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CALCULATION OF THREE DIMENSIONAL MAGNETIC FIELD DISTRIBUTION OF LARGE AIR-CORE REACTOR COILS

DISSERATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

by

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*****

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1996
ABSTRACT

A comprehensive review of previous research on the calculation methods of the magnetic field distribution of, and the current distribution in, air-core coils is presented first. An accurate modeling method, called the Volume Current Method (VCM), for calculating the 3D power-frequency magnetic field strength of air-core reactor coils with multiple layers connected in parallel is developed next. In VCM, a coil is modeled by a number of circular cylindrical conductor segments.

The analytical formulas for evaluating the magnetic field strength of finite circular cylindrical conductor segments with arbitrary oblique end planes carrying a quasi-static current are derived by utilizing a surface source modeling technique. The derived equations are useful for solving problems that involve the accurate calculation of the magnetic field distribution in and around a coil, the magnitude of forces on a coil, the current distribution in a coil, and the inductive properties of a coil. The calculation accuracy of the VCM is verified by the finite element method for a single-loop coil. The application ranges of two filament models are examined by the VCM.

Two methods for determining the current distribution among the coil layers are presented. One is based on the calculation of the magnetic field distribution in the coil, and the other one is based on the calculation of layer impedances. Their advantages and limitations are discussed.

In addition, a simplified modeling method for air-core coils is developed. It
uses three equivalent ideal current loops with their numbers of turns and positions flexible. The three flexible loop model proposed is not only simple but also gives accurate results even as close to the coil as at least \(0 \cdots 1.3)H\) (coil length) axially or \(0.2 \cdots 1.2)D_m\) (coil mean diameter) laterally from the outer surface of the coil, pending on the location of the traverse and the type of the coil. It is very useful for the fast evaluation of the magnetic field distribution of air-core coils.

The accurate and simplified modeling methods developed are essential for the design of air-core reactor coils, investigation of the magnetic field distribution and for the magnetic field management in power substations.
Dedicated to my family
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1. Q. Yu and S.A. Sebo, "Simplified Magnetic Field Modeling and Calculation of
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2. Q. Yu and S.A. Sebo, "Analytical Expressions for the Magnetic Field Strength
of General Circular Cylindrical Segments", Proceeding of the IEEE International


Research Reports


FIELDS OF STUDY

- Overall Field: Electrical Engineering
- Major Field: Electric Power Engineering
- Minor Field: Electromagnetics
- Minor Field: Computer and Information Science
5. SIMPLIFIED MODELING OF LARGE AIR-CORE REACTOR COILS

5.1 Introduction ................................................................. 109
5.2 Three Flexible Loop Model .............................................. 110
5.3 Accuracy of the Three Flexible Loop Model ..................... 115
5.4 An Alternative Implementation of the Circular Loop Model ... 118
5.5 Conclusions ................................................................. 122

6. CONCLUSIONS AND FUTURE WORK 123

APPENDICES

A. DERIVATION OF THE EXPRESSIONS OF $z_1$ AND $z_2$ 128
B. CHANGING THE INTEGRAL LIMITS FROM $\int_{-\varphi}^{2\pi - \varphi}$ TO $\int_{-\pi/2}^{\pi/2}$ 132
C. EXPRESSING HEUMAN'S LAMBDA FUNCTION IN TERMS OF GENERAL COMPLETE ELLIPTICAL INTEGRAL 135
D. AN ADDITION IDENTITY OF $cel$ AND ITS APPLICATION 136
E. MUTUAL INDUCTANCES OF COAXIAL CYLINDRICAL COILS OF INFINITESIMAL THIN THICKNESS 137
F. COEFFICIENT $f_H$ 140

BIBLIOGRAPHY 142
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A typical current-limiting reactor manufactured by Trench Electric [4].</td>
</tr>
<tr>
<td>2</td>
<td>Circular filament loop.</td>
</tr>
<tr>
<td>3</td>
<td>Conical filament helix.</td>
</tr>
<tr>
<td>4</td>
<td>Semi-infinite solenoid.</td>
</tr>
<tr>
<td>5</td>
<td>The geometry and local coordinate system of a finite straight segment of a circular cross-section conductor.</td>
</tr>
<tr>
<td>6</td>
<td>The front view of the finite straight conductor segment.</td>
</tr>
<tr>
<td>7</td>
<td>The vector $\vec{R}$.</td>
</tr>
<tr>
<td>8</td>
<td>Local coordinate systems of the conductor segment.</td>
</tr>
<tr>
<td>9</td>
<td>The helix of the conductor center line of the $j$th layer of the coil and the coil coordinate system.</td>
</tr>
<tr>
<td>10</td>
<td>Three different types of straight conductor segment models.</td>
</tr>
<tr>
<td>11</td>
<td>Geometry of a one-turn loop coil and the calculation traverses. (a) The top view of the coil, (b) The front view of the coil and traverses.</td>
</tr>
<tr>
<td>12</td>
<td>A single-layer coil and its related traverses.</td>
</tr>
<tr>
<td>13</td>
<td>Magnetic flux density of COIL1 along the lateral traverse through the center of the conductor (LCC traverse), (a) $B_{zc}$ vs. $y_c$, (b) $B_{res}$ vs. $y_c$.</td>
</tr>
<tr>
<td>14</td>
<td>Magnetic flux density of COIL1 along the lateral traverse at the side of the conductor (LSC traverse), (a) $B_{yc}$ vs. $y_c$, (b) $B_{zc}$ vs. $y_c$.</td>
</tr>
<tr>
<td>15</td>
<td>Resultant magnetic flux density of COIL1 along the lateral traverse at the side of the conductor (LSC traverse).</td>
</tr>
<tr>
<td>16</td>
<td>Magnetic flux density of COIL1 along the axial traverse through the center of the conductor (ACC traverse), (a) $B_{yc}$ vs. $z_c$, (b) $B_{res}$ vs. $z_c$.</td>
</tr>
<tr>
<td>17</td>
<td>Magnetic flux density of COIL1 along the axial traverse at the side of the conductor (ASC traverse), (a) $B_{yc}$ vs. $z_c$, (b) $B_{zc}$ vs. $z_c$.</td>
</tr>
</tbody>
</table>
Resultant magnetic flux density of COIL1 along the axial traverse at the side of the conductor (ASC traverse).

Resultant magnetic flux density components of COIL2 along the lateral traverse through the center of the conductor (LCC traverse) calculated by the VCM and the filament current loop model, respectively, and their discrepancy, (a) $B_{res}$ vs. $y_c$, (b) Discrepancy of two results vs. the normalized $y_c$ distance in terms of conductor radius ($R_s$) away from the center of the conductor.

Magnetic flux density components of COIL2 along the axial traverse through the center of the conductor (ACC traverse) calculated by the VCM and the filament current loop model, respectively, (a) $B_{yc}$ vs. $z_c$, (b) $B_{zc}$ vs. $z_c$.

Resultant magnetic flux density components of COIL2 along the axial traverse through the center of the conductor (ACC traverse) calculated by the VCM and the filament current loop model, respectively.

Discrepancy of the results between the filament current loop model and the VCM for COIL2 along the axial traverse through the center of the conductor (ACC traverse).

Discrepancy of the results between the filament current loop stack model and the VCM for COIL3 along the axial traverse at the side of the coil (ATS traverse) at $x_c = 0$ and $y_c = 1.0275$ m.

Discrepancy of the results between the filament current loop stack model and the VCM vs. the normalized $y_c$ distance in terms of coil mean diameter ($D_m$) away from the center of the conductor for COIL3 along the lateral traverse at the top of the coil (LTT traverse).

Approximate configuration of a coil with multiple layers connected in parallel.

The geometry of a double-layer coil.

Reactor coil and its models.

Errors of three simplified models along ATC traverse.

Errors of three simplified models along ATS traverse.

Errors of three simplified models along LTT traverse.

Errors of three simplified models along LTH traverse.

Errors of three simplified models along LTC traverse.

xiv
33 Errors in the resultant magnetic flux density calculations when the equivalent square loop is interior to the actual round loop. Traverses selected: L1 - perpendicular to loop element, L2 - diagonal. .......................... 120
34 Errors in the resultant magnetic flux density calculations when the equivalent square loop has the same area as the round loop. Traverses selected: L1 - perpendicular to loop element, L2 - diagonal. ......................... 121
35 Formulas used for evaluating magnetic field distribution of coil. ........ 126
36 Translation and rotation of coordinate systems. ............................... 128
37 Two single-layer coaxial coils. ............................................................. 137
### LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Typical skin depths $\delta$ of conductors made of non-magnetic materials at 60 Hz.</td>
</tr>
<tr>
<td>2</td>
<td>Impedance of a double-layer coil at 60 Hz (layer current is 1 A).</td>
</tr>
<tr>
<td>3</td>
<td>Ratio of layer current magnitudes in a double-layer coil connected in parallel, calculated by the Approximate Impedance Method and the Field Differential Method, respectively.</td>
</tr>
<tr>
<td>4</td>
<td>Critical distances of <em>FLEXIBLE</em> model to limit errors to 5%.</td>
</tr>
<tr>
<td>5</td>
<td>Coefficient $f_H$ versus the total number of turns in each layer of the coil.</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS

The following dimensions are used except specified: angular frequency in rad/sec, dielectric permittivity in farads/m, electric conductivity of conductor in S/m, electric current in A, electric surface current density in A/m, electric volume current density in A/m², inductance in henrys, length quantities and distance in m, magnetic field strength in A/m, magnetic flux density (or magnetic induction) in teslas, magnetic moment in A·m², magnetic moment density in A/m, magnetic permeability in henrys/m, magnetic surface charge density in webers/m², magnetic volume charge density in webers/m³, and resistance and reactance in ohms. It should be noted that all complex or phasor quantities in this dissertation should have the same definitions, that is, they are either all rms phasor quantities or all peak phasor quantities (1.1). The symbols used in this dissertation are listed and defined as follows except otherwise specified.

\[ A, B, C \]

\[ A_i = \frac{2A_i}{\rho \sqrt{(\rho + R_s)^2 + A_i^2}}, \quad i = 1, 2 \]

\[ A_{11} = -(r_b - \cos \theta_b R_s) \cos \varphi / \sin \theta_b, \quad \theta_b \neq 0 \]
\[ = 0, \quad \theta_b = 0, \]

\[ A_{12} = -(r_b - \cos \theta_b R_s) \sin \varphi / \sin \theta_b, \quad \theta_b \neq 0 \]
\[ = 0, \quad \theta_b = 0 \]

\[ A_{13} = -h_b - z, \]

\[ A_{21} = (r_t - \cos \theta_t R_s) \cos \varphi / \sin \theta_t, \quad \theta_t \neq 0 \]
\[ = 0, \quad \theta_t = 0 \]

\[ A_{22} = (r_t - \cos \theta_t R_s) \sin \varphi / \sin \theta_t, \quad \theta_t \neq 0 \]
\[ = 0, \quad \theta_t = 0 \]

\[ A_{23} = h_t - z. \]

\[ a \]

\[ a(\theta) \]

\[ a_{i0} = \rho^2 + R_s^2 + A_i^2, \quad i = 1, 2 \]

\[ a_{i1} = 2A_i A_{i3} - 2R_s \rho, \quad i = 1, 2 \]
\[ a_{i2} = 2A_{i1}A_{i3}, \quad i = 1, 2 \]
\[ a_{i3} = A_{i2}^2, \quad i = 1, 2 \]
\[ a_{i4} = 2A_{i1}A_{i2}, \quad i = 1, 2 \]
\[ a_{i5} = A_{i1}^2, \quad i = 1, 2 \]
\[ a_{i6} = \rho^2 + R_s^2, \quad i = 1, 2 \]
\[ a_{i7} = -2R_s\rho, \quad i = 1, 2 \]
\[ \text{amu} \quad \text{amplitude} \ u \]
\[ a_x, b_x, c_x \quad \text{calculation coefficients} \]
\[ a_y, b_y, c_y \quad \text{calculation coefficients} \]
\[ a_0 \quad x \text{ coordinate of the starting point of a conical helix} \]
\[ \vec{B} \quad \text{magnetic flux density (or magnetic induction)} \]
\[ B(\phi) = \sqrt{\rho^2 - 2R_s\rho \cos \phi + R_s^2} \]
\[ B_i = \frac{2(p + R_s)A_{i3}}{\rho(p - R_s)\sqrt{(p + R_s)^2 + A_{i3}^2}}, \quad i = 1, 2 \]
\[ \vec{B}_J \quad \text{magnetic flux density (or magnetic induction) due to } \vec{J} \]
\[ \vec{B}_{J_{sc}} \quad \text{magnetic flux density (or magnetic induction) due to } \vec{J}_{sc} \]
\[ \vec{B}_M \quad \text{magnetic flux density (or magnetic induction) due to } \vec{M} \]
\[ \vec{B}_{\text{max}} \quad \text{maximum value of } \vec{B} \]
\[ \vec{B}_{\text{min}} \quad \text{minimum value of } \vec{B} \]
\[ \vec{B}_R \quad \text{resultant value of } \vec{B} \]
\[ B_x, B_y, B_z \quad \text{three orthogonal components of } \vec{B} \]
\[ B_{xc} \quad \text{\(\hat{x}_c\) component of } \vec{B} \]
\[ B_{yc} \quad \text{\(\hat{y}_c\) component of } \vec{B} \]
\[ B_{zc} \quad \text{\(\hat{z}_c\) component of } \vec{B} \]
\[ b \quad \text{calculation coefficient} \]
\[ be \quad \text{Kelvin function} \]
\[ bei \quad \text{imaginary part of } be \]
\[ ber \quad \text{real part of } be \]
\[ b_{i0} = (\rho + R_s)^2 + (A_{i2} - A_{i3})^2, \quad i = 1, 2 \]
\[ b_{i1} = 4A_{i1}(A_{i3} - A_{i2}), \quad i = 1, 2 \]
\[ b_{i2} = 2(\rho^2 + R_s^2 + A_{i3}^2 - A_{i2}^2 + 2A_{i1}^2), \quad i = 1, 2 \]
\[ b_{i3} = 4A_{i1}(A_{i2} + A_{i3}), \quad i = 1, 2 \]
\[ b_{i4} = (\rho - R_s)^2 + (A_{i2} + A_{i3})^2, \quad i = 1, 2 \]
\[ C_s \quad \text{integral defined in Appendix E and used in inductance calculation} \]
\[ C_1, C_2 \quad \text{constants} \]
Calculation coefficient

\[ \text{cel}(\kappa', p, a, b) = \int_0^{\pi/2} \frac{a + (b-a) \sin^2 \varphi}{\sqrt{1+(1-p) \sin^2 \varphi}} \, d\varphi, \text{ where } \kappa^2 + \kappa'^2 = 1 \]

\[ \text{cel}(\kappa', p, a, b) = \int_0^{\pi/2} \frac{(a \cos^2 \varphi + b \sin^2 \varphi) \, d\varphi}{(\cos^2 \varphi + p \sin^2 \varphi) \sqrt{\cos^2 \varphi + \kappa'^2 \sin^2 \varphi}}, \]

general complete elliptic integral

\[ D_m \]
mean diameter of the coil which equals \[ \sqrt{(D_2^2 + D_1^2 + D_1 D_2)/3} \]

\[ D_1 \]
inner diameter of the coil

\[ D_2 \]
outer diameter of the coil

\[ d \]
distance between the field point and the center of a current loop

\[ d_i \]
thickness of the \( i \)th layer, \( i = 1, \ldots, N_{\text{layer}} \)

\[ d_l \]
differential line element

\[ d\nu \]
delta amplitude \( \nu \); Jacobian elliptic function

\[ dS, dS' \]
differential elements of area

\[ dt \]
differential element of the variable \( t \)

\[ du \]
differential element of the variable \( u \)

\[ dx_i' \]
differential element of the variable \( x_i' \), \( i = t, b \)

\[ dy_i' \]
differential element of the variable \( y_i' \), \( i = t, b \)

\[ dz' \]
differential element of the variable \( z' \)

\[ d\gamma_z \]
differential element of the variable \( \gamma_z \)

\[ d\vartheta \]
differential element of the variable \( \vartheta \)

\[ d\phi \]
differential element of the variable \( \phi \)

\[ d\phi' \]
differential element of the variable \( \phi' \)

\[ E \text{ or } E(\kappa) = E(\pi/2, \kappa), \text{ Legendre's complete elliptic integral of the second kind} \]

\[ E(\psi, \kappa) \]
Legendre's incomplete elliptic integral of the second kind, \( \psi = \text{am} \kappa \)

\[ E(\psi, \kappa') \]
Legendre's associated incomplete elliptic integral of the second kind, \( \psi = \text{am} \kappa' \)

\[ E_i(t) = (1 + t^2) [(\rho - R_s)^2 t^2 + (\rho + R_s)^2 t^2]. \]
\[ \sqrt{b_i t^4 + b_{i3} t^3 + b_{i2} t^2 + b_{i1} t + b_{i0}}, \quad i = 1, 2 \]

\[ E_{10}(t) = (1 + t^2) [(\rho - R_s)^2 t^2 + (\rho + R_s)^2 t^2]. \]
\[ \sqrt{b_i t^4 - b_{i3} t^3 + b_{i2} t^2 - b_{i1} t + b_{i0}}, \quad i = 1, 2 \]

\[ F \]
calculation coefficient, correction factor

\[ xix \]
\[ F(\psi, \kappa) \] Legendre's incomplete elliptic integral of the first kind, 
\[ \psi = \text{amu} \]
\[ F(\psi, \kappa') \] Legendre's associated incomplete elliptic integral of the 
first kind, \( \psi = \text{amu} \)
\[ F_i^\rho(\phi) = \frac{R_i^\rho(\phi)}{\sqrt{X_i(\phi)}}, \quad i = 1, 2 \]
\[ F_{i0}^\rho(\phi) = \frac{R_{i0}^\rho(\phi)}{\sqrt{X_{i0}(\phi)}}, \quad i = 1, 2 \]
\[ F_i^{\rho,\kappa}(\phi) \] represents \( F_i^\rho(\phi) \) and \( F_{i0}^\rho(\phi) \), \( i = 1, 2 \)
\[ F_{i0}^{\rho,\kappa}(\phi) \] represents \( F_i^\rho(\phi) \) and \( F_{i0}^\rho(\phi) \), \( i = 1, 2 \)
\[ G_i \] constant, \( i = t, b \)
\[ G_t = -\frac{I}{2} \cos \theta_t \]
\[ G_b = \frac{I}{2} \cos \theta_b \]
\[ g \] calculation constant
\[ H \] length of the coil
\[ \bar{H} \] magnetic field strength
\[ \bar{H}_J \] magnetic field strength due to \( \bar{J} \)
\[ \bar{H}_{J_{sc}} \] magnetic field strength due to \( \bar{J}_{sc} \)
\[ \bar{H}_{J_{sc},\rho} \] magnetic field strength due to \( \bar{J}_{sc}^\rho \)
\[ \bar{H}_{J_{sc}^{\rho},\phi} \] \( \rho \) component of \( \bar{H}_{J_{sc}^{\rho}} \)
\[ \bar{H}_{J_{sc}^{\rho},\phi} \] represents \( \bar{H}_{J_{sc}^{\rho},\phi} \) and \( \bar{H}_{J_{sc}^{\rho},\phi} \)
\[ \bar{H}_{J_{sc},\rho} \] \( \phi \) component of \( \bar{H}_{J_{sc}^{\rho}} \)
\[ \bar{H}_{J_{sc},e} \] magnetic field strength due to \( \bar{M} \)
\[ H_R \] resultant value of \( \bar{H} \)
\[ \bar{H}_i \] magnetic field strength of the \( i \)th segment in the \( j \)th layer,
\[ i = 1, \ldots, N_{j,\text{seg}}, \quad j = 1, \ldots, N_{\text{layer}} \]
\[ H_{i,x}, H_{i,y}, H_{i,z} \] \( \bar{x}_i, \bar{y}_i \) and \( \bar{z}_i \) components of \( \bar{H}_i \), \( i = 1, \ldots, N_{j,\text{seg}} \)
\[ H_{x_i}, H_{y_i}, H_{z_i} \quad j = 1, \ldots, N_{layer} \]
\[ \hat{x}_c, \hat{y}_c, \hat{z}_c \quad \text{components of} \quad \hat{H}, \quad i = 1, \ldots, N_{j,seg}, \]
\[ H_{\text{max}} \quad \text{maximum value of} \quad \hat{H} \]
\[ H_{\text{min}} \quad \text{minimum value of} \quad \hat{H} \]
\[ \hat{H}_x, \hat{H}_y, \hat{H}_z \quad \text{three orthogonal components of} \quad \hat{H} \quad \text{in the} \quad XYZ \quad \text{coordinate system} \]
\[ \hat{H}_{x_c}, \hat{H}_{y_c}, \hat{H}_{z_c} \quad \text{three orthogonal components of} \quad \hat{H} \quad \text{in the} \quad X_cY_cZ_c \quad \text{coordinate system} \]
\[ \hat{H}_{\rho_c}, \hat{H}_{\varphi_c}, \hat{H}_{\zeta_c} \quad \text{three orthogonal components of} \quad \hat{H} \quad \text{in the} \quad \rho_c\varphi_c\zeta_c \quad \text{coordinate system} \]
\[ \hat{H}_{\rho_{sc}}, \hat{H}_{\varphi_{sc}}, \hat{H}_{\zeta_{sc}} \quad \text{magnetic field strength due to} \quad \rho_{sc} \]
\[ \hat{H}_{\rho_{sc, x}}, \hat{H}_{\rho_{sc, y}}, \hat{H}_{\rho_{sc, z}} \quad \hat{x}_i, \hat{y}_i, \hat{z}_i \quad \text{components of} \quad \hat{H}_{\rho_{sc, i}}, \quad i = t, b \]
\[ \hat{H}_{\rho_{sc}} \quad \text{magnetic field strength due to} \quad \rho_{sc} \]
\[ \hat{H}_{\varphi_{sc}}, \hat{H}_{\zeta_{sc}} \quad \text{magnetic field strength due to} \quad \varphi_{sc}, \zeta_{sc} \]
\[ \hat{H}_{\rho_{sc, x}}, \hat{H}_{\rho_{sc, y}}, \hat{H}_{\rho_{sc, z}} \quad \text{magnetic field strength due to} \quad \rho_{sc} \]
\[ \hat{H}_{\rho_{sc}}, \hat{H}_{\varphi_{sc}}, \hat{H}_{\zeta_{sc}} \quad \text{represent} \quad \hat{H}_{\rho_{sc}} \quad \text{and} \quad \hat{H}_{\varphi_{sc}} \quad \text{magnetic field strength due to} \quad \rho_{sc}, \varphi_{sc} \]
\[ h_b \quad \text{distance between the origin of the local coordinate system} \]
\[ \text{XYZ} \quad \text{of the conductor segment and the center of the} \]
\[ \text{bottom end plane of the conductor segment, that is,} \]
\[ h_b = OO_b \]
\[ h_{ib} \quad h_b \quad \text{of the} \quad i \quad \text{th conductor segment in the} \quad j \quad \text{th layer,} \]
\[ i = 1, \ldots, N_{j,seg}, \quad j = 1, \ldots, N_{layer} \]
\[ h_{it} \quad h_t \quad \text{of the} \quad i \quad \text{th conductor segment in the} \quad j \quad \text{th layer,} \]
\[ i = 1, \ldots, N_{j,seg}, \quad j = 1, \ldots, N_{layer} \]
\[ h_p \quad \text{pitch of the coil} \]
\[ h_t \quad \text{distance between the origin of the local coordinate system} \]
\[ \text{XYZ} \quad \text{of the conductor segment and the center of the top} \]
\[ \text{end plane of the conductor segment, that is,} \]
\[ h_t = O_tO \]
\[ h_1 \quad \text{length of the top section in the} \quad FLEXIBLE \quad \text{simplified coil model} \]
\[ h_2 \quad \text{length of the bottom section in the} \quad FLEXIBLE \quad \text{simplified coil model} \]
\[ I \quad \text{complex current in a conductor} \]

xxi
\( \vec{I} \) current vector of the coil layers

\( I_{\text{coil}} \) total current of the reactor coil

\( I_i \) complex current in the \( i \)th layer, \( i = 1, \ldots, N_{\text{layer}} \)

\( I'_i \) complex current per unit axial length in the \( i \)th layer in \( A/m, \ I'_i = I_i N_i / l_i, \ i = 1, \ldots, N_{\text{layer}} \)

\( J^B_p \) \( p \)-th-order modified Bessel function of the first kind

\( i, j \) indexes

\( \vec{J} \) magnitude of \( \vec{J} \)

\( \vec{J} \) electric volume conducting current density in the coil conductor

\( J_a \) azimuthal current density of the coil

\( \vec{J}_{\text{sc}} \) equivalent electric surface current density

\( \vec{J}^b_{\text{sc}} \) equivalent electric surface current density on the bottom end plane of the conductor segment

\( \vec{J}^b_{\text{sc,z}}, \vec{J}^b_{\text{sc,y}}, \vec{J}^b_{\text{sc,z}} \) three orthogonal components of \( \vec{J}^b_{\text{sc}} \) in the \( XYZ \) coordinate system

\( \vec{J}^b_{\text{sc,x}}, \vec{J}^b_{\text{sc,y}}, \vec{J}^b_{\text{sc,z}} \) three orthogonal components of \( \vec{J}^b_{\text{sc}} \) in the \( X_i Y_i Z_i \) coordinate system

\( \vec{J}_{\text{sc}} \) equivalent electric surface current density on the end planes of the conductor segment, \( i = t, b \)

\( \vec{J}^t_{\text{sc}} \) equivalent electric surface current density on the top end plane of the conductor segment

\( \vec{J}^t_{\text{sc,x}}, \vec{J}^t_{\text{sc,y}}, \vec{J}^t_{\text{sc,z}} \) three orthogonal components of \( \vec{J}^t_{\text{sc}} \) in the \( XYZ \) coordinate system

\( \vec{J}^t_{\text{sc,x}}, \vec{J}^t_{\text{sc,y}}, \vec{J}^t_{\text{sc,z}} \) three orthogonal components of \( \vec{J}^t_{\text{sc}} \) in the \( X_i Y_i Z_i \) coordinate system

\( \vec{J}^w_{\text{sc}} \) equivalent electric surface current density on the cylindrical wall of the conductor segment

\( \vec{J}^w_{\text{sc}}, \vec{J}^b_{\text{sc}} \) represents \( \vec{J}^w_{\text{sc}}, \vec{J}^b_{\text{sc}} \) and \( \vec{J}^b_{\text{sc}} \)

\( \vec{J}_{\varphi, c} \) \( \varphi_c \) component of \( \vec{J} \)

\( \varphi = \sqrt{-1} \)

\( K \) or \( K(\kappa) \) \( = F(\pi / 2, \kappa) \), Legendre's complete elliptic integral of the first kind

\( K^B_p \) \( p \)-th-order modified Bessel function of the second kind

\( k \) index

\( k_e \) Kelvin function
kei: imaginary part of ke
ker: real part of ke
$L$: self inductance of the coil
$L_i$: self inductance of the $ith$ layer, $i = 1, \ldots, N_{layer}$
$L_{in,i}$: internal inductance of the $ith$ layer, $i = 1, \ldots, N_{layer}$
$L_{ex,i}$: external inductance of the $ith$ layer, $i = 1, \ldots, N_{layer}$
$l_i$: axial length of the $ith$ layer, $i = 1, \ldots, N_{layer}$
$\vec{M}$: magnetization vector or magnetic moment density
$\vec{M}_i$: magnetization vector or magnetic moment density in a region $i$
$M_{ij}$: mutual inductance between the $ith$ and $jth$ layers, $i,j = 1, \ldots, N_{layer}$
$\vec{M}_j$: magnetization vector or magnetic moment density in a region $j$
$\vec{m}$: magnetic moment vector
$m_x, m_y, m_z$: three orthogonal components of $\vec{m}$ in the $XcYcZc$ coordinate system
$N$: total number of turns of the coil
$N_j$: number of turns in the $jth$ layer, $j = 1, \ldots, N_{layer}$
$N_{j,seg}$: total number of segments in the $jth$ layer, $j = 1, \ldots, N_{layer}$
$N_{layer}$: total number of layers in a coil
$N_{seg}$: total number of segments per turn
$\hat{n}$: unit vector pointing in the normal direction of the coil given by the right-hand-rule
$\hat{n}_b$: unit vector normal to the bottom end plane and pointing outwards
$n_i = \frac{4R_{seg}}{(d-R_{seg})^2 + A_{3i}^2}$, $i = 1, 2$
$\hat{n}_{ij}$: unit vector normal to a surface and oriented from side $i$ to side $j$ of the surface
$\hat{n}_t$: unit vector normal to the top end plane and pointing outwards
$\hat{n}_w$: unit vector normal to the side wall and pointing outwards
$O$: origin of the $XYZ$ coordinate system
$O_b$: center of the bottom end plane or the origin of the $X_bY_bZ_b$ coordinate system
$O_c$ origin of the $X_cY_cZ_c$ coordinate system

$O_{ij}^*$ center point of the center line of the $i$th arc conductor segment of the $j$th layer, $i = 1, \cdots, N_j, \text{seg}$, $j = 1, \cdots, N_{layer}$

$O_{i-1j}^*, O_{ij}^*$ starting and ending points of the center line of the $i$th segment of the $j$th layer, $i = 1, \cdots, N_j, \text{seg}$, $j = 1, \cdots, N_{layer}$

$O_{kj}^*$ center of the $k$th turn in the $j$th layer, $k = 1, \cdots, N_j$, $j = 1, \cdots, N_{layer}$

$O_t$ center of the top end plane or the origin of the $X_bY_bZ_b$ coordinate system

$p$ parameter of cell $p_i \quad = \frac{\rho + R_s}{|\rho - R_s|} \sqrt{(\rho - R_s)^2 + A_{13}^2}, \quad i = 1, 2$

$p_A \quad = 1 + \kappa^2 \tan^2 \psi$

$Q$ rms phasor quantity of $q$, $q = \sqrt{2} \Re(Q e^{j\omega t})$

$\hat{Q}$ peak phasor quantity of $q$, $q = \Re(\hat{Q} e^{j\omega t})$

$q$ steady state sinusoidal instantaneous quantity

$\vec{R}$ magnitude of $\vec{R}$

$|\vec{R}|$ magnitude of $\vec{R}$

$R_i$ resistance of the $i$th layer, $i = 1, \cdots, N_{layer}$

$R_i^f(\phi) \quad = R_s \sin \phi (A_{11} \sin \phi + A_{12} \cos \phi + A_{13}), \quad i = 1, 2$

$R_i^{\theta \phi}(\phi) \quad = R_s \sin \phi (A_{11} \sin \phi + A_{12} \cos \phi - A_{13}), \quad i = 1, 2$

$R_i^{\theta \phi}(\phi)$ represents $R_i^f(\phi)$ and $R_i^{\theta \phi}(\phi)$, $i = 1, 2$

$R_i^{\theta \phi}(\phi)$ represents $R_i^f(\phi)$ and $R_i^{\theta \phi}(\phi)$, $i = 1, 2$

$R_i^{\theta \phi}(\phi) \quad = (\rho - R_s \cos \phi)(A_{11} \sin \phi + A_{12} \cos \phi + A_{13}), \quad i = 1, 2$

$R_i^{\theta \phi}(\phi) \quad = - (\rho + R_s \cos \phi)(A_{11} \sin \phi + A_{12} \cos \phi - A_{13}), \quad i = 1, 2$

$R_m$ mean radius of the coil, $R_m = D_m/2$

$R_{mi}$ mean radius of the $i$th layer, $i = 1, \cdots, N_{layer}$

$R_o$ outer radius of a coil

$R_{\text{round}}$ radius of a round loop

$R_s$ radius of the conductor segment

$R_{\text{square}}$ side length of a square loop

$r$ vector of the field point position $\bar{r} \quad = \rho_c/R_o$

$\bar{r}$ vector of the field point position

xxiv
vector of the source point position

$r_i^j$ radius of the conductor center line of the $j$th layer, $j = 1, \cdots, N_{\text{layer}}$

$r_b$ length of semimajor axis of the bottom end plane of the cylindrical conductor segment

$(r_c, \theta_c, \varphi_c)$ coordinates of the field point in the $r_c\theta_c\varphi_c$ coordinate system

total spherical coordinate system of the coil

$r_c\theta_c\varphi_c$ length of semimajor axis of the top end plane of the cylindrical conductor segment

