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STUDYING COSMOLOGICAL STRUCTURE FORMATION WITH NUMERICAL HYDRODYNAMIC SIMULATIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

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This dissertation investigates the numerics and physics involved in incorporating a hydrodynamic, baryonic species into simulations of cosmological structure formation. Such studies are key to fully understanding the process of galaxy formation, and they also offer the chance to make direct, detailed comparisons between observational data and theoretical models of large-scale structure.

I begin by describing a new numerical hydrodynamic technique known as Adaptive Smoothed Particle Hydrodynamics, or ASPH. ASPH is a modified form of Smoothed Particle Hydrodynamics (SPH), designed to allow resolution scales to adapt to locally anisotropic density evolution, thereby maximizing the potential resolution of a simulation. This is accomplished by allowing each ASPH node to sample in locally ellipsoidal, rather than spherical, interpolation volumes. A tensor derivation of ASPH is described, complete with the 2-D and 3-D formalisms, as well as a series of 2-D and selected 3-D test problems. ASPH's improved dynamic range in spatial resolution relative to SPH promises to be most useful for studying the process of large-scale structure formation, as limited dynamical range remains one of the outstanding computational challenges faced by cosmological investigations.
Next, a series of 2-D simulations of an idealized cosmology is presented, incorporating both baryons (10% by mass) and dark matter (90% by mass), varying both the resolution and input perturbations. These models are used to examine both the numerics and the general physical properties of gravitationally driven, hierarchical collapse in a mixed baryonic/dark matter fluid. These tests indicate that, under certain restrictions, such simulations converge with increasing resolution to a consistent solution. The dark matter achieves convergence provided that the relevant scales dominating nonlinear collapse are resolved. If the gas has a minimum temperature (as expected, for example, when intergalactic gas is heated by photoionization due to the ultraviolet background) and the corresponding Jeans mass is resolved, then the baryons also converge. However, if there is no minimum baryonic collapse mass or if this scale is not resolved, then the baryon results err in a systematic fashion. In such a case, as resolution is increased the baryon distribution tends toward a higher density, more tightly bound state. I attribute this to the fact that under hierarchical structure formation on all scales there is always an earlier generation of smaller scale collapses, causing shocks which irreversibly alter the state of the baryon gas. A simulation with finite resolution will always miss such an early generation of collapses below this resolution threshold, unless a physical scale is introduced below which structure formation is suppressed in the baryons. The models also show that the baryon/dark matter ratio follows a characteristic pattern, wherein collapsed structures possess a baryon enriched core (enriched by factors ~ 2 or more over the universal average), which is embedded within a dark matter halo, even without accounting for radiative cooling.
of the gas. The dark matter is unaffected by changing the baryon distribution (at least in the dark matter dominated case investigated here), allowing hydrodynamics to alter the distribution of visible material in the universe from that of the underlying mass.

Next, hierarchical structure formation based on scale free initial conditions is analyzed using a set of 3-D models without radiative cooling, with 5% of the mass density baryonic and 95% dark matter by mass. Three independent simulations based upon identical initial conditions are used, consisting of Gaussian distributed initial density fluctuations that have a power-spectrum $P(k) \propto k^{-1}$. The evolution is expected to be self-similar in time, and under certain restrictions the expected scalings for many properties of the distribution of collapsed objects are identified in all three realizations. The distributions of dark matter masses, baryon masses, and emission weighted temperatures scale quite reliably. However, the central high-density regions of these structures are dominated by resolution effects, and as a result such high-density cores must be excluded before the gas densities and luminosities demonstrate the expected scalings. The temperatures and luminosities of the groups are tightly correlated with the baryon masses, and these relations are well-represented by power-laws. The Press-Schechter (PS) mass distribution predicts the overall group dark matter and baryon masses fairly well, though PS may somewhat overpredict the baryon masses. Combining the PS mass distribution with the measured relations
for $T(M)$ and $L(M)$ predicts the temperature and luminosity distributions well. In general, the three simulations agree well for the properties of resolved groups, where a group is considered resolved if it contains more than 32 particles.

Finally, a set of 2-D scale-free models with radiative cooling is considered. For a variety of initial density perturbation spectra and cooling laws, the dissipated baryon objects obey the expected self-similar scaling relations reasonably well, in fact almost as well as the cases without radiative cooling. I attribute this somewhat surprising finding to the strong resolution dependence of radiative cooling. The radiative volume emissivity goes as $\epsilon \propto \rho^2$, and the gas density $\rho$ is quite sensitive to the numerical resolution. So long as the numerical resolution is inadequate, radiative cooling is almost entirely suppressed and no dissipated gas forms. Once the resolution passes the critical threshold where cooling can occur, the radiative dissipation becomes very effective and the fraction of cooled gas rapidly stabilizes. Additionally, simple scaling arguments reasonably predict the fraction of gas that cools in a collapsed object as a function of the object mass. This suggests that, so long as the assumption that objects of differing masses represent scaled versions of one another holds, the dissipated gas structures will also obey these relations, and analytic theories such as the PS mass function will hold. Overall these results are encouraging both for the use of numerical models to study galaxy formation, and for supporting the use of semi-analytic techniques to study the evolution of such dissipated structures.
Dedicated to Katie, who came along just when I needed her most,
and my friends who helped me get to this point.
There are few people indeed who can tackle a long project such as this alone, and I am no exception. There have been many who helped me along the way, though I'll probably only remember to mention a few here. Certainly I owe much to my original adviser here at OSU, Jens Villumsen. Jens started me out on this work, and believed that I could make it when few others did. Jens taught me a great deal about how science is actually done, and always emphasized understanding and physical reasoning above all else. Jens is an extremely intelligent and creative scientist, and I wish I had had longer to work with him. I also owe a great deal to my current adviser, David Weinberg. David came to OSU and unwittingly inherited me at an advanced stage, a strange position for any new faculty member to walk into. I have certainly learned a great deal from David, who is undoubtedly one of the most knowledgeable people in this field I have ever met. I am very grateful for all of the support and advice David has been able to provide me, and I would have been far worse off without his help. Yet another person I owe much to here at OSU is Rick Pogge, who stepped in to serve as my adviser when Jens left. Rick is an extremely versatile scientist who is always able to provide intelligent advice and direction on almost any subject, a quality which has always impressed me. Additionally, Rick has always been a good friend and mentor. During the year I spent working with Rick, he also taught me
much about how actual astronomical data is acquired, analyzed, and interpreted, and probably had a stronger influence on my computer programming style than any other single person.

I have also been fortunate enough to collaborate with some folks from outside OSU. I am indebted to both Paul Shapiro and Hugo Martel, who originated the concept of ASPH and collaborated with Jens and myself in the pioneering work developing this method. Many of the faculty in the astronomy department were kind enough to allow me to perform extended calculations on their workstations. I have had access to computer time at the Ohio Supercomputer Center, as well as the RZG computing center in Garching, Germany. I was able to spend two months working with Jens at the Max Planck Institut für Astrophysik in Garching, and I am grateful for their hospitality during that time.

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FIELDS OF STUDY

Major Field: Astronomy
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The universe we observe today exhibits a richness of structure, ranging over scales from subgalactic (< 1 kpc) to vast "voids", "filaments", and "sheets" on scales up to perhaps 100's of Mpc (Kirshner et al. 1981; Kirshner et al. 1987; Tully 1987; de Lapparent et al. 1988; Geller & Huchra 1989; Broadhurst et al. 1990). Galaxies, with an average interstellar density $\rho_{\text{ism}} \sim 10^{-24}$ gm/cm$^3$, represent a density enhancement of $\delta \rho_{\text{ism}} / \rho \sim 10^6$. This current situation is in marked contrast with the primordial structure at redshifts $z \sim 1000$, where the highly isotropic nature of the Cosmic Microwave Background Radiation (CMBR) indicates the density contrast is limited to $\delta \rho / \rho < 10^{-5}$ (Smoot et al. 1992). The prevalent theories to account for the presence and dramatic evolution of cosmological structure generically rely upon gravitational amplification of such small perturbations in the nearly homogeneous initial density distribution (Peebles 1980, 1993). Unfortunately, the inherent nonlinearity of gravitational collapse scenarios makes this a nearly intractable problem analytically. Investigators are therefore forced to use either numerical models and/or various analytic assumptions and approximations in order to follow the evolution of cosmological
structure from simple initial conditions. In either case, studying the evolution of structure in the universe is an initial-value problem, wherein the investigator evolves forward in time an initial density perturbation spectrum under a chosen cosmology.

Observations of dynamics on scales ranging from individual galaxies to large galactic clusters and beyond (Trimble 1987 and references therein) strongly suggest that the mass density of the universe is made up of at least two distinct components: an invisible collisionless "dark matter" species, evident only by its gravitational influence, and the visible baryonic matter making up the observed universe (gas, stars, galaxies, etc). Such observational estimates of the importance of this invisible dark matter indicate it may make up 90% of the global mass density. Theoretical prejudice toward believing that the universe is "flat" ($\Omega = 1$) combined with primordial nucleosynthesis bounds on the baryonic mass fraction $\Omega_{\text{bary}} h^2_{50} \in [0.04, 0.06]$ (Walker et al. 1991) also imply that the mass density of the universe is dominated by dark matter. The dynamics of such a mixed cosmological fluid, consisting of both collisionless dark matter and normal baryonic material, will be governed by the gravitational influences which couple both species. The baryonic component is additionally subject to more complex hydrodynamic interactions. Until recently, the majority of studies of the evolution of cosmological structure have concentrated solely on the collisionless dark matter, neglecting hydrodynamic forces and tacitly assuming that the baryonic material will simply follow the global gravitational potential. This has been primarily for the pragmatic reason that achieving the necessary dynamic ranges for the gravitational problem alone has strained contemporary computational capabilities, necessitating
the development of a variety of ingenious schemes to solve the gravitational problem efficiently (Efstathiou et al. 1985; Villumsen 1989). On large scales, one can make simple arguments that this assumption is reasonable. If we assume that the dark matter is the overwhelmingly dominant mass component, it will then dominate the gravitational potential. Hydrodynamic interactions are local in nature, and therefore the gravitational force should dominate the global evolution. Specifically, assuming a reasonable upper bound for the temperature of the intergalactic medium $T_{\text{igm}} \sim 10^8$ K implies a sound-speed limited communication distance of $\sim 10$ Mpc in a Hubble time. It is very difficult to conceive of a plausible mechanism whereby hydrodynamic effects could influence scales any larger than this limit. However, on smaller scales ($\ll 1$ Mpc), we can expect hydrodynamics to play an increasingly important role in the evolution of the baryons. In collapsing high-density regions, baryonic gas can undergo hydrodynamic interactions such as shock heating and radiative dissipation, which are precisely the sorts of processes believed to be responsible for forming the visible parts of galaxies. The fact that spiral galaxies have dynamically cold, rotationally supported disks requires hydrodynamic dissipation. Ignoring such effects implies we are neglecting potentially important physics in our models.

Understanding the distribution of baryonic material, and how it differs from that of dark matter, is a fundamental problem of modern cosmology. Everything that can be directly observed is composed of baryons (galaxies, clusters, X-ray emitting gas, etc). There are currently two major approaches to studying the evolution of structure in such a two phase cosmological fluid. One such class is represented by the semi-
analytic techniques (see for instance Cole 1991; White & Frenk 1991; Kauffmann et al. 1993; Lacey, Rocca-Volmerange, & Silk 1993; Cole et al. 1994). These techniques use phenomenological descriptions and approximations to make the inherently nonlinear problem of hierarchical structure formation analytically tractable. Such approaches are often based upon statistical analytic models such as Press-Schechter theory (Press & Schechter 1974) in combination with phenomenological approximations for the detailed behaviour of the dark matter and gas (including dissipation) based upon numerical work. The major advantage of these techniques are that they allow the researcher to study many different types of cosmologies and to gain a great deal of physical insight into such models. Probably their largest drawback is that they are based upon a series of approximations and assumptions, making the results dependent upon the validity of these assumptions.

The other major approach to studying large-scale structure is based upon numerical simulations, which attempt to directly model the nonlinear evolution of structure based upon first principles. The great advantage of this approach is that the physics going into such models is firmly established (i.e., the differential equations describing the relevant gravitational and hydrodynamical physics), and we can therefore try to go straight from such well-founded physics to the evolution of large-scale structure without the assumptions implicit in the semi-analytic approach. However, creating realistic numerical models of structure formation is extremely demanding, due to the large dynamic range in length, mass, and time scales required. Investigators must expend large amounts of time and effort designing and implementing clever numerical
techniques to even approach the problem, and even so, with current computational
technology one can only afford to perform a small number of experiments with dy­
namic resolution ranges substantially less than might be desired. Until recently, all such numerical models incorporated only gravitational physics, and as such are only directly applicable to the collisionless dark matter thought to make up the majority of the mass density in the universe. In order to relate the results of such collision­
less simulations to observations, we must make assumptions about how the observed baryonic structures relate to the hypothetical dark matter distribution. This problem has led to much debate as to exactly how the “light traces the mass”, as it is the luminous material we observe but the total mass distribution which is the fundamen­
tal dynamical quantity. While a variety of physically plausible schemes have been developed to identify simulated dark matter structures with galaxies and clusters (Bardeen et al. 1986; Brainerd & Villumsen 1992a, 1992b, 1993), a preferable solu­
tion is to directly incorporate an appropriate baryonic component within cosmological simulations. This then allows the baryons to be self-consistently evolved along with the dark matter, affording a basis for direct comparison of theory with observation. However, even a minimal treatment of the baryons necessitates the inclusion of much more complex physics than gravitation alone, such as pressure forces, shocks, and ra­
diative heating and/or cooling, as these are the major physical processes thought to play a role in the formation of large, bright galaxies (Rees & Ostriker 1977; Silk 1977; White & Rees 1978). This significantly increases the computational demands of such models, which already strain contemporary computational abilities. Nevertheless, the
potential rewards of such work are great, such as the ability to directly compare the consequences of any given cosmological model from first principles to observations. We can also use such numerical models to increase the reliability and utility of the semi-analytic approach, as many of the approximations underlying these techniques rely upon the results of detailed numerical investigations. In this dissertation I will expand upon and investigate these numerical approaches.

To date, there are three major classes of numerical techniques which have been used to attack the problem of modeling cosmological hydrodynamics, each with varying levels of expense and accuracy: the "sticky particle" method (Carlberg 1988; Carlberg & Couchman 1989), various implementations of "Smoothed Particle Hydrodynamics" (SPH) (Evrard 1988; Hernquist & Katz 1989), and traditional Eulerian finite-difference methods (Chiang et al. 1989; Cen et al. 1990). Of these potential approaches I will focus on SPH, which offers several advantages. First, SPH is a true hydrodynamic technique in that it solves for thermodynamic quantities in the system (temperature, pressure, etc.), avoiding more ad hoc assumptions as are required with a technique such as sticky particles. Secondly, SPH is highly adaptive in comparison with more traditional techniques, such as the Eulerian finite-difference methods, since SPH is fully Lagrangian and the computational nodes are allowed to freely move with the fluid. This adaptiveness is desirable in cosmological simulations, where the required dynamic range in resolution is large and there is no inherent geometry to the problem which can be exploited. There are of course disadvantages to using SPH as well. SPH is more computationally expensive as compared with the simpler sticky
particle method, and therefore the scale of the problems which can be studied is more restricted. Additionally, there has been little work done on estimating the error terms under SPH, such as exists for the Eulerian finite-difference methodology, which can make detailed quantitative interpretation of the results dubious (see, however, Evrard 1988 and Balsara 1995).

I begin this investigation by deriving and testing a new, highly adaptive form of SPH in Chapter 2, which I call “Adaptive” Smoothed Particle Hydrodynamics, or ASPH. ASPH is designed to maximize the resolution of a simulation for a given number of particles or nodes. In contrast to SPH, each ASPH node is allowed to adapt its resolution scale to the locally anisotropic evolution of the density. Therefore, in situations where physical anisotropy evolves (i.e., such as in gravitationally driven systems which generically form flattened sheets or filaments), ASPH is able to achieve a more accurate representation of the physics in the system than SPH with the same number of nodes. The basic idea of ASPH was originally conceived as part of a collaboration between Paul Shapiro, Hugo Martel (both at the University of Texas at Austin), and Jens Villumsen (then at Ohio State). The method as described in Chapter 2 expands upon that pioneering work through a rigorous tensor formulation of ASPH applicable in any number of dimensions, and a large set of test problems. In Chapter 3, I go on to model a number of idealized 2-D hierarchical cosmological models incorporating both dark matter and baryons (though without radiative cooling) at a variety of resolutions. Using these models I am able to demonstrate the criteria that must be met so that the physical results will converge with increasing numerical resolution. I also
identify some interesting physical behaviour relevant to generic structure formation, particularly with respect to cosmological shocks and the baryon to dark matter ratio. Then in Chapter 4, I study a set of 3-D scale-free simulations performed with two different contemporary SPH codes, again without radiative cooling. The scale-free nature of these systems allows me to test the numerical models against the analytic prediction of temporally self-similar evolution. The requirement that the physical properties of these systems obey self-similarity reveals a number of interesting aspects about these sorts of experiments, as well as allowing me to separate out the physics from the numerics for study. Finally, in Chapter 5 I use ASPH to model a series of 2-D scale-free models with radiative cooling. As these models are also scale-free, I can test the behaviour of the radiatively dissipated structures against the requirement of self-similar evolution, much as I did in the previous chapter with non-radiative models. The results of this chapter are relevant to numerical studies of galaxy formation, as radiative dissipation in the high-density baryon gas is required in order to form compact, dynamically cold structures, such as galaxies and stars.

On a stylistic note, I should point out that Chapters 2, 3, and 4 have been prepared as separate papers, suitable for independent publication. Although the ideas presented in each form an overall logical progression, the reader will likely note that each is written to be independent of the others. Only Chapter 5 has not yet been prepared as a separate publication, though the material in this chapter will also be presented so in the near future.
CHAPTER 2

Adaptive Smoothed Particle Hydrodynamics: Methodology

Published Work:

The following chapter has been submitted for publication in *The Astrophysical Journal* as Owen, J. M., Villumsen, J. V., Shapiro, P. R., & Martel, H. 1996, “Adaptive Smoothed Particle Hydrodynamics: Methodology II”. It is therefore formatted as a paper throughout. The only adaptations made here are that the figures and tables have been renumbered for continuity, and the references have been moved to the general bibliography at the end.

2.1 Introduction

Numerical modeling of hydrodynamics has become an increasingly important tool for astronomy. Problems ranging from stellar formation to the evolution of large scale cosmological structure have all been investigated using hydrodynamic simulations. A number of numerical techniques have been developed and utilized for these purposes, including the “sticky particle” method (Carlberg 1988; Carlberg & Couchman 1989), various implementations of Smoothed Particle Hydrodynamics (Evrard 1988; Hernquist & Katz 1989, hereafter HK89; Steinmetz & Müller 1993), and tra-
ditional Eulerian finite-difference methods (Chiang, Ryu, & Vishniac 1989; Cen et al. 1990). For many of these investigations (cosmological structure formation in particular) Smoothed Particle Hydrodynamics (SPH) has proven to be the technique of choice. First, SPH is a true hydrodynamic technique in that it solves for thermodynamic quantities in the system (temperature, pressure, etc.), avoiding more ad hoc assumptions as are required with a technique such as sticky particles. Secondly, SPH is highly adaptive in comparison with more traditional, grid-based techniques (such as the Eulerian finite-difference methods) since SPH is Lagrangian, and the computational nodes are allowed to move freely with the fluid. This adaptability is desirable in cosmological simulations, where the required dynamic range in resolution is large and there is no inherent symmetry to the problem which can be exploited. Additionally, the SPH dynamical equations governing the evolution of the fluid are easily derived from the Lagrangian conservation equations, which in combination with SPH's generalized interpolation scheme allows new physics (such as radiative effects or magneto-hydrodynamics) to be easily implemented. In many ways SPH can be thought of as a natural extension of traditional N-body techniques, where in order to simulate a continuous medium an interpolation kernel is convolved over the particle distribution, smearing out the discrete particle positions and information into a continuous representation. The primary disadvantage of SPH is that being a relatively
new technique, it is not as well understood in comparison with the more traditional grid-based methods. Still, on the whole, it has been found that SPH is a simple and robust method of studying hydrodynamic phenomena within the framework of traditional N-body investigations.

In Shapiro et al. (1996) (hereafter Paper I) we introduce Adaptive Smoothed Particle Hydrodynamics (ASPH). The ASPH algorithm is designed to maximize the resolution of an SPH simulation by allowing the resolution scales to adapt to intrinsically anisotropic density evolution, as well as reduce spurious preheating due to the presence of an artificial viscosity. Each ASPH node is allowed to sample locally ellipsoidal, rather than spherical, interpolation volumes, the size, shape, and orientation of which evolve in accordance with the evolution of the local density. ASPH can therefore adapt to anisotropic evolution within the simulated system, allowing optimal spatial resolution. This basic idea dates back to Bicknell & Gingold (1985), who first utilized ellipsoidal SPH kernels in order to study the tidal flattening of stars involved in close-encounters with black holes. They exploited the special geometry of that problem, simplifying the evolution of the ellipsoidal shapes. Shapiro, Martel, & Villumsen first began seriously investigating a generalized approach to ASPH (Shapiro et al. 1993; Shapiro, Martel, & Villumsen 1994; Martel et al. 1994), and have applied their ideas to a number of cosmological simulations.

In Paper I we introduce ASPH as a generalized hydrodynamic technique, wherein the ASPH algorithm is originally developed such that it does not rely upon any special geometry to the problem being studied. As a part of that technique we
also introduce an algorithm for implementing the artificial viscosity such that the spurious artificial viscous preheating endemic to SPH collapse simulations is greatly reduced. In this paper we present an alternative mathematical formalism for evolving the ASPH smoothing scale (expressed here as a tensor transformation), along with a more conservative approach to the problem of suppressing the artificial viscosity. There are several advantages of the formalism as developed in this work. The ASPH smoothing scale and its evolution are developed in a tensor notation which streamlines the mathematics, allowing a closed form derivation and a more efficient numerical implementation. In particular, this approach leads to general solutions for the ASPH tensor smoothing field, allowing simple dynamical equations to be derived with no special cases (such as are required with a geometric derivation of ASPH). Under this formalism deriving and implementing a 3-D form of ASPH is straightforward. In contrast to Paper I, the approach to the artificial viscosity does not require violating energy conservation, and also does not introduce any new free parameters that must be tweaked for different problems. While this approach to the artificial viscosity is not as successful at suppressing artificial viscous preheating in the Zel’dovich pancake problem, it does improve over the standard form and is general in its application, allowing us to test a wide variety of problems. We have developed a number of efficient numerical algorithms in order to make ASPH applicable to large scale problems, including an adaptive asynchronous time integrator, such that each ASPH node is allowed to possess its own timestep and current time. In Paper I we concentrated on the Zel’dovich pancake problem, as this represents the archetypical problem for
cosmological structure formation. In this work we investigate a number of problems, including the Zel’dovich pancake problem (to compare the formalism presented here with that of Paper I), the Sedov Blastwave solution, standard hydrodynamical tests such as the Riemann shocktube and the Woodward double blastwave problem, and a number of simulations designed to test the conservation of angular momentum under ASPH.

This paper is organized as follows. In §2.2 and §2.3 we describe the techniques of SPH and ASPH, deriving ASPH based upon SPH. In §2.4 we present various tests and comparisons of these techniques. In §2.5 we summarize the results of this paper and discuss directions for future work. In Appendix A we present the detailed numerical (A)SPH dynamical equations (both for proper and comoving coordinates). Appendix B presents the full mathematical formalism and derivation for defining and evolving the ASPH smoothing tensor. Finally, Appendix C outlines some of the major algorithms developed in order to efficiently implement ASPH numerically.

### 2.2 Standard SPH

ASPH is a generalization of SPH, and therefore throughout this paper we refer to “Standard” SPH as both a starting point and comparison for ASPH. Unfortunately such a standard version does not exist, as there are a variety of subtly different ways to implement SPH. Therefore in Appendix A we present the formalism for what we call Standard SPH. This discussion, in combination with what is discussed in the body of this paper, is complete, but brief. The reader who wishes a more in-depth introduction to SPH is referred to the reviews of Monaghan (1992) or Benz (1990), either of which
provide an excellent introduction to this subject. For the experienced practitioner of SPH, we simply state that our implementation of SPH consists of evolving the momentum and specific thermal energy for each node based upon the Lagrangian conservation equations, accounting for pressure and gravitational forces as well as radiative cooling. The mass density is updated using the summation approach. We implement a standard Monaghan-Gingold artificial viscosity (Monaghan & Gingold 1983) defined on a pairwise basis, with the shearing correction suggested by Balsara (1995). We have derived the (A)SPH dynamical equations based upon these choices for proper time-domain coordinates, as well as comoving coordinates expressed as a function of a power of the cosmological expansion factor (for use in cosmological simulations). Finally, we allow a variable smoothing scale which evolves in accordance with the continuity equation as outlined by Benz (1990). We discuss this SPH method of evolving the smoothing scale in §2.3.2.

2.3 SPH vs. ASPH

We will now introduce the basic notation and concepts necessary to understand ASPH and how it relates to SPH. As SPH and ASPH are so similar, much of our discussion is applicable to both. In such ambiguous situations we refer to the technique as (A)SPH. The major distinction between SPH and ASPH is the manner in which smoothing scales are defined and evolved, and therefore we will concentrate on this subject. This section is organized as follows: in §2.3.1 we discuss how smoothing scales are defined under both SPH and ASPH, as well as how they are used in order to make interpolated estimates of local quantities; in §2.3.2 we discuss how a variable
smoothing scale is evolved under SPH; finally in §2.3.3 we describe how the anisotropic smoothing of ASPH is justified and evolved. Note that we discuss ASPH descriptively here, simply presenting the resulting 2-D evolution equations without their derivation. We defer a detailed discussion of the mathematics upon which ASPH is formulated until Appendix B, where the complete 2-D and 3-D derivations may also be found.

2.3.1 Interpolation under SPH & ASPH

(A)SPH functions by numerically solving the Lagrangian conservation equations at a series of discrete points or nodes, which are interrelated through an interpolation scheme. In this way (A)SPH resembles the various flavors of Lagrangian finite-difference methods, the major distinction being the method of interpolation. Under a traditional Lagrangian finite-difference approach the interpolation nodes are required to be arranged on an underlying geometry or grid, while (A)SPH’s interpolation scheme makes no such restrictions.

Under the SPH formalism, the interpolated value of a quantity \( F \) at some spatial position \( \mathbf{r} \) is defined through the integral

\[
\langle F(\mathbf{r}) \rangle = \int F(\mathbf{r}') W(\mathbf{r} - \mathbf{r}', h) \, d\mathbf{r}'.
\]  

(2.1)

The function \( W(\mathbf{r}, h) \) is called the interpolation kernel, where \( h \) represents the SPH smoothing scale. The interpolation kernel has the properties

\[
\int W(\mathbf{r} - \mathbf{r}', h) \, d\mathbf{r}' = 1, 
\]

(2.2)

\[
\lim_{h \to 0} W(\mathbf{r} - \mathbf{r}', h) = \delta(\mathbf{r} - \mathbf{r}').
\]

(2.3)
Note from equation (2.3) that although the integration of equation (2.1) formally extends over all space, in fact the most important contributions to the integral occur within a few $h$ of the position $r$. This reflects the localized nature of hydrodynamic interactions.

In SPH applications, $h$ is defined as a scalar quantity associated with each discrete SPH node. Thus, SPH interpolation about any given node using such a scalar smoothing scale is isotropic in nature. In other words, each SPH node samples a spherical volume about itself, regardless of the physical conditions in which it is embedded. As we discuss in §2.3.3, in general this is not optimal. In order to implement an anisotropic smoothing scheme, we require a more generalized methodology for defining smoothing scales. We begin by noting that the interpolation kernel can be written as a function of $\eta = r/h$, such that we can restate $W(r, h) = W(r/h) = W(\eta)$. In general we refer to $\eta$ as the normalized position vector.

We can now introduce a generalized method of mapping from real to normalized position space ($r \rightarrow \eta$), which we define through a linear transformation $G$. In comparison with SPH, this relation is

\[
\text{SPH: } \eta = r/h \quad \rightarrow \quad \text{ASPH: } \eta = Gr.
\]

Clearly the $G$ tensor has the units of an inverse length-scale. The simplest generalization of the spherical smoothing implemented by SPH is to allow smoothing in elliptical (2-D) or ellipsoidal (3-D) volumes. Placing such a restriction upon $G$ implies that it must be a real, symmetric matrix. Our scheme under ASPH then is to associate such an ellipsoidal $G$ tensor with each computational node, taking the place
of the scalar smoothing scale $h$ of SPH. Under this formalism, SPH can be thought of as a special case of ASPH, where the $G$ tensor is diagonal and each diagonal element is equal to $1/h$.

Under this notation it is simple to define an ASPH kernel estimate. We need only replace the SPH method of defining $\eta$ with the ASPH method in the kernel function $W(\eta) = W(Gr)$. However, we also require the spatial gradient of the interpolation kernel $\nabla W$ in order to implement the (A)SPH dynamical equations. This can be expressed through equation (2.4), so long as we are neglecting $\nabla G$ terms (an important assumption we will return to in §2.3.4)

$$
\nabla W(Gr) = \frac{\partial W(Gr)}{\partial r} = \frac{\partial \eta \partial W}{\partial r \partial \eta} = G \frac{\eta \partial W}{\eta \partial \eta}.
$$

Before we can go on to express how to use $G$ in making kernel interpolations, we must deal with how to symmetrize such estimates. This issue arises for SPH with a variable smoothing scale as well. The problem can be understood by examining equation (2.1), which expresses the interpolated value of some general quantity $F$ in terms of a volume integral involving $h$. This equation makes no mention of how to deal with a smoothing scale which varies spatially $h(r)$. One could choose to either use $h(r)$ (known as a "gather" formalism) or $h(r')$ ("scatter"). However, it is advantageous to spatially symmetrize the kernel estimation process such that $W(r, r', h(r), h(r')) = W(r', r, h(r'), h(r))$ in order to symmetrize the (A)SPH dynamical equations, thereby ensuring rigorous conservation of quantities such as linear momentum. In order to accomplish this we adapt the symmetrization scheme of Hernquist & Katz (HK89), which defines a symmetrized kernel estimate as the average of both the gather and
scatter approaches. In terms of two discrete positions $\mathbf{r}_i$ and $\mathbf{r}_j$, the symmetrized kernel function $W_{ij}$ is

$$W_{ij} \equiv \frac{1}{2} \left[ W(\eta_i) + W(\eta_j) \right], \quad (2.5)$$

where

$$\eta_i \equiv G_i \mathbf{r}_{ij}, \quad \eta_j \equiv G_j \mathbf{r}_{ij}, \quad (2.6)$$

$$\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j. \quad (2.7)$$

A symmetrized gradient of the kernel can be similarly defined as

$$\nabla W_{ij} \equiv \frac{1}{2} \left[ \nabla W(\eta_i) + \nabla W(\eta_j) \right]. \quad (2.8)$$

These forms also arise naturally from a derivation of (A)SPH based upon the variational principle.

The interpolation relation of equation (2.1) can now be represented numerically by assigning known values for the general quantity $F(\mathbf{r})$ on a series of discrete nodes at positions $\mathbf{r}_j$ (each with an associated mass $m_j$, mass density $\rho_j$, and number density $n_j = \rho_j/m_j$), such that we have a discrete set $F_j$. The interpolated value of $F$ at position $\mathbf{r}_i$ is

$$F_i \equiv \langle F(\mathbf{r}_i) \rangle = \sum_j \frac{F_j}{n_j} W_{ij} = \sum_j F_j \frac{m_j}{\rho_j} W_{ij}. \quad (2.9)$$

Note that equation (2.9) basically represents a monte-carlo interpretation of equation (2.1). With this machinery in place, it is now possible to derive ASPH dynamical equations using the same approach as SPH. So long as we use the notational conventions outlined above (expressing quantities in terms of the normalized position.
vector $\eta$ rather than explicitly using $h$) the SPH and ASPH dynamical equations are identical. We present our complete set of (A)SPH dynamical equations derived in this fashion in Appendix A.

The form of the $G$ tensor depends upon the dimensionality in which it implemented. The $G$ tensor can be defined in terms of the underlying geometry as follows. In 2-D, consider a unit normalized position isocontour associated with a given $G$. In general such an isocontour represents an arbitrary ellipse in real position space, which is uniquely defined by a semi-major axis $h_1$, semi-minor axis $h_2$, and position angle $\psi$ associated with the semi-major axis. Since we are considering a unit isocontour, $h_1$ and $h_2$ are the smoothing scales along the primary axes of the ellipse. In terms of these geometrical quantities, the elements of the $G$ tensor are (see Appendix B)

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} h_1^{-1} \cos^2 \psi + h_2^{-1} \sin^2 \psi & (h_1^{-1} - h_2^{-1}) \cos \psi \sin \psi \\ (h_1^{-1} - h_2^{-1}) \cos \psi \sin \psi & h_1^{-1} \sin^2 \psi + h_2^{-1} \cos^2 \psi \end{pmatrix}. \quad (2.10)$$

The 3-D $G$ tensor can be similarly derived, as discussed in appendix B.4, resulting in equation (B.25).

We have discussed the interpolation kernel $W$ completely generally to this point. There are many possible choices for such a function, so long as the criteria of equations (2.2) and (2.3) are met and they can be expressed as functions of the normalized position $\eta$. The most popular forms currently in use with SPH are the Bi-Cubic Spline and the Gaussian kernels. In our investigation we use the Bi-Cubic Spline, which in $\nu$ dimension is

$$W_{\text{spline}}(\eta) = A_{\text{spline}}^{\nu} \begin{cases} 1 - 3/2 \eta^2 + 3/4 \eta^3, & 0 \leq \eta \leq 1; \\ 1/4 (2 - \eta)^3, & 1 < \eta \leq 2; \\ 0, & \eta > 2, \end{cases} \quad (2.11)$$
\[ \nabla W^{\nu-D}_{\text{spline}}(\eta) = A^{\nu-D}_{\text{spline}} \frac{G}{\eta} \left\{ \begin{array}{ll} -3\eta + 9/4 \eta^2, & 0 \leq \eta \leq 1; \\ -3/4 (2 - \eta)^2, & 1 < \eta \leq 2; \\ 0, & \eta > 2, \end{array} \right. \]  
(2.12)

\[ A^{1-D}_{\text{spline}} = \frac{2}{3}|G|, \quad A^{2-D}_{\text{spline}} = \frac{10}{7\pi}|G|, \quad A^{3-D}_{\text{spline}} = \frac{1}{\pi}|G|. \quad (2.13) \]

where \( A^{\nu-D}_{\text{spline}} \) represents the normalization constant required to meet the condition of equation (2.2).

### 2.3.2 Evolving the Smoothing Scale under SPH

Before going on to discuss how \( G \) is evolved under ASPH, we must begin with the rationale for evolving the variable smoothing scale under SPH. Most modern implementations of SPH allow an individual, time variable smoothing scale to be associated with each SPH node, such that \( h(r,t) \). The justification for implementing and evolving such a variable smoothing scale is based upon the philosophy that each SPH node should sample roughly the same number of “significant neighbors” (that is, the number of neighboring SPH nodes within some critical threshold distance, usually expressed as a multiple of the smoothing scale \( \eta \leq \eta^{\text{cut}} \)). This implies that each SPH node will always sample roughly the same amount of mass, which is consistent with SPH’s Lagrangian nature. This results in higher resolutions in dense regions, while still maintaining meaningful (if poorly resolved) measurements in low density regions. This condition can be stated mathematically in \( \nu \) dimensions as \( h_i \propto \rho_i^{-1/\nu} \) for a given node \( i \).
A well-defined standard method for evolving the SPH smoothing scale in order to meet this criterion is based upon the continuity equation (Benz 1990 and references therein), and can be expressed as

\[
\frac{Dh_i}{Dt} = \frac{1}{\nu \rho_i} \frac{D\rho_i}{Dt} = \frac{h_i}{\nu} (\nabla \cdot \mathbf{v})_i = -\frac{1}{\nu \rho_i} \sum_j m_j v_{ij} \cdot \nabla W_{ij}. \tag{2.14}
\]

### 2.3.3 Evolving G under ASPH

In order to understand the motivation for developing ASPH, it is necessary to understand the shortcomings of the SPH approach as defined by equation (2.14). The fundamental problem is that the SPH method is really only appropriate for isotropic density evolution. This can be understood through the following example. Consider a general 3-D system undergoing planar collapse in some arbitrary direction. The standard SPH smoothing scale will adapt to this collapse process as \( h \propto \rho^{-1/3} \). However, a planar collapse is really a 1-D problem, and ideally smoothing scales perpendicular to the plane of collapse should evolve as \( h_\perp \propto \rho^{-1} \), while smoothing scales parallel to the collapse should remain unchanged. In such a situation, SPH's approach will lead to less than optimal resolution along the direction of collapse (since \( h_\perp \) is shrinking too slowly), while nodes in the collapsing region will lose contact with neighbors parallel to the collapse (since \( h_\parallel \) is shrinking, but the inter-node spacing parallel to the plane of collapse remains unchanged). This demonstrates the weakness of the SPH approach, in that it cannot correctly adapt to anisotropic evolution of the density. In general problems of computational interest are not isotropic in nature, and in particular gravitational clustering scenarios generically result in strongly anisotropic
density evolution. ASPH seeks to address this problem by allowing an anisotropic
definition of the smoothing process, as well as a self-consistent method for evolving
this anisotropic smoothing function. The ASPH algorithm can be viewed as trying
to adapt to the physical or intrinsic dimensionality of a problem, rather than the
imposed geometrical dimensionality.

This line of reasoning suggests a straightforward approach to evolving the ASPH
smoothing transformation G. We can generalize the SPH philosophy of attempting to
maintain the same number of neighbors per node into an attempt to maintain the same
number of neighbors in all directions for each node. This can be rephrased to state
that we will try to keep the distribution of neighboring nodes isotropic in normalized
(η) space. In more physical terms each ASPH node attempts to always sample the
same Lagrangian volume as the system evolves, which is appropriate for ASPH’s
Lagrangian nature. This can be viewed as an attempt to track the local deformation
of a fluid element with an idealized shape corresponding to the underlying geometry
of the G tensor. At best such a picture represents an analogy, as a true fluid element
is not constrained to remain ellipsoidal, and the volumes defined by the collection of
G_i's interpenetrate. Nevertheless, the fluid element analogy is useful in order to gain
an intuitive sense as to the behaviour of the G tensors.

The deformation tensor \( \sigma (\sigma_{\alpha\beta} \equiv \partial v_\alpha / \partial r_\beta \) where (\alpha, \beta) refer to spatial directions)indicates how the velocity field varies spatially to first-order, such that \( v(r + dr) \approx
v(r) + \sigma dr \). This quantity predicts how a local volume of the fluid will deform with
time. If we visualize the ASPH interpolation volume as an embedded volume within

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the fluid, $\sigma$ predicts how this volume should be deformed by the local velocity field, such that the enclosed Lagrangian volume remains constant. As this is a first-order transformation, an initially ellipsoidal volume will be in general be mapped to a new ellipsoidal volume, guaranteeing that our ellipsoidal transformation for $G$ will remain appropriate. Under the (A)SPH formalism the deformation tensor can be estimated by

$$\langle \sigma_{\alpha\beta} \rangle_i = -\sum_j m_j (v_{ij})_a \frac{\partial W_{ij}}{\partial r_{\beta}}.$$  \hspace{1cm} (2.15)

We will defer a rigorous mathematical derivation of $G$ and its evolution to appendix B, and simply present the results for the 2-D case here. The evolution of the 2-D $G$ tensor (eq. [2.10]) is given by

$$\frac{DG}{Dt} = \begin{pmatrix} DG_{11}/Dt & DG_{21}/Dt \\ DG_{21}/Dt & DG_{22}/Dt \end{pmatrix}$$

$$= \begin{pmatrix} G_{21}(\dot{\theta} - \sigma_{21}) - G_{11}\sigma_{11} & G_{22}\dot{\theta} - G_{11}\sigma_{12} - G_{21}\sigma_{22} \\ -G_{11}\dot{\theta} - G_{21}\sigma_{11} - G_{22}\sigma_{21} & -G_{21}(\dot{\theta} + \sigma_{12}) - G_{22}\sigma_{22} \end{pmatrix},$$

$$\dot{\theta} = \frac{G_{11}\sigma_{12} - G_{22}\sigma_{21} - G_{21}(\sigma_{11} - \sigma_{22})}{G_{11} + G_{22}}.$$ \hspace{1cm} (2.17)

In order to demonstrate the connection of this evolution equation for $G$ with the SPH relation for $h$, consider the evolution of the determinant $|G|$.

$$\frac{D|G|}{Dt} = G_{11}\frac{DG_{22}}{Dt} + G_{22}\frac{DG_{11}}{Dt} - 2G_{21}\frac{DG_{21}}{Dt}$$

$$= -(G_{11}G_{22} - G_{21}^2)(\sigma_{11} + \sigma_{22})$$

$$= -|G|\nabla \cdot \mathbf{v}$$

$$= \frac{|G|}{\rho \frac{D\rho}{Dt}}.$$
For the special case of SPH, the determinant reduces to $|G|_{\text{SPH}} = h^{-\nu}$ in $\nu$ dimensions. In 2-D, the SPH evolution equation for $h$ (eq. [2.14]) yields

$$\frac{Dh^{-2}}{Dt} = -2h^{-3} \frac{Dh}{Dt} = \frac{h^{-2}}{\rho} \frac{D\rho}{Dt}. \quad (2.19)$$

Equation (2.18) shows that $|G|$ is directly proportional to the density ($|G| \propto \rho$). This also proves that the smoothing volume represented by $|G|^{-1}$ under ASPH evolves identically with its SPH counterpart, demonstrating that for the case of isotropic evolution ASPH formally reduces to SPH. Though derived here in 2-D, this result applies in general.

### 2.3.4 Stabilizing ASPH: Smoothing the G tensor field

The evolution equation for the $G$ tensor is derived based upon first-order arguments about the local velocity field – specifically, that the local velocity can be approximated by $v(r + dr) \approx v(r) + \mathbf{\sigma} dr$. Clearly, in complex simulations one can encounter situations where this approximation is not valid on "local" scales (scales of order a few $h$). A cosmological example of such a situation is the formation of a poorly resolved cluster at the intersection of several filaments, each of which contains gas streaming into the cluster. Ideally one would wish to resolve all mass scales adequately such that the "local" velocity field will always be well represented by $\mathbf{\sigma}$ on scales of $h$, but in reality computational limitations make this infeasible. Any simulation will always have a lower mass cutoff in the resolved mass distribution. In particular, a Cold Dark Matter like initial density fluctuation power-spectrum will generically have substantial power on small scales, and it is reasonable to assume that in such
situations one will have to deal with nonlinear velocity fields down to scales of $h$.

If we examine the assumptions underlying our derivation of the evolution equations for $G$, we find that we implicitly require the $G$ tensor field to be well-behaved on scales of a few smoothing lengths. This point can be demonstrated in two ways. First, consider the expression for the gradient of the kernel $\nabla W$ (eq. [2.4]). As with SPH, we neglect any $\nabla G$ terms, which formally should be included. Including these terms is somewhat problematic as they must be estimated numerically, and we have found that attempting to include them does little more than introduce noise. Neglecting such terms implies that we require $\nabla G$ to be negligible on scales of a few characteristic smoothing lengths. Another way of viewing this problem emerges if we use our idealized fluid element analogy. The smoothing volumes represented by neighboring $G$ tensors must necessarily overlap – this implies that the $G$ tensors of closely neighboring nodes should represent the same local idealized fluid element. Strong disorder in this $G$ tensor field on local scales is therefore inconsistent.

Arguments such as these lead us to conclude that we require the $G$ tensor field to be almost uniform on scales of a few smoothing lengths, and smoothly varying on larger scales. If this condition is not met, how would we expect this inconsistency to affect ASPH? Under our standard formulation of SPH, it will formally conserve mass (with the summation definition for the density), linear and angular momentum (because all pair interactions are symmetric and radial), while energy is conserved to second-order. ASPH utilizes the same dynamical equations as SPH, including the use of the symmetrized kernel function $W_{ij}$ (eq. [2.5]). Therefore, ASPH will rigorously
conserve mass and linear momentum, while energy will be conserved to second-order (see Appendix A for a discussion of the dynamical equations and these conservation properties). However, because the gradient of the kernel $\nabla W$ is not necessarily radial under ASPH, forces between interacting pairs of nodes will not in general be radial and therefore angular momentum will not be rigorously conserved. This makes the conservation of angular momentum under ASPH analogous to the conservation of energy — angular momentum will only be conserved to the order that the system is being solved (in general to second-order). ASPH is therefore vulnerable to errors in the angular momentum as well as the energy, and we might expect that as an ASPH simulation breaks down non-conservation of these quantities could be a symptom. As we discuss in §2.4.2, we find precisely this sort of behaviour.

