VARATIONS IN PRIMORDIAL NUCLEOSYNTHESIS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

The Ohio State University

1996

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Approved by

Adviser
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To my sons, Neil and Thomas.

We will be spending much more time together from now on.
ACKNOWLEDGEMENTS

I wish to thank my advisor, Bob Scherrer, for seeing me through this long process. It is perhaps a gross understatement to say that I couldn't have done it without him. I wish that I had learned to value his guidance earlier than I did.

I thank my parents and my parents-in-law for their helpfulness and for their steadfast loyalty to me, even though they don't quite know what to make of this cosmology stuff. I also thank my friends at Neil Avenue Mennonite Church who have been so supportive of me and my family. Above all I must acknowledge the importance in this effort of my wonderful wife, Ruth. She and I have learned so much together in recent times about partnership.

I got involved in this academic adventure because I am irresistibly attracted to great ideas. I am indebted to the people who have discovered and clarified these magnificent concepts. However, we should take care not to make the intellect our god; it has, of course, powerful muscles, but no personality.¹ Instead, I praise the true God who is the source of the marvelous universe we seek to understand.

¹Albert Einstein
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The extrapolation procedure for the Bulirsch-Stoer method.
CHAPTER I

Elements from the Early Universe

1.1 Primordial Nucleosynthesis

As implied by the name, theories of nucleosynthesis attempt to explain the formation of the nuclei of the elements of nature. These theories themselves are a synthesis of two branches of physics. Nuclear properties such as mass, internal structure, and reactions between nuclei are within the domain of nuclear physics. The determination of nuclear abundances is a major subfield of astrophysics. Nuclear properties and abundances are obviously related to some degree, but it is not obvious that knowing everything about the intrinsic properties of nuclei will enable a good accounting of the abundances; initial conditions and non-nuclear processes might be the dominant factors.

However, looking beyond the earth, the stars, which contain most of the ordinary matter in the universe, have been found to have similar compositions. In particular, hydrogen and helium make up the lion’s share of stellar and interstellar matter. (They are greatly underrepresented on earth, because, as light gases, they escape into space.) Typically, a star is by mass about 3/4 hydrogen gas and 1/4 helium. The “metals”, representing everything heavier in the periodic table, constitute a small fraction of
the total mass. This uniformity makes it a compelling idea that a single event would have established a uniform distribution of abundances throughout the universe before matter clumped into stars.

In the 1940s Alpher, Bethe, and Gamow [2] (the $\alpha \beta \gamma$ paper) proposed that nuclear processes occurring in the expanding early universe could be just the astrophysical setting to explain the abundances of essentially all the elements. It was soon realized, however, that heavy elements could not be produced within the primordial context and soon the focus switched to processes occurring within stars. This study has successfully accounted for the abundances of most of the elements [7]. Nevertheless, stellar nucleosynthesis could only account for a small portion of the helium observed in the universe, as simple estimates showed [20].

Attention shifted back to the primordial arena rather dramatically with the discovery of the microwave background radiation in 1965. This radiation, a relic of the early hot and dense universe, had been predicted as a crucial aspect of the theory on the basis of the helium abundance [3]. The fact that the Nobel Prize winning discovery of the radiation was serendipitous indicates the extent to which the primordial nucleosynthesis scenario was not seriously considered up to that point. Since that time the general notion that the universe really does trace back to a state of extreme temperature and density has steadily gained credence in the scientific community. The success in explaining the abundances of the lightest nuclei has made primordial nucleosynthesis a pillar of the whole cosmological enterprise.

The subject matter of this work contributes a small increment to the knowledge
accumulated over the past fifty years in this field of inquiry on the boundary between physics and astronomy. It is divided into three investigations in and around the standard big-bang model. A paper based on the work of Chapter III has been accepted for publication by The Astrophysical Journal [29]. A paper based on the work of Chapter IV has been submitted for publication to The Physical Review D [28]. A paper based on the work of Chapter II is in planning.

1.2 The Big-Bang Model

Science is conducted by constructing a model to describe some aspect of nature and proceeds by testing the model by comparing the predictions of the model with observations. When a model has achieved a certain degree of success in confrontation with the data it becomes a 'standard'; it is the one with which competing models are compared. The microwave background radiation discovery was a major impetus for the thorough exploration of primordial nucleosynthesis. The result has been that the big-bang model has achieved the status of a standard.

Before examining the model in more detail, take a brief look at some of the basic observations on which it is based. (Excellent comprehensive introductions to modern cosmology including primordial nucleosynthesis are found in Kolb & Turner [27] and Peebles [30].) First of all, the universe appears to be expanding uniformly in all directions. This is inferred from the fact that radiation from distant galaxies is shifted toward lower frequencies, and, furthermore, the redshift increases linearly with distance. Secondly, as already mentioned, the uniform sea of microwave radiation
pervading the universe if of tremendous importance for modern cosmology. Thirdly, the abundances of the light elements, and particularly of helium, is the focus of this work.

It is natural to ask how far back in time the universe has been expanding. In fact, this is the general line of reasoning which has lead to the Big-Bang Model. Extrapolating to earlier times, the universe was denser and all scales of distance were smaller. This includes the wavelengths of the photons of the background radiation. From quantum theory this tells us that the photon momenta and energies were larger in inverse proportion to the scale factor. Thus, the universe was hotter the farther back we go. Application of our knowledge of the physics of matter at high temperature and density allows us to go a long ways toward understanding the formation of galaxies and the characteristics of the relic radiation background. Primordial nucleosynthesis is the earliest probe of the theory that can be tested against observations. In order to fulfill this promise we need a model specific enough to be confronted with the data plus, of course, the data itself.

1.2.1 Assumptions

Denoting the given truths which form the foundation of any line of argument always involves some arbitrary choices to an extent. The most difficult thing to do, of course, is to include the hidden assumptions of which you are not aware. Having offered these caveats, the assumptions of the Big-Bang Model are usually listed in the following
three categories:

- Gravity

General Relativity is believed to be an accurate description for the gravitational attraction of matter. This work will not test that assumption in any way. It is not relevant to say more than the basic statement that Newtonian gravity will not suffice for understanding modern cosmology. It is certainly a very useful limiting approximation for many applications, but it cannot be used to predict the expansion of the universe.

- The distribution of matter and radiation

The theory of general relativity when applied to a general medium without simplifying assumptions involves a high degree of mathematical complexity. When Einstein made his assumptions that the universe is homogeneous, the same at all places, and isotropic, the same in all directions, the result was a great simplification in the equations of motion. It turns out that a single equation expresses the relation between the kinematics of space-time and the material content of the universe. A homogeneous and isotropic universe described by general relativity is called Friedmann-Robertson-Walker; the equation of motion is the Friedmann equation,

\[ H^2 = \frac{8}{3} \pi G \rho - \frac{k}{R^2} + \frac{\Lambda}{3}, \]

where the Hubble parameter is

\[ H \equiv \frac{\dot{R}}{R}. \]
The scale factor \( R \) denotes the radius of an arbitrary but definite comoving volume. The third term on the right hand side includes the cosmological constant \( \Lambda \). A nonzero value for \( \Lambda \) is an intriguing possibility with a fascinating history. However, it is not relevant for this work and will be considered negligible.

The first term on the right hand side of Equation (1.1) is all-important for this work. Two important limiting cases for the energy density \( \rho \) are non-relativistic matter, scaling as \( \rho \propto 1/V \propto 1/R^3 \), and extremely relativistic matter. This latter case includes massless particles such as photons or neutrinos. In this case the energy density falls with an extra power of the scale-factor, since the particle energies are equal to their momenta. Momenta are proportional to the inverse of the wavelength, which expands with the universe, so \( \rho \propto 1/R^4 \).

Because the terms on the right-hand side of Equation (1.1) scale differently with the scale-factor \( R \), the curvature term [second on the right, with \( k = (-1, 0, +1) \)] will be negligible at very early times. This is certainly the case in primordial nucleosynthesis. The curvature term might well be very significant today. The relative magnitudes of the density and curvature terms can be expressed in a concise way by dividing Equation (1.1) by \( H^2 \) (taking \( \Lambda = 0 \)) to obtain

\[
\Omega_0 + \Omega_k = 1, \tag{1.3}
\]

where \( \Omega_0 = \rho/\rho_{cl} \). The universe has a closed geometry with a density greater than the closure density

\[
\rho_{cl} = \frac{H^2}{8\pi G}. \tag{1.4}
\]
Estimates derived from various measures give a value for the density parameter today of $0.2 \leq \Omega_0 \leq 1$, so that it is not clear whether the universe is closed or not.

- Elementary particle physics

The Standard Model of elementary particle physics is at least an excellent approximation of nature at all laboratory energies yet probed. It serves very well within the context of modern cosmology, also. The electromagnetic forces are mediated by photons between electrically charged particles. The weak nuclear interactions are responsible for interchanging neutrons and protons as well as all neutrino interactions. The strong nuclear forces allow nucleons to combine into light nuclei during nucleosynthesis. Although these forces are believed to be residual forces of the quantum chromodynamic interactions between the constituent quarks of the nucleons, the complexity of the net force precludes calculation directly from first principles. In practice the relevant cross sections must be taken from nuclear accelerator measurements.

Even the weak interactions between neutrons and protons are subject to a significant uncertainty. Each cross-section is proportional to a common factor which must be measured. The weak interactions are well enough understood that only the complex internal structure of the nucleons prevents complete calculation from the theory.

The important principle here is that local physics applies at all times and places. This assumption enormously simplifies the study of cosmology. Although an assumption, the idea is a natural one. Alternatively, one might take the viewpoint that modern cosmology actually tests this assumption by applying the physics that we know from our present day laboratories to the early universe. The consistency of
theoretical predictions with observations tells us that the assumption is reasonable.

1.2.2 Predictions

Armed with a set of well-motivated and sufficiently definite assumptions we can set about discussing the consequent predictions of the theory. In order to describe the time development of important quantities and ultimately the nuclear abundances, we need to be clear about the initial state. It turns out that specifying the state of matter that would have existed at a temperature of about 10 MeV is both sufficient and possible given our understanding of the present universe. The corresponding initial time is about $10^{-2}$ s. The condition of thermal equilibrium is what saves us from having to know the details of the previous history. Of course, the one quantity that does survive from earlier times is the baryon-to-photon ratio $\eta$. We would like to be able to calculate this from a sufficiently definite theory of baryogenesis, but such a theory is still lacking. This makes $\eta$ the one relatively free parameter, although it can be somewhat bounded by present day observations.

In the early high temperature phase of the time evolution, the various particle distributions are given by thermal spectra. The very important quantity for the helium abundance is the neutron-proton (number density) ratio, which is conventionally denoted as $n/p$. The weak interaction rates affecting the neutron-proton ratio are

\begin{align}
\nu_e + n &\leftrightarrow p + e^- \\
e^+ + n &\leftrightarrow p + \bar{\nu}_e \\
n &\leftrightarrow p + \bar{\nu}_e + e^- \tag{1.5}
\end{align}
When the rates of interaction between neutrons and protons are much greater than the expansion rate, the number ratio of neutrons to protons is given by the equilibrium value

\[ \frac{n}{p} = e^{-Q/T} \]  

where \( Q = 1.293 \text{ MeV} \) is the neutron-proton mass difference.

At a temperature of \( T_{F} \sim 0.8 \text{ MeV} \) the weak rates can no longer keep up with the expanding universe, and the neutron-to-proton ratio freezes out. Free neutron decays decrease the neutron fraction somewhat after this point. At a temperature of \( T_{N} \sim 0.07 \text{ MeV} \) the production of deuterium begins to rapidly build by the process

\[ n + p \rightarrow D + \gamma. \]  

(1.7)

Since \(^4\text{He}\) is the most tightly bound of the light nuclei, reactions leading from deuterium act to incorporate essentially all the neutrons into helium. The most important of these reactions are

\[ \begin{align*}
D + D & \leftrightarrow ^3\text{H} + p \\
D + D & \leftrightarrow ^3\text{He} + n \\
^3\text{H} + D & \leftrightarrow ^4\text{He} + n \\
^3\text{He} + D & \leftrightarrow ^4\text{He} + p.
\end{align*} \]  

(1.8)

No appreciable nucleosynthesis proceeds beyond \(^4\text{He}\) because of mass gaps at \( A = 5 \) and \( A = 8 \) as well as freeze-out of the strong nuclear reactions shortly after \( T_{N} \). There results an astronomically significant leakage beyond helium to \(^7\text{Li}\), but little of anything heavier is synthesized.
Figure 1: The free and bound neutron fractions along with the equilibrium abundance. Also shown is the neutron fraction that would result if no nucleosynthesis occurred.
Figure 2: The standard model predictions for the primordial He$^4$ mass fraction $Y_P$ and the number densities relative to hydrogen of D, $^3$He, and $^7$Li. The values are for $N_\nu = 3$ and $\tau = 10.25$ min.
Figure 1 shows the neutron fraction as a function of time. Also shown is the equilibrium fraction. Here can be seen the interesting interplay between the various scales affecting the evolution. The near coincidence of the neutron-proton mass difference $Q$ and the freeze-out temperature $T_F$ results in a freeze-out neutron fraction intermediate between the limiting values of 0 and 1/2. The nucleosynthesis temperature $T_N$ corresponds to the time at which deuterium begins to rapidly build up. This occurs just before neutron decay would have acted to eliminate all of the neutrons. As it seems to have occurred in nature, the neutrons are saved to be incorporated almost exclusively into He$^4$. The asymptotic value of the bound value of $X_n = 0.12$ indicates a final primordial helium mass fraction of $Y_p = 0.24$.

