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GALOIS REPRESENTATIONS AND THE HECKE ACTION
ON THE MOD $p$ FARRELL COHOMOLOGY OF $GL_n(Z)$
IN THE RANGE $p - 1 \leq n < 2p - 2$

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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ABSTRACT

We are interested in the Hecke action on the torsion cohomology classes of the group of invertible integral matrices $\Gamma_n$ and its congruence subgroups. For matrices of dimension $p - 1$, it is known that for a coefficient module $M$ that is an admissible $\Gamma_{p-1}$-module of some characteristic $p$ and some level $N$, the level $pN$ Hecke eigenvectors in the Farrell cohomology $\hat{H}^*(\Gamma_{p-1}, M)$ have an attached Galois representation that is continuous, semi-simple, and unramified outside of $pN$. Being attached means that, for all prime $l$ not dividing $pN$, the Hecke polynomial at $l$ of the eigenvector is equal to the characteristic polynomial of the image of $\text{Frob}_l$ under the representation.

We find that, if the dimension $n$ of the matrices is inclusively between $p - 1$ and $2p - 3$ and $M$ is again admissible of characteristic $p$ and level $N$, then the system of eigenvalues of any Hecke eigenvector in the Farrell cohomology $\hat{H}^*(\Gamma_n, M)$ also appears as the system of eigenvalues of an eigenvector that is a tensor product of a Hecke eigenvector in the Farrell cohomology of $\Gamma_{p-1}$ and a Hecke eigenvector in the ordinary cohomology of $\Gamma_{n-(p-1)}$. If the eigenvector in the cohomology of $\Gamma_{n-(p-1)}$
has an attached Galois representation, then so does the original eigenvector in $\hat{H}^*(\Gamma_n, M)$.

From known results, it now follows that Hecke eigenvectors in $\hat{H}^*(\Gamma_n, M)$ for $n = p$ or $p + 1$ have attached Galois representations.
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CHAPTER 1

INTRODUCTION

We are interested in groups of integral matrices. Matrix groups are the best examples of groups since the purpose of any group should be to act on something and matrices are functions on vector spaces. Integral matrices are particular good to study, from a number theoretical point of view, because an integral matrix embodies the idea of a discrete action (as opposed to rational or complex matrices, which act continuously). For example, groups of permutations or symmetries, which arise in the study of zeroes of polynomials and are thus the first groups studied by number theory, are naturally expressed as groups of integral matrices. Matrices are such a fundamental concept, so thoroughly studied over the course of the centuries, with so many simple and elegant properties, that it is fascinating that they are yet so little understood. Even in the simplest case of two by two matrices, new and deep results are still discovered about them.

We wish to study groups of integral matrices by looking at their cohomology. The cohomology groups of a group are sets of vector spaces or $\mathbb{Z}$-modules, so the structure of the cohomology groups is simpler than the structure of the original
group. Yet since a group determines its cohomology, the cohomology must somehow be encapsulating information about the structure of the group. The particular property of cohomology that we will study in this dissertation is the striking and mysterious connection between the cohomology of groups of integral matrices and the representations of Galois groups. The fact that there is a connection between these two objects which arise from such different sources is evidence of how deep the subject of integral matrices goes.

Define $p$ to be a prime, $p - 1 < n < 2p - 2$, $\Gamma_n = GL_n(\mathbb{Z})$, and $Hecke(pN)$ to be the Hecke algebra of $\Gamma_n$ outside of $pN$. The only significant properties of $Hecke(pN)$, for the purposes of this introduction, are that $Hecke(pN)$ is a ring which naturally acts on the cohomology of $\Gamma_n$ and that $Hecke(pN)$ is generated by certain elements $T_n(l, k)$, with $l$ ranging over all primes not dividing $pN$ and $k$ ranging from 0 to $n$.

For a given $i \geq 0$ and $\Gamma_n$-module $M$, the cohomology group of $\Gamma_n$ of dimension $i$ with coefficient module $M$ is denoted $H^i(\Gamma_n, M)$. Instead of these standard cohomology groups, which we will refer to as the ordinary cohomology groups, we will use the less familiar Farrell cohomology groups, denoted $\hat{H}^i(\Gamma_n, M)$. The most important property of Farrell chomology is that it is equal to ordinary cohomology in high enough dimensions. In the case of $\Gamma_n$, $H^i(\Gamma_n, M) = \hat{H}^i(\Gamma_n, M)$ whenever $i > n(n - 1)/2$. Hence:
1.1 Result. Any statement in this dissertation that is true for Farrell cohomology \( \hat{H}^i(\Gamma_n, M) \) is true for ordinary cohomology \( H^i(\Gamma_n, M) \) if \( i > \text{vcd} \Gamma_n \), where \( \text{vcd} \Gamma_n = n(n - 1)/2 \).

So Farrell cohomology should be thought of as the high-dimensional part of ordinary cohomology. See [Br] for general background in cohomology of groups and Chapter X of [Br] in particular for information about Farrell cohomology. Superficially, the significant difference between ordinary and Farrell cohomology is that ordinary cohomology \( H^i(\Gamma_n, M) \) is defined for \( i \) a non-negative integer and Farrell cohomology \( \hat{H}^i(\Gamma_n, M) \) is defined for any integer \( i \). In terms of the resolutions which compute cohomology, ordinary cohomology is computed by an ordinary resolution starting at dimension 0 and increasing infinitely, while Farrell cohomology is computed by a complete resolution extending infinitely in both increasing and decreasing directions.

It happens that the \( p \)-parts of the Farrell cohomology \( \hat{H}^*(GL_n(\mathbb{Z})) \) are essentially alike for our values \( p - 1 \leq n < 2p - 2 \). Therefore it should be possible to extend results about the \( p \)-part of \( \hat{H}^*(GL_{p-1}(\mathbb{Z})) \) to the \( p \)-part of \( \hat{H}^*(GL_n(\mathbb{Z})) \), for these values of \( n \). Searching for interesting results that we could try to extend, we came upon the connection with Galois representations mentioned above. Now that we have defined \( \text{Hecke}(pN) \), we can state the connection explicitly. We are
interested in the connection as it appears in the following conjecture in [A1] (We use \( \Gamma'_m \) where Ash uses \( \Gamma \) and \( m \) instead of \( n \)):

Let \( F \) be a finite field of char \( p \). Let \( G_Q \) denote the absolute Galois group of \( \bar{Q} \) over \( Q \). Let \( \Gamma'_m \) be a congruence subgroup of \( \Gamma_m \).

**Conjecture B of [A].** Let \( m \geq 2, \Gamma'_m \) a subgroup of finite index in \( \text{Gl}_m(\mathbb{Z}) \), and \( (\Gamma'_m, S) \) a congruence Hecke pair of level \( N \). With \( p \) a prime and \( F \) as above, let \( V \) be an admissible \( FS \)-module. Suppose \( \beta \in H^1(\Gamma'_m, V) \) is an eigenclass for the action of the Hecke algebra \( \text{Hecke}(pN) \) with eigenvalues \( a(l, k) \in F \).

Then there exists a continuous semisimple representation \( \rho : G_Q \to \text{Gl}_m(F) \) unramified outside \( pN \) such that

\[
\sum_{k=0}^{k=m} (-1)^k l^{k(k-1)/2} a(l,k) X^k = \det(I - \rho(Frob_l)^{-1} X)
\]

for all \( l \) not dividing \( pN \).

The polynomial

\[
P(\beta, l) = \sum_{k=0}^{k=m} (-1)^k l^{k(k-1)/2} a(l,k) X^k.
\]

where \( a(l,k) \) is the eigenvalue defined by \( T_n(l,k) \beta = a(l,k) \beta \). is called the Hecke polynomial at \( l \) of the eigenvector \( \beta \). For brevity, if \( \beta \) is a \( \text{Hecke}(pN) \) eigenvector and \( \rho \) is a continuous and semisimple Galois representation, unramified outside of \( pN \), then we will say that \( \rho \) is attached to \( \beta \) if it has the property of the above
conjecture, i.e. that, for \( l \nmid pN \), the characteristic polynomial of \( \text{Frob}_l \) is the Hecke polynomial of \( \beta \) at \( l \). Sometimes, instead of saying that the Galois representation is attached to the eigenvector \( \beta \), we will say that the Galois representation is attached to the system of eigenvalues of \( \beta \).

Since we will use Farrell cohomology, instead of trying to prove Conjecture B exactly we will try to prove a slightly modified conjecture, call it Conjecture \( \hat{B} \), gotten by replacing the ordinary cohomology groups in Conjecture B with Farrell cohomology.

In [A], Ash proved that Conjecture \( \hat{B} \) holds for the Farrell cohomology of \( GL_{p-1}(\mathbb{Z}) \), i.e. that, if \( M \) is an admissible \( \mathbb{F} \) module of level \( N \), every Hecke eigenvector in \( \hat{H}^*(GL_{p-1}(\mathbb{Z}), M) \) has an attached Galois representation.

We will prove in this dissertation that, for \( M \) an admissible \( \mathbb{F} \) module of level \( N \) and \( p - 1 < n < 2p - 2 \), any system of eigenvalues that is realized in a Hecke\( (pN) \) eigenvector \( \mu \in \hat{H}^*(GL_n(\mathbb{Z}), M) \) is also realized in a vector of the form \( \alpha \otimes \beta \) with \( \alpha \) an eigenvector in \( \hat{H}^*(GL_{p-1}(\mathbb{Z}), M) \) and \( \beta \) an eigenvector in \( H^*(GL_{n-(p-1)}(\mathbb{Z}), M) \). Since \( \alpha \) has an attached Galois representation by [A], this means that \( \mu \) has an attached Galois representation if \( \beta \) does.

Before more explicitly stating our results, let us give more detail to the context. Generally speaking, the correspondence between Galois representation and Hecke eigenvectors in cohomology groups has been studied extensively. The literature
gives many results about the correspondence between modular forms or automorphic forms and Galois representations; these can often be restated as results about cohomology and Galois representations under the correspondence between forms and cohomology groups. For example, modular forms of weight 2 correspond to $H^*(Gl_2(\mathbb{Z}), \mathbb{C})$.

See [A2] for a discussion of the history. Major topics are the Eichler-Shimura theory from the 1950's and 1960's which showed that Galois representations could be attached to modular forms that are Hecke eigenforms. The Langlands program works with the problem for $Gl_n(\mathbb{Z}), n > 2$. Serre's Conjectures work with the converse problem: attaching Hecke eigenvectors to Galois representations.

Recently the subject has attracted a great deal of attention because Wiles proved Fermat's Last Theorem by attaching modular forms that were Hecke eigenforms to Galois representations. This shows how extremely complicated the question is even in the smallest matrix group $\Gamma_2$.

Within this broad field we have restricted ourselves to Conjecture B (and $\hat{B}$), the subtopic of attaching Galois representations to the mod $p$ cohomology of congruence subgroups of $Gl_n(\mathbb{Z})$. We describe the cohomology groups $H^*(\Gamma'_n, V)$ in Conjecture B as "mod $p$" because, since $V$ has char $p$, all the elements in $H^*(\Gamma'_n, V)$ have order $p$. See [A] for several results about Conjecture B. One result is that, to prove the conjecture for congruence subgroups $\Gamma'_n$, it is sufficient
to prove it for \( \Gamma_n = GL_n(\mathbb{Z}) \). This is true because a system of eigenvalues realized in \( H^i(\Gamma'_n, V) \), for a congruence Hecke pair \((\Gamma'_n, S)\) and some admissible \( V \), is also realized in \( H^i(GL_n(\mathbb{Z}), V') \), for some admissible \( V' \). Another result in [A1] is that the conjecture is true for the Farrell cohomology \( \hat{H}^*(GL_{p-1}(\mathbb{Z}), V) \). Additionally, in [A3] the conjecture is proved for \( H^*(GL_1(\mathbb{Z}), V) \) and \( H^*(GL_2(\mathbb{Z}), V) \). In [A-M] the conjecture is proved for certain cohomology classes in \( \hat{H}^*(GL_{m(p-1)}(\mathbb{Z}), V) \).

Now let us return to our statement that the cohomology groups \( \hat{H}^*(\Gamma_n, V) \) are essentially alike for \( n \) inclusively between \((p-1)\) and \((2p-3)\). For \( p-1 \leq n < 2p-2 \) and \( \Gamma_n = GL_n(\mathbb{Z}) \),

\[
\hat{H}^*(\Gamma_n, V) \cong \bigoplus_{P \in \mathcal{P}_n} \hat{H}^*(N_{\Gamma_n}(P), V),
\]

where \( \mathcal{P}_n \) is a set of representatives for the conjugacy classes of order \( p \) subgroups of \( \Gamma_n \).

For \( p-1 < n < 2p-2 \), \( \mathcal{P}_n \) naturally divides into two sets \( \mathcal{P}_n = \mathcal{P}_n(1) \sqcup \mathcal{P}_n(2) \), with \( \mathcal{P}_n(1) \) and \( \mathcal{P}_n(2) \) both naturally isomorphic to \( \mathcal{P}_{p-1} \). In this dissertation we call the groups in \( \mathcal{P}_n(1) \) the form 1 or the semisimple groups; each group in \( \mathcal{P}_n(1) \) is a group in \( \mathcal{P}_{p-1} \) extended by identity matrices. We call the groups in \( \mathcal{P}_n(2) \) the form 2 or the non–semisimple groups; each group in \( \mathcal{P}_n(2) \) is a group in \( \mathcal{P}_{p-1} \) “augmented” to dimension \( p \) matrices and then extended by identity matrices. It is because the cohomology groups depend on the \( \mathcal{P}_n \) and the \( \mathcal{P}_n \) are essentially
copies, or twisted copies, of \( \mathcal{CP}_{p-1} \) that we say that the cohomology groups are essentially alike.

Define

\[
\hat{H}^*_1(\Gamma_n, V) \cong \coprod_{P \in \mathcal{P}_n(1)} \hat{H}^*(N\Gamma_n(P), V) \quad \text{and} \\
\hat{H}^*_2(\Gamma_n, V) \cong \coprod_{P \in \mathcal{P}_n(2)} \hat{H}^*(N\Gamma_n(P), V).
\]

So \( \hat{H}^*(\Gamma_n, V) \cong \hat{H}^*_1(\Gamma_n, V) \oplus \hat{H}^*_2(\Gamma_n, V) \). We call \( \hat{H}^*_1(\Gamma_n, V) \) the form 1 or semisimple cohomology and \( \hat{H}^*_2(\Gamma_n, V) \) the form 2 or non-semisimple cohomology.

We prove in this dissertation that \( \text{Hecke}(pN) \) preserves \( \hat{H}^*_1(\Gamma_n, V) \) and \( \hat{H}^*_2(\Gamma_n, V) \), so we can speak of Hecke eigenvectors in \( \hat{H}^*_1(\Gamma_n, V) \) or \( \hat{H}^*_2(\Gamma_n, V) \) instead of \( \hat{H}^*(\Gamma_n, V) \). We separate cohomology into 4 cases.

Case 1 is the semisimple cohomology \( \hat{H}^*(\Gamma_{p-1}, V) \). Ash proved the conjecture for this case in \([A1]\). Ash proved it by inventing a ring \( R \) for which there is a natural map \( \text{Hecke}(pN) \rightarrow R \). He proved that \( R \)-modules, with \( \text{Hecke}(pN) \) action induced by pulling back the natural map, have Galois representations attached to every Hecke eigenvector. He could not prove that \( \hat{H}^*(\Gamma_{p-1}, V) \) itself was an \( R \)-module so he invented a helpful group \( \hat{H}^*(Z(\pi_0), M) \otimes_\mathbb{F} \mathbb{F}[C(\zeta)] \), let us call it \( K_{p-1} \), which is an \( R \)-module. Hence Hecke eigenvectors of \( K_{p-1} \) all have attached Galois representations. Ash defined an injective map \( \hat{H}^*(\Gamma_{p-1}, V) \hookrightarrow K_{p-1} \) and
proved that it is $Hecke(pN)$-invariant. Hence, for any eigenvector in $\hat{H}^*(\Gamma_{p-1}, V)$, its system of eigenvectors appears in the nonzero image of the eigenvector in $K_{p-1}$, and so the eigenvector has an attached Galois representation.

We summarize Case 1 in Section 2 of Chapter 3. When we summarize it we also invent another helpful group, $\hat{H}^*(\Gamma_{p-1}(P), M) \otimes_{\mathbb{F}} \mathbb{F}[\zeta]$, call it $K'_{p-1}$, which is related to $K_{p-1}$ and is also an $R$-module. This group is used in Case 4.

Case 2 is the non-semisimple cohomology $\hat{H}^2(\Gamma_p, V)$. This cohomology looks very much like the cohomology in Case 1, so we imitate the Case 1 proof that Ash invented. We invent a ring $R'$ such that all $R'$-modules have Galois representations attached to every Hecke eigenvector. Since we cannot prove that $\hat{H}^2(\Gamma_p, V)$ is an $R'$-module, we invent a helpful group $\hat{H}^*(Z(P), M) \otimes_{\mathbb{F}} \mathbb{F}[\zeta]$, call it $K_p$, which is an $R'$-module. Finally we prove that the natural map $\hat{H}^2(\Gamma_p, V) \rightarrow K_p$ is $Hecke(pN)$-invariant. This is proved in Section 3 of Chapter 3 and the final result is:

3.3.21 PROPOSITION. Let $\Gamma = Gl_p(\mathbb{Z})$ and $N$ a positive integer. Let $\alpha \in \hat{H}^1(\Gamma, M)$ be an $Hecke(pN)$-eigenvector. Then there is a continuous, semisimple, and unramified outside $pN$ Galois representation attached to $\alpha$.

Case 3 is the semisimple cohomology $\hat{H}^1(\Gamma_n, V)$ with $p-1 < n < 2p-2$. Using the Kunneth formula, we embed these cohomology groups into the tensor product.
of \( \hat{H}^\ast(\Gamma_{p-1}, V') \) and \( H^\ast(\Gamma_{n-(p-1)}, V'') \). We show that this embedding is in a sense \( \text{Hecke}(pN) \)-equivariant and that any system of Hecke eigenvalues realized by \( \mu \in \hat{H}_1^\ast(\Gamma_n, V) \) is realized by an eigenvector of the form \( \alpha \otimes \beta \) with \( \alpha \) a Hecke eigenvector in \( \hat{H}_1^\ast(\Gamma_{p-1}, V') \) and \( \beta \) a Hecke eigenvector in \( H^{i-j}(\Gamma_{n-(p-1)}, V'') \).

Here \( V' \) and \( V'' \) are modules defined in the course of the proof. Most of this is proved in Sections 1 and 2 of Chap 4.

Case 4 is the non-semisimple cohomology \( \hat{H}_2^\ast(\Gamma_n, V) \) with \( p-1 < n < 2p-2 \). Using the Kunneth formula, we embed these cohomology groups into the tensor product of \( K'_{p-1} \) and \( H^\ast(\Gamma_{n-(p-1)}, V'') \). We prove that any system of Hecke eigenvalues realized by \( \mu \in \hat{H}_1^\ast(\Gamma_n, V') \) is realized by a vector of the form \( \alpha \otimes \beta \) with \( \alpha \) a Hecke eigenvector in \( K'_{p-1} \) and \( \beta \) a Hecke eigenvector in \( H^{i-j}(\Gamma_{n-(p-1)}, V'') \).

Again \( V'' \) and also the coefficient module of \( K'_{p-1} \) are defined in the course of the proof. Most of this is proved in Sections 3 and 4 of Chapter 4.

The last parts of Case 3 and 4 are proved in Section 5 of Chapter 4. We define a group \( G \), which is different depending on whether we are dealing with Case 3 or Case 4. We show that, for any eigenvector in Case 3 or Case 4, its system of eigenvectors appears in an eigenvector of the form \( \alpha \otimes \beta \) with \( \alpha \) an eigenvector in either \( \hat{H}^\ast(\Gamma_{p-1}, M') \) or \( K'_{p-1} \), depending on whether we are dealing with Case 3 or Case 4 respectively, and \( \beta \) an eigenvector in \( H^\ast(\Gamma_{n-(p-1)}, M'') \). We finally prove:

Define \( m = n - (p-1) \). Remember the definition of \( P(\beta, l) \) from 1.2.
Proposition 4.5.5. Let $G$ be as in 4.5.3. Suppose $\alpha \otimes \beta \in G_1 \otimes H^*(\Gamma_m, M'')$ with $\alpha$ and $\beta$ both Hecke eigenvectors. Suppose there exists a continuous semi-simple representation $\tau : G_{\mathbb{Q}} \to GL(m, \mathbb{F})$, unramified outside $N$, such that $P(\beta, l) = \det(I - \tau(\text{Frob}_l)^{-1}X)$ for all $l \nmid N$. Let $\rho$ be a continuous semi-simple representation of $G_{\mathbb{Q}} \to GL(p-1, \mathbb{F})$ such that $P(\alpha, l) = \det(I - \rho(\text{Frob}_l)^{-1}X)$ for all $l \nmid pN$.

Then $\rho \otimes \tau : G_{\mathbb{Q}} \to GL(n, \mathbb{F})$ is a continuous semi-simple representation unramified outside $N$ such that $P(\alpha \otimes \beta, l) = \det(I - (\rho \otimes \tau)(\text{Frob}_l)^{-1}X)$ for all $l \nmid pN$.

Consequently, we have this result:

1.3 Main Theorem. Let $\mu \in \tilde{H}^i(\Gamma_n, M)$ be a Hecke($pN$) eigenvector for some admissible module $M$ and $p-1 \leq n \leq 2p-2$. The system of eigenvalues of $\mu$ appears in an eigenvector of the form $\alpha \otimes \beta$ where $\alpha$ has an attached Galois representation and $\beta$ is a Hecke($pN$) eigenvector in $H^j(\Gamma_{n-(p-1)}, M'')$ for some (positive) integer $j$ and some admissible module $M''$. If $\beta$ has a Galois representation then so does $\mu$. Hence theoretically we can find a Galois representation for $\mu$ by finding one for $\beta$.

We say "theoretically" above because our proof does not give a constructive way of creating $\beta$ from $\mu$. All we know is that $\beta$ exists. So, for a given $\mu$, Theorem
1.3 is not very practical. However, it is possible to say, as we do at the end of Section 5 of Chapter 4:

4.5.6 Theorem. Suppose $\Gamma_m$ has the property that every Hecke$(pN)$ eigenvector in $H^*(\Gamma_m, M)$, for any admissible module $M''$, has an attached Galois representation.

Then any Hecke$(pN)$ eigenvector of $\hat{H}^*(\Gamma_{p-1+m}, M)$, for any admissible module $M$, has an attached Galois representation.

In particular,

1.4 Corollary. For any admissible module $M$, any Hecke$(pN)$ eigenvector in $\hat{H}^*(\Gamma_{p-1}, M)$, $\hat{H}^*(\Gamma_p, M)$, or $\hat{H}^*(\Gamma_{p+1}, M)$ has an attached Galois representation.

Proof. This follows from 4.5.6 and from the fact that, as mentioned above, in [A3] it was proved that Hecke$(pN)$ eigenvectors have attached Galois representations for $H^*(\Gamma_1, M)$ and $H^*(\Gamma_2, M)$.

Remember that the $\hat{H}^*(\Gamma_{p-1}, M)$ case is not proved in this dissertation but in [A1]. We include it in the statement of 1.4 for completeness.

We can, for further completeness, in fact state that eigenvectors in $\hat{H}^*(\Gamma_n, M)$ have attached Galois representations for any $0 \leq n \leq p + 1$. This is true since $\hat{H}^*(\Gamma_n, M) = 0$ for $n < p - 1$.

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This dissertation deals only with the problem of $Hecke(pN)$ eigenvectors in $\hat{H}^*(\Gamma_n, M)$ for $p-1 \leq n < 2p-2$ because in those cases there is the injective map

$$\hat{H}^*(\Gamma_n, V) \hookrightarrow \bigoplus_{P \in \mathcal{P}_n} \hat{H}^*(Z_{\Gamma_n}(P), V)$$

and $\mathcal{P}_n$ is particularly simple. We use the cohomology of the centralizers $Z_{\Gamma_n}(P)$ to create helpful groups in Case 1 and Case 2 that are $R$-modules and so connected to Galois representations. Case 3 and Case 4 depend on Case 1 and Case 2 and also on the fact that we can find nice groups, related to the centralizers, that obey the Kunneth formula.

For further investigation, we suspect that we could use the techniques of this dissertation to give some results about $Hecke(pN)$ eigenvectors when $n \geq 2p-2$. Our basic idea in this dissertation is to use the Kunneth formula for the centralizers of matrices that are basically $(p-1)$-dim matrices extended by identity matrices. For $n \geq 2p-2$, there is no longer an injective map into the (direct sum of) the cohomology of the centralizers, so we can only possibly get results about eigenvectors with nonzero restriction to a centralizer. Our techniques then might apply to centralizers of $p$-groups which are groups of dimension $p-1$ extended by the identity matrix. Since [A-M] proves that, provided a certain conjecture is true, Galois representations can be attached to eigenvectors for $\Gamma_{m(p-1)}$ by turning the sums of the cohomology of the centralizers into a helpful $R$-module, then possibly
our techniques will work for centralizers of $p$-groups that are groups of rank $m$
and dimension $m(p-1)$ extended by an identity matrix.

Investigating this would mostly be a matter of analyzing the structure of the centralizers.
CHAPTER 2

PRELIMINARIES

§1 Notation and Definitions.

2.1.1 Notation and Definitions.

$p$ is an odd prime.

$n$ is an integer such that $p \leq n < 2p - 2$.

$m = n - (p - 1)$.

$N$ is a positive integer. In other contexts, this would be called the level.

$\mathbb{Z}$ as usual is the rational integers.

$\mathbb{Z}/p = \mathbb{Z}/p\mathbb{Z}$ is the field of order $p$, for any prime $p$.

$\mathbb{Q}_N$ is rational numbers with denominators prime to $N$.

$F$ is a finite field of char $p$ ($F$ may be enlarged. See §5 of Chapter 4).

$\Phi(x) = 1 + x + x^2 + \cdots + x^{p-1}$ the $p$-th cyclotomic polynomial.

$\zeta$ is the primitive $p$-th root of unity $e^{2\pi i/p}$.

$C(\zeta)$ = the ideal class group of $\mathbb{Q}[\zeta]$. For an ideal $J$, the element in the class group is denoted by double brackets: $[[J]] \in C(\zeta)$. 
For any integer \( r \geq 1 \) and ring \( R \), \( \Gamma_r(R) = GL_r(R) \) is the group of invertible matrices with entries in \( R \).

\[
\Gamma_r = \Gamma_r(\mathbb{Z})
\]

\[
\Gamma = \Gamma_n(\mathbb{Z})
\]

Regard \( \Gamma_{p-1} \subset \Gamma \) by embedding into the upper left corner: \( \Gamma_{p-1} \to \Gamma_{p-1} + I_m \).

For any matrix or group of matrices \( G \) and any invertible matrix \( m \) define \( G^m = m^{-1}Gm \).

\( \{e_1, e_2, \ldots, e_n\} \) is the standard basis of \( \mathbb{Z}^n \). Using this basis, \( \mathbb{Z}^n \) will always be thought of as a left \( \Gamma \)-module (so the vectors are column vectors). For \( (\mathbb{Z}/l)^n \), also a left \( \Gamma \)-module by reduction mod \( l \) of the coefficients of matrices, the standard basis is \( \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\} \).

For a vector \( u \), \( u^T \) is the transpose. We use this when we are specifying the components of a vector; it takes up less space to write a row vector then it does to write a column vector.

Usually a matrix and a vector will just be juxtaposed, e.g. \( Au \), but sometimes, when notation is complicated, they will be separated by a "\cdot", e.g. \( A \cdot u \). The meaning is the same in either case, normal matrix multiplication.

\( M \) is an admissible \( \Gamma \)-module of level \( N \). The definition of admissible \( \Gamma \)-module, adapted from Definition 1.4 of [A1], is a finite-dimensional right \( \mathbb{F}(\Gamma_n(\mathbb{Q}_N)) \)-module on which the matrices in \( \Gamma_n(\mathbb{Q}_N) \) of positive determinant act through
their reduction mod \( N \). Note that \( M \)'s being an admissible \( \Gamma \)-module means that \( M \) is also an admissible \( G \)-module for any subgroup \( G \subset \Gamma \).

\( \hat{H}^*(\Gamma, M) \) is the Farrell cohomology of \( \Gamma \) with coefficients in \( M \).

For \( g \in G\text{L}_n(\mathbb{C}) \) and \( g \) acting on \( M \), define the conjugation map

\[ g^* : \hat{H}^*(\Gamma, M) \to \hat{H}^*(\Gamma^g, M) \]

as the map induced on the cochain level by: \( g^{-1}hg \mapsto h, \ m \mapsto mg \). Induced maps on cohomology are contravariant for the group maps and covariant for the module maps. This is the standard conjugation map for right modules.

\( P \) (also \( P', P_I, P_0 \), etc) is always a subgroup of \( \Gamma \) of order \( p \) with \( \pi \) as generator (respectively \( \pi', \pi_I, \pi_0 \), etc). It may be that \( P \) satisfies other requirements; those will be mentioned as appropriate.

Normalizers and centralizers are with respect to whatever group is in the subscript. If there is no subscript, they are with respect to \( \Gamma \). So:

\[ Z(P) = Z^\Gamma(P) = \{ X \in \Gamma \mid XA = AX \ \forall A \in P \} = \text{ the centralizer of } P \text{ in } \Gamma. \]

\[ N(P) = N^\Gamma(P) = \{ X \in \Gamma \mid XAX^{-1} \in P \ \forall A \in P \} = \text{ the normalizer of } P \text{ in } \Gamma. \]

\( I_s \) is the identity matrix of rank \( s \), for any integer \( s \geq 1 \).

All matrices drawn to be block diagonal are square on the diagonals.

A * drawn inside a matrix indicates an unspecified matrix of the appropriate size with entries in the appropriate ring (usually either \( \mathbb{Z} \) or \( \mathbb{Q} \)).
A matrix $g$ is said to be \((r,s)\)-block diagonal if $g = \begin{pmatrix} *_r & 0 \\ 0 & *_s \end{pmatrix}$ for some integers $r, s \geq 1$. The subscripts here denote the size of square matrices. Similarly $g$ is \((r, s)\)-upper block triangular if $g = \begin{pmatrix} *_r & * \\ 0 & *_s \end{pmatrix}$.

2.1.2 Definition. Given any square matrix $A$ of dim $\geq p-1$, define this square matrix of dim $p-1$:

$$\text{cnr}[A] = \text{the top left (p-1)-corner of } A, \text{ i.e. } A = \begin{pmatrix} \text{cnr}[A] & * \\ * & * \end{pmatrix}$$

Also define these square matrices of dimension $p$, for any \((p-1)\)-dim vector $v$:

$$\text{acr}[A,v] = \begin{pmatrix} \text{cnr}[A] & v \\ 0 & 1 \end{pmatrix}$$

We will be particularly interested in $\text{acr}[A, e_1]$ and $\text{acr}[A, l e_1]$, where $l$ is a prime $\neq p$ and $e_1 = (1 \ 0 \ \ldots \ 0)^T$.

Think of $\text{cnr}[A]$ as the upper corner of $A$ and $\text{acr}[A,v]$ as the upper corner augmented.

There are a lot of square matrices of all sorts of dimensions in this thesis. We follow these size conventions:

If a square matrix has a subscript representing a number, then the subscript is the dimension. If there is no subscript, the matrix has dimension $n$. There are some exceptions: the images of the $\text{cnr}$ function and the $\text{acr}$ function have dim $p-1$ and $p$ respectively; matrices with tildes over them have dim $p-1$. 

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Generally matrix groups indicate size in the same way: if there is a number subscript then the subscript is usually the dimension; else the dimension is $n$. There are some exceptions: some matrix groups have dimension $n$ but also have subscripts. That is because in these cases the matrix groups are canonically isomorphic to matrix groups of the dimension of the subscripts. For reference here we mention that these groups are $G_{p-1}$, $Z_{p-1}(P)$, $G_m$, $U_{p-1}(P)$, and $G_m^1(P)$ (all defined in chapter 4).

We need this result:

2.1.3 Remark. Corollary X.7.4 of [Br]: Suppose $G$ is a group, $M$ is a $G$-module, and every elementary abelian $p$-subgroup of $G$ has rank $\leq 1$. Then

$$\hat{H}^*(G, M)_{(p)} \cong \prod_{P \in P_n} \hat{H}^*(N_G(P), M)_{(p)}$$

where $P_n$ is a set of representatives for the conjugacy classes of order $p$ subgroups of $G$.

The subscripted $(p)$ in this statement indicates the $p$-torsion part of the cohomology. The maps involved are the natural restriction maps.

Since $M$ is an $F$-module and $F$ has characteristic $p$, each $\hat{H}^*(\Gamma, M)$ is $p$-torsion. Since $\Gamma$ is an integral matrix group with dimension less than $2p - 2$, $\Gamma$ contains no subgroups of the form $\mathbb{Z}/p \times \mathbb{Z}/p$, i.e. no elementary abelian $p$-groups of $\Gamma$ has
rank > 1. So 2.1.3 gives:

\[ \hat{H}^\ast(\Gamma_r, M) \cong \bigoplus_{P \in \mathcal{P}_r} \hat{H}^\ast(N_{\Gamma_r}(P), M) \quad \text{for } p - 1 \leq r < 2p - 2 \]  

(2.1.4)

where the sum in each case is over a set of representatives of conjugacy classes of groups of order \( p \) in \( \Gamma_r \).

This statement 2.1.4 also holds for \( r < p - 1 \) but in an uninteresting way. For \( r < p - 1 \), \( \Gamma \) has no subgroups of order \( p \) so \( \hat{H}^\ast(\Gamma, M) = 0 \) and \( P_\Gamma = \emptyset \).

Since well-known facts about conjugacy classes of groups of order \( p \) show that \( \mathcal{P}_r \) are alike for different \( r \), then 2.1.4 gives a way to think of the cohomology groups \( \hat{H}^\ast(\Gamma_r, M) \) as being alike for different \( r \). Our plan of attack in this dissertation is to exploit this likeness to extend results about the cohomology of \( \Gamma_{p-1} \) to results about the cohomology of \( \Gamma_r, p - 1 < r < 2p - 2 \). We need to name the matrices that we will choose to be in \( \mathcal{P}_r \) so that the correspondence between different \( \mathcal{P}_r \) is clear.

2.1.5 Definition. A matrix \( \pi \in \Gamma, n \leq p - 1 \), of order \( p \) is of form 1 if \( \pi = cnr[\pi]I_m \), i.e. if

\[
\pi = \begin{pmatrix} cnr[\pi] & 0 \\ 0 & I_m \end{pmatrix}
\]

Necessarily \( cnr[\pi] \) has order \( p \) since \( \pi \) has order \( p \) and \( \pi \) is block diagonal. Note that the identity matrix, despite being block diagonal, is technically not of form 1 since it does not have order \( p \).
If \( n = p-1 \), then \( m = 0 \) and all matrices of order \( p \) trivially are of form 1. Since we restrict \( n \) to \( n < 2p - 2 \), we could have equivalently defined \( \pi \) to be of form 1 if it is \((p-1,m)\)-block diagonal. This is not a weaker condition because, since the bottom right corner has order \( p \) but dimension \( < p - 1 \), the bottom right corner must be the identity.

A group \( P \) of order \( p \) is of form 1 if some element is of form 1. Necessarily if one element is of form 1 then in fact all non-identity elements are of form 1.

Sometimes matrices of form 1 will be referred to as “the semisimple matrices”.

2.1.6 **Definition.** A matrix \( \pi \in \Gamma, \, n \leq p \), of order \( p \) is of form 2 if \( \pi \) is not \( \Gamma \)-conjugate to a matrix of form 1 and

\[
\pi = \begin{pmatrix}
\text{acr}[\pi, e_1] & 0 \\
0 & I_{m-1}
\end{pmatrix}
\]

A group \( P \) of order \( p \) is of form 2 if some element is of form 2. If \( P \) has an element of form 2, then none of its other elements is of form 2, but the other non-identity elements are \( \Gamma \)-conjugate to elements of form 2 (so by definition not conjugate to elements of form 1).

If \( P \) is of form 2, its generator \( \pi \) will always be chosen to be of form 2.

Sometimes matrices of form 2 will be referred to as “the non-semisimple matrices”.

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If \( \pi \) is of order \( p \), there is a rank \( p-1 \) subspace \( V \subset \mathbb{Z}^N \) such that \( \pi|_V \) does not have 1 as an eigenvalue. Hence form 1 and form 2 matrices can be characterized as follows: Let \( \pi \in \Gamma \) be of order \( p \). First, \( \pi \) is \( \Gamma \)-conjugate to form 1 if and only if \( \mathbb{Z}^n \) has a basis with \( m \) elements fixed by \( \pi \). Second, \( \pi \) is \( \Gamma \)-conjugate to form 2 if and only if any basis of \( \mathbb{Z}^n \) has no more than \((m - 1)\) elements fixed by \( \pi \).

We remark that being part of a basis is a stricter condition than being linearly independent. For any form 2 matrix, there are \( m \) linearly independent elements of \( \mathbb{Z}^n \) that are fixed by it. The characterization of form 2 matrices above means that these elements cannot be extended to a basis. The \( m \)-th fixed vector (i.e a vector in addition to the obvious vectors \( e_{p+1}, e_{p+2}, \ldots, e_n \)) of a form 2 matrix will show up often in the succeeding.

Now we look at the class group \( C(\zeta) \).

There is a 1-1 correspondence between conjugacy classes of groups of order \( p \) in \( \Gamma_{p-1} \) and elements in \( C(\zeta) \) as follows:

Given \([J] \in C(\zeta)\) with representative ideal \( J \), pick an integral basis of \( J \). Using this basis, \( J \cong \mathbb{Z}^{p-1} \) and \( J \) is a left \( \Gamma_{p-1} \)-module. Let \( \pi_{p-1}(J) \) be the matrix, with respect to this basis, representing the action of “multiplying by \( \zeta \)” on \( J \). Note that \( \pi_{p-1}(J) \) has order \( p \). The conjugacy class of the group generated by \( \pi_{p-1}(J) \) is the class associated with \([J] \in C\). See for example III.14 of [Ne] or 22.2 of [C-R] for details.
Now for every element $[[J]] \in C(\zeta)$, pick once and for all a representative ideal $J$ such that $J \not\subseteq (\zeta - 1)$ (for example pick $J$ to be a prime ideal that is prime to $Np$), and pick an integral basis of $J$ so that the first basis element is not in $(\zeta - 1)$. The condition on the basis is necessary in order to assert that the matrices $\pi_r(2, J)$, constructed in 2.1.7 below, are of form 2.

