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$p$-adic Finite Difference Equations

Dissertation

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

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ABSTRACT

General existence and uniqueness theorems are established for general first-order $p$-adic finite difference equations, general linear $p$-adic finite difference equations of arbitrary order, and general higher-order $p$-adic finite difference equations. Linear $p$-adic finite difference equations are then studied in greater depth and, in some cases, both necessary and sufficient conditions guaranteeing the existence of a unique solution are given. Finally the domains of analyticity for a few select non-linear $p$-adic finite difference equations are discussed.
I would like to dedicate this dissertation to my mother, who taught me how to communicate, to my father, who taught me how to think, and to Nancy Jo, who taught me how to live.
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CHAPTER I

Introduction

Finite difference equations, while certainly studied long before the 19th century, were firmly established as a theory in their own right by 1860. This was in large part due to the work of George Boole [B] towards developing a theory of finite difference equations paralleling that of differential equations. Historically, the theory of finite difference equations was motivated by two problems: (1) Find a 'nice' complex function "interpolating" a given function on the natural numbers (e.g. the Gamma function, the Beta function or, well before those, the exponentials); and (2) Describe the asymptotic properties of sequences (like $n!$ or the sequence of prime numbers). These questions change character completely in the $p$-adic domain. The second question makes no clear sense at all, and, whereas classically the focus of the first question was on the search for a nice interpolating function, $p$-adically the primary concern becomes one of existence as any continuous interpolation of a sequence to $\mathbb{Z}_p$ is necessarily unique. If the sequence is interpolatable, a further question might be: Can the interpolating function be extended to a larger domain?

Although not strictly motivated by the study of finite difference equations, significant work on the $p$-adic interpolation of sequences, to both $\mathbb{Z}_p$ and arbitrary domains
of $\mathbb{C}_p$, was done by Mahler [Mah] and Amice [A]. Nice conditions were provided by which one could determine to what extent a given sequence could be interpolated.

There has been a heavy focus in $p$-adic analysis in finding $p$-adic analogues to classical functions. Two examples that use finite difference equations look at the classical gamma function. Both Diamond [D] and Morita [Mo] studied the finite difference equation $y(n+1) = ny(n)$. As the sequence $n!$ cannot be interpolated directly, other methods were sought. Morita altered the recurrence relation so as to create a new sequence which could be interpolated but which still retained many of the properties of the original. Diamond used sums previously introduced by Leopoldt to provide a closed form solution to the related difference equation $\log y(n+1) = \log n + \log y(n)$, on the complement of $\mathbb{Z}_p$ in $\mathbb{C}_p$. Other specific sequences not related to finite difference equations have been studied in depth, most notably the values of the zeta function, which were successfully interpolated (with slight modifications) by Kubota and Leopoldt [K-L].

Since the focus of previous work in interpolation has not, in general, been on the study of finite difference equations, in this work I hope to partially fill that void. I use the methods of Mahler and Amice to classify those equations which generate interpolatable sequences. The intent is to provide means by which the interpolation question may be addressed without resorting to an attempt at expressing the solution in a 'nice' form. My focus is entirely on the $p$-adic interpolation of solutions to finite difference equations to $\mathbb{Z}_p$ and to balls of $\mathbb{C}_p$ centered at the origin.
In Chapter 2, I present numerous elementary results needed for the rest of the paper. Much of this can be found in Schikhof and Amice. Many of the results I have been unable to find in the literature but are certainly well-known. In Chapter 3, general existence and uniqueness theorems are established for general first-order finite difference equations, general linear finite difference equations of arbitrary order, and general higher-order finite difference equations. In Chapter 4, I explore linear finite difference equations in greater depth, providing in some cases both necessary and sufficient conditions guaranteeing the existence of a unique solution. Finally, in Chapter 5 I discuss the domains of analyticity for a few select non-linear finite difference equations.
CHAPTER II

Elementary Facts

2.1 The representation of elements.

In this chapter I will establish notation and present numerous elementary results needed in the sequel. I make no claim as to the originality of this material and, where applicable, I have given explicit references. When I refer to the natural numbers, I shall mean the set of non-negative integers \( \{0, 1, 2, \ldots \} \). Unless otherwise stated, \( p \) shall denote a fixed prime number. Also, I shall adhere to the following standard conventions.

- \( \mathbb{N} \) denotes the set of natural numbers.
- \( \mathbb{Q} \) denotes the field of rational numbers.
- \( \mathbb{R} \) denotes the field of real numbers.
- \( \mathbb{Z} \) denotes the field of integers.
- \( \mathbb{Z}_p \) denotes the ring of \( p \)-adic integers.
- \( \mathbb{Q}_p \) denotes the field of \( p \)-adic rational numbers.
- \( \mathbb{C}_p \) denotes the completion of the algebraic closure of \( \mathbb{Q}_p \).
- \( B_a(r) = \{ z \in \mathbb{C}_p : |z - a| \leq r \} \)
- $B_a^-(r) = \{ z \in \mathbb{C}_p : |z - a| < r \}$
- $E = B_0^-(p^{1/(1-r)})$
- $W$ is the group of all roots of unity of order prime to $p$.
- $U = \{ z \in \mathbb{C}_p : |z - 1| < 1 \} = B_1(1^-)$
- $T = \{ z \in \mathbb{C}_p : |z| = 1 \} = W \times U$
- $\text{ord} : \mathbb{C}_p^\times \to \mathbb{Q}$ is the $p$-adic ordinal map normalized by setting $\text{ord}(p) = 1$.

A detailed treatment of the construction and properties of $\mathbb{Z}_p$ and $\mathbb{C}_p$ may be found in Koblitz [K]. When necessary, elements $s$ of $\mathbb{Z}_p$ will be written in the form

\[ s = s_0 + s_1p + s_2p^2 + \cdots, \tag{2.1} \]

where $s_j$ is in $\{0, 1, \ldots, p - 1\}$ for $j$. This representation will be referred to as the cannonical $p$-adic expansion of $s$. Unfortunately, no such representation exists for elements of $\mathbb{C}_p$, but if I fix, once and for all, a homomorphism $r \mapsto p^r$ from $\mathbb{Q}$ to $\mathbb{C}_p^\times$ satisfying $\text{ord}(p^r) = r$ for all $r$ in $\mathbb{Q}$, I may then define functions $\pi$, $\omega$, and $\nu$ on $\mathbb{C}_p^\times$ as follows. $\pi(z) = p^{\text{ord}(z)}$ and is called the $p$-part of $z$. $\omega(z)$ is defined to be the unique element of $W$ closest to $z/\pi(z)$ and is called the Teichmüller part of $z$. Finally, $\nu(z) = \frac{s}{\pi(z)\omega(z)}$, and is called the principal unit part of $z$. The following Lemma is an easy consequence of these definitions.

**Lemma 2.1.1** $\pi$, $\omega$, and $\nu$ are continuous functions from $\mathbb{C}_p^\times$ to $\mathbb{C}_p^\times$.

**Proof** For $r$ in $\mathbb{Q}$,

\[ \pi^{-1}(p^r) = \{ w \in \mathbb{C}_p^\times : |w| = p^{-r} \}, \]
which is open in $\mathbb{C}_p^\times$, so $\pi$ is continuous. Continuity of $\omega$ on $T$ is clear from the
definition of $\omega$, and hence, as $\pi$ is continuous and nonzero, continuity of $\omega$ on $\mathbb{C}_p^\times$
follows from the relation

$$\omega(z) = \omega\left(\frac{z}{\pi(z)}\right).$$

Continuity of $\nu$ is immediate from the definition as both $\pi$ and $\omega$ are nonzero.

Now, for any non-vanishing continuous function $r : \mathbb{C}_p \to \mathbb{C}_p^\times$ I may define $r_\pi(x) = 
\pi(r(x))$, $r_\omega(x) = \omega(r(x))$, and $r_\nu(x) = \nu(r(x))$. By the above Lemma, $r_\pi$, $r_\omega$, and $r_\nu$
are continuous.

### 2.2 Some combinatorial identities.

I will often need to switch the order of summation in a multiple sum. The following
Lemma represents the primary means by which this procedure will be justified. In
the Lemma and thereafter, $B$ will denote a Banach space over $\mathbb{C}_p$. For a definition
and a brief introduction to $p$-adic Banach spaces, I refer the reader to section 13 of
Schikhof [S].

**Lemma 2.2.1** If $S$ is a countable subset of $B$ such that for every $\epsilon > 0$ there are
only finitely many $s \in S$ such that $|s| > \epsilon$, then

$$\sum_{s \in S} s$$

exists in $B$ and is independent of the ordering of the terms.
Proof This is obvious.

I will now present a number of simple identities involving the $\Delta$ operator, defined as follows. For any function $f$,

\[ \Delta f(x) = f(x+1) - f(x) \quad (2.3) \]

provided the expression on the right makes sense for all $x$ in the domain of $f$.

**Lemma 2.2.2** For any function $f$, and any natural number $n$,

\[ \Delta^n f(x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x+j) \quad (2.4) \]

provided the above expressions are meaningful.

Proof This follows by induction on $n$. If $n = 0$, both sides of the identity reduce to $f(x)$. If the identity is assumed to hold for all $0 \leq n \leq k$, then

\[
\begin{align*}
\Delta^{k+1} f(x) &= \Delta^k f(x+1) - \Delta^k f(x) \\
&= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} [f(x+1+j) - f(x+j)] \\
&= \sum_{j=1}^{k+1} (-1)^{k+1-j} \binom{k}{j-1} f(x+j) + \sum_{j=0}^{k} (-1)^{k+1-j} \binom{k}{j} f(x+j) \\
&= \sum_{j=0}^{k+1} (-1)^{k+1-j} \binom{k+1}{j} f(x+j). \quad (2.5)
\end{align*}
\]
Lemma 2.2.3

\[ \Delta^k \binom{x}{n} = \binom{x}{n-k} \quad (2.6) \]

for all \( k, n \geq 0 \) with the convention that \( \binom{z}{j} \) is identically zero whenever \( j < 0 \).

**Proof** This follows by induction on \( k \). If \( k = 0 \), both sides of the identity reduce to \( \binom{x}{n} \). If the identity is assumed to hold for all \( 0 \leq k \leq m \), then

\[
\Delta^{m+1} \binom{x}{n} = \Delta^m \left( \binom{x+1}{n} - \binom{x}{n} \right) \\
= \binom{x+1}{n-m} - \binom{x}{n-m} \\
= \binom{x+1}{n-m-1}. \quad (2.7)
\]

Lemma 2.2.4 For any function \( f : N \rightarrow B \),

\[
f(n) = \sum_{j=0}^{\infty} \Delta^j f(0) \binom{n}{j}. \quad (2.8)
\]

Also, if

\[
f(n) = \sum_{j=0}^{\infty} a_j \binom{n}{j} \quad (2.9)
\]

for all natural numbers \( n \), then

\[
a_k = \Delta^k f(0) \quad (2.10)
\]

for all \( k \geq 0 \).

**Proof** The first half of the Lemma follows by induction on \( n \). If \( n = 0 \), both sides of the identity reduce to \( f(0) \). If the identity is assumed to hold for all functions
$f : \mathbb{N} \to B$ and for all $0 \leq n \leq k$, then

$$f(k+1) = (\Delta f + f)(k)$$

$$= \sum_{j=0}^{\infty} \Delta^j(\Delta f + f)(0)\binom{k}{j}$$

$$= \sum_{j=0}^{\infty} \Delta^j f(0)\binom{k}{j-1} + \Delta^j f(0)\binom{k}{j}$$

$$= \sum_{j=0}^{\infty} \Delta^j f(0)\binom{k+1}{j}. \quad (2.11)$$

The second half of the Lemma may be derived by applying the $\Delta^k$ operator to both sides of equation 2.9 and by using Lemma 2.2.3. Note that the sum in equation 2.9 is finite so there is no problem with convergence. 

Lemma 2.2.5

$$\Delta^n f g(x) = \sum_{j=0}^{n} \binom{n}{j} \Delta^j f(x+j) \Delta^{n-j} g(x) \quad (2.12)$$

for all $n \geq 0$. Here $f$ and $g$ are assumed to be functions with some common domain space and range space for which the above expressions make sense. Note, however, that the multiplication in the range space need not be commutative.

**Proof** If $n = 0$, the statement of the Lemma reduces to the statement that

$$f g(x) = f g(x). \quad (2.13)$$

If

$$\Delta^k f g(x) = \sum_{j=0}^{k} \binom{k}{j} \Delta^j f(x+j) \Delta^{k-j} g(x) \quad (2.14)$$
for all $0 \leq k \leq n$, then

$$\Delta^{k+1} fg(x) = \Delta^k fg(x + 1) - \Delta^k fg(x)$$

$$= \sum_{j=0}^{k} \binom{k}{j} \left[ \Delta^j f(x + k + 1 - j) \Delta^{k-j} g(x + 1) - \Delta^j f(x + k - j) \Delta^{k-j} g(x) \right]$$

$$= \sum_{j=0}^{k} \binom{k}{j} \left[ \Delta^j f(x + k + 1 - j) \Delta^{k-j} g(x + 1) - \Delta^j f(x + k - j) \Delta^{k-j} g(x) \right]$$

$$= \sum_{j=0}^{k} \binom{k}{j} \left[ \Delta^j f(x + k + 1 - j) \Delta^{k-j+1} g(x) + \Delta^{j+1} f(x + k - j) \Delta^{k-j} g(x) \right]$$

$$= \sum_{j=0}^{k} \binom{k}{j} \Delta^j f(x + k + 1 - j) \Delta^{k-j+1} g(x)$$

$$+ \sum_{j=1}^{k+1} \binom{k+1}{j-1} \Delta^j f(x + k + 1 - j) \Delta^{k-j} g(x)$$

$$= \sum_{j=0}^{k+1} \binom{k+1}{j} \Delta^j f(x + k + 1 - j) \Delta^{k+1-j} g(x). \quad (2.15)$$

Hence, the statement of the Lemma holds for all $n \geq 0$. \qed

The final two lemmas of this section provide means for conveniently multiplying Mahler series. Mahler series will be formally introduced in section 2.4.

**Lemma 2.2.6**

$$\binom{x}{m} \binom{x}{n} = \sum_{k=\max(m,n)}^{m+n} \binom{k}{n} \binom{n}{k-m} \binom{x}{k}$$

$$\binom{x}{m} \binom{x}{n} = \sum_{k=\max(m,n)}^{m+n} \binom{k}{n} \binom{n}{k-m} \binom{x}{k} \quad (2.16)$$

for all $m, n \geq 0$.

**Proof** By Lemmas 2.2.5 and 2.2.3,

$$\Delta^k \binom{x}{m} \binom{x}{n} = \sum_{j=0}^{k} \binom{k}{j} \Delta^j \binom{x + k - j}{m} \Delta^{k-j} \binom{x}{n}$$
Hence,

\[
\Delta^k \binom{x}{m}(x) \bigg|_{x=0} = \binom{k}{k-n} \binom{n}{m-k+n} = \binom{k}{n} \binom{n}{k-m}.
\]  

(2.18)

In particular, \( \Delta^k \binom{x}{m}(x) \) vanishes when either \( k < n \) or \( k > m+n \). Also, by symmetry, \( \Delta^k \binom{x}{m}(x) \) vanishes whenever \( k < m \). Finally, by Lemma 2.2.4,

\[
\binom{x}{m}(x) = \sum_{k=\max(m,n)}^{m+n} \binom{k}{n} \binom{n}{k-m} \binom{x}{k}.
\]  

(2.19)

as desired.

**Lemma 2.2.7** If \( x \in \mathbb{Z}_p \), then

\[
\left( \sum_{n=0}^{\infty} a_n \binom{x}{n} \right) \left( \sum_{m=0}^{\infty} b_m \binom{x}{m} \right) = \sum_{k=0}^{\infty} \left[ \sum_{n=0}^{k} \sum_{m=0}^{k-n} a_n b_m \binom{k}{n} \binom{n}{k-n} \right] \binom{x}{k}.
\]  

(2.20)

provided the left hand side converges for all \( x \in \mathbb{Z}_p \).

**Proof** By Lemma 2.2.6,

\[
\left( \sum_{n=0}^{\infty} a_n \binom{x}{n} \right) \left( \sum_{m=0}^{\infty} b_m \binom{x}{m} \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{m+n} \sum_{k=\max(m,n)}^{m+n} a_n b_m \binom{k}{n} \binom{n}{k-m} \binom{x}{k}.
\]  

(2.21)

Hence, by Lemma 2.2.1, it suffices to show that for any \( \epsilon > 0 \)

\[
\left| a_n b_m \binom{k}{n} \binom{n}{k-m} \binom{x}{k} \right| < \epsilon.
\]  

(2.22)
for all but finitely many triples \((n, m, k)\). As the left hand side of 2.20 converges for all \(x \in \mathbb{Z}_p\), it converges when \(x = -1\). It follows that the sequences \((a_n)\) and \((b_n)\) are bounded, say by \(C\), and tend to zero as \(n\) tends to infinity. If \(N\) and \(M\) are chosen so that \(Ca_n < \epsilon\) and \(Cb_m < \epsilon\) for all \(n > N\) and all \(m > M\), then

\[
\left| a_n b_m \binom{k}{n} \binom{n}{k-m} (x) \right| < \epsilon \quad (2.23)
\]

if either \(n > N\) or \(m > M\). If both \(n \leq N\) and \(m \leq M\), then

\[
\left| a_n b_m \binom{k}{n} \binom{n}{k-m} (x) \right| = 0 \quad (2.24)
\]

for all \(k > N + M\). Hence there are at most \(NM(N + M)\) triples \((n, m, k)\) for which

\[
\left| a_n b_m \binom{k}{n} \binom{n}{k-m} (x) \right| \geq \epsilon. \quad (2.25)
\]

2.3 Limits and Norms

The following Lemma is not particularly interesting in its own right. It would appear to be a special case of a more general statement to the effect that exponential functions tend to zero sufficiently fast as to dominate the norm of any given rational function. Unfortunately, the more general statement is false.

Lemma 2.3.1 If \(0 \leq r < 1\), then

\[
\lim_{n \to \infty} \left| \frac{1}{n} \right| r^n = 0. \quad (2.26)
\]
Proof: \( \text{ord}(n) \leq \log_p(n) \), (the real base-\( p \) logarithm of \( n \)), so

\[
|n| \geq p^{-\log_p(n)} = \frac{1}{n},
\]

(2.27)

and

\[
\left| \frac{1}{n} \right| \leq n.
\]

(2.28)

The desired result follows.

The following result is Lemma 25.5 in Schikhof [S].

**Lemma 2.3.2** If \( n \in \mathbb{N} \) is written using the base \( p \),

\[
n = a_0 + a_1p + \cdots + a_sp^s,
\]

(2.29)

and if the sum of the digits \( s_n \) of \( n \) is defined by

\[
s_n = \sum_{j=0}^{s} a_j,
\]

(2.30)

then

\[
|n!| = p^{-\frac{s-n_s}{p-1}}.
\]

(2.31)

The next Lemma is a nice description of the \( p \)-norm of \( \binom{s}{n} \). When I refer to the number of carries resulting from an addition of two \( p \)-adic integers \( s \) and \( t \), I am assuming that \( s \) and \( t \) are written using their cannonical \( p \)-adic expansions,

\[
s = s_0 + s_1p + \cdots
\]

\[
t = t_0 + t_1p + \cdots,
\]

(2.32)

and then added term-wise from left to right. I will say that a carry occurs in the 0'th position if \( s_0 + t_0 \geq p \) and, for \( j > 0 \), that a carry occurs in the \( j \)'th position if either \( s_j + t_j \geq p \) or if a carry occurred in the \((j - 1)\)st position and \( s_j + t_j = p - 1 \).
Lemma 2.3.3 If \( x \) is a \( p \)-adic integer and \( n \) is a natural number, then

\[
\binom{x}{n} = p^{-c}
\]  

(2.33)

where \( c \) is the number of carries resulting from an addition of \( x - n \) and \( n \).

PROOF The proof is in two steps. First I will prove the statement of the Lemma for all natural numbers \( x \geq n \), then I will show that the function

\[
\phi : \mathbb{Z}_p \rightarrow \{0\} \cup \{p^{-n} : n \in \mathbb{N}\}
\]

\[
x \mapsto p^{-c},
\]

(2.34)

where \( c \) is the number of carries resulting from an addition of \( x - n \) and \( n \), is continuous.

If \( x \) is a natural number, then

\[
\binom{x}{n} = \frac{x!}{n!(x-n)!}.
\]  

(2.35)

Hence, by Lemma 2.3.2,

\[
\text{ord} \left( \binom{x}{n} \right) = \frac{a + b - c}{p - 1},
\]  

(2.36)

where \( a \), \( b \), and \( c \), are the sums of the digits in the base-\( p \) expansion of \( n \), \( x - n \), and \( x \), respectively. Suppose that

\[
n = a_0 + a_1 p + \cdots + a_s p^s
\]

\[
x - n = b_0 + b_1 p + \cdots + b_s p^s
\]

\[
x = c_0 + c_1 p + \cdots + c_s p^s.
\]  

(2.37)

If a string of carries begins at position \( M \) and ends at position \( M + m \), then

\[
a_M + b_M = p + c_M
\]
\[ a_{M+1} + b_{M+1} = (p-1) + c_{M+1} \]

\[ \vdots \]

\[ a_{M+m} + b_{M+m} = (p-1) + c_{M+m} \]

\[ 1 + a_{M+m+1} + b_{M+m+1} = c_{M+m+1}. \]  

(2.38)

Hence,

\[ \sum_{j=M}^{M+m+1} (a_j + b_j - c_j) \]
\[ \frac{p-1}{p-1} = m + 1 \]  

(2.39)

which is precisely the number of carries occurring in positions \( M \) through \( M + m + 1 \).

It follows that

\[ \frac{a + b - c}{p-1} \]  

(2.40)

is the total number of carries occurring in the base-\( p \) addition of \( n \) and \( x - n \). It only remains to be seen that \( \phi \), the 'number of carries' function defined above, is continuous on \( \mathbb{Z}_p \).

Fix \( \epsilon = p^{-N} \). If

\[ n = a_0 + a_1 p + \cdots + a_s p^s, \]

(2.41)

and if

\[ |x - y| < p^{-(s+N)}, \]

(2.42)

then

\[ |(x - n) - (y - n)| < p^{-(s+N)}, \]

(2.43)
and the first \(s + N + 1\) terms in the canonical \(p\)-adic expansions for \(x - n\) and \(y - n\) coincide. Hence, I may write

\[
\begin{align*}
n &= a_0 + a_1 p + \cdots + a_s p^s \\
x - n &= b_0 + b_1 p + \cdots + b_s p^s + \cdots + b_{s+N} p^{s+N} + b_{s+N+1} p^{s+N+1} + \cdots \\
y - n &= b_0 + b_1 p + \cdots + b_s p^s + \cdots + b_{s+N} p^{s+N} + c_{s+N+1} p^{s+N+1} + \cdots \quad (2.44)
\end{align*}
\]

If no carry occurs in the \((s + N)\)’th position in an addition of \(n\) and \(x - n\), then no carry occurs in any position after the \((s + N)\)’th and the same can be said about an addition of \(n\) and \(y - n\). Hence \(\phi(x) = \phi(y)\), and I am done. If a carry does occur in the \((s + N)\)’th in an addition of \(n\) and \(x - n\), then carries also occur in positions \(s\) through \(s + N\) so there are at least \(N + 1\) carries and the same holds for an addition of \(n\) and \(y - n\). Hence,

\[
|\phi(x) - \phi(y)| < p^{-N}, \quad (2.45)
\]

and \(\phi\) is continuous.

