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TWO GENERALIZATIONS OF
WITTEN-HELFFER-SJÖSTRAND THEORY

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

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To my parents
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# Table of Contents

DEDICATION .............................................................................................................................. ii

ACKNOWLEDGEMENTS .................................................................................................... iii

VITA ................................................................................................................................................ iv

CHAPTER

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Introduction</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 Review of Witten theory for a Morse function</td>
<td>6</td>
</tr>
<tr>
<td>1.2 Review of Helffer-Sjöstrand theory for a Morse function</td>
<td>10</td>
</tr>
</tbody>
</table>

| II |

| Witten-Helffer-Sjöstrand Theory for $S^1$-equivariant Cohomology | 15 |
| 2.1 Outline of Proof | 15 |
| 2.1.1 $S^1$-equivariant Cohomology and $S^1$-equivariant Hodge Theory | 15 |
| 2.1.2 Equivariant Morse Theory and Witten Deformation | 19 |
| 2.1.3 Local Expression of $\tilde{\Delta}^k(t)$ near Critical Orbits; 'Localized' Operators | 21 |
| 2.1.4 Main Theorem | 26 |
| 2.1.5 Helffer-Sjöstrand Theory for $S^1$-equivariant Cohomology | 30 |
| 2.2 Proof of Theorem 1 | 35 |
| 2.3 Proof of Morse Inequality | 57 |
| 2.4 Helffer-Sjöstrand Theory for $S^1$-equivariant cohomology | 69 |
| 2.4.1 Preliminaries | 70 |
| 2.4.2 Construction of $(C_\ast(M, f), \partial)$ | 71 |
| 2.4.3 Interpretation of $\Omega_{inv, sm}^\ast(M, t)$ as a Complex of Differential Forms on $M_{si}$ | 74 |
| 2.4.4 Helffer-Sjöstrand Theory | 78 |
CHAPTER I

Introduction

The purpose of this thesis is to extend the Witten-Helffer-Sjöstrand (WHS) theory from Morse functions to

(i) $S^1$-invariant Morse functions,

(ii) generalized Morse functions.

The Witten theory for a Morse function describes the asymptotic behavior, as $t \to \infty$, of the Witten deformation of de Rham complex $(\Omega^*(M), d(t))$ where $d(t) = e^{-tf}d e^{tf}$ and $f$ is a Morse function on the compact manifold $M^n$. For each $t$ the complex $(\Omega^*(M), d(t))$ calculates the de Rham cohomology of $M$. Let $g$ be a Riemannian metric on $M$, $\Delta_k(t) = d(t)d(t)^* + d(t)^*d(t)$ be the Laplacian on $k$-forms, $\sigma^k(t)$ be its spectrum. Witten has shown that

$$\sigma^k(t) = \sigma^k_{\text{small}}(t) \cup \sigma^k_{\text{large}}(t) \quad (1.1)$$

i.e. $\sigma^k(t)$ is decomposed into the small eigenvalues (which converge to 0) and the large eigenvalues (which tend to $\infty$ as $t \to \infty$). Also he showed that the number of small eigenvalues equals the number of critical points of index $k$. In fact, the corresponding eigenvectors localize at the critical points of index $k$ as $t \to \infty$. Hence
for \( t \) large enough one obtains a finite dimensional subcomplex \((\Omega^*_{\text{small}}(M, t), d(t))\), spanned by the small eigenvectors (i.e. the eigenvectors corresponding to small eigenvalues), which calculates the de Rham cohomology of \( M \). Furthermore, if \((f, g)\) satisfies the Morse-Smale condition (which is a generic property), then it provides \( M \) with a CW-complex structure whose open cells are the descending manifolds \( W_x^- \) of \( \text{grad } f \) associated to each critical point \( x \) of \( f \). Let \((C^*(M, f), \delta)\) be the cochain complex associated with this CW-complex. Helffer and Sjöstrand have shown that \((\Omega^*_{\text{small}}(M, t), d(t))\) converges to \((C^*(M, f), \delta)\) as \( t \to \infty \) by describing the asymptotics of \((\Omega^*_{\text{small}}(M, t), d(t))\) in terms of \((C^*(M, f), \delta)\).

As the de Rham theory is a fundamental tool for verifying the equality of homotopy invariants of smooth manifolds defined analytically and topologically, WHS theory seems to be a fundamental tool in establishing the equality of more subtle invariants such as the torsion for Riemannian manifolds defined analytically and topologically.

The first generalization is the WHS theory for an \( S^1 \)-invariant Morse function. Such a generalization can be interpreted as the WHS theory on the homotopy quotient \( M_{S^1} = M \times BS^1 / S^1 \) which is an infinite dimensional manifold. In general the WHS theory cannot be extended to the infinite dimensional setting, since it involves aspects of ellipticity which are not true in the infinite dimensions.

Let \( f \) be an \( S^1 \)-invariant function on the \( S^1 \)-manifold \( M^n \), \( g \) be an invariant metric on \( M \), \( X \) be the generating vector field of the \( S^1 \) action, \( \iota_X \) be the contraction of differential forms along \( X \), \( \Omega^*_\text{inv}(M) \) be the invariant forms and let

\[
\tilde{\Omega}^k_{\text{inv}}(M) = \Omega^k_{\text{inv}} \oplus \Omega^{k-2}_{\text{inv}} \oplus \cdots
\]  

(1.2)
Note that in general $\hat{\Omega}^k_{inv}(M) \neq 0$ even if $k > n$. It is well known that $\left(\hat{\Omega}^\ast_{inv}(M), d + i_X\right)$ calculates the $S^1$-equivariant cohomology of $M$, and so does

$$\left(\hat{\Omega}^\ast_{inv}(M), D(t) = e^{-tf}(d + i_X)e^{tf}\right)$$

In this case, it is shown that the spectrum of $\hat{\Delta}^k(t) = D(t)D(t)^* + D(t)^*D(t)$ can be separated:

$$\sigma_{inv,small}(t) \cup \sigma_{inv,finite}(t) \cup \sigma_{inv,large}(t)$$

Note that $\hat{\Delta}^k(t) : \hat{\Omega}^k_{inv}(M) \to \hat{\Omega}^k_{inv}(M)$. Also, $\hat{\Delta}^k(t)$ can be viewed as an elliptic operator on $\hat{\Omega}^k(M) = \Omega^k(M) \oplus \Omega^{k-2}(M) \oplus \cdots$ which leaves $\hat{\Omega}^k_{inv}(M)$ invariant.

The small eigenvalues tend to 0 as $t \to \infty$ while the finite and large eigenvalues tend to some non-zero constants and infinity respectively. Therefore, for $t$ sufficiently large, one obtains the finite dimensional subcomplex

$$\left(\hat{\Omega}^\ast_{inv,sm}(M, t), D(t)\right)$$

spanned by the eigenvectors corresponding to the small eigenvalues of $\hat{\Delta}^k(t)$, which calculates the $S^1$-equivariant cohomology of $M$. Also, by calculating $dim\left(\hat{\Omega}^\ast_{inv,sm}(M, t)\right)$, one verifies the $S^1$-equivariant Morse inequality.

The above can also be interpreted as the Witten theory for $M_{S^1}$. Associated to $f$ is a function $f_{S^1}$ in $M_{S^1}$. To each critical orbit $O$ of $f$ in $M$ is associated a critical manifold $O_{S^1}$ of $f_{S^1}$ in $M_{S^1}$ of the same index. The small eigenvectors of $\hat{\Delta}^k(t)$ can be interpreted as small eigenvectors localized at the critical manifolds $O_{S^1}$ in $M_{S^1}$.

One observes that

$$dim\left(\hat{\Omega}^k_{sm}(M, t)\right) = \sum_O dim H^{k-index} O(O_{S^1}, \theta^-) \quad (1.3)$$
as predicted by the Witten theory.

Also, a geometric complex can be constructed as follows. Let $f$ be a self-indexing $S^1$-invariant Morse function, $g$ an $S^1$-invariant metric so that $(f,g)$ satisfies the Morse-Smale condition, $f_{S^1}$ its associated function on $M_{S^1}$, $X_k = f_{S^1}^{-1}((-\infty, k + \frac{1}{2}])$. Then a boundary operator $\partial : H_*(X_k, X_{k-1}) \to H_{*+1}(X_{k-1}, X_{k-2})$ can be defined as usual. Let $C_k(M, f) = \oplus_{i=0}^n H_k(X_i, X_{i-1})$ and we obtain the complex $(C_*(M, f), \partial)$. Let $(C^*(M, f), \delta)$ be the dual complex of $(C_*(M, f), \partial)$. Integrating the small eigenvectors over cells in $C_k(M, f)$, it is shown that $\left(\Omega_{inv, sm}(M, t), D(t)\right)$ converges to $(C^*(M, f), \delta)$ as $t \to \infty$.

The second generalization is the WHS theory for a generalized Morse function $f$, which may include degenerate critical points of birth-death type. For such a critical point $y$ (of index $k$), one can choose local coordinates $(x_1, \cdots, x_n)$ s.t.

$$f(x) = f(y) = x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n-1}^2 + ax_n^3$$

where $a \neq 0$. The interest in generalized Morse functions comes from the following facts:

(a) a generalized Morse function provides a CW-complex structure for a Riemannian manifold just as a Morse function does.

(b) any two generalized Morse functions can be connected by a smooth family of generalized Morse functions while this is not true in the case of Morse functions.

In this case, it is shown that for $t$ large enough, the spectrum of $\Delta^k(t)$ can be separated as follows:

$$\sigma^k(t) = \sigma_{\text{small}}^k(t) \cup \sigma_{\text{large}}^k(t) \cup \sigma_{\text{very large}}^k(t)$$

(1.5)
While the small eigenvalues converge to 0 as $t \to \infty$, the large and very large eigenvalues tend to $\infty$ as $e^{2/3} \log t$ for some $e > 0$. Moreover,

\[
\begin{align*}
\text{Card} \left( \sigma^k_{\text{small}}(t) \right) &= m_k \\
\text{Card} \left( \sigma^k_{\text{large}}(t) \right) &= m'_{k-1} + m'_k
\end{align*}
\]

where $m_k, m'_k$ are respectively the number of non-degenerate and degenerate critical points of index $k$. So one obtains the finite dimensional subcomplex

\[
\left( \Omega^*_{\text{small}}(M, t) \oplus \Omega^*_{\text{large}}(M, t), d(t) \right)
\]

spanned by the small and large eigenvectors. Each birth-death point of index $k$ gives rise to two large eigenvectors one of which is a $k$-form, the other a $k+1$-form. As in the case of a Morse function, if $(f, g)$ satisfies the Morse-Smale condition, it provides $M$ with a CW-complex structure whose open cells are given by $W^-_x, W^-_y, W^-_z$. Here for any birth-death point $y$ the descending manifold $W^-_y = W^-_y \cup W^-_y$. Here $W^-_y, W^-_y$ are cells of dimension $k$ and $k+1$ respectively, and $k$ is the index of $y$. Let $(C^*(M, f), \delta)$ be the associated cochain complex. Again, it is shown that

\[
\left( \Omega^*_{\text{small}}(M, t) \oplus \Omega^*_{\text{large}}(M, t), d(t) \right)
\]

converges to $(C^*(M, f), \delta)$ as $t \to \infty$.

As the Witten-Helffer-Sjöstrand theory for a Morse function is an essential ingredient in the proof of equality of the analytic torsion and the Reidemeisder torsion [BFK], the above extensions may be used to relate

(i) $S^1$-equivariant analytic and combinatorial torsion,

(ii) Higher order analytic torsion and Higher order Reidemeisder torsion.
1.1 Review of Witten theory for a Morse function

In this section, we sketch Witten’s proof of Morse inequality as given in [W],[S],[ ].

Let $M^n$ be a compact orientable Riemannian manifold, $f$ be a Morse function on $M$, $m_i$ be the number of critical points of index $i$.

Define

$$P(M, t) = \sum_{i=0}^{\infty} t^i \dim H^i(M) = \sum_{i=0}^{\infty} t^i \beta_i \quad \text{(1.7)}$$

$$M(M, f, t) = \sum_{i=0}^{\infty} t^i m_i \quad \text{(1.8)}$$

Then we have

**Theorem (Morse Inequality)**

Formulation I:

$$M(M, f, t) - P(M, t) = (1 + t) Q(t) \quad \text{(1.9)}$$

where $Q(t) = \sum_{i=0}^{\infty} q_i t^i$ with $q_i \geq 0$.

Formulation II:

$$\sum_{i=0}^{k} (-1)^i m_i - \sum_{i=0}^{k} (-1)^i \beta_i \left\{ \begin{array}{ll} \geq 0 & \text{if } k \text{ is even} \\ \leq 0 & \text{if } k \text{ is odd} \end{array} \right. \quad \text{(1.10)}$$

**Remark** The above two formulations are in fact equivalent. Witten’s idea of proving Morse inequality ([W],[S]) is to use Witten deformation of de Rham complex to construct a complex of finite dimensional vector spaces and apply the following algebraic lemma:

**Lemma 1.1.1** Suppose $(C^k, d)$ is a complex of finite dimensional vector spaces, let
\[ M_i = \dim C^i, \beta_i = \dim H^i(C, d), \text{ then} \]
\[
\sum_{r=0}^{k} (-1)^r M_r - \sum_{r=0}^{k} (-1)^r \beta_r \begin{cases} 
\geq 0 & \text{if } k \text{ is even} \\
\leq 0 & \text{if } k \text{ is odd} 
\end{cases} \quad (1.11)
\]

To construct the complex of finite dimensional vector spaces, let \( c_1^k, \ldots, c_{n_k}^k \) be the critical points of \( f \) of index \( k \), \( x = (x_1, \ldots, x_n) \) be a coordinate system in a neighbourhood of \( x_j^k \) given by the Morse Lemma, i.e.
\[
f(x) = f(0) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2 \quad (1.12)
\]

Let \( g \) be a Riemannian metric on \( M \), which is given by the canonical Euclidean metric when represented in the above coordinate system. (Such a pair \((f, g)\) is said to be compatible.) Consider the de Rham complex \( (\Omega^*(M), d) \) which calculates the cohomology of \( M \).

Let
\[
d(t) = e^{-tJ} d e^{tJ} \quad (1.13)
\]
\[
d^*(t) = e^{tJ} d^* e^{-tJ} \quad (1.14)
\]
\[
\Delta(t) = d(t) d^*(t) + d^*(t) d(t) \quad (1.15)
\]

Since \( d(t) \) and \( d \) are conjugated by \( e^{tJ} \), \( (\Omega^*(M), d(t)) \) also calculates the cohomology of \( M \), which is also given by the harmonic forms of \( \Delta(t) \).

One can calculate \( \Delta(t) \) in the above coordinate system.
\[
\Delta(t) = dd^* + d^* d + 4t^2 x^2 + t \left\{ - \sum_{i=1}^{k} [dx_i, i(\partial_i)] + \sum_{i=k+1}^{n} [dx_i, i(\partial_i)] \right\} \quad (1.16)
\]

where \( dx_i \) is the exterior multiplication by \( dx_i \), while \( i(\partial_i) \) is the interior multiplication by \( \partial_i \). (Recall that the interior multiplication by a vector field \( X \) is a zero order operator \( i_X : \Omega^*(M) \rightarrow \Omega^{*+1}(M) \))
Now define the 'localized' operator $\overline{\Delta_j^k}(t) : \Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$ to be given by the above expression and notice that

$$
\overline{\Delta_j^k}(t) = U(t^{1/2})t\Delta_j^k(1)U(t^{-1/2})
$$

(1.17)

where $(U(\lambda)\omega)(x) = \lambda^{n/2}\omega(\lambda x)$. Since $U(\lambda)$ is an isometry, then $\overline{\Delta_j^k}(t)$ and $t\overline{\Delta_j^k}(1)$ are conjugated by an isometry.

Let $K_j^k = \overline{\Delta_j^k}(1)$ and $\{c_j\}_{1 \leq j \leq r}$ be all the critical points of $f$.

Let

$$
0 \leq c_1^{(k)} \leq c_2^{(k)} \leq \cdots \leq c_r^{(k)} \leq \cdots
$$

be all the eigenvalues of $\oplus_{j=1}^r K_j^k : \oplus_{j=1}^r \Omega^k(\mathbb{R}^n) \to \oplus_{j=1}^r \Omega^k(\mathbb{R}^n)$

Then

$$
0 \leq tc_1^{(k)} \leq tc_2^{(k)} \leq \cdots \leq tc_r^{(k)} \leq \cdots
$$

are all the eigenvalues of $\oplus_{j=1}^r \overline{\Delta_j^k}(t) : \oplus_{j=1}^r \Omega^k(\mathbb{R}^n) \to \oplus_{j=1}^r \Omega^k(\mathbb{R}^n)$

Let

$$
0 \leq E_1^{(k)}(t) \leq E_2^{(k)}(t) \leq \cdots \leq E_l^{(k)}(t) \leq \cdots
$$

be all the eigenvalues of $\Delta^k(t) : \Omega^k(M) \to \Omega^k(M)$ Then we have

**Theorem (Witten, Simon)** (cf. [S])

$$
\lim_{t \to \infty} \frac{E_l^{(k)}(t)}{t} = c_l^{(k)}
$$

(1.18)

Let us consider those $E_l^{(k)}(t)$ s.t. $\lim_{t \to \infty} \frac{E_l^{(k)}(t)}{t} = 0$.

Observe that $K_j^k$ has exactly one zero eigenvalue (whose corresponding eigenvector is a k-form) iff index $c_j = k$. 

So we have
\[
0 = \lim_{t \to \infty} \frac{E_1^{(k)}(t)}{t} = \lim_{t \to \infty} \frac{E_2^{(k)}(t)}{t} = \cdots = \lim_{t \to \infty} \frac{E_m^{(k)}(t)}{t} = 0 < \lim_{t \to \infty} \frac{E_{m+1}^{(k)}(t)}{t} \leq \cdots
\]
(1.19)

In fact, one can show that
\[
\left\{ \begin{array}{l}
\lim_{t \to \infty} E_1^{(k)}(t) = \lim_{t \to \infty} E_2^{(k)}(t) = \cdots = \lim_{t \to \infty} E_m^{(k)}(t) = 0 \\
\lim_{t \to \infty} E_{m+1}^{(k)}(t) = \lim_{t \to \infty} E_{m+2}^{(k)}(t) = \cdots = +\infty 
\end{array} \right.
\]
(1.20)

**Corollary 1.1.2** For any \(0 < a < b\), there exist \(t_0 > 0\) s.t. for any \(t \geq t_0\)
\[
0 \leq E_1^{(k)}(t) \leq \cdots \leq E_m^{(k)}(t) < a < b \leq E_{m+1}^{(k)}(t) \leq E_{m+2}^{(k)}(t) \leq \cdots
\]

**Definitions**

1. \(E(t)\) is a small (respectively large) eigenvalue of \(\Delta^k(t)\) if \(\lim_{t \to \infty} E(t) = 0\) (respectively \(\infty\)).

2. Define for \(t\) sufficiently large,
\[
\Omega^k_{\text{small}}(M, t) = \text{Span}\{\Psi(t) \in \Omega^k(M) \mid \Delta^k(t)\Psi(t) = E(t)\Psi(t) \text{ where } E(t) \text{ is a small eigenvalue}\}
\]
(1.21)

Similarly, \(\Omega^k_{\text{large}}(M, t)\) is defined to be the linear span of the eigenforms corresponding to the large eigenvalues of \(\Delta^k(t)\).

Then we have

**Corollary 1.1.3**

(i) For \(t\) sufficiently large,
\[
(\Omega^*(M), d(t)) = (\Omega^*_{\text{small}}(M, t), d(t)) \oplus (\Omega^*_{\text{large}}(M, t), d(t))
\]
(1.22)

where \((\Omega^*_{\text{large}}(M, t), d(t))\) is acyclic.
(ii) \((\Omega^*_{\text{small}}(M, t), d(t))\) calculates the cohomology of \(M\) with \(dim \Omega^k_{\text{small}}(M, t) = m_k < \infty\).

**Corollary 1.1.4 (Morse Inequality)**

\[
\sum_{i=0}^{k} (-1)^i m_i - \sum_{i=0}^{k} (-1)^i \beta_i \left\{ \begin{array}{ll}
\geq 0 & \text{if } k \text{ is even} \\
\leq 0 & \text{if } k \text{ is odd}
\end{array} \right.
\]  

**Proof** Apply Lemma 1.1.1 to the complex \((\Omega^*_{\text{small}}(M, t), d(t))\).

---

### 1.2 Review of Helffer-Sjöstrand theory for a Morse function

In this section, we review the Helffer-Sjöstrand theory as presented in [11S],[BZ].

**Definition:** Suppose \(f\) is a Morse function, \(g\) a Riemannian metric on \(M\), the pair \((f, g)\) is said to satisfy the Morse-Smale condition if for any two critical points \(x\) and \(y\), the ascending manifold \(W^+_x\) and the descending manifold \(W^-_y\), w.r.t. \(-\text{Grad}_g f\), intersect transversally.

**Definition:** The pair \((f, g)\) is said to be compatible if for any critical point \(c\) of \(f\), there exists a coordinate system \((x_1, \cdots, x_n)\) about \(c\) s.t.

\[
f(x) = f(c) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2
\]  

and \(g = \delta_{ij}\) when represented in the above coordinate system. Also, such coordinate system is said to be a compatible coordinate system.

**Proposition 1.2.1** (cf. [Sm]) For any pair \((f, g)\), there is a \(C^1\) approximation \(g'\) such that \(g = g'\) in a neighbourhood of the critical points of \(f\) and \((f, g')\) satisfies the Morse-Smale condition.
Definition: Suppose $f$ is a Morse function, then $f$ is said to self-indexing if

$$f(x) = k \quad \text{if } x \text{ is a critical point of index } k$$

Proposition 1.2.2 (cf. [M] §4) For any Morse function $f$, there exists a self-indexing Morse function $f'$ such that $f$ and $f'$ have the same critical points and corresponding indices.

Theorem 1.2.3 Suppose $f$ is a self-indexing Morse function and $g$ is a metric such that $(f, g)$ is compatible and satisfies the Morse-Smale condition, then

(i) $\{W^{-}_{x,j} \}_{0 \leq k \leq n; 1 \leq j \leq m_k}$ is a CW complex.

(ii) Let $(C_*(M,f), \delta)$ be the cellular chain complex of the above CW-complex.

$(C^*(M,f), \delta)$ be its dual cochain complex.

Then $Int : (\Omega^*(M), d) \to (C^*(M,f), \delta)$

$$Int(\omega_k)(W^{-}_{x,j}) = \int_{W^{-}_{x,j}} \omega \quad \text{for } \omega_k \in \Omega^k(M)$$

is a morphism of cochain complexes.

Proof: A proof of this theorem can be found in [L].

Hence the composition

$$(\Omega_{sma}^*(M,f,t), d(t)) \xrightarrow{\phi_t} (\Omega^*(M), d) \xrightarrow{Int} (C^*(M,f), \delta)$$

is also a morphism of cochain complexes.

Since $(f,g)$ is compatible, for any critical point $x_j^k$ of index $k$, choose a compatible coordinate system about $x_j^k$ so that $x_j^k$ has coordinate $(0, \cdots, 0)$ and the chart contains $B(0, \epsilon)$ the unit ball of radius $\epsilon$ in $R^n$. For $\epsilon > 0$, let $\rho \in C_0^\infty(\Re)$ be s.t.

$$\rho(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{\epsilon}{2} \\ 0 & \text{if } |x| \geq \epsilon \end{cases} \quad (1.25)$$
Define
\[ \Psi_{x_j^k}(t) = \beta(t) \rho(|x|) \left( \frac{2t}{\pi} \right)^{n/4} e^{-t(x_1^2 + \cdots + x_k^2)} dx_1 \wedge \cdots \wedge dx_k \] (1.26)
where \( \beta(t) \) is chosen s.t. \( \|\Psi_{x_j^k}(t)\| = 1 \).

Define \( J_k(t) : C^k(M, f) \to \Omega^k(M) \) by
\[ J_k(t) \left( e_{x_j^k} \right) = \Psi_{x_j^k}(t) \] (1.27)
where \( \{e_{x_j^k}\} \) is the dual basis of \( \{W_{x_j^k}\} \).

Let \( Q_k(t) \) be the orthogonal projection in \( L^2(\Lambda^k(M)) \) onto \( \Omega^k_{\text{small}}(M, t) \).

Define
\[ H_k(t) = (Q_k(t)J_k(t))^\ast(Q_k(t)J_k(t)) \] (1.28)
\[ \tilde{J}_k(t) = Q_k(t)J_k(t)(H_k(t))^{-1/2} \] (1.29)
Then \( \tilde{J}_k(t) : C^k(M, f) \to \Omega^k_{\text{small}}(M, t) \) is an isometry.

Define
\[ E_{x_j^k}(t) = \tilde{J}_k \left( e_{x_j^k} \right) \] (1.30)
Then \( \{E_{x_j^k}(t)\}_{1 \leq j \leq m_k} \) forms an orthonormal basis in \( \Omega^k_{\text{small}}(M, t) \).

**Proposition 1.2.4** (cf. [HS], [BZ]) There exists a neighbourhood \( U_{x_j^k} \) of \( x_j^k \) contained in the chart of compatibility s.t.

\[ E_{x_j^k}(t) = (\frac{2t}{\pi})^{n/4} e^{-t|x|^2} \left( dx_1 \wedge \cdots \wedge dx_k + O(t^{-1}) \right) \] (1.31)
on \( U_{x_j^k} \).

**Proposition 1.2.5** Let \( Int_k = Int |_{\Omega^k(M)} \), then
\[ Int_k e^{1f} \left( E_{x_j^k}(t) \right) = \left( \frac{2t}{\pi} \right)^{n-2k} e^{tk} \left( e_{x_j^k} + O(t^{-1}) \right) \] (1.32)
Hence, define $F^k(t) : \Omega_{small}^k(M, t) \to C^k(M, f)$ s.t.

$$F^k(t) \left( E_{x_j^k}(t) \right) = \left( \frac{\pi}{2t} \right)^{\frac{n-k}{4}} e^{-tk} \text{Int}_k \epsilon^J \left( E_{x_j^k}(t) \right)$$ (1.33)

and let

$$\left( \Omega_{small}^*(M, t), \tilde{d}(t) \right) = \left( \Omega_{small}^*(M, t), \epsilon^J \left( \frac{\pi}{2t} \right)^{1/2} \tilde{d}(t) \right)$$ (1.34)

then we have

**Theorem (cf. [BZ],[HS])** $F^* (t) : \left( \Omega_{small}^*(M, t), \tilde{d}(t) \right) \to \left( C^*(M, f), \delta \right)$ is a morphism of cochain complexes s.t.

$$F^*(t) = I + O(t^{-1})$$ (1.35)

w.r.t. the bases $\{ E_{x_j^k}(t) \}$ and $\{ c_{x_j^k} \}$.

As a consequence, we have

**Theorem (Helffer-Sjöstrand)(cf. [HS])**

$$< E_{x_j^{k+1}}(t), d(t) E_{x_j^k}(t) > = e^{-t} \sqrt{\frac{2t}{\pi}} \left( \sum \epsilon_\gamma + O(t^{-1}) \right)$$ (1.36)

where $i(x_j^{k+1}, x_j^k) = \sum \epsilon_\gamma$ is the incidence number between $x_j^{k+1}$ and $x_j^k$ as is defined in the Witten-Morse theory.