Figure 2 shows the abundances plotted for a baryon-to-photon ratio in the range $10^{-10} \leq \eta \leq 10^{-9}$. The $^4$He abundance increases gradually with $\eta$, while the abundances of D and $^3$He decline strongly with increasing $\eta$. This happens because a larger $\eta$ causes nucleosynthesis to begin earlier with a larger neutron-to-proton ratio and therefore more $^4$He. D and $^3$He are unburnt fuels of $^4$He accumulation. D especially declines dramatically as $^4$He increases only logarithmically. $^7$Li exhibits its characteristic 'trough' from the combination of two different production processes.

Not shown in this plot is the dependence of the abundances on two physical parameters $\tau$, the neutron lifetime, and $N_\nu$, the effective number of neutrino species. The $^4$He abundance is most sensitive to these parameters. The other abundances are affected by them as well, but the observational uncertainties in these rare nuclei are large in comparison. The $^4$He abundance is an increasing function of all three
parameters. An analytic fit for the primordial mass fraction is [42]

\[ Y_P = 0.228 + 0.010 \ln \eta_{10} + 0.012(N_\nu - 3) + 0.185 \left( \frac{\tau - 889}{889} \right). \]  \hspace{1cm} (1.9)

### 1.2.3 Observations

In testing the theoretical predictions of the standard model, it is important to compare with primordial abundances; that is we need a reliable observer when the universe was finished nucleosynthesizing after a few hours of expansion. In practice, we instead must infer primordial abundances from information coming from considerably later periods. Doing this requires having a reasonable picture of how the universe has evolved over the whole course of its history.

The overall story goes something like this. \(^4\)He is produced in stars during the course of stellar evolution, so this relatively small but significant amount must be subtracted to obtain the primordial abundance. This is done by studying the stars of lowest metallicity. The primordial component is found to be somewhere in the range 0.22-0.24. From Figure 2 this corresponds roughly to a range \(1 < \eta_{10} < 3\).

Deuterium in contrast is destroyed in any astrophysical environment that has been thought of. Consequently, its measured abundance must represent a lower limit to the primordial value. Observations typically give limits of \(D/H > 1 \times 10^{-5}\). This gives a rather strict upper limit to \(\eta\) of \(10^{-9}\). Since D typically burns into \(^3\)He, the measurement of the latter is an indication of the pre-solar \((D+^3\text{He})/H\) value. That is why this sum is plotted in Figure 2. Reduction by stellar burning, or astration, is a consideration. The typical result is \((D+^3\text{He})/H < 10^{-4}\). This gives a lower bound
to \( \eta \) very close to the upper bound from \(^4\)He. \(^7\)Li is a powerful probe of the baryon density because of its trough. Its upper limit of \(^7\)Li/H < 2 \times 10^{-10} \) puts it in a range roughly consistent with the other abundances as well.

The very interesting aspect of this general concordance for \( \eta \) in the range plotted in Figure 2 is that this range lies between the lower observational limit imposed by luminous matter in the universe and the upper limit implied by dynamical determinations of the cosmic mass density. These are most conveniently expressed as fractions of the closure density. With \( \Omega_l = \rho_l/\rho_{cl} \) it is found that luminous matter has \( \Omega_L \approx 0.01 \), while the total density corresponds to \( \Omega_0 \geq 0.2 \). The values of \( \eta \) coming from primordial nucleosynthesis correspond to \( \Omega_B \) situated between these two limits. The implication is the presence of both baryonic and nonbaryonic dark matter.

The field of primordial nucleosynthesis is quite interesting at this time, because the Big-Bang model is being strained by the observations. This situation has been slowly emerging over the last decade or so. The most recent studies disagree as to the severity of the situation. Hata, et al. [17] have made a detailed statistical analysis of the astrophysical data coupled with a fairly definite model of galactic chemical evolution. On this basis, they have proclaimed a crisis to be at hand in which theory cannot accommodate the observations with reasonable certainty. Copi, Schramm, and Turner [10] have described the state of affairs in milder terms, saying that 'tension' exists in the data. The \(^4\)He abundance favors a low value of \( \eta \), while the other abundances favor a high value. Regardless of whether the field is in true crisis or not, two basic interpretations can be given. Either observations will clear things up, or
the theory needs modification.

1.3 Variations on the Theme

The importance of primordial nucleosynthesis has been its use as a test of the standard cosmological model and as a probe of both cosmology and particle physics. The predictions of standard nucleosynthesis are sufficiently definite, and the primordial abundances can be inferred with enough confidence from observed abundances, that the comparison between the two sets is meaningful. The epoch of nucleosynthesis dates from a time $t \sim 10^{-2}$ s after the bang, and the standard model furnishes the earliest real test of the standard scenario. What has made the standard model especially attractive as a test is its degree of simplicity. The assumptions that go into its construction are well-motivated if not absolutely unavoidable. This gives an air of inevitability to its predictions.

The relative success of the standard model has not deterred efforts to construct variant models. It is these variations that turn primordial nucleosynthesis into a probe. If similar or even better success is achieved when confronting predictions with observations, then a model opens up a window for more possibilities in the organization of nature. The overwhelming story in these efforts has been the failure of variant models to perform much better than the standard model despite having more adjustable parameters to work with. Nevertheless, as mentioned in the last section, departures from the standard picture will be required if the latest hints of discrepancy persist.
In this work, three topics in primordial nucleosynthesis are explored. Each is a variation of previous studies, although the first (Chapter II) is a variation in a more general sense than just discussed. The subject of Chapter II emphasizes that the predictions of the standard model are based on estimates. Uncertainties and approximations are involved in making these estimates. If primordial nucleosynthesis is to be a stringent test of the standard cosmology, then the accuracy of these estimates must be understood. In this study, the commonly used integration algorithm of the calculations is replaced by a more modern and improved method in order to check for systematic error. The result is essentially a confirmation, with this more sophisticated approach, of a moderately important correction found recently for the helium abundance.

Chapters III and IV involve variations from the standard model. The assumptions of the model are relaxed or changed in some way. In Chapter III the assumption of homogeneity is relaxed by allowing the possibility of a nonuniform distribution of baryons. Here the variation functions as a probe of cosmology by attempting to expand the range of the cosmological parameter, the baryon-to-photon ratio $\eta$, beyond the narrow limits allowed (if allowance is made at all) by the standard model. This has a bearing on the dark matter issues mentioned above. The main new feature in this study is a very general method allowing for all types of fluctuation distribution for the simplest physical class of models. The result puts powerful limits on the mean value of $\eta$ for this class of models.

Chapter IV changes the standard model of nucleosynthesis by incorporating an
extension to the standard model of elementary particle physics. In this variation, a particle whose existence is reasonably well motivated yet undetected in nature is included. Furthermore, the particle is assumed to decay with a particular type of decay mode during a special time range. Here the analysis is partially observation driven, since a strong motive is to focus on a model variation capable of lowering the helium abundance. The result is to furnish a potential resolution of the helium problem mentioned in the last section.
2.1 Doing Things Right

As mentioned in Chapter 1, many variations from standard primordial nucleosynthesis exist which, however, lack the same degree of simplicity and, therefore, inevitability. Before considering them too seriously, it is imperative to scrutinize the accuracy of the predictions of the standard model. The uncertainties of the calculation can be thought of as coming from three basic sources: measurements of nuclear reaction rates, complications in dealing with higher order weak interaction effects, and systematic numerical error independent of the physics of the problem. In this chapter the last of these three will be addressed.

Kawano [24] and Kernan [26] have recently studied the accuracy of the standard computer code used in primordial nucleosynthesis by Wagoner [41, 39, 40]. Kawano examined the effect of integration step size and determined corrections to the abundances of D, $^3$He, $^4$He, and $^7$Li. Kernan performed even more precise calculations of the $^4$He abundance including the effect of modifying the integration algorithm. From the asymptotic limit of small step-size Kernan found the correction for error in the
Wagoner code, as a function of $\eta_{10} \equiv 10^{10}\eta$, to be

$$\Delta Y = +0.00174 - 0.00003(\ln \eta_{10}) \quad (2.1)$$

This result is significant enough, especially since it increases the helium prediction, that it would be nice to verify it by an appreciably different method to test for error systematic of Kernan's methodology. In fact, there are reasons for suspicion regarding the Kernan result, and these are discussed in Section 2.2. Section 2.3 presents the results of applying an improved integration algorithm. Suspicions notwithstanding, the error found in the Wagoner code is only slightly different than found by Kernan.

Note that, when referring to the Wagoner code, the statements contained in this work will apply equally well to the Kawano rendition [24]. Kawano did a great service in improving and clarifying the organization of the code, but the details of the calculation are essentially identical. Another note of interest is that, while the numerical techniques are somewhat specific to the physical problem (to be discussed later in the chapter), the physics originally was rather general. Compared to its usage today, the code was applied to a wide variety of astrophysical environments as implied by the title of Wagoner's 1969 paper, *Synthesis of the Elements within Objects Exploding from Very High Temperatures* [39].

### 2.2 The Wagoner Code

If the research field of primordial nucleosynthesis was just commencing at this time, it is unlikely that the computer calculations would take a form similar to that of the Wagoner code. The program is constructed to solve a particular system of equations,
and it shows evidence of a good deal of trial and error to obtain a custom fit. The integration algorithm could not be easily extracted for application to a quite different set of equations as is characteristic of the modern modular approach.

In mathematical language, the calculation involves the integration over the time \( t \) of a set of coupled ordinary differential equations subject to initial conditions:

\[
\dot{v} = f(v),
\]

where \( f \) is some function of the dependent variables. (It will be not be necessary to consider explicit dependence of the derivatives on the independent variable \( t \).) This is the so-called initial value problem. Methods for solving such equations have developed substantially since Wagoner wrote his code nearly three decades ago. Before looking at the Wagoner integration closely, let us review some of the basics of numerical methods for differential equations. (A more complete discussion can be found in reference [31].)

Minimizing the total error is, of course, a dominant consideration for precision computations. The error comes from two sources; roundoff and truncation. Roundoff is hardware dependent and will not be a major concern here. Truncation is associated with methodology which is the point of focus. The order of a method indicates the degree of truncation error.

A general notation, valid for a method of any order, for advancing the solution from \( t_n \) to \( t_{n+1} \equiv t_n + \Delta t \) is

\[
\frac{\Delta v}{\Delta t} = \frac{v_{n+1} - v_n}{\Delta t} = q,
\]
where $q$ is some linear combination of derivatives with various arguments. That is, one applies (2.3) according to the prescription

$$v_{n+1} = v_n + (\Delta t)q.$$ (2.4)

The starting point for integrating (2.2) is Euler's method:

$$q = f(v_n)$$ (2.5)

This is called a first order method with error of order $(\Delta t)^2$. [An $n$th order method has an error proportional to $(\Delta t)^{n+1}$]. Higher order methods employ more evaluations of the derivatives for each time step.

The general order Runge-Kutta formula is

$$q = \sum_i w_i k_i$$

$$k_i = f(v_n + \sum_{j=1}^{i-1} \beta_{ij} k_j).$$ (2.6)

The coefficients $w_i$ and $\beta_{ij}$ are chosen to match a Taylor series expansion up to a given order. A second order formula is

$$q = k_2$$

$$k_2 = f(v_n + \frac{1}{2}(\Delta t)k_1)$$

$$k_1 = f(v_n).$$ (2.7)

The Wagoner code uses the similar prescription

$$q = \frac{1}{2}(k_1 + k_2)$$

$$k_2 = f(v_n + (\Delta t)k_1)$$

$$k_1 = f(v_n).$$ (2.8)
That the use of this formula is improper or at least misleading is a topic taken up later in this section in conjunction with the phenomenon of stiffness. This has been called a second-order Runge-Kutta formula. It was, however, probably thought of originally as a predictor-corrector, which is a multi-step method given by the general formula

\[ q = \sum_i \beta_i f(v_{n+1-i}). \]  

(2.9)

Equation (2.5) is used to predict the new values of the variables, and (2.8) corrects the predictions. The one-step Runge-Kutta method was commonly considered to be a convenient starting procedure for the superior but more complex predictor-corrector approach. The terminology is somewhat nebulous with regard to the Wagoner code since the Runge-Kutta and predictor-corrector methods are really only distinct for orders greater than two.

