero

(1) Connection to the definition of reciprocal

(2) No connection to division by zero

Explanations with one degree of integration

19. Solving \( ax = b \)

(1) A procedural connection to related facts and to division. A procedural connection to the reciprocal

(2) No direct connection to division using a story-problem that can be represented by an equation and by a division problem

(3) No connection to the rate model of multiplication

(4) No connection to the properties of equality, associative, or multiplicative identity

Explanations with one degree of integration

24. Definition of \( a > b \)

(1) No connection between the geometric definition of \( a > b \) and the algebraic definition

Explanations with 0 degrees of integration

25. Multiplicative property of inequalities (first part)

(1) Some reference to the multiplicative property of equality

(2) Use of the multiplicative property of inequalities when solving inequalities

Explanations has two degrees of integration

26. The multiplicative property of inequalities (second part)
(1) Some reference to the multiplicative property of equality

(2) Mr. Kantor uses the second part of the multiplicative property when solving inequalities

Explanation with two degrees of integration

27. Solving $ax < b$

(1) Some reference to solving equations

(2) Use of the multiplicative property of inequalities

Explanation has two degrees of integration

28. Multiplication counting principle

(1) Some reference to multiplication as repeated addition or the number of elements in an array

Explanation with one degree of integration

29c. Classical definition of conditional probability

(1) This definition was related to the conditional probability formula

(2) This definition was not related to the second definition of conditional probability

Explanation with one degree of integration

31. Conditional probability formula

(1) A connection between $P(A \cap B)$ and $P(B \mid A)$ through $P(A \cap B) = P(A)P(B \mid A)$ Only a numerical connection between the definition of $P(A \cap B)$ and $P(A)P(B \mid A)$

(2) No further connections between these two concepts (e.g., when working with cards we can compute $P(A \cap B)$ using the definition of $P(A \cap B)$. This is another opportunity for using the multiplication counting principle

(3) No connection to the fact that the events $(A \cap B)$ and $(B \mid A)$ are the same event but their probabilities refer to different sample spaces

(4) No connection to $P(B \mid A) = \frac{n(B \mid A)}{n(A)} = \frac{n(B \mid A) / N}{n(A) / N} = \frac{n(A \cap B) / N}{n(A) / N} = \frac{P(A \cap B)}{P(A)}$

Explanation with one degree of integration.

32. $n!$
34. Permutation theorem
(1) Some reference to the multiplication counting principle
(2) No connection to multiplication
An explanation with one degree of integration

35. Meaning of division
(1) Reference to the quotitive model of division in addition to the partitive model
(2) No connection to the missing-factor model of division
Explanation with one degree of integration

36. Algebraic definition of division
(1) Relationship to multiplication
(2) No connection of the meaning of $a + b$ as a fraction ($\frac{a}{b}$) and hence to multiply numerator and denominator by $\frac{1}{b}$ to get $a \cdot \frac{1}{b}$
(3) No connection to the theorem of multiplication of fractions
(4) No connection to the missing-factor model for division
(5) No application of the repeated addition model of multiplication to $a \cdot \frac{1}{b}$ when $a$ is a whole number
Explanation with one degree of integration
APPENDIX I: Analysis of the story problem used by Mr. Kantor to illustrate the second property of inequalities (If $a > b$ then $ac < bc$ when $c < 0$)
A story-problem representation of the multiplicative property of inequalities

1. K: Let me give you a real life example where you can see if this makes sense. Let's say two people go to Las Vegas, they are gambling. Call one person W, one person L, winner and loser. Mr. W always wins the same amount every single day. Mr. L always loses the same amount every single day \([W > L]\). Now, this W is in a better situation, he is winning. The other guy is losing, correct? As I go into the future, this person [W] is always in a better position that this person: two days \([2W > 2L]\), ten days in the future, \([10W > 10L]\). Correct? This guy keeps winning the same amount every day, [and] this guy keeps losing the same amount every day. Now, look, what happens when we multiply both sides of this \([W > L]\) by a negative number?

2. S: It switches.

3. K: Negative three W is less than negative three L \([-3W < -3L]\). What does that mean? Why is this person now in a better situation? What does that mean?

4. S: Three days before they win or something.

5. K: Look, the first day it wins \([W > L]\), after two days is in a better situation, ten days is much better. So what does this mean? Emma, what does negative three mean?

6. E: Three days before.

7. K: Three days ago. Three days ago he hadn't won all the money \([-3W]\).... He hadn't lost three days work \([-3L]\). This guy \([-3L]\) would like go back in time. [Students are discussing] Look, look. Let's say that they ... [win or lose] exactly the same amount of money [every day].... Day zero. They each have a thousand dollars. Look, this guy W wins two hundred dollars a day. L loses a hundred dollars a day. W three days ago had how much money?

8. S: Four hundred.

9. K: Pick time zero. He's been gambling before time zero and gambles that [200]. Three days before time zero he had six hundred dollars less. L three days before he hadn't lost three hundred dollars. He had thirteen hundred dollars \(\begin{bmatrix} W \\ L \end{bmatrix} = \begin{bmatrix} $400 \\ $1300 \end{bmatrix}\). He [pointing out \(-3L\)] is in a better situation than he is [pointing out \(-3W\)]. True? ... Look, look, from point zero into the future this is always better than this \(\begin{bmatrix} 2W > 2L \\ 10W > 10L \end{bmatrix}\). If you go backwards, though. The loser going back in time [pointing out \(-3L\)]. He likes to do that.... The winner wouldn't. Why? [surprise].... If you went to Las Vegas for three days—for three days and you're losing money [pointing out at \(-3L\)], wouldn't you like to never have gone? (CO, 5, 10-19-94)

In speeches 1 though 7, Mr. Kantor constructed the following representation: two people, W and L, are gambling. Mr. W always wins the same amount of money per day and Mr. L always loses the same amount every single day \([W > L]\). As we go into the future two days, ten days, person W is in a better position than L, \(2W > 2L\) and \(10W > 10L\). Then if we multiply both sides of this inequality by \(-3\) we get \(-3W < -3L\) which means that W had not won all the money and L had not lost that money. This representation was confusing to students and I had some trouble understanding it. I
interpret it in the following way: Mr. Kantor was not very explicit about what \( W \) and \( L \) represents. He implied that \( W \) represents the person who is always winning and \( L \) the person who is always losing. It is not very clear if those variables also represent the money that those people win and lose per day, respectively. If this is the case then the amount of money that they have after one day is \( W \) and \( L \) and \( W > L \) because \( W \) is a positive amount and \( L \) is a negative amount. That amount is with respect to day zero \((t = 0)\). The winner has \( W \) dollars more than the day before and the loser has \( L \) dollars less than the day before \((t = 0)\). After two days the amount of money that they have (with respect to \( t = 0 \)) is \( 2W \) and \( 2L \), respectively, and \( 2W > 2L \). The same relationship holds for the money that they have after 10 days: \( 10W \) and \( 10L \), and then \( 10W > 10L \). Those representations, \( 2W > 2L \) and \( 10W > 10L \) are examples of the application of the first part of the multiplicative property of inequalities. But then the day before \( t = 0 \) the loser has \(-L\) dollars and the winner has \(-W\). \(-L\) is now a positive amount and \(-W\) is now a negative amount (with respect to \( t = 0 \)) and then \(-W < -L\). Three days ago \( W \) had \(-3W\) money (with respect to \( t = 0 \)) and \( L \) had \(-3L\) money (also with respect to \( t = 0 \)). Then it makes sense that \(-3W < -3L\) because \(-3W\) represents a negative amount of money and \(-3L\) represents a positive amount of money. Those representations, \( W > L, -3W < -3L \) illustrate the second part of the multiplicative property of inequalities. But when I thought I was understanding Mr. Kantor's representation (in the way just described) he added the following comments: at day zero each have a thousand dollars. \( W \) wins 200 dollars a day and \( L \) loses a hundred dollars a day. How much money did \( W \) have three days ago? How much money did \( L \) have three days ago? The loser is in a better situation because three days ago he had 1300 hundred dollars and the winner had 400 hundred dollars. In this example it is not clear what \( W \) and \( L \) represent. Do \( W \) and \( L \) represent the total amount of money that the winner and loser has, respectively, at any point in time? that is, \( W = 1000 + 200t \) and \( L = 1000 -100t \)? If this is the case then \( W > L \) and \( 2W > 2L \), provided that \( t \) is
positive, but now 2W and 2L do not represent the money that W and L have, respectively after two days. It is not clear what the physical interpretation would be for those expressions. The same holds for the money they had three days ago. With those values for W and L we can illustrate the second part of the multiplicative property of inequalities since \(-3W < -3L\) provided that \(t\) is positive but the money that they had three days before is not represented by \(-3W\) and \(-3L\) and it is not clear what those expressions, \(-3W\) and \(-3L\) represent. If we want W and L to represent the amount of money at any point in time then it is not clear how we can use them to illustrate the multiplicative property of inequalities.