**Proof:** Let

$$d(t) E_{x_j^k}(t) = \sum_j \lambda_{ji}(t) E_{x_j^{k+1}}(t)$$ (1.37)

for some real $\lambda_{ji}(t)$, $1 \leq j \leq m_{k+1}$.

Since $\delta \text{Int}_k \epsilon^J = \text{Int}_{k+1} \epsilon^J d(t)$, we have

$$\delta \left( \text{Int}_k \epsilon^J E_{x_j^k}(t) \right) = \sum_j \lambda_{ji}(t) \text{Int}_{k+1} \epsilon^J E_{x_j^{k+1}}(t)$$ (1.38)
By Proposition 1.2.5,

\[
\delta c_{x^k} = \left(\frac{2t}{\pi}\right)^{-1/2} t^l \left( \sum_j \lambda_{j\ell}(t)c_{\ell, j+1} + O(t^{-1}) \right)
\]  \hspace{1cm} (1.39)

Using the fact that \(\delta c_{x^k} = \sum_j i(x_j^{k+1}, x_j^k)c_{x_j^{k+1}}\), the Theorem follows by comparing coefficients in the above equation.
CHAPTER II

Witten-Helffer-Sjöstrand Theory for $S^1$-equivariant Cohomology

2.1 Outline of Proof

In this chapter, we shall use Witten’s method of proving Morse inequality (see §1.1) to verify the $S^1$-equivariant Morse inequality (cf. [B]). This is accomplished by using a family of finite dimensional subcomplexes $\Omega^*_\text{mv,um}(M,f,t)$, $t \in [0, \infty)$ of the Witten deformed $S^1$-equivariant de Rham complex, where $f$ is an $S^1$-invariant Morse function. Suppose furthermore that $f$ is self-indexing and $g$ is a metric on $M$ such that $(f,g)$ satisfies the Morse-Smale condition, then this family of subcomplexes is shown to converge to a geometric complex induced by the pair $(f,g)$ as $t \to \infty$. Let us first review on $S^1$-equivariant cohomology.

2.1.1 $S^1$-equivariant Cohomology and $S^1$-equivariant Hodge Theory
(a) \(S^1\)-equivariant Cohomology

Let \(M^n\) be a compact manifold, \(\mu: S^1 \times M \to M\) be a smooth action. Let \(X\) be the infinitesimal generator of the \(S^1\)-action and \(i_X\) be the contraction along the vector field \(X\).

Then \(d: \Omega^k(M) \to \Omega^{k+1}(M)\) and \(i_X: \Omega^k(M) \to \Omega^{k-1}(M)\)

Define

\[ \hat{\Omega}^k(M) \equiv \Omega^k(M) \oplus \Omega^{k-2}(M) \oplus \ldots \]  \hspace{1cm} (2.1)

So \(d + i_X: \hat{\Omega}^k(M) \to \hat{\Omega}^{k+1}(M)\). Note that

\[ (d + i_X)^2 = di_X + i_Xd = L_X \]  \hspace{1cm} (2.2)

where \(L_X\) is the Lie derivative of differential forms along the vector field \(X\). Define

\[ \Omega^*_\text{inv}(M) = \{ \omega \in \Omega^*(M) \mid L_X\omega = 0 \} \]

\[ \hat{\Omega}^*_\text{inv}(M) = \Omega^*_\text{inv}(M) \oplus \Omega^{k-2}(M) \oplus \ldots \]  \hspace{1cm} (2.3)

Then \(D \equiv d + i_X: \hat{\Omega}^*_\text{inv}(M) \to \hat{\Omega}^{k+1}_\text{inv}(M)\) and \((\hat{\Omega}^*_\text{inv}(M), D)\) is a differential complex.

Define the \(S^1\)-equivariant cohomology of \(M\) to be

Then \(D \equiv d + i_X: \Omega^*_\text{inv}(M) \to \Omega^{k+1}_\text{inv}(M)\) and \((\Omega^*_\text{inv}(M), D)\) is a differential complex.

Define the \(S^1\)-equivariant cohomology of \(M\) to be

(b) \(S^1\)-equivariant Cohomology with Coefficient in an Orientation Line Bundle

More generally, let \(E \to M\) be a vector bundle over \(M\) of rank \(k\), \(o(E) = \Lambda^{\text{rank } E}(E) \to M\) be its orientation line bundle, which is a flat vector bundle (cf. [BT]).

Let \(\Omega^*(M, o(E))\) be the space of \(o(E)\)-valued forms on \(M\).
Define $d: \Omega^k(M, o(E)) \to \Omega^{k+1}(M, o(E))$ by

$$d(\omega \otimes s) = d\omega \otimes s \quad (2.5)$$

where $s$ is a locally constant section of $o(E)$. Then $(\Omega^*(M, o(E)), d)$ is a cochain complex.

Define $i_x: \Omega^k(M, o(E)) \to \Omega^{k-1}(M, o(E))$ by

$$i_x(\omega \otimes s) = i_x\omega \otimes s \quad (2.6)$$

and $L_x: \Omega^k(M, o(E)) \to \Omega^k(M, o(E))$ by

$$L_x(\omega \otimes s) = L_x\omega \otimes s \quad (2.7)$$

Let

$$\Omega^k_{inv}(M, o(E)) = \{ \omega \in \Omega^k(M, o(E)) \mid L_x\omega = 0 \} \quad (2.8)$$

and

$$\hat{\Omega}^k_{inv}(M, o(E)) = \Omega^k_{inv}(M, o(E)) \oplus \Omega^{k-2}_{inv}(M, o(E)) \oplus \cdots \quad (2.9)$$

Then $D = d + i_x: \hat{\Omega}^k_{inv}(M, o(E)) \to \hat{\Omega}^{k+1}_{inv}(M, o(E))$ and $D^2 = 0$.

Hence define

$$H^*_S(M, o(E)) \simeq H^* \left( \hat{\Omega}^*_{inv}(M, o(E)), D \right) \quad (2.10)$$

(c) $S^1$-equivariant Hodge Theory

Suppose further that $g$ is an $S^1$-invariant metric on $M$. Then

$$D = d + i_x: \check{\Omega}^k(M) \to \check{\Omega}^{k+1}(M)$$

$$D^* = d^* + i^*_x: \check{\Omega}^{k+1}(M) \to \check{\Omega}^k(M) \quad (2.11)$$
and \( D(\tilde{\Omega}_{mn}^k(M)) \subset \tilde{\Omega}_{mn}^{k+1}(M), D^*(\tilde{\Omega}_{mn}^{k+1}(M)) \subset \tilde{\Omega}_{mn}^k(M) \).

Define

\[
\tilde{\Delta}^k = DD^* + D^*D : \tilde{\Omega}^k(M) \to \tilde{\Omega}^k(M)
\]  

(2.12)

Since \( \tilde{\Delta}^k = dd^* + d^*d + (di_X^* + i^*_X d + d^*i_X + i_X d^*) + i_X i^*_X + i^*_X i_X \) is elliptic,

\[
\tilde{\mathcal{H}}^k(M) = \{ \omega \in \tilde{\Omega}^k(M) \mid \tilde{\Delta}^k \omega = 0 \}
\]  

(2.13)

is finite dimensional. Also we have

\[
I = P + \tilde{\Delta}^k G \text{ on } \tilde{\Omega}^k(M)
\]  

(2.14)

where \( P \) is the projection to harmonic forms and \( G \) is the Green's operator or the parametrix of \( P \).

Therefore,

\[
I = P + D_{k-1}(D_{k-1}^* G) + D_k^*(D_k G)
\]  

(2.15)

Let \( \omega \in \tilde{\Omega}_{mn}^k(M) \), then

\[
\omega = P \omega \oplus D_{k-1}(D_{k-1}^* G \omega) \oplus D_k^*(D_k G \omega)
\]  

(2.16)

Since \( D(\tilde{\Omega}_{mn}^k(M)) \subset \tilde{\Omega}_{mn}^{k+1}(M) \) and \( D^*(\tilde{\Omega}_{mn}^{k+1}(M)) \subset \tilde{\Omega}_{mn}^k(M) \), every term in the above equation is in \( \tilde{\Omega}_{mn}^*(M) \), one can easily verify that

\[
\tilde{\Omega}_{mn}^k(M) = \tilde{\mathcal{H}}_{mn}^k(M) \perp D_k \left( \tilde{\Omega}_{mn}^{k-1}(M) \right) \perp D_k^* \left( \tilde{\Omega}_{mn}^{k+1}(M) \right)
\]  

(2.17)

Consequently, \( H_{N1}^k(M) \cong \tilde{\mathcal{H}}_{mn}^k(M) \) and \( \dim H_{N1}^k(M) < \infty \).
2.1.2 Equivariant Morse Theory and Witten Deformation

(a) Equivariant Morse Theory

Let $M$ be a compact manifold with a smooth $S^1$-action, $f$ be an $S^1$-invariant function on $M$. Let $O_x = \{gy \mid g \in S^1\}$ be the orbit of $x$. A submanifold $O$ of $M$ is called an orbit if $O = O_x$ for some $x \in M$.

Let $d^2_x f : T_x M \times T_x M \to \mathbb{R}$ be the Hessian of $f$ at $x \in O$. Since $f$ is invariant, it induces a symmetric bilinear form on $T_x M/T_x O$:

$$d^2_x f : (T_x M/T_x O) \times (T_x M/T_x O) \to \mathbb{R}$$

**Definition:** $O \subset M$ is called a non-degenerate critical orbit of $f$ if

(i) $O$ is an orbit which consists of critical points of $f$.

(ii) $d^2_x f$ is non-degenerate for some $x \in O$ (and hence for any $x \in O$).

We shall consider only invariant functions whose critical orbits are all non-degenerate. Such functions are in fact generic in the sense that they are open and dense in the $C^1$ topology. Also, since $M$ is compact, and since non-degenerate critical orbits are isolated, $f$ has only a finite number of critical orbits.

Now let $O$ be a critical orbit of $f$. $\nu(O)$ be the normal bundle of $O$ in $M$. Since $d^2_x f$ is symmetric and non-degenerate, let $\nu^-(O)$ be the subbundle spanned by the negative eigenvectors of $d^2_x f$ where $x \in O$. More precisely, using the metric $g$, we identify $T_x(M)/T_x(O)$ with $T_x(O)^\perp$. With the above identification, we regard $d^2_x f$ as a non-degenerate, symmetric bilinear form on $T_x(O)^\perp$. Let $H_x : T_x(O)^\perp \to T_x(O)^\perp$ be the linear map associated with the bilinear form $d^2_x f$ w.r.t. the metric $g$. Then
\( \nu^{-}(O) \) is the subbundle spanned by the eigenvectors corresponding to the negative eigenvalues of \( H_x \) where \( x \in O \).

Let \( \theta^{-} \) be the orientation line bundle of \( \nu^{-}(O) \).

**Definitions**

1. index \( O = \text{index } d^*_x f \) for any \( x \in O \)

2. 
   \[
   \mathcal{P}_{S^1}(M, t) \equiv \sum_{i=0}^{\infty} t^i \dim H_{S^1}^i(M) \tag{2.18}
   \]

3. 
   \[
   \mathcal{M}_{S^1}(M, f, t) \equiv \sum_{O \in \text{Crit Orbits}} t^{\text{index } O} \mathcal{P}_{S^1}(O, \theta^-, t) \tag{2.19}
   \]

   where \( \mathcal{P}_{S^1}(O, \theta^-, t) = \sum_{i=0}^{\infty} t^i \dim H_{S^1}^i(O, \theta^-) \).

   Then as suggested by Bott[13], we can formulate the \( S^1 \)-equivariant Morse Inequality as follows:

**Theorem (\( S^1 \)-equivariant Morse Inequality)**

\[
\mathcal{M}_{S^1}(M, f, t) - \mathcal{P}_{S^1}(M, t) = (1 + t) Q(t) \tag{2.20}
\]

where \( Q(t) = \sum_{i=0}^{\infty} t^i q_i \) with \( q_i \geq 0 \).

Our approach of proving the \( S^1 \)-equivariant Morse inequality is to apply the Witten deformation of the \( S^1 \)-equivariant de Rham complex, which is explained below, to obtain a subcomplex of finite dimensional vector spaces. Morse inequality is proved once again by applying the algebraic lemma.
(b) Witten Deformation of $S^1$-equivariant de Rham Complex

Let $M$ be an $S^1$-manifold and $g$ be an $S^1$-invariant metric on $M$.

Let $D(t) : \tilde{\Omega}^k_{inv}(M) \rightarrow \tilde{\Omega}^{k+1}_{inv}(M)$

$D(t) \equiv e^{-tJ}(d + i_X)e^{tJ} = e^{-tJ}de^{tJ} + i_X = d(t) + i_X$

Then $D(t)^* = e^{tJ}(d^* + i_X^*)e^{-tJ} = d(t)^* + i_X^*$

Define

$$\tilde{\Delta}^k(t) = D(t)D(t)^* + D(t)^*D(t) : \tilde{\Omega}^k_{inv}(M) \rightarrow \tilde{\Omega}^k_{inv}(M)$$

$$= d(t)d(t)^* + d(t)^*d(t) + (d^*i_X + i_X^*i_X) + (d^*i_X^* + i_Xd)$$

(2.21)

Clearly for any $t$, $(\tilde{\Omega}^*_{inv}(M), D(t))$ calculates the $S^1$-equivariant cohomology of $M$. Since $\tilde{\Delta}^k(t)$ (with domain and range $\tilde{\Omega}^*(M)$) is elliptic, the Hodge Decomposition Theorem remains true for any $t$, moreover for any $t$, the $\tilde{\Delta}^k(t)$-harmonic forms calculates $H^k_{S^1}(M)$.

2.1.3 Local Expression of $\tilde{\Delta}^k(t)$ near Critical Orbits; 'Localized' Operators

(a) Morse Lemma

Let $G$ be a compact Lie group, $H$ a closed subgroup, $\rho_i : H \rightarrow O(\mathbb{R}^{k_i})$ i=1,2 be two orthogonal representations. Denote by $\rho = \rho_1 \oplus \rho_2 : H \rightarrow O(\mathbb{R}^{k_1+k_2})$ the direct sum of $\rho_1$ and $\rho_2$. With the diagonal action of $G$ on $\mathbb{R}^{k_1+k_2} \times H\ G$, the quotient space $\mathbb{R}^{k_1+k_2} \times_H G$ becomes a bundle $E$ over $H\ G$.

$$E(\rho_1, \rho_2) = \mathbb{R}^{k_1+k_2} \times_H G \rightarrow H\ G$$
Note that the zero section of $E$ is an orbit of $G$ which has the isotopy group $H$. Let $\mu : E \times G \to G$ be the smooth action given by

$$\mu([v, g], g) = [v, g_1 g]$$

(2.22)

Also, let $h : E \to \mathbb{R}$ be the map defined by

$$h([v_1 \oplus v_2, g]) = -|v_1|^2 + |v_2|^2$$

(2.23)

The above example is called the standard model.

**Morse Lemma** Let $M$ be a $G$-manifold of dimension $n$, $f$ an invariant Morse function, $x$ a critical point of $f$ with orbit $O_x$, $G_x$ be the isotopy group of $x$. Suppose that

$$\hat{\partial}_{f}^2 : T_x M/T_x O \times T_x M/T_x O \to \mathbb{R}$$

is a symmetric non-degenerate bilinear form of index $k$. Then there exists two orthogonal representation $\rho_1 : G_x \to O(\mathbb{R}^k)$ and $\rho_2 : G_x \to O(\mathbb{R}^{n-k-\dim(G/G_x)})$ and a $G$-equivariant diffeomorphism

$$\phi : D(E(\rho_1, \rho_2)) \to U$$

where $D(E(\rho_1, \rho_2))$ is the unit disc bundle of $E$ and $U$ a compact neighbourhood of $O_x$, so that

(i) The zero section of $D(E(\rho_1, \rho_2)) \to G_x \backslash G$ is mapped onto $O_x$.

(ii) $(f - f(x)) \circ \phi = h$.

**Proof:** See [].
**Remark** Note that by using the product metric on $\mathbb{R}^{n-dim(G/G_x)} \times G$, it induces a metric on $E = \mathbb{R}^{n-dim(G/G_x)} \times G_x G$ which in turn, by using the above $G$-equivariant diffeomorphism $\phi$, induces a canonical metric on $U$.

(b) **Local Expression of $\hat{\Delta}^k(t)$ near Critical Orbits**

In the case of $G = S^1$, $G_x \cong 1, \mathbb{Z}_m$, or $S^1$. The following cases exhaust all the possibilities of the standard model.

**Case 1:** $G_x \cong 1$, $O_x \cong S^1$ and $U \cong D(E) \cong D^{n-1} \times S^1$ where $D^{n-1}$ is the unit disc in $\mathbb{R}^{n-1}$. Let index $O_x = l$

Let $x = (x_1, \cdots, x_{n-1})$, $\theta$ be the coordinates in $D^{n-1}$ and $S^1$ respectively. Then the function $f$ and the canonical metric $g$ can be expressed as

$$f \circ \phi(x, \theta) = f(0) - x_1^2 - \cdots - x_l^2 + x_{l+1}^2 + \cdots + x_{n-1}^2$$

(2.24)

$$dg = d^2 x_1 + \cdots + d^2 x_{n-1} + d^2 \theta$$

(2.25)

Recall that $i_X : \hat{\Omega}^{k-1}(M) \to \hat{\Omega}^k(M)$, $i_X^* : \hat{\Omega}^k(M) \to \hat{\Omega}^{k-1}(M)$

with

$$i_X^*(\omega_k, \omega_{k-2}, \omega_{k-4}, \cdots) = (i_X^* \omega_{k-2}, i_X^* \omega_{k-4}, \cdots) = (d\theta \wedge \omega_{k-2}, d\theta \wedge \omega_{k-4}, \cdots)$$

in $U$ (2.26)

For $\omega \in \hat{\Omega}^{k}_{run}(M)$, write

$$\omega = (\omega_k \oplus \omega_{k-1} \wedge d\theta) \oplus (\tilde{\omega}_{k-2} \oplus \tilde{\omega}_{k-3} \wedge d\theta)$$

in $U$ (2.27)

where $\tilde{\omega}_{k-2}$, resp. $\tilde{\omega}_{k-3}$, is the pullback of a form in $\hat{\Omega}^{k-2}(D^{n-1}) = \Omega^{k-2} \oplus \Omega^{k-4} \oplus \cdots$, resp. in $\hat{\Omega}^{k-3}(D^{n-1})$, by $\pi : D^{n-1} \times S^1 \to D^{n-1}$ and $\omega_k$, resp. $\omega_{k-1}$, is the pullback of a form in $\Omega^{k}(D^{n-1})$, resp. in $\Omega^{k-1}(D^{n-1})$, by the map $\pi$. 
Since
\[(di^*_{X} + i^*_X d)(\tilde{\omega}_{k-2} \oplus \tilde{\omega}_{k-3} \wedge d\theta) = d(i^*_{X}d\tilde{\omega}_{k-2} + i^*_X d(\tilde{\omega}_{k-3} \wedge d\theta) = d(d\theta \wedge \tilde{\omega}_{k-2}) + d\theta \wedge d\tilde{\omega}_{k-2} + 0 = -d\theta \wedge d\tilde{\omega}_{k-2} + d\theta \wedge d\tilde{\omega}_{k-2} = 0, \tag{2.28}\]
\[(di^*_{X} + i^*_X d)(\omega_k \oplus \omega_{k-1} \wedge d\theta) = 0.\]

Hence
\[
\begin{align*}
\begin{cases}
   di^*_{X} + i^*_X d = 0 \\
d^*i^*_X + i^*_X d^* = (di^*_{X} + i^*_X d)^* = 0
\end{cases}
\tag{2.29}
\]
Therefore,
\[
\tilde{\Delta}^k(t) = d(t)d(t)^* + d(t)^*d(t) + i^*_X i^*_X + i^*_X i_X \text{ in } U
\]
\[
= \tilde{P}^k(t) + i^*_X i^*_X + i^*_X i_X \tag{2.30}
\]
where \(\tilde{P}^k(t) = d(t)d(t)^* + d(t)^*d(t)\)

**Case 1:** \(G_x \cong \mathbb{Z}_m, O_x \cong S^1\) and \(U \cong D^{n-1} \times \mathbb{Z}_m S^1\)

Let \(p\) be the canonical projection
\[
p : D^{n-1} \times S^1 \to D^{n-1} \times \mathbb{Z}_m S^1 \cong U
\]
Note that the metric on \(D^{n-1} \times \mathbb{Z}_m S^1\) is induced by the product metric on \(D^{n-1} \times S^1\). Since \(p\) is locally a diffeomorphism, we can use on \(D^{n-1} \times \mathbb{Z}_m S^1\) the coordinates \((x_1, \cdots, x_{n-1}, \theta)\) of \(D^{n-1} \times S^1\). With respect to this coordinate system, \(\tilde{\Delta}^k(t)\) is given by the same expression as in Case 1.

**Case 2:** \(G_x \cong S^1, O_x = x\) and \(U \cong D^n \times S^1 S^1 \cong D^n\)

In this case,
\[
\tilde{\Delta}^k(t) = \tilde{P}^k(t) + (i^*_X i^*_X + i^*_X i_X) + (di^*_{X} + i^*_X d) + (d^*i^*_{X} + i^*_X d^*) \tag{2.31}
\]
(c) 'Localized' Operators

We define the corresponding Laplace operators in the standard models $\mathbb{R}^{n-1} \times S^1$, $E = \mathbb{R}^{n-1} \times \mathbb{Z}_m S^1$ and $\mathbb{R}^n$ respectively.

**Case 1:** Define $\overline{\Delta}_j(t) : \tilde{\Omega}_{inv}^k(\mathbb{R}^{n-1} \times S^1) \rightarrow \tilde{\Omega}_{inv}^k(\mathbb{R}^{n-1} \times S^1)$ given by (2.30), where the bar above the operator designates an operator in the standard model, $j$ corresponds to the critical orbit $O_j$, with index $O_j = l$ and

$$\tilde{\Delta}_j(t) = dd^* + d^* d + 4t^2 \omega^2 + t \left[ - \sum_{i=1}^{l}[dx_i, i\omega] + \sum_{i=l+1}^{n-1}[dx_i, i\omega] \right]$$

(2.32)

where

$$\epsilon(\omega) = \begin{cases} 
0 & \text{if } \omega \in \tilde{\Omega}_{inv}^k(\mathbb{R}^{n-1} \times S^1) \\
\omega & \text{if } \omega \in \tilde{\Omega}_{inv}^{k-2}(\mathbb{R}^{n-1} \times S^1) 
\end{cases}$$

(2.33)

Note that $\tilde{\Omega}_{inv}^k(\mathbb{R}^{n-1} \times S^1) = (\tilde{\Omega}^k(\mathbb{R}^{n-1}) \otimes \mathbb{R} \cdot 1) \oplus (\tilde{\Omega}^{k-1}(\mathbb{R}^{n-1}) \otimes \mathbb{R} \cdot d\theta)$.

Define $\mathcal{U}^{(n-1)}(\lambda) : \tilde{\Omega}_{inv}^*(\mathbb{R}^{n-1} \times S^1) \to \tilde{\Omega}_{inv}^*(\mathbb{R}^{n-1} \times S^1)$ by

$$(\mathcal{U}^{(n-1)}(\lambda) \omega)(x, \theta) = \lambda^{n-1/2} \omega(\lambda x, \theta)$$

(2.34)

Then

$$\overline{\Delta}_j(t) = \mathcal{U}^{(n-1)}(t^{1/2}) \left[ t \overline{K}_j + id \otimes \Delta_{S^1} + (iX i_X^* + i_X i_X^*) \right] \mathcal{U}^{(n-1)}(t^{-1/2})$$

(2.35)

where $\overline{K}_j = (\Delta_{\mathbb{R}^{n-1}} + 4x^2 + A_j) \otimes id$, $\Delta_M$ denotes the Laplace operator on the manifold $M$ and $A_j$ is the multiplication operator in parenthesis in (2.32).

**Case 1':** Define $\overline{\Delta}_j(t) : \tilde{\Omega}_{inv}^k(E) \to \tilde{\Omega}_{inv}^k(E)$ where $E = \mathbb{R}^{n-1} \times \mathbb{Z}_m S^1$ is a vector bundle over $S^1$. In the coordinate $(x, \theta)$, the operator is given by the same expression as in Case 1.
Case 2: Define $\tilde{\Delta}^k_j(t) : \tilde{\Omega}^k_{inv}(\mathbb{R}^n) \to \tilde{\Omega}^k_{inv}(\mathbb{R}^n)$ by

$$
\tilde{\Delta}^k_j(t) = \tilde{P}^k(t) + (i_X i_X^* + i_X^* i_X) + (d i_X^* + i_X d^*) + (d^* i_X + i_X d^*)
$$

Since $S^1$ acts by isometry, the $S^1$-action is given by an orthogonal representation $\mathcal{R}$ on $\mathbb{R}^n$ of the form

$$
\mathcal{R}(e^{i\theta}) = e^{im_1\theta} I_2 \oplus e^{im_2\theta} I_2 \oplus \ldots \oplus e^{im_q\theta} I_2 \oplus I_{n-2q}
$$

for some $q \in \mathbb{Z}$, and some $m_i \in \mathbb{Z}$ for $1 \leq i \leq q$. Here $I_k$ denotes the identity on $\mathbb{R}^k$.

Then

$$
X = (-m_1 x_2, m_1 x_1, -m_2 x_4, m_2 x_3, \ldots, -m_q x_{2q}, m_q x_{2q-1}, 0, \ldots, 0)
$$

and $i_X i_X^* + i_X^* i_X = \epsilon |X|^2$ where $\epsilon$ is defined as in Case 1 above.

In this case,

$$
\tilde{\Delta}^k_j(t) = \mathcal{U}^{(n)}(t^{1/2}) \left[ i K^k_j + \frac{1}{t} |X|^2 + (i_X^* d + di_X^* + i_X d^* + d^* i_X) \right] \mathcal{U}^{(n)}(t^{-1/2})
$$

2.1.4 Main Theorem

Let $O_1, \ldots, O_r$ be all the critical orbits of $f$,

$$
0 \leq E_1^{(k)} \leq E_2^{(k)} \leq \ldots \leq E_t^{(k)} \leq \ldots \quad \text{be all the eigenvalues of } \tilde{\Delta}^k(t)
$$

$$
0 \leq \tilde{r}_1^{(k)} \leq \tilde{r}_2^{(k)} \leq \ldots \leq \tilde{r}_t^{(k)} \leq \ldots \quad \text{be all the eigenvalues of } \oplus_{j=1}^r \tilde{K}^k_j
$$

$$
0 \leq \tilde{r}_1^{(k)}(t) \leq \tilde{r}_2^{(k)}(t) \leq \ldots \leq \tilde{r}_t^{(k)}(t) \leq \ldots \quad \text{be all the eigenvalues of } \oplus_{j=1}^r \tilde{\Delta}_j^k(t)
$$

Here $\tilde{\Delta}^k_j(t)$ acts on $\tilde{\Omega}^k_{inv}(M)$ and the other operators act on the corresponding spaces of invariant forms. Note that $\oplus_{j=1}^r \tilde{K}^k_j$ acts on the space

$$
\left( \oplus \tilde{\Omega}^k_{inv}(\mathbb{R}^{n-1} \times S^1) \right) \oplus \left( \oplus \tilde{\Omega}^k_{inv}(E) \right) \oplus \left( \oplus \tilde{\Omega}^k_{inv}(\mathbb{R}^n) \right)
$$
with as many copies in the individual summands as the number of critical orbits whose local structure is described by the corresponding standard model.