New procedures for integrating ordinary differential equations have incorporated sophisticated techniques for monitoring and controlling accuracy in an automatic way. This is achieved by regulating the stepsize at each step along the integration path so as to satisfy some specified error criterion. The stepper part of the routine chooses the largest stepsize consistent with the maximum allowed error per step. The error for a given stepsize is estimated by comparing, for each dependent variable, the change calculated at different orders. The difference between the calculations at different orders gives a measure of the error incurred for the step. The stepper accepts the result at the higher order if the overall error criterion is met. If not, then the step is retried at a shorter stepsize to decrease the error.
The major practical difficulty in performing the nucleosynthesis calculation concerns the \textit{stiffness} of the equations. Again, algorithms for the handling of stiff differential equations have advanced significantly since the advent of the Wagoner code. Stiffness occurs in a set of coupled equations when there are two or more very different scales of the independent variable on which the dependent variables are changing. The different scales are a consequence of processes in the physical system described by the equations having significantly distinct time scales. Stability of the solution can only be achieved by following the solution on the shortest scale regardless of whether that scale is relevant to the true solution. The continual admixture of error to the calculation can excite, typically, a transient which must be followed until it dies out, even though that transient solution does not contribute at all in the exact solution. In other words, the straight-forward application of the Euler formula (2.5), for instance, would commonly result in excessively small step-sizes and a prohibitively large total number of steps for a stiff problem.

The physics of the early universe environment gives rise to the mathematical quality of stiffness in the equations. The abundances evolve from a condition of thermal equilibrium at early times to constant freeze-out values at late times. The forward and reverse reactions which are dominant involve at most four different nuclides. These can be represented according to the general notation

\begin{equation}
N_i(A_i Z_i) + N_j(A_j Z_j) \leftrightarrow N_k(A_k Z_k) + N_l(A_l Z_l)
\end{equation}

(2.10)

where $A$ indicates the mass number of a particular isotope of element $Z$. The abundance of nuclide $i$ will be described in terms of its number fraction $Y_i$, where
$Y_i = X_i/A_i$, and $X_i$ is the mass fraction. The abundance of nuclide $i$ evolves according to the general equation

$$\dot{Y}_i = \sum_{j,k,i} N_i \left( R_{ik\rightarrow ij} Y_j^{N_i} Y_k^{N_i} - R_{ij\rightarrow ki} Y_j^{N_i} Y_i^{N_j} \right).$$

(2.11)

The sum is over all reactions involving nuclide $i$. The reaction rates, denoted by $R$, are temperature dependent functions whose specific form can be found in Reference [39]. They are determined by the thermally averaged reaction cross-sections and the baryon density. Included in their definition is a statistical factor correcting for overcounting of identical particles in the sum of Equation (2.11).

In the early high-temperature phase, the abundances track their equilibrium values. In perfect equilibrium the forward and reverse terms within the parenthesis of Equation (2.11) are exactly equal. However, they are individually very large, so that the imbalance created by any slight deviation from equilibrium restores the abundance rapidly to its equilibrium value. As the temperature falls throughout the expansion, the abundances continually readjust in attempting to reestablish equilibrium. Since the rates of reaction diminish much more rapidly than the expansion rate, the reactions find it increasingly harder to keep up with the expansion and the abundances freeze out of equilibrium, eventually reaching asymptotic values.

Although nature does not allow departure from equilibrium when the rates in Equation (2.11) are much greater than the expansion rate, a computer simulation does so through the inevitable introduction of error, whether from roundoff or truncation. The wonderful cure for the stiffness disease of instability is the use of implicit methods. The Euler Equation (2.5) is called explicit, meaning that it relies only on
past information. The *implicit* Euler Equation is

\[ q = f(v_{n+1}) \]  

(2.12)

To get the future information \( f(v_{n+1}) \) we need to find

\[ \Delta f \equiv f(v_{n+1}) - f(v_n) \approx \left. \frac{\partial f}{\partial v} \right|_{v_n} \cdot (v_{n+1} - v_n). \]  

(2.13)

This is an approximation introducing new error to the solution. Nevertheless, Equation (2.13) is very useful for solving stiff equations. A new expression for the derivatives is

\[ q = [\Gamma]_{v_n}^{-1} \cdot f(v_n). \]  

(2.14)

The price to be paid for taking the large step-sizes characteristic of the time variation of the true solution is the inversion at each step of the matrix

\[ \Gamma \equiv 1 - (\Delta t)J. \]  

(2.15)

The stability of a scheme for solving stiff equations is determined by the largest eigenvalue \( \lambda_{\text{max}} \) of the Jacobian matrix

\[ J \equiv \frac{\partial f}{\partial v}. \]  

(2.16)

The maximum stepsize to ensure stability is inversely related to \( \lambda_{\text{max}} \). Standard treatises of numerical methods discuss the stability of integration methods in detail [32].

Let's look at the Wagoner code in the light of these considerations. First of all, the Wagoner code does not make use of adaptive stepsize control. The stepsizes are indeed limited by a criterion preventing them from changing by too much in a given
step. The Kawano code uses the following prescription (slightly simplified from the original Wagoner mechanism): for each nuclide $i$ the maximum allowed time-step is given by

$$
\Delta t_{\max} = \frac{Y_i}{Y_i} c_T \left[ 1 + \left( \frac{\log(Y_i)}{\log(Y_{\text{min}})} \right)^2 \right].
$$

(2.17)

The maximum allowed time-step for the temperature is

$$
\Delta t_{\max} = \frac{|T|}{T} c_T.
$$

(2.18)

The constants $c_T$ and $c_T$ can be varied to control the time-step sizes and thus the number of steps taken. The actual time-step chosen is the smallest from Equations (2.17) and (2.18). The changes in the variables from the second order corrector step of Equation (2.8) could be compared with the first order step of Equation (2.5) to obtain an error estimate for use in regulating the time-step. However, this is effectively precluded by the arbitrary lower limit on the abundances of $Y_{\text{min}}$.

Looking deeper into the Wagoner method reveals some interesting surprises. In the following discussion it will be sufficient to describe the variables $v$ as being comprised of the temperature $T$ and the abundances $Y$. In actuality, the code also evolves a variable proportional to the baryon-to-photon ratio $\eta$ as well as the electron chemical potential $\phi_e$. However, $\phi_e$ is a relic of the days when the calculation was applied to much different physical circumstances and can safely be set to zero in the usual case of $\eta \ll 1$. (The value of $\phi_e$ is important if one is interested in following the electron density accurately even when it is very small.) The value of $\eta$ varies slowly in changing by a total factor of $4/11$ due to electron-positron annihilations. For present purposes
the variation of of $\eta$ can be either ignored or thought of as being lumped into the temperature evolution. The statements below can be generalized readily.

The first surprise for someone versed in numerical methods is that the temperature $T$ and the abundances $Y$ are treated differently! The variable $T$ is 'normal' and is evolved directly according to Equation (2.8). In contrast, the variables $Y$ are seen as 'stiff', so that the Jacobian of Equation (2.16) is limited to the abundance sector. However, in a strict sense, the entire set of variables $v$ is a coupled stiff set. Thus, all of the Jacobian elements involving the temperature are ignored in the code. The next section will address this issue in more detail.

Another surprise is found in the handling of the abundance derivatives. A direct way of producing a second-order method for handling stiff equations is to average the derivatives from the explicit and implicit Euler first-order Equations (2.5) and (2.12) to obtain

\[ q = \frac{1}{2} [f(v_n) + f(v_{n+1})] \]
\[ = \frac{1}{2} \left[ 1 + \Gamma|_{v_n} \right]^{-1} \cdot f(v_n). \quad (2.19) \]

This is not what is done in the Wagoner code. Instead, the abundance derivatives are calculated according to Equation (2.12) for both the predictor and corrector phases of Equation (2.8).

At this point it would seem that the temperature derivative is calculated at the beginning of a given time-step interval, while the corresponding abundance derivatives are effectively evaluated at the end of that interval. Looking ever more deeply reveals that the reality is even more complicated. The Wagoner code uses a trick which
depends on the specific form of the derivatives of Equation (2.11). First note that the derivatives of Equation (2.2) can be approximated, with $\Delta f$ as in Equation (2.13), by

$$q = f(v_n) + \gamma \Delta f,$$  \hspace{1cm} (2.20)

where the factor $\gamma$ is reasonably chosen anywhere in the range $0 \leq \gamma \leq 1$. For $\gamma = 0$ we get the explicit Euler scheme and for $\gamma = 1$ the implicit formula. A second-order result comes from taking the midpoint derivative corresponding to $\gamma = 1/2$. [The higher order Runge-Kutta formulas of Equation 2.6 have combinations of derivatives sprinkled throughout the time-step interval with various values of $\gamma$.]

Now look at the specific derivative form which is one term of Equation (2.11):

$$\dot{Y}_i \equiv f_i = C Y_k^{N_k} Y_i^{N_i}.$$ \hspace{1cm} (2.21)

For this form the change in the derivatives is

$$\Delta f_i = \left( \frac{\partial f_i}{\partial Y} \right) \cdot (\Delta Y) = \left( \frac{\partial f_i}{\partial Y_k} \right) (\Delta Y_k) + \left( \frac{\partial f_i}{\partial Y_i} \right) (\Delta Y_i).$$ \hspace{1cm} (2.22)

With some rearrangement we see that

$$\frac{\Delta Y_i}{\Delta t} = f_i + \gamma \Delta f_i = [1 - \gamma (N_k + N_i)] f_i + \gamma \left( \frac{\partial f_i}{\partial Y} \right) \cdot Y.$$ \hspace{1cm} (2.23)

The Wagoner trick is to choose

$$\gamma = \frac{1}{N_k + N_i}$$ \hspace{1cm} (2.24)
so that the term involving the derivative functions $f$ drops out. Then we can solve Equation (2.23) for $Y$ to get (with $\Delta Y \equiv Y - Y_0$)

$$Y = (1 - (\Delta t) J)^{-1} \cdot Y_0,$$

(2.25)

which is plugged into Equation (2.20) to get values for the derivatives.

The result (2.25) is mysterious for several reasons. First of all, the advanced derivatives of Equation (2.20) do not depend on the normal derivatives of Equation (2.11) when using the Wagoner trick. This is true for the form (2.21), but is not generally the case. Secondly, the order of the calculation becomes hard to define since $\gamma$ in Equation (2.24) takes on different values for the various reactions (2.10). For the most important reactions, the weak neutron-proton conversion processes, the value is $\gamma = 1$, so the truncation error for the entire calculation per step is effectively first order. The final mystery is the purpose of the Wagoner trick. Derivatives from Equation (2.19) would be just as easy to program and would make the method more transparent.

### 2.3 A Slightly Corrected Correction

In this work the Bulirsch-Stoer method is used to integrate the variables included in the Wagoner code. Appendix A describes some details of the Bulirsch-Stoer method as adapted to stiff systems of equations. Care has been taken to achieve a consistent comparison with the Wagoner code. The same input parameters are used. Of course, the final baryon-to-photon ratio $\eta_f$ should be equal for a proper comparison rather than the initial value $\eta_i$. In practice this is not a significant consideration.
The calculation is performed with double precision arithmetic throughout to ensure that roundoff error does not contaminate the results. In fact, high precision is important for stiff systems generally. In this case, the forward and reverse reaction rates are nearly equal until freeze-out, and significant error can be introduced when taking differences of the rates.

Rebuilding the code with Bulirsch-Stoer integration involves extracting the derivatives of the variables from the Wagoner code. For the abundances this involves a complication. As noted in the last section, the derivatives of Equation (2.11) are not explicitly calculated. However, they are closely related to the derivatives of the Jacobian matrix. One must take care to eliminate the $\gamma$ factors of Equation (2.24).

The initial values of the abundances need some thought. Ideally, all abundances should be set to their equilibrium values at high temperature for a smooth start to the integration. It turns out to be sufficient to set the neutron, proton, and deuterium abundances to equilibrium amounts, but to simply set all remaining abundances initially to zero. Note that the minimum value $Y_{\text{min}}$ of Equation (2.17) does not need to be used.

When these starting values and the derivatives, properly extracted from the Wagoner code, were applied to the Bulirsch-Stoer recipe (Numerical Recipes in Fortran, Second Edition [31, §16.6]), stability was still a problem. The reactions of the nucleosynthesis make for a volatile mixture. In fact, the warning is given that "certain stability restrictions on the stepsize ... come into play on some problems." [31, p. 739] It was possible to (almost) conquer this problem with a simple test for a runaway
abundance and a resetting of the stepsize to a smaller value in this eventuality.

The difference between the calculation with the new code and with the Wagoner code for the $^4$He abundance is given in Table 1. All results are for the common value of $\eta = 3.400 \times 10^{-10}$. The error per step for all variables is limited by the condition

$$\frac{\delta Y}{Y} < \epsilon.$$  \hspace{1cm} (2.26)

The number of steps $N$ can be seen to increase dramatically as the error criterion parameter $\epsilon$ is decreased. As a matter of interest, the proportion of the steps which had to be retried (because the error criterion was not met) slowly increased as $N$ increased, ranging from 17-25%.