For $[[Z(\zeta)]] \in C(\zeta)$, it is standard to choose $Z(\zeta)$ for the representative ideal and $\{\zeta, \zeta^2, \ldots, \zeta^{p-1}\}$ for the integral basis. So $\pi_{p-1}(Z(\zeta))$ is the rational canonical form matrix for $\Phi(x)$.

2.1.7 Definition. For each $[[J]] \in C(\zeta)$, $\tilde{\pi}_J = \pi_{p-1}(J) \in \Gamma_{p-1}$ is the corresponding element in $\Gamma_{p-1}$. Write $\pi_{p-1}(0)$ instead of $\pi_{p-1}(Z(\zeta))$.

If $p-1 < r < 2p - 2$, then

$$\pi_r(1, J) = \begin{pmatrix} \pi_{p-1}(J) & 0 \\ 0 & I_{r-p+1} \end{pmatrix} \quad \text{and} \quad \pi_r(2, J) = \begin{pmatrix} \text{acr}[\pi_{p-1}(J), e_1] & 0 \\ 0 & I_{r-p} \end{pmatrix}$$

Note that $\pi_r(1, J)$ is a form 1 matrix. By Theorem 74.3 of [C-R], $\pi_r(2, J)$ is not $\Gamma$-conjugate to a $(p-1, r-p+1)$-block diagonal matrix, so $\pi_r(2, J)$ is a form 2 matrix.

In future sections, whenever it is clear whether we are considering form 1 or form 2 matrices, we will define, locally for just that section, $\pi_J = \pi_n(1, J)$ or $\pi_n(2, J)$ accordingly.
2.1.8A Definition. 

\[ \mathcal{P}_{p-1} = \{ \langle \pi_J \rangle : J \in C \} \]

The brackets () above mean "group generated by". So \( \mathcal{P}_{p-1} \) is a set of subgroups of order \( p \).

2.1.8B Definition. For any \( r \) with \( p-1 < r < 2p-2 \),

\[ \mathcal{P}_r(1) = \left\{ \left\langle \begin{pmatrix} \pi_J & 0 \\ 0 & I_{r-p+1} \end{pmatrix} \right\rangle : J \in C \right\} = \left\{ \langle \pi_r(1, J) \rangle : J \in C \right\} \]

\[ \mathcal{P}_r(2) = \left\{ \left\langle \begin{pmatrix} \text{acr}[\pi_J, e_1] & 0 \\ 0 & I_{r-p} \end{pmatrix} \right\rangle : J \in C \right\} = \left\{ \langle \pi_r(2, J) \rangle : J \in C \right\} \]

Note the first is a set of form 1 groups and the second is a set of form 2 groups.

2.1.8C Definition. For \( p-1 < r < 2p-2 \),

\[ \mathcal{P}_r = \mathcal{P}_r(1) \bigcup \mathcal{P}_r(2) \]

2.1.9 Fact.

1) \( \mathcal{P}_{p-1} \) is a set of representatives of the conjugacy classes of order \( p \) subgroups of \( \Gamma_{p-1} \).

2) If \( p-1 < r < 2p-2 \), then \( \mathcal{P}_r \) is a set of representatives of the conjugacy classes of order \( p \) subgroups of \( \Gamma_r \).

3) If \( l \) is prime to \( p \), then \( \text{acr}[\pi_p(J), e_1] \sim_{\Gamma_{p-1}} \text{acr}[\pi_p(J), le_1] \).
Proof.

These facts are well-known, see for example Theorem 74.3 of [C-R]. Curtis and Reiner speak of \( \mathbb{Z} \)-isomorphisms of \( \mathbb{Z}(\mathbb{Z}/p) \)-modules instead of \( \mathbb{Z} \)-conjugacy classes of integral matrix groups of order \( P \). To apply their result to a matrix \( \pi \) of order \( p \), consider \( \mathbb{Z}^n \) to be a left \( \mathbb{Z}(\mathbb{Z}/p) \)-module via the action of \( \pi \). Then \( \mathbb{Z}^n \)'s being \( \mathbb{Z} \)-isomorphic to one of the modules in [C-R] means that \( \pi \) is \( \mathbb{Z} \)-conjugate to one of the matrices in \( \mathcal{P}_r \).

The third fact is not as obvious as the other two. The restatement of the relevant part of Theorem 74.3 of [C-R] in terms of matrices instead of modules is:

Let

\[
A = \begin{pmatrix}
\pi_{p-1} & 0 \\
0 & I_m
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
\text{acr}[\pi_{p-1}, u] & 0 \\
0 & I_{m-1}
\end{pmatrix}
\]

with \( u \) having the property that \( u \not\in (\pi_{p-1} - 1)\mathbb{Z}^{p-1} \). Then \( \begin{pmatrix}
\text{acr}[\pi_{p-1}, v] & 0 \\
0 & I_{m-1}
\end{pmatrix} \) is \( \Gamma \)-conjugate to \( A \) if \( v \in (\pi_{p-1} - 1)\mathbb{Z}^{p-1} \) and is conjugate to \( B \) otherwise.

Since the basis of \( J \) was chosen so that the first element is not in \( (\zeta - 1)\mathbb{Z}[\zeta] \), we have that, in terms of matrices, \( e_1 \) (and so also \( le_1 \)) is not in \( (\tilde{\pi}_J - 1)\mathbb{Z}^{p-1} \).

Note \( \mathcal{P}_r \) contains only form 1 and form 2 elements. Note the natural correspondence between \( \mathcal{P}_r \) and \( \mathcal{P}_s \) for \( p-1 < r, s < 2p - 2 \). Also note the 3-way natural correspondence between \( \mathcal{P}_{p-1} \), the form 1 elements of \( \mathcal{P}_r \), and the form 2 elements of \( \mathcal{P}_r \).
Now that the conjugacy classes \( \mathcal{P}_r \) have been fixed, let us return to considering the isomorphism of 2.1.3.

2.1.10 Definition. Define \( \hat{H}_1^*(\Gamma, M) \) and \( \hat{H}_2^*(\Gamma, M) \subset \hat{H}^*(\Gamma, M) \) by

\[
\hat{H}_1^*(\Gamma, M) \cong \bigoplus_{P \in \mathcal{P}_n(1)} \hat{H}^*(N(P), M)
\]
\[
\hat{H}_2^*(\Gamma, M) \cong \bigoplus_{P \in \mathcal{P}_n(2)} \hat{H}^*(N(P), M).
\]

So, by 2.1.3, \( \hat{H}^*(\Gamma, M) = \hat{H}_1^*(\Gamma, M) \oplus \hat{H}_2^*(\Gamma, M) \)

Sometimes we will call \( \hat{H}_1^*(\Gamma, M) \) the semisimple part of the cohomology and \( \hat{H}_2^*(\Gamma, M) \) the non-semisimple part. This is just because \( \mathcal{P}_n(1) \) consists of semisimple matrices and implies no sort of semisimplicity in the cohomology.

Since \( P \) is cyclic of finite order \( p \), then \( Z(P) \) has finite index in \( N(P) \). Moreover this index is prime to \( p \). Therefore restriction maps \( \hat{H}^*(N(P), M) \hookrightarrow \hat{H}^*(Z(P), M) \) for each \( P \), so, by 2.1.3,

\[
\hat{H}_1^*(\Gamma, M) \hookrightarrow \bigoplus_{P \in \mathcal{P}_n(1)} \hat{H}^*(Z(P), M)
\]
\[
\hat{H}_2^*(\Gamma, M) \hookrightarrow \bigoplus_{P \in \mathcal{P}_n(2)} \hat{H}^*(Z(P), M).
\]  \hspace{1cm} (2.1.11)

This particularly means that, for \( \alpha \in \hat{H}^*(\Gamma, M) \), then \( \alpha|_{Z(P)} \neq 0 \) iff \( \alpha|_{N(P)} \neq 0 \).

Proof that \( Z(P) \) has finite index prime to \( p \): For any \( 1 \leq i \leq p - 1 \) such that \( \pi \sim_{\Gamma} \pi^i \), pick an element \( c_i \in N(P) \) such that \( c_i \pi c_i^{-1} = \pi^i \). Any element in \( N(P) \) is an element in \( Z(P) \) times one of the \( c_i \). There are less than \( p \) of these \( c_i \)'s.

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2.1.12 Definition. An element $\alpha \in \hat{\mathcal{H}}^*(\Gamma, M)$ is primitive (on $P'$) if $\alpha|_{Z(P)} = 0$ for all $P \in \mathcal{P}_n$ such that $P \neq P'$.

Since $\mathcal{P}_n$ is a set of representatives of conjugacy classes of groups of order $p$, it would be good if being primitive were a property that was independent of choice of $\mathcal{P}_n$. In fact this is true: For any subgroup $P$ and any $g \in \Gamma$, the conjugation map $g^* : \hat{\mathcal{H}}^*(P, M) \cong \hat{\mathcal{H}}^*(P^g, M)$ is an isomorphism since trivially $gZ(P)g^{-1} = Z(gPg^{-1})$. So we could equivalently have defined "$\alpha$ primitive on $P'$" to be that $\alpha|_{Z(P)} = 0$ for all $P \subset \Gamma$ such that $P \not\sim \Gamma P'$ and think of $\alpha$ as being primitive on a conjugacy class, not a specific subgroup.

Since it is a direct sum by 2.1.10, then $\hat{\mathcal{H}}^*(\Gamma, M)$ is generated by its primitive elements.

Now we are ready to consider Hecke.

2.1.13 Hecke Notation and Definitions.

Hecke$(\Gamma, \mathbb{Q}_{pN}) = Hecke(pN) =$ the free abelian group on the set of double cosets $\{\Gamma s \Gamma \mid s \in \Gamma(\mathbb{Q}_{pN})$ and $\det s > 0\}$.

$$T_n(l, k) = \begin{pmatrix} I_{n-k} & 0 \\ 0 & lI_k \end{pmatrix}$$ for any numbers $n, l, \text{and } k$.

$l$ is a prime not dividing $pN$.

$k$ is an integer $0 \leq k \leq p$.

$s = T_n(l, k)$.
\[ s_r(i) = T_r(l, i) \] for any \( p - 1 \leq r \leq n \) and \( 0 \leq i \leq k \).

**Hecke** has a natural ring structure by section 3.1 of [Sh]. See 3.10 of [Sh] for the proof that \( \text{Gl}_n(\mathbb{Q}_pN) \) is contained in the commensurator of \( \Gamma \) (a fact needed in order to use 3.1 of [Sh]).

Each element of **Hecke** acts naturally on cohomology. Action here means a linear \( \mathbb{F} \)-action on \( \mathbb{F} \)-vector spaces.

2.1.14 **DEFINITION.** For \( c \in \Gamma_n(\mathbb{Q}_pN) \),

\[ T_c = \text{Tr}_{\Gamma} \text{Res}_{\Gamma}^* c^* : \hat{H}^*(\Gamma, M) \to \hat{H}^*(\Gamma, M) \]

where \( \text{Tr} \) is the transfer map, \( \text{Res} \) is the restriction map, and \( c^* \) is the conjugation map (see 2.1.1). Make this an action of **Hecke**(\( \Gamma, \mathbb{Q}_pN \)) on \( \hat{H}^*(\Gamma, M) \) by letting double cosets \( \Gamma c \Gamma \) act as \( T_c \) and extending by linearity to sums of cosets.

Also use the same definition of \( T_c \) for action on the ordinary cohomology \( H^*(\Gamma, M) \).

See [R-W] for proof that this is an action. In particular the action is independent of the choice of double coset representatives, i.e. \( T_c = T_{g'cg} \) for \( g', g \in \Gamma \). [R-W] uses a different definition of the action-- one involving cocycles-- but it is the same action as defined here (see Appendix A).
By Theorem 3.20 of [Sh], $\text{Hecke}(\Gamma, \mathbb{Q}_{pN})$ is a polynomial ring over $\mathbb{Z}$ with indeterminates $\{ \Gamma T_n(l,k) \Gamma \mid l \text{ prime}, l \nmid pN, 0 \leq k \leq n \}$. So any Hecke action is determined by how these generators $T_n(l,k)$ act.

Conversely, since Hecke is a polynomial ring, any arbitrary assignment of endomorphisms (on any group) to these generators $T_n(l,k)$ can be extended to an action (on the group) of the full Hecke ring (with the exception of course that the identity element $\Gamma I_n \Gamma$ cannot be assigned arbitrarily but must be assigned to the identity automorphism).

We will examine the Hecke action on $\hat{H}^*(\Gamma, M)$ by examining the action of an arbitrary generator. We need a name for an arbitrary generator, since $T_n(l,k)$ is unwieldy, so we named it $s$ in 2.12 above. We invented the notation for $s_r(i)$ because we will compare Hecke actions for different values of $n$ and $k$ but the same $l$, and $s_r(i)$ is less wieldy than $T_r(l,i)$.

We need to define some coset representatives:

2.1.15 Definition. For fixed $P$ and $s$,

$$D(P) = \text{a set of coset representatives of } (\Gamma \cap \Gamma^s) \backslash \Gamma / \mathbb{Z}(P) \text{ and}$$

$$D'(P) = \{ g \in D \mid P \subset \Gamma^s g \}$$

For easier reading, we will usually write $D(\pi)$ and $D'(\pi)$ in place of $D((\pi))$ and $D'((\pi))$, i.e. we write the matrix but mean the group generated by the matrix.
Since \( D'(P) \) is a subset of \( D(P) \), which is a complete set of coset representatives, we will sometimes refer to \( D'(P) \) as an "incomplete set of coset representatives". In §4 we will say a lot more about sets of coset representatives and choose them in 2.4.8 and 2.4.10 so that all elements of \( D'(P) \) have a specific form.

We are now ready to give the formula for the Hecke action that will be used for the rest of this paper.

Note first that, since \( H^*(\Gamma, M) \rightarrow \bigoplus_{\mathcal{P}} H^*(\mathcal{Z}(P), M) \) by 2.1.11, an element of \( H^*(\Gamma, M) \) is known if all its restrictions to the \( \mathcal{Z}(P)'s \) are known.

Exactly as in 6.3.2 of [A1], then II.9.5 of [Br] (adapted to Farrell cohomology and right modules) implies that the Hecke action \( T_s : \hat{H}^*(\Gamma, M) \rightarrow \hat{H}^*(\Gamma, M) \) from 2.1.14 obeys

\[
T_s(\alpha)|_{\mathcal{Z}(P)} = \sum_{g \in D(P)} T_{\mathcal{Z}(P) \cap g^{-1}(\Gamma \cap \Gamma^s)g} \text{Res}_{\mathcal{Z}(P) \cap g^{-1}(\Gamma \cap \Gamma^s)g} (g^* s^* \alpha)
\]

where \( D(P) \) is the set of representatives defined in 2.1.15 above, \( g^* \) and \( s^* \) are the conjugation maps, and \( |_{\mathcal{Z}(P)} \) means restriction to \( \mathcal{Z}(P) \).

Since \( g \in \Gamma \), then \( \mathcal{Z}(P) \cap g^{-1}(\Gamma \cap \Gamma^s)g = \mathcal{Z}(P) \cap \Gamma^g \). So

\[
T_s(\alpha)|_{\mathcal{Z}(P)} = \sum_{g \in D(P)} T_{\mathcal{Z}(P) \cap \Gamma^g} \text{Res}_{\mathcal{Z}(P) \cap \Gamma^g} (g^* s^* \alpha) \quad (2.1.16)
\]

The work in this dissertation will usually be on \( T_s|_{\mathcal{Z}(P)} \) instead of the full \( T_s \) operator.
§2 Some Information On Cohomology

Since the formula for \( T_\alpha \) has conjugation maps, \( \text{Res} \) maps, and \( \text{Tr} \) maps, some lemmas about these maps will be useful. In this section are technical lemmas about basic cohomology, all having proofs that involve resolutions.

This first lemma does not involve cohomology but will be helpful in figuring out domains and ranges of various cohomological maps after a conjugation map has been applied.

2.2.1 Lemma. Let \( P \subset \Gamma \) of order \( p \) and \( t \in \Gamma(Q_{pN}) \). Then \( Z(P) \cap t^{-1}\Gamma t = t^{-1}Z(tPt^{-1})t \cap \Gamma : \)

Proof. Let \( z \in Z(P) \cap t^{-1}\Gamma t \). Then \( z\pi = \pi z \Rightarrow tzt^{-1}t\pi t^{-1} = t\pi t^{-1}tzt^{-1} \Rightarrow tzt^{-1} \in Z(tPt^{-1}). \)

Conversely, let \( z \in Z(tPt^{-1}) \) and \( t^{-1}zt \in \Gamma \). Then \( zt\pi t^{-1} = t\pi t^{-1}z \Rightarrow t^{-1}zt\pi = \pi t^{-1}zt \Rightarrow t^{-1}zt \in Z(P). \)

If \( t \in \Gamma \), this lemma reduces to the obvious fact that \( Z(P) = t^{-1}Z(tPt^{-1})t \).

2.2.2 Lemma. Let \( H \subset G \subset G' \) be groups and \( M \) a right \( G' \)-module. Let \( t \in G' \). Then conjugation commutes with restriction in the following sense:

\[
\text{Res}_{H}^{G'} \circ t^* \alpha = t^* \circ \text{Res}_{H}^{G} \alpha \quad \text{for } \alpha \in \check{H}^*(G, M)
\]

This equality also holds for ordinary cohomology.
PROOF. Recall that conjugation \( t^* : \hat{H}^\ast(G, M) \to \hat{H}^\ast(G^t, M) \) is induced by \( G^t \to G, \ g^t \mapsto g \) and \( M \to M, \ m \mapsto mt \).

We will prove this lemma for ordinary cohomology, using a standard resolution. Since restriction and conjugation are natural maps, the statement's being true for ordinary cohomology implies, by dimension shifting, that it is true for Farrell cohomology. Here is a brief summary of the principle of dimension shifting:

For any group \( G \) and any \( G \)-module \( M \), then \( M \) is a quotient of a module \( \tilde{M} \) with the property that \( \hat{H}^i(G, \tilde{M}) = 0 \) for all \( i \). In [Br], this fact is summed up by the phrase the functor \( \hat{H} \) is effaceable. It is also true that \( M \) can be embedded in a module \( \tilde{M} \) with the property that \( \hat{H}^i(G, \tilde{M}) = 0 \) for all \( i \). In [Br], this fact is summed up by the phrase the functor \( \hat{H} \) is co-effaceable. Since short exact sequences of modules induce long exact sequences in cohomology, for any module \( M \) there are modules \( K \) and \( C \) such that \( \hat{H}^i(G, M) \cong \hat{H}^{i+1}(G, K) \) and \( \hat{H}^i(G, M) \cong \hat{H}^{i-1}(G, C) \) for all \( i \).

Therefore any map on cohomology which is equivariant on the long exact sequences (induced by short exact sequences of modules) is uniquely determined by its values on \( \hat{H}^i(G, M) \) for a fixed dimension \( i \) and for \( M \) ranging over all possible \( G \)-modules. In particular, if two maps are equivariant on long exact sequences and are equal, in some fixed dimension \( i \), for all \( G \)-modules \( M \), then the two maps are equal for all dimensions \( i \). In [Br], the property of being equivariant on long
exact sequences is summed up by the phrase the map is natural and the technique of proving something (about natural maps) by proving it just for one fixed $i$ is called dimension-shifting.

Since, by definition of Farrell cohomology, there exists $i$ such that $\hat{H}^i(G, M) = H^i(G, M)$ for all modules $M$, if two natural maps are equal on ordinary cohomology (for all modules $M$) then, by dimension-shifting, they are equal on Farrell cohomology.

See VI.5.4 of [Br] for more detailed information about dimension shifting. We remark that dimension shifting for Farrell cohomology is a little different from dimension-shifting for ordinary cohomology. Since ordinary cohomology is co-efaceable, but not effaceable, for any module $M$ there is a module $K$ such that $H^i(G, M) \cong H^{i+1}(G, K)$ but not necessarily a module $C$ such that $H^i(G, M) \cong H^{i-1}(G, C)$. So dimension-shifting only works to increase dimensions, and natural maps are uniquely defined, not by their values for an arbitrary fixed dimension $i$ but by their values for the particular dimension $0$ (ordinary cohomology is only defined for $i \leq 0$).

We return to the proof of the Lemma.

Define $\{F_i(G')\}$ to be the standard resolution of $G'$. It is also a resolution of $G$, and the conjugation action $\pi$ on the cochains $\text{Hom}_G(F_i(G'), M)$ is what induces the conjugation map on cohomology of $G$.
Looking at restriction and conjugation maps on $\text{Hom}_G(F_i(G'), M)$, we note that $\overline{\text{Res}^{-1}_{H}} \circ \overline{\text{Res}^G}$ is trivially true since $\overline{\text{Res}}$ actually has no effect whatsoever on cochains. So it is true on the induced cohomology maps.

2.2.3 LEMMA. Let $H \subset G \subset G'$ be groups and $M$ a $G'$-module. Let $t \in G'$.

Then conjugation commutes with transfer in the following sense:

$$\text{Tr}^{G'}_H \circ t^* \alpha = t^* \circ \text{Tr}^{G}_H \alpha \quad \text{for } \alpha \in H^*(G, M)$$

This equality also holds for ordinary cohomology.

PROOF. We will prove this lemma for ordinary cohomology, using a standard resolution. Since transfer and conjugation are natural maps the statement's being true for ordinary cohomology implies by dimension shifting that it is true for Farrell cohomology (See the proof of 2.2.2 above for some remarks about dimension-shifting).

As in 2.2.1, we again define $\{F_i(G')\}$ to be the standard resolution of $G'$, observe that $\{F_i(G')\}$ is also a resolution of $G$, and define that $\overline{\alpha}$ is the conjugation action on the cochains $\text{Hom}_G(F_i(G'), M)$ that induces the conjugation map $t^*$ on the cohomology of $G$.

On cochains, the transfer map for right modules is

$$\overline{(\text{Tr}^{G}_H \overline{\alpha})(x)} = \sum_a \overline{\alpha}(ax).a$$
for \( \alpha \) a cochain in \( \text{Hom}_H(F_i(G'), M) \), \( x \) a chain, and the sum running over any set of coset representatives of \( H \backslash G \) (these are right cosets since \( M \) is a right module).

So on cochains \( x \in F_i(G') \),

\[
(\text{Tr}_H^{G'} \circ \overline{\alpha})(x) = \sum_a (\overline{\alpha}(ax))(a) = \sum_a (tat^{-1}).ta
\]

where the second to last equality holds because if \( a \) runs over a set of coset representatives of \( G^t/H^t \) then \( tat^{-1} \) runs over a set of coset representatives of \( G/H \).

Since \( \text{Tr}_H^{G'} \circ \overline{\alpha} = \overline{\alpha} \) and \( \overline{\alpha} \) are equal on cochains, they induce equal maps on cohomology.

Now we examine how \( \text{Tr} \), \( \text{Res} \), and conjugation act on direct products.

2.2.4 REMARK. Suppose \( G \subset G' \), \( H \subset H' \), \( M' \) is a char \( p \) \( G' \)-module, \( M'' \) is a char \( p \) \( H' \)-module, and \( H' \) has \( p \)-finite cohomological dimension.

Then by the Kunneth formula (see X.5.8 of [Br]),

\[
\hat{H}^*(G \times H, M' \otimes_{\mathbb{Z}/p} M'') = \sum_{c_1 + c_2 = *} \hat{H}^{c_2}(G, M') \otimes_{\mathbb{Z}/p} H^{c_2}(H, M'')
\]

Note that \( H^{c_2}(H, M'') \) is ordinary cohomology, not Farrell cohomology. In order to use the Kunneth formula, we need a resolution over \( H' \) of finite length.

This, of course, does not exist in general. However, since \( M \) and \( M' \) have char \( p \), we
can use a resolution over \((\mathbb{Z}/p)[G]\) to get \(H^*(G, M)\), instead of using a resolution over \(\mathbb{Z}[G]\). Similarly for \(G', H, H'\), and their respective Farrell cohomologies: if the coefficient module has characteristic \(p\), then we can induce cohomology with a resolution that uses \(\mathbb{Z}/p\) instead of \(\mathbb{Z}\). Now note that, by VII.6.1 of [Br] adapted to resolutions over \((\mathbb{Z}/p)[H']\) instead of over \(\mathbb{Z}[H']\), the property that \(H'\) has \(p\)-finite cohomological dimension is equivalent to the existence of a resolution over \((\mathbb{Z}/p)[H']\) of finite length.

**2.2.5 Claim.** With the notation from 2.2.4 above and additionally with \(g \in G'\), \(h \in H'\), and \(\alpha \otimes \beta \in \check{H}^*(G \times H, M' \otimes_{\mathbb{Z}/p} M'')\) then

\[(gh)^*(\alpha \otimes \beta) = (g^*\alpha) \otimes (h^*\beta)\]

**Proof.** Observe

\[(gh)^* : \check{H}^*(G \times H, M' \otimes_{\mathbb{Z}} M'') \rightarrow \check{H}^*((G \times H)^{gh}, M' \otimes_{\mathbb{Z}} M'').\]

Since \((G \times H)^{gh} = G^g \times H^h\), it follows that the Kunneth formula holds for \(\check{H}^*((G \times H)^{gh}, M' \otimes_{\mathbb{Z}} M'')\). So \((g^*\alpha) \otimes (h^*\beta) \in \check{H}^*((G \times H)^{gh}, M' \otimes_{\mathbb{Z}} M'')\).

Consider the Kunneth Formula explicitly. The map is induced from the (co)chain level by using \(F(G) \otimes F(H)\) as a resolution over \(G \times H\), where \(F(G) = \{F_i(G)\}\) is a complete resolution over \(G\) (inducing the Farrell cohomology
of $G$ and $F(H) = \{F_i(H)\}$ is a resolution over $H$ of finite length (inducing the ordinary cohomology of $H$). Choose $F(G)$ to be a complete resolution of $G'$, hence a complete resolution of $G$, and $F(H)$ to be a resolution of $H'$ of finite length.

Again we remark that, since $H'$ only has $p$-finite cohomological dimension, these are $\mathbb{Z}/p$ resolutions, not $\mathbb{Z}$ resolutions.

Since $g \in G'$ and $h \in H'$, we have that the conjugation maps $g^*$ and $h^*$ on $\tilde{H}^*(G \times H, M' \otimes \mathbb{Z} M'')$ are induced by conjugation on the cochains on $F(G) \otimes F(H)$.

For $g$ an element in a group and $x$ a chain a corresponding resolution, use $g \cdot x$ to denote the conjugation action of $g$ on the chain. Suppose $x \otimes y \in F(G) \otimes F(H)$, $g \in G$, and $h \in H$. By definition of $F(G) \otimes F(H)$ then $(gh) \cdot (x \otimes y) = (g \cdot x) \otimes (h \cdot y)$.

Let $\alpha$ and $\beta$ be cochains from $\text{Hom}_G(F(G), M')$ and $\text{Hom}_H(F(H), M'')$ that induce $\alpha$ and $\beta$ respectively, and let $x \otimes y$ be a chain in $F(G) \otimes F(H)$. Then

$$[(gh)^*(\alpha \otimes \beta)](x \otimes y) = (\alpha \otimes \beta)[(gh)_*(x \otimes y)](gh)$$

$$= (\alpha \otimes \beta)[g \cdot x \otimes h \cdot y](gh) = [\alpha(g \cdot x) \otimes \beta(h \cdot y)]gh$$

$$= [\alpha(g \cdot x).g] \otimes [\beta(h \cdot y).h] = [g^* \alpha \otimes h^* \beta][x \otimes y].$$

So $(gh)^*(\alpha \otimes \beta) = (g^* \alpha) \otimes (h^* \beta)$.

2.2.6 Lemma. With the notation from 2.2.4 above and $G'' \subset G$, $H'' \subset H$, and $\alpha \otimes \beta \in \tilde{H}^*(G \times H, M' \otimes M'')$

Then $\text{Res}_{G'' \times H''}^G(\alpha \otimes \beta) = (\text{Res}_{G''}^G \alpha) \otimes (\text{Res}_{H''}^H \beta)$. 37
The Kunneth formula holds for $G'' \times H''$. As with 2.2.5, we look at the cochain level. Let $F(G) = \{F_i(G)\}$ be a complete resolution over $G$ and $F(H) = \{F_i(H)\}$ a resolution over $H$ (of finite length). So $F(G) \otimes F(H)$ is a complete resolution over both $G \times H$ and $G'' \times H''$. Let $\bar{\alpha}$ and $\bar{\beta}$ be cochains on $F(G)$ and $F(H)$ respectively that induce $\alpha$ and $\beta$ respectively. Restriction maps have absolutely no effect on cochains, so the equation is true on the cochain level and hence true on cohomology.

**2.2.7 Lemma.** With the notation from 2.2.4 above and with $H'' \subset H$ and $\alpha \otimes \beta \in \check{H}^*(G \times H'', M' \otimes M'')$

Then $\text{Tr}_{G \times H''}^G(\alpha \otimes \beta) = \alpha \otimes (\text{Tr}_{H''}^H \beta)$.

**Proof.** The Kunneth formula holds for $G \times H''$. Let $F(G) = \{F_i(G)\}$ be a complete resolution over $G$ and $F(H) = \{F_i(H)\}$ a resolution over $H$ of finite length (hence also a resolution over $H''$). Let $\bar{\alpha}$ and $\bar{\beta}$ be cochains in $\text{Hom}_G(F(G), M')$ and $\text{Hom}_H(F(H), M'')$ respectively that induce $\alpha$ and $\beta$. Let $x \otimes y$ be a chain in $F(G) \otimes F(H)$.

Recall that $[\text{Tr}_{G \times H''}^G(\bar{\alpha} \otimes \bar{\beta})](x \otimes y) = \sum_a (\bar{\alpha} \otimes \bar{\beta})[a.(x \otimes y)]a$, where $a$ is a set of coset representatives of $(G \times H'') \setminus (G \times H)$. We can choose these representatives $a$ to lie totally in $H$. By definition, since $a \in H$ then $a(x \otimes y) = x \otimes ay$ and, for
all $m' \otimes m'' \in M' \otimes M''$, $(m' \otimes m'')a = m' \otimes (m''a)$. So

$$
[Tr_{G \times H}^G \times H_\ast(\bar{\alpha} \otimes \bar{\beta})](x \otimes y) = \sum_a [\bar{\alpha}(x) \otimes \bar{\beta}(ay)]a = \bar{\alpha}(x) \otimes \sum_a \bar{\beta}(ay)a
$$

$$
= \bar{\alpha}(x) \otimes [Tr_{H_\ast}^H \bar{\beta}(y)]
$$

§3 Hecke Preserves Semisimple and Non-Semisimple Parts

Suppose $P$ and $P'$ are subgroups of $\Gamma$ of order $p$.

In order to show that the Hecke action preserves $H_1(\Gamma, M)$ (the semisimple cohomology) and $H_2(\Gamma, M)$ (the non-semisimple cohomology) we observe, as in Lemma 6.3.3 of [Al], an interesting relation that $P$ and $P'$ must satisfy in order for $T_s \alpha|_{Z(P)}$ not to be zero. Specifically:

2.3.1 Claim. Let $t \in \Gamma(\mathbb{Q})$ and let $\alpha \in \check{H}^\ast(\Gamma, M)$ be primitive on $P'$. Then

$$
Res_{Z(P) \cap \Gamma^t}^t t^* \alpha = 0 \text{ if } tPt^{-1} \not\subset \Gamma P'
$$

Proof.

See 2.1.12 for the definition of primitive.

Either $Z(P) \cap \Gamma^t$ has $p$-torsion or it doesn't.

i) If $Z(P) \cap \Gamma^t$ has no $p$-torsion then $\check{H}^\ast(Z(P) \cap \Gamma^t, M) = 0$. See X.7.3 of [Br].

In particular $Res_{Z(P) \cap \Gamma^t}^t t^* \alpha = 0$.

ii) If $Z(P) \cap \Gamma^t$ has $p$-torsion then $P \subset \Gamma^t$. (if not then $Z(P)$ contains a group of order $p$ which is not $P$, hence $Z(P)$ contains a copy of $\mathbb{Z}/p \times \mathbb{Z}/p$, but this is
impossible for $Z(P) \subset GL_n(\mathbb{Z})$ with $n < 2p - 2)$. So $tPt^{-1} \in \Gamma$.

\[
Res_{Z(P) \cap \Gamma}^\Gamma t^* \alpha = Res_{t^{-1}Z(tPt^{-1}) \cap \Gamma}^{t^{-1}Z(tPt^{-1})} t^* \alpha \quad \text{(See 2.2.1)}
\]

\[
= Res_{t^{-1}Z(tPt^{-1}) \cap \Gamma}^{t^{-1}Z(tPt^{-1})} t^* \alpha
\]

\[
= Res_{t^{-1}Z(tPt^{-1}) \cap \Gamma}^{t^{-1}Z(tPt^{-1})} o t^* o (Res_{Z(tPt^{-1})}^\Gamma \alpha) \quad \text{(See 2.2.12)}
\]

As $\alpha$ is primitive on $P$ and $tPt^{-1} \subset \Gamma$, by definition $Res_{Z(tPt^{-1})}^\Gamma \alpha = 0$ unless $tPt^{-1} \sim_\Gamma P'$.

Therefore, $Res_{Z(P) \cap \Gamma}^\Gamma t^* \alpha = 0$ if $tPt^{-1} \not\sim_\Gamma P'$

\[\blacksquare\]

2.3.2 COROLLARY. Recall formula 2.1.16 for $T_{s}(\alpha)|_{Z(P)}$. The $Res$ maps in the formula are zero for any $g$ such that

\[
sgP_g^{-1}s^{-1} \not\sim_\Gamma P' \quad \text{for all} \quad P' \in \mathcal{P}_n \text{ with } \alpha|_{Z(P')} \neq 0.
\]

In particular, if $P \not\sim_{(\Gamma, s)} P'$ for all $P' \in \mathcal{P}_n$ with $\alpha|_{Z(P')} \neq 0$ then $T_s \alpha|_{Z(P)} = 0$.

So it is not necessary for the sum in formula 2.1.16 to be over all $g \in D(P)$. At minimum, in order for $sgP_g^{-1}s^{-1} \sim_\Gamma P'$ for some $P'$, we need only sum over $g$ such that $sgP_g^{-1}s^{-1} \subset \Gamma$. Recall from 2.1.15 that $D'(P)$ is defined to be precisely those elements $g$ in $D(P)$ satisfying $sgP_g^{-1}s^{-1} \subset \Gamma$.

Thus 2.3.2 and formula 2.1.16 together give:

\[
T_s(\alpha)|_{Z(P)} = \sum_{g \in D'(P)} Tr_{Z(P)}^{Z(P)}(g^* s^* \alpha) (g^* s^* \alpha) \quad \text{(2.3.3)}
\]
2.3.4 Claim. Let $P$ be of form 2 and $P'$ of form 1 and let $R$ be a ring $\subset \mathbb{Q}$ such that all denominators in $R$ are prime to $p$. Then $\pi' \not\in \text{GL}_n(R)$.

Proof. Recall $\pi' = \begin{pmatrix} \text{cnr}[\pi'] & 0 \\ 0 & I_{m} \end{pmatrix}$, a $(p-1,m)$-block diagonal matrix, and $\pi = \begin{pmatrix} \text{acr}[\pi,e_1] & 0 \\ 0 & I_{m-1} \end{pmatrix}$, a $(p,m-1)$-block diagonal matrix (see Definition 2.1.2 and 2.1.5).

Imbed $\mathbb{Z}^n \subset R^n$. Let $(\mathbb{Z}^n,P)$ be $\mathbb{Z}^n$ made into a left $\mathbb{Z}(\mathbb{Z}/p)$-module by the action of $\pi$. Denote the basis by $\{e_1,e_2,\ldots,e_n\}$. Let $(R^n,P)$ be $R^n$ made into a left $\mathbb{Z}(\mathbb{Z}/p)$-module by the action of $\pi$, with the same basis $\{e_1,e_2,\ldots,e_n\}$. So $(\mathbb{Z}^n,P)$ is a submodule of $(R^n,P)$. Similarly define $(\mathbb{Z}^n,P')$ and $(R^n,P')$ with basis $\{e'_1,e'_2,\ldots,e'_n\}$.

Since these $\mathbb{Z}(\mathbb{Z}/p)$ modules have a chosen basis, any linear map between them is a matrix.

2.3.5 Lemma. If $f : (\mathbb{Z}^n,\pi) \hookrightarrow (\mathbb{Z}^n,\pi')$ is a $\mathbb{Z}(\mathbb{Z}/p)$-map then $p|\det(f)$.

Assuming this lemma, we continue the proof of the claim. Suppose $\exists \ r \in \Gamma(R)$ such that $r^{-1}\pi'r = \pi$. Then $r : (R^n,P') \sim (R^n,P)$ is an isomorphism of $\mathbb{Z}(\mathbb{Z}/p)$-modules which restricts to $r : (\mathbb{Z}^n,P') \hookrightarrow (R^n,P)$. Multiply by an appropriate scalar matrix $d \in \Gamma(R)$ so that $dr$ has $\mathbb{Z}$-coefficients. So $dr : (R^n,P') \sim (R^n,P)$ and $dr : (\mathbb{Z}^n,P') \hookrightarrow (\mathbb{Z}^n,P)$.
Now $p \mid \det(d\tau)$ by the lemma, but also $d\tau$ is invertible in $\Gamma(R)$. contradicting that $R$ does not allow $p$ in denominators. So $\pi' \not\in \Gamma(R) \pi$. 

Now we will prove the lemma.

**Proof of Lemma.** Suppose $f : (\mathbb{Z}^n, P') \to (\mathbb{Z}^n, P)$ is a $\mathbb{Z}(\mathbb{Z}/p)$-map. What does the matrix of $f$ look like? We are interested in the $p$-th row.

Since $\text{cnr}[\pi']$ and $\text{cnr}[\pi]$ have dim $p-1$ and order $p$, the characteristic polynomial of both is the irreducible separable polynomial $\Phi(x)$. So look at $\Phi(\pi)$ and $\Phi(\pi')$. They are both upper triangular matrices such that the upper left $(p-1)$-square block is zero and all the other diagonal elements are nonzero (in fact they are all $p$). So

$$\langle e_1, e_2, \ldots, e_{p-1} \rangle = \{ v \in (\mathbb{Z}^n, \pi) \mid \Phi(\pi)v = 0 \}$$

$$\langle e'_1, e'_2, \ldots, e'_{p-1} \rangle = \{ v \in (\mathbb{Z}^n, \pi) \mid \Phi(\pi')v = 0 \}$$

Since $f$ is a $\mathbb{Z}(\mathbb{Z}/p)$-map, it must send an element killed by $\Phi(\pi')$ to an element killed by $\Phi(\pi)$. So $f$ sends $\langle e'_1, e'_2, \ldots, e'_{p-1} \rangle$ into $\langle e_1, e_2, \ldots, e_{p-1} \rangle$.