**Theorem 1**

\[
\lim_{t \to \infty} \frac{E^{(k)}_i(t)}{t} = \lim_{t \to \infty} \frac{v^{(k)}_i(t)}{t} = \bar{\nu}^{(k)}_i
\]  

(2.40)

Now consider those eigenvalues of \( \bar{\Delta}^k(t) \) so that \( \lim_{t \to \infty} \frac{E^{(k)}_i(t)}{t} = 0 \), that is we need to count the zero eigenvalues of \( \bigoplus_{j=1}^{r} \bar{\Delta}_j \). It suffices to count in individual cases.

**Case 1:** \( \bar{K}_j : \tilde{\Omega}^k_{inv}(\mathbb{R}^{n-1} \times S^1) \to \tilde{\Omega}^k_{inv}(\mathbb{R}^{n-1} \times S^1) \)

For any \( \omega \in \tilde{\Omega}^k_{inv}(\mathbb{R}^{n-1} \times S^1) \), there exists \( \tilde{\omega}_k, \tilde{\omega}_{k-1} \) which are pullback of forms in \( \tilde{\Omega}^k(\mathbb{R}^{n-1}), \tilde{\Omega}^{k-1}(\mathbb{R}^{n-1}) \) respectively by the projection \( p : \mathbb{R}^{n-1} \times S^1 \to \mathbb{R}^{n-1} \) such that

\[
\omega = \tilde{\omega}_k \otimes 1 + \tilde{\omega}_{k-1} \otimes d\theta
\]  

(2.41)

that is

\[
\tilde{\Omega}^k_{inv}(\mathbb{R}^{n-1} \times S^1) = \left( \tilde{\Omega}^k(\mathbb{R}^{n-1}) \otimes \mathbb{R} \cdot 1 \right) \oplus \left( \tilde{\Omega}^{k-1}(\mathbb{R}^{n-1}) \otimes \mathbb{R} \cdot d\theta \right)
\]  

(2.42)

Let \( \Delta_M \) denote the Laplace operator on a manifold \( M \). Then

\[
\bar{K}_j = \left( \Delta_{\mathbb{R}^{n-1}} + 4x^2 + A_j \right) \otimes id
\]  

(2.43)

where \( A_j = -\sum_{i=1}^{ind_{x}} O_j \{dx_i, i_n\} + \sum_{i=ind_{x}}^{n-1} O_j + 1 \{dx_i, i_n\} \).

When regarded as an operator on the space of all forms in \( \mathbb{R}^{n-1} \), \( \Delta_{\mathbb{R}^{n-1}} + 4x^2 + A_j \) has exactly one zero eigenvalue with corresponding eigenform \( \omega_{l_j} \in \Omega^l_j(\mathbb{R}^{n-1}) \), where \( l_j = ind_{x} O_j \). This implies that \( \omega_{l_j} \otimes 1, \omega_{l_j} \otimes d\theta \) are eigenforms corresponding the eigenvalue zero in \( \tilde{\Omega}^*_{inv}(\mathbb{R}^{n-1} \times S^1) \). Since \( \bar{K}_j \) acts on \( \tilde{\Omega}^k_{inv}(\mathbb{R}^{n-1} \times S^1) \), we have
(i) If \( l_j \leq k \), then \( \tilde{K}_j^k \) has exactly one zero eigenvalue. This is because
\[
\begin{cases}
\text{If } l_j \equiv k \, (\text{mod } 2) , & \text{then } \omega_j \otimes 1 \text{ is the corresponding eigenform} \\
\text{If } l_j \not\equiv k \, (\text{mod } 2) , & \text{then } \omega_j \otimes d\theta \text{ is the corresponding eigenform}
\end{cases}
\]

(ii) If \( l_j > k \), then \( \tilde{K}_j^k \) has no zero eigenvalue.

**Case 1':** \( \tilde{K}_j^k : \tilde{\Omega}^k_{inv}(E) \rightarrow \tilde{\Omega}^k_{inv}(E) \)

The situation is pretty much the same, except that we have to restrict to those \( O_j \) whose \( \theta^- \) is trivial. (Indeed, if there exists such an eigenform \( \omega_j = g(x)dx_1 \wedge \cdots \wedge dx_j \), since it is invariant, \( g(x) \neq 0 \) for any \( x \in O_j \) which implies that \( \theta^- \) is trivial.)

**Case 2:** \( \tilde{K}_j^k : \hat{\Omega}^k_{inv}(\mathbb{R}^n) \rightarrow \hat{\Omega}^k_{inv}(\mathbb{R}^n) \) where \( \hat{K}_j^k = \Delta + 4x^2 + A_j \)

In this case, any eigenform is automatically invariant. Hence

(i) If \( l_j \leq k \) and \( l_j \equiv k \, (\text{mod } 2) \), then \( \tilde{K}_j^k \) has exactly one zero eigenvalue.

(ii) If \( l_j \leq k \) and \( l_j \not\equiv k \, (\text{mod } 2) \), then \( \tilde{K}_j^k \) has no zero eigenvalue.

(iii) If \( l_j > k \), then \( \tilde{K}_j^k \) has no zero eigenvalue.

**Definition**

\[
\hat{\Omega}^k_{inv,0}(M, t) = \text{Span} \left\{ \Psi(t) \in \hat{\Omega}^k_{inv}(M) \mid \hat{\Delta}^k(t)\Psi(t) = E(t)\Psi(t) \text{ and } \lim_{t \to \infty} \frac{E(t)}{t} = 0 \right\}
\]

**Corollary**

(i)

\[
dim \hat{\Omega}^k_{inv,0}(M, t) = m_k + m_{k-1} + \cdots + m_0 + m^f_k + m^f_{k-2} + m^f_{k-4} + \cdots
\]  \hspace{1cm} (2.44)

where

\[
\begin{cases}
m_i &= \text{number of critical orbits of index } i \text{ whose } \theta^- \text{ is trivial} \\
m^f_i &= \text{number of critical fixed points of index } i
\end{cases}
\]
(ii) \( \hat{\Omega}_{m,0}^*(M, t), D(t) \) is a complex of finite dimensional vector spaces which calculates the \( S^1 \)-equivariant cohomology of \( M \).

Using the complex \( \hat{\Omega}_{m,0}^*(M, t), D(t) \), one can verify the \( S^1 \)-equivariant Morse Inequality. However, one can also verify the \( S^1 \)-equivariant Morse inequality by using another subcomplex

\[
\hat{\Omega}_{m,sm}^*(M, t), D(t)
\]

which not only calculates the \( S^1 \)-equivariant cohomology of \( M \), but also is better connected with the geometric complex which calculates the \( S^1 \)-equivariant cohomology.

To introduce this subcomplex, note that the eigenvalues \( E(t) \) such that

\[
\lim_{t \to \infty} \frac{E(t)}{t} = 0 \tag{2.45}
\]

are of two types, namely the small eigenvalues and the finite type eigenvalues. It will be shown that for such eigenvalues where the above limit is zero, there exists a constant \( C \) independent of the eigenvalues and constants \( a \) which depend on the individual eigenvalue so that for \( t \) large enough,

\[
0 \leq E(t) \leq C \text{ and } \lim_{t \to \infty} E(t) = a
\]

**Definitions**

1. \( E(t) \) is a small eigenvalue iff \( \lim_{t \to \infty} E(t) = a = 0 \). Otherwise, i.e. if \( a \neq 0 \), \( E(t) \) is of finite type.

2. 
\[
\hat{\Omega}_{m,sm}^k(M, t) = \text{Span}\{\Psi(t) \in \hat{\Omega}_{m,0}^k(M) \mid \hat{\Delta}^k(t)\Psi(t) = E(t)\Psi(t) \text{ and } E(t) \text{ is a small eigenvalue}\}
\]

Clearly,

\[
\hat{\Omega}_{m,sm}^*(M, t) \subset \hat{\Omega}_{m,0}^*(M, t) \tag{2.46}
\]
Lemma

\[ \dim \hat{\Omega}^k_{\text{inv,sm}}(M, t) = m_k + m^f_k + m^l_{k-2} + m^l_{k-4} + \ldots \] (2.47)

Using the complex \( \left( \hat{\Omega}^*_{\text{inv,sm}}(M, t), D(t) \right) \), Morse inequality follows by applying the algebraic lemma mentioned in §1.1.

2.1.5 Helffer-Sjöstrand Theory for \( S^1 \)-equivariant Cohomology

Suppose \( f \) is an \( S^1 \)-invariant function on \( M \), \( g \) an \( S^1 \)-invariant metric on \( M \).

**Definition** \((f, g)\) is said to satisfy the Morse-Smale condition if for any critical orbits \( O_x \) and \( O_y \), \( W_x^- \) and \( W_y^+ \) intersect transversally, where \( W_x^- \) and \( W_y^+ \) are the descending and ascending manifold of \( O_x \) and \( O_y \) respectively.

**Definition** An \( S^1 \)-invariant Morse function is said to be self-indexing if for any critical orbit \( O_x \) (\( x \in O_x \)),

\[ f(x) = \text{index } x \] (2.48)

**Definition** The pair \((f, g)\) is said to be compatible if for any critical orbit \( O_x \) of \( f \), one can choose local coordinate system about \( O_x \) such that

(a) If \( O_x \cong S^1 \), then \( f \) and \( g \) are given by (2.24) and (2.25) respectively when represented in the above coordinate system.

(b) If \( O_x \cong \text{point} \), then when represented in the above coordinate system \((x_1, \ldots, x_n)\),

\[ f(x_1, \ldots, x_n) = f(x) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2 \] (2.49)

\[ dg = d^2 x_1 + \ldots + d^2 x_n \] (2.50)
Let $f$ be a self-indexing invariant Morse function such that $(f,g)$ is compatible and satisfies the Morse-Smale condition. Then a geometric complex, which calculates the $S^1$-equivariant cohomology of $M$, can be described as follows (see §2.4.2 for details).

Let $E = E_{S^1}$ be the universal principal bundle of $S^1$. $M_{S^1} = E \times M/S^1$ be the homotopy quotient of $M$.

Define $\tilde{f} : E \times M \to \mathbb{R}$ by $\tilde{f}(e,x) = f(x)$. Then $\tilde{f}$ descends to a function on the infinite dimensional manifold $M_{S^1}$ which is denoted by $f_{S^1}$. Let $X_k = f_{S^1}^{-1}(( - \infty, k + \frac{1}{2} ))$, then we have

\[ X_0 \subset X_1 \subset \cdots \subset X_n = M_{S^1} \]

Define $\partial : H_*(X_k, X_{k-1}) \to H_{*-1}(X_{k-1}, X_{k-2})$ by

\[ H_*(X_k, X_{k-1}) \to H_{*-1}(X_{k-1}) \xrightarrow{i_*} H_*(X_{k-1}, X_{k-2}) \]

\[ [\sigma] \to [\partial \sigma] \to i_*[\partial \sigma] \]

where $i_*$ is induced from the inclusion $X_{k-2} \subset X_{k-1}$

\[ H_*(X_{k-2}) \to H_*(X_{k-1}) \xrightarrow{i_*} H_*(X_{k-1}, X_{k-2}) \to H_{*-1}(X_{k-2}) \]

Define

\[ C_k(M, f) = \bigoplus_{i=0}^n H_k(X_i, X_{i-1}) \tag{2.51} \]

Also the above boundary map induces the linear map

\[ \partial : C_k(M, f) \to C_{k-1}(M, f) \]
with $\partial^2 = 0$. Therefore, we obtain the complex $(C_\ast(M, f), \partial)$ and its dual cochain complex

$$(C^\ast(M, f), \delta)$$

The complex $\left(\Omega^\ast_{inv, sm}(M, t), D(t)\right)$ can be interpreted as a complex of differential forms on $M_{S^1}$ which can be explained as follows (see [AB] and §2.4.3 for details):

Let $g \cong R$ be the Lie algebra of $S^1$, $g^*$ be the dual of $g$.

Let $Sy^*$ be the symmetric algebra generated by $g^*$ whose generator is denoted by $u$, $Ag^*$ be the exterior algebra generated by $g^*$ whose generator is denoted by $\theta$ with $\deg u = 2$, $\deg \theta = 1$.

Let $W(g) = Ag^* \otimes Sy^*$, called the Weil algebra, which is the algebra generated freely by $\theta$ and $u$ as a commutative graded algebra, i.e.

$$\omega_p \omega_q = (-1)^{pq} \omega_q \omega_p$$  \hspace{1cm} (2.52)

Define $D_0 : W(g) \rightarrow W(g)$ by

$$\begin{cases} 
D_0 \theta + u = 0 \\
D_0 u = 0
\end{cases}$$  \hspace{1cm} (2.53)

and is extended to $W(g)$ as a derivation. Observe that $D_0^2 = 0$ and the complex $(W(g), D_0)$ is a subcomplex of $(\Omega^\ast(E_{S^1}), D_0)$ where $D_0$ is the exterior derivative. $(W(g), D_0)$ is the de Rham model for $E_{S^1}$ whose homology calculates the cohomology of $E_{S^1}$.

Next let us describe the de Rham model for $B_{S^1}$. Consider the principal $S^1$-bundle over $B_{S^1}$, $S^1 \rightarrow E_{S^1} \rightarrow B_{S^1}$. Since $S^1$ acts on $E_{S^1}$, let $X$ be its generating vector field. Note that $\pi^* : \Omega^\ast(B_{S^1}) \rightarrow \Omega^\ast(E_{S^1})$ is an injection and $\omega \in \pi^*(\Omega^\ast(E_{S^1}))$ can be
characterized by
\[
\begin{align*}
i_X(\omega) &= 0 \\
L_X(\omega) &= (i_X D_0 + D_0 i_X)(\omega) = 0
\end{align*}
\] (2.54)

where \( D_0 \) is the exterior derivative. Hence define the basic subcomplex \( B_g \) of \( W(g) \) to be
\[
B_g = \{ \omega \in W(g) \mid i_X(\omega) = L_X(\omega) = 0 \}
\] (2.55)

Then \( B_g = \mathbb{R}[u] \cong H^*(B_{S^1}) \) and is called the de Rham model of \( B_{S^1} \).

For the de Rham model of \( M_{S^1} = E \times_{S^1} M \), consider
\[
\left( W(g) \otimes \Omega^*(M), D = D_0 \otimes \text{id} + (-1)^{deg\omega} \text{id} \otimes d \right)
\]
which is the de Rham model for \( E_{S^1} \times M \). Since \( S^1 \) acts on \( E_{S^1} \times M \) by diagonal action, let \( X \) be its generating vector field on \( E_{S^1} \times M \).

Define the basic subcomplex of \( W(g) \otimes \Omega^*(M) \) by
\[
\Omega^*_g(M) = \{ \omega \in W(g) \otimes \Omega^*(M) \mid i_X(\omega) = L_X(\omega) = 0 \}
\] (2.56)

Then \( H^*(\Omega^*_g(M), D) = H^*_{S^1}(M) \) and \( (\Omega^*_g(M), D) \) is the de Rham model for \( M_{S^1} \).

**Proposition** There exists
\[
\tilde{\lambda} : \left( \tilde{\Omega}^*_\text{inv}(M), D \right) \to \left( \Omega^*_g(M), D \right)
\]
which is an isomorphism between the two cochain complexes.

**Corollary**
\[
\tilde{\lambda}(t) : \left( \tilde{\Omega}^*_\text{inv}(M), D(t) = e^{-tf} D e^{tf} \right) \to \left( \Omega^*_g(M), D(t) = (e^{-tf})_{S^1} D(e^{tf})_{S^1} \right)
\]
is an isomorphism between the two cochain complexes.
Since \( \left( \tilde{\Omega}^*_\text{small}(M,t), D(t) \right) \subset \left( \tilde{\Omega}^*_\text{inv}(M), D(t) \right), \) \( \lambda(t) \) induces a corresponding 'small' subcomplex \( \left( \Omega^*_\text{small}(M,t), D(t) \right) \) which calculates the \( S^1 \)-equivariant cohomology of \( M \).

**Proposition**

\[
\text{Int} : \left( \Omega^*_g(M), D \right) \to (C^*(M,f), \delta)
\]

is a morphism of cochain complexes.

As a consequence, the composition

\[
\left( \Omega^*_g,\text{small}(M,t), D(t) \right) \xrightarrow{(c^t)_*} \left( (c^t)^*_g \Omega^*_g,\text{small}(M,t), D \right) \xrightarrow{\text{Int}} (C^*(M,f), \delta)
\]

is a morphism of cochain complexes. Finally, we can state the main theorem in this section (see §2.4.4):

**Theorem 2** Suppose \( f \) is an self-indexing invariant Morse function such that \((f,g)\) satisfies the Morse-Smale condition. Then there exists a morphism of cochain complexes

\[
F^*(t) = \text{Int}(c^t)_* : \left( \tilde{\Omega}^*_\text{small}(M,t), D(t) \right) \to (C^*(M,f), \delta)
\]

such that w.r.t. some suitably chosen bases (see §2.4.4 for the description of these bases),

\[
F^*(t) = 1 + O(t^{-1}) \tag{2.57}
\]

The following is an outline of this chapter. In §2.2 we prove Theorem 1, and in §2.3, we apply Theorem 1 to obtain \( \text{dim} \tilde{\Omega}^k_{\text{inv,sm}}(M,t) \) as described in the above lemma, hence finish the proof of Morse inequality. In §2.4 we show that \( \left( \tilde{\Omega}^*_\text{inv,sm}(M,t), D(t) \right) \) converges to \( (C^*(M,f), \delta) \) as \( t \to \infty \), i.e. we prove Theorem 2.
2.2 Proof of Theorem 1

We begin by recalling and introducing some notations. We have already defined the Laplacians $\tilde{\Delta}^k(t), \tilde{\Delta}_j^k(t), \overline{K}_j^k$ in §2.1. (Here, the bar above the operator designates an operator on the standard model.) In this section $k$ will be a fixed integer. For simplicity of notation, the superscript $k$ for eigenvalues and eigenvectors will be dropped.

Let

\[
0 < E_1(t) \leq E_2(t) \leq \cdots \leq E_l(t) \leq \cdots \quad \text{be all the eigenvalues of } \tilde{\Delta}^k(t)
\]

\[
\Psi_1(t), \Psi_2(t), \ldots, \Psi_l(t), \ldots \quad \text{be the corresponding normalized eigenvectors}
\]

\[
0 \leq \overline{\nu}_1(t) \leq \overline{\nu}_2(t) \leq \cdots \leq \overline{\nu}_l(t) \leq \cdots \quad \text{be all the eigenvalues of } \bigoplus_{j=1}^r \overline{\Delta}_j^k(t)
\]

\[
0 \leq \overline{\nu}_1 \leq \overline{\nu}_2 \leq \cdots \leq \overline{\nu}_l \leq \cdots \quad \text{be all the eigenvalues of } \bigoplus_{j=1}^r \overline{K}_j^k
\]

**Theorem 1**

\[
\lim_{t \to \infty} \frac{E_l(t)}{t} = \overline{\nu}_l
\]  

**(2.58)**

**Proof** We shall follow the argument of B.Simon (cf.[S] p219-222) and separate the proof into two parts.

In part I we shall prove

\[
\overline{\lim}_{t \to \infty} \frac{E_l(t)}{t} \leq \overline{\nu}_l
\]  

**(2.59)**

and in part II

\[
\underline{\lim}_{t \to \infty} \frac{E_l(t)}{t} \geq \overline{\nu}_l
\]  

**(2.60)**

**Part I:** For any $n$, let $\rho_n \in C^\infty_c(\mathbb{R}^n)$, $0 \leq \rho_n \leq 1$ be such that

\[
\rho_n(x) = \begin{cases} 
1 & \text{if } |x| \leq 1/2 \\
0 & \text{if } |x| \geq 1 
\end{cases}
\]  

**(2.61)**
For any critical orbit $O_j$, $1 \leq j \leq r$, define $J_j \in C^\infty(M)$ by

$$J_j(x) = \begin{cases} 
\rho_{n-1}(x) & \text{if } O_j \cong S^1 \text{ where } (x_1, \ldots, x_{n-1}, \theta) \text{ are the} \\
\rho_n(x) & \text{coordinates in } U_j \text{ as define in §2.1.3(b)}
\end{cases}$$

Define $J_0 = \sqrt{1 - \sum_{j=1}^{r} J_j^2}$.

It is clear that $\overline{\Delta}_j(t)$ acting on $L^2$-forms have discrete spectrum in Cases 1 and $1'$, and hence has a complete orthonormal basis of eigenvectors $\{\overline{\Psi}_{l,j}(t)\}_{l \in \mathbb{N}}$.

In proving Part I, we need several lemmas concerning the operator $\overline{\Delta}_j(t)$ corresponding to a critical fixed point $O_j$, whose proofs will be postponed at the end of this section.

**Lemma 2.2.1** For a critical fixed point $O_j$, The 'localized' operator $\overline{\Delta}_j(t)$ has discrete spectrum and has a complete orthonormal basis of eigenvectors $\{\overline{\Psi}_{l,j}(t)\}_{l \in \mathbb{N}}$ corresponding to the eigenvalues

$$0 \leq \overline{c}_{1,j}(t) \leq \overline{c}_{2,j}(t) \leq \cdots \leq \overline{c}_{l,j}(t) \leq \cdots$$

Let

$$0 \leq \overline{c}_{1,j} \leq \overline{c}_{2,j} \leq \cdots \leq \overline{c}_{l,j} \leq \cdots$$

be all the eigenvalues of $\overline{K}_j$.

**Lemma 2.2.2**

$$\lim_{t \to \infty} \frac{\overline{c}_{l,j}(t)}{t} = \overline{c}_{l,j} \quad \text{(2.62)}$$

In all cases, define $\Phi_{l,j}(t) \in C^\infty(M)$ by

$$\Phi_{l,j}(t) = J_j \overline{\Psi}_{l,j}(t) \quad \text{(2.63)}$$

Note that $\Phi_{l,j}(t)$ is localized at the critical orbit $O_j$. Also, using the identification of $U_j$ with a neighbourhood of zero section of the standard model, $\Phi_{l,j}(t)$ can be
considered as a form in $\hat{\Omega}_{\text{im}}^k(E)$. In the case that the critical orbit $O_j$ is a fixed point, then $E = \mathbb{R}^n$ and $\Phi_{t,j}(t)$ can be considered as a form in $\hat{\Omega}_{\text{im}}^k(\mathbb{R}^n)$.

**Lemma 2.2.3** For a critical fixed point $O_j$,

$$\lim_{t \to \infty} \langle \Phi_{t,j}(t), \Phi_{m,j}(t) \rangle = \delta_{lm}$$

Also we need (cf.[S])

**IMS Localization Formula**

$$J_j \tilde{\Delta}^k(t)J_j = \frac{1}{2}(J_j^2 \tilde{\Delta}^k(t) + \tilde{\Delta}^k(t)J_j^2) + |dJ_j|^2$$

Hence

$$\tilde{\Delta}^k(t) = \sum_{j=0}^{r} J_j \tilde{\Delta}^k(t)J_j - \sum_{j=0}^{r} |dJ_j|^2$$

The proofs of the above lemmas will be postponed at the end of this section.

Based on the above lemmas, we establish

**Proposition 2.2.4** For any critical orbit $O_j$, suppose $\overline{\Psi}_{l,j}, \overline{\Psi}_{m,j}$ are two eigenvectors of $\overline{\Delta}^k_j(t)$, then

$$\langle \Phi_{l,j}(t), \overline{\Delta}^k_j(t)\Phi_{m,j}(t) \rangle = \overline{\epsilon}_{l,j}(t)\delta_{lm} + o(t)$$

**Proof:** In Cases 1 and 1', we have (cf.[S])

$$\lim_{t \to \infty} \langle \Phi_{l,j}(t), \Phi_{m,j}(t) \rangle = \delta_{lm}$$

By Lemma 2.2.3, we have in all cases,

$$\langle \Phi_{l,j}(t), \Phi_{m,j}(t) \rangle = \delta_{lm} + \epsilon_{lm}(t)$$
where $\lim_{t \to \infty} \epsilon_{lm}(t) = 0$

$$< \Phi_{l,j}(t), \overline{\Delta}_j^k(t) \Phi_{m,j}(t) > = < \overline{\Psi}_{l,j}(t), J_j \overline{\Delta}_j^k(t) J_j \overline{\Psi}_{m,j}(t) >$$

$$= \frac{1}{2} < \overline{\Psi}_{l,j}(t), \left( J_j^2 \overline{\Delta}_j^k(t) + \overline{\Delta}_j^k(t) J_j^2 \right) \overline{\Psi}_{m,j}(t) >$$

$$+ < \overline{\Psi}_{l,j}(t), |dJ_j|^2 \overline{\Psi}_{m,j}(t) >$$

(by IMS Localization Formula)

$$= \frac{1}{2} (\overline{r}_{l,j}(t) + \overline{r}_{m,j}(t)) < \Phi_{l,j}(t), \Phi_{m,j}(t) > + O(1)$$

$$= \frac{1}{2} (\overline{r}_{l,j}(t) + \overline{r}_{m,j}(t)) \delta_{lm} + \frac{1}{2} (\overline{r}_{l,j}(t) + \overline{r}_{m,j}(t)) \epsilon_{lm}(t) + o(t)$$

(2.70)

By Lemma 2.2.2,

$$\lim_{t \to \infty} \frac{1}{2} \frac{(\overline{r}_{l,j}(t) + \overline{r}_{m,j}(t)) \epsilon_{lm}(t)}{t} = \frac{1}{2} \lim_{t \to \infty} \left( \frac{\overline{r}_{l,j}(t)}{t} + \frac{\overline{r}_{m,j}(t)}{t} \right) \lim_{t \to \infty} \epsilon_{lm}(t) = 0$$

(2.71)

This proves the Proposition. □

Now let $\{ \overline{\Psi}_l(t) \}_{t \in \mathbb{N}}$ be the eigenvectors corresponding to the eigenvalues $\{ \overline{r}_l(t) \}_{t \in \mathbb{N}}$

Then there exist $j_i, n_i$ so that

$$\overline{\Psi}_l(t) = \overline{\Psi}_{n_i,j_i}$$

(2.72)

Define $\Phi_l(t) = J_{j_i} \overline{\Psi}_{n_i,j_i}(t)$

The above Proposition leads to

Proposition 2.2.5

$$< \Phi_l(t), \overline{\Delta}_l^k(t) \Phi_m(t) >= \overline{r}_l(t) \delta_{lm} + o(t)$$

(2.73)

Proof: If $j_i \neq j_m$, then for sufficient large $t$, $\Phi_l(t), \Phi_m(t)$ have disjoint support.

Hence

$$< \Phi_l(t), \overline{\Delta}_l^k(t) \Phi_m(t) >= 0$$

(2.74)

If $j_i = j_m = j$, then for sufficient large $t$, $\Phi_l(t), \Phi_m(t)$ have support in $U_j$ and since
\[ \Delta^k(t), \Delta_j^k(t) \text{ agree on } U_j, \text{ we have} \]
\[ < \Phi(t), \Delta^k(t)\Phi(t) > = < \Phi_{n_j}(t), \Delta_j^k(t)\Phi_{n_j}(t) > \]
\[ = \bar{r}_{n_j}(t)\delta_{nm} + o(t) \text{ by Proposition 2.2.4} \quad (2.75) \]
\[ = \bar{r}_l(t)\delta_{nm} + o(t) \]

This proves the Proposition. \( \square \)

Since \( \Delta^k(t) \) operates on forms on a compact manifold, it has a discrete spectrum.