We therefore require a method of ensuring that the $G$ tensor field is well-behaved. The most obvious step to take is to smooth the $G$ tensor field using the ASPH formalism. We must be cautious in the implementation of such a scheme, however. Recall that the $G$ tensor has units of an inverse smoothing scale. Taking a straight ASPH estimate of $G$ is equivalent to taking the harmonic mean of the smoothing scales, which can be unstable toward small smoothing scales. Additionally, we would like to preserve the property of the determinant $|G|$ such that it evolves smoothly in accordance with the local density, as demonstrated in equation (2.18). Discontinuous changes in $G$ are equivalent to discontinuously changing an individual nodes contribution to the local density. After exploring several possible implementations for a smoothing scheme, we have settled on the following approach. Periodically each $G_i$
tensor is replaced by an averaged $G'_i$ calculated as

$$
\langle G^{-1} \rangle_i = \frac{\sum_j G^{-1}_{ij} W_{ij}}{\sum_j W_{ij}}, \quad G'_i = |G_i| \langle G^{-1} \rangle_i / \langle G^{-1} \rangle_i^{-1}.
$$

(2.20)

Note this scheme represents three modifications of an ordinary ASPH average. First, we average the quantity $G^{-1}$ in order to avoid the problems of a harmonic mean on the smoothing scale. Second, we force the determinant to be preserved $|G'_i| = |G_i|$. Finally, we force the normalization of the average to be unity by dividing by $\sum_j W_{ij}$. Formally this sum should be unity (eq. [2.2]), but in practice because we only sum over a finite number of nodes, this quantity typically deviates from that ideal. We have found that normalizing the average in this way increases the stability of the technique. The frequency with which we must enforce this smoothing process must be determined experimentally. We have found that smoothing roughly once or twice every characteristic timescale is generally adequate (where by characteristic timescale we mean the timescale setting the current timestep – see appendix C.2).

It is worth noting that there is an alternate method of viewing this problem and its solution. The evolution of the $G$ tensor is based upon attempting to follow the deformation of the local velocity field based on using the deformation tensor $\sigma_{\alpha\beta} = \partial v_\alpha / \partial x_\beta$, which is only correct to a first-order. If the local velocity field is not this obligingly simple, then the arguments we base our $G$ evolution equations on break down and there is no guarantee trouble will not ensue. This leads to the idea that smoothing on the deformation tensor $\langle \sigma \rangle$ could also be an equivalent method of dealing with this problem. Such a solution seems intuitively pleasing, as it would no longer involve directly fiddling with the $G$ tensors themselves. However, this is
basically equivalent to our adopted method of smoothing on the $G$ tensors, which has the additional advantage of directly guaranteeing the good behaviour of the $G$ tensor field. We should also point out that formally SPH also requires that $\nabla h$ be negligible on scales of a few $h$, and therefore standard SPH schemes utilizing a spatially variable smoothing scale should also ensure this behaviour in some way. Steinmetz & Müller (1993) find that the stability of SPH with a variable smoothing scale is indeed improved by spatially smoothing the $h$ field.

2.3.5 Artificial Viscosity under ASPH

In cosmological simulations SPH often shows evidence of nonphysical preheating in shock forming regions, particularly during collapse situations. This can result in poor resolution of shockfronts and related phenomena, as has been recently been pointed out by Kang et al. (1994) in a paper comparing several contemporary cosmological hydro-codes. This behaviour is due to the use of an artificial viscosity (denoted by $\Pi$) in the (A)SPH dynamical equations (eqs. [A.7] & [A.8]). Such an artificial viscosity term is required in the (A)SPH formalism both for stability and because without it (A)SPH is insufficiently dissipative to prevent the interpenetration of converging streams of gas, resulting in a poor representation of shock conditions. However, the artificial viscosity is by definition an artificial term, and ideally its use should be restricted solely to ongoing shocks, where it is required. This problem is particularly accentuated in cosmological studies, since the kinetic energy is typically much larger than the thermal, and the artificial viscosity functions by converting kinetic to thermal energy. Inspection of the standard Monaghan-Gingold (1983) form of the artificial
viscosity (eqs. [A.11] & [A.12]) shows that traditionally II is restricted to only be active for convergent flows within the material being modeled. While a convergent flow is a minimal requirement for the presence of shocks, clearly not all convergent flows necessarily result in the formation of shocks (e.g. homologous collapse). This overuse of II is what leads to the excessive heating of the material around shockfronts, a problem which can be especially troublesome in gravitational collapse scenarios, where the spurious preheating of the gas can become acute enough to interfere with the collapse process itself. As this is precisely the sort of scenario we are concerned with modeling well, we would like to improve upon this algorithm.

This line of reasoning led to the development of the algorithm outlined in Paper I for the suppression of the artificial viscosity. Through experimentation we determined that it is possible to delay the turn-on time for the artificial viscosity in the energy equation (which is the source of the spurious heating), but it is necessary to keep the artificial viscosity active in the momentum equation in order to prevent interpenetration of the (A)SPH nodes. That scheme yields excellent results insofar as suppressing the artificial viscous preheating of material, particularly in the Zel’dovich pancake scenario, for which it was developed. In this paper we adopt a different prescription for suppressing the artificial viscosity. Rather than attempting to delay the turn-on time for II and treating it distinctly between the momentum and energy equations, we instead adopt a different interpolation kernel to be used with II, which we denote as $W_{ij}^{II}$. We choose a form for $W_{ij}^{II}$ such that it is more spatially compact and has a sharper gradient than $W_{ij}$. We use the standard artificial viscosity turn-on criteria
(given in eq. [A.11]), and make no distinction between $\Pi$ in the momentum and energy equations. In effect, this approach can be thought of as restricting the influence of the artificial viscosity spatially, rather than temporally as the scheme of Paper I.

There are many possible choices that could be used for $W_\Pi^{ij}$. In order to successfully suppress the preheating problem while still stopping interpenetration, we want a kernel which is more spatially compact and possesses a stronger gradient as compared with the Spline kernel (eq. [2.11]). We use a simple variant of the Gaussian kernel, as given by

$$W^{\nu-D}_{\text{Gauss2}}(\eta) = A^{\nu-D}_{\text{Gauss2}} \exp \left[ -K (\eta \cdot \eta)^2 \right] = A^{\nu-D}_{\text{Gauss2}} \exp \left[ -K \eta^4 \right], \quad (2.21)$$

where the appropriate normalization constants are

$$A^{1-D}_{\text{Gauss2}} = \frac{2K^{1/4}|G|}{\Gamma(1/4)}, \quad A^{2-D}_{\text{Gauss2}} = \frac{2K^{1/2}|G|}{\pi^{3/2}}, \quad A^{3-D}_{\text{Gauss2}} = \frac{K^{3/4}|G|}{\pi \Gamma(3/4)}. \quad (2.22)$$

The gradient of this kernel is

$$\nabla W^{\nu-D}_{\text{Gauss2}}(\eta) = G \frac{\eta}{\eta} \frac{\partial W^{\nu-D}_{\text{Gauss2}}}{\partial \eta} = -4KW^{\nu-D}_{\text{Gauss2}} \eta^2 G \eta. \quad (2.23)$$

Through experimentation we have found setting $K = 1.5^4$ is a safe choice. Using this kernel as $W_\Pi^{ij}$ successfully prevents interpenetration in all of our test cases, while still reducing the preheating of the ASPH gas in collapse simulations (see the Zel’dovich pancake tests in §2.4.1). Comparison of these results with similar 2-D pancake runs in Paper I reveal that this form of $\Pi$ suppression is not as effective as that presented in Paper I, at least for this class of problems. However, this scheme does have the advantages that it is applicable in all situations (there are no problem dependent
parameters to fiddle), and because $\Pi$ is treated identically in the momentum and energy equations energy conservation is preserved. It is also interesting to note that although this scheme can be implemented under SPH as well as ASPH, in fact when this is done there is relatively little advantage. We attribute this to SPH's inability to optimally adapt the resolution scale to a planar shock, resulting in such poor resolution (relative to ASPH) that this sort of subtle improvement is ineffective.

2.4 Tests

In this section we discuss a set of test cases performed under both ASPH and SPH. These tests have been selected to compare the relative advantages of the two techniques, as well as to test questions about the validity of ASPH (in particular the angular momentum issue). We concentrate on 2-D simulations, for the pragmatic reason that we must run problems in at least 2-D for there to be a distinction between ASPH and SPH (in 1-D the two are formally identical), while 3-D simulations are much more computationally expensive. Granted that ASPH is valid in 2-D, the extension to 3-D represents no fundamental change. We do present a few relatively low-resolution 3-D versions of some of our tests, however, in order to demonstrate the validity of the technique in 3-D. We defer extensive testing of the 3-D formalism for now, though, as 2-D is adequate to demonstrate our major concerns. All simulation results presented as SPH utilize our "Standard" SPH formalism, outlined in Appendix A. All ASPH simulations have been performed using both smoothing of
the G tensor field (§2.3.4) and special treatment of the artificial viscosity (§2.3.5), unless otherwise noted. In appendix C we present a brief discussion of the specific algorithms used to perform these simulations.

We present two broad classes of test cases: those which are physically “1-D” (such as the Zel’dovich pancake problem, the Sedov Blastwave, and a few flavors of the Riemann Shocktube), and those which are physically “2-D” (rotating tests such as the Pseudo-Keplerian Disk and a collapsing disk with angular momentum).

2.4.1 “1-D” Test Cases

This class of test problems, although performed in higher dimensional frameworks, possess symmetries which allow them to considered physically 1-D. These tests are chosen for two reasons. First, they possess analytical solutions, allowing objective comparison and judgment of the results. Secondly, since they are physically 1-D they represent cases where ASPH can be expected to have an advantage over SPH, in that ASPH can recognize and adapt to the physical dimensionality of the problem, whereas SPH cannot. We present four examples of this class of problem: the Zel’dovich pancake (§2.4.1), the Sedov blastwave (§2.4.1), a simple 4:1 Riemann shocktube (§2.4.1), and a double-shocktube with two interacting strong shockfronts (§2.4.1). The first two cases are chosen to typify the sorts of problems we are interested in solving as well as demonstrating the distinction between ASPH and SPH, while the last two represent standard hydrodynamic test cases.
Zel'dovich Pancake Test

This class of problems consists of setting up 1-D plane-wave perturbations in an arbitrary dimension cosmological scenario (Zel'dovich 1970). This is a standard test case for cosmological codes (Efstathiou et al. 1985; Villumsen 1989). As this class of problems has been discussed extensively in Paper I, we need not go over the analytical properties here. We reexamine this problem primarily because the implementation of ASPH used for this work differs from that of Paper I. Pancake collapse represents a simplified example of the sort of cosmological structure formation scenarios we wish to investigate in general.

All of the Zel'dovich Pancake simulations we present are performed under an Einstein-de Sitter cosmology ($\Omega = 1, \Lambda = 0$), with equal baryonic and dark matter mass fractions ($\Omega_{\text{bary}} = \Omega_{\text{dm}} = 0.5$). The baryonic gas is assumed to be a pure hydrogen, adiabatic gas ($\mu = 1, \gamma = 5/3$). Since these simulations are carried out in a cosmological framework, they are evolved in comoving coordinates using a power of the expansion factor ($p = a^n$), rather than time, as the integration variable (see appendix A.2). The system is simulated in a periodic unit volume, using a 2-D Particle-Mesh (PM) gravity calculation to solve for the self-gravitation. We present both 2-D examples and a low-resolution 3-D example of this problem. In 2-D, the gravity obeys a $1/r$ force law, such that each pair of nodes interact gravitationally as though they are a pair of infinite, thin, parallel rods in 3-D. In this way the 2-D simulations can be thought of as a "slice" through an infinite 3-D simulation. Table 2.1 summarizes the major simulation parameters we use for our Zel’dovich pancake
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<th>3-D</th>
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<tr>
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Table 2.1: Simulation Parameters for Zel'dovich Pancake Simulations

Under this framework, we present ASPH and SPH simulations of 2-D Zel'dovich pancakes with \((k_x = 0, k_y = 1)\) and \((k_x = 2, k_y = 1)\), and a 3-D version with \((k_x = 0, k_y = 0, k_z = 1)\). All \(G\) tensors are initialized as SPH spherical tensors with the determinant \(|G|\) scaled appropriately for the local density. This is somewhat inconsistent for ASPH runs, since the linear evolution of an ASPH \(G\) tensor should only affect the geometry of the \(G\) tensors perpendicularly to the plane of collapse. However, at \(a = 1\) the initial conditions are almost uniform (the perturbations are small), and starting in this manner allows both the SPH and ASPH runs to use identical initial conditions.
Single-Wavelength 2-D Zel’dovich Pancakes  This case represents the simplest possible class of this problem: a single-wavelength pancake along one principal axis. Here we choose to use \((k_x = 0, k_y = 1)\), which corresponds to a collapse proceeding in the \(y\) direction – by symmetry, there should be no evolution in the \(x\) direction.

Figures 2.1–2.4 show examples of what we refer to as “kernel plots”. These are plots of smoothing scale isocontours about each (A)SPH node in a given region. In 2-D, SPH kernel plots are circles about each nodes position, whereas in general ASPH kernel plots are ellipses. We generically choose to plot the \(h = 0.2\) isocontour. Since the spline kernel extends to a cutoff radius of \(\eta_{\text{cut}} = 2\), this implies each node “sees” neighbors out to a contour 10 times that shown. We find that these sorts of figures are quite useful in order to gain an intuitive feel for how the \(G\) tensor field is adapting to the local fluid flow. In this case, we have plotted subregions centered about the pancake midplane at expansion factors \(a/a_c = 1.5\) and \(a/a_c = 2.5\). It is clear that ASPH is better able to deal with this sort of 1-D flow, since the ASPH \(G\) tensors can adapt the smoothing scales fully in the \(y\) direction, whereas SPH must adapt smoothing scales isotropically. In other words, the ASPH smoothing scale in the \(y\) direction is better able to evolve as the ideal \(\rho^{-1}\) (since the collapse is a 1-D process), whereas the SPH smoothing scales are constrained to evolve as \(\rho^{-1/2}\). This implies not only that ASPH is better able to sample along the physically interesting dimension where the collapse is occurring, but also avoids the undesirable effect of losing neighbor information parallel to the plane of the pancake. Since the SPH smoothing scale must shrink isotropically as each node falls into the pancake,
while the inter-node spacing parallel to the pancake is not changing, communication between the nodes parallel to the pancake begins to break down. This makes SPH more unstable to perturbations in the x direction, as is evidenced by the disorder seen in figure 2.4. It is also worth pointing out the while formally there should be no evolution of the ASPH smoothing scale parallel to the pancake plane, it is clear that this quantity is changing slightly. This is primarily due to the fact that we preserve the determinant $|G|$ when we smooth the $G$ tensor field.

Figure 2.5 shows half-wavelength profiles for the mass density $\rho$, specific thermal energy $u$, velocity $v$, and pressure $P$ for the baryonic component of these simulations, plotted as a function of distance from the pancake caustic $d_c$. This figure represents the system for $y \geq 0.5$, and can be thought of as looking parallel to the pancake in the $x$ direction through the system. Since there are $64^2$ (A)SPH nodes in these simulations, there are de facto 64 nodes per wavelength (since this is a physically 1-D problem), or 32 nodes per half-wavelength as plotted here. Bear in mind that all nodes in the region $y \geq 0.5$ are plotted, and therefore each of the "points" seen actually represents 64 overlapping points which share the same $x$ coordinate. It is clear where the symmetry in these simulations begins to break down, as these points begin to diverge. This symmetry breaking is particularly evident in the final results at $a/a_c = 2.5$ (figure 2.6) in the SPH simulation. It is clear ASPH's superior resolution along the direction of collapse allows a better representation of the physical state of the system. In particular, ASPH resolves peak densities in the plane of collapse $\sim 3$ times what SPH is capable of with the same number of nodes. Additionally, ASPH
Figure 2.1: "Kernel plot" ($h = 0.2$ smoothing scale isocontours) for the central region in the ($k_x = 0, k_y = 1$) ASPH 2-D Zel’dovich pancake simulation at $a/a_e = 1.5$. 
Figure 2.2: Kernel plot for the \((k_x = 0, k_y = 1)\) ASPH 2-D Zel’dovich pancake simulation at \(a/a_c = 2.5\).
Figure 2.3: Kernel plot for the \((k_x = 0, k_y = 1)\) SPH 2-D Zel'dovich pancake simulation at \(a/a_c = 1.5\).
Figure 2.4: Kernel plot for the \((k_x = 0, k_y = 1)\) SPH 2-D Zel'dovich pancake simulation at \(a/a_c = 2.5\).
proves much more stable against symmetry breaking than SPH, since ASPH nodes are able to sample effectively parallel to the pancake even under extreme collapse. Comparison of these plots with analogous figures in Paper I shows that the results are fundamentally the same, with the exception that the shock transition is more tightly constrained under the formalism of Paper I. This distinction is due to the difference in the methods used to suppress the artificial viscosity. The method described in Paper I is clearly more effective at capturing the sharp shock transition.

Figure 2.7 shows the evolution of global energies throughout these simulations. The Layzer-Irvine energy sum \( L.I. = a^4(K.E. + T.E.) + a\Phi - \int \Phi da \) (Efstathiou et al. 1985) demonstrates that both techniques (ASPH and SPH) conserve global energy equivalently, with fluctuations in this total of less than a percent the total fluctuation in its components.

**Multiple-Wavelength 2-D Zel'dovich pancakes** We now present a slightly modified pancaking problem: that of \( (k_x = 2, k_y = 1) \). In this case there are \( 5^{1/2} \) wavelengths in our computational volume, each tilted with respect to the principal axes. This problem represents no fundamental change from the previous example, but rather provides an example where the physical problem does not align with any special simulation symmetry (i.e., there is no alignment with either a simulation axis or any special direction with respect to the initial node seeding). The only real difference between this and the previous single-wavelength example is that now the resolution
Figure 2.5: Half-wavelength profiles of density $\rho$, specific thermal energy $u$, velocity $v$, and pressure $P$ as a function of distance from the pancake caustic $d_c$ for the $(k_x = 0, k_y = 1)$ 2-D Zel'dovich pancake simulations at $a/a_c = 1.5$. All quantities are converted to proper coordinates and are expressed in units of the critical density, pancake wavelength, and the Hubble time at the beginning of the simulation. The solid lines are the analytical expectations, the square points ASPH results, and the crossed points SPH.
Figure 2.6: Half-wavelength profiles of density $\rho$, specific thermal energy $u$, velocity $v$, and pressure $P$ as a function of distance from the pancake caustic $d_c$ for the $(k_x = 0, k_y = 1)$ 2-D Zel'dovich pancake simulations at $a/a_c = 2.5$. 
Figure 2.7: Evolution of the global energies (kinetic, thermal, potential, and Layzer-Irvine sum) for the \((k_x = 0, k_y = 1)\) 2-D Zel'dovich pancake simulations.
per wavelength is effectively reduced, since we have increased the number of waves in the box with the same number of particles. There are now effectively $64/5^{1/2} \approx 28$ nodes per wavelength, or $\sim 14$ nodes per half-wavelength.

Figures 2.8 and 2.9 present kernel plots for both the SPH and ASPH simulations at expansion $a/a_c = 2.5$. It is clear that ASPH has successfully adapted to the tilted geometry of this problem. Additionally, there is no evidence of the transverse perturbations evident in the single-wavelength case (as is particularly notable for SPH at $a/a_c = 2.5$, figure 2.4). This difference is due to the fact that in the $(k_x = 0, k_y = 1)$ case we are trying to stack up nodes exactly one on top of the other. Any deviation from this perfect line-up will be amplified, and lead to a breakdown in the symmetry. In the $(k_x = 2, k_y = 1)$ case we have broken that symmetry, and therefore the instability is greatly reduced. Figures 2.10 and 2.11 present half-wavelength profiles for both simulations at expansion factors $a/a_c = 1.5$ and $a/a_c = 2.5$. The results are fundamentally similar to those found in figures 2.5 and 2.6, in that ASPH again resolves central densities $\sim 3$ times that possible under SPH. The only difference is that it is evident that these results are lower-resolution that the single-wavelength case. Finally, figure 2.12 shows the evolution of the global energies for these simulations. Again there is little qualitative change from the single-wavelength case, with a total energy conservation violation of less than a percent.

**Single-Wavelength 3-D Zel’dovich Pancakes** We now revisit the single wavelength version of the problem in 3-D with $(k_x = 0, k_y = 0, k_z = 1)$. These 3-D simulations are much lower resolution than the 2-D cases – with only $32^3$ nodes in
Figure 2.8: Kernel plot for the \((k_x = 2, k_y = 1)\) ASPH 2-D Zel’dovich pancake simulations @ \(a/a_c = 2.5\).
Figure 2.9: Kernel plot for the \((k_x = 2, k_y = 1)\) SPH 2-D Zel'dovich pancake simulations \(a/a_c = 2.5\).
Figure 2.10: Half-wavelength physical profiles of density $\rho$, specific thermal energy $u$, velocity $v$, and pressure $P$ as a function of distance from the pancake caustic $d_c$ for the $(k_x = 2, k_y = 1)$ 2-D Zel'dovich pancake simulations at $a/a_c = 1.5$. Units and plotting conventions are the same as used in figure 2.5.
Figure 2.11: Half-wavelength physical profiles of density $\rho$, specific thermal energy $u$, velocity $v$, and pressure $P$ as a function of distance from the pancake caustic $d_c$ for the $(k_x = 2, k_y = 1)$ 2-D Zel'dovich pancake simulations at $a/a_c = 2.5$. 
Figure 2.12: Evolution of the global energies (kinetic, thermal, potential, and Layzer-Irvine sum) for the \((k_x = 2, k_y = 1)\) 2-D Zel'dovich pancake simulations.
3-D as compared with $64^2$ in 2-D, we have effectively lowered the spatial resolution by a factor of two and the mass resolution by a factor of 8. In figures 2.13–2.16 we present kernel plots of an ASPH and an SPH simulation at $a/a_c = 1.5$ and $a/a_c = 2.5$. For clarity we have selected out only a single plane of nodes in the central regions of the simulation to display. The surfaces are again $h = 0.2$ isocontours—note that in general ASPH kernels are ellipsoidal in shape, while SPH are spherical. It is evident that ASPH retains the ability to recognize and adapt to the 1-D nature of this collapse, flattening out the kernels as they fall into the pancake caustic. The SPH kernels, which are restricted to remain spherical, are not able to adapt as well to the ongoing collapse. Comparing figures 2.14 and 2.16 (at $a/a_c = 2.5$), it is clear that the SPH simulation is breaking the symmetry of the problem, as nodes begin to slip by one another and become disorganized in the pancake midplane. The ASPH simulation maintains the symmetry much more effectively.

We can see the physical effects of these differences in the profiles presented in figures 2.17–2.20. Looking at the density profiles at $a/a_c = 1.5$ in figures 2.17 and 2.19, we can see that the SPH simulation’s resolution is so poor at this time that it entirely misses the position of the shock transition. Comparing the density profiles at $a/a_c = 2.5$ in figures 2.18 and 2.20, the ASPH simulation follows the analytical profile essentially as far the nodes can resolve (down to scales $d_c/\lambda_{\text{pan}} \sim 10^{-3}$), whereas the SPH simulation fails to on scales of $d_c/\lambda_{\text{pan}} \sim 10^{-2}$, becoming quite disordered on smaller scales. At all times the shock transition is both more sharply defined and
Figure 2.13: Kernel plots for the ASPH 3-D Zel'dovich pancake simulation at $a/a_c = 1.5$. Shown is a thin slice ($x \in [0.4, 0.6], y \in [0.50, 52], z \in [0.4, 0.6]$) out of a unit volume simulation, rotated slightly for clarity. The $h = 0.2$ isocontours are drawn as wire-mesh surfaces.
Figure 2.14: Kernel plot for the ASPH 3-D Zel'dovich pancake simulation at $a/a_c = 2.5$. 
Figure 2.15: Kernel plot for the SPH 3-D Zel'dovich pancake simulation at $a/a_c = 1.5$. 
Figure 2.16: Kernel plot for the SPH 3-D Zel'dovich pancake simulation at $a/a_c = 2.5$. 
better localized under ASPH. Finally in figure 2.21 we present the evolution of the global energies throughout these 3-D simulations. As in 2-D, we find that both techniques conserve energy to better than 1%.

**Sedov Blastwave Test**

We will now turn to an entirely different class of test problems: that of an intense explosion in a gas. This problem possesses a set of well-known similarity solutions (Sedov 1959). We simulate this problem in both 2-D and 3-D, and compare the results to Sedov's solutions in the appropriate geometry. This problem represents a somewhat difficult case for Lagrangian techniques such as (A)SPH, since the void is dynamically important.

The formal initial condition for this problem is to introduce an intense point source of thermal energy into an initially pressureless, homogeneous gas. This immediately poses a problem for any (A)SPH formalism, as (A)SPH cannot represent a discontinuous energy distribution. Therefore, in order to initialize this problem we distribute a thermal energy spike amongst a small number of nodes in the gas, and then smooth this distribution. This results in an energy spike in the gas resembling the shape of our smoothing kernel, as this is the closest (A)SPH can come to representing a delta function. The remaining nodes are initialized with a small, but finite, internal energy in order to simulate an initially pressureless gas. All G tensors are initialized
Figure 2.17: Half-wavelength profiles of density $\rho$, specific thermal energy $u$, velocity $v$, and pressure $P$ as a function of distance from the pancake caustic $d_c$ for the ASPH ($k_x = 0, k_y = 0, k_z = 1$) 3-D Zel’дович pancake simulation at $a/a_c = 1.5$. All quantities are converted to proper coordinates and are expressed in units of the critical density, pancake wavelength, and the Hubble time at the beginning of the simulation. Solid lines represent the analytical expectations.
Figure 2.18: Half-wavelength profiles of density $\rho$, specific thermal energy $u$, velocity $v$, and pressure $P$ as a function of distance from the pancake caustic $d_c$ for the ASPH ($k_x = 0, k_y = 0, k_z = 1$) 3-D Zel’ dovich pancake simulation at $a/a_c = 2.5$. 
Figure 2.19: Half-wavelength profiles of density $\rho$, specific thermal energy $u$, velocity $v$, and pressure $P$ as a function of distance from the pancake caustic $d_c$ for the SPH ($k_x = 0, k_y = 0, k_z = 1$) 3-D Zel'dovich pancake simulation at $a/a_c = 1.5$. 
Figure 2.20: Half-wavelength profiles of density $\rho$, specific thermal energy $u$, velocity $v$, and pressure $P$ as a function of distance from the pancake caustic $d_c$ for the SPH ($k_x = 0$, $k_y = 0$, $k_z = 1$) 3-D Zel'dovich pancake simulation at $a/a_c = 2.5$. 
Figure 2.21: Evolution of the global energies (kinetic, thermal, potential, and Layzer-Irvine sum) for the 3-D Zel'dovich pancake simulations.
Table 2.2: Simulation Parameters for Sedov Blastwave Simulations

<table>
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<tr>
<th>Parameter(s)</th>
<th>2-D</th>
<th>3-D</th>
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<td>Periodic simulation volume</td>
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<td></td>
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<tr>
<td>$t_f$</td>
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<td></td>
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<td>$\in [1.0 \times 10^{-5}, 0.12]$</td>
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</table>

as identical, spherical SPH tensors appropriate for the undisturbed initial density $\rho_0$ and desired number of significant neighbors per smoothing length $N_h$. The major simulation parameters we use for our Sedov blastwave simulations are summarized in table 2.2.

2-D Sedov Blastwave Simulations We perform three different versions of this test in 2-D: one ASPH, one standard SPH, and one ASPH without smoothing of the $G$ tensors. Figures 2.22–2.24 present kernel plots for these three simulations at time $t = 0.12$. It is clear from these figure that both versions of ASPH are more successful at producing a round, azimuthally symmetric shockfront than the SPH run. The unsmoothed ASPH run produces the roundest shockfront, with ordinary ASPH nearly as good. ASPH is able to produce a better shape for the shock because the ASPH smoothing scale parallel to the shockfront is not shrinking, and therefore
communication between the nodes parallel to the shockfront is maintained. The effects of this difference are evident in figures 2.25–2.27, which show surface plots of the density in one quadrant of the blast for each of these runs. It is clear that the SPH density along the shock front shows much greater fluctuations compared with the ASPH runs, while unsmoothed ASPH maintains the most consistent symmetry.

Figure 2.28 presents a time-series of azimuthally averaged radial density profiles for the three simulations, binned in radial steps of equal numbers of particles. Shown are times in the interval \( t \in [0.02, 0.16] \). The SPH and unsmoothed ASPH runs have been offset for clarity. Since we use \( \gamma = 1.4 \), the strong-shock prediction for the density jump at the shock front is \( \rho_2 / \rho_1 = (\gamma + 1)/(\gamma - 1) = 6 \). The ASPH runs converge to a peak shock density \( \rho_2 > 5 \) fairly rapidly, while the SPH run does not achieve a similar peak density until the end of the simulation, by which time the shock has swept up most of the particles in the simulation. The reason for these differences is that the ASPH \( \mathbf{G} \) tensors are able to adapt to the radial nature of the problem, allowing the radial smoothing scale to adjust much more readily than SPH's isotropic approach allows. We can gauge how well ASPH's anisotropic smoothing is adapting to the predicted density jump by examining how elliptical the kernels are becoming. For the predicted density jump of \( \rho_2 / \rho_1 = 6 \), we would expect the ratio of the shortest to longest axis of an ASPH node in the shock front to be \( 6^{-1} \sim 0.167 \), while we in fact find in the simulation \( h_2/h_1 \sim 0.145 \) at the shock front. It is heartening that, as expected, the SPH results converge with the ASPH as the number of nodes increases. It should be noted, however, that even though the SPH radial profiles do eventually
Figure 2.22: Kernel plot for ASPH 2-D Sedov blastwave simulation at time $t = 0.12$. 

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Figure 2.23: Kernel plot for SPH 2-D Sedov blastwave simulation at time $t = 0.12$. 
Figure 2.24: Kernel plot for ASPH 2-D Sedov blastwave simulations at time $t = 0.12$, without smoothing of the $G$ tensors.
Figure 2.25: Surface plot of the mass density field $\rho$ in one quadrant of the expanding shockfront for the ASPH 2-D Sedov blastwave simulation at $t = 0.12$. 
Figure 2.26: Surface plot of the mass density for the SPH 2-D Sedov blastwave simulation at $t = 0.12$. 
Figure 2.27: Surface plot of the mass density for the ASPH 2-D Sedov blastwave simulation at $t = 0.12$ without smoothing of $G$ tensors.
catch up with ASPH, SPH never achieves as round a shockfront. Figure 2.29 presents radial profiles of the radial velocity \( v_r \), mass density \( \rho \), and pressure \( P \) for all three simulations at \( t = 0.12 \). Also plotted are the cylindrical Sedov solutions for these quantities (solid lines). The break in the pressure profile at \( r \sim 0.2 \) lies between the mass elements which mark the edge of the initial thermal energy spike. It is also somewhat curious to note that even though ASPH has the best overall shockfront shape, smoothed ASPH consistently has the superior radial profiles.

Finally, figure 2.30 presents the evolution of the global energies for these simulations. During the ASPH runs the total energy fluctuates peak to peak \( \Delta E/E \sim 2\% \) the fluctuation of its components, while SPH only suffers fluctuations of \( \Delta E/E \sim 0.2\% \). It is not unexpected for ASPH to have slightly worse energy conservation than SPH, because ASPH introduces more degrees of freedom into the problem. The difference is analogous to comparing SPH with a fixed smoothing scale to SPH with variable smoothing.

3-D Sedov Blastwave Simulations  We also examine 3-D simulations of the Sedov blastwave problem under both ASPH and SPH. In this case we use \( N = 32^3 \) (A)SPH nodes, which results in a linear resolution of only one quarter what we have in 2-D. We can therefore expect the 3-D simulations to be much lower-resolution than the previous 2-D case.

Figures 2.31–2.32 present kernel plots for the ASPH and SPH 3-D Sedov blastwave simulations at time \( t = 0.1 \). We only plot a thin slice through the volume (selecting out a single plane of nodes) for clarity. It is clear that the ASPH kernel shapes
Figure 2.28: Azimuthally averaged radial density profiles for ASPH (open squares), SPH (crosses), and unsmoothed ASPH (filled squares) 2-D Sedov blastwave simulations at times $t = 0.02, 0.04, 0.06, 0.08, 0.10,$ and $0.16$. The SPH and unsmoothed ASPH simulations have been offset radially for clarity – the ASPH curves show the correct radial position. Each point represents the average of a radial bin containing 100 nodes.
Figure 2.29: Azimuthally averaged radial profiles of mass density $\rho(r)$, pressure $P(r)$, and radial velocity $v_r(r)$ for the 2-D Sedov blastwave simulations at time $t = 0.12$. Shown are ASPH (open squares), SPH (crosses), and unsmoothed ASPH (filled squares). Each point represents an average for a radial bin containing 100 nodes. The solid lines show the cylindrical Sedov solutions.
Figure 2.30: Evolution of the global energies (kinetic, thermal, and total) for the 2-D Sedov blastwave simulations.
are indeed flattening radially as expected, while the SPH nodes show relatively little change in the expanding shockfront. This difference is expected, since in 3-D the ASPH $G$ tensors are still able to approach the $\rho^{-1}$ evolution of the smoothing scale in the radial direction, while the SPH smoothing scales are constrained to evolve much more slowly as $\rho^{1/3}$. Also, as is evident in 2-D, the ASPH nodes in 3-D retain their ability to adapt to the physical, spherical geometry of the problem well, avoiding the grid based artifacts evident in the SPH node distribution.

Figure 2.33 presents a time sequence of radial profiles for these simulations. Each point represents an average of 50 particles in a radial bin. The solid lines show the Sedov similarity solution predictions. The lower resolution of these simulations in comparison with the 2-D simulations is quite evident, as both simulations fall well short of the predicted density jump $\rho_2/\rho_1 = 6$. If as in the 2-D case we examine the typical compression of an ASPH kernel in the shock front, we find that typically the ratio of the shortest to longest axis is $h_3/h_1 \sim 0.4$, as compared with the theoretical prediction $6^{-1} \sim 0.167$. Nevertheless, it is clear that ASPH is still able to resolve the density jump more effectively than SPH, exceeding SPH's maximum density by 50% at $t = 0.1$ and localizing the shock more precisely. Figure 2.34 plots the radial profiles of the density, pressure, and velocity at $t = 0.1$ of both simulations against the analytical solutions. It is evident that ASPH is better able to reproduce the analytical profiles in all these quantities, though again the lack of resolution due to
Figure 2.31: Kernel plot for the ASPH 3-D Sedov blastwave simulation at time $t = 0.1$. Only a small slice through the expanding blastwave is shown for clarity: the region plotted is $(x \in [0,0.45], y \in [0,0.45], z \in [0,0.02])$ out of a unit volume.
Figure 2.32: Kernel plot for the SPH 3-D Sedov blastwave simulation at time $t = 0.1$. 
paucity of particles hurts both simulations. Finally, figure 2.35 shows the evolution of the global energies. In this case ASPH conserves energy to better than $\Delta E/E \sim 1\%$, while SPH conserves to $\Delta E/E \sim 0.1\%$.

**Riemann shocktube test**

The Riemann shocktube is a well-known test problem to which SPH codes are traditionally subjected (Monaghan & Gingold 1983; HK89; Rasio & Shapiro 1991). This problem is also examined in Paper I, but only with a 1-D code. We reexamine it here since the ASPH formalism differs from that of Paper I, and because the treatment here is 2-D. As this is such a well-known problem, we will only briefly outline the setup. The initial conditions consist of placing two regions of gas, differing in density and pressure, adjacent to one another across an interface. This discontinuity will collapse in a quasi-analytically known manner. Because this problem requires a fairly large number of nodes along the collapse dimension to get reasonable results, we have chosen to initialize it in a slightly different manner than previous problems. In this case, our computational volume is a rectangular strip (of aspect ratio 4:1) with the long axis aligned with the direction of collapse. This allows us to have more of our nodes in the direction of interest, while still maintaining a 2-D simulation. We should also note that since this simulation is periodic, we in fact have two shocktubes evolving in our volume. Therefore, even though we have 150 rows of particles along our $x$ dimension, we really only have 75 rows per shocktube. All G tensors are initialized as circular SPH G tensors appropriate for the local density. Table 2.3 presents the
Figure 2.33: Average radial density profiles for ASPH (open squares) and SPH (crosses) 3-D Sedov blastwave simulations at times $t = 0.01, 0.04, 0.06,$ and $0.10$. Each point represents an average of a radial bin containing 50 nodes. The solid lines show the Sedov solution.
Figure 2.34: Average radial profile of density $\rho(r)$, pressure $P(r)$, and radial velocity $v_r(r)$ for 3-D Sedov blastwave simulations at $t = 0.1$. The squares represent the ASPH result, crosses SPH, and the solid lines show the spherical Sedov solutions.
Figure 2.35: Evolution of the global energies (kinetic, thermal, and total) for the 3-D Sedov blastwave simulations.
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<tr>
<td>(N_h)</td>
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</tr>
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<td>(\Delta t)</td>
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</tr>
<tr>
<td>(h)</td>
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</tr>
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</table>

Table 2.3: Simulation Parameters for Riemann shocktube simulations

major simulation parameters for this test.

Figures 2.36 & 2.37 show kernel plots for a region including one of the shockfronts at \(t = 0.15\), the end state of the simulations. Note that the ASPH \(G\) tensors are only slightly deformed by the relatively gentle density evolution of this problem, with axis ratios in the range \(h_2/h_1 \in [0.63,1.0]\), \(<h_2/h_1> = 0.87\). Figure 2.38 shows the \(x\) profiles for the mass density \(\rho\), pressure \(P\), and velocity \(v_x\) for the regions shown in figures 2.36 & 2.37, along with the analytic solutions for these quantities. We plot all nodes in this region, so each point plotted actually represents many overlapping points as we project in the \(y\) direction through the system. There is little evidence for symmetry breaking, as the points overlap nicely. ASPH and SPH appear to solve this problem equivalently, with little distinction between the two. This is not surprising, as the density evolution is fairly gentle and therefore there is little need for large dynamic range in the resolution scale.
Figure 2.36: Kernel plots for 2-D Riemann shocktube ASPH simulation at $t = 0.15$. 
Figure 2.37: Kernel plots for 2-D Riemann shocktube SPH simulation at $t = 0.15$. 
Figure 2.38: Profiles of mass density $\rho(x)$, pressure $P(x)$, and velocity $u_x(x)$ along the $x$ axis for the ASPH (squares) and SPH (crosses) 2-D Riemann shocktube simulations at $t = 0.15$. Solid lines show the analytical profiles.
Two Interacting Blast Waves

We will now discuss a test problem popularized by Woodward & Colella (1984), which has become a fairly standard hydrodynamic test. This problem involves multiple interactions of strong shocks, rarefaction waves, and contact discontinuities, and is in general a rather stringent test. The initial conditions are discussed in detail in Woodward (1982) and Woodward & Colella (1984), so we only summarize here. A $\gamma = 1.4$ gas is initialized at unit density ($\rho = 1$) in a unit length. Three regions of differing pressure are established according to

$$P(x) = \begin{cases} 
1000, & x \in [0, 0.1]; \\
0.01, & x \in [0.1, 0.9]; \\
100, & x \in [0.9, 1]. 
\end{cases} \quad (2.24)$$

The system is evolved within reflecting boundary conditions. Note that these initial conditions result in two strong blast-waves (of Mach numbers ~ 170 and ~ 51 respectively) propagating toward one another through the low pressure gas, as well as two rarefaction waves moving backwards through the hot gas. In addition to Woodward & Colella's comparisons, this problem has also been investigated under 1-D SPH by Steinmetz & Müller (1993) and in 2-D under the free Lagrangian method by Whitehurst (1995). In order to mimic the reflecting boundary conditions, we use a periodic volume with the initial conditions of equation (2.24) mirrored about $x = 0$. Additionally, as in the previous Riemann shocktube this problem requires a fair degree of linear resolution in the direction of evolution, so we evolve the system in a rectangular strip of gas. Even using an exceedingly thin strip of gas we still only have marginal resolution along the direction of interest (256 nodes in the cases shown here). Table 2.4 summarizes our major simulation parameters for this test.

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<table>
<thead>
<tr>
<th>Parameter(s)</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Periodic simulation volume</td>
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</tr>
<tr>
<td>(\rho_0)</td>
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</tr>
<tr>
<td>(P(</td>
<td>x</td>
</tr>
<tr>
<td>(P(0.1 &lt;</td>
<td>x</td>
</tr>
<tr>
<td>(P(</td>
<td>x</td>
</tr>
<tr>
<td>(\gamma)</td>
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<tr>
<td>(t_f)</td>
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<tr>
<td>(N)</td>
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<td>(N_h)</td>
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</tr>
<tr>
<td>(dt)</td>
<td>([10^{-8}, 10^{-3}])</td>
</tr>
<tr>
<td>(h)</td>
<td>([10^{-5}, 0.125})</td>
</tr>
</tbody>
</table>

Table 2.4: Simulation Parameters for Double Blast Wave simulations

Figures 2.39–2.40 show kernel plots for the region \((x, y) \in ([0.55, 0.9], [0, 0.125])\) at the end state \(t = 0.038\) of these simulations. In accordance with the 1-D nature of this problem, the ASPH G tensors adapt primarily in the \(x\) direction, and remain marginally better ordered (closer to the initial lattice) than the SPH case. Figures 2.41–2.42 show the velocity and mass density profiles for these simulations at the same times slices shown in both Woodward & Colella (1984) and Steinmetz & Müller (1993), in order to facilitate comparison. Both ASPH and SPH represent this problem reasonably well, though note that at time \(t = 0.028\) (just as the two shockfronts collide) the ASPH simulation resolves a much higher density spike (of order \(\rho_{peak} \sim 30\)), as compared with SPH (which finds \(\rho_{peak} \sim 14\)). The predicted value for this density jump is 24, so ASPH slightly overshoots. The higher ASPH value is a direct result of ASPH’s ability to achieve superior spatial resolution in this small region.
Figure 2.39: Kernel plot for the ASPH 2-D Woodward double blastwave simulation at $t = 0.038$ in the sub-region ($x \in [0.55, 0.9], y \in [0, 0.125]$).

We also note that in the regions where there is evident disorder in these profiles (particularly behind the rarefaction wave moving through the initially $P = 1000$ gas) the system is hitting the maximum smoothing scale we can allow due to the narrow width of the simulation, so some fraction of this scatter is in fact likely an artifact. In general, as found by Müller & Steinmetz (1993), both SPH and ASPH seem capable of solving this problem, despite the commonly held belief that SPH cannot deal with strong shock phenomena.

2.4.2 “2-D” Test Cases

Each of the previous test cases represent idealized situations with no “handedness” to the problem (i.e. they are physically 1-D). Additionally, due to the physical symmetry of these 1-D problems, the $G$ tensors are able to align themselves such that all
significant interactions take place along one of their major axes. In this section we will investigate problems which break these symmetries and are truly 2-D, concentrating on rotational test cases in order to investigate the question of angular momentum conservation under ASPH (discussed in §2.3.4). Specifically we will examine a gas disk subjected to an external Keplerian potential (§2.4.2) and a self-gravitating disk undergoing radial collapse with rotation (§2.4.2).