Table 1: The Wagoner code $^4$He correction for $\eta_{10} = 3.400$

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$N$</th>
<th>$\Delta Y$</th>
<th>$E_{rw}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-4}$</td>
<td>349</td>
<td>.00189</td>
<td>$\pm .0005$</td>
</tr>
<tr>
<td>$10^{-5}$</td>
<td>1,751</td>
<td>.00173</td>
<td>$\pm .0001$</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>14,004</td>
<td>.00176</td>
<td>$\pm .00003$</td>
</tr>
</tbody>
</table>

The accumulated global error is much more difficult to assess than the local error per step. One must always allow for the possibility that regularities in the calculation cause errors to line up preferentially on one side of the true solution. Barring this possibility and assuming a random walk distribution of errors ($E \propto \sqrt{N}$) gives the error estimates listed in the table.

For the value of $\eta$ used in these calculations, the Kernan correction from Equation (2.1) is $\Delta Y = .00170$. From the most precise computation from Table 1 the
difference between the two corrections is less than .0001. The general purpose Ker­
nan correction of \( \Delta Y = .0017 \) probably should be replaced by \( \Delta Y = .0018 \). However, since third decimal place accuracy is sufficient given the observational uncertainties, this difference is not very significant.

It is fairly easy to isolate the source of the error estimated in Equation 2.1 and Table 1. As discussed in Chapter I the neutron-to-proton ratio at freeze-out is the critical quantity determining the helium abundance. Look at the expanding universe during the epoch in which the temperature was falling from \( T \sim 10 \text{ MeV} \) to \( T \sim .1 \text{ MeV} \). Baryons are in the form of protons or neutrons. Only a very tiny amount of deuterium has formed yet and even less of anything heavier. The neutron fraction will be denoted \( X_n \) and proton fraction \( X_p \). An excellent approximation for any time \( t \) during this epoch will be

\[
X_n(t) + X_p(t) = 1 \tag{2.27}
\]

We are interested in \(^4\text{He} \) ultimately. The helium abundance can be estimated to a first approximation by twice the neutron fraction when nuclei build up in earnest just after the epoch considered here. The differential equations describing \( X_n(t) \) and \( X_p(t) \) are

\[
\dot{X}_n = \lambda_{pn} X_p - \lambda_{np} X_n
\]

\[
\dot{X}_p = -\dot{X}_n. \tag{2.28}
\]

A formal solution is

\[
X_n(t) = \int_0^t dt' \lambda_{pn}(t') f(t, t')
\]
\[ f(t, t') = \exp \left[-\int_{t'}^{t} d\lambda(u) \lambda(u) \right] \]
\[ \lambda = \lambda_{pn} + \lambda_{np}. \]  

(2.29)

The toy model calculation of Bernstein, Brown, and Feinberg [6] (hereafter BBF) is useful for this purpose. With some simplifying approximations BBF find that the rates in Equation 2.29 can be written as

\[ \lambda_{np} = \frac{255}{(2y)^{5}} \left(12 + 6y + y^{2}\right) \]
\[ \lambda_{pn} = e^{-y} \lambda_{np}. \]  

(2.30)

Here \( y = Q/T \) with \( Q = 1.293 \) MeV. The neutron lifetime is taken to be \( \tau = 896 \) s. With \( T \propto 1/R \) and these rate functions applied to Equation (2.29) the resulting expression is

\[ X_{n}(\infty) = \int_{0}^{\infty} du \frac{e^{u}}{(1 + e^{u})^{2}e^{-K(u)}} \]
\[ K(u) = (0.251) \left[ \left(\frac{4}{y^{5}} + \frac{3}{y^{2}} + \frac{1}{y}\right) + \left(\frac{4}{y^{5}} + \frac{1}{y^{2}}\right)e^{-y}\right]. \]  

(2.31)

The result of integrating Equation (2.31) with a Romberg integration routine [31, §4.3] are shown in Table 2. The Wagoner code and the Bulirsch-Stoer code were modified to integrate just neutrons and protons with the BBF rates (2.30) and \( T \propto 1/R \). The difference from Table 2 shows the amount of error in the Wagoner code (with default step-size limitation constants) at neutron-proton ratio freeze-out. Indeed, the Bulirsch-Stoer integration gives the 'right' result given by the Romberg integration of Equation (2.31). (The BBF value from the same equation is puzzling. This could easily have resulted from an integration to a finite upper limit. The convergence is rather slow. The Romberg integration is designed for an infinite upper limit.)
Table 2: The neutron fraction for the BBF simplified model

<table>
<thead>
<tr>
<th>Equation(s)</th>
<th>Integration Procedure</th>
<th>$X_n(y = \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2.28)</td>
<td>modified Wagoner code</td>
<td>0.147</td>
</tr>
<tr>
<td>(2.28)</td>
<td>Bulirsch-Stoer</td>
<td>0.149</td>
</tr>
<tr>
<td>(2.29)</td>
<td>Romberg</td>
<td>0.149</td>
</tr>
<tr>
<td>(2.29)</td>
<td>BBF</td>
<td>0.151</td>
</tr>
</tbody>
</table>

The difference of $0.149 - 0.147 = 0.002$ translates into a helium deficit of $\Delta Y = 0.4$, considerably greater than the $\Delta Y \simeq 0.2$ from the complete calculation. Indeed, when the Wagoner and Bulirsch-Stoer codes were modified to evolve neutrons, protons, and deuterium, the result coming from taking $Y$ equal to the deuterium mass fraction was $\Delta Y \simeq 0.2$. The error in the deuterium evolution partially offsets the error in neutron fraction at freeze-out to give the Wagoner code a more accurate result.

The tiny difference between the Bulirsch-Stoer integration result of Table 1 and the Kernan correction (2.1) can only be due to the neglect of the Jacobian derivatives involving the temperature. Even though the Wagoner integration algorithm is, in essence, a first order method (as argued in the last section), convergence to the true value for any order method should always be achieved in the limit of zero step-size as long as the derivatives are calculated with sufficient accuracy. Neglecting the temperature Jacobians represents an approximation, however. This work shows that that approximation happens to be a very good one for the primordial nucleosynthesis equations.

In principle, the same difference of $\sim 0.001$ should be found by making the same
approximation using the Bulirsch-Stoer algorithm. Doing so adversely affected stability, however. In fact, the integration could not be completed for the error criteria of Table 1. It was even more interesting to find that stability was enhanced for less stringent error criteria of $\epsilon > 10^{-4}$. The Bulirsch-Stoer integration with the full Jacobian and large $\epsilon$, on the other hand, was very unstable. Although a thorough numerical analysis of this phenomenon was not performed, this behavior apparently illustrates a general feature of stiff systems: there is a trade-off between accuracy and stability. The puzzles of the Wagoner code may be explicable in these terms; that is, the code is apparently crafted to be very stable with moderate accuracy. The instability that Kernan observed [26, p. 181] may also be understandable in this light.

No attempt was made to craft a Bulirsch-Stoer based code competitive with the Wagoner code in the combination of speed, accuracy, and stability. Many improvements could be made to do so, however. Other integration algorithms could be used as well. (In fact, in this work the Adams-Gear routine DIVPAG of the IMSL libraries was used for an alternative comparison. The results were inconclusive, however, since warning messages during execution indicated that the error criterion was not being met.) Motivation for replacing the Wagoner code or its direct descendents is lacking, though. The Wagoner code has been a useful workhorse for nearly thirty years, and will probably continue in use for the foreseeable future.
CHAPTER III

Baryon Inhomogeneity

3.1 A Lumpy Early Universe

The dual assumptions of homogeneity and isotropy might seem to be a very special case. However, it is easy to argue that the early universe must have been extremely smooth as far as the total energy density is concerned. The very small fluctuations in the temperature of the microwave background radiation are an indication of near perfect uniformity in early epochs. Nevertheless, baryons constitute a very small component in terms of number density in our present universe and provided a small component of energy density around the epoch of nucleosynthesis. Sizeable fluctuations in baryons at that time are conceivable if compensated by slight and opposite fluctuations in radiation energy density.

We know that the abundances are dependent on \( \eta \), the baryon-to-photon ratio, and that the abundances are at least nearly consistent with observations for a very narrow range of \( \eta \). Could it be that combinations of different eta give consistent results as well? Furthermore, can baryon inhomogeneity solve one of the dark matter puzzles outlined in Chapter I? In fact, one of the most widely-investigated variations
in primordial nucleosynthesis is the possibility that the baryon density is inhomogeneous at the time of nucleosynthesis. In the simplest such inhomogeneous models, nucleosynthesis is assumed to take place independently in regions of different density, with the final element abundances derived by simply mixing the matter from these different regions [14, 5, 43, 8]. Such a treatment is valid if the dominant fluctuations are of sufficiently low amplitude and on a sufficiently large length scale that neither gravitational collapse nor particle diffusion is important.

If the length-scale of the fluctuations is sufficiently short, then particle diffusion before and during nucleosynthesis can significantly alter the element yields [4, 1, 38, 22]. On the other hand, if the fluctuation amplitudes are high enough, the highest density regions can undergo gravitational collapse. The elements in those regions are effectively removed from the final measured abundances. This scenario has also been extensively explored [34, 15, 8, 21].

This work considers only the simplest of these scenarios (no diffusion or collapse), but provides a significant generalization to previous work. All previous studies have assumed a particular form for the distribution of baryon-to-photon ratio $f(\eta)$. Here a method for treating the case of arbitrary $f(\eta)$ is developed. Obviously, a brute-force calculation of this case, examining all possible functions, would be impossible. Even the analysis of a limited subset of the infinite number of possibilities is an enormous computational task. However, by discretizing the problem, it can be turned into a problem in linear programming with a well-defined set of easily-obtained solutions.
3.2 Baryon Fluctuations

The most recent and complete treatment of inhomogeneous nucleosynthesis when post-nucleosynthesis mixing is the only important effect has been given by [8, hereafter COS], so it is helpful to follow their treatment. The elements produced in detectable quantities in primordial nucleosynthesis are $^4\text{He}$, D, $^3\text{He}$, and $^7\text{Li}$. The observational limits on the abundances of these elements have been discussed in great detail in a number of recent papers [42, 11, 9, 17, 16]. However, in order to compare results with the calculations of COS the abundance limits given there are used here as well:

$$0.221 \leq Y_p \leq 0.243,$$

$$(\text{D}/\text{H}) \geq 1.8 \times 10^{-5},$$

$$(\text{D} + ^3\text{He})/\text{H} \leq 1.0 \times 10^{-4}.$$

(3.1)

COS used two different bounds on the $^7\text{Li}$ abundance:

$$(^7\text{Li}/\text{H}) \leq 1.4 \times 10^{-10},$$

$$(^7\text{Li}/\text{H}) \leq 2.0 \times 10^{-10}.$$  

(3.2)

Note that the method of calculation is the main new idea in this work and can be applied to any set of limits.

Now assume that the distribution of densities is given by an unknown function $f(\eta)$. Previous studies have assumed a variety of functions for $f(\eta)$, including the gamma distribution [14, 43] and the lognormal distribution [5]. COS considered both of these as well as a Gaussian distribution for $\eta$. The objective here is to allow for an
arbitrary $f(\eta)$. The mean final value for the baryon-to-photon ratio, $\bar{\eta}$, is given by

$$\bar{\eta} = \int_0^\infty \eta \, f(\eta) \, d\eta$$

(3.3)

The final element abundances are mass-weighted averages of the element abundances produced in the individual regions. If $\langle X_A \rangle$ is the averaged mass fraction of nuclide $A$ measured today, then

$$\langle X_A \rangle = \int_0^\infty \left[ \frac{\eta}{\bar{\eta}} X_A(\eta) \right] f(\eta) \, d\eta,$$

(3.4)

where $X_A(\eta)$ is the mass fraction of nuclide $A$ produced in standard (homogeneous) nucleosynthesis with baryon-to-photon ratio $\eta$. Since $f$ represents a probability distribution, it must be normalized:

$$\int_0^\infty f(\eta) \, d\eta = 1.$$  

(3.5)

Since the distribution function expressing the statistics of the baryon fluctuations is not something well-motivated by ideas from basic physics, any choice of a distribution shape amounts to a parameterization of ignorance to form a model. Discussions of inhomogeneous primordial nucleosynthesis generally leave implicit the parameterization that corresponds with the choice a distribution or family of distribution functions. What is desirable is a more general method for sifting the logical possibilities, rather than simply trying one after another.