The min-max principle, Proposition 2.2.5 and Lemma 2.2.2 imply
\[ \lim_{t \to \infty} \frac{E(t)}{t} \leq \lim_{t \to \infty} \frac{\bar{r}_l(t)}{t} = \bar{r}_l \quad (2.76) \]

Part II:
\[ \lim_{t \to \infty} \frac{E(t)}{t} \geq \bar{r}_l \quad (2.77) \]

Proof: To prove (2.77), it suffices to show that for any \( c \in (\bar{r}_l, \bar{r}_{l+1}) \), there exists a symmetric operator \( R(t) \) of rank at most \( l \) such that
\[ \bar{\Delta}^k(t) \geq tc + R(t) + o(t) \quad (2.78) \]

If such operator exists, in order to derive (2.77) from (2.78), choose \( 0 \neq \Psi \in \hat{\Omega}_{inv}(M) \) such that
\[ \Psi(t) \in \text{Span}\{\Psi_1(t), \ldots, \Psi_{l+1}(t)\} \cap \ker R(t) \text{ and } ||\Psi(t)|| = 1 \]

Then
\[ (2.78) \Rightarrow < \Psi(t), \bar{\Delta}^k(t)\Psi(t) > \geq tc + o(t) \]
\[ \Rightarrow E_{l+1}(t) \geq < \Psi(t), \bar{\Delta}^k(t)\Psi(t) > \geq tc + o(t) \]
\[ \Rightarrow \lim_{t \to \infty} \frac{E_{l+1}(t)}{t} \geq c \quad \forall c \in (\bar{r}_l, \bar{r}_{l+1}) \]
\[ \Rightarrow \lim_{t \to \infty} \frac{E_{l+1}(t)}{t} \geq \bar{r}_{l+1} \]

To construct \( R(t) \), let \( f_j \) be an invariant function on \( M \) such that
\[
\begin{cases}
  f_j(x) = ||df(x)||^2 & \text{if } x \in U_j \\
  f_j(x) \geq c_j > 0 & \text{if } x \text{ is outside } U_j
\end{cases}
\]
Define $\tilde{\hat{\Delta}}_j^k(t) : \tilde{\Omega}_{inv}^k(M) \to \tilde{\Omega}_{inv}^k(M)$ by

$$\tilde{\hat{\Delta}}_j^k(t) = dd^* + d^*d + t^2f_j + tA + (i_X^*f_j + i_X^*i_X^*) + (i_X^*d^* + di_X^*) + (i_X^*d + di_X^*) \quad (2.79)$$

Observe that $\tilde{\hat{\Delta}}_j^k(t) = \hat{\Delta}^k(t)$ on $U_j$.

In order to show that $R(t)$ has rank at most $l$, we need

**Lemma 2.2.6**

$$\lim_{t \to \infty} \frac{E_{l,j}(t)}{t} = \bar{r}_{l,j} \quad (2.80)$$

where

$$0 \leq E_{l,j}(t) \leq E_{2,j}(t) \leq \cdots \leq E_{l,j}(t) \leq \cdots$$

are all the eigenvalues of $\tilde{\hat{\Delta}}_j^k(t)$.

**Proof of lemma 2.2.6:** By the proof of Part I,

$$\lim_{t \to \infty} \frac{E_{l,j}(t)}{t} \leq \bar{r}_{l,j} \quad (2.81)$$

Therefore, it suffices to show

$$\lim_{t \to \infty} \frac{E_{l,j}(t)}{t} \geq \bar{r}_l \quad (2.82)$$

Let $\Psi_{l,j}(t)$ be a normalized eigenvector of $\tilde{\hat{\Delta}}_j^k(t)$ corresponding to the eigenvalues $E_{l,j}(t)$.

Recall that

$$U_j \cong \begin{cases} D^{n-1} \times_{G_j} S^1 & \text{if } O_j \cong S^1 \\ D^n & \text{if } O_j \text{ is a critical fixed point} \end{cases}$$

Define

$$U'_j \cong \begin{cases} D^{n-1}(1/2) \times_{G_j} S^1 & \text{if } O_j \cong S^1 \\ D^n(1/2) & \text{if } O_j \text{ is a critical fixed point} \end{cases}$$
where $D^n(1/2)$ is the disc of radius $1/2$ in $\mathbb{R}^n$.

**Claim 1:**

$$\lim_{t \to \infty} \|\Psi_{t,j}(t)\|_{M^\prime_1} = 0 \quad (2.83)$$

Proof of Claim 1: Suppose the above is false, then there exists $\epsilon > 0, \{t_n\}_{n \in N}$ with $t_n \not\to \infty$ such that

$$< \Psi_{t,j}(t_n), \Psi_{t,j}(t_n) >_{M^\prime_1} \geq \epsilon \quad (2.84)$$

Since $\tilde{\Delta}_j(t)\Psi_{t,j}(t) = E_{t,j}(t)\Psi_{t,j}(t)$, we have

$$E_{t,j}(t) < \Psi_{t,j}(t), \Psi_{t,j}(t) > = < (dd^* + d^*d)\Psi_{t,j}(t), \Psi_{t,j}(t) > + t^2 < f_j\Psi_{t,j}(t), \Psi_{t,j}(t) > + < (tA + (B + B^*) + i_X i_X^* + i_X^* i_X)\Psi_{t,j}(t), \Psi_{t,j}(t) > \quad (2.85)$$

where $B = i_X^* d + d i_X^*$.

Since $dd^* + d^*d \geq 0$ and $< f_j\Psi_{t,j}(t), \Psi_{t,j}(t) > \geq c_j\|\Psi_{t,j}(t)\|_{M^\prime_1}^2$, therefore

$$E_{t,j}(t)\|\Psi_{t,j}(t)\|^2 \geq c_j t^2\|\Psi_{t,j}(t)\|_{M^\prime_1}^2 + < (tA + (B + B^*) + i_X i_X^* + i_X^* i_X)\Psi_{t,j}(t), \Psi_{t,j}(t) > \quad (2.86)$$

It is clear that $A, i_X i_X^* + i_X^* i_X$ are bounded operators. It will be shown later in this section in Lemma 2.2.0 that $B$ and $B^*$ are also bounded. Hence,

$$E_{t,j}(t)\|\Psi_{t,j}(t)\|^2 \geq c_j t^2\|\Psi_{t,j}(t)\|_{M^\prime_1}^2 - (t\|A\| + \|B + B^* + i_X i_X^* + i_X^* i_X\|)\|\Psi_{t,j}(t)\|^2 \quad (2.87)$$

Since $\|\Psi_{t,j}(t_n)\|_{M^\prime_1}^2 \geq \epsilon > 0$,

$$\frac{E_{t,j}(t_n)}{t_n} \geq c_j t_n \epsilon - \frac{1}{t_n}\|B + B^* + i_X i_X^* + i_X^* i_X\| \quad (2.88)$$

This contradicts

$$\lim_{n \to \infty} \frac{E_{t,j}(t_n)}{t_n} \leq \tilde{c}_{t,j} \quad (2.89)$$
Claim 1 is then proved. □

Recall that \( J_j \) has support in \( U_j \) and it is equal to 1 on \( U_j' \)

Define \( \phi_{l,j}(t) = J_j \Psi_{l,j}(t) \)

**Claim 2:**

\[
\lim_{t \to \infty} < \phi_{l,j}(t), \phi_{m,j}(t) > = \delta_{lm}
\]  

(2.90)

Proof of Claim 2:

\[
< \phi_{l,j}(t), \phi_{m,j}(t) > = \int_{U_j'} \phi_{l,j}(t) \wedge \phi_{m,j}(t) + \int_{M U_j'} \phi_{l,j}(t) \wedge \phi_{m,j}(t)
\]

\[
\leq \int_{U_j'} \Psi_{l,j}(t) \wedge \Psi_{m,j}(t) + \| \phi_{l,j}(t) \|_{M U_j'} \| \phi_{m,j}(t) \|_{M U_j'}
\]

\[
= < \Psi_{l,j}(t), \Psi_{m,j}(t) > - < \Psi_{l,j}(t), \Psi_{m,j}(t) >_{M U_j'}
\]

\[
+ \| J_j \Psi_{l,j}(t) \|_{M U_j'} \| J_j \Psi_{m,j}(t) \|_{M U_j'}
\]

\[
\leq \delta_{lm} + 2 \| \Psi_{l,j}(t) \|_{M U_j'} \| \Psi_{m,j}(t) \|_{M U_j'}
\]

(2.91)

Similarly, one shows that

\[
< \phi_{l,j}(t), \phi_{m,j}(t) > \geq \delta_{lm} - 2 \| \Psi_{l,j}(t) \|_{M U_j'} \| \Psi_{m,j}(t) \|_{M U_j'}
\]  

(2.92)

Claim 2 then follows from Claim 1. □

Observe that since \( \phi_{l,j}(t) \) has support in \( U_j \) it can be regarded as a form on the standard model.

**Claim 3:**

\[
< \phi_{l,j}(t), \Delta_j(t) \phi_{m,j}(t) > = E_{l,j}(t) \delta_{lm} + o(t)
\]  

(2.93)

Proof of Claim 3:

\[
< \phi_{l,j}(t), \Delta_j(t) \phi_{m,j}(t) > =< \phi_{l,j}(t), \Delta_j(t) \phi_{m,j}(t) >
\]

\[
= < \Psi_{l,j}(t), J_j \Delta_j(t) \Psi_{m,j}(t) >
\]

\[
= \frac{1}{2} < \Psi_{l,j}(t), (J_j^2 \Delta_j(t) + \Delta_j(t) J_j^2) \Psi_{m,j}(t) >
\]

\[
+ < \Psi_{l,j}(t), \| d J_j \|_{M U_j} \Psi_{m,j}(t) >
\]

\[
= \frac{1}{2} (E_{l,j}(t) + E_{m,j}(t)) < \phi_{l,j}(t), \phi_{m,j}(t) > + O(1)
\]

\[
= E_{l,j}(t) \delta_{lm} + \frac{1}{2} (E_{l,j}(t) + E_{m,j}(t)) \epsilon(t) + o(t)
\]

(2.94)
where \( \lim_{t \to \infty} \epsilon(t) = 0 \).

Since \( \lim_{t \to \infty} \frac{E_{t,j}(t)}{t} \leq \bar{r}_{t,j} \), Claim 3 is proved. \( \square \)

By Lemma 2.2.1, \( \Delta_j(t) \) has discrete spectrum. Lemma 2.2.2 together with the min-max principle imply

\[
\bar{r}_{t,j} = \lim_{t \to \infty} \frac{\bar{r}_{t,j}(t)}{t} \leq \lim_{t \to \infty} \frac{E_{t,j}(t)}{t} \quad (2.95)
\]

which in turn proves Lemma 2.2.6. \( \square \)

Now we show (2.78). By IMS Localization Formula,

\[
\hat{\Delta}^k(t) = \sum_{j=0}^{r} J_j \hat{\Delta}^k(t) J_j - \sum_{j=0}^{r} \|dJ_j\|^2 = J_0 \hat{\Delta}^k(t) J_0 + \sum_{j=1}^{r} J_j \hat{\Delta}^k(t) J_j + O(1) \quad (2.96)
\]

Denote \( d + i_{X}d \) by \( B \). Since \( dd^* + d^*d \) and \( i_{X}d \) are positive, we have

\[
J_0 \hat{\Delta}^k(t) J_0 \geq J_0 \left( t^2 \|d\|^2 + t^2 A + B + B^* \right) J_0 \quad (2.97)
\]

Also since \( J_0 \) has support away from \( \cup_{j=1}^{r} U_j \),

\[
|df(x)|^2 \geq \epsilon \quad \text{for some } \epsilon > 0 \text{ on support of } J_0
\]

Hence,

\[
J_0 \hat{\Delta}^k(t) J_0 \geq J_0 \left( ct^2 + t^2 A + B + B^* \right) J_0 \\
\geq tcJ_0^2 \left( ct + A + \frac{1}{t}(B + B^*) - c \right) J_0 \\
\geq tcJ_0^2 \text{ for sufficiently large } t \quad (2.98)
\]

Since \( \Delta^k_j(t) = \hat{\Delta}^k(t) \) on \( U_j \),

\[
\hat{\Delta}^k(t) \geq tcJ_0^2 + \sum_{j=1}^{r} J_j \Delta^k_j(t) J_j + O(1) \quad (2.99)
\]

Recall that \( E_{t,j}(t) \) are eigenvalues of \( \hat{\Delta}^k(t) \) with corresponding eigenvector \( \Psi_{t,j}(t) \). Note that in general \( E_{t,j}(t) \neq E_m(t) \) for any \( l, m \).
For any $j$, define $l_j$ as follows. Recall that

$$\lim_{t \to \infty} \frac{E_{l_j}(t)}{t} = \lim_{t \to \infty} \frac{\bar{r}_{l_j}(t)}{t} = \bar{r}_{l_j}$$

(2.100)

$l_j$ is chosen such that

$$\bar{r}_{l_j} < c < \bar{r}_{l_j+1}$$

Define $R_j(t) : \tilde{\Omega}_r^k(M) \to \tilde{\Omega}_r^k(M)$ such that on $\text{Span}\{\Psi_{l_j}(t)\}_{1 \leq t \leq l_j}$ and w.r.t. the basis $\{\Psi_{l_j}(t)\}_{1 \leq t \leq l_j}$,

$$R_j(t) = \begin{pmatrix} E_{l_j}(t) - tc & 0 & \cdots & 0 \\ 0 & E_{l_j}(t) - tc & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{l_j}(t) - tc \end{pmatrix}$$

(2.101)

and on $\left(\text{Span}\{\Psi_{l_j}(t)\}_{1 \leq t \leq l_j}\right)^\perp$, $R_j(t)$ is zero.

Then $\text{Rank} \ R_j(t) = l_j$ and $\Delta_j^k(t) \geq R_j(t) + tc$ for sufficiently large $t$.

Therefore,

$$\Delta_j^k(t) \geq tcJ_j^2 + \sum_{j=1}^r J_j R_j(t)J_j + tc \sum_{j=1}^r J_j^2 + O(1)$$

(2.102)

where

$$R(t) = \sum_{j=1}^r J_j R_j(t)J_j$$

(2.103)

Then $\text{Rank} \ R(t) \leq \sum_{j=1}^r \text{Rank} \ R_j(t) = \sum_{j=1}^r l_j = l$ and this proves (2.77).

This completes the proof of Theorem 1.

**Proof of Lemmas**

**Lemma 2.2.0** $B = i_X d + d i_X$ is a zero order operator. Hence it is a bounded operator in
(i) $L^2 \left( \hat{\Omega}^k_{inv}(M) \right)$ where $M$ is a compact $S^1$-manifold.

(ii) $L^2 \left( \hat{\Omega}^k_{inv}(\mathbb{R}^n) \right)$ where $\mathbb{R}^n$ is the standard model associated to a critical fixed point.

where $L^2(H)$ denotes the $L^2$ completion of the space $H$.

**Proof** Let $(x_1, \ldots, x_n)$ be a coordinate system on a manifold, $X = \sum_{i=1}^n X_i \partial_i$.

$X^*$ be the dual of $X$ and suppose that $X^* = \sum_{i=1}^n X_i^* dx_i$.

Case 1: $l < k$

We have

$$i_X^* d(f dx_{i_1} \wedge \cdots \wedge dx_{i_l}) = \left( \sum_{i,j} X_j^* \frac{\partial}{\partial x_j} dx_i \wedge dx_j \right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_l}$$

$$di_X^* (f dx_{i_1} \wedge \cdots \wedge dx_{i_l}) = - \left( \sum_{i,j} X_j^* \frac{\partial}{\partial x_j} dx_i \wedge dx_j \right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_l}$$

Therefore,

$$(i_X^* d + di_X^*)(f dx_{i_1} \wedge \cdots \wedge dx_{i_l}) = ext \left( - \sum_{i,j} \frac{\partial X_j^*}{\partial x_j} dx_i \wedge dx_j \right) (f dx_{i_1} \wedge \cdots \wedge dx_{i_l}) \quad (2.105)$$

where $ext$ denotes exterior multiplication.

Hence, for $\omega \in \hat{\Omega}^k_{inv}(M)$,

$$(i_X^* d + di_X^*)(\omega) = ext \left( - \sum_{i,j} \frac{\partial X_j^*}{\partial x_j} dx_i \wedge dx_j \right) (\omega) \quad (2.106)$$

Case 2: For $\omega \in \hat{\Omega}^k_{inv}(M)$,

$$(i_X^* d + di_X^*)(\omega) = 0 \quad (2.107)$$

(i) Since $B$ is a multiplication operator on a compact manifold, it is bounded in $L^2(\hat{\Omega}^k_{inv}(M))$.

(ii) On the standard model and using the canonical metric on $\mathbb{R}^n$,

$$\begin{cases}
X = (-m_1 x_2, m_1 x_1, -m_2 x_4, m_2 x_3, \cdots, -m_q x_{2q}, m_q x_{2q-1}, 0, \cdots, 0) \text{ for some } q \\
X_i^* = X_i
\end{cases}$$
From the above expression for $B$, it is clear that $B$ is bounded in $L_2(\Omega^{k}_{\text{inv}}(R^n))$. □

**Lemma 2.2.1**

$$\overline{\Delta}_j(t) = dd^* + d^*d + 4t^2x^2 + tA + (iX_iX + iX^*_iX^*) + (iX^*_iA + d^*_ix_iX^*) + (iX^*_iX + d^*_ix_iX)$$

has a complete orthonormal basis $\{\overline{\Psi}_{l,j}(t)\}_{l \in \mathbb{N}}$ of eigenvectors corresponding to the eigenvalues

$$\overline{\tau}_{1,j}(t) \leq \overline{\tau}_{2,j}(t) \leq \cdots \leq \overline{\tau}_{l,j}(t) \leq \cdots$$

with $\lim_{t \to \infty} \overline{\tau}_{l,j}(t) = \infty$

**Proof** Observe that $iX_iX + iX^*_iX^* = |X|^2 = \sum_{l=1}^n m_l^2(x_{2l-1}^2 + x_{2l}^2)$. Then,

$$L(t) = \left( L(t) - \lambda \right)^{-1}$$

is compact $\forall \lambda \in \rho(L(t))$

By Lemma 2.2.0, $\overline{\Delta}_j(t)$ is a perturbation of $L(t)$ by a bounded operator.

Since $L(t)$ is a Quantum harmonic oscillator, it has a compact resolvent, i.e.

$$(L(t) - \lambda)^{-1}$$

is compact $\forall \lambda \in \rho(L(t))$

such that $| \lambda_0 | > \| B + B^* \|$, then

$$\left( \overline{\Delta}_j(t) - i\lambda_0 \right)^{-1} = ((L(t) + B + B^*) - i\lambda_0)^{-1}$$

exists and is compact. Lemma 2.2.1 follows in view of the following Theorem.

**Theorem:** Suppose $A$ is a self-adjoint operator in a Hilbert space $H$ that is bounded from below (i.e. $A \geq cI$ for some $c \in R$), then the following are equivalent.

(i) $(A - \lambda)^{-1}$ is compact for some $\lambda \in \rho(A)$. 

(ii) There exists a complete orthonormal basis \( \{ \varphi_n \}_{n \in \mathbb{N}} \subset D(A) \) such that \( A \varphi_n = \lambda_n \varphi_n \) with
\[
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_t \leq \cdots
\]
and \( \lim_{n \to \infty} \lambda_t = \infty \).

**Lemma 2.2.2**

\[
\lim_{t \to \infty} \frac{t_{i,j}(t)}{t} = \bar{t}_{i,j}
\]

**Proof** Define

\[
\left( \mathcal{U}(\lambda) \right)(x) = \lambda^{n/2} \omega(\lambda x)
\]

Recall that

\[
\bar{P}_j(t) = dd^* + d^*d + 4t^2 \epsilon_x^2 + tA
\]

and \( \bar{K}_j = \bar{P}_j(1) \)

\[
\epsilon : \hat{\Omega}_{inv}^k(\mathbb{R}^n) \to \hat{\Omega}_{inv}^k(\mathbb{R}^n)
\]

such that

\[
\epsilon(\omega) = \begin{cases} 
0 & \text{if } \omega \in \Omega_{inv}^k(\mathbb{R}^n) \\
\omega & \text{if } \omega \in \hat{\Omega}_{inv}^{k-1}(\mathbb{R}^n)
\end{cases}
\]

A direct computation shows that

\[
\bar{\Delta}_j(t) = \mathcal{U}(\lambda)(t^{1/2}) t \left[ \bar{K}_j + \frac{1}{t^2} \epsilon_x^2 + \frac{1}{t} (i_X^*d + di_X^* + d^*i_X + d^*i_X) \right] \mathcal{U}(\lambda)(t^{-1/2})
\]

Let \( \beta = \frac{1}{t} \), then \( t \to \infty \Rightarrow \beta \to 0^+ \).

Let

\[
T(\beta) = \bar{K}_j + \beta^2 \epsilon_x^2 + \beta(i_X^*d + di_X^* + d^*i_X + d^*i_X)
\]

for \( \beta \in \mathbb{R} \subset \mathbb{C} \) such that \( \mathbb{R}^+ \cup \{0\} \subset \mathbb{R} \). Note that for \( \beta \in \mathbb{R} \), \( T(\beta) \) is self-adjoint.
Our strategy is to think of $T(\beta)$ as a perturbation of $K_j^k$ and apply the analytic perturbation theory to study the asymptotic behavior of the spectrum of $\Delta_j^k(t)$. (For introduction and proofs of the following statements, see [RS],[K])

Definition An operator-valued function $T(\beta)$ defined on a complex domain $\mathbb{R}$ is called an analytic family in the sense of Kato iff

(i) $\forall \beta \in \mathbb{R}$, $T(\beta)$ is closed and $\rho(T(\beta)) \neq \emptyset$ where $\rho(T(\beta))$ is the resolvent set of $T(\beta)$.

(ii) For any $\beta_0 \in \mathbb{R}$, there exists a $\lambda_0 \in \rho(T(\beta_0))$ such that $\lambda_0 \in \rho(T(\beta))$ for $\beta$ near $\beta_0$, and $(T(\beta) - \lambda_0)^{-1}$ is an analytic (i.e. holomorphic) operator-valued function of $\beta$ near $\beta_0$.

Theorem (Kato-Rellich) (cf.[RS] p22) Let $T(\beta)$ be an analytic family in the sense of Kato that is self-adjoint for $\beta$ real. Let $E_0$ be a discrete eigenvalue of $T(\beta_0)$ of multiplicity $m$. Then for $\beta$ near $\beta_0$, there exists $m$ not necessarily distinct single-valued functions, analytic near $\beta_0$, $E^{(1)}(\beta), \ldots, E^{(m)}(\beta)$ of eigenvalues of $T(\beta)$ near $\beta_0$ with $E^{(i)}(\beta_0) = E_0$. Also these are all the eigenvalues near $E_0$.

Suppose that we have shown that $T(\beta)$ is an analytic family in the sense of Kato. By the above Theorem of Kato-Rellich, for any $l \geq 1$, there exists $l'$ with $l \leq l'$ s.t.

$$\lim_{t \to \infty} \frac{\bar{e}_{l, j}^{(i)}(t)}{t} = \bar{e}_{l, j}$$

(2.114)

for some $l' \leq l$. In fact, one can show that

$$\lim_{t \to \infty} \frac{\bar{e}_{l, j}(t)}{t} = \bar{e}_{l, j}$$

(2.115)

In order to show that $T(\beta)$ is an analytic family in the sense of Kato, we have to
introduce a number of definitions.

Let \( t \) be a sesqui-linear form defined on a dense domain \( D(t) \) in a Hilbert space \( H \). Let \( t(u, u) \), \( u \in D(t) \) be the quadratic form associated with \( t(u, v) \). Define the numerical range of \( t \) to be the set

\[
\Theta(t) = \{ t(u) \mid u \in D(t), \|u\| = 1 \}
\]

Note that \( \Theta(t) \) is always a convex subset of \( \mathbb{C} \).

1. A quadratic form \( t \) is said to be sectorial if

\[
\Theta(t) \subset S_{r, \theta} = \{ z \in \mathbb{C} \mid |\arg(z - r)| \leq \theta \}
\]

where \( r \in \mathbb{R}, \theta < \frac{\pi}{2} \).

Example: A positive symmetric (quadratic) form is sectorial.

2. A sequence \( \{u_n\}_{n \in \mathbb{N}} \subset D(t) \) is \( t \)-convergent to \( u \in H \) denoted by

\[
u_n \overset{t}{\to} u \quad (\text{as } n \to \infty)
\]

if (i) \( u_n \to u \) in \( H \).

(ii) \( t(u_n - u_m) \to 0 \) as \( n, m \to \infty \).

3. A sectorial form \( t \) is said to be closed if

\[
u_n \overset{t}{\to} u \Rightarrow u \in D(t) \text{ and } t(u_n - u) \to 0
\]

4. A sectorial form is said to be closable if it has a closed extension.

Let \( T \) be an (unbounded) operator in \( H \) with a dense domain \( D(T) \). Let \( t = t(T) \) be the associated sesqui-linear form defined on \( D(T) \times D(T) \) such that

\[
t(u, v) = (Tu, v) \quad u, v \in D(T)
\]
Definition  $T$ is called sectorial if $t$ is sectorial.

It is shown in [K] that a sectorial operator $T$ is form-closable (i.e. $t(T)$ is closable). Hence, a symmetric operator bounded from below is form-closable.

Definition (b) A family $\{t(\beta)\}_{\beta \in R}$ of sesquilinear forms in $H$ is called an analytic family of type (b) if

(i) $t(\beta)$ is sectorial and closed with $D(t(\beta)) = D$ which is dense in $H$.

(ii) $t(\beta)(u)$ is analytic in $\beta$ for each fixed $u \in D$.

Representation Theorem (cf. [K] p. 322-3)

(i) Let $t(u, v)$ be a densely defined, closed, sectorial sesquilinear in $H$, then there exists a unique (m-)sectorial operator $T$ such that

$$D(T) \subset D(t) \text{ and } t(u, v) = (Tu, v) \quad \forall u \in D(T), v \in D(t)$$

(ii) $t \mapsto T(t)$ defined in (i) is a 1-1 correspondence between the set of densely defined, closed, sectorial forms and the set of all (m-)sectorial (hence closed) operators.

$T$ is self-adjoint iff $t$ is symmetric.

Definition (B) A family of operators $\{T(\beta)\}_{\beta \in R}$ is called an analytic family of type (B) if it is the associated family of (m)-sectorial operators to an analytic family $\{t(\beta)\}_{\beta \in R}$ of sesquilinear forms of type (b) given by the Representation Theorem.

Proposition If $\{T(\beta)\}_{\beta \in R}$ is an analytic family of type (B), then it is an analytic family in the sense of Kato.

In order to show our family

$$T(\beta) = \overline{K}_j^* + \beta^2 \epsilon X^2 + \beta(d^*_X + i^*_X d + i_X d^* + d^* i_X)$$  \hspace{1cm} (2.116)
of symmetric operators can be extended to an analytic family \( \{ T(\beta) \}_{\beta \in \mathbb{R}} \) in the sense of Kato, which is a particular family of closed operators, we will show that the associated family of forms \( \{ t(\beta) \}_{\beta \in \mathbb{R}} \), where \( t(\beta)(u,v) = (T(\beta)u,v) \) \( u,v \in D(T(\beta)) \), can be extended to an analytic family \( \{ \tilde{t}(\beta) \}_{\beta \in \mathbb{R}} \) of type (b). Hence, by the Representation Theorem, the associated family \( \{ T(\beta) \}_{\beta \in \mathbb{R}} \) of self-adjoint operators with domain \( D(T(\beta)) \) is an analytic family of type (B), hence an analytic family in the sense of Kato.