Before we begin this discussion, though, we should address why we do not examine an obvious test case for angular momentum. The simplest example of a problem with a well-defined angular momentum is a rigidly rotating bar, something with an equation of state like a solid so that it should simply maintain solid body rotation. There is, however, an unfortunate flaw with such a deceptively simple system, which is that it possesses an edge. (A)SPH is derived assuming that there are no distinct surfaces
Figure 2.41: Velocity ($v_x$) and mass density $\rho$ profiles plotted against $x$ for the ASPH simulation of the 2-D Woodward double blastwave problem. Shown are profiles for times $t = 0.01, 0.016, 0.026, 0.028, 0.030, 0.032, 0.034,$ and $0.038$, which are chosen for direct comparison to the corresponding figures in Woodward & Colella (1984) and Steinmetz & Müller (1993).
Figure 2.42: Velocity \((v_x)\) and mass density \(\rho\) profiles plotted against \(x\) for the SPH simulation of the 2-D Woodward double blastwave problem, as in Figure 2.42.
Figure 2.43: Evolution of the global energies (kinetic, thermal, and total) for the 2-D Woodward double blastwave simulations.
or edges present in a modeled system, but rather that all quantities vary smoothly. (A)SPH sampled averages, such as defined by equations (2.1)-(2.2), will be in error near an edge. Of particular concern for ASPH is the fact that the estimates for the spatial gradients of the velocity field \( \partial u_\alpha / \partial x_\beta \) (eq. [2.15]) will be in error near any edges, which ensures that the evolution of the \( G \) tensor will also be incorrect. Figure 2.44 presents images of the SPH estimates of the elements of \( \partial u_\alpha / \partial x_\beta \) for a 2-D bar rigorously in solid body rotation. If these estimates were correct, then the images should be uniform throughout. This figure clearly shows that, as expected, the SPH estimates are in error near the edges, by as much as 50%. So long as this is the case, the \( G \) tensors near the edges of the system will not be evolved correctly, resulting in a false torque about the surface, and thereby violating angular momentum conservation. We have in fact found this to be the case for ASPH simulations of such systems, and the magnitude of the angular momentum violation scales as the number of surface nodes in the system. It is interesting to note that SPH cannot deal with surfaces any better than ASPH, but because by construction SPH cannot violate global angular momentum conservation, such tests are meaningless. This is simply an example that global conservation alone does not guarantee a successful simulation. In fact SPH simulations which possess such edges must also be incorrect due to edge effects, but the error cannot show up in the global angular momentum - most likely this results in erroneous local transport of angular momentum. Fisher & Owen (1997) are currently investigating a modified technique based upon SPH designed to account and correct for such problems.
Figure 2.44: Gray-scale images of the SPH estimated components \( \frac{\partial v_x}{\partial x} \) and \( \frac{\partial v_y}{\partial x} \) for a bar in solid-body rotation with angular velocity \( \omega \). These quantities formally should be constant throughout the bar.
In order to investigate the angular momentum issue, it is necessary to study a non-periodic system, which requires the density fall to zero at some point. For this reason, we have investigated rotating disk systems in this paper, in an effort to have the density fall off smoothly to an edge. The fact that there is still an edge is somewhat worrisome, but by having the density fall off smoothly we hope to moderate this problem. Additionally, these sort of systems are more representative of the sorts of rotating problems we are likely to encounter in cosmological structure formation scenarios, such as if we were to model a disk galaxy.

Pseudo-Keplerian Disks

Our first rotational test is a type of Keplerian disk, modified to include pressure support. Since this is not a standard test case, we will describe the initial conditions in some detail. A gas disk is created in rotational and pressure balance with a fixed, external gravitational potential produced by a theoretical point mass at the disk's center. The initial radial density and pressure profiles of the disk are chosen arbitrarily. The disk is not self-gravitating. In order to make this problem computationally feasible, it is necessary to use a softened potential for the point mass

$$\Phi(r) = -\frac{GM}{(r^2 + r_c^2)^{1/2}} \Rightarrow \nabla(r) = -\frac{GMr}{(r^2 + r_c^2)^{3/2}},$$

(2.25)

where $M$ is the mass of the gravitating point mass and $r_c$ is the softening core radius. Note that the nodes in this gravitational potential are treated as point masses in 3-D, not as infinite rods (as was the case for the Zel'dovich pancake simulations). We are now simulating gas confined to a plane.
The forms we choose for the initial radial mass density and pressure profiles are

\[ \rho(r) = \rho_0 \left(1 - \frac{r^2}{r_d^2}\right), \quad (2.26) \]

\[ P(r) = K \rho^2(r) = K \rho_0^2 \left(1 - \frac{r^2}{r_d^2}\right)^2, \quad (2.27) \]

where \( \rho_0 \) is the central gas density, \( r_d \) is the outer radius of the gas disk, and \( K \) is a constant which sets the amount of pressure support. Based on these choices, the balancing rotational velocity of the gas is

\[ v_t^2(r) = \frac{GMr^2}{(r^2 + r_d^2)^{3/2}} - 4K \rho_0 \frac{r^2}{r_d^2}, \quad (2.28) \]

where \( v_t(r) \) is the supporting circular (or tangential) velocity. Equation (2.28) limits the possible range of the pressure constant \( K \) to

\[ K \in \left[0, \frac{GMr_d^2}{4\rho_0(r_c^2 + r_d^2)^{3/2}}\right]. \quad (2.29) \]

We generically choose the largest possible value for \( K \) in order to maximize the amount of pressure support. Note that maximizing \( K \) in this fashion sets \( v_t(r_d) = 0 \).

There is no physical motivation for choosing this problem to study. Rather, this test is designed solely for the purpose of testing ASPH. Since there is no preferred resolution direction, ASPH has no real advantage under this problem. Therefore ASPH’s extra degrees of freedom, rather than offering a real advantage, can only lead to trouble. For \( r > r_c \), this system possesses a strong radial velocity shear, which will tend to elongate the ASPH \( G \) tensors into the flow. This is a shear field that cannot be represented under the first-order treatment our \( G \) evolution derivation is based upon. The problem can be understood by considering the evolution of a
hypothetical fluid element in such a Keplerian potential: for \( r > r_c \) a true fluid element will distend and eventually be infinitely sheared around an arc of the disk. This creates a situation which our first-order derivation of \( \mathbf{G} \) and its evolution cannot ideally adapt to (an ellipse cannot be distorted to follow such a curving arc and remain an ellipse). Therefore our evolution equations for the \( \mathbf{G} \) tensor field must fail at some level for this case. Additionally, the imposed point mass potential is very strongly centralized, whereas our initial density profile is flat-topped. This situation is unstable, and although the system is born in radial balance, we can expect as it evolves it will rapidly deviate from the initial conditions. These effects will tend to force the ASPH \( \mathbf{G} \) tensor field to evolve under conditions it cannot simply, nor ideally, adapt to. Finally, our use of an external central force as the binding global potential will rigorously ensure angular momentum conservation for the gravitational interactions. Any angular momentum errors incurred are therefore due to either numerical problems or ASPH.

In order to initialize the (A)SPH node positions, we first select candidate positions quasi-randomly based on the Sobol sequence. These potential positions are subjected to a Monte-Carlo acceptance/rejection scheme, with a probability distribution appropriate to match the density profile of equation (2.26). Once a node position is selected, the specific thermal energy is uniquely identified by the required pressure profile in combination with the theoretical density. The \( \mathbf{G} \) tensors are initialized as SPH \( \mathbf{G} \) tensors with determinants \( |\mathbf{G}| \) scaled appropriately for the theoretical local density. The system is evolved until it is seen to settle into an apparently equilibrium
Table 2.5: Simulation Parameters for Pseudo-Keplerian Disk Simulations

<table>
<thead>
<tr>
<th>Parameter(s)</th>
<th>Value(s)</th>
</tr>
</thead>
<tbody>
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<td>Non-periodic volume</td>
<td>(x ∈ [-0.5, 0.5], y ∈ [-0.5, 0.5])</td>
</tr>
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<td>Gas disk centered on (x_d, y_d)</td>
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</tr>
<tr>
<td>Gravitating point mass M</td>
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</tr>
<tr>
<td>ρ_0</td>
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</tr>
<tr>
<td>γ</td>
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</tr>
<tr>
<td>r_c</td>
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</tr>
<tr>
<td>r_d</td>
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<tr>
<td>Rotation Period at core radius τ(r_c)</td>
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</tr>
<tr>
<td>t_f</td>
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</tr>
<tr>
<td>N</td>
<td>4096</td>
</tr>
<tr>
<td>N_h</td>
<td>2</td>
</tr>
<tr>
<td>dt</td>
<td>∈ [7.8125 × 10^{-6}, 0.001]</td>
</tr>
<tr>
<td>h</td>
<td>∈ [10^{-5}, 0.12]</td>
</tr>
</tbody>
</table>

distribution. Finally, in order to obtain meaningful measurements of the angular momentum, these simulations are evolved in a non-periodic computational volume. Table 2.5 summarizes our input numerical parameters for these simulations.

Figures 2.45 and 2.46 show kernel plots of the central core regions of both the ASPH and SPH simulations at t = 3.0. In order to emphasize the steady rotation field, we also plot arrows for each node indicating both the direction and relative magnitude of the velocity. Though grossly similar, there are interesting differences in these two simulations. It appears that the core of the ASPH simulation is relatively denser or more collapsed as compared with SPH. The ASPH kernels seem to be elongated into the direction of rotation, and there is an apparent trend for the ASPH
kernels’ ellipticity to increase with radius. The geometry of the G tensors ranges from $h_2/h_1 \in [0.5, 1.0]$, $<h_2/h_1> = 0.75$. This behaviour is understandable in terms of our smoothing algorithm. At all radii the rotational shearing field tends to elongate the G tensors into the direction of rotation. At “small” radii ($r/h \sim 1$), the requirement that each node sample several neighbors implies that each sees neighboring nodes which are significantly further around the arc of rotation, and therefore will be trying to elongate their G tensors in noticeably different directions. When the G tensor field is smoothed, averaging over many different G's, elongated in different directions will result in an average round shape. As we move out to larger radii ($r/h \gg 1$), this averaging process will progressively sample more and more “local” conditions. In this case, each ASPH node sees itself embedded within a coherently shearing field, with neighbors elongating in a similar manner. Averaging in this case retains the elliptical shape, though limits how elliptical each element can become. This sort of behaviour is desirable. When an ASPH node is in a region where on scales of $h$ there are conflicting signals dictating the evolution of G, the “safest” choice is for G to adopt a round shape, emulating SPH. This reflects the fact that the evolution of the G tensor is based upon approximating the local velocity field by the first-order argument $v(r+dr) \approx v(r)+σdr$. If this approximation is invalid, then the evolution equations for G break down. Only when there is a clear, unambiguous signal dictating the evolution of G should it be allowed to deviate from this “safe” choice.
Figure 2.45: Kernel plot for the ASPH 2-D Pseudo-Keplerian Disk simulations at $t = 3.0$. Arrows are drawn for each node indicating the direction and magnitude of their velocity.
Figure 2.46: Kernel plot for the SPH 2-D Pseudo-Keplerian Disk simulations at $t = 3.0$. 
In order to examine these different mass distributions in a more quantitative fashion, figures 2.47 and 2.48 show the distribution of $M(r)$ and $\rho(M)$ respectively, measured radially outward from the center of the potential. Note that these two figures are plotted subtly differently: figure 2.47 shows the mass fraction as a function of radius $M(r)/M_\text{tot}$, whereas figure 2.48 shows the density distribution as a function of the radially enclosed mass fraction $\rho(M_{\text{interior}}/M_\text{tot})$. Each figure plots many curves: the upper panels show the evolution of these functions from the beginning to the end of the simulations, while the lower panels only show the final few measurements. It is evident that in the beginning the mass distribution evolves fairly steadily up to a point, and then settles into an equilibrium (or at least slowly evolving) state. In figure 2.47 we can see that the total ASPH mass distribution is slightly more diffuse than SPH, indicating more mass has been thrown to the outer regions of the disk or become unbound. Figure 2.48 also shows that the core of the ASPH disk is more dense than the SPH by a little more than a factor of two, confirming the visual impression of the kernel plots.

Our primary interest in this problem is the evolution of the angular momentum. Figure 2.49 shows the function $dL_z/dM(M)$, which is the amount of angular momentum contained in shells as a function of the total mass interior to that shell. Figure 2.50 shows the integration of this function $L_z(M) = \int_0^M (dL_z/dM) dM$, which represents the total angular momentum contained within the mass fraction $M$. Since this problem should settle to a condition of axisymmetry, we expect that the radial angular momentum distribution should settle to a steady state. It is evident from
Figure 2.47: Azimuthally averaged radial mass profiles for the 2-D Pseudo-Keplerian Disk simulations. Each curve represent the fraction of the mass of the disk contained within the radius $r$ at a particular time. The top panels show the function $M(r)$ for times varying from the beginning of the simulation to the end ($t \in [0, 3]$). The bottom panels show only the last 6 measurements of $M(r)$, in the time range $t \in [2.5, 3]$. 
Figure 2.48: Azimuthally averaged radial density profiles for the 2-D Pseudo-Keplerian Disk simulations, expressed as a function of the enclosed mass fraction $\rho(M_{\text{interior}})$. The top panels show the function $\rho(M_{\text{interior}})$ for times varying from $t \in [0, 3]$, while the bottom panels only show the last few measurements at times $t \in [2.5, 3]$. 
figure 2.49 that the radial distribution of $dL_z/dM$ initially changes rather rapidly, and then settles for both simulations to a fairly steady configuration. The ASPH and SPH distributions of angular momentum are somewhat different, however. ASPH settles into a nearly linear distribution of $dL_z/dM$ with respect to increasing mass, while SPH characteristically maintains more angular momentum in the core region.

These differences are expected since ASPH finds a denser core, and therefore must lose more angular momentum from the core mass. Nevertheless, both simulations do converge to equilibrium, rotating configurations. Figures 2.51 & 2.52 show the evolution of the global energies and angular momentum. SPH of course conserves global angular momentum nearly exactly (by construction), while ASPH suffers an overall fluctuation $\Delta L_z/L_z \sim 3\%$.

Since we do not have an analytical expectation for this problem, it is difficult to know which technique better represents the "true" solution. However, we can state that ASPH appears to conserve the global angular momentum reasonably, and also does not seem to suffer dramatic local transport problems (as evidenced by the fact that the local radial angular momentum distribution establishes an equilibrium state).

It is worth noting that smoothing the $G$ tensors is critical for this problem. In all of the previous tests we found that smoothing $G$ has relatively little impact on the outcome. However, in this case, without smoothing the $G$ tensors rapidly become extremely distorted in the shearing velocity field, which leads to poor angular momentum conservation (angular momentum losses $\Delta L_z/L_z \sim 100\%$ in a single rotation or two). These extreme distortions in the individual $G$ tensors are unphysical, since
Figure 2.49: Azimuthally averaged radial measurements of angular momentum $\Delta L_z$ as a function of the enclosed mass for the 2-D Pseudo-Keplerian Disk simulations. These curves represent a radial measurement of $dL_z/dM$, where $M$ is interpreted as the radially enclosed mass. The top panels show the function $dL_z/dM$ for times varying from $t \in [0, 3]$, while the bottom panels only show the last few measurements at times $t \in [2.5, 3]$. 

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Figure 2.50: The total enclosed angular momentum $L_z$ as a function of the enclosed mass $M$ for the 2-D Pseudo-Keplerian Disk simulations, as measured radially from the center of the disk. These curves of $L_z(M)$ represent the integration $\int_0^M (dL_z/dM) dM$ of the function $dL_z/dM(M)$ presented in figure 2.49.
Figure 2.51: Evolution of the global energies (kinetic, thermal, potential, and total) for the 2-D Pseudo-Keplerian Disk simulations.
Figure 2.52: Evolution of the global angular momentum $L_\phi$ for the 2-D Pseudo-Keplerian Disk simulations.
they are trying to track neighboring ASPH nodes around the arc of rotation, which they cannot do. Smoothing the $G$ tensors limits this process and keeps the distortion manageable, though there may ultimately prove to be a better solution.

Collapsing Disk with Rotation

The Pseudo-Keplerian disk simulation offers encouraging evidence that ASPH can solve rotational problems while maintaining reasonable conservation of the angular momentum. However, it is not convincing in and of itself. Although the Pseudo-Keplerian disk does undergo significant radial evolution, this evolution is relatively gentle in time, and does not involve strongly distorting the $G$ tensors themselves. This is mainly due to the fact that the only strong signal to drive the evolution of the $G$ tensors is the shearing rotational field. We now wish to propose a more stringent test of angular momentum conservation, in which we can expect significant evolution of the $G$ tensors right down into the core, at least initially. In order to accomplish this, we simulate a modified form of a standard Maclaurin disk (Binney & Tremaine 1987), which is an analytically tractable class of rotating hydrodynamic disks. The modification we make to this problem is to take away the majority of its pressure support, making the system unstable to collapse. When the timescale for collapse is significantly shorter than the rotational timescale, the collapse process will dominate the $G$ tensor evolution. Additionally, a rotational problem undergoing
rapid and violent collapse presents a difficult problem both in terms of the transport and conservation of angular momentum, making this an overall much more difficult test.

As the Maclaurin disk is a well-documented problem, we will only briefly outline this setup here, emphasizing our modification of the pressure term. The Maclaurin disks represent a class of gas disks which have radial density, pressure, and circular velocity profiles

\[ \rho(r) = \rho_0 (1 - \frac{r^2}{r_d^2})^{1/2}, \quad (2.30) \]

\[ P(r) = f K \rho^3(r), \quad (2.31) \]

\[ v_t(r) = \Omega r, \quad (2.32) \]

\[ \Omega \in [0, \Omega_0], \quad \Omega_0^2 = \frac{\pi^2 G \rho_0}{2r_d}, \quad (2.33) \]

where \( \rho_0 \) is the central density, \( r_d \) the disk radius, \( K \) a constant normalizing the pressure, \( f \in [0, 1] \) a multiplicative fudge factor, \( \Omega \) the frequency of rotation, and \( \Omega_0 \) the natural frequency of the system. Our modification is the introduction of \( f \), which allows us to tweak the fraction of the required pressure support actually introduced into the system. If we set \( f = 1 \) we recover the traditional Maclaurin disk problem, for which solid body rotation provides radial balance. For \( f < 1 \), the pressure support is inadequate and the disk becomes unstable to collapse.

As with the Pseudo-Keplerian disk, this simulation is performed in a non-periodic computational volume in order make measurements of the global angular momentum meaningful. We solve for the self-gravity utilizing a non-periodic PM code. The gravity calculation treats the nodes as points in 3-D, so we are again considering gas
## Table 2.6: Simulation Parameters for Collapsing Disk Simulations

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<td>Gas disk centered on $(x_d, y_d)$</td>
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</tr>
<tr>
<td>$M_{\text{disk}}$</td>
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</tr>
<tr>
<td>$\Omega/\Omega_0$</td>
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</tr>
<tr>
<td>$f$</td>
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<tr>
<td>$\gamma$</td>
<td>$5/3$</td>
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<td>$dt$</td>
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</tr>
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<td>$h$</td>
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</tr>
<tr>
<td>PM mesh size $N_{\text{grid}}$</td>
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</tr>
</tbody>
</table>

confined to a plane in a 3-D space rather than infinite parallel rods. In order to seed the initial positions of the (A)SPH nodes to match the density profile of equation (2.30), we select candidate positions using a Sobol sequence which are subjected to a monte-carlo acceptance/rejection criterion. All $G$ tensors are initialized as round SPH $G$ tensors, scaled appropriately for the local theoretical density. Table 2.6 summarizes our major numerical parameters for these simulations.

The evolution of this system is quite complex. With $f = 0.001$, the system is highly unstable, undergoing immediate, radical collapse. As the overall radial collapse proceeds, the disk fragments into a collection of filaments due to the gravitational amplification of perturbations in the noisy initial density field. During this
filamentary stage, the ASPH $G$ tensors become quite elliptical, aligning themselves along the filaments. By $t = 0.1$ the system reaches maximum collapse, and this filamentary structure is destroyed. After this time, we find that the system settles into a two-phase structure. There is a hot, dense, rapidly rotating core surrounded by a diffuse, slowly expanding halo. Figures 2.53–2.58 present kernel plots for the ASPH and SPH simulations at times $t = 0.08, 0.1, 0.5$, corresponding to the pre-collapse, maximal-collapse, and post-collapse regimes. We also plot the velocity fields on these kernel plots as vector fields. In figure 2.53 the pre-collapse filamentary structure is evident. During this stage, the ASPH simulation exhibits $G$ tensors with ellipticities in the range $h_2/h_1 \in [0.28, 0.99]$, $\langle h_2/h_1 \rangle = 0.64$. Figure 2.54 shows the system at maximum collapse. There is evident clumpiness of the nodes, with a large range of smoothing scales. The velocity field demonstrates a rather complex structure, though an overall counterclockwise sense of rotation is maintained, reflecting our input angular momentum. Note that the ASPH simulation shows markedly elliptical kernels in this collapsing core region, with axis ratios in the range $h_2/h_1 \in [0.21, 0.98]$, $\langle h_2/h_1 \rangle = 0.59$. Despite this, comparing figures 2.54 & 2.57 demonstrates that both simulations correspond well. Finally in figure 2.56, we find that the system settles into orderly rotation about a dense core. The ASPH simulation exhibits basically round $G$ tensors in the core (with overall $G$ geometries in the range $h_2/h_1 \in [0.46, 1.0]$, $\langle h_2/h_1 \rangle = 0.92$) reminiscent of the Pseudo-Keplerian Disk, which is appropriate as there is no preferred resolution direction.
Figure 2.53: Kernel plot for the ASPH 2-D collapsing disk simulation at $t = 0.08$ (pre-collapse). The arrows associated with each node indicate the direction and magnitude of the velocity.
Figure 2.54: Kernel plot for the ASPH 2-D collapsing disk simulation at $t = 0.1$ (maximum collapse).
Figure 2.55: Kernel plot for the ASPH 2-D collapsing disk simulation at $t = 0.5$ (post-collapse).
Figure 2.56: Kernel plot for the SPH 2-D collapsing disk simulation at $t = 0.08$ (pre-collapse).
Figure 2.57: Kernel plot for the SPH 2-D collapsing disk simulation at $t = 0.1$ (maximum collapse).
Figure 2.58: Kernel plot for the SPH 2-D collapsing disk simulation at $t = 0.5$ (post-collapse).
Figures 2.59 and 2.60 show the evolution of the functions $M(r)$ and $\rho(M)$, analogously to figures 2.47 and 2.48 for the Pseudo-Keplerian disk simulations. In this case, though, we measure the radial coordinate out from the center of mass of the central disk, since this is a self-gravitating system and therefore the minimum of the potential moves with the center of mass. It is clear that the system rapidly forms a concentrated central disk containing $\sim 70\%$ of the mass following the maximum collapse. The radial density gradient of this central disk is also quite large, varying from $\rho \sim 10^3$ in the central regions to $\rho \sim 10^{-1}$ at the edge. The ASPH and SPH simulations both seem to agree well on this mass distribution.

Figure 2.61 shows the radial distributions of $dL_z/dM(M)$ throughout the simulations. It is evident that once the systems achieve equilibrium, the central $70\%$ of the mass (representing the collapsed disk) maintains a steady distribution of angular momentum. However, outside the collapsed disk this distribution becomes quite noisy for the hot, diffuse gas. It is also curious to note that it appears the ASPH simulation maintains slightly more angular momentum in the disk as compared with SPH. This is also evident in the cumulative distribution of $L_z(M) = \int_0^M (dL_z/dM) dM$ in figure 2.62. This trend is the opposite that found in the previous Pseudo-Keplerian disk, where SPH maintains a larger core angular momentum than ASPH. Regardless, it is clear that both ASPH and SPH settle to equilibrium distributions of the local angular momentum once the system settles and axisymmetry is achieved.
Figure 2.59: Azimuthally averaged radial mass profiles for the 2-D collapsing disk simulations. Each curve represents the fraction of the mass of the disk contained within the radius $r$. The top panels show the function $M(r)$ for times varying from the beginning of the simulation to the end ($t \in [0,0.5]$). The bottom panels show only the last measurements of $M(r)$, in the time range $t \in [0.35,0.5]$. 
Figure 2.60: Azimuthally averaged radial density profiles for the 2-D collapsing disk simulations, expressed as a function of the enclosed mass fraction $\rho(M_{\text{interior}})$. The top panels show the function $\rho(M_{\text{interior}})$ for times varying from $t \in [0, 0.5]$, while the bottom panels only show the last few measurements at times $t \in [0.35, 0.5]$. 

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Figure 2.61: Azimuthally averaged radial measurements of angular momentum $\Delta L_z$ as a function of the enclosed mass for the 2-D collapsing disk simulations. These curves represent a radial measurement of $dL_z/dM$, where $M$ is interpreted as the radially enclosed mass. The top panels show the function $dL_z/dM$ for times varying from $t \in [0, 0.5]$, while the bottom panels only show the last few measurements at times $t \in [0.35, 0.5]$. 
Figure 2.62: The total enclosed angular momentum $L_z$ as a function of the enclosed mass $M$ for the 2-D collapsing disk simulations, as measured radially from the center of the disk. The curves of $L_z(M)$ represent the integration $\int_0^M (dL_z/dM) dM$ of the function $dL_z/dM(M)$ presented in figure 2.61.
Finally, in figures 2.63 & 2.64 we present the time evolution of the global energies and angular momentum. It is apparent that at time $t = 0.1$ the system undergoes maximal collapse, at which time all the energies suffer a dramatic spike. After this time the energies achieve equilibrium values. The global angular momentum for the ASPH simulation varies by $\Delta L_z/L_z \approx 3\%$ throughout the simulation, while SPH varies by $\Delta L_z/L_z \approx 1\%$. This error is likely due to the PM code, which does not conserve angular momentum precisely. We therefore conclude that in this test ASPH again conserves angular momentum reasonably, both locally and globally.

2.5 Summary and Conclusions

We present a formalism and justification for allowing anisotropic smoothing under SPH, creating a variant we call Adaptive SPH or ASPH. In comparison with SPH, ASPH is designed to take full advantage of the potential resolution of a simulation for a given number of nodes by recognizing and adapting to anisotropic evolution within the system. While there is no substitute for increasing the number of computational nodes (and thereby increasing the mass resolution), ASPH is a relatively cheap method of fully utilizing the potential spatial resolution of a simulation for a given number of nodes. The ASPH algorithm can be viewed as allowing the local spatial resolution to evolve in accordance with the physical dimensionality of the density evolution, rather than being limited by the imposed simulation dimensionality. When a physically 1-D process (such as a planar shock) is modeled in 3-D under ASPH, ASPH is able to achieve comparable resolution for this problem as though it were simulated in 1-D, while standard SPH is limited by the imposed 3-D simulation framework. As an
Figure 2.63: Evolution of the global energies (kinetic, thermal, potential, and total) for the 2-D collapsing disk simulations.
Figure 2.64: Evolution of the global angular momentum $L_z$ for the 2-D collapsing disk simulations.
integral part of the ASPH formalism we also introduce algorithms for the suppression of the artificial viscosity. In contrast to the artificial viscosity algorithm presented in Paper I, the scheme presented here limits the spatial influence of the artificial viscosity through a judicious choice of the interpolation kernel used with II. While this is not as effective at suppressing the artificial viscous preheating in the Zel’dovich pancake problem as the scheme of Paper I, it is both more general and conservative. We present a set of simulations for physically 1-D problems implemented under 2-D and 3-D SPH and ASPH, using known analytical solutions to allow objective comparison of the results. The conclusion of this set of tests is that ASPH achieves a significant improvement relative to SPH in such intrinsically anisotropic situations.

We also investigate the potential shortcomings of the ASPH technique, the most serious of which is that while SPH is rigorously guaranteed to conserve angular momentum, ASPH is not. Through a set of 2-D rotational problems we show that, with reasonable precautions enforced, ASPH conserves the global angular momentum satisfactorily (of order a few percent) over many dynamical times for the types of rotating gas problems we are interested in. However, some might take issue with our rotational angular momentum tests, arguing that for a circularly symmetric problem (such as a rotating disk) random errors in the angular momentum will on average cancel out around the disk, leading to spurious angular momentum conservation. This viewpoint can be rephrased as proposing that at various points in the disk, eddies of random strength and orientation spontaneously appear and disappear. If such behaviour were occurring, we would expect to see small-scale variations or noise in the time evolution
of the angular momentum, on timescales of order the inter-node interaction timescale. Inspection of the evolution of the angular momentum for the various simulations presented in this paper does not demonstrate this sort of behaviour. Rather, the angular momentum tends to slowly and smoothly change its value over very long timescales, without apparent short timescale fluctuations. This suggests that for each integration pass through an ASPH simulation, the angular momentum is conserved to some finite accuracy, determined by how completely the net torque on each elliptical ASPH tensor cancels. In general this cancellation will not be perfect, since each node must interact with a finite number of neighbors and therefore we can expect some noise in this calculation. Under this viewpoint, the conservation of angular momentum under ASPH is a function of how reliably the system as a whole is being represented. Systems possessing discontinuities such as surfaces represent the sorts of problems which should not be approached under ASPH (or SPH for that matter). The assumptions under which (A)SPH is derived do not allow for such constructs, and the techniques must be modified before such systems can be properly simulated under them. This sort of problem is not an overriding concern for astrophysical problems, since such situations do not typically possess true surfaces. However, in simulations which form dense tightly packed rotating structures such as galaxies, the density gradient is strong enough that these surface problems could be an issue (both for ASPH and SPH). Future work to remedy this type of shortcoming may include fundamentally recasting (A)SPH, and/or incorporating surface terms into the formalism. Fisher & Owen (1997) are currently investigating one such reformulation of SPH.
We also investigate the behaviour of the angular momentum on local scales, looking for signs that ASPH might erroneously transport angular momentum due to false inter-particle torques on the ellipsoidal kernels. We find that in general once the system settles to an axisymmetric configuration, both ASPH and SPH cease to transport the angular momentum, establishing equilibrium distributions. We take this as heartening that ASPH does not result in drastic, spurious transport of the local angular momentum, though this issue still needs to be investigated further. It would be particularly interesting to investigate 3-D rotating problems, to see if spurious torques due to the ellipsoidal shapes of the ASPH kernels lead to incorrect transport of the angular momentum, perhaps even altering the overall direction of the total angular momentum vector. Initial investigations of such questions look promising, but the edge effect problem under ASPH makes it difficult to determine exactly what is going on. It is challenging to create a rotating 3-D problem with sufficient resolution such that any edge effects can be adequately isolated and suppressed. Until such investigations are completed and satisfactory, we would recommend caution employing ASPH in situations where the angular momentum is an issue. However, our emphasis is on using ASPH for cosmological investigations, where we are attempting to identify regions where the baryonic gas collapses and forms objects like galaxies and clusters. In these scenarios, the behaviour and transport of angular momentum is critical if we wish to study the internal structure of galaxies (on scales ~ 10kpc), but is of
secondary importance for simply determining where galaxies form and how massive they are. Such global investigations are where we wish to initially concentrate, and ASPH should prove to be a valuable tool for such work.

In general ASPH appears to be quite promising for a number of applications where SPH has been previously used. ASPH offers the ability to maximize the spatial resolution for a given number of computational nodes, at very little computational penalty compared to straight SPH. This can be understood by examining the mechanics of an ASPH implementation vs. SPH. Where SPH involves dividing a scalar distance by a smoothing scale, ASPH requires a matrix multiplication, incurring a computational penalty of a few operations. Determining the evolution of the G tensor relies solely upon evaluating $\sigma_{a\beta} = \partial v_a / \partial x_\beta$, the trace of which is required by SPH regardless. This quantity can be determined in the same loop over neighbors which calculates the normal SPH dynamical equations, and therefore the computational penalty is again light. The majority of the increased computational burden for ASPH is in fact due to the necessity of smaller timesteps and therefore more integration cycles, required by stability criteria for higher resolutions (see appendix C.2). This is a penalty which will always be necessitated by higher resolution, though, and should not reflect disfavorably on ASPH. For this reason it is critical to devise efficient integration schemes for use with large scale ASPH simulations. We outline our own asynchronous algorithm in appendix C.1, which allows each ASPH node to evolve at its own timescale, semi-independently of the simulation as a whole. It is important to remember, though, that while ASPH maximizes spatial resolution, the mass resolution is still set by the
number of nodes which can be represented. ASPH will show its greatest benefit for structures which are well resolved in mass by the simulation, and possess some inherent anisotropy. Systems which exhibit planar shocks, anisotropic gravitational collapse, and so forth are good candidates for such studies. In a cosmological context ASPH will show the greatest benefit for large scale structures such as filaments and pancakes. Small, marginally resolved clumps such as arise from CDM-like initial power-spectra, on the other hand, will not show as much benefit from ASPH's improved spatial adaptability.
CHAPTER 3

Baryons, Dark Matter, and the Jeans Mass in Simulations of Cosmological Structure Formation

Published Work:

The following chapter is currently in press for publication as Owen, J. M., & Villumsen, J. V. 1997, “Baryons, Dark Matter, and the Jeans Mass in Simulations of Cosmological Structure Formation”, The Astrophysical Journal, 481. Figures and tables have been renumbered for continuity, and the references have been moved to the general bibliography at the end.

3.1 Introduction

Hydrodynamics is thought to play a key role in the formation of the visible structures in the universe, such as bright galaxies and hot intracluster gas. For this reason there is a great deal of interest in incorporating hydrodynamical effects into cosmological structure formation simulations in order to make direct, quantitative comparisons of such simulations to observed data. In addition to gravitation, a cosmological hydrodynamical simulation must minimally account for pressure support, shock physics, and radiative cooling, as these are the fundamental physical processes thought to play a dominant role in the formation of large, bright galaxies (White & Rees 1978).
There is already a bewildering array of such studies published, including Cen & Ostriker (1992a,b), Katz, Hernquist, & Weinberg (1992), Evrard, Summers, & Davis (1994), Navarro & White (1994), and Steinmetz & Müller (1994), to name a few. In order to appreciate the implications of such ambitious studies, it is important that we fully understand both the physical effects of hydrodynamics under a cosmological framework and the numerical aspects of the tools used for such investigations. Basic questions such as how the baryon to dark matter ratio varies in differing structures (galaxies, clusters, and filaments) and exactly how this is affected by physical processes such as shock heating, pressure support, or radiative cooling remain unclear. It is also difficult to separate real physical effects from numerical artifacts, particularly given the current limitations on the resolution which can be achieved. For example, in a recent study of X-ray clusters Anninos & Norman (1996) find the observable characteristics of a simulated cluster to be quite resolution dependent, with the integrated X-ray luminosity varying as $L_x \propto (\Delta x)^{-1.17}$, core radius $r_c \propto (\Delta x)^{0.6}$, and emission weighted temperature $T_X \propto (\Delta x)^{0.35}$ (where $\Delta x$ is the gridcell size of the simulation). In a study of the effects of photoionization on galaxy formation, Weinberg, Hernquist, & Katz (1996) find that the complex interaction of numerical effects (such as resolution) with microphysical effects (such as radiative cooling and photoionization heating) strongly influences their resulting model galaxy population. In this paper we focus on separating physical from numerical effects in a series of idealized cosmological hydrodynamical simulations. This study is intended to be an
exploratory survey of hydrodynamical cosmology, similar in spirit to the purely gravitational studies of Melott & Shandarin (1990), Beacom et al. (1991), and Little, Weinberg, & Park (1991).

We will examine the effects of pressure support and shock heating in a mixed baryonic/dark matter fluid undergoing gravitationally driven hierarchical collapse. This problem is approached with two broad questions in mind: how stable and reliable is the numerical representation of the system, and what can we learn about the physics of such collapses? These questions have been investigated for purely gravitational systems in studies such as those mentioned above. In those studies numerically it is found that the distribution of collisionless matter converges to consistent states so long as the nonlinear collapse scale is resolved. Such convergence has not been demonstrated for collisional systems, however. It is neither clear that hydrodynamical simulations will demonstrate such convergence in general, nor, that even if they do that the nonlinear scale is the crucial scale which must be resolved. Hydrodynamical processes are dominated by localized interactions on small scales, allowing the smallest scales to substantially affect the state of the baryonic gas. As an example, consider a collisional fluid undergoing collapse. Presumably such a system will undergo shocking near the point of maximal collapse, allowing a large fraction of the kinetic energy of the gas to be converted to thermal energy. In a simple case such as a single plane-wave perturbation (the Zel'dovich pancake collapse), the obvious scale which must be resolved is the scale of the shock that forms around the caustic. However, in a hierarchical structure formation scenario there is a hierarchy of collapse scales, and for
any given resolution limit there is always a smaller scale that will undergo nonlinear collapse. The subsequent evolution of the gas could well depend upon how well such small scale interactions are resolved, and changes in the density and temperature of gas on small scales could in turn influence how it behaves on larger scales (especially if cooling is important).

In this paper we examine a series of idealized experiments, evolving a mixed fluid of baryons and collisionless dark matter (dark matter dominant by mass), coupled gravitationally in a flat, Einstein-de Sitter cosmology. The mass is seeded with Gaussian-distributed initial density perturbations with a power-law initial power spectrum. We perform a number of simulations, varying the resolution, the initial cutoff in the density perturbation spectrum, and the minimum allowed temperature for the baryons. Enforcing a minimum temperature for the baryons implies there will be a minimal level of pressure support, and therefore a minimum collapse scale (the Jeans mass), below which the baryons are pressure supported against collapse. From the numerical point of view, performing a number of simulations with identical initial physical conditions but varying resolution allows us to unambiguously identify resolution effects. By enforcing a Jeans mass for the baryons we introduce an intrinsic mass scale to the problem, which may or may not be resolved in any individual experiment. The hope is that even if the gas-dynamical results do not converge with increasing resolution in the most general case, the system will converge if the fundamental Jeans mass is resolved.
The effects of the presence (or absence) of a baryonic Jeans mass also raises interesting physical questions. Although we simply impose arbitrary minima for the baryon temperatures here, processes such as photoionization enforce minimum temperatures in the real universe by injecting thermal energy into intergalactic gas. The Gunn-Peterson test indicates that the intergalactic medium is highly ionized (and therefore at temperatures $T \gtrsim 10^4 \text{K}$) out to at least $z \lesssim 5$. Shapiro, Giroux, & Babul (1994) discuss these issues for the intergalactic medium. The dark matter, however, is not directly influenced by this minimal pressure support in the baryons, and therefore is capable of collapsing on arbitrarily small scales. Pressure support provides a mechanism to separate the two species, and since the dark matter dominates the mass density it can create substantial gravitational perturbations on scales below the Jeans mass. While there are many studies of specific cosmological models with detailed microphysical assumptions, the general problem of the evolution of pressure supported baryons in the presence of nonlinear dark matter starting from Gaussian initial conditions has not been investigated in a systematic fashion.

This paper is organized as follows. In §3.2 we discuss the particulars of how the simulations are constructed and performed. In §3.3 we characterize the numerical effects we find in these simulations, and in §3.4 we discuss our findings about the physics of this problem. Finally, §3.5 summarizes the major results of this investigation.
3.2 The Simulations

A survey such as this optimally requires a variety of simulations in order to adequately explore the range of possible resolutions and input physics. Unfortunately, hydrodynamical cosmological simulations are generally quite computationally expensive, and therefore in order to run a sufficiently broad number of experiments we restrict this study to 2-D simulations. There are two primary advantages to working in 2-D rather than 3-D. First, parameter space can be more thoroughly explored, since the computational cost per simulation is greatly reduced and a larger number of simulations can be performed. Second, working in 2-D enables us to perform much higher resolution simulations than are feasible in 3-D. While the real universe is 3-D and we must therefore be cautious about making specific quantitative predictions based on this work, 2-D experiments can be used to yield valuable qualitative insights into the behaviour of these systems. For similar reasons Melott & Shandarin (1990) and Beacom et al. (1991) also utilize 2-D simulations in their studies of purely gravitational dynamics.

The 2-D simulations presented here can be interpreted as a slice through an infinite 3-D simulation (periodic in \((x, y)\) and infinite in \(z\)). The particles interact as parallel rods of infinite length, obeying a gravitational force law of the form \(F_{\text{grav}} \propto 1/r\). The numerical technique used for all simulations is SPH (Smoothed Particle Hydrodynamics) for the hydrodynamics and PM (Particle-Mesh) for the gravitation. The code and technique are described and tested in Owen et al. (1996) [Chapter 2], so we will not go into much detail here. We do note, however, that while our code implements ASPH (Adaptive Smoothed Particle Hydrodynamics) as described in our
initial methods paper, we are not using the tensor smoothing kernel of ASPH for this investigation, but rather simple SPH. The results should be insensitive to such subtle technique choices since the goal is to compare simulation to simulation, so we employ simple SPH in order to separate our findings from questions of technique.

All simulations are performed under a flat, Einstein-de Sitter cosmology, with 10% baryons by mass ($\Omega_{\text{bary}} = 0.1, \Omega_{\text{dm}} = 0.9, \Lambda = 0$). Thus the mass density is dominated by the collisionless dark matter, which is linked gravitationally to the collisional baryons. The baryon and dark matter particles are initialized on the same perturbed grid, with equal numbers of both species. Therefore, initially all baryons exactly overlie the dark matter particles, and only hydrodynamical effects can separate the two species. The baryon/dark matter mass ratio is set by varying the particle mass associated with each species. The initial density perturbation spectrum is taken to be a power-law $P(k) = \langle |\delta \rho(k)/\bar{\rho}|^2 \rangle \propto k^n$ up to a cutoff frequency $k_c$. Note that since these are 2-D simulations, for integrals over the power spectrum this is equivalent in the 3-D to a power spectrum of index $n - 1$. In this paper we adopt a "flat" ($n = 0$) 2-D spectrum

$$
P_{2-D}(k) = A_{\text{norm}} \begin{cases} k^0 & : k \leq k_c \\ 0 & : k > k_c \end{cases} \Rightarrow P_{3-D}(k) = k^{-1} \tag{3.1}
$$

where $A_{\text{norm}}$ normalizes the power-spectrum. Note that using a flat cosmology and power-law initial conditions implies these simulations are scale-free, and should evolve self-similarly in time. We can choose to assign specific scales to the simulations in
order to convert the scale-free quantities to physical units. All simulations are halted after 60 expansion factors, at which point the nonlinear scale (the scale on which \( \delta \rho / \rho \sim 1 \)) is roughly \(1/8\) of the box size.

The Jeans length is the scale at which pressure support makes the gas stable against the growth of linear fluctuations due to self-gravitation – the Jeans mass is the amount of mass contained within a sphere of diameter the Jeans length. The Jeans length \( \lambda_J \) and mass \( M_J \) are defined by the well known formula (Binney & Tremaine 1987)

\[
\lambda_J = \left( \frac{\pi c_s^2}{G \rho} \right)^{1/2}, \tag{3.2}
\]

\[
M_J = \frac{4\pi}{3} \rho \left( \frac{1}{2} \lambda_J \right)^3 = \frac{\pi \rho}{6} \left( \frac{\pi c_s^2}{G \rho} \right)^{3/2}, \tag{3.3}
\]

where \( \rho \) is the mass density and \( c_s \) the sound speed. The baryons are treated as an ideal gas obeying an equation of state of the form \( P = (\gamma - 1)u \rho \), where \( P \) is the pressure and \( u \) is the specific thermal energy. Enforcing a minimum specific thermal energy (and therefore temperature) in the gas forces a minimum in the sound speed \( c_s^2 = \gamma P / \rho = \gamma(\gamma - 1)u \), which therefore implies we have a minimum Jeans mass through equation (3.3). Note that \( \rho \) is the total mass density (baryons and dark matter), since it is the total gravitating mass which counts, and therefore \( M_J \) as expressed in equation (3.3) represents the total mass contained within \( r \leq \lambda_J / 2 \). If we want the total baryon mass contained within this radius, we must multiply \( M_J \) by \( \Omega_{\text{bary}} / \Omega \).
It is also important to understand how the mass resolution is set for the baryons by the SPH technique. This is not simply given by the baryon particle mass, since SPH interpolation is a smoothing process typically extending over spatial scales of several interparticle spacings. In general the mass resolution for the hydrodynamic calculations can be estimated as the amount of mass enclosed by a typical SPH interpolation volume. If the SPH smoothing scale is given by \( h \) and the SPH sampling extends for \( \eta \) smoothing scales, then the mass resolution \( M_R \) is given by

\[
M_R = \frac{4}{3} \pi (\eta h)^3 \rho. \tag{3.4}
\]

This is probably something of an overestimate, since the weight for each radial shell in this interpolation volume (given by the SPH sampling kernel \( W \)) falls off smoothly towards \( r = \eta h \), but given the other uncertainties in this quantity equation (3.4) seems a reasonable estimate. Note that the resolution limit for the SPH formalism is best expressed in terms of a mass limit, appropriate for SPH's Lagrangian nature. For this reason we choose to express the Jeans limit in terms of the Jeans mass (eq. [3.3]) throughout this work, as the Jeans limit can be equally expressed in terms of a spatial or a mass scale. In N-body work it is common to express the mass resolution of an experiment in units of numbers of particles. In our simulations we use a bi-cubic spline kernel which extends to \( \eta = 2 \) smoothing lengths, and initialize the smoothing scales such that the smoothing scale \( h \) extends for two particle spacings. We therefore have a mass resolution in 2-D of roughly 50 particles, or equivalently in 3-D roughly 260 particles.
We perform simulations both with and without a minimum temperature (giving Jeans masses $M_J = 0, M_J > 0$), at three different resolutions ($N = N_{\text{bary}} = N_{\text{dm}} = 64^2, 128^2, \text{ and } 256^2$), and for three different cutoffs in the initial perturbation spectrum ($k_c = 32, 64, \text{ and } 128$). The initial density perturbations are initialized as Gaussian distributed with random phases and amplitudes, but in such a manner that all simulations have identical phases and amplitudes up to the imposed cutoff frequency $k_c$. The cutoff frequencies are the subset of $k_c \in (32, 64, 128)$ up to the Nyquist frequency for each resolution $k_{\text{Nyq}} = N^{1/2}/2$, so for each resolution we have $k_c(N = 64^2) = 32$, $k_c(N = 128^2) \in (32, 64)$, and $k_c(N = 256^2) \in (32, 64, 128)$. For each value of the minimum temperature we therefore have a grid of simulations which either have the same input physics at differing resolutions (i.e., $k_c = 32$ for $N \in [64^2, 128^2, 256^2]$), or varying input physics at fixed resolution (i.e., $N = 256^2$ for $k_c \in [32, 64, 128]$). This allows us to isolate and study both numerical and physical effects during the evolution of these simulations. In total we discuss twelve simulations.