So as a matrix $f = \begin{pmatrix} X_{p-1} & Y \\ 0 & Z_m \end{pmatrix}$ (this is $(p-1,m)$-block diagonal). The $p$-th row is zero for the first $p-1$ elements.

Define $u = (1 + \pi + \pi^2 + \cdots + \pi^{p-1})e_p = \Phi(\pi).e_p \in (\mathbb{Z}^n, P)$. Notice that $u$ is $\pi$-invariant. Let $u = \sum u_i e_i$. What is $u_p$? Examine $\pi$. Since $\pi$ fixes $e_p$, it must be that $\Phi(\pi).e_p$ has $p$-th entry equal to $p$. 42
Suppose $\exists e'_j \in (\mathbb{Z}^n, P')$, $j \geq p$, such that $p \nmid (the \ p-th \ entry \ of \ f(e_j))$. This corresponds to the $j$-th column of the $p$-th row of the matrix $f$. Since $e'_j$ is fixed by $\pi'$ (because $j \geq p$, look again at the matrix for $\pi'$) than $f(e'_j)$ is fixed by $\pi$ (because $f$ is a $\mathbb{Z}(\mathbb{Z}/p)$-map). Create $e$, an integral combination of $u$ and $f(e'_j)$ with $p$-th entry $= 1$. Then $e_1, \ldots, e_{p-1}, e, e_{p+1}, \ldots, e_n$ is a basis of $(\mathbb{Z}^n, P)$ with $e, e_{p+1}, \ldots, e_n$ all fixed by $\pi$, i.e. $P$ is form 1 after conjugating by this change of basis.

This is a contradiction, so $f(e'_j)$ has $p$-th entry divisible by $p$ for $j \geq p$. These $p$-th entries are the last $m$ elements of the $p$-th row of $f$. So $p$ divides every entry in the $p$-th row of the matrix of $f$. So $p | det \ f$.

Corollary 2.3.2 and Claim 2.3.4 immediately imply:

2.3.6 Corollary. The Hecke action preserves the form 1 and form 2 parts of $\hat{H}^*(\Gamma)$, i.e. $T_s(\hat{H}_1^*(\Gamma, M)) \subset \hat{H}_1^*(\Gamma, M)$ and $T_s(\hat{H}_2^*(\Gamma, M)) \subset \hat{H}_2^*(\Gamma, M)$.

The definitions of $\hat{H}_1^*(\Gamma, M)$ and $\hat{H}_2^*(\Gamma, M)$ are in 2.1.10.

Since Hecke preserves form 1 and form 2, any Hecke eigenvector is a form 1 eigenvector plus a form 2 eigenvector. So the problem of attaching Galois representations to eigenvectors in $\hat{H}^*(\Gamma, M)$ is the same as attaching Galois representations separately to eigenvectors in $H_1^*(\Gamma, M)$ and $H_2^*(\Gamma, M)$. 43
From now on, we will now deal separately with $H^*_1(\Gamma, M)$ and $H^*_2(\Gamma, M)$, the semisimple and non-semisimple cohomology.

§4 A Good Choice of Coset Representatives

Recall from 2.1.15 the definitions of $D(P)$ and $D'(P)$, sets of coset representatives. It is $D'(P)$ that we are interested in, since it appears in 2.1.16, the formula we are using for the Hecke action, but in order to manipulate $D'(P)$ it will be useful to define several other sets of coset representatives.

2.4.1 Definition. For a fixed $P$ and $s = T_n(l, k)$,

\[ \hat{D} = \text{a set of coset representatives of } (\Gamma \cap \Gamma^s) \backslash \Gamma \text{ with the property that } D \subset \hat{D}. \]

\[ \hat{D}'(P) = \{ g \in \hat{D} | P \subset \Gamma^{sg} \}. \text{ Note that necessarily } D'(P) \subset \hat{D}'(P). \]

Again for easier reading we write a matrix to indicate the group generated by the matrix, for example $\hat{D}'(\pi)$ instead of $\hat{D}'((\pi))$.

Note that we named the sets by putting a hat \(^\wedge\) on sets of single cosets and not putting a hat \(^\wedge\) on sets of double cosets. The prime \(^\prime\) indicates that the set obeys the $P \subset \Gamma^{sg}$ property and is an incomplete set of representatives, not a complete one.

In these definitions $s$ and $n$ are understood. There will be times when we will need to use cosets that involve matrices of rank smaller than $n$ and depend on a
diagonal matrix with a different number of 1's in it than $s$ has. For these situations additionally define, for $0 \leq i \leq k$, $p-1 \leq r \leq n$, and $P_r$ a group of order $p$ in $\Gamma_r$:

$$\hat{D}_r(i) = \text{a set of coset representatives of } (\Gamma_r \cap \Gamma_r^{s_r(i)}) \setminus \Gamma_r.$$  

$$\hat{D}'_r(i, P_r) = \{ g \in \hat{D}_r(i) \mid P_r \subset \Gamma_r^{s_r(i)g} \}.$$  

$D_r(i, P_r)$ is a set of coset representatives of $(\Gamma_r \cap \Gamma_r^{s_r(i)}) \setminus \Gamma_r / \Gamma_r(P_r)$. We insist that these representatives be chosen so that $D_r(i, P_r) \subset \hat{D}_r(i, P_r)$.  

$$D'_r(i, P_r) = \{ g \in D_r(i) \mid P_r \subset \Gamma_r^{s_r(i)g} \}. \text{ Again, we insist that the set be chosen so that } D'_r(i, P_r) \subset \hat{D}'_r(i, P_r).$$

The purpose of this section is to prove that elements in $D'(P), \hat{D}'(P), \hat{D}'_r(i, P_r)$, and $D'_r(i, P_r)$ can be chosen to be of a certain form: we want all these coset representatives, which lie in $\Gamma_n$ or $\Gamma_r$ with $p-1 \leq r < 2p-2$, to be obviously related to the coset representatives in the $n = p-1$ case. Once this is proved in 2.4.8 and 2.4.10, it will thereafter be assumed always that the elements are of this form. This affects the other sets of coset representatives too, since, for example, $D(P)$ must satisfy the condition $D'(P) \subset D(P)$.  

Remember that the idea of this dissertation is that the Hecke action on the cohomology of $\Gamma_n$ is almost the same for all $p-1 < n < 2p-2$. One step in connecting different values of $n$ was the definition of $\mathcal{P}_n$ ($n > p-1$) in 2.1.6: half of $\mathcal{P}_n$ is $\mathcal{P}_{p-1}$ with the generator matrices filled out by attaching an identity.
matrix and the other half is $\mathcal{P}_{p-1}$ with the generator matrices first augmented into matrices of dimension $p$ and then filled out by attaching identity matrices. So we think of $\mathcal{P}_n$ as, almost, $(p-1, m)$-block diagonal matrices with the $(p-1)$ block coming from $\mathcal{P}_{p-1}$ and the bottom block being trivial. This gives a way of considering the cohomology groups to be like each other for different values of $n$.

Now we will do something similar with $D'(P)$ by using the results about $\mathcal{P}_n$ to show that $D'(P)$ consists of matrices that are "almost" $(p-1, m)$-block diagonal, with the upper block being matrices from the $n = p-1$ case. This will give a way of considering the Hecke actions to be like each other for different values of $n$.

One way to think about a block diagonal matrix is to think of it as something which acts on a vector space and preserves certain subspaces. That is the approach we take. We make some appropriate vector space definitions:

2.4.2 Definition. Define $V_n(k) \subset (\mathbb{Z}/l\mathbb{Z})^n$ to be the full rank $k$ subspace defined by the first $n-k$ coordinates' being 0. So

$$V_n(k) = \left( \begin{array}{c} 0_{n-k} \\ *_k \end{array} \right)$$

The subscripts on $0_{n-k}$ and $*_k$ indicate the dimension.

2.4.3. Let $0 \leq i \leq p-1$, $0 \leq j \leq m$ (Recall $n = (p-1) + m$), and $i + j \leq n$. Define $V_n(i, j) \subset (\mathbb{Z}/l\mathbb{Z})^n$ to be the full rank $(i+j)$ subspace whose first $(p-1)-i$
coordinates are 0, next $i$ coordinates are arbitrary, next $m - j$ coordinates are 0, and last $j$ coordinates are arbitrary. So

$$V_n(i, j) = \begin{pmatrix} 0_{p-1-i} \\ *_i \\ 0_{m-j} \\ *_j \end{pmatrix}$$

Make the following generalization of Lemma 6.3.5 of [A1]:

2.4.4 LEMMA. With $s = T_n(l, k)$ and the natural left action of $\pi$ on $(\mathbb{Z}/l)^n$,

then

$$\{ \tilde{D}'(P) \} \overset{1-1}{\longleftrightarrow} \{ \text{full rank } k \text{ } \pi\text{-invariant subspaces of } (\mathbb{Z}/l)^n \}$$

$$g \longleftrightarrow g^{-1}V_n(k)$$

PROOF. For an arbitrary $n$ by $n$ matrix $\begin{pmatrix} A_{n-k} & B \\ C & D_k \end{pmatrix} \in \Gamma$ then

$$s^{-1} \begin{pmatrix} A_{n-k} & B \\ C & D_k \end{pmatrix} s = \begin{pmatrix} I_{n-k} & 0 \\ 0 & \frac{1}{l} I_k \end{pmatrix} \begin{pmatrix} A_{n-k} & B \\ C & D_k \end{pmatrix} \begin{pmatrix} I_{n-k} & 0 \\ 0 & lI_k \end{pmatrix}$$

$$= \begin{pmatrix} A_{n-k} & lB \\ \frac{1}{l} C & D_k \end{pmatrix}.$$}

So $(\Gamma \cap \Gamma^s) = \left\{ \begin{pmatrix} A_{n-k} & lB \\ C & D_k \end{pmatrix} \mid \text{det} = \pm 1; A, B, C, D \text{ have integer coefficients} \right\}$

So $(\Gamma \cap \Gamma^s)$ is the stabilizer of $V(k)$ in $\Gamma$, hence $(\Gamma \cap \Gamma^s) \setminus \Gamma$ corresponds to $k$-dim subspaces of $(\mathbb{Z}/l)^n$ by $(\Gamma \cap \Gamma^s) g \longleftrightarrow g^{-1}V(k)$. The condition in the definition of $D'(P)$ that $\pi \in \Gamma^s$, i.e. that $g \pi g^{-1} \in (\Gamma \cap \Gamma^s)$, is equivalent to stipulating that $g^{-1}V(k)$ be $\pi$-invariant. So the lemma is done. 

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Now suppose \( \pi \) is of form 1. Recall \( p \neq l \). Since \( \Phi(x) \) and \((x - \bar{I})\) are relatively prime in \( \mathbb{Z}/l \) (Here \( \bar{\cdot} \) denotes reduction of coefficients mod \( l \)), then

\[
< \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{p-1} > = \{ \bar{v} \in (\mathbb{Z}/l)^n \mid \Phi_p(\pi)\bar{v} = 0 \} \text{ and } \n
< \bar{e}_p, \ldots, \bar{e}_n > = \{ \bar{v} \in (\mathbb{Z}/l)^n \mid (\pi - I_n)\bar{v} = 0 \}
\]

2.4.5 Definition. Suppose \( \pi \) is of form 1 and \( W \) is any full \( \pi \)-invariant subspace of \((\mathbb{Z}/l)^n\). Define:

\[
W(\zeta) = \{ \bar{v} \in W \mid \Phi(\pi)\bar{v} = 0 \} \text{ and } W(1) = \{ \bar{v} \in W \mid (\pi - I_n)\bar{v} = 0 \}.
\]

Since \((\mathbb{Z}/l)^n = < \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{p-1} > \oplus < \bar{e}_p, \ldots, \bar{e}_n >\), and \( \Phi(x) \) and \((x - \bar{I})\) are relatively prime, then \( W = W(\zeta) \oplus W(1) \)

Moreover \( W(\zeta) = W \cap < \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{p-1} > \) and \( W(1) = W \cap < \bar{e}_p, \ldots, \bar{e}_n >\). In other words, \( W \) naturally breaks up into two subspaces compatible with \( \pi \)'s being block diagonal.

For \( \pi \) of form 1, classify all full \( \pi \)-invariant subspaces by defining

\[
\mathcal{W}(i) = \{ \text{ full } \pi \text{-invariant subspaces } W \text{ with } W(\zeta) \text{ of rank } i \}.
\]

So, under the correspondence of cosets with subspaces, \( \bar{D}'(P) = \bigsqcup_{i=0}^{k} \mathcal{W}(i) \).

Consider these \( \mathcal{W}(i) \). For the moment, fix \( i \) and set \( j = k - i \). For arbitrary \( W \in \mathcal{W}(i), W(\zeta) \) can be any full rank \( i \) \text{ cnr}(\pi)-invariant subspace of \((\mathbb{Z})^{p-1}\).
and \( W(1) \) can be any full rank \( j \) subspace of \((\mathbb{Z})^m\). However such subspaces are the ones that correspond to elements in \( \hat{D}'_{p-1}(i, \text{cnr}[\pi]) \) and \( \hat{D}_m(j) \). Hence the following correspondence:

\[
\{ g_{p-1} \hat{+} g_m \mid g_{p-1} \in \hat{D}'_{p-1}(i, \text{cnr}[\pi]) \text{ and } g_m \in \hat{D}_m(j) \} \longleftrightarrow \{ W(i) \} \quad \quad g_{p-1} \hat{+} g_m \quad \longrightarrow \quad (g_{p-1} \hat{+} g_m^{-1})V_n(i,j)
\]

So when \( \pi \) is of form 1, for each full rank \( k \) \( \pi \)-invariant subspace \( W \) we have a block diagonal element \( g' = g_{p-1} \hat{+} g_m \) in \( \Gamma \) such that \( W = g'^{-1}V_n(i,j) \) (with \( i \) depending on \( W \)). We can not choose \( g' \) to be the element \( g \) in \( \hat{D}'(P) \) that is associated to \( W \), since, by 2.4.4, such \( g \) obey \( W = g^{-1}V_n(k) \). We need to go from \( V_n(i,j) \) to \( V_n(k) \).

2.4.6 DEFINITION. Let \( 0 \leq i \leq p-1 \) and \( 0 \leq j \leq m \) (Recall \( n = (p-1) + m \)) with \( i + j = k \). Define the permutation matrix \( \sigma(i) \) as follows: Given arbitrary \( v \in \mathbb{Z}^n \),

\[
\text{Write } v = \begin{pmatrix} a_{p-1-i} \\ b_i \\ c_{m-j} \\ d_j \end{pmatrix} \quad \text{and define } \sigma(i).v = \begin{pmatrix} a_{p-1-i} \\ c_{m-j} \\ b_i \\ d_j \end{pmatrix}.
\]

Then \( W = g'^{-1}\sigma(i)^{-1}V_n(k) \). So, by 2.4.4, \((\Gamma \cap \Gamma^s)\backslash \sigma(i)g' \) is the coset in \((\Gamma \cap \Gamma^s)\backslash \Gamma \) associated with \( W \). The elements of \( D'(P) \) can therefore be chosen to be the product of a permutation matrix and a \((p-1,m)\)-block diagonal matrix.
2.4.7 Altered Definition. For \( \pi \) of form 1, \( D'(P) \) is still defined to be a set of representatives of cosets \((\Gamma \cap \Gamma') \backslash \Gamma \) corresponding to \( \pi \)-invariant subspaces.

However, we now additionally specify that

\[
D'(P) = \prod_{i=0}^{k} \left\{ \sigma(i)g' \mid g' = (g_{p-1} + g_m) \right\},
\]

with \( g_{p-1} \in D'_{p-1}(i, c\pi[i]) \) and \( g_m \in D_m(j) \),

where \( i + j = k \) and \( \sigma(i) \) depends on both \( i \) and \( k \).

Note that additionally \( i \leq p-1 \) and \( j \leq m \) since else either \( D'_{p-1}(i, c\pi[i]) \) or \( D_m(j) \) is empty.

Since this altered definition is for arbitrary \( 0 \leq k \leq n \) and \( p-1 \leq n \leq 2p-2 \), it also holds for \( D'_r(i, P_r) \). Since the other sets of coset representative we use are all subsets of \( D'(P) \) or \( D'_r(i, P_r) \), all coset representatives are of this form, when \( \pi \) is of form 1.

For a form 2 matrix, we will make the coset representative look like the representatives in the form 1 case times a matrix \( L \) which has useful properties, although it is not \((p-1, m)\)-block-diagonal:

Suppose \( n = p \). Remember \( l \) is a prime \( \neq p \). Let \( \pi \) be a form 2 matrix with \( \langle \pi \rangle \in P_n(2) \). So \( \pi = acr[\pi, e_1] \). Define \( \pi' = acr[\pi, le_1] \). As mentioned in 2.1.9, by Th 74.3 of [C-R] then \( \pi \sim_\Gamma \pi' \). Pick a matrix \( L_p \in \Gamma \) such that

\[
L_p^{-1} acr[\pi, e_1] L_p = acr[\pi, le_1].
\]
We claim $L_p$ is $(p-1,1)$-block upper triangular. In terms of $L_p$ acting on $\mathbb{Z}^p$, this is the same as saying $L_p.\langle e_1, e_2, \ldots, e_{p-1}\rangle \subset \langle e_1, e_2, \ldots, e_{p-1}\rangle$. Since $\Phi(\pi) = \begin{pmatrix} 0_{p-1} & * \\ 0 & p \end{pmatrix}$, then $\langle e_1, e_2, \ldots, e_{p-1}\rangle$ can be defined as the subspace of $\mathbb{Z}^n$ killed by $\Phi(\pi)$ and similarly $\langle e_1, e_2, \ldots, e_{p-1}\rangle$ also is the subspace of $\mathbb{Z}^n$ killed by $\Phi(\pi')$. By 2.4.8, it must be that $L_p.\langle e_1, e_2, \ldots, e_{p-1}\rangle$ is killed by $\Phi(\pi')$, i.e.

$$L_p.\langle e_1, e_2, \ldots, e_{p-1}\rangle \subset \langle e_1, e_2, \ldots, e_{p-1}\rangle.$$ 

Therefore $L_p$ is $(p-1,1)$-block upper triangular.

Since the determinant of the integral matrix $L_p$ is $\pm 1$, the bottom right corner is $\pm 1$. Multiplying by the scalar matrix $\text{diag}(-1, -1, \ldots, -1)$ if necessary (this won’t change 2.4.8), we can assume the bottom right corner is 1. Since $\pi$ and $\pi'$ have the same upper left $(p-1)$-corner, then $\text{cnr}[L_p] \in Z_{p-1}(\text{cnr}[\pi])$.

Instead of citing Th 74.3 of [C-R], we could have just explicitly constructed $L_p$: Define $f_1(x) = 1 + x + x^2 + \cdots + x^{p-1}$. Trivial calculations (given in detail in 3.1.3) show that $f_1(\text{cnr}[\pi]) \in Z_{p-1}(\text{cnr}[\pi])$. Define $g(x) = [f_1(x) - 1]/(x - 1)$, an integral polynomial. Define $L_p^{-1} = \begin{pmatrix} f_1(\text{cnr}[\pi]) & g(\text{cnr}[\pi]).e_1 \\ 0 & 1 \end{pmatrix}$. This value of $L_p$ works.
Now suppose \( n > p \) and \( \pi \) is of form 2. To find a matrix \( L = L_n \) such that \( L^{-1} \pi L = \begin{pmatrix} \text{cnr}[\pi, e_1] & 0 \\ 0 & I_{n-p} \end{pmatrix} \), simply use a matrix \( L_p \) as above such that \( L^{-1}_p \text{acr}[\pi, e_1] L_p = \text{cnr}[\pi, e_1] \) and extend by an identity matrix. This gives:

2.4.9 Definition. Given a form 2 matrix \( \pi = \begin{pmatrix} \text{acr}[\pi, e_1] & 0 \\ 0 & I_{n-p} \end{pmatrix} \) and a prime \( l \neq p \), define \( L \) to be a matrix such that

\[
L^{-1} \pi L = \begin{pmatrix} \text{acr}[\pi, e_1] & 0 \\ 0 & I_{n-p} \end{pmatrix}
\]

and

\[
L \text{ satisfies } L = \begin{pmatrix} L_p & 0 \\ 0 & I_{n-p} \end{pmatrix} \text{ with } L_p = \begin{pmatrix} Z_{p^{-1}} & * \\ 0 & 1 \end{pmatrix} \text{ and } Z_{p^{-1}} \in Z_{\Gamma_{p^{-1}}} (\text{cnr}[\pi]) .
\]

We can now use \( L \) to attack the coset representatives \( \hat{D}'(P) \). Suppose \( \pi \) is of form 2. Notice that \( A = \begin{pmatrix} \text{acr}[\pi, e_1] & 0 \\ 0 & I_{m-1} \end{pmatrix} \) and \( B = \begin{pmatrix} \text{cnr}[\pi] & 0 \\ 0 & I_m \end{pmatrix} \) act exactly the same way on \((\mathbb{Z}/l)^n\). So the \( A \)-invariant subspaces of \((\mathbb{Z}/l)^n\) are the same as the \( B \)-invariant subspaces. Since by Lemma 2.4.4 cosets exactly correspond to full invariant subspaces, we can pick \( D'(A) \) to be the same as the \( D'(B) \) specified in 2.4.7. However, notice that \( A \)-invariant subspaces are in 1-1 correspondence with \( \pi \)-invariant subspaces by \( W \leftrightarrow L^{-1}W \). Putting these facts together, we get:

2.4.10 Altered Definition. For \( \pi \) of form 2 and a prime \( l \neq p \), then \( \hat{D}'(P) \) is still defined to be a set of representatives of cosets \((\Gamma \cap \Gamma^*) \backslash \Gamma \) corresponding to
\(\pi\)-invariant subspaces. However, we now additionally specify that

\[
\tilde{D}'(P) = \prod_{i=0}^{k} \left\{ \sigma(i)g'L \mid g' = (g_{p-1} + g_m) \right. \\
\left. \text{with } g_{p-1} \in \tilde{D}'_{p-1}(i, \text{cnr}(\pi)) \text{ and } g_m \in \tilde{D}_m(j) \right\}.
\]

where \(i + j = k\), \(\sigma(i)\) depends on both \(i\) and \(k\), and \(L\) from 2.4.9 depends on \(l\) and \(\pi\).

Again, note that additionally \(i \leq p-1\) and \(j \leq m\) since else the sets are empty.

Again, the definitions of other sets of coset representatives involving matrices of form 2 are adjusted accordingly, since they are subsets.
CHAPTER 3

THE MINIMAL CASES

Any matrix with a tilde over it is a matrix of order \( p - 1 \).

§1 The Structure of the Centralizers \( Z(P) \)

The minimal matrices of order \( p \) are the form 1 matrices when \( n = p - 1 \) and the form 2 matrices when \( n = p \). All other matrices of order \( p \) are similar to one of these matrices extended by an identity matrix of appropriate size (see definition 2.1.8).

Remember our notation that, if \( \pi \) is a matrix of order \( p \), then \( P = \langle \pi \rangle \).

For \( n = p - 1 \) and \( \pi \) a matrix of order \( p \) (hence of form 1 since all matrices of order \( p \) are of form 1 when \( n = p - 1 \)), results about the structure of \( Z(P) \) are well-known. For convenience these results, in the form most useful to this dissertation, are summed up and proved in Appendix C.

Let \( n = p \) and \( \pi \) a matrix of form 2, i.e. \( \pi = \text{acr}[\pi, e_1] = \begin{pmatrix} \text{cnr}[\pi] & e_1 \\ 0 & 1 \end{pmatrix} \) with \( \text{cnr}[\pi] \) a matrix of order \( p \).
3.1.1 LEMMA. For $\pi$ a matrix of form 2 and dimension $p$

1) If $Z \in M_p(\mathbb{Q}_N)$ (matrices of dimension $p$ with $\mathbb{Q}_N$-entries) and $Z$ commutes with $\pi$, then $Z = \begin{pmatrix} Z_{p-1} & b \\ c & d \end{pmatrix}$ for some matrix $Z_{p-1} \in M_{p-1}(\mathbb{Q}_N)$ that commutes with $\text{cnr}[\pi]$ and some $b$ uniquely determined by $Z_{p-1}$, $\text{cnr}[\pi]$, and $d$. Either $Z_{p-1} \in \Gamma_{p-1}(\mathbb{Q})$ or $Z_{p-1} = 0$.

2) Furthermore $Z$ can be written as $Z = f(\pi) + r\Phi(\pi)$ for some unique polynomial $f$ with $\mathbb{Q}$-coefficients and degree less than $p-1$ and for some unique $r \in \mathbb{Q}$. This unique polynomial has the property that $f$ depends only on $Z_{p-1}$, the coefficients of $f$ are all in $\mathbb{Q}_N$, and $r \in \mathbb{Q}_N$. The bottom right corner $d$ of $Z$ obeys $d = f(1) + r\Phi(1)$.

3) If $Z \in \Gamma_p(\mathbb{Q}_p\mathbb{N})$, i.e. $Z \in M_p(\mathbb{Q}_p\mathbb{N})$ and $\det Z \in \mathbb{Q}_p^\times$, then $f(1) \not\equiv 0 \mod p$ in $\mathbb{Q}_p$. Hence also $f(1) + r\Phi(1) \not\equiv 0 \mod p$ in $\mathbb{Q}_p$.

PROOF. 1) Suppose $\begin{pmatrix} Z_{p-1} & b \\ c & d \end{pmatrix} \in M_p(\mathbb{Q}_N)$ and commutes with $\pi$. Then

\[
\begin{pmatrix} Z_{p-1} & b \\ c & d \end{pmatrix} \begin{pmatrix} \text{cnr}[\pi] & e_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \text{cnr}[\pi] & e_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Z_{p-1} & b \\ c & d \end{pmatrix} \Rightarrow
\]

$Z_{p-1}\text{cnr}[\pi] = \text{cnr}[\pi]Z_{p-1} + e_1c$,

$(Z_{p-1} - dI_{p-1})e_1 = (\text{cnr}[\pi] - I_{p-1})b$,

$c\text{cnr}[\pi] = c$,

and $ce_1 + d = d$. 

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Any \((p-1)\)-dim matrix of order \(p\) has as its eigenvalues exactly the primitive \(p\)-th roots of unity. Since \(\text{cnr}[\pi]\) does not have \(1\) as an eigenvalue, we get that 
\[ c \cdot \text{cnr}[\pi] = c \implies c = 0. \]
Then \(Z_{p-1} \cdot \text{cnr}[\pi] = \text{cnr}[\pi] Z_{p-1}\), so \(Z_{p-1}\) commutes with \(\text{cnr}[\pi]\). By Appendix C, either \(Z_{p-1} = 0\) or \(Z_{p-1} \in \Gamma_{p-1}(\mathbb{Q})\).

Since \(\text{cnr}[\pi]\) has characteristic polynomial \(\Phi(x)\), then 
\[ \text{det}(\text{cnr}[\pi] - I_{p-1}) = p. \]
In particular \(\text{cnr}[\pi] - I_{p-1}\) is nonsingular. So 
\[ (Z_{p-1} - dI_{p-1})e_1 = (\text{cnr}[\pi] - I_{p-1})b \implies \]
\(b\) must be defined by 
\[ b = (\text{cnr}[\pi] - I_{p-1})^{-1}(Z_{p-1} - dI_{p-1})e_1. \]

2) So \(Z = \begin{pmatrix} Z_{p-1} & b \\ 0 & d \end{pmatrix} \in Z_{M_p(\mathbb{Q})}(P).\)

Since \(Z_{p-1} \in Z_{\Gamma_p(\mathbb{Q})}(\text{cnr}[\pi])\), by Appendix C there is a unique polynomial \(f \in \mathbb{Q}[x]\) of degree less than \(p-1\) such that \(f(\text{cnr}[\pi]) = Z_{p-1}\). Also by Appendix C, \(f \in \mathbb{Q}_N[x]\).

Notice that \(\Phi(\pi) = \begin{pmatrix} 0_{p-1} & v_\Phi \\ 0 & p \end{pmatrix}\) for some vector \(v_\Phi \in Z^{p-1}\) (Specifically \(v_\Phi\) is 
\[ \text{cnr}[\pi]^{p-2}e_1 + 2\text{cnr}[\pi]^{p-3}e_1 + 3\text{cnr}[\pi]^{p-4}e_1 + \cdots + (p-1)e_1. \]

Since \(f(\pi) + (d - f(1))p^{-1}\Phi(\pi)\) has upper left corner \(Z_{p-1}\) and bottom right corner \(d\), it must be equal to \(Z\) by part (1). This polynomial is unique because, first, any other choice of \(f\) will give a different upper left corner and, second, the coefficient of \(\Phi(\pi)\) is unique since it is determined by \(f(1)\) and \(d\).
Notice $d = f(1) + r \Phi(1)$ is trivially true since $\pi$ is $(p-1, 1)$-upper block triangular with lower right corner equal to 1.

If $p \nmid N$, then $r = (d - f(1)p^{-1}) \in \mathbb{Q}_N$ and part (2) is done.

If $p \mid N$, we need to prove that $r = (d - f(1)p^{-1}) \in \mathbb{Q}_N$. The only part to prove is that the denominator of $r$ is prime to $p$. Let us examine $v_\Phi$ more carefully. We claim that not all the entries of $v_\Phi$ are divisible by $p$. Notice $\pi \Phi(\pi) = \Phi(\pi)\pi \Rightarrow$

$$\begin{pmatrix} \text{cnr}[\pi] & e_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0_{p-1} & v_\Phi \\ 0 & p \end{pmatrix} = \begin{pmatrix} 0_{p-1} & v_\Phi \\ 0 & p \end{pmatrix} \begin{pmatrix} \text{cnr}[\pi] & e_1 \\ 0 & 1 \end{pmatrix} \Rightarrow$$

the vector $u = \begin{pmatrix} v_\Phi \\ p \end{pmatrix}$ is $\pi$-invariant. If all the entries in $v_\Phi$ are divisible by $p$, then $(1/p)u \in \mathbb{Z}_p$ is a $\pi$-invariant vector with $p$-th coefficient equal to 1. Then $\{e_1, e_2, \ldots, e_{p-1}, (1/p)u\}$ is an integral basis of $\mathbb{Z}_p$, contradicting that $\pi$ is of form 2.

So some entry of $v_\Phi$ is prime to $p$, say the $j$-th entry. So the $(j, p)$-th entry of $\Phi(\pi)$ is prime to $p$. Therefore, if $Z = f(\pi) + r \Phi(\pi)$ has all denominators prime to $p$, then it must be that the denominator of $r$ is prime to $p$.

3) Since $f(\text{cnr}[\pi]) = Z_{p-1}$, this is just Claim 2 of Appendix C.

3.1.2 Corollary. For $\pi$ a matrix of form 2 and dimension $p$,

$$Z_{\Gamma_p(\mathbb{Q}_N)}(P) \cong \{g(x) \in \mathbb{Q}_N(x)/(x^p-1) \mid \det g(\pi) \in \mathbb{Q}_N^\times \} \text{ by } g(\pi) \mapsto g(x).$$

$$Z_{\Gamma_p}(P) \cong \{g(x) \in \mathbb{Z}(x)/(x^p-1) \mid \det g(\pi) = \pm 1 \} \text{ by } g(\pi) \mapsto g(x).$$

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Proof. Obviously any polynomial in $\pi$ commutes with $\pi$. The respective determinant conditions are the necessary and sufficient conditions for such polynomials to be in $\Gamma_p(\mathbb{Q}_N)$ or $\Gamma_p$ respectively. Since $\pi^p - 1 = 0$, we get a map

$$\{g \in \mathbb{Q}_N[x]/(x^p - 1) \mid \det g(\pi) \in \mathbb{Q}_N^\times \} \to Z_{\Gamma_p}(\mathbb{Q}_N)(P).$$

So we just need to prove that every matrix has a unique polynomial mod $x^p - 1$ associated with it. This is immediate from Lemma 3.1.1 above.

For the second isomorphism we need to show that the unique mod $x^p - 1$ polynomial $g(x) = f(x) + r\Phi(x)$ associated to $Z$ is integral. This is true by part (2) of Lemma 3.1.1 above.

Recall that $Z_{\Gamma_p}(P)$ acts trivially on $\hat{H}^\bullet(Z_{\Gamma_p}(P), M)$. What about the matrices in $Z_{\Gamma_p(\mathbb{Q}_N)}(P)$ that project onto $Z_{\Gamma_{p-1}}(\text{cnr}[\pi])$, i.e. matrices with upper left corners in $Z_{\Gamma_{p-1}}(\text{cnr}(\pi))$? How do they act?

3.1.3 Claim. For any positive integer $N$ and any $d \in \mathbb{Q}_N^\times$, there exists a matrix $D$ in $Z_{\Gamma_p(\mathbb{Q}_N)}(P)$ with lower right corner equal to $d$ and $\text{cnr}[D] \in \Gamma_p$.

For $d \in \mathbb{Z}$, then $D$ can be chosen to be in $M_p(\mathbb{Z})$.

Proof. To prove such matrices exist, we construct them. Note that any rational matrix that commutes with $\pi$ satisfies (1) of Lemma 3.1.1 with $N = 0$ (so $\mathbb{Q}_N = \mathbb{Q}$).
Consider the polynomials $1, 1 + x, 1 + x + x^2, \ldots, 1 + x + x^2 + \cdots + x^{p-2}$. These polynomials are cyclotomic units and well-known to be invertible in $\mathbb{Z}[x]/\Phi(x)$. For example, here is an inductive proof of invertibility:

Certainly 1 itself is trivially a unit in the field $\mathbb{Z}[x]/\Phi(x)$. Since $\Phi(x)/(1 + x)$ has remainder 1, there exists $g(x)$ and $r$ such that $(1 + x)f(x) = -1 + r\Phi(x)$. So $1 + x$ is a unit. Since $\Phi(x)/(1 + x + x^2)$ has remainder 1 or $1 + x$ and both these possible remainders are units, $1 + x + x^2$ is a unit. For $1 + x + x^2 + x^3$, the remainders are 1 or $1 + x$ or $1 + x + x^2$, all units. Continuing this argument, all the polynomials are units.

So by Appendix C, $1, 1 + \pi, 1 + \pi + \pi^2, \ldots, 1 + \pi + \pi^2 + \cdots + \pi^{p-2}$ all have upper left corner in $\mathbb{Z}_{\Gamma_{p-1}}(\text{cnr}[\pi])$. Since the bottom right corner of any matrix $f(\pi)$ is $f(1)$, the bottom right corners of these matrices range from 1 to $p - 1$. Adding $\Phi(x)$ to any of these polynomials increases the bottom right corner by $p$ and leaves the upper right corner unchanged. So, by adding integral multiples of $\Phi(x)$, we get integral matrices $M(d)$ with upper left corner in $\mathbb{Z}_{\Gamma_{p-1}}(\text{cnr}[\pi])$ and bottom right corner any non-zero integer $d$. The determinant of $M(d)$ is $\pm d$.

Now to get any rational number $r/s \in \mathbb{Q}$; $r, s \in \mathbb{Z}$, just define $M(r/s) = M(r)M(s)^{-1}$. If $r/s \in \mathbb{Q}_N^\times$, $M(r/s) \in \Gamma(\mathbb{Q}_N)$. ■
3.1.4 Claim. Let \( pr : Z_{\Gamma_p(Q_{pN})}(P) \rightarrow Z_{\Gamma_{p^{-1}}(Q_N)}(\text{cnr}(\pi)) \) be the projection map onto the upper left corner, a surjective homomorphism by 3.1.1.

Then the action of \( pr^{-1}(Z_{\Gamma_{p^{-1}}(\text{cnr}[\pi])}) \) on \( \check{H}^*(Z_{\Gamma_p}(P), M) \) factors through \( (\mathbb{Z}/pN)^\times \cong (\mathbb{Q}_{pN}/pN)^\times \) via the surjective homomorphism \( \begin{pmatrix} Z_{p^{-1}} & b \\ 0 & d \end{pmatrix} \rightarrow d \mod pN \) in \( \mathbb{Q}_{pN} \).

Proof. Since matrices in \( Z_{\Gamma_p(Q_{pN})}(P) \) commute with \( Z_{\Gamma_p}(P) \) (because they are all polynomials in \( \pi \)), their conjugation actions on \( \check{H}^*(Z_{\Gamma_p}(P), M) \) depend solely on how they act on \( M \) (because in terms of resolutions, conjugation acts trivially on all chains). So we need to prove that if two matrices \( Z \) and \( Z_1 \) have the same lower right corner \( \mod pN \) in \( \mathbb{Q}_{pN} \), then \( ZZ_1^{-1} \) acts trivially on \( M \), hence trivially on \( \check{H}^*(Z_{\Gamma_p}(P), M) \). Let \( Z = \begin{pmatrix} \bar{Z} & b \\ 0 & d \end{pmatrix} \) and \( Z_1 = \begin{pmatrix} \bar{Z}_1 & b_1 \\ 0 & d_1 \end{pmatrix} \) with \( \bar{Z}, \bar{Z}_1 \in Z_{\Gamma_{p^{-1}}(\text{cnr}[\pi])} \) and \( d \equiv d_1 \mod pN \). Then \( ZZ_1^{-1} \) has bottom right corner \( d/d_1 \equiv 1 \mod pN \). Let \( d/d_1 = 1+mpN, m \in \mathbb{Q}_{pN} \). Define \( Y = ZZ_1^{-1} \Phi(\pi) \), a matrix that commutes with \( \pi \). Then \( \text{cnr}[Y] = \bar{Z} \bar{Z}_1^{-1} \) is given by an integral polynomial in \( \text{cnr}[\pi] \) (since that is true for both \( \bar{Z} \) and \( \bar{Z}_1^{-1} \) by Appendix C) and the bottom right corner of \( Y \) is 1 so by 3.1.1(2) all of \( Y \) is integral and so, by 3.1.2, \( Y \in Z_{\Gamma_p}(P) \). Since \( M \) is admissible, the action on \( M \) factors \( \mod N \) so \( ZZ_1^{-1} \) acts the same way as \( Y \), i.e. trivially.
The map to $d$ is a homomorphism since the matrices are block triangular. It is surjective since the matrices in 3.1.3 are in $\mathbb{Z}_{\Gamma Q_{pN}}(\pi)$ for $d \in (\mathbb{Z}/pN)\times$.