Now define \( t(\beta)(u,v) = (T(\beta)u,v) \) \( u,v \in D(T(\beta)) \)

Let

\[
T(\beta) = T^{(0)} + \beta T^{(1)} + \beta^2 T^{(2)}
\]

i.e.

\[
\begin{align*}
T^{(0)} &= \overset{k}{K}_j \\
T^{(1)} &= d_i X + i_i X d + d_i^* i X + i_i X d^* \\
T^{(2)} &= \epsilon X^2
\end{align*}
\]

Then

\[
t(\beta)(u,v) = (T^{(0)}u,v) + \beta (T^{(1)}u,v) + \beta^2 (T^{(2)}u,v)
\]

Choose \( D = D(T^{(0)}) = D(\overset{k}{K}_j) \) such that \( D(T^{(1)}), D(T^{(2)}) \supset D \) with \( D \) dense in \( H \).

Observe that \( T^{(0)} \) is densely defined, symmetric, bounded from below, hence form-closable. Therefore \( t^{(0)} \) is densely defined, sectorial and closable. To show that its closure \( \{ \tilde{t}(\beta) \} \) is an analytic family of type (b), we need the following result.

**Theorem** Let \( \{ t^{(n)} \}_{n \in \mathbb{N} \cup \{0\}} \) be a sequence of sesqui-linear forms in \( H \). Let \( t^{(0)} \) be densely defined, \( D(t^{(0)}) = D \), sectorial and closable. Let \( t^{(n)} \) be relatively bounded
w.r.t. $t^{(0)}$ i.e.

$$D(t^{(m)}) \supset D(t^{(0)}) \text{ and } |t^{(n)}(u)| \leq a_n\|u\|^2 + b_n |t^{(0)}(u)| \quad \forall u \in D(t^{(0)})$$

and

$$|t^{(n)}(u)| \leq e^{n-1}(a\|u\|^2 + bRe t^{(0)}(u)) \text{ for some } a, b, c > 0 \quad (2.120)$$

Then the form

$$t(\beta)(u) := \sum_{n=0}^{\infty} \beta^n t^{(n)}(u), \quad D(t(\beta)) = D$$

is defined for $|\beta| < \frac{1}{b+c}$, sectorial and closable for $|\beta| < \frac{1}{b+c}$.

Let $\tilde{t}(\beta)$ be the closure (i.e. the smallest closed extension) of $t(\beta)$. Then $\{\tilde{t}(\beta)\}_{\beta \in \mathbb{H}}$ is an analytic family of type (b) for $|\beta| < \frac{1}{b+c}$ with $D(\tilde{t}(\beta)) = D(\tilde{t}^{(0)})$.

To finish the proof that our family $\{t(\beta)\}$ can be extended to $\{\tilde{t}(\beta)\}$ an analytic family of type (b), it suffices to prove (2.120) for $n=1,2$.

$$|t^{(1)}(u)| = |(T^{(1)}u, u)| \leq \|T^{(1)}\|\|u\|^2$$

$$|t^{(2)}(u)| \leq C(x^2 u, u) = C(\langle \Delta + x^2 + A \rangle u, u) = C(\langle T^{(0)}(u, u) + \|A\|\|u\|^2)$$

Hence, we proved

**Proposition 2.2.7** The family

$$T(\beta) = K^\delta_j + \beta^2 eX^2 + \beta(d^*i_X + i_Xd^* + di_X^* + i_X^*d)$$

is an analytic family in the sense of Kato.

Therefore, we have proved that for any $l \geq 1$, there exists $l'$ with $l \leq l'$ s.t.

$$\lim_{t \to \infty} \frac{\tilde{v}_{\ell,j}(t)}{t} = \tilde{v}_{\ell,j} \quad (2.121)$$
To show that

$$\lim_{t \to \infty} \frac{\bar{v}_{l,j}(t)}{t} = \bar{v}_{l,j}$$  \hspace{1cm} (2.122)

Let $T'(\beta) = \overline{K}_j + \beta^2 \epsilon X^2$ and $B_1 = i_X^* d + d i_X^* + i_X d^* + d^* i_X$.

Then

$$T(\beta) = T'(\beta) + \beta B_1$$  \hspace{1cm} (2.123)

Since $T'(\beta)$ is a family of harmonic oscillator operators, it has eigenvalues

$$0 \leq \lambda_{1,j}(\beta) \leq \lambda_{2,j}(\beta) \leq \cdots \leq \lambda_{l,j}(\beta) \leq \cdots$$

satisfying

$$\lim_{\beta \to 0} \lambda_{l,j}(\beta) = \bar{v}_{l,j}$$  \hspace{1cm} (2.124)

Since $T(\beta) = T'(\beta) + \beta B_1$ and $B_1$ is bounded, by the min-max principle

$$\lim_{t \to \infty} \frac{\bar{v}_{l,j}(t)}{t} = \bar{v}_{l,j}$$  \hspace{1cm} (2.125)

This completes the proof of Lemma 2.2.2. \hspace{1cm} □

**Lemma 2.2.3** For a critical fixed point $O_j$, $\Phi_{l,j}(t) = J_j \Psi_{l,j}(t)$.

$$\lim_{t \to \infty} < \Phi_{l,j}(t), \Phi_{m,j}(t) > = \delta_{lm}$$  \hspace{1cm} (2.126)

**Proof:** Recall that

$$\overline{\Delta_j}(t) = U^{(n)}(t^{1/2}) t \left[ \overline{K}_j + \frac{1}{2} \epsilon X^2 + \frac{1}{4} (i_X^* d + d i_X^* + i_X d^* + d^* i_X) \right] U^{(n)}(t^{-1/2})$$

$$= U^{(n)}(t^{1/2}) \left[ t \frac{1}{4} \left[ T(\frac{1}{t}) \right] \right] U^{(n)}(t^{-1/2})$$

$$T(\beta) = \overline{K}_j + \beta^2 \epsilon X^2 + \beta (i_X^* d + d i_X^* + i_X d^* + d^* i_X)$$  \hspace{1cm} (2.127)

Let $\{ \Gamma_i(\beta) \}_{i \in N}$ be the eigenvectors of $T(\beta)$.
Define a 2-parameter family of operators

\[ \tilde{\Delta}_j(t, \beta) = U^{(n)}(t^{1/2}) [tT(\beta)] U^{(n)}(t^{-1/2}) \]  

with eigenvectors \( \{ U^{(n)}(t^{1/2}) \Gamma_i(\beta) \}_{i \in \mathbb{N}} \)

Then

\[ \tilde{P}_j(t) = U^{(n)}(t^{1/2}) \left[ tK_j \right] U^{(n)}(t^{-1/2}) = \tilde{\Delta}_j(t, 0) \text{ with eigenvector } U^{(n)}(t^{1/2}) \Gamma_i(0) \]

\[ \tilde{\Delta}_j(t) = \tilde{\Delta}_j(t, 1) \text{ with eigenvector } \Psi_{t,j}(t) = U^{(n)}(t^{1/2}) \Gamma_i(1) \]

**Theorem (Kato-Rellich)** (cf. [RS] p15) Let \( T(\beta) \) be an analytic family in the sense of Kato. Let \( E_0 \) be a non-degenerate discrete eigenvalue of \( T(\beta_0) \). Then for \( \beta \) near \( \beta_0 \), there is exactly one eigenvalue \( E(\beta) \in \sigma(T(\beta)) \) near \( E_0 \) and this eigenvalue is isolated and non-degenerate. \( E(\beta) \) is an analytic function of \( \beta \) near \( \beta_0 \), and there is an analytic eigenvector \( \Gamma(\beta) \) for \( \beta \) near \( \beta_0 \).

**Case 1:** For simplicity, assume \( E_0 \) is a non-degenerate eigenvalue of \( T(0) = \tilde{K}_j \), By the above theorem,

\[ \lim_{\beta \to 0} \Gamma_i(\beta) \rightleftharpoons \Gamma_i(0) \]  

(2.129)

Observe that \( \Gamma_i(0) \) is an eigenvector of \( \tilde{K}_j \), so

\[ \lim_{t \to +\infty} < J_t U^{(n)}(t^{1/2}) \Gamma_i(0), J_t U^{(n)}(t^{1/2}) \Gamma_m(0) > = \delta_{im} \]  

(2.130)

The above equality in fact is the corresponding statement for the Witten deformation of de Rham complex on \( \mathbb{R}^n \) and can be shown directly.

Note that

\[ < \Phi_{t,j}(t), \Phi_{m,j}(t) > = < J_t \Psi_{t,j}(t), J_t \Psi_{t,m,j}(t) > \]

(2.131)
Similarly,

\[
< J_j \mathcal{U}^{(n)}(\Gamma_0), J_j \mathcal{U}^{(n)}(1/2) (\Gamma_m(0) - \Gamma_m(\beta)) > \leq ||J_j \mathcal{U}^{(n)}(\Gamma_0)|| ||J_j \mathcal{U}^{(n)}(1/2) (\Gamma_m(0) - \Gamma_m(\beta)) ||^2 \to 0
\]  

(2.132)

Hence,

\[
\lim_{t \to \infty} < \Phi_{\lambda_j}(t), \Phi_{m_j}(t) >= \lim_{t \to \infty} < J_j \mathcal{U}^{(n)}(1/2) \Gamma_0, J_j \mathcal{U}^{(n)}(1/2) \Gamma_m(0) >= \delta_{\lambda m}
\]  

(2.133)

Hence,

\[
\lim_{t \to \infty} < \Phi_{\lambda_j}(t), \Phi_{m_j}(t) >= \lim_{t \to \infty} < J_j \mathcal{U}^{(n)}(1/2) \Gamma_0, J_j \mathcal{U}^{(n)}(1/2) \Gamma_m(0) >= \delta_{\lambda m}
\]  

(2.134)

Case 2: More generally, suppose \( E_0 \) is an eigenvalue of multiplicity \( m \). By the calculation in Case 1, it is clear that it suffices to show that \( \Gamma_i(\beta) \) is an analytic family and hence \( \lim_{\beta \to 0} \Gamma_i(\beta) = \Gamma_i(0) \).

Since \( T(\beta) \) is an analytic family in the sense of Kato, then

\[
P(\beta) = \frac{-1}{2\pi i} \int_{|E-E_0| = \epsilon} \frac{1}{T(\beta) - E} dE
\]  

(2.135)

for some \( \epsilon > 0 \) small enough, is analytic in \( \beta \) near \( \beta_0 = 0 \). Hence

\[
\text{Rank } P(\beta) = \text{Rank } P(0) = m
\]  

(2.136)

Theorem (cf. [RS] p22) Let \( R \) be a connected, simply connected region of \( \mathbb{C} \) containing \( 0 \). Let \( P(\beta) \) be a projection-valued analytic function in \( R \). Then there exists an analytic family \( \{ U(\beta) \}_{\beta \in R} \) of invertible operators such that

\[
P(\beta) = U(\beta)P(0)U^{-1}(\beta)
\]  

(2.137)
Moreover, if \( P(\beta) \) is self-adjoint for \( \beta \in R \cap R, \) \( U(\beta) \) can be chosen to be unitary.

Now the family \( \{ U(\beta) \}_{\beta \in R} \) described in the above theorem exists for family \( P(\beta) \) defined in (A.2).

Define

\[
\hat{T}(\beta) = U^{-1}(\beta)T(\beta)U(\beta)
\]

Then range of \( P(0) \) is invariant under \( \hat{T}(\beta) \) for all \( \beta \) near 0.

**Proposition** (cf. [RS] p71) Let \( T(\beta) \) be an analytic \( n \times n \) matrix-valued function in a neighbourhood of 0. Suppose \( T(\beta) \) is self-adjoint for all \( \beta \) real. Then

(a) Suppose \( E_0 \) is an eigenvalue of multiplicity \( m \) for \( T(0) \), \( E_1(\beta), \cdots, E_k(\beta) \) be the distinct eigenvalues of \( T(\beta) \) near \( \beta_0 \), \( P_1(\beta), \cdots, P_k(\beta) \) be the corresponding projections. Then \( P_i(\beta) \) are analytic near 0.

(b) There exist analytic vector-valued functions \( \gamma_1(\beta), \cdots, \gamma_n(\beta) \) in a neighbourhood of 0 such that

\[
(i) \gamma_i(\beta) \text{ are eigenvectors of } T(\beta)
\]

\[
(ii) \langle \gamma_i(\beta), \gamma_j(\beta) \rangle = \delta_{ij} \text{ for all } \beta \text{ real}
\]

Now apply the above Proposition to \( \hat{T}(\beta) \mid_{\text{Range } P(0)} \). Since \( E_0 \) has multiplicity \( m \) as an eigenvalue of \( T(0) \), there exist analytic functions of eigenvectors \( \gamma_1(\beta), \cdots, \gamma_m(\beta) \) of \( \hat{T}(\beta) \) with \( \langle \gamma_i(\beta), \gamma_j(\beta) \rangle = \delta_{ij} \) for all \( \beta \) real.

Let

\[
\Gamma_{l(i)}(\beta) = U(\beta)\gamma_i(\beta)
\]

be an eigenvector of \( T(\beta) \), where \( l(i) \) depends on \( i \) with

\[
\langle \Gamma_{l(i)}(\beta), \Gamma_{l(j)}(\beta) \rangle = \langle \gamma_i(\beta), \gamma_j(\beta) \rangle = \delta_{ij} \text{ for } \beta \text{ real}
\]
Finally, we have

\[
\lim_{\beta \to 0} \Gamma_{\beta}(0) = \lim_{\beta \to 0} U(\beta) \gamma_n(\beta) = U(0) \gamma_n(0) = \Gamma_{\beta}(0)
\]

which completes the proof of Lemma 2.2.3.

For the sake of the proof of Morse inequality in §2.3, we give the following definition.

**Definition (A)** Let \( T(\beta) \) be a family of closed operators, defined on \( \mathbb{R} \) with \( \rho(T(\beta)) \neq \emptyset \) \( \forall \beta \in \mathbb{R} \). Then \( T(\beta) \) is an analytic family of type (A) iff

(i) Domain of \( T(\beta), D(T(\beta)) = D \) \( \forall \beta \in \mathbb{R} \)

(ii) \( \forall \Psi \in D, T(\beta)\Psi \) is a vector-valued analytic function of \( \beta \).

It is proved in [K] that an analytic family of type (A) is an analytic family in the sense of Kato.

### 2.3 Proof of Morse Inequality

So far we have obtained the complex

\[
(\hat{Q}_{n,0}^*(M, t), D(t))
\]

which is spanned by the eigenvectors of \( \hat{\Delta}(t) \) corresponding to the eigenvalues \( E(t) \) so that

\[
\lim_{t \to \infty} \frac{E(t)}{t} = 0
\]
This complex calculates the $S^1$-equivariant cohomology of $M$. As a consequence of Theorem 1 proved in §2.2,

$$\dim \tilde{\Omega}_{inv,0}^k(M,t) = m_k + m_{k-1} + \cdots + m_0 + m^f_k + m^f_{k-2} + \cdots < \infty$$ (2.143)

where $m_i$ is the number of critical orbits of index $i$ whose $\theta^-$ is trivial and $m^f_i$ is the number to critical fixed points of index $i$.

We want to show further that such eigenvalues which satisfy (2.142) are bounded.

Recall that $\tilde{\Delta}^k_j(t)$ is the 'localized' Laplace operator on the standard model $\mathbb{R}^{n-1} \times \mathbb{Z}_m$ $S^1$ or $\mathbb{R}^n$. Consider those critical orbits with normalized eigenvector $\overline{\Psi}_{1,j}(t)$ corresponding to the smallest eigenvalue $\tau^k_{1,j}(t)$ of $\tilde{\Delta}^k_j(t)$ which satisfies $\lim_{t \to \infty} \frac{\tau_{1,j}(t)}{t} = 0$.

(Such eigenvector exists for the critical orbit $O_j$ iff $\text{index} O_j \leq k$ and $\theta^-$ trivial in Cases 1 and 1'; and iff $\text{index} O_j \leq k$ and $\text{index} O_j \equiv k(\text{mod} 2)$ in Case 2)

Let $\{J_j\}_{0 \leq j \leq r}$, a partition of unity such that $\sum_{j=0}^{r} J_j^2 = 1$.

For those $j$ such that $\overline{\Psi}_{1,j}(t)$ exists, let

$$\Phi^k_{1,j}(t) = J_j \overline{\Psi}^k_{1,j}(t)$$ (2.144)

Let

$$\mathcal{E}_k(t) = \text{Span}\{\Phi^k_{1,j}(t) \mid \text{index} O_j \leq k, \theta \text{ trivial in Cases 1 and 1' with } \theta^- \text{ trivial} \}$$

Then

$$\dim \mathcal{E}_k(t) = \dim \tilde{\Omega}_{inv,0}^k(M,t) = m_k + m_{k-1} + \cdots + m_0 + m^f_k + m^f_{k-2} + \cdots$$ (2.145)

**Lemma 2.3.1** There exists a constant $C$ so that
\[ \| \hat{\Delta}^k(t) \omega \| \leq C \| \omega \| \text{ for any } \omega \in E_k(t) \]

**Proof** It suffices to show \[ \| \hat{\Delta}^k(t) \Phi_{1,j}^k(t) \| \leq C \]

Observe that \[ \hat{\Delta}^k(t) \Phi_{1,j}^k(t) = \hat{\Delta}^k(t) \left( J_j \overline{\Psi}_{1,j}^k(t) \right) = \left( \hat{\Delta}_j^k(t) J_j \right) \overline{\Psi}_{1,j}^k(t) \]

Since \[ \left( \hat{\Delta}_j^k(t) J_j \right) (\omega) = \left( J_j \hat{\Delta}_j^k(t) \right) (\omega) + (\Delta J_j) \omega + 2 \nabla J_j \nabla \omega \]

We have

\[ \| \hat{\Delta}^k(t) \Phi_{1,j}^k(t) \| \leq \| J_j \hat{\Delta}_j^k(t) \overline{\Psi}_{1,j}^k(t) \| + \| (\Delta J_j) \overline{\Psi}_{1,j}^k(t) \| + 2 \| \nabla J_j \nabla \overline{\Psi}_{1,j}^k(t) \| \]

(2.146)

Let \[ \overline{\Delta}_j^k(t) \overline{\Psi}_{1,j}^k(t) = \overline{r}_{1,j}^k(t) \overline{\Psi}_{1,j}^k(t) \]. Let \( C \geq 0 \) be such that

\[ | \overline{r}_{1,j}^k(t) | \leq C/3 \text{ and } \max | \Delta J_j | \leq C/3 \]

Then

\[ \| \hat{\Delta}^k(t) \Phi_{1,j}^k(t) \| \leq \| \overline{\Delta}_j^k(t) \overline{\Psi}_{1,j}^k(t) \| + \frac{C}{3} \| \overline{\Psi}_{1,j}^k(t) \| + 2 \| \nabla J_j \nabla \overline{\Psi}_{1,j}^k(t) \| \]

(2.147)

Hence it suffices to estimate \( \nabla J_j \nabla \overline{\Psi}_{1,j}^k(t) = \sum_{i=1}^n \frac{\partial J_j}{\partial x_i} \frac{\partial \overline{\Psi}_{1,j}^k(t)}{\partial x_i} \). In fact we are going to show

\[ \lim_{t \to \infty} \| \nabla J_j \nabla \overline{\Psi}_{1,j}^k(t) \| = 0 \]

(2.148)

**Case 1** \( \overline{\Delta}_j^k(t) : \hat{\Omega}_{int}^k(\mathbb{R}^{n-1} \times S^1) \to \hat{\Omega}_{int}^k(\mathbb{R}^{n-1} \times S^1) \).

Let \( l_j \) denote the index of \( O_j \), \( C^{(1)} = \{ x \in \mathbb{R} | \frac{1}{2} \leq x \leq 1 \} \)

Let \( C^{(n-1)} \times S^1 = \{ x_1, \ldots, x_{n-1}, 0 \} \in \mathbb{R}^{n-1} \times S^1 | \frac{1}{2} \leq | x_i | \leq 1, i = 1, \ldots, n-1 \} \).

Clearly \( \text{supp}(\nabla J_j) \subset C^{(n-1)} \times S^1 \). Since \( J_j \) and \( \overline{\Psi}_{1,j}^k(t) \) are invariant, we have

\[ \nabla J_j \nabla \overline{\Psi}_{1,j}^k(t) = \sum_{i=1}^{n-1} \frac{\partial J_j}{\partial x_i} \frac{\partial \overline{\Psi}_{1,j}^k(t)}{\partial x_i} \]

(2.149)
Recall that
\[
\overline{\Psi}_{1,j}^k(t) = U^{(n-1)}(t^{1/2}) (\Omega_0(x_1) \cdots \Omega_0(x_{n-1})) \, dx_1 \wedge \cdots \wedge dx_j
\]
where \(\Omega_0\) is the ground state of \(-d^2/dx^2 + x^2\).

Then
\[
\left\| \frac{j \overline{\psi}_{1,j}^k(t)}{dx_j} \right\|_{C(J \times N \times S)} = t^{1/2} \left\| t^{1/4} \Omega_0(t^{1/2}x_1) \cdots t^{1/4} \Omega_0(t^{1/2}x_{n-1}) \right\|_{C(J \times N \times S)}
\]
\[
= t^{1/2} \left\| U^{(1)}(t^{1/2}) \Omega_0(x_1) \right\|_{C(J \times N \times S)} \cdots \left\| U^{(1)}(t^{1/2}) \Omega_0'(x_j) \right\|_{C(J \times N \times S)}
\]
\[
\leq t^{1/2} \left\| U^{(1)}(t^{1/2}) \Omega_0'(x_j) \right\|_{C(J \times N \times S)} \xrightarrow{t \to \infty} 0
\]  
(2.151)

Hence,
\[
\lim_{t \to \infty} \| \nabla J_j \nabla \overline{\Psi}_{1,j}^k(t) \| = 0
\]  
(2.152)

Case 1' can be handled in the same way.

Case 2 \(\overline{\Delta}_j^k(t) : \hat{\Omega}_{in}^k(\mathbb{R}^n) \to \hat{\Omega}_{in}^k(\mathbb{R}^n)\)

\[
\overline{\Delta}_j^k(t) = dd^* + d^*d + t^2 \left\| df \right\|^2 + tA + (i_X i^*_X + i^*_X i_X) + (d^* i_X + i_X d^*) + (di^*_X + i^*_X d)
\]
\[
= U^{(n)}(t^{1/2}) \left[ \overline{\Delta}_j^k + \beta^2 eX^2 + \beta (d^* i_X + i_X d^*) + (di^*_X + i^*_X d) \right] U^{(n)}(t^{-1/2})
\]  
(2.153)

where \(\beta = \frac{1}{t}\) and the definition of \(\overline{\Delta}_j^k\) is given in \(\S 2.1.3(c)\).

Then
\[
\overline{\Psi}_{1,j}^k(t) = U^{(n)}(t^{1/2}) \Gamma_1(\beta)
\]  
(2.154)

where \(\Gamma_1(\beta)\) is the eigenvector of \(T(\beta)\) defined in \(\S 2\). Recall that
\[
T(\beta) = \overline{\Delta}_j^k + \beta^2 eX^2 + \beta (d^* i_X + i_X d^*) + (di^*_X + i^*_X d)
\]  
(2.155)
Since \( \text{supp}(\nabla J_j) \subset C^{(n)} = \{ x \in \mathbb{R}^n \mid \frac{1}{2} \leq |x| \leq 1 \} \), we have

\[
\left\| \frac{\partial \Gamma_{1}^{(n)}(t)}{\partial x_i} \right\|_{C^{(n)}} = \left\| \frac{\partial}{\partial x_i} \left( t^{n/4} \Gamma_{1}(\beta)(t^{1/2}x) \right) \right\|_{C^{(n)}} = t^{n/4} \left\| \frac{\partial}{\partial x_i} \frac{\partial \Gamma_{1}(\beta)}{\partial x_i} (t^{1/2}x) \right\|_{C^{(n)}} = t^{1/2} \left( \int_{\mathbb{R}^n} \left| \frac{\partial \Gamma_{1}(\beta)}{\partial x_i} (t^{1/2}x) \right|^2 d^n x \right)^{1/2} \leq t^{1/2} \left( \int_{\mathbb{R}^n} \left| \frac{\partial \Gamma_{1}(\beta)}{\partial x_i} (t^{1/2}x) \right|^2 d^n x \right)^{1/2}
\]

where \( \Gamma_{1}(\beta, 0) \) is the groundstate of \( T(\beta, 0) \) and

\[
T(\alpha, \beta) = \overline{K}_{j} + \alpha \epsilon X^2 + \beta (d^* i_{X} + i_{X} d^* + d i_{X} + i_{X} d)
\]

Note that \( T(\beta) = T(\beta^2, \beta) \) and \( \Gamma_{1}(\beta) = \Gamma_{1}(\beta^2, \beta) \)

Note that \( \Gamma_{1}(\beta^2, 0) \) is the groundstate of

\[
\overline{K}_{j} + \beta^2 \epsilon X^2
\]

which is the quantum harmonic oscillator. Since \( \Gamma_{1}(\beta^2, 0) \) decays exponentially, we have

\[
t^{1/2} \left( \int_{\mathbb{R}^n} \left| \frac{\partial \Gamma_{1}(\beta^2, 0)}{\partial x_i} \right|^2 d^n x \right)^{1/2} \xrightarrow{t \to \infty} 0
\]

Claim: There exists a constant \( C \) so that

\[
\left\| \Gamma_{1}(\beta) - \Gamma_{1}(\beta^2, 0) \right\| \leq C \mid \beta \mid
\]

If the Claim is true, then

\[
t^{1/2} \left( \int_{\mathbb{R}^n} \left| \frac{\partial \Gamma_{1}(\beta)}{\partial x_i} - \frac{\partial \Gamma_{1}(\beta^2, 0)}{\partial x_i} \right|^2 d^n x \right)^{1/2} \leq t^{1/2} \left\| \Gamma_{1}(\beta) - \Gamma_{1}(\beta^2, 0) \right\| \leq \frac{C}{t^{1/2}} \xrightarrow{t \to \infty} 0
\]
hence (2.152) is true and Lemma 2.3.1 is proved.