As a quick check, observe that the number of carries when adding \(p^n - j\) and \(j\) is always positive provided \(0 < j < p^n\). By the Lemma, this is equivalent to the well known fact that

\[
p \left| \binom{p^n}{j} \right| \quad (2.46)
\]

whenever \(0 < j < p^n\).

The next Lemma might be more clearly phrased as a statement about the domain of convergence of the exponential function, namely that the aforementioned domain is precisely the set \(E\) as defined in section 2.1. The chosen wording of the Lemma,
however, more closely reflects the way in which the result will be used.

**Lemma 2.3.4** As \( n \) tends to infinity, the expression

\[
\frac{r^n}{|n!|}
\]

(2.47)

tends to zero if \( 0 < r < p^{-\frac{1}{r-1}} \), tends to infinity if \( r > p^{-\frac{1}{r-1}} \), and has no limit if \( r = p^{-\frac{1}{r-1}} \).

**Proof** This is simply a restatement of Theorem 25.6 in Schikhof [S]. □

### 2.4 The Mahler series for a continuous function on \( \mathbb{Z}_p \).

In this section I will use \( B \) to denote a Banach space over \( \mathbb{C}_p \), \( |x| \) to denote the norm of an element \( x \) of \( B \), and \( \|f\|_{\mathbb{Z}_p} \) to denote the supremum norm over \( \mathbb{Z}_p \) of a function \( f \) from \( \mathbb{Z}_p \) to \( B \).

**Lemma 2.4.1** If \( B \) is a Banach space over \( \mathbb{C}_p \) and \( f : \mathbb{Z}_p \to B \) is continuous, then

\[
\lim_{n \to \infty} \|\Delta^n f\|_{\mathbb{Z}_p} = 0.
\]

(2.48)

**Proof** For any \( g : \mathbb{Z}_p \to B \), \( \|\Delta g\|_{\mathbb{Z}_p} \leq \|g\|_{\mathbb{Z}_p} \) so it suffices to show that the sequence \( (\|\Delta^n f\|_{\mathbb{Z}_p}) \) has a subsequence tending to zero. As \( \mathbb{Z}_p \) is compact, \( f \) is bounded and uniformly continuous. Hence, there is a natural number \( N \) such that

\[
|f(x) - f(y)| \leq p^{-1}\|f\|_{\mathbb{Z}_p},
\]

(2.49)

whenever

\[
|x - y| \leq p^{-N}.
\]

(2.50)
If $p \neq 2$, the inequality in (2.49) may be directly rewritten as

$$|f(x) + (-1)^p f(y)| \leq p^{-1}||f||_{z_p},$$

(2.51)

and if $p = 2$, then

$$|f(x) + (-1)^p f(y)| = |f(x) - f(y) + pf(y)| \leq p^{-1}||f||_{z_p},$$

(2.52)

so (2.51) also holds whenever

$$|x - y| \leq p^{-N}.$$  

(2.53)

It follows from Lemma 2.2.2 that,

$$|\Delta^p f(x)| = \left| \sum_{j=0}^{p} (-1)^{p-j} \binom{p}{j} f(x + j) \right| \leq \max\{p^{-1}||f||_{z_p}, |f(x + p) + (-1)^p f(x)| \} \leq p^{-1}||f||_{z_p}. $$

(2.54)

Repeated application of this reasoning yields the desired subsequence. 

Lemma 2.4.2 If $B$ is a Banach space over $\mathbb{C}_p$, then a function $f : \mathbb{Z}_p \rightarrow B$ is continuous if and only if

$$f(x) = \sum_{n=0}^{\infty} \Delta^n f(0) \left( \begin{array}{c} x \\ n \end{array} \right)$$

(2.55)

for all $x$ in $\mathbb{Z}_p$.

PROOF By the first part of Lemma 2.2.4, equation 2.55 is already known to hold for all natural numbers $x$. If $f$ is continuous, it follows from Lemma 2.4.1 that the
The left hand side of equation 2.55 is continuous. Hence equation 2.55 holds for all \( x \) in \( \mathbb{Z}_p \). Now if equation 2.55 holds, then, in particular, it holds when \( x = -1 \). As \( \binom{-1}{n} = (-1)^n \), it must be the case that \( \Delta^n f(0) \) tends to zero as \( n \) tends to infinity. As before, it follows that the left hand side of the equation is a continuous function and therefore, that \( f \) is a continuous function.

The series given for \( f \) in Lemma 2.4.2 is called the Mahler series for \( f \). It follows from the second part of Lemma 2.2.4 that the Mahler series of a continuous function on \( \mathbb{Z}_p \) is unique. One advantage of expressing functions in this way is the fact that the supremum norm of a function over \( \mathbb{Z}_p \) may be easily determined from the coefficients of its Mahler series as stated in the following Lemma.

**Lemma 2.4.3** If \( f : \mathbb{Z}_p \to B \) has Mahler series

\[
f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n},
\]

then

\[
\|f\|_{\mathbb{Z}_p} = \sup_n |a_n|.
\]  

**Proof** It follows immediately from equation 2.56 that

\[
\|f\|_{\mathbb{Z}_p} \leq \sup_n |a_n|.
\]

Also, by Lemmas 2.2.2 and 2.2.4,

\[
a_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(j).
\]
Hence

\[ \|f\|_{Z_p} \geq \sup |a_n| \]  \hspace{1cm} (2.60)

as desired.  

\[ \boxed{\text{Lemma 2.5.1}} \]

If \( B \) is a Banach space over \( C_p \), and \( f : Z_p \rightarrow B \) is a continuous function, then the finite difference equation

\[ \begin{align*}
\Delta y &= f(x) \\
y(0) &= 0
\end{align*} \]  \hspace{1cm} (2.61)

has a unique continuous solution \( y : Z_p \rightarrow B \).

**Proof** Uniqueness is immediate from the fact that equation 2.61 determines \( y \) on the natural numbers. Existence follows from the results of the previous section. Specifically, if \( f \) has Mahler series

\[ f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}, \]  \hspace{1cm} (2.62)

then

\[ y(x) = \sum_{n=1}^{\infty} a_{n-1} \binom{x}{n} \]  \hspace{1cm} (2.63)

is the desired solution.  

The unique continuous solution to the finite difference equation of Lemma 2.5.1 is called the *indefinite sum* of \( f \) and will be denoted by \( Sf \) or, more often, by

\[ x \mapsto \sum_{j=0}^{x-1} f(j). \]  \hspace{1cm} (2.64)
2.6 Analytic functions

In this section, and henceforth unless stated otherwise, $B$ will denote a Banach space over $\mathbb{C}_p$, $|x|$ will denote the norm of an element $x$ of $B$, and $\|f\|$ will denote the supremum norm over $B_0(1)$ of a function $f$ from $B_0(1)$ to $\mathbb{C}_p$. The primary function of this section is to provide a definition of the term analytic which, although narrow in scope, is sufficient for the purposes of this paper. Much of the material in this section has been taken, almost entirely without modification, from Schikhof [S], sections 22 and 23. The primary difference is one of context. Where Schikhof refers to a field $K$, I use a Banach space over $\mathbb{C}_p$.

A function represented by a power series on a ball in $\mathbb{C}_p$ will be said to be analytic on that ball. Likewise, a function represented by a power series on $\mathbb{Z}_p$ will be said to be analytic on $\mathbb{Z}_p$. An easy induction argument shows that a function analytic on a ball in $\mathbb{C}_p$ has a unique power series expansion about any given point, so there is no ambiguity in referring to the power series expansion of a function about a point.

The radius of convergence of a power series $\sum a_n x^n$ is defined by

$$\rho = \left( \limsup_{n \to \infty} |a_n|^{1/n} \right)^{-1}. \quad (2.65)$$

The radius of convergence of a power series, as just defined, behaves as one would expect. Surprisingly, the 'boundary' behavior displayed by a power series is much nicer than in the classical case. The following Lemma provides the details.

**Lemma 2.6.1** If $\rho$ is the radius of convergence of a power series $\sum a_n x^n$, then $\sum a_n x^n$ converges on $\{x \in \mathbb{C}_p : |x| < \rho\}$ and diverges on $\{x \in \mathbb{C}_p : |x| > \rho\}$. For each
positive real number \( r \) with \( r < \rho \), the convergence is uniform on \( \{ x \in \mathbb{C}_p : |x| \leq r \} \). On the 'boundary' \( \{ x \in \mathbb{C}_p : |x| = \rho \} \), the power series either converges everywhere, or converges nowhere.

**Proof** Contrary to the classical case, a series in a non-Archimedean Banach space converges if and only if the terms tend to zero. Hence \( \sum a_n x^n \) converges if and only if

\[
\lim_{n \to \infty} |a_n| |x|^n = 0. \tag{2.66}
\]

In particular, convergence depends only on the norm of \( x \) and not on \( x \) itself. This establishes the statement concerning boundary behavior. Now if \( 0 < \tau < \sigma < \rho \), then, by the definition of \( \rho \), there is an \( N \) such that

\[
|a_n|^{1/n} \leq \sigma^{-1} \tag{2.67}
\]

for all \( n \geq N \). It follows that

\[
|a_n x^n| \leq \left( \frac{\sigma}{\rho} \right)^n \tag{2.68}
\]

for all \( n \geq N \) and all \( x \) in \( \mathbb{C}_p \) with \( |x| \leq \tau \). This establishes convergence on \( \{ x \in \mathbb{C}_p : |x| < \rho \} \) and uniform convergence on \( \{ x \in \mathbb{C}_p : |x| \leq \tau \} \). Finally, if \( |x| > \rho \), then there are infinitely many \( a_n \) such that

\[
|a_n|^{1/n} > |x|^{-1} \tag{2.69}
\]

and, hence, such that

\[
|a_n x^n| > 1. \tag{2.70}
\]
The next lemma is the analytic analog to lemma 2.4.3. Again, it would follow directly from Lemma 42.1 in Schikhof [S] if it were not for the switch to the realm of Banach spaces.

**Lemma 2.6.2** If \( r > 0 \) is the norm of an element of \( C_p \) and \( f : B_0(r) \to B \) is analytic on \( B_0(r) \) with power series

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n,
\]

then

\[
\sup_{B_0(r)} |f(x)| = \sup_n |a_n| r^n. \tag{2.72}
\]

**Proof** One direction follows immediately from the strong triangle inequality. Namely that

\[
\sup_{B_0(r)} |f(x)| \leq \sup_n |a_n| r^n. \tag{2.73}
\]

For the other direction I may assume, with no loss of generality, that \( r = 1 \). Indeed, if the lemma holds whenever \( r = 1 \), and if \( s = |\lambda| \neq 1 \) and \( g : B_0(s) \to B \) is analytic on \( B_0(s) \) with power series

\[
g(x) = \sum_{n=0}^{\infty} b_n x^n, \tag{2.74}
\]

then the function

\[
f : B_0(1) \to B \\
x \mapsto g(\lambda x) \tag{2.75}
\]

is analytic on \( B_0(1) \) and has power series

\[
f(x) = \sum_{n=0}^{\infty} (\lambda^n b_n) x^n, \tag{2.76}
\]
so that
\[ \sup_{B_0(x)} |g(x)| = \sup_{B_0(t)} |f(x)| = \sup_n |b_n| s^n. \] (2.77)

Assuming, then, that \( r = 1 \), it follows that \( \sum a_n x^n \) converges when \( x = 1 \) and, hence, that
\[ \lim_{n \to \infty} a_n = 0. \] (2.78)

I may therefore set
\[ D = \sup_n |a_n| = \max_n |a_n|, \] (2.79)

and
\[ M = \min\{m : |a_m| = D\}. \] (2.80)

It now suffices to demonstrate the existence of an \( x \) in \( B_0(1) \) for which \( |f(x)| \) is as close to \( D \) as desired. If \( D = 0 \), then \( f \) is identically zero and I am done. If \( M = 0 \), then \( |f(0)| = D \) and, again, I am done. If \( D \) and \( M \) are both nonzero, I may, for each \( \epsilon \) with \( 0 < \epsilon < D \), choose an \( x \) in \( B_0(1) \) so that
\[ \max\left\{ 1 - \frac{\epsilon}{D}, \frac{|a_0|}{D}, \ldots, \frac{|a_{M-1}|}{D} \right\} < |x^M| < 1. \] (2.81)

I claim that, under these assumptions,
\[ |a_M x^M| > |a_m x^m| \] (2.82)

for all \( m \neq M \). If \( m > M \), this is clear as both \( |a_M| \geq |a_m| \) and \( |x^M| > |x^m| \). Also, if \( m < M \), then
\[ |a_M x^M| > D \frac{|a_m|}{D} = |a_m| \geq |a_m x^m| \] (2.83)
as desired. Finally,

\[ |f(x)| = |a_M x^M| > D \left(1 - \frac{\epsilon}{D}\right) = D - \epsilon, \tag{2.84} \]

and I am done.

The remaining two lemmas of this section provide conditions under which the uniform limit of analytic functions is again analytic.

**Lemma 2.6.3** If \( r > 0 \) is the norm of an element of \( C_\nu \) and \( f_n : B_0(r) \to B \) are analytic functions which converge uniformly on \( B_0(r) \), then the limiting function \( f \) is analytic on \( B_0(r) \).

**Proof** It suffices to demonstrate this for the case that \( r = 1 \). If \( H(B_0(1)) \) denotes the vector space of functions \( f : B_0(1) \to B \), analytic on \( B_0(1) \), endowed with the supremum norm over \( B_0(1) \), and if \( C_0(N \to B) \) denotes the Banach space, under the supremum norm, of sequences of elements of \( B \) which tend to zero, then, by Lemma 2.6.2, \( H(B_0(1)) \) and \( C_0(N \to B) \) are isometrically isomorphic via the map

\[ \sum_{n=0}^{\infty} a_n x^n \mapsto (a_n : n \in N). \tag{2.85} \]

Hence \( H(B_0(1)) \) is a Banach space and I am done.

**Lemma 2.6.4** If \( r > 0 \) and \( f_n : B^-_0(r) \to B \) are analytic functions which converge uniformly on \( B_0(r_0) \) for each \( 0 < r_0 < r \), then the limiting function \( f \) is analytic on \( B^-_0(r) \).
PROOF As the power series expansion of a function about the origin is unique, this follows directly from Lemma 2.6.3.

2.7 The Mahler series of an analytic function

The purpose of this section is to give a nice characterization of those Mahler series which represent analytic functions. The main result is given in Lemma 2.7.3. Prior to that, I present two lemmas needed in the proof of the Lemma 2.7.3. The first provides a formula for the supremum over $B_0(r)$ of the norm of $(a_n)$, and the second provides means for passing between a Mahler series and a power series. The second lemma, in particular, is simply a modified version of exercise 52C in Schikhof [S].

Lemma 2.7.1 If $r \geq 1$ is the norm of an element of $C_n$, and $n$ is a fixed natural number, then

$$\sup_{x \in B_0(r)} \left| \binom{x}{n} \right| = \frac{r^n}{n!}. \quad (2.86)$$

PROOF By Lemma 2.6.2,

$$\sup_{x \in B_0(r)} \left| \binom{x}{n} \right| = \max_{0 \leq m \leq n} \left| \frac{\tau(m, n)}{n!} \right| r^m. \quad (2.87)$$

where the $\tau(m, n)$ are the polynomial coefficients of $(x)_n$. Now the $\tau(m, n)$ are all integers and $\tau(n, n) = 1$, so the maximum of $|\tau(m, n)|$ is achieved when $m = n$, and is equal to one. The maximum of $r^m$ is also achieved when $m = n$, as it was assumed that $r \geq 1$. Hence,
\[
\max_{0 \leq m \leq n} \frac{\tau(m,n)}{n!} r^m = \frac{r^n}{|n!|} \tag{2.88}
\]
as desired. \[\square\]

**Lemma 2.7.2** If \(s(m,n)\) and \(t(m,n)\) are defined by the identities

\[
x^n = \sum_{m=0}^{\infty} s(m,n) \binom{x}{m} \tag{2.89}
\]

and

\[
\binom{x}{n} = \sum_{m=0}^{\infty} t(m,n)x^m, \tag{2.90}
\]

with the additional convention that \(s(m,n) = t(m,n) = 0\) whenever \(m\) or \(n\) is less than zero, then for all \(m,n \geq 0\):

(i) \(s(m,n)\) is an integer divisible by \(m!\).

(ii) \(s(m,n) = 0\) if \(m > n\).

(iii) \(n! t(m,n)\) is an integer.

(iv) \(t(m,n) = 0\) if \(m > n\).

**Proof** Statements (iii) and (iv) are immediate consequences of the identity

\[
\binom{x}{n} = \frac{1}{n!} \prod_{j=0}^{n-1} (x - j). \tag{2.91}
\]

Statements (i) and (ii) are obvious when \(n = 0\). Suppose, therefore, that both statements hold for some \(n \geq 0\). From the identity in 2.2.6, it follows that

\[
\sum_{m=0}^{\infty} s(m,n+1) \binom{x}{m} = x^{n+1}
\]
\[
\begin{align*}
\sum_{m=0}^{\infty} s(m, n) \binom{x}{m} & = x \left( \sum_{m=0}^{\infty} s(m, n) \binom{x}{m} \right) \\
& = \sum_{m=0}^{\infty} s(m, n) \left[ m \binom{x}{m} + (m + 1) \binom{x}{m+1} \right] \\
& = \sum_{m=0}^{\infty} m[s(m, n) + s(m - 1, n)] \binom{x}{m}. \quad (2.92)
\end{align*}
\]

Hence,
\[
s(m, n + 1) = m[s(m, n) + s(m - 1, n)]. \quad (2.93)
\]

As
\[
(m - 1)! \left| s(m, n) + s(m - 1, n) \right|, \quad (2.94)
\]
it follows that
\[
m! \left| s(m, n + 1) \right|. \quad (2.95)
\]

Also, as \(s(m, n) = s(m - 1, n) = 0\) whenever \(m > n + 1\), it follows that \(s(m, n + 1) = 0\) whenever \(m > n + 1\). This establishes the validity of claims (i) and (ii) and concludes the proof of the Lemma.

**Lemma 2.7.3** If \(r \geq 1\) is the norm of an element of \(C_n\), and \(f : \mathbb{Z}_p \to B\) is continuous, with Mahler series
\[
f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}, \quad (2.96)
\]
then the following statements are equivalent.

(i) \(f\) extends to an analytic function on \(B_0(r)\).

(ii) \(\lim_{n \to \infty} \left| \frac{a_n}{n!} r^n \right| = 0\).
(iii) \( \sum_{n=0}^{\infty} a_n \binom{x}{n} \) converges for all \( x \) in \( B_0(r) \).

(iv) \( x \mapsto \sum_{n=0}^{\infty} a_n \binom{x}{n} \), is an analytic extension of \( f \) to \( B_0(r) \).

**Proof** I shall first show that condition (i) implies the other three. If \( f \) extends to the analytic function

\[
x \mapsto \sum_{n=0}^{\infty} b_n x^n,
\]

valid on all of \( B_0(r) \), and if \( s(m,n) \) is as defined in Lemma 2.7.2, then I claim that

\[
\sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \sum_{m=0}^{n} b_n s(m,n) \binom{x}{m} = \sum_{m=0}^{\infty} \left( \sum_{n=m}^{\infty} b_n s(m,n) \right) \binom{x}{m}
\]

for all \( x \) in \( B_0(r) \). By Lemma 2.2.1 it suffices to show that, for any \( \epsilon > 0 \),

\[
|b_n s(m,n) \binom{x}{n}| < \epsilon
\]

for all but finitely many pairs \((m, n)\). Also, as \( s(m, n) = 0 \) whenever \( m > n \), I may restrict my attention to those pairs for which \( m \leq n \). Now, as \( r \) was assumed to be the norm of an element of \( C \), it follows that

\[
\lim_{n \to \infty} |b_n| r^n = 0.
\]

Finally, if \( N \) is chosen so that

\[
|b_n| r^n < \epsilon
\]

for all \( n > N \), then

\[
|b_n s(m,n) \binom{x}{m}| \leq \left| \frac{b_n s(m,n)}{m!} \right| r^m
\]
for all \( n > N \), and the claim is established. Here, I am using Lemma 2.7.1 for the first inequality, Lemma 2.7.2 (and specifically the fact that \( s(m, n)/m! \) is integral) for the second inequality, and the assumptions that \( r \geq 1 \) and that \( m \leq n \) for the third inequality.

As the Mahler series of a continuous function is unique, Equation 2.98 yields the formula

\[
a_m = \sum_{n=m}^{\infty} b_n s(m, n),
\]

(2.103)

and simultaneously establishes conditions (iii) and (iv). Now

\[
\left| \frac{a_m}{m!} \right| r^m = r^m \left| \sum_{n=m}^{\infty} \frac{b_n s(m, n)}{m!} \right|
\leq \sup_{n \geq m} \left| \frac{b_n s(m, n)}{m!} \right| r^m
\leq \sup_{n \geq m} |b_n| r^n,
\]

(2.104)

and this last expression tends to zero as \( m \) tends to infinity. This concludes the verification of the implications (i)\( \rightarrow \)(ii), (i)\( \rightarrow \)(iii), and (i)\( \rightarrow \)(iv).

If condition (ii) holds, then, by Lemma 2.7.1, the series in (iii), not only converges for all \( z \) in \( B_0(r) \), but converges uniformly on that set. Hence, by Lemma 2.6.3, the limiting function is analytic on \( B_0(r) \). This establishes the implications (ii)\( \rightarrow \)(iii), and (ii)\( \rightarrow \)(iv).
Now condition (iv) obviously implies condition (i), and, hence, implies conditions (ii) and (iii) as well, so it only remains to be seen that condition (iii) implies any of the other conditions. I shall argue that (iii) implies (ii).

It suffices to show that, for some $x$ in $B_0(r)$,

$$\left|\binom{x}{n}\right| = \frac{r^n}{|n!|},$$

for all natural numbers $n$. As

$$\binom{x}{n + 1} = \binom{x}{n} \left(\frac{x - n}{n + 1}\right),$$

this is equivalent to the statement that $|x - n| = r$ for all $n$ in $N$. If $r > 1$ then any $x$ with $|x| = r$ will work. If $r = 1$, one must choose an $x$ whose representative in the residue class field is not in the prime subfield $\mathbb{F}_p$. As I am working over a sufficiently 'large' field, ($\mathbb{C}_p$ in this case), such $x$'s abound. Hence condition (ii) follows from condition (iii) and the equivalence of all four statements has been established.

When a function, initially defined on $\mathbb{Z}_p$, satisfies any of the equivalent conditions in the above Lemma, I shall say that the function itself is analytic on $B_0(r)$, rather than using the more precise statement that the function may be extended to an analytic function on $B_0(r)$. As the analytic extension is unique, there is no ambiguity in doing so.