§2. Semisimple Type: Review of Galois Representations

From formula 2.3.3, we see that each Hecke generator acts on $\hat{H}^*(\Gamma, M)$ as the sum of the elements in $D'(P)$ acting individually. From §4 of Chapter 2, we see that $D'(P)$ and other related sets of coset representatives correspond to subspaces. Hence the approach we take is to think of each Hecke element as actually being a sum of subspaces. The subspaces involved are, for $s = T_n(l, k)$, the full $k$-dimensional $\pi$-invariant subspaces of $(\mathbb{Z}/l)^n$.

In Section 6 of [A1], working with the case $n = p-1$, Ash turned these invariant subspaces into a ring $R$: Noticing that the subspaces corresponded to the elements of the ray class group $E$ of $Q(\zeta)$ of conductor $N$, he set $R = F[E]$. Then he found a Galois representation over $R$ attached to $Hecke(\Gamma, Q_{pN})$ (in the sense that for $l$ prime to $pN$ the characteristic polynomial of $Frob_l$ equals the Hecke polynomial of $R$ at $l$) and showed that, for any finite dimensional $R$-module $M$, the Galois representation over $R$ induces Galois representations over $F$ that are continuous, semisimple, unramified outside of $pN$, and attached to the Hecke eigenvectors of $M$. He then showed that, when $M$ is an admissible module, the cohomology
group $\hat{H}_1^*(\Gamma_{p-1}, M) = \hat{H}^*(\Gamma_{p-1}, M)$ is an $R$-module. Thus he proved that Galois representations over $\mathbb{F}$ exist for all the Hecke eigenvectors in $\hat{H}_1^*(\Gamma_{p-1}, M)$.

We are going to modify this proof in order to get Galois representations for $\hat{H}_2^*(\Gamma_n, M)$, $n = p$. It is because the proof for $\hat{H}^*(\Gamma_{p-1}, M) = \hat{H}_1^*(\Gamma_{p-1}, M)$ can be modified for $\hat{H}_2^*(\Gamma_p, M)$ that we have grouped $\hat{H}_1^*(\Gamma_{p-1}, M)$ and $\hat{H}_2^*(\Gamma_p, M)$ together under the heading "minimal cases". The difference between the $\hat{H}_2^*(\Gamma_p, M)$ case and the case $\hat{H}_1^*(\Gamma_{p-1}, M)$ that Ash treated is that dimension $p$ has more subspaces than dimension $(p-1)$ so $R$ has to be enlarged to hold the extra spaces.

We now summarize the results from Section 6 of [A1] that we will use in §3. We will change his notation as necessary so it matches our notation.

3.2.1 Definitions.

$K = \mathbb{Q}(\zeta)$. By "an ideal in $K$" we mean an ideal in the ring of algebraic integers of $K$. Remember that we denote the class group of $K$ by $C(\zeta)$, and, for an ideal $J$, its image in the class group is denoted by double brackets $[[J]]$.

$K_N^\times = \{ u \in K \mid u$ is a unit at all the valuations dividing $N \}$. The elements in $K_N^\times$ are exactly polynomials in $\zeta$ such that the polynomials and the inverses of the polynomials both have coefficients in $\mathbb{Q}_N[x]$. 

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$E$ = the class group of ideals in $K$ prime to $N$ modulo the group of principal ideals $(y)$ with $y \equiv 1 \mod N$. The image of an ideal $J$ in $E$ will be denoted by single brackets $[J]$.

$R = \mathbb{F}[E]$.

d = the degree of a prime ideal in $K$ above $l$.

$L(t) =$ the set of squarefree products of exactly $t$ prime ideals in $K$ above $l$. For some values of $t$, such as any value $t > p-1 = [\mathbb{Q}(\zeta) : \mathbb{Q}]$, then $L(t)$ will be the empty set. Naturally $L(t)$ is defined for all positive integers $t$. Extend this definition to all $t \in \mathbb{Q}$ by defining $L(t)$ to be the empty set for all the other numbers. In the succeeding we use the convention that a sum over an empty set is defined to be zero.

Recall from 2.1.7 that, for each $I \in C(\zeta)$, then $\pi_I = \pi_{p-1}(I)$.

For each $I \in C(\zeta)$, fix matrices $\bar{A}_I \in \Gamma_{p-1}(\mathbb{Q}_{pN})$ that have the property $\pi_0 = \pi_I^{\bar{A}_I}$. The existence of such $\bar{A}_I$ is well-known, for example it follows from the fact that algebraic integers with norm prime to $pN$ are units in $\mathbb{Q}_{pN}[\zeta]$. We say nothing more about $\bar{A}_I$ now, but actually we want $\bar{A}_I$ to satisfy stricter conditions than merely being a $\mathbb{Q}_{pN}$ matrix that conjugates $\pi_I$ into $\pi_0$. We will return to $\bar{A}_I$ in 3.3.13.
\( \psi : \text{Hecke}(pN) \rightarrow R \) is the homomorphism

\[
T(l, k) \mapsto \sum_{\lambda \in \mathcal{L}(k/d)} [\lambda]
\]

Remember \( \hat{H}^*(\Gamma_{p-1}, M) = \hat{H}^1(\Gamma_{p-1}, M) \) since in dimension \( p-1 \) there are no form 2 elements. So 2.1.10 gives

\[
\hat{H}^*(\Gamma_{p-1}, M) \cong \bigoplus_{I \in \mathcal{C}(\zeta)} \hat{H}^i(N_{\Gamma_{p-1}}(\pi_I), M)
\]

Now define

\[ F : \bigoplus_{I \in \mathcal{C}(\zeta)} \hat{H}^i(Z(\pi_I), M) \cong \hat{H}^i(Z(\pi_0), M) \otimes \mathbb{F}[C(\zeta)] \]

by \( (0, \ldots, \tilde{A}^{-1}_I \alpha, \ldots, 0) \mapsto \alpha \otimes I \), where the expression appears in the \( I \)-th place.

Write \( \alpha_I \) for the \( I \)-th component of \( \alpha \) in \( \hat{H}^i(Z(\pi_0), M) \otimes \mathbb{F}[C(\zeta)] \).

3.2.2 Result. In Theorem 6.1.2 of [A1], Ash proves the existence of \( \rho \), a \( p-1 \) dimensional representation from \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) to \( R^X \) such that

\[
det(I - \rho(Frob_l)^{-1} X) = \sum_{k=0, \ldots, p-1} (-1)^k \frac{k^2}{2} \sum_{\lambda \in \mathcal{L}(k/d)} [\lambda] X^k
\]

Consider finite dimensional \( R \)-modules to be \( \text{Hecke}(pN) \) modules via the map \( \psi \). In Theorem 6.1.4 of [A1], Ash uses Theorem 6.1.2 of [A1] to prove that any Hecke eigenvector has an attached continuous, semisimple, and unramified outside of \( pN \) Galois representation.

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3.2.3 Result. Under the isomorphism $F$, Ash proves that the Hecke action, for an operator $s_{p^{-1}}(i) = T_{p^{-1}}(l, i)$ is

$$F[T_{s_{p^{-1}}(i)}(\alpha)]_l = \sum_{g_{p^{-1}} \in D'_{p^{-1}(i), \tilde{\pi}_I}} (\tilde{A}_I^I g_{p^{-1}} s_{p^{-1}}(i)^* \tilde{h}^* \tilde{A}_J^{-1*})(F(\alpha)_J).$$

where, for each $g_{p^{-1}}$, $[[J]]$ is the unique class in $C(\zeta)$ with the property that

$$\tilde{\pi}_I^{(s_{p^{-1}}(i) g_{p^{-1}})^{-1}} \sim_{\Gamma_{p^{-1}}} \tilde{\pi}_I$$

and $\tilde{h}$ is any matrix in $\Gamma_{p^{-1}}$ such that $\tilde{\pi}_I = \tilde{h} s_{p^{-1}}(i) g_{p^{-1}}$.

Remembering from 2.4.1 that $D'_{p^{-1}(i), \tilde{\pi}_I}$ is a set of coset representatives of $(\Gamma_{p^{-1}} \cap \Gamma_{p^{-1}(k)}) \backslash \Gamma_{p^{-1}} / \Gamma_{p^{-1}(\pi_I)}$ such that $\pi_I \subset \Gamma_{p^{-1}}^{s_{p^{-1}}(i)}$, we observe that $[[J]]$ and so $\tilde{h}$ exist for all $g_{p^{-1}}$.

Notice that $g_{p^{-1}}$ depends on $i, \tilde{\pi}_I$, and $\tilde{\pi}_J$; and $\tilde{h}$, which is not uniquely defined, depends on $g_{p^{-1}}, i, \tilde{\pi}_I$, and $\tilde{\pi}_J$.

3.2.4 Result. In 6.4.3 of [A1], Ash turns $\hat{H}^*(Z(\pi_0), M) \otimes_F F[C(\zeta)]$ into an $R$-module by defining:

For $v \in \hat{H}^*(Z(\pi_0), M) \otimes_F F[C(\zeta)]$ and $\lambda \in E$, then

$$(\lambda \cdot v)_I = y^* v_J$$

where $J$ is defined by $[[J]] = [[I^{-1}]]$, and $y$ is any matrix in $Z_{\Gamma_{Q_N}}(\pi_0)$ such that the algebraic number $\bar{y} \in K_N^X$ associated to $y$ obeys $(\bar{y}) = I^{-1} J \lambda$.

Extend by linearity to all of $R$. 

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Since the cohomology involved is the cohomology of $Z(\tilde{\pi}_0)$, to find the algebraic number $\tilde{y}$ associated to $y$ first write $y$ as a polynomial in $\tilde{\pi}_0$, for example $y = f(\tilde{\pi}_0)$, and define $\tilde{y} = f(\zeta)$.

Now by 3.2.2, with the Hecke action induced by the map $\psi$, every Hecke eigenvector of $\hat{H}^*(Z(\tilde{\pi}_0), M) \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)]$ has an attached Galois representation.

With the Hecke action on $\hat{H}^*(Z(\tilde{\pi}_0), M) \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)]$ induced by the map $\psi$, $F$ is Hecke equivariant. This is true because invariant subspaces (associated to the Hecke action on $\hat{H}^*(\Gamma_{p-1}, M)$ because of the sets $D'(P)$) correspond to square-free ideals (associated to the Hecke action on $\hat{H}^*(Z(\tilde{\pi}_0), M) \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)]$ because of the sets $\Lambda(k/d)$ used in $\psi$) and because, if we set $y = \tilde{A}^{-1}_f h s_{p-1} g_{p-1} \tilde{A}_f$, then the algebraic number $\tilde{y}$ associated to $y$ obeys $(\tilde{y}) = I^{-1} J \lambda$, where $\lambda$ is the square-free ideal that corresponds to the same $\tilde{\pi}_f$-invariant subspace that $g_{p-1}$ corresponds to.

Since $F$ is Hecke-invariant and injective, any Hecke eigenvector in $\hat{H}^*(\Gamma_{p-1}, M)$ has an attached Galois representation.

This concludes our summary of [A1]. The difference between our notation here and the notation in [A1] is that we use $C(\zeta)$ instead of $C$, $\tilde{A}_f$ instead of $A_f$, $\tilde{\pi}_I$ instead of $\pi_I$, $Z(\tilde{\pi}_I)$ instead of $Z_I$, $g_{p-1}$ instead of $g$, $s_{p-1}(k)$ instead of $s$, $\tilde{h}$ instead of $h$, $\mathcal{L}(t)$ instead of $L(t)$, and $D'_{p-1}(k, \tilde{\pi}_I)$ instead of $D'$. 66
For later use, we would like to create an $R$-module that is just a slight variation of one used in 3.2.4:

**3.2.5 Proposition.** Let

$$
\Upsilon = \Upsilon_{p-1}(\tilde{\pi}_0) = \{ z \in \mathbb{Z}(\tilde{\pi}_0) \mid z = f(\tilde{\pi}_0) \text{ with } f(1) = 1 \}
$$

Define an $R$-action on $\hat{H}^*(\Upsilon, M) \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)]$ by:

For $u \in \hat{H}^*(\Upsilon, M) \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)]$ and $\lambda \in E$, then

$$(\lambda \cdot u)_J = y^* v_J$$

where $J$ is defined by $[[J]] = [[I\lambda^{-1}]]$, and $y$ is any matrix in $\mathbb{Z}_{(\mathbb{Q})}((\tilde{\pi}_0))$ such that the algebraic number $\tilde{y} \in K_N^*$ associated to $y$ obeys $(\tilde{y}) = I^{-1}J\lambda$ and $y$ can be written $y = f(\tilde{\pi}_0)$ with $f(1) = 1$.

Then for the Hecke action on $\hat{H}^*(\Upsilon, M) \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)]$ induced by pulling back the map $\psi$, any Hecke eigenvector has an attached continuous, semisimple, and unramified outside of $pN$ Galois representation.

Moreover the natural restriction map

$$
\hat{H}^*(Z(\tilde{\pi}_0), M) \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)] \hookrightarrow \hat{H}^*(\Upsilon, M) \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)]
$$

is an $R$-map.
\textbf{Proof.}

The existence of Galois representations is a simple corollary of 3.2.2. The restriction maps are obviously $R$-equivariant since the only difference between this new $R$-action and the $R$-action on $\tilde{H}^* (Z(\tilde{\pi}_0), M) \otimes_{\mathbb{F}} \mathbb{F} [C(\zeta)]$ in 3.2.4 is the additional requirement on $y$ that it can be written $y = f(\tilde{\pi}_0)$ with $f(1) = 1$.

The only thing that needs to be proved is that this $R$-action is in fact well-defined. We must prove:

i) $y = f(\tilde{\pi}_0)$ with $f(1) = 1$ exists.

ii) $y^*$ acts trivially when $y$ comes from an ideal $\lambda$ that is principal and congruent to $1 \mod N$.

iii) if $y$ and $z$ are both generators of $I^{-1} J \lambda$ (so both come from the ideal $\lambda$), then $y^*$ and $z^*$ are equal on $H^i (H, M)$.

For (i), suppose we have found $y$ obeying $(\tilde{y}) = I^{-1} J \lambda$. Since $\lambda$ is prime to $pN$, not just $N$, then $\tilde{y} \in K_{pN}^\times$. So by Claim 1 of Appendix C, then $y = f(\tilde{\pi}_0)$ with $f \in \mathbb{Q}_{pN}[x]$ and by Claim 2 of Appendix C then $f(1) \not\equiv 0 \mod p$. If $f(1) \not\equiv 1 \mod p$, pick a matrix $\text{cnr}[D] = g(\tilde{\pi}_0) \in Z_{\Gamma_{p^{-1}}}(\tilde{\pi}_0)$ from 3.1.3 such that $g(1)f(1) \equiv 1 \mod p$ in $\mathbb{Q}_{pN}$. Replace $y$ with $\text{cnr}[D]y$. Since $\text{cnr}[D]$ corresponds to an algebraic unit, $\tilde{y}$ also corresponds to a generator of $I^{-1} J \lambda$. Now let $h(\tilde{\pi}_0) = y$. If $h(1) \neq 1$, then add $\mathbb{Q}_{pN}$-multiples of $\Phi(x)$ until it is 1. This
changes $h$ but not $y$ since $\Phi(\pi_0) = 0$. So it is always possible to choose $y$ with $f(1) = 1$.

ii) If $\bar{y} \cong 1 \mod N$, then $\bar{y} = 1 + Ng(\zeta)$ with $g(\zeta) \in \mathbb{Q}_N[x]$. So $y = 1 + Ng(\pi_0)$. Since $M$ is admissible, then matrices act through their reduction mod $N$ and so $y$ acts on $M$ as the identity.

iii) If $y$ and $z$ are both associated with $\lambda$, then $yz^{-1} \in \Upsilon$ and so acts trivially on $\hat{H}^*(\Upsilon, M)$.

§3 Non-Semisimple Type: Galois Representations

In this section $n = p$. As always, $\Gamma = GL_n(\mathbb{Z})$. Let $P = \langle \pi \rangle$ be a subgroup of order $p$ and of form 2 and $s = T_n(l, k)$ for some $k$ and some prime $l \neq p$. Let $\alpha \in \hat{H}^2(\Gamma, M)$.

First we are going to give the correspondence between $T_s$ and subspaces. Then we will give the enlargement of $R$.

Recall the definitions of $D(P)$ and $D'(P)$ from 2.1.15, the improved definition of $\tilde{D}'(P)$ from 2.4.7, and that $D'(P) \subset \tilde{D}'(P)$.

Recall from 2.3.3 that

$$T_s(\alpha)|Z(P) = \sum_{g \in D'(P)} Tr^Z_{Z(P) \cap \Gamma_s} Res^\Gamma_{Z(P) \cap \Gamma_s}(sg)^* \alpha.$$
We comment that if an integral matrix, say $z$, has the property that its upper right $(n-k)$-by-$k$ corner $\equiv 0 \mod l$, then $f(z)$, for any integral polynomial $f$, also has this property. It happens that:

i) $g\pi g^{-1} \subset s^{-1}\Gamma s$ by definition of $g \in D'(P)$,

ii) as mentioned in 2.4.4, $\Gamma^s \cap \Gamma = \text{all matrices in } \Gamma \text{ with upper right } (n-k)\text{-by-}k$ corner $\equiv 0 \mod l$, and

iii) all elements in $Z(P)$ are integral polynomials in $\pi$ (by 3.1.2) (Hence elements in $gZ(P)g^{-1}$ are integral polynomials in $g\pi g^{-1}$).

Therefore $gZ(P)g^{-1} \subset s^{-1}\Gamma s$. So the transfer map vanishes and the formula for the Hecke action is

$$T_s(\alpha)|Z(P) = \sum_{g \in D'(P)} \text{Res}^{Z(P)}_{D'(P)} (sg)^* \alpha \quad (3.3.1)$$

In the non-minimal case there will be a problem with associating $T_s$ directly with subspaces because $D'(P)$ will not correspond with subspaces. It is the cosets $(\Gamma^s \cap \Gamma)/\Gamma$ that correspond to subspaces, not the double cosets $(\Gamma^s \cap \Gamma)/\Gamma \setminus Z(P)$. That is why $\widehat{D}'(P)$, a set of single cosets, was invented and the results in §4 of Chapter 2 are awkwardly stated in terms of the ugly $\widehat{D}'(P)$ instead of the desirable $D'(P)$.

Happily, in this minimal case of form 2 and $n = p$ (also the case in [A1] of form 1 and $n = p-1$), there are no such problems. Let $g \in \Gamma$ such that $gPg^{-1} \subset s^{-1}\Gamma s$
and let \( z \in Z(P) \). Since \( Z(P) \subset \Gamma g \) then \( gz = (gzg^{-1})g \subset (s^{-1} \Gamma s)g \). So \( \Gamma g \cap \Gamma g = \Gamma g \). So \( D'(P) \) can be considered to be a subset of a set of coset representatives of \( \Gamma(s) \backslash \Gamma \). In other words, \( D'(P) = \hat{D}'(P) \).

Recall from altered definition 2.4.10 that, for \( s = T_p(l, k) \) and \( P \) of form 2.

\[
\hat{D}'(P) = \bigcup_{i=0}^{k} \left \{ \sigma(i)g'L \mid g' = (g_{p-1} + g_m) \text{ with } g_{p-1} \in \hat{D}'_{p-1}(i, \text{cnn}[\pi]) \right \}
\]

and \( g_m \in \hat{D}_m(j) \). where \( i + j = k \), \( \sigma(i) \) depends on both \( i \) and \( k \), and \( L \) depends on \( l \) and \( \pi \).

Since \( m = 1 \) in this case, \( j \) can only be either 0 or 1. Recall from its definition in 2.1.13 that \( s_1(0) = (1) \) and \( s_1(1) = (l) \). So \( \hat{D}_1(0) = 1 \) and \( \hat{D}_1(1) = 1 \) since, for \( j = 0 \) or 1, \( (\Gamma_1^{p-1(j)} \cap \Gamma_1) = \Gamma_1 \). Note also that \( \sigma(k-1) = I_p \). So

\[
D'(P) = \hat{D}'(P)
\]

\[
= \left \{ \sigma(k)g'L \mid g' = (g_{p-1} + 1) \text{ with } g_{p-1} \in \hat{D}'_{p-1}(i, \text{cnn}[\pi]) \right \}
\]

and \( g_m \in \hat{D}_m(j) \). (3.3.2)

Using the notation of §4 of chapter 2, the elements \( g \in D'(P) \) correspond to \( \pi \)-invariant \( k \)-dimensional subspaces of \((\mathbb{Z}/l)^p\) by \( g \leftrightarrow g^{-1}V_p(k) \).

3.3.3 Definitions.

\[
C = (\mathbb{Z}/pN) \times .
\]

\[
E' = E \times C.
\]
\[ R' = \mathbb{F}[E']. \]

As \( E \) and \( C \) are canonically embedded in \( E' \), we consider all elements of \( E \) and \( C \) to be elements in \( E' \). For any integer \( m \) prime to \( pN \), let \( c(m) \in C \) be reduction mod \( Np \).

3.3.4 Definition. The homomorphism \( \psi' : \text{Hecke}(N) \to R' \) is

\[
T(l, k) \mapsto \sum_{\Lambda \in \mathcal{L}(k/d)} [\Lambda] + \sum_{\Lambda \in \mathcal{L}(k-1/d)} c(l)[\Lambda]
\]

The Hecke polynomial attached to \( \psi' \) at \( l \) is

\[
\sum_{k=0}^{p} (-1)^k l^{k(k-1)/2} \psi'(T(l,k)) X^k =
\]

\[
\sum_{k=0}^{p} (-1)^k l^{k(k-1)/2} \left( \sum_{\Lambda \in \mathcal{L}(k/d)} [\Lambda] + \sum_{\Lambda \in \mathcal{L}(k-1/d)} c(l)[\Lambda] \right) X^k =
\]

\[
\sum_{k=0}^{p-1} (-1)^k l^{k(k-1)/2} \left( \sum_{\Lambda \in \mathcal{L}(k/d)} \Lambda X^k \right) \left[ 1 - l^k c(l)X \right] = \quad (3.3.5)
\]

\[
\sum_{k=0}^{p-1} (-1)^k l^{k(k-1)/2} \left( \sum_{\Lambda \in \mathcal{L}(k/d)} \Lambda X^k \right) \left[ 1 - c(l)X \right] = \quad (3.3.6)
\]

Statement 3.3.5 follows because \( \mathcal{L}(-1/d) \) and \( \mathcal{L}(p/d) \) are both empty (it must be that \( d \leq p-1 \); \( d \) cannot be 1 because \( p \) is the only prime that splits completely in \( Q(\zeta) \) and \( l \neq p \)).

Statement 3.3.6 follows because first \( l^d \equiv 1 \mod p \) and because second \( \mathcal{L}(k/d) \neq 0 \Rightarrow d \mid k \).
3.3.7 Definition. \( \tau : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{Gal}(\mathbb{Q}(e^{2\pi i/pN})/\mathbb{Q}) \sim C \) is the standard 1-dim representation sending \( \text{Frob}_l^{-1} \mapsto (l \mod Np) = c(l) \). So

\[
det(1 - \tau(\text{Frob}_l)^{-1}X) = 1 - c(l)X.
\]

Define \( \rho' = \rho \oplus \tau : G \to \text{Gl}(p, \mathbb{R}') \), where \( \rho \) is the same \( \rho \) as in 3.2.2. Then \( \rho' \) is a \( p \)-dimensional representation with, since determinants multiply in direct sums,

\[
det(I_p - \rho'(\text{Frob}_l)^{-1}X) = \text{the Hecke polynomial 3.3.6 attached to } \psi' \text{ at } l.
\]

3.3.8 Result. Lemma 6.1.3 and Theorem 6.1.4 of [A1] follow immediately with \( R', E', \) and \( \phi' \) in place of \( R, E, \) and \( \phi \). The content of these is that if a Hecke module is a finite dimensional \( R' \)-module (with the \( R' \) action extending the Hecke action) then, using \( \rho' \), any Hecke eigenvector has a continuous, semisimple Galois representation, unramified outside \( pN \).

The rest of this section is devoted to putting an \( R' \)-module structure on \( \hat{H}_2^*(\Gamma, M) \) so we can use 3.3.8 to immediately conclude the existence of Galois representations for Hecke eigenvectors.

Return to the definition of the Hecke action in 3.3.1 and the specification of \( D'(P) \) in 3.3.2.

Note that

\[
sg = s\sigma(i)g'L = \sigma(i)s^{\sigma(i)}g'L.
\]

(3.3.9)
So in particular

\[ g^* s^* \alpha = (s^{\sigma(i)} g' L)^* \sigma(i)^* \alpha = (s^{\sigma(i)} g' L)^* \alpha \]

(the last step since \( \sigma(i) \in \Gamma_p \) implies it acts trivially on \( \alpha \in H^*(\Gamma_p, \mathcal{M}) \))

and

\[ \Gamma^s g = \left( \sigma(i)^{-1} \Gamma \sigma(i) \right)^{s^{\sigma(i)} g' L} = \Gamma^{s^{\sigma(i)} g' L}. \]

In our particular example 3.3.2, \( m = 1 \) and \( i = k \) or \( k - 1 \). Note that \( \sigma(k - 1) = I_p \).

3.3.10 DEFINITION.

\[ d = s^{\sigma(k)} = \sigma(k)^{-1} s \sigma(k) = \begin{pmatrix} I_{p-1-k} & 0 & 0 \\ 0 & U_k & 0 \\ 0 & 0 & 1 \end{pmatrix} = s_{p-1}(k + 1). \]

Note \( s = s^{\sigma(k-1)} = \sigma(k-1)^{-1} s \sigma(k-1) = \begin{pmatrix} I_{p-k+1} & 0 & 0 \\ 0 & U_{k-1} & 0 \\ 0 & 0 & l \end{pmatrix} = s_{p-1}(k-1 + l). \)

Now the Hecke action 3.3.1 is

\[ T_s(\alpha)|_{Z(P)} = \sum_{g_{p-1} \in \bar{B}_{p-1}(k, \text{cnr}[\pi])} Res^{d_{g' L}}_{Z(P)} (dg' L)^* \alpha \]

\[ + \sum_{g_{p-1} \in \bar{B}_{p-1}(k-1, \text{cnr}[\pi])} Res^{g_{g' L}}_{Z(P)} (sg' L)^* \alpha \quad (3.3.11) \]

Recall from 2.1.11 that

\[ \hat{H}_2^*(\Gamma, \mathcal{M}) \cong \bigoplus_{P_j \in \mathcal{P}_p(2)} \hat{H}^*(N(P_j), \mathcal{M}) \leftrightarrow \bigoplus_{P_j \in \mathcal{P}_p(2)} \hat{H}^*(Z(P_j), \mathcal{M}) \]

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We would like to interpret Hecke as acting on this direct sum instead of on $\hat{H}^*_2(\Gamma, M)$. That is because in order to make $\hat{H}^*_2(\Gamma, M)$ into an $R$-module we would have to think of a connection between $\Gamma$ and $R$, and nothing has come to mind. On the other hand, $R'$ and $Z(P)$ are connected to each other since the generators of $R$ can be thought of as ideals in $\mathbb{Q}(\zeta)$ and the elements in $Z(P)$ can be thought of as elements in $\mathbb{Q}(\zeta)$ (by using the map $\pi \mapsto \zeta$). We can use their common connection to $\mathbb{Q}(\zeta)$ to connect them to each other.

The first problem, easily surmounted, with forcing Hecke to act on the direct sum is that in general $dg'L$ and $sg'L$ do not, for $P \in \mathcal{P}_p(2)$, conjugate $Z(P)$ into $Z(P')$ for some $P' \in \mathcal{P}_p(2)$. Instead, they conjugate $Z(P)$ to a conjugate of some $Z(P')$. So the Hecke action doesn't send $\hat{H}^*(Z(P), M)$ to other $\hat{H}^*(Z(P'), M)$ with $P' \in \mathcal{P}_p(2)$.

We will solve this problem the same way it was solved in [A1]. We need to define more objects analogous to those in §2.

3.3.12 Notation.

Recall from 2.1.9 that $\mathcal{P}_p(2) = \left\{ \pi_p(2, I) \right\}_{I \in C(\zeta)}$. For the rest of this section let

$$\pi_I = \pi(2, I) = \text{acr}[\pi_{p-1}(I), e_1] = \begin{pmatrix} \pi_{p-1}(I) & e_1 \\ 0 & 1 \end{pmatrix}.$$

$$\pi_0 = \text{acr}[\pi_{p-1}(0), e_1].$$

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Let $P_I = \langle \pi_I \rangle$ and $P_0 = \langle \pi_0 \rangle$.

3.3.13 Definition. All form 2 matrices of order $p$ are conjugate in $\Gamma_p(\mathbb{Q}_{pN})$ (the class group of $K_{pN}$ is 1). For each $[I] \in C(\zeta)$, define $A_I \in \Gamma_p(\mathbb{Q}_{pN})$ by

$$\pi_0 = A_I^{-1} \pi_I A_I$$

Since $A_I$ must preserve the unique $(p-1)$-dim subspace of $\mathbb{Z}^P$ that $\pi_I$ and $\pi_0$ preserve, $A_I$ is $(p-1,1)$-block upper triangular. Multiply $A_I$ by an appropriate scalar so the bottom right corner of $A_I$ is 1.

Note that $\text{cnr}[A_I]$ satisfies

$$\text{cnr}[\pi_0] = \text{cnr}[A_I]^{-1} \text{cnr}[\pi_I] \text{cnr}[A_I]$$

This is exactly the definition of $\tilde{A}_I$ from 3.2.1. From now on, instead of fixing $\tilde{A}_I$ to be any matrix that conjugates $\tilde{\pi}_I$ into $\tilde{\pi}_0$, first fix a set of matrices $A_I$ for all $I$ and then define $\tilde{A}_I = \text{cnr}[A_I]$.

3.3.14 Definition. Let $g \in D'(P)$. By the definition of $D'(P)$, then $(sg)P(sg)^{-1} \subseteq \Gamma$. By 3.3.2, either $g = \sigma(k)g'L$ or $g = g'L$. So, from 3.3.9, $(dg'L)P(dg'L)^{-1} \in \Gamma$ for the first type of $g$ and $(sg'L)P(sg'L)^{-1} \in \Gamma$ for the second type of $g$. Call these first type $g$ and second type $g$.

Let $P = P_I$. For each $g \in D'(P)$, define $J$ to be the ideal such that $(sg)P(sg)^{-1} \sim \Gamma P_J$ and choose an $h \in \Gamma$ such that

$$P = P_J^{h^dg'L}$$

for first type $g = \sigma(k)g'L$.
\[ P = P_j^{h'gL} \text{ for second type } g = g'L \]

Note that either \( A Jhhdg'LA_i^{-1} \) or \( A JhhdgL^{-1} \) respectively commutes with \( P_0 \).

Note that \( h \) is \((p-1,1)\)-block upper triangular since \((sg)P(sg)^{-1}\) and \(P_j\) are both \((p-1,1)\)-block upper triangular groups of order \(p\). We additionally insist that the lower right corner of \( h \) be 1 (If the corner is \(-1\) for our first choice of \( h \), then multiply \( h \) by \( diag(-1,-1,\ldots,-1) \)).

Looking at all the upper left corners in 3.3.14, we see that

\[ \text{cnr}[P] = \pi_j^{\text{cnr}[h]s_{p-1}(k)g_{p-1} \text{cnr}[L]} \text{ for first type } g \]

\[ \text{cnr}[P] = \pi_j^{\text{cnr}[h]s_{p-1}(k-1)g_{p-1} \text{cnr}[L]} \text{ for second type } g \]

Since \( \text{cnr}[L] \in Z_{\Gamma_{p-1}}(\text{cnr}[\pi]) \) by 2.4.9 the upper left corner of \( L \) does not contribute to these equations; so in fact

\[ \text{cnr}[P] = \pi_j^{\text{cnr}[h]s_{p-1}(k)g_{p-1}} \text{ for first type } g \]

\[ \text{cnr}[P] = \pi_j^{\text{cnr}[h]s_{p-1}(k-1)g_{p-1}} \text{ for second type } g \quad (3.3.15) \]

However, this is exactly the definition of \( \tilde{h} \) from 3.2.3. So \( \text{cnr}[h] = \tilde{h} \) for the \( \tilde{h} \) associated to those particular values of \( \text{cnr}[P] \), \( \pi_j \), \( g_{p-1} \), and \( k-1 \).
Since $h \in \Gamma$, then $h^*\alpha = \alpha$ for all $\alpha \in \hat{H}^*_2(\Gamma, M)$. So 3.3.11 can be written as

$$T_s(\alpha)|_{Z(P)} = \sum_{g_{p-1} \in B_{p-1}(k, \text{cnr}[x])} \text{Res}_{Z(P)}^{\Gamma \cdot \text{hdg}'L} (\text{hdg}'L)^* \alpha + \sum_{g_{p-1} \in B_{p-1}(k-1, \text{cnr}[x])} \text{Res}_{Z(P)}^{\Gamma \cdot \text{hsq}'L} (\text{hsq}'L)^* \alpha \quad (3.3.16)$$

Choosing $h$ might now allow us to think of Hecke as acting on the direct sum $\bigoplus_{P_j \in \mathcal{P}_k(2)} \hat{H}^*(Z(P_j), M)$ since now the conjugation maps $(\text{hdg}'L)^*$ and $(\text{hsq}'L)^*$ permute the groups $P_j \in \mathcal{P}_k(2)$. However it will be easier to define the $R'$-action if all the cohomology groups in the sum are the same group.

3.3.17 DEFINITION.

$$F' : \bigoplus_{I \in C(\zeta)} \hat{H}^*(Z(P_I), M) \cong \hat{H}^*(Z(P_0), M) \otimes F[C(\zeta)]$$

given by $(0, \ldots, A_I^{-1} \alpha, \ldots, 0) \mapsto \alpha \otimes I$, where the expression is appearing in the $I$-th place. For an element $\alpha \in \hat{H}^*(Z(P_0), M) \otimes F[C(\zeta)]$, we denote the $I$-th component by $\alpha_I$. Also define

$$F' \circ \text{pr} : \hat{H}^*_2(\Gamma, M) \hookrightarrow \hat{H}^*(Z(P_0), M) \otimes F[C(\zeta)]$$

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Now let us look at the Hecke action under the isomorphism $F' \circ pr$: For $\alpha \in \hat{H}^*(\Gamma, M) \mapsto J$-th component $\alpha_J \otimes J$ and for $P = P_I$, the Hecke action 3.3.16 is

$$F'[T_*(\alpha)]_I = \sum_{g_{p-1} \in \mathcal{O}'_{p-1}(k, \text{cnr}[\pi]) \atop g_1 = g_{p-1} + 1} A_I^* \left[ \text{Res}_{Z(P_I)}^{\text{hdg}'L} (\text{hdg}'L)^*(\alpha) \right] + \sum_{g_{p-1} \in \mathcal{O}'_{p-1}(k-1, \text{cnr}[\pi]) \atop g_1 = g_{p-1} + 1} A_I^* \left[ \text{Res}_{Z(P_I)}^{\text{hs}'L} (\text{hs}'L)^*(\alpha) \right]$$

$$= \sum_{g_{p-1} \in \mathcal{O}'_{p-1}(k, \text{cnr}[\pi]) \atop g_1 = g_{p-1} + 1} A_I^* (\text{hdg}'L)^* \left[ \text{Res}_{Z(P_J)}^F (\alpha) \right] + \sum_{g_{p-1} \in \mathcal{O}'_{p-1}(k-1, \text{cnr}[\pi]) \atop g_1 = g_{p-1} + 1} A_I^* (\text{hs}'L)^* \left[ \text{Res}_{Z(P_J)}^F (\alpha) \right]$$

$$= \sum_{g_{p-1} \in \mathcal{O}'_{p-1}(k, \text{cnr}[\pi]) \atop g_1 = g_{p-1} + 1} A_I^* (\text{hdg}'L)^* A_{J^{-1}}^*(\alpha_J) + \sum_{g_{p-1} \in \mathcal{O}'_{p-1}(k-1, \text{cnr}[\pi]) \atop g_1 = g_{p-1} + 1} A_I^* (\text{hs}'L)^* A_{J^{-1}}^*(\alpha_J) \quad (3.3.18)$$

(here we used the commutativity of restriction and conjugation (see 2.2.2))

3.3.19 Claim. The following defines an action of $R'$ on $H^i(Z(P_0), M) \otimes \mathbb{F}[C]$: For $\lambda$ an ideal in $K$ prime to $N$ and $v \in H^i(Z(P_0), M) \otimes \mathbb{F}[C]$, define $(\lambda,v)_I = y^* v_J$ where $[[J]] = [[I \lambda^{-1}]]$ and $y \in Z_{\Gamma_p(Q_N)}(P_0)$ is chosen so that $y$ "corresponds" (as in 3.2.4) to an algebraic number $\bar{y}$ that satisfies $(\bar{y}) = I^{-1}J\lambda$ and so that the bottom right corner of $y$ is 1.
For \( c(l) \in R' \) define \( (c(l) \cdot v)_I = u^* v_I \) where \( u \in \mathbb{Z}_{\Gamma_p(Q_N)}(\pi_0) \) is any matrix with \( \text{cnr}[u] \in \mathbb{Z}_{\Gamma_{p-1}}(\text{cnr}[\pi_0]) \) and bottom right corner congruent to \( I \mod Np \).

Extend by linearity to all of \( R' \).

**Proof.** By “corresponding to an algebraic number” we mean as usual that if we write \( y = f(\pi_0) \) with \( f(x) \) a polynomial in \( \mathbb{Q}_N[x] \) (which we can do by 3.1.2) then \( f(\zeta) \) is the algebraic number \( \tilde{y} \) corresponding to \( y \).

For any ideal prime to \( pN \) there exists an acceptable \( y \): the same \( y \) used in 3.2.4. The condition in 3.2.4 that \( f(\pi_0) \) correspond to \( \tilde{y} \) and obey \( f(1) = 1 \) is the same as \( f(\pi_0) \)’s corresponding to \( \tilde{y} \) and having bottom right corner equal to 1.

On the other hand \( y^* \) is well defined since, if \( y_1 \) is another such matrix, set \( z = y_1 y^{-1} \in \mathbb{Z}_{\Gamma_p(Q_N)}(\pi_0) \). The bottom right corner of \( z \) is 1 and \( \text{cnr}[z] = \text{cnr}[y_1] \cdot \text{cnr}[y]^{-1} \) corresponds to a unit in the algebraic integers. So, by 3.1.2, \( z \in Z(\pi_0) \) and so acts trivially on \( \tilde{H}^*(Z(P_0), M) \).