Suppose that one wanted to test all possible functions $f$. One way to do this would be to divide the range in $\eta$ into discrete bins, and to approximate the integrals in equations (3.3) and (3.4) as sums:

$$\bar{\eta} = \sum_i \eta_i \, f_i \, \Delta \eta_i$$

(3.6)
and

\[ (X_A) = \sum_i \left[ \frac{\eta_i}{\bar{\eta}} X_{A_i} \right] f_i \Delta \eta_i, \tag{3.7} \]

where the \( \eta \) dependence of \( f \) and \( X_A \) is expressed through their dependence on the bin number \( i \). The bins need not all be of equal size, hence the factor \( \Delta \eta_i \). The normalization constraint (equation 3.5) becomes

\[ \sum_i f_i \Delta \eta_i = 1. \tag{3.8} \]

Now one could imagine doing a Monte Carlo simulation, scanning through all possible distributions \( f_i \) which satisfy the constraint in equation (3.8). In practice, this is impossible. For example, if one divided the range in \( \eta \) into 100 bins, and divided \( f_i \) into 1000 "units" of magnitude 0.001 to be distributed among these bins, one would have to calculate \( C(1099,1000) \sim 10^{144} \) sets of element abundances, an intractable problem. However, most discussions of inhomogeneous nucleosynthesis center on a much simpler problem: given a set of constraints such as those given in equations (3.1)-(3.2), and a function or family of functions \( f \) for the distribution of \( \eta \), what are the largest and smallest allowed values for \( \eta \)? If we express the problem this way, discretizing it as in equations (3.6) and (3.7), then the question of maximizing or minimizing the value of \( \eta \) is reduced to a problem in linear programming. (A discussion of both the theory and practice of linear programming can be found in, for example, Reference [31, §10.8], from which some of the following discussion is taken).

The fundamental problem in linear programming is the following: given a set of
$N$ non-negative independent variables $x_j$, and a set of $M$ constraints of the form:

$$
\sum_{j=1}^{N} a_j x_j \leq b,
$$

or

$$
\sum_{j=1}^{N} a_j x_j \geq b,
$$

or

$$
\sum_{j=1}^{N} a_j x_j = b,
$$

maximize or minimize the function

$$
z = \sum_{j=1}^{N} c_j x_j
$$

The linear programming nature of the discretized problem above becomes clearer the quantities $p_i$ are defined to be

$$
p_i \equiv f_i \Delta \eta_i.
$$

Then equations (3.6)-(3.8) become

$$
\bar{\eta} = \sum_i \eta_i p_i,
$$

$$
\langle X_A \rangle = \sum_i \left[ \frac{\eta_i}{\bar{\eta}} X_{Ai} \right] p_i,
$$

and

$$
\sum_i p_i = 1.
$$

Equations (3.14) - (3.16) are now clearly in the form of a linear programming problem: the $N$ independent variables are the $p_i$'s, equations (3.15) and (3.16) provide the constraint equations, and $\bar{\eta}$ given by equation (3.14) is the quantity to be maximized.
or minimized. Note that equation (3.15) contains \( \bar{\eta} \) in the denominator, so that the bounds \( \langle X_A \rangle < X_{ub} \) and \( \langle X_A \rangle > X_{lb} \) do not immediately translate into linear programming constraints of the form given by equations (3.9)-(3.11). (The subscripts \( ub \) and \( lb \) denote the observational upperbound and lowerbound, respectively.) However, using equations (3.14) and (3.15), these bounds as can be rewritten as

\[
\sum_i \eta_i [X_{Ai} - X_{ub}] p_i \leq 0,
\]
and

\[
\sum_i \eta_i [X_{Ai} - X_{lb}] p_i \geq 0,
\]
which are in the form of equations (3.9) and (3.10).

An additional complication is the fact that the constraints on all of the elements other than \(^4\text{He}\) are expressed in terms of number ratios to hydrogen, \((A/H) = n_A/n_H\), rather than as mass fractions, \(X_A\), while the prescription for mixing the various element abundances uses the mass fractions. However, it is easy to translate the number ratio bounds into a suitable form. Recall that

\[
\left( \frac{A}{H} \right) = \frac{X_A/A}{X_H},
\]
where \(X_H\) is the \(^1\text{H}\) mass fraction. Then an observational upper bound on \((A/H)\) of the form \((A/H) \leq (A/H)_{ub}\) can be written in the form

\[
\frac{\langle X_A \rangle / A}{\langle X_H \rangle} \leq \left( \frac{A}{H} \right)_{ub},
\]
where both \(\langle X_A \rangle\) and \(\langle X_H \rangle\) are given by equation (3.15). Substituting for \(\langle X_A \rangle\) and \(\langle X_H \rangle\) from equation (3.15), we obtain

\[
\sum_i \eta_i \left[ \frac{X_{Ai}}{A} - \left( \frac{A}{H} \right)_{ub} X_{Hi} \right] p_i \leq 0.
\]
Note that the expression for $\bar{\eta}$ has dropped out of the equation, but now the inequality includes a sum over $X_{H_i}$. For the lower bound $(A/H) \geq (A/H)_{lb}$ a similar expression results:

$$\sum_i \eta_i \left[ \frac{X_{A_i}}{A} - \left( \frac{A}{H} \right)_{lb} X_{H_i} \right] p_i \geq 0. \quad (3.22)$$

Equations (3.21) and (3.22) are both in the form of acceptable linear programming constraints.

### 3.3 Baryon Bounds

The first step in determining the bounds on $\eta$ is the calculation of the abundances over a broad range of $\eta$ subdivided into a large number of small intervals. The abundances were generated by running the primordial nucleosynthesis code of Wagoner [41] as updated and reorganized by Kawano [24], with a neutron lifetime of $\tau = 887$ sec. [This differs slightly from the neutron lifetime in COS, and the small correction factor to the $^4$He production used by COS is ignored here, but these are small differences which do not significantly affect the results]. $\eta$ was varied from $10^{-13}$ to $10^{-7}$ and this interval divided into 600 equally-spaced logarithmic bins. The abundances were then calculated for 601 values of $\eta$. Also included was a bin corresponding to zero baryon density and zero element production. Figure 3 shows the abundances plotted over the indicated range in $\eta$.

The next step in determining the baryon bounds is the application of the calculated abundances to the linear programming equations derived in the last section. A variety of methods exist for solving linear programming problems. The simplex method was
Figure 3: Abundances vs. $\eta$. 
used in this work to determine the form for $p_i$ which maximizes or minimizes $\bar{\eta}$ for the abundance constraints given in equations (3.1) and (3.2). This method uses an intuitive geometric analogy to efficiently sort through possibilities in search of an optimal vector of values. (See Reference [31, §10.8] for a description.)

Before the presentation of specific results, note an important general result of linear programming theory: given a set of $N$ variables and $M$ constraints, the solution which maximizes or minimizes $z$ has at least $N - M$ of the variables $x_j$ equal to zero [31]. If there exist $m$ constraints on the element abundances (e.g., equations (3.1) and (3.2) give $m = 5$ constraints), and the normalization condition (equation 3.8) provides one additional constraint, then at most $M = m + 1$ of the $p_i$'s are non-zero; in this case $M = 6$. Taking the continuum limit of equations (3.6)-(3.8) gives an interesting result of this work: for a set of $m$ constraints on the element abundances such as those given in equations (3.1) and (3.2), and an arbitrary distribution of $\eta$ given by the function $f(\eta)$, the largest and smallest possible values for $f_j$ occur when $f(\eta)$ is the sum of at most $m + 1$ delta functions.

Upon the attempt to minimize $\bar{\eta}$, one finds that almost all of $p_i$ is concentrated in the lowest bin (i.e., the bin corresponding to zero baryon density). In fact, there is no "lowest value" for $\bar{\eta}$; the mean baryon-to-photon ratio can be arbitrarily small, a point recently emphasized by Jedamzik, Mathews, and Fuller [23]. For example, one could take $f(\eta) = p_1 \delta(\eta - \eta_0) + p_2 \delta(\eta)$, where $\eta_0$ is a value for $\eta$ which gives acceptable nucleosynthesis for the homogeneous case. Mixing the correct homogeneous $\eta$ with the baryon-free regions gives the correct element abundances regardless of the values
of $p_1$ and $p_2$, but by taking the limit $p_1 \to 0, p_2 \to 1$, the value for $\bar{\eta}$ can be made arbitrarily small.

A more interesting question is the upper bound on $\bar{\eta}$. Using either lithium bound in equation (3.2) results in only two non-zero bins, straddling the upper bound on $\eta$ in the homogeneous model. It was possible to resolve this function further by rerunning the nucleosynthesis code using 1000 bins between $\eta = 10^{-10}$ and $\eta = 10^{-9}$. Using the first lithium bound in equation (3.2) gives:

$$f(\eta) = (0.21)\delta(\eta - 3.31 \times 10^{-10}) + (0.79)\delta(\eta - 3.33 \times 10^{-10}), \quad (3.23)$$

which gives $\bar{\eta} = 3.33 \times 10^{-10}$. In this case, it is the limit on $^7\text{Li}$ which is saturated. The second lithium bound in equation (3.2) yields the solution:

$$f(\eta) = (0.58)\delta(\eta - 3.40 \times 10^{-10}) + (0.42)\delta(\eta - 3.44 \times 10^{-10}), \quad (3.24)$$

with $\bar{\eta} = 3.42 \times 10^{-10}$. For this case, the $^4\text{He}$ limit is saturated. In fact, given the precision used to calculate the various element abundances, the difference between these bounds and the homogeneous upper bound is not significant. It must be concluded that the homogeneous upper bound on $\eta$ cannot be exceeded for any distribution $f(\eta)$. These results are consistent with the claim that $f(\eta)$ should be at most a sum of $m + 1$ delta functions; in this case $m + 1 = 6$, while the optimum solution is the sum of only two delta functions.

To summarize, in this simplest model for inhomogeneous nucleosynthesis, there is no lower bound on $\eta$, as expected [23]. Previous studies which assumed particular functional forms for $f(\eta)$ all produced a fairly narrow range in the allowed values for
\(\eta\) [14, 5, 43, COS]. However, this occurred because all of the functions \(f(\eta)\) in these papers were unimodal, i.e., characterized by a single maximum. Hence, they cannot approximate the sort of solution which has two large peaks at \(\eta = 0\) and \(\eta = \eta_0\) (where \(\eta_0\) gives acceptable abundances in the homogeneous model). The upper bound on \(\eta\) is, for all practical purposes, the homogeneous upper bound on \(\eta\). This is consistent with the results of COS; for all three functions they examined, it is clear from their figures that the upper bound on \(\eta\) is no larger than that obtained when the variance of \(f(\eta)\) goes to zero. The importance of the results given here is that they give the most general upper and lower bounds on \(\eta\) for any density distribution, essentially bringing to a close the two-decade-long investigations of these simplest inhomogeneous models.

Since inhomogeneities allow for the reduction of \(\eta\) by an arbitrary amount, is it tempting to think that a reduction in the \(^4\)He abundance might also be possible. This can be tested by using the D, \(^3\)He, and \(^7\)Li limits given above, while minimizing the value of \(^4\)He. A linear programming method cannot be applied directly to minimize \(^4\)He, because the expression for \(^4\)He contains a factor proportional to \(\eta\). However, \(\eta\) can be assigned a fixed value to provide another constraint equation. Then \(\eta\) becomes a variable parameter just as in the homogeneous case. However, using this method, no significant reduction in \(^4\)He results. Again, the results apply to arbitrary distributions of \(\eta\).

Linear programming is not a technique usually applied to astrophysical problems, although it has been previously used in galactic dynamics [37]. This technique of
discretizing the problem and using linear programming could be applied to any problem with constraints on integrals of an unknown function. The technique cannot be applied to inhomogeneous nucleosynthesis when particle diffusion is significant. However, it could be applied to the case when collapse of high density regions is important. To address this case would involve many particular model assumptions, but, for example, equation (21) of COS can easily be put into the form of a linear programming constraint.
CHAPTER IV
Decaying Particles

4.1 Unstable Exotica in the Early Universe

As discussed in Chapter I, the addition of a new particle species tends to cause trouble for primordial nucleosynthesis. The number and types of particles contributing to the energy density is crucial to the helium abundance in particular. The particle adds to the energy density, thereby increasing the expansion rate at a given temperature. It is the competition between the expansion rate and neutron-proton weak interaction rates that determines the neutron-to-proton ratio at freeze-out. A greater expansion rate will result in earlier freeze-out to preserve a greater proportion of neutrons, and, therefore, more helium as a result.

For a decaying particle the effect on nucleosynthesis is more complicated than for a stable particle, as there are various subeffects. These effects have been investigated by many authors (see, for example, [25, 12] and references therein). Extensions to the standard model of elementary particle physics provide an assortment of candidates for such a particle. The gravitino is an especially interesting case to consider [35].
The interesting effect that this work pursues, first noted by Scherrer and Turner [36], can occur if the particle decays into products which interact electromagnetically and go into thermal equilibrium with the cosmic blackbody background. If such decays occur well after the neutrinos have dropped out of thermal equilibrium (at a temperature of a few MeV) then the neutrinos do not share in the heating of the blackbody background. For lifetimes much shorter than the neutrino decoupling temperature, the neutrinos share fully in the heating of the blackbody background. In the intermediate regime (lifetimes on the order of 0.1 sec), an interesting effect can occur. Because the electron neutrino couples more strongly to the $e^-e^+$ pairs than do the $\mu$ and $\tau$ neutrinos, it decouples at a slightly lower temperature. Thus, for lifetimes in this intermediate range, it is possible for the decaying particle to heat the electron neutrinos more than the $\mu$ and $\tau$ neutrinos. This results in a net decrease in the expansion rate relative to the weak ($n \leftrightarrow p$) rates, giving a reduction in the $^4$He abundance [36].