In the sequel, it will be desirable to be able to compute the supremum of an analytic function over the 'closed' ball $B_0(r)$ given only the Mahler coefficients of the function. The following Lemma provides the required formula. One should note the similarities between it and Lemma 2.6.2.
Lemma 2.7.4 If $r \geq 1$ is the norm of an element of $\mathbb{C}_p$, and if $f$ is analytic on $B_0(r)$ and has Mahler series

\[ f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}, \quad (2.107) \]

then

\[ \sup_{B_0(r)} |f(x)| = \sup_{n} \left| \frac{a_n}{n!} \right| r^n. \quad (2.108) \]

PROOF That the left hand side is not greater than the right hand side is obvious. Also, if

\[ f(x) = \sum_{n=0}^{\infty} b_n x^n \quad (2.109) \]

for all $x$ in $B_0(r)$, then, by Lemma 2.6.2,

\[ \sup_{B_0(r)} |f(x)| = \sup_{n} |b_n| r^n. \quad (2.110) \]

Hence, it suffices to show that

\[ \sup_{n} \left| \frac{a_n}{n!} \right| r^n \leq \sup_{n} |b_n| r^n. \quad (2.111) \]

From the proof of Lemma 2.7.3, specifically, inequality 2.104,

\[ \left| \frac{a_m}{m!} \right| r^m \leq \sup_{n \geq m} |b_n| r^n \leq \sup_{n} |b_n| r^n. \quad (2.112) \]

The second condition in Lemma 2.7.3 is occasionally not so easy to check. For this reason, it is desirable to have alternatives. The next Lemma provides one such alternative. Roughly, it allows one to apply a kind of 'nth-root-test' to the Mahler coefficients of an analytic function in order to determine the radius of convergence.
Lemma 2.7.5 If $\epsilon$ is the norm of an element of $C_p$ which satisfies

$$0 < \epsilon < p^{-\frac{1}{p-1}} \quad (2.113)$$

and if

$$|a_n| \leq E\epsilon^n \quad (2.114)$$

for some $E > 0$ and for all $n \geq 0$, then

$$x \mapsto \sum_{n=0}^{\infty} a_n \left( \frac{x}{n} \right) \quad (2.115)$$

is analytic on $B_0 \left( \epsilon^{-1} p^{-\frac{1}{p-1}} \right)$. Conversely, if $\epsilon$ is the norm of an element of $C_p$ which satisfies

$$0 < \epsilon < p^{-\frac{1}{p-1}} \quad (2.116)$$

and if

$$x \mapsto \sum_{n=0}^{\infty} a_n \left( \frac{x}{n} \right) \quad (2.117)$$

is analytic on $B_0 \left( \epsilon^{-1} p^{-\frac{1}{p-1}} \right)$, then, for every $\epsilon_0 > \epsilon$, there is an $E > 0$ such that

$$|a_n| \leq E\epsilon_0^n \quad (2.118)$$

for all $n \geq 0$.

Proof Suppose that $0 < \epsilon < p^{-\frac{1}{p-1}}$, that $r$ is the norm of an element of $C_p$, and that

$$1 \leq r < \epsilon^{-1} p^{-\frac{1}{p-1}}.$$ If, in addition, $|a_n| \leq E\epsilon^n$, then

$$\left| \frac{a_n}{n!} \right| r^n \leq \frac{E(\epsilon r)^n}{|n!|} \quad (2.119)$$
which, by Lemma 2.3.4, tends to zero as \( n \) tends to infinity. Hence, by Lemma 2.7.3

\[
x \mapsto \sum_{n=0}^{\infty} a_n \binom{x}{n}
\]

(2.120)

is analytic on \( B_0(r) \) and the first statement of the Lemma follows. Now, with the same assumptions on \( \epsilon \), suppose that

\[
x \mapsto \sum_{n=0}^{\infty} a_n \binom{x}{n}
\]

(2.121)

is analytic on \( B_0 \left( \epsilon^{-1} p^{\frac{1}{n-1}} \right) \), and let \( \epsilon_0 > \epsilon \) be arbitrary. To prove the last statement of the Lemma, it suffices to show that \( |a_n|/\epsilon_0^n \) tends to zero as \( n \) tends to infinity. By Lemma 2.3.4,

\[
\lim_{n \to \infty} \frac{\left( \epsilon_0 \epsilon^{-1} p^{\frac{1}{n-1}} \right)^n}{|n!|} = \infty,
\]

(2.122)

and, by Lemma 2.7.3,

\[
\lim_{n \to \infty} \frac{|a_n|}{n!} \left( \epsilon^{-1} p^{\frac{1}{n-1}} \right)^n = 0.
\]

(2.123)

Hence,

\[
\lim_{n \to \infty} \frac{|a_n|}{\epsilon_0^n} = 0,
\]

(2.124)

as this limit is simply the quotient of the two previous limits.

\[\blacksquare\]

### 2.8 The indefinite sum of an analytic function

The following Lemma provides conditions under which the indefinite sum of an analytic function is again analytic.
Lemma 2.8.1 If \( r > 1 \), then \( f \) is analytic on \( B_0^-(r) \) if and only if \( Sf \) is analytic on \( B_0^-(r) \).

**Proof** \( f(x) = Sf(x+1) - Sf(x) \), so one direction is immediate. Now if

\[
f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}
\]

(2.125)

is analytic on \( B_0^-(r) \), then, by Lemma 2.7.3,

\[
\lim_{n \to \infty} \left| \frac{a_n}{n!} \right| r_0^n = 0,
\]

(2.126)

for all \( 0 < r_0 < r \). Hence, if \( r_0, r_1 \) are fixed, with \( 0 < r_0 < r_1 < r \), then

\[
\lim_{n \to \infty} \left| \frac{a_{n-1}}{n!} \right| = \lim_{n \to \infty} \left| \frac{a_{n-1}}{(n-1)!} \right| r_1^{n-1} \cdot \left| \frac{1}{n} \right| \left( \frac{r_0}{r_1} \right)^n \cdot r_1
\]

(2.127)

by Lemma 2.3.1. This establishes the other direction and concludes the proof of the lemma.

The result of Lemma 2.8.1 is sharp in that the indefinite sum \( Sf \) of an analytic function on \( B_0(1) \) need not be analytic. One such function is provided in the following example.

**Example 2.8.1** If a function \( f: \mathbb{Z}_p \to B \) has Mahler series

\[
f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n},
\]

(2.128)

then the indefinite sum \( Sf \) of \( f \) has Mahler series

\[
Sf(x) = \sum_{n=1}^{\infty} a_{n-1} \binom{x}{n}.
\]

(2.129)
By Lemma 2.7.3, if I wish to demonstrate the existence of a function \( f \) which is analytic on \( B_0(1) \), but whose indefinite sum is not analytic on \( B_0(1) \), I need only exhibit a sequence \( (a_n) \) for which

\[
\lim_{n \to \infty} \left| \frac{a_n}{n!} \right| = 0, \quad (2.130)
\]

but for which

\[
\lim_{n \to \infty} \left| \frac{a_{n-1}}{n!} \right| \neq 0. \quad (2.131)
\]

If I set

\[
a_n = \begin{cases} 
\rho \left( \frac{n_{k-1}}{p^{k-1}} \right) & \text{if } n = p^k - 1 \\
0 & \text{otherwise}
\end{cases}, \quad (2.132)
\]

then, by Lemma 2.3.2,

\[
\left| \frac{a_n}{n!} \right| = \begin{cases} 
\rho^{-k} & \text{if } n = p^k - 1 \\
0 & \text{otherwise}
\end{cases}, \quad (2.133)
\]

whereas

\[
\left| \frac{a_{n-1}}{n!} \right| = \begin{cases} 
1 & \text{if } n = p^k \\
0 & \text{otherwise}
\end{cases}. \quad (2.134)
\]

Hence \( a_n \) has the desired properties.

\( \Box \)
CHAPTER III

General Sufficiency Results

3.1 The General First Order Equation

Typically, the cornerstone for any theory of equations consists of one or more results which provide conditions sufficient to guarantee the existence of solutions to the equations under consideration. The results of this chapter are intended to provide that foundation for the $p$-adic theory of finite difference equations. In this chapter, I have made every effort to pattern the development of $p$-adic finite difference equations after the classical theory of differential equations. Theorem 1 is the best example of this, as both the statement and the proof are strikingly similar to those of the standard result on the existence of a continuously differentiable solution to the classical first order differential equation contained in many introductory texts. The following Lemma is one such classical result. It, and the following outline of proof, are simply watered down versions of the statement and proof of Theorem 6 in Chapter 7 of Marsden [Mar].
Lemma 3.1.1 If $f : [-a, a] \times [x_0 - r, x_0 + r] \to \mathbb{R}$ is a continuous function, and if there is a positive constant $K$ such that

$$|f(t,x) - f(t,y)| \leq K|x - y|$$ (3.1)

for all $t$ in $[-a, a]$ and all $x$ and $y$ in $[x_0 - r, x_0 + r]$, then there is a unique continuously differentiable map $x : [-b, b] \to [x_0 - r, x_0 + r]$ such that

$$\frac{dx}{dt} = f(t, x(t)), \quad x(0) = x_0$$ (3.2)

where $0 < b < \min\{a, r/C, 1/K\}$ and $C = \sup_{[-a,a] \times [x_0 - r, x_0 + r]} |f(t,x)|$.

The method of proof is to construct a sequence of functions which uniformly converge to a solution of the corresponding integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s)) \, ds.$$ (3.3)

Specifically, one takes

$$x_1(t) = x_0,$$ (3.4)

and

$$x_{n+1}(t) = x_0 + \int_0^t f(s, x_n(s)) \, ds$$ (3.5)

for all natural numbers $n$.

The following theorem provides conditions sufficient to guarantee the existence of a continuous solution $y : Z_p \to B$ to the finite difference equation

$$\Delta y(x) = f(x, y(x))$$

$$y(0) = y_0.$$ (3.6)
In the theorem, and thereafter, if $B$ is a Banach space, I shall use $D_R(B)$ (abbreviated to $D_R$ when $B$ is clear from context) to denote the set \( \{ y \in B : |y| \leq R \} \).

**Theorem 1** If \( 0 < R \leq \infty \), \( f : \mathbb{Z}_p \times D_R(B) \to D_R(B) \) is continuous, \( y_0 \in D_R(B) \), and for some \( A < 1 \),

\[
|f(x, y_1) - f(x, y_2)| \leq A|y_1 - y_2|
\]  

(3.7)

for every \( x \) in \( \mathbb{Z}_p \) and for all \( y_1 \) and \( y_2 \) in \( D_R(B) \), then there is a unique continuous function \( y : \mathbb{Z}_p \to D_R(B) \) satisfying the finite difference equation

\[
\Delta y(x) = f(x, y(x)) \\
y(0) = y_0
\]

(3.8)

**PROOF** Uniqueness follows immediately since the set of natural numbers is dense in \( \mathbb{Z}_p \) and since (3.8) completely determines \( y \) on this set. To see existence, observe that equation (3.8) is satisfied if and only if the function \( x \mapsto y(x) \) is a fixed point of the operator \( \Phi : C(\mathbb{Z}_p, D_R(B)) \to C(\mathbb{Z}_p, D_R(B)) \), defined as follows.

\[
\Phi(y) : x \mapsto y_0 + \sum_{j=0}^{x-1} f(j, y(j)) = y_0 + Sf(x, y(x))
\]

(3.9)

Since the \( S \) operator preserves continuity, (Lemma 2.5.1), \( \Phi(y) \) is continuous whenever \( y \) is continuous.

Now for each natural number \( n \), define \( y_n : \mathbb{Z}_p \to D_R(B) \) inductively as follows.

\[
y_0(t) = y_0 \\
y_{n+1}(t) = \Phi(y_n)(t)
\]

(3.10)

If the sequence \( \{y_n\} \) is uniformly Cauchy, then the pointwise limit

\[
y(t) = \lim_{n \to \infty} y_n(t)
\]

(3.11)
is a continuous solution to the finite difference equation (3.8). By the strong triangle inequality it is enough to show that

$$
\lim_{N \to \infty} \left[ \sup_{s \in \mathbb{Z}_p} |y_{N+1}(s) - y_N(s)| \right] = 0 
$$

(3.12)

We have

$$
\sup_{s \in \mathbb{Z}_p} |y_{N+1}(s) - y_N(s)| \leq \sup_{t \in \mathbb{Z}_p} |f(t, y_N(t)) - f(t, y_{N-1}(t))| 
\leq A \sup_{t \in \mathbb{Z}_p} |y_N(t) - y_{N-1}(t)|. 
$$

(3.13)

The result follows as it was assumed that $A < 1$.

A few remarks regarding the possible analyticity of the unique solution to equation 3.8 are in order. It would be natural at this point to hope, that by requiring $f$ to be analytic in each variable, one could conclude that $y$ is also analytic. Indeed, there are only three things one must verify in order to extend the proof of Theorem 1 to cover the question of analyticity. Namely, that the uniform limit of analytic functions is again analytic, that the $S$ operator preserves analyticity, and that the sequence of successive approximations is uniformly Cauchy. The first two requirements are satisfied as long as one is careful about what one means by them. For instance, by Lemma 2.4.2, the polynomials are dense in the continuous functions on $\mathbb{Z}_p$, so it is not the case that the uniform limit of analytic functions on $\mathbb{Z}_p$ is again analytic on $\mathbb{Z}_p$. Fortunately, by Lemma 2.6.3, if the convergence is required to be uniform on $B_0(1)$, then it is true that the limiting function is analytic on $B_0(1)$. By the same token, the $S$ operator does not preserve analyticity in general, but it does preserve analyticity provided that the functions in question are analytic on balls strictly larger than $B_0(1)$. 
This was precisely the content of Lemma 2.8.1 and Example 2.8.1. Sadly, however, the third requirement is not satisfied. Even for simple examples, the sequence of functions constructed in the proof of Theorem 1 need not be uniformly Cauchy. The following is one such example.

**Example 3.1.1** Let $p = 3$ and $B = \mathbb{C}_p$ and consider the finite difference equation

\[
\begin{align*}
\Delta y &= 3y^2 + 1 \\
y(0) &= 0.
\end{align*}
\]  

(3.14)

In the notation of Theorem 1,

\[
\begin{align*}
f(x, y) &= 3y^2 + 1 \\
y_0 &= 0,
\end{align*}
\]

(3.15)

and $R$ may be taken to be any fixed real number with $1 \leq R \leq 3$. Also,

\[
|(3y^2 + 1) - (3z^2 + 1)| = |3(y + z)(y - z)|
\]

\[
\leq \frac{R}{3}|y - z|
\]

(3.16)

for all $y$ and $z$ in $B_0(R)$. Hence, if I set

\[
A = \frac{R}{3},
\]

(3.17)

and insist that $R < 3$, then $f$ satisfies the required Lipschitz condition. Now the first three terms of the sequence $(y_n)$ of successive approximations to the continuous solution to equation 3.14 are as follows.

\[
y_0(x) = 0 \\
y_1(x) = x \\
y_2(x) = \sum_{j=0}^{x-1}(3j^2 + 1) = x^3 - \frac{3}{2}x^2 + \frac{3}{2}x
\]

(3.18)
One need compute no farther to see that inequality 3.13 in the proof of Theorem 1 does not hold if the supremums are computed over some closed ball $B_0(r)$ instead of over $\mathbb{Z}_p$. Indeed, if $r \geq 1$, then

$$\sup_{x \in B_0(r)} |y_2(x) - y_1(x)| = r^3, \quad (3.19)$$

whereas

$$\sup_{x \in B_0(r)} |y_1(x) - y_0(x)| = r. \quad (3.20)$$

Of course, this does not prove that the sequence $(y_n)$ is not eventually uniformly Cauchy, but it does show that the proof of Theorem 1 does not apply to the question of analyticity in this case. In section 5.4 I will give a complete proof of the fact that the continuous solution to equation 3.14 is not analytic on $\mathbb{Z}_p$. □

### 3.2 The General Linear Equation

The result of this section follows from the more general Theorem 3 presented in the next section. It is given here as a separate result both to emphasize how the Lipschitz condition on $f$ translates into a statement about the norms of the coefficients of higher order linear equations, and to demonstrate how higher order equations may be reduced to first order equations. In this section $B$ will now denote a Banach algebra with norm $\|\|$ satisfying

$$|vw| \leq |v||w| \quad (3.21)$$

for all $v$ and $w$ in $B$. As before, $\|\|_{p_n}$ will be used to denote the supremum norm.
of a function over $\mathbb{Z}_p$ and $||| \|$ will now be used to denote the maximum norm of a component of a vector in $B^n$.

**Theorem 2** If $s : \mathbb{Z}_p \rightarrow B$ is continuous, and for each $j$ in $\{0, \ldots, n-1\}$, $b_j$ is an element of $B$, and $r_j : \mathbb{Z}_p \rightarrow B$ is a continuous function which satisfies

$$||r_j||_{\mathbb{Z}_p} < 1,$$

then the finite difference equation,

$$\Delta^n y + r_{n-1}\Delta^{n-1}y + \ldots + r_0 y = s$$
$$\Delta^j y(0) = b_j \text{ for each } j$$

has a unique continuous solution $y : \mathbb{Z}_p \rightarrow B$.

**Proof** The method of proof is to replace equation 3.23 by an equivalent first order equation and then to apply Theorem 1. Let $y$ denote the vector $\langle y_{n-1}, \ldots, y_0 \rangle$ in $B^n$, and consider the finite difference equation

$$\Delta y(x) = (s(x) - \sum_{j=0}^{n-1} \alpha_j^{-1} r_j(x) y_j(x), \alpha_{n-2} \alpha_{n-1} y_{n-1}(x), \ldots, \alpha_0 \alpha_1^{-1} y_1(x))$$

$$y(0) = (\alpha_{n-1} b_{n-1}, \ldots, \alpha_0 b_0),$$

where the constants $\alpha_0, \ldots, \alpha_{n-1}$ are elements of $\mathbb{C}_p^\times$ to be determined later.

If $y : \mathbb{Z}_p \rightarrow B^n$ is a solution to 3.24, then

$$\Delta y_0 = \alpha_0 \alpha_1^{-1} y_1$$
$$\Delta y_1 = \alpha_1 \alpha_2^{-1} y_2$$
$$\vdots$$
$$\Delta y_{n-2} = \alpha_{n-2} \alpha_{n-1}^{-1} y_{n-1}$$
$$\Delta y_{n-1} = s - \sum_{j=0}^{n-1} \alpha_j^{-1} r_j y_j,$$
and
\[ \Delta^0 y_0 = \alpha_0 \alpha_0^{-1} y_0 \]
\[ \Delta^1 y_0 = \alpha_0 \alpha_1^{-1} y_1 \]
\[ \Delta^2 y_0 = \alpha_0 \alpha_2^{-1} y_2 \]
\[ : \]
\[ \Delta^{n-1} y_0 = \alpha_0 \alpha_{n-1}^{-1} y_{n-1} \]
\[ \Delta^n y_0 = \alpha_0 \alpha_{n-1}^{-1} s - \sum_{j=0}^{n-1} \alpha_0 \alpha_{n-1}^{-1} \alpha_j^{-1} r_j y_j \]
\[ = \alpha_{n-1}^{-1} \left[ \alpha_0 s - \sum_{j=0}^{n-1} r_j \Delta^j y_0 \right]. \]

Hence, if \( \alpha_{n-1} = 1 \) and \( y = \alpha_0^{-1} y_0 \), then
\[ \Delta^n y = \alpha_0^{-1} \Delta^n y_0 \]
\[ = s - \sum_{j=0}^{n-1} r_j \alpha_0^{-1} \Delta^j y_0 \]
\[ = s - \sum_{j=0}^{n-1} r_j \Delta^j y, \]

so that \( y \) is a solution to equation 3.23.

In the notation of Theorem 1, \( R = \infty \) and the function \( f : \mathbb{Z}^p \times B^n \to B^n \) for equation 3.24 is given by the formula
\[ f(x, y) = \langle s(x) - \sum_{j=0}^{n-1} \alpha_j^{-1} r_j(y) y_j(x), \alpha_{n-2}^{-1} y_{n-1}(x), \ldots, \alpha_0 \alpha_1^{-1} y_1(x) \rangle. \]

If nonzero constants \( \alpha_0, \ldots, \alpha_{n-1} \), with \( \alpha_{n-1} = 1 \), may be found so that \( f(x, y) \) satisfies the Lipschitz condition of Theorem 1, then equation 3.24 and, hence, equation 3.23 will have a unique continuous solution as desired. For \( y \) and \( z \) in \( B^n \),
\[ \| f(x, y) - f(x, z) \| \]
\[ \leq \max \left\{ \max_j \| \alpha_j^{-1} r_j \|_{x^p}, |\alpha_{n-2}^{-1} y_{n-1}|, \ldots, |\alpha_0 \alpha_1^{-1}| \right\} \| y - z \|. \]
The $\alpha_j$’s must therefore be chosen so that

$$|\alpha_0| < |\alpha_1| < \cdots < |\alpha_{n-1}| = 1, \quad (3.32)$$

and so that, for each $j$,

$$\|r_j\|_{Z_p} < \alpha_j. \quad (3.33)$$

As the set of norms of elements of $B$ is dense in $\mathbb{R}^+$, such $\alpha_j$’s may be found.

3.3 The General Finite Difference Equation

Theorem 3 is the natural generalization of Theorem 1 to arbitrary higher order finite difference equations. The proof is identical to that of Theorem 1 albeit with somewhat more complicated notation due to the increase in order.

**Theorem 3** For $0 < R \leq \infty$, if $f : Z_p \times D_R(B^n) \to D_R(B)$ is continuous, and for each $j$ in $\{0, \ldots, n - 1\}$, $b_j$ is an element of $D_R(B)$, and if there is an $A < 1$ such that,

$$|f(x, y) - f(x, z)| \leq A\|y - z\| \quad (3.34)$$

for all $y$ and $z$ in $D_R(B^n)$ and for all $x$ in $Z_p$, then the finite difference equation,

$$\Delta^n y = f(x, y, \Delta y, \ldots, \Delta^{n-1} y)$$

$$\Delta^j y(0) = b_j \quad \text{for each} \quad j \quad (3.35)$$

has a unique continuous solution $y : Z_p \to D_R(B)$. 

The key observation to be made, is that a continuous function \( y : \mathbb{Z}_p \to B \), is a solution to equation 3.35 if and only if it is a fixed point of the operator

\[
\Phi : C(\mathbb{Z}_p \to B) \to C(\mathbb{Z}_p \to B)
\]

\[
y \mapsto \Phi y,
\]

(3.36)

where

\[
\Phi y(x) = \sum_{j=0}^{n-1} \left( \frac{x}{j} \right) b_j + S^n f(x, y(x), \Delta y(x), \ldots, \Delta^{n-1} y(x)).
\]

(3.37)

Now \( \Phi \) is a continuous operator on the complete metric space \( C(\mathbb{Z}_p \to B) \) and, for all \( y \) and \( z \) in \( C(\mathbb{Z}_p \to B) \),

\[
\| \Phi y - \Phi z \|_{\mathbb{Z}_p} \\
\leq \sup_{x \in \mathbb{Z}_p} | f(x, y(x), \Delta y(x), \ldots, \Delta^{n-1} y(x)) - f(x, z(x), \Delta z(x), \ldots, \Delta^{n-1} z(x)) | \\
\leq A \max_j \| \Delta^j y - \Delta^j z \|_{\mathbb{Z}_p} \\
\leq A \| y - z \|_{\mathbb{Z}_p}.
\]

(3.38)

Hence, \( \Phi \) has a unique fixed point in \( C(\mathbb{Z}_p \to B) \).
CHAPTER IV

Linear Equations

In Chapter III, I presented the best results I have for the most general finite difference equation. In this chapter, I show how these results may be improved for linear equations.

I study two questions left unresolved by Theorem 1. In 4.1, I consider the problem of specifying necessary and sufficient conditions for the existence of a continuous solution to a linear finite difference equation of order 1, and give some very partial results on higher order equations. In 4.2, I give sufficient conditions on a linear difference equation which guarantee the analyticity of the solutions.