$S_i^j$ length of the arc $O_{i-1}^jO_i^j$, $i = 1, \cdots, N_{j, \text{seg}}$,

$t$ calculation variable

$ar{U}$ transpose of $U$

$u = F(\psi, \kappa)$, incomplete elliptic integral of the first kind

$V$ voltage applied to the reactor coil

$\bar{V}$ voltage vector of the coil layers

$X_i$ reactance of the $i$th layer, $i = 1, \cdots, N_{\text{layer}}$

$X_{\text{in}, i}$ internal reactance of the $i$th layer, $i = 1, \cdots, N_{\text{layer}}$

$X_i(\phi) = a_{i0} + a_{i1} \cos \phi + a_{i2} \sin \phi + a_{i3} \cos^2 \phi + a_{i4} \sin \phi \cos \phi + a_{i5} \sin^2 \phi, i = 1, 2$

$X_{\text{io}}(\phi) = a_{i0} - a_{i1} \cos \phi - a_{i2} \sin \phi + a_{i3} \cos^2 \phi + a_{i4} \sin \phi \cos \phi + a_{i5} \sin^2 \phi, i = 1, 2$

$XYZ$ local rectangular coordinate system of the conductor segment

$X_bY_bZ_b$ local rectangular coordinate system of the bottom end plane of the conductor segment

$X_cY_cZ_c$ global rectangular coordinate system of the coil

$X_iY_iZ_i$ local rectangular coordinate system of the top or bottom
$X_i^j Y_i^j Z_i^j$

eend plane of the conductor segment, $i = t, b$

local rectangular coordinate system of the $i$th segment of the $j$th layer, $i = 1, \ldots, N_{j,seg}$, $j = 1, \ldots, N_{layer}$

$X_i Y_i Z_t$

local rectangular coordinate system of the top end plane of the conductor segment

$X_0 Y_0 Z_0$
rectangular coordinate system

$X_1 Y_1 Z_1$
rectangular coordinate system

$X_2 Y_2 Z_2$
rectangular coordinate system

$(x, y, z)$
coordinates of the field point in the $XYZ$ coordinate system

$(x', y', z')$
coordinates of the source point in the $XYZ$ coordinate system

$\hat{x}, \hat{y}, \hat{z}$
unit vectors of the $X, Y$ and $Z$ axes

$(x_b, y_b, z_b)$
coordinates of the field point in the $X_b Y_b Z_b$ coordinate system

$(x'_b, y'_b, z'_b)$
coordinates of the source point in the $X_b Y_b Z_b$ coordinate system

$(x_c, y_c, z_c)$
coordinates of the field point in the $X_c Y_c Z_c$ coordinate system

$(x', y', z')$
coordinates of the source point in the $X_c Y_c Z_c$ coordinate system

$\hat{x}_c, \hat{y}_c, \hat{z}_c$
unit vectors of the $X_c, Y_c$ and $Z_c$ axes

$(x_{loop}, y_{loop}, z_{loop})$
coordinates of the center of a current loop

$(x_e, y_e, z_e)$
coordinates of the ending point of a straight filament segment in the $X_e Y_e Z_e$ coordinate system

$(x_i, y_i, z_i)$
coordinates of the field point in the $X_i Y_i Z_i$ coordinate system, $i = t, b$

$(x'_i, y'_i, z'_i)$
coordinates of the source point in the $X_i Y_i Z_i$ coordinate system, $i = t, b$

$\hat{x}_i, \hat{y}_i, \hat{z}_i$
unit vectors of the $X_i, Y_i$ and $Z_i$ axes, $i = t, b$

$\hat{x}'_i, \hat{y}'_i, \hat{z}'_i$
unit vectors of $X'_i, Y'_i$ and $Z'_i$ axes, $i = 1, \ldots, N_{j,seg}$, $j = 1, \ldots, N_{layer}$

$(x_s, y_s, z_s)$
coordinates of the starting point of a straight filament segment in the $X_s Y_s Z_s$ coordinate system

$(x_t, y_t, z_t)$
coordinates of the field point in the $X_t Y_t Z_t$ coordinate system

$(x'_t, y'_t, z'_t)$
coordinates of the source point in the $X_t Y_t Z_t$ coordinate system

xxvi
coordinates of the field point in the $X_0Y_0Z_0$ coordinate system

coordinates of the field point in the $X_1Y_1Z_1$ coordinate system

coordinates of the field point in the $X_2Y_2Z_2$ coordinate system

$Y_i^p(t) = -4tR_s[(A_{i2} + A_{i3})t^2 + 2A_{i1}t + A_{i3} - A_{i2}], \ i = 1, 2$

$Y_{10}^p(t) = -4tR_s[(A_{i2} + A_{i3})t^2 - 2A_{i1}t + A_{i3} - A_{i2}], \ i = 1, 2$

$Y_{10}^{p, p}(t)$ represents $Y_i^p(t)$ and $Y_{10}^p(t), \ i = 1, 2$

$Y_{10}^q(t)$ represents $Y_{10}^p(t)$ and $Y_{10}^q(t), \ i = 1, 2$

$Y_{10}^{q, q}(t) = -2[\rho + R_s + (\rho - R_s)t^2]$

$[(A_{i2} + A_{i3})t^2 + 2A_{i1}t + A_{i3} - A_{i2}], \ i = 1, 2$

$Y_{10}^{q, q}(t) = 2[\rho + R_s + (\rho - R_s)t^2]$

$[(A_{i2} + A_{i3})t^2 - 2A_{i1}t + A_{i3} - A_{i2}], \ i = 1, 2$

$\tilde{Z}$
impedance matrix of the coil

$\tilde{Z}$
inverse of the matrix $\tilde{Z}$

$Z_i$
self impedance of the $i$th layer, $i = 1, \ldots, N_{layer}$

$\tilde{z}_j$

$z_i = A_{i1}\sin(\phi) + A_{i2}\cos(\phi) + A_{i3} + z, \ i = 1, 2$

$z_i^0$
z coordinate of the starting point of the conductor center line of the $j$th layer, $j = 1, \ldots, N_{layer}$

$\alpha$

$\alpha_c$

$\alpha_i = \sqrt{\frac{-4R_s\rho}{(\rho - R_s)^2}}$

$\beta$

$\beta_{i1}$
counterclockwise rotated angle between the $Z_c$ and $Z_i^j$ axes, $i = 1, \ldots, N_{j, seg}, \ j = 1, \ldots, N_{layer}$

$\beta_{i2}$
counterclockwise rotated angle between the $X_c$ and $X_i^j$ axes, $i = 1, \ldots, N_{j, seg}, \ j = 1, \ldots, N_{layer}$

$\beta_1$
counterclockwise rotated angle between the $Z$ axes of two rectangular coordinate systems

$\beta_2$
counterclockwise rotated angle between the $X$ axes of two rectangular coordinate systems

$\gamma$

propagation constant
\[ \gamma_z = z' - z \]
\[ \gamma_{z1} = z' - \bar{z}_1 \]
\[ \gamma_{z2} = z' - \bar{z}_2 \]

- \( \delta \) skin depth of the conductor
- \( \epsilon \) dielectric permittivity
- \( \epsilon_0 = \frac{1}{36\pi} \times 10^{-9} \) F/m, dielectric permittivity of free space
- \( \theta \) polar angle of a helix in radians
- \( \theta_b \) slope angle of the bottom end plane of the cylindrical conductor segment
- \( \theta_i^{ij} \) polar angle of the point \( O_i^{ij} \), \( i = 1, \ldots, N_{j,\text{seg}}, \quad j = 1, \ldots, N_{\text{layer}} \)
- \( \theta_{i-1}^i, \theta_i^i \) polar angles of the points \( O_{i-1}^i \) and \( O_i^i \), respectively, \( i = 1, \ldots, N_{j,\text{seg}}, \quad j = 1, \ldots, N_{\text{layer}} \)
- \( \theta_{ib}^i \) \( \theta_b \) of the \( i \)th segment of the \( j \)th layer, \( i = 1, \ldots, N_{j,\text{seg}}, \quad j = 1, \ldots, N_{\text{layer}} \)
- \( \theta_{it}^i \) \( \theta_t \) of the \( i \)th segment of the \( j \)th layer, \( i = 1, \ldots, N_{j,\text{seg}}, \quad j = 1, \ldots, N_{\text{layer}} \)
- \( \theta_t \) slope angle of the top end plane of the cylindrical conductor segment
- \( \kappa \) modulus of Jacobian elliptic functions and integrals
- \( \kappa' = \sqrt{1 - \kappa^2} \), complementary modulus of Jacobian elliptic functions and integrals
- \( \kappa_i = \sqrt{\frac{4R_s\rho}{(\rho + R_s)^2 + A_{13}^2}}, \quad i = 1, 2 \), modulus of Jacobian elliptic integrals
- \( \kappa_i' = \sqrt{\frac{(\rho - R_s)^2 + A_{13}^2}{(\rho + R_s)^2 + A_{13}^2}}, \quad i = 1, 2 \), complementary modulus of Jacobian elliptic integrals
- \( \Lambda_0(\beta, \kappa) \) Heuman's Lambda function
- \( \Lambda_0(\psi, \kappa) \) Heuman's Lambda function
- \( \Lambda_0(\psi_i, \kappa_i) \) Heuman's Lambda function, \( i = 1, 2 \)
- \( \mu \) magnetic permeability
- \( \mu_i \) magnetic permeability in a region \( i \)
- \( \mu_j \) magnetic permeability in a region \( j \)
- \( \mu_0 \) magnetic permeability of free space, \( \mu_0 = 4\pi \times 10^{-7} \) H/m
\[ \Pi(\kappa_0, \kappa) \] Legendre's complete elliptic integral of the third kind, where \( \alpha^2 = \kappa_0 \) and \( \psi = \pi/2 \) in \( \Pi(\psi, \alpha^2, \kappa) \)

\[ \Pi(\psi, \alpha^2, \kappa) \] Legendre's incomplete elliptic integral of the third kind, \( \psi = \text{am} \)

\( \hat{\rho}', \hat{\varphi}' \) unit vectors in the directions of \( \hat{\rho} \) and \( \hat{\varphi} \), respectively

\( (\rho, \varphi, z) \) coordinates of the field point in the \( \rho\varphi Z \) coordinate system

\( (\rho', \varphi', z') \) coordinates of the source point in the \( \rho\varphi Z \) coordinate system

\( \hat{\rho}, \hat{\varphi}, \hat{z} \) unit vectors of \( \rho, \varphi \) and \( Z \) axes

\( (\rho_c, \varphi_c, z_c) \) coordinates of the field point in the \( \rho_c\varphi_c Z_c \) coordinate system

\( \rho_i \) inner radius of the \( i \)th equivalent layer, \( i = 1, \cdots, \text{Niayer} \)

\( \rho_i' \) outer radius of the \( i \)th equivalent layer, \( i = 1, \cdots, \text{Niayer} \)

\( \rho_{sc} \) equivalent magnetic surface charge density

\( \rho_{sc}^b \) equivalent magnetic surface charge density on the bottom end plane of the conductor segment

\( \rho_{sc}^t \) equivalent magnetic surface charge density on the top end plane of the conductor segment

\( \rho_{sc}^{t,b} \) represents \( \rho_{sc}^t \) and \( \rho_{sc}^b \)

\( \rho_{sc}^{\varphi} \) equivalent magnetic surface charge density on the cylindrical side wall surface of the conductor segment

\( \rho_v \) equivalent magnetic volume charge density

\( \rho_1, \rho_2 \) calculation coefficients

\( \rho\varphi Z \) local cylindrical coordinate system of the conductor segment

\( \rho_c\varphi_c Z_c \) global cylindrical coordinate system of the coil

\( \sigma \) electric conductivity of conductor

\( \phi \) \( \varphi' - \varphi \)

\( \varphi \) \( \varphi \) coordinate of the field point in the \( \rho\varphi Z \) coordinate system

\( \varphi' \) \( \varphi \) coordinate of the source point in the \( \rho\varphi Z \) coordinate system

\( \psi \) parameter of Heuman's Lambda function

\[ \psi_i = \sin^{-1} \sqrt{\frac{A_{12}^2}{A_{12}^2 + (\rho - R_s)^2}}, \; i = 1, 2 \]

\( \Omega \) Nagaoka factor which takes into account the end effects
of a coil in the inductance calculation

\( \omega \)
angular frequency of the current source in rad/sec

\( \Re \)
real part of a complex number

\( \Im \)
imaginary part of a complex number

\( \nabla^2 \)
Laplacian

*  conjugate of a complex number
1.1 Statement of the Problem

Studies related to the calculation, measurement and management of magnetic field distributions at power frequency (60 Hz in North America) are being conducted in many countries. To some extent, these have been initiated due to the public concern about possible health effects of low-frequency magnetic fields on people [1, 2].

Air-core reactor coils, such as short-circuit current-limiting reactor coils, line trap reactor coils, neutral grounding reactor coils, shunt reactor coils, smoothing reactor coils, harmonic filter reactor coils, capacitor-bank reactor coils (for inrush current-limiting and resonant frequency control purposes) and duplex reactor coils are widely used in electric power system facilities for various reasons and purposes [3]. The main reason is the protection of equipment in power system facilities by the following features: reduction or elimination of the destructive effects of fault currents, separation of the high frequency carrier signals from unwanted noise signals for equipment control purposes, compensation for capacitive VARs, and improvement of power quality. The normal operating currents of current-limiting reactors and line traps are the same as the line currents which can be up to several thousand amperes. The magnetic field strength in the surrounding area of large air-core reactor coils could be very strong, say 50 mT (50 G). Also, there are cases where the current-limiting reactors are located very close to ground level. Therefore, studying methods
for calculating the magnetic field distribution of these large coils is a practical and important problem. Very briefly, the problem statements are as follows:

- Develop an accurate method to evaluate the magnetic field distribution inside and outside the coil. That is essential for solving coil design related problems, such as calculating the magnitude of forces on a coil, the current distribution in the conductors of a coil and inductive properties of a coil.

- Develop a simplified method for computing the magnetic field strength quickly at field points not in the immediate vicinity of the reactor coil. That will be significant in finding the contributions of these large coils to the electromagnetic environment of a substation at or close to ground level efficiently.

Appropriate methods for the calculation of the magnetic field distribution around reactor coils made of circular cross-section conductors have not been found in the literature (refer to Chapter 2).

1.2 Configuration of an Air-Core Reactor

Air-core reactor coils are important components of the substations of 50 or 60-Hz electric power transmission systems. A leading power system coil manufacturer in North America is Trench Electric Company. The prototype of the coil used in this study is based on a typical Trench Electric reactor coil, as shown in Figure 1, which is essentially a multiple parallel connected air-core coil made of conductors of circular cross section.

From Figure 1, it can be observed that a “Trench” reactor coil consists of several coaxial and concentric winding groups or packages encapsulated with epoxy impregnated filament fiberglass. Also, fiberglass spacers are used to keep a uniform space between the packages. The number of the packages ranges from 1 to 12. The typical
Figure 1: A typical current-limiting reactor manufactured by Trench Electric [4].
number of packages of the current-limiting reactor coil is about 7. Each package is comprised of two to five layers of small diameter aluminum conductors. Each conductor is individually insulated by polyester film. In the coil, each layer resembles a solenoid coil. All the layers are parallelly connected and are terminated on multi-spoked aluminum spiders, which are made of aluminum bars, to allow a more even current distribution among the layers [4, 5, 6].

The line trap coils have the same structure as the current-limiting reactors except that they have a much smaller number of layers than the current-limiting reactors.

The typical mean diameter of a reactor coil is about 1 m. Some reactor coils might have a diameter in the range of 0.5 m to 2.0 m. The typical ratio of the reactor coil length to the reactor coil mean diameter \((H/D_m)\) ranges from 0.5 to 2.0. Some special reactor coils might have a length over diameter ratio up to 7. The typical length of the reactor coil varies between 0.3 m to 3.5 m. The typical conductor diameter is around or less than 0.01 m. The coil is mounted either in the horizontal or in the vertical position. The clearance between a reactor coil and the ground plane can be as low as 0.5 m. The rated current of a reactor coil varies from 30 A to 4000 A. The normal operating currents of current-limiting reactor coils and line trap reactor coils are the same as the line currents. As another example, the reactor coils employed in the filter circuits of high voltage DC converter stations have currents smaller than the line currents.

1.3 Objective of the Research

The main objective of this research is to develop appropriate methods for calculating accurately the three-dimensional power frequency magnetic field distribution produced by large air-core reactor coils at arbitrary field points. For the field points that are inside or in the very close vicinity of the reactor coil, it is desired to de-
velop an accurate modeling method which can take into consideration the effects of the actual shapes of the coil and the conductor, the current distribution in the coil layers and the volume current distribution in the conductor. For the field points that are outside and not in the very close vicinity of the reactor coil, it is desired to develop a simplified modeling method in order to save computation time. This method should be not only simple and easy to implement, but also give accurate results in the regions of practical interests.

In this dissertation, the magnetic flux density (or magnetic induction) and the magnetic field strength (or magnetizing force) are defined in the same way as in the "1995 Annual Book of ASTM Standards" [7]. They are quoted in the following:

- The magnetic flux density or induction \( \mathbf{B} \) is defined as the magnetic vector quantity which at any point in a magnetic field is measured either by the mechanical force experienced by an element of electric current at the point, or by the electromotive force induced in an elementary loop during any change in flux linkages with the loop at the point. Its unit is tesla (T, 1 tesla = 10^4 gauss).

- The magnetic field strength or magnetizing force \( \mathbf{H} \) is defined as the magnetic vector quantity at a point in a magnetic field which measures the ability of electric currents or magnetized bodies to produce magnetic induction at the given point. Its preferred unit is amperes per meter (A/m).

In the regions free of magnetized matter, the magnetic field strength is proportional to the magnetic flux density, that is, \( \mathbf{H} = \mathbf{B}/\mu_0 \), where \( \mu_0 = 4\pi \times 10^{-7} \) H/m is the permeability of free space.

For the magnetic field of a single-phase reactor coil, the space components of the magnetic field strength or magnetic flux density are in time phase with each
The magnetic field strength or magnetic flux density at any point can be described in terms of a single direction in space and its time-varying magnitude. For the magnetic field of a three-phase reactor coil, the space components of the magnetic field strength or magnetic flux density may not be in time phase with each other. The magnetic field strength or magnetic flux density at any point can be described by the magnetic field strength or magnetic flux density ellipse, i.e., by the magnitude and direction of the semi-major axis, and the magnitude and direction of the semi-minor axis.

Refer to [8] for other definitions given in the following. The maximum value of the magnetic field strength $H_{\text{max}}$ (or magnetic flux density $B_{\text{max}}$) at a point is defined as the value of the semi-major axis of the magnetic field strength (or magnetic flux density) ellipse at that point. It is the largest value of the magnetic field strength (or magnetic flux density) which can be measured at that point with a single-axis meter. The maximum value of the magnetic field strength (or magnetic flux density) takes into consideration the phase angles of the magnetic field strength (or magnetic flux density) components.

The minimum value of the magnetic field strength $H_{\text{min}}$ (or magnetic flux density $B_{\text{min}}$) at a point is defined as the value of the semi-minor axis of the magnetic field strength (or magnetic flux density) ellipse at that point. It is the smallest value of the magnetic field strength (or magnetic flux density) which can be measured at that point with a single-axis meter. The minimum value of the magnetic field strength (or magnetic flux density) takes into consideration the phase angles of the magnetic field strength (or magnetic flux density) components.

The resultant values of magnetic field strength $\vec{H}$ and magnetic flux density $\vec{B}$
are defined, respectively, by

\[ H_R = (H_x^2 + H_y^2 + H_z^2)^{1/2} = (H_{\text{max}}^2 + H_{\text{min}}^2)^{1/2}, \]

\[ B_R = (B_x^2 + B_y^2 + B_z^2)^{1/2} = (B_{\text{max}}^2 + B_{\text{min}}^2)^{1/2}, \]

where \( H_x, H_y \) and \( H_z \) are the three orthogonal magnetic field strength components of \( \vec{H} \), and \( B_x, B_y \) and \( B_z \) are the three orthogonal magnetic flux density components of \( \vec{B} \). It should be noted that the resultant value of \( \vec{H} \) or \( \vec{B} \) is always greater than or equal to its maximum value. If the field is linearly polarized, \( H_{\text{min}} = 0 \) and \( H_R = H_{\text{max}} \), and \( B_{\text{min}} = 0 \) and \( B_R = B_{\text{max}} \). If the field is circularly polarized, \( H_{\text{max}} = H_{\text{min}} \) and \( H_R = \sqrt{2}H_{\text{max}} \), and \( B_{\text{max}} = B_{\text{min}} \) and \( B_R = \sqrt{2}B_{\text{max}} \).

All AC quantities used in this dissertation are phasor quantities, that is, complex quantities. The vector quantities used are all space vectors. The relationships between a steady state sinusoidal instantaneous quantity \( q \) and its phasor quantities \( \bar{Q} \) and \( \dot{Q} \) are defined by

\[ q = \sqrt{2} \Re(Q e^{\omega t}) = \Re(\dot{Q} e^{\omega t}), \tag{1.1} \]

where \( \Re \) is the real part of a complex number, \( \omega \) is the angular frequency of \( q \), \( Q \) is referred to as the \textit{rms} phasor quantity of \( q \) and \( \dot{Q} \) is called the \textit{peak} phasor quantity of \( q \).

The \textbf{pitch} of a helical coil is defined as the distance measured axially from a point on one turn to a corresponding point on the next turn. The \textbf{pitch angle} of a helical coil is the arctangent of the ratio of its pitch over the mean circumference of the projection of the helical coil.

1.4 Contributions of the Research

In this dissertation, the modeling methods for calculating the magnetic field strength of large air-core reactor coils are studied. These studies include accurate
and simplified modeling methods.

Compared with the existing methods, the major contributions of this dissertation are:

1. An accurate modeling method has been developed which can calculate the magnetic field strength at arbitrary field points of a multi-layer parallel connected air-core coil comprised of circular cross-section conductors, carrying a quasi-static current.

2. A surface source model has been developed. It enables the volume current distribution problem to be replaced by a surface source distribution problem.

3. Analytical formulas for evaluating the magnetic field strength of a general circular cylindrical segment with the end planes of arbitrary slope angles have been derived. The presented formulas can be used to compute the magnetic field strength of coils of arbitrary shape with circular cross-section conductors, that is, they are not limited to helical coils.

4. The major singularity problem has been overcome by appropriate algebraic manipulations.

5. Very simple and concise analytical expressions have been developed for evaluating the magnetic field strength of a circular cylindrical bar (which has nonoblique top and bottom surfaces) in terms of the elliptic integral of the first kind and Heuman's Lambda function.

6. The analytical equations for the circular cylindrical bar have been further simplified by using the definition of the general complete elliptic integral given by Bulirsch [9] and its addition theorem. The simplified expres-
sion needs to compute only one general complete elliptic integral, while in the traditional method three elliptic integrals need to be evaluated for Heuman's Lambda function. The computation efficiency is thus significantly enhanced.

7. Two methods have been proposed for determining the current distribution in multiple coil layers connected in parallel. One is the field differential method which utilizes the accurate method (called the Volume Current Method) developed in this dissertation for evaluating the magnetic field strength inside the coil conductors. The other one is the approximate impedance method which is based on calculating the impedance matrix of coil layers to determine the current distribution.

8. In the approximative impedance method, analytical equations have been derived for evaluating internal impedances of coil layers. They take the skin effect, proximity effect and the curvature of windings into consideration.

9. A simplified model of three flexible equivalent current loops for large air-core reactor coils has been developed. It is useful for the fast evaluation of the three-dimensional power frequency magnetic field distribution of air-core reactor coils.

This research will help not only the management of power system magnetic field distributions, but will also assist power system coil manufacturers in designing reactor coils. In addition, the accuracy of some existing reactor coil models can be evaluated by comparing their results with those of the proposed accurate calculation method.
1.5 Outline of the Dissertation

Chapter 2 of this dissertation gives a comprehensive review of related literature. Section 2.2 summarizes the previous work published in the open literature in developing methods for calculating the magnetic field distribution of air-core coils, which include analytical, semi-analytical and numerical methods. The calculation equations of magnetic field strength of some basic modeling elements are presented there as well. Section 2.3 outlines the previous studies in determining the current distribution in coil layers.

In Chapter 3, an accurate modeling approach, i.e., the Volume Current Method (VCM), as well as formulas for computing the magnetic field strength of a helical coil made of circular cross-section conductors are developed and described. In the VCM, a coil is modeled by an appropriate number of straight conductor segments. Section 3.2 derives the equations for evaluating the magnetic field strength due to a general circular cylindrical conductor segment of finite length. Section 3.3 develops concise and simple equations used for computing the magnetic field strength due to a circular cylindrical bar of finite length, which is a special case of a general circular cylindrical segment with non-oblique end planes. Section 3.4 describes the methods for dividing a helical coil into a number of straight conductor segments and discusses three possible types of straight modeling segments to replace the original arc conductor segment. The transformation of coordinates and vector components between the global coordinate system of the coil and the local coordinate system of the conductor segment are described as well. Section 3.5 presents and discusses the calculation results obtained by applying the developed VCM to several simple coils.

In Chapter 4, two methods for determining the current distribution among the layers of a coil are proposed. Those two methods, that is, the field differential
method and approximate impedance method, are described in Sections 4.2 and 4.3, respectively. The calculation results of some simple coils are presented and discussed in Sections 4.4 and 4.5.

In Chapter 5, a simplified magnetic field model for large air-core reactor coils is proposed. It uses three flexible equivalent current loops with their numbers of turns and positions flexible. The three flexible loop model is described in Section 5.2. Its calculation accuracy of the three dimensional power frequency magnetic flux density is analyzed in Section 5.3. The accuracy of the use of square loops to implement round loops is investigated in Section 5.4.

Chapter 6 deals with the conclusions of this study. Work to be done in the future is also discussed.
CHAPTER 2
LITERATURE REVIEW

2.1 Introduction

The methods for calculating the three-dimensional (3D) magnetic field distribution in the vicinity of an air-core coil have been widely studied for more than a half century. Many papers, presenting and discussing a variety of calculation approaches, have been published.

Section 2.2 gives an overall review of the calculation methods for evaluating the magnetic field strength or magnetic flux density with known current distribution in coils. Generally, those methods can be classified into three types: analytical methods, semi-analytical methods and numerical methods, which are reviewed in Sections 2.2.1, 2.2.2 and 2.2.3, respectively. The magnetic field strength calculation formulas of some basic modeling elements are presented in 2.2.4. In Section 2.3, the methods for studying the current distribution among coil layers are discussed.

2.2 Coil Magnetic Field Distribution Calculation Methods

2.2.1 Analytical Methods

Analytical methods usually are the obvious choice for the problems which can fit coordinate systems, that is, which have a high degree of symmetry. They exploit the property of symmetry such that a compact and general solution can be given.

The analytical methods developed can be further classified into two types. One is based on approximate models, such as ideal conductors and thin sheets, etc.; the
other type is the precise calculation which takes the exact geometry of the coil and the volume current distribution into consideration.

a. Approximate Calculations

Due to the complex geometries of the coil systems, certain approximations have been adopted in many previous studies, such as,

- **filament approximation**, which neglects the wire cross-section of the coil and assumes that the current is concentrated along the conductor center line;

- **thin sheet approximation**, which assumes that the thickness of each layer of the coil is infinitesimally thin, the pitch of the coil is negligible and the coil is axisymmetric and made of thin sheet conductors;

- **solenoid approximation**, which not only ignores the small pitch distance between the turns of coils and the actual geometry of the cross-section of the conductor, but also the space between layers, and assumes the coil is axisymmetric and the current is uniformly distributed in the coil;

- **equivalent current loop approximation**, which uses the equivalent ideal current loops (referred to as the magnetic dipole model in some books and papers) to simulate the magnetic field distribution of current-carrying systems with complex geometries.

Perkins and Brown [10] derived a formula based on the Biot-Savart law for calculating the magnetic flux density of a helical coil of infinitesimal cross-section. The integrals involved were solved by the Weddle method of integration. Katyl [11] looked at a coil as an array of identical wire cells of negligible cross-section. Each cell is made of an arbitrary number of wire segments carrying the same current. The
fast Fourier transform was used to estimate the magnetic field strength of the coil. Bayazitoglu and Sathuvalli [12] obtained analytical equations for the magnetic field gradient of a conical helix with the filament assumption.

Fawzi et al. [13] presented an approach for computing the forces between two coaxial infinitesimally thin circular coils. The method is based on the application of Bartky's transformation to the basic force expression.

Brown and Flax [14] proposed a method for calculating the magnetic field strength of thick solenoids by the superposition of magnetic fields of four semi-infinite solenoids with zero inner radius.

Garrett [15, 16] developed two sets of formulas for evaluating the magnetic flux densities, forces and mutual inductances of various current systems. These current systems include ideal circular loops, cylindrical or annular current sheets and thick-walled axially symmetric coils whose current density is uniform or a function of the cylindrical radius only. One formula is based on zonal harmonic series and another one is based on elliptic integrals. The formulas for thick solenoids are derived from a cylindrical current sheet by Gaussian numerical quadrature. Rusinov [17] systematically outlined and discussed these two methods developed by Garrett for calculating the magnetic flux density of solenoids. It was pointed out by him that the zonal harmonics method is more efficient than the elliptical integral method and is particularly suitable for calculating the magnetic flux density inside the winding. The second method with elliptical integral expressions has advantages in solving the field problems where the winding consists of many radial sections carrying different currents. It is more efficient for calculating mutual inductances and forces in coaxial solenoid systems.

Shaked et al. [18] studied the current and magnetic field distributions in the frequency range of DC to 1kHz for a pancake coil with rectangular cross-section,
applying the solenoid assumption. The effect of the insulation thickness was taken into consideration by a filling factor.

In [19], a coil was simulated by a filament current loop stack model. In the filament current loop stack model, a coil is modeled by a stack of series coaxial current-carrying ideal loops by ignoring the pitch angle of the winding. The number of current loops corresponds to the number of turns. The current loops have the same cross-sectional area as that of the conductor forming the coil. They are placed apart at a distance which equals the pitch of the coil. The magnetic flux density of a coil is calculated by the integration of the magnetic flux density of each current-carrying loop. The magnetic flux density induced from an air-core reactor coil has been calculated as a function of the field point location, axial length, diameter and value of ampere-turns of the reactor, and the height of the reactor coil above the ground. The effect of the non-perfectly conducting earth plane was also included by using the images of the current loops.

A computer program called COILB has been developed at The Ohio State University which is based on the filament current loop stack model [20]. This program can calculate the magnetic flux density of a single-phase reactor coil, a split reactor coil, three-phase stacked reactor coils and three-phase side-by-side reactor coils, where each coil can have its arbitrary winding direction. In [20], the accuracy of the program COILB was also studied by comparing its results along different traverses with those of a proprietary computer program of a major coil manufacturer for a single coil, then two stacked coils, and a set of three-phase stacked coils. The percentage discrepancies are less than 5% when the field points are kept a certain distance away, such as, 0.5D_m from the outer surface of the reactor coil and the discrepancies are less when the field point is further away from the coil.
In *SUBCALC*¹, a coil is modeled by a single coaxial current loop placed at the center of the coil [20]. However, the single current loop model is not sufficient for modeling large air-core reactor coils. It can only successfully model "medium length" coils. That is, when the length-to-diameter ratio of the coil is around 0.8 ~ 1.2 and the field point is at a distance at least about twice the mean radius of the coil away from the outside surface of the coil.

The principles for determining the equivalent current loop model of some magnetostatic current carrying systems were presented in [21]. At a relatively far distance, the calculation of the 3D magnetic field strength of a practical current-carrying system with complex geometry is thus simplified considerably.

b. Precise Methods

The above approximate methods are justified only for field points that are outside of the winding and are not very close to the coil due to the existence of the singularity problem with the filament assumption and/or due to ignoring the pitch angle of the coil, the volume current distribution in the coil and the actual shape of conductors. However, the exact analytical solutions are only available for extremely simple geometries, e.g., a segment or part of practical coils. For the practical problems of complex geometries, usually semi-analytical methods are used.

The previous research work in developing the analytical formulas of some modeling elements of coils is reviewed and discussed in details in the following.