For the simulations with a minimum temperature, there is an ambiguity in assigning a global Jeans mass with that temperature. The density in equation (3.3) is formally the local mass density, and therefore the Jeans mass is in fact position dependent through $\rho(\tau)$. Throughout this work we will refer to the Jeans mass at any given expansion factor as the Jeans mass defined using a fixed minimum temperature and the average background density, making this mass scale a function of time only. This is equivalent to taking the zeroth order estimate of $M_J$, giving us a well defined characteristic mass scale. In terms of this background density, Figure 3.1 shows the
baryon Jeans mass (in units of the resolved mass via equation [3.4]) as a function of expansion. Note that for a given simulation $M_R$ remains fixed, and it is the Jeans mass which grows as $M_J \propto \rho^{-1/2} \propto a^{3/2}$. It is apparent that the $N = 256^2$ simulations resolve the Jeans mass throughout most of the evolution, the $N = 128^2$ simulations resolve $M_J$ by $a/a_i \sim 15$, and the $N = 64^2$ simulation does not approach $M_J/M_R \sim 1$ until the end of our simulations at $a/a_i \sim 60$. The specific value of $T_{\text{min}}$ used in this investigation is chosen to yield this behaviour. We discuss physically motivated values for this minimum temperature in §3.5.

3.3 Numerical Resolution and the Jeans Mass

3.3.1 Dark Matter

We will begin by examining the dark matter distribution, as this is a problem which has been examined previously. Figures 3.2 and 3.3 show images of the dark matter overdensity $\rho_{\text{dm}}/\bar{\rho}_{\text{dm}}$ for the $M_J = 0$ simulations. In order to fairly compare with equivalent images of the SPH baryon densities, the dark matter information is generated by assigning a pseudo-SPH smoothing scale to each dark matter particle, such that it samples roughly the same number of neighboring dark matter particles as the SPH smoothing scale samples in the baryons. We then use the normal SPH summation method to assign dark matter densities, which are used to generate these images. The panels in the figure are arranged with increasing simulation resolution $N$ along rows, and increasing cutoff frequency $k_c$ down columns. The diagonal panels represent each resolution initialized at its Nyquist frequency for $P(k)$. Note that the physics of the problem is constant along rows, and numerics is constant along columns. If
Figure 3.1: The ratio of the Jeans mass to the resolved mass \((M_J/M_R)\) as a function of expansion for each of the three resolutions used in this paper \((N = 64^2, 128^2, 256^2)\) for the \(M_J > 0\) case. The Jeans mass is calculated using the average background density of the universe at each expansion.
resolution were unimportant, the results along rows should be identical. Likewise, since the numerics is held constant along columns, only physical effects can alter the results in this direction.

Comparing the dark matter densities along the rows of Figure 3.2, it is clear that the structure becomes progressively more clearly defined as the resolution increases. This is to be expected, since the higher resolution simulations can resolve progressively more collapsed/higher density structures. The question is whether or not the underlying particle distribution is systematically changing with resolution. In other words, do the simulations converge to the same particle distribution on the scales which are resolved? Figure 3.3 shows this same set of dark matter overdensities for the $M_J = 0$ simulations, only this time each simulation is degraded to an equivalent $N = 64^2$ resolution and resampled. This is accomplished by selecting every $n$th node from the higher resolution simulations, throwing away the rest and suitably modifying the masses and smoothing scales of the selected particles. Note that now the dark matter distributions look indistinguishable for the different resolution experiments, at least qualitatively. This similarity implies that the high frequency small scale structure has minimal effect on the larger scales resolved in this figure. Looking down the columns of Figure 3.2 it is clear that increasing $k_c$ does in fact alter the dark matter particle distribution, such that the large scale, smooth filaments are progressively broken up into smaller clumps aligned with the overall filamentary structure. These differences are lost in the low-res results of Figure 3.3, implying that these subtle changes do not significantly affect the large scale distribution of the dark matter.
Figure 3.2: Dark matter overdensities ($\rho_{\text{dm}}/\langle \rho_{\text{dm}} \rangle$) for $M_J = 0$ simulations. Panels arranged with increasing resolution along rows ($N = 64^2, 128^2, 256^2$), and increasing cutoff in initial input perturbation spectrum down columns ($k_c = 32, 64, 128$). All simulations are shown at the final time slice (expansion factor $a/a_i = 60$), with grey scale intensity scaled logarithmically with dark matter density.
Figure 3.3: Dark matter overdensities ($\rho_{dm}/\bar{\rho}_{dm}$) for $M_J = 0$ simulations as in Figure 3.2, except here the densities are calculated after resampling the simulations down to equivalent $N = 64^2$ resolutions.
In Figures 3.4 and 3.5 we show the mass distribution functions for the dark matter overdensity \( f(\rho_{\text{dm}}/\bar{\rho}_{\text{dm}}) \). Figure 3.4 includes all particles from each simulation (as in Figure 3.2), while Figure 3.5 is calculated for each simulation degraded to equivalent \( N = 64^2 \) resolutions (comparable to Figure 3.3). The panels are arranged as in Figure 3.2, with \( M_J = 0 \) and \( M_J > 0 \) overplotted as different line types. It is clear that the varying Jeans mass in the baryons has negligible effect on the dark matter, a point we will return to in §3.4. The full resolution results of Figure 3.4 show a clear trend for a larger fraction of the mass to lie at higher densities with increasing resolution. There is also a similar though weaker trend with increasing \( k_c \). However, examining the resampled results of Figure 3.5 it appears that the results of all simulations converge, bearing out the visual impressions of Figures 3.2 and 3.3. For the dark matter, with increasing resolution more information is gained about the highest density/most collapsed fraction of the mass, but so long as the pertinent nonlinear scales are resolved the results converge. The underlying particle distribution does not depend upon the numerical resolution, similarly to the results discussed in Little et al. (1991).

### 3.3.2 Baryons

We now turn our attention to the baryon distribution. Figures 3.6, 3.7, 3.8, and 3.9 show images of the baryon overdensity for \( M_J = 0 \) and \( M_J > 0 \) at expansions \( a/a_r = 30 \) and \( a/a_r = 60 \). There is a pronounced trend for the collapsed filaments and clumps to become progressively more strongly defined as the simulation resolution improves – even more so than we see in the dark matter. The tendency to break up filaments into small scale clumps with increasing \( k_c \) is also clearly evident for the
Figure 3.4: Normalized dark matter overdensity distribution functions \( f(\rho_{dm}/\bar{\rho}_{dm}) \) for \( M_J = 0 \) (solid lines) and \( M_J > 0 \) (dotted lines) simulations. These results are presented at the full resolution for each simulation.
Figure 3.5: Normalized dark matter overdensity distribution functions $f(\rho_{dm}/\bar{\rho}_{dm})$ for $M_J = 0$ (solid lines) and $M_J > 0$ (dotted lines) simulations. In this case we plot the results when each simulation is first degraded to equivalent $N = 64^2$ resolution.
The $M_J = 0$ case. Additionally, the presence of a nonzero Jeans mass visibly influences the baryon density distribution in Figures 3.8c and 3.9. This is particularly evident in the high resolution $N = 256^2$ column, where the increased pressure support creates a "puffier" distribution, wiping out the smallest scale structures in the baryons. Recall from Figure 3.1 that we naively expect the presence of the pressure support for $M_J > 0$ to affect both $N = 128^2$ and $N = 256^2$ at $a/a_i = 30$, but not $N = 64^2$. Comparing the results of Figures 3.6 and 3.8, we indeed see this trend. By $a/a_i = 60$, the effects of the Jeans mass are clearly evident (comparing Figures 3.7 and 3.9) for $N = 128^2$ and $N = 256^2$, though $N = 64^2$ still appears relatively unaffected.

Figures 3.10 and 3.11 show images of the baryon densities for the $k_z = 32$ simulations, but in this case resampled to $N = 64^2$ resolutions analogous to Figure 3.3. At expansion $a/a_i = 30$ (Figure 3.10), we see that for $M_J = 0$ the baryons appear to be systematically more tightly collapsed with increasing simulation resolution, even though they have all been resampled to the same sampling resolution to produce this image. This supports the view that the baryon distribution is fundamentally changing with increasing simulation resolution, in contrast with the dark matter. The $N = 128^2$ and $N = 256^2$ $M_J > 0$ simulations, however, demonstrate very similar baryon density images, though $N = 64^2$ still appears different at $a/a_i = 30$. At $a/a_i = 60$ (Figure 3.11) we again see for $M_J = 0$ a clear trend with simulation resolution, while the $M_J > 0$ runs look remarkably similar to one another.
Figure 3.6: Baryon overdensities $\rho_{\text{bary}}/\bar{\rho}_{\text{bary}}$ for $M_J = 0$ simulations at $a/a_i = 30$. Panels arranged as in Figure 3.2.
Figure 3.7: Baryon overdensities $\rho_{\text{bary}}/\langle \rho_{\text{bary}} \rangle$ for $M_J=0$ runs at expansion $a/a_i=60$. Panels arranged as in Figure 3.2.
Figure 3.8: Baryon overdensities $\rho_{\text{bary}}/\bar{\rho}_{\text{bary}}$ for $M_j > 0$ simulations at $a/a_i = 30$. Panels arranged as in Figure 3.2.
Figure 3.9: Baryon overdensities $\rho_{\text{bary}}/\langle \rho_{\text{bary}} \rangle$ for $M_j > 0$ simulations at $a/a_i = 60$. Panels arranged as in Figure 3.2.
Figure 3.10: Baryon overdensities for $k_c = 32$ simulations at $a/a_i = 30$, resampled to $N = 64^2$ resolution as in Figure 3.3. Note these panels represent the same simulations as the top rows of the previous figures. Panels are arranged with increasing simulation resolution ($N = 64^2, 128^2, 256^2$) along rows, and increasing baryon Jeans mass ($M_J = 0, M_J > 0$) down columns.
Log$_{10}$ Baryon Overdensity for $k_c=32$ Runs at expansion $a/a_i=60.0$

All Runs degraded to $N=64^2$ resolution

Figure 3.11: Baryon overdensities for $k_c = 32$ simulations at $a/a_i = 60$, resampled to $N = 64^2$ resolution.
Figures 3.12 and 3.13 show the full resolution mass distribution functions of the baryon overdensities $f(\rho_{\text{bary}}/\bar{\rho}_{\text{bary}})$ for all simulations at $a/a_i = 30$ and $a/a_i = 60$, respectively. The $M_J = 0$ functions show a strong trend to transfer mass from low to high densities with increasing resolution, and a similar though weaker trend with $k_c$. However, even at full resolution the $M_J > 0$ simulations show very similar density distributions once $M_J$ is resolved. The $M_J > 0$ simulations also appear to be relatively insensitive to $k_c$, suggesting that the increased small scale power is being wiped out by the pressure support. Figures 3.14 and 3.15 show these same baryon density distribution functions, only for all simulations degraded to $N = 64^2$ resolutions. These bear out our previous observations. In the case with no Jeans mass, there is no sign of convergence in the baryon distribution as the resolution is increased. However, when a Jeans mass is present, then the baryon distributions do converge once the Jeans mass is resolved.

Figure 3.16 presents a more quantitative way to measure this convergence problem. In this figure we calculate the Kolmogorov-Smirnov statistic $D(\rho_{\text{bary}})$, comparing the baryon density distribution for each simulation to the others at the same expansion and Jeans mass. We do not expect these simulations to exactly reproduce one another, and therefore there is little point in assigning significance to the exact quantitative value of $D$. However, the K-S statistic does provide objective measures of how similar or dissimilar these distributions are, and therefore we might expect to learn something by comparing their relative values. Comparing the upper panels of Figure 3.16 we can see that at $a/a_i = 30$ the $N = 128^2$ and $N = 256^2$ simulations are more similar
Figure 3.12: Normalized baryon overdensity distribution functions $f(\rho_{\text{bary}}/\bar{\rho}_{\text{bary}})$ for $M_J = 0$ (solid lines) and $M_J > 0$ (dotted lines) at $a/a_i = 30$. Panels arranged as in Figure 3.4.
Figure 3.13: Normalized baryon overdensity distribution functions \( f(\rho_{\text{bary}}/\bar{\rho}_{\text{bary}}) \) for \( M_J = 0 \) (solid lines) and \( M_J > 0 \) (dotted lines) at \( a/a_i = 60 \).
Figure 3.14: Normalized baryon overdensity distribution functions $f(\rho_{\text{bary}}/\bar{\rho}_{\text{bary}})$ for $M_J = 0$ (solid lines) and $M_J > 0$ (dotted lines) at $a/a_i = 30$. All simulations degraded to equivalent $N = 64^2$ low resolutions before sampling.
Figure 3.15: Normalized baryon overdensity distribution functions $f(\rho_{\text{bary}}/\bar{\rho}_{\text{bary}})$ for $M_J = 0$ (solid lines) and $M_J > 0$ (dotted lines) at $a/a_i = 60$. All simulations degraded to equivalent $N = 64^2$ low resolutions before sampling.
for $M_J > 0$ than for $M_J = 0$, while the $N = 64^2$ simulation remains relatively distinct in both cases. At $a/a_i = 60$, however, we can see that for $M_J > 0$ all the simulations appear comparable, while for $M_J = 0$ they remain distinct for the different resolutions.

We therefore have a subtly different picture for the numerical behaviour of the dark matter and baryons. The critical resolution scale for the dark matter is the scale of nonlinearity. So long as this scale is resolved, the dark matter distribution can be expected to converge to a consistent state on resolved scales. Unfortunately, the distribution and state of the baryonic particles appears in general to be sensitive to the numerical resolution. However, it is possible and physically plausible to define a fundamental collapse scale in the form of the Jeans mass for the baryons, below which baryonic structure formation is suppressed. This scale can now be treated as the critical baryonic resolution scale, and we do find that once this threshold is reached the baryon distribution will reliably converge as well.

3.4 Hydrodynamics and the Baryon Distribution

3.4.1 Shocks and Temperatures

The results of the previous section indicate that hydrodynamical interactions on small scales can significantly alter the final state of the baryons in ways which propagate upward and affect larger scales. The tendency for a simulation with a given finite resolution is to underestimate the "true" fraction of high density, collapsed baryons. A likely cause for this trend is the presence of small scale, unresolved shocks in the baryon gas. Because shocks provide a mechanism for transferring the gas's kinetic energy to thermal energy, it is reasonable to expect that the fashion and degree to
Figure 3.16: Kolmogorov-Smirnov statistic $D$ comparing $f(\rho_{\text{bary}})$ between simulations. Each line type corresponds to one simulation which is compared to each simulation listed on the ordinate axis, where the simulations are denoted as $N : k_e$. Note that the K-S statistic for comparing an individual simulation to itself is formally $D = 0$, but for the sake of clarity we have interpolated over these points in this plot. The panels are arranged with Jeans mass $M_J$ increasing along rows, and expansion $a/a_i$ increasing down columns.
which the baryons collapse will be dependent upon when and how strongly they undergo shocks. In this section we investigate the thermal state of the baryons, with the goal of understanding the pattern and importance of shocking in the gas.

In the top row of Figure 3.17 we show the 2-D mass distribution function of the baryons in terms of their overdensity and temperature $f(\rho_{\text{bary}}/\bar{\rho}_{\text{bary}}, T)$ for the $M_J = 0$ simulations at $a/a_c = 60$. The various resolutions share some gross properties in the $\rho - T$ plane. The low density gas tends for the most part to be cool, though there is a tail of low density material with temperatures up to $T \lesssim 10^5\text{K}$. The high density gas is at relatively high temperatures, with most of the material near $T \sim 10^6\text{K}$. However, there is a notable trend for the highest density material to be somewhat cooler with increasing simulation resolution. This effect is similar to the behaviour seen in simple 1-D collapse such as the Zel'dovich pancake (Shapiro & Struck-Marcell 1985). The highest density gas is the fraction which collapses earliest, when the background density is highest. Such gas is placed on a lower adiabat than gas which falls in at later times, and thus remains cooler. In our case this means that since higher resolution simulations can resolve higher density clumps (which therefore form at earlier times), we should tend to see the temperature of the highest density material fall with increasing resolution.

The high temperature gas is heated by shocks as it falls into the dark matter dominated potential wells. In order to isolate shock heating from simple adiabatic compression heating, we calculate the distribution of the temperature in units of the
Figure 3.17: Baryon mass distribution for the $M_f = 0$ simulations as a function of overdensity and temperature $f(\rho_{\text{bary}}/\bar{\rho}_{\text{bary}}, T)$ (upper row) and $f(\rho_{\text{bary}}/\bar{\rho}_{\text{bary}}, T/T_{ad})$ (lower row). $T_{ad} = T_0 (\rho/\rho_0)^{\gamma-1}$ is defined as the temperature the gas would have due solely to adiabatic processes. Panels arranged as in Figure 3.12.
adiabatic temperature \( T_{\text{ad}} \), given by

\[
T_{\text{ad}} = T_0 \left( \frac{\rho_{\text{bary}}}{\rho_0} \right)^{\gamma^{-1}}.
\]  

(3.5)

\( T_{\text{ad}} \) represents the temperature the gas would be at if it were only heated through simple \( PdV \) work. Since the only non-adiabatic process we allow is shock heating, only gas which has undergone shocking should be at \( T/T_{\text{ad}} > 1 \). In the bottom row of Figure 3.17 we calculate the distribution \( f(\rho_{\text{bary}}/\bar{\rho}_{\text{bary}}, T/T_{\text{ad}}) \) for the \( M_f = 0 \) simulations at \( a/a_i = 60 \). The high density fraction of the gas is clearly strongly shocked in all cases, with \( T/T_{\text{ad}} \sim 10^7 - 10^9 \). There is a clear trend for \( T/T_{\text{ad}} \) in the high density gas to fall with resolution, indicating that the highest density fraction of the gas is less strongly shocked as the resolution increases. Though we do not show the results at fixed resolution and increasing \( k_c \) here, there are also subtle trends evident with \( k_c \) in both the \( \rho - T \) and \( \rho - T/T_{\text{ad}} \) planes. Generally the temperature/shocking distribution of moderately overdense material grows wider with increasing \( k_c \).

It appears that shocks are indeed the key physical mechanism distinguishing the different resolution experiments. We find that in general most of the baryonic material is processed through shocks at some point. We note a general pattern in which the highest density gas in low resolution experiments is characteristically more strongly shocked than the highest density gas in higher resolution experiments. The physical inference of these trends is that the larger the region which collapses, the stronger the resulting shock. The underlying physical mechanism for this property is easily understood. Since the highest density material represents the gas which collapses earliest, this is also the gas which falls into the shallowest potential wells. As the
structure continues to grow, these potential wells deepen. Gas which infalls at later times therefore picks up more kinetic energy, which in turn leads to stronger shocking and higher temperatures. Once shocking occurs, the state of the baryon gas is discontinuously and irreversibly altered. In order to properly represent the physical state of the gas, a simulation must resolve the smallest scales on which shocks are occurring. This is why enforcing a Jeans mass allows convergence, since establishing a minimum Jeans mass implies there is a minimum scale on which baryonic structures can form, forcing a minimum scale for shocking.

3.4.2 Comparing the Baryon & Dark Matter Distributions

One of the most fundamental questions we can address is how the distributions of dark matter and baryons compare to one another. Comparing the dark matter and baryon density fields for the $M_J = 0$ case in Figures 3.2 and 3.7, there is a distinct impression that the baryons tend to be more tightly clustered than the dark matter on all collapsed scales. The situation is a bit more complex for the $M_J > 0$ case in Figure 3.9. Comparing the $(N = 256^3, k_c = 128)$ distributions, it is evident that the $M_J > 0$ baryons show a more diffuse structure than that of the $M_J = 0$ case, to the point that some of the smallest scale structures are entirely suppressed. Bear in mind that the dark matter evolves essentially independently of the baryons in this dark matter dominated case, so the small scale structures still form in the overall mass distribution – the baryons are simply excluded from them. The large scale structures such as the filaments and the largest knots are still quite prominent in the $M_J > 0$ baryon distribution, just as for the $M_J = 0$ case. These patterns suggest
that the baryons are generically more clustered than the dark matter, down to the scale set by the Jeans mass. At this scale and lower, the dark matter continues to form collapsed structures, whereas the baryons are held out of these structures by the pressure support enforced by the minimum temperature.

In Figure 3.18 we calculate the baryon to dark matter number density ratio as a function of baryonic overdensity. The baryon to dark matter ratio is defined as $n_{\text{baryon}}/n_{\text{dm}} = \Omega_{\text{dm}} \rho_{\text{baryon}} / \Omega_{\text{baryon}} \rho_{\text{dm}}$, so that $n_{\text{baryon}}/n_{\text{dm}} > 1$ corresponds to baryon enrichment, while $n_{\text{baryon}}/n_{\text{dm}} < 1$ implies baryon depletion. There is a clear trend for the highest density material to be baryon enriched, implying that the cores of the most collapsed structures are relatively enriched in baryons compared with the universal average. This trend persists even in the $M_f > 0$ simulations, though it is not as pronounced as in the $M_f = 0$ case. There is no evidence that a significant fraction of the baryons exist in regions which are dark matter enhanced. In all simulations underdense material appears to lie near the universal average mixture $n_{\text{baryon}}/n_{\text{dm}} \sim 1$. We also note a trend with resolution, such that the higher the resolution of the simulation, the greater the baryon enrichment found in overdense regions.

A simple physical picture can account for these trends. So long as the density evolution is in the linear regime ($\delta \rho/\rho \ll 1$), the dark matter and baryons evolve together, remaining at the universal mix of $n_{\text{baryon}}/n_{\text{dm}} \sim 1$. During this linear phase the pressure support (barring any imposed minimum pressure) is orders of magnitude less important than the gravitational term, so the baryon/dark matter fluid evolves as a pressureless gas. Once nonlinear collapse begins ($\delta \rho/\rho \gtrsim 1$), the baryons rapidly
Figure 3.18: Average baryon to dark matter mixture as a function of baryon overdensity at $a/a_i = 60$. The baryon to dark matter mixture is defined as $n_{\text{bary}}/n_{\text{dm}} = \Omega_{\text{dm}} \rho_{\text{bary}} / \Omega_{\text{bary}} \rho_{\text{dm}}$, so that $n_{\text{bary}}/n_{\text{dm}} > 1$ represents baryon enriched material, while $n_{\text{bary}}/n_{\text{dm}} < 1$ is baryon depleted. In each panel the solid line shows the measured average baryon to dark matter mixture, while the dashed lines represent the mixtures such that 10% of the mass at each overdensity is above and below the enclosed region. The dotted line shows the universal average $n_{\text{bary}}/n_{\text{dm}} = 1$. The top and bottom rows represent the $M_f = 0$ and $M_f > 0$ simulations, respectively.
fall inward with the dark matter until they collide near the potential minimum. At this point the baryon gas shocks, converting the majority of its kinetic energy into thermal energy, and it stops, forming a hot pressure supported gas at the bottom of the potential well. In the case with a minimum pressure support, the collapse proceeds until the pressure term (due to the increase in density) builds sufficiently to impede the baryons infall, at which point the baryons slow, separate from the infall, and shock. In either case the dark matter forms a more diffuse structure supported by velocity dispersion. This process leads to the generic patterns noted above: on scales below which the collapse has become nonlinear, the baryons tend to be characteristically more clustered than the dark matter, at least down to the minimal point set by the Jeans scale. In either case the critical factor determining exactly when the baryons separate from the general inflow is the point at which shocking sets in. We also know from the numerical observations that this process is resolution dependent, and in fact the baryon enrichments we see for the $M_J = 0$ case in Figure 3.18 must represent lower limits to the "true" baryon enrichment. The enrichments noted for the $M_J > 0$ simulations should be reliable, to the extent that the specific minimum temperature chosen is reasonable.

It is somewhat puzzling to note that our measured positive biasing of the baryons in collapsed structures is at odds with previously published results. In a study of the cluster formation under the standard $\Omega = 1$ Cold Dark Matter (CDM) model, Evrard (1990) finds that while outside of the cluster environment the baryons and dark matter simply track the universal average mix, the baryon fraction within the
cluster is in fact somewhat lowered. Kang et al. (1994) examine a larger volume of an $\Omega = 1$ CDM cosmology, and find that not only are the overdense regions baryon depleted, but that their underdense, void structures are baryon enriched. There are several possible explanations for this disagreement. One possibility is that this represents a geometric effect, in that our experiments are 2-D, while these other studies employ fully 3-D simulations. In our simulations, the "filaments" actually represent walls, and the most collapsed knots are best interpreted as cross-sections through tubular filaments. The processes of collapsing to a plane, a line, or a point are certainly different processes, and the isotropy of pressure support makes these structures progressively more difficult to form. In a 1-D planar collapse, for instance, it is well known that the central collapse plane will be baryon enriched, while the question of whether or not a cluster is baryon enriched or depleted is still hotly debated. We see some evidence for this effect in Figure 3.18. Looking particularly at the upper dashed lines in this figure (representing the baryon enrichment at which 90% of the mass at that overdensity lies below) we note our most extreme enrichments occur at moderate overdensities, roughly in the range $\rho_{\text{bary}}/\bar{\rho}_{\text{bary}} \sim 10^1 - 10^2$. This extremely baryon enriched material represents the "filaments" in our simulations (walls in 3-D). It is also possible that resolution effects play a role here. As pointed out previously, we find a strong resolution dependence, such that finite resolution tends to underestimate the fraction of high density, collapsed baryonic material. Evrard (1990) uses $16^3$ SPH nodes to represent his baryon component, which for the scale of his box is equivalent to our lowest resolution simulations. Kang et al. (1994) use an entirely
different technique to simulate the hydrodynamics, which relies upon a fixed grid to represent the baryons. This limits their spatial resolution so that typical clusters are only a handful of cells across. It is also important to compare these quantities in the same manner. In Figure 3.18 we calculate the baryon to dark matter mixture in a manner which follows the baryon mass, since we sample at the positions of the baryon particles. This naturally gives the greatest weight to the most prominent baryonic structures. Kang et al. (1994) calculate this distribution in a manner which is volume weighted, which will tend to give the greatest weight to underdense, void like regions. Since the baryon fraction appears to be a function of environment, these differences can be significant. Without further study, it is difficult to know the true reason for this discrepancy, or how the actual baryon/dark matter ratio should evolve.

3.5 Discussion

The results of this investigation can be broken into two broad categories: what is revealed about the physics of hierarchical collapse in a mixed baryonic/dark matter fluid, and what is learned about the numerics of simulations of this process. We find that the dark matter converges to a consistent state on resolved scales, so long as the nonlinear collapse scales are well resolved. Increasing the resolution of the experiment does not fundamentally alter the dark matter distribution, but simply yields more detailed information about the small scale collapsed structures. This is in agreement with previous, purely collisionless studies, though we demonstrate this here including a collisional component.
The numerical story is quite different for the collisional baryonic gas. We find that in the case where we do not impose a fundamental physical resolution scale in the baryons, the simulation results do not converge with increasing resolution. Rather, as the numerical resolution of the experiment is increased, the collapsed fraction of the baryons is systematically altered toward a higher density, more tightly bound, and less strongly shocked state. The physical reason for this behaviour is the presence of shocks, which allows the evolution on small scales to affect the overall state of the baryonic mass. With improving resolution the simulation is able to resolve the collapse of smaller structures at earlier times. The smaller scale (and therefore earlier) the resolved collapse, the weaker the resulting shock is found to be. This effect is most obvious in Figure 3.17, where there is a systematic trend of higher density/more weakly shocked material with increasing resolution.

The fact that the dark matter converges in general with resolution, while the baryons do not, highlights a fundamental difference in the physics of these two species. While both dark matter and baryon fluids react to the global and local gravitational potential, the baryons are additionally subject to purely local hydrodynamical phenomena – most prominently shocking in this case. Once strong shocking sets in these hydrodynamical effects can rise to rival the gravitational force on the baryonic fluid, allowing the baryons to be strongly influenced by interactions on small scales in ways which the dark matter is not. This implies that such small scale interactions can be just as important as the large scale forces in determining the final state of the baryons. In other words, for the dark matter there is no back reaction from small
to large scales, whereas the baryons are strongly influenced by interactions on small scales. In the coupling of these physical processes, gravitation dominates the large scale structure, but hydrodynamics affects the local arrangements and characteristics of the baryonic gas. If we want the quantitative results of such studies to be reliable, we must have reason to believe that the localized hydrodynamical processes are adequately resolved.

This gloomy picture is alleviated by an important physical effect: the Jeans mass. Introducing a minimum temperature (and therefore pressure support) into the baryons creates a fundamental length/mass scale, below which the baryons are supported by pressure against any further collapse or structure formation. We find that once we introduce such a minimal scale into the baryonic component, the simulation results converge as this scale is resolved. This convergence holds even though the dark matter component continues to form structures below the baryon Jeans scale. Although the Jeans scale is dependent upon the local density, we find that the global Jeans scale defined using the background density is adequate to define the critical resolution necessary for the hydrodynamics to converge. This therefore describes an additional resolution scale necessary for hydrodynamical simulations to meet, much as the nonlinear mass scale represents the crucial resolution necessary for purely gravitational systems. Furthermore, our experiments indicate that equation (3.4) is a reasonable estimate of an SPH simulation's true mass resolution, since we find that the threshold $M_R \lesssim M_J$ marks the point at which convergence is achieved.
In these experiments we have tested the effects of the Jeans mass in an idealized framework by simply imposing an arbitrary minimum temperature into our system, but there is reason to believe that such minimum temperatures should exist in the real universe. Based upon observations such as the Gunn-Peterson test (Gunn & Peterson 1965), it is known that the IGM is highly ionized out to redshifts \( z \lesssim 5 \), which implies a minimum temperature for the IGM of at least \( T \gtrsim 10^4 \). Assuming an Einstein-de Sitter cosmology, a minimum temperature of \( T \sim 10^4 \) requires a minimum spatial resolution (via eq. [3.2])

\[
\lambda_J \sim 0.777 (1 + z)^{-3/2} \left( \frac{\mu}{0.6} \right)^{-1/2} \left( \frac{T}{10^4 \text{K}} \right)^{1/2} h^{-1} \text{Mpc},
\]

(3.6)

which equates to a baryon mass resolution of (eq. [3.3])

\[
M_R \lesssim \Omega_{\text{bary}} M_J \sim 6.82 \times 10^{10} \Omega_{\text{bary}} (1 + z)^{-3/2} \left( \frac{\mu}{0.6} \right)^{-3/2} \left( \frac{T}{10^4 \text{K}} \right)^{3/2} h^{-1} \text{M}_\odot.
\]

(3.7)

This limit can also be expressed in terms of a minimum circular velocity, which has the advantage of being independent of redshift. The minimum circular velocity can found as a function of the minimum temperature by relating the kinetic energy necessary for dynamical support to the internal energy for equivalent pressure support (Thoul & Weinberg 1996), yielding

\[
v_{\text{circ}} = \left( \frac{2kT}{\mu m_p} \right)^{1/2} \sim 16.6 \left( \frac{\mu}{0.6} \right)^{-1/2} \left( \frac{T}{10^4 \text{K}} \right)^{1/2} \text{km/sec}.
\]

(3.8)

In our \( M_J > 0 \) simulations if we choose to call the scale at which RMS mass fluctuation is \( \Delta M/M \sim 0.5 \) to be \( 8 \text{ h}^{-1} \text{Mpc} \) at the final expansion, then our box scale is \( L = 64 \text{h}^{-1}\text{Mpc} \) and the minimum temperature corresponds to \( T_{\text{min}} \sim 10^6 \text{K} \). While there
are some suggestions that the intergalactic medium could be heated to temperatures as hot as $10^6$K (through mechanisms such as large scale shocks of the IGM), clearly these simulations do not meet our criteria if we wish to consider photoionization as setting the minimum temperature. It is also not clear that the current generation of large-scale hydrodynamical cosmological simulations meet this criterion, but it should be achievable.

It is still unclear whether or not in the case with no minimum temperature imposed the baryon distribution will eventually converge. It is well known that in a purely gravitational system, as structure builds and smaller dark matter groups merge into larger structures, the dark matter "forgets" about the earlier small scale collapses as such small structures are incorporated into larger halos and disrupted. This is why the dark matter results converge once the nonlinear mass scale is resolved. While it is evident from studies such as this that the baryons maintain a longer memory of their previous encounters, it seems likely that as the baryon gas is progressively processed through larger scale and stronger shocks, at some point the previous evolution should become unimportant. At exactly what level this transition is reached remains uncertain, however, as we see no evidence for such convergence here.

Radiative cooling must be accounted for in order to model processes such as galaxy formation, and the inclusion of radiative cooling can only exacerbate the non-convergence problems we find here. The amount of energy per unit mass dissipated by radiative cooling is proportional to the density, and we have already noted that the trend with finite resolution is to underestimate the local gas density and overestimate
the temperature. Given these tendencies, it is not difficult to envision problems for finite resolution simulations which will tend to underestimate the effectiveness of radiative cooling in lowering the temperature (and therefore pressure support) of the shocked gas. This could lead to perhaps drastic underestimates of the fraction of cold, collapsed baryons for a given system, and therefore strongly influence the inferred galaxy formation. Evrard et al. (1994) note this effect when comparing their high and low resolution 3-D simulations. They find that altering their linear resolution by a factor of two (and therefore the mass resolution by a factor of eight) changes the measured total amount of cold collapsed baryons by a factor of \( \sim 3 \). They attribute this change to just the sort of problems we discuss here. Weinberg, Hernquist, & Katz (1996) report similar findings and interpretation for simulations with a photoionizing background. It therefore seems likely it is all the more important to resolve the minimum mass scale set by the minimum temperature in systems with radiative cooling.

We find that the majority of the baryonic mass undergoes strong shocking so long as the nonlinear mass scale exceeds the Jeans mass. At infinite resolution in the \( M_j = 0 \) case, it is possible that all of the baryonic material undergoes shocking. As anticipated from previous investigations, the highest density collapsed fraction is characteristically less shocked as compared with later infalling material from larger regions. The underlying cause for this behavior is the fact that potential wells deepen as structure grows. The highest density material is that which collapses earliest due to the smallest scale perturbations. This material falls into relatively shallow potential
wells, and is only weakly shocked. As the structures continue to grow, progressively larger scales go nonlinear and collapse. The potential wells deepen and infalling material gains more energy, resulting in stronger shocking and higher temperatures.

Hydrodynamics can also play an important role in determining the distribution of the baryon mass, particularly in collapsed structures. In the absence of external mechanisms to heat the baryons (such as energy input from photoionization), during the linear phase of structure growth the baryons evolve as a pressureless fluid and simply follow the dominant dark matter. Once nonlinear collapse sets in, the baryons fall to the potential minimum, shock, convert their kinetic energy to thermal energy, and settle. In contrast, the dark matter simply passes though the potential minimum and creates a more diffuse structure supported by the anisotropic pressure of random velocities. This difference gives rise to a characteristic pattern in the baryon/dark matter ratio. Wherever the evolution is still linear, the baryons and dark matter simply remain at the universal mix. With the onset of nonlinear collapse, the baryons fall to the minimum of the potential well where they form a baryon enriched core, surrounded by a dark matter rich halo. We find that even in the absence of radiative cooling the cores of collapsed structures can become baryon enriched by factors of $n_{\text{bary}}/n_{\text{dm}} \sim 2$ or more, though this value is likely resolution and dimension dependent. If the thermal energy of the baryons is raised to the point that it rivals the potential energy during the collapse, the baryons will become pressure supported and stop collapsing at that point. In all cases we find that the dark matter is relatively unaffected by the baryon distribution. This is due to the fact that the dark matter
dominates the mass density, and therefore the gravitational potential. In general it appears that under a dark matter dominated scenario hydrodynamics can substantially alter the characteristics of the baryonic material (and therefore the visible universe), such that it does not directly follow the true mass distribution which is dominated by the the dark matter.
CHAPTER 4

Hydrodynamic Cosmological Simulations with Scale-Free Initial Conditions

Published Work:

The following chapter is in the process of being submitted for publication in the \textit{The Astrophysical Journal} as Owen, J. M., Evrard, A. E., Hernquist, L, & Katz, N. 1997, "Hydrodynamic Cosmological Simulations with Scale-Free Initial Conditions." It is therefore formatted as a paper throughout. The only adaptations made here are that the figures and tables have been renumbered for continuity, and the references have been moved to the general bibliography at the end.

4.1 Introduction

The evolution of large-scale cosmological structure from scale-free initial conditions has been well studied, and it provides many useful insights into the general problem of gravitationally driven, hierarchical structure formation. So long as the background cosmology, input physics, and initial conditions remain scale-free, such systems should evolve self-similarly in time. In reality the universe certainly imposes fixed physical scales (such as breaks in the initial power-spectrum of density fluctuations, or through microphysical processes such as the photoionization transition at $T \sim 10^4$K), and
therefore disrupts formal self-similarity at some level. However, for many interesting
mass ranges (such as those applicable to clusters of galaxies) the dominant physical
processes are essentially scale-free, and therefore the evolution of structure on such
scales should be reasonably self-similar. Self-similar systems are particularly useful
in cosmology, because in general structure formation is analytically intractable due
to the inherent nonlinearity of gravitational clustering. If structure forms in a self-
similar manner, though, we can use this property to predict how a given distribution
of properties should evolve over time, even though we cannot precisely predict what
the quantitative values of those properties should be. For example, even though
we may not know the precise form of the distribution \( f(M) \), we can predict how this
distribution should evolve with time, \( f(M, t_1) \rightarrow f(M, t_2) \). Kaiser (1986) exploits this
property in order to predict the temporal evolution of the hot intracluster medium in
galaxy clusters. Numerically modeling self-similar scenarios provides a particularly
powerful tool, since the numerical model can provide detailed information about the
state of the system at a particular time, and then self-similarity can be used to scale
this state to any desired time. Efstathiou et al. (1988) use collisionless scale-free N-
body simulations to study structure formation for a variety of input power-spectra,
testing their results for the expected self-similar scalings as well as comparing against
analytical predictions such as those of Press & Schechter (1974). In this study we
present a set of hydrodynamic simulations based on scale-free initial conditions, in
the hopes that we will be able to identify self-similar scaling in the properties of the
baryon gas as well as the dark matter.
In the absence of radiative cooling, adding hydrodynamical processes to the purely gravitational sorts of systems studied previously does not introduce any new physical scales, and therefore the properties of the baryons should scale self-similarly as well as the dark matter. This is not to say that the two species should evolve identically to one and other, however. The physics governing the details of collapse in the two species (with the baryons subject to added processes such as shock heating and pressure support) are quite different. Self-similarity does imply that given the detailed differences in the arrangement of the dark matter and baryons in a spectrum of structures at a given time, corresponding structures at any other time should be arranged in the same manner. These differences are what make this problem worthwhile to investigate including the baryon component. Basic observational questions such as the cluster X-ray luminosity function critically depend upon how the baryons wind up distributed in collapsed, cluster scale objects. Additionally, there is reason to suspect that numerical effects may be more (or at least differently) evident in the baryons than the dark matter. Several independent investigations (Anninos & Norman 1996; Weinberg, Hernquist, & Katz 1996; Owen & Villumsen 1997a [Chapter 3]; to name merely a few) have found evidence for complicated numerical artifacts in the properties of baryons in collapsed structures—particularly in the innermost, high-density regions. However, there is no a priori reason to believe such numerical effects should scale self-similarly, so we should be able to use self-similar scaling of the physical properties of the system to separate them from purely numerical artifacts. The self-similar scaling identified in the collisionless experiments of Efstathiou et al.

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(1988) was a key test and important success for the use of N-body methods as a tool for investigating cosmological structure formation. This paper attempts to apply similar tests to hydrodynamical simulations.

Finally, we should point out that radiative cooling is an important process in the formation and evolution of galaxies, though not so much for structures on the scale of galaxy clusters. Therefore the results of this paper are most relevant to cluster studies, which are the focus of many recent hydrodynamical studies such as Bryan et al. (1994), Cen et al. (1995), Crone & Evrard (1996), Pinkney et al. (1996), etc. While in general imposing a realistic radiative cooling law violates self-similarity, it is possible to construct artificial cooling laws which do not. In a subsequent investigation we will present the results of self-similar studies including radiative cooling, in order to more directly address the issue of galaxy formation.

4.2 The Simulations

We present three 3-D hydrodynamical simulations based upon the same initial conditions. The background cosmology is a flat, Einstein-de Sitter universe with $\Omega_{\text{bary}} = 0.05$ and $\Omega_{\text{dm}} = 0.95$. The initial density perturbations are Gaussian distributed with random phases according to a power-law $P(k) \propto k^{-1}$, and initialized such that the linearly predicted mass fluctuations will go nonlinear within a top-hat window of radius $R_{\text{th}} \sim 0.2$ the box size at the final expansion. The random phases and amplitudes of the density fluctuations are seeded similarly to those used in Katz, Hernquist, and Weinberg (1992) (hereafter KHW) and Hernquist, Katz, and Weinberg (1995), except that here we use a pure $n = -1$ power-spectrum for the density fluctuations.
rather than that of a Cold Dark Matter (CDM) model. If we choose to match the parameters of the CDM model with our scale-free experiments (such that the box size is $L_0 = 11.11h^{-1} \text{Mpc}$, $H_0 = 50 \text{ km/sec/Mpc}$, and normalization $\sigma_8 \sim 0.7$ in a top-hat of radius $8h^{-1} \text{Mpc}$), then the final expansion of our scale-free simulations correspond to an expansion $a/a_0 = 0.4$, or redshift $z = 1.5$. Throughout this paper we define the final expansion at which the simulations are halted to be $a_f \equiv 1$, which in combination with this choice for the correspondence to physical scales implies the expansion of the universe today is $a_0 = 2.5$. Note that if we choose to characterize our computational box by these parameters then we are considering a very small cosmological volume indeed, and the “clusters” formed within this volume are really more representative of poor clusters or galactic groups rather than classic galaxy clusters such as Coma. It is important to stress, though, that due to the scale-free nature of these experiments this is simply one plausible choice for comparing these simulations with physically motivated models. We could in principle choose to associate the nonlinear scale at some epoch in the simulations to any physical time, which would simply rescale the corresponding box size and final redshift of the experiment.

Within this framework we perform a low-resolution ($N_{\text{bary}} = N_{\text{dm}} = 32^3$) experiment with TreeSPH (Hernquist & Katz 1989; Hernquist, Weinberg, & Katz 1996), and two high-resolution $N_{\text{bary}} = N_{\text{dm}} = 64^3$ experiments using TreeSPH and P3MSPH (Evrard 1988). We examine each simulation at expansions spaced logarithmically with an interval of $\Delta \log a = 0.2$. At each expansion we identify groups of particles using the “friends-of-friends” algorithm (see, e.g., Barnes et al. 1985; the specific implemen-
tation used here can be found at "http://www-bpcc.astro.washington.edu/tools/FOF/"), compute global average properties for each group (such as the dark matter and baryon mass, the emission weighted temperature, luminosity, and so forth), and check for self-similar behaviour in the distribution of properties among these groups. Though there are more sophisticated group finding algorithms available, friend-of-friends provides a simple, unbiased, and unambiguous definition of a group. Additionally, in order to test for self-similar behaviour it is necessary to identify the groups in a manner which maintains the self-similarity of the system. Friends-of-friends with a fixed linking length essentially identifies objects of a given overdensity, which meets this criterion. We use a linking length \( l \sim 0.2\Delta x_p \) (where \( \Delta x_p \) is the initial interparticle spacing), which corresponds roughly to objects of overdensity \( \delta\rho/\rho \sim 250 \).