4.1 Necessary and Sufficient Conditions

The primary focus of this section is on linear finite difference equations of the form

\[ \Delta y(x) = R(x)y(x) + s(x), \]
\[ y(0) = y_0. \]  

(4.1)

where \( R \) is a function from \( \mathbb{Z}_p \) into the \( d \times d \) matrices over \( \mathbb{C}_p \), \( s \) is a function from \( \mathbb{Z}_p \) to \( \mathbb{C}_p^d \), and \( y_0 \) is an element of \( \mathbb{C}_p^d \). The intent is to provide nice necessary
and sufficient conditions which guarantee the existence of a continuous solution \( y : \mathbb{Z}_p \to \mathbb{C}_p \) to this equation. For the one-dimensional \((d = 1)\) homogeneous case, I do just that. For the one-dimensional non-homogeneous case, I obtain necessary and sufficient conditions only under certain boundedness assumptions. Finally, for higher dimensional equations, I am only able to provide such conditions in the case of an homogeneous equation and a constant matrix \( R \).

4.1.1 The Converse to Theorem 1

In the following example I examine a simple finite difference equation for which the conditions of Theorem 1 fail. This, and the discussion which follows, is intended to motivate, in part, the development of conditions which are both necessary and sufficient to guarantee the existence of a continuous solution to the general first order homogeneous linear finite difference equation in dimension one.

**Example 4.1.1** Consider the finite difference equation

\[
\begin{align*}
\Delta y &= y \\
y(0) &= y_0.
\end{align*}
\]

(4.2)

Here

\[
y(n + 1) = 2y(n),
\]

(4.3)

so

\[
y(n) = y_0 2^n
\]

(4.4)
for each natural number \( n \). Unfortunately the function \( n \mapsto 2^n \) cannot be extended continuously to \( \mathbb{Z}_p \). Indeed, if there were such an extension, then, in particular, we would have

\[
\lim_{n \to \infty} 2^{pn} = 2^0 = 1, \tag{4.5}
\]

but

\[
2^p \equiv 2 \pmod{p} \tag{4.6}
\]

for all primes \( p \) and so

\[
2^{pn} \equiv 2 \pmod{p} \not\equiv 1 \pmod{p} \tag{4.7}
\]

for any prime \( p \) and any natural number \( n \). Hence, this finite difference equation has a continuous solution on \( \mathbb{Z}_p \) only in the case that \( y_0 = 0 \). Namely, the identically zero solution. \( \Box \)

In light of the previous paragraph it seems reasonable to ask the question, "For which elements \( a \) of \( C_p \) does the function \( y(n) = a^n \) on the natural numbers, extend to a continuous function on \( \mathbb{Z}_p \)?" This question is answered by Schikhof [S] in the following lemma.

**Lemma 4.1.1** If \( K \) is a complete non-Archimedean non-trivially valued field with residue class field \( k \) of characteristic \( p \), then for any element \( a \) of \( K \), the function

\[
y : \mathbb{N} \to K \quad n \mapsto a^n \tag{4.8}
\]

can be continuously extended to a function on \( \mathbb{Z}_p \) if and only if

\[
|1 - a| < 1. \tag{4.9}
\]
This lemma immediately provides us with necessary and sufficient conditions on elements $a$ of $\mathbb{C}_p$ such that the finite difference equation

$$\begin{align*}
\Delta y &= ay \\
y(0) &= y_0
\end{align*} \quad (4.10)
$$

has a continuous solution on $\mathbb{Z}_p$. Indeed, any solution to (4.10) satisfies

$$y(n) = (a + 1)^n y_0 \quad (4.11)$$

for all natural numbers $n$. So, from the lemma we see that equation (4.10) has a continuous solution on $\mathbb{Z}_p$ if and only if $|a| < 1$.

It is interesting that this is precisely the condition that Theorem 1 imposes on equation 4.10 to guarantee the existence of a continuous solutions on $\mathbb{Z}_p$. One might, therefore, hope that the Lipschitz condition of Theorem 1 is both necessary and sufficient. The following example, however, demonstrates that this is not the case.

**Example 4.1.2** Suppose $p \neq 2$ and consider the finite difference equation

$$\begin{align*}
\Delta y &= r(x)y \\
y(0) &= 1
\end{align*} \quad (4.12)$$

where

$$r(x) = \begin{cases} 
-(x + 1) & \text{if } |x| = 1 \\
-2 & \text{if } |x| < 1
\end{cases} \quad (4.13)$$

Then in the notation of Theorem 1

$$f(x, y) = r(x)y \quad (4.14)$$

so that

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = |r(x)| \quad (4.15)$$
and (3.7) is not satisfied for any \( A < 1 \). Nonetheless, equation (4.12) does have a continuous solution on \( \mathbb{Z}_p \), for if we rewrite it in the form

\[
\begin{align*}
y(0) &= 1 \\
y(n + 1) &= \begin{cases} 
-ny(n) & \text{if } p \nmid n \\
-y(n) & \text{if } p \mid n
\end{cases}
\end{align*}
\]

we see immediately that \( y(x) \) is really just Morita's gamma function \( \Gamma_p(x) \) in disguise.

\[ \Box \]

For the next few subsections, the primary goal will be to provide 'nice' necessary and sufficient conditions which guarantee the existence of a continuous solution to the general linear first order finite difference equation in dimension one. The culmination of these efforts is presented in Theorem 5. I will then conclude the section with a brief discussion on systems of linear equations and the reduction of higher order equations to systems of first order equations.

### 4.1.2 The First Order Homogeneous Linear Equation

Here I examine the homogeneous case in detail. For that case, and then occasionally thereafter, I need the following two Lemmas. The first is a characterization of the functions \( r_\pi \) and \( r_\omega \) introduced in section 2.1, and the second is useful in estimating distances between products of elements.

**Lemma 4.1.2** If \( r : \mathbb{Z}_p \rightarrow \mathbb{C}_p^\times \) is continuous, then \( r_\pi \) and \( r_\omega \) have finite range and hence, are periodic with period \( p^n \) for some natural number \( n \).
Proof Both \( r_r \) and \( r_w \) are continuous and map \( \mathbb{Z}_p \), a compact space, into a discrete subspace of \( \mathbb{C}_p^\times \). It follows that they both have finite range. Now, again using compactness of \( \mathbb{Z}_p \), \( r_r \) and \( r_w \) are uniformly continuous functions from \( \mathbb{Z}_p \) into a finite discrete topological space. The second assertion of the lemma then follows immediately from the definition of uniform continuity.

Lemma 4.1.3

a) If \( t, u, v, \) and \( w \) are elements of \( \mathbb{C}_p \), then

\[
|tv - uw| \leq \max\{|tu||v - w|, |w||t - u|\}. \quad (4.17)
\]

b) If \( t, u, v, \) and \( w \) are elements of \( B_0(1) \), then

\[
|tv - uw| \leq \max\{|v - w|, |t - u|\}. \quad (4.18)
\]

c) If \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) are elements of \( B_0(1) \) with \( |a_j - b_j| < \delta \) for each \( j \), then

\[
\left| \prod_{j=1}^n a_j - \prod_{j=1}^n b_j \right| < \delta. \quad (4.19)
\]

Proof The inequality

\[
|tv - uw| \leq |tv - tw + tw - uw|
\]

\[
\leq \max\{|t||v - w|, |w||t - u|\} \quad (4.20)
\]

establishes part (a), and part (b) is simply a special case of part (a). Finally, part (c) follows from part (b) by an easy induction argument.

Now consider the finite difference equation

\[
y(x + 1) = r(x)y(x) \quad \text{and} \quad y(0) = a, \quad (4.21)
\]
where \( r \) is an arbitrary function from \( \mathbb{Z}_p \) into \( \mathbb{C}_p \) and \( a \) is any element of \( \mathbb{C}_p \). By Theorem 1, this equation has a continuous solution provided \( r \) is a continuous function with

\[
|r(x) - 1| < 1 \tag{4.22}
\]

for all \( x \) in \( \mathbb{Z}_p \). Here, one may go further and give precise conditions on \( r \) and \( a \) under which (4.21) admits a continuous solution on \( \mathbb{Z}_p \).

The recurrence in (4.21) dictates that any continuous solution \( y : \mathbb{Z}_p \to \mathbb{C}_p \) must be either non-vanishing or identically zero. Hence, if \( a = 0 \), then \( y \equiv 0 \) is the unique solution to (4.21) independently of \( r \), and, if \( a \neq 0 \), then a necessary condition on \( r \) for the existence of a continuous solution is that \( r \) be non-vanishing.

If \( a \neq 0 \) then, as remarked above, \( y \) must be non-vanishing and we may write

\[
r(x) = \frac{y(x + 1)}{y(x)}. \tag{4.23}
\]

So a further necessary condition is that \( r(x) \) be continuous.

If equation (4.21) admits a non-vanishing continuous solution, then the equation

\[
\begin{align*}
y(x + 1) &= r(x)y(x) \\
y(0) &= b
\end{align*} \tag{4.24}
\]

admits a non-vanishing continuous solution for any \( b \) in \( \mathbb{C}_p^\times \). Indeed, if \( \alpha \) is a solution to (4.21), then \((b/a)\alpha\) is a solution to (4.24).

The set of non-vanishing continuous functions \( r : \mathbb{Z}_p \to \mathbb{C}_p^\times \) for which (4.21) admits a continuous solution for some, and hence, all nonzero choices of \( a \), is a group under multiplication. Indeed, if \( \alpha \) is a non-vanishing solution to (4.21) and \( \beta \) is a
non-vanishing solution to

\[ y(x + 1) = s(x)y(x) \]
\[ y(0) = b, \tag{4.25} \]

then \(1/\alpha\) is a solution to

\[ y(x + 1) = y(x)/s(x) \]
\[ y(0) = 1/b, \tag{4.26} \]

and \(\alpha\beta\) is a solution to

\[ y(x + 1) = r(x)s(x)y(x) \]
\[ y(0) = ab. \tag{4.27} \]

If (4.21) admits a non-vanishing continuous solution then, as remarked earlier, \(r\) is a non-vanishing continuous function and has a decomposition

\[ r = r_\pi r_\omega r_\nu \tag{4.28} \]

where each factor is continuous and non-vanishing. By Theorem 1, the equation

\[ y(x + 1) = r_\nu(x)y(x) \]
\[ y(0) = a \tag{4.29} \]

has a continuous solution and hence, by the previous paragraph, the equation

\[ y(x + 1) = r_\pi(x)r_\omega(x)y(x) \]
\[ y(0) = a \tag{4.30} \]

must also have a continuous solution. Conversely, if \(r\) is any non-vanishing continuous function for which equation (4.30) admits a non-vanishing continuous solution, then equation (4.21) must also admit a non-vanishing continuous solution.

By Lemma 4.1.2, \(r_\pi r_\omega\) is a periodic function with period \(p^n\) for some natural number \(n\). If

\[ L = \prod_{j=0}^{p^n-1} r_\pi(j)r_\omega(j), \tag{4.31} \]
then equation (4.30) implies that

$$y(mp^n) = aL^n$$  \hspace{1cm} (4.32)

for all natural numbers $m$, and continuity of $y$ implies that

$$a = y(0) = a \cdot \lim_{r \to \infty} L^r.$$  \hspace{1cm} (4.33)

If $a$ is nonzero, then, as $L$ is of the form $\eta p^r$, where $\eta$ is a root of unity of order prime to $p$, and $r$ is rational (with $p^r$ as defined in section 2.1), it must be the case that $L = 1$. Furthermore, as $r_\omega(j)$ always has norm one, we may conclude that

$$\prod_{j=0}^{p^n-1} r_\omega(j) = 1.$$  \hspace{1cm} (4.34)

Conversely, it is clear that these conditions on $r_\omega$ and $r_\omega$ guarantee not only that equation (4.30) admits a continuous solution $y : \mathbb{Z}_p \to C_p^*$, but that $y$ itself is periodic with the same period as $r_\omega r_\omega$.

In summary, we have the following.

**Theorem 4** If $a \in C_p$, and $r : \mathbb{Z}_p \to C_p$ is arbitrary, then the finite difference equation

$$\Delta y(x) = (r(x) - 1)y(x)$$

$$y(0) = a$$  \hspace{1cm} (4.35)

has a continuous solution $y : \mathbb{Z}_p \to C_p$ if and only if either $a = 0$, in which case $y \equiv 0$, or $a \neq 0$, $r$ is continuous and non-vanishing, $r_\omega$ and $r_\omega$ are periodic with period $p^n$ for some natural number $n$, and the product over any period of $r_\omega$ and of $r_\omega$ is one.

In the sequel, when I wish to assert that a non-vanishing continuous function $r : \mathbb{Z}_p \to C_p$ satisfies the conditions of the above lemma, I will simply say that $r$ has
a nonzero continuous indefinite product. The assorted conclusions that can be drawn from that statement will be used without further comment.

**Example 4.1.3** It was noted earlier that Morita's gamma function $\Gamma_p(x)$ satisfies equation 4.35 with $a = 1$, and

$$r(x) = \begin{cases} -x & \text{if } |x| = 1 \\ -1 & \text{if } |x| < 1 \end{cases}. \quad (4.36)$$

As an easy check, observe that, for this particular choice of $r(x)$, $r_\omega \equiv 1$, $r_\omega$ is periodic with period $p$, and, over each period, $r_\omega$ runs through the $p-1$'st roots of unity hitting each exactly once, except for $-1$, which is taken on exactly twice. As the product of the $p-1$'st roots of unity is $-1$, it follows from the Lemma that $\Gamma_p(x)$ is a continuous function on $\mathbb{Z}_p$. Of course, all I have really done here is reprove Wilson's Theorem, nevertheless, Theorem 4 does shed some more light onto why Morita's definition works. □

### 4.1.3 The First Order Non-homogeneous Linear Equation

In this section I will examine the finite difference equation

$$\Delta y(x) = r(x)y(x) + s(x)$$

$$y(0) = a, \quad (4.37)$$

where $r$ and $s$ are arbitrary functions from $\mathbb{Z}_p$ into $\mathbb{C}_p$, and $a$ is any element of $\mathbb{C}_p$. In particular, conditions for continuity of $y$, analogous to those of the previous section, are sought.
Example 4.1.4 Suppose that \( h : \mathbb{Z}_p \to \mathbb{C}_p \) is continuous and consider the finite difference equation

\[
y(x + 1) = r(x)y(x) + [h(x + 1) - r(x)h(x)]
y(0) = h(0)
\]  

(4.38)

It is obvious that this equation has been contrived specifically to admit the solution \( y = h \). Considering that my goal is to produce necessary and sufficient conditions under which equation 4.37 has a continuous solution, and considering that the existence of a continuous solution to equation 4.38 imposes no condition whatsoever on \( r \), it would seem that the results of the previous section have no bearing on the present situation. This is misleading. Recall that, as regards equation 4.35, there was a unique initial condition for which a continuous solution could be found irrespective of the choice of \( r(x) \). Namely, \( y(0) = 0 \). It turns out that equation 4.38 displays similar behavior. Indeed, the expression

\[
y(x) = (a - h(0)) \prod_{j=0}^{x-1} r(j) + h(x)
\]  

(4.39)

is a formal solution to the finite difference equation

\[
y(x + 1) = r(x)y(x) + [h(x + 1) - r(x)h(x)]
y(0) = a.
\]  

(4.40)

Hence, this equation has a continuous solution \( y : \mathbb{Z}_p \to \mathbb{C}_p \) if and only if either \( a = h(0) \), or \( r \) is continuous and non-vanishing and has a nonzero continuous indefinite product.
More generally, the following holds.

**Lemma 4.1.4** If the finite difference equation

\[
\begin{align*}
y(x + 1) &= r(x) y(x) + s(x) \\
y(0) &= a
\end{align*}
\]

has continuous solutions for at least two distinct choices of initial condition, then \( r \) and \( s \) are continuous and \( r \) has a nonzero continuous indefinite product. Conversely, if these latter conditions hold for \( r \) and \( s \), then there are continuous solutions for all choices of initial condition.

**Proof** If \( r : \mathbb{Z}_p \rightarrow \mathbb{C}_p \) and \( s : \mathbb{Z}_p \rightarrow \mathbb{C}_p \) are arbitrary functions, \( a_1 \) and \( a_2 \) are distinct elements of \( \mathbb{C}_p \), and \( y_1 : \mathbb{Z}_p \rightarrow \mathbb{C}_p \) and \( y_2 : \mathbb{Z}_p \rightarrow \mathbb{C}_p \) are continuous functions which satisfy

\[
\begin{align*}
y_j(x + 1) &= r(x) y_j(x) + s(x) \\
y_j(0) &= a_j
\end{align*}
\]

for \( j = 1, 2 \) respectively, then \( y_1 - y_2 \) satisfies the finite difference equation

\[
\begin{align*}
y(x + 1) &= r(x) y(x) \\
y(0) &= a_1 - a_2 \neq 0.
\end{align*}
\]

That \( r \) is necessarily continuous and non-vanishing and admits a nonzero continuous indefinite product then follows from Lemma 4, and continuity of \( s \) follows from continuity of \( r \) and the recurrence relation in equation 4.42.

For the second statement of the Lemma, suppose that \( r \) and \( s \) have the desired properties and let \( a \in \mathbb{C}_p \) be arbitrary. It then follows that the function \( y : \mathbb{N} \rightarrow \mathbb{C}_p \),
given by the relation

\[ y(n) = \left[ \prod_{j=0}^{n-1} r(j) \right] \left[ a + \sum_{k=0}^{n-1} \left( \frac{s(k)}{r(k) \prod_{j=0}^{k-1} r(j)} \right) \right] \]  

(4.44)

extends to a continuous function on \( \mathbb{Z}_p \). Also, as

\[ y(n) = a \prod_{j=0}^{n-1} r(j) + \sum_{k=0}^{n-1} s(k) \prod_{j=k+1}^{n-1} r(j), \]  

(4.45)

for all natural numbers \( n \), an easy induction argument shows that this function satisfies equation 4.41 for all \( x \) in \( N \) and, hence, for all \( x \) in \( \mathbb{Z}_p \).

Continuing in my investigation of the finite difference equation

\[ y(x + 1) = r(x)y(x) + s(x) \]

\[ y(0) = a, \]  

(4.46)

I now focus on the case where \( r \) does not admit a continuous indefinite product. I shall start with the simplest such example. Namely, that of a constant function \( r(x) = p \).

Let \( p \) be an element of \( \mathbb{C}_p \) and \( s \) be a continuous function from \( \mathbb{Z}_p \) to \( \mathbb{C}_p \) and consider the finite difference equation

\[ y(x + 1) = p y(x) + s(x) \]

\[ y(0) = a, \]  

(4.47)

By Lemma 4.1.4, equation 4.47 has a continuous solution for every choice of \( a \) if and only if the function \( r(x) \equiv p \) has a nonzero continuous indefinite product. By Lemma 4.1.1, this is the case if and only if \( |p - 1| < 1 \). Hence, if \( |p - 1| \geq 1 \), there is at most one continuous function \( y : \mathbb{Z}_p \to \mathbb{C}_p \) satisfying the recurrence

\[ y(x + 1) = p y(x) + s(x). \]  

(4.48)
In fact, there is exactly one such continuous function. A proof of this, together with a formula for the solution, is provided in the following.

**Lemma 4.1.5** If \( s : \mathbb{Z}_p \to \mathbb{C}_p \) is a continuous function with Mahler series

\[
s(x) = \sum_{k=0}^{\infty} s_k \binom{x}{k},
\]

and if \( \rho \) is any element of \( \mathbb{C}_p \setminus \mathbb{U} \), then the function \( y : \mathbb{Z}_p \to \mathbb{C}_p \), given by the formula

\[
y(x) = \sum_{j=0}^{\infty} \left[ \sum_{k=j}^{\infty} -s_k (\rho - 1)^{j-k} \right] \binom{x}{j},
\]

is the unique continuous solution to the recurrence relation

\[
y(x + 1) = \rho y(x) + s(x).
\]

**Proof** By Theorem 4 and Lemma 4.1.4, and by the choice of \( \rho \), it is clear that any continuous solution to 4.51 must be unique. Hence, it need only be shown that 4.50 provides such a solution. Convergence of the sums involved follows from continuity of \( s \), and the verification that this function satisfies the recurrence relation is straightforward. Indeed, if \( y \) is continuous with Mahler series

\[
y(x) = \sum_{j=0}^{\infty} a_j \binom{x}{j},
\]

then

\[
y(x + 1) = \Delta y(x) + y(x) = \sum_{j=0}^{\infty} (a_j + a_{j+1}) \binom{x}{j}.
\]

Hence, equation 4.51 is satisfied if and only if

\[
a_j + a_{j+1} = \rho a_j + s_j
\]
or, equivalently, if and only if
\[ a_{j+1} = (\rho - 1)a_j + s_j \tag{4.55} \]
for all natural numbers \( j \). For the purposes of the Lemma, it suffices to note that, for each \( j \),
\[ \sum_{k=j+1}^{\infty} -s_k(\rho - 1)^{j-k} = (\rho - 1) \sum_{k=j}^{\infty} -s_k(\rho - 1)^{j-k-1} + s_j. \tag{4.56} \]

Although, in the above Lemma, uniqueness followed easily from previous results, it is still of some interest to note that uniqueness might have been obtained directly. Recall that any formal solution
\[ y(x) = \sum_{j=0}^{\infty} a_j \binom{x}{j}, \tag{4.57} \]
to equation 4.51, must satisfy
\[ a_{j+1} = (\rho - 1)a_j + s_j \tag{4.58} \]
for each natural number \( j \). It follows that
\[ a_j = (\rho - 1)^ja_0 + \sum_{k=0}^{j-1}(\rho - 1)^{j-k-1}s_k. \tag{4.59} \]
If \( y \) is required to be continuous, then, as \( |\rho - 1| > 1 \), it must be the case that
\[ \lim_{j \to \infty} \frac{a_j}{(\rho - 1)^j} = 0. \tag{4.60} \]
Combining this with equation 4.59 yields
\[ \lim_{j \to \infty} \left[ a_0 + \sum_{k=0}^{j-1}(\rho - 1)^{-k-1}s_k \right] = 0. \tag{4.61} \]
Hence

\[ a_0 = \sum_{k=0}^{\infty} -s_k (\rho - 1)^{-k-1}, \tag{4.62} \]

and the initial condition is uniquely determined.

For what follows, I am going to want to know that the process by which one starts with a constant \( \rho \) in \( \mathbb{C}_p \setminus \mathbb{U} \) and a continuous function \( s : \mathbb{Z}_p \to \mathbb{C}_p \), and obtains the unique continuous solution to the recurrence relation

\[ y(x + 1) = \rho y(x) + s(x), \tag{4.63} \]

is a uniformly continuous one. More precisely, I need the following.

**Lemma 4.1.6** The map \( \phi \) from \( \mathbb{C}_p \setminus \mathbb{U} \times C(\mathbb{Z}_p \to \mathbb{C}_p) \) to \( C(\mathbb{Z}_p \to \mathbb{C}_p) \) which takes any pair \( (\rho, s) \) to the unique \( y \) which satisfies the relation

\[ y(x + 1) = \rho y(x) + s(x), \tag{4.64} \]

is uniformly continuous on subsets of \( \mathbb{C}_p \setminus \mathbb{U} \times C(\mathbb{Z}_p \to \mathbb{C}_p) \) which are bounded in the second component. Here, both domain and range are viewed as metric spaces under the supremum norm.