**Proof of Claim:** To prove the Claim, we will use the analytic perturbation techniques as presented in §2.2. Recall

$$T(\alpha, \beta) = K_j + \alpha \epsilon X^2 + \beta (d^* i_X + i_X d^* + di_X^* + i^*_X d)$$  \hspace{1cm} (2.160)

Consider Analytic perturbation in the Sobolev space $H_1(\Lambda^*(\mathbb{R}^n))$:

For any $\alpha \in [0, \alpha_0]$, $T(\alpha, 0) = K_j + \alpha \epsilon X^2$ is self-adjoint with a dense domain $D_{H_0}(T(\alpha, 0)) \subset H_0 = H_0(\Lambda^*(\mathbb{R}^n)) = L^2(\Lambda^*(\mathbb{R}^n))$ and $\rho_{H_0}(T(\alpha, 0)) \neq \emptyset$. Let $\lambda \in \rho_{H_0}(T(\alpha, 0))$. Observe that by restricting the domain to $H_1 = H_1(\Lambda^*(\mathbb{R}^n))$, we have

$$(T(\alpha, 0) - \lambda)^{-1} \in B(H_1)$$

(where $B(H_1)$ denotes the space of bounded linear maps in $H_1$). Therefore $T(\alpha, 0) - \lambda I$ is defined on $D_{\alpha} = (T(\alpha, 0) - \lambda I)^{-1}(H_1)$ which is dense in $H_1$ and so $T(\alpha, 0)$ is closed as an unbounded operator in $H_1$. Hence $\rho_{H_0}(T(\alpha, 0)) = \rho_{H_1}(T(\alpha, 0))$ and $T(\alpha, 0)$ as an unbounded operator in $H_1$ has discrete spectrum. Since by Lemma 0 in §2.2 $B = d^* i_X + i_X d^* + di_X^* + i^*_X d \in B(H_1)$, we conclude that for any fixed $\alpha \{T(\alpha, \beta)\}$ is a family of closed operators with domain $D_{H_1}(T(\alpha, \beta)) = D_{H_1}(T(\alpha, 0)) = D_{\alpha}$. Since $T(\alpha, \beta)$ has non-empty resolvent set, $\{T(\alpha, \beta)\}$ is an analytic family of type (A) (see the end of §2.2 for the relevant definitions), hence an analytic family in the sense of Kato, and we can apply analytic perturbation theory.

Let $\Gamma_1(\alpha, 0)$ be the eigenvector corresponding to the eigenvalue $E_1(\alpha, 0)$. Assume further that $\|\Gamma_1(\alpha, 0)\|_1 = 1$ in $H_1(\Lambda^*(\mathbb{R}^n))$. In view of the analytic perturbation theory in $H_1$, one obtains $\Gamma_1(\alpha, \beta)$, the normalized eigenvector of $T(\alpha, \beta)$ corresponding to $E_1(\alpha, \beta)$ and $\Gamma_1(\alpha, \beta)$ is an analytic functions in $\beta$. 


In the calculation which follows, we shall always refer to the ground state of the operators $T(\alpha, \beta)$, thus for ease of notations, the subscript 1 will be dropped. Also we shall use the notations $T_\alpha(\beta) = T(\alpha, \beta)$ etc.

Let $\epsilon_0 > 0$ be sufficiently small. For $E$ s.t. $|E - E_\alpha(0)| = \epsilon_0$, there exists $C_1$ s.t. $\|(T_\alpha(0) - E)^{-1}\|_1 \leq C_1$ for all $\alpha \in [0, \alpha_0]$.

Since
\[
(T_\alpha(\beta) - E)^{-1} = (T_\alpha(0) + \beta B - E)^{-1} = \sum_{n=0}^{\infty} (-\beta)^n (T_\alpha(0) - E)^{-1} [B(T_\alpha(0) - E)^{-1}]^n
\]

by choosing $|\beta| \leq \beta_0$ with $\beta_0 \|B\|_1 C_1 \leq \frac{1}{2}$, the above series is convergent and $(T_\alpha(\beta) - E)^{-1}$ exists.

Now
\[
P_\alpha(\beta) = \frac{-1}{2\pi i} \oint_{|E - E_\alpha(0)| = \epsilon_0} \frac{1}{T_\alpha(\beta) - E} \, dE
\]
is the projection onto the eigenspace spanned by $\Gamma_\alpha(\beta)$.

Then
\[
\Gamma_\alpha(\beta) = < \Gamma_\alpha(0), P_\alpha(\beta) \Gamma_\alpha(0) >^{-1/2} P_\alpha(\beta) \Gamma_\alpha(0)
\]
and we have
\[
\|(\Gamma_\alpha(\beta) - \Gamma_\alpha(0))\|_1 = \|< \Gamma_\alpha(0), P_\alpha(\beta) \Gamma_\alpha(0) >^{-1/2} P_\alpha(\beta) \Gamma_\alpha(0) - \Gamma_\alpha(0)\|_1
\]
\[
\leq \|< \Gamma_\alpha(0), P_\alpha(\beta) \Gamma_\alpha(0) >^{-1/2} -1 \| P_\alpha(\beta) \Gamma_\alpha(0) \|_1 + \|P_\alpha(\beta) - P_\alpha(0)\|_1 \|\Gamma_\alpha(0)\|_1
\]
\[
\leq |< \Gamma_\alpha(0), P_\alpha(\beta) \Gamma_\alpha(0) >^{-1/2} -1 | \|P_\alpha(\beta)\|_1 + \|P_\alpha(\beta) - P_\alpha(0)\|_1
\]
(2.163)

But
\[
P_\alpha(\beta) - P_\alpha(0)
\]
\[
= \frac{-1}{2\pi i} \oint_{|E - E_\alpha(0)| = \epsilon_0} \frac{1}{T_\alpha(0) + \beta B - E} - \frac{1}{T_\alpha(0) - E} \, dE
\]
\[
= \frac{-1}{2\pi i} \oint_{|E - E_\alpha(0)| = \epsilon_0} \left\{ \sum_{n=0}^{\infty} (-\beta)^n (T_\alpha(0) - E)^{-1} [B(T_\alpha(0) - E)^{-1}]^n - (T_\alpha(0) - E)^{-1} \right\} \, dE
\]
\[
= \frac{-1}{2\pi i} \oint_{|E - E_\alpha(0)| = \epsilon_0} \beta \sum_{n=1}^{\infty} (-1)^n \beta^{n-1} (T_\alpha(0) - E)^{-1} [B(T_\alpha(0) - E)^{-1}]^n \, dE
\]
(2.164)
Therefore,
\[ \|P_n(\beta) - P_n(0)\|_1 \leq \frac{|\beta|}{2\pi} \int_{E - \Gamma_n(0) = 0} \sum_{n=1}^{\infty} (|\beta| \|B\|_1 \|(T_n(0) - E)^{-1}\|_1)^{n-1} \|(T_n(0) - E)^{-1}\|_1 \|B\|_1 dE \]
\[ = C_2 |\beta| \]
(2.166)

where \( C_2 \) is defined by the last equality. Similarly, \(|< \Gamma_n(0), P_n(\beta)\Gamma_n(0) >^{1/2} - 1| \leq C_3 |\beta| \) for some \( C_3 > 0 \). Hence \( \|\Gamma_n(\beta) - \Gamma_n(0)\|_1 \leq C |\beta| \) with \( C = C_2 + C_3 \). By choosing \( \alpha = \beta^2 \), the Claim follows. This completes the proof of Lemma 2.3.1. □

Recall that \( \tilde{\Omega}_{inv,0}(M, t) \) are spanned eigenvectors of \( \tilde{\Delta}(t) \) whose corresponding eigenvalues \( E(t) \) satisfy
\[ \lim_{t \to \infty} \frac{E(t)}{t} = 0 \]
(2.167)

**Corollary 2.3.2** The eigenvalues of \( \tilde{\Delta}^k(t) \) in \( \tilde{\Omega}_{inv,0}(M, t) \) are bounded, i.e., there exists \( C \) such that
\[ E(t) \leq C \]

Hence, the eigenvalues of \( \tilde{\Delta}^k(t) \) are seperated by some constant \( C \) so that those greater than \( C \) will tend to infinity as \( t \to \infty \).

Let
\[ \mathcal{F}_k(t) = \text{Span}\{\Phi^k_{\text{trivial}}(t) \mid \text{index}O_j = k \text{ in Cases 1 and 1'} \text{ and } \theta^\text{trivial} \}
\[ \text{index}O_j \leq k \text{ and } \text{index}O_j \equiv k \text{ (mod 2) in Case 2} \}
\]

Then \( \mathcal{F}_k(t) \subset \mathcal{E}_k(t) \).

**Lemma 2.3.3**
\[ \lim_{t \to \infty} \|\tilde{\Delta}^k(t) |_{\mathcal{F}_k(t)} \| = 0 \]
(2.168)

where \( \|\tilde{\Delta}^k(t) |_{\mathcal{F}_k(t)} \| \equiv \max\{\|D\tilde{\Delta}^k(t)\omega\| \mid \omega \in \tilde{\Omega}^k(M), \|\omega\| = 1\} \). **Proof:** Recall
that
\[ \left\| \Delta^k(t) \Phi^k_{1,j}(t) \right\| \leq \left\| J_j \Delta^k_j (t) \Psi^k_{1,j}(t) \right\| + \left\| (\Delta J_j) \Psi^k_{1,j}(t) \right\| + 2 \left\| \nabla J_j \nabla \Psi^k_{1,j}(t) \right\| \]
\[ = \left\| J_j \Delta^k_j (t) \Psi^k_{1,j}(t) \right\| + \left\| (\Delta J_j) \Psi^k_{1,j}(t) \right\| + 2 \left\| \nabla J_j \nabla \Psi^k_{1,j}(t) \right\| \] (2.169)

Therefore it suffices to show each term above tends to zero as \( t \to \infty \).

Since we have shown in Lemma 2.3.1 that
\[ \lim_{t \to \infty} \left\| \nabla J_j \nabla \Psi^k_{1,j}(t) \right\| = 0 \] (2.170)

and since \( \hat{\alpha}_{1,j}(t) = 0 \) and \( \Psi^k_{1,j}(t) \) exponentially decaying in Cases 1 and \( \text{I} \) with index \( O_j = k \) and \( \theta \) trivial, the first two terms tend to zero as \( t \to \infty \). Hence it suffices to show that the first two terms tend to zero as \( t \to \infty \) in Case 2.

Let \( \text{index} O_j = l_j \),
\[ \omega(t) = \mathcal{U}^{(n)}(t^{1/2}) \phi_0 = t^{n/2} \Omega_0(t^{1/2} x_1) \cdots \Omega_0(t^{1/2} x_n) dx_1 \cdots dx_j \] (2.171)

Recall
\[ \Delta^k_j(t) = \mathcal{U}^{(n)}(t^{1/2}) t \left[ \hat{K}_j + \beta^2 \epsilon X^2 + \beta (i_X^* d + di_X^* + i_X d^* + d^* i_X) \right] \mathcal{U}^{(n)}(t^{1/2}) \] (2.172)

where \( \beta = \frac{1}{t} \). Let
\[ T(\beta) = \hat{K}_j + \beta^2 \epsilon X^2 + \beta (i_X^* d + di_X^* + i_X d^* + d^* i_X) \] (2.173)

Then \( \frac{\hat{\alpha}_{1,j}(t)}{t} \) is the smallest eigenvalue of \( T(\beta) \).

Let
\[ \left\{ \begin{array}{l}
\chi_{1,j}(t) = \sum_{i=1}^\infty a_i \beta^i \\
\Gamma_{1,j}(\beta) = \sum_{i=0}^\infty \phi_i \beta^i
\end{array} \right. \] (2.174)

Claim: \( a_1 = 0 \)
Proof of Claim: Let \( B = i_X^* d + dt_X^* + i_X d^* + d^* i_X \).

Then
\[
T(\beta) \Gamma^k_{1,j}(t) = \frac{r_{i,j}(t)}{t} \Gamma^k_{1,j}(t)
\]
\[
\Rightarrow \overline{K}_j \phi_0 + (\overline{K}_j \phi_1 + B \phi_0) + \ldots = (a_1 \phi_0) + \ldots
\]
\[
\Rightarrow \begin{cases}
K_j \phi_0 = 0 \\
K_j \phi_1 + B \phi_0 = a_1 \phi_0
\end{cases}
\]
\[
\Rightarrow < \phi_0, K_j \phi_1 > + < \phi_0, B \phi_0 > = a_1 < \phi_0, \phi_0 >
\]

But \( ||\Gamma^k_{1,j}(\beta)|| = 1 \) which implies that \( < \phi_0, \phi_1 > = 0 \). Since
\[
\overline{K}_j \left( < \phi_0, \phi_0 > \right) \subset < \phi_0, \phi_0 >
\]
we have \( < \phi_0, K_j \phi_1 > = 0 \).

Therefore,
\[
a_1 = \frac{< \phi_0, B \phi_0 >}{< \phi_0, \phi_0 >} = < \phi_0, B \phi_0 >
\]

Since \( \phi_0 \in \Omega^k(\mathbb{R}^n) \), then \( B \phi_0 \in \Omega^{k-2}(\mathbb{R}^n) \cup \Omega^{k+2}(\mathbb{R}^n) \). This implies \( < \phi_0, B \phi_0 > = 0 \).

Therefore \( a_1 = 0 \) and this finishes the proof of the Claim.

Now the above Claim implies that
\[
\overline{r}^k_{1,j}(t) = t \sum_{i=1}^\infty u_i, \beta^i
\]
\[
= \frac{u_2}{t} + \frac{u_3}{t^2} + \ldots \xrightarrow{t \to 0} 0
\]

Finally,
\[
|| (\Delta J \overline{\Psi}^k_{1,j}(t)) || \leq || (\Delta J \mathcal{U}^a(t^{1/2}) \Gamma^k_{1,j}(\beta)) ||
\]
\[
\leq || (\Delta J) \mathcal{U}^a(t^{1/2}) (\Gamma^k_{1,j}(\beta) - \phi_0) || + || (\Delta J) \mathcal{U}^a(t^{1/2}) \phi_0 ||
\]
\[
= || (\Delta J) \mathcal{U}^a(t^{1/2}) (\Gamma^k_{1,j}(t) - \phi_0) || + || (\Delta J) \phi(t) || \xrightarrow{t \to 0} 0
\]

Hence, the first two terms also tend to zero in Case 2 and this completes the proof of Lemma 2.3.2. \( \square \)
From the previous Lemmas, it is clear that the eigenvalues of \( \hat{\Delta}^k(t) \) in \( \tilde{\Omega}^k_{inv,0}(M,t) \) are divided into two classes. Namely, those which are bounded from below by a positive constant, and those which tend to zero as \( t \to \infty \).

**Definitions**

1. \( E(t) \) is a small eigenvalue of \( \hat{\Delta}^k(t) \) if \( \lim_{t \to \infty} E(t) = 0 \).

2. \[ \tilde{\Omega}^k_{inv,sm}(M,t) = \text{Span}\{ \Psi(t) \in \tilde{\Omega}^k_{inv}(M) \mid \hat{\Delta}^k(t)\Psi(t) = E(t)\Psi(t) \text{ and } E(t) \text{ is a small eigenvalue} \} \]

**Corollary 2.3.4**

\[ \dim \tilde{\Omega}^k_{inv,sm}(M,t) = \dim \mathcal{F}_k(t) = m_k + m_k^f + m_{k-2}^f + \cdots \]  \( \text{(2.178)} \)

**Lemma 2.3.5** Let \( M = \dim \tilde{\Omega}^k_{inv,sm}(M,t) \), then

\[ \sum_{i=0}^{k} (-1)^i M_i - \sum_{i=0}^{k} (-1)^i b_i \begin{cases} \geq 0 & \text{if } k \text{ is even} \\ \leq 0 & \text{if } k \text{ is odd} \end{cases} \]  \( \text{(2.179)} \)

**Proof:** This follows from the finite dimensional analogue of Hodge decomposition theorem.

**Proof of \( S^1 \)-equivariant Morse Inequality**

Let us calculate \( \mathcal{P}_{S^1}(O_j, \theta^-, t) \) in different cases.

(i) \( O_j \) as in Case 1 and Case 1' with \( \theta^- \) trivial.

Then \( \mathcal{P}_{S^1}(O_j, \theta^-, t) = \mathcal{P}_{S^1}(O_j, t) \)

Let \( O_j \cong S^1, StabO_j \cong \mathbb{Z}_m \) for some \( m \in \mathbb{Z} \). Then \( H^*_\mathbb{Z}(O_j) \) can be calculated from the cochain complex

\[ 0 \to \tilde{\Omega}^0_{inv}(S^1) \xrightarrow{d+i} \tilde{\Omega}^1_{inv}(S^1) \xrightarrow{d+i} \tilde{\Omega}^2_{inv}(S^1) \xrightarrow{d+i} \tilde{\Omega}^3_{inv}(S^1) \xrightarrow{d+i} \cdots \]
Note that $\tilde{\Omega}^{2k}_{inv}(S^1) = \mathbb{R}$ and $\tilde{\Omega}^{2k+1}_{inv}(S^1) = \mathbb{R}d\theta$ for any $k$, and we have

$$H^*_S(O_j) = \begin{cases} \mathbb{R} & \text{if } \ast = 0 \\ 0 & \text{if } \ast \neq 0 \end{cases} \quad (2.180)$$

Therefore $\mathcal{P}_{S^1}(O_j, t) = 1$.

(ii) $O_j$ as in Case 1 and Case 1' with $\theta^-$ non-trivial.

Let $O_j \cong S^1$, $\text{Stab} O_j \cong \mathbb{Z}_m$. Then $H^*_S(O_j, \theta^-)$ can be calculated from the cochain complex

$$0 \longrightarrow \tilde{\Omega}^0_{inv}(S^1, o(E)) \overset{d_1}{\longrightarrow} \tilde{\Omega}^1_{inv}(S^1, o(E)) \overset{d_2}{\longrightarrow} \tilde{\Omega}^2_{inv}(S^1, o(E)) \longrightarrow \cdots$$

But

$$\begin{cases} \tilde{\Omega}^{2k}_{inv}(S^1, o(E)) \cong \Omega^0_{inv}(S^1, o(E)) \cong 0 \\ \tilde{\Omega}^{2k+1}_{inv}(S^1, o(E)) \cong \Omega^1_{inv}(S^1, o(E)) \cong 0 \end{cases} \quad (2.181)$$

Hence, $H^*_S(O_j, \theta^-) \cong 0$ and $\mathcal{P}_{S^1}(O_j, \theta^-, t) = 0$.

(iii) $O_j$ as in Case 2 with $O_j = x_j$ and $\text{Stab} O_j \cong S^1$.

In this case, the $S^1$-action is trivial and $X = 0$. $H^*_S(x_j)$ can be calculated from the cochain complex

$$0 \longrightarrow \Omega^0(x_j) \overset{d}{\longrightarrow} \Omega^1(x_j) \overset{d}{\longrightarrow} \Omega^0(x_j) \overset{d}{\longrightarrow} \Omega^1(x_j) \overset{d}{\longrightarrow} \cdots$$

So

$$H^*_S(x_j) = \begin{cases} \mathbb{R} & \text{if } \ast \text{ is even} \\ 0 & \text{if } \ast \text{ is odd} \end{cases} \quad (2.182)$$

Then

$$\mathcal{P}_{S^1}(O_j, t) = 1 + t^2 + t^4 + \cdots \quad (2.183)$$

Hence we have proved
Lemma 2.3.6

(i) If \( O_j \cong S^1 \), then
\[
P_{S^1}(O_j, t) = \begin{cases} 
1 & \text{if } \theta^- \text{ is trivial} \\
0 & \text{otherwise} 
\end{cases} \tag{2.184}
\]

(ii) If \( O_j \cong x_j \) and \( \text{Stab}O_j \cong S^1 \), then
\[
P_{S^1}(O_j) = 1 + t^2 + t^4 + \cdots \tag{2.185}
\]

Recall that \( M_i = \dim \tilde{\Omega}^i_{\text{inv, sm}}(M, t) = m_i + m_i^I + m_{i-2}^I + \cdots \)

Observe that by Lemma 2.3.6, the \( S^1 \)-equivariant Morse inequality is equivalent to
\[
\sum_{i=0} M_i t^i - \sum_{i=0} \beta_i t^i = (1 + t) \sum_{i=0} q_i t^i = \sum_{i=0} (q_i + q_{i+1}) t^i \quad \text{with} \quad q_i \geq 0 \tag{2.186}
\]

Lemma 2.3.7 Let \( q_i \) be defined in (2.186) (whose non-negativity imply the Morse inequality). Then
\[
q_k = \begin{cases} 
\sum_{i=0}^k (-1)^i M_i - \sum_{i=0}^k (-1)^i \beta_i & \text{if } k \text{ is even} \\
\sum_{i=0}^k (-1)^i \beta_i - \sum_{i=0}^k (-1)^i M_i & \text{if } k \text{ is odd} 
\end{cases} \tag{2.187}
\]

Proof: It can be proved by induction on \( k \).

Finally, an application of Lemma 2.3.5 implies \( q_k \geq 0 \) for all \( k \geq 0 \). This completes the proof of the \( S^1 \)-equivariant Morse inequality.

2.4 Helffer-Sjöstrand Theory for \( S^1 \)-equivariant cohomology

Recall that in Part I, we have the complex of finite dimensional vector spaces
\[
\left( \tilde{\Omega}^*_{\text{inv, sm}}(M, t), D(t) \right)
\]
which calculates the $S^1$-equivariant cohomology of $M$. Here $D_t = e^{-tI}(d + i_X)e^{tI}$.

We want to show that $(\hat{\Omega}_{inv, sm}^*(M, t), D(t))$ converges to a geometric complex $(C^*(M, f), \delta)$ as $t \to \infty$, where this geometric complex also calculates the $S^1$-equivariant cohomology of $M$. Our strategy is as follows: in §2.4.2, we construct a geometric chain complex $(C_*(M, f), \partial)$ (and its dual $(C^*(M, f), \delta)$) from the filtration of the homotopy quotient $M_{S^1}$ induced by $f$. In §2.4.3 we interpret $(\hat{\Omega}_{inv, sm}^*(M, t), D(t))$ as a cochain complex of differential forms in $M_{S^1}$. In §2.4.4, we deduce the Helffer-Sjöstrand theory by integration of differential forms in $M_{S^1}$. We begin with some preliminaries about equivariant Morse theory in §2.4.1.

2.4.1 Preliminaries

Suppose $f$ is an $S^1$-invariant function in $M^n$, $g$ an $S^1$-invariant metric on $M$.

**Definition** $(f, g)$ is said to satisfy the Morse-Smale condition if for any two critical orbits $O_x$ and $O_y$, $W^{-}_x$ and $W^{+}_y$ intersect transversally where $W^{-}_x, W^{+}_y$ are the descending and the ascending manifold of $O_x$ and $O_y$ respectively.

**Definition** An $S^1$-invariant Morse function $f$ is said to be self-indexing if for any critical orbit $O_x$,

$$f(x) = \text{index } O_x \quad (2.188)$$

**Definition** *(Whitney Pre-stratification)* Let $M$ be a smooth manifold, $\mathcal{C} = \{W_\alpha\}_{\alpha \in A}$. Then $(M, \mathcal{C})$ is a Whitney pre-stratification of $M$ if

(i) $\mathcal{C}$ is a partition of $M$ into pairwise disjoint submanifolds $W_\alpha$ of $M$ and $\mathcal{C}$ is locally finite.
(ii) $C$ satisfies the frontier condition, i.e.

$$W_{\alpha} \cap W_{\beta} \neq \emptyset \Rightarrow W_{\beta} \subset W_{\alpha}$$

(iii) For any $\alpha, \beta$, $(W_{\alpha}, W_{\beta})$ satisfies Whitney condition (b).

For any $x \in W_{\alpha} \cap W_{\beta}$, there exists a coordinate chart $U$ and $\phi: U \rightarrow \phi(U) \subset R^n$ s.t. if $y_i \in W_{\beta} \cap U, x_i \in W_{\alpha} \cap U$ are two sequences of points with $\lim_{i \rightarrow \infty} x_i = \lim_{i \rightarrow \infty} y_i = x$ and if

$$T_{\phi(x_i)}(\phi(W_{\alpha} \cap U)) \rightarrow T \subset T_{\phi(x)}(\phi(U))$$

$$L_{\phi(x_i)\phi(y_i)} \rightarrow L \subset T_{\phi(x)}(\phi(U))$$

then $L \subset T$.

**Proposition 2.4.1.1** Suppose $f$ is an $S^1$-invariant Morse function s.t. $(f, g)$ satisfies the Morse-Smale condition, let $C = \{W_O^+ \mid O$ is a critical orbit of $f\}$, then $(M, C)$ is a Whitney pre-stratification of $M$.

By a theorem of M. Ferrarotti[F], we have

**Corollary 2.4.1.2** Suppose $(f, g)$ satisfies the Morse-Smale condition, then

$$\int_{W_O^-} d\omega = \int_{W_O^+} \omega$$

### 2.4.2 Construction of $(C_*(M, f), \partial)$

Let $f$ be a self-indexing $S^1$-invariant Morse function in $M$.

Define $\tilde{f}: E \times M \rightarrow R$ by

$$\tilde{f}(e, x) = f(x) \quad (2.189)$$
Recall that $S^1$ acts on $E \times M$ by diagonal action, the homotopy quotient of $M$ is defined as the quotient space

$$M_{S^1} = E \times_{S^1} M \equiv E \times M / S^1$$

Since $f$ is $S^1$-invariant, $\tilde{f}$ descends to a function on $M_{S^1}$ which is denoted by $f_{S^1}$.

Recall that by a nondegenerate smooth function on a manifold, we mean a smooth function whose set of critical points consists of a finite union of smooth submanifolds (critical submanifolds) and for each such manifold, the Hessian is nondegenerate in the normal directions. For example, an $S^1$-invariant Morse function is a nondegenerate function.

**Proposition 2.4.2.1** (i) If $f$ is non-degenerate on $M$, then $f_{S^1}$ is non-degenerate on $M_{S^1}$.

(ii) If $O$ is a non-degenerate critical orbit of $f$ in $M$, then $f_{S^1}$ has corresponding non-degenerate critical manifold $E \times_{S^1} O = O_{S^1}$ and

$$\text{index } O = \text{index } O_{S^1} \quad (2.190)$$

Consequently, let $O_1^k, \ldots, O_{m_k}^k$ be the critical orbits of $f$ of index $k$, then $(O_1^k)_{S^1}, \ldots, (O_{m_k}^k)_{S^1}$ are the critical manifolds of $f_{S^1}$ of index $k$.

Let $X_k = f_{S^1}^{-1} \left( (-\infty, k + \frac{1}{2}] \right)$, then

$$X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = M_{S^1}$$

Define $\partial : H_* (X_k, X_{k-1}) \to H_{*-1} (X_{k-1}, X_{k-2})$ by

$$H_* (X_k, X_{k-1}) \to H_{*-1} (X_{k-1}, X_{k-2}) \xrightarrow{\partial} H_{*-1} (X_{k-1}, X_{k-2})$$
\[ \sigma \rightarrow [\partial \sigma] \rightarrow i_* [\partial \sigma] \]

where \( i_* \) is induces from the inclusion \( X_{k-2} \subset X_{k-1} \)

\[
H_*(X_{k-2}) \rightarrow H_*(X_{k-1}) \xrightarrow{i_*} H_*(X_{k-1}, X_{k-2}) \rightarrow H_{*+1}(X_{k-2})
\]

Since the above sequence is exact, it is easy to see that \( \partial^2 = 0 \).

Define

\[
C_*(M, f) = \bigoplus_{i=0}^n H_k(X_i, X_{i-1})
\]

(2.191)

Then the above map \( \partial \) induces the boundary homomorphism

\[
\partial : C_k(M, f) \rightarrow C_{k-1}(M, f)
\]

with \( \partial^2 = 0 \) and therefore \((C_*(M, f), \partial)\) is a chain complex.

Note that

\[
H_*(X_k, X_{k-1}) \cong \bigoplus_{\text{index } \omega = 0} H_*(DN^-(O) \times S^1 E, SN^-(O) \times S^1 E)
\cong \bigoplus_{\text{index } \omega = 0} H_*^{s1}(DN^-(O), SN^-(O))
\]

(2.192)

where \( N^-(O) \) is the negative normal bundle of the critical orbit \( O \).