While it is true that in two days \(W = 1000 + 200(2) = 1400\) and \(L = 1000 - 100(2) = 800\) and \(1400 > 800\), that is not a representation of the property being discussed. The same holds for the money that they had three days ago. While it is true that \(W = 1000 + 200(-3) = 400\) and \(L = 1000 - 100(-3) = 1300\) and \(400 < 1300\), this is not a representation of the multiplicative property of inequalities. The first interpretation illustrates the multiplicative property of inequalities but that is not what Mr. Kantor explicitly constructed even if that is what he meant. When I asked him about this representation during the interviews he was more explicit and he provided appropriate representations (see cell 26 of Table F.1). We can conclude, then, that Mr. Kantor attempted to construct a story-problem representation. but that he did not do so even when he knew some. Probably the reason was that in that moment he was making it up, and did not reflect on what he was doing or on the degree of explicitness needed to help some students understand it.

After this teaching event Mr. Kantor provided more examples of how to solve inequalities of the form \(ax < b\) in a review format (\(5X \geq 10, -3Y < 300, -4A < -124\)). However, some students were still having difficulties with applying the multiplicative property of inequalities. Mr. Kantor tried again to give some physical meaning to that property by returning to the problem of the winner and the loser, after teaching how to solve \(-4A < -124\). He said:
We should somehow understand when we talk about the example about the person that always wins and the person that always loses. The person goes back in time. The loser is better off if they go back in time. The loser is better off than the winner. It reverses the process. The winner is always better off than the loser when you go forth in time \([W > L]\). When you go backwards in time the opposite is true \([-3W < -3L]\). The loser is better off if they go backwards in time. He hasn't lost that money and the winner hasn't won that money. If you go back three days the loser hasn't lost for three days and the winner hasn't won for three days. So the loser is in a better position there—Over time. But I'm saying the winner doesn't want to back in time. . . . It's the same thing. If we both went to Las Vegas. And you always win and I always lose. I'm gonna say I wish I never went. You are not gonna say you wish you didn't because you made money. (CO, 6, 10-20-94)

However, we notice that Mr. Kantor's representation was not explicit at all about the multiplicative property of inequalities and it is likely that students did not construct conceptual knowledge when exposed to that representation.
APPENDIX J: Description and analysis of Mr. Kantor's pictorial representations

for $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$
In this appendix, I describe and analyze in detail Mr. Kantor's knowledge of pictorial representations about \( P(B \mid A) = \frac{P(A \cap B)}{P(A)} \). To better understand his struggles I begin with the pictorial representation that Mr. Kantor constructed for illustrating the classical definition of \( P(A \mid B) \) (i.e., \( \frac{n(A \mid B)}{N} \) or \( \frac{\text{Area} (A \mid B)}{\text{Area} (S)} \)). The representation is depicted in Figure J.1. His thinking when constructing the representation for the definition of \( P(A \mid B) \) was as follows:

Here are other things. Let's say, you take part of it and might say if that's the probability, this is the probability that \( B \) occurred [pointing to region GHFE] ... and right in here. Ah, let's see, you are gonna call that the probability of \( A \) [given \( B \)] That probability is that area; that area compared to this area, I mean. If I call these the successes for \( A \), then this is the probability of \( A \) given \( B \), I mean, that area [region CDFE] compared to this total area [region GHFE] is probability of \( A \) given \( B \). (I, 28, 06-06-95)

We notice that he began saying that the area CDFE represents \( P(A \mid B) \) but that he changed his representation to the ratio of the area CDFE and GHFE. However, we notice that he wrote \( P(A \mid B) \) in the region CDFE. I speculate that he probably did not have a very explicit thinking or notation to differentiate the cases in which it is appropriate to take the

\[ \text{Successes for } A \]

\[ \begin{array}{ccc}
\text{E} & \text{F} \\
\text{C} & \text{D} \\
\text{G} & \text{H} \\
\end{array} \]

\[ P(A \mid B) \]

\[ P(B) \]

Figure J.1 The definition of \( P(A \mid B) \)

\[ \frac{P(A \cap B)}{P(A)} \]

\[ \frac{P(A \cap B)}{P(B)} \]

---

1When Mr. Kantor and I were discussing the second definition of conditional probability we used the two representations \( P(B \mid A) = \frac{P(A \cap B)}{P(A)} \) and \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} \).
region as the probability and the cases in which it is not. This is important from a teaching perspective. As teachers, we need to use appropriate language and correct representations when explaining mathematical concepts and procedures to students. Since the events $A \cap B$ and $A \cup B$ are represented by the same area but their probabilities refer to different sample spaces it is not correct to take $P(A \mid B)$ as $A(A \mid B)$. That is, while the event $A \mid B$ is the same as the event $A \cap B$, $P(A \mid B) \neq P(A \cap B)$. In this case it is more appropriate to take the area of the region CDFE as $P(A \cap B)$ rather than as $P(A \mid B)$.

I made a tentative conclusion after analyzing all the representations (pictorial and verbal) constructed by Mr. Kantor about conditional probability: it seems that he has a good informal understanding of the representation of the definition of probability of an event but that his knowledge is not very explicit as to appropriate ways of representing it. Mr. Kantor did not construct a pictorial representation for $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$ the first time I asked him (Actually I asked him to represent $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$). He said that he could represent $P(A \cap B) = P(B) \cdot P(A \mid B)$ instead. I asked Mr. Kantor to construct a representation for $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$ two more times. I wanted to know whether he could represent $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$ using a picture similar to Figure J.2. Figure J.2 does not represent $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$ directly. We cannot represent $\frac{P(A \cap B)}{P(A)}$ using areas for both $P(A \cap B)$ and $P(A)$ because that expression is a quotient. Using Figure J.2 we can reason that $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$ because $P(B \mid A) = \frac{\text{Area} (B \mid A)}{\text{Area} (A)} = \frac{\text{Area} (A \cap B) / \text{Total area}}{\text{Area} (A) / \text{Total area}} = \frac{P(A \cap B)}{P(A)}$ and conclude that $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$. Another way to get to the final result is as follows: Without losing generality, we can assume that the total area equals 1 and then we can have $P(A \cap B) = \text{Area} (A \cap B)$ and $P(A) = \text{Area} (A)$ and as a consequence we have $P(B \mid A) = \frac{\text{Area} (B \mid A)}{\text{Area} (A)} = \frac{\text{Area} (A \cap B)}{\text{Area} (A)}$. 

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Area \( (A \cap B) \) \[
\frac{\text{Area} \ (A \cap B)}{\text{Area} \ (A)} = \frac{P(A \cap B)}{P(A)}
\]

Searching for that construction turned out to be a kind of difficult journey for me and Mr. Kantor. Since his representations were not always totally appropriate nor explicit, I wanted him to be very explicit about them, and I followed up on his representations for \( P(B | A) = \frac{P(A \cap B)}{P(A)} \). That situation added another difficulty to the difficult topic of conditional probability. The first time I asked him to construct that representation, his comments seemed to suggest that he did not have a pictorial representation for \( P(A | B) = \frac{P(A \cap B)}{P(B)} \) because he said "if you do it the other way, where it is probability of \( A \) given \( B \) \( [P(A | B)] \) times probability of \( B \) \( [P(B)] \) gives you this \( [P(A \cap B)] \)." Then he constructed, as discussed in Chapter 5, a representation that illustrated, at least implicitly, the conditional probability formula, \( P(A \cap B) = P(B) \cdot P(A | B) \), (Figure 5.9a).