**Proof** First, if \( s \) is a fixed continuous function from \( \mathbb{Z}_p \) to \( \mathbb{C}_p \) with Mahler series

\[ s(x) = \sum_{k=0}^{\infty} s_k \binom{x}{k}, \tag{4.65} \]

and if \( \rho_1 \) and \( \rho_2 \) are elements of \( \mathbb{C}_p \) with

\[ |\rho_1 - \rho_2| < \delta, \tag{4.66} \]
and

\[ |\rho_j - 1| \geq 1 \tag{4.67} \]

for each \( j \), then

\[
\left| \frac{1}{\rho_1 - 1} - \frac{1}{\rho_2 - 1} \right| = \frac{|\rho_2 - \rho_1|}{|\rho_1 - 1||\rho_2 - 1|} < \delta. \tag{4.68}
\]

Hence, by equation 4.50 and lemma 4.1.5,

\[
\|\phi(\rho_1, s) - \phi(\rho_2, s)\|_{z_\rho} \leq \sup_{0 \leq j \leq k < \infty} \left| s_k \left( \frac{1}{(\rho_1 - 1)^{k+1-j}} - \frac{1}{(\rho_2 - 1)^{k+1-j}} \right) \right| \\
\leq \|s\|_{z_\rho} \delta. \tag{4.69}
\]

Next, if \( \rho \in C_p \setminus U \) is fixed, and if

\[
s = \sum_{k=0}^{\infty} s_k \binom{x}{k}, \tag{4.70}
\]

and

\[
t = \sum_{k=0}^{\infty} t_k \binom{x}{k}, \tag{4.71}
\]

then

\[
\|\phi(\rho, s) - \phi(\rho, t)\|_{z_\rho} \leq \sup_{0 \leq j \leq k < \infty} |(s_k - t_k)(\rho - 1)^{j-k-1}| \\
\leq \|s - t\|_{z_\rho}. \tag{4.72}
\]

Finally, if \( |\rho_1 - \rho_2| < \delta \) and \( \|s - t\|_{z_\rho} < \delta \), and if \( s \) and \( t \) are bounded in norm by \( M \), then

\[
\|\phi(\rho_1, s) - \phi(\rho_2, t)\|_{z_\rho} \leq \max \left\{ \|\phi(\rho_1, s) - \phi(\rho_2, s)\|_{z_\rho}, \|\phi(\rho_2, s) - \phi(\rho_2, t)\|_{z_\rho} \right\} \tag{4.73}
\]
\[ \leq \max \{ \delta \| s \|_{x_p}, \delta \} \]
\[ \leq \max \{ \delta M, \delta \}. \tag{4.74} \]

I am now ready to replace the constant \( p \) with an arbitrary step function. In the sequel, when I say that a natural number \( n \) is a period of a step function \( r \), I shall mean that \( n = p^c \) for some \( c \geq 0 \) and that \( r(x + p^c) = r(x) \) for all \( x \) in the domain of \( r \). Now it follows from Theorem 4 and Lemma 4.1.4 that if

\[ r(x) = \begin{cases} \rho_0 & : x \in p^r \mathbb{Z}_p \\ \vdots & \\ \rho_{p^r-1} & : x \in (p^r - 1) + p^r \mathbb{Z}_p \end{cases}, \tag{4.75} \]

and if \( s : \mathbb{Z}_p \to \mathbb{C}_p \) is an arbitrary continuous function, then the finite difference equation

\[ y(x + 1) = r(x)y(x) + s(x) \]
\[ y(0) = a \tag{4.76} \]

has a continuous solution for every value of \( a \), if and only if

\[ \prod_{j=0}^{p^r-1} \rho_j \in \mathbb{U}. \tag{4.77} \]

In contrast to this, the following holds.

**Lemma 4.1.7** If \( r(x) \) is given by formula 4.75, and if \( s : \mathbb{Z}_p \to \mathbb{C}_p \) is continuous, then there is a unique continuous function \( y : \mathbb{Z}_p \to \mathbb{C}_p \) satisfying the recurrence relation

\[ y(x + 1) = r(x)y(x) + s(x), \tag{4.78} \]
if and only if
\[
\prod_{j=0}^{p^r-1} \rho_j \in \mathbb{C}_p \setminus \mathbb{U}.
\] (4.79)

**Proof** The 'only if' direction follows from the comments immediately preceding the statement of the Lemma. For the 'if' direction, observe that any function \( y : \mathbb{Z}_p \to \mathbb{C}_p \) which satisfies equation 4.78 is determined by its behavior on \( p^r \mathbb{Z}_p \) as follows.

\[
y(1 + p^r x) = \rho_0 y(p^r x) + s(p^r x)
y(2 + p^r x) = \rho_0 \rho_1 y(p^r x) + \rho_1 s(p^r x) + s(p^r x + 1)
y(3 + p^r x) = \rho_0 \rho_1 \rho_2 y(p^r x) + \rho_1 \rho_2 s(p^r x) + \rho_2 s(p^r x + 1) + s(p^r x + 2)
\]

\[\vdots\]

\[
y(j + p^r x) = \left[ \prod_{k=0}^{j-1} \rho_k \right] y(p^r x) + \sum_{m=0}^{j-1} \left[ \prod_{k=m+1}^{j-1} \rho_k \right] s(p^r x + m)\] (4.80)

\[\vdots\]

\[
y(p^r + p^r x) = \left[ \prod_{k=0}^{p^r-1} \rho_k \right] y(p^r x) + \sum_{m=0}^{p^r-1} \left[ \prod_{k=m+1}^{p^r-1} \rho_k \right] s(p^r x + m)\]

Setting
\[
w(x) = y(xp^r),
\] (4.81)

and
\[
t(x) = \sum_{m=0}^{p^r-1} \left[ \prod_{k=m+1}^{p^r-1} \rho_k \right] s(p^r x + m),
\] (4.82)

the last line of 4.80 may be rewritten as
\[
w(x + 1) = \left[ \prod_{j=0}^{p^r-1} \rho_j \right] w(x) + t(x).
\] (4.83)

The desired result is thus seen to be an easy consequence of Lemma 4.1.5. □

I would now like to pause for a brief overview. Of central interest has been the precise nature of the set \( S \) of continuous solutions to the recurrence relation
\[
y(x + 1) = r(x) y(x) + s(x),
\] (4.84)
when \( r \) and \( s \) are fixed continuous functions from \( \mathbb{Z}_p \) to \( \mathbb{C}_p \). Specifically, I have been attempting to determine whether \( S \) is empty, has a unique element, or has infinitely many elements. Recall that, by Lemma 4.1.4, these are the only possibilities. So far, \( S \) has never been seen to be empty, and the determination of whether \( S \) is finite or infinite has relied solely on whether or not \( r \) admits a continuous indefinite product. Of those functions \( r \) which do not admit a continuous indefinite product, I have only investigated the behavior of step functions. It is to the remainder that I now turn. It is convenient, however, to first introduce some notation.

- \( \mathcal{P} = \{ r \in C(\mathbb{Z}_p \to \mathbb{C}_p) : r \) admits a nonzero continuous indefinite product.\}
- \( \mathcal{P}^c \) is the complement of \( \mathcal{P} \) in \( C(\mathbb{Z}_p \to \mathbb{C}_p) \).
- \( \hat{\mathcal{P}} = \mathcal{P} \cap C(\mathbb{Z}_p \to B_0(1)) \).
- \( \hat{\mathcal{P}}^c = \mathcal{P}^c \cap C(\mathbb{Z}_p \to B_0(1)) \).
- \( \mathcal{D} \) is the set of step functions in \( C(\mathbb{Z}_p \to \mathbb{C}_p) \) for which the product over any period is in \( U \).
- \( \mathcal{D}^c \) is the set of step functions in \( C(\mathbb{Z}_p \to \mathbb{C}_p) \) for which the product over any period is in \( \mathbb{C}_p \setminus U \).
- \( \hat{\mathcal{D}} = \mathcal{D} \cap C(\mathbb{Z}_p \to B_0(1)) \).
- \( \hat{\mathcal{D}}^c = \mathcal{D}^c \cap C(\mathbb{Z}_p \to B_0(1)) \).

Now the set \( S \) of continuous solutions to

\[
y(x + 1) = r(x)y(x) + s(x),
\]

has already been shown to be infinite in the case that \( r \) is an element of \( \mathcal{P} \). For functions \( r \) in the complement \( \mathcal{P}^c \) of \( \mathcal{P} \), the analysis of \( S \) is more complicated. For
this reason, I restricted my attention to the set \( D^c \) of step functions in \( \mathcal{P}^c \) in the hopes that \( D^c \) was indeed a dense subset of \( \mathcal{P}^c \) and that this fact could be used to complete the classification of the set \( S \) in the case that \( r \) was an element of \( \mathcal{P}^c \). Specifically, having already shown that \( S \) contained a unique element whenever \( r \) was an element of \( D^c \), (Lemma 4.1.7), I had hoped that the same held for all functions \( r \) in \( \mathcal{P}^c \) and, furthermore, that one could express the unique solution to equation 4.85 as a uniform limit of the unique solutions \( y_j \) corresponding to a sequence \( r_j \) of step functions in \( D^c \) converging to \( r \).

Unfortunately, this was not to be the case. While it is true that \( D^c \) is dense in \( \mathcal{P}^c \), (Lemma 4.1.17), in the absence of any restriction on the size of \( ||r||_{z_p} \), it may happen that the sequence of solutions \( y_j \) to equation 4.85 corresponding to a sequence \( r_j \) of step functions converging to \( r \), may not be uniformly Cauchy. Indeed, it may happen that, for certain \( r \) in \( \mathcal{P}^c \), the set \( S \) of solutions to equation 4.85 is empty. If, however, I require \( r \) to be an element of \( \hat{D}^c \), so that \( ||r||_{z_p} \leq 1 \), and if I approximate \( r \) by functions from \( \hat{D}^c \), then the entire line of reasoning as outlined above goes through. The following sequence of Lemmas, culminating in Theorem 5, provide the necessary details.

**Lemma 4.1.8** For any step function \( r : \mathbb{Z}_p \rightarrow \mathbb{C}_p \), if the product over some period is in \( U \), then the product over all periods is in \( U \) and, hence, the same an be said of \( \mathbb{C}_p \setminus U \).

**Proof** This follows immediately from the fact that both \( U \) and \( \mathbb{C}_p \setminus U \) are closed under the taking of \( p \)'th powers. \( \blacksquare \)
Lemma 4.1.9 \( \mathcal{D}, \mathcal{D}^e, \hat{\mathcal{D}}, \) and \( \hat{\mathcal{D}}^e \) may also be defined as the set of step functions contained in \( \mathcal{P}, \mathcal{P}^e, \hat{\mathcal{P}}, \) and \( \hat{\mathcal{P}}^e \) respectively.

Proof This has essentially already been established as it is an easy consequence of Lemmas 4.1.4, 4.1.8, and Theorem 4. The fundamental idea here, is that the property of admitting a nonzero continuous indefinite product is equivalent, for step functions, to the requirement that the product over any period be in \( U. \)

The following example and Lemma are intended to demonstrate the desirability of restricting further discussion on functions which admit a nonzero continuous indefinite product to those continuous functions which are bounded by 1.

Example 4.1.5 In this example I shall demonstrate that, even if a continuous function \( r \) from \( \mathbb{Z}_p \) to \( \mathbb{C}_p \) satisfies the condition

\[
\prod_{j=0}^{p^{e-1}} r(j) \in U
\]  

(4.86)

for all sufficiently large natural numbers \( e \), it need not have a nonzero continuous indefinite product. Specifically, the function \( r \), given below, does not admit a nonzero continuous indefinite product, as it has a zero at \(-1\), but does satisfy condition 4.86 above. Note that, in the formula, \( p \neq 2 \) and \( j \) ranges over the positive integers.

\[
r(x) = \begin{cases} 
  p^{-1} & : x \in \mathbb{Z}_p \setminus (-1 + p\mathbb{Z}_p) \\
  p^j(p^{e-2}) & : x \in (-1 + p^j\mathbb{Z}_p) \setminus ((-1 + p^{j+1}\mathbb{Z}_p) \cup ((p^j - 1) + p^{j+1}\mathbb{Z}_p)) \\
  p^j(p^{e-2}+1) & : x \in (p^j - 1) + p^{j+1}\mathbb{Z}_p \\
  0 & : x = -1 
\end{cases}
\]  

(4.87)

Observe that \( r \) is continuous as it has been constructed by first dividing \( \mathbb{Z}_p \setminus \{-1\} \) into disjoint open balls clustering about \( \{-1\} \) and then letting \( r \) remain constant on
each of those balls with the provision that the values of $r$ on balls 'close' to $\{-1\}$ tend to $r(-1)$. Now the verification of condition 4.86 is as follows. If $e \geq 2$, and $n$ ranges over the natural numbers $\{0, 1, \ldots, p^e - 1\}$, then $n$ is in $\mathbb{Z}_p \setminus (-1 + p\mathbb{Z}_p)$ exactly $p^{e-1}(p-1)$ times. If $1 \leq j \leq e - 1$, then $n$ is in $(p^j - 1) + p^{j+1}\mathbb{Z}_p$ exactly $p^{e-j-1}$ times and $n$ is in $(-1 + p^j\mathbb{Z}_p) \setminus ((-1 + p^{j+1}\mathbb{Z}_p) \cup ((p^j - 1) + p^{j+1}\mathbb{Z}_p))$ exactly $p^{e-j} - 2p^{e-j-1}$ times. Also, $n$ is in $(-1 + p^e\mathbb{Z}_p) \setminus ((-1 + p^{e+1}\mathbb{Z}_p) \cup ((p^e - 1) + p^{e+1}\mathbb{Z}_p))$ exactly once. One may verify that this covers all possible cases for $n$. It is comforting to note that

$$p^{e-1}(p-1) + \left[ \sum_{j=1}^{e-1} p^{e-j-1} + (p^{e-j} - 2p^{e-j-1}) \right] + 1 = p^e. \quad (4.88)$$

Combining the above with the formula given for $r$, one finds that

$$\prod_{j=0}^{p^e-1} r(j) = p^M, \quad (4.89)$$

where

$$M = -p^{e-1}(p-1) + \sum_{j=1}^{e-1} j(p-2) + \sum_{j=1}^{e-1} p^{e-j-1}$$

$$+ \sum_{j=1}^{e-1} j(p-2)(p^{e-j} - 2p^{e-j-1})$$

$$+ e(p-2) + 1. \quad (4.90)$$

In fact, it is easily shown that $M = 0$ for all $e \geq 2$, hence,

$$\prod_{j=0}^{p^e-1} r(j) = 1 \in \mathbb{U}, \quad (4.91)$$

which is precisely what was required. $\square$
Lemma 4.1.10 A function \( r \in C(\mathbb{Z}_p \to B_0(1)) \) admits a nonzero continuous indefinite product if and only if

\[
\prod_{j=0}^{p^e-1} r(j) \in \mathbb{U}
\]

for all sufficiently large natural numbers \( e \).

Proof Recall from Theorem 4 that if \( r \) admits a nonzero continuous indefinite product, then \( r_\pi \) and \( r_\omega \) are periodic with period \( p^n \) for some natural number \( n \) and the product of \( r_\pi r_\omega \) over any period is one. Hence, for any \( c \geq n \),

\[
\prod_{j=0}^{p^e-1} r(j) = \prod_{j=0}^{p^e-1} r_\pi(j) \in \mathbb{U}.
\]

Conversely, suppose that

\[
\prod_{j=0}^{p^e-1} r(j) \in \mathbb{U}
\]

for all sufficiently large natural numbers \( e \). This, together with the assumption that \( \|r\|_{\mathbb{Z}_p} \leq 1 \), implies that \( |r(j)| = 1 \) for all natural numbers \( j \). In particular, \( r \) is bounded away from zero and \( r_\pi \) and \( r_\omega \) are well-defined. By Lemma 4.1.2, \( r_\pi \) and \( r_\omega \) are automatically periodic so it merely remains to check that the product of each function over any period is one. Now if \( x \in \mathbb{U} \), then \( \pi(x) = \omega(x) = 1 \). Hence, for all sufficiently large natural numbers \( e \),

\[
\prod_{j=0}^{p^e-1} r_\pi(j) = \pi \left( \prod_{j=0}^{p^e-1} r(j) \right) = 1
\]

and similarly for \( r_\omega \). By Lemma 4.1.8, it follows that the product of \( r_\pi \) and \( r_\omega \) over any period is one. This concludes the proof of the Lemma.
Lemma 4.1.11 If \( r_1 \) and \( r_2 \) are continuous functions from \( \mathbb{Z}_p \) to \( B_0(1) \), and if

\[
\|r_1 - r_2\|_{\mathbb{Z}_p} < 1,
\]

then either, both functions are in \( \hat{\mathcal{P}} \), or both functions are in \( \hat{\mathcal{P}}^c \).

**Proof** If \( r_1 \) and \( r_2 \) satisfy the hypotheses of the lemma, then it follows from part (c) of Lemma 4.1.3 that for any natural number \( c \),

\[
\left| \prod_{j=0}^{p^c-1} r_1(j) - \prod_{j=0}^{p^c-1} r_2(j) \right| < 1.
\]  

(4.97)

The desired result is thus seen to be a consequence of Lemma 4.1.10.

Lemma 4.1.12 \( \hat{\mathcal{P}} \) and \( \hat{\mathcal{P}}^c \) are clopen subsets of \( C(\mathbb{Z}_p \rightarrow B_0(1)) \).

**Proof** This is an immediate consequence of Lemma 4.1.11.

Lemma 4.1.13 \( C(\mathbb{Z}_p \rightarrow B_0(1)) \) is a clopen subset of \( C(\mathbb{Z}_p \rightarrow \mathbb{C}_p) \).

**Proof** If two elements of \( C(\mathbb{Z}_p \rightarrow \mathbb{C}_p) \) are less than 1 unit apart, and if one of them is in \( C(\mathbb{Z}_p \rightarrow B_0(1)) \), then so must the other be.

Lemma 4.1.14 \( \hat{\mathcal{D}} \) is dense in \( \hat{\mathcal{P}} \) and \( \hat{\mathcal{D}}^c \) is dense in \( \hat{\mathcal{P}}^c \).

**Proof** This follows from Lemmas 4.1.12 and 4.1.13 and the fact that every continuous function may be uniformly approximated by step functions.
Lemma 4.1.15 \( \mathcal{P} \) is open in \( C(\mathbb{Z}_p \to \mathbb{C}_p) \).

PROOF Elements of \( \mathcal{P} \) are bounded away from infinity and bounded away from zero.

Suppose, therefore, that \( r \in \mathcal{P} \), and choose \( \delta > 0 \) sufficiently small so that

\[
|r(x)| > \delta
\]

(4.98)

for all \( x \in \mathbb{Z}_p \). I claim that the '\( \delta \)-ball' about \( r \) in \( C(\mathbb{Z}_p \to \mathbb{C}_p) \) is again contained inside \( \mathcal{P} \).

If \( s \) is any continuous function from \( \mathbb{Z}_p \) to \( \mathbb{C}_p \) such that

\[
\|s - r\|_{\mathbb{Z}_p} < \delta,
\]

(4.99)

then it follows from the assumption on \( \delta \) that

\[
|r(x)| = |s(x)|
\]

(4.100)

for all \( x \in \mathbb{Z}_p \). In particular, \( s_{\tau} \) is well-defined and identically equal to \( r_{\tau} \). As \( \mathcal{P} \) is a multiplicative group it only remains to show that \( s/s_{\tau} \) is in \( \mathcal{P} \). But \( s/s_{\tau} \) and \( r/r_{\tau} \) are both in \( C(\mathbb{Z}_p \to B_0(1)) \) and, again by the assumption on \( \delta \),

\[
\|s/s_{\tau} - r/r_{\tau}\|_{\mathbb{Z}_p} < \delta^{-1}\|s - r\|_{\mathbb{Z}_p} < 1.
\]

(4.101)

As \( r/r_{\tau} \) is in \( \hat{\mathcal{P}} \), it follows from Lemma 4.1.11 that \( s/s_{\tau} \) is in \( \hat{\mathcal{P}} \) and, hence, in \( \mathcal{P} \). This concludes the proof of the Lemma.
Lemma 4.1.16 \( \mathcal{P} \) is not closed in \( C(\mathbb{Z}_p \to \mathbb{C}_p) \).

**Proof** This is clear from Example 4.1.5. Indeed, the function \( r \) given in that example is not in \( \mathcal{P} \) but may be uniformly approximated by step functions in \( \mathcal{P} \). Left open however, is the question as to whether the set of functions \( r \) which satisfy condition 4.86 for all sufficiently large natural numbers \( c \), is closed. Hence, it is perhaps desirable to have on hand an example of a sequence of step functions which satisfy condition 4.86 and, hence, as they are step functions, are necessarily in \( \mathcal{P} \), but which converge uniformly to a function which does not satisfy condition 4.86 and is therefore not an element of \( \mathcal{P} \). One such example is given below. Note that \( n \) represents a natural number, \( r_n \) has period \( p^n \), and \( j \) is intended to range over the natural numbers \( \{1, \ldots, n-1\} \).

\[
    r_n(x) = \begin{cases} 
        p^{-1} & : x \in \mathbb{Z}_p \setminus p\mathbb{Z}_p, \\
        p^j & : x \in p^j\mathbb{Z}_p \setminus p^{j+1}\mathbb{Z}_p, \\
        p^{n+p^{n-1}(p-1)-(n-1)} & : x \in p^n\mathbb{Z}_p 
    \end{cases}
\]  

(4.102)

Lemma 4.1.17 \( \mathcal{D}^c \) is dense in \( \mathcal{P}^c \), but \( \mathcal{D} \) is not dense in \( \mathcal{P} \).

**Proof** The first statement follows from Lemma 4.1.15 and the second follows from Lemma 4.1.16. In the latter case, slightly more may be said, namely, that the uniform closure of \( \mathcal{D} \) does contain, but is not contained in, \( \mathcal{P} \).
The following Lemma provides one of the key steps in the proof of Theorem 5. It is essentially the statement that, if two step functions in \( \hat{D}^c \) are 'close', then the corresponding unique solutions to equation 4.85 are 'close' as well.

**Lemma 4.1.18** If \( \rho \) and \( \tau \) are elements of \( \hat{D}^c \) which satisfy

\[
\|\rho - \tau\|_{\mathbb{Z}_p} < \delta, \tag{4.103}
\]

and if \( s : \mathbb{Z}_p \to B_0(1) \) is continuous with Mahler series

\[
s(x) = \sum_{j=0}^{\infty} s_j \binom{x}{j}, \tag{4.104}
\]

and if \( y_1 \) and \( y_2 \) are the unique elements of \( C(\mathbb{Z}_p \to \mathbb{C}_p) \) which satisfy

\[
y_1(x + 1) = \rho(x)y_1(x) + s(x) \tag{4.105}
\]

and

\[
y_2(x + 1) = \tau(x)y_1(x) + s(x) \tag{4.106}
\]

respectively, then

\[
\|y_1 - y_2\|_{\mathbb{Z}_p} < \delta. \tag{4.107}
\]

**Proof** If

\[
\rho(x) = \rho_j \text{ when } x \in j + p^r \mathbb{Z}_p, \tag{4.108}
\]

and

\[
\tau(x) = \tau_j \text{ when } x \in j + p^r \mathbb{Z}_p, \tag{4.109}
\]
it follows from equation 4.80 that

\[ y_1(j + p^r x) = \left[ \prod_{k=0}^{j-1} \rho_k \right] y_1(p^r x) + \sum_{m=0}^{j-1} \left[ \prod_{k=m+1}^{j-1} \rho_k \right] s(p^r x + m) \]  

(4.110)

and, likewise,

\[ y_2(j + p^r x) = \left[ \prod_{k=0}^{j-1} \tau_k \right] y_2(p^r x) + \sum_{m=0}^{j-1} \left[ \prod_{k=m+1}^{j-1} \tau_k \right] s(p^r x + m) \]  

(4.111)

Hence,

\[ |y_1(j + p^r x) - y_2(j + p^r x)| \leq \max \left\{ \left[ \prod_{k=0}^{j-1} \rho_k \right] |y_1(p^r x)|, \left[ \prod_{k=0}^{j-1} \tau_k \right] |y_2(p^r x)| \right\} \]

(4.112)

Repeated application of Lemma 4.1.3, together with the boundedness assumptions on \( \rho, \tau, \) and \( s, \) yield

\[ |y_1(j + p^r x) - y_2(j + p^r x)| \leq \max\{\delta, |y_1(p^r x) - y_2(p^r x)|\}. \]  

(4.113)

If \( \tilde{y}_1 = y_1(p^r x) \) and \( \tilde{y}_2 = y_2(p^r x), \) then

\[ \tilde{y}_1(x + 1) = \left[ \prod_{k=0}^{p^r-1} \rho_k \right] \tilde{y}_1(x) + \sum_{m=0}^{p^r-1} \left[ \prod_{k=m+1}^{p^r-1} \rho_k \right] s(p^r x + m) \]  

(4.114)

and

\[ \tilde{y}_2(x + 1) = \left[ \prod_{k=0}^{p^r-1} \tau_k \right] \tilde{y}_2(x) + \sum_{m=0}^{p^r-1} \left[ \prod_{k=m+1}^{p^r-1} \tau_k \right] s(p^r x + m) \]  

(4.115)

Finally, by the proof of Lemma 4.1.6, specifically equation 4.74,

\[ \| \tilde{y}_1 - \tilde{y}_2 \|_{x^p} \leq \delta. \]  

(4.116)

The desired result follows.
Theorem 5 If \( r \) and \( s \) are continuous, then the set \( S \) of continuous solutions to

\[
y(x + 1) = r(x)y(x) + s(x);
\]

is infinite if and only if \( r \in \mathcal{P} \), has a unique element if \( r \in \mathcal{D} \), also has a unique element if \( r \in \hat{\mathcal{D}} \), and may be empty if \( r \in \mathcal{P}^c \).