We need to show that this action defined on ideals factors through the ray class group \( E \). Suppose \( \lambda \) is a principal ideal congruent to 1 mod \( N \). So \( \lambda = (\tilde{y}) \) and \( \tilde{y} - 1 = N\tilde{a} \) for an \( \tilde{a} \in \mathbb{Q}_N[\zeta] \). Pick a \( y = f(\pi_0) \) with \( f(1) = 1 \) corresponding to \( \tilde{y} \).

Consider the polynomial \( g(x) = [f(x) - 1]/N \). So \( g(\zeta) = [f(\zeta) - 1]/N = \tilde{a} \). Since \( \tilde{a} \in \mathbb{Q}_N[\zeta] \) then \( g(x) \in \mathbb{Q}_N[x] \). So all the coefficients of \( f(x) - 1 \) are divisible by \( N \) and \( f(\pi_0) - 1 \) is \( \equiv 0 \mod N \).
Since $y - I_p$ commutes with $Z(P_0)$ and acts trivially on $M$ (Since $M$ is admissible, action of matrices factors through $N$), $y - I_p$ acts trivially on $H^*(Z(P_0), M)$. So $\lambda$ has the same effect as the identity.

The action of $c(l)$ is well-defined by 3.1.3 and 3.1.4.

Since all the matrices $y$ and $u$ commute with each other, the ring structure of $R'$ is preserved.

Since $H^*(Z(P_0), M) \otimes_F F[C(\zeta)]$ is an $R$-module, the map

$$\psi' : T(l, k) \mapsto \sum_{\Lambda \in \mathcal{L}(k/d)} [\Lambda] + \sum_{\Lambda \in \mathcal{L}(k-1/d)} c(l)[\Lambda]$$

from 3.3.4 gives a Hecke action on $H^*(Z(P_0), M) \otimes_F F[C(\zeta)]$.

Immediately by 3.3.8, any Hecke eigenvector in $H^*(Z(P_0), M) \otimes_F F[C(\zeta)]$ has a continuous semisimple Galois representation unramified outside of $pN$.

3.3.20 Claim. The map $F' \circ \text{pr} : \hat{H}^2(\Gamma, M) \hookrightarrow \hat{H}^*(Z(P_0), M) \otimes_F F[C(\zeta)]$ is Hecke invariant.

Proof. First consider the sums. The map $\psi'$ changes a Hecke element into a sum over ideals in various $\mathcal{L}(k/t)$; the Hecke action on $\hat{H}^*(\Gamma, M)$ is a sum over invariant subspaces.
We can use $D_{p-1}(i, \pi_{p-1})$ and $L(i/d)$ interchangeably because, as noted in 6.3.5 of [A1], by elementary number theory, for $n = p-1$,

\[ L(k/d) \nspace\leftrightarrow\nspace\text{full} \ n-k\text{-dim} \ \pi_{p-1}\text{-invariant subspaces of } (\mathbb{Z}/l)^{p-1}. \]

Let $P = P_I$. The Hecke action 3.3.18 is:

\[ F'(T_s(\alpha))|_I = \sum_{g_{p-1} \in \mathcal{B}_{p-1}(k, \text{cnr}[\pi])} (A_{j-1}^{-1}hdg'LA_I)^*\alpha_J \]

\[ + \sum_{g_{p-1} \in \mathcal{B}_{p-1}(k-1, \text{cnr}[\pi])} (A_{j-1}^{-1}hsg'LA_I)^*\alpha_J \]

In order to prove that this is the same as $\psi'(T_s)[(pr \circ F')\alpha]$, we need to show that, in the first sum, $A_{j-1}^{-1}hdg'LA_I = y$ for $y$ a matrix coming from the action of the ideal $[\lambda] \in L(k/d)$ associated to the subspace corresponding to $g$; and similarly in the second sum $A_{j-1}^{-1}hsg'LA_I = c(l)y$ for $y$ a matrix coming from the action of the ideal $[\lambda] \in L(k-1/d)$ associated to the subspace corresponding to $g$.

Consider the terms in the first sum first.

Define $y = A_{j-1}^{-1}hdg'LA_I$. Define $u = A_I^{-1}LA_I$ so $y = A_{j-1}^{-1}hdg'A_Iu$. We will show that this $y$ obeys the necessary conditions.

Since all the matrices in the product that forms $y$ are in $\Gamma(\mathbb{Q}_N)$, so is $y$ itself.

It is immediate from the definition of $h$, $L$, $A_I$, and $A_J$ that $y$ commutes with $\pi_0$.

Write $y = A_{j-1}^{-1}hdg'A_Iu$. Since $\text{cnr}[L] \in Z_{\Gamma_{p-1}}(\tilde{\pi}_I)$, then $\text{cnr}[A_I^{-1}LA_I] \in Z_{\Gamma_{p-1}}(\tilde{\pi}_0)$. So, comparing the upper left corners of $y = A_{j-1}^{-1}hdg'A_Iu$ with the
matrices in 3.2.4, we see that \( \text{cnr}[A_I] = \tilde{A}_I, \text{cnr}[A_J]^{-1} = \tilde{A}_J^{-1}, \text{cnr}[g] = g_{p-1}, \)
\( \text{cnr}[u] \) corresponds to an algebraic unit, and \( \text{cnr}[h] = \tilde{h} \) (see 3.3.15) for the case \( i = k \). So by 3.2.4, the algebraic number \( \tilde{y} \) associated to \( \text{cnr}[y] \) has the property that \( (\tilde{y}) = I^{-1}J\lambda \) for \( \lambda \) the ideal in \( \mathcal{L}(k/d) \) associated to the subspace associated to \( g_{p-1} \). Since \( y \) and \( \text{cnr}[y] \) yield the same algebraic number, \( y \) is associated to an algebraic number that is a generator of \( I^{-1}J\lambda \).

The bottom right corner of \( y \) is 1 since the matrices used to define \( y \) are all \((p-1,1)-\)block upper diagonal with 1 in their bottom right corners.

So \( y \) satisfies all the necessary conditions.

Now consider the terms in the second sum.

Define \( y_1 = A_I^{-1}h d g' L A_I \). Define \( u = A_I^{-1} L A_I \) so \( y_1 = A_I^{-1} h d g' A_I u \).

Again \( y_1 \in Z_{\Gamma(Q_N)}(\pi_0) \). Again comparing the upper left corners of our matrices here with the matrices in 3.2.4, we see that \( \text{cnr}[A_I] = \tilde{A}_I, \text{cnr}[A_J]^{-1} = \tilde{A}_J^{-1}, \text{cnr}[g] = g_{p-1}, \text{cnr}[u] \) corresponds to an algebraic unit, and \( \text{cnr}[h] = \tilde{h} \) (see 3.3.15) for the case \( i = k - 1 \). So by 3.2.4, the algebraic number \( \tilde{y} \) associated to \( \text{cnr}[y] \) has the property that \( (\tilde{y}) = I^{-1}J\lambda \) for \( \lambda \) the ideal in \( \mathcal{L}(k-1/d) \) corresponding to the subspace associated to \( g_{p-1} \). So \( y_1 \) is associated to an algebraic number that generates \( I^{-1}J\lambda \).

Now we come to a problem. The bottom right corner of \( y_1 \) is \( l \) instead of 1. Pick a matrix \( w \in Z_{Q_p}(\pi) \) such that \( \text{cnr}[w] \) is an algebraic unit and the bottom right
corner of $w$ is equal to $l$ (this is possible by 3.1.3). Replace $y_1$ with $y = y_1w^{-1}$.

Note that, since the upper left corners of $y$ and $y_1$ differ by a matrix corresponding to an algebraic unit, the ideals generated by the numbers associated to $y$ and $y_1$ are the same. So $y$ is associated to an algebraic number that generates the correct ideal.

Note that $w$ satisfies all the necessary conditions to be a matrix coming from the action of $c(l)$. So the composite conjugation map $(A_J^{-1}hdg^\prime LA_I)^* = w^*y^*$ is the same is the action of $c(l)[\lambda]$.

So this action of $R'$ on $\tilde{H}^*(\Gamma, M)$ extends the Hecke action. Thus the following is proved:

3.3.21 Proposition. Let $\Gamma = GL_p(\mathbb{Z})$ and $N$ a positive integer. Let $\alpha \in \tilde{H}^i(\Gamma, M)$ be an Hecke($pN$)-eigenvector. Then there is a continuous, semisimple, and unramified outside $pN$ Galois representation attached to $\alpha$. 
CHAPTER 4
THE NON-MINIMAL CASES

§1 Semisimple Type: The Structure of $Z(P)$

The semisimple non-minimal matrices are $P$ of form 1 with $n > p-1$.

4.1.1 Definition. Let $P = \langle \pi \rangle$ be of form 1, so $\pi = \begin{pmatrix} \text{cnr}[\pi] & 0 \\ 0 & I_m \end{pmatrix}$. Define

$$Z_{p-1}(P) = \begin{pmatrix} Z_{\Gamma_{p-1}}(\text{cnr}[\pi]) & 0 \\ 0 & I_m \end{pmatrix} \quad \text{and} \quad G_m = \begin{pmatrix} I_{p-1} & 0 \\ 0 & \Gamma_m \end{pmatrix}$$

We intend to prove $Z(P) = Z_{p-1}(P) \times G_m$. To do this we need a more intrinsic definition of $Z_{p-1}(P)$ and $G_m$.

4.1.2 Definition. Let $P = \langle \pi \rangle$ be $\Gamma$-conjugate to a group of form 1. Define

$$V(\zeta, P) = \{ v \in \mathbb{Z}^n \mid \Phi(\pi).v = 0 \} \quad \text{and} \quad V(1, P) = \{ v \in \mathbb{Z}^n \mid (\pi - I_n).v = 0 \}$$

i.e. $V(\zeta, P)$ is the maximal subspace of $\mathbb{Z}^n$ such that $\pi$ restricted to this subspace has powers of $\zeta$ (excluding 1) as its eigenvalues; $V(1, P)$ is the maximal subspace such that $\pi$ has 1 as its eigenvalues.

Notice, since $\pi$ is conjugate form 1, that $\mathbb{Z}^n = V(\zeta, P) \oplus V(1, P)$.
If $P$ is of form 1 (instead of just conjugate), then

$$V(\zeta, P) = \langle e_1, e_2, \ldots, e_{p-1} \rangle \subset \mathbb{Z}^n \text{ and } V(1, P) = \langle e_p, e_{p+1}, \ldots, e_n \rangle \subset \mathbb{Z}^n$$

Now use these subspaces to define some groups:

4.1.3 DEFINITION. Let $P = \langle \pi \rangle$ be a group that is $\Gamma$-conjugate to a form 1 group.

$$Z_{p-1}(P) = \{ Z \in Z(P) \mid Zu \subset V(\zeta, P) \forall u \in V(\zeta, P) \text{ and } Zu = u \forall u \in V(1, P) \}.$$  

$$G_m(P) = \{ Z \in Z(P) \mid Zu = v \forall u \in V(\zeta, P) \text{ and } Zu \subset V(1, P) \forall u \in V(1, P) \}.$$  

If $P$ is of form 1, then these definitions agree with those in 4.1.1 above. The fact that the matrices in $Z_{p-1}(P)$ and $G_m$ are all $(p-1,m)$-block diagonal for $P$ of form 1 is equivalent to the fact that they preserve the spaces $V(\zeta, P) = \langle e_1, \ldots, e_{p-1} \rangle$ and $V(1, P) = \langle e_p, \ldots, e_n \rangle$.

4.1.4 RESULT. Let $P = \langle \pi \rangle$ be of form 1. From Appendix C,

$$Z_{\Gamma, p-1}(\text{cnr}[\pi]) = \{ f(\text{cnr}[\pi]) \mid f \in \mathbb{Z}[x]/\Phi(x), \det f(\text{cnr}[\pi]) = \pm 1 \}$$

$$\cong \{ \text{ the units of the algebraic integers of } \mathbb{Q}(\zeta) \}$$

Therefore

$$Z_{p-1}(P) = \{ f(\pi) \mid f \in \mathbb{Z}[x]/\Phi(x), \det f(\pi) = \pm 1 \}$$

$$\cong \{ \text{ the units of the algebraic integers of } \mathbb{Q}(\zeta) \}$$

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Before we can prove $Z(P)$ is the direct product of $Z_{p-1}(P)$ and $G_m(P)$, note what happens when $Z_{p-1}(P)$ and $G_m(P)$ are conjugated:

4.1.5 Lemma. If $t \in \Gamma$ and $P$ is of form 1,

$$Z_{p-1}(tPt^{-1}) = tZ_{p-1}(P)t^{-1} \quad \text{and} \quad G_m(tPt^{-1}) = tG_m(P)t^{-1}$$

Proof. Just note $V(1, tPt^{-1}) = tV(1, P)$ and $V(\zeta, tPt^{-1}) = tV(\zeta, P)$.

4.1.6 Lemma. If $t \in \Gamma(\mathbb{Q}_N)$, $P$ is of form 1, and $tPt^{-1} \subset \Gamma$, then

$$Z_{p-1}(P) = t^{-1}Z_{p-1}(tPt^{-1})t$$

Proof. Let $f(\pi) \in Z_{p-1}(P)$. Since $f(\pi)$ is an integral polynomial and $t\pi t^{-1}$ an integral matrix, $tf(\pi)t^{-1} = f(t\pi t^{-1})$ is an integral matrix. Since conjugation does not change determinant, $tf(\pi)t^{-1}$ has determinant 1. So we get $tf(\pi)t^{-1} \in Z_{p-1}(tPt^{-1})$. Therefore $Z_{p-1}(P) \subset t^{-1}Z_{p-1}(tPt^{-1})t$. The converse is exactly the same.

Now here is the important lemma which as a corollary proves $Z(P)$ with $\pi$ conjugate to form 1 is a direct product:

4.1.7 Lemma. Suppose $\pi$ and $\pi'$ are both of form 1. If $t\pi t^{-1} = \pi'$ with $t \in \Gamma(\mathbb{Q}_N)$, then $t$ is $(p-1, m)$-block diagonal, i.e. $t = \begin{pmatrix} *_{p-1} & 0 \\ 0 & *_m \end{pmatrix}$

Proof. Since $\pi$ and $\pi'$ are both form 1, $V(\zeta, P) = V(\zeta, P') = \langle e_1, \ldots, e_{p-1} \rangle$ and $V(1, P) = V(1, P') = \langle e_{p-1}, \ldots, e_n \rangle$. By 4.1.2, $V(\zeta, P') = tV(\zeta, P)$
and $V(1, P') = tV(1, P)$. So $t$ preserves the subspaces $\langle e_1, e_2, \ldots, e_{p-1} \rangle$ and $\langle e_p, \ldots, e_n \rangle$. Hence $t$ is $(p-1, m)$-block diagonal.

4.1.8 Corollary. If $\pi$ is $\Gamma$-conjugate to a matrix of form 1,

$$Z(P) = Z_{p-1}(P) \times G_m(P)$$

Proof. This will be true if it is true for all form 1 matrices, since $Z_{p-1}(tP t^{-1}) = tZ_{p-1}(P)t^{-1}$ and $G_m(tP t^{-1}) = tG_m(P)t^{-1}$. Let $\pi$ be a form 1 matrix and $z \in Z(P)$. By 4.1.7 above (with $\pi = \pi'$), $z = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$. Looking at $\pi$, we see that $z_2 \in \Gamma_m$ and $z_1 \in Z_{\Gamma_{p-1}}(\text{cnr}[\pi])$. So $z \in Z_{p-1}(P) \times G_m(P)$.

§2 Semisimple Type: Hecke Action as a Tensor

For this section, $P = \langle \pi \rangle$ and $P' = \langle \pi' \rangle$ are in $\mathcal{P}_n(1)$ (so in particular are of form 1) and $\alpha \in \hat{H}_1^*(\Gamma, M)$.

The purpose of this section is to examine the Hecke action and show that, for certain coefficient modules the Hecke action on the cohomology group can be viewed as the tensor of smaller Hecke rings on smaller cohomology groups.

Elements in $\hat{H}_1^*(\Gamma, M)$ are uniquely determined by their restrictions to $\hat{H}^*(Z(P))$ for $P \in \mathcal{P}_n(1)$. Since $Z(P) = Z_{p-1}(P) \times G_m \cong Z_{\Gamma_{p-1}}(\text{cnr}[\pi]) \times \Gamma_m$, 88
and $\Gamma_m$ is $p$-torsion free (hence has $p$-finite cohomological dimension), then, as mentioned in 2.2.4, for $M = M' \otimes_F M''$, X.5.8 of [B] implies the Kunneth formula

$$\hat{H}^c(Z(P), M) \cong \bigoplus_{c_1 + c_2 = c} \hat{H}^c_1(Z_{p-1}(\text{cnr}[\pi]), M') \otimes H^{c_2}(\Gamma_m, M'').$$

Looking at the Kunneth formula raises the question of whether the Hecke action on $\hat{H}^*_1(\Gamma, M)$ somehow respects the direct product decomposition of $Z(P)$ so that the Hecke action can be written in terms of actions on $\hat{H}^*_1(Z_{p-1}(\text{cnr}[\pi]), M')$ and $H^*_1(\Gamma_m, M'')$. Pursuing this question is the purpose of this chapter.

Since $Z(P)$ is a $(p-1, m)$-block diagonal group for all $P \in \mathcal{P}_n(1)$, "respecting the direct product" turns out to mean that the formula for $T_s |_{Z(P)}$ must be in terms of $(p-1, m)$-block diagonal matrices acting on $(p-1, m)$-block diagonal groups.

Recall by 2.3.3 that

$$T_s(\alpha) |_{Z(P)} = \sum_{g \in D'(P)} Tr^{Z(P)}_{\mathcal{G}(P) \cap \Gamma \ast s} Res^{\uparrow \uparrow}_{Z(P) \cap \Gamma \ast s}(sg)^s \alpha.$$  

Remember the discussions of coset representatives in §4 of Chapter 2. The conclusion of that section was 2.4.7: For $s = T_n(l, k)$ and $P$ of form 1,

$$\hat{D}'(P) = \prod_{i=0}^{k} \left\{ \sigma(i)g' \mid g' = (g_{p-1} + g_m) \right\},$$  

with $g_{p-1} \in \hat{D}'_{p-1}(i, P)$ and $g_m \in \hat{D}_m(k - i)$. 

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where $\sigma(i)$ is a permutation matrix which depends only on $k$ and $i$.

Now let us go to $D'(P)$. What happens when we go from $\hat{D}'(P)$, which is an incomplete set of coset representatives of $(\Gamma \cap \Gamma^s) \backslash \Gamma$, to $D'(P)$, which is an incomplete set of representatives of $(\Gamma \cap \Gamma^s) \backslash \Gamma / Z(P)$? We call these sets incomplete because we only permit representatives $g$ such that $P \subset \Gamma^g$, instead of including representatives of all cosets.

4.2.1 Lemma. For $P$ of form 1, $D'(P)$ has the following form:

$$D'(P) = \prod_{i=0}^{k} \left\{ \sigma(i)g' \mid g' = (g_{p-1} + I_m) \text{ with } g_{p-1} \in D'_{p-1}(i, P) \right\}.$$ 

Proof. For ease in reference, let us give a temporary name to the right side of the equation we want to prove:

$$\mathcal{R} = \prod_{i=0}^{k} \left\{ \sigma(i)g' \mid g' = (g_{p-1} + I_m) \text{ with } g_{p-1} \in D'_{p-1}(i, P) \right\}$$

By definition, $D'(P)$ is any subset of $\hat{D}'(P)$ such that: if we set elements $g, h \in \hat{D}'(P)$ to be equivalent if $g \in (\Gamma \cap \Gamma^s) h Z(P)$, i.e. equivalent if they are in the same double coset, then $D'(P)$ is a complete set of representatives for this equivalence relation.

Note that $\mathcal{R}$ is a subset of $\hat{D}'(P)$ because, by Lemma 6.3.5 of [A1], $D'_{p-1}(i, P) = \hat{D}'_{p-1}(i, P)$. We previously mentioned this in the remarks prior to 3.3.2.
Now that we know that we do in fact have a subset of $\hat{D}'(P)$, we need to prove, first, that the elements in $\mathcal{R}$ all lie in distinct double cosets, and, second, that $\mathcal{R}$ contains a representative for every double coset that has a representative in $D'(P)$.

Let $g \in \hat{D}'(P)$. Suppose $g \in (\Gamma \cap \Gamma^*) h Z(P)$ for some $h \in \hat{D}'(P)$. We claim that when we write $g$ and $h$ in their standard form as permutation matrices times block diagonal matrices, it is the same permutation matrix for both of them, i.e. if the subspace $g^{-1}V_n(k)$ which corresponds to $g$ is in $\mathcal{W}(i)$, the subspace $h^{-1}V_n(k)$ corresponding to $h$ is also in $\mathcal{W}(i)$:

Let $h = \gamma g z$ with $\gamma \in \Gamma \cap \Gamma^*$ and $z \in Z(P)$. Since $\Gamma \cap \Gamma^*$ is the stabilizer of $V_n(k)$, then $h^{-1}V_n(k) = z^{-1}g^{-1}V_n(k)$. Since $z$ is $(p-1,m)$-block diagonal, it preserves $(e_1, e_2, \ldots, e_{p-1})$ and $(e_p, e_{p+1}, \ldots, e_n)$ and its restriction to $(e_1, e_2, \ldots, e_{p-1})$ is an invertible matrix. So the rank of $g^{-1}V_n(k) \cap (e_1, e_2, \ldots, e_{p-1})$ is the same as the rank of $z^{-1}g^{-1}V_n(k) \cap (e_1, e_2, \ldots, e_{p-1})$. So $g$ and $h$ both use the permutation matrix $\sigma(i)$. Write $g = \sigma(i)g'$ with $g' = g_{p-1}+g_m$ and $h = \sigma(i)h'$ with $h' = h_{p-1}+h_m$.

So we have

$$g' = \sigma(i)^{-1}\gamma \sigma(i)h'z \text{ for some } \gamma \in \Gamma \cap \Gamma^* \text{ and } z \in Z(P) \quad (4.2.2)$$
As observed in 2.4.4,

\[ \Gamma \cap \Gamma^s = \{ \left( \begin{array}{cc} A_{n-k} & lB \\ C & D_k \end{array} \right) \in \Gamma | A_{n-k}, B, C, D_k \text{ all integral coefficients} \} \]

Therefore

\[ \sigma(i)^{-1}(\Gamma \cap \Gamma^s)\sigma(i) = \left\{ \left( \begin{array}{ccc} A_{p-i} & lB & * \\ C & D_i & * \\ & E_{m-k+i} & lB \\ & C & D_{k-i} \end{array} \right) \in \Gamma \right\} \quad (4.2.3) \]

where \( A_{n-k}, B, C, D_k, * \) range over integral matrices. Note that \( \sigma(i)^{-1}\gamma\sigma(i) \) must be \((p-1, m)\)-block diagonal since \( g', h', \) and \( z \) all are. So, for \( \sigma(i)^{-1}\gamma\sigma(i) \), the *'s in 4.2.3 are zero, and \( \sigma(i)^{-1}\gamma\sigma(i) \in \left[ \Gamma_{p-1} \cap \Gamma_{s-p-1}^{(i)} \right] + \left[ \Gamma_m \cap \Gamma_m^{(k-i)} \right] \). Now looking at the blocks in 4.2.2 and remembering the structure of \( Z(P) \) from 4.1.8 (mainly that it is \((p-1, m)\)-block diagonal), we see that

\[ g \in (\Gamma \cap \Gamma^s)hZ(P) \implies \]

\[ g_{p-1} \in \left( \Gamma_{p-1} \cap \Gamma_{s-p-1}^{(i)} \right) h_{p-1} Z_{p-1}(\text{cfr}[\pi]) \text{ and } g_m \in \left( \Gamma_m \cap \Gamma_m^{(k-i)} \right) h_m \Gamma_m \]

So if \( g \) and \( h \) are both in \( \mathcal{R} \), then, by definition of \( \mathcal{R} \), \( g_{p-1} = h_{p-1} \) (since they lie in a common double coset and \( D'_{p-1}(i, P) \) admits only one element from each double coset) and \( g_m = h_m = I_m \). Also we know \( g \) and \( h \) have the same \( \sigma(i) \). So if \( g \) and \( h \) are both in \( \mathcal{R} \) then \( g = h \). Hence the elements in \( \mathcal{R} \) all lie in distinct cosets.
To show \( \mathcal{R} \) contains all the coset reps it should, note that if \( g = \sigma(i)(g_{p-1}+g_m) \in \hat{D}'(P) \), then

\[
g_{p-1} \in \hat{D}'_{p-1}(i, P) = D'_{p-1}(i, P) \implies g_{p-1}+I_m \in \mathcal{R} \text{ and } g \in (\Gamma \cap \Gamma^s)(g_{p-1}+I_m)Z(P).
\]

Now that the elements of \( D'(P) \) are in a good form, we can force the matrices in formula 2.3.3 to be block diagonal:

4.2.4 Definition. Define

\[
d(i, k) = \sigma(i)^{-1} s \sigma(i) = \begin{pmatrix}
I_{p-1-i} & 0 & 0 & 0 \\
0 & I_i & 0 & 0 \\
0 & 0 & I_{m-k+i} & 0 \\
0 & 0 & 0 & I_{k-i}
\end{pmatrix} = s_{p-1}(i) + s_m(k - i)
\]

See 4.1.9 for definition of \( s_{p-1}(i) \) and \( s_m(k - i) \). So, for \( g = \sigma(i)g' \in D'(P) \),

\[
sg = s \sigma(i) g' = \sigma(i) d(i, k) g'.
\]

Since \( \sigma(i) \in \Gamma \), this means

\[
\Gamma^s g = \Gamma^{d(i, k)g'} \text{ and } (sg)^* \alpha = (d(i, k)g')^* \alpha \text{ for } \alpha \in \hat{H}^*(\Gamma, M)
\]

(4.2.5)

Now look again at 2.3.3: For \( \alpha \in H_1^*(\Gamma, M) \),

\[
T_s(\alpha)_{Z(P)} = \sum_{g \in D'(P)} Tr_{Z(P)}^{Z(P) \cap \Gamma^s g} Res_{Z(P) \cap \Gamma^s g}^{\Gamma^s g} (sg)^* \alpha
\]

\[
= \sum_{\sigma(i)g' \in D'(P)} Tr_{Z(P) \cap \Gamma^{d(i, k)g'}}^{Z(P) \cap \Gamma^{d(i, k)g'}} Res_{Z(P) \cap \Gamma^{d(i, k)g'}}^{\Gamma^{d(i, k)g'}} (d(i, k)g')^* \alpha
\]

(4.2.6)
Note \( g' \) and \( d(i,k) \) are both \((p-1,m)\)-block diagonal.

Now all the matrices in the formula for \( T_\alpha \) are block-diagonal, but we still have the problem that the Hecke action is defined on \( \hat{H}^*(\Gamma,M) \) and \( \Gamma \) is not a block diagonal group. We will replace \( \Gamma \) by the group of block diagonal matrices:

4.2.7 Definition. Define

\[
B = B_n(Z) = (p-1,m)\text{-block diagonal matrices in } \Gamma.
\]

\[
G_{p-1} = \Gamma_{p-1} \overset{+}{\times} I_m \quad \quad \text{and} \quad \quad G_m = I_{p-1} \overset{+}{\times} \Gamma_m, \quad \text{so} \quad B = G_{p-1} \times G_m.
\]

\[
B(Q_N) = G_{p-1}(Q_N) \times G_m(Q_N) = (p-1,m)\text{-block diagonal matrices in } \Gamma(Q_N),
\]

i.e.

\[
G_{p-1}(Q_N) = \Gamma_{p-1}(Q_N) \overset{+}{\times} I_m \quad \quad \text{and} \quad \quad G_m(Q_N) = I_{p-1} \overset{+}{\times} \Gamma_m(Q_N).
\]

We pick this group \( B \) because it is a direct product, it is a subgroup of \( \Gamma \), and by 4.1.8 it contains all of \( Z(P) \) for all \( P \in \mathcal{P}_n(1) \) (in fact it contains all of \( N(P) \)). Obviously any element of order \( p \) in \( B \) is of form 1. By 4.1.7, form 1 matrices that are conjugate in \( \Gamma \) must be conjugate in \( B \). So \( \mathcal{P}(1) \) is a set of representatives of the conjugacy classes of order \( p \) subgroups of \( B \).

As with \( \Gamma \) in 2.1.4 and 2.1.11,

\[
\hat{H}^*(B,M) \cong \bigoplus_{P \in \mathcal{P}(1)} \hat{H}^*(N(P),M) \hookrightarrow \bigoplus_{P \in \mathcal{P}(1)} \hat{H}^*(Z(P),M)
\]

where \( Z(P) = Z_B(P) = Z_\Gamma(P) \). So

\[
\hat{H}^*(\Gamma,M) \overset{\text{Res}}{\hookrightarrow} \hat{H}^*(B,M) \hookrightarrow \bigoplus_{P \in \mathcal{P}(1)} \hat{H}^*(Z(P),M)
\]  

(4.2.8)
where the first map is injective because the composition of the two maps is the injective map in 2.1.11.

4.2.9 Definition. For a $B(\mathbb{Q}_N)$-module $M'$, define a $\text{Hecke}(\Gamma, \mathbb{Q}_N)$-action on $\hat{H}^*(B, M')$ by defining, for $\alpha \in \hat{H}^*(B)$ and $s \in T_n(l, k)$

$$T_s(\alpha) = \sum_{i=0}^{i=k} \text{Tr}_{B \cap B d(i, k)}^{B} \text{Res}_{B \cap B d(i, k)}^{d(i, k)} \beta^{d(i, k)} d(i, k)^* \alpha$$

Since $\text{Hecke}(\Gamma, \mathbb{Q}_N)$ is a free abelian algebra generated by the $s = T_n(l, k)$, to prove 4.2.9 is an action we need to prove $T_s T_{s'} = T_{s'} T_s \forall s, s' \in T_n(l, k)$. Instead of doing this directly, we will prove that the above is the sum of other actions that we know are well-defined, namely the usual Hecke action on $B$ with the usual Hecke algebra.

Consider $\text{Hecke}(B, \mathbb{Q}_N) = \{ \text{formal sums of double cosets } B \backslash B(\mathbb{Q}_N) / B \}$. Since $B$ is a direct product of $\Gamma_{p-1}$ and $\Gamma_m(\mathbb{Z})$, $\text{Hecke}(B, \mathbb{Q}_N)$ is the free abelian algebra generated by direct products of all $s_{p-1}(i)$ and $s_m(j)$, i.e. the free abelian algebra generated by $d(i, k)$ (see 4.2.4) for $0 \leq k \leq n$, $i \leq k$, and $l$ running over primes not dividing $pN$. The usual action of $\text{Hecke}(B, \mathbb{Q}_N)$ on $\hat{H}^*(B, M')$ denote the action of the coset represented by $d(i, k)$ as $S_{d(i, k)}$ is

$$S_{d(i, k)}(\alpha) = \text{Tr}_{B \cap B d(i, k)}^{B} \text{Res}_{B \cap B d(i, k)}^{d(i, k)} \beta^{d(i, k)} d(i, k)^* \alpha \quad (4.2.10)$$

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This is defined for any \( B(\mathbb{Q}_N) \)-module \( M \) (\( M \) does not need to be admissible). As with 2.1.3, Appendix A is the proof that this is the same as the natural Hecke\((B, \mathbb{Q}_N)\) action as defined in [R-W].

Since 4.2.9 defines the action of \( T_s \) to be the sum of \( S_{d(i,k)} \), 4.2.9 does define a valid action.

Just as with \( \hat{H}^*(\Gamma, M) \), this action 4.2.9 is determined because of 4.2.8 by its restrictions to the \( \hat{H}^*(Z(P), M) \)'s. We can use on \( \hat{H}^*(B, M) \) the same standard formula used on \( \hat{H}^*(\Gamma, M) \) in Lemma 6.1 of [A] to get, for \( \alpha \in H^*(B, M) \),

\[
T_s(\alpha)|_{Z(P)} = \sum_{i=0}^{k} \sum_{g' \in g'(Z(P) \cap B^{d(i,k)}g')} \text{Res}^{B^{d(i,k)}g'}_{Z(P) \cap B^{d(i,k)}g'} (d(i, k)g')^* \alpha
\]  

(4.2.11)

where \( g' \) varies over \( (B \cap B^{d(i,k)}) \backslash B / Z(P) \). Now we compare this double coset with \( D'(P) \):

For ease of reference, let us temporarily define

\[
R(i) = \{ \sigma(i)g' \mid g' = (g_{p-1} I_m) \text{ with } g_{p-1} \in D'_{p-1}(i, P) \}.
\]

So \( D'(P) = \bigsqcup_{i=0}^{k} R(i) \). Since \( B \) acts transitively on each \( \mathcal{W}(i) \) and \( B^{d(i,k)} = \left[ \Gamma_{p-1} \cap \Gamma_{p-1}^{s_{p-1}(i)} \right] + \left[ \Gamma_m \cap \Gamma_{s_{m}(k-i)} \right] \) is the stabilizer in \( B \) of \( V_n(i, k - i) \), representatives \( g' \in B^{d(i,k)}g' \backslash B \) correspond to subspaces \( g'^{-1}V_n(i, k - i) \in \mathcal{W}(i) \).

Now suppose \( g' \in (B \cap B^{d(i,k)}g') \backslash B / Z(P) \). Since \( (d(i, k)g')Z(P)(d(i, k)g')^{-1} \cap \Gamma \) has \( p \)-torsion only if \( (d(i, k)g')P(d(i, k)g')^{-1} \subset \Gamma \) (because \( \Gamma \) contains no groups
isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$ and hence only if $(d(i, k)g')P(d(i, k)g')^{-1} \subset B$, then

$$\text{Res}_{Z(P) \cap B^{d(i, k)}g'}^B(d(i, k)g')^* \alpha = 0 \text{ if } P \not\subset B^{d(i, k)}g'.$$

But note, as in 2.4.4, that $P \subset B^{d(i, k)}g'$ is exactly the condition on $g'$ that means $g'^{-1}V(i, k)$ is a $\pi$-invariant subspace. So, eliminating $g'$ such that $P \not\subset B^{d(i, k)}g'$, the second sum in 4.2.11 runs over $g'$ corresponding to full $k$-dimensional $\pi$-invariant subspaces of $(\mathbb{Z}/l)^n$ with identifications due to $Z(P)$, i.e. $g'$ runs over $\sigma(i)^{-1}R(i)$.

The double sum $\sum_{i=0}^{i=k} \sum_{g' \in \sigma(i)R(i)}$ is exactly the sum over $\sigma(i)g' \in D'(P)$.

So we get, for $\alpha \in \hat{H}^*(B, M),

$$T_{\sigma}(\alpha)|_{Z(P)} = \sum_{\sigma(i)g' \in D'(P)} T_{\sigma}^{Z(P)}(P)_{Z(P) \cap B^{d(i, k)}g'}, \text{Res}_{Z(P) \cap B^{d(i, k)}g'}^B(d(i, k)g')^* \alpha. \quad (4.2.12)$$

4.2.13 Proposition. $\text{Res}_B^\Gamma: \hat{H}^*_\Gamma(\Gamma, M) \hookrightarrow \hat{H}^*(B, M)$ is Hecke($\Gamma, \mathbb{Q}_N$)-invariant for any $\Gamma(\mathbb{Q}_N)$-module $M$.

Proof. For ease of notation, denote $t = d(i, k)g'$. Since $Z(P)$ and $t$ are all $(p-1, m)$-block diagonal, $tZ(P)t^{-1} \cap \Gamma = tZ(P)t^{-1} \cap B$.

So the Hecke action 4.2.6 on $\alpha \in \hat{H}^*_\Gamma(\Gamma)$ is

$$T_{\sigma}(\alpha)|_{Z(P)} = \sum_{\sigma(i)g' \in D'(P)} T_{\sigma}^{Z(P)}(P)_{Z(P) \cap B^i}, \text{Res}_{Z(P) \cap B^i}^\Gamma t^* \alpha.$$  

However

$$\text{Res}_{Z(P) \cap B^i}^\Gamma t^* \text{Res}_B^\Gamma \alpha = \text{Res}_{Z(P) \cap B^i}^\Gamma t^* \alpha = \text{Res}_{Z(P) \cap B^i}^\Gamma t^* \alpha$$

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(The first equality is true by 2.2.2, the lemma that restriction commutes with conjugation).

So for $\alpha \in \hat{H}_1^*(\Gamma)$,

$$T_s(\text{Res}_B^\Gamma \alpha)|_{Z(P)} = T_s(\alpha)|_{Z(P)} = [\text{Res}_B^\Gamma T_s(\alpha)]|_{Z(P)}$$

Since they have the same values on the $\hat{H}^*(Z(P))$’s, $T_s(\text{Res}_B^\Gamma \alpha) = \text{Res}_B^\Gamma T_s(\alpha)$.

Therefore, with respect to the new action on $\hat{H}^*(B)$, $\text{Res}_B^\Gamma$ is $\text{Hecke}(\Gamma, \mathbb{Q}_N)$-equivariant. $lacksquare$

Proposition 4.2.13 is unusual because it shows that the restriction map from $\Gamma$ to $B$ is Hecke-invariant with respect to the usual Hecke actions and a natural map of the usual Hecke algebras $(s_\alpha(k) \mapsto \sum_{i=0}^k d(i,k))$. In general there is no natural map of the standard Hecke algebras that allows restriction from a group to a subgroup to be Hecke-invariant. The Hecke map is sort of a sum over subspaces (single cosets) and in general the subspaces (single cosets) in one algebra can’t be matched in 1-1 correspondence with the subspaces (single cosets) of another. The reason it works here is that in this case we only sum over “invariant” subspaces (the terms in the action are zero for cosets corresponding to non-invariant spaces) and while the general subspaces do not match up the invariant subspaces do.
From now on we will examine the Hecke action on $\hat{H}^*(B, M)$, not $\hat{H}^*(\Gamma, M)$, and $M$ need only be an admissible $F(B(Q_N))$-module, not an admissible $F(\Gamma(Q_N))$-module.

Now the Hecke action is defined on a block diagonal group and the formula is in terms of block diagonal matrices, so everything is taken care of except the module $M$. In order to use the Kunneth formula, $M$ must be a tensor product.

For the rest of this section we assume:

4.2.14 ALTERED DEFINITION. $M$ is an admissible right $B$-module such that

$M = M' \otimes_F M''$ with $M'$ a $G_{p-1}(Q_N)$-module and $M''$ a $G_m(Q_N)$-module.