In order to understand the regimes we can probe, we must first identify what our mass resolution limits are. Since SPH is a Lagrangian technique, the lower limit on the hydrodynamic interactions is best expressed as a mass limit, essentially set by a multiple of the particle mass. In each simulation the SPH smoothing scales are evolved such that each particle samples roughly 32 of its neighbors, which provides a reasonable lower-limit on the SPH mass resolution. The gravitational force resolution is essentially set by some multiple of the softening length, so unfortunately this is not Lagrangian in the same way as the hydrodynamic resolution. However, in each run the gravitational softening length is \( L_{\text{soft}} = L_{\text{box}}/1111 \) (corresponding to 10 \( h^{-1} \) kpc in the KHW CDM simulation), so on the scale of the objects we will be examining we may consider the gravitational resolution to be unrestrictive and the minimal resolved
mass is therefore set by the 32 particle limit. The upper limit on the mass range we are sensitive to is set by our box size. The larger a group is, the more statistically rare it is. We can therefore only expect to find objects up to a certain size at any given expansion within our finite simulation volume. The Press-Schechter mass function (Press & Schechter 1974) provides a rough estimate of this limit. In Figure 4.1 we plot these mass limits over a range of expansions. We also show the evolution of various multiples of a "characteristic" mass $M_\ast$, where $M_\ast$ is defined as the mass of an object with a linearly predicted overdensity of $\delta M/M = 1.68$. This is chosen as an appropriate comparison mass for our friends-of-friends identified groups, as is discussed in more detail in §4.3.1. Based on Figure 4.1 we can reasonably expect the mass range $[0.2 M_\ast, 2.0 M_\ast] \delta M/M = 1.68$. This is chosen as an appropriate comparison mass for our friends-of-friends identified groups, as is discussed in more detail in §4.3.1. Based on Figure 4.1 we can reasonably expect the mass range $[0.2 M_\ast, 2.0 M_\ast]$ to be accessible for expansions $\log a \in [-0.4, 0]$, though the $32^3$ simulation may be dicey on the low end at early expansions. Note that this only represents about a factor of 2.5 in expansion, so we are fairly restricted in the range of expansions which are accessible. Future experiments with more particles could certainly improve on this range, but these sorts of particle numbers are fairly representative of the experiments published to date.

4.3 Self-similar evolution

4.3.1 Expected scalings

Before we discuss in detail the results of the simulations, it is worthwhile to summarize what the expected scalings are. Kaiser (1986) provides a good overview of this topic, so we only briefly cover the salient points here. Under our scale-free assumption for the input physics and the background cosmology, the only available physical scale
Figure 4.1: Mass resolution limits as a function of expansion. The dotted lines are 32 times the particle mass for the different simulation resolutions (assuming a unit mass divided up into $N = 32^3$ and $64^3$ particles). The dashed lines show various multiples of $M_*$, the characteristic mass for an object with a linearly predicted overdensity of $\Delta M/M = 1.68$. The solid lines show the group masses at which the Press-Schechter formalism predicts statistically there will be 1 ($M_1$), 10 ($M_{10}$), and 100 ($M_{100}$) groups of that mass in our simulation volume. All masses scaled such that the total mass is 1.
is set by the amplitude of density fluctuations, e.g., the length-scale on which rms fluctuations are of order $\delta \rho/\rho \sim 1$ at a given expansion. This nonlinear scale can be equivalently expressed as a mass $M_{nl}$ or a length $R_{nl}$. So long as the system obeys temporal self-similarity, any dimensionless statistic set by the density field must be a function of this nonlinear scale only ($M/M_{nl}$ or $R/R_{nl}$). We test for self-similar scaling in the simulations by following the evolution of dimensionless distributions such as $M/M_*$ or $T/T_*$ as functions of the expansion $a$, where the variables subscripted with * are the “characteristic” quantities.

The evolution of these characteristic quantities can be parameterized in terms of the power-law index of the density perturbations $n$, where $P(k) \propto k^n$. The characteristic length and mass scales follow the corresponding nonlinear scales, given by

$$R_* \propto R_{nl} \propto a^{(5+n)/(3+n)},$$

$$M_* \propto M_{nl} \propto a^{6/(3+n)}.$$  \hspace{2cm} (4.1, 4.2)

The characteristic density evolves as the background density

$$\rho_* \propto \bar{\rho} \propto a^{-3},$$

while the characteristic temperature and luminosity are defined through combinations of these parameters.
Our initial conditions have a spectral index \( n = -1 \), so these scalings become \( R_* \propto a^2 \), \( M_* \propto a^3 \), \( \rho_* \propto a^{-3} \), \( T_* \propto a^1 \), and \( L_* \propto a^{1/2} \). Note that these scalings are expressed in proper coordinates.

Given the relative scalings, we must now choose reasonable normalizations for our characteristic parameters. Since we are identifying objects with an rms overdensity \( \sigma = \delta \rho / \rho \sim 250 \), we will try to select parameters characteristic of this scale. The spherical top-hat model for the collapse of an isolated perturbation predicts that an overdense region will collapse when it reaches a linearly predicted overdensity of \( \sigma = \delta \rho / \rho = 1.68 \) – its actual density contrast at this point is of order 200, roughly that of our fiducial objects identified with friends-of-friends. Our initial density perturbations are normalized so that the linearly predicted overdensity in a top-hat of radius 0.2 \( L_{\text{box}} \) at the final expansion is of order \( \sigma(a_f) \sim 1 \), and we know \( \sigma^2 \propto R^{-(n+3)} = R^{-2} \). Using this information we can derive characteristic quantities for an object of linearly predicted overdensity \( \sigma = 1.68 \) in species \( X \) as

\[
R_* \sim 0.120 a^2 \, L_f^{\text{box}},
\]

\[
M_* \sim 7.25 \times 10^{-3} \Omega_X a^3 \, M_f^{\text{box}},
\]

\[
\rho_* \sim 1.00 \Omega_X a^{-3} \, M_f^{\text{box}} (L_f^{\text{box}})^{-3},
\]

\[
T_* \sim 3.02 \times 10^{-3} a \, M_f^{\text{box}} (L_f^{\text{box}})^{-1},
\]

\[
L_* \sim 2.10^{-5} a^{1/2} (M_f^{\text{box}})^3 (L_f^{\text{box}})^{-4}.
\]
Note that if we relate these simulations to the KHW CDM simulations as above, these quantities become $R_* \sim 0.533 a^2 \, h^{-1} \, \text{Mpc}$, $M_* \sim 2.76 \times 10^{12} \Omega_X a^3 h^{-1} \, M_\odot$, $\rho_* \sim 4.34 \times 10^{12} \Omega_X a^{-3} h^{-2} \, M_\odot \, \text{Mpc}^{-3}$, $T_* \sim 8.04 \times 10^5 \, a \, \text{K}$, and $L_* \sim 2.67 \times 10^{25} \, a^{1/2} \, M_\odot^2 \, \text{Mpc}^{-3} \, \text{K}^{1/2}$.

### 4.3.2 Global scaling in the simulations

Perhaps the most obvious statistic we can measure is the differential distribution of the mass in groups, $f(M)$. Figure 4.2 shows $f(M)$ for each of these simulations, plotting the dark matter and baryons separately. Note that we are computing the amount of mass contained in groups in each mass range, not the number of groups. We measure $f(M)$ at four different expansions, and scale the results self-similarly to $a = a_f \equiv 1$ for comparison. We also plot the Press-Schechter (PS) mass function for comparison as the thin solid lines. Note that the PS mass function is a prediction with no free parameters based upon an analytic approximation, not a fit to the data. The measured distributions $f(M)$ at different expansions appear to link up quite nicely, indicating that mass is following the expected self-similar behaviour reasonably well (though somewhat better in the dark matter than the baryons). As we might anticipate from Figure 4.1, it is clear that the simulations probe different regions of the overall mass function at different expansions. At early times only the highest mass objects have collapsed, and therefore we tend to see only the high end of $f(M)$ represented. As the simulations evolve we progressively resolve smaller and smaller objects in the overall mass range, until by the end we begin to lose the high mass end due to the limited computational size of the box.
Figure 4.2: The differential mass distribution function of group masses $f(M)$ (baryons and dark matter plotted separately) for expansions $a = 0.25, 0.4, 0.63,$ and $1.0$. All quantities are scaled self-similarly to $a = a_f = 1$. Each expansion is represented by a different line type, with the Press-Schechter prediction plotted as the thin solid line. The masses are scaled such that the total mass in each species is 1.
In general the mass distribution follows the shape of the PS prediction quite well, though the baryon mass in groups tends to be a bit low compared with both the PS prediction and the dark matter results. This depression in the baryon mass in groups across the entire measured mass range suggests that the baryon fraction in collapsed objects slightly underrepresents the universal baryon to dark matter mixture, at least within a density contrast of $\delta \rho/\rho \sim 250$. This supports similar findings by previous studies (Evrard 1990; Kang et al. 1994a), suggesting that the baryon fraction in collapsed objects is depleted compared with the universal mixture, at least in the absence of cooling. Still, these results must be treated with some caution, since there are indications that numerical effects due to finite resolution can drive the baryon to dark matter ratio in this direction (Owen & Villumsen 1997a [Chapter 3]).

In order to more directly demonstrate the self-similar scaling of the group masses, Figure 4.3 measures the cutoff mass $M_f$ such that 90% of the total mass is contained in groups of mass $M \leq M_f$. The solid lines shows the self-similar scaling solution fitted to the high-resolution experiments ($64^3$ TreeSPH and P3MSPH) in the expansion range $\log a \in [-0.6, -0.2]$. Each point is measured by summing the cumulative mass function, locating the two mass points which bracket our desired mass fraction, and linearly interpolating (in log space) between these points. The careful observer will note in this and subsequent figures that the $\log a = -0.1$ output is missing from the $32^3$ TreeSPH measurements. That timeslice was inadvertently overwritten and lost in the process of transferring and analyzing the data.
Figure 4.3: Evolution of the group mass $M_f(a)$, such that at each expansion 90% of the total mass in species $X$ (baryons and dark matter plotted separately) is contained in groups of mass $M_X \leq M_f$. The points depict the results of each simulation at specific expansions, and the lines show the predicted power-law $M \propto a^3$ fitted to the high-resolution experiments for $\log a \in [-0.6, -0.2]$
When considering this figure (as well as the following similar figures), it is worthwhile to keep the mass resolution limits of these simulations in mind. Our measured value for $M_f$ is only valid so long as the simulations can sample $f(M)$ in a statistically meaningful way near the chosen mass fraction. In a hierarchical structure formation scenario, all of the mass may be bound in structures down to arbitrarily small mass scales (Bond et al. 1991). However, any simulation with finite mass resolution can only represent bound objects down to this resolution limit – the mass which should be bound in structures below this scale is simply unbound in the simulations. Likewise, the finite simulation volume implies we can only statistically represent structures up to an upper mass limit. As the simulations evolve we begin to lose the upper end of the mass function because the power saturates on the scale of the box. We have chosen a fairly large mass fraction (90%) in order to sample the distribution back to early expansions, when only the upper end of the group mass distribution is accessible. However, as the systems evolve we begin to lose the high mass end of the distribution, which is why the final few points at large expansions in Figure 4.3 fall below the extrapolation from earlier epochs. Aside from these last two points, the group masses scale quite well in the range of expansions $\log a \in [-0.6, -0.2]$. The dark matter group masses seem to obey self-similar scaling somewhat better than the baryons, but the differences are minor. The dark matter and baryon measurements in Figure 4.3 are normalized to the total amount of mass in their own species, so the fact that the baryon points at each epoch tend to be slightly lower than the corresponding dark matter measurements supports the trend noted in the $f(M)$ distributions.
in Figure 4.2 – the fraction of baryons in collapsed structures is slightly depressed compared with the collapsed fraction of dark matter. Note also that while both of the high-resolution experiments seem to agree on the masses quite well, there is a trend for the low-resolution $32^3$ TreeSPH simulation to find systematically lower baryon masses. This suggests there may be resolution artifacts which tend to suppress the baryon masses of collapsed structures, even though the dark matter masses appear to be unaffected.

In Figure 4.4 we plot the 90% mass evolution of the group temperatures. Note that the upper row uses emission weighted temperatures for the objects, while the lower row uses mass weighted estimates. For Bremsstrahlung radiation the volume emissivity is $\epsilon \propto \rho^2 T^{1/2}$, so the emission weighted temperature for group $i$ is

$$T_i = \frac{\int \epsilon T \, dV}{\int \epsilon \, dV} = \frac{\int \rho^2 T^{3/2} \, dV}{\int \rho^2 T^{1/2} \, dV} = \frac{\sum_j m_j \rho_j T_j^{3/2}}{\sum_j m_j \rho_j T_j^{1/2}},$$

represented as a sum over all particles $j$ which are members of group $i$. The mass weighted temperature can be simply calculated as an average over the SPH nodes which are members of group $i$: $T_i = \frac{1}{N_j} \sum_j T_j$. Note also that in the left column we use all of the particles in a group to define the temperature, while in the right column we use only particles up to an upper density limit $\rho/\bar{\rho} \leq 2000$. The reasons for this are discussed in more detail below, when we consider the gas density evolution. In order to calculate the 90% mass evolution we sort the groups in order of $T_i$ and find the cutoff $T_f$ such that 90% of the total baryon mass is contained in groups with temperatures $T \leq T_f$ (counting the unbound mass as groups with $T < T_f$). The solid line shows the self-similar solution fitted to the high-resolution results for $\log \alpha \in [-0.6, -0.2]$. [195]
In each case we find that the high-resolution TreeSPH temperatures scale quite well. Additionally, while the $32^3$ TreeSPH temperatures are only clearly defined for $\log a \in [-0.2, 0]$, they are consistent with the $64^3$ high-resolution results. The failure of the low-resolution TreeSPH experiment at early expansions is not surprising, once one takes into account that with $N = 32^3$ particles a typical $M_*$ group will not be represented by more than 32 particles until expansion $\log a \geq -0.3$ (see Figure 4.1). The most notable trend in this figure is the distinction between the P3MSPH and TreeSPH results. In all cases we find lower temperatures for the P3MSPH objects than in TreeSPH, though the mass weighted temperatures agree more closely. The emission weighted temperature tends to be most strongly weighted toward the central, high-density cores of collapsed objects. This is simply because the emission goes as $\epsilon \propto \rho^2$, and the density is largest in cores. The fact that the mass weighted temperatures (as well as the emission weighted where the highest density gas is excluded) agree more closely between the P3MSPH and TreeSPH experiments indicates that the difference noted in the total emission weighted temperatures in the upper right panel are largely due to differences in the cores of these objects. The group cores in the P3MSPH simulation are characteristically cooler than in the TreeSPH case – a point we will return to below. Despite this normalization difference, the P3MSPH measurements scale as well as the TreeSPH, the only exception being the last two points in the emission weighted temperature without an upper density cutoff.
Figure 4.4: Evolution of the emission weighted group temperature $T_f(a)$, such that at each expansion 90% of the total mass is contained in groups with temperature $T \leq T_f$. The upper row calculates the emission weighted temperature, and the bottom row the mass weighted. In the left column we use all particles in each group to define the average, while in the right only SPH particles with $\rho/\bar{\rho} \leq 2000$ are allowed to contribute. Point types and line defined as in figure 4.3.
Figure 4.5 shows the 90% mass scaling of the group luminosities, calculated similarly to the temperature scaling in Figure 4.4. For each group the total luminosity is defined as \( L = \int \rho^2 T^{1/2} \, dV \), which becomes a sum over the member SPH nodes \( L = \sum_j m_j \rho_j T_j^{1/2} \). As with the temperature, we sort the groups in order of increasing luminosity, and find the luminosity \( L_f \) such that 90% of the total baryon mass is contained in groups with \( L \leq L_f \). The solid line shows the expected \( L \propto a^{1/2} \) scaling fitted to the high-resolution data for \( \log a \in [-0.6, -0.2] \). It is evident that the total group luminosity scales quite poorly. There is also a rather large resolution effect: the low-resolution TreeSPH luminosities are fainter than those of the high-resolution case by an order of magnitude. The poor scaling of the luminosity is quite disheartening, as the luminosity function is a basic observational prediction we would like to obtain from these sorts of simulations. We must also ask why the luminosity is scaling so poorly: we already know that the mass and temperature scale reasonably, which suggests that the gas density is the culprit.

Figure 4.6 tests the scaling of the group gas density, though in a slightly different fashion than the previous tests for mass, temperature, and luminosity. The density needs to be treated distinctly because the groups are essentially selected by their overdensities, so in a sense the density should scale by construction. In this figure we calculate median densities \( \rho_{x\%} \) for each group, such that \( x\% \) of the SPH nodes in the group are at densities \( \rho_j \leq \rho_{x\%} \). We then take all of the groups in a given mass range and find the average of \( \rho_{x\%} \), where the average is weighted by the group mass. We also impose the additional constraint that a group must contain more
Figure 4.5: Evolution of the group luminosity \( L_f(a) \), such that at each expansion 90\% of the total mass is contained in groups with luminosities \( L \leq L_f \). Point types and line defined as in figure 4.3.
than 32 particles to be considered – this is why the low-resolution experiment is missing from the early expansions in the low mass range. In Figure 4.6 we plot $\rho_{10\%}$, $\rho_{50\%}$, and $\rho_{90\%}$, progressively probing from the outskirts of each group inward. We also consider two mass ranges for selecting the groups: $M \in [0.2 M_*, 2.0 M_*]$ and $M \in [2.0 M_*, 20.0 M_*]$, representing the low and high ends of mass range we believe are accessible. As before, the solid lines indicate the self-similar evolution law fitted to the high-resolution experiments for $\log a \in [-0.6, -0.4]$.

Several trends are immediately obvious in the evolution of $\rho$. The density follows self-similar scaling best for low density cuts (such as $\rho_{10\%}$), suggesting that the outer regions of the groups scale more effectively than the inner, high-density cores. Further, as we take more restrictive low density cuts not only does the self-similar scaling improve but the scatter between the different simulations tightens substantially. Note also that although $\rho_{10\%}$ agrees well for the two high-resolution experiments, the core densities diverge by $a = a_f$. This is the underlying cause for the divergence in the emission weighted temperatures between the two high-resolution experiments, noted in Figure 4.4. There is also a tendency for the high mass range to scale better than the low mass, though even for the highest (and presumably the best resolved) mass range $\rho_{90\%}$ scales quite poorly. Taken together these trends suggest that the densities in the inner, highest density regions of the groups are dominated by numerical effects, and therefore these regions are effectively unresolved. However, if we cut out the high-density cores of the groups and only consider the outer regions, we obtain a much better representation of the physics. There doesn’t appear to be a magic cutoff.
Figure 4.6: Evolution of the group median density \( \rho_{X\%}(a) \). \( \rho_{X\%} \) is defined as the average density for groups in a given mass range, where \( \rho_{X\%} \) for each group \( i \) is the density such that \( X\% \) of the mass in the group is at densities \( \rho \leq \rho_{X\%} \). The left column shows the results for averaging over groups in the mass range \( 0.2 M_\odot \leq M \leq 2.0 M_\odot \), while the right column averages over groups in the range \( 2.0 M_\odot \leq M \leq 20.0 M_\odot \). The top row shows \( \rho_{10\%} \), the middle \( \rho_{50\%} \), and the bottom \( \rho_{90\%} \), progressively sampling from the outskirts of each group inward. Point types and lines defined as in figure 4.3.
density threshold, below which the density scales reliably and above which numerical effects dominate. Rather, we have a continuum of behaviour, in which the density varies smoothly from well-behaved in the outer low-density surroundings of the groups to essentially unresolved in the inner high-density cores. This behaviour is at once somewhat reassuring and disconcerting. On the one hand we have identified self-similar scaling in the group densities so long as we restrict ourselves to the low to moderate density regions of well-resolved groups. However, one could argue it is the innermost high-density regions which are the most interesting, and therefore finding that the results in the high-density cores are unreliable is somewhat discouraging.

The total luminosity of a group is dominated by the high-density core, so the results of Figure 4.6 suggest that the poor scaling of the luminosity noted previously is likely due to the poor representation of the core density. However, the gas density in moderate to low density regions in the groups does scale well, so we might hope that the luminosity in these restricted regions will more closely follow self-similar scaling. In Figure 4.7 we recalculate the 90% mass luminosity, only allowing mass points that fall below an upper density cutoff of $\rho/\bar{\rho} \leq 2000$ to contribute to the luminosity of each group. The scaling is still not spectacular, but there is definite improvement compared with Figure 4.5, particularly at late expansions. The high-resolution experiments show far less evolution of the luminosity with expansion, though the fit to the expected $L \propto a^{1/2}$ scaling is still mediocre. It is also notable that the scatter between the different simulations is considerably reduced. By the final expansion even the low and high-resolution experiments agree, in stark contrast with Figure 4.5.
Figure 4.7: Evolution of the group luminosity $L_f(a)$ as in figure 4.5, except here only particles with densities $\rho/\bar{\rho} \leq 2000$ are allowed to contribute to the luminosity.
4.3.3 Group Temperatures and Luminosities

X-ray observations of hot gas in galaxy clusters typically measure the temperature and luminosity of such objects (see for instance Edge et al. 1990; Henry et al. 1995; Burns et al. 1996), so in this section we will focus on these quantities. In Figures 4.8 and 4.9 we plot the emission weighted temperature and luminosity for each group identified in the high-resolution TreeSPH experiment at $a = a_f$ against its baryon mass. We include only resolved groups (i.e., those with more than 32 particles). It is evident that the temperature and luminosity of a group is well correlated with the mass, and in fact appear to form relatively tight power-laws. If we make the assumption that at any given time the internal structure of groups with differing masses should be similar (i.e., the density profile of 0.5 $M_\ast$ group is a scaled version of that found in an $M_\ast$ group), then simple scaling arguments predict $T \propto M^{2/3}$ and $L \propto M^{4/3}$. Kaiser (1990) uses this sort of reasoning to make a detailed comparison between observed X-ray luminosity functions and predictions based upon self-similar models in combination with the Press-Schechter mass function. It is worth emphasizing, though, that while self-similarity rigorously predicts the temporal evolution $f(M, T, \ldots, t_1) \rightarrow f(M, T, \ldots, t_2)$, it does not necessarily tell us about the detailed structure or arrangements of objects at any given time. In other words, while we do know from our scale-free condition that an $M_\ast$ group at time $t_1$ must be similar to an $M_\ast$ group at $t_2$, we do not necessarily know that a 0.5 $M_\ast$ group is similar to an $M_\ast$ case. However, recent N-body experiments suggest that collapsed objects built through hierarchical structure formation do tend toward a universal density profile
(Navarro, Frenk, & White 1995, 1996, 1997; Cole & Lacey 1996) over roughly two orders of magnitude in mass, though these studies also find some evidence that low mass halos tend to be denser. It therefore seems reasonable to adopt the assumption that objects will be similar over our limited mass range, from which the predictions $T \propto M^{2/3}$ and $L \propto M^{4/3}$ follow.

The solid lines in Figures 4.8 and 4.9 represent the fits for these expected power-laws to the data – the dotted line in Figure 4.8 shows the mass weighted best fit power-law $T \propto M^{0.5}$. In Figure 4.8 there are two obvious fairly low mass groups lying well above the prevalent mass-temperature relation. Close inspection of these outliers reveals them to be low density objects in the process of merging into larger structures. We just happen to be catching them in the process of being strongly shock heated and having their atmospheres stripped. The low-temperature outlier seems to represent the beginning of the spread for unresolved groups. This one just happens to barely make it above our low-mass cutoff. While both the total and “shell” luminosities in Figure 4.9 are well correlated with the mass, based on the results of Figures 4.6 and 4.7 it is not surprising that the shell luminosities form a better power-law of the mass once we exclude the unresolved high-density cores.

Based on the tight correlation of the individual group masses with their temperatures and luminosities, it seems reasonable to construct average group mass-temperature and mass-luminosity relations. Figure 4.10 computes the average mass-temperature relation at four different expansions for each simulation. Each of the curves is scaled assuming self-similarity to $a = a_f$, so theoretically the curves should
Figure 4.8: Individual group masses vs. emission weighted temperatures for the $64^3$ TreeSPH simulation at the final expansion $a = a_f = 1$. The solid lines show the fit for the expected scaling $T \propto M^{2/3}$, while the dotted line shows the best fit power-law $T \propto M^{1/2}$. Only groups with more than 32 particles are shown.
Figure 4.9: Individual group masses vs. luminosities for the $64^3$ TreeSPH simulation at the final expansion $a = a_f \equiv 1$. The solid lines show the fit for the expected scaling $L \propto M^{4/3}$. The expected and best fit power-laws for the luminosities are indistinguishable. Only groups with more than 32 particles are shown.
join up and form one continuous sequence. We can see that the mass-temperature relations do in fact follow a reasonably continuous power-law. A least-squares fit to these relations gives us \( T \propto M^{0.5 \pm 0.1} \) for the \( 64^3 \) TreeSPH simulation, \( T \propto M^{0.4 \pm 0.1} \) in \( 64^3 \) P3MSPH, and \( T \propto M^{0.47 \pm 0.2} \) for \( 32^3 \) TreeSPH, all in good agreement with one another and consistent with \( T \propto M^{1/2} \). Due to resolution effects (as will be elaborated on in §4.4), we expect the temperature should tend to be overestimated for low-mass and presumably less-resolved systems. This would push the measured relation for \( T(M) \) to shallower slopes, so for \( T \propto M^{\alpha_T} \) we should take this measurement to represent a lower limit \( \alpha_T \geq 0.5 \) for the "true" (i.e., infinite resolution) value. Note this is consistent with the prediction \( T \propto M^{2/3} \).

In Figure 4.11 we consider the average mass-luminosity relations. In the left column we use all the particles in a group to define its luminosity, while the right plots the shell luminosities using only particles with densities \( \rho/\bar{\rho} \leq 2000 \). As in Figure 4.10, each expansion is scaled assuming self-similarity to \( a = a_f \), so we expect the curves to form a continuous relation. Clearly the total mass-luminosity relations in the left column fail this test, while the shell luminosities in the right column do form a continuous sequence. Qualitatively it appears that including the unresolved cores in the total luminosity pushes the apparent power-law to a steeper slope, presumably because the cores of higher-mass objects are at least better resolved. We again use least-squares to fit a power-law to the mass-luminosity relations: for the total luminosities in the left column we find \( L \propto M^{1.4 \pm 0.1} \) for \( 64^3 \) TreeSPH and P3MSPH, and \( L \propto M^{1.3 \pm 0.2} \) for \( 32^3 \) TreeSPH. Measuring the shell luminosities from the right hand
Figure 4.10: Average mass vs. emission weighted temperature relations at expansions $a = 0.25, 0.4, 0.63,$ and 1.0. All quantities are scaled self-similarly to $a = a_f = 1$. The top row shows the results for $64^3$ TreeSPH, the middle $64^3$ P3MSPH, and the bottom $32^3$ TreeSPH. Only groups with more than 32 particles contribute to the averages.
column gives us $L \propto M^{1.340.1}$ for $64^3$ TreeSPH and P3MSPH, and $L \propto M^{1.240.2}$ in $32^3$ TreeSPH. As we might expect from Figures 4.5 and 4.7, the scatter in the mass-luminosity relation tightens substantially when we exclude the high-density cores. However, considering the somewhat underwhelming scaling of the shell luminosities in Figure 4.7, it is still surprising that the luminosity relations from the different expansions link up as well as they do. The mass scaling test in Figure 4.3 indicates that the group mass distribution at the different expansions scales well, while the luminosity scaling in Figure 4.7 shows that we tend to overestimate the true group luminosities at early expansions. We would therefore predict that the relation between $L$ and $M$ for an infinite resolution realization might be shallower than what we measure, so if $L \propto M^{\alpha L}$ these results give us an upper limit $\alpha_L \lesssim 1.3$.

As interesting comparison to these measurements can be found in Navarro, Frenk, & White (1995), which presents a hydrodynamical study of X-ray clusters selected from a large-scale CDM simulation. In this experiment the authors adopt a rather different approach than that used here. Rather than simulate hydrodynamics for every object in their volume, they instead select out a few interesting cluster scale structures from a large volume collisionless CDM simulation. They then resimulate the development of these selected objects at high-resolution, including a baryonic component. The advantage of this approach is that they can simulate the development of such interesting structures consistently within a very large volume ($180 \, h^{-1} \, \text{Mpc}$), thereby correctly representing the long-wavelength gravitational power while still achieving high-resolution in these interesting regions. The major disadvantage is that they can
Figure 4.11: Average mass vs. luminosity relations at expansions $a = 0.25, 0.4, 0.63,$ and 1.0. All quantities are scaled self-similarly to $a = a_f = 1$. The top row shows the results for $64^3$ TreeSPH, the middle $64^3$ P3MSPH, and the bottom $32^3$ TreeSPH. The left column shows the luminosities if the values for individual groups are calculated using all particles in each group, while in the right column only particles with densities $\rho/\bar{\rho} \leq 2000$ are allowed to contribute to the luminosity. Only groups with more than 32 particles contribute to the averages.
only afford to study a few structures within the volume at reasonable resolution, in this case eight. Though this low-number limit makes it difficult to pin down the power-law relations as in Figures 4.8–4.11, they find that their measured mass, temperature, and luminosity relations are consistent with the expected scalings $T \propto M^{2/3}$ and $L \propto M^{4/3}$.

Finally, Figure 4.12 computes the number distribution function of the groups by temperature and luminosity, $f(T)$ and $f(L)$. We measure the distributions at four different expansions shifted self-similarly to $a = a_f$, so the curves should overlap. Note that the total number of $M_*$ groups in the simulation volume falls with expansion, so it is necessary to normalize by the number of nonlinear volumes contained within our simulation volume, such that we are considering the number density distribution as $n/R_s^{-3}$. The thin solid line shows the prediction if we assume the Press-Schechter mass function and use the expected power-laws $T \propto M^{2/3}$ and $L \propto M^{4/3}$. We empirically determine the normalization of these power-laws by fitting them to the average $T(M)$ and $L(M)$ curves depicted in Figure 4.10. This is the only fitting that goes into the PS prediction (changing this normalization will only slide the resulting PS relations horizontally in this figure), so the PS curves should be treated as a prediction, rather than a fit. As we found with the masses in Figure 4.2, the measured distributions in temperature and luminosity form a continuous sequence for the various expansions, indicating that the distributions are obeying the expected self-similar scaling. The PS formalism predicts $f(M)$ well in figure 4.2 and the power-law relations connecting $M \to T$ and $M \to L$ are quite tight, so not surprisingly we also find reasonable
agreement between the PS prediction and the measured results for $f(T)$ and $f(L)$. It appears that PS somewhat overestimates the number of high-temperature objects, while it fits the distribution of shell luminosities quite well. There is some indication that the PS formalism underestimates the number of highly luminous objects for the total luminosity, but this could well be due to numerical artifacts.

4.4 Direct Comparison of the Simulations

In the previous sections we find that the simulations demonstrate reasonable self-similar scaling, so long as we are careful to take into account the numerical limitations of each experiment. Beyond the simple scaling tests we have considered thus far, it is also interesting to ask whether or not simulations using different resolutions and implementations get the same answer, and if not, how and why they diverge from one another. In several of the previous figures we can see evidence of numerical effects, even in objects which we would consider to be reasonably resolved. For instance, in the mass scaling test of Figure 4.3 there is a trend for the baryon mass in collapsed objects to increase with improving resolution, even though the dark matter masses appear to be stable. Likewise, the total luminosities and central densities of the collapsed structures are also resolution dependent. In this section we will make direct comparisons of individual groups in the simulations in order to better understand the precise regimes over which the results are reliable, as well as how the simulations tend to err as numerical effects become important.
Figure 4.12: The differential number distribution functions of emission weighted group temperatures $f(T)$ and luminosities $f(L)$ for expansions $\alpha = 0.25, 0.4, 0.63,$ and 1.0. All quantities are scaled self-similarly to $a = a_f = 1$. The left column shows the distribution of emission weighted temperatures $f(T)$, the middle $f(L)$ using all particles in each group, and the right $f(L)$ when the luminosities are computed using only particles with densities $\rho/\bar{\rho} \leq 2000$. The thin solid lines show the Press-Schechter prediction which results from combining the PS mass distribution with the expected scalings $T \propto M^{2/3}$ and $L \propto M^{4/3}$. 

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Since each simulation is based upon identical initial conditions, it is possible to identify the same objects in each realization. We use the algorithm described in Weinberg et al. (1996) to match corresponding objects in pairs of simulations. Under this scheme, groups in each simulation are first sorted in decreasing order of mass. Then the first (most massive) group from the first simulation is matched to the most massive group within $L_{\text{box}}/55.55$ (corresponding to $200 \, h^{-1} \, \text{kpc}$ in the KHW CDM simulation) that remains unmatched in the second. The results are insensitive to changes of this length criterion by factors of at least 4. This process is repeated for progressively less massive objects until the end of the list is reached. In this manner we compare objects between $64^3$ TreeSPH and $64^3$ P3MSPH, as well as $64^3$ TreeSPH and $32^3$ TreeSPH. Comparing the high and low-resolution TreeSPH experiments allows us to directly investigate the effects of resolution, while comparing the two high-resolution simulations should tell us something about the variances introduced by minor differences in the SPH implementations. In this section we only consider the results of the simulations at the final expansion, $a = a_f$. At this point we have 1884, 1856, and 471 groups in the dark matter and 1227, 1673, and 257 groups in the baryons for the $64^3$ TreeSPH, $64^3$ P3MSPH, and $32^3$ TreeSPH simulations, respectively. Of these, we find matches for 1485 in the dark matter and 1065 in the baryons between $64^3$ TreeSPH-$64^3$ P3MSPH, and 471 in the dark matter and 165 in the baryons for $64^3$ TreeSPH-$32^3$ TreeSPH.
Figure 4.13 compares the group masses at $a = a_0$, where we plot the $64^3$ TreeSPH-$64^3$ P3MSPH comparison in the upper panels and the $64^3$ TreeSPH-$32^3$ TreeSPH comparison in the lower. The open circles represent groups which have been matched between the two simulations, while the differing point types along the axes are groups which remain unmatched. The dotted lines show the 32 particle mass limit, representing our estimate of the mass resolution. The dark matter masses clearly match quite well for all of the simulations, with almost all of the resolved groups falling nicely along the diagonal line. This is the expected result, as it is well established that purely collisionless N-body simulations converge once the nonlinear mass scale is resolved. Examining the baryon masses in the $64^3$ TreeSPH-$32^3$ TreeSPH comparison, the largest mass groups appear to match well. However, there is evidence for a resolution effect as we go to progressively smaller (and therefore more poorly resolved) objects, such that the baryon mass is suppressed with poorer resolution. This effect is plausible, since as resolution degrades, the SPH kernels extend over a larger volume, thereby increasing the volume over which the pressure force due to the hot, dense gas in the object extends. This could have the effect of causing infalling gas to shock earlier in the collapse, leading to more extended, less massive structures. Comparing the $64^3$ TreeSPH-$64^3$ P3MSPH runs, the baryon masses for resolved objects concur very nicely. Only below the resolution limit do we see any discrepancy between these experiments, at which point we know numerics dominates the results anyway.
Figure 4.13: Group by group comparison of the group masses in dark matter (left column) and baryons (right column), in different simulations at $a = a_f = 1$. The top row compares $64^3$ TreeSph to $64^3$ P3MSPH; the lower row compares $64^3$ TreeSph to $32^3$ TreeSph. In each panel the circles represent groups which are matched between the simulations, while the points lying along the axes show groups which remain unmatched. The dotted lines show the 32 particle mass threshold. Groups are matched by first sorting them in decreasing order of mass, and then matching the first group from the first simulation to the first group in the second within a distance $L_{\text{box}}/55.55$ which has not yet been matched to a more massive group.
The density scaling in Figure 4.6 suggests that the gas density is very sensitive to the numerical resolution. In Figure 4.14 we directly compare the gas densities in the groups, with mass averaged densities in the left column and median densities in the right. The median plotted here is the 50% gas density, corresponding to the middle row of Figure 4.6. As anticipated, the gas density seems to be quite sensitive to the numerics, with definite systematics to the deviation. The densities found in the $32^3$ TreeSPH run always underestimate those found in the higher resolution $64^3$ TreeSPH experiment, with errors ranging to nearly an order of magnitude in the average density, and smaller errors of order 0.5 dex in the median. This is due to the fact that the mass averaged density of a group is dominated by the high density gas in the unresolved core. Comparing the average density in the two high-resolution experiments, we find that for low to moderate mass objects the average density matches reasonably well. However, the highest mass objects in P3MSPH systematically find higher average densities than in TreeSPH. These differences vanish in the median density measurement, indicating that the variances in the average are likely restricted to the high-density, unresolved cores. The most puzzling aspect of this discrepancy is that it is the high-mass, and therefore best resolved, objects where we find the largest errors. This could indicate that perhaps the P3MSPH run achieves somewhat higher-resolution in the cores of the most massive objects than TreeSPH. However, the poor scaling of the $\rho_{90\%}$ in the high-mass P3MSPH test (in the lower
right-hand panel of Figure 4.6) suggests that in fact neither experiment is resolving these high-density cores well, and the distinction in Figure 4.14 is more likely just an indication that these two codes fail in the cores differently.

Figure 4.15 plots the emission weighted temperature comparison. The fact that the group temperature scales well in Figure 4.4 suggests that the temperature is one of the more reliably measured quantities, and this comparison bears that impression out. Temperatures in the $32^3$ TreeSPH simulation very nearly match those of the high-resolution experiment, though the low-resolution case tends to be slightly hotter. Similarly, while the $64^3$ P3MSPH and $64^3$ TreeSPH temperatures correlate very tightly, the P3MSPH objects tend to be somewhat cooler. This difference is due to a combination of two effects. First, the P3MSPH simulation is started with a slightly cooler temperature distribution than the TreeSPH. Second, as noted previously the P3MSPH experiment tends to find higher core densities than TreeSPH, which has two effects: first, the cores of such collapsed structures are in fact cooler than the outer regions (see, e.g., Owen & Villumsen 1997a [Chapter 3] or Shapiro & Struck-Marcel 1985), and increasing the core densities increases the core luminosities, making the emission weighted temperature even more sensitive to the core temperature; second, as the density increases the core temperature must drop in order to maintain the same degree of pressure support, making the cores cooler still. Together, these effects force the P3MSPH emission weighted temperatures to be cooler than the TreeSPH, whereas the mass weighted temperatures (or temperatures where the core are excluded by an upper density cutoff) agree more closely.
Figure 4.14: Group by group comparison of the gas density at $a = a_f \equiv 1$. The left column uses the mass weighted average of the density, while the right shows the median density. The median density for each group is defined to be the gas density such that 50% of the SPH particles in the group are at that density or lower, corresponding to the middle row of Figure 4.6. Only groups containing more than 32 particles are shown.
Figure 4.15: Group by group comparison of the emission weighted group temperatures at $a = a_f = 1$. Only groups containing more than 32 particles are shown.
The scaling results in Figures 4.5 and 4.7 suggest that the luminosity is very sensitive to numerical artifacts, as is also evident in the direct comparison of Figure 4.16. In this figure we impose three different upper thresholds on the gas density of particles allowed to contribute to the luminosity: the left column shows the total group luminosities using all particles from each group, the middle column enforces a relatively high density cutoff of $\rho/\bar{\rho} \leq 5000$, and the right column uses a more restrictive upper density threshold of $\rho/\bar{\rho} \leq 2000$. Clearly the groups in $32^3$ TreeSPH tend to underestimate the total luminosities compared with the $64^3$ results. However, as we impose upper limits on the density, the high mass groups begin to agree. Further, the more restrictive the upper limit we enforce, the greater the mass range of groups which concur. We see a similar trend in the $64^3$ TreeSPH–$64^3$ P3MSPH comparison. Based on the previous figures, it is not surprising that the upper-end of the luminosity distribution diverges between the two high-resolution experiments. However, once we impose upper density cutoffs, the shell luminosities agree very well, indicating that the luminosity differences between the two experiments are restricted to the collapsed cores of the groups.

Taken together, the trends noted in Figures 4.13–4.16 suggest that numerical artifacts in otherwise resolved objects are most evident in the central, highest density regions. The most direct effect of using finite resolution is that we underestimate the central gas density, and to a lesser extent to the total baryon mass. In agreement with previous investigations, we find that the temperature is the most robust measurement, though there is a slight tendency to overestimate the temperature given
Figure 4.16: Group by group comparison of the group luminosities at $a = a_f \equiv 1$. In the left column all particles in each group are used, in the middle only particles with $\rho/\bar{\rho} \leq 5000$, and in the right $\rho/\bar{\rho} \leq 2000$. Only groups containing more than 32 particles are shown.
finite resolution. Since the total luminosity is dominated by the central, high-density gas (being proportional to $L \propto M \rho T^{1/2}$), these effects combine to cause at times drastic underestimates of the total luminosity. The evident stability of the temperature is likely due to the fact that the temperature of the collapsed gas is dominated by the initial strong shock the gas undergoes as it falls into the dark matter dominated potential wells. The post-shock evolution of the temperature as the gas settles into the high-density cores of the collapsed structures is relatively mild by comparison. The initial shock of the infalling gas raises its temperature to of order the virial temperature – typically a change of several orders of magnitude. By contrast, the adiabatic post-shock compression of the gas evolves the temperature as $T_{\text{adiab}} \propto \rho^{7/4} = \rho^{2/3}$. Even if the simulation underestimates the density of the highest density material by a factor of two, this only represents an error of $T_{\text{sim}}/T_{\text{actual}} \sim 2^{2/3} \sim 1.59$ in the temperature.

It is possible to draw an analogy between the numerical artifacts associated with hierarchical collapse such as shown here, and the more idealized collapse of a 1-D perturbation (the Zel'dovich pancake). In hydrodynamic simulations of the collapse of a single, 1-D perturbation (Shapiro & Struck-Marcel 1985; Owen et al. 1997b [Chapter 2]) it is the pressure which the gas is forced to drive toward the correct solution. This is easily understandable – as the perturbation collapses the potential well becomes well defined, and the post-shock gas must maintain the correct pressure in order to balance the gravitational potential. As one considers progressively smaller scales in the core of the pancake caustic, finite resolution eventually forces the gas density to
be underestimated, and the corresponding temperature is driven up in order to strike the correct pressure balance and keep the post-shock gas in hydrostatic equilibrium. Based upon this we might expect more generalized hierarchically built structures to demonstrate similar behaviour, and indeed we do find that where the gas density is underestimated the temperature tends to be slightly overestimated. However, when we examine the pressure in the experiments presented here, we find similar behaviour to that of the density: the pressure scales poorly in the cores and quite well in the outer regions of groups. This could indicate that either the gas in these structures is not being allowed to achieve hydrostatic equilibrium, or the amount of mass (and hence the gravitational potential) in the cores is not correct.

We should point out that evidence of these sorts of resolution effects is neither a novel result nor particular to SPH. Kang et al. (1994b) perform a comparison of several cosmological hydro-codes (both SPH and Eulerian based), and also find that in collapsed objects the central density can be quite resolution dependent, while the temperature profile (being more flat-topped) is more reliably represented. Curiously, they also find that when averaged over cosmologically large volumes, which statistically include both over and underdense regions, the density is more reliably estimated than the temperature. Since we only consider the properties of collapsed structures in this paper, we cannot comment on this latter result. More recently, Anninos & Norman (1996) study the collapse and formation of a typical galaxy cluster size object in an Einstein-de Sitter cosmology ($\Omega_{\text{bary}} = 0.06$) using a nested Eulerian grid code to simulate the hydrodynamics. Even at the most refined subgrid (effectively a 512$^3$
grid covering the cluster) they do not find convergence of the integrated luminosity or central density profile, while the emission weighted temperature is more reliably represented. In a study of the effects of photoionization on structure formation, Weinberg et al. (1996) find that the cooling balance (and therefore the eventual collapsed fraction) in marginally resolved structures can be strongly affected by the numerical tendency to underestimate the gas density. Owen & Villumsen (1997a [Chapter 3]) in a survey of hierarchical structure formation scenarios using 2-D SPH simulations also find that in general the gas density distribution does not converge with increasing numerical resolution. However, they do find that once a minimum temperature is imposed in the gas the gas distribution does converge if the corresponding Jeans mass is resolved. The one confusing study is that of Navarro et al. (1995) mentioned previously. In this case the investigators find that the properties of their X-ray clusters show evidence of convergence at their highest resolutions. Since they only simulate a small number of objects, they are able to use much higher resolutions for these objects than achieved here ($\sim 10^4$ SPH nodes for a single cluster, as opposed to $\sim 10^2 - 10^3$ nodes for typical objects in our experiments). It is possible that at these resolutions the cluster properties really do converge, though this seems at odds with the results found by Anninos & Norman (1996).

Comparing the two high-resolution experiments, it appears that the results of these experiments are relatively insensitive to the minor implementation differences between TreeSPH and P3MSPH. The only significant distinction we note is that the high-mass objects in P3MSPH tend to have higher core densities than their corollar-
ies in TreeSPH. This density difference then causes the temperature discrepancy. We also find, though, that these high-density cores are in fact unresolved, so the results in these regions should not be taken very seriously anyway. Once we restrict ourselves to regions which do seem to be effectively resolved, the P3MSPH and TreeSPH experiments agree very closely.