2.2.2 Semi-analytical Methods

For a given current distribution in a homogeneous medium, the magnetic field strength can always be calculated by the Biot-Savart formula [22]. However, the

¹A computer program developed by Enertech Consultants, Inc. for calculating the magnetic field distribution in power substations.
precise evaluation of the Biot-Savart formula by the numerical integration is very time
consuming due to the volume integrals involved. Concise analytical expressions exist
with single integrals or closed forms only for some simple conductor arrangement.
Hence, the magnetic field analysis of an electrical device is generally performed by
decomposing the current carrying conductors into many elements of suitable shape.
The analytical expressions are developed for evaluating the magnetic field strength
or magnetic flux density produced by each element. The total magnetic field strength
is obtained by summing over partial field components generated by each element.

Halacsy and Dunks [23] divided a helical coil into finite straight segments and
employed Biot-Savart's expression for a straight line of infinitesimal cross-section.
The total magnetic field strength equals the vector summation of the magnetic field
strengths from each segments.

Azzerboni et al. [24] presented analytical expressions for estimating the magnetic
flux density due to a conductor of annular arc shape with rectangular cross-section.
In another publication, Azzerboni et al. [25] derived an equation for computing the
magnetic flux density of a current slab with either constant or linear current distribu-
tion functions.

In the analytical expressions derived by the integral method for calculating the
magnetic field strength of straight conductors and circular arc conductors with rect-
angular cross-section, some singularities exist when the field points are at some spe-
cial places in the windings of the coil. Feng [26] developed a method for eliminating
those singularities.

An analytical method for calculating the magnetic vector potential due to tetra-
hedral conductors carrying a uniform current density was also presented [27]. The
developed expressions can be used in conjunction with finite element analysis.

Weggel and Schwartz [28] deduced a very simple analytical formula for the mag-
netic flux density of a rectangular-cross-section prism with arbitrary oblique end planes, carrying uniform current. They claimed that the published formulas possess an advantage over existing integral formulas in avoiding a singularity problem.

Urankar has derived the analytical expressions of the vector potential and magnetic field strength of a finite arc segment carrying a constant azimuthal current for a filament conductor [29], a thin sheet conductor [30], and the conductors with rectangular cross-section [31] and n-sided polygonal cross-section [32], respectively. Corresponding expressions for a straight conductor segment were also given which is a special case of an arc segment. In [33], the results of [30] and [31] were extended to include the case with constant three-dimensional current density vector.

Conventionally, Jacobian elliptical integrals involved in above expressions were evaluated individually, which are of the same modulus but different angles. The angle depends on the arc length and the position of the field point. So, the same algorithms are repeated a number of times. In [34] and [35], a new compact algorithm was developed for elliptic integrals of three kinds with an extended integration range. The algorithm was based on the modified Landen and Bartky's transformations. All the integrals thus can be computed quickly. It was claimed that their calculation time was reduced by a factor of 2.5 to 3.

Ciric [36] proposed a surface source modeling technique on the basis of the Amperian model for magnetized media which replaces a given volume distribution of static or quasi-static electric currents by equivalent distributions of fictitious magnetic charges, surface electric currents and magnetization. Therefore, the magnetic field strength due to a given volume current can be determined by calculating the magnetic field strength from the corresponding equivalent magnetic charges and surface electric currents. This modeling approach showed a distinct advantage in its flexibility in dealing with the problems of volume current distribution. For certain
geometries, it substantially simplifies the magnetic field strength calculation with a volume current source and greatly reduces the amount of computation time. Based on this modeling technique, Ciric [37] presented an analysis for the straight segments of rectangular cross-section. A surface source model of a current-carrying straight conductor segment of arbitrary polygonal cross-section was also developed and the analytical formulas for calculating its magnetic field strength were derived accordingly [38, 39].

Nevertheless, it can be noticed from the above discussions that precise calculation formulas for a straight conductor segment of circular cross-section carrying a quasi-static volume current have not been developed yet. Even though the formulas obtained by Urankar [32] and Ciric [39] for straight conductor segments of arbitrary polygonal cross-section can be used to approximate circular cylindrical conductor segments, many polygonal side segments might need to be used for an accurate calculation. Therefore, using the formulas for polygonal cross-section to approximately compute the magnetic field strength of a straight conductor segment of circular cross-section is complicated and time-consuming.

In [40], an algorithm for evaluating the Biot-Savart integration in the case of bars and arcs was introduced. The proposed algorithm is based on a mixed analytical-numerical evaluation strategy. It subdivided the bar cross-section into many elements and then summed the field contribution of the elements. The contribution of each element was approximately calculated by a current filament formula. The author also simplified Urankar's expressions in [30] for stranded arcs and developed calculation formulas for solid arcs.

Diserens [41] compared the computation time of three algorithms for a finite arc conductor with rectangular cross-section which use different basic elements for integration, that is, the thin filament formula, the thin cylinder formula and the
infinitesimal wedge formula. It was concluded that the infinitesimal wedge algorithm was the fastest one and the filament algorithm is the slowest one due to the fact that it needs two numerical integrations in the radial and axial directions, respectively.

2.2.3 Numerical Methods

The finite element method (FEM), the boundary element method (BEM) and the moment method (MM) are three popular and powerful numerical methods which have been widely used for electromagnetic analysis. However, all of the above methods have certain limitations in modeling helical coils. Lavers [42] gave a review of the computational methods that are typically applied to model and solve electromagnetic problems in power engineering. A comprehensive list of references was also provided.

FEM uses the differential forms of the governing equations and needs to approximate the entire space by using surface elements in two-dimensional (2D) models and volume elements in 3D models. This results in a large sparse system matrix.

Currently, FEM can only be able to model a practical large helical coil as a stack of current loops with volume current successfully, i.e., as a 2D axisymmetric problem. Even through theoretically FEM can simulate any helical winding in a 3D model, the capability of computers limits its application for many practical problems due to the extremely large amount of mesh elements generated. In addition, the difficulties in solving 3D problems that FEM encounters also come from the vector nature of the field itself. There are at least three unknowns at each node. The system matrix in turn is increased dramatically compared with a 2D problem for the same node number. Currently, FEM can only successfully and precisely model a coil of either very simple geometry or a coil with certain approximation. Further research is needed for modeling and solving complex 3D problems involving exterior infinite free space
Logeais [43] compared the accuracy of 2D and 3D FEMs. It was pointed out that the 2D model is less accurate in terms of theoretical modeling, however, in spite of that, its numerical accuracy is better. The 3D model has a small modeling error, but a large numerical error can be the result of the computation capability of the computers available.

BEM is usually appropriate for coils having core materials. It enforces the boundary conditions at the boundary surface of core material. The BEM system matrices are usually full and non-symmetric. The field components due to the coil conductor currents have to be found by other methods, such as, the Biot-Savart’s law [44] and the Legendre polynomial method [45].

MM is a traditional way to obtain the accurate solution of boundary value problems with integral description. MM employs the expansion functions and weighting functions with appropriate boundary conditions to obtain a set of linear equations. The computation usually involves a fully populated system matrix [46]. Hence, it is difficult to solve a problem precisely with complex geometric shape by using MM.

In [47], Brauer proposed an asymptotic boundary condition (ABC) to simulate the open boundaries of 3D magnetostatic finite element models. The matrix sparsity could be reserved. The developed ABC method is applied to a simple air-core circular coil made of circular conductors and a simple air-core square coil made of rectangular cross-section, respectively, to calculate their magnetic flux density and inductances.

Ishihara [48] presented an approximate 3D FEM that combined axisymmetric fields with 2D fields to analyze magnetic fields of a three-phase delta-type shunt reactor with silicon steel-core.

In [49], Kanai proposed a fast and stable converging method which used FEM to solve the magnetic field of an iron-core toroidal coil. The effects of the gradual
saturation characteristics of the core were also considered.

In [44], a BEM for 3D magnetic field analysis was developed. In this method, a reduced scalar potential is selected as an unknown variable to simplify the calculation procedures of boundary conditions. The magnetic field intensity due to the current density is obtained by the Biot-Savart’s law. The method is applied to an iron-core coil.

Zisserman [45] developed a Legendre polynomial BEM for calculating the magnetic field strength of a coil with iron core. The core was assumed to have constant permeability. There, the scalar potential due to the current-carrying conductors was calculated by using the Legendre polynomial method, that is, the method of Garret [15, 16]. The coefficients in the polynomial expansion only depend on the conductor geometry and current density. The author expanded the potential form arising from the iron core in terms of the Legendre polynomials also. So the effects of iron can be included in the Legendre coefficients.

Krstajic [50] proposed an extended BEM which includes surface elements and volume elements to calculate 3D non-linear magnetostatic field strength of a non-air-core coil.

Lipinski [51] applied the integral transform method to calculate the magnetic field produced by a spherical coil.

Lean and Wexler [52] used BEM with parametric representation of surfaces and sources to solve some 2D problems. An approach was introduced to solve the singularities of the Green functions over arbitrarily-shaped geometries. In [53, 54], BEM is also employed to calculate the magnetostatic field strength of coils.
2.2.4 Magnetic Field Strength Expressions of Simple Modeling Elements

This subsection outlines the formulas for calculating the magnetic field strength from several simple current-carrying elements.

a. Finite straight filament

A finite straight filament segment has \((x_s, y_s, z_s)\) and \((x_e, y_e, z_e)\) as the rectangular coordinates of its two ends. The line carries a static current \(I\) which flows from \((x_s, y_s, z_s)\) to \((x_e, y_e, z_e)\). At the field point \((x_c, y_c, z_c)\), the magnetic field strength, given by [10], is

\[
H_{zc} = \frac{I}{4\pi} \left\{ \left( y_c - y_s + x_s \frac{y_e - y_s}{x_e - x_s} \right) \left[ \frac{4a_x x_e + 2b_x}{(4a_x c_x - b_x^2)\rho_2} - \frac{4a_x x_s + 2b_x}{(4a_x c_x - b_x^2)\rho_1} \right] \\
+ \frac{ye - ys}{xe - xs} \left[ \frac{2b_x xe + 4cx}{(4a_x cx - b_x^2)\rho_2} - \frac{2b_x x_e + 4cx}{(4a_x cx - b_x^2)\rho_1} \right] \\
- \left( x_c - x_s + y_s \frac{x_e - x_s}{ye - ys} \right) \left[ \frac{4a_y ye + 2by}{(4a_y c_y - b_y^2)\rho_2} - \frac{4a_y y_e + 2by}{(4a_y c_y - b_y^2)\rho_1} \right] \\
- \frac{xe - xs}{ye - ys} \left[ \frac{2by ye + 4cy}{(4a_y cy - b_y^2)\rho_2} - \frac{2by y_s + 4cy}{(4a_y cy - b_y^2)\rho_1} \right] \right\},
\]  

(2.1)

where

\[
a_x = 1 + \left( \frac{ye - ys}{xe - xs} \right)^2 + \left( \frac{ze - zs}{xe - xs} \right)^2, \\
b_x = 2 \left[ --xc - yc \frac{ye - ys}{xe - xs} - xc \frac{ze - zs}{xe - xs} + ye - ys \left( ys - xs \frac{ye - ys}{xe - xs} \right) \right] \\
+ \frac{ze - zs}{xe - xs} \left( zs - xs \frac{ze - zs}{xe - xs} \right), \\
c_x = xc^2 + ye^2 + ze^2 + \left( ys - xs \frac{ye - ys}{xe - xs} \right)^2 + \left( zs - xs \frac{ze - zs}{xe - xs} \right)^2 \\
- 2ye \left( ye - ys \frac{ye - ys}{xe - xs} \right) - 2ze \left( ze - zs \frac{ze - zs}{xe - xs} \right), \\
\rho_1 = \left[ (xc - xs)^2 + (yc - ys)^2 + (zc - zs)^2 \right]^{1/2}, \\
\rho_2 = \left[ (xc - xe)^2 + (yc - ye)^2 + (zc - ze)^2 \right]^{1/2}.
\]
\( a_y, b_y, c_y, a_z, b_z, c_z, H_{x_c} \) and \( H_{y_c} \) can be obtained by cyclic permutation of \( x, y, z \) in the above equations. A good feature of the above equation is that it does not need to transfer coordinates and vector components between the local coordinate system of the conductor segment and the global coordinate system of the coil.

b. Circular filament loop

\[
\begin{align*}
H_{\rho_c} &= -\frac{I}{2\pi \rho_c} \frac{z_c}{[(\rho_c + R_m)^2 + z_c^2]^{1/2}} \left\{ K - \left[ \frac{\rho_c^2 + R_m^2 + z_c^2}{(\rho_c - R_m)^2 + z_c^2} \right] E \right\}, \\
H_{\phi_c} &= 0, \\
H_{z_c} &= \frac{I}{2\pi} \frac{1}{[(\rho_c + R_m)^2 + z_c^2]^{1/2}} \left\{ K - \left[ \frac{\rho_c^2 - R_m^2 + z_c^2}{(\rho_c - R_m)^2 + z_c^2} \right] E \right\},
\end{align*}
\]

(2.2)

For a circular filament loop, located in the \( z_c = 0 \) plane, of radius \( R_m \) carrying a static current \( I \), as shown in Figure 2, the magnetic field strength at a field point \((\rho_c, \phi_c, z_c)\) equals [55]:

Figure 2: Circular filament loop.
where $K$ and $E$ are complete elliptic integrals of the first and second kind, respectively, and

$$
\kappa^2 = \frac{4R_m \rho_c}{(\rho_c + R_m)^2 + z_c^2}.
$$

When the field point is on the $z$-axis, that is, $\rho_c = 0$, the above equation is simplified to

$$
\vec{H} = \frac{R_m^2 I}{2(R_m^2 + z_c^2)^{3/2}} \hat{z}_c.
$$

When the field point is very far from the loop, that is, $r_c \gg R_m$, Equation (2.2) reduces to

$$
\vec{H} = \frac{R_m^2 I}{4r_c^3} \left[ 2 \cos \theta_c \hat{r}_c + \sin \theta_c \hat{\theta}_c \right],
$$

where $r_c = \sqrt{\rho_c^2 + z_c^2}$, $\hat{r}_c$ and $\hat{\theta}_c$ are the unit vectors of the spherical coordinate system of the loop, as shown in Figure 2.

c. Conical filament helix

For a conical helix, with pitch $h_p$ and the total number of turns $N$, that carries a static current $I$ as shown in Figure 3, $\alpha_c$ is the semiangle of the cone on which the helix is wound, and $(a_0, 0, 0)$ is the starting point of the helix. At the field point $(x_c, y_c, z_c)$, the magnetic field strength [12] is

$$
\vec{H} = \frac{I}{4\pi} \int_{\theta=0}^{\theta=2N\pi} (\vec{dS} \times \vec{R})/|\vec{R}|^3,
$$

where

$$
(d\vec{S} \times \vec{R})_{x_c} = \left[ a_0(bz_c + c) \sin \theta + a_0z_c \cos \theta + a_0(bz_c - c) \theta \cos \theta \
- a_0 \theta^2 \cos \theta - y_c \right] d\theta,
$$

$$
(d\vec{S} \times \vec{R})_{y_c} = \left[ a_0z_c \sin \theta - a_0(bz_c + c) \cos \theta + a_0(bz_c - c) \theta \cos \theta \
- a_0 \theta^2 \sin \theta + x_c \right] d\theta,
$$

$$
(d\vec{S} \times \vec{R})_{z_c} = \left[ a_0(bx_c + y_c) \sin \theta + a_0(by_c - x_c) \cos \theta - a_0 by_c \theta \sin \theta
$$

25
\[-a_0 b x_c \theta \cos \theta + 2 a_0^2 b \theta + a_0^2 b^2 \theta^2 + a_0^2 \right] d\theta,
\]
\[|\vec{F}| = \left\{ (x_c - a(\theta) \cos \theta)^2 + (y_c - a(\theta) \sin \theta)^2 + (z_c - c \theta)^2 \right\}^{1/2},
\]
\[a(\theta) = a_0 (1 + b \theta),
\]
\[b = h_p \sin \alpha_c / (2a_0 \pi),
\]
\[c = h_p \cos \alpha_c / (2\pi).
\]

For the right circular helix, \(\alpha_c = 0\). Then \(b = 0\), \(a(\theta) = a_0\) and \(c = h_p / (2\pi)\).

\[d. \textbf{Thin sheet cylinder}
\]

In [17], the equations were presented for calculating the magnetic field strength of a circular cylindrical current layer of infinitesimal thickness with radius \(R_m\), height \(H\), and azimuthal current per unit length \(J_a\), which were derived by Garrett. At the
field point \((\rho_c, \varphi_c, z_c)\), the magnetic flux density equals

\[
H_{\rho_c} = -\frac{J_a R_m}{\pi} \left[ f_\rho(\rho_c, z_+) - f_\rho(\rho_c, z_-) \right],
\]

\[
H_{z_c} = \frac{J_a R_m}{\pi} \frac{R_m}{R_m + \rho_c} \left[ f_z(\rho_c, z_+) - f_z(\rho_c, z_-) \right],
\]

where \(z_\pm = z_c \pm H/2\); and

\[
f_\rho(\rho_c, z_c) = \frac{1}{\sqrt{(\rho_c + R_m)^2 + z_c^2}} \left[ \frac{K(\kappa) - E(\kappa)}{\kappa^2} - \frac{1}{2} K(\kappa) \right],
\]

\[
f_z(\rho_c, z_c) = \frac{z_c}{\sqrt{(\rho_c + R_m)^2 + z_c^2}} \left\{ K(\kappa) + \frac{R_m - \rho_c}{2\rho_c} [\Pi(\kappa_0, \kappa) - K(\kappa)] \right\},
\]

where \(\kappa = 2\sqrt{\rho_c R_m}/\sqrt{(\rho_c + R_m)^2 + z_c^2}\); \(\kappa_0 = 2\sqrt{\rho_c R_m}/(\rho_c + R_m)\); \(E, K\) and \(\Pi(\kappa_0, \kappa)\) are the complete elliptic integrals of the first, second and third kind.

e. Semi-infinite solenoid

A semi-infinite long solenoid with zero inner radius, \(R_o\) outer radius and uniform azimuthal current density \(J_a\) is shown in Figure 4, where it extends from \(z_c = 0\) to

Figure 4: Semi-infinite solenoid.
$z_c = \infty$. The magnetic field strength at the field point $(\rho_c, \varphi_c, z_c)$ has the form of [14]

$$H_{\rho_c} = \frac{J_a R_o}{2\pi} \int_0^\pi F \cos \theta \, d\theta + \frac{J_a R_o}{2\pi} \int_0^\pi \rho \cos^2 \theta \sin^{-1} \left[ \frac{1 - \rho \cos \theta}{(\rho^2 + \rho^2 \sin^2 \theta)^{1/2}} \right] \, d\theta,$$

$$H_{\varphi_c} = 0,$$

$$H_{z_c} = \frac{J_a R_o}{4} (1 - \bar{r} + |1 - \bar{r}|) - \frac{\bar{z} J_a R_o}{2\pi} \int_0^\pi \ln \left[ \frac{2(1 - \bar{r} \cos \theta + F)}{|\bar{z}| + (\bar{z}^2 + \bar{r}^2)^{1/2}} \right] \, d\theta$$

$$+ \frac{J_a R_o \bar{z} \bar{r}^2}{2\pi} \int_0^\pi \frac{\sin^2 \theta \, d\theta}{(1 + \bar{r}^2 - 2\bar{r} \cos \theta) F}$$

$$+ \frac{J_a R_o \bar{z} \bar{r}}{2\pi |\bar{z}|} \int_0^\pi \frac{\sin \theta \tan^{-1} \left[ \frac{(1 - \bar{r} \cos \theta)|\bar{z}|}{\bar{r} \sin \theta F} \right] \, d\theta},$$

where $\bar{r} = \rho_c/R_o$, $\bar{z} = z_c/R_o$ and $F = (\bar{z}^2 + \bar{r}^2 + 1 - 2\bar{r} \cos \theta)^{1/2}$.

### 2.3 Coil Layer Current Distribution Calculation Methods

The coils which are being studied consist of parallel connected layers. The skin effect, proximity effect and mutual induction among the layers of a coil result in unequal current distribution among layers, even though the technique of dropping one or more turns at the ends of the outer layers of a coil combined with the application of multipled spoked spiders is employed to allow a more uniform current distribution among layers.

The current distribution among coil layers which are parallelly connected can be determined if the impedance of each layer can be estimated. The skin effect, proximity effect and mutual induction of the coil layers affect the impedance values of layers. Skin effect exists whenever conductors carry alternating currents. The current in an isolated conductor tends to flow near the surface of the conductor, where the magnetic field strength is the strongest. When more than two conductors are in relatively close proximity, those conductors are electromagnetically coupled together. The alternating current in one conductor may establish a field in another
conductor and in turn induces currents in the second conductor. The apparent resistance of that conductor is increased accordingly. This phenomenon is called proximity effect.

A classical one-dimensional Helmholtz equation was employed to calculate the approximate power losses in each layer of a coil, where the end effects and the curvature of the coil were neglected [56].

Shaked [18] considered the non-uniform current distribution in the radial direction in the conductors of a pancake coil. The analytical expression for the current distribution in the coil was derived based on a 2D model. It was assumed that the coil studied was an infinitely long hollow cylinder of uniform metal with a radial thickness equal to the difference of the outside and inside radii of the real coil.

El-Markabi and Freeman [57] proposed a method for calculating the impedance of a cylindrical coil of finite length based on the transmission line modeling technique. The model was based on the assumption that the coil is very thin in the radial direction.

Lotfi et al. [58] developed an equation for evaluating the AC resistance of coil windings, which took both the skin effect and proximity effect into account. They concluded that the proximity effect is negligible only if the inter-conductor spacing is greater than twice the diameter of the conductor.

Ryff [59] presented an approach for predicting the current distribution and power losses in the individual turns of helical solenoids of non-magnetic materials. A circuit representation of a coil was established, where each turn was divided into a large number of circular subconductors. A system matrix equation was derived with the assumption that the current density in each subconductor was constant. The voltage drops, current distributions and power losses were calculated for a 5-turn and 14-turn solenoids, respectively. Good agreement between calculated results and experimental
results was observed.

Lammeraner and Stafl [60] derived analytical expressions for calculating the current density and the power losses in the coil winding where the both the axial and radial field components were considered. In the derivation, it was assumed that the axial magnetic field strength decreased linearly from the inner surface of the winding to zero in the outer surface, that is, the zero space between layers was assumed.

Four approximate analytical methods were studied and discussed in [61] for determining the current density and power loss distributions in multilayer sheet windings. The approximate methods included one-, two-, and three-dimensional approximate solutions, where the end effects in the sheet windings were completely ignored in all methods. The assumptions used in each method were described. The limitation and application range of each method was discussed.

Burke and Fawzi [62] proposed a proximity impedance matrix to account for the proximity effects on the eddy current losses of air-core reactor coils and on the current distribution among the parallel layers of the reactor coil in the lumped circuit model of the reactor coils. The proximity impedance matrix was added to the original impedance matrix where only the skin effect was considered. The curvature of the winding turns was ignored in the derivation.

Nakata et al. [63] developed an axisymmetric finite element model for analyzing induced currents in multiple parallely connected conductors. The induced currents include both eddy and circulating currents. In the axisymmetric finite element model, a helical conductor was modeled by a number of ring conductors, where the number of the ring conductors is equal to the number of turns. The calculation accuracy of the method was verified by measured results.

Kofier and Reisinger [64] presented two numerical methods to calculate the
inductances of the air gap windings of electric machines. One uses the magnetic energy and the other one utilizes the flux linkages of the windings. In both methods, the magnetic flux density was calculated by the Biot-Savart's law with filament assumption.

Nasar et al. [65] determined eddy current losses in a tubular linear induction machine, where the end effects on the winding impedance are considered by assuming the "ladder" distribution of the axial flux.

2.4 Conclusions

A comprehensive review of the methods has been presented; these methods were used for calculating the magnetic field strength from an air-core coil and for determining the current distribution among coil layers. It has been found that the accurate formulas for computing the magnetic field distribution of large air-core reactor coils with circular cross-section conductors, at field points either inside or outside the reactor coil, are not available yet in the open literature. At the present time, exact magnetic field evaluation expressions exist only for air-core coils made of rectangular or polygonal cross-section conductors. The formulas based on filament and thin sheet assumptions may lead to significant calculation errors at field points inside the reactor coils and in the very close vicinity of the reactor coils. Using many filaments to simulate the effects of volume current distribution is time-consuming. In addition, there is only a limited number of papers which discuss the current distribution among coil layers. This suggests that the appropriate studies on the precise calculation of the magnetic field strength of large multiple layer air-core reactor coils made of circular cross-section conductors might have not been conducted yet.

Hence, it is necessary to solve the following problems in this study:
• develop a method which can accurately evaluate the magnetic field distribution inside and outside an air-core reactor coil made of circular cross-section conductors if the currents flowing in the coil layers are known;

• develop an approach which can determine the current distribution in an air-core reactor coil with multiple layers connected in parallel;

• develop a simplified modeling method which can quickly and accurately evaluate the magnetic field distribution at field points which are outside an air-core reactor coil and are at least a small distance away from the reactor coil.
CHAPTER 3

MAGNETIC FIELD DISTRIBUTION OF HELICAL COILS IN CASE OF ROUND CONDUCTORS

3.1 Introduction

In this chapter an accurate modeling method for large air-core reactor coils, called Volume Current Method (VCM), is proposed. In VCM, the coil is modeled by an appropriate number of finite circular cylindrical conductor segments with arbitrary oblique end planes as shown in Figure 5, due to the complex geometry of the coil. The total magnetic field strength equals the vectorial summation of the magnetic field strength of each conductor segment that a helical coil is modeled of.

Figure 5: The geometry and local coordinate system of a finite straight segment of a circular cross-section conductor.
The accurate expressions for evaluating the magnetic field strength of a finite circular cylindrical conductor segment with either oblique or non-oblique end planes, carrying a uniform static current, are derived and presented in Sections 3.2 and 3.3. The approaches for dividing a coil into a required number of conductor segments and obtaining the geometric parameters of segments and the local coordinates of field points are discussed in Section 3.4. The proposed VCM is based on the assumption that the currents flowing in the conductors of the coil are known. The algorithms for determining the currents flowing in the conductors are described in Chapter 4. The calculation results for several simple coils are presented, analyzed and discussed in Sections 3.5 and 3.6.

3.2 Magnetic Field Strength of a General Circular Cylindrical Conductor of Finite Length

In the derivations of the equations for evaluating the magnetic field strength due to a general circular straight conductor segment of finite length, a surface source model is developed which enables the given volume current distribution to be replaced by equivalent surface sources. A volume integral problem thus can be simplified to have only single integral expressions.

3.2.1 Surface Source Modeling

For a magnetic material with a given volume distribution of magnetic moment density \( \vec{M} \) (or magnetization vector), the magnetic flux density produced by \( \vec{M} \) satisfies the equation [66]:

\[
\vec{B}_M = \mu_0 (\vec{H}_M + \vec{M}),
\]

(3.1)

where \( \vec{H}_M \) is the magnetic field strength produced by \( \vec{M} \) and \( \mu_0 \) is the magnetic permeability of free space. The field lines of \( \vec{H}_M \) can be substituted by that of the equivalent magnetic volume and surface charge distributions, \( \rho_v \) and \( \rho_{sc} \), that is,
\( \vec{H}_M = \vec{H}_{\rho v} + \vec{H}_{\rho sc} \). By using this substitution, Equation (3.1) becomes

\[
\vec{B}_M = \mu_0(\vec{H}_{\rho v} + \vec{H}_{\rho sc} + \vec{M}).
\]  

(3.2)

At the same time, the magnetic flux density produced by \( \vec{M} \) (\( \vec{B}_M \)) can be replaced by that of the equivalent volume and surface current distributions, \( \vec{J} \) and \( \vec{J}_{sc} \), in free space, that is,

\[
\vec{B}_M = \vec{B}_J - \vec{B}_{J_{sc}}.
\]  

(3.3)

Therefore, for a non-magnetic body with a piecewise continuous volume distribution of stationary or quasi-stationary current density \( \vec{J} \), the effect of \( \vec{J} \), in terms of magnetic field strength, can be replaced by an equivalent piecewise continuous magnetic moment density \( \vec{M} \), the distribution of equivalent magnetic volume charge density \( \rho_v \), magnetic surface charge density \( \rho_{sc} \) and equivalent electric surface current density \( \vec{J}_{sc} \) [36]. By using Equations (3.2) and (3.3), this substitution can be illustrated by the following equation:

\[
\vec{H}_J = \mu_0^{-1}(\vec{B}_M + \vec{B}_{J_{sc}})
\]

\[
= \vec{M} + \vec{H}_{\rho v} + \vec{H}_{\rho sc} + \vec{H}_{J_{sc}},
\]  

(3.4)

where

\[
\vec{J} = \text{curl} \, \vec{M},
\]

\[
\vec{J}_{sc} = -\hat{n}_{ij} \times ((\vec{M}_j - \vec{M}_i)),
\]  

(3.5)

\[
\rho_v = -\text{div} (\mu \vec{M}),
\]

\[
\rho_{sc} = -\hat{n}_{ij} \cdot (\mu_j \vec{M}_j - \mu_i \vec{M}_i);
\]

\( \hat{n}_{ij} \) is the unit vector normal to a surface and is oriented from side \( i \) to side \( j \) of the surface; \( \mu_i \) and \( \mu_j \) are the permeabilities in regions \( i \) and \( j \), respectively; and \( \mu \) is the
magnetic permeability of the material to be substituted. Such equivalence is valid for both inside and outside the material from the macroscopical point of view, such as magnetic field strength components, forces and torques. However, it is only valid for linear and isotropic materials.

\( \vec{M} \) can be arbitrarily chosen as long as it satisfies \( \vec{J} = \text{curl} \vec{M} \) and is piecewise continuous. This implies that for a given volume current distribution, more than one model can be constructed. The appropriate choice of \( \vec{M} \), however, is important in simplifying the calculation. If \( \vec{M} \) is chosen to have zero divergence everywhere, then the equivalent volume charge density \( \rho_v \) becomes zero. Only surface sources exist then. The volume integration is thus avoided.

![Figure 6: The front view of the finite straight conductor segment.](image)

Referring to Figure 5, \( XYZ \) and \( \rho\varphi Z \) are the local rectangular and cylindrical coordinate systems of the conductor segment, respectively. The origin is at \( O; (\rho, \varphi, \)
and \((\rho', \varphi', z')\) are the cylindrical coordinates of the field point and the source point in the \(\rho \varphi Z\) coordinate system, respectively. Furthermore, \((x, y, z)\) and \((x', y', z')\) are the rectangular coordinates of the field and source points in the \(XYZ\) system, respectively. \(R_s\) is the radius of the cylindrical conductor segment. The front view of the segment is shown in Figure 6. In Figure 6, \(h_t\) and \(h_b\) are the distances between the origin \(O\) and the centers of the top and bottom end planes of the cylindrical conductor segment, respectively; \(r_t\) and \(r_b\) are the lengths of semi-major axes of the top and bottom end planes, respectively; and \(\theta_t\) and \(\theta_b\) are the slope angles of the top and bottom end planes, respectively. Referring again to Figure 5, \(\hat{n}_w, \hat{n}_t\) and \(\hat{n}_b\) are the unit vectors normal to the side wall, top and bottom surfaces, respectively, and all pointing outwards.

It is assumed in the following discussion that

- \(\vec{J}\) is uniformly distributed in the conductor. This assumption is justified by the fact that the diameter of the conductor is smaller than or comparable to the skin depth of the conductor at 60 Hz and is much smaller than the total thickness of the coil in most of the practical cases (refer to Table 1 [67]).

- The conductor segment is symmetrical about the \(YZ\) plane (see Figure 5).