![Diagram of conditional probability](image)

Figure J.2 A pictorial representation for \( P(B | A) = \frac{P(A \cap B)}{P(A)} \)

I formulated a tentative hypothesis that Mr. Kantor did not know a pictorial representation for \( P(B | A) = \frac{P(A \cap B)}{P(A)} \). To check it, I asked him a second time in a later interview to construct a representation illustrating that formula and I told him that he had
already constructed a representation for \( P(A \cap B) = P(A)P(B \mid A) \). Figures 1-6 (Table J.1) illustrate his attempts to construct a pictorial representation during interview 31.

Figure 1 (Table J.1) illustrates Mr. Kantor's first attempt: the region ABCD represents the probability of A, the region BGHC is the probability of not A, and the region ABFE is the probability of B given A. While the region ABCD can represent \( P(A) \), assuming the area of the region AGHD equals 1, I do not see how the region ABFE can represent \( P(B \mid A) \). What is clear to me is that the region ABFE can represent \( P(A \cap B) \). It is not evident to me how to use those probabilities to represent \( P(B \mid A) = \frac{P(A \cap B)}{P(A)} \). Mr. Kantor, probably had the same feelings because he made another attempt to achieve the purpose of representing \( P(B \mid A) = \frac{P(A \cap B)}{P(A)} \). Figures 2 and 3 (Table J.1) illustrate his second attempt. His thinking was as follows "that's the probability of A [ABCD], that's the probability of B given A [AE]. When you multiply them together you get this [AEFB or \( P(A \cap B) \)]" (Figure 3, Table J.1). We notice that this is an implicit representation of why \( P(A \cap B) = P(A)P(B \mid A) \). However, it is not quite explicit and I think it needs some elaboration for students to understand it. Without losing generalizability, we can assume that AGHD is a unit square. Then the area ABCD and the length AB both represent \( P(A) \).

<table>
<thead>
<tr>
<th>Area ABCD</th>
<th>Area AGHD</th>
<th>Area ABCD/1</th>
<th>Area ABFE</th>
<th>Area AGHD/1</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABCD</td>
<td>AGHD</td>
<td>ABCD</td>
<td>ABFE</td>
<td>ABFE</td>
</tr>
</tbody>
</table>

\[ P(A) = \frac{\text{Area ABCD}}{\text{Area AGHD}} = \frac{\text{Area ABCD}}{1} = \text{Area ABCD} \]

\[ P(A \cap B) = \frac{\text{Area ABFE}}{\text{Area AGHD}} = \frac{\text{Area ABFE}}{1} = \text{Area ABFE} \]

The region ABFE also represents the event \( B \mid A \). Since AD measures 1 unit, AE can be taken as representing \( P(B \mid A) \) and ED as \( P(B' \mid A) \) assuming a line model for conditional probability. By the area model of multiplication we know that \( AB \cdot AE = \text{Area ABFE or } P(A)P(B \mid A) = P(A \cap B) \). As we saw in Chapter 5, the representation depicted in Figure 3 is similar to the one he had provided before for \( P(A \cap B) = P(A)P(B \mid A) \) (Figure 5.9a). Since I wanted to know if
Table J.1 Mr. Kantor's pictorial representations for $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$ during the second round of attempts
Mr. Kantor knew how to represent $P(B \mid A)$ with area using the same picture ($P(B \mid A) = \frac{\text{Area ABFE}}{\text{Area ABCD}}$), I asked him if we could represent $P(B \mid A)$ with area. He then constructed the representation depicted in Figure 4 (Table J.1). His thinking was as follows:

You can do it also with area. If I call this a unit square, the probability of $A$ is this length $[DC]$. Then this area is also the probability of $A \{ABCD\}$ ... and then we can call this the probability of $B$ [the length DE], and this is the probability of $B$ given $A \{EFCD\}$. This area $[EFCD]$ compared to this area $[ABCD]$ is the probability of $B$ given $A$. (I, 31, 06-09-95)

During the construction of this representation Mr. Kantor said and wrote that the area $EFCD$ represented $P(B \mid A)$ and then he later said that it was the quotient of the area $EFCD$ and $ABCD$ that represented $P(B \mid A)$. Probably what he meant was that $EFCD$ represents the event $B \mid A$. But $EFCD$ also represents $A \cap B$ and hence $A$ and $B$ are independent because $\frac{\text{Area $EFCD$}}{\text{Area $ABCD$}} = P(B)P(A) = P(A \cap B)$. We notice also that while the representation of $P(B \mid A)$ as $\frac{\text{Area $EFCD$}}{\text{Area $ABCD$}}$ is correct, it does not make, explicitly, the connection to $P(A \cap B)$. Since Mr. Kantor was not very explicit about why the region $EFCD$ represents $P(A \cap B)$ I asked him and he said "I just multiplied probability of $A$ times probability of $B$" (Figure 5, Table J.1). Rather that helping him to make the mental connection that I was thinking he relabeled $DE$ as $P(B \mid A)$ (See Figure 6, Table J.1) and said: "because we are looking at the probability of $B$. We are only looking at this section [region $EFCD$]. It's the probability of $B$ given that $A$ has already occurred." It seems that he knows that we can take a fraction of $AD$ as representing $P(B \mid A)$. In the case of independent events $DE$ represents both $P(B)$ and $P(B \mid A)$ but it seems that this was not very clear to Mr. Kantor. As we saw before, Mr. Kantor said that $P(B \mid A)$ can also be represented as the comparison of area $EFCD$ to area $ABCD$, but he seemed to face the same kind of difficulty mentioned above: the region $EFCD$ represents both the event $B \mid A$ and $A \cap B$. Mr. Kantor did not construct the connection, and I asked him if he wanted to think more about the representation, and he said "Not now." When I asked Mr. Kantor another
pictorial representation for the definition of conditional probability \( P(A \mid B) = \frac{\text{Area } (A \cap B)}{\text{Area } (B)} \) he attempted to construct one for \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} \). After one failed attempt he constructed an appropriate representation for both \( P(A \cap B) \) and \( P(B \mid A) \) using areas for each probability. Figure J.3 illustrates the pictorial representations constructed during the third time. Mr. Kantor's thinking during his first attempt (see Figure J.3a) was as follows:

That area is the probability of \( B \) [ABCD].... [The] probability of \( B \) I shade it this way. The probability of \( A \) [Area EIHD] I shade it this way.... This has a total area of one.... So this probability of \( B \) is a length [AB] and it is an area since it has a unit width... This is the probability of \( A \) [DE] as well as that area; so then \( A \) given \( B \) is double crossed [Area of EFCD], and it is this, its length times its width. Wait a minute, that's not right. That's right. Mmh, OK. What are we doing wrong? ... Oh. I see. Yeah, that's, this [Area of EFCD] is the probability of \( A \) and \( B \). That's not the probability of \( A \) given \( B \).... [P(A) and P(B) are the areas and the lengths because the figure is a square of area 1].... I don't think that works. (I, 33, 08-25-95)
It seems to me that Mr. Kantor had a cognitive conflict about the region EFCD representing \( P(A \mid B) \) and \( P(A \cap B) \). This was because he is taking the regions as probabilities. If we take the regions as events, then there is no conflict. The region EFCD represent both events \( A \mid B \) and \( A \cap B \). Again, he was close to constructing a representation illustrating both \( P(A \mid B) \) and \( P(A \cap B) \). We then continued working with representations about probability. It was when we were working with finding a story-problem representation for \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} \) that Mr. Kantor had an insight on the pictorial representation for that formula (Figure J.3b). His thinking was as follows:

\[
P(J \text{ and } H) = \frac{P(J \text{ and } H)}{P(H)} \cdot \text{See corresponding story-problem Appendix F, Table F1, cell 30}
\]