This possibility of reducing the primordial $^4$He production is quite interesting, because, as mentioned in Chapter I, recent calculations suggest that the standard model of big bang nucleosynthesis can be made consistent with current observations only if the "true" primordial $^4$He abundance is larger than currently believed, or if primordial nucleosynthesis is modified to give a smaller production of $^4$He [11, 17, 9]. However, the treatment of this effect in references [36, 35] was quite crude. The neutrinos were assumed to decouple suddenly when their interaction rates dropped below the expansion rate. Entropy dumped into the system before decoupling was
shared equally by the neutrinos, photons, and $e^-e^+$ pairs, while entropy released after
decoupling went entirely into the photons and $e^-e^+$ pairs.

This work incorporates a more careful investigation, involving a more accurate
treatment of the transfer of energy from the decaying particle to the neutrinos. The
linearized energy-transfer equation given by Rana and Mitra [33], based on earlier
work by Herrera and Hacyan [18, 19] is used. The calculation does not include the
full distortion in the spectrum, but instead treats each species of neutrino as a black
body with a single temperature. Although this is not as sophisticated as the treatment
given by Dodelson, Gyuk, and Turner [12] it represents a significant improvement over
other previous studies of this problem. (Dodelson, Gyuk, and Turner examined only
decaying neutrinos, which do not have a large enough energy density to make this an
interesting effect).

4.2 Entropy Production

This work follows the work of Reference [36] in analyzing the consequences of particles
producing electromagnetic entropy in the course of their decay. This section presents
some basic relations from that study and a focus on the parameter space of interest in
this study. The calculation begins by adding to the standard cosmological model an
X particle which decays exclusively into particles that thermalize rapidly compared
with the expansion timescale. This particle adds a contribution to the total energy
density driving the expansion according to the Friedmann equation

$$H \equiv \frac{\dot{R}}{R} = \left(\frac{8}{3\pi G} \rho \right)^{1/2}. \quad (4.1)$$
where $R$ is the expansion scale factor. Here the total density $\rho = \rho_{std} + \rho_X$ with $\rho_{std}$ representing the energy density in the standard model. The $X$ density $\rho_X$ evolves according to the equation

$$\dot{\rho}_X = -\Gamma \rho_X - 3H \rho_X,$$  \hspace{1cm} (4.2)

where $\Gamma$ is the decay rate, $\Gamma \equiv 1/\tau$, and $\tau$ is the $X$ particle lifetime. This equation has the solution

$$\rho_X = \rho_{X0} \left( \frac{R}{R_0} \right)^{-3} e^{-t/\tau}. \hspace{1cm} (4.3)$$

The interesting range of particle lifetimes turns out to be $10^{-2} \text{sec} < \tau < 2 \text{ sec}$. Following reference [36], the particle number density can be quantified by the parameter $r$, which gives the number density of the $X$ particles relative to photons before $e^- e^+$ annihilation. The only quantity which determines the energy density of the $X$ particles as a function of temperature is the energy density parameter $r m_x$, where $m_x$ is the particle mass. In all of the calculations of this work the parameter is taken to be $r m_x = 10^4 \text{ MeV}$.

For this choice of $r m_x$ and particle lifetimes above, the density of the $X$ particles dominates the expansion prior to decay. More importantly, the entropy produced by the decay dominates the previously-existing radiation during nucleosynthesis. When these two conditions are satisfied, the evolution of the element abundances is independent of $r m_x$, and the final results will be essentially a function only of $\tau$ [36]. For the range of lifetimes considered here, these conditions will be satisfied for $r m_x \gtrsim 10^3 \text{ MeV}$, so this is the range over which these results will be applicable (See ref. [36]). For comparison, a massive tau neutrino has a maximum $r m_x$ of about 1 MeV (for
\( m_\nu \sim 3 - 5 \text{ MeV} \), so a very different region of parameter space is being examined from that discussed in reference [12].

The decay of the X particle heats the electromagnetic plasma. Let \( \rho_\gamma \) and \( p_\gamma \) be the energy density and pressure of the photons and all relativistic particles in thermal equilibrium with the photons. Then the evolution of \( \rho_\gamma \) is given by

\[
\dot{\rho}_\gamma = \Gamma \rho_X - 3H(\rho_\gamma + p_\gamma),
\]

(4.4)

from which the photon temperature evolves according to

\[
\dot{T} = \dot{\rho}_\gamma \sqrt{\frac{d\rho_\gamma}{dT}},
\]

(4.5)

where \( \rho_X = (r m_X)n_\gamma \), with \( n_\gamma \) being the number density of photons. The ratio of the final to initial entropy per comoving volume is

\[
\frac{S_f}{S_i} = 0.36 g_*^{-3/4} \left( \frac{r m_X}{\text{MeV}} \right) \left( \frac{\tau}{\text{sec}} \right)^{1/2},
\]

(4.6)

where \( g_* \) is an average of the effective number of relativistic degrees of freedom in thermal equilibrium during the X decay. This formula presumes that decays occur prior to electron-positron annihilation. The entropy serves to inflate the photon number density. Therefore, the following relation will hold:

\[
\epsilon \equiv \frac{\eta}{\eta'} = \frac{S_f}{S_i}.
\]

(4.7)

Here \( \eta' \) is the (final) baryon-to-photon ratio diluted by X decay, and \( \eta \) is the value it would have had in the standard model. As an example, for an early decay of \( \tau = 10^{-2} \text{ s}, g_* = 10.75 \). With the X particle density parameter of \( r m_X = 10^4 \text{ MeV} \)
one obtains $e = 61$. The initial value of $\eta$ must be adjusted by this factor in the decay calculation in order to achieve a consistent comparison with the standard calculation at the same final value of $\eta$.

### 4.3 Differential Neutrino Heating

For early decays the neutrinos are strongly coupled to the electromagnetic plasma, so that they are included in Equation (4.4), and $T_\nu = T_\gamma$. For late decays the neutrinos are fully decoupled and evolve according to

$$\dot{\rho}_\nu = -4H\rho_\nu, \quad (4.8)$$

so that $T_\nu \propto 1/R$. For the intermediate case of decays occurring while the neutrinos are freezing out, the situation becomes more interesting while also becoming much more complicated. The difficulty involves the determination of the neutrino phase space distributions $f_i(i = e, \mu, \tau)$. These are needed for calculating the neutrino energy densities. In general, the energy density for a particle species with $g$ spin degrees of freedom is

$$\rho = \frac{g}{(2\pi)^3} \int E f(E) d^3p. \quad (4.9)$$

The distribution function for the electron neutrino is also necessary for calculating the neutron-proton weak interaction rates which are critical for the $^4$He calculation.

As an example, for the reaction

$$\nu_e + n \rightarrow p + e \quad (4.10)$$
the rate is

$$
\Gamma_{\nu n-pe} = \int \left( \prod_{\sigma=n,p,e} dQ_\sigma \right) [f_\nu(1 - f_\nu)] |A_{\nu n-pe}^2| (2\pi)^4 \delta^4(p_\nu + p_n - p_p - p_e) \quad (4.11)
$$

where

$$
dQ_\sigma = \frac{1}{(2\pi)^3} \frac{d^3p_\sigma}{2E_\sigma} \quad (4.12)
$$

is the Lorentz invariant phase space element. The rate is an integral over phase space of the transition probability $|A|^2$ while enforcing energy-momentum conservation. Note that Fermi statistics requires the Pauli blocking factor $1 - f_e$ for electrons in the final state.

While close to thermal equilibrium with photons and electrons, the neutrinos are accurately characterized by a Fermi-Dirac energy spectrum

$$
f(E, t) = \left[ \exp \left( \frac{E}{T(t)} \right) + 1 \right]^{-1} \quad (4.13)
$$

where the neutrino temperature $T_\nu$ is very nearly equal to the photon-electron temperature $T = T_\gamma$. This spectrum applies equally well to antineutrinos, since the neutrino chemical potentials have implicitly been assumed to vanish. Electrons and positrons are tightly coupled to photons throughout the epoch of interest here, and so also are characterized by the distribution (4.13) with $E^2 = p^2 + m^2$. Electron-positron chemical potentials can be neglected. The energy density for electrons(positrons) is given by (4.9), where the spin degrees of freedom for electrons is $g = 2$. The standard neutrinos being considered here are massless, $m = 0$, and have only left-handed states, $g = 1$. 
As freeze-out proceeds the neutrino spectrum becomes distorted relative to the equilibrium shape. The neutrino interactions are strongly energy dependent, so the high energy tail of the spectrum maintains thermal contact longer than the low energy end. With complete generality, the neutrino spectrum can be characterized by a non-equilibrium spectrum

\[ f_\nu(E, t) = \left[ \exp\left( \frac{E}{T_\nu} \right) + 1 \right]^{-1} \]  \hspace{1cm} (4.14)

The effective neutrino temperature \( T_\nu \), a function of time \( t \) and energy \( E \), may be parameterized as

\[ \bar{T}_\nu(E, t) = T_\nu(t)[1 + \delta_\nu(E)] \]  \hspace{1cm} (4.15)

Figure 4 schematically pictures this relation for a spectrum tilt proportional to energy.

The simple model illustrated in Figure 4 demonstrates how neutrino freeze-out can be thought of in terms of two distinct features: a divergence of the entire spectrum on average and a tilt with respect to that average. Note that the effective photon temperature is a function of time and is constant with respect to energy.

To accurately calculate the neutrino energy densities and neutron-proton weak interaction rates one must evolve the neutrino distribution functions \( f_i(i = e, \mu, \tau) \) according to the Boltzmann equation. For the FRW cosmology this is

\[ \dot{L}[f_i(E, t)] = C(E, t) \]

\[ \dot{L} = E \frac{\partial}{\partial t} - HE^2 \frac{\partial}{\partial E} \]  \hspace{1cm} (4.16)

where \( \dot{L} \) is the Liouville operator. Equation 4.16 is the dynamical equation of motion generalizing Newton's second law, \( \dot{p} = F \). The collision term \( C \) is evaluated using
basic ideas from quantum field theory. It expresses the effect of interactions of the particle of interest with all other particles. The reactions to be considered here are all of the form

\[ i + j \leftrightarrow k + l. \tag{4.17} \]

For such two-body reactions the collision term is

\[
C_{ij\rightarrow kl}(E,t) = \frac{1}{2} \int \left( \prod_{\sigma=j,k,l} dQ_\sigma \right) (\Delta F)|A_{ij\rightarrow kl}|(2\pi)^4 \delta^4(p_i + p_j - p_k - p_l). \tag{4.18}
\]

In Equation (4.18) the distribution difference factor $\Delta F$ is defined to be

\[
\Delta F \equiv F_{klij} - F_{ijkl}, \tag{4.19}
\]

where

\[
F_{ijkl} = f_if_j(1-f_k)(1-f_l). \tag{4.20}
\]
The presence of the unknown function $f_{i}$ in $\Delta F$ makes Equation (4.16) an integro-differential equation. In general, the distributions for particles $j, k, l$ are also described by Boltzmann equations of the same form (4.16) to make a coupled set of integro-differential equations.

Equation (4.16) is a complex one to solve. An iterative procedure is necessary in order to transform the problem into a tractable one. What is desirable is a suitable approximation to the full general solution. Exact solutions, besides being difficult to obtain, can actually obscure the salient physical principles. A good approximate solution can aid in the discovery of a better one later.

To this end, in references [36, 35], the authors used a sudden decoupling approximation to deal with neutrino heating. In this approximation the neutrinos are assumed to have blackbody spectra with temperatures $T_{\nu i}, i = e, \mu, \tau$. In Equation (4.15) this amounts to taking $\delta_{\nu} \equiv 0$, so that the effective temperature is the energy independent dashed line in Figure 4. What remains is to find the neutrino temperature as a function of time [$T_{\nu} \equiv T_{\nu}(t)$].

The temperature of an individual neutrino species is set equal to the photon temperature until it decouples and thereafter the neutrino temperature decreases as $1/R$. The decoupling condition used in references [36, 35] is $\Gamma_{\nu i}/H < 1$ where $\Gamma_{\nu i} = A_{\nu i} T^{5} (i = e, \mu, \tau)$. For consistency with derivations to follow, $\Gamma_{\nu i}$ is expressed in the form

$$\Gamma_{\nu i} = \alpha_{\nu i} G_{F}^{2} T^{5}, \quad (4.21)$$

where $G_{F}$ is the Fermi coupling constant, and $\alpha_{\nu i}$ is a dimensionless constant of order
unity. The values for $A_{\nu_i}$ used in references [36, 35] correspond to $\alpha_{\nu_e} = 0.3$ and $\alpha_{\nu_{\mu,\tau}} = 0.07$. This is obviously a very crude approximation and is considered here only for comparison with the new calculations.