Let \(\vec{M}\) is chosen as

\[
\vec{M} = \begin{cases} 
\varrho' \frac{1}{2} J \rho' & \text{inside the conductor segment,} \\
0 & \text{outside the conductor segment.}
\end{cases}
\]  

(3.6)

It can be shown that \(\vec{M}\) satisfies Equation (3.5). From Figures 5 and 6, it can be seen that the normal unit vectors of the cylindrical side-wall surface, the top and the bottom end surfaces are given by

\[
\hat{n}_w = \hat{\rho'} ,
\]
<table>
<thead>
<tr>
<th>Conductor</th>
<th>Conductivity (MS/m)</th>
<th>$\delta$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>35.4</td>
<td>0.011</td>
</tr>
<tr>
<td>Brass</td>
<td>15.9</td>
<td>0.0163</td>
</tr>
<tr>
<td>Chromium</td>
<td>38.0</td>
<td>0.010</td>
</tr>
<tr>
<td>Copper</td>
<td>58.0</td>
<td>0.0085</td>
</tr>
<tr>
<td>Gold</td>
<td>45.0</td>
<td>0.0097</td>
</tr>
<tr>
<td>Silver</td>
<td>61.5</td>
<td>0.0083</td>
</tr>
<tr>
<td>Zinc</td>
<td>18.6</td>
<td>0.0151</td>
</tr>
</tbody>
</table>

Table 1: Typical skin depths $\delta$ of conductors made of non-magnetic materials at 60 Hz.

\[
\hat{n}_t = -\sin \theta_t \hat{y} + \cos \theta_t \hat{z}, 
\]

\[
\hat{n}_b = -\sin \theta_b \hat{y} - \cos \theta_b \hat{z}. 
\]

By substituting Equations (3.6) and (3.7) into Equations (3.5), it can be obtained that

\[
\rho_v = 0, 
\]
\[
\rho^{uv}_{sc} = 0, 
\]
\[
J^{w}_{sc} = \frac{1}{2} J R_{s} \hat{z}, 
\]
\[
\rho^{t}_{sc} = -\frac{1}{2} \mu_0 J \sin \theta_t \hat{x}', 
\]
\[
J^{t}_{sc} = -\frac{1}{2} J (\cos \theta_t \hat{x}' \hat{z} + \cos \theta_t \hat{y}' \hat{y} + \sin \theta_t \hat{y}' \hat{z}), 
\]
\[
\rho^{b}_{sc} = -\frac{1}{2} \mu_0 J \sin \theta_b \hat{x}', 
\]
\[
J^{b}_{sc} = \frac{1}{2} J (\cos \theta_b \hat{x}' \hat{z} + \cos \theta_b \hat{y}' \hat{y} - \sin \theta_b \hat{y}' \hat{z}), 
\]
where $\rho_{sc}^w$ and $\vec{J}_{sc}^w$ are the equivalent magnetic surface charge density and electric surface current density on the cylindrical side-wall surface, $\rho_{sc}^t$ and $\vec{J}_{sc}^t$ are the equivalent magnetic surface charge density and electric surface current density on the top surface, and $\rho_{sc}^b$ and $\vec{J}_{sc}^b$ are the equivalent magnetic surface charge density and electric surface current density on the bottom surface.

The magnetic field strength due to $\vec{J}$ thus equals the vectorial summation of those generated by $\vec{M}$, $\vec{J}_{sc}^{w,t,b}$, and $\rho_{sc}^{w,t,b}$, that is,

$$\vec{H}(\vec{r}) = \vec{M}(\vec{r}) + \vec{H}_{\rho_{sc}^w}(\vec{r}) + \vec{H}_{\rho_{sc}^t}(\vec{r}) + \vec{H}_{J_{sc}^w}(\vec{r}) + \vec{H}_{J_{sc}^t}(\vec{r}) + \vec{H}_{J_{sc}^b}(\vec{r}).$$  (3.9)

In the above equation, most of the magnetic field strength contribution comes from $\vec{J}_{sc}^w$. Since $\vec{M}$ is always perpendicular to $\vec{J}$, or to the longitudinal wall of the conductor, it can be shown that there are no contributions from the end surfaces of an internal segment of a layer if it shares the common end surfaces with its adjacent conductor segments. The magnetic field strength contributions from the end surfaces are canceled out by that of its adjacent conductor segments. It is necessary to include the magnetic field strength contributions from the end planes but only for the first and last segments of a layer. As a result, for the internal segments,

$$\vec{H}(\vec{r}) = \vec{M}(\vec{r}) + \vec{H}_{J_{sc}^w}. \quad (3.10)$$

For the first and last segments,

$$\vec{H}(\vec{r}) = \begin{cases} \vec{M}(\vec{r}) + \vec{H}_{J_{sc}^w} + \vec{H}_{J_{sc}^t} + \vec{H}_{J_{sc}^b} + \vec{H}_{\rho_{sc}^w} + \vec{H}_{\rho_{sc}^t} + \vec{H}_{\rho_{sc}^b}, & \text{if } \theta_{t,b} = 0, \\ \vec{M}(\vec{r}) + \vec{H}_{J_{sc}^w} + \vec{H}_{J_{sc}^t} + \vec{H}_{J_{sc}^b} + \vec{H}_{\rho_{sc}^w} + \vec{H}_{\rho_{sc}^t} + \vec{H}_{\rho_{sc}^b}, & \text{if } \theta_{t,b} \neq 0. \end{cases} \quad (3.11)$$

In practical cases, $\theta_t$ and $\theta_b$ of the first and last segments are equal to zero.

### 3.2.2 Expressions of Magnetic Field Strength of a Conductor Segment

The general expressions for computing $\vec{H}_{J_{sc}^w}$, $\vec{H}_{J_{sc}^t}$, $\vec{H}_{J_{sc}^b}$, $\vec{H}_{\rho_{sc}^w}$, and $\vec{H}_{\rho_{sc}^t}$ in Equation (3.9) are deduced in this section.
The magnetic field strength contributions due to the electric surface current density $j_{\text{sc}}^{w,t,b}$ and magnetic surface charge density $\rho_{\text{sc}}^{t,b}$ can be evaluated by the following surface integrations:

$$\vec{H}_{j_{\text{sc}}^{w,t,b}} = \frac{1}{4\pi} \int \frac{j_{\text{sc}}^{w,t,b}(r') \times \vec{R}}{|\vec{R}|^3} dS', \quad (3.12)$$

$$\vec{H}_{\rho_{\text{sc}}^{t,b}} = \frac{1}{4\pi \mu_0} \int \frac{\rho_{\text{sc}}^{t,b}(r') \vec{R}}{|\vec{R}|^3} dS', \quad (3.13)$$

where $\vec{R} = \vec{r} - \vec{r}'$; $\vec{r}$ is the vector of field point position and $\vec{r}'$ is the vector of source point position; $dS'$ is the differential surface element and the surface integrations are conducted over the surfaces where the discontinuity of $\vec{M}$ exists.

a. Expression of $\vec{H}_{j_{\text{sc}}^{w}}$

![Figure 7: The vector $\vec{R}$.](image)

In the cylindrical coordinate system, it can be shown that the vector $\vec{R}$ which points from the source point to the field point, as shown in Figure 7, has the form of

$$\vec{R} = \vec{r} - \vec{r}',$$

$$= (\rho - \rho \cos\phi) \hat{\rho} - \rho \sin\phi \hat{\phi} - \gamma z \hat{z}, \quad (3.14)$$

where

$$\phi = \phi' - \phi,$$
\[ \gamma_z = z' - z. \]

The magnitude of \( \tilde{R} \) has the form of

\[
| \tilde{R} | = \sqrt{\rho^2 - 2\rho \rho' \cos \phi + \rho'^2 + \gamma_z^2}
= \sqrt{B^2(\phi) + \gamma_z^2}, \tag{3.15}
\]

where \( \rho' = R_s \), and

\[ B^2(\phi) = \rho^2 - 2R_s\rho \cos \phi + R_s^2. \]

Combining \( J_{sc}^w \) given in (3.8) with \( \tilde{R} \) given in (3.14) yields

\[ J_{sc}^w \times \tilde{R} = \frac{JR_s}{2} \left[ \rho' \sin \phi \hat{p} + (\rho - \rho' \cos \phi) \hat{\phi} \right]. \tag{3.16} \]

Inserting Equation (3.16) into Equation (3.12) with \( dS' = R_s dz' d\phi' \) gives

\[
\tilde{H}_{J_{sc}^w} = \frac{JR_s^2}{8\pi} \int_{-\pi}^{2\pi} \int_{z_1}^{z_2} \frac{R_s \sin \phi \hat{p} + (\rho - R_s \cos \phi) \hat{\phi}}{|\tilde{R}|^3} dz' d\phi' \\
= \frac{JR_s^2}{8\pi} \int_{-\pi}^{2\pi} \int_{z_1}^{z_2} \frac{R_s \sin \phi \hat{p} + (\rho - R_s \cos \phi) \hat{\phi}}{|\tilde{R}|^3} d\gamma_z d\phi, \tag{3.17}
\]

where \( z_1 \) and \( z_2 \) are referred to Figure 5, \( \gamma_{z1} = \tilde{z}_1 - z \) and \( \gamma_{z2} = \tilde{z}_2 - z \). By using the variable substitution

\[ \gamma_z = B(\phi) \tan t, \quad |t| \leq 90^\circ, \]

\( d\gamma_z \) and \( |\tilde{R}| \) (see (3.15)) give

\[ d\gamma_z = B(\phi) \sec^2(t) dt, \]

\[ |\tilde{R}| = B(\phi) \sec t. \]

The formula of \( \tilde{H}_{J_{sc}^w} \) thus becomes

\[
\tilde{H}_{J_{sc}^w} = \frac{JR_s^2}{8\pi} \int_{-\pi}^{2\pi} \frac{R_s \sin \phi \hat{p} + (\rho - R_s \cos \phi) \hat{\phi}}{B^2(\phi)} (\sin t_2 - \sin t_1) d\phi \\
= \frac{JR_s^2}{8\pi} \int_{-\pi}^{2\pi} \frac{R_s \sin \phi \hat{p} + (\rho - R_s \cos \phi) \hat{\phi}}{B^2(\phi)} \left\{ \sum_{i=1}^{2} \frac{(-1)^i (\tilde{z}_i - z)}{\sqrt{B^2(\phi) + (\tilde{z}_i - z)^2}} \right\} d\phi, \tag{3.18}
\]
where

\[ \ddot{z}_i - z = A_{i1} \sin \phi + A_{i2} \cos \phi + A_{i3}, \]

\[ A_{11} = \begin{cases} -(r_b - \cos \theta_b R_s) \cos \phi / \sin \theta_b, & \theta_b \neq 0, \\ 0, & \theta_b = 0, \end{cases} \]

\[ A_{12} = \begin{cases} -(r_b - \cos \theta_b R_s) \sin \phi / \sin \theta_b, & \theta_b \neq 0, \\ 0, & \theta_b = 0, \end{cases} \]

\[ A_{21} = \begin{cases} (r_t - \cos \theta_t R_s) \cos \phi / \sin \theta_t, & \theta_t \neq 0, \\ 0, & \theta_t = 0, \end{cases} \]

\[ A_{22} = \begin{cases} (r_t - \cos \theta_t R_s) \sin \phi / \sin \theta_t, & \theta_t \neq 0, \\ 0, & \theta_t = 0, \end{cases} \]

\[ A_{13} = -h_b - z, \]

\[ A_{23} = h_t - z. \]

The procedures for obtaining \( \ddot{z}_1 \) and \( \ddot{z}_2 \) are described in Appendix A.

It can be observed that the above equation becomes singular when the field point is located on the wall surface of the conductor segment. Some mathematical manipulations are necessary to avoid the singularity problem.

Equation (3.18) can be rewritten as follows:

\[
H_{j_{sc,\phi}} = \frac{JR_s}{8\pi} \sum_{i=1}^{2} (-1)^i \int_{-\varphi}^{2\pi-\varphi} \frac{R_s \sin \phi}{a_{i6} + a_{i7} \cos \phi} \times \frac{A_{i1} \sin \phi + A_{i2} \cos \phi + A_{i3}}{\sqrt{a_{i0} + a_{i1} \cos \phi + a_{i2} \sin \phi + a_{i3} \cos^2 \phi + a_{i4} \sin \phi \cos \phi + a_{i5} \sin^2 \phi}} d\phi
\]

\[
= \frac{JR_s}{8\pi} \sum_{i=1}^{2} (-1)^i \int_{-\varphi}^{2\pi-\varphi} \frac{R_s \sin \phi}{a_{i6} + a_{i7} \cos \phi} \times \frac{R_s \cos \phi}{X_i(\phi)} d\phi \tag{3.19}
\]

\[
= \frac{JR_s}{8\pi} \sum_{i=1}^{2} (-1)^i \int_{-\varphi}^{2\pi-\varphi} \frac{F_i^\rho(\phi)}{X_i(\phi)} d\phi,
\]

\[
H_{j_{sc,\phi}} = \frac{JR_s}{8\pi} \sum_{i=1}^{2} (-1)^i \int_{-\varphi}^{2\pi-\varphi} \frac{\rho - R_s \cos \phi}{a_{i6} + a_{i7} \cos \phi} \times
\]

42
\[
\frac{A_{i1}\sin\phi + A_{i2}\cos\phi + A_{i3}}{\sqrt{a_{i0} + a_{i1}\cos\phi + a_{i2}\sin\phi + a_{i3}\cos^2\phi + a_{i4}\sin\phi\cos\phi + a_{i5}\sin^2\phi}}
\]

\[
= \frac{JR_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \int_{-\phi}^{2\pi-\phi} \frac{R_i^\phi(\phi)}{X_i(\phi)} d\phi
\]

\[
= \frac{JR_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \int_{-\phi}^{2\pi-\phi} F_i^\phi(\phi) d\phi,
\]

where \(R_i^\phi(\phi)\) are rational functions, and with \(i = 1, 2, \ldots, n\).

\[
F_i^\phi(\phi) = \frac{R_i^\phi(\phi)}{X_i(\phi)},
\]

\[
R_i^\phi(\phi) = \frac{a_{i0} + a_{i1}\cos\phi}{a_{i0} + a_{i1}\cos\phi},
\]

\[
X_i(\phi) = a_{i0} + a_{i1}\cos\phi + a_{i2}\sin\phi + a_{i3}\cos^2\phi + a_{i4}\sin\phi\cos\phi + a_{i5}\sin^2\phi,
\]

\[
a_{i0} = \rho^2 + R_s^2 + A_{i3}^2,
\]

\[
a_{i1} = 2A_{i2}A_{i3} - 2R_s\rho,
\]

\[
a_{i2} = 2A_{i1}A_{i3},
\]

\[
a_{i3} = A_{i2}^2,
\]

\[
a_{i4} = 2A_{i1}A_{i2},
\]

\[
a_{i5} = A_{i1}^2,
\]

\[
a_{i6} = \rho^2 + R_s^2,
\]

\[
a_{i7} = -2R_s\rho.
\]

Referring to Appendix B, Equations (3.19) and (3.20) can be changed into the integrals with limits \([-\pi/2, \pi/2]\):

\[
H_{\phi, \rho, \phi} = \frac{JR_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \int_{-\pi/2}^{\pi/2} \left[ F_i^{\phi, \phi}(\sin\phi, \cos\phi) + F_i^{\phi, \phi}(\sin\phi, -\cos\phi) \right] d\phi
\]

\[
= \frac{JR_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \int_{-\pi/2}^{\pi/2} \left[ F_i^{\phi, \phi}(\phi) + F_i^{\phi, \phi}(\phi) \right] d\phi,
\]

(3.21)
where

\[ F_{i0}^{\rho,\phi}(\phi) = \frac{R_{i0}^{\rho,\phi}(\phi)}{\sqrt{X_{i0}(\phi)}}, \]

\[ R_{i0}^{\rho}(\phi) = \frac{R_s \sin(\phi (A_{i1} \sin \phi + A_{i2} \cos \phi - A_{i3}))}{a_{i6} - a_{i7} \cos \phi}, \]

\[ R_{i0}^{\phi}(\phi) = \frac{-(\rho + R_s \cos \phi)(A_{i1} \sin \phi + A_{i2} \cos \phi - A_{i3})}{a_{i6} - a_{i7} \cos \phi}, \]

\[ X_{i0}(\phi) = a_{i0} - a_{i1} \cos \phi - a_{i2} \sin \phi + a_{i3} \cos ^2 \phi + a_{i4} \sin \phi \cos \phi + a_{i5} \sin ^2 \phi. \]

Let \( t = \tan(\phi/2) \) which is monotonic in the range \([-\pi/2, \pi/2]\), then (3.21) becomes

\[
\int_{-\pi/2}^{\pi/2} F_{i0}^{\rho,\phi}(\phi) \, d\phi = \int_{-1}^{1} S_i^{\rho,\phi}(t) \, dt,
\]

\[
\int_{-\pi/2}^{\pi/2} F_{i0}^{\phi,\rho}(\phi) \, d\phi = \int_{-1}^{1} S_i^{\phi,\rho}(t) \, dt.
\]

In order to eliminate the singularity existing in the integral \( \int_{-1}^{1} S_i^{\rho,\phi}(t) \, dt \), let \( t \) be substituted by \( 1/t \) in the integral \( \int_{-1}^{1} S_i^{\rho,\phi}(t) \, dt \), thus (3.21) equals

\[
H_{J_{x,\rho,\phi}} = \frac{JR_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \int_{-1}^{1} \left[ \frac{Y_i^{\rho,\phi}(t)}{E_i(t)} + \frac{Y_{i0}^{\rho,\phi}(t)}{E_{i0}(t)} \right] \, dt, \quad (3.22)
\]

where with \( i = 1, 2, \)

\[
Y_i^{\rho}(t) = -4tR_s \left[ (A_{i1} + A_{i3})t^2 + 2A_{i1}t + A_{i3} - A_{i2} \right],
\]

\[
Y_i^{\phi}(t) = -2 \left[ (\rho - R_s)t^2 + \rho + R_s \right] \left[ (A_{i2} + A_{i3})t^2 + 2A_{i1}t + A_{i3} - A_{i2} \right],
\]

\[
E_i(t) = (1 + t^2) [(\rho - R_s)^2 t^2 + (\rho + R_s)^2] \sqrt{b_{i4}t^4 + b_{i3}t^3 + b_{i2}t^2 + b_{i1}t + b_{i0}},
\]

\[
Y_{i0}^{\rho}(t) = -4tR_s \left[ (A_{i2} + A_{i3})t^2 - 2A_{i1}t + A_{i3} - A_{i2} \right],
\]

\[
Y_{i0}^{\phi}(t) = 2 \left[ (\rho - R_s)t^2 + \rho + R_s \right] \left[ (A_{i2} + A_{i3})t^2 - 2A_{i1}t + A_{i3} - A_{i2} \right],
\]

\[
E_{i0}(t) = (1 + t^2) [(\rho - R_s)^2 t^2 + (\rho + R_s)^2] \sqrt{b_{i4}t^4 - b_{i3}t^3 + b_{i2}t^2 - b_{i1}t + b_{i0}},
\]

\[
b_{i0} = (\rho + R_s)^2 + (A_{i2} - A_{i3})^2,
\]

44
\[ b_{i1} = 4A_{i1}(A_{i3} - A_{i2}), \]
\[ b_{i2} = 2(\rho^2 + R_s^2 + A_{i3}^2 - A_{i2}^2 + 2A_{i1}^2), \]
\[ b_{i3} = 4A_{i1}(A_{i3} + A_{i2}), \]
\[ b_{i4} = (\rho - R_s)^2 + (A_{i2} + A_{i3})^2. \]

Substituting the coefficients \( b_{ij} (j = 0, 4) \) into \( E_i(t) \) and \( E_{io}(t) \), then they become:

\[
E_i(t) = (1 + t^2)[(\rho - R_s)^2t^2 + (\rho + R_s)^2] \cdot \{(\rho - R_s)^2 + (A_{i2} + A_{i3})^2\}t^4
\]
\[ + 4A_{i1}(A_{i2} + A_{i3})t^3 + 2(\rho^2 + R_s^2)t^2 + 4A_{i1}^2t^2 + 2(A_{i3}^2 - A_{i2}^2)t^2
\]
\[ + 4A_{i1}(A_{i3} - A_{i2})t + (\rho + R_s)^2 + (A_{i2} - A_{i3})^2 \}\]^{1/2}
\[ = (1 + t^2)[(\rho - R_s)^2t^2 + (\rho + R_s)^2] \cdot \{(\rho - R_s)^2t^4 + (\rho + R_s)^2
\]
\[ + [(A_{i2} + A_{i3})t^2 + 2A_{i1}t + A_{i3} - A_{i2}] + 4A_{i1}^2t^2
\]
\[ + (\rho + R_s)^2 + (\rho - R_s)^2t^4 + 2(\rho^2 + R_s^2)t^2 \}\]^{1/2},

(3.23)

\[
E_{io}(t) = (1 + t^2)[(\rho - R_s)^2t^2 + (\rho + R_s)^2] \cdot \{(\rho - R_s)^2 + (A_{i2} + A_{i3})^2\}t^4
\]
\[ - 4A_{i1}(A_{i2} + A_{i3})t^3 + 2(\rho^2 + R_s^2)t^2 + 4A_{i1}^2t^2 + 2(A_{i3}^2 - A_{i2}^2)t^2
\]
\[ - 4A_{i1}(A_{i3} - A_{i2})t + (\rho + R_s)^2 + (A_{i2} - A_{i3})^2 \}\]^{1/2}
\[ = (1 + t^2)[(\rho - R_s)^2t^2 + (\rho + R_s)^2] \cdot \{(\rho - R_s)^2t^4 + (\rho + R_s)^2
\]
\[ - 4A_{i1}[A_{i2} + A_{i3}]t^2 + A_{i3} - A_{i2}] + 4A_{i1}^2t^2
\]
\[ + (\rho + R_s)^2 + (\rho - R_s)^2t^4 + 2(\rho^2 + R_s^2)t^2 \}\]^{1/2}
\[ = (1 + t^2)[(\rho - R_s)^2t^2 + (\rho + R_s)^2] \cdot \{(\rho - R_s)^2t^4 + (\rho + R_s)^2
\]
\[ + [(A_{i2} + A_{i3})t^2 - 2A_{i1}t + A_{i3} - A_{i2}] + 2(\rho^2 + R_s^2)t^2 \}\]^{1/2}.

(3.24)
It can be seen that $E_i(t)$ and $E_{io}(t)$ are not equal to zero for any values of $\rho$. Therefore, Equation (3.22) is free from any singularity problem mathematically.

In summary, when $\theta_{t,b} \neq 0$,

$$
H_{j^{sc},p,\epsilon} = \frac{JR_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \int_{-1}^{1} \left[ \frac{Y_{i}^{\rho,\epsilon}(t)}{E_i(t)} + \frac{Y_{io}^{\rho,\epsilon}(t)}{E_{io}(t)} \right] dt, \tag{3.25}
$$

where with $i = 1, 2$,

- $Y_i^\rho(t) = -4tR_s [(A_{i2} + A_{i3})t^2 + 2A_{i1}t + A_{i3} - A_{i2}]$,
- $Y_{io}^\rho(t) = -2[(\rho - Rs)t^2 + \rho + Rs][(A_{i2} + A_{i3})t^2 + 2A_{i1}t + A_{i3} - A_{i2}]$,
- $E_i(t) = (1 + t^2)[(\rho - Rs)^2t^2 + (\rho + Rs)^2] \cdot \{(\rho - Rs)^2t^2 + (\rho + Rs)^2\}
+ [(A_{i2} + A_{i3})t^2 + 2A_{i1}t + A_{i3} - A_{i2}]^2 + 2(\rho^2 + Rs^2)t^2 \}^{1/2},$
- $Y_{io}(t) = -4tR_s [(A_{i2} + A_{i3})t^2 - 2A_{i1}t + A_{i3} - A_{i2}]$,
- $Y_{io}^\rho(t) = 2[(\rho - Rs)t^2 + \rho + Rs][(A_{i2} + A_{i3})t^2 - 2A_{i1}t + A_{i3} - A_{i2}]$,
- $E_{io}(t) = (1 + t^2)[(\rho - Rs)^2t^2 + (\rho + Rs)^2] \cdot \{(\rho - Rs)^2t^2 + (\rho + Rs)^2\}
+ [(A_{i2} + A_{i3})t^2 - 2A_{i1}t + A_{i3} - A_{i2}]^2 + 2(\rho^2 + Rs^2)t^2 \}^{1/2}.$

b. Expressions of $\vec{H}_{j^{sc}}$ and $\vec{H}_{p^{sc}}$

In this subsection, the equations for computing the magnetic field strength contributions from the two end planes are derived.

The derivations of $\vec{H}_{j^{sc}}$ and $\vec{H}_{p^{sc}}$ are conducted in the local coordinate systems of the top and bottom end planes $(X_tY_tZ_t$ and $X_bY_bZ_b)$, respectively, as shown in Figure 8, where

- $O_t$ and $O_b$ locate at the center points of the end planes, respectively;
- both $X_t$ and $X_b$ are parallel to the $X$ axis;
• the planes of $X_tO_tY_t$ and $X_bO_bY_b$ coincide with the top and bottom end planes, respectively;

• $Z_t$ and $Z_b$ coincide with the normal vectors of the top and bottom end plates, respectively.

Hence, it is necessary to change the coordinates of the field point and the expressions of the field sources $J_{ac}^{t,b}$ and $\rho_{ac}^{t,b}$ into the corresponding ones in the $X_tY_tZ_t$ or $X_bY_bZ_b$ coordinate system.

Suppose $(x_i, y_i, z_i)$ and $(x_i', y_i', z_i')$ are the coordinates of the field and source points in the $X_iY_iZ_i$ system, where $i = t, b$. Then $z_i'$ equals zero for the field sources in the end planes. Symbols $r_t$ and $r_b$ are the length of semi-major axis of the top and bottom end planes of the cylindrical conductor segment, respectively.

![Figure 8: Local coordinate systems of the conductor segment.](image)

For a field point which has the rectangular coordinates $(x, y, z)$ in the $XYZ$ coordinate system or cylindrical coordinates $(\rho, \varphi, z)$ in the $\rho\varphi Z$ coordinate system,
its corresponding coordinates in the $X_t Y_t Z_t$ and $X_b Y_b Z_b$ coordinate systems are:

$$
\begin{align*}
\begin{bmatrix}
x_t \\
y_t \\
z_t
\end{bmatrix} &= \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_t & \sin \theta_t \\
0 & -\sin \theta_t & \cos \theta_t
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} + \begin{bmatrix}
0 \\
-h_t \sin \theta_t \\
-h_t \cos \theta_t
\end{bmatrix}, \\
\text{and} \\
\begin{bmatrix}
x_b \\
y_b \\
z_b
\end{bmatrix} &= \begin{bmatrix}
1 & 0 & 0 \\
0 & -\cos \theta_b & \sin \theta_b \\
0 & -\sin \theta_b & -\cos \theta_b
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} + \begin{bmatrix}
0 \\
h_t \sin \theta_b \\
-h_b \cos \theta_b
\end{bmatrix}.
\end{align*}
$$

By inverting the above matrices, it gives the $(x, y, z)$ coordinates expressed in terms of $(x_t, y_t, z_t)$ and $(x_b, y_b, z_b)$ coordinates, respectively.

$$
\begin{align*}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} &= \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_t & -\sin \theta_t \\
0 & \sin \theta_t & \cos \theta_t
\end{bmatrix}
\begin{bmatrix}
x_t \\
y_t \\
z_t
\end{bmatrix} + \begin{bmatrix}
0 \\
h_t \\
l_h
\end{bmatrix}, \\
&= \begin{bmatrix}
1 & 0 & 0 \\
0 & -\cos \theta_b & -\sin \theta_b \\
0 & \sin \theta_b & -\cos \theta_b
\end{bmatrix}
\begin{bmatrix}
x_b \\
y_b \\
z_b
\end{bmatrix} + \begin{bmatrix}
0 \\
h_b \\
l_h
\end{bmatrix}.
\end{align*}
$$

The relationships among different coordinate systems are also valid for the relationship among the vector components in different coordinate systems. Therefore,

$$
\begin{align*}
\begin{bmatrix}
J^t_{sc,xt} \\
J^t_{sc,yt} \\
J^t_{sc,zt}
\end{bmatrix} &= \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_t & \sin \theta_t \\
0 & -\sin \theta_t & \cos \theta_t
\end{bmatrix}
\begin{bmatrix}
J^t_{sc,x} \\
J^t_{sc,y} \\
J^t_{sc,z}
\end{bmatrix}, \\
\begin{bmatrix}
J^b_{sc,xb} \\
J^b_{sc,yb} \\
J^b_{sc,zb}
\end{bmatrix} &= \begin{bmatrix}
1 & 0 & 0 \\
0 & -\cos \theta_b & \sin \theta_b \\
0 & \sin \theta_b & -\cos \theta_b
\end{bmatrix}
\begin{bmatrix}
J^b_{sc,x} \\
J^b_{sc,y} \\
J^b_{sc,z}
\end{bmatrix}.
\end{align*}
$$
Substituting Equations (3.8), (3.28) and (3.29) into the above equations, it is obtained that

\[
\begin{bmatrix}
J_{sc,xt}^t \\
J_{sc,yt}^t \\
J_{sc,zt}^t
\end{bmatrix}
= 
\begin{bmatrix}
-j\frac{1}{2}\cos\theta_t & 0 & 0 \\
0 & -j\frac{1}{2} & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix}
= 
\begin{bmatrix}
-j\frac{1}{2}\cos\theta_t & 0 & 0 \\
0 & -j\frac{1}{2}\cos\theta_t & \frac{j}{2}\sin\theta_t \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x'_t \\
y'_t \\
z'_t
\end{bmatrix},
\] (3.30)

\[
\begin{bmatrix}
J_{sc,xb}^t \\
J_{sc,yb}^t \\
J_{sc,zb}^t
\end{bmatrix}
= 
\begin{bmatrix}
j\frac{1}{2}\cos\theta_b & 0 & 0 \\
0 & j\frac{1}{2} & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix}
= 
\begin{bmatrix}
j\frac{1}{2}\cos\theta_b & 0 & 0 \\
0 & j\frac{1}{2}\cos\theta_b & \frac{j}{2}\sin\theta_b \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x'_b \\
y'_b \\
z'_b
\end{bmatrix},
\] (3.31)

and

\[
\rho_{sc}^t = -\mu_0 J \sin\theta_t x_t',
\] (3.32)

\[
\rho_{sc}^b = -\mu_0 J \sin\theta_b x_b'.
\] (3.33)

The vector \( \vec{R} \), which points from the source point to the field point, equals

\[
\vec{R} = (x_i - x'_i) \hat{x}_i + (y_i - y'_i) \hat{y}_i + z_i \hat{z}_i
\] (3.34)

and its magnitude is

\[
|\vec{R}| = \sqrt{(x_i - x'_i)^2 + (y_i - y'_i)^2 + z_i^2}.
\]

Combining (3.30), (3.31) and (3.34), it gives

\[
\vec{J}_{sc}^i \times \vec{R} = G_i [z_i y'_i \hat{x}_i - z_i x'_i \hat{y}_i + (x'_i y_i - y'_i x_i) \hat{z}_i],
\]

49
where \( G_t = -\frac{j}{2} \cos \theta_t \) and \( G_b = \frac{j}{2} \cos \theta_b \).