Proof The first claim is simply a restatement of Lemma 4.1.4 and the second is Lemma 4.1.7. Now if \( r \) is in \( \hat{\mathcal{D}} \), then, by Lemma 4.1.14, there is a sequence \( (r_n) \) of step functions in \( \hat{\mathcal{D}} \) which converge uniformly to \( r \). Lemma 4.1.7 guarantees the existence of unique continuous solutions \( y_n \) to each of the recurrence relations

\[
y_n(x + 1) = r_n(x)y_n(x) + s(x),
\]

and, if \( \|s\|_{\mathcal{L}_p} \leq 1 \), it follows from Lemma 4.1.18 that the sequence of these solutions must converge, say, to a function \( y : \mathbb{Z}_p \to \mathbb{C}_p \). By taking limits on both sides of equation 4.118, one may easily see that \( y \) has the required properties. If \( \|s\|_{\mathcal{L}_p} > 1 \), I may choose an \( \alpha \) in \( \mathbb{C}_p \) with \( \alpha \geq \|s\|_{\mathcal{L}_p} \), and then apply the above procedure to produce the unique continuous function \( \tilde{y} \) which satisfies the equation

\[
\tilde{y}(x + 1) = r(x)\tilde{y}(x) + \alpha^{-1}s(x).
\]

The function \( y = \alpha \tilde{y} \) is then the unique element of \( S \), and the third claim is established.

The only remaining task is to produce continuous functions \( r \) and \( s \), such that no continuous function \( y \) is a solution to the finite difference equation

\[
y(x + 1) = r(x)y(x) + s(x).
\]
Necessarily, \( r \) will not have a nonzero continuous indefinite product, will not be a step function, and will not by bounded by 1, and \( s \) will not be identically zero. Consider, therefore, the recurrence relation

\[
y(x + 1) = \alpha xy(x) + \beta,
\]

where \( \alpha \) and \( \beta \) are arbitrary elements of \( C_p \). Now if

\[
y(x) = \sum_{j=0}^{\infty} a_j \binom{x}{j}
\]

is a continuous solution to equation 4.121, then

\[
y(x + 1) = \sum_{j=0}^{\infty} (a_{j+1} + a_j) \binom{x}{j},
\]

and

\[
\alpha xy(x) + \beta = \beta + \sum_{j=1}^{\infty} \alpha j(a_j + a_{j-1}) \binom{x}{j}.
\]

It follows that

\[
a_1 + a_0 = \beta
\]

\[
a_{j+1} + a_j = \alpha j(a_{j-1} + a_j) \text{ for } j \geq 1,
\]

and, hence, that

\[
a_0 = a_0
\]

\[
a_1 = \beta - a_0
\]

\[
a_{j+1} = (\alpha j - 1)a_j + \alpha ja_{j-1} \text{ for } j \geq 1.
\]

An easy induction argument shows that

\[
a_n = (-1)^n \left[ a_0 - \beta \sum_{j=0}^{n-1} (-1)^j j! \alpha^j \right].
\]

Now, \( y \) is continuous if and only if \( a_n \) tends to zero as \( n \) tends to infinity. If \( \beta \) is nonzero, this is possible precisely when \( j! \alpha^j \) tends to zero as \( j \) tends to infinity, or, alternatively, when

\[
|\alpha| < p^{\frac{1}{n^2}}.
\]
In particular, equation 4.121 has no continuous solution when $\alpha = 1/p$ and $\beta = 1$. This concludes the proof of the theorem.

4.1.4 First Order Linear Systems

This section will focus on the finite difference equation

$$y(x + 1) = R(x)y(x) + s(x),$$

(4.129)

where $R$ is a function from $\mathbb{Z}_p$ into the $d \times d$ matrices over $\mathbb{C}_p$ and $s$ is a function from $\mathbb{Z}_p$ to $\mathbb{C}_p^d$. I will observe the convention of writing elements of $\mathbb{C}_p^d$ as column vectors throughout. When $R$ and $s$ are fixed, I will use $S$ to denote the set of continuous functions $y : \mathbb{Z}_p \to \mathbb{C}_p^d$ which satisfy equation 4.129, and $S_{\text{hom}}$ to denote the set of continuous solutions to the corresponding homogeneous equation

$$y(x + 1) = R(x)y(x).$$

(4.130)

**Lemma 4.1.19** $S_{\text{hom}}$ is a finite dimensional subspace of $C(\mathbb{Z}_p \to \mathbb{C}_p^d)$, and $S$ is either empty or an additive coset of $S_{\text{hom}}$ in $C(\mathbb{Z}_p \to \mathbb{C}_p^d)$. In particular, if $S$ is nonempty, then its dimension depends only on $R$ and not on $s$.

**Proof** $S_{\text{hom}}$ contains the identically zero function, hence, is nonempty. Also, the map $y \mapsto y(0)$, from $S_{\text{hom}}$ to $\mathbb{C}_p^d$, is an injective linear transformation, so $S_{\text{hom}}$ is a
finite dimensional subspace of $C(Z_p \to \mathbb{C}_p^d)$. Finally, if $y_0$ is any element of $\mathcal{S}$, then

$$\mathcal{S} = \mathcal{S}_{\text{hom}} + y_0. \quad (4.131)$$

In light of this lemma it makes sense to talk about the set of functions $R$ from $Z_p$ into the $d \times d$ matrices over $\mathbb{C}_p$ for which $\mathcal{S}_{\text{hom}}$ has dimension $j$. I will use $\mathcal{dP}_j$ to denote this set and $\mathcal{dP}_j'$ to denote the set $\mathcal{dP}_j \cap C(Z_p \to \text{Mat}_{d \times d}(B_0(1)))$. When $d$ is clear from context I will also denote these sets by $\mathcal{P}_j$ and $\mathcal{P}_j'$ respectively. If a function $R$ is an element of $\mathcal{dP}_d$ for some natural number $d$, then I will say that $R$ admits a nonsingular continuous indefinite product. Note that, when $d = 1$, this coincides with the earlier definition. When it exists, the indefinite product of a function $x \mapsto R(x)$ will be written

$$x \mapsto \prod_{j=0}^{x-1} R(j), \quad (4.132)$$

where the product is to be thought of as a right-to-left product. Specifically, if $x$ is a natural number,

$$\prod_{j=0}^{x-1} R(j) = R(x-1)R(x-2)\cdots R(0). \quad (4.133)$$

Having introduced this notation, it would be nice give a partial characterization of the sets $\mathcal{dP}_j$ and $\mathcal{dP}_j'$. Having accomplished this, it would then hopefully be possible to determine conditions on $R$ and $s$ which force $\mathcal{S}$ to be nonempty. The point of this is that, if $R$ is an element of $\mathcal{dP}_j$ and if $s$ is chosen so that $\mathcal{S}$ is nonempty, then, by the foregoing discussion, there is a $j$-dimensional affine subspace of $\mathbb{C}_p^d$ of initial conditions which yield continuous solutions to the finite difference equation 4.129. Observe that
significant work towards this goal has already been done for the case \( d = 1 \). Indeed \( \mathcal{P}_0, \mathcal{P}_1, \hat{\mathcal{P}}_0, \) and \( \hat{\mathcal{P}}_1 \) are simply new names for \( \mathcal{P}^c, \mathcal{P}, \hat{\mathcal{P}}^c, \) and \( \hat{\mathcal{P}} \) respectively. Also, \( \mathcal{S} \) is guaranteed to be nonempty, independently of \( s \), provided that \( R \) is in any of the sets \( \mathcal{P}_1, \hat{\mathcal{P}}_0, \) or \( \hat{\mathcal{P}}_1 \). Of course, when \( d > 1 \), the functions \( R \) no longer have a nice decomposition analogous to that given in section 2.1, due to the fact that matrix multiplication is not commutative. If that is the only obstacle to extending the results of the previous section to higher dimensional systems of equations, then perhaps there is hope. In any case, a detailed study of systems of linear finite difference equations lies beyond the scope of this paper. I will, however, discuss the case where \( R \) is a constant matrix so as to provide some hint as to what propositions might hold in a more general setting.

Before proceeding directly to the main theorem of this section, a few results on the nature of the set \( \mathcal{P}_d \), of so-called 'functions which admit a nonsingular continuous indefinite product.', are in order. In particular, I will show that the terminology is justified in spite of the rather abstract way in which the definition of \( \mathcal{P}_d \) was formulated.

**Lemma 4.1.20** If \( R \in \mathcal{P}_d \) then \( R(x) \) is nonsingular for all \( x \) in \( \mathbb{Z}_p \).

**Proof** Assume to the contrary that \( R(x_0) \) is singular. Now the map \( y \mapsto y(x_0) \) from \( \text{Shom} \) to \( C_\mathcal{P}^d \) is always an injective linear transformation. But the assumption that \( R \) is in \( \mathcal{P}_d \), by definition, implies that \( \text{Shom} \) has dimension \( d \), hence, this map is surjective as well. In particular, there is a function \( y \) in \( \text{Shom} \) such that \( y(x_0) \) is a nonzero
vector in the kernel of the linear transformation \( R(x_0) \). It follows that \( y(x_0 + n) \) is the zero vector for each natural number \( n \), and, hence, that \( y \) is identically zero. This contradicts the fact that \( y(x_0) \) is nonzero. ■

**Lemma 4.1.21** If \( R \in \mathcal{P}_d \) then the function

\[
k \mapsto \prod_{j=0}^{k-1} R(j),
\]

from \( N \) to \( \text{Mat}_{d \times d}(\mathbb{C}_p) \), may be extended to a nonsingular continuous function on \( \mathbb{Z}_p \). Conversely, any function \( R \), from \( N \) to \( \text{Mat}_{d \times d}(\mathbb{C}_p) \), which has a nonsingular continuous indefinite product in the literal sense, may be extended to a continuous function on \( \mathbb{Z}_p \) which is, in turn, an element of \( \mathcal{P}_d \).

**Proof** If \( R \in \mathcal{P}_d \) and if \( y_0 \) is an element of \( \mathbb{C}_p \), then, by the definition of \( \mathcal{P}_d \), the function

\[
k \mapsto \left[ \prod_{j=0}^{k-1} R(j) \right] y_0,
\]

from \( N \) to \( \mathbb{C}_p \), may be extended to a non-vanishing continuous function on \( \mathbb{Z}_p \). The continuous extension to \( \mathbb{Z}_p \) of the function

\[
k \mapsto \prod_{j=0}^{k-1} R(j),
\]

from \( N \) to \( \text{Mat}_{d \times d}(\mathbb{C}_p) \), may then be obtained by letting \( y_0 \) range over the standard base for \( \mathbb{C}_p \). To simplify notation, I will call this extension \( \Pi R \). To see that \( \Pi R \) is nonsingular, define a new function \( y \) from \( \mathbb{Z}_p \) to \( \mathbb{C}_p \) by setting

\[
y(x) = \det[(\Pi R)(x)].
\]
Now $y$ is clearly continuous and $y(0) = 1$ so $y$ is not identically zero. Also,

$$y(x + 1) = \det[R(x)]y(x) \quad (4.138)$$

for all $x$ in $\mathbb{Z}_p$, so $y$ must be non-vanishing for otherwise it would vanish on a dense subset of $\mathbb{Z}_p$. It follows that $(\Pi R)(x)$ is nonsingular for each $x$ in $\mathbb{Z}_p$.

For the converse, suppose that $R$ is any function from $\mathbb{N}$ into the $d$ by $d$ matrices over $\mathbb{C}_p$. Also, suppose that the function

$$k \mapsto \prod_{j=0}^{k-1} R(j), \quad (4.139)$$

from $\mathbb{N}$ to $\text{Mat}_{d \times d}(\mathbb{C}_p)$ extends to a nonsingular continuous function on $\mathbb{Z}_p$. As before, $\Pi R$ will be used to denote the extension to $\mathbb{Z}_p$. Now, keeping in mind that the product in 4.139 is a right-to-left product, one may write

$$R(k) = [(\Pi R)(k + 1)] [((\Pi R)(k))^{-1} \quad (4.140)$$

for all natural numbers $k$, hence $R$ itself may be extended to a continuous function on $\mathbb{Z}_p$. Finally, if $y_0$ is any vector in $\mathbb{C}_p^d$, then the function

$$y : \mathbb{Z}_p \to \mathbb{C}_p^d \quad x \mapsto [(\Pi R)(x)] y_0 \quad (4.141)$$

is continuous and satisfies the finite difference equation

$$y(x + 1) = R(x)y(x) \quad y(0) = y_0. \quad (4.142)$$

Hence $R$ belongs to $d\mathcal{P}_d$ as required.

This concludes the justification of the terminology. The next Lemma, and the Theorem which follows are the promised characterization of the constant matrices in $d\mathcal{P}_d$. 
Lemma 4.1.22 If $R$ is an $n \times n$ matrix with entries in $\mathbb{C}_p$ and $w$ in $\mathbb{C}_p^n$ is arbitrary, then the finite difference equation
\[
\Delta y = Ry
\]
\[
y(0) = w
\] (4.143)
has a continuous solution
\[
y : \mathbb{Z}_p \to \mathbb{C}_p^n
\] (4.144)
if and only if $w$ is a linear combination of generalized eigenvectors $v$ of $R$ whose corresponding eigenvalues $\lambda_v$ each satisfy
\[
|\lambda_v| < 1.
\] (4.145)

Proof If $A = (a_{ij})$ is a Jordan normal form for $R$, $M$ a nonsingular matrix such that
\[
A = M^{-1}RM,
\] (4.146)
and $y = Mz$, equation (4.143) may be written as
\[
\Delta z = Az
\]
\[
z(0) = M^{-1}w = v.
\] (4.147)
Observe that for each $j$ in $\{1, \ldots, n\}$ the $j^{th}$ column of $M$ is a generalized eigenvector of $R$ with corresponding eigenvalue $a_{jj}$. Hence, as $w = Mv$, the initial value $w$ will be a linear combination of generalized eigenvectors with sufficiently small eigenvalues if and only if, the $j^{th}$ component of $v$ is zero for all $j$ such that
\[
|a_{jj}| \geq 1.
\] (4.148)
To show that this latter condition is equivalent to the existence of a continuous solution to equation (4.147), it suffices to consider only the case where the matrix $A$ is a single Jordan block. Therefore, consider the finite difference equation

$$\Delta z = \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ & & \ddots & \vdots \\ 0 & & & 1 \\ \end{bmatrix} z$$

$$z(0) = v.$$  \hspace{1cm} (4.149)

The theorem will have been established if it can be shown that this simpler equation admits a continuous solution on $\mathbb{Z}_p$ if and only if, either $v = 0$ or $\lambda < 1$.

An easy induction argument shows that any solution $z(s)$ to equation (4.149) must satisfy

$$z(m) = \begin{bmatrix} (\lambda + 1)^m \binom{m}{1} (\lambda + 1)^{m-1} & \cdots & (\lambda + 1)^{m-n+1} \\ (\lambda + 1)^m & \cdots & (\lambda + 1)^{m-n+2} \\ 0 & \ddots & (\lambda + 1)^m \\ \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

$$\hspace{1cm} (4.150)$$

for all natural numbers $m$. In the special case that $\lambda = -1$ I am adopting the convention

$$\binom{m}{j}(\lambda + 1)^{m-j} = \begin{cases} 1 & \text{if } m = j \\ 0 & \text{otherwise} \end{cases}.$$

$$\hspace{1cm} (4.151)$$

Now by Lemma 4.1.1, equation (4.150) can be extended to a continuous solution on $\mathbb{Z}_p$ for an arbitrary choice of initial value $v$ provided $|\lambda| < 1$. However, if $|\lambda| \geq 1$, then (4.150) determines a continuous function on $\mathbb{Z}_p$ if and only if $v$ is the zero vector.

Indeed, if $v = 0$ then $z$ vanishes on the natural numbers and thus extends to the identically zero function on $\mathbb{Z}_p$. Also, if $\lambda = -1$ then $z(0) = v$ and $z(m) = 0$ for all sufficiently large natural numbers $m$, hence, in this case, $z$ extends to a continuous
function on $\mathbb{Z}_p$ only when $v = 0$. Finally, if $|\lambda| \geq 1$ and $\lambda \neq -1$, then

$$z_1(m) = (\lambda + 1)^m \sum_{j=0}^{n-1} v_j (\lambda + 1)^{-j} \binom{m}{j}$$

which, by Lemma (4.1.1), can only be extended to a continuous function on $\mathbb{Z}_p$ provided the polynomial

$$f(m) = \sum_{j=0}^{n-1} v_j (\lambda + 1)^{-j} \binom{m}{j}$$

is identically zero. By the uniqueness of the Mahler expansion of a continuous function, this polynomial is identically zero if and only if each of the $v_j$'s are zero. This concludes the proof of the Lemma.

\textbf{Theorem 6} A constant $d \times d$ matrix $R$ is in $\mathcal{P}_j$ if and only if the direct sum of the generalized eigenspaces of $R$, whose corresponding eigenvalues are in $U$, has dimension $j$. In particular, $R$ admits a nonsingular continuous indefinite product if and only if all of its eigenvalues are in $U$.

\textbf{Proof} This is simply a useful restatement of the Lemma. It should be noted that the $R$ of the Lemma and the $R$ of this Theorem differ by the identity matrix.

I shall conclude this section with a brief discussion about what Theorem 6 says about the general higher order linear equation with constant coefficients. In van der Put [V], Theorem 3.1, it is shown, among other things, that if $K$ is a subfield of $\mathbb{C}_p$ with discrete valuation, and if $w_0, \ldots, w_{s-1}$ are constants in $K$, then the dimension of the space of solutions $y : \mathbb{Z}_p \rightarrow K$ to the finite difference equation

$$\Delta^s y + w_{s-1} \Delta^{s-1} y + \cdots + w_0 y = 0$$

\textbf{(4.154)}
is equal to the number of zeros in \( B_0(1) \), counting multiplicity, of the polynomial

\[
f(x) = x^s + w_{s-1}x^{s-1} + \cdots + w_0. \tag{4.155}
\]

His proof relies heavily on the algebraic properties of \( K \) and does not apply to the case where \( K = \mathbb{C}_p \). The result, however, does hold when \( K = \mathbb{C}_p \) and is, in fact, an easy Corollary of Theorem 6. To see this, consider the two finite difference equations

\[
\Delta^s y + w_{s-1} \Delta^{s-1} y + \cdots + w_0 y = 0 \tag{4.156}
\]

and

\[
\Delta y = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 1 \\ \vdots \\ -w_0 & -w_1 & \cdots & -w_{s-1} \end{bmatrix} y \tag{4.157}
\]

where, in the second equation,

\[
y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{s-1} \end{bmatrix}. \tag{4.158}
\]

It is easy to see that any solution \( y : \mathbb{Z}_p \rightarrow \mathbb{C}_p \) to equation 4.156 yields a solution \( y : \mathbb{Z}_p \rightarrow \mathbb{C}_p \) to equation 4.157, namely

\[
y = \begin{bmatrix} y \\ \Delta y \\ \vdots \\ \Delta^{s-1} y \end{bmatrix}, \tag{4.159}
\]

and, conversely, that any solution \( y : \mathbb{Z}_p \rightarrow \mathbb{C}_p \) to equation 4.157 yields a solution \( y : \mathbb{Z}_p \rightarrow \mathbb{C}_p \) to equation 4.156, namely

\[
y = y_0. \tag{4.160}
\]
By Theorem 6, the space of solutions to equation 4.157 is equal to the number of
eigenvalues, counting multiplicity, of the matrix

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & \\
& & \\
-w_0 & -w_1 & \cdots & -w_{s-1}
\end{bmatrix}
\]  

(4.161)

which have norm less than one. This, in turn, is well known to be equal to the number
of zeros, counting multiplicity, of the polynomial

\[
f(x) = x^s + w_{s-1}x^{s-1} + \cdots + w_0,
\]

(4.162)

which lie in the 'open' unit ball \(B_0^2(1)\). As the solution spaces of the two equations
have the same dimension, this provides the desired result.

4.2 Analyticity Results For Linear Equations

In this section I present an extension of Theorem 2 to cover the question of whether
solutions to certain linear finite difference equations are analytic. Recall that Theorem
2 provided conditions which were sufficient to guarantee the existence of a continuous
solution to the general higher order linear equation, but failed to address either the
question of whether these conditions were necessary, or the question of whether the
solutions were analytic. In section 4.1 I spent considerable time examining the former
question, and in this section I will examine the latter. I will not attempt to provide
conditions which are necessary to guarantee the existence of analytic solutions to
linear finite difference equations.
The proof of Theorem 2 relied heavily on the result of Theorem 1. It was pointed out in the discussion following the proof of Theorem 1, that the proof did not extend directly to cover the question of analyticity, but it was observed that much of the proof went through if certain functions were required to be analytic on balls strictly larger than \( B_0(1) \). In Lemma 2.7.5, this latter condition was shown to be closely tied to the degree of exponential decay of the Mahler coefficients of the functions in question. The following quantitative Lemma provides a more precise statement of these ideas.

**Lemma 4.2.1** If \( r_j : \mathbb{Z}_p \to B \) and \( s : \mathbb{Z}_p \to B \) have Mahler expansions

\[
r_j(x) = \sum_{n=0}^{\infty} r_{jn} \binom{x}{n} \tag{4.163}
\]

and

\[
s(x) = \sum_{n=0}^{\infty} s_n \binom{x}{n} \tag{4.164}
\]

which, for some \( 0 < \epsilon < 1 \), satisfy

\[
|s_n| \leq \epsilon^n \tag{4.165}
\]

and

\[
|r_{jn}| \leq \epsilon^{n+d-j} \tag{4.166}
\]

for all \( n \geq 0 \) and for all \( j \) such that \( 0 \leq j \leq d-1 \), then the unique continuous solution

\[
y(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \tag{4.167}
\]
to the finite difference equation
\[
\Delta^d y + r_{d-1} \Delta^{d-1} y + \ldots + r_0 y = s
\]
\[
\Delta^j y(0) = a_j \in B_0(1) \quad (j \in \{0, 1, \ldots, d - 1\}),
\]
must satisfy
\[
|a_{d+k}| \leq \epsilon^k
\]
for all \( k \geq 0 \).