4.5 Summary

We analyze three 3-D hydrodynamical simulations evolved from scale-free initial conditions in order to study self-similar evolution of hierarchical structure formation including a gaseous component. Due to the expense of such experiments, we only consider a single model using an $n = -1$ power-spectrum of density fluctuations. We evolve three independent simulations based upon these identical initial conditions: two high-resolution experiments performed with $64^3$ dark matter and SPH particles, one performed with TreeSPH and one with P3MSPH, and one low-resolution run performed with $32^3$ dark matter and SPH particles under TreeSPH. Because both the physics and initial conditions of these experiments are scale-free, we can make use of the powerful prediction that these systems should evolve self-similarly over time. Using temporal self-similarity in combination with the known initial power-spectrum index $n$, we know how characteristic quantities such as the mass, temperature, and luminosity should evolve. We test for self-similar evolution in the simulations by identifying objects consisting of groups of particles of a given overdensity at various times and examining how the properties of these distributions of groups evolve. Due to the mass resolution limits of our experiments, a reasonable fraction of the total expected
mass distribution of structures is only accessible for roughly $\Delta \log a \sim 0.4$, or about a factor of 2.5 in expansion. During this rather restricted interval we identify the expected scalings for the mass and temperatures of the groups, while we find that the densities and total luminosities scale much more poorly. Upon further investigation we find that it is the central high-density regions of the groups which are causing the apparent poor scalings, and if we impose upper overdensity cutoffs for the particles considered in each group (thereby restricting ourselves to density "shells" around each group), the resulting densities and luminosities do scale reasonably.

The group temperatures and luminosities form quite tight power-laws of the mass. Restricting the luminosity contributions to shells outside the core of each group greatly improves the power-law relation between luminosity and mass, and also tends to force $L(M)$ toward a shallower slope. Taking resolution effects into account, we estimate that the measured power-law index for $T(M)$ is best viewed as a lower limit, while that for $L(M)$ is an upper limit. Representing these power-laws as $T \propto M^{\alpha_T}$ and $L \propto M^{\alpha_L}$, we empirically determine these exponents to be $\alpha_T \gtrsim 0.5$ and $\alpha_L \lesssim 1.3$. These estimates are consistent with the prediction $\alpha_T = 2/3$ and $\alpha_L = 4/3$ which results by assuming that objects of differing masses have similar structures.

We find that the Press-Schechter prediction for the mass distribution function works quite well for both the dark matter and baryon masses of the groups, though PS tends to overpredict the amount of baryon mass contained in objects at all scales. This confirms the agreement found in purely collisionless studies such as Efstathiou et al. (1988), though we confirm the agreement between the PS prediction and the
numerical results here for the baryonic component as well. The agreement between the
PS prediction and the numerical results is all the more remarkable for the fact that this
is a prediction, not a fit, as there are no free parameters. Since the power-law relation
between the baryon mass and the group temperature and luminosity is quite tight,
not surprisingly when we use these relations to map the PS mass prediction to the
temperature and luminosity distributions it also matches the numerical results quite
well. This demonstrates the predictive power of combining numerical simulations with
the self-similar requirement. Self-similarity can tell us how to map the distribution of
a quantity over time, but cannot predict the precise form of that distribution at any
given time. However, numerical simulations can predict the form of the distribution
over a restricted range of expansions set by our resolution limits. Combining these
two approaches, numerical simulations predict the detailed form of the distribution
for interesting quantities such as the temperature or luminosity function, and then
self-similarity allows us to scale this distribution to any point in time.

By direct comparison of the properties of groups in our different experiments, we
find that in general the obvious mass limitations outlined in Figure 4.1 are reasonable
estimates of the mass and expansion ranges we can probe. However, we also find that
the central high-density cores of the collapsed objects are unresolved and therefore
dominated by numerical effects, even for objects which we would otherwise predict
to be adequately resolved. The tendency with improving resolution for any given
group is to find a greater central density, increased total luminosity, increased baryon
mass, and slightly decreased central temperature. The most affected quantities are
the central gas density and luminosity, which can vary by up to an order of magnitude when the SPH particle number is increased by a factor of eight. The most robust quantities are the group mass and temperature, which we find to be only very weakly dependent on resolution. Similar numerical effects have been noted in previous studies for a variety of hydrodynamical techniques (Kang et al. 1994; Anninos & Norman 1996; Weinberg et al. 1996; Owen & Villumsen 1997a [Chapter 3]).

The fact that Efstathiou et al. (1988) identify the correct self-similar scaling for classical collisionless N-body codes represents one of the great successes for such techniques. In this study we find that (under certain restrictions) this self-similar behaviour is still maintained when hydrodynamical processes are added to such experiments, lending credence to results based upon such simulations. However, there are several complex numerical effects which come into play for these sorts of simulations beyond those found in purely collisionless investigations, which must be accounted for if the eventual results are to be believed. Additionally, we have omitted one important physical process in these experiments: radiative cooling. While radiative cooling is probably not important for the hot intergalactic gas found in clusters and groups of galaxies (outside of possible cooling flows in the cores of such objects), it is crucial to the process of galaxy formation. Understanding the process of galaxy formation is a major goal in studies of cosmological structure formation today, and many investigators are already using hydrodynamical simulations in order to approach this problem. Unfortunately introducing a physically motivated cooling law (such as that for a primeval H/He gas) in general certainly violates the scale-free requirement for
self-similar evolution. It is possible, however, to construct artificial cooling laws which do maintain the scale-free requirement for a given power-spectrum of initial density fluctuations. This forms the basis of study we are currently undertaking to examine self-similar behaviour in simulations which incorporate radiative cooling, which should have direct application to numerical studies of galaxy formation.
CHAPTER 5

Hydrodynamic Cosmological Simulations with Scale-Free Initial Conditions II: Radiative Cooling

5.1 Introduction

In Chapter 4 we find that, within certain numerical restrictions, hydrodynamical cosmological simulations of structure formation incorporating gravitational, pressure, and shock processes scale reliably. While this lends credence to the results of studies designed to study the evolution of the gas in large-scale objects such as galaxy clusters, there is one physical process crucial to studying galaxy formation which we neglected: radiative energy losses from the gas. Radiative cooling is key in the standard model for the formation of galaxies (White & Rees 1978) – some such dissipative process is required in order to form such tightly bound, overdense objects as galaxies. Studying the formation and evolution of galaxies is one of the prime motivators for doing hydrodynamical cosmological simulations, so in this Chapter we consider the effects of including the physics of radiative cooling in such simulations.

Our approach in this investigation will differ from that of Chapter 4 in a few important respects. First, in the scale-free study without cooling we consider a set of 3-D simulations. While fully 3-D studies are necessary in order to make direct,
quantitative comparisons of model results with observations, this is not required for an idealized study such as this. Rather, in this case we are more interested in characterizing the qualitative behaviour of both the physical systems we are considering as well as the numerical model. Therefore for this study we will consider 2-D simulations such as those used in Chapter 3. This choice affords us the luxury of performing more experiments than used in Chapter 4, as well as going to higher effective resolutions than are feasible in 3-D. The disadvantage of working in 2-D is that it is unclear how directly the results apply to the 3-D systems we are really interested in. However, one can argue that the numerical issue is the resolution which is achieved in any region. Therefore, since it is reasonable to expect a 2-D model will be able to adapt either similarly or slightly more effectively than 3-D, we might expect that wherever the 2-D model fails then 3-D should fail as well. It is, however, dicier going the other direction. Just because we find reliable scaling in the properties of a 2-D object does not necessarily imply that the 3-D counterpart will scale as well. This approach does represent a first cut approach to the problem though, and although there are no guarantees, if we can identify reasonable scaling in 2-D it at least lends credence to the 3-D results.

5.2 Expected Scalings for 2-D Perturbations

While working in 2-D affords us a number of computational advantages, studying cosmological structure in 2-D represents a number of rather subtle changes from the traditional 3-D cases which are familiar to most researchers. It is important to understand that these models really represent imposing 2-D perturbations in a
formally 3-D universe. In other words, the correct conceptual way to view the setup for these simulations is that we are taking an initially homogeneous 3-D universe consisting of both baryons and dark matter, and initializing only $k_x$ and $k_y$ density perturbations in Fourier space. This results in structure evolving in the $x$ and $y$ directions alone - matter continues to simply expand with the Hubble flow in the $z$ direction. Therefore the most collapsed structures that form are infinitely long filaments, rather than distinct collapsed groups or galaxies. When we measure the "mass" of one of these 2-D objects, we are in fact measuring a mass per unit length, or linear density $\theta \equiv m/\ell$. Similarly, the mass of each particle in the simulations is actually a mass per unit length, and the particles interact gravitationally as a group of infinite, thin rods.

To illustrate some of the more important distinctions from the general 3-D case, consider the collapse and formation of an isolated filament. If the total mass per unit length of this object is $\Theta$, and we have another infinite, thin rod of mass per unit length $\theta$ at a distance $r$, then their mutual gravitational force is $F = G\Theta\theta/r \propto 1/r$, and the gravitational potential is $\Phi(r) = G \ln r$. The circular velocity $v_c^2 = r d\Phi/dr = G\Theta$ is independent of distance. In order to define a temperature which corresponds to this circular velocity (see Thoul & Weinberg 1996 for the 3-D analogy), consider an isothermal atmosphere of gas about the filament in hydrostatic equilibrium. If the gas obeys a profile $\rho(r) = \rho_1(r/r_1)^{-n}$ and the pressure is given by $P(r) = kT(\mu m_p)^{-1}\rho(r)$, then the requirement of hydrostatic equilibrium becomes

$$\frac{G\Theta}{r} 2\pi r \, dr \, \rho(r) = \frac{dP}{dr} 2\pi r \, dr,$$

(5.1)
which after some manipulation yields

\[ T = -\frac{\mu m_p}{nk} G \Theta = -\frac{\mu m_p}{nk} \nu_c^2. \quad (5.2) \]

Inspection reveals that typically our 2-D objects obey \( \rho(r) \propto r^{-2} \) (where \( \rho \) is the mass per volume), so we recover the 3-D result \( T = \mu m_p (2k)^{-1} \nu_c^2 \).

As in Chapter 4, we wish to analyze these simulations in terms of the population of collapsed structures which form. However, we must reformulate the scaling relations for 2-D perturbations. The rms mass fluctuations smoothed with a window function \( W(R) \) are defined as

\[ \sigma_M^2(R) = \left\langle \frac{\delta M}{M} \right\rangle^2_R = \int d^2 k \left( \frac{W_k}{V_W} \right)^2 \delta_k^2 = \int_0^\infty dk 2\pi k \left( \frac{W_k}{V_W} \right)^2 \delta_k. \quad (5.3) \]

If we consider a 2-D power-spectrum of perturbations \( P(k) \propto k^\alpha \), and evaluate this expression for a typical window function (such as the top-hat or gaussian), we find

\[ \sigma_M^2(a, R) \propto a^2 R^{-(n+2)} . \quad (5.4) \]

Note that the length scale \( R \) in this expression is in fact the comoving length scale \( R^* \). The characteristic length scales as the nonlinear length scale, for which equation (5.4) yields

\[ R^* \propto a^{2/(2+n)} \Rightarrow R_* \propto a R^* \propto a^{(4+n)/(2+n)} \quad (5.5) \]

Since the underlying physics is 3-D, the background density still scales as \( a^{-3} \) and we have

\[ \rho_* \propto \rho \propto a^{-3} . \quad (5.6) \]
The characteristic linear density $\theta_\ast$ (analogous to the characteristic mass $M_\ast$ in 3-D) scales as

$$\theta_\ast^c \propto \rho_\ast^c (R_\ast^c)^2 \propto a^{4/(2+n)} \Rightarrow \theta_\ast \propto a^{-1} \theta_\ast^c \propto a^{(2-n)/(2+n)}. \quad (5.7)$$

Based on equation 5.2 we know that the supporting temperature scales directly with $\theta$, so

$$T_\ast \propto \theta_\ast \propto a^{(2-n)/(2+n)}, \quad (5.8)$$

and finally the luminosity per unit length scales as

$$\frac{L}{\ell} \propto \theta_\ast \rho_\ast T_\ast^{1/2} \propto a^{-(6+9n)/(4+2n)}. \quad (5.9)$$

Note that we are defining the luminosity as proportional to the square-root of the temperature. Even though we will be assuming a different temperature dependence for radiative cooling in the following section, whenever we refer to the "luminosity" we will still use $L \propto T^{1/2}$, since this corresponds to the physical process of Bremsstrahlung emission in the real universe.

### 5.3 Radiative Cooling Laws

Unfortunately, a realistic cooling function such as arises from a primordial H/He plasma (see, e.g., Katz, Weinberg, & Hernquist 1996) will impose an arbitrary physical scale into the system, violating our scale-free requirement for self-similar evolution. This is because, in general, the cooling time $t_C$ for a characteristic nonlinear structure of mass $M_{nl}$ will be a function of time $t_C(M_{nl}, a)$. Additionally, the standard cooling
law has distinct, fixed features such as the photoionization transition at $T \sim 10^4 K$ or the H and He bumps at $T \sim 10^4 K - 10^{5.5} K$. We therefore cannot use the standard physical cooling law and maintain the scale-free nature of the problem.

However, for a given scale-free power-law of initial density fluctuations, $P(k) \propto k^n$, it is possible to determine a power-law cooling relation $\Lambda(T)/n_H^2 \propto T^\beta$ such that the cooling time for a characteristic mass (such as the nonlinear mass $M_{\text{nl}}$) is a fixed fraction of the Hubble time. So long as the cooling law meets this requirement, the physical system will remain scale-free, and the rigorous prediction of self-similar evolution for the system holds. Under this condition, at a fixed expansion factor the ratio of the cooling time to the Hubble time may vary with mass, but it will depend only on $M/M_{\text{nl}}$, and thus will still scale properly in time. Not surprisingly, the cooling exponent $\beta$ depends upon whether we are considering 2-D or 3-D perturbations, so we will give both cases here.

The radiative contribution to the time rate of change of the specific thermal energy $u$ is (see Eq. [A.3])

$$\left. \frac{Du}{Dt} \right|_R = \frac{\Lambda(T)}{\rho}.$$  \hspace{1cm} (5.10)

We will parameterize the radiative cooling function $\Lambda(T)$ as a power-law

$$\frac{\Lambda(T)}{n_H^2} = [\mu m_p (1 + n_{\text{He}}/n_H)]^2 \frac{\Lambda(T)}{\rho^2},$$

$$\equiv \ A_0 T^\beta,$$

$$= \ A_0 \left[ \frac{(\gamma - 1) \mu m_p u}{k} \right]^\beta,$$

$$\equiv \ A_1 u^\beta.$$
In these terms the cooling time $t_C$ can be expressed as

$$t_C = u \left( \frac{D u}{D t} \right)_R^{-1} = u \left( \frac{\Lambda(T)}{\rho} \right)^{-1} = u(\rho A_1 u^\beta)^{-1} = A_1^{-1} \rho^{-1} u^{1-\beta}. \quad (5.11)$$

The requirement to maintain self-similarity is that the cooling time for a characteristic \( \theta_\ast \) (or \( M_\ast \)) object must be a fixed fraction \( \hat{t}_C \) of the Hubble time \( t_H = H_0 (a/a_0)^{3/2} \).

Note that this implies scaling for other masses as well, even though the value of \( \hat{t}_C \) may be different. For 2-D perturbations we know that the specific thermal energy and the density scale like \( u_\ast \propto a^{(2-n)/(2+n)} \) and \( \rho_\ast \propto a^{-3} \), so fixing \( t_C^* = \hat{t}_C t_H \) yields

$$A_1^{-1} \rho_0^{-1} \left( \frac{a}{a_0} \right)^3 u_0^{-1} \left( \frac{a}{a_0} \right)^{(1-\beta)(2-n)/(2+n)} = \hat{t}_C H_0^{-1} \left( \frac{a}{a_0} \right)^{3/2}, \quad (5.12)$$

implying,

$$A_1^{-1} \rho_0^{-1} u_0^{1-\beta} = f H_0^{-1} \left( \frac{a}{a_0} \right)^{-3/2-(1-\beta)(2-n)/(2+n)}. \quad (5.13)$$

Our specification that \( t_C^* \) remain the same fraction of the Hubble time at all times requires that the \( a \) dependence on the right-hand side of equation (5.13) vanish, which finally gives us

$$\beta_{2-D} = \frac{3}{2} \frac{2 + n}{2 - n} + 1. \quad (5.14)$$

The 3-D derivation is very similar, except that for 3-D perturbations the specific thermal energy scales as \( u_\ast \propto a^{(1-n)/(3+n)} \). Following the same arguments as above yields

$$\beta_{3-D} = \frac{3}{2} \frac{3 + n}{1 - n} + 1. \quad (5.15)$$
Now that we know the appropriate index for the cooling power-law, we need only specify the normalization. According to equation (5.12), the normalization $A_1$ is proportional to

$$A_1 \propto \frac{H_0}{\dot{t}_C \rho_0 u_0^{\beta-1}} \propto \frac{H_0}{\dot{t}_C \rho_0 T_0^{\beta-1}},$$

(5.16)

where $\rho_0$ and $T_0$ are chosen as characteristic of a fiducial group at expansion $a = a_0$. We will adopt the convention that the characteristic density is of order $\rho_0 \sim 1000 \bar{\rho}_{\text{bary}}$, as this is typical of the objects we select. A well-defined choice for the characteristic temperature is the support temperature for an object with the nonlinear mass: $T_{\text{nl}} = \mu m_p (2k)^{-1} G \theta_{\text{nl}}$. Then by choosing a cooling fraction $\dot{t}_C \equiv t_C/t_H$, we completely specify the normalization of the cooling law. Of course, the population of objects we identify at any given time will include a spectrum of structures, and therefore the specific ratio of the cooling time to the Hubble time will be a function of mass. However, the ratio of the cooling to Hubble time will remain a fixed function of $M/M_{\text{nl}}$ only, so we will still meet the scale-free requirement for self-similar evolution of the population as a whole.

5.4 The Simulations

In both Chapters 3 and 4 we consider simulations at a variety of numerical resolutions, in order to directly investigate the numerical effects of finite resolution on the measured results. Although resolution effects are also an important consideration when radiative cooling becomes important (see, e.g., Frenk et al. 1996), for this investigation we choose instead to fix the resolution at one value and vary the physical model.
Each experiment is performed using $N_{\text{bary}} = N_{\text{dm}} = 128^2$ computational nodes, on a $512^2$ PM grid to evaluate the gravitational interaction. As in Chapter 4 the background cosmology is a flat, Einstein-de Sitter universe (consistent with our scale-free requirement for the physics) with $\Omega_{\text{bary}} = 0.05$ and $\Omega_{\text{dm}} = 0.95$. The initial density perturbations are Gaussian distributed with random phases according to a power-law $P(k) \propto k^n$, where we consider cases for $n = -1, 0, \text{ and } +1$. These 2-D power-spectra dynamically correspond to 3-D spectra of indices $n_{3-D} = n_{2-D} - 1 = -2, -1, \text{ and } 0$ (in terms of quantities such as the rate that $R_{\text{eff}}$ grows, for instance). For each $n$ the density perturbations are initialized according to the same random number seed, so that we form analogous structures in each realization. Additionally, for each $n$ we vary the amplitude of the cooling law by varying $\dot{t}_C$ ($\dot{t}_C = t_C/t_H$, see equation [5.12]). In all we consider a total of 12 different physical scenarios (3 different power-spectra, with 4 different cooling amplitudes for each). Due to the strong resolution dependence of radiative cooling we use ASPH simulations in this investigation, though in some cases we also perform an SPH version for comparison. The overall simulation parameters are summarized in Table 5.1.

We should point out that these simulations are integrated using a power of the expansion ($p = a^\alpha$), rather than time, as the independent variable (see appendix A.2, or Efstathiou et al. 1985). The exponent $\alpha$ is chosen such that the rms displacement of the particles per step $\Delta p$ will be roughly equal throughout the evolution the system. Through arguments similar to those used above in §5.2, it is simple to show that for an initial power-spectrum of 2-D density fluctuations $P(k) \propto k^n$ the optimum value
### Table 5.1: Simulation parameters for 2-D scale-free radiative cooling simulations.

All but the last two rows are the generic scale-free parameters, while the last two show one possible choice of absolute scalings to associate the simulation models with real scales in the universe.
for this exponent is $\alpha = 2 / (2 + n)$. Additionally, since these simulations are entirely scale-free, we can choose to identify them with any set of physical scales in the real universe. The last two rows of Table 5.1 gives one plausible choice for such scalings. For instance, the final output expansion for the $n = 0$ models could be considered to be at redshift $z = 1.5$ in a $15 \, h^{-1} \, \text{Mpc}$ box. It is important to emphasize, however, that these values are not unique, and any consistent set of corresponding scalings could be substituted. The choice of simulation starting and ending points are motivated numerically rather than physically, and are chosen such that for each power-spectrum $n$ the initial rms mass fluctuation on the scale of the particle resolution grid ($128^3$) is $\left< \delta \rho / \rho \right> \sim 0.5$, and the models are stopped once the nonlinear scale approaches $R_{\text{NL}} \sim 10\% - 20\%$ of the box size.

We are most interested in the properties of the distribution of collapsed structures which form in our simulations. In order to find such objects in an unbiased manner we use the "friends-of-friends" algorithm (such as in Chapter 4; see, e.g., Barnes et al. 1985). We find objects using two linking parameters: $\ell \sim 0.2 \, \Delta x_p$ (where $\Delta x_p$ is the initial interparticle spacing), corresponding to the usual choice in 3-D to identify objects of average overdensity $\delta \rho / \rho \sim 250$; and $\ell \sim 0.05 \, \Delta x_p$, which through experimentation we find picks out the radiatively cooled gas in the cores of collapsed objects. Before we go on to discuss the properties of the distribution of these objects, however, it is important to understand the mass regimes are we are sensitive to in each experiment. Clearly our low-mass limit is set by the particle number. Since each ASPH node is set to sample out roughly three interparticle spacings in two
smoothing scales, we can safely state that any object with fewer than $\pi 3^2 \sim 28$ particles is unresolved. The upper mass limit on the mass spectrum of objects we can expect to find in an experiment is a function of the initial density perturbation spectrum ($n$), the amplitude of perturbations (as a function of expansion: $\theta_{nt}(a)$), and the box size. As in Chapter 4 we can use the Press-Schechter (Press & Schechter 1974; hereafter PS) mass function to predict the spectrum of objects we expect to find in our simulation volume at any particular time. However, there are two subtle distinctions in using PS theory in 2-D rather than the better known case of 3-D perturbations. First, it is necessary to rederive the PS theory for 2-D perturbations, as is given in equation (5.17) below. Second, in 3-D we know that friends-of-friends with a linking parameter $\ell = 0.2\Delta x_p$ finds objects with an overdensity of roughly $\delta \equiv \delta \rho / \rho \sim 250$. The well-known solution for the collapse of an isolated 3-D perturbation (the "top-hat" collapse) tells us that an object with an actual overdensity of $\delta \sim 250$ corresponds to a linearly predicted mass fluctuation of $\delta_{\text{linear}} = 1.68$. We can therefore use $\delta_{\text{linear}}$ in the PS mass distribution to predict the properties of the population of objects identified by friends-of-friends. In 2-D, however, the well-known top-hat solution does not apply, so we do not have an a priori prediction for the appropriate value of $\delta_{\text{linear}}$. Instead, we measure the differential group mass distribution $f(\theta)$ in the simulations and fit the PS model by varying the value of $\delta_{\text{linear}}$. The results can be seen in Figures 5.4-5.6 below.
Figures 5.1–5.3 plot the mass limits for each initial fluctuation power-spectrum (\( n = 0, +1, \text{and} -1 \)), in a format similar to Figure 4.1 in Chapter 4. There is one important but subtle difference between these figures and Figure 4.1, however. Since we are now working in 2-D, these "masses" are in fact total linear densities \( \theta \propto m/\ell \).

Further, Figures 5.1–5.3 are plotted in comoving coordinates, since the linear density for an isolated, collapsed object will continue to fall as \( \theta \propto a^{-1} \) due to the Hubble expansion in the \( z \) direction. In the \( n = 0 \) case (Figure 5.1) we can expect that a mass range \( \theta^c \in [1, \theta_\text{nl}, 10 \theta_\text{nl}] \) should be accessible over a range of expansions \( \log a/a_f \in [-0.9, 0] \). Similarly, Figures 5.2 and 5.3 indicate a reasonable mass range should be accessible over \( \log a/a_f \in [-1, 0] \) for \( n = +1 \) and \( \log a/a_f \in [-0.4, 0] \) for \( n = -1 \). Clearly we are most restricted for the \( n = -1 \) simulations, which is not surprising since it is not evolved as far as the other two cases presented here: \( R_\text{nl}/L_\text{box}(a_f) \sim 0.04 \) for \( n = -1 \), as opposed to \( R_\text{nl}/L_\text{box} \sim 0.1 \) for \( n = 0 \) and \( n = +1 \). There is no fundamental reason that the \( n = -1 \) models are halted at \( R_\text{nl}/L_\text{box}(a_f) \sim 0.04 \). This was in fact inadvertent, and these models could easily be restarted and evolved to a similar point as the \( n = 0 \) and \( +1 \) cases. Due to this limitation, in the following investigation we may expect to see much more restricted scaling in the \( n = -1 \) models than in the other cases.

In Figures 5.4–5.6 we calculate the mass distribution of objects \( f(\theta) \) over a range of expansions for the simulations without cooling (\( \hat{t}_C \equiv \infty \)), as well as for the cooling models with \( \hat{t}_C = 1 \) and \( \hat{t}_C = 0.1 \). At each expansion \( f(\theta) \) is calculated and shifted (assuming the self-similar relation \( \theta \propto a^{(2-n)/(3+n)} \) holds) to \( a = a_f \) for comparison.
Figure 5.1: Comoving $\theta$ resolution limits as a function of expansion for the $n = 0$ simulations. In this 2-D context, $\theta$ is the corollary of mass in 3-D: $\theta \propto m/\ell$. The dotted line shows the low mass limit, assumed to be 28 times the particle "mass" (actually the linear density per particle $\theta = L_{\text{box}}^2 \bar{\rho}/N$). The dashed lines show various multiples of the nonlinear mass scale $\theta_{nl}$, while the solid lines show the PS prediction such that statistically we would expect to find 1 ($\theta_1$), 10 ($\theta_{10}$), and 100 ($\theta_{100}$) objects of that mass in the simulation volume. All $\theta$'s are scaled such that the total linear density in the volume is 1.
Figure 5.2: Comoving $\theta$ resolution limits as a function of expansion for the $n = +1$ simulations. Line types and definitions as in Figure 5.1.
Figure 5.3: Comoving $\theta$ resolution limits as a function of expansion for the $n = -1$ simulations. Line types and definitions as in Figure 5.1.
giving the heavy solid lines. In each figure, the left column gives the results for the
dark matter linked with $\ell = 0.2 \Delta x_p$, the middle column the baryons linked with
$\ell = 0.2 \Delta x_p$, and the right column the baryons linked with $\ell = 0.05 \Delta x_p$. Each panel
also shows the 2-D PS prediction, given by the relation

$$f(\theta) \, d\theta = \sqrt{\frac{2}{\pi}} \left(\frac{2 + n}{4}\right)^{(2+n)/4} \left(\frac{\theta}{\theta_*}\right)^{(2+n)/2} \exp\left(-\frac{1}{2} \left(\frac{\theta}{\theta_*}\right)^{(2+n)/2}\right) \, d\theta,$$

(5.17)

where $f(\theta) \, d\theta$ represents the mass fraction of groups in the mass range $[\theta, \theta + d\theta]$. Here $\theta_*$ is defined as $\theta_* = \rho R_{nl}^2 (\delta_* / \delta_{nl})^{4/(2+n)}$, where $R_{nl}$ and $\delta_{nl}$ are some fiducial
normalization scale (presumably the nonlinear scale so that $\delta_{nl} = 1$), and $\delta_*$ is the
linearly predicted fluctuation amplitude appropriate for the objects in question. As
mentioned above, in 3-D we know th at for objects with actual overdensities of $\delta \sim 250$
we can use $\delta_* = 1.68$. However, in 2-D we don’t necessarily know what value for $\delta_*$ is
appropriate, so instead we fit the PS prediction to the measured curves for $f(\theta)$ with
$\delta_*$ as a free parameter. The thin lines in Figures 5.4–5.6 show the resulting PS curves,
while Table 5.2 gives the values determined for $\delta_*$. Based upon this table, it appears
that in 2-D there is not a single value for $\delta_*$ which corresponds uniquely to a given
linking parameter (and therefore actual overdensity). In general, the measured curves
for $f(\theta)$ seem to link up well, which implies that the mass function of the objects is
obeying the correct scaling. The only exception is the population of objects selected
with $\ell = 0.05 \Delta x_p$ in the no cooling models (the upper right panel). This is because
in the no cooling case, without radiative dissipation almost no material reaches the
overdensities selected by this small linking parameter. Aside from this case, the PS
curves fit the measured distributions quite well, though with a fair amount of scatter.
Table 5.2: Fitted values for the linearly predicted mass fluctuation amplitude $\delta_*$ corresponding to each linking parameter $\ell$ as a function of power-spectral index $n$.

<table>
<thead>
<tr>
<th>$\delta_*$</th>
<th>$n = -1$</th>
<th>$n = 0$</th>
<th>$n = +1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell = 0.2 \Delta x_p$</td>
<td>0.25</td>
<td>0.7</td>
<td>3</td>
</tr>
<tr>
<td>$\ell = 0.05 \Delta x_p$</td>
<td>0.4</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Note that the PS prediction with the same value of $\delta_*$ fits the both the dark matter and baryonic objects selected with $\ell = 0.2 \Delta x_p$. This implies that the baryon to dark matter ratio in these structures is near the universal average. It is also interesting that the PS function fits the dissipated baryon structures in the rightmost column as well. Here we plot the PS prediction using the same $\delta_*$ as for the large objects (the thin solid lines), as well as with a larger value of $\delta_*$ fitted to these dissipated groups (dotted lines). This is interesting because, although we can predict that PS theory should work reasonably for objects constructed purely gravitationally, we do not necessarily know that PS should apply to structures which form through the additional process of radiative dissipation. Equivalently, this can also be interpreted as indicating that a nearly constant fraction of the baryons selected with $\ell = 0.2 \Delta x_p$ is cooling with mass.

5.5 Testing for Self-Similar Evolution

As a more direct measure of how well the group masses are scaling, Figures 5.7–5.9 plot the characteristic cutoff group $\theta_{70\%}$, defined so that 70% of the mass in the simulations is contained in groups with $\theta \leq \theta_{70\%}$, as a function of expansion.
Figure 5.4: Differential mass distribution function $f(\theta)$ for $n = 0$ simulations at $a = a_f$. The left column shows the dark matter groups linked with $\ell = 0.2 \, \Delta x_p$, the middle column shows the baryons linked with $\ell = 0.2 \, \Delta x_p$, and the right column shows the baryon groups linked with $\ell = 0.05 \, \Delta x_p$. Each heavy solid line represents the measured $f(\theta)$ for a different expansion - all expansions are shifted assuming self-similarity to the final time $a = a_f$. The thin solid lines show the PS mass function assuming $\delta_\star = 0.6$ (which is fitted as the appropriate comparison for objects identified with linking parameter $\ell = 0.2 \, \Delta x_p$), while the thin dotted line in the right column is the PS mass function using $\delta_\star = 1$. 

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Figure 5.5: Differential mass distribution function $f(\theta)$ for $n = +1$ simulations at $a = a_f$. Columns and line types as defined in Figure 5.4, except that the thin solid line represents the PS prediction using $\delta_s = 3$ (fitted as appropriate for objects identified with linking parameter $\ell = 0.2 \Delta x_p$), while the thin dotted lines in the right column show the PS function using $\delta_s = 4$. 

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Figure 5.6: Differential mass distribution function $f(\theta)$ for $n = -1$ simulations at $a = a_f$. Columns and line types as defined in Figure 5.4. In this case, the PS prediction shown by the thin solid lines uses $\delta_* = 0.35$, while the thin dotted lines use $\delta_* = 0.6$. 

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Note that for each initial spectral index \( n \) we have several models, including the four possible cooling normalizations \( (\hat{t}_C = \infty, 200, 1, \text{ and } 0.1) \) as well as, in a few instances, both SPH and ASPH simulations of the same physical system, such as the \( \hat{t}_C = 0.1 \) case for both \( n = 0 \) and \( +1 \). We plot the results for each distinct \( n \) in the same figure, differentiating between models in the same panel with different point and line types. This test is simplest to perform in comoving coordinates, because the total proper \( \theta \) in the system is a function of expansion \( (\theta_{\text{tot}} \propto a^{-1}) \), while it is fixed in comoving coordinates. The lines show the expected scaling \( \theta^c \propto a^{4/(2+n)} \) (Eq. [5.7]), with the normalization of the line fitted to the data. We plot both the dark matter and baryon groups using \( \ell = 0.2 \Delta x_p \), while only the baryons are plotted with \( \ell = 0.05 \Delta x_p \). This is because there are no dark matter groups identifiable with such a small linking parameter – only with dissipation through radiative cooling do any baryons collapse to such dense groupings.

While these figures are comparable to the 3-D version in Figure 4.3, there is one important computational distinction in how the mass fractions are calculated in the 2-D vs. the 3-D case. In the 3-D simulations we typically have several hundreds to a few thousand identified objects, which is sufficient to create a rather smooth and well-behaved cumulative mass function. However, in the 2-D simulations we only have of order several tens to a few hundred well-resolved objects, mostly due to the smaller number of particles used. The 2-D raw cumulative mass functions therefore tend to be much jumpier than those in the 3-D simulations, due to these counting statistics. This can lead to noise in measurements of cutoff fractions in the mass
Figure 5.7: Evolution of the comoving group $\theta_{70\%}(a)$ for the $n = 0$ models, such that at each expansion $70\%$ of the mass in the simulation is contained in groups with $\theta^c \leq \theta_{70\%}$. The points show the measured simulation results at each timeslice, while the lines show the expected scaling fitted to each model. Each individual model — distinguished either by the chosen cooling fraction $\dot{t}_C = t_C/t_H$ ($\dot{t}_C = \infty$, 200, 1, or 0.1) and/or the simulation technique (either ASPH or SPH) is assigned a distinct point and line type, as shown in the legend.
Figure 5.8: Evolution of the comoving group $\theta_{70\%}(a)$ for the $n = +1$ models, such that at each expansion 70% of the mass in the simulation is contained in groups with $\theta^c \leq \theta_{70\%}$. The points show the measured simulation results at each timeslice, while the lines show the expected scaling fitted to each model.
Figure 5.9: Evolution of the comoving group $\theta_{70\%}(a)$ for the $n = -1$ models, such that at each expansion 70% of the mass in the simulation is contained in groups with $\theta^c \leq \theta_{70\%}$. The points show the measured simulation results at each timeslice, while the lines show the expected scaling fitted to each model.
function, such as is used in figures like 5.7–5.9. We deal with this problem by fitting a polynomial to the cumulative mass function using general linear least-squares, and then interpolating for the desired mass fraction cutoff in this smooth polynomial fit. As in Chapter 4, we again adopt the convention that all the unbound mass in each simulation is counted as belonging to bound structures below our resolution limit.

The expected scalings are $\theta \propto a^2 \ (n = 0)$, $\theta \propto a^{4/3} \ (n = +1)$, and $\theta \propto a^4 \ (n = -1)$. The lines plotted show these expected slopes, where only the normalization of each line type is fitted to the associated model (i.e., solid lines with the no cooling model, dotted lines with the $t_C = 200$ models, etc.) Both the $n = 0$ and $+1$ models (Figures 5.7 and 5.8) demonstrate good scaling of the group $\theta$ distribution over a range $\log a/a_f \in [-0.8, 0]$ for $n = 0$ and $\log a/a_f \in [-1.4, 0]$ for $n = +1$, similar to the ranges we expect based on Figures 5.1 and 5.2. The scaling holds for objects identified both with $\ell = 0.2 \Delta x_p$ (dark matter and baryons), as well as the cooled baryon gas identified with $\ell = 0.05 \Delta x_p$. At earlier times the scalings fail due to the lack of resolved objects – not enough material has accreted into collapsed structures to resolve the population yet. We also find that at times later than those plotted here the scaling fails again, in this case because the scale of nonlinearity is approaching the size of our box, which leads to a saturation effect. In the $n = -1$ case (Figure 5.9), we only see reasonable scaling for the $\ell = 0.2 \Delta x_p$ groupings for $\log a/a_f \in [-0.2, 0]$, or about a factor of 1.6 in expansion. The dissipated gas identified with $\ell = 0.05 \Delta x_p$
does not scale convincingly at all. This is not surprising, since the \( n = -1 \) models are not evolved as far as the \( n = 0 \) and \( +1 \) cases, and therefore the population of objects which forms is resolved for a much shorter time.

In Figures 5.10–5.12 we test the 70% mass evolution of the emission weighted group temperatures. The emission weighted temperature for a 2-D structure is defined similarly to the 3-D case (Eq. 4.11): we take the volume emissivity to be of a form typical of Bremsstrahlung radiation \( \varepsilon \propto \rho^2 T^{3/2} \), so that the emission weighted temperature for group \( i \) is defined as

\[
T_i = \frac{\int \varepsilon T \, dA}{\int \varepsilon \, dA} = \frac{\int \rho^2 T^{3/2} \, dA}{\int \rho^2 T^{1/2} \, dA} = \frac{\sum_j \theta_j \rho_j T_j^{3/2}}{\sum_j \theta_j \rho_j T_j^{1/2}}, \tag{5.18}
\]

represented as a sum over particles \( j \) (with individual linear densities \( \theta_j \)) which are members of group \( i \). At each expansion we sort the resolved groups in a given simulation by temperature, and determine the cutoff \( T_{70\%} \) such that 70% of the mass in the simulation is contained in structures with \( T \leq T_{70\%} \). The solid lines in Figures 5.10–5.12 show the expected scalings \( T \propto \theta^{(2-n)/(2+n)} \) (Eq. [5.8]), which becomes \( T \propto a^1 \) (\( n = 0 \)), \( T \propto a^{1/3} \) (\( n = +1 \)), and \( T \propto a^3 \) (\( n = -1 \)).

Considering first the \( n = 0 \) case in Figure 5.10, we see that in general the emission weighted temperatures for the ASPH simulations scale reasonably for nearly all of the simulations, both for \( \ell = 0.2 \Delta x_p \) and \( \ell = 0.05 \Delta x_p \). The effect of the radiative cooling is also very evident: the collapsed gas in the \( t_C = 0.1 \) case is nearly 1.5 orders of magnitude cooler than in the case without cooling. Additionally, the dense, dissipated gas picked out with \( \ell = 0.05 \Delta x_p \) is typically about a half an order of magnitude cooler than the \( \ell = 0.2 \Delta x_p \) gas in the models with cooling. In contrast
with the $\theta^c$ scalings, though, the temperature scalings seem to hold over a more restricted range of expansions: roughly $\log a/a_f \in [-0.6,0]$, or about a factor of 4. The SPH cooling $\dot{\xi}_C = 0.1$ simulation does not scale as convincingly, but rather holds a nearly constant temperature throughout – this is particularly evident in the cooled gas component ($\ell = 0.05 \Delta x_p$ in the lower panel). The $n = +1$ case in Figure 5.11 consists of only ASPH simulations, so we cannot comment on this distinction between the techniques. Interestingly, the temperature for the hot+cold baryon objects found with $\ell = 0.2 \Delta x_p$ (the upper panel) scales less effectively than the temperature of the cooled component found with $\ell = 0.05 \Delta x_p$ (lower panel). This difference suggests that the cooled gas cores of the structures in the $n = +1$ models scale correctly, and that it is the hot, diffuse gas in the outer regions that does not. This result seems contrary to what we find in the no cooling 3-D tests of Chapter 4, and it could be due to hot gas that is not really physically bound being erroneously counted as parts of collapsed structures by friends-of-friends with the larger linking parameter.

In Figure 5.12 we examine the temperature scaling for the $n = -1$ simulations. Based on the $\theta^c$ scaling, it is not surprising that the ASPH simulations only seem to demonstrate the expected scalings over a small range of expansions near the end: $\log a/a_f \in [-0.2,0]$, the same as for $\theta^c$. The SPH simulation does not convincingly show the proper scaling, but since we effectively resolve such a small range of expansions this is not terribly significant.
Figure 5.10: Evolution of the emission weighted group temperature $T_{70\%}(a)$ for the $n = 0$ models, such that at each expansion 70% of the mass in the simulation is contained in groups with $T \leq T_{70\%}$. Point and line types as defined in Figure 5.7, such that solid lines represent the models without cooling, dotted lines cooling with $\dot{t}_c = 200$, short dashes $\dot{t}_c = 1$, long dashes $\dot{t}_c = 0.1$, and dot-dashes SPH with $\dot{t}_c = 0.1$. 
Figure 5.11: Evolution of the emission weighted group temperature $T_{70\%}(a)$ for the $n = +1$ models, such that at each expansion 70\% of the mass in the simulation is contained in groups with $T \leq T_{70\%}$. Point and line types as defined in Figure 5.8, such that solid lines represent the models without cooling, dotted lines cooling with $i_C = 200$, short dashes $i_C = 1$, and long dashes $i_C = 0.1$. 

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Figure 5.12: Evolution of the emission weighted group temperature $T_{70\%}(a)$ for the $n = -1$ models, such that at each expansion 70% of the mass in the simulation is contained in groups with $T \leq T_{70\%}$. Point and line types as defined in Figure 5.9, such that solid lines represent the models without cooling, dotted lines cooling with $\tilde{t}_C = 200$, short dashes $\tilde{t}_C = 1$, long dashes $\tilde{t}_C = 0.1$, and dot-dashes SPH with $\tilde{t}_C = 0.1$. 

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Figures 5.13–5.15 measure the 70% mass scaling of the group luminosities. The
ccharacteristic luminosity at each expansion is calculated similarly to the characteristic
temperatures in the previous figures. At each expansion we first define the total group
luminosity as \( L/l = \int \rho^2 T^{1/2} dA = \sum_j \theta_j \rho_j T_j^{1/2} \). Note that this is actually a luminosity
per unit length \( L/l \) – the total luminosity for an infinite filament formally diverges,
much like the mass. We then sort the resolved groups by their luminosities, and find
the cutoff value \( L_{70\%}/l \) such that 70% of the mass is contained in objects with \( L/l \leq L_{70\%}/l \). The expected scaling for the luminosities goes as \( L/l \propto a^{-(6+9n)/(4+2n)} \) (Eq.
[5.9]), so that for \( n = 0 \) we have \( L/l \propto a^{-3/2} \), for \( n = +1 \), \( L/l \propto a^{-5/2} \), and for \( n = -1 \),
\( L/l \propto a^{3/2} \). Note that in our range of \( n \) the expected evolution of the luminosity varies
quite strongly – from decreasing with expansion for \( n = 0 \) and \( +1 \), to increasing for
\( n = -1 \). It is also interesting to note that the luminosity does not change significantly,
even as we drastically change the strength of the cooling. This consistency is because
as the radiative cooling is increased and the typical temperatures of the objects drops,
this loss is compensated for in the luminosity by the increased gas densities.

In the 3-D scale-free runs without cooling, we find that the total group luminosities
scale very poorly compared with the self-similar solution (Figure 4.5), and we might
expect to find similar poor behaviour in the 2-D models. Surprisingly, though, this
turns out not to be the case. The \( n = 0 \) simulations are the 2-D corollaries of
\( n = -1 \) in 3-D (at least in terms of integrals over the power-spectrum), yet the
\( L_{70\%}/l \) evolution in Figure 5.13 matches the expected scaling reasonably well for all
the \( n = 0 \) models, over a similar range of expansions as \( \theta^c \) and \( T \). This is true both
for the hot+cold baryon groups in the left column and for the cooled baryon clumps on the right. The $n = +1$ runs also demonstrate good scaling for $L/l$ in Figure 5.14. Only the $n = -1$ runs in Figure 5.15 fail to find reasonable scaling in the luminosity, though this is likely due to the fact they are not sufficiently evolved.

Why then do we find good scaling for luminosity in these simulations, whereas in the 3-D case we do not? One possibility is that this is a 2-D vs. 3-D trait. Collapsing to a line is certainly a different physical process than collapsing to a point – the fully 3-D collapse results in much more dramatic density enhancement, as well as the ability to form more complex geometries and anisotropies. Another possibility is resolution. These 2-D runs have greater linear resolution and dynamic range than the high-resolution 3-D runs in Chapter 4, as we can see by comparing the number of nodes per fluctuation wavelength at equivalent density fluctuation amplitudes.