If I change it, probability of a jack given that you have a heart.... Yeah, this is where I got mixed up. This, that area [EFCD] is the probability of A given B when compared to this [area ABCD].... Yes. So this is the probability of A given B when compared to anything that's shaded [that way] ... and it is this \( P(A \cap B) \) when compared to total area. Total area is 1. (l, 33, 08-25-95)

While Mr. Kantor did not know a pictorial representation about conditional probability, he was able to construct a pictorial representation for a special case because he probably had some good implicit knowledge about probability and conditional probability in particular. The special case was \( P(A \mid B) = \frac{\text{Area EFCD}}{\text{Area ABCD}} \) which leads to

\[
\frac{\text{Area EFCD}}{\text{Total area}} = \frac{P(A \cap B)}{P(B)}.
\]

For these reasons (special case and difficulties in constructing the pictorial representation), I judged his representations for \( P(B \mid A) = \frac{P(A \cap B)}{P(A)} \) as partially correct.
APPENDIX K: Excerpt for $\frac{3}{4}$, $\frac{3}{5}$, and $4 \times 5$
The representation of $\frac{3}{4}$, $\frac{3}{5}$, and $4 \times 5$

1. K: The length of the side of the square is one unit. Therefore the area of the square is.
2. S: One square unit.
3. K: That makes sense? The size is one unit and it’s a square. Its area is side times side. That gives you one square unit…. This vertical side here is A to E…. Divide that side into four equal lengths…. You make three cuts at certain points and label those B, C, D…. Draw horizontal lines through B, C, D and entirely through the square. This will cut the square into four rectangles of the same size. Marian, the area of each rectangle is how much? [silent]…. If this square is one square unit and we cut it into four equal parts. The area of one of the rectangles is
4. S: One fourth.
5. K: What’s the length from B to E?
7. K: Three fourths. A to E is one, each of these are equal parts. That will be three fourths.
8. K: Take the bottom side and cut that into five equal parts…. Each of these of the same length and label them F, G, H, I, J. Draw vertical lines going through the square…. How many rectangles have you created?
10. K: Twenty rectangles. Four by five…. What’s the area of each rectangle? [Silent] If the square is one square unit [and] you cut it into twenty equal parts, what’s the area of one of those? One twentieth. [He helps some students that are behind and kind of confused, other students are laughing, others talking]. OK. Let’s go on…. What’s the length for EH?
12. S: Why?
13. K: The whole side is one. That’s a fifth, a fifth, a fifth, a fifth, a fifth. We have one, two, three fifths…. We cut the line into five equal parts. Each of those is a fifth. We go one, two, three fifths from E to H. (CO, 1, 10-13-94)
APPENDIX L: Mr. Kantor's pedagogical events about why $21 + \frac{3}{4} = 21 \cdot \frac{4}{3}$
Mr. Kantor's pedagogical events to illustrate why $21 + \frac{3}{4} = 21 + \frac{4}{3}$

1. K: Now, let me give you another example. Twenty one divided by three fourths $21 + \frac{3}{4}$, and let's see that this is the situation ... I wanna take some of my students out to lunch and I know that I can buy a pop per seventy five cents per person, seventy five cents a can. I've got twenty one bucks so I have to figure out how many people I can buy pop. That gives me the answer, doesn't it? Tells me how often seventy five cents goes into twenty one dollars, right? How do you do that problem? ... Can a first grader do this problem?

2. S: With a calculator
3. K: Can a first grader do that problem without a calculator?
4. S: Yeah
5. K: How? ...
6. S: Twenty one times one third, twenty one times four thirds
7. K: ... The first grader goes twenty one times four over three?
8. S: I don't think so ...
9. A: Three on the bottom divided by twenty one.
10. S: Twenty one divided by three [fourths] is the same thing as twenty times
11. K: The first grader doesn't understand division.
12. S: Yeah [students talking at the same time]
13. K: Let me make this clear.... A first grade can get close to the answer by doing this. A first grader knows that seventy five cents is less than a dollar, all right? ... If you ask a first grader how many cans of pop can you buy, first grader knows you can buy more than how many?
15. K; Twenty one [21]. Now, if you buy twenty one cans of pop, how much money do you have left over? ... The kid knows he got twenty one quarters left over, it's the same thing. So what you have left over? You bought your twenty one cans. You have twenty one quarters left over, all right?. Now, you got twenty one quarters, how do you figure out how many cans of pop you can buy now? ...
17. K: I'm saying can a first grader do it?
18. K: Let's put it this way. What a first grader will not do? A first grader will not give up because they want the cans of pop, all right? They know it's three quarters in the machine and they get twenty one quarters, what are they gonna do? [Students talking at the same time]
19. S: Twenty eight.
20. K: How many times do three quarters go into twenty one quarters? [Students talking]
21. S: You just put them into groups
22. K: In groups of three. And they come up with seven groups.... So look, listen ... we start with twenty one dollars and end up with twenty one plus seven. You end up with twenty eight cans of pop. They can do that division problem $21 + \frac{3}{4}$ without dividing. But you have a calculator you can do it by dividing, you can take twenty one and divide it by point seven five.
23. S: And get twenty eight.
24. K: But now tell me this. We know by the algebraic definition of division that that's the same thing $21 + \frac{3}{4} = 21 + \frac{4}{3}$. How do we know that from a concrete standpoint that that's the same thing?
25. S: You can do it in the calculator. You get the same thing.
26. S: You can try it, it works.
27. K: With a concrete example.
28. S: I just told you. You can do it.
29. S: You like to make thing more complicated than they are Mr. Kantor.
30. S: Right [Students talking at the same time]
31. K: ... There is a way you can figure out, that you can look at this problem \( \frac{21 + \frac{3}{4}}{4} \) and you can say that's the same thing as taking twenty one divided by three fourths [Students talking. They don't see the point]

32. S: Oh, boy.

33. K: I want everyone to listen because I do want you guys to tells us how to do it. What we did we take twenty one dollars divided by seventy five cents ... you can buy twenty eight cans of pop, seventy five cents a piece. I'm saying to do the same thing as twenty one times four thirds. You can conceptualize. I want you to understand that concretely. [You don't have just] to think abstractly. Dividing by a fraction is the same thing as multiplying by its reciprocal. You know in this case that this \( \frac{21}{\frac{4}{3}} \) will give you the same answer, why?

34. S: You can do it and it works. What else do you want? [students discussing that. They don't see the point]

35. S: ... because division is multiplication.

36. K: ... I want you guys to tell me, what does this represent in concrete? What does that represent?

37. S: Twenty one times four.

38. K: Twenty one times four quarters so what are you doing? You take the twenty one dollars, what are you doing?


40. K: To me the easiest way to do this problem.... I take twenty one dollars, I go to the bank and I say give me twenty one dollars worth in quarters. That's what that is, eighty four quarters and then what's this \( \frac{84}{3} \)? I put it into groups of three and my first graders can do that. I end up with twenty eight groups of three \( \frac{84}{3} = 28 \), all right? Look, it's important that you understand these mean the same thing. [This is an example where they work. (CO, 12, 11-01-94)]