The contour of the top end plane has an equation in the \( X_tY_tZ_t \) system:

\[
\frac{x_t^2}{R_s^2} + \frac{y_t^2}{r^2} = 1.
\]  

(3.35)

Similarly, the contour of the bottom end plane has the following equation in the \( X_bY_bZ_b \) system:

\[
\frac{x_b^2}{R_s^2} + \frac{y_b^2}{r^2} = 1.
\]  

(3.36)

As a result, with

\[
T_{1i} = (x'_i - x_i)^2 + z'_i^2,
\]

\[
T_{2i} = r_i (1 - \frac{x_i^2}{R_s^2})^{1/2},
\]  

(3.37)

\( \bar{H}_{J_{is}} \) equals (refer to (3.12))

\[
\bar{H}_{J_{is}} = \frac{1}{4\pi} \int_{-R_s}^{R_s} \int_{-T_{2i}}^{T_{2i}} \frac{\bar{J}_{sc} \times \bar{R}}{|\bar{R}|^3} dy'_i dx'_i,
\]

where \( i = t, b \). With \( y'_t - y_t = t \), the above equation becomes

\[
\bar{H}_{J_{is}} = \frac{G_i}{4\pi} \int_{-R_s}^{R_s} \int_{-T_{2i}}^{T_{2i}} \frac{z_i t \hat{x}_i - z_i t \hat{z}_i + z_i y'_i \hat{y}_i - z_i x'_i \hat{y}_i + (x'_i - x_i) y_i \hat{z}_i}{[T_{1i} + t^2]^{3/2}} dt dx'_i.
\]

By employing the integral identity,

\[
\int \frac{B t + C}{[A + t^2]^{3/2}} dt = \frac{C t - AB}{A\sqrt{t^2} + A},
\]  

(3.38)

\( \bar{H}_{J_{is}} \) \((i = t, b)\) becomes

\[
\bar{H}_{J_{isc}} = \frac{G_i}{4\pi} \int_{-R_s}^{R_s} \left[ \frac{y_i (T_{2i} - y_i) - T_{1i}}{T_{1i}\sqrt{(T_{2i} - y_i)^2 + T_{1i}}} \right] dt.
\]

50
\[ H_{J_{3c}yi} = \frac{-G_i}{4\pi} \int_{-R_s}^{R_s} \left[ \frac{x'_i(T_{2i} - y_i)}{T_{1i} \sqrt{(T_{2i} - y_i)^2 + T_{1i}}} \right] dx'_i, \]

\[ H_{J_{3c}zi} = \frac{G_i}{4\pi} \int_{-R_s}^{R_s} \left[ \frac{y_i(x'_i - x_i)(T_{2i} - y_i) + x_i T_{1i}}{T_{1i} \sqrt{(T_{2i} - y_i)^2 + T_{1i}}} \right] dx'_i. \]

Similarly, substituting (3.32) and (3.33) into (3.13), \( H_{\rho_{sc}} \) equals

\[ \bar{H}_{\rho_{sc}} = -\frac{J \sin \theta_i}{8\pi} \int_{-R_s}^{R_s} \int_{-T_{2i}}^{T_{2i}} \frac{x'_i \bar{H}}{|\bar{R}|^3} dy'_i dx'_i \]

\[ = -\frac{J \sin \theta_i}{8\pi} \int_{-R_s}^{R_s} \int_{-T_{2i}}^{T_{2i}} \left[ (x'_i - x_i)^2 + (y'_i - y_i)^2 + z'^2 \right]^{3/2} dy'_i dx'_i. \]

With \( y'_i - y_i = t \), the above equation can be simplified into (by employing the same integral identity given in (3.38)),

\[ H_{\rho_{sc}zi} = \frac{J \sin \theta_i}{8\pi} \int_{-R_s}^{R_s} \left[ \frac{x'_i}{\sqrt{(T_{2i} - y_i)^2 + T_{1i}}} \right] dx'_i, \]

\[ H_{\rho_{sc}yi} = -\frac{J \sin \theta_i}{8\pi} \int_{-R_s}^{R_s} \left[ \frac{x'_i}{\sqrt{(T_{2i} - y_i)^2 + T_{1i}}} \right] dx'_i, \]

\[ H_{\rho_{sc}zi} = -\frac{J \sin \theta_i}{8\pi} \int_{-R_s}^{R_s} \left[ \frac{x'_i(T_{2i} - y_i)}{T_{1i} \sqrt{(T_{2i} - y_i)^2 + T_{1i}}} \right] dx'_i, \]

\[ + \frac{x'_i(T_{2i} + y_i)}{T_{1i} \sqrt{(T_{2i} + y_i)^2 + T_{1i}}} \right] dx'_i, \]

(3.40)
where \( i = t, b, \) and \( T_{1i} \) and \( T_{2i} \) are given in (3.37), respectively.

Equations (3.39) and (3.40) have the singularity at field points on the edges of the end planes. Due to the fact that most of the field strength contribution comes from the equivalent electric surface current along the side wall, the singularity can be avoided by shifting slightly the position of the end plane or changing slightly the slope angle of the end plane where the singularity exists, that is, making the singular point off the edge of the end plane.

The single integral equations given in (3.25), (3.39) and (3.40) can be evaluated by a numerical integration method, such as, the Gaussian Quadrature method.

### 3.3 Magnetic Field Strength of a Finite Circular Cylindrical Bar

A circular cylindrical bar of finite length is a special case of a general circular cylindrical segment with non-oblique end planes, that is, \( \theta_t = \theta_b = 0 \). Normally, the conductor size of air-core reactor coils is much smaller than the coil diameter. Therefore, a helical coil can be usually modeled by circular cylindrical bar segments instead of general circular cylindrical segments.

It can be seen from Equation (3.18) that if \( \theta_b = 0, A_{11} = 0 \) and \( A_{12} = 0 \), and if \( \theta_t = 0, A_{21} = 0 \) and \( A_{22} = 0 \). Hence, for a circular straight bar whose \( \theta_b \) and \( \theta_t \) equal zero, \( A_{11} = 0 \) and \( A_{12} = 0 \) \((i = 1, 2)\). Substituting the values of \( A_{11} \) and \( A_{12} \) into Equations (3.19) and (3.20), the expressions for the following coefficients become:

\[
\begin{align*}
  a_{i0} & = \rho^2 + R_s^2 + A_{i3}^2, \\
  a_{i1} & = -2R_s\rho, \\
  a_{i2} & = a_{i3} = a_{i4} = a_{i5} = 0, \\
  a_{i6} & = \rho^2 + R_s^2,
\end{align*}
\]

(3.41) \hspace{1cm} (3.42) \hspace{1cm} (3.43) \hspace{1cm} (3.44)
\[ a_{i7} = -2R_s \rho = a_{i1}, \quad (3.45) \]

where \( A_{i3} \) is a constant (refer to (3.18)).

Moreover, for the bar conductor segment, there exists the \( \phi \) symmetry about the field point. Hence, \( \phi \) can be set to zero. As \( A_{i1}, a_{i2} \) and \( a_{i4} \) are equal to zero, \( F_i^\phi (\phi) \) becomes an even function of \( \phi \) and \( F_i^\phi (\phi) \) becomes an odd function of \( \phi \) (see (3.19) and (3.20)). Thus, Equations (3.19) and (3.20) become:

\[
H_{J_{\phi c p}} = 0, \quad (3.46)
\]

\[
H_{J_{\phi c p}} = \frac{JR_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \int_0^{2\pi} \frac{A_{i3}(\rho - R_s \cos \phi)}{(a_{i6} + a_{i1} \cos \phi)\sqrt{a_{i0} + a_{i1} \cos \phi}} d\phi
\]

\[
H_{J_{\phi c p}} = \frac{JR_s^2}{4\pi} \sum_{i=1}^{2} (-1)^i \int_0^{\pi} \frac{A_{i3}(\rho - R_s \cos \phi)}{(a_{i6} + a_{i1} \cos \phi)\sqrt{a_{i0} + a_{i1} \cos \phi}} d\phi. \quad (3.47)
\]

It can be proved that \( H_{J_{\phi c p}} = 0 \) if \( \rho = 0 \).

If \( A_{i3} = 0 \) (\( i = 1 \) or 2), the term corresponding to the \( i \) in the summation of (3.47) becomes zero.

If \( A_{i3} \neq 0 \) and \( \rho \neq 0 \), then \( H_{J_{\phi c p}} \) can be rearranged as

\[
\tilde{H}_{J_{\phi c p}} = \frac{JR_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \left[ -\frac{2A_{i3}R_s}{a_{i1}} \int_0^{\pi} \frac{d\phi}{\sqrt{a_{i0} + a_{i1} \cos \phi}} + 2A_{i3} \left( a_{i6} a_{i1} R_s + \rho \right) \int_0^{\pi} \frac{d\phi}{(a_{i6} + a_{i1} \cos \phi)\sqrt{a_{i0} + a_{i1} \cos \phi}} \right]. \quad (3.48)
\]

Due to the fact that \( a_{i0} > 0, a_{i1} < 0 \) and \( a_{i0} > -a_{i1} \), the following equation exists

\[
\int_0^{\pi} \frac{d\phi}{\sqrt{a_{i0} + a_{i1} \cos \phi}} = \frac{2}{\sqrt{\rho^2 + R_s^2 + A_{i3}^2 + 2R_s \rho}} K(\kappa_i), \quad (3.49)
\]

where \( K \) is Legendre's complete elliptic integral of the first kind [68] and the argument \( \kappa_i \)

\[
\kappa_i = \frac{4R_s \rho}{(\rho + R_s)^2 + A_{i3}^2}, \quad i = 1, 2, \quad (3.50)
\]

53
is the modulus of Jacobian elliptic integrals.

For changing the integral limits from \( \int_0^\pi \) to \( \int_0^{\pi/2} \) in the second integral of (3.48), a change of the variable \( \phi \) by \( \phi/2 \) is used. It gives,

\[
\int_0^\pi \frac{d\phi}{(a_{i6} + a_{i1}\cos\phi)\sqrt{a_{i0} + a_{i1}\cos\phi}} = \int_0^{\pi/2} \frac{2d\phi}{(a_{i6} + a_{i1}\cos\phi)\sqrt{a_{i0} + a_{i1}\cos\phi}}
\]

\[
= \int_0^{\pi/2} \frac{2d\phi}{[(\rho - R_s)^2 + 4R_s\rho \sin^2\phi]\sqrt{(\rho - R_s)^2 + A_{i3}^2 + 4R_s\rho \sin^2\phi}}
\]

\[
= \frac{2}{(\rho - R_s)^2(\rho - R_s)^2 + A_{i3}^2} \int_0^{\pi/2} \frac{d\phi}{(1 - \alpha^2 \sin^2\phi)\sqrt{1 + n_\ell^2 \sin^2\phi}}, \quad (3.51)
\]

where

\[
n_\ell^2 = \frac{4R_s\rho}{(\rho - R_s)^2 + A_{i3}^2} > 0,
\]

\[
\alpha^2 = \frac{-4R_s\rho}{(\rho - R_s)^2} < 0.
\]

By referring to [68], it is found that

\[
\int_0^{\pi/2} \frac{d\phi}{(1 - \alpha^2 \sin^2\phi)\sqrt{1 + n_\ell^2 \sin^2\phi}} = \sqrt{1 - \kappa_i^2} \int_0^K \frac{dn^2 u}{1 - \alpha_i^2 \sin^2 u} du,
\]

\[
\int_0^K \frac{dn^2 u}{1 - \alpha_i^2 \sin^2 u} du = \frac{\pi(\kappa_i^2 - \alpha_i^2)\Lambda_0(\psi_i, \kappa_i)}{2\sqrt{\alpha_i^2(1 - \alpha_i^2)(\kappa_i^2 - \alpha_i^2)}},
\]

where \( \Lambda_0(\psi_i, \kappa_i) \) is Heuman’s Lambda function, \( \alpha_i \) is the parameter of the elliptic integral of the third kind, and

\[
\alpha_i^2 = \frac{-4R_s\rho A_{i3}^2}{(\rho - R_s)^2[(\rho + R_s)^2 + A_{i3}^2]} < 0,
\]

\[
\psi_i = \sin^{-1}\sqrt{\frac{\alpha_i^2}{\alpha_i^2 - \kappa_i^2}}.
\]

Therefore, Equation (3.51) becomes

\[
\int_0^\pi \frac{d\phi}{(a_{i6} + a_{i1}\cos\phi)\sqrt{a_{i0} + a_{i1}\cos\phi}} = \frac{\pi}{|A_{i3}(\rho^2 - R_s^2)|} \Lambda_0(\psi_i, \kappa_i). \quad (3.52)
\]
It is seen that the expression of $\tilde{H}_{jyc}$ for a circular cylindrical bar has a concise solution by substituting (3.52) and (3.49) into (3.48):

$$\tilde{H}_{jyc} = \frac{\varphi J R_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \left[ \frac{2 A_{i3}}{\rho \sqrt{(\rho + R_s)^2 + A_{i3}^2}} K(\kappa_i) + \frac{\pi (\rho - R_s) A_{i3}}{\rho |A_{i3} (\rho - R_s)|} \Lambda_0(\psi_i, \kappa_i) \right],$$

where $\rho \neq R_s$, $\kappa_i$ ($i = 1, 2$) is given in (3.50), and

$$\psi_i = \sin^{-1} \sqrt{\frac{A_{i3}^2}{A_{i3}^2 + (\rho - R_s)^2}}.$$  

If $\rho = R_s$, (3.47) becomes

$$\tilde{H}_{jyc} = \frac{\varphi J R_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \int_0^{\pi} \frac{A_{i3}}{R_s \sqrt{2 R_s^2 + A_{i3}^2 - 2 R_s^2 \cos \phi}} d\phi.$$  

Referring to [68] again, there exists

$$\int_0^{t} \frac{d\phi}{\sqrt{a - b \cos \phi}} = g F(A, \kappa), \quad a > b > 0,$$

where

$$g = \frac{2}{\sqrt{a + b}},$$

$$\kappa^2 = \frac{2b}{a + b},$$

$$A = \sin^{-1} \sqrt{\frac{b (1 - \cos t)}{\kappa^2 (a - b \cos t)}}, \quad 0 < t \leq \pi.$$  

Substituting $a = 2R_s^2 + A_{i3}^2$ and $b = 2R_s^2$ into (3.55), it is obtained that $A = \pi/2$ and (3.54) becomes

$$\tilde{H}_{jyc} = \frac{\varphi J R_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \frac{2 A_{i3}}{R_s \sqrt{4 R_s^2 + A_{i3}^2}} K,$$

where

$$\kappa_i = \frac{4R_s^2}{4R_s^2 + A_{i3}^2}, \quad i = 1, 2.$$
For Equation (3.53), the conventional way is to evaluate $K$ and $\Lambda_o$ individually and separately. $\Lambda_o(\psi, \kappa_i)$ can be evaluated by

$$\Lambda_o(\psi, \kappa_i) = \frac{2}{\pi} \left[ EF(\psi, \kappa_i) + KE(\psi, \kappa'_i) - KF(\psi, \kappa'_i) \right], \quad (3.57)$$

where $\kappa'_i$ is the complementary modulus of Jacobian elliptic integrals,

$$\kappa'^2_i = \frac{(\rho - R_\lambda)^2 + A_{23}^2}{(\rho + R_\lambda)^2 + A_{23}^2}; \quad (3.58)$$

$F(\psi, \kappa_i)$ is Legendre's incomplete elliptic integral of the first kind; $F(\psi, \kappa'_i)$ is the associated incomplete elliptic integral of the first kind; $K = F(\pi/2, \kappa_i)$ is a complete elliptic integral of the first kind; $E(\psi, \kappa_i)$ is Legendre's incomplete elliptic integral of the second kind; $E(\psi, \kappa'_i)$ is the associated incomplete elliptic integral of the second kind and $E = E(\pi/2, \kappa_i)$ is a complete elliptic integral of the second kind.

The new approach is to unify the expressions of $K$ and $\Lambda_o$ into one integral by using the definition of the general complete elliptic integral [9]. Then $K$ and $\Lambda_o$ can be evaluated simultaneously and the redundant computations are thus avoided.

In [9], the general complete elliptic integral is defined as

$$cel(\kappa', p, a, b) = \int_0^{\pi/2} \frac{a + (b - a) \sin^2 \phi}{[1 + (1 - p) \sin^2 \phi] \sqrt{1 - \kappa'^2 \sin^2 \phi}} d\phi,$$

$$= \int_0^{\pi/2} \frac{(a \cos^2 \varphi + b \sin^2 \varphi) d\varphi}{(\cos^2 \varphi + p \sin^2 \varphi) \sqrt{\cos^2 \varphi + \kappa'^2 \sin^2 \varphi}}, \quad (3.59)$$

where the relationship between the modulus and the complementary modulus of elliptic integral equals $\kappa^2 + \kappa'^2 = 1$, the parameters $a, b$ and $p$ can be arbitrary real numbers, and $0 < |\kappa'| < \infty$. According to the above definition,

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \kappa'^2 \sin^2 \phi}} = cel(\kappa', 1, 1, 1), \quad (3.60)$$

$$\Lambda_o(\psi, \kappa) = \frac{2}{\pi} \sin \psi \sqrt{1 + \kappa^2 \tan^2 \psi} \int_0^K \frac{\sin^2 u du}{1 + \kappa^2 \tan^2 \psi \sin^2 u}$$

$$= \frac{2}{\pi} \sqrt{P_\Lambda} \sin \psi \ cel(\kappa', P_\Lambda, 1, \kappa'^2), \quad (3.61)$$

56
where \( p_A = 1 + k^2 \tan^2 \psi \). Equation (3.61) is proved in Appendix C. The evaluation of \( \text{cel} \) is based on Bartky's transformation which is given in [9].

It can be shown that the following addition theorem exists (refer to Appendix D):

\[
AK(\kappa) + BA_0(\psi, \kappa) = \text{cel}(\kappa', p_A, A + B' \, A_{p_A} + B' \kappa'^2),
\]  

where \( A \) and \( B \) are constants. By utilizing Equations (3.60), (3.61) and (3.62), (3.53) can be solely expressed in term of the general complete elliptic integral \( \text{cel} \).

In summary, when \( A_{i3} \neq 0 \ (i = 1, 2) \), \( \rho \neq 0 \) and \( \rho \neq R_s \),

\[
\tilde{H}_{Jw} = \varphi \frac{JR_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \text{cel}(\kappa_i', p_i, A_i + B_i, A_i p_i + B_i \kappa_i'^2),
\]  

where with \( i = 1 \) and \( 2 \),

\[
A_i = \frac{2A_{i3}}{\rho \sqrt{(\rho + R_s)^2 + A_{i3}^2}}, \quad B_i = \frac{2(\rho + R_s)A_{i3}}{\rho(\rho - R_s)\sqrt{A_{i3}^2 + (\rho + R_s)^2}}, \\
p_i = \frac{\rho + R_s}{|\rho - R_s|} \frac{(\rho - R_s)^2 + A_{i3}^2}{(\rho + R_s)^2 + A_{i3}^2}, \\
\kappa_i' = \sqrt{\frac{(\rho - R_s)^2 + A_{i3}^2}{(\rho + R_s)^2 + A_{i3}^2}}.
\]

When \( A_{i3} \neq 0 \ (i = 1, 2) \), \( \rho \neq 0 \) and \( \rho = R_s \),

\[
\tilde{H}_{Jw} = \varphi \frac{JR_s^2}{8\pi} \sum_{i=1}^{2} (-1)^i \frac{2A_{i3}}{\sqrt{4R_s^2 + A_{i3}^2}} \text{cel}(\kappa_i', 1, 1, 1),
\]  

where with \( i = 1 \) and \( 2 \),

\[
\kappa_i' = \frac{A_{i3}^2}{\sqrt{4R_s^2 + A_{i3}^2}}.
\]

When \( \rho = 0 \), \( \tilde{H}_{Jw} = 0 \). When \( A_{i3} = 0 \ (i = 1 \text{ or } 2) \), the corresponding \( i \) terms in the summations of Equations (3.63) and (3.64) become zero.
\[ \tilde{H}_{t,b} \rho_{sc} = 0. \] The expressions for evaluating \( \tilde{H}_{t,b} \rho_{sc} \) can be found in (3.39), where \( G_t = -J/2 \) and \( G_b = J/2 \).

The above equations are very simple and concise compared with (3.19) and (3.20) or (3.25). It can also be shown that there is no mathematical singularity associated with Equations (3.63) and (3.64).

### 3.4 Magnetic Field Strength of an Air-Core Helical Coil

A helical coil is modeled as a number of cylindrical conductor segments. Geometry parameters of each individual conductor segment and the coordinates of field points in the local coordinate system are required for calculating the magnetic field strength of an air-core helical coil by using the equations derived in Sections 3.2 and/or 3.3. These geometry parameters include the length of the segment, \( h_t \) and \( h_b \), the radius of the conductor segment and the oblique angles of two end planes, \( \theta_t \) and \( \theta_b \).

In this section, the methods for dividing a helical coil into a number of straight conductor segments are discussed. The transformation of coordinates and vector components between the local coordinate system of the conductor segment and the global coordinate system of the coil are described.

#### 3.4.1 Dividing a Helical Coil into Segments

The center line of the conductor of a coil layer forms a helix, as shown in Figure 9. In the figure, \( \rho_c \phi_c Z_c \) and \( X_c Y_c Z_c \) are the global cylindrical and rectangular coordinate systems of the coil, respectively. The origin is at \( O_c \). Assuming \( r_j^0 \) is the radius of the conductor center line of the \( j \)th layer and \( z_j^0 \) is the \( z \) coordinate in the \( \rho_c \phi_c Z_c \) or \( X_c Y_c Z_c \) coordinate system of the starting point of the conductor center line of the \( j \)th layer, where \( j = 1, \cdots, N_{\text{layer}} \) and \( N_{\text{layer}} \) is the total number of layers.
in the coil. Then the coordinates of the starting point of the center line of the $j$th layer equal $(r_i^j, 0, z_i^j)$ in the $\rho_c\varphi_cZ_c$ coordinate system. The current in the coil flows from the bottom end to the top end, complying with the right hand rule. For a point on the helix of the conductor center line of the $j$th layer, it satisfies the following equations in the $X_cY_cZ_c$ coordinate system

$$
\begin{align*}
x &= r_i^j \cos \theta, \\
y &= r_i^j \sin \theta, \\
z &= \frac{h_p}{2\pi} \theta + z_i^j,
\end{align*}
$$

where $h_p$ is the pitch of the helix and $\theta$ is the polar angle in radians measured in the $X_cO_cY_c$ plane and counted consecutively in each layer (refer to Figure 9). The
above equation is also called the equation of the helix.

Let $N_{seg}$ be the number of conductor segments per turn, that is, each turn is divided into $N_{seg}$ arc segments and $i$ be the segment number counted consecutively in each layer. In other words, the segment number of the last segment in a layer is equal to the $N_{seg}$ times the total number of turns in the layer.

For the $i$th segment of the $j$th layer, the polar angle of its starting point $O_{i-1}^j$ on the center line of the conductor segment, is $\theta_{i-1}^j = 2\pi(i-1)/N_{seg}$ and the polar angle of its ending point $O_i^j$ on the center line, is $\theta_i^j = 2\pi i/N_{seg}$. Therefore, the coordinates of $O_{i-1}^j$ and $O_i^j$ in the $X_cY_cZ_c$ coordinate system are $(r_i^j \cos \theta_{i-1}^j, r_i^j \sin \theta_{i-1}^j, h_p \theta_{i-1}^j(2\pi)^{-1} + z_d^j)$ and $(r_i^j \cos \theta_i^j, r_i^j \sin \theta_i^j, h_p \theta_i^j(2\pi)^{-1} + z_d^j)$, respectively.

Each original arc conductor segment can be replaced by three possible different types of straight conductor segments given in Figure 10. They are referred to as Type A, Type B and Type C conductor segment, respectively. In the figure, $O_i^{ej}$ is the center point of the center line of the $i$th arc conductor segment. $O_k^j$ is the center point of the $k$th turn of the $j$th layer, where $k = 1, \cdots, N_j$, $j = 1, \cdots, N_{layer}$ and $N_j$ is the total number of turns in the $j$th layer. $O_k^j$ has the coordinates of $(0, 0, z_d^j + kh_p - h_p/2)$.

a. Straight Conductor Segment Type A

For the Type A segment given in Figure 10(a), it is assumed that

- the end planes of the original arc conductor segment coincide with that of the straight segment;
- $h_{it}^j = h_{ib}^j = \frac{1}{2} \overrightarrow{O_{i-1}^j O_i^j}$, where $\overrightarrow{O_{i-1}^j O_i^j}$ is the distance between $O_{i-1}^j$ and $O_i^j$, $h_{it}^j$ and $h_{ib}^j$ are the $h_t$ and $h_b$ of the $i$th conductor segment in the $j$th layer, $i = 1, \cdots, N_{j, seg}$ and $j = 1, \cdots, N_{layer}$.
Figure 10: Three different types of straight conductor segment models.
The major advantage of Type A segment is that it shares the common end planes with its adjacent segments. However, the length of its center line is shorter than that of the original arc conductor segment. Therefore, the volume of the straight conductor segment is less than that of the original arc conductor segment. This implies that this model might need a larger number of conductor segments compared with Type B and Type C segments (which are described below) for a given calculation accuracy. The more the number of conductor segments is used, the smaller the volume difference will be.

b. Straight Conductor Segment Type B

For the Type B segment given in Figure 10(b), it is assumed that

- the center point of the straight conductor segment coincides with the center of the original arc conductor segment \( O_{ij} \);
- the center line of the straight conductor segment parallels the straight line that connects points \( O_{i-1}^j \) and \( O_i^j \);
- the length of the center line of the straight conductor segment is equal to that of the original arc conductor segment;
- the diameter of the straight conductor segment is the same as the coil conductor diameter;
- the slope angle of the top end plane \( \theta_{it}^j \) is determined by the smaller angle between the line \( O_i^j O_i^j \) and the line \( O_{i-1}^j O_i^j \);
- the slope angle of the bottom end plane \( \theta_{ib}^j \) is determined by the smaller angle between the line \( O_{i-1}^j O_i^j \) and the line \( O_{i-1}^j O_i^j \);
- \( h_{it}^j = h_{ib}^j = \frac{1}{2} S_i^j \), where \( S_i^j \) is the length of the arc \( O_{i-1}^j O_i^j \).
It can been seen from the above assumptions that the conductor segment of Type B can give approximately equal volume as the original arc conductor segment. Therefore, it can be predicted that using Type B segment model can give better calculation accuracy than using the conductor segment of Type A for the same number of conductor segments per turn.

The expressions for determining $h_{it}^j$, $h_{ib}^j$, $\theta_{it}^j$ and $\theta_{ib}^j$ ($i = 1, \ldots, N_{j,seg}$, $j = 1, \ldots, N_{layer}$) are given in the following.

The differential length of an arc on the helix equals
\[
dl = \sqrt{x^2(\theta) + y^2(\theta) + z^2(\theta)} \, d\theta
= \sqrt{r_a^2 + \frac{h_p \theta}{2\pi}^2} \, d\theta.
\]

Thus the length of the arc $O_{i-1}^j O_i^j$, $S_i^j$, is equal to
\[
S_i^j = \int_{\theta_{i-1}^j}^{\theta_{i}^j} \sqrt{r_a^2 + \frac{h_p \theta}{2\pi}^2} \, d\theta
= \left[ \frac{\theta}{2} \sqrt{\frac{h_p^2 \theta^2}{4\pi^2} + r_a^2 + \frac{\pi r_a^2}{h_p} \arsh \frac{h_p \theta}{2\pi r_a}} \right]_{\theta_{i-1}^j}^{\theta_{i}^j},
\]
where $h_p \neq 0$, $i = 1, \ldots, N_{j,seg}$, $j = 1, \ldots, N_{layer}$. If $h_p = 0$,
\[
S_i^j = (\theta_{i}^j - \theta_{i-1}^j) r_a^j,
\]
where $i = 1, \ldots, N_{j,seg}$, $j = 1, \ldots, N_{layer}$.

As a result, if $h_p \neq 0$,
\[
h_{it}^j = h_{ib}^j = \left[ \frac{\theta}{4} \sqrt{\frac{h_p^2 \theta^2}{4\pi^2} + r_a^2 + \frac{\pi r_a^2}{h_p} \arsh \frac{h_p \theta}{2\pi r_a}} \right]_{\theta_{i-1}^j}^{\theta_{i}^j},
\]
where $i = 1, \ldots, N_{j,seg}$, $j = 1, \ldots, N_{layer}$. If $h_p = 0$,
\[
h_{it}^j = h_{ib}^j = (\theta_{i}^j - \theta_{i-1}^j) r_a^j/2,
\]

63
where \( i = 1, \cdots, N_{j,\text{seg}}, j = 1, \cdots, N_{\text{layer}} \).

It can be derived that the vectors \( \mathbf{O}_i \), \( \mathbf{O}_{i-1} \), \( \mathbf{O}_{i} \), \( \mathbf{O}_{i-1} \) and \( \mathbf{O}_{i} \), \( \mathbf{O}_{i-1} \) have the following expressions:

\[
\mathbf{O}_{i-1} \mathbf{O}_i = r_i \left( \cos \theta_i - \cos \theta_{i-1} \right) \hat{x} + r_i \left( \sin \theta_i - \sin \theta_{i-1} \right) \hat{y} \\
+ \frac{h_p}{2\pi} (\theta_i - \theta_{i-1}) \hat{z},
\]

(3.68)

\[
\mathbf{O}_i \mathbf{O}_{i-1} = r_i \left( \cos \theta_i - \cos \theta_{i-1} \right) \hat{x} + r_i \left( \sin \theta_i - \sin \theta_{i-1} \right) \hat{y} \\
+ \frac{h_p}{2\pi} (\theta_i - \theta_{i-1}) \hat{z},
\]

(3.69)

\[
\mathbf{O}_{i-1} \mathbf{O}_{i} = r_i \left( \cos \theta_{i-1} - \cos \theta_i \right) \hat{x} + r_i \left( \sin \theta_{i-1} - \sin \theta_i \right) \hat{y} \\
+ \frac{h_p}{2\pi} (\theta_{i-1} - \theta_i) \hat{z},
\]

(3.70)

where \( \theta_i \) is the polar angle of the point \( O_{i} \), \( i = 1, \cdots, N_{j,\text{seg}} \) and \( j = 1, \cdots, N_{\text{layer}} \). By substituting the values of \( \theta_i \), \( \theta_{i-1} \) and \( \theta_{i-1} \) into the above equations, the angle between \( \mathbf{O}_{i-1} \mathbf{O}_i \) and \( \mathbf{O}_i \mathbf{O}_{i-1} \), i.e., \( \theta_{ii} \), equals

\[
\cos \theta_{ii} = \frac{r_i^2 \left( 1 - \cos \frac{2\pi}{N_{\text{seg}}} \right) + \frac{h_p^2}{2N_{\text{seg}}^2}}{\sqrt{\left[ 2r_i^2 \cos \frac{\pi}{N_{\text{seg}}} + \frac{h_p^2}{4N_{\text{seg}}^2} \right] \cdot \left[ 2r_i^2 \cos \frac{\pi}{N_{\text{seg}}} + \frac{h_p^2}{4N_{\text{seg}}^2} \right]}}
\]

(3.71)

and the angle between \( \mathbf{O}_{i-1} \mathbf{O}_i \) and \( \mathbf{O}_i \mathbf{O}_{i-1} \), i.e., \( \theta_{ii} \), can be found to be equal to \( \theta_{ii} \), where \( i = 2, \cdots, N_{j,\text{seg}} - 1 \) and \( j = 1, \cdots, N_{\text{layer}} \).

For the first segment in the \( j \)th layer, \( \theta_{1b} = 0 \) and \( \theta_{1b} \) can be found from the above equation, \( j = 1, \cdots, N_{\text{layer}} \). Similarly, for the last segment in the \( j \)th layer, \( \theta_{ib} = 0 \) and \( \theta_{ib} \) can be found from the above equation, where \( i = N_{j,\text{seg}} \) and \( j = 1, \cdots, N_{\text{layer}} \).

c. Straight Conductor Segment Type C

The conductor segment of Type C has the same assumptions as Type B segment except that it uses the bar segment model instead of the general segment model, that
is, its end planes are non-oblique. Therefore, $\theta_{it}^j = \theta_{ib}^j = 0$ for all conductor segments, where $i = 1, \cdots, N_{j, \text{seg}}$ and $j = 1, \cdots, N_{\text{layer}}$.

From the expressions given in Section 3.3, it can be seen that using Type C segment can greatly simplify the calculations compared with the expressions for the general conductor segments. Using Type C segment can also give satisfactory results in most practical cases (see Section 3.5).

3.4.2 Transformation between the Local Coordinate System of Conductor Segments and the Global Coordinate System of the Coil

When calculating the magnetic field strength due to a conductor segment, it is necessary to change the coordinates of a field point given in the global coordinate system of the coil into the coordinates in the local coordinate system of the conductor segment. In addition, the magnetic field strength of the conductor segment calculated is a vector given in the three directions of the local rectangular coordinate system of the conductor segment. It is necessary to change these three components into the components expressed in the directions of the global rectangular coordinate system. Transformation between the local coordinate system of a conductor segment and the global coordinate system of the coil, therefore, needs to be discussed.

The local coordinate system $X_i^j Y_i^j Z_i^j$ of the $i$th segment in the $j$th layer is chosen as follows ($i = 1, \cdots, N_{\text{seg}} \cdot N_j$ and $j = 1, \cdots, N_{\text{layer}}$) (see Figure 10):

- let the origin of the $X_i^j Y_i^j Z_i^j$ coordinate system coincide with $O_i^j$;
- $X_i^j$ axis is chosen to coincide with the line $O_k^j O_i^j$ and points from $O_k^j$ to $O_i^j$;
- $Z_i^j$ axis is parallel to the line $O_{i-1}^j O_i^j$ and points from $O_{i-1}^j$ to $O_i^j$.