**Proof** First, it is easy to see that equation 4.168 does satisfy the hypotheses of Theorem 2 and, hence, has a unique continuous solution
\[
y : x \mapsto \sum_{n=0}^{\infty} a_n \binom{x}{n}.
\]
By Lemmas 2.2.3 and 2.2.7, it follows that
\[
\Delta^j y(x) r_j(x) = \left( \sum_{n=0}^{\infty} a_{n+j} \binom{x}{n} \right) \left( \sum_{n=0}^{\infty} r_{j,n} \binom{x}{n} \right)
\]
\[
= \sum_{k=0}^{\infty} \left[ \sum_{0 \leq m, n \leq m+k} a_{m+j} r_{j,n} \binom{k}{m} \binom{m}{k-n} \right] \binom{x}{k},
\]
and, by applying equation 4.168, that
\[
a_{d+k} = s_k - \sum_{0 \leq m, n \leq m+k} a_{m+j} r_{j,n} \binom{k}{m} \binom{m}{k-n}
\]
for all \( k \geq 0 \). Hence,
\[
|a_{d+k}| \leq \max_{0 \leq m, n \leq m+k} \{ \epsilon^k, \epsilon^{d-j} |a_{m+j}| \}
\]
for all \( k \geq 0 \). The desired result will follow from this last inequality, together with induction on \( k \geq 0 \). If \( k = 0 \), then, by 4.173,
\[
|a_{d+0}| \leq \max_{0 \leq j \leq d-1} \{ 1, \epsilon^{d-j} |a_{j}| \} = \epsilon^0.
\]
If \(|a_{\nu+d}| \leq \epsilon^\nu\) for all \(\nu\) with \(0 \leq \nu \leq k\), then, again by 4.173,

\[
|a_{(k+1)+d}| \leq \max_{0 \leq m, n \leq k+1 \leq m+n} \{\epsilon^{k+1}, \epsilon^{n+d-j}|a_{m+j}|\}, \tag{4.175}
\]

and I am done provided that

\[
\max_{0 \leq m, n \leq k+1 \leq m+n} \{\epsilon^{n+d-j}|a_{m+j}|\} \leq \epsilon^{k+1}. \tag{4.176}
\]

To see this, there are two cases to consider, namely the case where \(m + j \in \{0, \ldots, d-1\}\) and the case where \(m + j \in \{d, \ldots, k + d\}\). In the former case, the fact that \(m\) is 'small' implies that \(n\) is 'large' and, hence, that \(\epsilon^{n+d-j}\) is 'small'. Specifically, if \(0 < m + j < d - 1\), then

\[
n \geq (k + 2) - (d - j) \tag{4.177}
\]

and

\[
\epsilon^{n+d-j}|a_{m+j}| \leq \epsilon^{n+d-j} \leq \epsilon^{k+2}. \tag{4.178}
\]

In the latter case the induction hypothesis provides a bound on \(|a_{m+j}|\) which yields the desired inequality. Specifically, if \(d \leq m + j \leq k + d\), then

\[
|a_{m+j}| \leq \epsilon^{m+j-d} \tag{4.179}
\]

so that

\[
\max_{0 \leq m, n \leq k+1 \leq m+n} \{\epsilon^{n+d-j}|a_{m+j}|\} \leq \epsilon^{k+1} \tag{4.180}
\]

as required.
Theorem 7 If \( s : \mathbb{Z}_p \to B \) is analytic on \( B_0^{-}(r) \) for some \( r > 1 \), and if the functions \( r_j : \mathbb{Z}_p \to B \), with Mahler series
\[
r_j(x) = \sum_{n=0}^{\infty} r_{j,n} \binom{x}{n},
\]
have Mahler coefficients which satisfy the inequality
\[
|r_{j,n}| \leq \left( r^{-1} p^{-\frac{1}{r-1}} \right)^{n+d-j}
\]
for each \( j \) in \( \{0, 1, \ldots, d-1\} \) and for all \( n \geq 0 \), and finally, if \( a_0, \ldots, a_{d-1} \) in \( \mathbb{C}_p \) are arbitrary, then the continuous solution \( y : \mathbb{Z}_p \to \mathbb{C}_p \), to the finite difference equation
\[
\Delta^d y + r_{d-1} \Delta^{d-1} y + \cdots + r_0 y = s \\
\Delta^j y(0) = a_j \quad (j \in \{0, 1, \ldots, d-1\}),
\]
is analytic on \( B_0^{-}(r) \).

PROOF If \( \bar{r} \) is the norm of an element of \( \mathbb{C}_p \) and satisfies
\[
1 < \bar{r} < r,
\]
and if
\[
\epsilon = \bar{r}^{-1} p^{-\frac{1}{r-1}},
\]
then \( \epsilon \) is also the norm of an element of \( \mathbb{C}_p \) and satisfies the inequality
\[
r^{-1} p^{-\frac{1}{r-1}} < \epsilon < p^{-\frac{1}{r-1}}.
\]
In particular, by the second part of Lemma 2.7.5, if \( \epsilon_0 \) is the norm of an element of \( \mathbb{C}_p \) which satisfies the inequality
\[
\epsilon < \epsilon_0 < p^{-\frac{1}{r-1}},
\]
and if

\[ s(x) = \sum_{n=0}^{\infty} s_n \binom{x}{n}, \]

then there is an \( E > 0 \) such that

\[ |s_n| \leq E \epsilon_0^n \]

for all \( n \geq 0 \). Furthermore, \( E \) may be chosen to be the norm of an element of \( \mathbb{C}_p \) by making it larger if necessary.

Now one may choose an element \( \mu \) in \( \mathbb{C}_p^\times \) such that

\[ |\mu| = M = \max\{E, |a_0|, \ldots, |a_{d-1}|\}, \]

and then define

\[ \sigma = \mu^{-1}s, \]

and

\[ \alpha_j = \mu^{-1}a_j \]

for each \( j \) in \( \{0, \ldots, d-1\} \). It follows from Lemma 4.2.1 that the Mahler coefficients \( b_n \), of the unique continuous solution

\[ z : \mathbb{Z}_p \to B \]

\[ x \mapsto \sum_{n=0}^{\infty} b_n \binom{x}{n} \]

to the finite difference equation

\[ \Delta^d z + r_{d-1} \Delta^{d-1} z + \cdots + r_0 z = \sigma \]

\[ \Delta^j z(0) = \alpha_j \quad (j \in \{0, 1, \ldots, d-1\}) \]

satisfy the inequality

\[ |b_{d+k}| \leq \epsilon_0^k \]
for all \( k \geq 0 \). In particular, if

\[
E_0 = \max \left\{ \epsilon_0^{-d}, \max_{0 \leq j \leq d-1} |b_j| \epsilon_0^{-j} \right\}, 
\]

then

\[
|b_n| \leq E_0 \epsilon_0^n
\]

for all \( n \geq 0 \). Indeed, if

\[
O Q
\]

then

\[
|\sigma_n| = |\mu^{-1}s_n| \leq E^{-1}E \epsilon_0^n = \epsilon_0^n,
\]

\[
|r_{j,n}| \leq \epsilon_0^{n+d-j} < \epsilon_0^{n+d-j}
\]

for all \( n \geq 0 \) and all \( j \) in \( \{0, \ldots, d-1\} \), and

\[
|\alpha_j| = |\mu^{-1}|a_j| \leq 1,
\]

so that each \( \alpha_j \) is in \( \mathcal{B}_0(1) \).

Now, if

\[
y(x) = \mu z(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n},
\]

then \( y \) is a solution to the finite difference equation

\[
\Delta^d y + r_{d-1} \Delta^{d-1} y + \cdots + r_0 y = s
\]

\[
\Delta^j y(0) = a_j \quad (j \in \{0, 1, \ldots, d-1\}),
\]

and

\[
|a_n| = |\mu b_n| \leq M E_0 \epsilon_0^n
\]
for all \( n \geq 0 \). Hence, by the first part of Lemma 2.7.5, \( y \) is analytic on \( B_0^- \left( \epsilon_0^{-1} p^{-\frac{1}{p-1}} \right) \).

As this holds for all \( \epsilon_0 \) which are norms of elements of \( C_p \) and which satisfy the inequality

\[
\epsilon < \epsilon_0 < p^{-\frac{1}{p-1}}, \quad (4.205)
\]

it follows that \( y \) is analytic on \( B_0^- (\bar{r}) \). Finally, as \( \bar{r} \) may be chosen arbitrarily close to \( r \), \( y \) is analytic on \( B_0^- (r) \). ■

I shall conclude this section with an example to demonstrate that Theorem 7 is not the best possible result of its type.

**Example 4.2.1** Let \( p = 3 \) and consider the linear finite difference equation

\[
\begin{align*}
\Delta y(x) &= 3xy(x) \\
y(0) &= 1.
\end{align*} \quad (4.206)
\]

In the notation of Theorem 7, \( d = 1 \) and \( r_0(x) = 3x \) so that \( r_{0,1} = 3 \) and \( r_{0,n} = 0 \) if \( n \neq 1 \). In particular,

\[
|r_{0,1}| = \left( 3^{-\frac{1}{1+1}} \right)^2, \quad (4.207)
\]

and inequality 4.182 is not satisfied for any \( r > 1 \). Nonetheless, I will show that the unique continuous solution

\[
y(x) = \sum_{n=0}^{\infty} a_n \left( \begin{array}{c} x \\ n \end{array} \right), \quad (4.208)
\]

to equation 4.206, which is guaranteed to exist by Theorem 2, is analytic on \( B_0^- (3^{1/6}) \).

As a side note, \( B_0^- (3^{1/6}) \) is in fact the maximal domain of analyticity for \( y \). Specifically, I have shown that

\[
|a_n| = 3^{-2n/3} \quad (4.209)
\]
for all \( n \equiv 0 \pmod{9} \). Hence,

\[
\left| \frac{a_n}{n!} \right| 3^{n/6} = 3^{-1/2} \tag{4.210}
\]

provided \( n \) is an integral power of 3 with \( n \geq 9 \). It then follows from Lemma 2.7.3 that \( y \) is not analytic on \( B_0(3^{1/6}) \). The details are somewhat messy and will therefore be postponed until the end of this section.

Continuing with the example, it follows from Lemma 2.2.7 that

\[
3xy(s) = \sum_{n=1}^{\infty} 3n(a_{n-1} + a_n) \binom{x}{n} \tag{4.211}
\]

and, hence, that

\[
\begin{align*}
    a_0 &= 1 \\
    a_1 &= 0 \\
    a_{n+1} &= 3n(a_{n-1} + a_n)
\end{align*} \tag{4.212}
\]

for all \( n \geq 1 \). I now claim that

\[
\text{ord}(a_n) \geq \left\lfloor \frac{2n + 1}{3} \right\rfloor \tag{4.213}
\]

for all \( n \geq 0 \). This is immediate for \( n = 0, 1 \), and if \( k \geq 1 \) and (4.213) holds for all \( 0 \leq n \leq k \), then

\[
\begin{align*}
\text{ord}(a_{k+1}) &\geq 1 + \min(\text{ord}(a_{k-1}), \text{ord}(a_k)) + \text{ord}(k) \\
&\geq \left\lfloor \frac{2k + 2}{3} \right\rfloor + \text{ord}(k) \\
&\geq \left\lfloor \frac{2k + 3}{3} \right\rfloor.
\end{align*} \tag{4.214}
\]

Hence, (4.213) holds for all \( n \geq 0 \). Combining this with the formula for the \( p \)-order of \( n! \) given in Lemma 2.3.2, I find that

\[
\text{ord} \left( \frac{a_n}{n!} \right) \geq \left\lfloor \frac{2n + 1}{3} \right\rfloor - \left( \frac{n - s_n}{2} \right)
\]
\[ \geq \frac{2n - 1}{3} - \frac{n - 1}{2} = \frac{n + 1}{6}. \quad (4.215) \]

for all \( n \geq 1 \). Finally, if \( r < 3^{1/6} \), then

\[ \lim_{n \to \infty} \left| \frac{a_n}{n!} \right| r^n = \lim_{n \to \infty} 3^{-1/6} \left( \frac{r}{3^{1/6}} \right)^n = 0 \quad (4.216) \]

and hence, by Lemma 2.7.3, \( y \) is analytic on \( B_0(3^{1/6}) \). \( \square \)

As promised, I shall conclude this section with a proof that equation 4.209 holds for all \( n \equiv 0 \) (mod 9). By the previous example, this would imply that the unique continuous solution \( y : \mathbb{Z}_p \to \mathbb{C}_p \) to the finite difference equation

\[ \Delta y(x) = 3xy(x) \]
\[ y(0) = 1. \quad (4.217) \]

is not analytic on \( B_0(3^{1/6}) \). The starting point is, of course, the recurrence relation satisfied by the \( a_n \)'s, namely

\[ a_0 = 1 \]
\[ a_1 = 0 \]
\[ a_{n+1} = 3n(a_{n-1} + a_n). \quad (4.218) \]

I will use induction on \( k \geq 0 \) to simultaneously establish the following 9 relations.

\[ \text{ord}(a_{3k}) = 6k \]
\[ \text{ord}(a_{3k+1}) \geq 6k + 3 \]
\[ \text{ord}(a_{3k+2}) = 6k + 1 \]
\[ \text{ord}(a_{3k+3}) = 6k + 2 \]
\[ \text{ord}(a_{3k+4}) = 6k + 3 \]
\[ \text{ord}(a_{3k+5}) = 6k + 3 \]
\[ \text{ord}(a_{3k+6}) \geq 6k + 5 \]
\[ \text{ord}(a_{3k+7}) = 6k + 5 \]
\[ \text{ord}(a_{3k+8}) \geq 6k + 6 \quad (4.219) \]
In particular, this will establish that

\[ |a_n| = 3^{-2n/3} \tag{4.220} \]

for all \( n \equiv 0 \pmod{9} \) as desired. The first 9 values of \( a_n \) are given below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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</tr>
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<td>1</td>
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<td>766422</td>
</tr>
<tr>
<td>8</td>
<td>16936857</td>
</tr>
</tbody>
</table>

\( \begin{align*}
\text{ord}(a_n) &= 1 + 6(n - 1) + 5 = 6n, \\
a_{2n+1} &= 3(9n + 1)(a_n + a_{n+1}), \\
\text{ord}(a_{2n+1}) &\geq 6n + 3, \\
a_{2n+2} &= 3(9n + 1)(a_n + a_{n+1}), \\
\text{ord}(a_{2n+2}) &= 6n + 1, \tag{4.222}
\end{align*} \)

and, by the last relation,

\[ a_{2n+2} = 3^{6n+1}(b_n + \text{'higher order terms'}) \tag{4.223} \]
where \( b_n \in \{1, 2\} \) and 'higher order terms' are simply arbitrary elements of \( 3 \mathbb{Z}_3 \).

Continuing in this fashion, one may then obtain the relations:

\[
\begin{align*}
    a_{9n+3} &= 3(9n + 2)(a_{9n+1} + a_{9n+2}) \\
    &= 3(9n + 2)(3^{6n+1}(b_n + \text{h.o.t.})) \\
    &= 3^{6n+2}(2b_n + \text{h.o.t.}), \\
    a_{9n+4} &= 3(9n + 3)(a_{9n+2} + a_{9n+3}) \\
    &= 3(9n + 3)(3^{6n+1}(b_n + \text{h.o.t.})) \\
    &= 3^{6n+3}(b_n + \text{h.o.t.}), \\
    a_{9n+5} &= 3(9n + 4)(a_{9n+3} + a_{9n+4}) \\
    &= 3(9n + 4)(3^{6n+2}(2b_n + \text{h.o.t.})) \\
    &= 3^{6n+3}(2b_n + \text{h.o.t.}), \quad (4.224)
\end{align*}
\]

and

\[
\begin{align*}
    a_{9n+6} &= 3(9n + 5)(a_{9n+4} + a_{9n+5}) \\
    &= 3(9n + 5)(3^{6n+3}(3b_n + \text{h.o.t.})) \quad (4.225)
\end{align*}
\]

so that

\[
\begin{align*}
    \text{ord}(a_{9n+3}) &= 6n + 2, \\
    \text{ord}(a_{9n+4}) &= 6n + 3, \\
    \text{ord}(a_{9n+5}) &= 6n + 3, \\
    \text{and } \text{ord}(a_{9n+6}) &\geq 6n + 5. \quad (4.226)
\end{align*}
\]
Finally, it follows that

\[ a_{9n+7} = 3(9n + 6)(a_{9n+5} + a_{9n+6}), \]
\[ \text{ord}(a_{9n+7}) = 6n + 5, \]
\[ a_{9n+8} = 3(9n + 7)(a_{9n+6} + a_{9n+7}), \] (4.227)

and, hence, that

\[ \text{ord}(a_{9n+8}) \geq 6n + 6. \] (4.228)
CHAPTER V

Nonlinear Equations

5.1 General Comments

Theorem 1 provides a powerful tool for determining whether or not nonlinear equations have continuous solutions. However, the problem of determining the domains of analyticity of these solutions has not been addressed except to point out that the proof of Theorem 1 does not yield useful information regarding the analyticity of the continuous solution to $\Delta y = 3y^2 + 1$. The purpose of this chapter is therefore to partially address this situation. I shall begin with Theorem 8 as it represents my only attempt to provide a statement on the analyticity of solutions to nonlinear finite difference equations in general. The remainder of the chapter will then focus on equations of the form $\Delta y = qy^r + 1$. In particular, I shall prove that the continuous solution $y : \mathbb{C}_p \rightarrow \mathbb{C}_p$ to the finite difference equation

$$\Delta y = 3y^2 + 1$$
$$y(0) = 0,$$

is not analytic on $B_0(1)$. 

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5.2 An Existence Theorem

In this section \( \| \| \) will be used to denote the supremum of a function over the 'closed' unit ball \( B_0(1) \). Prior to the main theorem, some remarks on the composition of a power series with a Mahler series are in order.

Lemma 5.2.1 If \( \eta(m_1, \ldots, m_k : m) \) denotes the coefficient of \( \binom{x}{m} \) in the Mahler expansion for
\[
x \mapsto \binom{x}{m_1} \cdots \binom{x}{m_k},
\]
then \( \eta(m_1, \ldots, m_k : m) = 0 \) whenever \( m \) is either strictly less than any of the \( m_j \)'s or strictly greater than the sum of the \( m_j \)'s.

Proof This was shown to be the case for \( k = 2 \) in Lemma 2.2.6. At this point, I could proceed by induction on \( k \). It is more desireable, however, to argue as follows. By the definition of \( \eta(m_1, \ldots, m_k : m) \),
\[
\binom{n}{m_1} \cdots \binom{n}{m_k} = \sum_{m=0}^{n} \eta(m_1, \ldots, m_k : m) \binom{n}{m} \tag{5.3}
\]
for all natural numbers \( n \). If \( X \) is a fixed set of cardinality \( n \), then the left hand side of this equation is precisely the number of ordered \( k \) -tuples \( (W_1, \ldots, W_k) \) of subsets of \( X \) of cardinalities \( m_1, \ldots, m_k \) respectively. If \( \mathcal{O} \) denotes this collection of ordered \( k \) -tuples of subsets of \( X \), then
\[
\binom{n}{m_1} \cdots \binom{n}{m_k} = \sum_{m=0}^{n} \sum_{W \in \mathcal{P}(X)} \sum_{\bigcup_{j=1}^{k} W_j = W} 1 \tag{5.4}
\]
where \(|W|\) and \(\mathcal{P}(X)\) denote the cardinality of \(W\) and the power set of \(X\) respectively. Now the innermost sum is simply the number of ways to cover a set \(W\) of cardinality \(m\) with subsets \(W_1, \ldots, W_k\) of cardinalities \(m_1, \ldots, m_k\) respectively. In particular, it depends only on the cardinality of \(W\) and not on the set itself. I will therefore denote the value of the innermost sum by \(\tilde{\eta}(m_1, \ldots, m_k : m)\). It follows that

\[
\binom{n}{m_1} \cdots \binom{n}{m_k} = \sum_{m=0}^{\infty} \binom{n}{m} \tilde{\eta}(m_1, \ldots, m_k : m)
\]

and, hence, by Lemma 2.2.4,

\[
\eta(m_1, \ldots, m_k : m) = \tilde{\eta}(m_1, \ldots, m_k : m). \tag{5.6}
\]

I have shown that \(\eta(m_1, \ldots, m_k : m)\) is precisely the number of ways to cover a set \(W\) of cardinality \(m\) with an ordered collection of subsets \(W_1, \ldots, W_k\) of cardinalities \(m_1, \ldots, m_k\) respectively. If \(m\) is strictly less than any of the \(m_j\)'s or is strictly greater than the sum of the \(m_j\)'s, then this number is clearly zero.

**Lemma 5.2.2** If \(y : \mathbb{Z}_p \to B_0(1)\) is continuous with Mahler series

\[
y(x) = \sum_{m=0}^{\infty} a_m \binom{x}{m}, \tag{5.7}
\]

and if \(g : B_0(1) \to \mathbb{C}_p\) is analytic with power series expansion about \(a_0\),

\[
g(x) = \sum_{k=0}^{\infty} b_k (x - a_0)^k, \tag{5.8}
\]
then
\[ g(y(x)) = b_0 + \sum_{m=1}^{\infty} \left[ \sum_{k=1}^{\infty} \sum_{(m_1, \ldots, m_k) \mid m \leq m_j \leq \sum m_j} b_k a_{m_1} \cdots a_{m_k} \eta(m_1, \ldots, m_k : m) \right] \begin{pmatrix} x \\ m \end{pmatrix} \] (5.9)

for all \( x \) in \( \mathbb{Z}_p \).

**Proof** If \( g \equiv 0 \), there is nothing to show, so assume \( g \) is nonzero. In that case,

\[
g(y(x)) = \sum_{k=0}^{\infty} b_k \left[ \sum_{m=1}^{\infty} a_m \left( \begin{array}{c} x \\ m \end{array} \right) \right]^k
= b_0 + \sum_{k=1}^{\infty} \sum_{m_1=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} b_k a_{m_1} \cdots a_{m_k} \left( \begin{array}{c} x \\ m_1 \\ \vdots \\ m_k \end{array} \right)
= b_0 + \sum_{k=1}^{\infty} \sum_{(m_1, \ldots, m_k) \mid m = \text{max}(m_j)}^{\infty} b_k a_{m_1} \cdots a_{m_k} \eta(m_1, \ldots, m_k : m) \left( \begin{array}{c} x \\ m \end{array} \right),
\] (5.10)

provided that, for any \( \epsilon > 0 \),

\[
|b_k a_{m_1} \cdots a_{m_k}| < \epsilon
\] (5.11)

for all but finitely many choices of \( k, m \), and \( k \)-tuples \( (m_1, \ldots, m_k) \) such that

\[
1 \leq m_j \leq m \leq \sum_{i=1}^{k} m_i
\] (5.12)

for all \( 1 \leq j \leq k \). This may be verified as follows. Set

\[
R = \|g\| = \max_k |b_k|,
\] (5.13)

choose \( K \) so that

\[
|b_k| < \epsilon
\] (5.14)
for all \( k > K \), and choose \( M \) so that

\[
R|a_m| < \epsilon 
\]

(5.15)

for all \( m > M \). I claim that condition 5.11 is satisfied if either \( k > K \) or \( m > KM \).