This time Mr. Kantor does not ask students to construct a story-problem whose solution is represented by 21 divided by three fourths. He constructs the representation: "I wanna take some of my students out to lunch and I know that I can buy a pop per seventy five cents ... I get twenty one bucks so I have to figure out how many people I can buy pop" and then he asks students how they would do the problem (speech 1). A student says using a calculator (speech 2). Speeches 3 through 13 show Mr. Kantor's efforts to make students try to solve the problem without algorithmic computation. Speeches 13 through 22 describe the interchange between Mr. Kantor and his students for solving that problem without algorithmic computation in the following way: with 21 dollars they can get 21 cans if each one costs 75 cents and having 21 quarters left with which they can get 7 more cans buying then 28 cans of pop in total. That way is used to illustrate how to get the answer to 21 divided by 3/4 without carrying out the computation. In speech 24 he comes
back to the point of the lesson and asks students why $21 + \frac{3}{4} = 21 \cdot \frac{4}{3}$. A student says that we get the same thing if we do it in the calculator thus showing that s/he does not understand the meaning of the question even after having seen two representations before (speech 25). Mr. Kantor insists on trying to make students understand his question but a student replies "I just told you. You can do it" and another says "you like to make things more complicated than they are Mr." and still another student agrees with his/her fellows by saying "right" (speeches 28-30). When Mr. Kantor says that there is another way to look at the division problem and see that it is the same thing, a student just exclaims "oh boy" (speech 32). Mr. Kantor asks again why $21 + \frac{3}{4} = 21 \cdot \frac{4}{3}$ and a student says "you can do it and it works. What else do you want?" (speech 34). Another student says "because division is multiplication" (speech 35). Then Mr. Kantor constructs the following explanation to show why $21 + \frac{3}{4} = 21 \cdot \frac{4}{3}$. "I take twenty one dollars, I go to the bank and I say give me twenty one dollars worth in quarters. That's what that is, eighty four quarters and then what's this $\left\lfloor \frac{84}{3} \right\rfloor$? I put it into groups of three ... I end up with twenty eight groups of three $\left\lfloor \frac{84}{3} = 28 \right\rfloor$, all right? Look, it's important that you understand these mean the same thing. [This is an] example where they work" (speech 40). We notice that his explanation illustrates directly why $21 + \frac{3}{4} = \frac{21 \cdot 4}{3}$, not why $21 + \frac{3}{4} = 21 \cdot \frac{4}{3}$. 
APPENDIX M: Other explanations for teaching how to solve $ax = b$
Excerpt M.1: Explanation for solving $\frac{3}{4}B = 15$

That's an equation, what operation? Addition, subtraction, multiplication, or division? … If I apply the inverse, what do I do? … Multiply both sides by four thirds \( \left( \frac{\frac{4}{3}}{\frac{3}{4}} \right) \). Apply the inverse which is the reciprocal. I get B equals twenty. I can do the same thing by doing the related operation. What's the related operation? … What do I divide both sides by? … Three fourths…. Here are two different types [of multiplication equations]. This one has a whole number \([40R = 600]\). This one has a fraction \(\frac{3}{4}B = 15\)…. When it's a whole number, would you prefer to multiply by the reciprocal or divide? … Divide…. If it's a fraction, what would you prefer to do? Multiply by the reciprocal or divide? … Multiply by the reciprocal…. Here \([40R = 600]\), divide both sides by forty but here, \(\frac{3}{4}B = 15\), I don't want to divide by three fourths. That's more complicated. I much rather take this and multiply both sides by its reciprocal…. Dividing by a fraction is the same thing as multiplying by the reciprocal…. So we are gonna do it anyway if you divide by. It just takes an extra step. (CO, 3, 10-17-94)
What formula comes to mind when you talk about volume of a box? ... Base times height, what other formula? ... Volume equals length times width times height \([V = lwh]\), and see how many of those quantities you already know.... You know the volume is gonna be 500 cubic centimeters \([500 \text{ cm}^3]\).... Carry units through the problem.... The length and width are twelve point five centimeters and five centimeters and the height is the unknown \([500 \text{ cm}^3 = (12.5 \text{ cm})(5 \text{ cm})h]\). One equation, one unknown, you should be able to solve for that unknown. Probably the best thing is to multiply these together, sixty two point five centimeters square \([62.5 \text{ cm}^2]h\). What does that represent by the way? ... The area of the base.... How do we find the height? ... Divide both sides by this \([62.5 \text{ cm}^2]\). Use related facts, that divided by this has to be equal \(h\). Multiply both sides by the reciprocal.... If you multiply both sides by the reciprocal of this, you get this, \(\frac{500 \text{ cm}^3}{62.5 \text{ cm}^2}\). Carry units through.... Because you get centimeters times centimeters times centimeters on the top.... On the bottom centimeters times centimeters cancels out two of these. Five hundred divided by this is eight.... that's the height, \(\frac{500 \text{ cm}^3}{62.5 \text{ cm}^2} = 8 \text{ cm} = h\).
APPENDIX N: The teaching of finding the probability of getting two diamonds when selecting two cards
The teaching of finding the probability of getting two diamonds when selecting two cards

1. K: Some people still have a hard time with the concept given, what that really means. Let me show you one of those. Probability that you pull a diamond out of the first draw and a diamond out of the second draw [pointing to \( P(D_1 \text{ and } D_2) \)], and remember, that's the probability to pull a diamond out of the first draw times the probability you pull a diamond out of the second draw given you pull one out of the first [pointing to \( P(D_1 \text{ and } D_2) = P(D_1) \cdot P(D_2 \text{ given } D_1) \)].

2. S: What does that mean, though? [Students are discussing the meaning of given]

3. K: OK. Let me explain this again. Let me show you what these numbers are. The probability of pulling out in the first draw is this, there are thirteen diamonds in the fifty two cards, all right? \( \frac{13}{52} \). Do you agree that once you pull a diamond out, the next time you have less chance of pulling a diamond out than the first? [Some students say yes, others no]. There are twelve diamonds in there and fifty one cards \( \frac{12}{51} \). The point is this, listen, listen, this probability means, this probability means the probability you get a diamond on the second draw. That means something has already happened, you have to account for that. If it's the second draw, there has already been a first draw, it hasn't been put back yet; so this probability depends on that there are only fifty one cards in there and there is only twelve diamonds in there.

To explain the meaning of conditional probability using the verbal representation "given" Mr. Kantor selected the situation of drawing two cards out of a deck of 52. The first card is not put back before drawing the second card. He wants to represent \( P(D_1 \text{ and } D_2) \) as \( P(D_1) \cdot P(D_2 \mid D_1) \) where \( D_1 \) represents the event "to get a diamond on the first draw", and \( D_2 \) represents the event "to obtain a diamond on the second draw" to explain the meaning of \( P(D_2 \mid D_1) \). First, he gives the verbal representation for \( P(D_1 \text{ and } D_2) \) "that's the probability to pull a diamond out of the first draw times the probability you pull a diamond out of the second draw given you pull one out of the first" (Speech 1). A student asks what given means (Speech 2) and Mr. Kantor replies "The probability of pulling out in the first draw is this, there are thirteen diamonds in the fifty two cards, all right? \( \frac{13}{52} \). There are twelve diamonds in there and fifty one cards \( \frac{12}{51} \). This probability means the probability you get a diamond on the second draw. That means something has already happened, you have to account for that. If it's the
second draw there has already been a first draw, it hasn't been put back yet; so this probability depends on that there are only fifty one cards in there and there is only twelve diamonds in there, OK?" (speech 3).
APPENDIX O: Other pedagogical events constructed by Mr. Kantor to teach the rate model for multiplication
Problem 2. Use the reciprocal of the rates to do the multiplication

\[
\frac{6 \text{ dollars}}{\text{lb}} \cdot \frac{30 \text{ shrimp}}{\text{lb}} = \frac{\text{shrimp}}{\text{dollar}}
\]

Mr. Kantor's Explanation. Six dollars per pound is the same as one pound per six dollars

\[
\left[ \frac{\text{lb}}{6 \text{ dollars}} \right] \cdot \frac{30 \text{ shrimp}}{\text{lb}} = 5 \frac{\text{shrimp}}{\text{dollar}}.
\]

Now, if you multiply that out, you get 30 divided by six which is five, and the units, now the top and the bottom and cancel those out. The units carried through the problem. You can reduce a rate pound just as any other multiplication or fraction problem.... You are left with a number which is five, and shrimp per dollar that you carry through the problem. Now, she got the right answer [the students are having trouble in understanding the problem because of the way was presented in the book (the first representation)]. Six is in the denominator, it is one pound per six dollars.... They [the authors of the book] want you to look at the problem and realize that your units wouldn't make any sense. They want you to think about the reciprocal.... Now, this is kind of hard for people to understand. Six dollars over pound and when I take the reciprocal pound over six dollars. I can rewrite this way one sixth pound per dollar. Do you understand that can be done? (CO, 2, 10-14-94)

Problem 3. Suppose a laser printer prints 5 pages per minute. How long will it take to print 2400 documents with 3 pages per document?