Recall that

(i) if the orientation line bundle of \( N^-(O) \) is non-trivial, then

\[
H_*^{s1}(DN^-(O), SN^-(O)) \cong 0
\]

(2.193)

(ii) If \( O \cong S^1 \) with orientation line bundle trivial, then

\[
H_*^{s1}(DN^-(O), SN^-(O)) \cong \begin{cases} 
\mathbb{R} & \text{if } *=\text{index } O \\
0 & \text{otherwise}
\end{cases}
\]

(2.194)
(iii) If $O \cong \text{point}$, then

$$H^S_*(DN^-(O), SN^-(O)) \cong \begin{cases} \mathbb{R} & \text{if } * = \text{index } O + 2k, k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(2.195)

Hence we have

**Proposition 2.4.2.2**

\[ \dim C_k(M, f) = m_k + m_{k-2} + m_{k-4} + \cdots = \dim \tilde{\Omega}_{inv, sm}^k(M, t) \quad < \infty \]

(2.196)

### 2.4.3 Interpretation of $\tilde{\Omega}_{inv, sm}^*(M, t)$ as a Complex of Differential Forms on $M_{S^1}$

Recall that $H^*_S(M) = H^* \left( \tilde{\Omega}_{inv}^*(M), d + i_X \right)$.

First define $(\Omega_{inv}^*(M)[u], d_X)$ by

\[
\begin{align*}
\Omega_{inv}^k(M)[u] &= \{ \sum_{\text{deg} \varphi_i + 2i = k} \varphi_i u^i \mid \varphi_i \in \Omega_{inv}^i(M) \} \\
\{ d_X \varphi = d \varphi + i_X(\varphi)u, \quad \varphi \in \Omega_{inv}^*(M) \} \\
d_X u &= 0
\end{align*}
\]

(2.197)

Therefore,

\[
\begin{align*}
d_X \left( \sum_{\text{deg} \varphi_i + 2i = k} \varphi_i u^i \right) &= \sum_{\text{deg} \varphi_i + 2i = k} (d_X \varphi_i) u^i \\
&= \sum_{\text{deg} \varphi_i + 2i = k} (d \varphi_i + i_X(\varphi_i)u) u^i
\end{align*}
\]

(2.198)

Clearly

\[
(\Omega_{inv}^*(M)[u], d_X) \cong \left( \tilde{\Omega}_{inv}^*(M), d + i_X \right)
\]

(2.199)

Our aim in this section is to introduce a complex $(\Omega_{g}^*(M), D)$ of differential forms on $M_{S^1}$ and an isomorphism of cochain complexes

\[
\lambda : (\Omega_{inv}^*(M)[u], d_X) \to (\Omega_{g}^*(M), D)
\]
Using the above isomorphism, we can regard $\Omega^*_{\text{mor}}(M)[u]$ (hence $\hat{\Omega}^*_{\text{mor}}(M)$) as differential forms on $M_{\text{mor}}$. Let $g \cong \mathbb{R}$ be the Lie algebra of $G = S^1$, $g^*$ be the dual of $g$.

Let $Sg^*$ be the symmetric algebra generated by $g^*$ whose generator is denoted by $u$, $\Lambda g^*$ be the exterior algebra generated by $g^*$ whose generator is denoted by $\theta$ with $\deg u = 2$, $\deg \theta = 1$.

Define $W(g) = \Lambda g^* \otimes Sg^*$, called the Weil algebra, which is the algebra generated freely by $\theta$ and $u$ as a commutative graded algebra i.e.

$$\omega_p \omega_q = (-1)^{pq} \omega_q \omega_p \quad (2.200)$$

Define $D : W(g) \to W(g)$ by

$$\begin{cases} D\theta + u = 0 \\ Du = 0 \end{cases} \quad (2.201)$$

and is extended to $W(g)$ as an anti-derivation. Observe that $D^2 = 0$ and

$$H^*(W(g), D) \cong \mathbb{R} \quad (2.202)$$

Consider

$$\left(W(g) \otimes \Omega^*(M), D = D \otimes \text{id} + (-1)^{d_\Omega^*} \text{id} \otimes d \right)$$

and define the basic subcomplex of $W(g) \otimes \Omega^*(M)$

$$\Omega^*_{g}(M) = \{ \omega \in W(g) \otimes \Omega^*(M) \mid i_X(\omega) = L_X(\omega) = 0 \} \quad (2.203)$$

For $\omega \in W(g) \otimes \Omega^*(M)$, let $\omega = \sum_k u^k a_k + \sum_k u^k \theta b_k$ where $a_k, b_k \in \Omega^*(M)$. Then

$$\begin{cases} i_X(\omega) = \sum_k u^k i_X(a_k) + \sum_k u^k (b_k - \theta i_X(b_k)) = 0 \\ L_X(\theta) = \sum_k u^k L_X(a_k) + \sum_k u^k L_X(b_k) = 0 \end{cases} \Leftrightarrow \begin{cases} b_k = -i_X(a_k) \\ L_X(a_k) = 0 \end{cases} \quad (2.204)$$
Therefore we have

$$\Omega_g^* (M) = \{ \omega = \sum_k u^k (a_k + \theta b_k) \in W(g) \otimes \Omega^* (M) \mid L_X(a_k) = 0, b_k = -i_X(a_k) \}$$

(2.205)

Define $\lambda : (\Omega^*_{inv}[u], d_X) \to (\Omega^*_g(M), D)$ such that

$$\begin{align*}
\lambda(\varphi) &= \varphi - \theta i_X(\varphi) \\
\lambda(u) &= u
\end{align*}$$

(2.206)

and extend as a ring homomorphism. Observe that it is also a ring isomorphism.

**Proposition 2.4.3.1**

$$\lambda : (\Omega^*_{inv}(M)[u], d_X) \to (\Omega^*_g(M), D)$$

is an isomorphism between the two cochain complexes.

**Proof** It suffices to show that $\lambda d_X (\sum_k u^k a_k) = D \lambda (\sum_k u^k a_k)$.

$$\lambda d_X (\sum_k u^k a_k) = \lambda \left( \sum_k u^k (da_k + i_X(a_k)) \right)$$

$$= \lambda \left( \sum_k u^k (da_k + i_X(a_k)) \right)$$

$$= \sum_k u^k [(da_k + i_X(a_k)) - \theta i_X(da_k + i_X(a_k))]$$

$$= \sum_k u^k da_k + i_X(a_k) - \sum_k u^k \theta i_X(da_k)$$

(2.207)

Also

$$D \lambda (\sum_k u^k a_k) = D \left( \sum_k u^k (a_k - \theta i_X(a_k)) \right)$$

$$= D \left( \sum_k u^k a_k - \sum_k u^k \theta i_X(a_k) \right)$$

$$= \sum_k u^k da_k + \sum_k u^{k+1} i_X(a_k) + \sum_k u^k \theta di_X(a_k)$$

$$= \sum_k u^k (da_k + i_X(a_k - 1) - \sum_k u^k \theta i_X(da_k))$$

(2.208)

Since $a_k \in \Omega^*_g(M)$, we have $L_X(a_k) = (di_X + i_X d)(a_k) = 0$. Hence, $\lambda d_X = D \lambda$. □

It is possible to interpret $(\Omega^*_g(M), D)$ (hence $(\Omega^*_g(M), d + i_X)$) as a subcomplex of differential forms on $M_{S^1}$. For this purpose assume that one can describe $M_{S^1}$ as an infinite dimensional Hilbert manifold, which is the base of the principal fibration

$$S^\infty \times M \to S^\infty \times M/\Sigma$$
Here $S^\infty$ is the unit sphere in the separable Hilbert space $l_2(\mathbb{C})$. $S^1 = \{ e^{i\alpha} \in \mathbb{C} | \alpha \in \mathbb{R} \}$ acts freely on $S^\infty$ by

$$\mu(e^{i\alpha}, (z_1, z_2, \cdots)) = (e^{i\alpha}z_1, e^{i\alpha}z_2, \cdots) \quad (2.209)$$

and consider the diagonal action on $S^\infty \times M$.

Identify the element $\theta \in W^1(g)$ respectively $u \in W^2(g)$ with the restriction of the form $\sum_i z_i d\bar{z}_i$ respectively $\sum_i d z_i \wedge d\bar{z}_i$ to $S^\infty$. Then one can regard $W(g) \otimes \Omega^*(M)$ as a subcomplex of differential forms on $S^\infty \times M$. Via this interpretation, $\Omega^*_g(M)$ consists of invariant forms on $S^\infty \times M$ which are pullback of smooth forms on $S^\infty \times M / S^1 = M_{S^1}$.

**Proposition 2.4.3.2** The following diagram is commutative

$$\begin{array}{ccc}
\Omega^*_{\text{in}}(M)[u] & \xrightarrow{\lambda} & \Omega^*_g(M) \\
\downarrow e^{ij} & & \downarrow (e^{ij})_{S^1} \\
\Omega^*_{\text{in}}(M)[u] & \xrightarrow{\lambda} & \Omega^*_g(M)
\end{array}$$

where $(e^{ij})_{S^1}$ is defined as follows: let $e^{ij}: W(g) \otimes \Omega^*(M) \to W(g) \otimes \Omega^*(M)$

$$e^{ij} \left( \sum_k a_k^j (a_k + \theta b_k) \right) = \sum_k a_k^j (e^{ij}a_k + \theta e^{ij}b_k) \quad (2.210)$$

Then

$$\lambda_{e^{ij}}(\sum_k a_k^j) = \lambda(e^{ij}) \lambda(\sum_k a_k) \quad (2.211)$$

**Proof**

$$\lambda e^{ij}(\sum_k a_k^j) = \lambda(e^{ij}) \lambda(\sum_k a_k) = e^{ij} \lambda(\sum_k a_k) = (e^{ij})_{S^1} \lambda(\sum_k a_k) \quad (2.212)$$

**Remark:** The map $(e^{ij})_{S^1}$ is reminiscent of the multiplication of forms by the function $(e^{ij})_{S^1}$ on $M_{S^1}$ induced by $e^{ij}$.
Corollary 2.4.3.3 For any \( t \),

\[
\lambda(t) : \left( \Omega^*_{inv}(M)[u], d_X(t) = e^{-if} d_X e^{if} \right) \rightarrow \left( \Omega^*_{g}(M), D(t) = (e^{-if})_* D(e^{if}) \right)
\]

is an isomorphism between the two complexes.

Corollary 2.4.3.4 Since \( \left( \Omega^*_{inv.sm}(M, t), D(t) \right) \subset \left( \Omega^*_{inv}(M)[u], e^{-if} d_X e^{if} \right) \), \( \lambda(t) \) induces a corresponding small complex \( \left( \Omega^*_{g.sm}(M, t), D(t) \right) \) whose homology is the \( S^1 \)-equivariant cohomology of \( M \).

Finally, observe that by using the isomorphism \( \lambda \), a 'small eigenvector' in \( \left( \Omega^*_{inv.sm}(M), D(t) \right) \) is identified to a differential form on \( M_{S^1} \) which is localized at \( E \times S^1 \) \( O = O_{S^1} \).

2.4.4 Helffer-Sjöstrand Theory

In the previous sections, we have described the complexes \( \left( \Omega^*_{g.sm}(M, t), D(t) \right) \) and \( (C^*(M, f), \partial) \) the dual cochain complex of \( (C^*(M, f), \partial) \).

Recall that

\[
C_k(M, f) = \bigoplus_{i=0}^\mu H_k(X_i, X_{i-1}) = \bigoplus_{O \in \text{Crit}(f)} H_k(DN^{-}(O) \times_{S^1} E, SN^{-}(O) \times_{S^1} E)
\]

where \( \text{Crit}(f) \) is the set of critical orbits of \( f \) and \( E = E_{S^1} \).

Consider \( H_k(DN^{-}(O) \times_{S^1} E, SN^{-}(O) \times_{S^1} E) \)

Case 1: \( O \cong S^1, \text{stab } O = 1, \text{index } O = 1 \)

In this case, \( N^{-}(O) \) is a trivial bundle over \( O \).

\[
H_k(DN^{-}(O) \times_{S^1} E, SN^{-}(O) \times_{S^1} E) \cong H_{k-\text{index } O}(O \times_{S^1} E) \cong H_{k-\text{index } O}(\text{point})
\]
Let \( y \in O \times S^1 E, D^l \rightarrow DN^-(O) \times S^1 E \xrightarrow{\pi_0} O \times S^1 E \), then \([\pi_0^{-1}(y)]\) generates \(H_{\text{index}}(DN^-(O) \times S^1 E, SN^-(O) \times S^1 E)\).

**Case 1**: \( O \cong S^1, \text{stab } O \cong \mathbb{Z}_m \) for some \( m > 1 \) with \( N^-(O) \) orientable (otherwise \( H_*(DN^-(O) \times S^1 E, SN^-(O) \times S^1 E) = 0 \)).

In this case, \( H_*(DN^-(O) \times S^1 E, SN^-(O) \times S^1 E) \) is similarly described as in Case 1.

**Case 2**: \( O = \text{point}, \text{stab } O = S^1, N^-(O) \) orientable (otherwise \( H_*(DN^-(O) \times S^1 E, SN^-(O) \times S^1 E) = 0 \)).

\[
H_k(DN^-(O) \times S^1 E, SN^-(O) \times S^1 E) \cong H_{k-\text{index}}(O \times S^1 E) = H_{k-\text{index}}(\mathbb{C}P^\infty)
\] (2.215)

But \( H_*(\mathbb{C}P^\infty) = \oplus_{i=0}^\infty H_{2i}(\mathbb{C}P^\infty) = \oplus_{i=0}^\infty \mathbb{R}[\mathbb{C}P^i] \).

Let \( D^l \rightarrow N^-(O) \times S^1 E \xrightarrow{\pi_0} O \times S^1 E \). Then \([\pi_0^{-1}(\mathbb{C}P^i)]\) generates \(H_k(DN^-(O) \times S^1 E, SN^-(O) \times S^1 E)\) where \( 2i + \text{index } O = k \).

**Proposition 2.4.4.1**

\[
\text{Init} : \left( \Omega^*_g(M, D) \right) \rightarrow (C^*(M, f), \delta) \quad \omega \rightarrow \int_\alpha \omega
\]

is a morphism of cochain complexes.

**Proof** Recall that \( W_x^- \) denotes the descending manifold associated with the critical orbit \( O_x \) of \( x \in M \). Then we have

\[
W_x^- \rightarrow W_x^- \times S^1 E \xrightarrow{\pi_x} O_x \times S^1 E
\]

Also we can use

\[
\begin{cases}
\pi_x^{-1}((\mathbb{C}P^i)) & \text{if } O_x = \text{point} \\
\pi_x^{-1}(y) & \text{if } O_x \cong S^1 \text{ and } y \in O_x \times S^1 E
\end{cases}
\] (2.216)
to represent the relative homology classes in $H_\ast(X_k, X_{k-1})$. The proposition follows from Corollary 2.4.1.2. □

As a consequence, the composition

$$\left(\Omega_{g, \text{smal}l}(M, t), D(t)\right) \xrightarrow{(\alpha, t)} \left(\Omega_{g, \text{smal}l}(M, t), D\right) \xrightarrow{\text{int}} (C_\ast(M, f), \delta)$$

is also a morphism of cochain complexes.

Next we define $\Psi_{O_j}(t)$.

**Case 1**: $O_j \cong S^1$, $\text{stab} O_j \cong 1$, index $O_j = k$

Let $U_j \cong D^{n-1} \times S^1$ be an open neighbourhood of $O_j$ s.t. $(x_1, \ldots, x_{n-1}, 0)(\in D^{n-1} \times S^1)$ is compatible coordinate system about $O_j$.

Define

$$\Psi_{O_j}(t) = \beta(t)\rho(|x|)\left(\frac{2t}{\pi}\right)^{n-1/4} e^{-t(x_1^2 + \cdots + x_{n-1}^2)} dx_1 \wedge \cdots \wedge dx_k$$

(2.217)

where $|x| = \sqrt{x_1^2 + \cdots + x_{n-1}^2}$, $\rho \in C_0^\infty(\mathbb{R})$ such that

$$\rho(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{\epsilon}{2} \\ 0 & \text{if } |x| \geq \epsilon \end{cases}$$

(2.218)

for some $\epsilon > 0$ small enough and $\beta(t)$ chosen so that $\|\Psi_{O_j}(t)\| = 1$.

**Case 1’**: $O_j \cong S^1$, $\text{stab} O_j \cong \mathbb{Z}_m$ for some $m > 1$.

Let $U_j$ be an open neighbourhood of $O_j$ s.t.

$$U_j \cong D^{n-1} \times_{\mathbb{Z}_m} S^1$$

(an orientable bundle over $O_j$)

Let $p : D^{n-1} \times S^1 \to D^{n-1} \times_{S^1} S^1$ be the canonical projection. Since $\mathbb{Z}_m$ acts by isometry, the standard metric on $D^{n-1} \times S^1$ in Case 1 induces a metric on $D^{n-1} \times_{S^1} S^1$. 

via p. Let \( \Delta_{j,i}^k(t) \) denote the 'localized' operator in Case \( i, i = 1, 1' \). Then

\[
p \Delta_{j,0}^k(t) = \Delta_{j,1}^k(t)p
\]

(2.219)

Hence, if \( \Psi(t) \) is the small eigenvector of \( \Delta_{j,1}^k(t) \), then \( p^*(\Psi) \) is the small eigenvector of \( \Delta_{j,0}^k(t) \), which is \((\frac{2\pi}{\beta})^{-1/4}e^{-t(x_1^2 + \cdots + x_{n-1}^2)}dx_1 \wedge \cdots \wedge dx_k\).

Hence, define

\[
\Psi_{O_j^l}(t) = \beta(t)p(|x|)(\frac{2\pi}{\beta})^{-1/4}e^{-t(x_1^2 + \cdots + x_{n-1}^2)}dx_1 \wedge \cdots \wedge dx_k
\]

(2.220)

as in Case 1.

**Case 2:** \( O_{j}^l = x_{j}^l \) a critical fixed point of index \( l_j \) where \( l_j \leq k \) and \( l \equiv k (mod \ 2) \).

Let \( U_j \cong D^n \) be an open neighbourhood of \( x_{j}^l \) s.t. \( (x_1, \cdots, x_n) \in D^n \) is a compatible coordinate system about \( x_{j}^l \).

Recall that in Case 2, the 'localized' operator

\[
\Delta_{j}^k(t) = \Delta^k + 4t^2x^2 + tA + iX^*_X + iX^*i_X + (di_X^* + iX^*d) + (d^*i_X + i_Xd^*)
\]

(2.221)

Let \( \overline{\Psi}_{j,l}^k(t) \) be the normalized ground state of \( \Delta_{j}^k(t) \).

Define

\[
\Psi_{O_j^l}(t) = \beta_1(t)p(|x|)\overline{\Psi}_{j,l}^k(t)
\]

(2.222)

where \( |x| = \sqrt{x_1^2 + \cdots + x_n^2} \) and \( \beta_1(t) \) is chosen s.t. \( \|\Psi_{O_j^l}(t)\| = 1 \).

Let \( C_k(M, J) \) be generated by

\[
\{ \sigma_{O_j}^k \mid \text{Case 1 and 1': } O_{j}^l \cong S^1, l = k \\
\text{Case 2: } O_{j}^l = x_{j}^l, l \equiv k (mod \ 2), l \leq k \}
\]

Let \( \{ e_{O_j}^k \} \) be the dual basis of \( \{ \sigma_{O_j}^k \} \).
Define $J_k(t) : C^k(M, f) \to \hat{\Omega}^k(M)$ s.t.

$$J_k(t) \left( c_{O_j}^k \right) = \Psi_{O_j}^k(t) \quad (2.223)$$

Define

$$Q_k(t) : \hat{\Omega}^k(M) \to \hat{\Omega}^k_{inv, sm}(M, t)$$

to be the orthogonal projection onto $\hat{\Omega}^k_{inv, sm}(M, t)$.

Let

$$H_k(t) = (Q_k(t)J_k(t))^* (Q_k(t)J_k(t))$$
$$\tilde{J}_k(t) = Q_k(t)J_k(t)H_k^{-\frac{1}{2}}(t) \quad (2.224)$$

Then $\tilde{J}_k(t) : C^k(M, f) \to \hat{\Omega}^k_{inv, sm}(M, t)$ is an isometry.

Define

$$E_{O_j}^k(t) = \tilde{J}_k(t) \left( c_{O_j}^k \right) \quad (2.225)$$

**Proposition 2.4.4.2** There exists neighbourhood $U_{O_j}^k$ of $O_j^k$ contained in the chart of compatibility s.t.

(i) $O_j^k \cong S^1$

$$E_{O_j}^k(t) = \left( \frac{2t}{\pi} \right)^{n-1/4} e^{-t(x_1^2 + \cdots + x_{n-1}^2)} \left( dx_1 \wedge \cdots \wedge dx_k + O(t^{-1}) \right) \quad \text{on } U_{O_j}^k \quad (2.226)$$

(ii) $O_j^l = x_j^l \ (l \equiv k (mod 2), l \leq k)$

$$E_{x_j}^k(t) = \left( \frac{2t}{\pi} \right)^{n-1/4} e^{-t(x_1^2 + \cdots + x_n^2)} \left( dx_1 \wedge \cdots \wedge dx_l + O(t^{-1}) \right) \quad \text{on } U_{x_j}^k \quad (2.227)$$

**Proposition 2.4.4.3**

(i) $O_j^k \cong S^1$

$$Int_k(c^{ij})_{S^1} \left( \lambda E_{O_j}^k(t) \right) = \left( \frac{2t}{\pi} \right)^{n-1-2k} c^{ij} \left( c_{O_j}^k + O(t^{-1}) \right) \quad (2.228)$$
(ii) $O_j^l = x_j^l$ ($l \equiv k (\text{mod} \ 2), l \leq k$)

$$\text{Int}_k(t) = \left( \frac{2^k}{\pi} \right)^{\frac{n}{4} - \frac{nu}{2}} e^{it} \left( c^k + O(t^{-1}) \right)$$  \quad (2.229)

**Proof** We prove (ii), (i) can be proved similarly. To show (ii), it suffices to integrate on $\sigma_{x_j^l}^k$. With the identification of $\tilde{\Omega}^*(M) \cong \Omega_{\text{mv}}^*(M)[u]$, we have

$$\int_{\sigma_{x_j^l}^k} e(t) \sim \int_{\sigma_{x_j^l}^k} \left( \lambda E_{x_j^l}^k(t) \right) = \left( \frac{nu}{\pi} \right)^{\frac{n}{4} - \frac{nu}{2}} e^{it} \left( c^k + O(t^{-1}) \right)$$

where in the above computation, we let $i = \frac{k - l}{2}$. $\square$

In view of Proposition 2.4.4.3, define

$$\hat{E}_{O_j^l}^k(t) = \left( \frac{\pi}{2^l} \right)^{\frac{n}{4} - \frac{nu}{2}} e^{-tk} E_{O_j^l}^k(t) \quad \text{if } O_j^l \cong \mathbb{S}^1$$

$$\hat{E}_{x_j^l}^k(t) = \left( \frac{\pi}{2^l} \right)^{\frac{n}{4} - \frac{nu}{2}} e^{-tk} E_{x_j^l}^k(t) \quad \text{if } O_j^l = x_j^l$$

Then we have

$$\left\{ \begin{array}{ll}
\text{Int}_k(t) = \left( \frac{\pi}{2^l} \right)^{\frac{n}{4} - \frac{nu}{2}} e^{-tk} E_{O_j^l}^k(t) + O(t^{-1}) & \text{if } O_j^l \cong \mathbb{S}^1 \\
\text{Int}_k(t) = \left( \frac{\pi}{2^l} \right)^{\frac{n}{4} - \frac{nu}{2}} e^{-tk} E_{x_j^l}^k(t) + O(t^{-1}) & \text{if } O_j^l = x_j^l
\end{array} \right. \quad (2.231)$$

**Remark:** Observe that $\{ \hat{E}_{O_j^l}^k(t), \hat{E}_{x_j^l}^k(t) \}$ is still an approximately orthogonal basis localized at the corresponding critical orbits of $f$.

Finally, we have proved
Theorem 2

\[ F^*(t) = Int(e^{t\mathcal{I}})_{\mathbf{S}^1} : \left( \Omega_{\mathbf{g}, \text{small}}^*(M, t), \mathcal{D}(t) \right) \to (C^*(M, f), \delta) \]

is a morphism of cochain complexes such that

\[ F^*(t) = I + O(t^{-1}) \quad (2.232) \]

w.r.t. the bases \( \left\{ \lambda(\tilde{E}^k_{\mathcal{O}}(t)), \lambda(\tilde{E}^k_{\mathcal{D}}(t)) \right\} \) and \( \{ e^k_{\mathcal{O}}, e^k_{\mathcal{D}} \} \).
CHAPTER III

Witten-Helffer-Sjöstrand Theory for a Generalized Morse Function

3.1 Introduction

In this chapter, we extend the Witten-Helffer-Sjöstrand theory (cf. [W], [H-S]) from Morse functions on compact manifold to generalized Morse functions. Such a generalized Morse function has all critical points either non-degenerate or of birth-death type, i.e., in some neighbourhood of the critical point and with respect to a convenient coordinate system, the function can be written

\[
 f(x_1, x_2, \ldots, x_n) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + ax_n^3 \quad (3.1)
\]

for some \( a \neq 0 \).

The interest of generalized Morse function comes from the following theorem due to H. Chaltin (cf. [I]).
**Theorem:** If \( \pi : E \to B \) is a smooth bundle with fibre a compact manifold \( M \), then there exists \( f : E \to \mathbb{R} \) so that for any \( t \in B \)

\[
f_t = f \big|_{\pi^{-1}(t)} : M_t = \pi^{-1}(t) \to \mathbb{R}
\]

is a generalized Morse function.

It is easy to see that in general one cannot have such a statement with \( f_t \) a Morse function.

Now, let us state the results of this chapter. Let the eigenvalues of the operator

\[
-\frac{d^2}{dx^2} + 9x^4 - 6x
\]

be

\[
0 < c_1 < c_2 \leq \ldots \leq c_l \leq \ldots
\]

(See Lemma 3.2.2 in §3.2 for proof.)

Suppose \( M^n \) is a compact orientable Riemannian manifold, \( f \) be a generalized Morse function on \( M \).

Suppose \( x_1^k, \ldots, x_m^k \) are all the non-degenerate critical points of \( f \), of index \( k \), \( y_1^k, \ldots, y_n^k \) are all the critical points of birth-death type, of index \( k \). Also, let \( a_j^{(k)} \in \mathbb{R} \) be associated with \( y_j^k \) so that in some neighbourhood of \( y_j^k \) and with respect to a suitable oriented co-ordinate system,

\[
f(x_1, \ldots, x_n) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + a_j^{(k)}x_n^3 \quad (3.2)
\]
Suppose for simplicity that

\[ |a_{1}^{(0)}| < |a_{2}^{(0)}| < \ldots < |a_{m_{0}^{(0)}}| < |a_{1}^{(1)}| < \ldots < |a_{m_{1}^{(1)}}| < \ldots < |a_{m_{n-1}^{(n-1)}}| \]  (3.3)

(in fact, the Witten-Helffer-Sjöstrand theory is very similar with minor modifications without assuming (3.3))

Also, let \( g \) be a Riemannian metric on \( M \) so that in the above co-ordinate system near the critical points \( x_{j}^{k} \) or \( y_{j}^{k} \), \( g \) is the canonical metric on \( \mathbb{R}^{n} \).