The expression of the vector $O_{i-1}^j O_i^j$ is given in (3.68). The vector $O_k^j O_i^j$ has
the form of

$$O_k^j O_i^j = r_d^j \cos \theta_i^j \hat{x}_c + r_d^j \sin \theta_i^j \hat{y}_c + \left[ \frac{h_p \theta_i^j}{2 \pi} - (2k - 1)h_p/2 \right] \hat{z}_c. \quad (3.72)$$

The angle between \( \hat{z}_c \) and \( \hat{z}_i \), that is, \( \beta_{i1}^j \), is equal to the angle between the vectors \( \hat{z}_c \) and \( O_{i-1}^j \hat{O}_i^j \):

$$\cos \beta_{i1}^j = \frac{h_p}{N_{seg}} \left[ \frac{2r_d^j}{2} - 2r_d^j \cos \frac{2\pi}{N_{seg}} + \frac{h_p^2}{N_{seg}} \right]^{1/2}, \quad (3.73)$$

where \( \beta_{i1}^j \) is independent of \( i \) and \( j = 1, \ldots, N_{layer} \). The angle between \( \hat{x}_c \) and \( \hat{x}_i \), that is, \( \beta_{i2}^j \), equals the angle between the vectors \( \hat{x}_c \) and \( O_k^j \hat{O}_i^j \):

$$\cos \beta_{i2}^j = \frac{r_d^j \cos \frac{(2i-1)\pi}{N_{seg}}}{\left\{ r_d^j + \left[ \frac{(2i-1)h_p}{2N_{seg}} - \frac{(2k-1)h_p}{2} \right]^2 \right\}^{1/2}}. \quad (3.74)$$

It can be seen from the above equations that if \( h_p = 0 \), then \( \beta_{i1}^j = \pi/2 \) and \( \beta_{i2}^j = (2i - 1)\pi/N_{seg} \).

According to Equations (A.1) and (A.2), the coordinates \( (x, y, z) \) of a field point in the local coordinate system \( X_i^j Y_i^j Z_i^j \), which has the coordinates \( (x_c, y_c, z_c) \) in the global coordinate system, is found to be

$$\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} =
\begin{bmatrix}
  \cos \beta_{i2}^j & \sin \beta_{i2}^j & 0 \\
  -\cos \beta_{i1}^j \sin \beta_{i2}^j & \cos \beta_{i1}^j \cos \beta_{i2}^j & \sin \beta_{i1}^j \\
  \sin \beta_{i1}^j \sin \beta_{i2}^j & -\sin \beta_{i1}^j \cos \beta_{i2}^j & \cos \beta_{i1}^j
\end{bmatrix}
\begin{bmatrix}
  x_c - r_d^j \cos \theta_i^j \\
  y_c - r_d^j \sin \theta_i^j \\
  z_c - h_p \theta_i^j/(2\pi) - z_o
\end{bmatrix}, \quad (3.75)$$

where \( i = 1, \ldots, N_{j,seg}, j = 1, \ldots, N_{layer} \), \( \beta_{i1}^j \) and \( \beta_{i2}^j \) are given in Equations (3.73) and (3.74), respectively.
A local vector \( H_{i,x}^j \hat{x}_i + H_{i,y}^j \hat{y}_i + H_{i,z}^j \hat{z}_i \) can be changed into a global one 
\( H_{i,x}^i \hat{x}_c + H_{i,y}^i \hat{y}_c + H_{i,z}^i \hat{z}_c \) by using the following matrix expression:

\[
\begin{bmatrix}
H_{i,x}^i \\
H_{i,y}^i \\
H_{i,z}^i
\end{bmatrix} = \begin{bmatrix}
\cos \beta_{i1}^j - \sin \beta_{i1}^j \cos \beta_{i2}^j & \sin \beta_{i1}^j \sin \beta_{i2}^j \\
\sin \beta_{i1}^j \cos \beta_{i2}^j & \sin \beta_{i2}^j - \sin \beta_{i1}^j \cos \beta_{i2}^j \\
0 & \sin \beta_{i1}^j \cos \beta_{i2}^j
\end{bmatrix}
\begin{bmatrix}
H_{i,x}^j \\
H_{i,y}^j \\
H_{i,z}^j
\end{bmatrix}
\]

which is obtained by inversing the equations given in (A.2).

### 3.5 Quantitative Evaluation of the Volume Current Method

The magnetic flux density of several simple coils has been calculated at various field points by different modeling methods for the purposes of

- verifying the calculation accuracy of the proposed new modeling method, the Volume Current Method (VCM) and the equations derived;
- comparing VCM with the filament modeling methods which ignore the actual cross section of the conductor and have been widely used in coil magnetic field strength calculations;
- investigating the application range of the filament model for an acceptable calculation error;
- studying the accuracy of the planar filament current loop stack model which ignores the pitch angle of the coil.

The calculation results are presented and analyzed in the following subsections.

The sample coils used in this study include

- **COIL1**: a one-turn current loop coil, whose mean radius is 1.015 m and the conductor radius is 0.015 m;
Figure 11: Geometry of a one-turn loop coil and the calculation traverses. (a) The top view of the coil, (b) The front view of the coil and traverses.
- COIL2: a one-turn current loop coil, whose mean radius is 1.005 m and the radius of the cross section of the conductor is 0.005 m;

- COIL3: a single-layer helical coil, whose mean radius is 1.005 m, the height is 1.05 m, the conductor radius is 0.005 m, the number of turns is 100 and the pitch of the coil is 0.0105 m.

Figure 12: A single-layer coil and its related traverses.

The currents flowing in all three coils are assumed to be 1 A. All coils are assumed to be made of copper.

The geometry of COIL1 or COIL2 is shown in Figure 11(a), where the center of the loop is located at the origin. Referring to Figure 11(b), four typical traverses are used for the calculations using COIL1 or COIL2, represented by the abbreviations ACC, ASC, LCC and LSC, respectively, which mean:

- ACC: axial traverse, center of conductor,
• ASC: axial traverse, side of conductor,

• LCC: lateral traverse, center of conductor,

• LSC: lateral traverse, side of conductor.

The geometry of COIL3 is shown in Figure 12. In Figure 12, the abbreviations ATS, LTC and LTT are the three traverses used in the calculations using COIL3 which mean

• ATS: axial traverse, side of coil,

• LTC: lateral traverse, center of coil,

• LTT: lateral traverse, top of coil,

3.5.1 Verification of the VCM

In this subsection, the calculation accuracy of the VCM is tested by using a simple single-loop coil. The results are compared with those obtained by the finite element method (FEM).

a. Test Case

The magnetic flux density of COIL1 (a single-loop coil) was calculated by the VCM with different numbers of conductor segments per turn along the four different traverses ACC, ASC, LCC and LSC (refer to Figure 11), respectively. In the VCM, the Type C conductor segment model was used.

b. Reference Case

The magnetic flux density of COIL1 was calculated by the FEM along the same traverses used in the test case. In the FEM, a 2D axisymmetrical finite element
model at 60 Hz was used. Maxwell\textsuperscript{1} 2D Field Simulator was utilized to obtain the FEM results.

For a coil, like COIL1, which is simple and of axisymmetrical geometry, the 2D axisymmetrical finite element model can give very accurate results. Therefore, the results obtained by the FEM for COIL1 are used as the reference data.

c. Results

The results by using the VCM and FEM are compared and illustrated in Figures 13 to 18 for the resultant magnetic flux density $B_{res}$ and the magnetic flux density in the $y_c$ and/or $z_c$ axial directions, i.e., $B_{y_c}$ and/or $B_{z_c}$ components, respectively.

Figures 13 to 18 show that the magnitude of the magnetic flux density on the surface of the conductor is much higher than at internal field points. The resultant magnetic flux density decreases with the distance away from the conductor. The resultant magnetic flux density along ACC traverse is symmetrical about the $y_c$ axis (see Figure 16). This is consistent with the expected results. However, the resultant magnetic flux density along the LCC traverse is not completely symmetrical about the conductor center line in the $z_c$ axial direction (see Figure 13). Field points close to the inner part of the loop experience higher magnetic flux density values than field points near the outer part, or the top or the bottom part of the loop by comparing Figure 13 with Figure 16.

From Figures 13 to 18, it can be seen that in general, the agreement is good between the VCM and the FEM when the number of conductor segments per turn is at least ten. Good agreement is shown for the resultant magnetic flux density and

\textsuperscript{1}It is the software package of Ansoft Corporation.
for each individual axial component of magnetic flux density as well, even though a test case which is worse than most real cases was chosen.

The actual reactor coils are usually made of aluminum whose skin depth is about 0.011 m (see Table 1). The typical conductor size of the reactor coils is less than its skin depth. The current distribution in the conductors of reactor coils is thus nearly uniform. However, in the test case, the conductor radius of the COIL1 is 0.015 m which is larger than the skin depth of the conductor (0.0085 m for copper, refer to Table 1). The existence of the skin effect causes only small differences between the results of the VCM and the FEM at field points inside the conductor (refer to Figures 13 and 16).

The calculation accuracy of the VCM for the real cases must be even better than in the test case, because the current is more uniformly distributed in the conductor for most reactor coils than the test case. Hence, it can be concluded that the VCM can adequately and accurately model air-core reactor coils.

d. Parametric Analysis

Appropriate Number of Conductor Segments per Turn The smaller number of conductor segments per turn is used, the more computation time is saved. However, it can be seen from Figures 13 to 18 that when the number of conductor segments per turn is decreased, discrepancies between the VCM and FEM become larger, that is, the accuracy of the VCM is decreased. Figures 14(a), 16(a), 17 and 18 show that four conductor segments per turn are not adequate for modeling a round loop. From Figures 13 to 18, it can be concluded that in general, about 10 conductor segments per turn are adequate for modeling a round loop.

Effects of Conductor Size By comparing Figure 13(b) with Figure 19(a), and
Figure 16(b) with Figure 21, it can be concluded that with the same conductor current, but smaller conductor size, the resultant magnetic flux density at the surface of the conductor is higher. In addition, with smaller conductor size, the magnetic flux density value drops faster with the distance away from the conductor surface.

e. Conclusions

The new modeling method VCM developed can accurately calculate the 3D magnetic field strength of air-core coils at arbitrary field points. Generally, about 10 conductor segments per turn are adequate for modeling a round loop. In the following comparison tests, the results obtained by the VCM, therefore, are used as the "standard" ones.

3.5.2 Comparison with the Filament Model

Due to the simplicity of the filament model and its calculation formulas, it has been extensively utilized to model air-core coils. It can be derived mathematically that an infinitely long ideal filament line can accurately model an infinitely long circular cylindrical bar as long as field points are outside the conductor. However, for finite conductors, the application range of the filament model might be smaller than for infinitely long conductors. In this subsection, the calculation results of a single-loop coil by using the ideal filament current loop model is compared with that by using the VCM, the reference ("standard") method.

a. Test Case

The magnetic flux density of COIL2 (a single-loop coil) was calculated by the ideal filament current loop model along the LCC and ACC traverses (refer to Figure 11), respectively. In the ideal filament current loop model, the actual coil loop
is replaced by an ideal filament current loop which coincides with the center line of the actual coil conductor.

b. Reference Case

The magnetic flux density of COIL2 calculated by the VCM along the same traverses given in the above test case is used as the reference results. In the VCM, the Type C conductor segment model was used again.

c. Results

The resultant magnetic flux density components calculated by the ideal filament current loop model and the VCM along the LCC traverse vs. \( y_c \), and the discrepancy between them vs. the normalized \( y_c \) distance in terms of conductor radius away from the center of the conductor are plotted in Figure 19. In Figure 19(b), the discrepancy between the filament current loop model and the VCM is calculated by the formula:

\[
\text{error}(\%) = 100\% \times \left| \frac{B(\text{Filament current loop model}) - B(\text{VCM})}{B(\text{VCM})} \right|
\]

where \( R_s \) is the radius of the conductor and \( R_m \) is the mean radius of the coil. \( B_{yc} \), \( B_{zc} \) and \( B_{res} \) along ACC traverse calculated by the VCM and the filament current loop model vs. \( z_c \) are plotted in Figures 20 and 21, respectively. The discrepancies of values of \( B_{yc} \), \( B_{zc} \) and \( B_{res} \) between the VCM and the filament current loop model along ACC traverse are computed and shown in Figure 22.

Figures 19 to 22 show that the agreement between the filament current loop model and the VCM is good only for field point locations outside the conductor, having a small distance away from the surface of the conductor. The filament current loop model gives the values of the magnetic flux density inside the conductor much higher than the values given by the VCM. This is due to the singularity problem that the filament model exhibits.
When a field point is at least three times the conductor radius away from the center of the conductor, the discrepancy between the filament current loop model and the VCM (here, the reference method) is below 2% (refer to Figure 22).

d. Conclusions

It is found that the ideal filament current loop model is applicable only for field point locations outside the conductor, having a distance at least three times the conductor radius away from the center of the conductor.

3.5.3 Comparison with the Planar Filament Current Loop Stack Model

The Planar Filament Current Loop Stack Model has been implemented in the code COILB (refer to Section 2.2.1) for modeling helical reactor coils. Here, its accuracy and appropriate application range are studied by comparing with the VCM.

a. Test Case

The magnetic flux density of COIL3, i.e., a single-layer helical coil, was calculated by using the code COILB (which is based on the filament current loop stack model) along the traverses ATS, LTC and LTT (see Figure 12), respectively.

b. Reference Case

The magnetic flux density of the same coil and along the same traverses given in the above test cases was calculated by the VCM. The VCM is used here as the reference method. In the VCM, the Type C conductor segment model was used.

c. Results

The discrepancies between the filament current loop stack model and the VCM are calculated by the following equation:

\[
\text{error(\%)} = 100\% \times \left| \frac{B(\text{COILB}) - B(\text{VCM})}{B(\text{VCM})} \right|.
\]
The two typical plots are shown in Figures 23 and 24. The discrepancies between the filament current loop stack model and the VCM along the ATS traverse at $x_c = 0$ and $y_c = 1.0275 \text{ m}$ are shown in Figures 23. Figure 24 gives the discrepancies between the filament current loop stack model and the VCM along the LTT traverse (at $x_c = 0$ and $z_c = 0.556 \text{ m}$) vs. the normalized $y_c$ distance in terms of coil mean diameter away from the center of the conductor. In Figure 24, $R_m$ and $D_m$ are the coil mean radius and diameter, respectively.

It can be seen in Figure 23 that the error of $B_{yc}$ at locations close to the center area of the coil ($z_c$ is close to zero) is very large. But at those locations, the value of $B_{yc}$ component is very small compared to the values of $B_{zc}$ and $B_{res}$. The errors of $B_{zc}$ and $B_{yc}$ at the locations close to the end of the coil ($z_c$ is around 0.5 m to 0.6 m) are very large. Figure 24 shows that when the field points are at least 0.2 times the coil mean diameter (or 80 times the conductor radius) away from the outer surface of the coil, the errors of all magnetic flux density components are less than 2.5%.

For the calculation results given in Figures 23 and 24, all field points are at least four times the conductor radius away from the closest conductor center. From Section 3.5.2, it is known that within that range, if the cross section of the conductor is ignored, the errors are below 2%. Therefore, the excess error shown in Figures 23 and 24 are caused because the pitch angle of the coil in the current loop stack model is ignored. In COIL3, the only space between coil turns is the thin conductor insulation thickness. This implies that in the close vicinity of the coil, the pitch angle of the coil must be taken into consideration in the model and can't be ignored. Otherwise, large errors will be introduced even if the interturn space of the coil is very small. The planar filament current loop stack model is applicable to modeling a helical coil satisfactorily only at field points which are at least about 0.2 times the coil mean
diameter away from the outer surface of the coil, pending on the pitch angle of the coil. When the pitch angle increases, field points must be further away from the coil for the appropriate application of the planar filament current loop stack model.

d. Conclusions

It is found that the planar filament current loop stack model is appropriate for modeling a helical coil only at field points which are at least about 0.2 times the coil mean diameter away from the outer surface of the coil. The above stated minimal distance will increase when the pitch angle of the coil increases. Otherwise, the pitch angle of the coil must be considered.
Figure 13: Magnetic flux density of COIL1 along the lateral traverse through the center of the conductor (LCC traverse), (a) $B_{zc}$ vs. $y_c$, (b) $B_{res}$ vs. $y_c$. 
Figure 14: Magnetic flux density of COIL1 along the lateral traverse at the side of the conductor (LSC traverse), (a) $B_{yc}$ vs. $yc$, (b) $B_{zc}$ vs. $yc$. 

79
Figure 15: Resultant magnetic flux density of COIL1 along the lateral traverse at the side of the conductor (LSC traverse).
Figure 16: Magnetic flux density of COIL1 along the axial traverse through the center of the conductor (ACC traverse), (a) $B_{yc}$ vs. $z_c$, (b) $B_{res}$ vs. $z_c$. 
Figure 17: Magnetic flux density of COIL1 along the axial traverse at the side of the conductor (ASC traverse), (a) $\vec{B}_{yc}$ vs. $z_c$, (b) $\vec{B}_{zc}$ vs. $z_c$. 

82
Figure 18: Resultant magnetic flux density of COIL1 along the axial traverse at the side of the conductor (ASC traverse).
Figure 19: Resultant magnetic flux density components of COIL2 along the lateral traverse through the center of the conductor (LCC traverse) calculated by the VCM and the filament current loop model, respectively, and their discrepancy, (a) $B_{res}$ vs. $y_c$, (b) Discrepancy of two results vs. the normalized $y_c$ distance in terms of conductor radius ($R_s$) away from the center of the conductor.
Figure 20: Magnetic flux density components of COIL2 along the axial traverse through the center of the conductor (ACC traverse) calculated by the VCM and the filament current loop model, respectively, (a) $B_{yc}$ vs. $zc$, (b) $B_{zc}$ vs. $zc$. 

85
Figure 21: Resultant magnetic flux density components of COIL2 along the axial traverse through the center of the conductor (ACC traverse) calculated by the VCM and the filament current loop model, respectively.

Figure 22: Discrepancy of the results between the filament current loop model and the VCM for COIL2 along the axial traverse through the center of the conductor (ACC traverse).
Figure 23: Discrepancy of the results between the filament current loop stack model and the VCM for COIL3 along the axial traverse at the side of the coil (ATS traverse) at \( x_c = 0 \) and \( y_c = 1.0275 \) m.

Figure 24: Discrepancy of the results between the filament current loop stack model and the VCM vs. the normalized \( y_c \) distance in terms of coil mean diameter \((D_m)\) away from the center of the conductor for COIL3 along the lateral traverse at the top of the coil (LTT traverse).
3.6 Overall Conclusions

An accurate modeling method, the Volume Current Method (VCM), for calculating the three-dimensional magnetic field strength of an air-core helical coil with volume current distribution is proposed in this chapter. The analytical expressions of the magnetic field strength of a circular cylindrical conductor segment, which is the basic modeling element of large air-core reactor coils, have been derived. The derived equations are valid for arbitrary field point locations either inside or outside the coil. The derived equations are useful for solving problems that involve the accurate calculation of the magnetic field distribution in and around a coil, the magnitude of forces on a coil, the current distribution in the conductors of a coil and inductive properties of a coil, where the coil is made of circular cross-section conductor. The modeling method developed is applicable to any helical coils made of circular cross-section conductors, carrying static or quasi-static currents. The only requirement is that either the conductor radius is smaller or comparable to its skin depth, or the conductor is composed of stranded subconductors. Therefore, the assumption of the uniform current distribution in the coil conductors is valid.

Section 3.4.1 described three possible modeling segments (Types A, B and C) and gave the formulas for finding the geometric parameters of different modeling segments, such as, the radius, the lengths ($h_t$ and $h_b$) and the slope angles of the end surfaces ($\theta_t$ and $\theta_b$). Section 3.4.2 gave the formulas for changing the given coordinates of a field point in the global coordinate system of the coil into the coordinates in the local coordinate system of each modeling segment. The matrix equations are also given in that section for transforming vector components obtained in the local coordinate system of the conductor segment back into the vector components in the global coordinate system of the coil.
The equations used for evaluating the magnetic field strength due to a modeling segment of Type A or B are given in Section 3.2. The equations used for calculating the magnetic field strength due to a modeling segment of Type C are given in Section 3.3.

From the calculation results presented in Section 3.5, it can be concluded that

1. the proposed modeling method VCM and derived equations can be used to evaluate the quasi-static magnetic field distribution of an air-core helical coil at arbitrary field points accurately;

2. when field points are in the immediate vicinity of the coil, e.g., within a range of three times the conductor radius away from the center of the conductor, the VCM must be employed in order to give accurate results; beyond that range, the filament model can be utilized to save computation time;

3. the effects of pitch angle must be taken into consideration when the field point is in the very close vicinity of the coil, say, within a distance of at least 0.2 times the coil mean diameter away from the outer surface of the coil, pending on the pitch angle of the coil;

4. when the interspace between the turns of the coil is only caused by the thin conductor insulation thickness, the minimum distance away from the outer surface of the coil for ignoring the pitch angle of the coil and having less than 2.5% calculation error is 0.2 times the coil mean diameter;

5. beyond the above stated critical distance which will increase with the increase of the coil pitch angle, a helical coil can be accurately modeled by a stack of planar filament current loops to save computation time.
CHAPTER 4
CURRENT DISTRIBUTION AMONG MULTIPLE COIL LAYERS CONNECTED IN PARALLEL

4.1 Introduction

Two methods for determining the current distribution among coil layers are proposed in this chapter. One of them is the field differential method, and the other one is the approximate impedance method.

The field differential method can be used to obtain the current distribution in each layer of a coil accurately by using the differential form of Ampere's law. It utilizes the method presented in Chapter 3 to accurately evaluate the magnetic field strength or magnetic flux density inside the conductor. The approximate impedance method is based on computing the impedance matrix of the coil layers. It employs analytical expressions derived for approximately computing layer impedances. It is assumed in both methods that the effects of the currents flowing in the spider on the current distribution of each coil layer are ignored.

The calculation accuracy of the field differential method and the approximate impedance method is verified by a single-layer coil and two simple multi-layer coils, respectively. The advantages, disadvantages and limitations of each method are discussed.
4.2 Field Differential Method

From Ampere's law, it is known that the current density in the conductor of a coil is approximately equal to

\[ \vec{J} = \dot{\phi}_c \left( \frac{\partial H_{\rho_c}}{\partial z_c} - \frac{\partial H_{z_c}}{\partial \rho_c} \right), \]

(4.1)

where the center axis of the coil is along \( z_c \) axis and \( J_{\rho_c} \) and \( J_{z_c} \) are ignored. It is legitimate to ignore \( J_{\rho_c} \) and \( J_{z_c} \) components when the pitch angle of the coil is small. The current value in a conductor is the integration of its current density over the conductor cross section. Chapter 3 has presented a method which can accurately estimate the magnetic field strength or magnetic flux density inside a coil conductor when the current amplitudes in coil conductors are available. Therefore, the current distribution among coil layers can be obtained by iteratively solving the magnetic field strength inside coil conductors and Equation (4.1) until the difference between the current values of two consecutive iterations can be ignored.

4.3 Approximate Impedance Method

Due to the fact that all layers are connected in parallel, the current in each layer of a coil can be determined if the impedance matrix of the coil is available. A method for approximately calculating the impedance matrix of layers is proposed in this section. The corresponding equations are also presented.

4.3.1 Impedance of Coil Layers

The impedance of a layer consists of resistance, internal reactance and external reactance. The internal inductance is related to the flux linkage with the conductor itself while the external inductance is related to the flux linkage with the external
area of the conductor. The resistance plus the internal reactance is referred to as internal impedance.

Accurate evaluation of the internal impedance of a layer requires the calculation of the magnetic field strength at field points all over the conductor. That would be very time consuming. In addition, the value of the external inductance of a layer is much larger than its resistance and internal inductance, respectively, of the layer. Therefore, some simplifications are used in deriving the approximate equations for estimating the internal impedance of a layer. Equations for calculating the mutual inductance between layers are also given in the following.

a. Internal Impedance

The formulas for finding an impedance are based on the relationship between the magnetic flux density (or magnetic field strength) and the energy expression as well as the relationship between the current density and the energy expression, which are given by the conservation of energy principle. The analytical solutions are available only for some simplified cases. For the simplified cases, the expressions of magnetic flux density can be obtained by solving the Helmholtz equation with associated boundary conditions.

The Helmholtz equation for linear materials has the form:

\[ \nabla^2 \vec{H} = \gamma^2 \vec{H}, \]  

where \( \omega \) is the radian excitation frequency; \( \gamma \) is the propagation constant and has the expression \( \gamma^2 = \omega \mu \sigma \) at low frequencies where the displacement current can be ignored; and \( \mu \) and \( \sigma \) are the permeability and conductivity of the conductor of the layer. The propagation constant \( \gamma \) can also be expressed by

\[ \gamma = \frac{1 + j}{\delta}, \]
where $\delta$ is the skin depth and is defined by

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}}.$$  

The above equations are valid for quasi-static problems with linear materials.

The coil studied is assumed to be symmetric, long, and tightly wound. Therefore, each layer of the coil is replaced by a long cylindrical sheet conductor as shown in Figure 25. The thickness of conductor insulation is assumed to be very thin and negligible compared with the size of conductor. The inner space between turns thus can be ignored. Each layer carries a complex current $I_i$ and has a thickness of $d_i$, where $i (= 1, \ldots, N_{layer})$ is the layer number counted from inside out. The innermost layer is numbered 1, the outermost layer is numbered $N_{layer}$. The $Z_c$ axis coincides with the center axis of the coil and $\rho_c$ points in the radial direction of the coil. The thickness of a layer equals the diameter of the layer conductor. Symbols $\rho_i$ and $\rho'_i$
are the inner and outer radii of the "equivalent" layer \( i \), respectively. The magnetic field strength \( \vec{H} \) is assumed to be independent of \( \phi_c \) and \( z_c \), that is, \( \vec{H} = H_z(\rho_c) \hat{z}_c \).

Equation (4.2) becomes

\[
\nabla^2 H_z = j\omega\mu_o H_z. 
\]

In the cylindrical coordinate system \( \rho_c\phi_cZ_c \) of the coil, the above equation can be expanded as

\[
\frac{\partial^2 H_z}{\partial \rho_c^2} + \frac{1}{\rho_c} \frac{\partial H_z}{\partial \rho_c} = \gamma^2 H_z. 
\]  \hspace{1cm} (4.3)

By multiplying both sides of Equation (4.3) with \( \rho_c^2 \), it is changed into the form of Bessel's equation of order 0 with the variable \( \gamma \rho_c \).

Since the variable \( \gamma \rho_c \) is a complex number, the general solution of Equation (4.3) takes the form

\[
H_z(\rho_c) = C_1 I_0^B(\gamma \rho_c) + C_2 K_0^B(\gamma \rho_c),
\]  \hspace{1cm} (4.4)

where \( I_0^B \) and \( K_0^B \) are the zero-order modified Bessel functions of the first and second kind, respectively, and \( C_1 \) and \( C_2 \) are arbitrary constants which can be specified by enforcing the appropriate boundary conditions.

It is assumed that

\[
H_z(\rho_c = \rho_i) = H_i, \\
H_z(\rho_c = \rho'_i) = H'_i, \\
H_z(\rho_c = \rho_N^{layer}) = 0.
\]

The tangential component of the magnetic field strength must be continuous across each surface of the coil layers. So that on the inner and outer surfaces of the \( i \)th layer, the following two equations exist

\[
H_i = C_1 I_0^B(\gamma \rho_i) + C_2 K_0^B(\gamma \rho_i), \\
H'_i = C_1 I_0^B(\gamma \rho'_i) + C_2 K_0^B(\gamma \rho'_i).
\]
By solving the above two equations, it is obtained that

\[ C_1 = \frac{H_i K_0^B(\gamma \rho_i') - H_i' K_0^B(\gamma \rho_i)}{I_0^B(\gamma \rho_i)K_0^B(\gamma \rho_i') - I_0^B(\gamma \rho_i')K_0^B(\gamma \rho_i)}, \]

\[ C_2 = \frac{-H_i I_0^B(\gamma \rho_i') + H_i' I_0^B(\gamma \rho_i)}{I_0^B(\gamma \rho_i)K_0^B(\gamma \rho_i') - I_0^B(\gamma \rho_i')K_0^B(\gamma \rho_i)}. \]

From Ampere's law, \( H_i \) and \( H_i' \) are given by

\[ H_i = \sum_{j=i}^{N_{\text{layer}}} I_j', \quad \text{(4.5)} \]

\[ H_i' = \sum_{j=i+1}^{N_{\text{layer}}} I_j', \quad \text{(4.6)} \]

where \( I_j' \) is the current per unit axial length,

\[ I_j' = \frac{I_j N_j}{l_j}, \]

\( N_j \) is the number of turns in the \( j \)th layer, and \( l_j \) is the axial length of the \( j \)th layer.

Here it is assumed that the inner space between the layers is small compared with the radius of the coil. So that the space between layers does not affect the magnetic field distribution.