Mr. Kantor's explanation. Everything is correct. The only problem I have is that, it comes out without units at the end.... Let me show you that solution. First of all, you start with 2400 documents. Then what you are going to do is to convert that to pages, and after [that] to change that to how much time it takes to type them. The way I do this is by using rates. If I multiply that by three pages per document what happens? Documents cancel out in the top and bottom. I end up with 7200 pages. It's exactly the same thing I started with, but in a different form. 2400 documents is 7200 pages—same amount of work. I just did a conversion. This fraction has a value of one. Three pages is the same thing as a document. When I multiply this by one, it doesn't get any bigger. It stays the same size. I just converted from documents to pages.... When you multiply that by a rate, it doesn't get any bigger.... It carries the units through. I did exactly the same thing that Anthony did, but I carried the units through the problem. Then I multiply this by. I want to end up with time. So I want to get rid of this. So I want pages down here and five minutes up here. Pages cancel out and when I multiply by that I get fourteen, oh wrong way. One minute per five pages. Then I divided by five and I get fourteen forty minutes.... Really, all I've done, I took this and multiplied it by one in this form. It's not getting any bigger. 2400 documents is the same thing as 7200 pages which is the same thing as 1440 minutes of work.... Do you have any questions on that? The process makes some sense? (CO, 2, 10-14-94)

Explanation 4. Units are part of your problem. When you do area, you do length times the width. If the units are feet and feet, you end up with feet square. But you multiply the units together.... No, no. If you are comparing a giraffe 18 feet to a, let's say, a person six feet tall \( \frac{18 \text{ ft}}{6 \text{ ft}} = 3 \), in that case the feet does cancel out. The giraffe is three times as big as the person; but if you are doing area such as eighteen feet by six feet you multiply feet together and get square feet \( 18 \text{ ft} \cdot 6 \text{ ft} = \text{ft}^2 \). Depends on what the problem is. When you are in physics you deal with acceleration. A unit for acceleration would be \( \text{ft per second square} \), Velocity is feet per second, area is square units, volume is cubic units. So units are part of your problem.... They are just like variables. In fact, if I have three \( x \) over one times six \( y \) over two \( x \), those can cancel out \( \frac{3x \cdot 6y}{1 \cdot 2x} \). Same thing if I have units in there. [A student asks why?]
Because it is a multiplication of fractions problem.... You tell me this. If you don't agree with this then we have a problem. Is three feet the same as three times a foot? \[3 \text{ ft} = 3 \cdot \text{ ft}\] [yes] So you do have a multiplication problem. That's part of your problem. You just can't get rid of it. How can you just get rid of it? ... Oh, fine, you are considering and then at the end you put down the units. There are formulas in physics that have 12 different variables with twelve different units. You cannot go through and realize at the end what the units would be, especially if you are doing conversions in the meantime.... You have to work through this way. For example, let's say we have a problem of acceleration. A person is accelerating at four feet per second square and the person says what is the acceleration in inches per hour square?

\[
\left[\frac{4 \text{ ft}}{	ext{sec}^2}\right] \quad \text{You want to end up with inches per hour square} \quad [\text{students are talking at the same time. They are proposing to solve the problem using multiplication and division}] \quad (\text{CO, 2, 10-14-94})
\]

**Problem 5.** Convert \[
\frac{4 \text{ ft}}{	ext{sec}^2} \] into \[
\frac{\text{in}}{	ext{hr}^2}
\]

**Mr. Kantor's explanation.** You see how can you get rid of all those units? \[
\left[\frac{4 \text{ ft}}{	ext{sec}^2}\right]
\]
I want to end up with inches \[
\left[\frac{\text{in}}{	ext{hr}^2}\right]
\]
So what conversion do I use so I can put inches on the problem? ...

Twelve inches over one foot \[
\left[\frac{4 \text{ ft}}{\text{sec}^2} \right] \quad \frac{12\text{in}}{1\text{ft}} \quad \frac{\text{in}}{\text{hr}^2}
\]
Look, I'm getting rid of the feet, but I got inches in my problem because of what I want to end up with [in], do you agree with that? Is that true? ... I started with seconds square and want to end up with hours square, what I have to eventually do? ... There are these many [3600] seconds in an hour. OK? Now, I got seconds square here, I have these many seconds to get an hour \[
\left[\frac{4 \text{ ft}}{\text{sec}^2} \right] \quad \frac{12\text{in}}{1\text{ft}} \quad \frac{3600\text{sec}}{1\text{hr}} \quad \frac{\text{in}}{\text{hr}^2},
\]
so I take one of those but I need to take another one of those, too. \[
\left[\frac{4 \text{ ft}}{\text{sec}^2} \right] \quad \frac{12\text{in}}{1\text{ft}} \quad \frac{3600\text{sec}}{1\text{hr}} \quad \frac{3600\text{sec}}{1\text{hr}} = \frac{\text{in}}{\text{hr}^2}
\]
I am not saying that you have to be able to do problems that are this complicated but it shows you why we want you to be able to use rates because all I did was to convert this, feet per second square into inches per hour square. [It seems to me that you are making it more complicated that it needs to be.] Not if you really care what this acceleration is in inches per hour square.... Usually the problems are so simple that you can figure it out on your own but if it is this complicated you can't do that [the whole process] on your own. You saw how you tried to do it, you were gonna do it by sixty square, that's not it. Thirty six hundred square. It's a huge number inches per hour square. (CO, 2, 10-14-94)
APPENDIX P: Other explanations about how to solve inequalities of the form $ax < b$
Explanation for how to solve \(-12 < 48X\)

Now, how do I solve that problem? ... Multiply both sides by what? What happens to this \([\text{the sign}]\)
\[
\begin{bmatrix}
-12 < 48 \cdot X \\
\frac{1}{48}
\end{bmatrix}
\]
... I multiplied both sides by a positive number so the sense does not change:

the sense does not change; the sense does not change [pointing to the sign]. [The sense only switches]
When this \([1/48]\) is negative.... Always when it's negative.... And here is negative twelve, multiply ... negative point two five is less than \(X\). \(X\) is greater than negative point two five and again what's an easy way to use to check? ... Zero. Everybody sees that zero is a solution of this \([-25 < X]\). Plug it in here \([-12 < 48 \cdot X]\). I get zero is greater than negative twelve. It works, it checks. (CO, 5, 10-19-94)

Explanation for how to solve \(-10u < 0\)

Look at this problem.... Think what the answer is. \(u\) is what? ... That's the answer \(u > 0\) because, how do you solve this? \([-10u < 0]\).... How do you undo this? \([-10u < 0]\).... Not add.... When you multiply by a negative number, what happens to the sense? ... And it is trivial this times this is zero. This times

this is \(u\), \(u\) is greater than zero

\[
\begin{bmatrix}
\frac{-1}{10} < 0 \\
\frac{-1}{10}
\end{bmatrix}
\]

It seems obvious but the method it does \(u > 0\)

works with this, too.... To get rid of that \([-10]\), look, multiplication. I undo multiplication by ... and I check it. Remember, here is my original problem.... I check it, try something bigger than zero. I try ten. Negative one hundred is less than zero \([-10u < 0]\). It works. (CO, 5, 10-19-94)

Explanation for how to solve \(10−5a < 10\)

Tim, how do you do that? ... Tim, what is my first step to do it? When we talked about general ways of solving equations and inequalities, our overall strategy is always to do what? ... Before we do something to both sides we do simplify each side. Negative fifty \(a\) is less than ten \([-50a < 10]\) and then what we do? ...