For an improved approximation, this work follows the treatment of Rana and Mitra [33] based on the earlier work of Herrera and Hacyan [18, 19]. They also treated the neutrino-antineutrino pairs as a perfect blackbody, so that the effective temperature is again the flat (dashed line) spectrum in Figure 4. The improvement in this work is to evolve the neutrino temperatures in a continuous way by using their calculation of the rate of energy transfer from the electron-positron pairs to the neutrinos through scattering and annihilation. A description of the energy transfer calculation follows a brief summary of the method.

Let $u_{\nu_i}$ be the total rate of energy density transfer from the $e^-e^+$ pairs to a single neutrino species $\nu_i$ ($i = e, \mu, \tau$) via annihilations and scatterings. Then equation (4.4) is modified to

$$\dot{\rho}_{\gamma} = \Gamma \rho_X - \sum_{i=e,\mu,\tau} u_{\nu_i} - 3H(\rho + p)_{\gamma}$$

while the neutrino density evolves as

$$\dot{\rho}_{\nu_i} = u_{\nu_i} - 4H\rho_{\nu_i},$$

In the limit of small temperature difference between the electrons and neutrinos, $u_{\nu_i}$ is given by [33]

$$u_{\nu_i} = 2I_{e^-\nu_i}(T_\gamma - T_{\nu_i})$$

where $I_{e^-\nu_i}$ is a function of $T_\gamma$ which gives the rate of energy density transfer per unit temperature difference from electrons into a particular neutrino species $\nu_i$, and
the factor of 2 comes from including both electrons and positrons. In the limit of large temperatures \( T \gg m_e \), \( I \) has the form

\[
I_{e-\nu_i} = \beta_{e-\nu_i} G_F^2 T_\gamma^8,
\]

(4.25)

where \( \beta_{e-\nu_i} \) is a dimensionless constant of order unity. [Equation (4.25) comes from the fact that \( I_{e-\nu_i} \) is approximately equal to the interaction rate \( \Gamma = n(\sigma v) \sim G_F^2 T_\gamma^8 \) times the heat capacity \( \sim T^3 \).] Three distinct processes contribute to the net value of \( I_{e-\nu_i} \):

\[
\begin{align*}
e^- + e^+ & \leftrightarrow \nu_i + \bar{\nu}_i \\
e^- + \nu_i & \rightarrow e^- + \nu_i \\
e^- + \bar{\nu}_i & \rightarrow e^- + \bar{\nu}_i.
\end{align*}
\]

(4.26)

The first represents electron-positron annihilation into neutrino-antineutrino pairs along with the inverse annihilation. The last two represent electron scattering off of neutrinos and antineutrinos, respectively. The values for \( \beta_{e-\nu_i} \) can be derived from the high-temperature limit of the results of Rana and Mitra [33] and are listed in Table 3, which includes both the total \( \beta_{e-\nu_i} \) used in equation (4.25), as well as the contribution to \( \beta_{e-\nu_i} \) from the individual reactions. It is obvious from the numbers in Table 3 that \( e^- e^+ \) annihilation is the dominant energy transfer mechanism.

To see where the coefficients in Table 3 come from, start by integrating the Boltzmann equation (4.16) over the neutrino momentum distribution of interest. The result is

\[
\dot{\rho} + 4H \rho = u(t)
\]

(4.27)
Table 3: The coefficients for the energy transfer rates in equations (4.25) and (4.46), where $\nu_i$ denotes a $\mu$ or $\tau$ neutrino. Included is the breakdown of $\beta_{\nu_e \rightarrow \nu_i}$ into specific reactions as well as the total $\beta_{\nu_e \rightarrow \nu_i}$ used in equations (4.25) and (4.46).

<table>
<thead>
<tr>
<th>Process</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^- e^+ \leftrightarrow \nu_e \bar{\nu}_e$</td>
<td>0.99</td>
</tr>
<tr>
<td>$e^- \nu_e \rightarrow e^- \nu_e$</td>
<td>0.14</td>
</tr>
<tr>
<td>$e^- \bar{\nu}_e \rightarrow e^- \bar{\nu}_e$</td>
<td>0.04</td>
</tr>
<tr>
<td>$e^- \rightarrow \nu_e$</td>
<td>1.17</td>
</tr>
<tr>
<td>$e^- e^+ \leftrightarrow \nu_i \bar{\nu}_i$</td>
<td>0.21</td>
</tr>
<tr>
<td>$e^- \nu_i \rightarrow e^- \nu_i$</td>
<td>0.02</td>
</tr>
<tr>
<td>$e^- \bar{\nu}_i \rightarrow e^- \bar{\nu}_i$</td>
<td>0.02</td>
</tr>
<tr>
<td>$e^- \rightarrow \nu_i$</td>
<td>0.25</td>
</tr>
<tr>
<td>$\nu_e \bar{\nu}_e \leftrightarrow \nu_i \bar{\nu}_i$</td>
<td>0.42</td>
</tr>
<tr>
<td>$\nu_e \nu_i \rightarrow \nu_e \nu_i$</td>
<td>0.07</td>
</tr>
<tr>
<td>$\nu_e \bar{\nu}_i \rightarrow \nu_e \bar{\nu}_i$</td>
<td>0.01</td>
</tr>
<tr>
<td>$\nu_e \rightarrow \nu_i$</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Here Equation (4.23) has been regained along with a general form for the energy transfer function $u$:

$$ u(t) = \frac{1}{(2\pi)^3} \int d^3 p C(E, t). \quad (4.28) $$

For the general two-body reaction (4.17) this becomes

$$ u_i(t) = \int \left( \prod_{\sigma=i,j,k,l} dQ_\sigma \right) E_i(\Delta F_{ijkl}|A_{ijkl}|(2\pi)^4 \delta^4(p_i + p_j - p_k - p_l). \quad (4.29) $$

The total energy transfer function is obtained by summing over all relevant reactions. Focus on the annihilation process of the reactions (4.26), since it is the dominant energy transfer mechanism. One term of the distribution difference factor in Equation (4.29) is

$$ F_{e^- e^+ \nu_i \bar{\nu}_i} = f_{e^-} f_{e^+} (1 - f_{\nu_i})(1 - f_{\bar{\nu}_i}). \quad (4.30) $$
Another general expression for the neutrino particle distributions is

\[ f_\nu = f_{eq}[1 + \phi(E)] \]  

(4.31)

where \( f_{eq} \) is the equilibrium distribution for a fermion at temperature \( T_\gamma \). Note that if \( \phi \equiv 0 \) then the distribution factor and with it the collision term vanish. In the formalism of Rana & Mitra the distributions of electrons and neutrinos are both taken to be of Fermi-Dirac form, separated in temperature by

\[ \Delta T_i \equiv T_\gamma - T_{\nu i}. \]  

(4.32)

In the limit of a small temperature difference \( \Delta T_i \) one finds that

\[ \phi_{\nu i} \simeq -\left( \frac{\Delta T_i}{T_\gamma^2} \right) E_{\nu i}. \]  

(4.33)

The distribution difference factor in this approximation becomes

\[ \Delta F \simeq F_{e^- e^+ \nu_i \bar{\nu}_i} \left( \frac{\Delta T_i}{T_\gamma^2} \right) (E_{\nu i} + E_{\bar{\nu}_i}). \]  

(4.34)

It is instructive to apply the approximation of Maxwell-Boltzmann statistics, in which

\[ f \simeq e^{-\beta E} \ll 1 \]  

(4.35)

where \( \beta \equiv 1/T \). Using energy conservation, the distribution difference factor is easy to find:

\[ \Delta F \simeq f_e - f_{e^+} - f_\nu f_\bar{\nu} \]

\[ \simeq \exp[-\beta(E_e^- + E_{e^+})] - \exp[-\beta(E_\nu + E_{\bar{\nu}})] \]

\[ \simeq f_e - f_{e^+} \left( (\Delta \beta) E_{\text{total}} \right) \]

\[ \simeq f_e - f_{e^+} \left( \frac{\Delta T_i}{T_\gamma^2} \right) (E_\nu + E_{\bar{\nu}}). \]  

(4.36)
The calculations to follow, however, do not rely on the Maxwell-Boltzmann approximation.

When one applies Equation (4.34) to Equation (4.29) the result is

\[ u(t) = I \Delta T \]  

(4.37)

where

\[ I = \frac{1}{T^2} \int \left( \prod_{\sigma=e^{-}, e^{+}, \nu_i, \bar{\nu}_i} dQ_\sigma \right) E_{\nu_i} F_{e^{-} e^{+} \nu_i \bar{\nu}_i} (E_{\nu_i} + E_{\bar{\nu}_i}) \times |A|_{e^{-} e^{+} \nu_i \bar{\nu}_i}^2 (2\pi)^4 \delta^4(p_e^- + p_e^+ - p_{\nu_i} - p_{\bar{\nu}_i}). \]

(4.38)

The expression for the transition probability is

\[ |A|_{e^{-} e^{+} \nu_i \bar{\nu}_i} = 4G_F^2 s^2 \left( \frac{s}{s - 4m_e^2} \right)^{1/2} \left( C_1 + C_2 \cos \theta + C_3 \cos^2 \theta \right), \]

(4.39)

where

\[ C_1 = (g_V^2 + g_A^2) + (g_V^2 - g_A^2) \left( \frac{4m_e^2}{s} \right), \]

\[ C_2 = 4g_V g_A \left( 1 - \frac{4m_e^2}{s} \right)^{1/2}, \]

\[ C_3 = (g_V^2 + g_A^2) \left( 1 - \frac{4m_e^2}{s} \right). \]

(4.40)

The quantities \( g_V \) and \( g_A \) are, respectively, the vector and axial vector coupling coefficients. The values for the various electron-neutrino discussed here are included in Table 4. The calculations in this work use \( \sin^2 \theta_W = 0.23 \).

The 12 dimensional integral in (4.38) can be reduced to a four dimensional expression. The result from Reference [19] with a somewhat altered notation is

\[ I = K(z)G_F^2 T^8, \]

(4.41)
Table 4: Vector and axial vector coupling coefficients. Here \( a = 2 \sin^2 \theta_W \), where \( \theta_W \) is the Weinberg angle.

<table>
<thead>
<tr>
<th>Process</th>
<th>( g_V )</th>
<th>( g_A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^-e^+ \leftrightarrow \nu_e \bar{\nu}_e )</td>
<td>+0.5 + a</td>
<td>+0.5</td>
</tr>
<tr>
<td>( e^-\nu_e \rightarrow e^-\nu_e )</td>
<td>+0.5 + a</td>
<td>+0.5</td>
</tr>
<tr>
<td>( e^-\bar{\nu}_e \rightarrow e^+\bar{\nu}_e )</td>
<td>-0.5 + a</td>
<td>-0.5</td>
</tr>
<tr>
<td>( e^-e^+ \leftrightarrow \nu_i \bar{\nu}_i )</td>
<td>-0.5 + a</td>
<td>-0.5</td>
</tr>
<tr>
<td>( e^-\nu_i \rightarrow e^-\nu_i )</td>
<td>-0.5 + a</td>
<td>-0.5</td>
</tr>
<tr>
<td>( e^-\bar{\nu}_i \rightarrow e^-\bar{\nu}_i )</td>
<td>-0.5 + a</td>
<td>+0.5</td>
</tr>
</tbody>
</table>

where \( z = m_e/T \),

\[
K(z) = \left( \frac{1}{2\pi} \right) ^5 \frac{1}{2^5} \int_1^\infty dy (y^2 - 1)^{1/2} y^2 \\
\times \int_z^\infty du (u^2 - z^2)^{1/2} u^8 J(u, y; z), \tag{4.42}
\]

and

\[
J(u, y; z) = \int_0^1 dx \int_0^1 dx' \frac{(2C_1 + C_3) + C_3(3x^2x'^2 - x^2 - x'^2)}{[\cosh(\xi_1) + \cosh(\xi_3x)][\cosh(\xi_1) + \cosh(\xi_3x)]} \tag{4.43}
\]

with

\[
\xi_1 = \frac{1}{2} uy \\
\xi_2/\xi_1 = \left[ 1 - \left( \frac{1}{y} \right)^2 \right]^{1/2} \\
\xi_3/\xi_1 = z \left[ 1 - \left( \frac{z}{u} \right)^2 \right]^{1/2}. \tag{4.44}
\]

Similar expressions have been derived for the scattering processes of Equation (4.26).

The coefficient \( K(z) \) is shown in Figure 5 for the electron-neutrino total heating.
function including the annihilation and scattering reactions of Equation (4.26). The function drops rapidly to zero for $z \geq 1$, corresponding to temperatures less than the electron mass ($T \leq m_e \simeq 0.5 \text{ MeV}$). Thus the functions (4.41) follow a $T^8$ power law behavior at high temperatures, and then fall off rapidly at $T \sim m_e$. Neutrino freeze-out is effectively finished by this time, so the high-temperature limit $z \ll 1$ will be a good approximation for this work. The constants $\beta$ in Table 3 are the values $K(0)$.