The 2-D simulations are normalized so that at $a_0$ the rms fluctuations are $\sigma_0 \sim 0.7$ at $R_0 = 8/15$ box lengths, implying the nonlinear scale is $R_{nl}(2-D) = R_0 \sigma_0^{2/(2+n)} \sim 0.373$ box lengths (for $n = 0$). Similarly, the 3-D simulations are normalized at $a_0$ so that the rms fluctuations are $\sigma_0 \sim 0.7$ at $R_0 = 8/11.11$ box lengths, so that the nonlinear scale is $R_{nl}(3-D) = R_0 \sigma_0^{2/(3+n)} \sim 0.504$ box lengths. We therefore effectively have $128 \times 0.373 \sim 48$ particles per nonlinear wave in 2-D, vs. $64 \times 0.504 \sim 32$ particles per nonlinear wave in 3-D. In other words, these 2-D simulations are roughly equivalent to a model with more than 3 times the number of particles used in the high resolution 3-D models. We certainly see evidence in Figures 4.5 and 4.16 that resolution is an important issue in estimating the luminosity. We can most likely rule out a simple
technique difference (ASPH vs. SPH). If we examine the 2-D cases where we perform both SPH and ASPH versions (such as the cooling \( t_C = 0.1, n = 0 \) runs in the bottom panels of Figure 5.13), we note that the SPH runs exhibit reasonable scaling for the luminosity — about as good as in the ASPH case. There is a difference in that the SPH groups tend to be more luminous than those found in the ASPH model. It is possible that the problems in 3-D are exacerbated by SPH's inability to adapt to locally anisotropic evolution — this shortcoming is worse for SPH in general 3-D collapses than 2-D since SPH's resolution scale adapts as \( \rho^{-1/\nu} \) in \( \nu \) dimensions. It seems more likely, though, that the increased linear resolution due to the difference in the number of particles per boxlength is the more important resolution effect. Whatever the explanation for this discrepancy with our 3-D findings, we can conclude that the luminosity obeys the expected self-similar scalings well for these 2-D cooling runs.

Figures 5.16–5.18 test the scaling of the group gas densities. Since our friends-of-friends identification algorithm essentially selects objects by their overdensity, we can expect that in some sense the density should scale by construction. For this reason we test the density scaling in the same manner used for the 3-D investigation. We begin by identifying the local density \( \rho_x \% \) for each group, such that \( x \% \) of the gas nodes in that group are at densities \( \rho \leq \rho_x \% \). We then average this density for all groups in a given characteristic \( \theta \) range (such as \( \theta \in [1 \theta_{\text{al}}, 10 \theta_{\text{al}}] \)), and plot this value as a function of expansion. For the three rows of these figures, we consider three possible densities: \( \rho_{10\%}, \rho_{50\%}, \) and \( \rho_{90\%} \), progressively probing from the outskirts of each group inward toward the core. We also work in two distinct ranges of \( \theta \) for each
Figure 5.13: Evolution of the group luminosity $L_{70\%}/l(a)$ for the $n = 0$ models, such that at each expansion 70\% of the mass in the simulation is contained in groups with $L/l \leq L_{70\%}/l$. Point and line types as defined in Figure 5.7, such that solid lines represent the models without cooling, dotted lines cooling with $t_C = 200$, short dashes $t_C = 1$, long dashes $t_C = 0.1$, and dot-dashes SPH with $t_C = 0.1$. 
Figure 5.14: Evolution of the group luminosity $L_{70\%}/l_1(a)$ for the $n = +1$ models, such that at each expansion 70\% of the mass in the simulation is contained in groups with $L/l \leq L_{70\%}/l$. Point and line types as defined in Figure 5.8, such that solid lines represent the models without cooling, dotted lines cooling with $t_C = 200$, short dashes $t_C = 1$, and long dashes $t_C = 0.1$. 

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Figure 5.15: Evolution of the group luminosity $L_{70\%}/L(a)$ for the $n = -1$ models, such that at each expansion 70% of the mass in the simulation is contained in groups with $L/l \leq L_{70\%}/l$. Point and line types as defined in Figure 5.9, such that solid lines represent the models without cooling, dotted lines cooling with $t_C = 200$, short dashes $t_C = 1$, long dashes $t_C = 0.1$, and dot-dashes SPH with $t_C = 0.1$. 
linking parameter $\ell$, resulting in the four columns. The left-most columns use the larger linking parameter $\ell = 0.2 \Delta x_p$ (including both the hot and cold components of collapsed structures), while the right two columns use the smaller linking parameter $\ell = 0.05 \Delta x_p$ (restricting the selection to only the cooled gas cores in the simulations with radiative cooling).

Considering first the $n = 0$ case in Figure 5.16, we note that the $\ell = 0.2 \Delta x_p$ columns qualitatively reproduce what we found in the 3-D scale-free tests in Figure 4.6. The lower density cuts (probing the outer regions of the objects) scale most effectively, whereas as we move toward higher-density cuts (and therefore inward to smaller radii) the gas density progressively scales more poorly. Additionally, the objects with larger $\theta$ tend to scale better than the lower mass objects, as we would expect since the more massive objects are better resolved. There are, however, subtle differences from the 3-D case. For one, the relative change in the scaling between $\rho_{10\%}$ and $\rho_{90\%}$ is not as dramatic in 2-D as in 3-D: $\rho_{10\%}$ in 2-D does not scale quite as well as in 3-D, while $\rho_{90\%}$ scales more effectively in 2-D than 3-D. The $\ell = 0.2 \Delta x_p$, $n = +1$ models demonstrate similar traits, though the $n = -1$ densities more closely resemble the 3-D case. The fact that the core densities in 2-D scale at least somewhat explains why the 2-D luminosities scale more effectively than their 3-D counterparts, as noted above. There are resolution effects which can at least partially account for the fact that the outer and core densities more closely follow each other in 2-D than in 3-D. In both 2-D and 3-D we typically find the radial density profile of an object roughly follows $\rho(r) \propto r^{-2}$. In 2-D, the number of particles in a radial
shell goes as $N_{2,D}(r) \propto \rho(r) \, dA = 2\pi r \rho(r) \, dr \propto r^{-1}$, while in 3-D this becomes $N(r) \propto \rho(r) \, dV = 4\pi r^2 \rho(r) \, dr \propto r^0$. The numerical importance of this difference arises because in these sorts of Lagrangian hydrodynamical techniques the resolution follows the particle density. Therefore, in comparison with the 3-D objects, in 2-D we tend to have relatively poorer resolution for large $r$ (the region probed by $\rho_{10\%}$), and somewhat enhanced resolution for small $r$ (tested by $\rho_{90\%}$). These trends work counter to, and to some extent mollify, the more pronounced resolution differences noted in the core/halo comparison in 3-D. These numerical issues are rather subtle effects, though. The salient point to recognize is that qualitatively we do see similar behaviour in the 2-D gas densities (at least for the population of objects selected by $\ell = 0.2 \, \Delta x_p$) as found in the 3-D investigations.

Examining the density evolution for the cooled baryon structures for $n = 0$ (the two right columns in Figure 5.16), we can see the same general trends noted with the larger linking parameter. Not surprisingly, the density scaling is not as robust for these cooled objects: $\rho_{10\%}$ only scales well for $\log a/a_f \in [-0.5, 0]$, for instance. The gas density for the $n = +1$ cooled objects scales almost as well as the those found with the larger linking parameter, however, so this difference is not universal. The $n = -1$ cooled structures demonstrate very poor scaling, even for the outermost $\rho_{10\%}$. It is possible that the $n = -1$ cooled structures would begin to demonstrate better scaling if evolved to larger expansions. Even the $\ell = 0.2 \, \Delta x_p$, $n = -1$ structures do not show terribly impressive scaling over the range of expansions shown.
Figure 5.16: Evolution of the group density $\rho_{X\%}(a)$ for the $n = 0$ simulations. $\rho_{X\%}$ is defined as the average of $\rho_{X\%}^i$ for groups in a given range of $\theta$, where $\rho_{X\%}^i$ for each group $i$ is the density such that $X\%$ of the mass in the group is at densities $\rho \leq \rho_{X\%}^i$. The left two columns show the results in two $\theta$ ranges for groups selected with $\ell = 0.2 \Delta x_p$, while the right two columns are for groups selected with $\ell = 0.05 \Delta x_p$. From the top to the bottom rows we show the averages for $\rho_{10\%}$, $\rho_{50\%}$, and $\rho_{90\%}$, respectively. Point and line types as defined in Figure 5.7, such that solid lines represent the models without cooling, dotted lines cooling with $t_C = 200$, short dashes $t_C = 1$, long dashes $t_C = 0.1$, and dot-dashes SPH with $t_C = 0.1$. 

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Figure 5.17: Evolution of the group density $\rho_{X\%}(a)$ for the $n = +1$ simulations. $\rho_{X\%}$ is defined as the average of $\rho'_{X\%}$ for groups in a given range of $\theta$, where $\rho'_{X\%}$ for each group $i$ is the density such that $X\%$ of the mass in the group is at densities $\rho \leq \rho'_{X\%}$. The left two columns show the results in two $\theta$ ranges for groups selected with $\ell = 0.2 \Delta x_p$, while the right two columns are for groups selected with $\ell = 0.05 \Delta x_p$. From the top to the bottom rows we show the averages for $\rho_{10\%}$, $\rho_{50\%}$, and $\rho_{90\%}$, respectively. Point and line types as defined in Figure 5.8, such that solid lines represent the models without cooling, dotted lines cooling with $t_C = 200$, short dashes $t_C = 1$, and long dashes $t_C = 0.1$. 

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Figure 5.18: Evolution of the group density $\rho_{X\%}(a)$ for the $n = -1$ simulations. $\rho_{X\%}$ is defined as the average of $\rho_{X\%}^i$ for groups in a given range of $\theta$, where $\rho_{X\%}^i$ for each group $i$ is the density such that $X\%$ of the mass in the group is at densities $\rho \leq \rho_{X\%}^i$. The left two columns show the results in two $\theta$ ranges for groups selected with $\ell = 0.2 \Delta x_p$, while the right two columns are for groups selected with $\ell = 0.05 \Delta x_p$. From the top to the bottom rows we show the averages for $\rho_{10\%}$, $\rho_{50\%}$, and $\rho_{90\%}$, respectively. Point and line types as defined in Figure 5.9, such that solid lines represent the models without cooling, dotted lines cooling with $t_C = 200$, short dashes $t_C = 1$, long dashes $t_C = 0.1$, and dot-dashes SPH with $t_C = 0.1$. 

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5.6 Testing the Cooled Mass Fraction Scaling

The primary motivator for this investigation is to verify how well the radiatively cooled baryon mass obeys self-similar scaling, since getting the properties of the radiatively dissipated baryon gas component correct is a key requirement for a successful model of galaxy formation. Therefore, in this section we concentrate on the distribution of this cooled baryon mass in more detail. In the previous scaling tests we approached this problem by simply identifying baryonic objects with a very small linking parameter, which we determined by experimentation picks out just the radiatively cooled gas. While this is a reasonable first approach to objectively identify the dissipated gas, we would like some more precise tools for examining the effects of radiative cooling on typical baryon structures in these models. We begin by running friends-of-friends using our standard linking parameter of $\ell = 0.2 \Delta x_p$ on all particles in the simulations. In the previous sections we selected objects using friends-of-friends in the dark matter and baryons separately, but now we are interested in examining the properties of the complete structures, simultaneously including the dark matter with both hot and radiatively dissipated gas. Using Equation (5.2) it is possible to assign a characteristic support temperature for each object based purely on its mass (or $\theta$) and the assumption of hydrostatic equilibrium: $T_c = \mu m_p (2k)^{-1} G \theta$. In Figures 5.19–5.21 we plot average baryon temperature distribution functions for objects in our models, with temperatures in each object scaled to this characteristic temperature. These figures are constructed by first computing individual distributions of the
baryon mass in terms of $T/T_c$ for each object, then averaging these individual profiles in ranges of $\theta$ (one high and one low) that should be resolved at $a = a_f$. Note that these figures are all calculated at $a = a_f$.

In both the $n = 0$ and $n = +1$ models in Figures 5.19 and 5.20, it appears that some gas that is substantially hotter than $T_c$ is being linked using $\ell = 0.2 \Delta x_p$. While this hot gas clearly must be shocked, it probably represents a different population from the interior, cooler gas, which is at $T \lesssim T_c$. This extra population of hot gas may help explain the behavior noted in Figure 5.10, where we find that the emission weighted temperatures of objects identified with the smaller linking parameter actually scales better than with the large $\ell$, counter to our expectations.

The effects of radiative cooling on the temperature distribution of the gas in these groups is also apparent in the $f(T/T_c)$ profiles, particularly for the $n = -1$ models in Figure 5.21. In the runs with cooling, a significant fraction of the baryon gas is cooled to temperatures $T \lesssim 10^{-2}T_c$, whereas little to no gas exists at these temperatures in models without cooling. There is also a tendency for gas in larger groups to cool more effectively, particularly notable in the $n = 0$ models. This could be a real physical trend, but it is probably due at least in part to resolution effects. From Chapter 3 we know that with increasing resolution there is a tendency to find higher densities, and the efficiency of radiative cooling is directly proportional to the density (much like the luminosity). Since more massive objects are also better resolved, we should expect that resolution effects will tend to make radiative cooling more effective with increasing mass. It is also interesting to note that these temperature distributions

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are fairly continuous, while in typical, realistic Cold Dark Matter (CDM) simulations (see, e.g., Katz, Hernquist, & Weinberg 1992) the temperature distribution of gas in such objects is more nearly bimodal. In such CDM experiments with realistic cooling laws, the dissipated gas is typically much cooler than the surrounding hot gas halo, having cooled efficiently to $T \sim 10^4 K$ from typical shocked temperatures of $T \sim 10^4 - 10^5 K$. This difference is likely due to our use of idealized, featureless power-law cooling relations, and/or possibly a dimensionality effect. This issue will be investigated further in future work.

Figure 5.22 provides a more direct measure of how the efficiency of radiative cooling varies with $\theta$. In order to construct this figure, we first determine the fraction of mass $f_M$ in each object that has $T \leq \chi T_c$, where $\chi$ is a specified cutoff indicated on the right-axis labels. We bin the groups in log $\theta$, and determine the average of $f_M$ in each bin. This test is designed to give us an idea of how much baryon mass cools (and to what extent) as a function of the total mass in the object. While this is an important physical question, we also face the daunting task of separating the physical dependence on $\theta$ from the numerical resolution effects. The analytic prediction of self-similarity only formally applies to the time evolution of the system, and it does not directly address the problem of how different structures at the same time relate to one another. In other words, while self-similarity tells us how to map a distribution from expansions $a_1$ to $a_2$ ($f(\theta, T, \rho, \ldots, a_1) \rightarrow f(\theta, T, \rho, \ldots, a_2)$), it does not tell us how to relate a $\theta_*$ object to a $0.5 \theta_*$ object at the same expansion. However, as we did with the temperatures and luminosities in Chapter 4, we can make the additional
Figure 5.19: Average baryon temperature distributions for groups in the $n = 0$ simulations. Temperatures are scaled to $T_c = \mu m_p(2k)^{-1}G\theta$ ($\theta$ is the total mass in the group, baryons + dark matter). For each group (as identified with $\ell = 0.2 \, \Delta x_p$) the baryon mass is binned as a function of $T/T_c$. We then average these profiles for groups in a given range of $\theta$, as shown in the column headings. All measurements are made at $a = a_f$. 

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Figure 5.20: Average baryon temperature distributions for groups in the $n = +1$ simulations, defined as in Figure 5.19. All measurements are made at $a = a_f$. 

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Figure 5.21: Average baryon temperature distributions for groups in the $n = -1$ simulations, defined as in Figure 5.19. All measurements are made at $a = a_f$. 

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assumption that objects of different masses are simply scaled versions of one another, so that they all share a similar density profile. Under this assumption, we can make predictions about how characteristic quantities evolve with $\theta$, such as $\rho \propto \theta^0$, $T \propto \theta^1$, and $R \propto \theta^{1/2}$. We will assume that the amount of mass which cools in an object will scale with the inverse of the cooling time for that object: $\theta_{\text{cool}} \propto (t_C/t_H)^{-1}$. From equation (5.16) the cooling time scales as $t_C/t_H \propto \rho^{-1}T^{1-\theta}$. We assume that the density is similar for different $\theta$, we are comparing objects at the same time, and that the temperature scales as $T \propto \theta$. Therefore, we have

$$\theta_{\text{cool}} \propto \left(\frac{t_C}{t_H}\right)^{-1} \propto \rho T^{\theta-1} \propto T^{\theta-1} \propto \theta^{\theta-1}. \quad (5.19)$$

For the power-spectra used in our experiments, this becomes $\theta_C \propto \theta^{3/2}$ ($n = 0$), $\theta_C \propto \theta^{1/2}$ ($n = +1$), and $\theta_C \propto \theta^{-1/2}$ ($n = -1$). So we predict that physically the cooling should become more effective with increasing $\theta$, varying most rapidly for $n = +1$ and most slowly for $n = -1$. Unfortunately, this is the same direction we expect the numerical trend to work, so the two will be rather difficult to distinguish.

Clearly, for the rather large temperature cutoff $T/T_c \leq 0.5$ (in the upper row) the cooled mass fraction as a function of total mass is roughly constant in all models. Of course, only the models without cooling show any mass at temperatures hotter than this — all of the bound baryon mass in the models with cooling is at temperatures below than this cutoff. Imposing a more restrictive cutoff $T/T_c \leq 0.1$ (middle row), in the $n = 0$ and $+1$ models we still find that the cooled mass fraction is nearly independent of total mass. In the $n = -1$ run without cooling, though, we see that
Figure 5.22: Fraction of baryon mass per group with $T/T_c \leq \chi$ as a function of total group $\theta$ at $a = a_f$ for $n = 0, +1, and -1$ simulations. Line types as defined in Figures 5.7-5.9, such that solid lines represent the models without cooling, dotted lines cooling with $i_C = 200$, short dashes $i_C = 1$, long dashes $i_C = 0.1$, and dot-dashes SPH with $i_C = 0.1$. 

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for objects with \( \log \theta/\theta_{\text{tot}} \leq -2 \) there is essentially no bound baryon mass cooler than \( T/T_c \leq 0.1 \), whereas for larger masses we find a roughly constant (though small) fraction of baryons cooler than this cutoff. This trend most likely represents a numerical threshold. Finally, in the lower row we test a very cool cutoff of \( T/T_c \leq 0.005 \). In the \( n = 0 \), cooling \( \dot{t}_C = 1 \) and \( \dot{t}_C = 0.1 \) models we see a gradual transition that is roughly consistent with the expected slope \( \theta_C \propto \theta^{3/2} \). This suggests that the measured evolution is at least consistent with being physical rather than numerical in origin. The cooling \( n = +1 \) models show a more steep function, though again the slope is roughly consistent with the expected \( \theta_C \propto \theta^{11/2} \). In the \( n = -1 \) models we expect to see a relatively gentle evolution, \( \theta_C \propto \theta^{1/2} \). For objects above the threshold \( \log \theta/\theta_{\text{tot}} \geq -2 \), we do indeed roughly find this behaviour. However, we also see some evidence for a steep transition (particularly for the \( \dot{t}_C = 1 \) run near \( \log \theta/\theta_{\text{tot}} \sim -2.1 \)), where it appears the systems cross a numerical threshold from no cooling to relatively strong cooling. This suggests that for small \( \theta \) objects cooling is numerically suppressed, but once we pass the numerical resolution threshold we rapidly find the expected solution. In general, these findings support the idea that there is a critical resolution threshold for radiative cooling in the baryons, such that below this numerical limit cooling is rather ineffective, whereas above it cooling rapidly converges to a fixed efficiency (presumably the "true" physical value). The fact that this transition is so rapid is somewhat heartening, indicating that once we reach the critical resolution such that radiative cooling can occur, it rapidly converges to a stable level. These findings are
similar to those proposed based upon a more restricted sample of 3-D simulations performed with TreeSPH by Weinberg et al. (1996), where the authors note a marked sensitivity of the efficiency of radiative cooling to resolution.

We will now apply a direct test to the cooled baryon mass fraction, in order to see how well it obeys the expected self-similar scaling. We begin by taking each object identified at a given expansion, and calculating the fraction of the baryon mass in that object which has cooled below an arbitrary fraction of the characteristic support temperature, $T \leq \chi T_c$. Then, we determine the average of this fraction for all groups in a given characteristic range of mass and plot these values as a function of expansion in Figures 5.23–5.25. Since both the $\theta$ and $T$ distributions should scale with time, if the cooled fractions are indeed scaling self-similarly we would expect this fraction to remain constant (for a given $n$ and cooling model $t_C$) throughout the evolution, resulting in horizontal lines in these plots. For the $n = 0$ models in Figure 5.23, in general we find excellent scaling of the cooled mass fraction. We test two $\theta$ ranges, and while in the models with cooling we do find that characteristically more mass cools for larger objects (as expected based on the previous discussion), the models exhibit very stable cooled mass fractions for $\log a/a_f \in [-0.8, 0]$. For instance, in the lower right-hand panel of Figure 5.23 we can see that for the ASPH $t_C = 0.1$ models, roughly 80% of the baryon mass in each object consistently cools to $T \leq 0.05T_c$. The scaling also holds for the $n = +1$ models in Figure 5.24, and, remarkably, even for the $n = -1$ runs in Figure 5.25. Comparing the SPH and ASPH versions of the $n = 0$, $t_C = 0.1$ models in the bottom row of Figure 5.23, we can see that there is a
tendency for the ASPH run to find a somewhat higher fraction of cooled mass. This is not surprising, since ASPH should have the resolution edge on SPH, and this is the direction we would expect the resolution effect to push the results. It does appear that the ASPH and SPH runs converge to a consistent answer by the later expansions, as the collapsed objects continue to grow and the resolution of the SPH run thereby improves.

Finally, because the issue of the mass fraction of cooled baryon gas is so important and the scaling results of Figures 5.23–5.25 are surprisingly good, we present an alternate test of the cooled mass fraction scaling. In Figures 5.26–5.28 we calculate the scaling of \( T_{10\%} \), \( T_{50\%} \), and \( T_{90\%} \) for baryonic objects selected with \( \ell = 0.2 \Delta x_p \) and \( \ell = 0.05 \Delta x_p \). We begin by calculating \( T_{x\%}^i \) for each object \( i \) at a given expansion, such that \( x\% \) of the mass in that object is at temperatures \( T \leq T_{x\%}^i \). We then find the overall temperature \( T_{x\%} \) such that 70\% of the mass at that expansion is contained in structures with \( T_{x\%}^i \leq T_{x\%} \). Note that this quantity is defined similarly to our gas density test in Figures 5.16–5.18, and therefore allows us to test the temperature scaling at various characteristic regions in our population of objects (such as the outskirts vs. the cores). However, since the temperature actually rises as we move out from the core of a group, the regions probed in these figures are in the opposite sense compared with the density tests: i.e., \( T_{10\%} \) probes the innermost cores, while \( T_{90\%} \) probes the outer environs of the groups.
Figure 5.23: Average fraction of baryon mass in groups with $T/T_c \leq \chi$ as a function of expansion for $n = 0$ simulations. For each group at a given expansion, the fraction of baryon mass with $T/T_c \leq \chi$ is calculated. Then the average value of this fraction is found for a range of total group $\theta$, as given in the column heading. Point and line types as defined in Figure 5.7, such that solid lines represent the models without cooling, dotted lines cooling with $t_C = 200$, short dashes $t_C = 1$, long dashes $t_C = 0.1$, and dot-dashes SPH with $t_C = 0.1$. 

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Figure 5.24: Average fraction of baryon mass in groups with $T/T_c \leq \chi$ as a function of expansion for $n = +1$ simulations, calculated as in Figure 5.23. Point and line types as defined in Figure 5.8, such that solid lines represent the models without cooling, dotted lines cooling with $i_C = 200$, short dashes $i_C = 1$, and long dashes $i_C = 0.1$. 

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Figure 5.25: Average fraction of baryon mass in groups with $T/T_c \leq \chi$ as a function of expansion for $n = -1$ simulations, calculated as in Figure 5.23. Point and line types as defined in Figure 5.9, such that solid lines represent the models without cooling, dotted lines cooling with $t_C = 200$, short dashes $t_C = 1$, long dashes $t_C = 0.1$, and dot-dashes SPH with $t_C = 0.1$. 

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For the $n = 0$ models in Figure 5.26, we find very similar trends to those noted in the gas density. The characteristic temperatures of the outer regions of the groups ($T_{90\%}$) scale very well for both linking parameters; however, while the core temperatures ($T_{10\%}$) scale rather poorly for the $\ell = 0.2 \Delta x_p$ objects, the core temperatures of the dissipated baryon objects as identified by $\ell = 0.05 \Delta x_p$ actually scale rather reasonably. We find similar behaviour in both the $n = +1$ (Figure 5.27) models and even the $n = -1$ (Figure 5.28) runs. This tends to support the conclusions based on Figures 5.23–5.25. It appears that the total masses (and even temperatures) of the radiatively dissipated baryon structures reasonably follow the analytic self-similar scalings.

5.7 Summary and Conclusions

In this chapter we examine the effects, both physical and numerical, of radiative cooling in a set of scale-free cosmological simulations of hierarchical structure formation, including both a dark matter and baryonic component (95% dark matter by mass, 5% baryonic). The radiative dissipation of energy from the primordial baryonic component of the cosmological gas is a key ingredient in the formation of compact, high-density structures such as galaxies, and therefore in order to obtain meaningful information about galaxy formation based on hydrodynamical cosmological simulations it is important to characterize how well such simulations represent this process. In this investigation we set up a number of scale-free, idealized experiments, which allow us to test the numerical representation against the requirement of self-similar evolution. In this way we can characterize the quality and utility of such numer-
Figure 5.26: Evolution of the group temperature $T_{X\%}(a)$ for the $n = 0$ simulations. $T_{X\%}$ is defined as the temperature such that 70% of the mass in the system is contained in groups with $T_{X\%}^i \leq T_{X\%}$, where $T_{X\%}^i$ for each group $i$ is the temperature such that $X\%$ of the mass in the group is at temperatures $T \leq T_{X\%}^i$. The left column is calculated for objects identified with linking parameter $\ell = 0.2 \Delta x_p$, while the right column uses $\ell = 0.05 \Delta x_p$. From the top to the bottom rows we show $T_{10\%}$, $T_{50\%}$, and $T_{90\%}$, respectively. Point and line types as defined in Figure 5.7, such that solid lines represent the models without cooling, dotted lines cooling with $\dot{t}_C = 200$, short dashes $\dot{t}_C = 1$, long dashes $\dot{t}_C = 0.1$, and dot-dashes SPH with $\dot{t}_C = 0.1$. 

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Figure 5.27: Evolution of the group temperature $T_{X\%}(a)$ for the $n = \pm 1$ simulations, defined similarly to Figure 5.26. Point and line types as defined in Figure 5.8, such that solid lines represent the models without cooling, dotted lines cooling with $\dot{t}_C = 200$, short dashes $\dot{t}_C = 1$, and long dashes $\dot{t}_C = 0.1$. 

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Figure 5.28: Evolution of the group temperature $T_{\chi \%}(a)$ for the $n = -1$ simulations, defined similarly to Figure 5.26. Point and line types as defined in Figure 5.9, such that solid lines represent the models without cooling, dotted lines cooling with $\dot{i}_C = 200$, short dashes $\dot{i}_C = 1$, long dashes $\dot{i}_C = 0.1$, and dot-dashes SPH with $\dot{i}_C = 0.1$. 

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ical hydrodynamical models that include radiative effects as tools to explore more
generalized scenarios of galaxy formation. We derive a family of artificial cooling
laws specifically designed to maintain the scale-free nature of the model physics. We
then perform a number of 2-D simulations, using three different spectral indices \( n \)
for the initial density perturbation spectra, and several different cooling strengths
\( \Delta C \) for each spectrum. We find that the population of objects identified with a rela-
tively large linking parameter \( \ell = 0.2 \Delta x_p \) (which picks up both dark matter and
hot + cold baryon gas) exhibits scaling properties similar to those found in the 3-D
hydrodynamical runs without cooling, presented in Chapter 4.

The main thrust of this investigation is to explore the properties of the radiatively
dissipated, high-density baryon gas. Based on the resolution investigations in both
Chapters 3 and 4, we initially have reason to suspect such radiatively cooled structures
may be highly susceptible to numerical effects, and that it will prove very difficult to
isolate the expected self-similar scalings (i.e., the physical behaviour) in light of these
numerical difficulties. We examine the cold gas using two different approaches. First,
we use friends-of-friends with a very small linking parameter \( \ell = 0.05 \Delta x_p \) to select
small, high-density objects, and apply the same sorts of scaling tests used with the
larger objects. We empirically determine that this linking parameter picks out very
high-density, low temperature gas that only forms in models with radiative cooling.
We find that the properties of this cooled gas in fact scale reasonably - in all cases
almost as well as found in the larger groupings. Most significantly, the masses (or \( \theta \) in
2-D) of these cooled, condensed objects follow the analytically expected scaling quite
well, implying that these simulations correctly track the condensation of gas into galaxy-like objects. The temperatures and luminosities of these objects also scale fairly well, although the internal densities appear to be more sensitive to resolution effects.

We then proceed to test the scaling of the cooled gas fraction in a different manner. First, we identify structures using friends-of-friends with linking parameter $\ell = 0.2 \Delta x_p$ using all particles, both dark matter and baryons. Then, we associate a characteristic support temperature $T_c = \mu m_p (2k)^{-1} G \theta$ with each object, and determine the fraction of mass which cools to various fixed fractions of this temperature. Testing in this manner reveals that once cooling switches on in the gas, the mass fraction of cooled gas scales very well for all of the models we examine, over a range of initial conditions and cooling laws. This physical scaling appears to hold for both the time scaling (for which we rigorously know the required evolution), as well as the distribution of the cooled gas at any particular time (which we test based upon the assumption that structures at different mass scales are scaled versions of one another). Ironically, although such scaling seemed dubious based upon our initial resolution tests, it is probably explicable by the extraordinary sensitivity of the radiative cooling efficiency to the numerical resolution. Based upon tests such as those presented in Figure 5.22, it appears that so long as numerical resolution suppresses radiative cooling, there is essentially no cooling at all. Once we pass the numerical threshold where cooling can become effective, it very rapidly comes on and converges to a stable efficiency. We therefore have a more encouraging picture of the utility of
such investigations than originally surmised based upon the previous chapters. Basic-
ically, the process of radiative dissipation is so sensitive to our numerical limitations
that so long as the numerical model is inadequate, we essentially form no dissipated
structures. As the numerics reach the point that radiative effects come into play,
the numerical representation of the physics converges so rapidly that once we form
dissipated structures, the gross properties of those structures are relatively stable.

Figure 5.22 also demonstrates one of the more interesting physical results of this
investigation. Through one simple assumption (that objects of differing masses simply
represent scaled versions of one another), we are able to predict how the fraction of
cooled mass scales with total mass in an object to a reasonable accuracy. If this is true
in general, then it should be possible to extend simple analytic mass functions such
as the Press-Schechter theory to encompass dissipated structures, such as galaxies.
There is no a priori justification for making this assumption, but the results of these
models seem to support the conclusion over a limited mass range. This also explains
why we are able to achieve reasonable fits to the mass spectrum of dissipated objects
in the rightmost columns of Figures 5.4--5.6 using simple PS theory. Of course, we
do not know over what mass range this assumption might hold for realistic structure
formation models, if at all, nor if a realistic cooling law will disrupt this relation. In
a series of studies of collisionless Cold Dark Matter (CDM) models, Navarro, Frenk
& White (1995; 1996; 1997) find that hierarchically constructed dark matter halos
tend toward a universal density profile over roughly two orders of magnitude in mass,
with only minor deviations in this range. If this is also true of the baryon component,
then there is reason to believe these results for the dissipated gas may hold for CDM models as well. Validation of this sort of idea may be very useful for semi-analytic approximations to study the evolution of large-scale structure and galaxy formation (Cole 1991; White & Frenk 1991; Kauffmann et al. 1993; Lacey, Rocca-Volmerange, & Silk 1993; Cole et al. 1994).

We should point out a few notes of caution about these results. Two obvious shortcomings of this study are that it is restricted to 2-D models, and we are using a set of purely artificial cooling laws. 3-D collapses can achieve both more radical density evolution and more complex, anisotropic geometries than we find in 2-D. This is evident by comparing the density evolution in the 2-D models (Figure 5.16) to the 3-D case in the previous chapter (Figure 4.6). The distinction between the core gas densities ($\rho_{90\%}$) and the gas densities in the outer regions of the groups ($\rho_{10\%}$) is more marked in 3-D, and therefore the radiative cooling in the cores of general 3-D structures may be somewhat more extreme and complex than we find in 2-D. Perhaps a more important restriction, however, is the simplicity of the radiative cooling laws used in this investigation. A realistic cooling law (as can be seen in Katz et al. 1996) possesses several highly nonlinear features, such as the photoionization transition at $T \sim 10^4K$ and the H/He bumps at $T \in [10^4K, 10^5K]$. It is possible that such strong changes in the efficiency of the cooling could make the numerical job of representing the radiative physics more demanding than is the case here. Granted these caveats, though, these results are encouraging for the careful use of numerical hydrodynamical simulations in order to study the complex but fascinating process of galaxy formation.
The potential rewards of such studies are great, since galaxies represent our major tracer of large-scale structure in the universe. As we continue to improve our understanding of galaxy formation and how it relates to large-scale structure formation in general, we will be able to increasingly tightly constrain cosmological theories based upon such observations, which is the ultimate goal of all such scientific inquiries.
Appendix A

(A)SPH Dynamical Equations

In this appendix we briefly describe the (A)SPH dynamical equations, with the exception of the evolution of the G tensor which is treated in detail in Appendix B. In §A.1 we present the set of proper coordinate, time dependent equations we use to evolve non-cosmological simulations – §A.2 presents the set of comoving dynamical equations, expressed in terms of a power of the cosmological expansion factor, rather than time, as the independent variable. We express all dynamical equations in terms of the normalized position vector $\eta$, such that they are equally applicable to SPH and ASPH.

A.1 (A)SPH Equations in Proper Space

The (A)SPH dynamical equations are based upon solving discretized forms of the Lagrangian conservation equations at the positions of each (A)SPH node. We use the following forms of the Lagrangian conservation equations

\[
\frac{D \rho}{D t} = -\rho \nabla \cdot \mathbf{v}, \quad (A.1)
\]

\[
\frac{D \mathbf{v}}{D t} = -\nabla P \rho + \mathbf{g}, \quad (A.2)
\]

\[
\frac{D \mathbf{u}}{D t} = -\frac{P}{\rho} \mathbf{v} + \frac{1}{\rho} \left[ \Gamma(T) - \Lambda(T) \right], \quad (A.3)
\]
which represent the conservation laws for mass, momentum, and energy. Note we make use of the convention that the Lagrangian derivative is $D/Dt \equiv \partial/\partial t + (\mathbf{v} \cdot \nabla)$.

We define $\rho$ as the mass density, $\mathbf{v}$ velocity, $P$ pressure, $g$ gravitational acceleration, $u$ specific thermal energy, $T$ temperature, $\Gamma(T)$ the photoionization heating rate, and $\Lambda(T)$ the radiative cooling rate. These forms of the conservation equations account for pressure and gravitational forces, as well as the lowest order influences of radiation. We neglect the influence of viscous drag, heat conduction, and radiation pressure. These other effects could in principle be included at the penalty of complicating the equations. However, throughout this work we have chosen to follow a similar philosophy to that of Evrard (Evrard 1988; Evrard et al. 1994) in keeping the dynamical system relatively "clean", avoiding the introduction of too many complicating parameters. The physical processes accounted for here are thought to be the dominant processes in the formation of cosmological structures such as large galaxies (Rees & Ostriker 1977; Silk 1977; White & Rees 1978).

We use the equation of state appropriate for an ideal gas, which yields the relations

$$P = (\gamma - 1)\rho u,$$  \hspace{1cm} (A.4)

$$kT = (\gamma - 1)\mu m_p u,$$  \hspace{1cm} (A.5)

with $\gamma \equiv c_P/c_V$ the ratio of the specific heats, $k$ Boltzmann’s constant, $\mu$ the average atomic weight (in atomic units), and $m_p$ the mass of a proton.
Granted these forms of the conservation equations and an equation of state, the evolution equations for a given node $i$ in proper coordinates are

$$\frac{Dv_i}{Dt} = v_i, \quad (A.6)$$

$$\frac{Dv_i}{Dt} = -\sum_j m_j \left[ \left( \frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla W_{ij} + \Pi_{ij} \nabla W_{ij} \Pi_j \right] + g_i, \quad (A.7)$$

$$\frac{Du_i}{Dt} = \frac{P_i}{\rho_i^2} \sum_j m_j v_{ij} \cdot \nabla W_{ij} + \frac{1}{2} \sum_j m_j \Pi_{ij} v_{ij} \cdot \nabla W_{ij} + \frac{\Gamma(T_i) - \Lambda(T_i)}{\rho_i} \quad (A.8)$$

$$\frac{Du_i}{Dt} = \sum_j m_j v_{ij} \cdot \nabla W_{ij}, \quad (A.9)$$

$$\frac{D\rho_i}{Dt} = \sum_j m_j W_{ij}, \quad (A.10)$$

Note that there is no direct assumption about the dimensionality of the problem in these expressions. This exemplifies one of the nice properties of (A)SPH based techniques, in that they are easily implemented in any number of dimensions. It is only in the evolution of the smoothing scales that the dimensionality of the problem becomes important, as is discussed in the body of the paper (§2.3.2 and §2.3.3) and in detail in appendix B.

Inspection of the above set of equations reveals that equations (A.9) and (A.10) are redundant. Either are capable of specifying the density. If we choose equation (A.9), then the density becomes another parameter to be evolved from a given set
of initial conditions. This approach has the disadvantage of not ensuring absolute mass conservation. If we instead choose to use the summation approach of equation (A.10), then mass conservation is rigorously enforced. We nominally use the summation technique to update the density. However, our code utilizes an asynchronous integration algorithm, which requires intermediate estimates of the mass density. For this reason we also calculate equation (A.9) for use in making these estimates, and therefore include it here.

The term $\Pi_{ij}$ in the momentum and energy equations (eqs. [A.7] and [A.8]) represents an artificial viscosity, which is present to account for shock conditions in the gas. Without this artificial viscosity, (A)SPH is insufficiently dissipative to prevent interpenetration of the nodes, and shock conditions will be poorly represented. We adopt a standard SPH form of the artificial viscosity due to Monaghan & Gingold (1983), which is defined on a pair-by-pair basis as

\[
\Pi_i = \begin{cases} 
\rho_i^{-1}(\gamma \Pi c_s \mu_i + \beta \Pi \mu_i^2), & (v_i - v_j) \cdot (r_i - r_j) = v_{ij} \cdot r_{ij} < 0; \\
0, & \text{otherwise},
\end{cases}
\]  

(A.11)

\[
\mu_i = \frac{(v_i - v_j) \cdot (r_i - r_j)}{h_i \left(\frac{|r_i - r_j|^2}{h_i^2} + \zeta^2\right)} = \frac{v_{ij} \cdot \eta_i}{\eta_i \cdot \eta_i + \zeta^2},
\]  

(A.12)

where $\alpha_{\Pi}$ and $\beta_{\Pi}$ are numerical constants of order unity, $c_s^2 \equiv (dP/d\rho)_S = \gamma P_i / \rho_i$ the sound speed, and $\zeta$ a numerical factor required to avoid divergences. Note that this artificial viscosity is easily represented as a function of $\eta$, so that it may be directly implemented under ASPH. This form of artificial viscosity represents a hybrid of the usual Von Neumann-Richtmyer artificial viscosity $\Pi = \beta_{\Pi} \rho l^2 (\nabla \cdot v)^2$ and a bulk viscosity $\Pi = -\alpha_{\Pi} \rho l c_s \nabla \cdot v$ (where $l$ is the characteristic resolution scale for the shock),
and has units of \( P/\rho^2 \). In order to maintain the symmetry of the SPH equations we must use a symmetrized version of the artificial viscosity, which can be defined analogously to equation (2.5) as \( \Pi_{ij} = (\Pi_i + \Pi_j)/2 \).

The Monaghan-Gingold formulation of the artificial viscosity outlined is very successful at reproducing shock conditions in SPH simulations, but has the unfortunate drawback of producing a great deal of artificial shear viscosity. This can cause spurious transport of angular momentum in rotating systems, and is therefore of concern to us in this paper since we are concerned with testing the angular momentum properties of ASPH. In order to deal with this problem, we implement a multiplicative correction factor \( f \) suggested by Balsara (1995)

\[
\bar{\Pi}_{ij} = \Pi_{ij} \frac{f_i + f_j}{2}, \quad (A.13)
\]

\[
f_i = \frac{|(\nabla \cdot \mathbf{v})_i|}{|(\nabla \cdot \mathbf{v})_i| + |(\nabla \times \mathbf{v})_i| + 0.0001c_i/k_i}. \quad (A.14)
\]

Note that in a shear-free flow \((\nabla \cdot \mathbf{v} \neq 0, \nabla \times \mathbf{v} = 0)\), \( f = 1 \) and we recover the normal Monaghan-Gingold form for \( \Pi_{ij} \). However, in a pure shear flow \((\nabla \cdot \mathbf{v} = 0, \nabla \times \mathbf{v} \neq 0)\), \( f = 0 \) and the artificial viscosity is suppressed. Clearly this correction term cannot completely fix the problem of artificial shear viscosity (such as in a combined compressive, shearing flow), but at least it moderates the problem.

Artificial viscosity is a necessary evil for many hydrodynamic techniques, and ideally should be used as little as possible. The artificial viscosity is present solely to account for shock conditions and ensure that (A)SPH nodes do not interpenetrate. In the definition of equation (A.11) \( \Pi \) is restricted to be active solely for convergent flows. While a convergent flow is a minimal requirement for the presence of a shock,
not all convergent flows necessarily indicate shock conditions. Overuse of $\Pi$ can lead to spurious heating of the system, resulting in poor resolution of shocks. This issue is of great concern for gravitational collapse simulations, which is precisely the sort of situation we are concerned with modeling in a gravitationally dominated cosmological structure simulation. In an effort to overcome these shortcomings, we utilize a distinct interpolation kernel with the artificial viscosity ($W_{ij}^{\Pi}$ vs. $W_{ij}$, see eqs. [A.7] and [A.8]). In concert with ASPH's finer resolution, a judicious choice for $W_{ij}^{\Pi}$ allows significant improvement in the resolution of shocks. This approach reduces the spurious preheating problem endemic to the standard SPH implementation of the artificial viscosity, allowing the capture of the shock jump over a few nodes. This is discussed in §2.3.5. The Zel'dovich pancake test case in §2.4.1 is one example of the improvement that can be gained by this approach.

For the case of SPH, these forms of the dynamical equations rigorously ensure the conservation of mass (with the summation approach to density), linear momentum (due to the symmetrization of the momentum equation), and angular momentum (because all node interactions are radially symmetric on a pair-by-pair basis) – energy is only conserved to second-order. It is worth noting, though, that some of these conservation properties are somewhat artificially enforced. For instance, we allow a variable smoothing scale $h(r)$, and therefore formally the expression for the gradient of the kernel should include $\nabla h$ terms. These terms are almost universally neglected by practitioners of SPH, as they must be numerically estimated and it can be argued they are of secondary importance (Evrard 1988). However, we note that if such terms
are included then the expression for the gradient of the kernel becomes

\[ \nabla W(r, h) = \frac{\partial W(r, h)}{\partial r} + \frac{\partial W(r, h)}{\partial h} \nabla h. \]  \hspace{1cm} (A.15)

This expression for the \( \nabla W \) is not necessarily radial for interacting pairs of nodes, and therefore if these \( \nabla h \) terms are included angular momentum is not rigorously conserved.

For the case of ASPH, using these forms of the dynamical equations again rigorously enforces the conservation of mass and linear momentum. However, for an ellipsoidal kernel \( \nabla W \) is not necessarily radial, and therefore angular momentum conservation is not guaranteed. Based on our experience with testing ASPH in rotational problems, we have found that so long as one does not violate the basic assumptions upon which ASPH is derived (the system studied should be smooth in nature, and the \( G \) tensor field should be well-behaved on scales of a few \( h \)), angular momentum will be conserved acceptably (of order a few percent over several dynamical times). We discuss this issue in some detail in \$2.3.4, and test the conservation of angular momentum in \$2.4.2.