Multiply both sides by negative one over fifty.... What else is in parenthesis? It's a negative and when I do that, Jeff, what happens? ... Change the sense

\[
\begin{bmatrix}
\frac{-1}{50} < 0 \\
\frac{1}{50}
\end{bmatrix}
\]

It was less than. Now is greater than \([Is always gonna change?]\). If I multiply by a negative number. I multiplied by a negative number.... This is \(a\) is greater than negative, I think point two \([a > -2]\). (CO, 5, 10-19-94)

Explanation for how to solve \(5X \geq 10\)

Multiply both sides by one fifth. You get \(X\) is greater than or equal to two \([X \geq 2]\). How do I check that answer? ... Say ten. Does ten work? Five times ten is fifty, that's greater than or equal to ten.... Check something that's inside of an inequality, inside the shaded area and also check the end point. Martin, does the end point work? (CO, 6, 10-20-94)

Explanation for how to solve \(-2 \leq -n\)

What do I do to solve this one \([-2 \leq -n]\)? You want to find out what \(n\) is. What's the coefficient of this? ... This is negative one times \(n\) \([-1 \cdot n]\). So the coefficient is negative one, what's the reciprocal of negative one? ... Negative one. What do I multiply both sides by? ... You multiply both sides by the reciprocal of
this. It is its own reciprocal. It's like one and one are reciprocals. Negative one and negative one are reciprocals ... So I multiply both sides by negative one and I get two over here. What about the inequality between them? Does that make sense, that it switches? \([2 \geq n]\). OK. ... Another way you can look at this, if this makes it less confusing. If I graph the opposite of \(n\). Graph all things that make that true. Everything that's greater than or equal to negative two. Now, the opposite of that is what \(n\) is. Here is zero. The opposite switches everything over. Now I have positive two. Everything changes to this way. (See figure). What's the opposite of this? [how is this related to that?] (CO, 6, 10-20-94)

\[n\]

\[-n\]

\[2 \geq n\]

\[0\]

\[-2\]

[A student asked Mr. Kantor to explain the process again] The opposite of \(n\) is greater than or equal to negative two, correct? The opposite of \(n\) ... everything up this way, correct? Now, when I graph what \(n\) is. \(n\) is the opposite of this \([-n]\), correct? \(n\) is the opposite of that. Now, how do I find what the opposite of this number \([-2]\) is? How do I find it using a compass? ... It's on the other side of zero. I just draw an arc [he draws an arc with center at zero and radius 2] and that's where the opposite of this number is. That's where two is. It's on the opposite side, OK? Here is negative one. I wanna find the opposite of this. Same thing. I draw an arc from zero to there. It's on the other side. It's where one is. So that's the opposite of this one [one is the opposite of \(-1\)]. The opposite of this one \([-2]\) goes over here. So all these numbers. I put this one over. I put this one over (see figure). Well, take something that's over here, let's say this is number five, where is the opposite of that? ... On the other side. Draw an arc to the other way. You flip it over, OK? you flip it over. Now, it's just like this. What did I start with? What I started with was this. Everything.... When I flip this, what happens? ... they all flip like this ... around zero and all flips over. So now I have \(n\) being less than or equal to two, \([2 \geq n]\) you see that? (CO, 6, 10-20-94)

\[n\]

\[-n\]

\[2 \geq n\]

\[0\]

\[-2\]

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APPENDIX Q: Mr. Kantor's questions
<table>
<thead>
<tr>
<th>Content curriculum event</th>
<th>Procedural questions</th>
<th>Conceptual questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. Rule for multiplying fractions (CO, 1, 10-13-94)</td>
<td>The length of the side of the square is one unit. Therefore the area of the square is:</td>
<td></td>
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<tr>
<td></td>
<td>The area of each rectangle is how much?</td>
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<tr>
<td></td>
<td>What's the length from B to E?</td>
<td></td>
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<td></td>
<td>How many rectangles have you created?</td>
<td></td>
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<tr>
<td></td>
<td>What's the area of each rectangle?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>What's the length for EH?</td>
<td></td>
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<tr>
<td></td>
<td>Each of those has an area of what?</td>
<td></td>
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<tr>
<td></td>
<td>We know that another way to find the area of a rectangle is what?</td>
<td></td>
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<tr>
<td></td>
<td>If you cut something into fourths ... and then cut those into fifths, what do you end up with?</td>
<td></td>
</tr>
</tbody>
</table>
| 7. Rate model for multiplication (CO, 1, 10-13-94) & (CO, 2, 10-14-94) | What's fifty five times 12?                                                           | Rates can be multiplied by other quantities. Here is a rate \[
|                          | Does that [660 mi] look bigger than this? [12 hr]                                      | \[
|                          | If I multiply that by three pages per document what happens? Documents cancel out in the top and bottom. | [55 \text{ mi} \over 12 \text{ hr}], and I'm going to \[
|                          | If you don't agree with this then we have a problem. Is three feet the same as three times a foot? \[3 \text{ ft} = 3 \cdot 12 \text{ ft}]. | multiply it by another quantity \[
|                          | How can you just get rid of it? ... Oh, fine, you are considering and then at the end you put down the units. | [12 \text{ hr}]. Who says you can do that? |
|                          | A person is accelerating at four feet per second square and the person says what is the acceleration in inches per hour square? | What does that mean, rates can be multiplied by other quantities? |

(To be continued)

Table Q.1 Questions asked by Mr. Kantor during teaching of algebraic multiplication

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I want to end up with inches
\[
\frac{4 \text{ ft}}{\text{sec}^2} \frac{\text{in}}{\text{hr}^2}.
\]
So what conversion do I use so I can put inches on the problem?

How can you get rid of all those units?
\[
\frac{4 \text{ ft}}{\text{sec}^2}.
\]

You try to do too much all at once. I started with seconds square and want to end up with hours square, what do I have to eventually do?

12. The multiplicative identity of 1 (CO, 2, 10-14-94)

When you look at the operation of addition, what's the identity?

What do I add to that to end up with the same thing I started with \([A + = A]\)?

Multiplication has an identity also. If I start with this \([A]\), what do I multiply by \([A \cdot = A]\)?

14a. Definition of reciprocal (CO, 2, 10-14-94)

1. What is an inverse?

2. Now, what's the inverse in multiplication?

3. If I take something \([B]\) and make it twice as big \([B \cdot 2]\) what do I multiply that by to get back to where I started from?

14b. The reciprocal of \(a\) is \(1/a\) (CO, 3, 10-17-94)

If that's the case, how do I solve for \(R\)?
\[
\frac{3}{4 - R} = 1
\]

The other way is simply by related facts, what's \(R\)?

(To be continued)
Table Q.1 (Continued)

15. Reciprocal of zero
   (CO, 2, 10-14-94)
   Every number has an opposite but not every number has a reciprocal. What number does not have a reciprocal?

18. Solving \( ax = b \)
   (CO, 3, 10-17-94) &
   (CO, 4, 10-18-94)
   Now, if you look at an equation that involves what operation? \([40R = 600]\)
   What's the identity in multiplication?
   What's the inverse?
   What's the related operation?
   If you apply the inverse to solve this, what do you use? What do you multiply by? Inverse is multiplying by?
   So I can undo the multiplication. I can reverse the process by applying the inverse or I can reverse the process by

   \[
   \text{Solving } \frac{3}{4} = 15
   \]
   That's an equation, what operation? Addition, subtraction, multiplication, or division?
   Now, if I apply the inverse what do I do?
   I can do the same thing by doing the related operation. What's the related operation?
   What do I divide both sides by?
   When it's a whole number, would you prefer to multiply by the reciprocal or divide?
   If it's a fraction, what would you prefer to do? Multiply by the reciprocal or divide?
   What formula comes to mind when you talk about volume of a box?
   Base times height, what other formula?

   (To be continued)
What does that represent by the way?
How do we then find the height?

23. $a > b$
(CO, 5, 10-19-94)
Here is zero, negative three is here, positive four is here. [what's the relationship between those two?]

24. The multiplicative property of inequalities (first part)
(CO, 5, 10-19-94)
Five and seven, what's the relationship between those two numbers? ... Are they equal, less than, greater than?
If I double five and get ten, double seven and get fourteen ... what's the relationship then?

25. The multiplicative property of inequalities (Second part)
(CO, 5, 10-19-94)
Now, if I take and multiply each of these by negative one, negative ten, negative fourteen $[-10 \ -14]$, we get negative ten and negative fourteen, what do I put in between those?

$[-6 < 8]$ When I multiply both sides by negative, multiply both sides by negative... what's the relationship?

If I have an inequality and I multiply both sides by a negative number, what happens?

What happens when we multiply both sides of this $[W > L]$ by a negative number?

L loses a hundred dollars a day. W three days ago had how much money?

Negative three W is less than negative three L $[-3W < -3L]$. What does that mean? Why is this person now in a better situation? What does that mean? ... what does negative three mean?