Hence, for \( \rho_i \leq \rho_c \leq \rho_i' \), \( i = 1, \ldots, N_{\text{layer}} \), the magnetic field strength inside the conductor equals

\[ H_{zc}(\rho_c) = C_1 I_0^B(\gamma \rho_c) + C_2 K_0^B(\gamma \rho_c), \quad \text{(4.7)} \]

where \( C_1 \) and \( C_2 \) become

\[ C_1 = \frac{\left( \sum_{j=i}^{N_{\text{layer}}} I_j' \right) K_0^B(\gamma \rho_i') - \left( \sum_{j=i+1}^{N_{\text{layer}}} I_j' \right) K_0^B(\gamma \rho_i)}{I_0^B(\gamma \rho_i)K_0^B(\gamma \rho_i') - I_0^B(\gamma \rho_i')K_0^B(\gamma \rho_i)}, \quad \text{(4.8)} \]

\[ C_2 = \frac{-\left( \sum_{j=i}^{N_{\text{layer}}} I_j' \right) I_0^B(\gamma \rho_i') + \left( \sum_{j=i+1}^{N_{\text{layer}}} I_j' \right) I_0^B(\gamma \rho_i)}{I_0^B(\gamma \rho_i)K_0^B(\gamma \rho_i') - I_0^B(\gamma \rho_i')K_0^B(\gamma \rho_i)}. \quad \text{(4.9)} \]
From the differential form of Ampere's law, the current density in the conductor
is of the form
\[ \mathbf{J} = -\frac{\partial \mathbf{H}_{\text{ez}}}{\partial \rho_c} \phi_c. \]

Thus
\[ J_{\phi c} = -C_1 \gamma I_1^B(\gamma \rho_c) + C_2 \gamma K_1^B(\gamma \rho_c), \quad (4.10) \]

where
\[ I_0^B(x) = I_1^B(x), \]
\[ K_0^B(x) = K_1^B(x). \]

From the principle of the conservation of energy, the internal impedance of the
ith layer is defined by
\[ R_i = \frac{1}{\sigma I_i \cdot I_i^*} \int_{\mathcal{V}} \mathbf{J} \cdot \mathbf{J}^* \, d\mathcal{V} = \frac{2\pi l_i}{\sigma I_i \cdot I_i^*} \int_{\rho_i}^{\rho_i^t} \rho_c \mathbf{J} \cdot \mathbf{J}^* \, d\rho_c, \quad (4.11) \]
\[ L_i = \frac{\mu}{I_i \cdot I_i^*} \int_{\mathcal{V}} \mathbf{H}_i \cdot \mathbf{H}_i^* \, d\mathcal{V} = \frac{2\pi \mu l_i}{I_i \cdot I_i^*} \int_{\rho_i}^{\rho_i^t} \rho_c \mathbf{H}_i \cdot \mathbf{H}_i^* \, d\rho_c, \quad (4.12) \]

where \( i = 1, \ldots, N_{\text{layer}}. \)

By substituting Equations (4.10) into Equation (4.11), the expression for the
resistance becomes
\[ R_i = \frac{2\pi l_i |\gamma|^2}{\sigma |I_i|^2} \int_{\rho_i}^{\rho_i^t} \rho_c \left[ \left| C_1 \right|^2 I_1^B(\gamma \rho_c) I_1^{B*}(\gamma \rho_c) - C_1 \gamma C_2 K_1^B(\gamma \rho_c) K_1^{B*}(\gamma \rho_c) - C_1^2 C_2 I_1^B(\gamma \rho_c) K_1^{B*}(\gamma \rho_c) \right. \right. \]
\[ \left. \left. - C_1^2 C_2 K_1^B(\gamma \rho_c) I_1^{B*}(\gamma \rho_c) + |C_2|^2 K_1^B(\gamma \rho_c) K_1^{B*}(\gamma \rho_c) \right] \right) d\rho_c. \quad (4.13) \]

The integrals of the Bessel functions have the following properties [69]:
\[ \int I_1^B(x) I_1^{B*}(x) \, dx = \gamma \Re \left[ x^* I_0^B(x) I_1^{B*}(x) \right], \]
\[ \int K_1^B(x) K_1^{B*}(x) \, dx = -\gamma \Re \left[ x^* K_0^B(x) K_1^{B*}(x) \right], \]
\[ \int I_1^B(x) K_1^{B*}(x) \, dx = \frac{1}{2} \Re \left[ x^* I_0^B(x) K_1^{B*}(x) + x^* I_1^B(x) K_0^{B*}(x) \right], \]
\[ \int I_1^{B*}(x) K_1^B(x) \, dx = -\frac{1}{2} \Re \left[ x^* I_0^{B*}(x) K_1^B(x) + x I_1^{B*}(x) K_0^B(x) \right]. \]
Equation (4.13) then becomes

\[
R_i = \frac{2\pi l_i}{\sigma |l_i|^2} \left\{ |C_1|^2 \Re \left[ x^* I_0^B(x) I_1^B*(x) \right] - |C_2|^2 \Re \left[ x^* K_0^B(x) K_1^B*(x) \right] + \Re \left[ j C_1 C_2^* \left( x I_0^B(x) K_1^B*(x) + x^* I_1^B(x) K_0^B*(x) \right) \right] \right\} \gamma_{\rho_i}.
\]  

(4.14)

Similarly, substituting Equation (4.7) into Equation (4.12) and employing the following properties of Bessel functions:

\[
\begin{align*}
\int I_0^B(x) I_0^{\ast}(x) \, dx &= j \Re \left[ x I_0^B(x) I_1^{\ast}(x) \right], \\
\int K_0^B(x) K_0^{\ast}(x) \, dx &= -j \Re \left[ x K_0^B(x) K_1^{\ast}(x) \right], \\
\int I_0^B(x) K_0^{\ast}(x) \, dx &= \frac{1}{2} \Re \left[ x I_1^B(x) K_0^{\ast}(x) + x^* I_0^B(x) K_1^{\ast}(x) \right], \\
\int I_0^{\ast}(x) K_0^B(x) \, dx &= -\frac{1}{2} \Re \left[ x^* I_1^B(x) K_0^B(x) + x I_0^B(x) K_1^B(x) \right],
\end{align*}
\]

it is found that the internal reactance of a layer has the form of

\[
X_{in,i} = \frac{\pi l_i \omega \mu_0}{|l_i|^2} \left\{ |C_1|^2 \Re \left[ x I_0^B(x) I_1^{\ast}(x) \right] - |C_2|^2 \Re \left[ x K_0^B(x) K_1^{\ast}(x) \right] - \Re \left[ j C_1 C_2^* \left( x I_0^B(x) K_1^{\ast}(x) + x^* I_1^B(x) K_0^{\ast}(x) \right) \right] \right\} \gamma_{\rho_i}.
\]  

(4.15)

The modified Bessel functions \( I_p^B(\gamma \rho_c) \) and \( K_p^B(\gamma \rho_c) \) \((p = 0 \text{ or } 1)\) can be evaluated \([69]\) by either the following series expressions

\[
I_p^B(z) = \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 - \frac{4p^2 - 1^2}{1! 8z} + \frac{(4p^2 - 1^2)(4p^2 - 3^2)}{2!(8z)^2} - \ldots \right\}
\]  

(4.16)

\[
+ e^{(p+\frac{1}{2})\pi} \frac{e^{-z}}{\sqrt{2\pi z}} \left\{ 1 + \frac{4p^2 - 1^2}{1! 8z} + \frac{(4p^2 - 1^2)(4p^2 - 3^2)}{2!(8z)^2} + \ldots \right\},
\]

when \(|z|\) is large enough and \(0 \leq \text{phase} \ z \leq \pi,

\[
I_p^B(z) = \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 - \frac{4p^2 - 1^2}{1! 8z} + \frac{(4p^2 - 1^2)(4p^2 - 3^2)}{2!(8z)^2} - \ldots \right\}
\]  

(4.17)
\[ + e^{-j(p+\frac{1}{2})\pi} \frac{e^{-z}}{\sqrt{2\pi z}} \left\{ 1 + \frac{4p^2 - 1^2}{1! 8z} + \frac{(4p^2 - 1^2)(4p^2 - 3^2)}{2! (8z)^2} + \ldots \right\}, \]

when \(|z|\) is large enough and \(-\pi < \text{phase } z < 0\),

\[ K_p^B(z) \doteq \sqrt{\frac{\pi}{2z}} e^{-z} \left\{ 1 + \frac{4p^2 - 1^2}{1! 8z} + \frac{(4p^2 - 1^2)(4p^2 - 3^2)}{2! (8z)^2} + \ldots \right\} \]

\[ + \frac{(4p^2 - 1^2) \ldots [4p^2 - (2p - 3)^2]}{(p - 1)! (8z)^{p-1}} \],

when \(|z|\) is large enough and \(-\pi < \text{phase } z < \pi\),

or by the Kelvin functions \(b_{p}^\text{e}\) and \(k_{p}^\text{e}\)

\[ I_p^B(\gamma \rho_c) = j^{-p} \left[ \text{ber}_p\left(\frac{\sqrt{2}\rho_c}{\delta}\right) + j \text{bei}_p\left(\frac{\sqrt{2}\rho_c}{\delta}\right) \right], \]

\[ K_p^B(\gamma \rho_c) = j^{p} \left[ \text{ker}_p\left(\frac{\sqrt{2}\rho_c}{\delta}\right) + j \text{kei}_p\left(\frac{\sqrt{2}\rho_c}{\delta}\right) \right]. \]

When \(\Re(z) > 0\), the second terms in (4.16) and (4.17) can be ignored.

Functions \(b_{p}^\text{e}(x)\) and \(k_{p}^\text{e}(x)\) \((x \text{ is real})\) have very simple expressions when \(x \gg 1\) and the order \(p\) is not very high. Due to the fact that the skin depths of aluminum conductors and copper conductors at 60 Hz are about 0.011 m and 0.009 m, respectively, therefore, \(\sqrt{2}\rho_c/\delta \gg 1\) in general for reactor coils. For \(x \gg 1\), Kelvin functions of order 0 and 1 have the following approximate expressions:

\[ b_{p}^\text{e}(x) \approx \frac{\sqrt{x}}{\sqrt{2\pi x}}, \]

\[ a_{p}^\text{e}(x) \approx \frac{\sqrt{x}}{\sqrt{2\pi}}, \]

\[ \text{ker}_p(x) \approx \frac{\sqrt{\frac{2\pi}{3}}}{\sqrt{\frac{2\pi}{3}}}, \]

\[ \text{kei}_p(x) \approx \frac{-\sqrt{\frac{2\pi}{3}}}{\sqrt{\frac{2\pi}{3}}}, \]

where

\[ \Phi_p(x) = \frac{x}{\sqrt{2}} - \frac{\pi}{8} + \frac{p\pi}{2}, \]
\[ \Psi_p(x) = \frac{x}{\sqrt{2}} + \frac{\pi}{8} + \frac{p\pi}{2}. \]

Substituting Equation (4.19) into (4.20), it can be derived that

\[ I_0^B(\gamma\rho_c) = I_1^B(\gamma\rho_c) = \frac{\sqrt{2}e^{i(1+j)}e^{-j\frac{\pi}{8}}}{\sqrt{2\pi x}} |x=\sqrt{2}\rho_c, \quad (4.21) \]

\[ K_0^B(\gamma\rho_c) = K_1^B(\gamma\rho_c) = \frac{\sqrt{2}e^{i(1+j)}e^{-j\frac{\pi}{8}}}{\sqrt{2\pi x}} |x=\sqrt{2}\rho_c. \quad (4.22) \]

By utilizing the above equations, the formulas for evaluating the internal impedance can be simplified to (refer to (4.14) and (4.15))

\[ R_i = \frac{l_i}{\sqrt{2\sigma|I_i|^2}} \left\{ |C_1|^2 e^{\frac{2\pi}{\sqrt{2}}} - |C_2|^2 e^{\frac{2\pi}{\sqrt{2}}} e^{\frac{2\pi}{\sqrt{2}}} + 2\Re(j\pi C_1C_2^* e^{\frac{2\pi}{\sqrt{2}}}) \right\}^{\frac{2\rho_i}{\delta}}, \quad (4.23) \]

\[ X_{in,i} = \frac{\delta^2 l_i\mu_0}{2\sqrt{2}|I_i|^2} \left\{ |C_1|^2 e^{\frac{2\pi}{\sqrt{2}}} - |C_2|^2 e^{\frac{2\pi}{\sqrt{2}}} e^{\frac{2\pi}{\sqrt{2}}} - 2\Re(j\pi C_1C_2^* e^{\frac{2\pi}{\sqrt{2}}}) \right\}^{\frac{2\rho_i}{\delta}}, \quad (4.24) \]

where \( i = 1, \cdots, N_{layer} \), and \( C_1 \) and \( C_2 \) are given in (4.8) and (4.9), respectively. When using (4.23) and (4.24) to evaluate the internal impedance, it needs to evaluate \( |C_1|^2 e^{\frac{2\pi}{\sqrt{2}}} \) by \( |C_1|^2 e^{\frac{2\pi}{\sqrt{2}}} \cdot |C_1|^2 e^{\frac{2\pi}{\sqrt{2}}} \) and evaluate \( |C_2|^2 e^{\frac{2\pi}{\sqrt{2}}} \) by \( |C_2|^2 e^{\frac{2\pi}{\sqrt{2}}} \cdot |C_2|^2 e^{\frac{2\pi}{\sqrt{2}}} \) in order to avoid causing the overflow or underflow of the computer.

The effects of finite length of coil, that is, the end effects of coil in inductance calculations can be taken into account by a factor given by Nagaoka [70]. The equation for computing the Nagaoka factor is described in the following.

b. Self and Mutual Inductances between Layers

By referring to Appendix E, the self inductance of a coil of infinitesimal thin thickness, which has a radius of \( R_m \), an axial length of \( H \) and the total number of the turns \( N \), can be evaluated by the following formula:

\[ L = \frac{4\pi\mu_0 N^2 R_m^3}{H^2} \left[ C_s(R_m, R_m, H) - \frac{2}{3\pi} \right]. \quad (4.25) \]
The method for calculating $C_s$ can be found in Appendix E. The above equation can be used to evaluate the external inductance of a layer or the self inductance of a layer if the internal inductance can be ignored compared with the external inductance which is the case in most of practical cases.

Nagaoka [70] has given a factor which takes account of the end effects of coil. For an infinitesimally thin cylindrical coil of height $H$ and radius $R_m$, the Nagaoka factor has a form of complete elliptic integrals of the first and second kinds

$$
\Omega = \frac{4}{3\pi} \left[ \frac{R_m^2 + (H/2)^2}{H/2} \right] \left[ \frac{H/2}{R_m^2} (K - E) + E - \frac{R_m}{\sqrt{R_m^2 + (H/2)^2}} \right].
$$

(4.26)

Alternatively, it can be expressed in terms of $C_s$ [71]:

$$
\Omega = \frac{4R_m}{H} \left[ C_s(R_m, R_m, H) - \frac{2}{3\pi} \right].
$$

(4.27)

$\Omega$ can be applied to the calculation of the internal inductance to take into account the end effects of coil. The effects of the space between the turns, if they can’t be ignored, can be taken into account by $\Delta L$ [72]

$$
\Delta L = -4\pi R_m N (f_G + f_H) \times 10^{-7},
$$

(4.28)

where

$$
f_G = \frac{5}{4} - \log_e \left( \frac{h_p}{R_s} \right),
$$

and $f_H$ is given in Appendix F.

Therefore, the self inductance of the $i$th layer is equal to

$$
L_i = L_{in,i} + L_{ex,i} + \Delta L,
$$

(4.29)

where the formulas for calculating $L_{in,i}$, $L_{ex,i}$ and $\Delta L$ can be found in (4.15),(4.25) and (4.28). Usually, $L_{in,i}$ is small compared with $L_{ex,i}$ and may be ignored in the impedance calculation due to the fact that the thickness of the layer is usually small.
compared with the radial size of the coil. The resistance values are important in determining the current distribution among the layers and can't be ignored, even though they are small compared with the reactance values. The current distribution among coil layers is determined mainly by the reactances of those coils whose layers are weakly coupled, e.g., $M_{ij}/\sqrt{L_iL_j} \leq 0.8$ [56].

The mutual inductance between the layers $i$ and $j$ of mean radii $R_{mi}$ and $R_{mj}$, respectively, can be computed by (see Appendix E)

$$M_{ij} = \frac{4\pi\mu_0 N_i N_j (R_{mi}R_{mj})^{3/2}}{l_i l_j} \sum_{i=1}^{i=2} (-1)^{i+1} C_s(R_{m1}, R_{m2}, s_i),$$

where $i, j = 1, \ldots, N_{layer}$, $s_1 = (l_i + l_j)/2$ and $s_2 = (l_i - l_j)/2$.

4.3.2 Algorithm for Determining Current Distribution among Layers

It can be noticed from Section 4.3.1 that $R_i$ and $X_i$ can be computed by using Equations (4.14) and (4.15) only if the current values of $I_1, \ldots, I_{N_{layer}}$ are available. Since the current values $I_1, \ldots, I_{N_{layer}}$ are initially unknown, an iterative method can be applied to determine the current distribution among coil layers.

Let $V$ be the voltage applied to the coil, $I_{coil}$ be the total current in the coil, $\hat{Z}$ be the impedance matrix of the coil, and $\bar{V}$ and $\bar{I}$ be the current and voltage vectors of the coil layers, respectively, where

$$\bar{V} = \begin{bmatrix} V \\ \vdots \\ V \end{bmatrix} \quad \text{and} \quad \bar{I} = \begin{bmatrix} I_1 \\ \vdots \\ I_{N_{layer}} \end{bmatrix}.$$

There exist

$$\hat{Z}\bar{I} = \bar{U}^T V,$$
\[ \bar{U} \bar{I} = I_{\text{coil}}, \]

where \( \bar{U} = [1 \cdots 1] \) and \( \bar{U}^T \) is the transpose of \( \bar{U} \). Then it can be derived that

\[ \bar{I} = \bar{Z}^{-1} \bar{U}^T (\bar{U} \bar{Z}^{-1} \bar{U}^T)^{-1} I_{\text{coil}}, \tag{4.31} \]

where \( \bar{Z}^{-1} \) is the inverse of the matrix \( \bar{Z} \).

The procedures for obtaining the current distribution among layers are outlined in the following:

1. The initial values of currents in layers are set to be equal, that is, equal to \( I_{\text{coil}} / N_{\text{layer}} \).

2. Calculate the impedance matrix \( \bar{Z} \)
   \[
   \bar{Z} = \begin{bmatrix}
   Z_1 & M_{12} & \cdots & M_{1j} & \cdots & M_{1N_{\text{layer}}} \\
   \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
   M_{i1} & \cdots & Z_i & M_{ij} & \cdots & M_{iN_{\text{layer}}} \\
   \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
   M_{N_{\text{layer}}1} & M_{N_{\text{layer}}2} & \cdots & M_{N_{\text{layer}}j} & \cdots & Z_{N\text{layer}}
   \end{bmatrix},
   \]

   where \( Z_i \) is the self impedance of the \( i \)th layer (\( i = 1, \cdots, N_{\text{layer}} \)).

3. Solve the matrix equation (4.31) to obtain the current vector \( \bar{I} \).

4. Go to Step 2 with the newly obtained current values.

5. Stop if the convergence criterion is satisfied.

The convergence criterion can be chosen as that the maximum change of the current values estimated between two consecutive iterations is ignorable. The external inductance and mutual inductance of each layer, in the absence of magnetic materials, depend on the geometry of the coil only and are independent of the current values of the layers. Hence, they need to be evaluated only once at the beginning of the calculation.
4.4 Calculation Results

In this section, the calculation accuracy of the field differential method and the approximate impedance method is tested by calculating several simple coils. The accuracy of the results are discussed as well.

4.4.1 Verification of the Field Differential Method

The current of a simple single-layer helical coil, COIL3 (refer to Section 3.5 for the geometry of the coil) is calculated by the field differential method and is compared to the initially assumed one. For a single-layer coil, there is no effect from other layers. The calculated coil current should be exactly same as the initially assumed one, if the calculation method used is accurate. The test procedure is described in the following.

Suppose the total current in COIL3 is 1.0 A. First, the magnetic field strength at field points across a cross-section of the coil conductor is calculated by the VCM method developed in Chapter 3. The conductor cross section tested is chosen to be near the center part of the coil. Then the current density across that cross section is evaluated by Equation (4.1). Finally, the total current flowing through the conductor cross section is obtained by the area integration of the current density across that cross section. The calculated coil current is equal to 1.0024 A.

Excellent agreement indicates that the field differential method can be used to accurately determine the current distribution in the coil. The field differential method is also very flexible in determining the current distribution in coil layers. It doesn’t have any constrains on the interlayer space of the coil and the length of the coil layer.
4.4.2 Verification of the Approximate Impedance Method

In this subsection, the impedance values of a double-layer coil connected in parallel are calculated by using the approximate impedance method. The results are compared with the impedance values of a double-layer sheet coil, which is made of the same conductor material and is of the same layer lengths and widths, obtained by the finite element method. The ratio of layer current magnitudes are computed by the approximate impedance method accordingly and compared with the value obtained by the field differential method.

![Diagram](image)

**Figure 26:** The geometry of a double-layer coil.

Figure 26 shows the geometry of the tested coil, which is a double-layer coil connected in parallel. The coil is made of copper conductors. Each layer has 156
turns. The mean radius of the inner layer is 0.652 m and the outer layer has a mean radius 0.657 m. The radius of the conductor is 0.002 m and the length of each layer equals 0.7 m.

<table>
<thead>
<tr>
<th>Elements</th>
<th>Impedance Values (Ω)</th>
<th>Approximate Impedance Method</th>
<th>Finite Element Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z_{11}</td>
<td>0.64 + j11.83</td>
<td>0.75 + j11.18</td>
<td></td>
</tr>
<tr>
<td>Z_{22}</td>
<td>0.62 + j11.96</td>
<td>0.76 + j11.29</td>
<td></td>
</tr>
<tr>
<td>Z_{12}</td>
<td>j11.85</td>
<td>j11.13</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Impedance of a double-layer coil at 60 Hz (layer current is 1 A).

Its impedance matrix and the layer current values have been calculated by the approximate impedance method. The impedance matrix of a double-layer sheet coil of the same geometry (including layer length and width) and made of the same conductor material is also calculated by the finite element method where a 2D axi-symmetrical model is used. For the finite element method, convergence is reached after 15 passes. Table 2 compares their impedance values at 60 Hz with a layer current of 1 A.

It can be observed from Table 2 that the consistency between the impedance values obtained by the approximate impedance method and the finite element method is reasonably good. The resistance values calculated by the approximate impedance method are slightly lower than that calculated by the finite element method. This is because in the finite element model used, a coil made of sheet conductors is assumed. While in the approximate impedance method, a solenoid coil is assumed where each
layer consists of an array of series connected turns. A sheet winding suffers from severe end effects. As a result, the current tends to concentrate near the ends where the inductance is lower than the center part. This causes more losses than a solenoid (non-sheet) coil.

The current value in each layer has also been calculated by the approximate impedance method and the field differential method, respectively. The ratio of the layer current magnitudes is tabulated in Table 3. The magnitude of the total coil current is assumed to be 1 A, and its phase angle is zero. The initial current in each layer is set to be 0.5 A in each case. Good agreement between two results can be observed.

From Table 3, it is also noticed that the outer layer has a slightly higher current magnitude than the inner layer even though the reactance value of the outer layer is slightly higher than that of the inner layer. This further verifies the statement mentioned before that the current distribution in coil layers is not determined by their reactance only when the layers are tightly coupled.

<table>
<thead>
<tr>
<th></th>
<th>Approximate Impedance Method</th>
<th>Field Differential Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>I_1/I_2</td>
<td>$</td>
</tr>
</tbody>
</table>

Table 3: Ratio of layer current magnitudes in a double-layer coil connected in parallel, calculated by the Approximate Impedance Method and the Field Differential Method, respectively.

It can be concluded that the approximate impedance method is not only simpler and faster than the finite element method in calculating the current distribution in
the solenoid coil, but also can give good results for the coil whose layer thickness is thin, and the pitch angle, the space between layers and the number of layers are small.

4.5 Conclusions and Discussion

Two methods for estimating the current distribution in coil layers have been proposed. One is the field differential method which utilizes the volume current method developed in Chapter 3 for accurately evaluating the magnetic field strength inside the coil conductors. The other one is the approximate impedance method which is based on calculating the impedance matrix of coil layers to determine the current distribution. In the approximate impedance method, analytical equations have been derived for evaluating internal impedances of coil layers. They take the skin effect, proximity effect and the curvature of windings into consideration.

The first method is very flexible and accurate in determining the current distribution in coil layers. It doesn’t have any constrains on the interlayer space of the coil and the length of the layer as the second method does. However, the second method requires less computation time than the first one.

The calculation accuracy of the field differential method is verified by calculating the current value of a single-layer coil. The calculated current value is consistent with the initially assumed one.

From the approximate impedance method, it is known that once the impedance matrix of the coil is accurately evaluated, the current distribution among layers can also be determined accurately. The impedance values of a three-layer coil and a double-layer coil have been calculated. The results are verified by and compared with the data given in a technical paper and obtained by the finite element method for a sheet winding, respectively. In general, reasonably good agreement has been reached.
observed. The layer current values of a double-layer coil have also been calculated by
the approximate impedance method. The results are consistent with the expectation.

The approximate impedance method is simple in determining the current dis-
tribution among the coil layers. It can give good results when the layer thickness is
thin, and the pitch angle of the coil, the space between layers and the number of lay-
ers are small. However, it can be expected that the calculation error might increase
when the conditions described above are not satisfied. The limitations of the ap-
proximate impedance method come from the basic assumptions used in deriving the
expressions for computing the internal impedance of a layer. It is assumed that the
axial field decreases to zero from the inner surface of the coil to the outer surface of
the coil and the space between the layers doesn’t change the field distribtuion. These
assumptions are not valid when the number of layers is large or the space between
layers is large. In those cases, the field differential method proposed in Section 4.2
must be used to determine the current distribution among layers.
CHAPTER 5

SIMPLIFIED MODELING OF LARGE AIR-CORE REACTOR COILS

5.1 Introduction

The accurate modeling method presented in Chapters 3 and 4 for large air-core reactor coils can give very accurate results no matter where field points are located. However, in order to save computation time, it is necessary to develop a simplified modeling method instead of using the accurate one when field points are not inside or in the very close vicinity of the reactor coil. The simplified modeling method developed should not only simplify the calculation greatly, but also give accurate results in the regions of practical interest.

In this chapter, a simplified magnetic field model for large air-core reactor coils is proposed. It uses three flexible equivalent current loops with their numbers of turns and positions flexible. The accuracy of its calculation of the three dimensional power-frequency magnetic flux density is analyzed.

Based on typical length-to-diameter ratios \(H/D_m\) (refer to Chapter 1), the following three different types of coils are used in this study:

- "short" coil with \(H/D_m = 0.5\),
- "medium length" coil with \(H/D_m = 0.9\),
- "long" coil with \(H/D_m = 2.0\).
Five typical traverses are used for the calculations, represented by abbreviations ATC, ATS, LTT, LTH and LTC (refer to Figure 27(a)), respectively, which mean:

- ATC: axial traverse, center,
- ATS: axial traverse, side,
- LTT: lateral traverse, top,
- LTC: lateral traverse, center,
- LTH: lateral traverse, halfway between LTT and LTC.

The model developed not only simplifies the calculation greatly but also gives a calculation error of less than 5% at the designated field points which are at least $(0 \cdots 1.3)H$ axially or $(0.2 \cdots 1.2)D_m$ laterally away from the outer surface of the coil, depending on the location of the traverse and the type of the coil.

5.2 Three Flexible Loop Model

The simplest (but inadequate) model of a reactor coil is a single coaxial current loop placed at the center of the coil, as shown in Figure 27(b) (referred to as SINGLE model). It is assumed that the loop current is equal to the actual coil current $I$ times the total number of actual turns of the reactor coil, $N$. The diameter of the loop is $D_m$. Large errors are generated when field points are relatively close to the coil and the coil represented is in the "long" category. This is due to the fact that a single current loop can not and does not reflect the geometry changes of the coil. Multiloop models must be used to reflect the geometry shape of the coil.

The use of three coaxial current loops of fixed turns and positions to model a reactor coil can enforce the shape of the coil. As shown in Figure 27(c), where this three fixed loop model is referred to as 3FIXED model, there are always two loops with $N/12$ turns each at the top and bottom of the coil to model the longitudinal
Figure 27: Reactor coil and its models.
shape of the coil, and a third loop always with $5N/6$ turns placed at the center of the coil. The 3FIXED model can improve the calculation accuracy slightly for the "long" coil, but the results are still not satisfactory, especially for the field points along the lateral traverses.

Studying the calculation accuracy of the single loop model it was found that its calculation accuracy can be greatly improved if the coil being modeled is a "medium length" coil, as opposed to a "long" coil. That suggests that if a "long" coil is divided into several sections which are of "medium length" and each section is modeled by an equivalent loop, the overall accuracy will be improved. Three loops usually are sufficient due to the fact that the practical $H/D_m$ values are limited. More loops can be used if an extremely long coil is being studied. Thus, a three flexible loop model (called three flexible magnetic dipole model in [73, 20]) is proposed as shown in Figure 27(d) (referred to as FLEXIBLE model), where a coil is divided into three sections and an equivalent current loop is located at the center of each section, that is:

- The center section has a length of $0.8D_m$. It represents the part of the actual coil that is closest to the field point. It is as symmetric about the position of the field point in the axial direction as possible. Its number of turns is $N0.8D_m/H$.

- The top section is the section between the top of the center section and the top of the actual coil. Its length is $h_1$, its number of turns is $Nh_1/H$.

- The bottom section is the section between the bottom of the center section and the bottom of the actual coil. Its length is $h_2$, its number of turns is $Nh_2/H$.

Comments related to the use of the three flexible loop model are as follows:
• In general, there exists the relationship \(0.8D_m + h_1 + h_2 = H\).

• When the field point is at the lateral side, close to the center area of the coil to be modeled, all three loops are used.

• When the field point is at the lateral side and its axial distance from the top or bottom of the actual coil is less than \(0.8D_m\), the three loops are reduced to two because the length of the top or bottom section \((h_1, \text{ or } h_2)\) is reduced to zero. Only two sections of the length of \(0.8D_m\) and \(H - 0.8D_m\), respectively, are defined.

• When the length of the coil is shorter than \(0.8D_m\), the three loop model becomes one loop model automatically.

• When the field point is in the axial area, i.e., above the top or below the bottom of the coil, the coil is divided into three sections of equal length. Hence, each current loop has \(N/3\) turns.

• When the field point is far from the coil, e.g., the distance from the outer surface of the coil is larger than, say, \(5D_m\), the single loop model is used.

The magnetic moment vector of a current loop is \(\vec{m} = NI\pi D_m^2 \hat{n}/4\), where \(I\) is in amperes; \(\hat{n}\) is the unit vector pointing in the normal direction of the coil given by the right-hand-rule; and \(D_m = [(D_1^2 + D_1 D_2 + D_2^2)/3]^{1/2}\) is the effective mean diameter of the coil in meters, and \(D_1\) and \(D_2\) are the inside and outside diameters of the coil in meters.

In rectangular coordinates, the magnetic flux density components at a far zone field point \((x_c, y_c, z_c)\) produced by an electric current loop whose center is located at \((x_{loop}, y_{loop}, z_{loop})\) with three orthogonal moment components \((m_x, m_y, m_z)\) can...
be calculated by the following equations [74]:

\[
B_{xc} = \frac{\mu_0}{4\pi} \left\{ 3(x_c - x_{loop}) \left[ m_x(x_c - x_{loop}) + m_y(y_c - y_{loop}) + m_z(z_c - z_{loop}) \right] + m_z(z_c - z_{loop}) \right\} / d^3
\]

\[
B_{yc} = \frac{\mu_0}{4\pi} \left\{ 3(y_c - y_{loop}) \left[ m_x(x_c - x_{loop}) + m_y(y_c - y_{loop}) + m_z(z_c - z_{loop}) \right] + m_z(z_c - z_{loop}) \right\} / d^3
\]  

\[
B_{zc} = \frac{\mu_0}{4\pi} \left\{ 3(z_c - z_{loop}) \left[ m_x(x_c - x_{loop}) + m_y(y_c - y_{loop}) + m_z(z_c - z_{loop}) \right] + m_z(z_c - z_{loop}) \right\} / d^3
\]

(5.1)

where \( d = [(x_c - x_{loop})^2 + (y_c - y_{loop})^2 + (z_c - z_{loop})^2]^{1/2} \) is the distance between the field point and the center of the current loop; the distances and coordinates are in meters; and the magnetic flux density components \( B_{xc}, B_{yc}, \) and \( B_{zc} \) are in teslas.

The above equations are based on the far zone field assumption. In other words, the above equations are accurate only if the radius of the coil is much smaller than the distance between the field point and the center of the current loop. However, for the FLEXIBLE model, this is not the case when a field point is close to the center of the top or bottom surface of the coil and the length of the coil is less than the radius of the coil.

In order to improve the accuracy of the magnetic flux density calculations in the areas above or below the end surfaces, a correction factor, \( F \), for the \( B_z \) expression in (5.1) is introduced if a vertical coil is considered. \( F \) takes into account the difference between the exact solution and the far-zone solution:

\[
F = [1 + D_m^2/(4d^2)]^{-3/2}.
\]

(5.2)

\( F \) is to be used for field points which are on the axial side and close to the surface of the coil, where \( B_z \) dominates other components when \((D_m/2) > d\). For field points on the lateral side, \( F \) can be ignored due to the fact that the distance between a
field point and the center of the current loop is always larger than the radius of the current loop.

5.3 Accuracy of the Three Flexible Loop Model

The magnetic flux density has been calculated along 5 traverses, ATC, ATS, LTT, LTH and LTC for three different types of coils (short, medium length, long) by using the three simplified models SINGLE, 3FIXED and FLEXIBLE outlined above, respectively, and compared with that of the code COILB (see Chapter 2).

It is known from Section 3.5 that when a field point is about four times the wire radius away from a reactor coil, the program COILB can give reasonably accurate results. The accuracy becomes higher when the field point is further away from the reactor coil.

The discrepancies of the resultant magnetic flux density of the three simplified models vs. the normalized distances away from the outer surface of the coil are
Figure 29: Errors of three simplified models along ATS traverse.

Figure 30: Errors of three simplified models along LTT traverse.
Figure 31: Errors of three simplified models along LTH traverse.

Figure 32: Errors of three simplified models along LTC traverse.
calculated and plotted in Figures 28 to 32. The minimum distances away from the surface of the coil to limit the discrepancies to 5% for the FLEXIBLE model are summarized in Table 4. It can be seen that at field points whose distance from the coil is at least $(0 \cdots 1.3)H$ axially or $(0.2 \cdots 1.2)D_m$ laterally from the outer surface of the coil, pending on the location of the traverse and the type of the coil, the calculation error of the FLEXIBLE model is less than 5%. That gives very satisfactory calculation accuracy.

5.4 An Alternative Implementation of the Circular Loop Model

In SUBCALC (refer to Chapter 2) every element is approximated by a number of straight line current-carrying segments. Their magnetic field contributions are then computed by the Biot-Savart law. Therefore, sometimes it is convenient to use a square current loop to substitute each round loop in the simplified loop model.

To investigate the accuracy of the use of square loops as compared to round loops, the resultant magnetic flux density of a one-turn square loop is calculated

<table>
<thead>
<tr>
<th>$H/D_m$</th>
<th>Axial Traverses</th>
<th>Lateral Traverses</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ATC</td>
<td>ATS</td>
</tr>
<tr>
<td>0.5</td>
<td>No Limit</td>
<td>1.3 H</td>
</tr>
<tr>
<td>0.9</td>
<td>0.5 H</td>
<td>0.3 H</td>
</tr>
<tr>
<td>2.0</td>
<td>1.0 H</td>
<td>0.8 H</td>
</tr>
</tbody>
</table>

Table 4: Critical distances of FLEXIBLE model to limit errors to 5%.
and compared to results related to a one-turn round loop along two representative traverses.

If the square loop is just interior to the round loop and its size in meters is

\[ R_{\text{square}} = \sqrt{2} R_{\text{round}} \]

where \( R_{\text{round}} \) is the radius of the round loop in meters, the error of the resultant magnetic flux density of the square loop compared with that of the round loop is rather large, as shown in Figure 33. If the square loop has the same area as the round loop and of the size of

\[ R_{\text{square}} = (\pi R_{\text{round}}^2)^{1/2} \]

the error of the resultant magnetic flux density of the square loop compared with that of the round loop becomes small and diminishes very quickly when the field point is moved away from the coil, as illustrated in Figure 34.