In section 5 we will show how results for the Hecke action on $\hat{H}^*(B, M' \otimes M'')$ (and thus results about $\hat{H}^*(\Gamma, M' \otimes M'')$) can be applied to a general $M$ that is not a tensor product.

For any $(p-1,m)$-block diagonal matrix $h = h_{p-1} + h_m$, regard $h_{p-1}$ as an element in $G_{p-1}(Q_N)$ by putting an identity block in the bottom corner; similarly regard $h_m$ to be in $G_m(Q_N)$.

Then $h$ acts on $a' \otimes a'' \in M' \otimes M''$ by $(a' \otimes a'')h = a'h_{p-1} \otimes a''h_m$.

By the Kunneth Formula (see X.5.8 of [B]),

$$\hat{H}^*(B, M) \cong \sum_{c_1 + c_2 = \ast} \hat{H}^{c_1}(G_{p-1}, M') \otimes H^{c_2}(G_m, M'').$$

(4.2.15)
Note that $H^\omega(G_m, M''')$ is ordinary cohomology, not Farrell cohomology. Using this isomorphism, every element of $\check{H}^*(B, M)$ will be written as a tensor product. Also, we identify $G_{p-1}$ with $\Gamma_{p-1}$ and $G_m$ with $\Gamma_m$.

Our goal is to use 4.2.15 to write the Hecke action on $\alpha \in \check{H}^*(B, M)$ as a tensor product of actions on $\check{H}^*(\Gamma_{p-1}, M')$ and $H^*(\Gamma_m, M''')$. The first step is to move the conjugation maps in the formula 4.2.12 for the Hecke action into the tensor product.

Fix $j = k - i$.

4.2.16 Claim. Let $\alpha \otimes \beta \in \check{H}^*(B, M)$ and $\sigma(i) g' \in D'(P)$ with $g' = g_{p-1} + I_m$.

Then

$$(d(i,k)g')^*(\alpha \otimes \beta) = g_{p-1}^* s_{p-1}(i)^* \alpha \otimes s_m(j)^* \beta.$$

Proof. Immediate from 2.2.5.

So formula 4.2.12 has become: For $\alpha = \alpha \otimes \beta \in \check{H}^*(B, M),$

$$T_s(\alpha)|_P = \sum_{\sigma(i) g' \in D'} T_{s(P)}^{Z(P) \cap B^{d(i,k)} g'} \text{Res}^{B^{d(i,k)} g'}_{Z(P) \cap B^{d(i,k)} g'} \left[ g_{p-1}^* s_{p-1}(i)^* \alpha \otimes s_m(j)^* \beta \right]. \quad (4.2.17)$$

4.2.19 Claim. $T_{s(P)}^{Z(P) \cap B^{d(i,k)} g'} \text{Res}^{B^{d(i,k)} g'}_{Z(P) \cap B^{d(i,k)} g'} \left[ g_{p-1}^* s_{p-1}(i)^* \alpha \otimes s_m(j)^* \beta \right] = \text{Res}_{s_{P-1}(P)}^{s_{P-1}(i) g_{P-1}} \alpha \otimes T_{s_{P-1}(i) g_{P-1}}^{G_m} \text{Res}^{G_m}_{G_m \cap G_m^{s_m(j)}} \text{Res}^{G_m^{s_m(j)}}_{G_m \cap G_m^{s_m(j)}} s_m(j)^* \beta.$
Proof. Remember $g' = g_{p - 1} + I_n$.

We wish to apply 2.2.6, so we look at $Z(P) \cap B^{d(i, k)} g'$. Since $P$ is in this intersection by the definition of $g' \in D'(P)$, so is all of $Z_{p - 1}(P)$ (See Lemma 6.3.3 of [A1]. It is because $Z_{p - 1}(P)$ is polynomials in $\pi$ by 4.1.4 and the upper left block of $B^{d(i, k)} g$ is all matrices that are lower block triangular mod $l$). So $Z(P) \cap B^{d(i, k)} g' = [Z_{p - 1}(P) \times G_m] \cap B^{d(i, k)} g' = Z_{p - 1}(P) \times [G_m \cap B^{d(i, k)} g'] = Z_{p - 1}(P) \times [G_m \cap G_m^{s_m(j)} I_m] = Z_{p - 1}(P) \times [G_m \cap G_m^{s_m(j)}]$.

So $Z(P) \cap B^{d(i, k)} g' = Z_{p - 1}(P) \times (G_m \cap G_m^{s_m(j)})$. So by Lemma 2.2.6,

$$
Res_{Z(P) \cap B^{d(i, k)} g'}^B g^{p - 1}(i) s_{p - 1}(i) \alpha \otimes s_m(j) \beta
$$

For ease of reference, let us write this as

$$
Res_{Z(P) \cap B^{d(i, k)} g'}^B g^{p - 1}(i) s_{p - 1}(i) \alpha \otimes s_m(j) \beta
$$

Now since $Z(P) \cap B^{d(i, k)} g' = Z_{p - 1}(P) \times (G_m \cap G_m^{s_m(j)})$, by 2.2.7 then

$$
Tr_{Z(P) \cap B^{d(i, k)} g'} \left[ \alpha \otimes \beta \right] = \alpha' \otimes \beta'.
$$

Applying 4.2.19 to formula 4.2.17, we get, for $\mu = \alpha \otimes \beta \in H^*(B, M)$,

$$
T_s(\mu) |_{Z(P)} = \sum_{\sigma(i) g' \in D'(P)} \left( Res_{Z_{p - 1}(P)}^{G_m} (s_{p - 1}(i) g_{p - 1})^* \alpha \otimes \left( Tr_{G_m \cap G_m^{s_m(j)}} Res_{G_m \cap G_m^{s_m(j)}}^{G_m} (s_m(j) \beta) \right) \right).
$$

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From the decomposition of $D'(P)$ given in lemma 4.2.1, a sum $\sum_{i=0}^{k} g' \in D'(P)$ is the same as a sum $\sum_{i=0}^{k} g' = g_{p-1} + f_m$. So we rewrite 4.2.20 as:

$$T_s(\mu) |z(P) = \sum_{i=0}^{k} \sum_{g_{p-1} \in D'_{p-1}(P)} \text{Res}_{Z_{p-1}(P)} G_{p-1}^{i} g_{p-1}^{i} s_{p-1}(i)^* \alpha \otimes \left( T_R G_m \big| G_m \cap G_m^{j} \right) \text{Res}_{Z_{p-1}(P)} G_m^{j} s_m(j)^* \beta \right). \quad (4.2.21)$$

The formula of the Hecke action on $\hat{H}^*(\Gamma_{p-1}, M')$ in 2.3.3 is that, for $\alpha \in \hat{H}^*(\Gamma_{p-1}, M')$,

$$T_s(\mu) |z(cnr[\pi]) = \sum_{g_{p-1} \in D'_{p-1}(i)} T_R Z_{(cnr[\pi])} \big| \Gamma_{p-1}^{g_{p-1} s_{p-1}(i)} \text{Res}_{Z_{(cnr[\pi])} \cap \Gamma_{p-1}^{g_{p-1} s_{p-1}(i)}} \Gamma_{p-1}^{g_{p-1} s_{p-1}(i)} g_{p-1}^{i} s_{p-1}(i)^* \alpha.$$ 

By Lemma 6.3.4 of [A1] the transfer map vanishes in this formula (since $Z(cnr[\pi]) \cap \Gamma_{p-1}^{g_{p-1} s_{p-1}(i)} = Z(cnr[\pi])$).

Moreover, the Hecke action on $H^*(\Gamma_m, M'')$, from 4.1.13, is, for $\beta \in \hat{H}^*(\Gamma_m, M'')$,

$$T_s(j) \beta = T_R \Gamma_m \big| \Gamma_m \cap \Gamma_m^{s_m(j)} \text{Res}_{\Gamma_m \cap \Gamma_m^{s_m(j)}} \Gamma_m^{s_m(j)} s_m(j)^* \beta$$

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So, using the isomorphisms $Z_{p-1}(P) \cong Z\langle \text{cnr} [\pi] \rangle$ and $G_m \cong \Gamma_m$, 4.2.21 becomes: for $\alpha \otimes \beta \in \hat{H}^*(B, M)$

$$T_s(\alpha \otimes \beta) \mid_{Z(P)} = \sum_{i=0}^{i=k} \left[ T_{s_{p-1}(i)} \alpha \mid_{Z\langle \text{cnr} [\pi] \rangle} \right] \otimes T_{s_m(j) \beta}$$

$$= \sum_{i=0}^{i=k} \text{Res}_{Z\langle \text{cnr} [\pi] \rangle \times \Gamma_m} \left( T_{s_{p-1}(i)} \alpha \otimes T_{s_m(j) \beta} \right)$$

(4.2.22)

(The second equality because of 2.2.6)

Remember that the reason we are examining the restriction of elements to $Z(P)$ is that elements are uniquely defined by their values in the $\hat{H}^*(Z(P), M)$’s. So 4.2.22 implies:

4.2.23 CONCLUSION. For $\mu = \sum_b \alpha_b \otimes \beta_b \in \hat{H}^*(B, M' \otimes M'')$ and $s = T_n(l, k)$,

$$T_s(\mu) = \sum_b \sum_{i=0}^{i=k} T_{s_{p-1}(i)} \alpha_b \otimes T_{s_m(j) \beta_b}$$

where $T_{s_{p-1}(i)} \alpha_b$ is the Hecke action of $T_{p-1}(l, i)$ on $\hat{H}^*(\Gamma_{p-1}, M')$ and $T_{s_m(j) \beta_b}$ is the Hecke action of $T_{m}(l, j)$ on $H^*(\Gamma_m, M'')$.

COROLLARY 4.2.24. For $\mu \in \hat{H}_1^*(\Gamma, M)$, $\text{Res}_B \mu = \sum \alpha_b \otimes \beta_b$, $M = M' \otimes M''$ as a $B(\mathbb{Q}_N)$-module,

$$T_s(\mu) \mid_B = \sum_b \sum_{i=1}^{i=k} T_{s_{p-1}(i)} \alpha_b \otimes T_{s_m(j) \beta_b}$$

with $T_{s_{p-1}(i)}$ and $T_{s_m(j)}$ the Hecke actions on $\hat{H}^*(\Gamma_{p-1}, M')$ and $H^*(\Gamma_m, M'')$. 

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§3 Non-Semisimple Type: The Structure of $Z(P)$

The non-simple non-minimal matrices are the matrices $P$ in $\Gamma_n$ of form 2 with $n > p$. For this section define $\pi$ to be of form 2 and $n > p$.

Since for $\text{acr}[^{\pi},e_1] = \begin{pmatrix} \text{cnr}[^{\pi}] & e_1 \\ 0 & 1 \end{pmatrix}$ the eigenvalue 1 appears with multiplicity 1, the space of 1-eigenvectors has dimension 1.

When we look at the structure of $Z(P)$ it will be convenient for us to regard 1-eigenvectors of $\text{acr}[^{\pi},e_1]$ as arbitrary integral multiples of some fixed primitive 1-eigenvector. We now pick this fixed eigenvector, which previously appeared in 3.1.1.

4.3.1 Definition. Let $u = \begin{pmatrix} v_\phi \\ p \end{pmatrix} \in \mathbb{Z}^p$ be the 1-eigenvector of $\text{acr}[^{\pi},e_1]$ with $p$-th coefficient equal to $p$. Specifically $u = \Phi(\text{acr}[^{\pi},e_1]).e_p$.

Remember $\Phi(x)$ is the $p$-th cyclotomic polynomial. Note $v_\phi \in \mathbb{Z}^{p-1}$ and that in fact $u_\phi$ is the upper right corner of the matrix $\Phi(\text{acr}[^{\pi},e_1])$ (since any matrix applied to $e_p$ yields the $p$-th column of the matrix).

Notice $u$ is certainly fixed by $\text{acr}[^{\pi},e_1]$: Since $\text{acr}[^{\pi},e_1]^p = 1$, then $\text{acr}[^{\pi},e_1]\Phi_p(\text{acr}[^{\pi},e_1]) = \Phi_p(\text{acr}[^{\pi},e_1])$, so in fact $\Phi_p(\text{acr}[^{\pi},e_1])$ applied to any vector will give a fixed vector (although for most choices of vectors it is zero; we know $u$ is nonzero since its $p$-th entry is $p$). This vector $u$ must be primitive since, if it was a nonunit multiple of a primitive vector, the primitive vector would
have the property that its $p$-th entry is equal to 1. However, that means that the primitive vector could be extended to a full integral basis of $Z^p$, contradicting that $\text{acr}[\pi, e_1]$ is not conjugate to a form 1 matrix.

Since $u$ is primitive, in particular $v_\phi$ cannot be evenly divided by $p$.

As an example of a primitive 1-eigenvector, let

$$\pi_{p-1}(0) = \begin{pmatrix} 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix}$$

be the standard matrix of order $p-1$ associated with the ideal $Z[\zeta]$ itself. The 1-eigenvector $u$ for $\text{acr}[\pi_{p-1}(0), e_1]$ is $(p-1, p-2, \ldots, 2, 1, p)^T$.

**Claim 4.3.2.** Suppose $P$ is of form 2. Then

1) $Z_\Gamma(P) = \text{all matrices } Z \text{ of the form } Z = \begin{pmatrix} Z_p & B \\ C & D_m \end{pmatrix}$ such that:

a) $Z$ is integral with determinant $= \pm 1$.

b) $Z_p$ is $(p-1,1)$-block diagonal, commutes with $\text{acr}[\pi, e_1]$, and has nonzero determinant prime to $p$. The upper left $(p-1)$-block of $Z_p$ is in $Z_{\Gamma_{p-1}}(\text{cnr}[\pi])$.

c) $C$ has the property that each row is an integral multiple of $e_1^T$.

d) $B$ has the property that each column is a 1-eigenvector of $\text{acr}[\pi, e_1]$, i.e. an integral multiple of $u$. In particular all the entries in the last row of $B$ are integral multiples of $p$.

e) $D_m$ has no conditions on it. It can be anything, subject to (a).
2) \(Z_{\Gamma(Q_N)}(P)\) is the same as \(Z_{\Gamma}(P)\) except with \(Q_N\)-matrices instead of integral matrices and without any condition on the determinant of \(Z_p\). I.e.

\(Z_{\Gamma(Q_N)}(P) = \) all matrices of the form \(Z = \begin{pmatrix} Z_p & B \\ C & D_m \end{pmatrix}\) with \(Z\) a \(Q_N\)-matrix with determinant prime to \(N\); \(Z_p\) is \((p-1,1)\)-block diagonal, commutes with \(acr[\pi,e_1]\), and has upper left block in \(Z_{\Gamma^{-1}(Q_N)}(cnr[\pi])\); \(C\) has the property that each row is a \(Q_N\) multiple of \(e_1^T\); and \(B\) has the property that each column is a \(Q_N\) multiple of \(u\).

**Proof.** 1) Suppose \(Z = \begin{pmatrix} Z_p & B \\ C & D_m \end{pmatrix} \in Z_{\Gamma}(P)\)

Then \(\begin{pmatrix} Z_p & B \\ C & D_m \end{pmatrix} \begin{pmatrix} acr[\pi,e_1] & 0 \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} acr[\pi,e_1] & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} Z_p & B \\ C & D_m \end{pmatrix}\)

\[\implies acr[\pi,e_1]^{-1}Z_pacr[\pi,e_1] = Z_p,\]

\[(acr[\pi,e_1] - I_p)B = 0,\]

\[C(acr[\pi,e_1] - I_p) = 0,\]

\[D_m = D_m\]

Note \((acr[\pi,e_1] - I_p)B = 0\) means that each column of \(B\) is a 1-eigenvector of \(acr[\pi,e_1]\).

Note \(C(acr[\pi,e_1] - I_p) = 0\) means each column of \(C^T\) is a 1-eigenvector in \(Z^P\) of \(acr[\pi,e_1]^T = \begin{pmatrix} cnr[\pi] & 0 \\ e_1^T & 1 \end{pmatrix}\). The 1-eigenspace is 1-dimensional since 1
is an eigenvalue of $acr[\pi, e_1]$ of multiplicity 1. By inspection, $e_p$ is a primitive 1-eigenvector.

Now consider the equation $acr[\pi, e_1]^{-1}Z_pacr[\pi, e_1] = Z_p$. There is no reason so far to think that $Z_p$ is invertible: it is just an integral matrix. Since it commutes with $acr[\pi, e_1]$, by 3.1.1 then $Z_p$ is $(p-1,1)$-block upper triangular. So, since $C$ is zero in the first $(p-1)$ columns, $Z$ is $(p-1,m)$-block upper triangular. Hence $cnr[Z_p]$ is invertible, i.e. $cnr[Z_p] \in \Gamma_{p-1}$.

For the bottom right corner of $Z_p$, notice that in the $p$-th row of $Z$ the first $p-1$ entries are 0 and the last $n-p$ entries are divisible by $p$. Since $Z \in \Gamma$, each row must have greatest common factor 1. So the $p$-th entry in the $p$-th row of $Z$, i.e. the bottom right element on $Z_p$, must be prime to $p$. So $Z_p$ has determinant prime to $p$.

Notice that, when the determinant of $Z_p$ is $\pm1$, $Z_p \in Z_{\Gamma_p}(acr[\pi, e_1])$

2) For $Z_{\Gamma(Q_N)}(P)$, just repeat the proof of (1), but with $Q_N$ matrices.

An immediate consequence of 4.3.2 is that the matrices in $Z(P)$ and $Z_{\Gamma(Q_N)}(P)$ are $(p-1,m)$-block diagonal. It will be more convenient for us to write the matrices $Z$ in $(p-1,m)$-block diagonal form than to use the size $p$ and size $n-p$ blocks of 4.3.2.

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Since by 3.1.1 (with \( N = 0 \)) integral matrices in \( \Gamma_P(\mathbb{Q}) \) that commute with \( \text{acr}[\pi, e_1] \) can be uniquely expressed as integral polynomials in \( \text{acr}[\pi, e_1] \), a re-statement of 4.3.2 in \((p - 1, m)\)-block diagonal form is that a matrix picture of a typical matrix in \( Z_\Gamma(P) \) is

\[
Z = \begin{pmatrix}
   f(\text{cnr}[\pi]) & v_f + a_1 v_\Phi & a_2 v_\Phi & a_3 v_\Phi & \ldots & a_m v_\Phi \\
   0 & f(1) + a_1 p & a_2 p & a_3 p & \ldots & a_m p \\
   \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & 0 & 0 & \ddots & D'_{m-1}
\end{pmatrix}
\]

(4.3.3)

where \( f \) is a unique integral polynomial of degree less than \( p \), \( \det(f(\text{cnr}[\pi])) = \pm 1 \), \( f(1) \) is an integer prime to \( p \), \( v_f \) depends on \( f \), the \( n_i \) are arbitrary integers, the \( a_i \) are arbitrary integers, \( \left( \begin{smallmatrix} v_\Phi \\ p \end{smallmatrix} \right) \) is a primitive 1-eigenvector \( u \) of \( \text{acr}[\pi, e_1] \), and \( D'_{m-1} \) is any matrix that causes the determinant of the whole thing to work out to be \( \pm 1 \). Perhaps some choices of \( f, n_i, \) and \( a_i \) will be bad and no such \( D'_{m-1} \) will exist.

For matrices in \( Z_{\Gamma(\mathbb{Q}_N)}(P) \) the picture is the same, except with \( \mathbb{Q}_N \) coefficients:

- \( f \) is a unique \( \mathbb{Q}_N \) polynomial of degree less than \( p \); \( \det(f(\text{cnr}[\pi])) \) prime to \( N \);
- \( f(1) \in \mathbb{Q}_N \); and \( n_i \) and \( a_i \) are in \( \mathbb{Q}_N \).

A particularly important consequence of 4.3.2, which is easier to see when 4.3.2 is rewritten in the form 4.3.3, is that, if \( Z \in Z_{\Gamma(\mathbb{Q}_N)}(P) \), then \( Z \) is uniquely determined by its top left \((p-1)\) and bottom right \( m \) blocks. Moreover any matrices
\[ f(\text{cnr}[\pi]) \in Z_{\Gamma_{p-1}(\mathbb{Q}_N)}(\text{cnr}[\pi]) \text{ and } D \in \Gamma_m(\mathbb{Q}_N) \] can be uniquely extended to a matrix in \( Z_{\Gamma(\mathbb{Q}_N)}(P) \) if and only if \( f(1) \equiv 1 \text{ mod } p \) in \( \mathbb{Q}_N \).

Similarly for \( \mathbb{Z} \) matrices instead of \( \mathbb{Q}_N \) matrices.

Instead of working with all of \( Z(P) \), it will be more convenient to work with the following subgroup of index prime to \( p \):

4.3.4 Definition. Let \( P \) be of form 2.

\[ Z'(P) = \{ Z \in Z(P) \mid \text{the } (p, p)-\text{th entry } \equiv 1 \text{ mod } p \text{ in } \mathbb{Z} \}. \]

\[ Z'(P, \mathbb{Q}_N) = \{ Z \in Z_{\Gamma(\mathbb{Q}_N)}(P) \mid \text{the } (p, p)-\text{th entry } \equiv 1 \text{ mod } p \text{ in } \mathbb{Q}_p \}. \]

Note that, in the typical picture 4.3.2 above, the \( (p, p) \)-th entry is \( f(1) \).

We will decompose \( Z'(P) \) into a direct product, as we decomposed \( Z(P) \) in the form 1 case in 4.1.8 above.

4.3.5 Definitions. Let \( P = \langle \pi \rangle \) be of form 2. We define some subgroups of \( \Gamma_m \) and \( \Gamma_{p-1} \):

\[ \Gamma_m^1 = \{ \gamma \in \Gamma_m \mid \text{the first row is congruent to } (1 \ 0 \ 0 \ \ldots \ 0) \text{ mod } p \}. \]

\[ \Gamma_m^1(\mathbb{Q}_N) = \{ \gamma \in \Gamma_m(\mathbb{Q}_N) \mid \text{the first row is congruent to } (1 \ 0 \ 0 \ \ldots \ 0) \text{ mod } p \text{ in } \mathbb{Q}_N \}. \]

\[ \Upsilon_{p-1}(P) = \{ z \in Z_{\Gamma_{p-1}}(\text{cnr}[\pi]) \mid z = f(\text{cnr}[\pi]) \text{ with } f(1) = 1 \}. \]

\[ \Upsilon_{p-1}(P, \mathbb{Q}_N) = \{ z \in Z_{\Gamma_{p-1}(\mathbb{Q}_N)}(\text{cnr}[\pi]) \mid z = f(\text{cnr}[\pi]) \text{ with } f(1) = 1 \}. \]
Note that the bottom right $m$ corners of matrices in $Z'(P)$ (respectively $Z'(P, \mathbb{Q}_N)$) are all in $\Gamma^1_m$ (respectively $\Gamma^1_m(Q_N)$).

Remember the important fact about $\Upsilon_{p-1}(P)$ from 3.2.5: $\hat{H}^*(\Upsilon, M) \otimes F[\zeta]$ admits an $R$-module structure and the corresponding $\text{Hecke}(\Gamma_{p-1}, \mathbb{Q}_N)$ action has the property that Hecke eigenvectors have Galois representations.

Now we define some subgroups in $\Gamma$ and $\Gamma(Q_N)$ to correspond to the groups in 4.3.5:

4.3.6 Definition. Define $G^1_m(P, \mathbb{Q})$ to be the image of a particular embedding of $\Gamma^1_m(Q_N)$ into $Z'(P, \mathbb{Q}_N)$.

Define $G^1_m(P) \subset Z'(P)$ to be the image of the embedding restricted to $\Gamma^1_m$.

Here is the embedding:

Suppose $D_m = \begin{pmatrix} 1 + a_1p & a_2p & a_3p & \ldots & a_mp \\ \ast & \ast & \ast & \ldots & \ast_{m-1} \\ \vdots & & & & \\ \ast & & & & \end{pmatrix} \in \Gamma^1_m(Q_N)$.

Then $D_m \mapsto \begin{pmatrix} I_{p-1} & a_1v_\Phi & a_2v_\Phi & \ldots & a_mv_\Phi \\ 0 & & & & D_m \end{pmatrix}$

where $v_\Phi$ is the $\mathbb{Z}^{p-1}$ vector from 4.3.1.

From the typical picture 4.3.3, $G^1_m(P, \mathbb{Q})$ is indeed in $Z'(P, \mathbb{Q}_N)$. If also $D_m \in \Gamma^1_m$ then the image is, from 4.3.3, in $Z'(P)$. 110
Clearly the map is injective. Since, by 4.3.3, matrices in \( Z'(P, Q_N) \) are determined by their top left \((p-1)\) corner and bottom right \(m\) corner, the map is a homomorphism.

4.3.7 Definition. Define \( U_{p-1}(P, Q) \) to be the image of a particular embedding of \( \Gamma_{p-1}(\text{cnr}[\pi], Q_N) \) into \( Z'(P, Q_N) \).

Define \( U_{p-1}(P) \subset Z'(P) \) to be the image of the same embedding restricted to \( \Gamma_{p-1}(\text{cnr}[\pi]) \).

Here is the embedding:

Suppose \( Z_{p-1} = f(\text{cnr}[\pi]) \in \Gamma_{p-1}(\text{cnr}[\pi], Q_N) \). Choose \( f \) so \( f(1) = 1 \). Define

\[
Z_{p-1} \mapsto \begin{pmatrix}
f(\text{acr}[\pi, e_1]) & 0 \\
0 & I_{m-1}
\end{pmatrix}.
\]

Note \( f(\text{acr}[\pi, e_1]) = \begin{pmatrix} f(\text{cnr}[\pi]) & v_f \\ 0 & 1 \end{pmatrix} \).

From 4.3.3 the image of \( Z_{p-1} \) is in \( Z'(P, Q_N) \). Clearly the map is injective and a homomorphism.

4.3.8 Definition. \( \Delta(Q_N) = \text{scalar matrices in } \Gamma(Q_N) \).

Note that \( Z_{\Gamma(Q_N)}(P) = \Delta(Q_N)Z'(P, Q_N) \):

If \( \begin{pmatrix} Z_{p-1} & * \\ 0 & D_m \end{pmatrix} \in Z_{\Gamma(Q_N)}(P) \) with \( Z_{p-1} = f(\text{cnr}[\pi]) \), then

\[
\begin{pmatrix} Z_{p-1} & * \\ 0 & D_m \end{pmatrix} = \text{diag}(d) \begin{pmatrix} \frac{1}{d}Z_{p-1} & * \\ 0 & \frac{1}{d}D_m \end{pmatrix} \in \Delta(Q_N)Z'(P, Q_N)
\]

with \( d \) chosen so that \( d \equiv f(1) \mod p \) and \( d \) so that prime to \( N \).
4.3.9 LEMMA. For \( P = \langle \pi \rangle \) of form 2,

\[
Z'(P, \mathbb{Q}_N) = U_{p-1}(P, \mathbb{Q}_N) \times G^1_m(P, \mathbb{Q}_N)
\]

\[
Z'(P) = U_{p-1}(P) \times G^1_m(P).
\]

PROOF. This is clear, again since, by 4.3.3, matrices in \( Z'(P, \mathbb{Q}_N) \) are determined by their top left \((p-1)\) corner and bottom right \(m\) corner. ■

§4 Non-Semisimple Type: Hecke Action as a Tensor

The purpose of this section, as with section 2, is to examine the Hecke action and show that, for certain coefficient modules the Hecke action on the cohomology group can be viewed as the tensor of smaller Hecke rings on smaller cohomology groups.

Since the index of \( Z'(P) \) in \( Z(P) \) is prime to \( p \) (it is \( p-1 \)), \( \hat{H}^*(Z(P), M) \hookrightarrow \hat{H}^*(Z'(P), M) \). So we change 2.1.11 to

\[
\hat{H}^*_2(\Gamma, M) \hookrightarrow \sum_{P \in \mathbb{P}_n(2)} \hat{H}^*(Z'(P), M) \tag{4.4.1}
\]

Therefore we know any element in \( \hat{H}^*_2(\Gamma, M) \) if we know its restrictions to all of the \( Z'(P) \).

Proceeding as in 2.1.16, we use formula II.9.5 of [Br] and get, for \( \alpha \in \hat{H}^*_2(\Gamma, M) \) and \( s = T_n(l, k) \),

\[
T_s(\alpha)|_{Z'(P)} = \sum_g T_{Z'(P) \cap \Gamma^g} \text{Res}_{Z'(P) \cap \Gamma^g} (g^* s^* \alpha) \tag{4.4.2}
\]
where \( g \) runs over a set of representatives of \( (\Gamma \cap \Gamma^s)/\Gamma \backslash Z'(P) \).

Since

\[
\text{Res}_{Z'(P) \cap \Gamma^s}^\Gamma g^* s^* \alpha = 0 \implies \text{Res}_{Z'(P) \cap \Gamma^s}^\Gamma g^* s^* \alpha = 0,
\]

from 2.3.2 we can choose \( g \) to run only over the representatives such that \( P \subset \Gamma^s \).

Recall that, for \( s = T_n(l, k) \) and \( P \) of form 2, the set \( \tilde{D}'(P) \) is all elements \( g \) from a complete set of representatives \( g \) of \( (\Gamma \cap \Gamma^s)/\Gamma \) such that \( P \subset \Gamma^s \). The elements in \( \tilde{D}'(P) \) have the special form, given in 2.4.10, of

\[
\tilde{D}'(P) = \prod_{i=0}^{k} \left\{ \sigma(i)g'L \mid g' = (g_{p-1} + g_m) \text{ with } g_{p-1} \in \tilde{D}'_{p-1}(i, \text{cnr}[\pi]) \right\}
\]

and \( g_m \in \tilde{D}_m(k - i) \)

where \( \sigma(i) \) is the permutation matrix from 2.4.6 that depends only on \( i \) and \( k \), and \( L \) is the special matrix from 2.4.9 that obeys

\[
L^{-1} \left( \text{acr}[\pi, e_1] + I_{n-p} \right) L = \text{acr}[\pi, le_1] + I_{n-p}
\]

and depends only on \( l \) and \( \pi \).

Recall further that the set \( D'(P) \), defined in 1.1.14, is all elements \( g \) from a complete set of representatives of \( (\Gamma \cap \Gamma^s)/\Gamma \backslash Z(P) \) such that \( P \subset \Gamma^s \). We choose representatives so that \( D'(P) \subset \tilde{D}'(P) \).

4.4.3 Claim. For \( P \) non-minimal of form 2 and \( D'(P) \) as above,

\[
D'(P) = \prod_{i=0}^{k} \left\{ \sigma(i)g'L \mid g' = (g_{p-1} + I_m) \text{ with } g_{p-1} \in D'_{p-1}(i, \text{cnr}[\pi]) \right\}.
\]
Furthermore \( D'(P) \) = all elements from a set of coset representatives of \((\Gamma \cap \Gamma^g)/\Gamma \setminus Z'(P)\) such that \( P \subset \Gamma^g \).

**Proof.** We prove the first equality. We must prove that the right side of the equation contains exactly one element from each coset represented in \( D'(P) \). Compare the equality with 4.2.1, which is the analogous equality for \( P \) of form 1.

We will prove the equality by reducing it to the form 1 case. First note that

\[ P \subset \Gamma^g \text{ if and only if } \langle \text{cnr}[\pi]+I_m \rangle \subset \Gamma^{(gL^{-1})} : \quad (4.4.4) \]

If \( P \subset \Gamma^g \), then \((gL^{-1})LPL^{-1}(gL^{-1})^{-1} \subset \Gamma^g \). Since \( \Gamma \cap \Gamma^g \) is exactly those matrices with upper \((n-k)\) by \(k\) corner \(\equiv 0 \mod l\), then, reducing mod \(l\), we get that \(gL^{-1}(\text{cnr}[\pi]+I_m)(gL^{-1})^{-1} \subset \Gamma^g \).

Now we prove the equality. Since \( \Gamma \cap \Gamma^g \) is the stabilizer of \( V_n(k) \) (see 2.4.2), two elements \( g, h \in \Gamma \) are in the same \((\Gamma \cap \Gamma^g)/\Gamma \) coset if and only if \( g^{-1}V_n(k) = h^{-1}V_n(k) \). So \( g \in (\Gamma \cap \Gamma^g)hZ(P) \) if and only if there exists \( z \in Z(P) \) such that \( z^{-1}h^{-1}V_n(k) = g^{-1}V_n(k) \).

Writing \( g = \sigma(i)g'L \) and \( h = \sigma(j)h'L \), we get that \( g \in (\Gamma \cap \Gamma^g)hZ(P) \) if and only if

\[ z^{-1}L^{-1}h'^{-1}\sigma(j)^{-1}V_n(k) = L^{-1}g'^{-1}\sigma(i)^{-1}V_n(k) \quad \text{if and only if} \]

\[ L^{-1}(z^L)^{-1}h'^{-1}\sigma(j)^{-1}V_n(k) = L^{-1}g'^{-1}\sigma(i)^{-1}V_n(k) \quad \text{if and only if} \]

\[ (z^L)^{-1}h'^{-1}\sigma(j)^{-1}V_n(k) = g'^{-1}\sigma(i)^{-1}V_n(k). \quad (4.4.5) \]
However, since $L$ conjugates $\pi$ into $acr[\pi, le_1] + I_{n-p}$ and since it is true that $acr[\pi, le_1] + I_m \cong cnr[\pi] + I_m \mod l$, $z^L$ is congruent mod $l$ to a matrix in $Z(cnr[\pi] + I_m)$, namely the matrix gotten by changing the upper right corner of $z^L$ to zero (so that we are left with a $(p-1, m)$-block diagonal matrix). We can replace $z^L$ with this block diagonal matrix in 4.4.5 because the equation deals with actions mod $l$. So 4.4.5 is true if and only if

$$\sigma(i)g' \in (\Gamma \cap \Gamma^s) [\sigma(j)h'] Z(cnr[\pi] + I_m),$$

(4.4.6)

i.e., $g$ and $h$ are in the same $(\Gamma \cap \Gamma^s)/\Gamma \setminus Z(P)$ double coset if and only if $\sigma(i)g'$ and $\sigma(j)h'$ are in the same $(\Gamma \cap \Gamma^s)/\Gamma \setminus Z(cnr[\pi] + I_m)$ double coset.

So, together with 4.4.4, we get a 1-1 correspondence between $D'(P)$ and $D'(cnr[\pi] + I_m)$. So, by 4.2.1, the first equality of the claim is true.

For the second equality we need to prove that, if two elements $g$ and $h$ are in the same $(\Gamma \cap \Gamma^s)/\Gamma \setminus Z(P)$ double coset, they are in the same $(\Gamma \cap \Gamma^s)/\Gamma \setminus Z'(P)$ double coset. So suppose $g = khy$ with $k \in \Gamma \cap \Gamma^s$ and $y \in Z(P)$.

Write $y = \begin{pmatrix} f(cnr[\pi]) \\ 0 \\ B \\ D_m \end{pmatrix}$ with $f(1) = (1, 1)$th entry of $D_m$. Add integral multiples of $l$ to $f(x)$ in order to create a $g(x)$ such that $g(1) \equiv 1 \mod p$. Then $g(cnr[\pi]) \equiv f(cnr[\pi]) \mod l$.

Pick a matrix $D^1_m$ such that $D^1_m \in \Gamma^1_m$ and $D^1_m \cong D_m \mod l$. Such a $D^1_m$ exists because the projection map $\Gamma^1_m \to \Gamma_m(Z/l)$ is surjective onto the matrices.
of determinant $\pm 1 \mod l$. This is a well known fact; for example, by Theorem VII.8 of [Ne], the unimodular integral matrices congruent to the identity mod $p$ (i.e. the principal congruence subgroup of $SL_n(\mathbb{Z})$ of level $p$) map surjectively onto the unimodular $\mathbb{Z}/l$ matrices.

Define $y^1 = \left( \begin{array}{c} g(cnr[\pi]) \\ 0 \end{array} \right) B_1 \in Z'(P)$, where $y^1$ exists by 4.3.2 with $B_1$ depending on $g(x)$, $\pi$, and the first row of $D_m^1$. Since we have that $g(cnr[\pi]) \equiv f(cnr[\pi]) \mod l$, $D_m \equiv D_m^1 \mod l$, and the first rows of $D_m^1$ and $D_m \equiv (1 \ 0 \ 0 \ 0 \ldots) \mod p$; by 4.3.3, $B \equiv B^1 \mod l$.

So $y^1 \equiv y \mod l$. Therefore $g^{-1}V_n(k) = y^{-1}h^{-1}k^{-1}V_n(k) = y^{-1}h^{-1}V_n(k) = (y^1)^{-1}h^{-1}V_n(k)$. Since they define the same subspace in $\mathbb{Z}/l^n$, $g$ must differ from $hy^1$ by an element in the stabilizer of $V_n(k)$, so $g = \gamma hy^1$ for some $\gamma \in \Gamma \cap \Gamma^g$ and with $y^1 \in Z'(P)$.

So, by 4.4.3, the sum in 4.4.2 can be chosen to run over $D'(P)$. We get:

for $\alpha \in \tilde{H}_2^*(\Gamma, M)$ and $s = T_n(l, k)$,

$$T_s(\alpha)|Z'(P) = \sum_{g = \sigma(i)g' \in D'(P)} T_{gZ'(P)\cap \Gamma^s}^{\Gamma^g} \text{Res}_{Z'(P)\cap \Gamma^s}^{\Gamma^g} (g^*s^*\alpha)$$ (4.4.7)

Recall the definition of $d(i, k)$ from 4.2.4. Remember that $\sigma(i) = \sigma(i)d(i, k)$ and so, since $\sigma(i) \in \Gamma$, $\Gamma^{\sigma(i)} = \Gamma$ and $\sigma(i)^* = \text{identity map}$. Applying these facts to 4.4.7 we get:

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\[
T_s(\alpha)|_{Z'(P)} = \sum_{\sigma(i)g' \in \mathcal{D}'(P)} \text{Tr}_{Z'(P)\cap \Gamma'(1,k)g'L} \text{Res}_{Z'(P)\cap \Gamma'(1,k)g'L} (d(i,k)g'L)^* \alpha
\]

(4.4.8)

In §2, it was at this point in the section that we defined the block diagonal group \(B\) which had the nice property that it contained the centralizers \(Z(P, Q_N)\) for all \(P \in \mathcal{P}_n(1)\). Unfortunately, for form 2 matrices there doesn’t seem to be any analogous block diagonal group. Fortunately a subgroup of each of the centralizers, namely \(Z'(P, Q_N)\), is naturally isomorphic to a block diagonal group. So instead of defining and working with a big group \(B\) we will work with the direct sums of the centralizers. First, as we did with the \(n = p\) form 2 matrices, we will need to define some useful matrices so that elements can be conjugated into their proper places.