![Graph](image)

**Figure 5:** The temperature dependent coefficient for the electron-neutrino heating function. Here $z = m_e/T$.

In calculating the relative temperatures of $\nu_e$, $\nu_\mu$, and $\nu_\tau$ one must also consider the
rate of energy transfer from the $\nu_e$ into the other two neutrino types. This neutrino-neutrino direct coupling was neglected in reference [33]. Again, these rates can be modeled in terms of an energy loss term $w_{\nu_i}$, $i = \mu, \tau$, where

$$w_{\nu_i} = 2I_{\nu_e \rightarrow \nu_i}(T_{\nu_e} - T_{\nu_i}) \quad (4.45)$$

and $I_{\nu_e \rightarrow \nu_i}$ is a function of $T_{\nu_e}$ which can be written in the form

$$I_{\nu_e \rightarrow \nu_i} = \beta_{\nu_e \rightarrow \nu_i} G_F^2 T_{\nu_e}^8 \quad (4.46)$$

Again, there are three processes which contribute to the net energy transfer from $\nu_e$ into $\nu_i$:

$$\begin{align*}
\nu_e + \bar{\nu_e} & \rightarrow \nu_i + \bar{\nu_i} \\
\nu_e + \nu_i & \rightarrow \nu_e + \nu_i \\
\nu_e + \bar{\nu_i} & \rightarrow \nu_e + \bar{\nu_i}. \quad (4.47)
\end{align*}$$

The standard matrix elements (see, e.g., reference [13]) have been used to calculate differential cross sections for these three processes. These cross sections can then be substituted into the integral expressions for $I$ corresponding to those given in reference [6] to derive the values for $\beta_{\nu_e \rightarrow \nu_i}$ in equation (4.46). These values are given in Table 1. Again, $\nu_e\bar{\nu_e}$ annihilation is the dominant energy transfer mechanism. With this additional energy loss term, equation (4.23) becomes:

$$\begin{align*}
\dot{\rho}_{\nu_e} &= u_{\nu_e} - \sum_{i=\nu,\tau} w_{\nu_i} - 4H \rho_{\nu_e}, \quad (4.48) \\
\dot{\rho}_{\nu_i} &= u_{\nu_i} + w_{\nu_i} - 4H \rho_{\nu_i} \quad (i = \mu, \tau). \quad (4.49)
\end{align*}$$
In deriving these equations, two major approximations have been made: first, the distortion in the energy density spectrum of the neutrinos has been neglected in treating the spectrum as a blackbody and considering only the change in the total energy density. This is relatively unimportant for the \( \mu \) and \( \tau \) neutrinos, since their only effect on nucleosynthesis is via their contribution to the total energy density, but it could be important for the electron neutrinos, since the weak rates are highly energy dependent. Second, equations (4.24) and (4.45) are strictly valid only in the limit of \( (T_\gamma - T_{\nu_\mu})/T_\gamma \ll 1 \) and \( (T_{\nu_e} - T_{\nu_\mu})/T_{\nu_e} \ll 1 \), respectively; the energy transfer rate is approximated by extrapolating these equations into a regime where the temperature differences are large. For instance, with \( \tau = 1 \) sec, and for expansion times \( t < 1 \) sec (the relevant time range for the freeze-out of the \( n \leftrightarrow p \) reactions), one finds that \( (T_\gamma - T_{\nu_\mu})/T_\gamma < 0.5 \), \( (T_\gamma - T_{\nu_e})/T_\gamma < 0.35 \) and \( (T_{\nu_e} - T_{\nu_\mu,\tau})/T_{\nu_e} < 0.25 \). The temperature differences are smaller for shorter lifetimes. For \( \tau < 0.1 \) sec \( (T_\gamma - T_{\nu_\mu,\tau})/T_\gamma < 0.2 \), \( (T_\gamma - T_{\nu_e})/T_\gamma < 0.1 \) and \( (T_{\nu_e} - T_{\nu_\mu,\tau})/T_{\nu_e} < 0.1 \). The approximation is expected to be quite accurate for \( \tau < 0.1 \) sec, but less accurate for larger lifetimes.

Figure 6 shows the evolution of the neutrino temperatures as a function of time for the case of \( \tau = 0.1 \) sec, including the effect of \( e^-e^+ \) annihilation. The temperatures are displayed relative to the photon temperature in terms of the fraction \( T_{\nu_e}/T_\gamma \). We see that for \( t < \tau \), both \( T_{\nu_e}/T_\gamma \) and \( T_{\nu_\mu,\tau}/T_\gamma \) decrease as the particle decays, but the electron neutrinos are held closer to the photon temperature than the \( \mu \) and \( \tau \) neutrinos, as in reference [3]. The decrease in \( T_{\nu_e}/T_\gamma \) which occurs at \( t > 10 \) sec is the
standard effect from the annihilation of the $e^-e^+$ pairs and has nothing to do with the particle decay.

An interesting “rebound” effect occurs between $10^{-1}$ and 1 sec. The neutrino temperatures actually increase briefly relative the photon temperature before the $e^-e^+$ entropy dump sets in. This is a real effect rather than an artifact of the approximations used. It arises when the rate of energy transfer to the photons (from the decaying particle) is greater than the rate of energy transfer from the photons to the neutrinos. Consider, for example, the (unphysical) limit where the energy of the
decaying particles is transferred instantaneously into the photons. After a sharp increase in the photon temperature, the ratio $T_\nu/T_\gamma$ would increase as photon energy is transferred to the neutrinos.

Figure 7 shows the final temperature ratios, multiplied by $(11/4)^{1/3}$ to factor out the effect of $e^-e^+$ annihilation, as a function of X particle lifetime. Also shown for comparison are the results of the sudden decoupling approximation used in reference [36]. This plot demonstrates significant differences between these two treatments. The sudden decoupling approximation produces a large difference between the electron and other neutrino temperatures in the region $\tau = 0.1$ sec. In the more exact treatment, the differential heating effect is reduced, but it persists over a larger range in lifetime, for $\tau$ as large as 1 sec. This is what one might have expected, since the more exact treatment allows for energy transfer from photons to neutrinos at late times, when the sudden decoupling approximation assumes that the neutrinos are already decoupled.

### 4.4 Helium Reduction

These equations have been incorporated into the primordial nucleosynthesis computer code of Wagoner [41, 39, 40], as modified by Kawano [24]. Figure 8 shows the result of calculating, for a decaying particle with energy density $\nu m_X = 10^4$ MeV and lifetime $\tau$, the change in the primordial helium mass fraction $\Delta Y$ as a function of $\tau$ for three values of the baryon to photon ratio: $\eta = 10^{-10}, 10^{-9.5},$ and $10^{-9}$. One sees that $\Delta Y$ is nearly independent of $\eta$ and can be as large as $-0.012$. Note that $\Delta Y < 0$ for
Figure 7: The final ratios $(\frac{11}{4})^{\frac{1}{3}}\left(\frac{T_{\alpha}}{T_{\gamma}}\right)$ as a function of particle lifetime $\tau$ for a decaying particle with a density parameter $\rho m_{X} = 10^{4}$ MeV (solid curves). For comparison, the corresponding ratios are also shown for the sudden decoupling approximation (dashed curves).
\[ \tau < 1.5 \text{ sec}; \text{ for larger lifetimes the decaying } X \text{ particle, rather than its thermalized decay products, dominates the expansion during the freeze-out of the } n \leftrightarrow p \text{ reactions, resulting in a larger expansion rate and a net increase in } ^4\text{He}. \]

Figure 8: The change in the primordial $^4\text{He}$ abundance, $\Delta Y$, as a function of decaying particle lifetime $\tau$ for a decaying particle with a density parameter $\tau m_X = 10^4 \text{ MeV}$ and three different baryon-to-photon ratios: $\eta = 10^{-10}$ (dashed curve), $\eta = 10^{-9.5}$ (solid curve) and $\eta = 10^{-9}$ (dotted curve).

Figure 9 shows, for comparison, the $\Delta Y$ value for $\eta = 10^{-9.5}$ using the sudden decoupling approximation from reference [36]. The more exact treatment produces a smaller decrease in the primordial helium, but the effect occurs over a larger range in particle lifetime; $\Delta Y < -0.005$ for $\tau = 0.05 - 1.2 \text{ sec}$. This is not surprising, since the more exact treatment produces a smaller differential heating effect spread
out over a larger range in $\tau$. Part of the difference, however, is that the actual total interaction rates used to compute decoupling in references [36, 35] are different from those used here. So Figure 9 also gives the helium abundance corresponding to sudden decoupling upon changing the total interaction rates in equation (4.21) to give the same freeze-out temperatures as those derived in reference [33] ($\alpha_{\nu_e} = 1.5$, $\alpha_{\nu_e,\tau} = 0.33$). The curve is shifted over to correspond more closely to the more exact treatment, but it retains a much sharper and deeper minimum.

Figure 9: The change in the primordial $^4$He abundance, $\Delta Y$, for $\eta = 10^{-9.5}$, as a function of decaying particle lifetime $\tau$ for a decaying particle with a density parameter $\rho m_X = 10^4$ MeV (solid curve) compared with the results of the sudden decoupling approximation (dashed curve) and the sudden decoupling approximation with the interaction rates updated to correspond to the ones used here (dotted curve).
Of the $\nu_T$ decay modes considered by Dodelson, Gyuk, and Turner [12], this present work most closely resembles $\nu_T$ decay into sterile plus electromagnetic decay products. No reduction in $^4$He is evident for this case in reference [12], but this is to be expected, because the $\nu_T$ energy density is too low to lead to significant differential neutrino heating. Dodelson, Gyuk, and Turner [12] do see a significant reduction in $^4$He for decay modes which include a $\nu_e$ in the final state, but this reduction occurs for entirely different reasons: the additional $\nu_e$ decay products increase the $n \leftrightarrow p$ weak rates, keeping them in thermal equilibrium longer.

The results of this work indicate that the decrease in $^4$He production due to differential neutrino heating from a decaying particle, first discussed in reference [36], holds up under a more exact treatment. The effect is not as large as had been previously estimated, but it occurs over a longer range in particle lifetime. The change in $Y$ can be as large as $-0.01$ for $\tau$ in the range $0.1 - 0.7$ sec, and as large as $-0.005$ for $\tau = 0.05 - 1.2$ sec. A decrease on the order of 0.01 can resolve current problems with standard primordial nucleosynthesis [11, 17, 9], although it is perhaps not the most plausible mechanism for resolving these problems. A more detailed treatment of the spectral distortion using the full machinery of reference [12] seems justified, although the results would probably not be significantly altered in such a treatment. Even the treatment in reference [12] would need to be modified to treat this problem correctly, since that treatment also assumes small temperature differences.
CHAPTER V

Assessment

The three investigations of this work are quite different in detail, but are united in comprising a nice span of the important roles of primordial nucleosynthesis. Chapter II concentrated on its role as a test of the standard cosmology. The standard model will offer a progressively more stringent test as its predictions are sharpened. The study of the numerical integration of the evolution equations with the application of improved techniques is a step in that direction. No significant new source of error was uncovered. Rather, the results confirmed previous calculations and thus have given greater validity to them.

Chapter III focused on the role of indicator of the baryon density. Primordial nucleosynthesis predictions are sensitive to this cosmological parameter, resulting in an effective probe of the composition of the universe. The variation in this case also confirmed previous findings, but again with an improved approach. For the class of inhomogeneous models considered, the determined baryon bounds offer the ‘last word’ in a certain sense because of the generality of the methodology.

Chapter IV narrowed in on a parameter regime not accurately analyzed in previous research efforts. Here the role of primordial nucleosynthesis as a probe of elementary
particle physics became manifest. The particular model of entropy producing decays with differential neutrino heating was shown to offer a potential resolution to the helium problem outlined in Chapter I with the improved treatment. It would have to be recognized that the scenario relies on a very special set of properties. This gives a sense of implausibility despite the success of the model.

Taken together, these variations are suggestive of the maturity of primordial nucleosynthesis as an aspect of physical science. Each of the three continues along paths already well traveled in previous research studies. Truly new, unexplored ideas are hard to find in a mature discipline. Variant models and variant techniques will continue to become more refined and specialized in the future.

Nevertheless, there is plenty of excitement in the field. Ironically, this is because the standard model is beginning to look quite bad. As agreement between theory and observation wanes, variant models will become increasingly interesting. Perhaps observational issues are at the heart of the discrepancy. What is certain is that the 'tension' in the data is a very healthy thing. The big bang cosmology is very much alive as it points toward new understanding of nature.
APPENDIX A

Bulirsch-Stoer Integration

Figure 10 illustrates the basic idea behind the Bulirsch-Stoer method. A large interval $H$ is progressively traversed with finer subdivisions. The resulting final values form a sequence of numbers, which can be thought of as a function of step-size. A polynomial fit to the function is followed by an extrapolation to zero step-size.

Figure 10: The extrapolation procedure for the Bulirsch-Stoer method.
BIBLIOGRAPHY