Therefore, in terms of accuracy, it is *practically feasible* to use a square loop instead of a round loop. The center of the square loop should coincide with the center of the round loop and the size of each segment forming the loop is given by \( (\pi R_{\text{round}}^2)^{1/2} \). The current in each segment equals the current in the round loop.
Figure 33: Errors in the resultant magnetic flux density calculations when the equivalent square loop is interior to the actual round loop. Traverses selected: L1 - perpendicular to loop element, L2 - diagonal.
Figure 34: Errors in the resultant magnetic flux density calculations when the equivalent square loop has the same area as the round loop. Traverses selected: L1 - perpendicular to loop element, L2 - diagonal.

(b) $R_{\text{square}} = (\pi R_{\text{round}}^2)^{1/2}$. 

121
5.5 Conclusions

A model with three flexible equivalent current loops is proposed for the fast evaluation of the three dimensional power-frequency magnetic flux density distribution of air-core reactor coils. The magnetic flux density was calculated along 5 typical traverses for three different types of coils. The three flexible loop model proposed is not only simple and easy to be implemented but also gives accurate results even as close to the reactor coil as at least \((0.1\cdots1.3)H\) axially or \((0.2\cdots1.2)D_m\) laterally from the outer surface of the coil, depending on the location of the traverse and the type of the coil (see Table 4).

In addition, it is concluded that using appropriate square loops instead of round loops will not decrease the accuracy in the area of practical interest. This gives an alternative implementation of the proposed model.

The simplified model developed is very useful for the fast evaluation of the three dimensional power-frequency magnetic field distribution of air-core reactor coils. The model can be applied easily to the calculation of the three dimensional power-frequency magnetic field strength or magnetic flux density distribution of substations. The model developed is very efficient in terms of CPU time.
CHAPTER 6
CONCLUSIONS AND FUTURE WORK

The statement of the problem of this research, the configuration of a typical air-core reactor coil used throughout the dissertation, the objective and the contributions of this research are described in Chapter 1. The outline of this dissertation is also given there.

An overall literature review of previous research on the calculation methods of the magnetic field distribution of and the current distribution in air-core coils is presented in Chapter 2, which covers more than 60 references.

In Chapter 3, an accurate modeling method, the Volume Current Method (VCM), for calculating the three-dimensional magnetic field strength of an air-core helical coil is proposed. The analytical formulas for evaluating the magnetic field strength components contributed from each finite circular cylindrical conductor segment carrying static or quasi-static current are derived, which is the basic modeling element of air-core coils. A surface source modeling technique is utilized in the derivation and the corresponding surface source model is developed as well. Thus using multiple thin filament segments or a conductor segment with polygonal cross section to approximate a circular cylindrical conductor segment in the conventional calculation methods is avoided. The derived equations are valid for arbitrary field points either inside the coil or outside the coil. They are applicable to coils carrying static or quasi-static currents, made of either circular cross-section solid conductors whose radii are...
smaller or comparable to their skin depths, or made of circular cross-section cables which are composed of stranded subconductors. Therefore, the current distribution in the cross sections of the coil conductors or cables is nearly uniform.

The volume current method is useful for solving problems that involve the accurate calculation of the magnetic field distribution in and around a coil, the magnitude of forces on a coil, the current distribution in the conductors of a coil and inductive properties of a coil, where the coil is made of circular cross-section conductors.

For verification purposes, the magnetic flux density of a single-loop coil calculated by the proposed accurate method is compared with that of using a finite element model. Good agreement between the two sets of results is observed. The profiles of the magnetic field distribution of a single-loop coil, the appropriate number of conductor segments per turn needed and the effects of the conductor size on the the magnetic field distribution are analyzed and discussed. The calculation accuracy of the filament current loop model for modeling a current loop made of circular cross-section conductor and the planar filament current loop stack model for modeling a helical coil made of circular cross-section conductor are also examined by the VCM developed. It has been concluded that when field points are in the very close vicinity of the coil, e.g., within a range of three times the conductor radius away from the center of the conductor, the volume current model must be employed in order to give accurate results; beyond that range, the filament model can be utilized to save computation time. The effects of the coil pitch angle must be taken into consideration when the field point is in the immediate vicinity of the coil. When the interspace between the turns of the coil is only caused by the thin conductor insulation thickness, the minimum distance away from the outer surface of the coil for ignoring the pitch angle and having less than 2.5% calculation error is 0.2 times the coil mean diameter; beyond that range (which depends
on the interspace between the coil turns), a helical coil can be accurately modeled by a stack of planar filament current loops to save computation time.

In Chapter 4, two methods for determining the current distribution among the coil layers are presented. One, called the field differential method, is based on the calculation of the magnetic field strength or magnetic flux density inside the coil and the other one, called the approximate impedance method, is based on the calculation of layer impedances. The analytical expressions for calculating the layer impedance are derived. The calculation accuracy of both methods are tested by several simple coils. The advantages and limitations of each method are discussed. It is concluded that the field differential method is very flexible and accurate in determining the current distribution in coil layers. The approximate impedance method is simple and fast. But it is only appropriate for the coil whose layer thickness is thin, length is relatively long compared with the coil mean diameter, and pitch angle of the coil, space between layers and number of layers are small.

In Chapter 5, a model with three flexible equivalent ideal current loops is proposed for the fast evaluation of the three dimensional power-frequency magnetic flux density distribution of air-core reactor coils. The magnetic flux density was calculated along 5 typical traverses for three different types of coils. The three flexible loop model proposed is not only simple and easy to implement but also gives satisfactorily accurate results as close to the reactor coil as at least \((0 \cdots 1.3)H\) (coil length) axially or \((0.2 \cdots 1.2)D_m\) (coil mean diameter) laterally from the outer surface of the coil, depending on the location of the traverse and the type of the coil. Each circular current loop in the model can also be alternatively implemented by a square current loop whose area equals to that of the round current loop. The developed simplified model can be applied easily to the calculation of the three-dimensional
power-frequency magnetic field distribution of substations.

As stated above, for evaluating the magnetic field distribution from a helical coil, different modeling methods should be utilized according to field points to save computation time and efforts. As summarized in Figure 35, when field points are inside the coil or within a distance of three times conductor radius away from the center of the conductor, the VCM developed must be used; when field point locations are outside of the conductor and are at least three times the conductor radius away from the center of the conductor, the filament model can be employed; when field points are at least 0.2 times the coil mean diameter (pending on the pitch angle of the coil) away from the outer surface of the coil, the planar filament current loop stack model can be used to obtain reasonably accurate results; when field points are further away, at least \((0 \cdots 1.3)H\) axially or \((0.2 \cdots 1.2)D_m\) laterally from the outer
surface of the coil, depending on the location of the traverse and the type of the coil, the three flexible loop model proposed can be used for fast computation.

In this dissertation, however, only single-phase reactor coils are considered in the accurate calculation of the magnetic field strength of the coil. The magnetic field distributions of split reactor coils, three-phase stacked or three-phase side-by-side reactor coils can be further determined based on the study of single-phase reactor coils. The resultant magnetic field strength at any point will be the superposition of the appropriate field components from each coil, but the effects of adjacent coils need to be included in the calculation of the current distribution in the coils. The effects of the spiders on the magnetic field distribution and the current distribution in the coil need to be studied. Comparison studies related to measurements and the application of various calculation methods for actual reactor coils of different types should be made as well. The study of the optimal arrangement of the multiphase coils is an interesting topic and might need extensive work. The goal is to arrive at an arrangement that gives the minimum magnetic field strength in the surrounding area of the coils while keeping mechanical forces within allowable limits.
APPENDIX A

DERIVATION OF THE EXPRESSIONS OF $z_1$ AND $z_2$

The expressions of $z_1$ and $z_2$ are derived in the following, which are the $z$ axial coordinates of the points on the edges of the top and bottom end planes of a conductor segment (see Figure 5), respectively.

Let $(x, y, z)$ be the coordinates of a point $P$ in the $XYZ$ rectangular coordinate system. Its corresponding cylindrical coordinates $(\rho, \varphi, z)$ are given by

$$\rho = \sqrt{x^2 + y^2},$$

$$\varphi = \tan^{-1} \frac{y}{x}.$$

![Figure 36: Translation and rotation of coordinate systems.](image)

Assume that rectangular coordinate system $X_0Y_0Z_0$ is the translation of the coordinate system $XYZ$ and that rectangular coordinate system $X_1Y_1Z_1$ is the rotation
of the coordinate system $X_0Y_0Z_0$ with counterclockwise rotated angles $\beta_1$ (view from the $X_0$ axis) from the $Z_0$ axis and $\beta_2$ from the $X_0$ axis, respectively, as shown in Figure 36. The origin $O$ of the $XYZ$ system has the coordinates $(x^0, y^0, z^0)$ in the $X_0Y_0Z_0$ system. Therefore, the coordinates of the point $P$ in the $X_0Y_0Z_0$ system are

$$x_0 = x + x^0 = \rho \cos \varphi + x^0,$$

$$y_0 = y + y^0 = \rho \sin \varphi + y^0,$$

$$z_0 = z + z^0. \quad (A.1)$$

It can be shown that the coordinates $(x_0, y_0, z_0)$ in the $X_1Y_1Z_1$ system become

$$x_1 = x_0 \cos \beta_2 + y_0 \sin \beta_2,$$

$$y_1 = -x_0 \cos \beta_1 \sin \beta_2 + y_0 \cos \beta_1 \cos \beta_2 + z_0 \sin \beta_1, \quad (A.2)$$

$$z_1 = x_0 \sin \beta_1 \sin \beta_2 - y_0 \sin \beta_1 \cos \beta_2 + z_0 \cos \beta_1.$$

From the above equations, if $\beta_2 = 0$, $x^0 = 0$, $y^0 = 0$ and $z^0 \neq 0$, the coordinates of $(x, y, z)$ or $(\rho, \varphi, z)$ in the $X_1Y_1Z_1$ system are equal to

$$x_1 = \rho \cos \varphi = x,$$

$$y_1 = \rho \sin \varphi \cos \beta_1 + (z + z^0) \sin \beta_1 = y \cos \beta_1 + (z + z^0) \sin \beta_1, \quad (A.3)$$

$$z_1 = -\rho \sin \varphi \sin \beta_1 + (z + z^0) \cos \beta_1 = -y \sin \beta_1 + (z + z^0) \cos \beta_1.$$

Referring to Figure 8 and (A.3), it can be obtained that the relationship of coordinates between the $XYZ$ coordinate system and the $X_tY_tZ_t$ coordinate system of the top end plane has the following form

$$x_t = x,$$

$$y_t = y \cos \theta_t + (z - h_t) \sin \theta_t, \quad (A.4)$$

$$z_t = -y \sin \theta_t + (z - h_t) \cos \theta_t.$$
where \((x_t, y_t, z_t)\) are the coordinates of the point \(P\) in the \(X_tY_tZ_t\) coordinate system.

Similarly, the relationship of coordinates between the \(XYZ\) coordinate system and \(X_bY_bZ_b\) coordinate system of the bottom end plane can be written as

\[
x_b = x_t
\]
\[
y_b = -y_t \cos \theta_b + (z_t + h_b) \sin \theta_b,
\]
\[
z_b = -y_t \sin \theta_b - (z_t + h_b) \cos \theta_b,
\]

where \(\beta_1 = \pi - \theta_b\) and \((x_b, y_b, z_b)\) are the coordinates of the point \(P\) in the \(X_bY_bZ_b\) coordinate system.

The contour of the top end plane is described in the \(X_tY_tZ_t\) system by Equation (A.6):

\[
\frac{x_t^2}{R_s^2} + \frac{y_t^2}{R_s^2} = 1.
\]

(A.6)

The contour of the bottom end plane has the following equation in the \(X_bY_bZ_b\) coordinate system:

\[
\frac{x_b^2}{R_s^2} + \frac{y_b^2}{R_s^2} = 1.
\]

(A.7)

then, the equations of the contours of the top and bottom end planes in the \(XYZ\) coordinate system have the following expressions

\[
\frac{x^2}{R_s^2} + \frac{[y \cos \theta_t + (z - h_t) \sin \theta_t]^2}{r_t^2} = 1,
\]

\[
\frac{x^2}{R_s^2} + \frac{[-y \cos \theta_b + (z + h_b) \sin \theta_b]^2}{r_b^2} = 1,
\]

by substituting Equations (A.4) and (A.5) into Equations (A.6) and (A.7), respectively. Then, \(z\) can be obtained by solving the above equations.

Therefore, \(\tilde{z}_2\), which is the \(z\) coordinate of a point on the contour of the top end plane, is equal to

\[
\tilde{z}_2 = \pm \sqrt{r_t^2(1 - x^2/R_s^2)} - y_t \cos \theta_t \sin \theta_t + h_t.
\]

(A.8)
Due to the fact that $x = R_s \cos \varphi$ and $y = R_s \sin \varphi$, $\tilde{z}_2$ becomes

$$\tilde{z}_2 = \frac{r_t - R_s \cos \theta_t}{\sin \theta_t} \sin \varphi + h_t. \quad (A.9)$$

The reason for taking “+” in (A.8) is due to the fact that when $\varphi \in [0, \pi)$, $\tilde{z}_2 \geq h_t$ and when $\varphi \in [\pi, 2\pi)$, $\tilde{z}_2 \leq h_t$.

Similarly, $\tilde{z}_1$, which is the $z$ coordinate of a point on the contour of the bottom end plane, is given by

$$\tilde{z}_1 = -\frac{r_b - R_s \cos \theta_b}{\sin \theta_b} \sin \varphi - h_b. \quad (A.10)$$
APPENDIX B

CHANGING THE INTEGRAL LIMITS FROM $\int_{-\varphi}^{2\pi-\varphi} F_i^p(\sin\phi, \cos\phi) \, d\phi$ TO $\int_{-\pi/2}^{\pi/2} F_i^p(\sin\phi, \cos\phi) \, d\phi$

In order to change the limits of the integrals given in Equations (3.19) and (3.20) to the limits within $[-\pi/2, \pi/2]$, various possible cases need to be discussed.

If $\varphi \in [0, \pi/2)$, then $-\pi/2 < -\varphi < 0$ and $3\pi/2 < 2\pi - \varphi < 2\pi$. Thus

$$\int_{-\varphi}^{2\pi-\varphi} F_i^p(\sin\phi, \cos\phi) \, d\phi = \int_{-\varphi}^{\pi/2} F_i^p(\sin\phi, \cos\phi) \, d\phi$$

$$+ \int_{\pi/2}^{3\pi/2} F_i^p(\sin\phi, \cos\phi) \, d\phi$$

$$+ \int_{3\pi/2}^{2\pi-\varphi} F_i^p(\sin\phi, \cos\phi) \, d\phi.$$

Employing a variable change $\phi = \varphi - \pi$ in the integral $\int_{\pi/2}^{3\pi/2} F_i^p(\sin\phi, \cos\phi) \, d\phi$ and $\phi = \varphi - 2\pi$ in the integral $\int_{-\pi/2}^{\varphi} F_i^p(\sin\phi, \cos\phi) \, d\phi$, the integration yields

$$\int_{-\varphi}^{2\pi-\varphi} F_i^p(\sin\phi, \cos\phi) \, d\phi = \int_{-\varphi}^{\pi/2} F_i^p(\sin\phi, \cos\phi) \, d\phi$$

$$+ \int_{-\varphi}^{\pi/2} F_i^p(\sin\phi, \cos\phi) \, d\phi$$

$$+ \int_{-\pi/2}^{\pi/2} F_i^p(-\sin\phi, -\cos\phi) \, d\phi$$

$$= \int_{-\varphi}^{\pi/2} F_i^p(\sin\phi, \cos\phi) \, d\phi$$

$$+ \int_{-\pi/2}^{\pi/2} F_i^p(-\sin\phi, -\cos\phi) \, d\phi. \quad (B.1)$$

If $\varphi \in [\pi/2, 3\pi/2)$, then $-3\pi/2 < -\varphi < -\pi/2$ and $\pi/2 < 2\pi - \varphi < 3\pi/2$. Therefore,

$$\int_{-\varphi}^{2\pi-\varphi} F_i^p(\sin\phi, \cos\phi) \, d\phi = \int_{-\varphi}^{\pi/2} F_i^p(\sin\phi, \cos\phi) \, d\phi$$

132
Applying a variable change \( \phi = \phi + \pi \) to the integral \( \int_{-\pi/2}^{\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi \)
and \( \phi = \phi - \pi \) to the integral \( \int_{\pi/2}^{2\pi-\phi} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi \), then

\[
\int_{-\varphi}^{2\pi-\varphi} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi = \int_{-\varphi}^{\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi
+ \int_{-\pi/2}^{\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi
+ \int_{-\varphi}^{\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi
+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi
+ \int_{-\pi/2}^{\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi.
\]

If \( \varphi \in [3\pi/2, 2\pi) \), then \(-2\pi < -\varphi \leq -3\pi/2 \) and \(0 < 2\pi - \varphi \leq \pi/2 \). Hence,

\[
\int_{-\varphi}^{2\pi-\varphi} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi = \int_{-\varphi}^{-3\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi
+ \int_{-3\pi/2}^{-\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi
+ \int_{-\pi/2}^{\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi.
\]

Using a variable change \( \phi = \phi + 2\pi \) for the integral \( \int_{-\varphi}^{-3\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi \)
and \( \phi = \phi + \pi \) in the integral \( \int_{-3\pi/2}^{-\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi \), consequently,

\[
\int_{-\varphi}^{2\pi-\varphi} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi = \int_{-\varphi}^{\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi
+ \int_{-\pi/2}^{\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi
+ \int_{-\varphi}^{\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi
+ \int_{-\pi/2}^{\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi
+ \int_{-\pi/2}^{\pi/2} F_i^{p,q}(\sin \phi, \cos \phi) \, d\phi.
\]

133
Therefore, it can be concluded from the above discusses that for $\varphi \in [0, 2\pi)$,

$$
\int_{-\varphi}^{2\pi-\varphi} F_i^{\rho \varphi}(\sin \varphi, \cos \varphi) d\varphi = \int_{-\pi/2}^{\pi/2} F_i^{\rho \varphi}(\sin \varphi, \cos \varphi) d\varphi + \int_{-\pi/2}^{\pi/2} F_i^{\rho \varphi}(-\sin \varphi, -\cos \varphi) d\varphi.
$$

(B.4)
APPENDIX C

EXPRESSING HEUMAN’S LAMBDA FUNCTION IN TERMS OF GENERAL COMPLETE ELLIPTICAL INTEGRAL

In order to evaluate Heuman’s Lambda function in a simpler way than in the conventional method (refer to Section 3.3), it is rewritten in the following in terms of general complete elliptical integral.

Heuman’s Lambda function has the following definition [9]

\[ \Lambda_0(\psi, \kappa) = \frac{2}{\pi} \sin \psi \sqrt{1 + \kappa^2 \tan^2 \psi} \int_0^K \frac{\dn^2 u}{1 + \kappa^2 \tan^2 \psi \sn^2 u}. \]  

(C.1)

Let \( \sin \varphi = \sn u \) and \( p_\Lambda = 1 + \kappa^2 \tan^2 \psi \), then there exists

\[ \cos \varphi \, d\varphi = \sqrt{(1 - \sn^2 u)(1 - \kappa^2 \sn^2 u)} \, du, \]

that is,

\[ \sqrt{1 - \sn^2 u} \, d\varphi = \sqrt{(1 - \sn^2 u)(1 - \kappa^2 \sn^2 u)} \, du. \]

Since \( \kappa^2 \sn^2 u + \dn^2 u = 1 \) [68], the above equation can be simplified to

\[ d\varphi = \dn u \, du. \]

By using \( \sn K = 1 \) and \( \dn^2 u = 1 - \kappa^2 \sn^2 u \), (C.1) becomes (refer to (3.59))

\[ \Lambda_0(\psi, \kappa) = \frac{2}{\pi} \sqrt{p_\Lambda} \sin \psi \int_0^{\pi/2} \frac{1 - \kappa^2 \sin^2 \varphi}{(1 + \kappa^2 \tan^2 \psi \sn^2 u) \sqrt{1 - \kappa^2 \sin^2 \varphi}} \, d\varphi \\
= \frac{2}{\pi} \sqrt{p_\Lambda} \sin \psi \int_0^{\pi/2} \frac{1 + (\kappa'^2 - 1) \sin^2 \varphi}{(1 + (1 - p_\Lambda) \sn^2 u) \sqrt{1 - \kappa^2 \sin^2 \varphi}} \, d\varphi \\
= \frac{2}{\pi} \sqrt{p_\Lambda} \sin \psi \, \text{cel}(\kappa', p_\Lambda, 1, \kappa'^2). \]
APPENDIX D

AN ADDITION IDENTITY OF cel AND ITS APPLICATION

According to Equation (3.59), it can be proved that the following addition equation exist

\[
\begin{align*}
A \text{cel}(\kappa', 1, 1, 1) + B \text{cel}(\kappa', p, a, b) &= \int_{0}^{\pi/2} \frac{d\varphi}{\sqrt{\cos^2 \varphi + \kappa'^2 \sin^2 \varphi}} + B \int_{0}^{\pi/2} \frac{(a \cos^2 \varphi + b \sin^2 \varphi) d\varphi}{(\cos^2 \varphi + p \sin^2 \varphi) \sqrt{\cos^2 \varphi + \kappa'^2 \sin^2 \varphi}} \\
&= \int_{0}^{\pi/2} \frac{(A + Ba) \cos^2 \varphi + (Ap + Bb) \sin^2 \varphi) d\varphi}{(\cos^2 \varphi + p \sin^2 \varphi) \sqrt{\cos^2 \varphi + \kappa'^2 \sin^2 \varphi}} \\
\end{align*}
\]

From the above addition equation and (3.61), it can be obtained that

\[
\begin{align*}
AK(\kappa) + B A_{\lambda}(\psi, \kappa) &= A \text{cel}(\kappa', 1, 1, 1) + \frac{2}{\pi} \sqrt{p_{\lambda}} \sin \psi B \text{cel}(\kappa', p_{\lambda}, 1, \kappa'^2) \\
&= \text{cel}(\kappa', p_{\lambda}, A + B', A p_{\lambda} + B' \kappa'^2),
\end{align*}
\]

where

\[
B' = \frac{2}{\pi} \sqrt{p_{\lambda}} \sin \psi B,
\]

\[
p_{\lambda} = 1 + \kappa^2 \tan^2 \psi.
\]

136
APPENDIX E

MUTUAL INDUCTANCES OF COAXIAL CYLINDRICAL COILS OF INFINITESIMAL THIN THICKNESS

In the approximate impedance method for determining the current distribution in the coil (see Section 4.3), the mutual inductance between and the self inductances of the coil layers need to be evaluated.

Equations for calculating the self and mutual inductances of coaxial cylindrical coils of infinitesimal radial thickness have been presented in [72], [75] and [71], respectively. Grover [72] also gave an expression for evaluating the self inductance of helices of round wire which took the pitch of windings and size of conductors into consideration.

![Figure 37: Two single-layer coaxial coils.](image)

In most practical cases, the diameter of the conductor of reactor coil is very small compared with the radius of the reactor coil. The conductors of reactor coil are
only covered with thin insulating material and the turns are crowded together. The thickness of coil layer and the space occupied by insulating material are of very small importance in inductance calculation. Therefore, Fawzi's formulas [71], which are outlined in the following, can be employed for calculating the mutual inductances. Fawzi's formula is based on Bartky's transformation and can be evaluated more efficiently and easily by computer than that given either in [72] or in [75].

For two coaxial single-layer coils as shown in Figure 37, $R_{m1}$ and $R_{m2}$ are the radii of the two coils, respectively; $s$ is the separation distance between the two coils; and $l_1$ and $l_2$ are the axial lengths of the two coils, respectively. The mutual inductance between them is given

$$M_{12} = \frac{2\pi \mu_0 N_1 N_2 (R_{m1} R_{m2})^{3/2}}{l_1 l_2} \sum_{i=1}^{i=4} (-1)^{i+1} C_s(R_{m1}, R_{m2}, s_i), \quad (E.1)$$

where $N_1$ and $N_2$ are the number of turns of the two coils; and $C_s(R_{m1}, R_{m2}, s_i)$ is defined by

$$C_s(R_{m1}, R_{m2}, s_i) = \frac{2}{\pi} \int_0^{\pi/2} \left( A \sin^2 \theta + B \cos^2 \theta + C \frac{\sin^2 \theta}{\sin^2 \theta + \kappa^2 \cos^2 \theta} \right) \frac{1}{\sqrt{\sin^2 \theta + \kappa^2 \cos^2 \theta}} d\theta, \quad (E.2)$$

where

$$q = \frac{R_{m1} - R_{m2}}{R_{m1} + R_{m2}}, \quad 0 < |q| < 1,$$

$$\kappa^2 = \frac{(R_{m1} - R_{m2})^2 + s_i^2}{(R_{m1} + R_{m2})^2 + s_i^2}, \quad 0 < \kappa^2 < 1,$$

$$A = \sqrt{1 - \kappa^2} \left[ -r + \frac{1}{3(1 - \kappa^2)} \right],$$

$$B = \frac{\kappa^2}{3\sqrt{1 - \kappa^2}},$$

$$C = r \sqrt{1 - \kappa^2},$$

$$r = \frac{s_i^2}{4R_{m1} R_{m2}},$$

138
and

\begin{align*}
    s_1 &= (l_1 + l_2)/2 + s, \\
    s_2 &= (l_1 - l_2)/2 + s, \\
    s_3 &= (-l_1 - l_2)/2 + s, \\
    s_4 &= (-l_1 + l_2)/2 + s.
\end{align*}

\( C_s(R_{m1}, R_{m2}, s_i) \) can be evaluated by the following algorithm

\begin{align*}
    \alpha_0 &= 1, \quad \beta_0 = |\alpha|, \quad q_0 = |q|, \\
    A_0 &= A, \quad B_0 = B, \quad C_0 = C, \quad D_0 = 0, \\
    \alpha_{j+1} &= \alpha_j + \beta_j, \\
    \beta_{j+1} &= 2\sqrt{\alpha_j \beta_j}, \\
    q_{j+1} &= q_j + \frac{\alpha_j \beta_j}{q_j}, \\
    A_{j+1} &= \frac{B_j}{\alpha_j} + A_j, \\
    B_{j+1} &= 2(B_j + \beta_j A_j), \\
    C_{j+1} &= C_j + \frac{D_j}{q_j}, \\
    D_{j+1} &= 2(C_j \frac{\alpha_j \beta_j}{q_j} + D_j).
\end{align*}

If the computation is terminated at an index \( j = l \) such that \( |1 - \beta_l/\alpha_l| \leq 10^{-D} \), then

\[ C_s(R_{m1}, R_{m2}, s_i) = \frac{A_l \alpha_l + B_l}{2 \alpha_l^2} + \frac{C_l \alpha_l + D_l}{\alpha_l (\alpha_l + q_l)} + O(10^{-D}), \quad (E.3) \]

where \( D \) is the number of the significant digits used in the calculations and \( O(10^{-D}) \) is the error that the algorithm yields.
APPENDIX F

COEFFICIENT $f_H$

The coefficient $f_H$ used in the inductance calculation (refer to Section 4.3), which takes the partial effects of the space between the turns into consideration, is given in Table 5 [72]:

<table>
<thead>
<tr>
<th>Num. of Turns</th>
<th>$f_H$</th>
<th>Num. of Turns</th>
<th>$f_H$</th>
<th>Num. of Turns</th>
<th>$f_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>31</td>
<td>0.3087</td>
<td>110</td>
<td>0.3278</td>
</tr>
<tr>
<td>2</td>
<td>0.1137</td>
<td>32</td>
<td>0.3095</td>
<td>120</td>
<td>0.3285</td>
</tr>
<tr>
<td>3</td>
<td>0.1663</td>
<td>33</td>
<td>0.3102</td>
<td>130</td>
<td>0.3291</td>
</tr>
<tr>
<td>4</td>
<td>0.1973</td>
<td>34</td>
<td>0.3109</td>
<td>140</td>
<td>0.3296</td>
</tr>
<tr>
<td>5</td>
<td>0.2180</td>
<td>35</td>
<td>0.3115</td>
<td>150</td>
<td>0.3301</td>
</tr>
<tr>
<td>6</td>
<td>0.2329</td>
<td>36</td>
<td>0.3121</td>
<td>160</td>
<td>0.3305</td>
</tr>
<tr>
<td>7</td>
<td>0.2443</td>
<td>37</td>
<td>0.3127</td>
<td>170</td>
<td>0.3309</td>
</tr>
<tr>
<td>8</td>
<td>0.2532</td>
<td>38</td>
<td>0.3132</td>
<td>180</td>
<td>0.3312</td>
</tr>
<tr>
<td>9$^a$</td>
<td>0.2604</td>
<td>39</td>
<td>0.3137</td>
<td>190</td>
<td>0.3315</td>
</tr>
</tbody>
</table>

$^a$This table is continued on the next page

Table 5: Coefficient $f_H$ versus the total number of turns in each layer of the coil.
Continued from the previous page:

<table>
<thead>
<tr>
<th>Num. of Turns</th>
<th>$f_H$</th>
<th>Num. of Turns</th>
<th>$f_H$</th>
<th>Num. of Turns</th>
<th>$f_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.2664</td>
<td>40</td>
<td>0.3142</td>
<td>200</td>
<td>0.3318</td>
</tr>
<tr>
<td>11</td>
<td>0.2715</td>
<td>41</td>
<td>0.3147</td>
<td>220</td>
<td>0.3323</td>
</tr>
<tr>
<td>12</td>
<td>0.2758</td>
<td>42</td>
<td>0.3152</td>
<td>240</td>
<td>0.3327</td>
</tr>
<tr>
<td>13</td>
<td>0.2795</td>
<td>43</td>
<td>0.3156</td>
<td>260</td>
<td>0.3330</td>
</tr>
<tr>
<td>14</td>
<td>0.2828</td>
<td>44</td>
<td>0.3160</td>
<td>280</td>
<td>0.3333</td>
</tr>
<tr>
<td>15</td>
<td>0.2857</td>
<td>45</td>
<td>0.3164</td>
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<td>0.3336</td>
</tr>
<tr>
<td>16</td>
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<td>0.3168</td>
<td>350</td>
<td>0.3341</td>
</tr>
<tr>
<td>17</td>
<td>0.2906</td>
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<td>400</td>
<td>0.3346</td>
</tr>
<tr>
<td>18</td>
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<td>0.3175</td>
<td>450</td>
<td>0.3349</td>
</tr>
<tr>
<td>19</td>
<td>0.2946</td>
<td>49</td>
<td>0.3179</td>
<td>500</td>
<td>0.3351</td>
</tr>
<tr>
<td>20</td>
<td>0.2964</td>
<td>50</td>
<td>0.3182</td>
<td>550</td>
<td>0.3354</td>
</tr>
<tr>
<td>21</td>
<td>0.2980</td>
<td>55</td>
<td>0.3197</td>
<td>600</td>
<td>0.3356</td>
</tr>
<tr>
<td>22</td>
<td>0.2994</td>
<td>60</td>
<td>0.3210</td>
<td>650</td>
<td>0.3357</td>
</tr>
<tr>
<td>23</td>
<td>0.3008</td>
<td>65</td>
<td>0.3221</td>
<td>700</td>
<td>0.3358</td>
</tr>
<tr>
<td>24</td>
<td>0.3020</td>
<td>70</td>
<td>0.3230</td>
<td>750</td>
<td>0.3360</td>
</tr>
<tr>
<td>25</td>
<td>0.3032</td>
<td>75</td>
<td>0.3238</td>
<td>800</td>
<td>0.3361</td>
</tr>
<tr>
<td>26</td>
<td>0.3043</td>
<td>80</td>
<td>0.3246</td>
<td>850</td>
<td>0.3362</td>
</tr>
<tr>
<td>27</td>
<td>0.3053</td>
<td>85</td>
<td>0.3253</td>
<td>900</td>
<td>0.3362</td>
</tr>
<tr>
<td>28</td>
<td>0.3062</td>
<td>90</td>
<td>0.3259</td>
<td>950</td>
<td>0.3363</td>
</tr>
<tr>
<td>29</td>
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<td>0.3264</td>
<td>1000</td>
<td>0.3364</td>
</tr>
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<td>0.3269</td>
<td>$\infty$</td>
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</tr>
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</table>
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142


[70] H. Nagaoka, “The Inductance Coefficients of Solenoids,” *The Journal of the College of Science, Imperial University of Tokyo, Japan*, vol. 27, no. 6, pp. 1–33, 1909.