4.4.9 Notation.

Recall from 2.1.9 that \(\mathcal{P}_p(2) = \{\pi_p(2, I)\}_{I \in \mathcal{C}(\zeta)}\). For the rest of this section let

\[
\pi_I = \pi(2, I) = \begin{pmatrix}
\text{acr}[\pi_{p-1}(I), e_1] & 0 \\
0 & I_{m-1}
\end{pmatrix}
\]

\[
\pi_0 = \begin{pmatrix}
\text{acr}[\pi_{p-1}(0), e_1] & 0 \\
0 & I_{m-1}
\end{pmatrix}.
\]

Let \(P_I = \langle \pi_I \rangle\) and \(P_0 = \langle \pi_0 \rangle\).
4.4.10 Definitions. For all $\pi_j \in \mathcal{P}_n(2)$,

Define $A_{J,p} = \text{the matrices in 3.3.13 that were called "}A_J\text{" there. Recall that } A_{J,p} \in \Gamma_p(\mathbb{Q}_p N) \text{ and has the property that it is } (p-1,1)\text{-block upper triangular, the bottom right corner is } 1, \text{ and }

\text{acr}[\pi_0, e_1] = \text{acr}[\pi_j, e_1]^{A_j}

Define $A_J = A_{J,p} + I_{n-p}$. So $A_J \in \Gamma(\mathbb{Q}_p N)$, it is $(p-1,m)$-block upper triangular, the $(p,p)$-th element is $1$, and

$\pi_0 = \pi_j^{A_J}$.

4.4.11 Definition. Let $P = P_I$. Let $g = \sigma(i)g'L \in D'(P)$, so $\pi_I \subset \Gamma^{d(i,k)g'L}$. When considering $\pi_0 = (\pi_j)h^{d(i,k)g'L}$ for some $(\pi_j) \in \mathcal{P}(2)$

and $A_J^{-1}hd(i,k)g'L A_I \in Z'(P_0, \mathbb{Q}_N)$.

We need to prove that such an $h$ exists.

Note that $\pi_J$ is uniquely defined by $g$, $s$, and $\pi_I$, since $\pi_I^{d(i,k)g'L}$ is conjugate to exactly one element in $\mathcal{P}_n(2)$ (the conjugate $\pi_I^{d(i,k)g'L}$, which is in $\Gamma$ by the definition of $D'(P)$, must be $\Gamma$-conjugate to form 2 since $\pi$ is being conjugated by matrices with determinant prime to $p$, see 2.3.4).

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So there exists $h \in \Gamma$ such that $\pi_I = (\pi_J)^{hd(i,k)g'L}$. Note that $A_J^{-1}hd(i,k)g'L\Lambda_I$ commutes with $P_0$, and so, since all the matrices in the product are in $\Gamma(\mathbb{Q}_pN)$, $A_J^{-1}hd(i,k)g'L\Lambda_I \in Z_{\Gamma(\mathbb{Q}_pN)}(P_0)$.

Suppose it happens that, for our first choice of $h$, the $(p,p)$-th entry of $A_J^{-1}hd(i,k)g'L\Lambda_I \neq 1$ mod $p$ in $\mathbb{Q}_N$. Let $d$ denote the $(p,p)$-th entry of $A_J^{-1}hd(i,k)g'L\Lambda_I$. By Claim 2 of Appendix C, $d \neq 0$ mod $p$ in $\mathbb{Q}_pN$. Pick a $(p-1)$ integral matrix $f(\alpha r_1') \in \mathbb{Z}$ from 3.1.3 with $f(\alpha r_1') \in \mathbb{Z}_{p-1}$ and $f(1) = e \equiv 1/d$ mod $p$. Now find a matrix $E \in \Gamma_m$ with $(1,1)$-th entry equal to $e$. Use 4.3.3 to extend $f(\alpha r_1')$ and $E$ into a matrix $U \in Z(\pi_J)$. Now replace $h$ by $Uh$. For our new value of $h$ then it continues to be true that $\pi = (\pi_J)^{hd(i,k)g'L}$ but now the $(p,p)$-th entry of $A_J^{-1}hd(i,k)g'L\Lambda_I$ is congruent to 1 mod $p$, and so $A_J^{-1}hd(i,k)g'L\Lambda_I \in Z'(P_0,\mathbb{Q}_N)$.

So $h$ always exists, although it is not unique. In fact, given $h$ that satisfies 4.4.11, $\gamma h$ satisfies 4.4.11 if and only if $\gamma \in Z'(P_J)$.

So we rewrite 4.4.8 as: for $\alpha \in \tilde{H}_2^*(\Gamma, M)$ and $s = T_n(l,k)$,

$$T_s(\alpha)|_{Z'(P_I)} = \sum_{\sigma(i)g'L \in D'(P_I)} Tr_{Z'(P_I)}^{Z'(P_I)\cap\pi^{hd(i,k)g'L}} Res_{Z'(P_I)\cap\pi^{hd(i,k)g'L}}^{\pi^{hd(i,k)g'L}}(hd(i,k)g'L)^*\alpha \quad (4.4.12)$$
4.4.13 Definition.

\[ F'' : \bigoplus_{I \in C(\zeta)} H^i(Z'(\pi_I), M) \cong H^i(Z'(\pi_0), M) \otimes_F \mathbb{F}[C(\zeta)] \]

given by \((0, \ldots, A_I^{-1} \alpha, \ldots, 0) \mapsto \alpha \otimes I\), where the expression \(A_I^{-1} \alpha\) is appearing in the \(I\)-th place. For an element \(\alpha \in H^i(Z'(\pi_0), M) \otimes_F \mathbb{F}[C(\zeta)]\), we denote the \(I\)-th component by \(\alpha_I\).

Since it has an inverse, it is clear that \(F''\) is an isomorphism, provided it is well-defined. The only question about its being well-defined is whether \(A_I^{-1} Z'(\pi_I) A_I \subset Z(\pi_0)\). Obviously anything in \(A_I^{-1} Z'(\pi_I) A_I\) commutes with \(\pi_0\), but why are the elements in \(A_I^{-1} Z'(\pi_I) A_I\) all integral matrices with \((p, p)\)-th entry \(\equiv 1 \mod p\)?

Let \(Z \in Z'(\pi_I)\) with typical picture as in 4.3.3. Consider \(A_I^{-1} Z A_I\). Since the bottom right \(m\) corner of \(A_I\) is \(I_m\), the bottom right corner of \(A_I^{-1} Z A_I\) is the same as the bottom right corner of \(Z\). Since the upper left \((p-1)\) corner of \(A_I\) conjugates \(\mathfrak{c n r}[\pi_I]\) into \(\mathfrak{c n r}[\pi_0]\) and the upper left corner of \(Z\) is an integral polynomial in \(\mathfrak{c n r}[\pi_I]\), the upper left corner of \(A_I^{-1} Z A_I\) is an integral polynomial in \(\mathfrak{c n r}[\pi_0]\).

The rest of \(A_I^{-1} Z A_I\) is totally determined by its upper left and bottom right corners, as shown in 4.3.3. In particular the rest of \(A_I^{-1} Z A_I\) is also integral, so all of \(A_I^{-1} Z A_I\) is integral.
Now we see what happens to the Hecke action 4.4.10 under $F''$.

4.4.14 **Definition.** Let $M$ be an admissible $Z_{\Gamma(\mathbb{Q}_N)}(P_0)$-module. For $\alpha \in \mathcal{H}^{\ast}(Z'(P_0), M) \otimes_F F[C(\zeta)], s = T_n(l, k)$, and $I \in F[C(\zeta)]$, define

\[
(T_s \alpha)_I = \sum_{\sigma(i)g' \in D'(P)} \text{Tr} Z'(P_0)_{\mathbb{Z}'(P_0) \cap \mathbb{Z}'(P_0)_{A_J^{-1}hd(i, k)g'L_AI}} \text{Res}_{Z'(P_0)_{\mathbb{Z}'(P_0) \cap \mathbb{Z}'(P_0)_{A_J^{-1}hd(i, k)g'L_AI}}(A_{\gamma}^{-1}hd(i, k)g'L_{A_I})^*(\alpha_I)
\]

where $h$ and $I$ are defined in 4.4.11; they are elements that make certain conjugation relations hold. Both $h$ and $I$ depend on $J$ and $\sigma(i)g'L$.

Let us check that this map is well-defined. The problem is the matrices on the right hand side that are not uniquely defined. The matrices $A_J, A_I, and L$ have been fixed once and for all.

For $h$, then, as noted above in 4.4.11, if $h$ is replaced by $\gamma h$ then $\gamma \in Z'(P_I, M)$. Then $\gamma^{A_J} \in Z'(P_0, M)$ so

\[
(A_{\gamma}^{-1}hd(i, k)g'L_{A_I})^* = (A_{\gamma}^{-1}hd(i, k)g'L_{A_I})^*(\gamma^{A_J})^* = (A_{\gamma}^{-1}hd(i, k)g'L_{A_I})^*
\]

For the groups, note $Z'(P_0)_{\gamma^{A_I}} = Z'(P_0)$ since $\gamma^{A_J} \in Z'(P_0, M)$.

For $g'$, suppose $\sigma(i)g'L$ is replaced by $\sigma(i)g'Lz$ with $z \in Z'(P_I)$. Then $z^{A_I} \in Z'(P_0)$, so $A_{\gamma}^{-1}hd(i, k)g'L_{zA_I})^* = (A_{\gamma}^{-1}hd(i, k)g'L_{A_I})^* and Z'(P_0)z^{A_I} = Z'(P_0)$. So this substitution for $g$ has no effect. On the other hand, suppose $\sigma(i)g'L$ is
replaced by $\gamma \sigma(i) g'Lz$ with $\gamma \in \Gamma \cap \Gamma'$. Define $\gamma' = \gamma^{-1} \in \Gamma$. Since $\sigma(i) g'L$ has been changed, $h$ must be changed too, call it $h'$. Then

$$A_j^{-1} h' \gamma' s \sigma(i) g'L A_I = A_j^{-1} h' \gamma' d(i, k) g'L A_I$$

So, by setting $h' = h \gamma$, we get the same term $A_j^{-1} h d(i, k) g'L A_I$ as we did for the original $\sigma(i) g'L$. So this substitution for $g$ has no effect either.

So $T_s$ is a well-defined map. However, it is not clear that 4.4.14 defines a $\text{Hecke}(\Gamma, \mathbb{Q_N})$ action: it may be that $T_s(T_i \alpha) \neq T_i(T_s \alpha)$. For the moment, just think of all of the $T_s$ as being well-defined maps with no assumptions about their commuting. We will address commutativity after we move the maps to

$$(\hat{H}^c_1(T_{p-1}(\text{cbr}[\pi_0]), M')) \otimes \mathbb{F}[C(\zeta))] \otimes H^c_1(\Gamma_m, M'').$$

4.4.15 **PROPOSITION.** If $M$ is an admissible $\Gamma(\mathbb{Q_N})$ module,

$$\hat{H}_2^*(\Gamma, M) \to \hat{H}^*(Z'(P_0), M) \otimes \mathbb{F}[C(\zeta)]$$

is $T_s$-equivariant for all $s = T_n(l, k)$, $l$ prime with $l \nmid pN$, $k$ from 0 to $n$.

**PROOF.** For this proof, let $T_s$ denote the $\text{Hecke}$ action on $\hat{H}^*(\Gamma, M)$ and $S_s$ the $\text{Hecke}$ action on $\hat{H}^*(Z'(P_0), M) \otimes \mathbb{F}[C(\zeta)]$. Let $f$ be the map $\hat{H}_2^*(\Gamma, M) \to \hat{H}^*(Z'(\pi_0), M) \otimes \mathbb{F}[C(\zeta)]$, so $[f(\alpha)]_l = A_I^* \text{Res}_{Z'(P_1)} \alpha$. We need to prove that, for $\alpha \in \hat{H}^*(\Gamma, M)$, then $f(T_s \alpha) = S_s f(\alpha)$. 

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For ease of notation, set $t = h d(i, k) g^' L$ for this proof. From the definition of $h$ in 4.4.7, $\pi_J = \pi^J_I$.

By 4.4.10,

$$T_s(\alpha)|_{Z'(P_I)} = \sum_{\sigma(i)g' L \in D'(P_I)} Tr^t_{Z'(P_I) \cap \Gamma^t_i, Res^t_{Z'(P_I) \cap \Gamma^t_i}(t)^* \alpha}.$$  

So, composing this with $F''$ in order to get the map $f$, we have:

$$f(T_s(\alpha)) = \sum_{\sigma(i)g' L \in D'(P_I)} A^t_I Tr^t_{Z'(P_I) \cap \Gamma^t_i, Res^t_{Z'(P_I) \cap \Gamma^t_i}(t)^* \alpha}$$
$$= \sum_{\sigma(i)g' L \in D'(P_I)} Tr^t_{Z'(P_I) \cap \Gamma^t A_I, Res^t_{Z'(P_I) \cap \Gamma^t A_I}(t A_I)^* \alpha}$$

(4.4.16)

(the last follows from the commutativity of transfer and conjugation (see 2.2.3) and the commutativity of restriction and conjugation (see 2.2.2)).

Observe that if $x \in Z'(P_0)$ then $x^{(t A_I)^{-1}}$ commutes with $P_J$, with $P_J$'s being the group determined by $I$ and $g$ from 4.4.11. So $Z'(P_0) \cap \Gamma^t A_I = Z'(P_0) \cap Z'(P_J)^{t A_I}$. But $Z'(P_J) = Z'(P_0)^{A_J^{-1}}$. For ease of notation, set $H = Z'(P_0) \cap Z'(P_0)^{A_J^{-1} t A_I}$.

So the equation above becomes:

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\[ f(T_s(\alpha))_I = \sum_{\sigma(t) \sigma' \delta \in D'(P_1)} Tr^Z(P_0) Res^\Gamma_{Z'(P_0)} (tA_I)^* \alpha \]

\[ = \sum_{D'(P_1)} Tr^Z(P_0) Res^\Gamma_{Z'(P_0)} (tA_I)^* \alpha \]

\[ = \sum_{D'(P_1)} Tr^Z(P_0) Res^\Gamma_{Z'(P_0)} (tA_I)^* \alpha \]

\[ = \sum_{D'(P_1)} Tr^Z(P_0) Res^\Gamma_{Z'(P_0)} (tA_I)^* \alpha \]

\[ = S_s(f(\alpha)). \]

For the rest of this section we assume:

4.4.17 ALTERED DEFINITION. \( M \) is an admissible right \( Z_{T(\mathbb{Q}_N)}(P_0) \)-module such that:

1) As a \( Z'(P_0, \mathbb{Q}_N) \) module, \( M = M' \otimes M'' \) with \( M' \) a \( U_{p-1}(P_0, \mathbb{Q}_N) \) module and \( M'' \) a \( G_{m}(P, \mathbb{Q}_N) \) module.

2) The action of \( \Delta(\mathbb{Q}_N) \) on \( M \) respects the tensor product decomposition of \( M \) in (1). We mean by this that \( M' \) admits a \( \Delta(\mathbb{Q}_N)U_{p-1}(P_0, \mathbb{Q}_N) \) action (and \( M'' \) admits a \( \Delta(\mathbb{Q}_N)G_{m}(P_0, \mathbb{Q}_N) \) action), compatible with the actions in (1), such that for all \( m' \otimes m'' \in M \) and \( d \in \Delta(\mathbb{Q}_N) \),

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\[(m' \otimes m'')d = (m'd) \otimes (m''d).\]

Since, by 4.3.9, \(Z'(P_0) = U_{p-1}(P_0) \times G^1_m(P_0)\), we can use the Kunneth formula as in 4.2.15 (see X.5.8 of [Br]) to say

\[
\hat{H}^*(Z'(P_0), M) \cong \sum_{c_1 + c_2 = \ast} \hat{H}^{c_1}(U_{p-1}(P_0), M') \otimes H^{c_2}(G^1_m(P_0), M'') \quad \text{and} \quad
\hat{H}^*(Z'(P_0, Q_N), M) \cong \sum_{c_1 + c_2 = \ast} \hat{H}^{c_1}(U_{p-1}(P_0, Q_N), M') \otimes H^{c_2}(G^1_m(P_0, Q_N), M'')
\]  

(4.4.18)

for \(M = M' \otimes M''\) with \(M'\) a \(Z'(P_0, Q_N)\) module and \(M''\) a \(G^1_m(P_0, Q_N)\) module.

We remark, as we did for 4.2.15, that \(H^{c_2}(G^1_m(P_0), M'')\) is ordinary cohomology, not Farrell cohomology. We needed to define the Kunneth formula for both \(\hat{H}^*(Z'(P_0), M)\) and \(\hat{H}^*(Z'(P_0, Q_N), M)\) because we will want to use 2.2.5 and that means we need a direct product that includes \(Z'(P_0)\) and matrices that we are conjugating with. We will also conjugate by matrices in \(\Delta(Q_N)\), which is not wholly in the direct product \(Z'(P_0, Q_N)\); We invent a separate technical lemma to show how these matrices respect the Kunneth formula.

**4.4.19 Lemma.** Let \(M\) be a tensor \(M' \otimes M''\) as in 4.4.17, so \(\hat{H}^*(Z'(P_0), M)\) obeys the Kunneth formula 4.4.18. If \(d \in \Delta(Q_N)\) and \(\alpha \otimes \beta \in \hat{H}^*(Z'(P_0), M)\), then

\[
d^*(\alpha \otimes \beta) = (d^* \alpha) \otimes (d^* \beta)
\]

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where $d^*$ as usual is the conjugation map.

**Proof.** We need to examine the isomorphism

$$\hat{H}^*(Z'(P_0), M) \cong \sum_{c_1+c_2=\ast} \hat{H}^{c_1}(U_{p-1}(P_0), M') \otimes H^{c_2}(G_m(P_0), M'')$$

on the cochain level. Let $F(U) = \{F_i(U)\}$ be a complete resolution of $U_{p-1}(P_0)$ and $F(G) = \{F_i(G)\}$ be a $p$-finitely non-zero resolution of $G_m^1(P_0)$. Then

$$f : \text{Hom}_{U(P_0)}(F(U), M') \otimes \text{Hom}_{G^1(P_0)}(F(G), M'')$$

$$\rightarrow \text{Hom}_{U(P_0) \times G^1(P_0)}(F(U) \otimes F(G), M' \otimes M'')$$

is defined by $f(\alpha \otimes \beta)(x \otimes y) = \alpha(x) \otimes \beta(y)$ for all chains $x \otimes y$.

Since $d$ commutes with $Z'(P_0)$, the conjugation map is induced by the identity map on $Z'(P_0)$ and the module map $m \mapsto md$. So $d^*$ can be defined using the resolution $\text{Hom}(F(U) \otimes F(G), M' \otimes M'')$; it is

$$d^*(\bar{\mu})(x \otimes y) = \bar{\mu}(x \otimes y).d$$

for any cochain $\bar{\mu} \in \text{Hom}_{U(P_0) \times G^1(P_0)}(F(U) \otimes F(G), M' \otimes M'')$ and chains $x \otimes y$ (In general it is tricky to define conjugation maps using arbitrary resolutions. Usually we use the standard resolution of a group big enough to contain what we are conjugating by). The conjugation map is defined the same way for $\text{Hom}_{U(P_0)}(F(U), M')$ and $\text{Hom}_{G^1(P_0)}(F(G), M'')$. 

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Hence the lemma is clearly true in terms of cochains, so true for cohomology.

Now note 4.4.18 gives the isomorphisms

\[
\bigoplus_{J \in \mathcal{C}(\zeta)} H^i(Z'(P_J), M) \cong H^i(Z'(P_0), M) \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)] \\
\cong \sum_{c_1 + c_2 = *} \hat{H}^{c_1}(U_{p-1}(P_0), M') \otimes H^{c_2}(G^1_m(P_0), M'') \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)] \\
\cong \left( \hat{H}^{c_1}(U_{p-1}(P_0), M') \otimes \mathbb{F}[C(\zeta)] \right) \otimes H^{c_2}(G^1_m(P_0), M'') \quad (4.4.20)
\]

4.4.21 Lemma. Let (\alpha \otimes J) \otimes \beta \in \left[ \hat{H}^*(U_{p-1}(P_0), M') \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)] \right]$ for some $\alpha \in \hat{H}^*(U_{p-1}(P_0), M')$ and $\beta \in H^*(G^1_m(P_0), M'')$. Recall that the Hecke action defined in 4.4.14 above is:

\[
(T_s \alpha)_I = \sum_{\sigma(i) \in D'(P_I)} T_{\sigma(i)}^{Z'(P_0)} \hat{Z}'(P_0) \cap Z'(P_0) \otimes_{A_J^{-1} \text{hd}(i,k) g'LA_I} \left( T_{\sigma(i)}^{G^1_m(P_0)} \cap (G^1_m(P_0))^{y_1 \in \mathbb{F}} \right) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qa...
where $y_1$ is in $\Delta_n(Q_N)U_{p-1}(P_0,Q_N)$ and is $(p-1,m)$-block diagonal with upper left corner $\tilde{A}_j^{-1}\tilde{h}s_{p-1}(k)g_{p-1}\cpr[L]\tilde{A}_I$ and bottom right corner either $I_m$ or $\diag_m(l)$; and $y_2$ is in $\Delta_n(Q_N)G_m(P_0,Q_N)$ and is $(p-1,m)$ block diagonal with bottom right corner in $\Gamma_m^1s_m(j) \cap \Gamma_m^1(Q_N)$ and upper left corner either $I_{p-1}$ or $\diag_{p-1}(l)$.

**Proof.** We will start with 4.4.14 and slowly work our way through the conjugation map, the restriction map, and the transfer map.

Remember $\tilde{A}_j^{-1}\tilde{h}s_{p-1}(k)g_{p-1}\cpr[L]\tilde{A}_I$ is the upper left $(p-1)$ corner of the term that appeared in 3.3.18, the definition of Hecke action for the minimal case of a form 2 matrix with $n=p-1$. For ease of notation, set $t = A_j^{-1}hd(i,k)g'LA_I$ for this proof. So $\cpr[t] = \tilde{A}_j^{-1}\tilde{h}s_{p-1}(k)g_{p-1}\cpr[L]\tilde{A}_I$.

Notice that all the matrices in the product $t$ are $(p-1,m)$-upper block triangular. Let $h_2$ be the bottom right corner of $h$. The bottom right corner of $t$ is $h_2s_m(j)$ (since $h$ and $d(i,k)$ are the only matrices that are not $I_m$ in the bottom right corner) and by the definition of $h$ in in 4.4.11 then $h_2 \in \Gamma_m$ and $h_2s_m(j) \in \Gamma_m^1(Q_N)$. Define $l_1 = l$ if the $(p,p)$-th entry of $d(i,k)$ is $l$ (i.e. if $i > k$) and $l_1 = 1$ otherwise (i.e. if the $(p,p)$-th entry of $d(i,k)$ is 1), and define $d = \diag_n(l_1)$. Then

$$d^{-1}t \in Z'(P,Q_N)$$

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Since $d^{-1}t$ is in a group that is a direct product, we want to write $d^{-1}t$ as an element $z_1$ in $U_{p-1}(P_0, \mathbb{Q}_N)$ times an element $z_2$ in $G^1_m(P_0, \mathbb{Q}_N)$. The $G^1_m(P_0, \mathbb{Q}_N)$ element is completely determined by the bottom right $m$ corner of $d^{-1}t$; the bottom right corner is $\text{diag}_m(l_1^{-1}) h_2 s_m(j)$. So the $G^1_m(P_0, \mathbb{Q}_N)$ element in $d^{-1}t$ is $z_2 = \begin{pmatrix} I_{p-1} & 0 \\ 0 & \text{diag}_m(l_1^{-1}) h_2 s_m(j) \end{pmatrix}$. Now looking at upper left $(p-1)$-corners, we see that the $U_{p-1}(P_0, \mathbb{Q}_N)$ element of $d^{-1}t$ is $z_1 = \begin{pmatrix} \text{diag}_{p-1}(l_1^{-1}) \text{cnr}[t] & 0 \\ 0 & I_m \end{pmatrix}$.

Define $y_1 = dz_1 = \begin{pmatrix} \text{cnr}[t] & 0 \\ 0 & \text{diag}_m(l_1) \end{pmatrix}$ and $y_2 = dz_2 = \begin{pmatrix} \text{diag}_{p-1}(l_1) & 0 \\ 0 & s_m(j) \end{pmatrix}$.

Note that $y_1$ and $y_2$ are matrices with properties as specified in 4.4.21.

Now, by 2.2.5 and 4.4.19,

$$t^*(\alpha \otimes \beta) = (d^{-1}t)^*d^*(\alpha \otimes \beta) = (z_1 z_2)^*d^*(\alpha \otimes \beta)$$

$$= (z_1^* d^* \alpha) \otimes (z_2^* d^* \beta) = (y_1^* \alpha) \otimes (y_2^* \beta) \quad (4.4.22)$$

This takes care of conjugation.

Since $U_{p-1}(P_0, \mathbb{Q}_N)$ consists of polynomials in $\pi_0$,

$$\pi_0^t = \pi_0 \implies U_{p-1}(P_0) \subset Z'(P_0)^t$$

So $Z'(P_0) \cap Z'(P_0)^t = U_{p-1}(P_0) \times \left[ Z'(P_0)^t \cap G^1_m(P_0) \right]$. So, by 2.2.6,

$$\text{Res}_{Z'(P_0)^t \cap Z'(P_0)^t} \left( y_1^* \alpha \otimes y_2^* \beta \right) = \text{Res}_{U_{p-1}(P_0)^t \times G^1_m(P_0)^t} \left[ U_{p-1}(P_0) \times \left[ Z'(P_0)^t \cap G^1_m(P_0) \right] \right] \left( y_1^* \alpha \otimes y_2^* \beta \right)$$

$$= y_1^* \alpha \otimes \left( \text{Res}_{Z'(P_0)^t \cap G^1_m(P_0)} y_2^* \beta \right) = y_1^* \alpha \otimes \left( \text{Res}_{G^1_m(P_0)^t \cap G^1_m(P_0)} y_2^* \beta \right).$$

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This takes care of restriction.

Set $\alpha' \otimes \beta' = y_i^* \alpha \otimes \left( \text{Res}_{(G_m^1)^{p_2}}^G \text{Res}_{(G_m^1)^{p_2}}^G y_i^* \beta \right)$. Now by 2.2.7,

\[
T_{(G_m^1)^{p_2}}^G \left( \alpha' \otimes \beta' \right) = T_{(G_m^1)^{p_2}}^G \left( \alpha' \otimes \beta' \right)
\]

This takes care of transfer.

Finally, note that by 4.4.3

\[
\sum_{\sigma(i)g' \in D'(P)} \sum_{i=0}^{i=k} g_{i+1} \in D'_{p-1}(i, \text{cnr}[\pi_j])
\]

4.4.23 Claim. For $s = T_n(l, k)$ and

\[
\mu = \sum_b \alpha_b \otimes \beta_b \in \left[ \hat{H}^*(U_{p-1}(P_0), M') \otimes_F \mathbb{F}[C(\zeta)] \right] \otimes H^*(G_m^1(P_0), M'')
\]

then

\[
T_s(\mu) = \sum_b \sum_{i=0}^{k} (T_{s_{p-1}(i)} \alpha_b) \otimes T_{s_m(k-i)} \beta_b
\]

where $T_{s_{p-1}(i)} \alpha_b$ is the Hecke action of $T_{p-1}(l, i)$ on $\hat{H}^*(\Gamma_{p-1}(P_0), M') \otimes_F \mathbb{F}[C(\zeta)]$ and $T_{s_m(j)} \beta_b$ is the Hecke action of $T_m(l, j)$ on $H^*(\Gamma_m^1(P_0), M'')$ and $T_{s_m(j)} \beta_b$ is the Hecke action of $T_m(l, j)$ on $H^*(\Gamma_m^1(P_0), M'')$. and $T_{s_m(j)} \beta_b$ is the Hecke
action of $h_2 T_m(l, j)$ on $H^* (\Gamma_m^1 (P_0), M'')$, where $h_2$ is any element in $\Gamma_m$ such that $h_2 T_m(l, j) \in \Gamma_m^1 (\mathbb{Q}_N)$.

**Proof.** Note that the Hecke action on $\hat{H}^* (\Upsilon_{p-1} (P_0), M') \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)]$ is defined in 3.3.22; the Hecke action on $H^* (\Gamma_m^1, M'')$ is defined in 2.1.14; and the Hecke action on $\left[ \hat{H}^* (U_{p-1} (P_0), M') \otimes_{\mathbb{F}} \mathbb{F}[C(\zeta)] \right] \otimes H^* (\Gamma_m^1 (P_0), M'')$ is defined in 4.4.12.

The claim follows as a consequence of Lemma 4.4.21. Note that under the isomorphism $U_{p-1} (P_0, \mathbb{Q}_N) = \Upsilon_{p-1} (P_0, \mathbb{Q}_N)$, then $y_1^*$ acting on $\hat{H}^* (U_{p-1} (P_0), M')$ becomes $(\text{cnr}[y_1])^*$ acting on $\hat{H}^* (\Upsilon_{p-1} (P_0), M')$. Under the isomorphism $G_m^1 (P_0, \mathbb{Q}_N) = \Gamma_m^1 (\mathbb{Q}_N)$, then $y_2^*$ acting on $H^* (G_m^1 (P_0), M'')$ becomes $[h_2 s_m (j)]^*$ acting on $H^* (\Gamma_m^1, M'')$. Hence the claim follows provided that we can prove that $y_1$ has the correct upper left $(p-1)$ corner, i.e. that

\[(\tilde{A}_j^{-1} \tilde{h} s_{p-1} (k) g_{p-1} \text{cnr}[L] \tilde{A}_I)^* = (\tilde{A}_j^{-1} \tilde{h}' s_{p-1} (k) g_{p-1} \tilde{A}_I)^* \]  \hspace{1cm} (4.4.24)

where the left side of this equation is $y_1^*$ from 4.4.21 defined for form 2 cohomology; the right side is the Hecke action defined in 3.2.3 (defined for form 1 cohomology), with the symbol $\tilde{h}'$ used in place of $\tilde{h}$, since the form 1 and form 2 variations of $h$ are not the same.

The definitions for $\tilde{h}$ and $\tilde{h}'$ are:

By 4.4.11, $\tilde{h}$ satisfies

\[\tilde{\pi}_I = \pi_j^{\tilde{h} s_{p-1} (k) g_{p-1} \text{cnr}[L]}\]
and, by 3.2.3, \( \tilde{h}' \) is any matrix that satisfies

\[
\tilde{\pi}_I = \pi_j^{h's_{p-1}(k)g_{p-1}}
\]

Remember from the definition of \( L \) in 2.4.9 that \( \text{cnr}[L] \in \mathbb{Z}_{p-1}(\text{cnr}[\pi_I]) \).

Write

\[
L = L_1L_2 \text{ with } L_1 = f(\pi_I) \text{ and } L_2 = \begin{pmatrix} I_{p-1} & * \\ 0 & I_m \end{pmatrix}.
\]

From the definition of \( h \) in 4.4.11,

\[
hsgL = hsgf(\pi_I)L_2 = f(\pi_J)hsgL_2.
\]

So the upper left corner of \( f(\pi_J)h \) satisfies the definition of \( \tilde{h}' \), and we can replace \( \tilde{h}' \) in 4.4.24 with \( \text{cnr}[f(\pi_J)]\tilde{h} \). Now, since \( \text{cnr}[hsgL] = \text{cnr}[f(\pi_J)hsg] \), 4.4.24 holds.

\[\square\]

§5 Tensoring Galois Representations

We wish to use the results of the previous section to make statements about attaching Galois representations to Hecke eigenvectors.

Since the previous section only gives information about cohomology when the coefficient module is a tensor product, we first want to show that any system of Hecke eigenvalues occurs in a cohomology group with a coefficient module that is a tensor product.
Consider the form 1 case of $\hat{H}^*_1(\Gamma, M)$. By 4.2.13, the map $\text{Res}^\Gamma_B : \hat{H}^*_1(\Gamma, M) \rightarrow \hat{H}^*(B, M)$ is Hecke-invariant. So any system of Hecke eigenvalues occurring in $\hat{H}^*_1(\Gamma, M)$ also occurs in $\hat{H}^*(B, M)$.

4.5.1 Claim. Let $M$ be an admissible $B(\mathbb{Q}_N)$-module. Any system of Hecke eigenvalues occurring in $\hat{H}^*(B, M)$ occurs, possibly after enlargement of $F$, in $\hat{H}^*(B, M' \otimes M'')$ for some $M' \otimes M''$ as in 4.2.14, i.e., with $M'$ an admissible $\Gamma_{p-1}(\mathbb{Q}_N)$-module and $M''$ an admissible $\Gamma_m(\mathbb{Q}_N)$-module.

Proof. Enlarge $F$ so it contains all the eigenvalues of Hecke acting on $\hat{H}^*(B, M)$.

First we need $M$ to be irreducible. If $M$ is not irreducible, then, since $M$ is finite dimensional, there is a short exact sequence of admissible modules

$$0 \rightarrow M_1 \xrightarrow{f_1} M \xrightarrow{f_2} M_2 \rightarrow 0$$

with $M_2$ irreducible and $M_1$ of smaller dimension than $M$.

This induces the long exact sequence

$$\cdots \rightarrow \hat{H}^i(B, M_2) \xrightarrow{f_2^*} \hat{H}^i(B, M) \xrightarrow{f_1^*} \hat{H}^i(B, M_1) \rightarrow \cdots.$$

On the cochain level, $f_1^*$ and $f_2^*$ are the maps $f_1$ and $f_2$ on the modules and the identity on the group $B$. So these maps are equivariant for transfer, restriction, and conjugation maps, hence equivariant for the Hecke action (Recall the definition of Hecke action in 4.2.9).
Consider a Hecke eigenvector $\alpha \in \hat{H}^*(B, M)$. If $f_1^*(\alpha) \neq 0$, then $f_1^*(\alpha) \in \hat{H}^*(B, M_1)$ is a Hecke eigenvector with the same system of eigenvalues. If $f_1(\alpha) = 0$, look at $K = f_2^{-1}(F\alpha)$. This is a finite dimensional vector space, non-empty by exactness, and preserved by all $T_s$ since $\alpha$ is a simultaneous eigenvector. Since $F$ is contains all eigenvalues, $K$ must contain eigenvectors for each $T_s$. Since the $T_s$ commute, there must be a simultaneous eigenvector, call it $\alpha'$. Since $f_2(\alpha') = \alpha$, then $\alpha'$ must have the same system of eigenvalues as $\alpha$.

So $\alpha \in \hat{H}^*(B, M)$ a Hecke eigenvector $\implies$ the same system of eigenvalues either shows up in $\hat{H}^*(B, M_2)$, with $M_2$ irreducible, or in $\hat{H}^*(B, M_1)$, with $M_1$ of smaller dimension. By induction on dimension of $M$, $\alpha$'s system of eigenvalues appears in $\hat{H}^*(B, M_1)$ for some irreducible module $M_1$.

So assume $M$ is irreducible. Since $M$ is admissible, the action of $B(\mathbb{Q}_N)$ on $M$ factors through a finite group. Therefore any statement that is true for modules acted on by a finite group is true for admissible modules.

Enlarge $F$ so that it is a splitting field for the action of $B(\mathbb{Q}_N)$ on $M$. By Theorem 9.14 of [H], $B(\mathbb{Q}_N)$'s being a direct product and and $M$'s being irreducible together imply $M$ is a tensor product. $\blacksquare$
Now consider the form 2 case of \( \hat{H}_2^*(\Gamma, M) \). We will treat this similarly to the way we treated the form 1 case. By 4.4.15, the map

\[
\hat{H}_2^*(\Gamma, M) \to \hat{H}^*(Z'(P_0), M) \otimes_F \mathbb{F}[C(\zeta)]
\]

is Hecke-invariant. So any system of Hecke eigenvalues occurring in \( \hat{H}_2^*(\Gamma, M) \) also occurs in \( \hat{H}^*(Z'(P_0), M) \otimes_F \mathbb{F}[C(\zeta)] \).

4.5.2 Claim. Let \( M \) be an admissible \( Z_{\Gamma(\mathbb{Q}_N)}(P_0) \)-module. Any system of Hecke eigenvalues occurring in \( \hat{H}^*(Z'(P_0), M) \otimes_F \mathbb{F}[C(\zeta)] \) occurs, possibly after enlargement of \( F \), in \( \hat{H}^*(Z'(P_0), M' \otimes M'') \otimes_F \mathbb{F}[C(\zeta)] \) for some \( M' \otimes M'' \) as in 4.4.17, i.e. with \( M' \) an admissible \( \Gamma_{p-1}(\mathbb{Q}_N) \)-module, \( M'' \) an admissible \( \Gamma_\mathfrak{m}^1(\mathbb{Q}_N) \)-module, and the scalar matrices \( \Delta(\mathbb{Q}_N) \) respecting this tensor product decomposition.

Proof. For ease of notation, set \( H^*(M) = \hat{H}^*(Z'(P_0), M) \otimes_F \mathbb{F}[C(\zeta)] \) for this proof.

Enlarge \( F \) so it contains all the eigenvalues of Hecke acting on \( H(M) \).

First we need \( M \) to be irreducible. If \( M \) is not irreducible, then, since \( M \) is finite dimensional, there is a short exact sequence of admissible modules

\[
0 \to M_1 \xrightarrow{f_1} M \xrightarrow{f_2} M_2 \to 0
\]

with \( M_2 \) irreducible and \( M_1 \) of smaller dimension than \( M \).
This induces the long exact sequence

\[ \cdots \to \hat{H}^i(Z'(P_0), M_2) \overset{f_2^*}{\to} \hat{H}^i(Z'(P_0), M) \overset{f_2}{\to} \hat{H}^i(Z'(P_0), M_1) \to \cdots \]

Taking direct sums, we get the long exact sequence

\[ \cdots \to H^i(M_2) \overset{f_2^*}{\to} \hat{H}^i(M) \overset{f_2}{\to} \hat{H}^i(M_1) \to \cdots \]

On the cochain level, \( f_1^* \) and \( f_2^* \) are the maps \( f_1 \) and \( f_2 \) on the modules and
the identity on the group \( Z'(P_0) \). So these maps are equivariant for transfer, restriction, and conjugation maps. Comparing with the definition of Hecke action
4.4.14, \( f_1^* \) and \( f_2^* \) must be Hecke-invariant.

Since we have a long exact sequence that is Hecke-invariant, we get, by exactly
imitating 5.4.1, that \( M \) can be assumed to be an irreducible \( Z_{\Gamma(\mathbb{Q}_N)}(P_0) \)-module.