If you go backwards, though. The loser going back in time [pointing out $-3L$]. He likes to do that.... The winner wouldn't. Why?

(To be continued)
Table Q.1 (Continued)

26. Solving \( ax < b \)  

-5x < 10 (CO, 5, 10-19-94)

1. How do I solve that problem?
2. How do I undo multiplication?
3. Division or multiplying by
4. As soon as I do that what happens to it? 
   [to the sign, he multiplied both sides by \((-1/5)\) ]
5. How would you check this?
6. What's the easiest one to pick, greater than negative two?

-12 < 48X (CO, 5, 10-19-94)

1. How do I solve that problem?
2. Multiply both sides by what?
3. What happens to this [the sign]?
4. What's an easy way to use to check?

-10u < 0 (CO, 5, 10-19-94)

1. Think what the answer is. u is what?
2. How do you solve this [-10u < 0]?
3. How do you, how do you undo this? [-10u < 0]
4. When you multiply by a negative number what happens to the sense?

10 : -5a < 10 (CO, 5, 10-19-94)

1. How do you do that?
2. What is my first step to do it?
3. When we talked about general ways of solving equations and inequalities, our overall strategy is always to do what?
4. [-50a < 10 ] and then what we do?
5. What else is in parenthesis? It's a negative and when I do that, James, what happens?

5X ≥ 10 (CO, 6, 10-20-94)

1. [X ≥ 2]. How do I check that answer?
2. Say ten. Does ten work?
3. Anton, does the end point work?

(To be continued)
1. What do I do to solve this one \[-2 \leq -n\]?
2. What's the coefficient of this? [pointing negative \(n\)]
3. What's the reciprocal of negative one?
4. What do I multiply both sides by?
5. What about the inequality between them?
6. Does that make sense, that it switches? \([2 \geq n]\)
7. What's the opposite of this?

27. Multiplication counting principle

To figure out how many different schedules are possible.... For example, if you simply look at English and music, English and music, how many different schedules are possible?

What are the five?

So, think about all the other option you have. How many different schedules are possible?

Emma, let's talk about the total number.

And next we can look at, global language

\[
\begin{array}{ccc}
1 & 4 & 3 \\
\text{English} & \text{music} & \text{math GL} \\
\end{array}
\]

how many branches?

28. Classical definition of conditional probability

You are looking at only those people that take Spanish and the percentage of time those people branch up, so what do these percentages add up to?

There are only 55 people that get to this point. How do those people branch up?

Those 55 people, how many even go in each of these paths?

And, again, they have to add up to a hundred percent, right?
Now the percentage of people taking choir.

Music? ... you got music for us?
(CO, 8, 10-25-94)

Now all the blondes, how many of you
run track because that's what I'm trying to
find....

Do you agree that once you pull a diamond
out, the next time you have less chance of
pulling a diamond out than the first?
(CO, 10, 10-27-94)

30. Conditional
probability formula
(CO, 8, 10-25-94) &
(CO, 9, 10-26-94)

I wanna know what percentage of people
takes Spanish and Choir. How do you
figure that out?

How do you figure it out Emma?

How much of the time you draw two
hearts, if you look at that overhead?

31. n!
(CO, 9, 10-26-94)

Out of a standard deck of playing cards
how many different poker hands are
possible? ... How many different hands
could you get? How would you calculate?

It gives you a hand. A hearts, two
diamonds, five spades, three diamonds and
J hearts. In how many different ways can
that hand be arranged in a row from left to
right?

Anna, ten factorial [10!], what do you get?
... Ten factorial?

\[
\binom{10}{7} = \frac{10!}{7! \cdot 3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 720
\]

I can reduce before I multiply, cross out all
these at a time. So you end up with....

(To be continued)
Table Q.1 (Continued)

How do you do one hundred and twenty one factorial divided by one hundred and nineteen factorial?

Does everyone see that if you write all these down (121!), you write all these down (119!) you cancel everything but that [121 \cdot 120]?

Is a hundred and twenty one factorial the same thing as a hundred and twenty one times a hundred and twenty times a hundred and nineteen factorial?

A hundred and twenty one times a hundred and twenty what is it?

33. Permutation theorem

In softball, there are 10 people who can bat. In how many ways can the manager of a softball team arrange the batting order

How many people can bat first?

Once you pick up a leader up there, how many can bat second?

Why?

Why is that we do it that way?

There is ten people on the line.

Why is that ten factorial represents the number of different

34. Meaning of division

You have five dollars, pencils is fifty cents a piece, how many times does fifty cents go into five dollars?

I wanna take some of my students out to lunch and I know that I can buy a pop per seventy five cents per person, seventy five cents a can. I've got twenty one bucks so I have to figure out how many people I can buy pop. That gives me the answer, doesn't it?

Can someone give me a concrete example of seven divided by two, a situation in which seven divided by two gives you the answer?

Give me a problem, make up a problem where five divided by one half represents that problem. Five divided by one half.

Give me a story problem where I can express that, directly as five divided by one half.

(To be continued)
Tells me how often seventy five cents goes into twenty one dollars, right? How do you do that problem?

Can a first grader do this problem? Can a first grader do that problem without a calculator?

Let me give you a start. Let's say that one half represents a half a dollar. Give me a story problem with five divided by one half represents that problem.

That will be a half divided by five. I want five divided by a half. I want a situation where I'm trying to figure out how many times one half goes into five.

What's a problem that talks about a half going into five, Jacinta? What would be a problem? ... A problem?

How? [a first grader can do the problem without a calculator]

What's the algebraic definition of division?

Now, if a is seven and b is two. Seven divided by two is the same thing as what?

Can you get the same answer taking seven times one half?

If you do seven times one half, what happens?

When you were at elementary school you understood that dividing by two is the same as taking half of something, right?

I have to go the bank and get fifty cent pieces for five dollar bills. For every dollar bill they give what?

Doesn't it make sense that you get the same answer as five times two? ... Why?

Using the same situation explain why.

(To be continued)
Table Q.1 (Continued)

<table>
<thead>
<tr>
<th>Question</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Now, in the context of that problem why does five times two give me the answer?</td>
<td>I can find out how many times it goes into five. The process of doing that, how can I do it? [multiplying five times two]</td>
</tr>
<tr>
<td>If you ask a first grader how many cans of pop can you buy, first grader knows you can buy more than how many?</td>
<td>Now, you got twenty one quarters. how do you figure out how many cans of pop you can buy now? [without dividing]</td>
</tr>
<tr>
<td>Now, if you buy twenty one cans of pop, how much money do you have left over?</td>
<td>We know by the algebraic definition of division that that's the same thing</td>
</tr>
<tr>
<td>I'm saying can a first grader do it?</td>
<td>$\frac{21 \cdot \frac{3}{4}}{\frac{4}{3}}$ How do we know that from a concrete standpoint that that's the same thing?</td>
</tr>
<tr>
<td>They know it's three quarters in the machine and they get twenty one quarters, what are they gonna do?</td>
<td>With a concrete example. Dividing by a fraction is the same thing as multiplying by its reciprocal. You know in this case that this $\frac{21 \cdot \frac{4}{3}}{3}$ will give you the same answer, why?</td>
</tr>
<tr>
<td>How many times do three quarters go into twenty one quarters?</td>
<td>What does this represent in concrete? What does that represent?</td>
</tr>
<tr>
<td></td>
<td>Twenty one times four quarters so what are you doing? You take the twenty one dollars, what are you doing?</td>
</tr>
</tbody>
</table>

Table Q.1 Questions asked by Mr. Kantor during teaching of algebraic multiplication
Coding schema

I classified Mr. Kantor questions formulated during his teaching of algebraic multiplication as procedural or conceptual. I categorized as procedural those questions that refer either to symbolic representations of mathematical ideas and its syntax or to the rules, algorithms, or procedures needed to solve mathematical problems or questions (Hiebert & Lefevre, 1986). Conceptual questions were questions that asked for an explanations of why a relationship existed or that asked for the construction of a pictorial or story-problem representation whose construction reflects some understanding. Other questions such as "do you understand?," "does that make sense?," "do you understand that can be done?," "do you have any questions?" belong to another category and are not listed.