As there are at most finitely many \( k \)-tuples \((m_1, \ldots, m_k)\) satisfying conditions 5.12 for any given pair \((k, m)\), it will follow that there are at most finitely many legal choices of \( k, m \) and \((m_1, \ldots, m_k)\) for which condition 5.11 fails. If \( k > K \), then condition 5.11 follows immediately from the fact that no \( a_{m_j} \) has norm greater than 1. If \( m > KM \), then, as

\[
\sum_{j=1}^{k} m_j \geq m, 
\]

there is at least one \( m_j \) with \( m_j > M \) and I am done.

\[\square\]

**Theorem 8** If \( a_0 \in B_0(1) \), and \( g : B_0(1) \rightarrow C_p \) is analytic on \( B_0(1) \) with

\[
\|g\| = \epsilon < p^{-\frac{1}{p+1}},
\]

(5.17)

then the unique continuous solution \( y : Z_p \rightarrow C_p \), to the finite difference equation

\[
\Delta y(x) = g(y(x))
\]

\[
y(0) = a_0,
\]

(5.18)

is analytic on \( B_0^- \left( \epsilon^{-1} p^{-\frac{1}{p+1}} \right) \).

**Proof** The method of proof will be to plug the Mahler series for \( y \) into the Taylor series for \( g \) about \( a_0 \), and then to use induction to show that the Mahler coefficients of
$y$ exhibit sufficiently strong exponential decay. Suppose, therefore, that $g : B_0(1) \to \mathbb{C}_p$ is analytic on $B_0(1)$ with power series expansion about $a_0$,

$$g(x) = \sum_{k=0}^{\infty} b_k (x - a_0)^k. \tag{5.19}$$

If

$$\|g\| = \epsilon < p^{-\frac{1}{p-1}}, \tag{5.20}$$

then

$$|b_k| \leq \epsilon \tag{5.21}$$

for all $k \geq 0$, and

$$|g(x) - g(y)| \leq \sup_k |b_k| |x - y| \left| \frac{(x - a_0)^k - (y - a_0)^k}{(x - a_0) - (y - a_0)} \right| \leq \epsilon |x - y| \tag{5.22}$$

for all $x$ and $y$ in $B_0(1)$. In particular, $g$ satisfies a sufficiently strong Lipschitz condition on $B_0(1)$ and hence, by Theorem 1 (with $R = 1$), there is a unique continuous function $y : \mathbb{Z}_p \to \mathbb{C}_p$ which satisfies the finite difference equation

$$\Delta y(x) = g(y(x))$$

$$y(0) = a_0. \tag{5.23}$$

If $y$ has Mahler series

$$y(x) = \sum_{m=0}^{\infty} a_m \left( \begin{array}{c} x \\ m \end{array} \right), \tag{5.24}$$

then

$$\Delta y(x) = \sum_{m=0}^{\infty} a_{m+1} \left( \begin{array}{c} x \\ m+1 \end{array} \right) \tag{5.25}$$
and, by Lemma 5.2.2,

\[ a_1 = b_0 \]

\[ a_{m+1} = \sum_{k=1}^{\infty} \sum_{m_1 + \ldots + m_k = m} b_k a_{m_{1}} \ldots a_{m_k} \eta(m_1, \ldots, m_k : m) \]  

(5.26)

for each \( m \geq 1 \). By Lemma 2.7.5, it suffices to show that

\[ |a_m| \leq \epsilon^m \]  

(5.27)

for all \( m \geq 0 \). By assumption,

\[ |a_0| \leq 1. \]  

(5.28)

Also,

\[ |a_1| = |b_0| \leq \epsilon. \]  

(5.29)

Finally, if

\[ |a_j| \leq \epsilon^j \]  

(5.30)

for \( 1 \leq j \leq m \), then, as \( \eta(m_1, \ldots, m_k : m) \) is \( p \)-integral,

\[ |a_{m+1}| \leq \sup |b_k a_{m_1} \ldots a_{m_k}| \]

\[ \leq \epsilon^{1+\Sigma m_j} \]

\[ \leq \epsilon^{1+m} \]  

(5.31)

as desired. \( \blacksquare \)
5.3 A Special Case

In this section I will examine the finite difference equation

\[
\Delta y = qy^2 + 1
\]

\[
y(0) = 0.
\]  \hspace{1cm} (5.32)

By Theorem 1, this equation has a unique continuous solution \( y : \mathbb{Z}_p \to \mathbb{C}_p \) if \(|q| < 1\).

I will show that, by further restricting the norm of \( q \), this solution may be shown to be analytic on some \( B_0^*(r) \) with \( r > 1 \). Note that the presence of the ' +1' in equation 5.32 prohibits the application of Theorem 8. Before stating the main result, I need the following simple Lemma.

**Lemma 5.3.1** If \( m_1, m_2, \) and \( m \) are integers with

\[
m \leq m_1 + m_2,
\]  \hspace{1cm} (5.33)

then

\[
\left\lfloor \frac{m_1}{2} \right\rfloor + \left\lfloor \frac{m_2}{2} \right\rfloor \geq \left\lfloor \frac{m-1}{2} \right\rfloor.
\]  \hspace{1cm} (5.34)

**Proof** If not, then

\[
\frac{m_1 + m_2}{2} - 1 = \frac{m_1 - 1}{2} + \frac{m_2 - 1}{2}
\]

\[
\leq \left\lfloor \frac{m_1}{2} \right\rfloor + \left\lfloor \frac{m_2}{2} \right\rfloor
\]

\[
\leq \left\lfloor \frac{m-1}{2} \right\rfloor - 1
\]

\[
\leq \frac{m-3}{2}
\]

\[
\leq \frac{m_1 + m_2}{2} - \frac{3}{2},
\]  \hspace{1cm} (5.35)

which is absurd.

\[\blacksquare\]
Theorem 9  If

\[ |q| = \epsilon < p^{-\frac{n}{n-1}}, \]  

(5.36)

then the unique continuous solution \( y : \mathbb{Z}_p \to \mathbb{C}_p \) to the finite difference equation

\[ \Delta y = qy^2 + 1 \]
\[ y(0) = 0. \]  

(5.37)

is analytic on \( B_0^\infty (\epsilon^{-1/2} p^{-1/(n-1)}) \). In particular, \( y \) is analytic on \( B_0 (1) \).

Proof  By Lemma 5.2.2, if

\[ y(x) = \sum_{m=0}^{\infty} a_m \binom{x}{m}, \]  

(5.38)

then \( a_0 = 0, a_1 = 1 \), and

\[ a_{m+1} = \sum_{1 \leq n_1, n_2 \leq n_1 + m_2} q a_{n_1} a_{n_2} \sigma(n_1, n_2 : m) \]  

(5.39)

for all \( m \geq 1 \). I will use induction on \( m \) to show that

\[ |a_m| \leq \epsilon^{\frac{m}{2}} \]  

(5.40)

for all \( m \geq 0 \). If \( m = 0, 1 \), the inequality holds, so assume that \( m \geq 1 \) and that

\[ |a_k| \leq \epsilon^{\frac{k}{2}} \]  

(5.41)

for \( 0 \leq k \leq m \). By Lemma 5.3.1, it follows that

\[ |a_{m+1}| \leq \sup_{1 \leq n_1, n_2 \leq n_1 + m_2} |q a_{n_1} a_{n_2}| \]
\[ \leq \epsilon^{1 + \left( \frac{m+1}{2} \right) + \left( \frac{m_2}{2} \right)} \]
\[ \leq \epsilon^{1 + \left( \frac{m+1}{2} \right)} \]
\[ = \epsilon^{\frac{m+1}{2}}. \]  

(5.42)
Finally, as
\[ c^{p/2} \leq c^{m-1} = c^{-1/2} (c^{1/2})^m, \quad (5.43) \]
the desired result follows from Lemma 2.7.5. ■

As a final note, I should point out that equations 5.39 through 5.43 are valid for all \( \epsilon \leq 1 \). Condition 5.39 is needed only in the last step where I appeal to Lemma 2.7.5. I shall use these observations in the sequel.

5.4 A Counterexample Revisited

In this section I will show that the result of Theorem 9 is sharp. Specifically, I will prove that the unique continuous solution to the finite difference equation

\[ \Delta y = 3y^2 + 1 \]
\[ y(0) = 0, \quad (5.44) \]

of Example 3.1.1, is not analytic on the closed unit ball \( B_0(1) \) of \( \mathbb{C}_3 \). Note that, using the notation of Theorem 9,

\[ |q| = 3^{-1} = p^{2/3}, \quad (5.45) \]

so this example does indeed fall right on the boundary of the set of equations for which Theorem 9 guarantees a solution analytic on \( B_0(1) \).

The proof I will give is extremely specific to equation 5.44, and, hence, is rather unenlightening. Therefore, some preliminary remarks on the procedure by which I came by this result are in order. Recall, from Lemma 2.7.3, that the unique continuous
solution

\[ x \mapsto \sum_{m=0}^{\infty} a_m \binom{x}{m}, \quad (5.46) \]

to equation 5.44, is analytic on \( B_0(1) \) if, and only if,

\[ \lim_{m \to \infty} ord \left( \frac{a_n}{n!} \right) = \infty. \quad (5.47) \]

With this in mind, I used Mathematica, together with the relations

\[ a_0 = 0 \]
\[ a_1 = 1 \]
\[ a_{m+1} = \sum_{1 \leq m_1 \leq m} 3a_{m_1}a_{m_2} \binom{m}{m_2} \binom{m_2}{m - m_1}, \quad (5.48) \]

from the proof of Theorem 9 and the statement of Lemma 2.2.6, to obtain the following table.

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This, of course, led immediately to the conjecture that

\[ ord \left( \frac{a_n}{n!} \right) = 0 \quad (5.50) \]
whenever \( n \) was a nonnegative integral power of 3, and, hence, that the solution to equation 5.44 was not analytic on \( B_0(1) \). Unfortunately, a direct proof by induction of this conjecture was not practical so further investigation was merited. Setting

\[
a_n = \sum_{j=0}^{\infty} a_{n,j} 3^j,
\]

(5.51)

the canonical \( p \)-adic expansion of \( a_n \), I generated the table below in which the element in the \( n \)’th row and \( j \)’th column is \( a_{n,j} \).

<table>
<thead>
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<th>( j )</th>
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<th>2</th>
<th>3</th>
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</tbody>
</table>

(5.52)

Ultimately, what I was able to show is that the alternating sequence of 1’s and 2’s in the odd rows along the slope-\((-2)\) diagonal, continues indefinitely. From this it follows that, for odd \( n \),

\[
ord(a_n) = \left\lfloor \frac{n}{2} \right\rfloor.
\]

(5.53)

Now, by Lemma 2.3.2, if \( n \) is a positive integral power of 3, then

\[
ord(n!) = \frac{n-1}{3-1} = \left\lfloor \frac{n}{2} \right\rfloor.
\]

(5.54)

Hence, a rigorous demonstration that the aforementioned sequence of 1’s and 2’s continues indefinitely, would constitute proof that the unique continuous solution to
equation 4.2.1 is not analytic on the 'closed' unit ball of $\mathbb{C}_3$. This is precisely the content of the following Lemma.

**Lemma 5.4.1** If the unique continuous solution $y : \mathbb{Z}_p \to \mathbb{C}_p$, to the finite difference equation

$$
\Delta y = 3y^2 + 1 \\
y(0) = 0,
$$

(5.55)

has Mahler series

$$
y(x) = \sum_{m=0}^{\infty} a_m \binom{x}{m},
$$

(5.56)

then the canonical $p$-adic expansion of $a_m$, for odd natural numbers $m$, is given by

$$
a_m = \begin{cases} 
3^{|\mathbb{F}|} + \text{(higher order terms)} & \text{if } m \equiv 1(4) \\
2 \cdot 3^{|\mathbb{F}|} + \text{(higher order terms)} & \text{if } m \equiv 3(4)
\end{cases}
$$

(5.57)

**Proof** I will use induction on $m$. First recall that, by the comments following the proof of Theorem 9,

$$
3^{|\mathbb{F}|} | a_m,
$$

(5.58)

for all $m \geq 0$, and that

$$
a_0 = 0 \\
a_1 = 1 \\
a_{m+1} = 3 \sum_{1 \leq m_1, m_2 \leq m} a_{m_1} a_{m_2} \binom{m}{m_2} \binom{m_2}{m-m_1},
$$

(5.59)

for all $m \geq 1$. To start the induction argument, I need to verify equation 5.57 for both $a_1$ and $a_3$. Here is the computation of $a_3$.

$$
a_2 = 3 \left( a_1 a_1 \binom{1}{1} \binom{1}{0} \right)
= 3.
$$
\[ a_3 = 3 \left( a_1 a_1 \binom{2}{1} \binom{1}{1} + a_1 a_2 \binom{2}{2} \binom{2}{1} + a_2 a_1 \binom{2}{1} \binom{1}{0} + a_2 a_2 \binom{2}{2} \binom{2}{0} \right) \]
\[ = 3(2 + 6 + 6 + 9) \]
\[ = 2 \cdot 3 + 1 \cdot 3^2 + 2 \cdot 3^3. \quad (5.60) \]

Now suppose that \( k \geq 2 \), and that equation 5.57 holds for all odd \( m \) less than \( 2k + 1 \).

From above,
\[ a_{2k+1} = 3 \sum_{1 \leq i, j \leq 2k} a_i a_j \binom{2k}{j} \binom{j}{2k-i} \quad (5.61) \]

but, by (5.58),
\[ 3 \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{j}{2} \right\rfloor a_i a_j \quad (5.62) \]

for all \( i, j \geq 0 \), and, by Lemma 5.3.1,
\[ \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{j}{2} \right\rfloor \geq k \quad (5.63) \]

whenever \( i + j \geq 2k + 1 \). Hence, equation 5.61 may be rewritten as
\[ a_{2k+1} = \left\lceil 3 \sum_{1 \leq i, j \leq 2k} a_i a_j \binom{2k}{j} \binom{j}{2k-i} \right\rceil + 3^{k+1} b, \quad (5.64) \]

for some integer \( b \). Now if \( i + j = 2k \), then \( i \) and \( j \) have the same parity. If they are both even, then
\[ \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{j}{2} \right\rfloor = k, \quad (5.65) \]

so that, again,
\[ 3^k a_i a_j. \quad (5.66) \]

If \( i \) and \( j \) are both odd, then,
\[ \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{j}{2} \right\rfloor = k - 1, \quad (5.67) \]
and, by the induction hypothesis,

$$a_ia_j = \begin{cases} 3^{k-1} + \text{(higher order terms)} & \text{if } i + j \equiv 2(4) \\ 2 \cdot 3^{k-1} + \text{(higher order terms)} & \text{if } i + j \equiv 0(4) \end{cases}$$

Combining these observations with the fact that

$$\binom{j}{2k-i} = 1$$

whenever $i + j = 2k$, I obtain the equation

$$a_{2k+1} = \begin{cases} 3^k \left[ \sum_{1 \leq i \leq 2k-1, \ i \equiv 1(3)} \binom{2k}{j} \right] + \text{(higher order terms)} & \text{if } 2k \equiv 2(4) \\ 2 \cdot 3^k \left[ \sum_{1 \leq i \leq 2k-1, \ i \equiv 1(3)} \binom{2k}{j} \right] + \text{(higher order terms)} & \text{if } 2k \equiv 0(4) \end{cases}$$

Finally,

$$2^{2k} = (1 + 1)^{2k} - (1 - 1)^{2k} = \sum_{j=0}^{2k} \binom{2k}{j} - \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} = 2 \sum_{j=0}^{2k} \binom{2k}{j},$$

so that

$$\sum_{1 \leq i \leq 2k-1} \binom{2k}{j} = 2^{2k-1} \equiv 2(3)$$

for all $k \geq 1$. Hence,

$$a_{2k+1} = \begin{cases} 2 \cdot 3 \left[ \frac{2k+1}{2} \right] + \text{(higher order terms)} & \text{if } 2k + 1 \equiv 3(4) \\ 3 \left[ \frac{2k+1}{2} \right] + \text{(higher order terms)} & \text{if } 2k + 1 \equiv 1(4) \end{cases}$$

as required. \(\blacksquare\)
5.5 A More General Special Case

In this section I extend the result of Theorem 9 to the finite difference equation

\[ \Delta y = q y^r + 1 \]
\[ y(0) = 0. \]  \hspace{1cm} (5.74)

The following Lemma will be needed in the proof of the main result. It is simply a generalization of Lemma 5.3.1.

Lemma 5.5.1 If \( m \) and \( s \) are integers, \( r \) is a natural number, and \( \{m_1, \ldots, m_r\} \) are integers which satisfy the inequality

\[ 1 \leq m_j \leq m \leq \sum_{j=1}^{r} m_j, \]  \hspace{1cm} (5.75)

then

\[ \sum_{j=1}^{r} \left\lfloor \frac{m_j + s}{r} \right\rfloor \geq \left\lfloor \frac{m - (r - 1)^2 + sr}{r} \right\rfloor. \]  \hspace{1cm} (5.76)

Proof If not, then

\[ \left( \frac{1}{r} \sum_{j=1}^{r} m_j \right) + s - r + 1 = \sum_{j=1}^{r} \frac{m_j + s - (r - 1)}{r} \leq \sum_{j=1}^{r} \left\lfloor \frac{m_j + s}{r} \right\rfloor \leq \frac{m - (r - 1)^2 + sr}{r} - 1 \leq \left\lfloor \sum_{j=1}^{r} \frac{m_j - (r - 1)^2 + sr - r}{r} \right\rfloor \leq \left( \frac{1}{r} \sum_{j=1}^{r} m_j \right) + s - r + 1 - \frac{1}{r} \]  \hspace{1cm} (5.77)

which is absurd. \( \blacksquare \)
Theorem 10 If
\[ |q| = \epsilon < p^{-\frac{r-1}{r}}, \quad (5.78) \]
then the unique continuous solution \( y : \mathbb{Z}_p \rightarrow \mathbb{C}_p \) to the finite difference equation
\begin{align*}
\Delta y &= qy^r + 1 \\
y(0) &= 0.
\end{align*}
\quad (5.79)
is analytic on \( B_0^- (\epsilon^{-1/r} p^{-1/(p-1)}) \). In particular, \( y \) is analytic on \( B_0(1) \).

Proof By Lemma 5.2.2, if
\[ y(x) = \sum_{m=0}^{\infty} a_m \binom{x}{m}, \quad (5.80) \]
then \( a_0 = 0, a_1 = 1, \) and
\[ a_{m+1} = \sum_{\{m_1, \ldots, m_r\} \leq m} qa_{m_1} \cdots a_{m_r} \eta(m_1, \ldots, m_r : m) \quad (5.81) \]
for all \( m \geq 1 \). I will use induction on \( m \) to show that
\[ |a_m| \leq \epsilon^{\frac{m+r-2}{r}} \quad (5.82) \]
for all \( m \geq 0 \). If \( m = 0,1 \), the inequality holds, so assume that \( m \geq 1 \) and that
\[ |a_k| \leq \epsilon^{\frac{k+r-2}{r}} \quad (5.83) \]
for \( 0 \leq k \leq m \). By Lemma 5.5.1, it follows that
\[ |a_{m+1}| \leq \sup_{1 \leq m_j \leq m \leq \sum m_j} |qa_{m_1} \cdots a_{m_r}| \leq \epsilon^{\frac{1+\sum \frac{m+r-2}{r}}{r}} \leq \epsilon^{\frac{(m+1)+r-2}{r}}. \quad (5.84) \]
Finally, as

$$\epsilon^{\left[ m + \frac{1}{2} \right]} \leq \epsilon^{\frac{m}{r}} = \epsilon^{-1/r} \left( \epsilon^{1/r} \right)^m,$$

(5.85)

the desired result follows from Lemma 2.7.5.
CHAPTER VI

Conclusion

6.1 Summary

In this thesis I have covered a broad spectrum of topics ranging from general theorems such as Theorem 3, in which I provided conditions sufficient to guarantee the existence of a unique continuous solution to the finite difference equation

\[ \Delta^n y = f(x, y, \Delta y, \ldots, \Delta^{n-1} y) \]
\[ \Delta^j y(0) = c_j \text{ for each } j, \] (6.1)

to specific results such as Theorem 9, in which I discussed the domain of analyticity of the unique continuous solution to the finite difference equation

\[ \Delta y = qy^2 + 1 \]
\[ y(0) = 0. \] (6.2)

Certainly the most complete results, in that very little more may be said, are given in Theorems 4 and 5 in which I discussed first order linear homogeneous and non-homogenous equations respectively. I then briefly discussed higher order linear equations with the primary intent of shedding more light on the result of Theorem 5. Next, I turned to the question of analyticity. In theorems 7 and 8 I gave conditions
sufficient to guarantee solutions analytic in $B_\infty(r)$ for the general higher order linear equation and the general first order nonlinear equation with no $y$-dependence respectively. Finally, as neither theorem 7 nor theorem 8 provided conditions which were necessary for the existence of analytic solutions, I examined several specific examples chosen specifically to demonstrate the difficulty of the question of necessity.

### 6.2 Future Work

The field of $p$-adic finite difference equations is relatively unexplored territory. It has been, therefore, the primary goal of this thesis to lay the necessary groundwork for future research in this area. As stated above, much progress has already been made towards a complete classification of those first order linear equations which have continuous solutions. In that case, all that is required is further investigation into equations of the form

$$y(x + 1) = r(x)y(x) + s(x); \quad (6.3)$$

where $r$ is not bounded by 1, is not a step function, and does not admit a continuous indefinite product. For systems of first order linear equations and the related higher order linear equations, much remains to be done. For constant matrices $R$, the behavior of the finite difference equation

$$\Delta y = Ry$$

$$y(0) = w$$

is fully understood, (lemma 4.1.22), so possible next steps would be an investigation of the behavior of this equation for step functions $R : \mathbf{Z}_p \rightarrow \text{Mat}_{d \times d}(\mathbf{C}_p)$, followed by
some statement, analogous to lemma 4.1.18, to the effect that close \( R \)'s yield close spaces of admissible initial conditions.

Possibilities for further research into the question of analyticity abound. Specifically, more work needs to be done to close the gap between conditions which suffice to guarantee that solutions are analytic and conditions which are necessary for the existence of analytic solutions. Even a compendium of finite difference equations for which the precise domain of analyticity were known would add much to the existing theory. In particular, it would be nice to obtain a complete classification of those pairs \((q,r)\) for which the solution to finite difference equation

\[
\Delta y = qy^r + 1 \\
y(0) = 0
\]  

(6.5)

is analytic on \( B_0(1) \). Towards this goal I have already produced vast quantities of numerical data but, as of yet, have no conjecture as to what form such a classification might take.

One avenue of investigation which I have omitted from this thesis in an attempt to narrow the focus of this work is the study of equations which have continuous solutions on \( \mathbb{Z}_p \) which have domains of analyticity strictly smaller than \( B_0(1) \). For such equations, one may often find solutions which are locally analytic on \( B_0(1) \) and may then talk about the degree of analyticity of said solutions.

In closing, I'd like to return once more to the finite difference equation

\[
\Delta y = 3y^2 + 1 \\
y(0) = 0.
\]  

(6.6)

During the course of my research, I generated more raw data related to this equation than to any other. As a consequence I arrived at the rather striking conjecture that
if the continuous solution $y : \mathbb{Z}_p \to \mathbb{C}_p$ to this equation has Mahler series

$$y(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n},$$

then

$$L(i) = \lim_{k \to \infty} \frac{a_{3^k+i}}{(3^k + i)!}$$

exists for all $i \geq 0$. For concreteness, here is a short table of conjectural values for $L(i)$ modulo $3^5$.

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<th>$L(i)$</th>
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<td>171</td>
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</table>

As of yet, I have neither a closed formula for $L(i)$, nor do I have a conjecture as to the significance of this function as it may or may not shed light on the nature of the original equation. Nonetheless, I have noted what I believe to be similar behavior in the Mahler coefficients for several other equations and, hence, feel that this is an important observation.
References