Enlarge \( \mathbb{F} \) so that it is a splitting field for the action of \( Z_{\Gamma(\mathbb{Q}_N)}(P_0) \) on \( M \).
Consider how the scalar matrices \( \Delta(\mathbb{Q}_N) \) act on \( M \). Let \( d \in \Delta(\mathbb{Q}_N) \). Since \( \mathbb{F} \)
contains all eigenvalues, \( d \) commutes with every element in \( Z_{\Gamma(\mathbb{Q}_N)}(P_0) \), and \( M \)
is irreducible, it must be that \( d \) acts as a scalar on \( M \) (i.e \( d \) has exactly one
eigenvalue and every element in \( M \) is an eigenelement).

So, since \( Z_{\Gamma(\mathbb{Q}_N)}(P_0) = \Delta(\mathbb{Q}_N)Z'(P_0, \mathbb{Q}_N) \), \( M \) is an irreducible \( Z'(P_0, \mathbb{Q}_N) \)-module. Enlarge \( \mathbb{F} \) so that it is a splitting field for the action of \( B(\mathbb{Q}_N) \) on \( M \).
Then by Theorem 9.14 of [H], \( M \) is a tensor product. Write \( M = M' \otimes_{\mathbb{F}} M'' \) with
with \( M' \) an admissible \( \Upsilon_{p-1}(\mathbb{Q}_N) \)-module and \( M'' \) an admissible \( \Gamma_m^1(\mathbb{Q}_N) \)-module.
To show $M' \otimes_F M''$ has a compatible $\Delta(Q_N)$ structure, we can artificially force a structure. Since $\Delta(Q_N)$ commutes with both $\Upsilon_{p-1}(Q_N)$ and $\Gamma_m^1(Q_N)$ and has empty intersection with them, any $\Delta_m(Q_N)$-structure on $M'$ and $M''$ is compatible with their internal $\Upsilon_{p-1}(Q_N)$- and $\Gamma_m^1(Q_N)$-structures. Let $d \in \Delta(Q_N)$. Let $f \in F$ be the element such that $m.d = m.f \ \forall \ m \in M$. Make $M'$ and $M''$ into $\Delta_m(Q_N)$-modules by defining $m'.d = m'.f \ \forall \ m' \in M'$ and $m''.d = m'' \ \forall \ m'' \in M''$. This is a compatible $\Delta_m(Q_N)$-structure since

$$(m' \otimes m'')d = (m' \otimes m'').f = m'f \otimes m'' = m'.d \otimes m''.d$$

So from now on assume $M = M' \otimes M''$.

Now let us restate the results of the previous section in a convenient form.

4.5.3 DEFINITIONS. For the form 1 case of $\hat{H}^*(B, M)$ and $n \geq p$, define

$G = \hat{H}^*(B, M)$.

$G_1 = \hat{H}^*(\Upsilon_{p-1}, M').$

$G_2 = H^*(\Gamma_m^1, M'').$

For the form 2 case of $\hat{H}^*(Z'(P_0), M) \otimes_F F[C(\zeta)]$ and $n > p$, define

$G = \hat{H}^*(Z'(P_0), M) \otimes_F F[C(\zeta)]$.

$G_1 = \hat{H}^*(\Upsilon_{p-1}, M') \otimes_F F[C(\zeta)]$.

$G_2 = H^*(\Gamma_m^1, M'')$. 

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Then $G$ is a $\text{Hecke}(\Gamma, \mathbb{Q}_p N)$-module, $G_1$ is a $\text{Hecke}(\Gamma_{p-1}, \mathbb{Q}_p N)$-module, and $G_2$ is a $\text{Hecke}(\Gamma_m, \mathbb{Q}_p N)$-module.

By the Kunneth formula (see 4.2.15 and 4.4.18), $G \cong G_1 \otimes G_2$.

The main result of section 4, achieved in 4.2.23 and 4.4.23, is that the map $G \to G_1 \otimes G_2$ is $\text{Hecke}$-equivariant under the map

$T_s \mapsto \sum_{i=0}^{k} T_{s_{p-1}(i)} \otimes T_{s_m(k-i)}$.

Since 4.5.1 and 4.5.2 showed that all the $\text{Hecke}$ systems of eigenvalues that we are interested in do in fact occur in some $G$, we want to connect eigenvectors in $G$ with eigenvectors in $G_1$ and $G_2$.

4.5.4 Claim. If $\mu \in G$ is a $\text{Hecke}$ eigenvector, then, possibly after an enlargement of $F$, its system of eigenvalues shows up in an element of the form $\alpha \otimes \beta \in G_1 \otimes H^*(\Gamma_m, M'')$ with $\alpha \in G_1$ a $\text{Hecke}(\Gamma_{p-1}, \mathbb{Q}_p N)$-eigenvector and $\beta \in H^*(\Gamma_m, M'''')$ a $\text{Hecke}(\Gamma_m, \mathbb{Q}_p N)$-eigenvector. In the form 1 case then $M''' = M''$; in the form 2 case then it is just some admissible module.

Proof. Since $G, G_1$, and $G_2$ are all finite dimensional $F$-spaces, $\text{Hecke}$ factors through a finite group in each case and so can be regarded as a finite group of commuting transformations.

Enlarge $F$ so that it contains all the eigenvalues of all the transformations.
Now the result for the form 1 case follows immediately from Appendix B. For the form 2 case, the result almost follows except with \( \beta \in H^*(\Gamma_m(1), M'') \) instead of \( H^*(\Gamma_m, M'') \). We remark that the Hecke algebra \( \text{Hecke}(\Gamma_m(1), \mathbb{Q}) \) for \( \Gamma_m(1) \) is a polynomial ring which has as its indeterminates the set \( \{ h_2 T_m(l, i) \}_{l, i} \) with \( l \) ranging over all primes not dividing \( pN \), \( i \) ranging from 0 to \( m \), and \( h_2 \in \Gamma_m \) chosen in each case so that \( h_2 T_m(l, i) \in \Gamma_m(1)(\mathbb{Q}) \) (the exact choice of \( h_2 \) does not matter since any choice yields the same element in the Hecke algebra).

By Lemma 1.6 of [A1], a system of Hecke eigenvalues that occurs in some \( \hat{H}^i(\Gamma_m(1), M) \) also occurs in \( \hat{H}^i(\Gamma_m, M') \) for some admissible module \( M' \).

So we can assume that any eigenvector \( \mu \in G \) has the form \( \mu = \alpha \otimes \beta \).

Fix an eigenvector \( \mu = \alpha \otimes \beta \).

Now to make it easier to manipulate attached Galois representations, recall the notation from 1.1 that, if \( \mu \) is any Hecke(\( pN \)) eigenvector in any Hecke(\( pN \))-module, \( P(\mu, l) \) denotes the Hecke polynomial of \( \mu \) at \( l \), which is defined by:

\[
P(\mu, l) = \sum_{i=0}^{p-1} (-1)^k k(k-1)/2 m(i, l) X^i
\]

where \( m(i, l) \) is the eigenvalue for \( T_n(l, i) \) of \( \mu \).
Fix an \( l \) and denote the eigenvalues of \( T_{p-1}(l, i) \otimes T_m(l, j) \) on \( \alpha \otimes \beta \) by \( a_i b_j \).

Consider the Hecke polynomial of \( \alpha \otimes \beta \) for this \( l \). Since eigenvalues multiply in tensor products, it is

\[
P(\alpha \otimes \beta, l) = \sum_{k=0}^{n} (-1)^k l^k(k-1)/2 \left( \sum_{i+j=k \leq p-1} a_i b_j \right) X^k
\]

On the other hand,

\[
P(\alpha, l)P(\beta, l) = \left( \sum_{i=0}^{p-1} (-1)^i l^i(k-1)/2 a_i X^i \right) \left( \sum_{j=0}^{m} (-1)^j l^j(k-1)/2 b_j X^j \right)
\]

\[
= \sum_{k=0}^{n} \sum_{i+j=k} (-1)^{(i+j)} \left[ \left( \frac{1}{2} i(i-1) + \frac{1}{2} j(j-1) \right) a_i b_j X^k \right] = \sum_{k=0}^{n} \sum_{i+j=k} (-1)^k l^k(k-1)/2 l^i a_i b_j X^k.
\]

From 6.1 of [A] and 3.2.5, the action of \( T_{p-1}(l, i) = 0 \) for both types of \( G_1 \), and so \( a_i = 0 \), unless \( d \mid i \), where \( d \) is the smallest integer such that \( l^d \equiv 1 \mod p \). So

\[
P(\alpha \otimes \beta, l) \equiv P(\alpha, l)P(\beta, l) \mod p.
\]

Note that characteristic polynomials multiply in the tensor product of representations. Putting all these facts together gives

**Proposition 4.5.5.** Let \( G \) be as in 4.5.3 Suppose \( \alpha \otimes \beta \in G_1 \otimes H^*(\Gamma_m, M'') \) with \( \alpha \) and \( \beta \) both Hecke eigenvectors. Suppose there exists a continuous semi-simple representation \( \tau : G_Q \to G l(m, \mathbb{F}) \) unramified outside \( N \) such that

\[
P(\beta, l) = \det(I - \tau(Frob_l)^{-1}X) \text{ for all } l \nmid N.
\]

Let \( \rho \) be a continuous semi-simple representation of \( G_Q \to G l(p-1, \mathbb{F}) \) such that \( P(\alpha, l) = \det(I - \rho(Frob_l)^{-1}X) \text{ for all } l \nmid pN. \)
Then $\rho \otimes \tau : G_Q \rightarrow GL(n,\mathbb{F})$ is a continuous semi-simple representation unramified outside $N$ such that $P(\alpha \otimes \beta, l) = \det(I - (\rho \otimes \tau)(\text{Frob}_{l})^{-1}X)$ for all $l \nmid pN$.

Note that the representation $\rho$ always exists, by Theorem .2 of [A] for $G_1 = \hat{H}^*(\Gamma_{p-1}, M')$ or by 3.2.5 for $G_1 = \hat{H}^*(\Gamma_{p-1}(P_0), M') \otimes \mathbb{F}[C(\zeta)]$.

A corollary of 4.5.5 is:

4.5.6 Theorem. Suppose $\Gamma_m$ has the property that, for any admissible module $M''$, every Hecke($pN$) eigenvector in $H^*(\Gamma_m, M)$ has an attached Galois representation.

Then any Hecke($pN$) eigenvector of $\hat{H}^*(\Gamma_{p-1+m}, M)$, for any admissible module $M$, has an attached Galois representation.
APPENDIX A
EQUIVALENCE OF DEFINITIONS OF HECKE ACTIONS

Let $G$ be a group and $\Gamma$ a subgroup of $G$.

Define $\tilde{\Gamma} = \{\alpha \in G \mid [\Gamma : \Gamma^\alpha \cap \Gamma] < \infty\}$, the commensurator subgroup of $\Gamma$.

Let $\Delta$ be a semigroup with $\Gamma \subset \Delta \subset \tilde{\Gamma}$. Then $\Delta$ defines the (associative) Hecke ring

$$\mathcal{R}(\Gamma, \Delta) = \left\{ \sum n_\alpha \Gamma \alpha \Gamma \mid \alpha \in \Delta, \ n_\alpha \in \mathbb{Z} \right\}$$

As an additive group $\mathcal{R}(\Gamma, \Delta)$ is the free abelian group on the set of double cosets.

It is easily seen that $\Gamma^\alpha \cap \Gamma$'s having finite index in $\Gamma$ is equivalent to $\Gamma$'s having finite index in $\Gamma \alpha \Gamma$. In fact, for $\alpha \in \Delta$,

$$\text{Fix } \Gamma = \prod_{i=1}^{n} \Gamma(\alpha) \alpha_i.$$  Then $\Gamma \alpha \Gamma = \prod_{i=1}^{n} \Gamma \alpha \alpha_i$.

For any $\gamma \in \Gamma$, define $\tau_i(\gamma) \in \Gamma$ and $i(\gamma) \in \{1, 2, \ldots, n\}$ by

$$\alpha \alpha_i \gamma = \tau_i(\gamma) \alpha \alpha_{i(\gamma)}.$$
Such $\tau_i(\gamma)$ and $i(\gamma)$ exist and are uniquely defined since \{\alpha i\} is a set of right coset representatives of $\Gamma \alpha \Gamma$.

Since we use a lot of ordered $(r + 1)$-tuples in the sequel, define $\tilde{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_r)$.

Define $F_i(\Gamma) = \{F_i(\Gamma)\}$ to be the standard (homogenous) resolution of $\Gamma$ over $\mathbb{Z}$. This means that for $r \geq 0$,

$$F_r(\Gamma) = \text{the free abelian group on the set } \Gamma \times \Gamma \times \cdots \times \Gamma \text{ (r+1 factors)}.$$ 

Turn $F_r(\Gamma)$ into a left $\Gamma$-module by defining $\gamma \cdot (\gamma_0, \gamma_1, \ldots, \gamma_r) = (\gamma \gamma_0, \gamma \gamma_1, \ldots, \gamma \gamma_r)$.

For $r = -1$, $F_{-1} = \mathbb{Z}$, a trivial left $\Gamma$-module. For $r < -1$, $F_r = 0$. The boundary maps, for $r > 0$, are $\delta(\gamma_0, \gamma_1, \ldots, \gamma_r) = \sum_{i=0}^r (\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_r)$ where $\hat{\gamma}_i$ means "omit $\gamma_i"$ (This defines $\delta$ on ordered $\Gamma$-tuples; extend linearly to all of $F_r(\Gamma)$).

For $r = 0$, the boundary map is the augmentation map $\delta(\gamma) = 1$.

For a right $\Gamma$-module $M$, an $r$-cochain $\tilde{f}$ is a linear map $\tilde{f} : F_r(\Gamma) \to M$ with

$$\tilde{f}(\gamma \gamma_0, \ldots, \gamma \gamma_r) = \gamma \cdot \tilde{f}(\gamma_0, \ldots, \gamma_r) \text{ for all } \gamma, \gamma_0, \ldots, \gamma_r \in \Gamma.$$ 

The additive group of all cochains is denoted $C^r(\Gamma, M) = \text{Hom}_{\Gamma}(F_r(\Gamma), M)$. It has a boundary map induced by the boundary map of $F(\Gamma)$. The cohomology induced by the boundary map is the group cohomology $H^*(\Gamma, M)$.

The standard way of defining the action of a Hecke ring on cohomology is this one from 2.1 of [R-W] (using the notation as above):
Let $M$ be a right $\Delta$-module. For $\Gamma \alpha \Gamma$ and $\tilde{f} \in C^r(\Gamma, M)$,

$$T_{\alpha} \tilde{f}(\gamma_0, \gamma_1, \ldots, \gamma_r) = \sum_{i=1}^{n} \tilde{f}(\tau_i(\gamma_0), \tau_i(\gamma_1), \ldots, \tau_i(\gamma_r)) \cdot \alpha \alpha_i.$$  

Since [R-W] uses left modules, they write $\alpha_i^{-1} \alpha^{-1}$ on the left instead of $\alpha \alpha_i$ on the right. Using right modules doesn’t change anything significantly since every right module can be changed into a left module by taking the opposite module. Also [R-W] calls $\alpha_i$ what we here call $\alpha \alpha_i$.

In [R-W] it is proved that, extending linearly to all of $\mathcal{R}(\Gamma, \Delta)$, this definition of $T_{\alpha}$ for $\alpha \in \Delta$ induces a well-defined action of $\mathcal{R}(\Gamma, \Delta)$ on $H^*(\Gamma, M)$. An interesting point about this is that this does not give a well-defined action on the set of cochains $C^r(\Gamma, M)$. If we change the representative of $\Gamma \alpha \Gamma$ from $\alpha$ to some $\gamma \alpha \gamma'$, then the cochain $T_{\alpha} \tilde{f}$ may change. However, if $\tilde{f}$ is a cocycle, the new cochain is equal to the old one modulo coboundaries from $C^{r-1}$.

Now recall the definition of the Hecke action that we use in this dissertation. Call it $S_{\alpha}$ to distinguish it from $T_{\alpha}$:

For $\Gamma \alpha \Gamma$ and $\tilde{f} \in H^*(\Gamma, M)$ we define $S_{\alpha} \tilde{f} = Tr_{\Gamma \alpha \Gamma}^\Gamma Res_{\Gamma \alpha \Gamma}^{\Gamma} \alpha^* f$

where $Tr$ and $Res$ are the usual transfer and restriction maps, and $\alpha^*$ is the conjugation map induced by $\Gamma \alpha \rightarrow \Gamma$ and $M \rightarrow M \alpha$.

The purpose of this section is to show:
Claim. \( T_\alpha = S_\alpha \)

Proof. We need to express the \( S_\alpha \) definition in terms of cochains, since \( T_\alpha \) is defined that way. Now \( \text{Res} \) and \( \alpha^* \) have obvious cochain definitions, but the definition of \( Tr \) is more complicated. Let us discuss transfer:

We follow III.9(D) of [Br], expanding his statements. For abstractedness, let \( H \subset K \) be groups with \( [K : H] \) finite and consider \( Tr^K_H \). Define \( F(H) \) to be the standard resolution of \( H \) and \( F(K) \) the standard resolution of \( K \). Since both \( F(H) \) and \( F(K) \) are resolutions of \( H \), \( Hom_H(F(K), M) \) and \( Hom_H(F(H), M) \) must induce the same cohomology groups by I.7 of [Br]. “Same” means there is a canonical isomorphism between them. We want to make this isomorphism explicit. This means, see I.7 of [Br], finding a chain complex map \( \rho \), necessarily unique up to homotopy, from \( F(K) \) to \( F(H) \) which is the identity for \( F_{-1}(H) = F_{-1}(K) = \mathbb{Z} \).

Decompose \( K \) into right cosets \( K = \bigsqcup_{i=1}^n H.h_i \) Define the function \( \sigma : K \to H \) which “takes the \( H \) part of an element” by, for all \( k \in K \), \( k = \sigma(k)h_i \) for some \( i \) depending on \( k \). Notice how similar \( \sigma \) is to \( \tau \) above.

Now define the chain complex map \( \rho : F(K) \to F(H) \) by \( \mathbb{Z} \to \mathbb{Z} \) is the identity in the \( \tau = -1 \) position and otherwise

\[(k_0, k_1, \ldots, k_r) \mapsto (\sigma(k_0), \sigma(k_1), \ldots, \sigma(k_r))\]
It is immediate from the definition that \( \rho \) is a left \( H \)-map and commutes with the boundary maps. So it is under this map \( \rho \) that \( F(K) \) and \( F(H) \) give isomorphic cohomology groups.

Considering \( F(K) \) as a resolution of both \( K \) and \( H \), we observe III.9(A) of [Br] defines the transfer map as

\[
\text{Hom}_H(F(K), M) \to \text{Hom}_K\left(F(K), \text{Hom}_H(\mathbb{Z}K, M)\right) \to \text{Hom}_K(F(K), M)
\]

\[
[f : (\tilde{\kappa}) \to m] \mapsto [g : (\tilde{\kappa}) \to f\tilde{\kappa}] \mapsto [Tr_f : (\tilde{\kappa}) \to \sum f\tilde{\kappa}(h_i).h_i]
\]

Here \( \tilde{\kappa} \) stands for an ordered \( K \) \( r \)-tuple. Remember \( K = \coprod H_i. \) The first map is the Shapiro isomorphism (III.6.2 of [Br]) so \( f\tilde{\kappa} \) is defined by \( f\tilde{\kappa}(k) = f(k\tilde{\kappa}) \). So the final map \( Tr_f \) is \( Tr_f(\tilde{\kappa}) = \sum f(h_i\tilde{\kappa}).h_i \)

This definition of transfer is not good for us because it uses \( F(K) \) as the resolution of \( K \) instead of the standard resolution. To define the transfer map from \( C^r(H, M) \) to \( C^r(K, M) \), compose the map \( \rho \ F(H) \to F(K) \) with the above map from \( \text{Hom}_H(F(K), M) \) to \( \text{Hom}_K(F(K), M) \). So for \( f \in C^r(H, M) \) and \( \tilde{\kappa} \in C^r(K, M) \), then

\[
Tr^K_H f(\kappa_0, \kappa_1, \ldots, \kappa_r) = \sum_{i=0}^{n} f(\sigma(h_i\kappa_0), \sigma(h_i\kappa_1), \ldots, \sigma(h_i\kappa_r)).h_i
\]

Note \((\sigma(h_i\kappa_0), \sigma(h_i\kappa_1), \ldots, \sigma(h_i\kappa_r)) \in C^r(H, M)\).

Now that we know how transfer acts on the chain level we can attack \( S_\alpha \). We are back in the former situation of a group \( \Gamma \) and \( \alpha \in \Delta \).
Let $f \in C^r(\Gamma, M)$. Then $\alpha^* f \in C^r(\Gamma^\alpha, M)$ is defined by

$$\alpha^* f(\gamma) = f(\alpha \gamma \alpha^{-1}). \alpha$$

$\text{Res}^{F^\alpha}_{\Gamma \cap \Gamma^\alpha} \alpha^* f$ has the same definition as $\alpha^* f$ but with the domain restricted from $C^r(\Gamma^\alpha, M)$ to $C^r(\Gamma \cap \Gamma^r)$.

So for $f \in C^r(\Gamma, M)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_r) \in F_r(\Gamma)$,

$$T_{\Gamma \cap \Gamma^\alpha} \text{Res}^{F^\alpha}_{\Gamma \cap \Gamma^\alpha} f(\gamma_0, \gamma_1, \ldots, \gamma_r) = \sum_{i=0}^n f(\alpha_{i+1}^{-1} \sigma(\alpha_i \gamma_0) \alpha_i, \alpha \sigma(\alpha_i \gamma_1) \alpha^{-1}, \ldots, \alpha \sigma(\alpha_i \gamma_r) \alpha^{-1}). \alpha \alpha_i$$

In this case $\sigma$ is associated to $T_{\Gamma \cap \Gamma^\alpha}$. Since $\Gamma = \bigsqcup (\Gamma \cap \Gamma^\alpha) \alpha_i$, then $\sigma(\alpha_i \gamma) \in \Gamma \cap \Gamma^\alpha$ is uniquely defined by $\alpha_i \gamma = \sigma(\alpha_i \gamma) \alpha_i(\gamma)$. However, then $\alpha \alpha_i \gamma = \alpha \sigma(\alpha_i \gamma) \alpha^{-1} \alpha \alpha_i(\gamma)$, so, by definition of $\tau$, it must be that $\alpha \sigma(\alpha_i \gamma) \alpha^{-1} = \tau_i(\gamma)$

So $S_\alpha = T_\alpha$. 

\[\square\]
Let \( \{A_s\}, \{B_t\} \) be sets of commuting linear transformations on the finite dimensional \( \mathbb{F} \)-spaces \( M \) and \( N \) respectively with \( \mathbb{F} \) a finite field of char \( p \) big enough to contain all the eigenvalues in the sets \( \{A_s\} \) and \( \{B_t\} \). Assume both sets include the identity transformation.

All tensors in this appendix are over \( \mathbb{F} \), i.e \( \otimes = \otimes_\mathbb{F} \).

Let \( \{A_s\} \otimes \{B_t\} = \{\sum f_i A \otimes B | f_i \in \mathbb{F}, A \in \{A_s\}, B \in \{B_t\}\} \), i.e formal sums over \( \mathbb{F} \) of elements in \( \{A_s\} \times \{B_t\} \). Note \( \{A_s\} \otimes \{B_t\} \) naturally acts on \( M \otimes_\mathbb{F} N \) by \( (A \otimes B)(m \otimes n) = (Am) \otimes (Bn) \).

**Lemma.** Let \( S \subset \{A_s\} \otimes \{B_t\} \). For any simultaneous eigenvector of \( S \) in \( M \otimes N \), its systems of eigenvalues occurs in a vector of the form \( \alpha \otimes \beta \) with \( \alpha \) and \( \beta \) simultaneous eigenvectors of \( \{A_s\} \) and \( \{B_t\} \) respectively.

**Proof.** If the linear transformations were diagonalizable, we would prove this by finding simultaneous eigenbases for \( M \) and \( N \). Since they are not necessarily
diagonolizable, instead decompose $M$ and $N$ into the direct sums of invariant $\{A_s\}$ and $\{B_t\}$ "Jordan"-spaces.

For the purposes of this proof we define that, for a linear transformation $A$ on a finite dimensional $F$-vector space $V$, $V$ is a Jordan space if $A$ only has one eigenvalue, i.e. if $A$ has minimal polynomial $(x - a)^n$ for some $n$ and some $a$. If we happen to know that the single eigenvalue is $a$, we will more specifically call $V$ an $a$-Jordan space for $A$.

By the theory of Jordan canonical forms, if $F$ contains all the eigenvectors of $A$ on a vector space $V$ then $V$ can be decomposed into $A$-invariant maximal Jordan spaces, $V = \bigoplus V_a$ with each $V_a$ defined as

$$V_a = \{v \in V | (A - aI)^n v = 0 \text{ for some } n\}$$

Here $I$ is the identity transformation.

Now consider $\{A_s\}$ and $M$. We will inductively define a decomposition of $M$ into $\{A_s\}$-invariant simultaneous Jordan spaces. Start the induction by picking an arbitrary $A_{s_0}$. Write $N = \bigoplus N_a$, each $N_a$ a maximal Jordan space. Since maximal Jordan spaces are defined as satisfying a polynomial in $A_{s_0}$, and $\{A_s\}$ is a commuting set of transformations, each $N_a$ is invariant for all of $\{A_s\}$.

Now suppose that we have picked a finite $C \subset \{A_s\}$ and decomposed $M = \bigoplus M(i)$ with each $M(i)$ invariant for all of $\{A_s\}$ and each a $M(i)$ a Jordan space
for all of $C$ (but $M(i)$ will not necessarily have the same eigenvalue for all elements in $C$). Pick an $a \in \{A_s\} - C$.

Since each $M(i)$ is $A$-invariant, write $M(i) = \bigoplus M(i)_a$ with each $M(i)_a$ a maximal Jordan subspace of $M(i)$ for $A$. As above, $M(i)_a$ must be invariant for all of $\{A_s\}$. For any element in $C$, $M(i)_a$, as a subspace of a Jordan space, must be a Jordan space itself. So $M = \bigoplus \bigoplus M(i)_a$ is the decomposition we want.

So we can inductively decompose $M$ into smaller and smaller spaces. Since $M$ is finite-dimensional, this process must stop eventually, and we will have spaces that are Jordan spaces for all of $\{A_s\}$. Similarly for $N$ and $\{B_t\}$. Define

$$M = \bigoplus M(i) \quad \text{and} \quad N = \bigoplus N(j)$$

to be two decompositions into simultaneous Jordan spaces.

Note that

$$M \otimes N = \bigoplus M(i) \otimes N(j)$$

with each $M(i) \otimes N(j)$ invariant for all of $\{A_s\} \otimes \{B_t\}$

We claim that each Jordan space has a simultaneous eigenvector. We will just prove it for $M(i)$ since $N(j)$ is the same, and again we will prove inductively. Pick $A_{s_0} \in \{A_s\}$. Let $V$ be the space of $A_{s_0}$ eigenvectors in $M(i)$. This is non-zero since $F$ is big enough to contain all eigenvalues. Since the transformations commute, $V$ is preserved by all of $\{A_s\}$. Now assume that we have a subspace $V$.
of $M(i)$ which is preserved by all of $\{A_s\}$ and is a space of eigenvectors for any transformation in some finite set $C$. Pick $A \in \{A_s\} - C$. Define $V'$ to be the space of $A$ eigenvectors in $V$. Then $V'$ is preserved by all of $\{A_s\}$. So by induction we can find a simultaneous eigenvector $\alpha$.

We claim each $M(i) \otimes N(j)$ is a Jordan space for $\{A_s\} \otimes \{B_t\}$. For $A \otimes B$, suppose $(x - a)^m$ is the minimal polynomial for $A \in \{A_s\}$ acting on $M(i)$ and $(x - b)^n$ is the minimal polynomial for $B \in \{B_t\}$ acting on $N(j)$. Then the minimal polynomial for $A \otimes B$ acting on $M(i) \otimes N(j)$ divides $(x - ab)^{m+n}$. So $M(i) \otimes N(j)$ is a Jordan space for $A \otimes B$ with eigenvalue $ab$. Since $A \otimes B$ was arbitrary, $M(i) \otimes N(j)$ is a Jordan space for all of the primitive elements in $\{A_s\} \otimes \{B_t\}$.

Now consider sums. Consider $(A \otimes B) + (A' \otimes B')$. Suppose the minimal polynomial for $(A \otimes B)$ on $M(i) \otimes N(j)$ divides $(x - ab)^m$ and the minimal polynomial for $(A' \otimes B')$ divides $(x - a'b')^n$. Then, since $(A \otimes B)$ and $(A' \otimes B')$ commute as actions on $M(i) \otimes N(j)$, the minimal polynomial of $(A \otimes B) + (A' \otimes B')$ divides $(x - (ab + a'b'))^{m+n}$. So $M(i) \otimes N(j)$ is a Jordan space for any $(A \otimes B) + (A' \otimes B')$. Repeating this argument we inductively get that $M(i) \otimes N(j)$ is a Jordan space for all of $\{A_s\} \otimes \{B_t\}$.

Now we are ready to prove the lemma.

Let $v \in M \otimes N$ be a simultaneous eigenvector for $S$. Let $v = \sum v_{(i,j)}$ be the decomposition of $v$ with respect to the decomposition $M \otimes N = \bigoplus M(i) \otimes N(j)$.
above. Since each $M(i) \otimes N(j)$ is fixed by $S$, it must be that each $u_{(i,j)}$ is a simultaneous eigenvector. So, without loss of generality, $v \in M(i) \otimes N(j)$ for some $M(i), N(j)$.

Let $\alpha$ be an eigenvector for $\{A_s\}$ in $M(i)$ and $\beta$ an eigenvector for $\{B_t\}$ in $N(j)$. Then $\alpha \otimes \beta$ is an eigenvector for all of $\{A_s\} \otimes \{B_t\}$, in particular for $S$. Since $M(i) \otimes N(j)$ is a Jordan space for $S$, $\alpha \otimes \beta$ and $v$ have the same system of eigenvalues. ■

Now let's apply this claim to 4.5.4. For the non-minimal semisimple cohomology (the form 1 case) recall that the situation is that we have transformations $T_{s_{p-1}(i)}$ which act on $H^*(\Gamma_{p-1}, M')$ and transformations $T_{s_{m}(j)}$ which act on $H^*(\Gamma_m, M'')$. We have that $H^*_1(\Gamma, M) \cong H^*(\Gamma_{p-1}, M') \otimes H^*(\Gamma, M'')$. For the set $S$ we have Hecke, since each element $T_s \in Hecke$ acts on the tensor product $H^*(\Gamma_{p-1}, M) \otimes H^*(\Gamma_m, N)$ as $T_s = \sum T_{s_{p-1}(i)} \otimes T_{s_{m}(j)}$.

For the non-minimal non-semisimple cohomology (the form 2 case), we have $T_{s_{p-1}(i)}$ acting on $\bigoplus H^*(\Gamma_{p-1}(P_0), M')$ and $T_{s_{m}(j)}$ acting on $H^*(\Gamma_m, M'')$. We have $H^*_2(\Gamma, M) \leftrightarrow \bigoplus H^*(\Gamma_{p-1}(P_0), M') \otimes H^*(\Gamma_m, M'')$ and each element $T_s$ in Hecke acting on $\bigoplus H^*(\Gamma_{p-1}(P_0), M') \otimes H^*(\Gamma_m, M'')$ as $T_s = \sum T_{s_{p-1}(i)} \otimes T_{s_{m}(j)}$.

So in both the semisimple and the non-semisimple case we can apply this lemma.
APPENDIX C
CENTRALIZER OF $P$ IN $\Gamma_{p-1}$

Let $\pi \in GL_{p-1}(\mathbb{Z})$ have order $p$, $p$ odd prime, $N$ positive integer.

CLAIM 1. Let $Z \in M_{p-1}(\mathbb{Q})$ be a non-zero matrix that commutes with $\pi$. Then:

1) There is a unique polynomial $f \in \mathbb{Q}[x]$ of degree less than $p-1$ such that $Z = f(\pi)$. Either $Z = 0$ or $Z \in \Gamma_{p-1}(\mathbb{Q})$.

2) If the entries in $Z$ have denominator prime to $N$, i.e. if $Z \in M_{p-1}(\mathbb{Q}_N)$, then the coefficients of $f$ are in $\mathbb{Q}_N$

3)

$$Z_{\Gamma_{p-1}}(\pi) \cong \{ f \in \mathbb{Z}[x]/\Phi[x] \mid \det f(\pi) = \pm 1 \} \text{ by } f(\pi) \leftrightarrow f(x)$$

$$Z_{\Gamma_{p-1}(\mathbb{Q}_N)}(\pi) \cong \{ f \in \mathbb{Q}_N[x]/\Phi[x] \mid \det f(\pi) \in \mathbb{Q}_N^\times \} \text{ by } f(\pi) \leftrightarrow f(x)$$

PROOF.

Define $\pi_0 = \begin{pmatrix} 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix}$

So $\pi_0$ is the standard rational canonical form matrix for the polynomial $\Phi(x)$. 

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Now find a matrix \( g \in \Gamma_{p-1}(\mathbb{Q}) \) such that \( \pi_0 = g \pi g^{-1} \). Multiply by an appropriate scalar matrix so \( g \) has integral coefficients.

Consider the \( p-1 \) dimensional \( \mathbb{Q} \)-vector space \( \mathbb{Q}[\zeta] \). Pick the basis \( 1, \zeta, \ldots, \zeta^{p-2} \). Then \( \pi_0 \), viewed as a rational matrix, naturally acts (on the left) on \( \mathbb{Q}[\zeta] \). Looking at \( \pi_0 \), it is apparent that this action is in fact multiplication by \( \zeta \). So \( \pi \) also acts on \( \mathbb{Q}[\zeta] \) as multiplication by \( \zeta \) after a changing the basis by conjugating by \( g \). Call this basis associated to \( \pi \{ e_i \} \). Note that each \( e_i \) is an integral combination of powers of \( \zeta \).

Let \( Z \in M_{p-1}(\mathbb{Q}) \) commute with \( \pi \). Find an element \( e \in \mathbb{Q}[\zeta] \) such that \( Ze \neq 0 \). Since \( Ze \) is in \( \mathbb{Q}[\zeta] \) then, using the field structure of \( \mathbb{Q}[\zeta] \), \( Z \) sends \( e \) to some element times \( e \).

Define \( v \in \mathbb{Q}[\zeta] \) by \( Ze = ve \).

Notice that \( Z \pi^i e = v \pi^i e \) for all \( i \).

Define \( Y \in M_{p-1}(\mathbb{Q}) \) to be the matrix “multiply by \( v \)” with respect to the basis \( \{ e_i \} \).

Then \( Y \) and \( Z \) are rational matrices which have the same values on the set of \( \mathbb{Q} \)-linearly independent vectors \( e, \zeta e, \zeta^2 e, \ldots, \zeta^{p-2} e \). Hence \( Y = Z \).

So \( Z \) is “multiplication by \( v \)” for some necessarily unique \( v \in \mathbb{Q}[\zeta] \). Since \( \mathbb{Q}(\zeta) \) is a field, either \( v = 0 \) (then \( Z = 0 \)) or \( v \) is invertible (then \( Z \) is invertible).
Since \( v \) is uniquely expressed as a rational combination of powers of \( \zeta \) less than \( p-1 \), there is a unique poly \( f \) such that \( v = f(\zeta) \), ie \( Z = f(\pi) \).

Since \( \Phi(\pi) = 0 \), instead of saying \( Z = f(\pi) \) with \( f(x) \) a unique poly of degree less than \( p-1 \), we can say that \( Z = f(\pi) \) with \( f \) a unique element of \( \mathbb{Q}[x]/\Phi(x) \).

For part (2), we need to prove \( f \) has the right coefficients. Assume \( Z \neq 0 \), so \( Z \in \text{GL}_{p-1}(\mathbb{Q}) \).

First suppose \( Z \in M_{p-1}(\mathbb{Z}) \). Then \( Z \) satisfies its characteristic polynomial, which is monic with integral coefficients. So \( f(\zeta) \) is an algebraic integer. However, the powers of \( \zeta \) form an integral basis of the algebraic integers so \( f(x) \in \mathbb{Z}[x] \).

Now suppose \( Z \in M_{p-1}(\mathbb{Q}_N) \). So now \( Z \) satisfies a polynomial which is monic with \( \mathbb{Q}_N \) coefficients or equivalently an integral polynomial with leading coefficient prime to \( N \). Notice that powers of \( \zeta \) form a \( \mathbb{Q}_N \)-basis of such elements (i.e. elements of \( \mathbb{Q}[\zeta] \) which satisfy an integral polynomial with leading coefficient prime to \( N \)): Let \( v \) be such an element and let the leading coefficient be \( a \). Then \( av \) is an algebraic integer. So \( v = (1/a)(av) \) is in the set of \( \mathbb{Q}_N \) combinations of powers of \( \zeta \).

So \( Z \in \Gamma_{p-1}(\mathbb{Q}_N) \) means that \( f \) has coefficients in the desired ring.

For part (3), note that being invertible is equivalent to saying that \( Z \) satisfies the given determinant conditions so by parts (1) and (2) the left side of each isomorphism maps injectively into the right side.
Conversely, if \( f(x) \in \mathbb{Z}[x] \) or \( \mathbb{Q}_N[x] \), then, since \( \pi \) has integer coefficients, the matrix \( f(\pi) \) has entries in \( \mathbb{Z} \) or \( \mathbb{Q}_N \) respectively.

\[ \text{Claim 2. If } Z = f(\pi) \in Z_{\Gamma_{p-1}(\mathbb{Q}_N)}(\pi) \text{ then } \]
\[
f(1) \not\equiv 0 \mod p \text{ in } \mathbb{Q}_pN
\]

\textsc{Proof.} By part (3) of Claim 1 above, \( f \in \mathbb{Q}_pN[x] \) and there is a polynomial \( g \in \mathbb{Q}_pN[x] \) (the polynomial associated to \( Z^{-1} \)) such that \( f(x)g(x) = 1 + h(x)\Phi(x) \) for some \( h \in \mathbb{Q}_pN[x] \).

Since \( \Phi(1) = p \), then \( f(1)g(1) = 1 + h(1)p \). Since \( f \), \( g \), and \( h \) all have denominators prime to \( p \), then \( f(1) \not\equiv 0 \mod p \) in \( \mathbb{Q}_pN \). \[ \square \]
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