We parenthetically note that the obvious solution to the angular momentum issue is to simply radialize the ASPH force in pairwise interactions between nodes. This would then guarantee the global conservation of angular momentum, just as with SPH. However, we have found that such an approach yields unphysical results in practice. For instance, using such radialized forces in the Sedov problem leads to artificial, angular shapes to the shock front. The problem is easily understood, since forcing the ASPH inter-node forces to be radial means that direction of the mutual
forces between nodes differs from the true gradient of the smoothing kernel. The only forces which will be correctly aligned are those along the principle axes of the smoothing ellipses.

A.2 (A)SPH Equations in Comoving Coordinates

Our goal with this work is to use (A)SPH simulations to investigate the formation and evolution of cosmological structure. Such studies are most naturally implemented in comoving, rather than proper, coordinates. It is also advantageous to convert from using time dependent equations to using a power of the universal metric or expansion factor $a$, such that we evolve in terms of an independent variable $p = a^q$. This is a standard formalism used in collisionless cosmological simulations (Efstathiou et al. 1985; Villumsen 1989), and we wish to similarly adapt (A)SPH. In order to accomplish this, we define several comoving quantities analogous to their proper counterparts. We denote comoving positions by $x$, comoving “velocities” (in terms of $p$ rather than $t$) as $w \equiv dx/dp$, and the comoving specific thermal energy by $\epsilon$. All other comoving quantities are denoted by a superscript $c$ to avoid confusion. For the purposes of clarity, it is worthwhile to compile a list of conversions for quantities between the proper time-dependent and comoving frames. Table A.2 provides such a summary. Note that in this table we relate non-relativistic quantities, appropriate for use with our Newtonian approximation to the cosmology. We also refer to velocities as peculiar velocities ($v^p$ vs. $v$) with reference to the overall expansion of space. While some of these quantities may appear a bit strangely defined, recall that we are only redefining the meaning of position in going from the proper to comoving frame – both time and
mass remain unchanged. The conversions of table A.2 result based on these choices.

The comoving forms of the Lagrangian conservation equations, as transformed from their proper forms (eqs. [A.1] to [A.3]) are

\[ \frac{Dp^c}{Dp} = -\rho^c \nabla_\perp \cdot \mathbf{w}, \quad (A.16) \]

\[ \frac{Dw}{Dp} = -2A(p)w + B(p)g^c - \frac{\rho^c}{\rho^c} \nabla_\perp P^c, \quad (A.17) \]

\[ \frac{De}{Dp} = -\frac{1}{\dot{a}} \left( 2\varepsilon + 3 \frac{P^c}{\rho^c} \right) - \frac{P^c}{\rho^c} \nabla_\perp \cdot \mathbf{w} + \frac{a}{\dot{p}} \Gamma(T) - \Lambda(T), \quad (A.18) \]

where we use the definitions

\[ A(p) \equiv \frac{1 + \alpha + a\dot{a}\dot{\rho}^{-2}}{2\alpha a^\alpha}, \quad B(p) \equiv (\alpha^2 a^{2\alpha + 1}\dot{\rho}^2)^{-1}, \quad (A.19) \]

and the evolution of the expansion factor is given by

\[ \dot{a} = \left( \frac{8\pi G\bar{\rho}}{3} \right)^{\frac{1}{2}} \left( \frac{1}{\Omega_i} - 1 + \frac{1}{a} \right)^{\frac{1}{2}}, \quad (A.20) \]

where \( \bar{\rho} \) is the average comoving mass density and \( \Omega_i \) the initial value of the cosmological density parameter at \( a = 1 \), defined as the beginning of the simulation.
Transforming these comoving forms of the conservation equations to the (A)SPH formalism, we find for the (A)SPH dynamical equations

\[
\frac{D\mathbf{x}_i}{Dp} = \mathbf{w}_i, \quad (A.21)
\]

\[
\frac{D\mathbf{w}_i}{Dp} = -2A(p)\mathbf{w}_i + B(p)g^c_i - \dot{p}^{-2} \sum_j m_j \left[ (\gamma - 1) \left( \frac{\varepsilon_i}{\rho_i^c} + \frac{\varepsilon_j}{\rho_j^c} \right) \nabla_x W_{ij} + \Pi_i^c \nabla_x W_{ij}^{\Pi} \right], \quad (A.22)
\]

\[
\frac{D\varepsilon_i}{Dp} = \sum_j m_j \mathbf{w}_{ij} \cdot \left[ (\gamma - 1) \frac{\varepsilon_i}{\rho_i^c} \nabla_x W_{ij} + \frac{1}{2} \Pi_i^c \nabla_x W_{ij}^{\Pi} \right] - \frac{1}{p^a} (3\gamma - 1) \varepsilon_i + \frac{a}{p^a} \frac{\Gamma(T_i) - \Lambda(T_i)}{\rho_i^c}, \quad (A.23)
\]

\[
\frac{D\rho_i^c}{Dp} = -\rho_i^c \nabla_x \cdot \mathbf{w}_i = \sum_j m_j \mathbf{w}_{ij} \cdot \nabla_x W_{ij}, \quad (A.24)
\]

\[
\rho_i^c = \sum_j m_j W_{ij}, \quad (A.25)
\]

where we have explicitly used the ideal gas equation of state (eq. [A.4]).

The treatment of the comoving artificial viscosity (\(\Pi^c\)) deserves some attention. Since the artificial viscosity has units of \(P^{\text{visc}}/\rho^2\), it transforms to comoving coordinates as \(\Pi = a^5\Pi^c\). A comoving artificial viscosity term defined analogously to equations (A.11) and (A.12) is therefore

\[
\Pi_i^c = \begin{cases} 
    a^{-2}(\rho_i^c)^{-1}(-\alpha_{\Pi} c_i \mu_i + \beta_{\Pi} \mu_i^2), & \mathbf{v}_{ij} \cdot \mathbf{r}_{ij} = a^2 \hat{p} \mathbf{x}_{ij} \cdot \mathbf{w}_{ij} + a \dot{a} x_{ij}^2 < 0; \\
    0, & \text{otherwise},
\end{cases} \quad (A.26)
\]

\[
\mu_i = \frac{a \mathbf{x}_{ij} \cdot (\hat{p} \mathbf{w}_{ij} + \dot{a} \mathbf{x}_{ij})}{h_i^c \left( \frac{\varepsilon_i^c}{\rho_i^c} + \zeta^2 \right)} = \frac{a \eta_i^c \cdot (\hat{p} \mathbf{w}_{ij} + \dot{a} \mathbf{x}_{ij})}{\eta_i^c \cdot \eta_i^c + \zeta^2}, \quad (A.27)
\]

where the sound speed is given by

\[
c_i = \left( \frac{\gamma P_i}{\rho_i} \right)^{1/2} = (\gamma(\gamma - 1)u_i)^{1/2} = a[\gamma(\gamma - 1)e_i]^{1/2}. \quad (A.28)
\]

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Note that in equation (A.26) the artificial viscosity is switched on only for flows which are convergent in proper coordinates rather than comoving. This accounts for the extra term $a \dot{x}_i$. 
Appendix B

Defining and Evolving the ASPH G Tensor

In this appendix we present the detailed mathematical derivation of the G tensor and its derivative $DG/Dt$. We begin by presenting these derivations and justifications in a completely general, dimension-free formalism. Then in appendices B.3 and B.4 we present the specific cases of 2-D and 3-D, respectively.

B.1 G as a linear transformation

The G tensor is defined as a linear transformation which maps from real position space to normalized position space ($r \rightarrow \eta$). In this section we will use a superscript letter in parenthesis to denote different coordinate frames. A superscript $(r)$ implies the ordinary, positional frame within which $r$ is defined. A superscript $(k)$ implies the primary coordinate frame of our smoothing kernel in which the G tensor is diagonal. These two frames are related by a rotational transformation. We can find the representation of the G tensor in the real frame through a similarity transform

$$G^{(r)} = T_r^{(k\rightarrow r)}G^{(k)}T_r^{(r \rightarrow k)}.$$  \hspace{1cm} (B.1)

Here $G^{(r)}$ and $G^{(k)}$ are the G tensor represented in the overall and kernel frames, and $T_r^{(r \rightarrow k)}$ and $T_r^{(k \rightarrow r)}$ are the rotational transformations to and from the kernel
frame, respectively. The advantage of this relation is that $G$ is trivial to define in the kernel coordinate frame. In this frame, $G^{(k)}$ is diagonal, with each diagonal element corresponding to the inverse smoothing scale along that cardinal direction.

**B.2 The evolution equation $DG/Dt$**

We will now derive the evolution equation for $G$ in arbitrary dimension. The evolution is most easily derived in terms of the inertial moment tensor $I$ of the smoothing kernel, which is related to the $G$ tensor by

$$I = G^{-2}. \quad (B.2)$$

Note that $I$ has units of length squared. Its eigenvalues are in fact the squares of the lengths of the principle axes of $G$. The fact that $G$ is a symmetric matrix implies that $I$ is also symmetric.

Adopting the convention that an unprimed quantity is at time $t$, and primed at time $t + Dt$, the deformation tensor $\sigma_{ik} = \partial v_i / \partial x_k$ maps position space as

$$x'_i = x_i + Dt \sigma_{ik} x_k. \quad (B.3)$$

To first-order we then have

$$x'_i x'_j = x_i x_j + x_i \sigma_{ji} x_l Dt + x_j \sigma_{ik} x_k Dt, \quad (B.4)$$

which implies that $I$ evolves as

$$\frac{DI_{ij}}{Dt} = \sum_l \sigma_{jl} I_{li} + \sum_k \sigma_{ik} I_{kj} = (\sigma I)_{ji} + (\sigma I)_{ij}, \quad (B.5)$$
so that
\[
\frac{DI}{Dt} = \sigma I + I^t\sigma^t = \sigma I + I\sigma^t = \sigma G^{-2} + G^{-2}\sigma^t,
\] (B.6)
where we denote a transposed matrix by a superscript \(^t\). This relation is derived using the fact that \(G\) and \(I\) are symmetric, but places no restrictions on the form of \(\sigma\).

We can use equation (B.6) to find the evolution of \(G\) as follows.

\[
\Delta I = (G + \Delta G)^{-2} - G^{-2} = (\sigma G^{-2} + G^{-2}\sigma^t)\Delta t.
\] (B.7)

Working this out, keeping only first-order times we get

\[
G\Delta G + \Delta GG = -(G^2\sigma + \sigma^tG^2)\Delta t,
\] (B.8)

\[
(\Delta G + G\sigma\Delta t)G^{-1} + G^{-1}(\Delta G + \sigma^tG\Delta t) = 0,
\] (B.9)

which can be rewritten

\[
(\Delta G + G\sigma\Delta t)G^{-1} + ((\Delta G + G\sigma\Delta t)G^{-1})^t = 0.
\] (B.10)

Equation (B.10) implies that the term \((\Delta G + G\sigma\Delta t)G^{-1}\) represents an antisymmetric rotation matrix, which we denote by \(R\). The general solution for \(DG/Dt\) is then

\[
(\Delta G + G\sigma\Delta t)G^{-1} = \Delta R \Rightarrow \Delta G = RG - G\sigma\Delta t,
\] (B.11)

or

\[
\frac{DG}{Dt} = \frac{DR}{Dt}G - G\sigma,
\] (B.12)

where the infinitesimal rotational transformation \(R\) is uniquely specified by

\[
G^{-1}\frac{DR}{Dt} + \left(G^{-1}\frac{DR}{Dt}\right)^t = G\sigma - \sigma^tG.
\] (B.13)
The infinitesimal rotation angles may either be solved for by expanding equation (B.13) or by setting the off-axis elements of $DG/Dt$ from equation (B.12) equal. Either method yields (in $\nu$ dimensions) $\nu$ independent equations in $\nu$ unknowns. Once these rotational angles are found, the solution for $DG/Dt$ is completely specified.

B.3 ASPH in 2-D

We now present the full ASPH formalism for the 2-D case. The results of this section have already been given in the body of the paper.

B.3.1 The 2-D G tensor

In order to define the $G$ tensor it is helpful to start from the underlying geometry. We will specify the geometry of the $\eta = 1$ isocontour (which in 2-D is in general an ellipse) and define $G$ in terms of this geometry. We define the components of this isocontour as follows: $h_1$ represents the semi-major axis, $h_2$ the semi-minor axis, and $\psi$ the position angle associated with the semi-major axis. $G$ is therefore defined in its primary (or kernel) frame by

$$G^{(k)} = \begin{pmatrix} h_1^{-1} & 0 \\ 0 & h_2^{-1} \end{pmatrix}.$$  

(B.14)

In the primary frame of $G$, $h_1$ lies along the $x^{(k)}$ axis. $\psi$ represents the angle of rotation for the transformation between the kernel and real frames. The rotational transformations relating the real and kernel frames are then

$$T_r^{(r \rightarrow k)} = \begin{pmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix}, \quad T_r^{(k \rightarrow r)} = \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix}.$$  

(B.15)
We can apply the similarity transform of equation (B.1) to find the representation of $G^{(r)}$

$$G^{(r)} = T_r^{(k \rightarrow r)} G^{(k)} T_r^{(r \rightarrow k)} \quad (B.16)$$

$$= \begin{pmatrix}
 h_1^{-1} \cos^2 \psi + h_2^{-1} \sin^2 \psi & (h_1^{-1} - h_2^{-1}) \cos \psi \sin \psi \\
 (h_1^{-1} - h_2^{-1}) \cos \psi \sin \psi & h_1^{-1} \sin^2 \psi + h_2^{-1} \cos^2 \psi
\end{pmatrix}. $$

### B.3.2 The 2-D G evolution equation

We now present the form of the 2-D G evolution equation. First we need the following 2-D forms for $G$, $\sigma$, and $R$.

$$G \equiv \begin{pmatrix}
 G_{11} & G_{21} \\
 G_{21} & G_{22}
\end{pmatrix}, \quad (B.17)$$

$$\sigma \equiv \begin{pmatrix}
 \sigma_{11} & \sigma_{12} \\
 \sigma_{21} & \sigma_{22}
\end{pmatrix} = \begin{pmatrix}
 \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\
 \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y}
\end{pmatrix}. \quad (B.18)$$

$$\frac{D R}{D t} = \begin{pmatrix}
 0 & \dot{\theta} \\
 -\dot{\theta} & 0
\end{pmatrix}. \quad (B.19)$$

We then have for $D G / D t$ via equation (B.12)

$$\frac{D G}{D t} = \frac{D R}{D t} G - G \sigma \quad (B.20)$$

$$= \begin{pmatrix}
 DG_{11}/D t & DG_{21}/D t \\
 DG_{21}/D t & DG_{22}/D t
\end{pmatrix}$$

$$= \begin{pmatrix}
 G_{21}(\dot{\theta} - \sigma_{21}) - G_{11}\sigma_{11} & G_{22}\dot{\theta} - G_{11}\sigma_{12} - G_{21}\sigma_{22} \\
 -G_{11}\dot{\theta} - G_{21}\sigma_{11} - G_{22}\sigma_{21} & -G_{21}(\dot{\theta} + \sigma_{12}) - G_{22}\sigma_{22}
\end{pmatrix},$$

$$\dot{\theta} = \frac{G_{11}\sigma_{12} - G_{22}\sigma_{21} - G_{21}(\sigma_{11} - \sigma_{22})}{G_{11} + G_{22}}, \quad (B.21)$$

where we have solved for $\dot{\theta}$ by setting $DG_{12}/D t = DG_{21}/D t$. 

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B.4 ASPH in 3-D

B.4.1 The 3-D G tensor

Once again we specify the form of the 3-D G tensor in terms of the geometry of the $\eta = 1$ isocontour. In order to specify rotations in 3-D space we adopt the so-called "xyz" convention as outlined in Goldstein (1981), which has the advantage of being non-degenerate for infinitesimal rotations. We choose to represent the rotation angles to transform from the real frame to the kernel's principal frame as $(\omega, \psi, \chi)$, where $\omega$ is the yaw angle about the $z$ axis, $\psi$ is pitch angle about the intermediate $y$ axis, and $\chi$ is the bank or roll angle about the $x$ axis in the kernel's frame. For the sake of notational compactness we also adopt the convention that an angle subscripted by 1 represents the cosine of that angle, and a subscript 2 represents the sine (i.e. $\psi_1 \equiv \cos \psi, \psi_2 \equiv \sin \psi$). The full rotational transformations $T_r^{(r \rightarrow k)}$ and $T_r^{(k \rightarrow r)}$ can now be written as

$$T_r^{(r \rightarrow k)} = \begin{pmatrix} \psi_1 \omega_1 & \psi_1 \omega_2 & -\psi_2 \\ \chi_2 \psi_2 \omega_1 - \chi_1 \omega_2 & \chi_2 \psi_2 \omega_2 + \chi_1 \omega_1 & \psi_1 \chi_2 \\ \chi_1 \psi_2 \omega_1 + \chi_2 \omega_2 & \chi_1 \psi_2 \omega_2 - \chi_2 \omega_1 & \psi_1 \chi_1 \end{pmatrix}, \quad (B.22)$$

$$T_r^{(k \rightarrow r)} = \begin{pmatrix} \psi_1 \omega_1 & \chi_2 \psi_2 \omega_1 - \chi_1 \omega_2 & \chi_1 \psi_2 \omega_1 + \chi_2 \omega_2 \\ \psi_1 \omega_2 & \chi_2 \psi_2 \omega_2 + \chi_1 \omega_1 & \chi_1 \psi_2 \omega_2 - \chi_2 \omega_1 \\ -\psi_2 & \psi_1 \chi_2 & \psi_1 \chi_1 \end{pmatrix}. \quad (B.23)$$

We identify $(h_1, h_2, h_3)$ as the smoothing scales along the $(x^{(k)}, y^{(k)}, z^{(k)})$ axes in the kernel’s primary frame, such that $h_1 \geq h_2 \geq h_3$. The G tensor is therefore given in the kernel's primary frame by

$$G^{(k)} = \begin{pmatrix} h_1^{-1} & 0 & 0 \\ 0 & h_2^{-1} & 0 \\ 0 & 0 & h_3^{-1} \end{pmatrix}, \quad (B.24)$$
and applying the similarity transform (eq. [B.1]) we find for $G^{(r)}$

$$G^{(r)} = T_r^{(k-r)}G^{(k)}T_r^{(r-k)} \equiv \begin{pmatrix} G_{11} & G_{21} & G_{31} \\ G_{21} & G_{22} & G_{32} \\ G_{31} & G_{32} & G_{33} \end{pmatrix}, \quad (B.25)$$

where the six unique matrix elements are given by

$$G_{11} = h_1^{-1}\omega_1^2\psi_1^2 + h_2^{-1}(\omega_1\psi_2x_2 - \omega_2x_1)^2 + h_3^{-1}(\omega_1\psi_2x_1 + \omega_2x_2)^2, \quad (B.26)$$

$$G_{21} = h_1^{-1}\omega_1\omega_2\psi_1^2 + h_2^{-1}(\omega_1\psi_2x_2 - \omega_2x_1)(\omega_2\psi_2x_2 + \omega_1x_1) + h_3^{-1}(\omega_2\psi_2x_1 - \omega_1x_2)(\omega_1\psi_2x_1 + \omega_2x_2), \quad (B.27)$$

$$G_{31} = -h_1^{-1}\omega_1\psi_1\psi_2 + h_2^{-1}\psi_1x_2(\omega_1\psi_2x_2 - \omega_2x_1) + h_3^{-1}\psi_1x_1(\omega_1\psi_2x_1 + \omega_2x_2), \quad (B.28)$$

$$G_{22} = h_1^{-1}\omega_2^2\psi_1^2 + h_2^{-1}(\omega_2\psi_2x_2 + \omega_1x_1)^2 + h_3^{-1}(\omega_2\psi_2x_1 - \omega_1x_2)^2, \quad (B.29)$$

$$G_{32} = -h_1^{-1}\omega_2\psi_1\psi_2 + h_2^{-1}\psi_1x_2(\omega_2\psi_2x_2 + \omega_1x_1) + h_3^{-1}\psi_1x_1(\omega_2\psi_2x_1 - \omega_1x_2), \quad (B.30)$$

$$G_{33} = h_1^{-1}\psi_2^2 + h_2^{-1}\psi_1^2x_2^2 + h_3^{-1}\psi_1^2x_1^2. \quad (B.31)$$
Clearly these expressions for the elements of $G$ are quite unwieldy, and would be most computationally expensive if we needed to evaluate these expressions each time we wished to use the $G$ tensor. Fortunately, however, we need only use these expressions when we are initializing the $G$ matrix. Once we have the numerical values for these elements, we need not concern ourselves with the geometry in order to use or evolve $G$.

### B.4.2 The 3-D $G$ evolution equation

Finally, we present the form of the 3-D $G$ evolution equation. We define $G$, $\sigma$, and $\mathbf{R}$ as

$$
G \equiv \begin{pmatrix}
G_{11} & G_{21} & G_{31} \\
G_{21} & G_{22} & G_{32} \\
G_{31} & G_{32} & G_{33}
\end{pmatrix},
$$

(B.32)

$$
\sigma \equiv \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial \psi_x}{\partial x} & \frac{\partial \psi_x}{\partial y} & \frac{\partial \psi_x}{\partial z} \\
\frac{\partial \psi_y}{\partial x} & \frac{\partial \psi_y}{\partial y} & \frac{\partial \psi_y}{\partial z} \\
\frac{\partial \psi_z}{\partial x} & \frac{\partial \psi_z}{\partial y} & \frac{\partial \psi_z}{\partial z}
\end{pmatrix},
$$

(B.33)

$$
\frac{d\mathbf{R}}{dt} = \begin{pmatrix}
0 & \dot{\theta} & -\dot{\phi} \\
-\dot{\gamma} & 0 & \dot{\phi} \\
\dot{\gamma} & -\dot{\phi} & 0
\end{pmatrix},
$$

(B.34)

where we have used $(\dot{\gamma}, \dot{\theta}, \dot{\phi})$ as the infinitesimal rotation angles for $\mathbf{R}$. Then through equation (B.12) we have for $\frac{DG}{Dt}$

$$
\frac{DG}{Dt} = \frac{d\mathbf{R}}{dt} G - G\sigma = \begin{pmatrix}
\frac{DG_{11}}{Dt} & \frac{DG_{12}}{Dt} & \frac{DG_{13}}{Dt} \\
\frac{DG_{21}}{Dt} & \frac{DG_{22}}{Dt} & \frac{DG_{23}}{Dt} \\
\frac{DG_{31}}{Dt} & \frac{DG_{32}}{Dt} & \frac{DG_{33}}{Dt}
\end{pmatrix},
$$

(B.35)

where the individual elements are

$$
\frac{DG_{11}}{Dt} = -G_{11}\sigma_{11} + G_{21}(\dot{\gamma} - \sigma_{21}) - G_{31}(\dot{\theta} + \sigma_{31}),
$$

(B.36)

$$
\frac{DG_{12}}{Dt} = G_{22}\dot{\gamma} - G_{32}\dot{\theta} - G_{11}\sigma_{12} - G_{21}\sigma_{22} - G_{31}\sigma_{32},
$$

(B.37)
\[
\frac{DG_{13}}{Dt} = G_{32} \dot{\gamma} - G_{33} \dot{\phi} - G_{11} \sigma_{13} - G_{21} \sigma_{23} - G_{31} \sigma_{33}, \quad (B.38)
\]

\[
\frac{DG_{21}}{Dt} = -G_{11} \gamma + G_{31} \dot{\phi} - G_{21} \sigma_{11} - G_{22} \sigma_{21} - G_{32} \sigma_{31}, \quad (B.39)
\]

\[
\frac{DG_{22}}{Dt} = G_{32} (\dot{\phi} - \sigma_{32}) - G_{21} (\gamma + \sigma_{12}) - G_{22} \sigma_{22}, \quad (B.40)
\]

\[
\frac{DG_{23}}{Dt} = -G_{31} \gamma + G_{33} \dot{\phi} - G_{21} \sigma_{13} - G_{22} \sigma_{23} - G_{33} \sigma_{33}, \quad (B.41)
\]

\[
\frac{DG_{31}}{Dt} = G_{11} \dot{\theta} - G_{21} \dot{\phi} - G_{31} \sigma_{11} - G_{32} \sigma_{21} - G_{33} \sigma_{31}, \quad (B.42)
\]

\[
\frac{DG_{32}}{Dt} = G_{21} \dot{\theta} - G_{22} \dot{\phi} - G_{31} \sigma_{12} - G_{32} \sigma_{22} - G_{33} \sigma_{32}, \quad (B.43)
\]

\[
\frac{DG_{33}}{Dt} = G_{31} (\dot{\theta} - \sigma_{13}) - G_{32} (\dot{\phi} + \sigma_{23}) - G_{33} \sigma_{33}. \quad (B.44)
\]

In order to solve for the rotation angles ($\gamma, \dot{\theta}, \dot{\phi}$), we could set the symmetric off-axis elements of $DG/DT$ equal and solve the resulting system of linear equations.

\[
\dot{\gamma} = \frac{\gamma_c \gamma_d - \gamma_a \gamma_e}{\gamma_a \gamma_c - \gamma_d \gamma_e}, \quad (B.45)
\]

\[
\dot{\theta} = \frac{\gamma_b \gamma_d - \gamma_a \gamma_e}{\gamma_a \gamma_c - \gamma_d \gamma_e}, \quad (B.46)
\]

\[
\dot{\phi} = \frac{G_{31} \gamma + G_{21} \dot{\theta} + C}{G_{22} + G_{33}}, \quad (B.47)
\]

where we have defined for convenience:

\[
\gamma_a \equiv (G_{11} + G_{22})(G_{22} + G_{33}) - G_{31}^2, \quad (B.48)
\]

\[
\gamma_b \equiv (G_{22} + G_{33})G_{32} + G_{21}G_{31}, \quad (B.49)
\]

\[
\gamma_c \equiv (G_{11} + G_{33})(G_{22} + G_{33}) - G_{21}^2, \quad (B.50)
\]

\[
\gamma_d \equiv (G_{22} + G_{33})A + G_{31}C, \quad (B.51)
\]

\[
\gamma_e \equiv (G_{22} + G_{33})B - G_{21}C, \quad (B.52)
\]

\[
A \equiv G_{11} \sigma_{12} - G_{21} (\sigma_{11} - \sigma_{22}) + G_{31} \sigma_{32} - G_{22} \sigma_{21} - G_{32} \sigma_{31}, \quad (B.53)
\]

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\[ B \equiv G_{11}\sigma_{13} + G_{21}\sigma_{23} - G_{31}(\sigma_{11} - \sigma_{33}) - G_{32}\sigma_{21} - G_{33}\sigma_{31}, \quad (B.54) \]
\[ C \equiv G_{21}\sigma_{13} + G_{22}\sigma_{23} - G_{32}(\sigma_{22} - \sigma_{33}) - G_{31}\sigma_{12} - G_{33}\sigma_{32}. \quad (B.55) \]

We note that in the process of solving for these angles one must be careful never to divide by off-axis elements of $G$, as only the diagonal elements of $G$ are guaranteed not to be zero.

This now completely specifies $DG/Dt$ in closed (if somewhat ungainly) form. As before, only half (3) of the off-axis elements of $DG/Dt$ need be evaluated, as $DG/Dt$ is formally symmetric. We present the full expressions in equations (B.36) - (B.44), since these forms are needed to find $(\gamma, \dot{\theta}, \dot{\phi})$. 
Appendix C

Numerical Algorithms

In this appendix we will briefly discuss some of the numerical algorithms we have developed in the process of implementing ASPH, including time integration (§C.1), timestep criteria (§C.2), and neighbor selection (§C.3). While these sorts of numerical details are not fundamental, we consider this discussion worthwhile as it is imperative to code ASPH efficiently if we wish to be able to perform large-scale, dynamic simulations. This issue is particularly crucial for ASPH because the increased spatial resolution (as compared with SPH) implies correspondingly shorter timescales (and therefore timesteps), increasing the computational demands for integrating the system. It is critical to an ASPH simulation that the time integration be performed with sufficient temporal resolution, or the Courant and related timestep criteria imply that the spatial resolution is compromised. The processes of integration and significant neighbor selection typically dominate the computational time for (A)SPH simulations, so we focus on these issues here.

C.1 Asynchronous Time Integration

In a typical cosmological structure simulation, regions of vastly different densities and temperatures (and therefore timescales) can generically be expected to evolve.
This follows directly from the gravitational instability which is the driving source for this structure, as gravity tends to form collapsed, dense, hot structures from initially cool, nearly homogeneous gas. In order to efficiently integrate such systems, it is advantageous to decouple regions with such differing timescales and evolve them independently, such that regions with small timesteps can be followed without the penalty of having to advance the entire system at such smaller timescales. This is possible for (A)SPH due to the local nature of hydrodynamic interactions, as reflected by the local sampling of the (A)SPH interpolation kernel. This suggests the development of an asynchronous integrator, such that each (A)SPH node can evolve with its own timestep and current time. Clearly, since the evolution of each node depends upon the state of its neighbors, neighboring nodes cannot be completely decoupled. However, by its nature (A)SPH is a smooth technique, and we expect that neighboring nodes will be in physically similar states, and thereby possess similar timescales. This implies such an asynchronous integration approach is consistent for (A)SPH. We also enforce criteria in the integration algorithm which insure that coupled nodes will remain nearly synchronous.

Other investigators have also developed asynchronous integrators, such as Hernquist & Katz (HK89) who implement an asynchronous version of the second-order accurate time-centered leapfrog algorithm in their TREESPH code. We choose instead to develop our integration algorithm around a second-order Runge-Kutta scheme, which eases the conceptual complexity of the algorithm. This is primarily because Runge-Kutta maintains a node's information at a single time, rather than having
the variables and their derivatives at differing times such as required by the leapfrog scheme. The generalized second-order Runge-Kutta algorithm can be quantified as follows. In order to integrate a quantity $x$ at time $t$ through a timestep $dt$ we use

$$x(t + dt) = x(t) + \left( a_1 \frac{dx(t)}{dt} + a_2 \frac{dx(t + \chi dt)}{dt} \right) dt,$$  \hspace{1cm} (C.1)$$

where

$$a_1 = 1 - \frac{1}{2\chi}, \quad a_2 = \frac{1}{2\chi}, \quad \chi \in [0.5, 0.9],$$  \hspace{1cm} (C.2)$$

and we have limits on $\chi$ in equation (C.2) for the sake of stability. Note that this scheme requires that we be able to evaluate the derivatives for a given node $i$ at an intermediate time $t_n = t_i + \chi_i dt_i$. This requires that we have synchronous information at time $t_n$ for all of the nodes significant to $i$ (all nodes within a few $\delta$). Fortunately, we need only integrate this intermediate neighbor information to first-order in order to maintain the second-order accuracy of equation (C.1), such that we can extrapolate the set of information for all significant nodes $j$ via

$$x_j(t_n) \approx x_j(t_j) + \frac{dx_j}{dt}(t_j)(t_n - t_j).$$  \hspace{1cm} (C.3)$$

The flexibility in the value of $\chi$ in equation (C.1) allows us to use this same set of synchronous neighbor information at $t_n$ for several nodes $i$, so long as $t_n$ falls within the desired range for $\chi_i$. This sort of recycling is important, as the process of selecting neighbors and creating this synchronous information can become computationally expensive if required too often.
We will now outline the asynchronous integration algorithm. All nodes start synchronously at a time $t_0$, with an allowed range of timesteps $[dt_{\text{min}}, dt_{\text{max}}]$. The goal of this algorithm is to advance all nodes asynchronously to a "goal time" $t_{\text{goal}} = t_0 + dt_{\text{max}}$, at which time they will all be synchronous again. Note the inclusion of gravity introduces a complication, in that while hydrodynamic interactions are local in nature, gravitation is global. Therefore, while localized regions can be decoupled for the (A)SPH interactions, this is not possible for the gravitational forces. Fortunately, the gravitational timescale (given roughly by the gravitational dynamical time $dt_{\text{grav}} = e_{\text{grav}}/(G\rho)^{1/2}$, where $e_{\text{grav}}$ is the fraction of the gravitational timescale we use) is typically the longest, least restrictive of the physical timescales, and we can therefore set our overall $dt_{\text{max}} = dt_{\text{grav}}$, and solve the gravitational problem synchronously. In order to determine the gravitational forces for the hydrodynamic integrations in the interval $[t_0, t_{\text{goal}}]$, we estimate the purely gravitational forces at $t_0$, $t_0 + 0.5 \times dt_{\text{max}}$, and $t_{\text{goal}}$, and parabolically interpolate for the forces in this interval. This is equivalent to integrating the gravitational problem synchronously using the time-centered version of equation (C.1). The asynchronous algorithm for the hydrodynamic interactions is:

1. All nodes start out synchronous at time $t_0$, each with a current time $t_i$, timestep $dt_i \in [dt_{\text{min}}, dt_{\text{max}}]$, and "target time" $t_i^{\text{targ}} = t_i + dt_i$.

2. A "local goal time" $t_i^{\text{goal}} = \min(t_i^{\text{targ}}) \forall i$ is identified, which is simply the minimum available target time. All nodes which share this $t_i^{\text{goal}}$ are located and placed in a sequential list.

3. The list of nodes due for integration is broken up into spatially correlated
batches, such that a single set of neighbor information can be used for one such batch of nodes, rather than having to recalculate the neighbor information for each node. Such batches are identified as those which share the same gridcell, as discussed in §C.3.

4. We loop over each batch of nodes. For each batch, a list of synchronous neighbor information \((r_j, v_j, u_j, \ldots)\) is constructed at both an intermediate time \(t_n \in [t_i, t^{\text{goal}}]\) and the end time \(t^{\text{goal}}\).

5. Each node in a batch is integrated to second-order via equation (C.1) using the intermediate neighbor information at \(t_n\). Once integrated, each nodes derivatives are updated using the neighbor information at \(t^{\text{goal}}\). The individual timesteps are also updated, creating a new set of potential target times \(t_i^{\text{targ}}\).

6. Once all nodes with the currently targeted local goal time \(t^{\text{goal}}\) have been integrated, we make another pass through all of these nodes. For each of the just integrated nodes the potential neighbors are found, and the minimum target time for all these causally connected nodes is determined. The timestep for each of these linked nodes (including the neighboring nodes, whether they were just integrated or not) is then reset such that they all have this same minimum target time. In this way we ensure that nodes which are causally connected will be integrated together synchronously, and that regions of small timescales will not sweep through those with larger timescales before they can adapt.
7. We now loop back to step 2, and repeat this process until all nodes have been advanced to \( t_{\text{goal}} \), at which time the system is again synchronous.

Finally, we note that for the purposes of efficiency it is useful to try and keep as many nodes evolving at the same timestep and target time as possible, such that the size of the batches that can be integrated together will be increased. In an effort to achieve this, we force all timesteps to be integer multiples of the minimum timestep. Of course, forcing all nodes which are significant neighbors to one and other to share the same target time (as described in step 6 above) is also quite helpful for this purpose.

C.2 Timestep Criteria

As our integration scheme is based upon a second-order Runge-Kutta algorithm, we have a good deal of flexibility in how to choose our timesteps. Typically, Runge-Kutta integrators use an accuracy limited criterion to determine the step-size. However, in order to determine such an accuracy limited timestep requires trial integrations, which can become prohibitively expensive under (A)SPH. We therefore set the integration timestep by using physical arguments about the timescales in the system, leading to criteria such as the Courant time. In general these criteria underestimate the necessary timestep as compared with an accuracy limited scheme, but the simplicity and speed with which these physical criteria can be evaluated make up for the added integration cycles required.
The basic requirement of our timestep is that for a given smoothing scale \( h \), the timestep should be small enough such that the fastest signal (of velocity \( v_\star \)) can only propagate across a given fraction \( \epsilon \) of \( h \), implying \( \Delta t \leq \epsilon h/v_\star \). Since each ASPH node possesses an anisotropic smoothing scale embodied by \( G \), we choose the smallest smoothing scale associated with \( G \) to use for this criterion. This is given by the inverse of the maximum eigenvalue of \( G \). There are three basic timescales which set our timestep choice. The first of these is the familiar sound-speed Courant condition, which can be expressed for a given node \( i \) by

\[
\Delta t^c_i = \epsilon^c \frac{h^\text{min}_i}{c^i_i},
\]

where \( \epsilon^c \) represents the fractional multiplier for the Courant condition, \( h^\text{min}_i \) is the minimum smoothing scale associated with \( G_i \), and \( c^i_i = (\gamma P_i/\rho_i)^{1/2} = [(\gamma - 1)u_i]^{1/2} \) is the local sound speed. Our second timescale is set by the local divergence of the velocity field. We need only concern ourselves with the divergence of the velocity, since (A)SPH is a Lagrangian technique and it is only the relative, rather than the bulk, velocity of the nodes which is significant. The eigenvalues of the symmetric part of the local deformation tensor essentially measure this quantity on the scale of \( h \), so we can set this timescale as

\[
\Delta t^\nu_i = \epsilon^\nu \min \left\{ \text{Eigenvalue} \left[ \frac{1}{2}(\sigma_{\alpha\beta} + \sigma_{\beta\alpha})_i \right] \right\}.
\]

Our final timescale limits how rapidly the \( G \) tensor is allowed to evolve (or equivalently the rate of density evolution), which can be set by placing a limit on the
evolution of the density. This is accomplished through

\[ \Delta t_i^e = \epsilon^e \frac{\rho_i}{D \rho_i / Dt}. \] (C.6)

Together, these three relations determine our timestep for a given node \( i \), such that \( \Delta t_i = \min(\Delta t_i^e, \Delta t_i^*). \) We have found experimentally that using \( \epsilon^e = \epsilon^* = \epsilon^p = 0.1 \) is successful. Such choices are in fact quite conservative, but our asynchronous approach to the integration allows us to be somewhat generous here.

### C.3 Neighbor selection

Hydrodynamic interactions are strongly local in nature, which is why (A)SPH kernel estimates only sample neighboring nodes out to a few smoothing scales. For this reason, developing an efficient method of identifying only those nodes which are within a few \( h \) of a given node's position can potentially greatly increase the speed of an (A)SPH code. For example, if we have a simulation of \( N \) nodes, each of which sample roughly \( N_n \ll N \) significant neighbors, ideally the computation time should scale as \( O(NN_n) \). If the problem were treated as a global interaction, such that no effort were made to identify only those neighbors which are significant for any given interaction, the computational time would scale like \( O(N^2) \). This is an enormous difference, and therefore for large-scale (A)SPH simulations it is imperative that an efficient neighbor finding algorithm be developed. ASPH presents two complications such an algorithm must deal with. First, since each node samples ellipsoidal regions, we would like to develop an algorithm which can recognize this anisotropy when selecting candidate neighbors. Secondly, our symmetrization scheme (eq. [2.5]) requires that we not only
find all nodes \( j \) which fall under the influence of the node in question \( i \), but also any nodes which may happen to lie outside of the cutoff normalized radius \( \eta_i > \eta_{\text{cut}} \) and yet still influence \( i \) because \( \eta_j < \eta_{\text{cut}} \). The criteria for determining whether or not a given node \( j \) should be counted as significant for \( i \) can be quantified as

\[
\eta_{\text{min}}^{j} \equiv \min(\eta_i, \eta_j) \leq \eta_{\text{cut}}. \tag{C.7}
\]

We have developed an algorithm for finding a group of such potential nearest neighbors, which relies on a two-stage culling process. The first step is based upon the popular gridcell method, whereby the simulation volume is divided up into a collection of subvolumes or gridcells. Each node is then associated with the gridcell within which it happens to fall, allowing a fast but crude spatial localization of the nodes (at least to the resolution scale of the gridcell size). Ideally, the gridcell size should be related to the smoothing scale, such that the length of a gridcell is roughly the radius of influence for a given node's influence (i.e., a few \( h \)). However, in our implementations of both SPH and ASPH the smoothing scales vary, and for ASPH there isn't even a single, unique smoothing scale per node. We deal with these issues by establishing a hierarchy of gridlevels. On successive gridlevels the linear gridcell size is halved, such that

\[
\Delta^g = \frac{\Delta^0}{2^g}, \tag{C.8}
\]

where \( \Delta^g \) represents the length of one side of a gridcell on gridlevel \( g \), and \( \Delta^0 \) represents the top-most gridcell size (on gridlevel \( g = 0 \)). Note that under this convention the gridcell sizes decrease with increasing gridlevel, and for \( N_g \) total gridlevels, \( g \in [0, N_g - 1] \). Each node is now associated with a particular gridlevel and gridcell on
that level. The appropriate gridlevel for a given node $i$ is defined to be the "deepest" (maximum $g$) level on which that node can only influence at most one gridcell in any direction, implying

$$\Delta_i^g \geq \eta^\text{cut} h_i^\text{max},$$

(C.9)
giving us

$$g_i \leq \log_2 \left( \frac{\Delta^0}{\eta^\text{cut} h_i^\text{max}} \right),$$

(C.10)

where we have defined $h_i^\text{max}$ to be the maximum smoothing scale associated with node $i$. Once each node is associated with a gridlevel and gridcell in this fashion, the search algorithm for finding a list of potential neighbors for node $i$ goes as follows. Beginning with the topmost ($g = 0$) gridlevel, we identify which gridcell contains node $i$, and build a list of all nodes on this gridlevel which are in this or any immediately adjacent gridcells. We descend through gridlevels $g < g_i$ and repeat this process. This stage picks up any nodes with smoothing scales greater than our node in question $h_j^\text{max} \geq h_i^\text{max}$. For $g > g_i$, we are now dealing with nodes which possess smoothing scales $h_j^\text{max} < h_i^\text{max}$. In this case, we must check all adjacent gridcells out to a radius equivalent to the gridcell size on level $g_i$ (a radius in gridcells on gridlevel $g$ of $2^{g-g_i}$).

Once we have descended through all gridlevels in this fashion, we will have a list of potential neighbor nodes for node $i$ guaranteed to include all nodes which meet the criteria $\eta_{ij}^\text{min} \leq \eta^\text{cut}$. Note that this list of potential neighbors is in fact appropriate for all nodes which are members of $i$'s gridcell. This provides us with a logical definition for the groups of nodes which are to be integrated together, as described in §C.1.
Batches of nodes which are defined as those nodes which are assigned to the same gridcell should all have the same target time for integration, and the same set of neighbor information can be used for them all.

While this gridcell search is quick and efficient, it is still possible to further cull the resulting list of potential neighbors to a smaller set. There are two reasons for this. First, the volume per gridcell on each gridlevel decreases as $2^\nu$ in $\nu$ dimensions, which is not a very fine scale. Secondly, we have not yet capitalized upon the anisotropy of the smoothing scales, but rather have used the maximum smoothing scale associated with each node to select neighbors. We therefore implement a second culling stage to the neighbor selection process. For each $G_i$, the maximum smoothing scale in each dimension $h_i^{\text{max}} \equiv (h_x^{\text{max}}, h_y^{\text{max}}, h_z^{\text{max}})_i$ can be used to more finely cull the potential neighbor list. This process is slightly complicated by our symmetrization scheme and the fact that we want our list of potential neighbors to apply to all nodes in a particular gridcell, rather than a single node. We employ the following culling algorithm, wherein we denote nodes which are members of our target gridcell with the subscript $i$ and the full list of potential neighbors as $j$. (The set $i$ is therefore a subset of $j$.)

1. For all nodes $i$, identify the minimum and maximum coordinates $(x_i^{\text{min}}, x_i^{\text{max}})_i \equiv (\min(x_i)_i \forall i, \max(x_i)_i \forall i)$, and the minimum and maximum coordinates influenced by these nodes $(x_i h^{\text{min}}, x_i h^{\text{max}})_i \equiv (\min(x_i - \eta^{\text{cut}} h_i^{\text{max}})_i \forall i, \max(x_i + \eta^{\text{cut}} h_i^{\text{max}})_i \forall i)$.

2. Loop over all nodes $j$.
3. For each \( j \) verify whether or not the node is significant to any \( i \) node (fulfilling the criterion \( \eta_i < \eta^{\text{cut}} \)) by verifying that \( x_j \in [x_i^{\min}, x_i^{\max}] \).

4. Then check whether any \( i \) node can count as significant to \( j \) (fulfilling the criterion \( \eta_j < \eta^{\text{cut}} \)) by verifying the volumes defined by \( [x_j - \eta^{\text{cut}} h_j^{\max}, x_j + \eta^{\text{cut}} h_j^{\max}] \) and \( [x_i^{\min}, x_i^{\max}] \) overlap.

5. If a node \( j \) fails both of these tests, then remove it from the list of candidate neighbors.

Together these two steps quickly create a relatively small list of candidate neighbor nodes for a given gridcell. Note that this algorithm is guaranteed to find all significant neighbors, so long as the topmost gridcell size meets the criterion \( \Delta^0 \geq \eta^{\text{cut}} h^{\max} \).
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