Consider the Witten deformation of the de Rham complex \((\Omega^{r}(M), d(t))\) with

\[ d(t) = e^{-tf}d_{c}t^{f} : \Omega^{r}(M) \rightarrow \Omega^{r}(M) \]

Consider the deformed Laplacian

\[ \Delta(t) = d(t)d^{*}(t) + d^{*}(t)d(t) \]  (3.4)

When the above canonical coordinates near the critical points are used,

\[ \Delta(t) = \Delta + t^{2} \left| df \right|^{2} + tA \]  (3.5)

where

\[ A = \sum_{i,j=1}^{n} \frac{\partial^{2}f}{\partial x_{i} \partial x_{j}} [dx_{i}, i_{o_{j}}] \]  (3.6)

and \( i_{o_{j}} \) denotes the contraction along the vector field \( \partial_{j} \), \( dx_{i} \) is the exterior multiplication by the form \( dx_{i} \) and \([dx_{i}, i_{o_{j}}]\) denotes the commutator \( dx_{i}i_{o_{j}} - i_{o_{j}}dx_{i} \).

There are two cases.

**Case 1:** \( x_{j}^{k} \) is non-degenerate.
\begin{align*}
f(x) &= f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2 \quad (3.7) \\
|df|^2 &= 4(x_1^2 + \ldots + x_k^2) \quad (3.8) \\
A &= -2 \sum_{i=1}^k [dx_i, i_{h_i}] + 2 \sum_{i=k+1}^n [dx_i, i_{h_i}] \quad (3.9)
\end{align*}

**Case 2:** \(y_j^k\) is of birth-death type.

\begin{align*}
f(x) &= f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + a_j^{(k)} x_n^3 \quad (3.10) \\
|df|^2 &= 4(x_1^2 + \ldots + x_{n-1}^2) + 9(a_j^{(k)})^2 x_n^4 \quad (3.11) \\
A &= -2 \sum_{i=1}^k [dx_i, i_{h_i}] + 2 \sum_{i=k+1}^{n-1} [dx_i, i_{h_i}] + 6a_j^{(k)} [dx_n, i_{h_n}] \quad (3.12)
\end{align*}

For each critical point \(c\), define the 'localized' operator \(\overline{\Delta}_c(t) : C^\infty(A^*(\mathbb{R}^n)) \to C^\infty(A^*(\mathbb{R}^n))\) which is given by (3.5) where \(A\) is (3.9) if \(c = x_j^k\) is nondegenerate, respectively (3.12) if \(c = y_j^k\) is birth-death. The Laplace operator \(\Delta\) in (3.5) is the canonical Laplace operator corresponding the canonical metric in \(\mathbb{R}^n\). The operator \(\overline{\Delta}_c(t)\) then extends uniquely to a self-adjoint positive unbounded operator in \(L^2(A^*(\mathbb{R}^n))\).

Now suppose \(\overline{\Delta}(t)\) is the 'localized' operator associated to a critical point of birth-death type. Since \(L^2(A^*(\mathbb{R}^n)) \cong L^2(A^*(\mathbb{R}^{n-1})) \otimes L^2(A^*(\mathbb{R}^n))\), \(\overline{\Delta}(t)\) can be written as

\[
\overline{\Delta}(t) = \{ \Delta_{\mathbb{R}^{n-1}} + 4t^2(x_1^2 + \ldots + x_{n-1}^2) + 2t \sum_{i=1}^{n-1} \epsilon_i [dx_i, i_{h_i}] \} \otimes id \\
+ id \otimes \{ \Delta_{\mathbb{R}^n} + 9(a_j^{(k)})^2 t^2 x_n^4 + 6a_j^{(k)} t x_n [dx_n, i_{h_n}] \}
\]
where

\[ \chi_i = \begin{cases} 
-1 & \text{if } 1 \leq i \leq k \\
1 & \text{if } k + 1 \leq i \leq n - 1 
\end{cases} \]  

(3.13)

and \( \Delta_{\mathbb{R}^{n-1}}, \Delta_{\mathbb{R}} \) are the Laplace operators on \( \mathbb{R}^{n-1} \) and \( \mathbb{R} \) respectively.

By Corollary 3.2.2 in §3.2, \( \Delta_R + 9(a_j^{(k)})^2 x_n^3 + 6a_j^{(k)} x_n [dx_n, i_n] : L^2(\Lambda^*(\mathbb{R})) \to L^2(\Lambda^*(\mathbb{R})) \) has discrete spectrum with eigenvalues

\[ 0 < c_1(\| a_j^{(k)} \| t)^{2/3} < c_2(\| a_j^{(k)} \| t)^{2/3} \leq c_3(\| a_j^{(k)} \| t)^{2/3} \leq \ldots \]

Each eigenvalue has a multiplicity of 2 with corresponding eigenvectors consisting of a 0-form and a 1-form.

Let \( \Delta^k(t) = \Delta(t) |_{L^2(\Lambda^k(M))} \).

Let \( 0 \leq E_1(t) \leq E_2(t) \leq \ldots \leq E_l(t) \leq \ldots \) be all the eigenvalues of \( \Delta^k(t) \).

Suppose for simplicity that

\[ c_1(\| a_j^{(n-1)} \| t)^{2/3} < c_2(\| a_j^{(0)} \| t)^{2/3} \]  

(3.14)

This together with (3.3) imply that \( c_1(\| a_j^{(k)} \| t)^{2/3} < c_2(\| a_j^{(k-1)} \| t)^{2/3} \) for all \( k \).

**Theorem 3 (Quasi-classical limit of eigenvalues)**

\[
\lim_{t \to \infty} E_1(t) = \ldots = \lim_{t \to \infty} E_{m_1}(t) = 0
\]

\[
\lim_{t \to \infty} \frac{E_{m_1+1}(t)}{t^{2/3}} = c_1(\| a_j^{(k-1)} \| t)^{2/3} < \lim_{t \to \infty} \frac{E_{m_1+2}(t)}{t^{2/3}} = c_1(\| a_j^{(k-1)} \| t)^{2/3} < \ldots
\]

\[
\ldots < \lim_{t \to \infty} \frac{E_{m_k+m_k+1}(t)}{t^{2/3}} = c_1(\| a_j^{(k)} \| t)^{2/3}
\]

\[
\left( \lim_{t \to \infty} \frac{E_{m_k+m_k+1}(t)}{t^{2/3}} = c_2(\| a_j^{(k-1)} \| t)^{2/3} < \ldots \right)
\]
**Remarks:** 1. In fact, the eigenvectors corresponding to $E_1(t), \ldots, E_{m_k}(t)$ are localized at the non-degenerate critical points of index $k$, while the eigenvectors corresponding to $E_{m_k+1}(t), \ldots, E_{m_{k+1}+1}(t)$ are localized at the birth-death critical points of index $k-1$ and $k$. However, the eigenvectors are not necessarily localized at a single critical point.

2. If all the critical points of $f$ are non-degenerate, then the above theorem should be formulated as follows (cf. [S] p219):

**Theorem:**

$$\lim_{t \to \infty} E_1(t) = \ldots = \lim_{t \to \infty} E_{m_k}(t) = 0$$

$$0 < \lim_{t \to \infty} \frac{E_{m_k+1}(t)}{t} \leq \lim_{t \to \infty} \frac{E_{m_{k+1}}(t)}{t} \leq \ldots$$

Let us index $a_1^{(0)}, \ldots, a_{m_0}', a_1^{(1)}, \ldots, a_{m_1}'$, by $b_1, \ldots, b_N$ where $N = \sum_{k=0}^{\infty} m_k'$.

Also, for $0 \leq l \leq n-1, 1 \leq j \leq m_l'$ let

$$I_j^{(l)}(\epsilon) = [c_1(|a_j^{(l)}|)^{2/3} - \epsilon, c_1(|a_j^{(l)}|)^{2/3} + \epsilon]$$

Choose an $\epsilon$ small enough so that the family of intervals

$$\{[0, \epsilon], I_j^{(l)}(\epsilon), [c_2(|a_1^{(0)}|)^{2/3} - \epsilon, \infty)\}_{0 \leq t \leq n-1, 1 \leq j \leq m_l'}$$

is pairwise disjoint. The pairwise disjointness is satisfied if $\epsilon$ satisfies

$$0 < \epsilon < \min_i \left( \frac{c_1}{2} |a_1^{(0)}|^{2/3}, \frac{c_1}{2} (|b_{i+1}|^{2/3} - |b_i|^{2/3}), \frac{1}{2} (c_2 |a_1^{(0)}|^{2/3} - c_1 |a_{m_0}'|^{2/3}) \right)$$

As a consequence of Theorem 3, for $t$ sufficiently large and $\epsilon$ satisfying the above disjointness condition (3.15), we have

$$Spcc(t^{-2/3} \Delta^k(t)) \subset [0, \epsilon] \cup (\bigcup_{t \leq k, 1 \leq j \leq m_l'} I_j^{(l)}(\epsilon)) \cup [c_2(|a_1^{(0)}|)^{2/3} - \epsilon, \infty)$$

(3.16)
and

\[
\begin{cases}
\text{Card} \left( \text{Spec} \left( t^{-2/3} \Delta^k(t) \right) \cap [0, \varepsilon] \right) = m_k \\
\text{Card} \left( \text{Spec} \left( t^{-2/3} \Delta^k(t) \right) \cap I^0_j(\varepsilon) \right) = 1 \\
\text{(for } l = k - 1, k; 1 \leq j \leq m_i)\end{cases}
\]

Define

\[\Omega^k_{\text{small}}(M, t) = \text{Span} \{ \psi \in L^2(\Lambda^k(M)) | \Delta^k(t) \psi(t) = E(t) \psi(t), t^{-2/3} E(t) \in [0, \varepsilon] \} \]

\[\Omega^k_{\text{large}, k,j}(M, t) = \text{Span} \{ \psi \in L^2(\Lambda^k(M)) | \Delta^k(t) \psi(t) = E(t) \psi(t), t^{-2/3} E(t) \in [c_1(|a^0_j|)^{2/3} - \varepsilon, c_1(|a^0_j|)^{2/3} + \varepsilon] \}
\]

\[\Omega^k_{\text{large}}(M, t) = \text{Span} \{ \psi \in L^2(\Lambda^k(M)) | \Delta^k(t) \psi(t) = E(t) \psi(t), t^{-2/3} E(t) \in [c_2 - \varepsilon, \infty] \}
\]

\[\left( \Omega^*_0(M, t), d(t) \right) = \left( \Omega^*_\text{small}(M, t), d(t) \right) \perp \left( \Omega^*_{\text{large}, k,j}(M, t), d(t) \right) \]

**Corollary**

\[\left( \Omega^*(M), d(t) \right) = \left( \Omega^*_\text{small}(M, t), d(t) \right) \perp \left( \Omega^*_{\text{large}, k,j}(M, t), d(t) \right) \perp \left( \Omega^*_{\text{large}}(M, t), d(t) \right) \]

and \(\left( \Omega^*_\text{small}(M, t), d(t) \right), \left( \Omega^*_0(M, t), d(t) \right)\) are complexes of finite dimensional vector spaces which calculate the de Rham cohomology of \(M\).

**Remark:** Observe that \(\left( \Omega^*_\text{large}, k,j(M, t), d(t) \right)\) has dimension 2.

As in the Helffer-Sjöstrand theory for a generic pair \((f,g)\), \(\left( \Omega^*_0(M, t), d(t) \right)\) converges as \(t \to \infty\) to a geometric complex, which can be described as follows.

Let \(f\) be a self-indexing generalized Morse function, i.e.

\[
\begin{cases}
f(x^k_j) = k & \text{if } x^k_j \text{ is a non-degenerate critical point of index } k \\
f(y^k_j) \in (k, k + 1) & \text{if } y^k_j \text{ is a birth-death critical point of index } k
\end{cases}
\]
Let \( W_x^\rightarrow \) be the descending manifold of a non-degenerate critical point \( x_j \). For a birth-death critical point \( y_j \), choose an open neighbourhood \( U_{y_j} \) and a suitable co-ordinate s.t.

\[
    f(x) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + a_j^{(k)} x_n^3
\]

(3.18)

Let

\[
    W_{y_j}^{-,0} = \{ x \in M | \gamma_x(t) \in U_{y_j} \cap \mathbb{R}^k \text{ for some } t \in \mathbb{R}\}
\]

where \( \gamma_x \) is the trajectory of \( \text{Grad } f \) s.t. \( \gamma_x(0) = x \}

\[
    W_{y_j}^{-,1} = \{ x \in M | \gamma_x(t) \in U_{y_j} \cap (\mathbb{R}^k \times \mathbb{R}_-) \text{ for some } t \in \mathbb{R}\}
\]

where \( \mathbb{R}^k \times \mathbb{R}_- = \{ (x_1, \ldots, x_k, 0, \ldots, 0, x_n) \in \mathbb{R}^n | x_n < 0 \} \}

when \( a > 0 \) with obvious modifications when \( a < 0 \). Then \( W_{y_j}^{-,0}, W_{y_j}^{-,1} \) are manifolds diffeomorphic to \( \mathbb{R}^k, \mathbb{R}^{k+1} \) respectively. Note that \( W_{y_j}^{-,0} \cap W_{y_j}^{-,1} = \emptyset \)

Define the descending manifold

\[
    W_{y_j}^- = W_{y_j}^{-,0} \cup W_{y_j}^{-,1}
\]

which is then a manifold with boundary diffeomorphic to \( \mathbb{R}^{k+1}_+ \). The ascending manifold \( W_{y_j}^+ \) is defined similarly.

Suppose the ascending and descending manifolds for any two critical points intersect transversally, then \( \{ W_{x_j}^-, W_{y_j}^{-,0}, W_{y_j}^{-,1} \} \) form a CW-complex (see §3.3 for more details). While the incidence number between \( W_{x_j}^- \) and \( W_{x_{k+1}}^- \) is given by the intersection number between the ascending and descending manifolds in \( f^{-1}(k + \frac{1}{2}) \) i.e. between \( W_{x_j}^+ \cap f^{-1}(k + \frac{1}{2}) \) and \( W_{x_{k+1}}^- \cap f^{-1}(k + \frac{1}{2}) \), the incidence number is 1 between \( W_{y_j}^{-,0} \) and \( W_{y_j}^{-,1} \). However, those between \( W_{x_j}^- \) and \( W_{y_j}^{-,i} \) may be non-trivial.
Let us denote the above described chain complex by \((C_\ast (M, f), \partial)\) (with \(C_k(M, f) = \text{Span}\{W_{y_j}^{-}, W_{y_j}^{0}, W_{y_j}^{-1}\}\)), its dual cochain complex by \((C^\ast (M, f), \delta)\).

Also, let us rescale the complex \((\Omega_0^\ast (M, t), d(t))\) to be

\[
(\Omega_0^\ast (M, t), \tilde{d}(t)) = \left(\Omega_{\text{small}}^\ast (M, t), e^{\sqrt{\frac{\pi}{2t}}} d(t)\right) \bot \left(\Omega_{\text{large}, j, k}^\ast (M, t), d(t)\right)
\]

**Theorem 4** There exists \(f^\ast (t) : (\Omega_0^\ast (M, t), \tilde{d}(t)) \to (C^\ast (M, f), \delta)\) which is a morphism of co-chain complexes s.t.

\[
f^\ast (t) = I + O(t^{-1}) \quad (3.20)
\]

w.r.t. some suitably chosen bases.

**Definitions:**

1. (i) Suppose \(x_j^k, x_{j+1}^{k+1}\) are two non-degenerate critical points, \(\gamma\) be a generalized trajectory between \(x_j^{k+1}\) and \(x_j^k\), i.e. \(\gamma\) is a piecewise smooth curve with singularities at the birth-death points \(y_1, \ldots, y_{n(\gamma)}\) and

\[
\gamma = \gamma_{x_j^{k+1}, y_1} \cup \{y_1\} \cup \gamma_{y_1 y_2} \cup \{y_2\} \cup \ldots \cup \gamma_{y_{n(\gamma)} x_j^k}
\]

(3.21)

where \(\gamma_{y_i y_{i+1}}\) is a trajectory between \(y_i\) and \(y_{i+1}\). Then one can associate \(\epsilon = \pm 1\) to the trajectories \(\gamma_{x_j^{k+1}, y_1}, \gamma_{y_1 y_{i+1}}\) as in the Witten-Morse theory. Then define

\[
\epsilon^\text{new}_\gamma = (-1)^{n(\gamma)} \epsilon_{x_j^{k+1}, y_1} \left(\prod_{l=1}^{n(\gamma)-1} \epsilon_{y_{i+1} y_{i+1}}\right) \epsilon_{\gamma_{y_{n(\gamma)} x_j^k}}
\]

(3.22)

(ii) Suppose \(x_j^k\) is an non-degenerate critical point and \(y_j^k\) a birth-death critical point, \(\gamma\) be a generalized trajectory between them. With the above notation for \(\gamma\) and \(y_1 = y_j^k\), define

\[
\epsilon^\text{new}_\gamma = (-1)^{n(\gamma)} \left(\prod_{l=1}^{n(\gamma)-1} \epsilon_{y_j y_{i+1}}\right) \epsilon_{\gamma_{y_{n(\gamma)} x_j^k}}
\]

(3.23)
2. (i) The (generalized) incidence number between two critical points is defined as follows:

\[
I(x_i^{k+1}, x_j^k) = \sum_{\gamma} \epsilon_{\gamma}^{new} \tag{3.24}
\]

\[
I(y_i^k, x_j^k) = \sum_{\gamma} \epsilon_{\gamma}^{new} \tag{3.25}
\]

where \( \gamma \) is a generalized trajectory between the initial and end point.

(ii) Here we recall that the (ordinary) incidence number between two critical points (non-degenerate or birth-death) is

\[
i(x_i^{k+1}, x_j^k) = \sum_{\gamma} \epsilon_{\gamma} \tag{3.26}
\]

\[
i(y_i^k, x_j^k) = \sum_{\gamma} \epsilon_{\gamma} \tag{3.27}
\]

where \( \gamma \) is a trajectory between the two critical points.

**Remark:** Observe that in general

\[
\epsilon_{\gamma}^{new} \neq \epsilon_{\gamma} \tag{3.28}
\]

for a trajectory between two critical points. For example, if \( \gamma \) is a trajectory between a birth-death point \( y_i^k \) and a non-degenerate critical point \( x_j^k \), then \( \epsilon_{\gamma}^{new} = -\epsilon_{\gamma} \).

With the above definition, we can reformulate Theorem 4 in the same way that Helffer and Sjöstrand formulated their theorem:

**Theorem 4':** There exist orthonormal bases \( \{E_{x_j^k}(t)\} \) of \( \Omega_{small}^k(M, t) \), \( \{E_{y_j^k}(t), E_{y_j^k}(t)\} \) of \( \Omega_{large,k,j}^*(M, t) \) s.t.

\[
< E_{x_j^k}(t), d(t)E_{x_j^k}(t) > = e^{-t} \left( \sqrt{\frac{t}{\pi}} \sum_{\gamma} \epsilon_{\gamma}^{new} + O(t^{-1/2}) \right) \tag{3.29}
\]
\[ <E_{y_j}^1(t), d(t)E_{y_j}^0(t)> = \sqrt{\epsilon_1(a_j^k)^{1/3}t^{1/3}} + O(t^{1/6}) \] (3.30)

Also for \( t \) sufficiently large, we have

\[ <E_{y_j}^1(t), d(t)E_{y_j}^0(t)> = 0 \text{ if } j_1 \neq j_2 \] (3.31)

and

\[ <E_{x_i}(t), d(t)E_{y_j}^0(t)> = <E_{y_j}^1(t), d(t)E_{x_i}(t)> = 0 \] (3.32)

where \( \sum \epsilon_{y_j}^{u,v} = I(x_i^{k+1}, x_j^{k}) \) is the incidence number between \( x_i^{k} \) and \( x_i^{k+1} \) defined above.

Inside the complex \( (C^*(M, f), \delta) \), there is a subcomplex \( (C^*_{nd}(M, f), \delta) \) such that

\[ \dim C^*_{nd}(M, f) = m_k \]

where \( m_k \) is the number of non-degenerate critical points of index \( k \). Note that this subcomplex is not generated by the non-degenerate critical points, since the latter in general do not generate a subcomplex. Instead the subcomplex is obtained by applying Lemma 3.3.7 repeatedly as is done in §3.3. See §3.3 for details.

**Theorem 4**: \( f^*(t) |_{\Omega^*_{\text{mult}}(M,t)}: \Omega^*_{\text{mult}}(M,t, \tilde{d}(t)) \rightarrow (C^*(M, f), \delta) \) is an injective homomorphism of co-chain complexes whose image complex converges to \( (C^*_{nd}(M, f), \delta) \) in \( (C^*(M, f), \delta) \) as \( t \rightarrow \infty \), more precisely,

\[ f^k(t) \left( E_{x_j^k}(t) \right) = \hat{e}_{x_j^k} + O(t^{-1}) \text{ in } C^*(M, f) \] (3.33)

where \( \hat{e}_{x_j^k} = \epsilon_{x_j^k} + \sum_{l} I(y_l^k, x_j^k) \delta_{y_l^k} \in C^k_{nd}(M, f). \)
Also, using similar consideration, one can extend the result to any representation \( \rho : \pi_1(M) \to GL(V) \) (cf.[BZ]) or any representation \( \rho : \pi_1(M) \to GL(W_A) \) where \( W_A \) is finite type Hilbert module over a finite von Neumann algebra \( A \) (cf.[BFKM]).

### 3.2 Witten Deformation for a Generalized Morse Function

Let \( f \) be a generalized Morse function on \( M^n \), \( y \) be a critical point of birth-death type. Let \((U_y, \varphi)\) be a chart s.t. \( y \in U_y \), and

\[
    f(\varphi^{-1}, x) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + ax_n^3
\]

(3.34)

Let \( g \) be a Riemannian metric on \( M \) s.t. \((\varphi^{-1})^*(g) = \delta_{ij}\).

Define

\[
    d(t) = e^{-tf} d\epsilon^
u J
\]

(3.35)

\[
    \Delta(t) = d(t) d^* (t) + d^* (t) d(t)
\]

(3.36)

Then in the coordinate system \((U_y, \varphi)\),

\[
    \Delta(t) = \Delta + t^2 \left[ |df|^2 + 2t \sum_{i=1}^{n-1} \epsilon_i [dx_i, i\theta_i] + 6ux_n t dx_n, i\theta_n \right]
\]

(3.37)

where

\[
    |df|^2 = 4(x_1^2 + \ldots + x_{n-1}^2) + 9a^2 x_n^4
\]

(3.38)

\[
    \epsilon_i = \begin{cases} 
    -1 & \text{if } 1 \leq i \leq k \\
    1 & \text{if } k + 1 \leq i \leq n - 1
\end{cases}
\]

(3.39)
Define $\Delta(t) : L^2(\Lambda^*(\mathbb{R}^n)) \to L^2(\Lambda^*(\mathbb{R}^n))$ to be given exactly by the above expression.

Recall that

$$\Delta(t) = \{ \Delta_{\mathbb{R}^{n-1}} + 4t^2(x_1^2 + \ldots + x_{n-1}^2) + 2t \sum_{i=1}^{n-1} \epsilon_i [d{x_i}, i_{a_i}] \} \otimes id$$

$$\otimes id \otimes \{ \Delta_{\mathbb{R}} + 9t^2 a^2 x_n^3 + 6atx_n [dx_n, i{x_n}] \}$$

Observe that the first term is exactly the Witten deformed Laplacian on $L^2(\Lambda^*(\mathbb{R}^{n-1}))$ for the classical Morse theory (cf. [S], [W]) and that the two operators in parenthesis commute with each other. Hence to study the spectrum of $\Delta(t)$, it suffices to find out the spectrum of $\Delta + 9t^2 a^2 x^1 + 6atx [dx, i_{x^1}]$ on $L^2(\Lambda^*(\mathbb{R}))$. Note that

$$[dx, i_{\frac{\partial}{\partial x}}](f) = -f$$

$$[dx, i_{\frac{\partial}{\partial x}}](f dx) = f dx$$

Therefore

$$\left\{ \Delta + 9t^2 a^2 x_1^3 + 6atx [dx, i_{\frac{\partial}{\partial x}}] \right\}(f) = \left( -\frac{d^2}{dx^2} + 9t^2 a^2 x_1^3 - 6atx \right)(f)$$

$$\left\{ \Delta + 9t^2 a^2 x_1^3 + 6atx [dx, i_{\frac{\partial}{\partial x}}] \right\}(f dx) = \left( -\frac{d^2}{dx^2} + 9t^2 a^2 x_1^3 + 6atx \right)(f) dx$$

Define $\mathcal{R} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$(\mathcal{R}f)(x) = f(-x)$$

Then

$$\mathcal{R}^{-1} \left( -\frac{d^2}{dx^2} + 9t^2 a^2 x_1^3 + 6atx \right) \mathcal{R} = -\frac{d^2}{dx^2} + 9t^2 a^2 x_1^3 - 6atx$$

Hence, it is sufficient to consider

$$P(at) = -\frac{d^2}{dx^2} + 9t^2 a^2 x_1^3 - 6atx$$
where \( P(t) \equiv -\frac{d^2}{dx^2} + 9t^2 x^4 - 6t x \) for \( t \in \mathbb{R} \).

Define \( U(\lambda) : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \), \( \lambda > 0 \)

\[
(U(\lambda) f)(x) = \lambda^{1/2} f(\lambda x)
\]

(3.47)

**Lemma 3.2.1** For \( t > 0 \),

\[
P(t) = U(t^{1/3}) \left( t^{2/3} P(1) \right) U(t^{-1/3})
\]

(3.48)

**Proof:** Clearly the above equation is equivalent to

\[
U(t^{-1/3}) P(t) U(t^{1/3}) = t^{2/3} P(1)
\]

(3.49)

By the definition of \( U(\lambda) \) and direct computation,

\[
\left[ U(t^{-1/3}) \frac{d^2}{dx^2} U(t^{1/3}) f \right](x) = t^{2/3} \frac{d^2 f}{dx^2}(x)
\]

(3.50)

\[
\left[ U(t^{-1/3}) 9t^2 x^4 U(t^{1/3}) f \right](x) = t^{2/3} (9x^4 f(x))
\]

(3.51)

\[
\left[ U(t^{-1/3}) 6t x U(t^{1/3}) f \right](x) = t^{2/3} (6x f(x))
\]

(3.52)

Hence

\[
U(t^{-1/3}) \left( -\frac{d^2}{dx^2} + 9t^2 x^4 - 6t x \right) U(t^{1/3}) = t^{2/3} \left( -\frac{d^2}{dx^2} + 9x^4 - 6x \right)
\]

(3.53)

**Lemma 3.2.2** (i) \( P(1) \) has compact resolvent, hence has discrete spectrum.

\[
0 \leq \epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_t \leq \ldots
\]

(ii) The smallest eigenvalue of \( P(1) \) is strictly positive and is simple, i.e.

\[
0 < \epsilon_1 < \epsilon_2 \leq \ldots
\]
(iii) Let \( \Xi_1 \) be a normalized eigenfunction of \( P(1) \) corresponding to the smallest eigenvalue \( c_1 \). Then one can choose \( \Xi_1 \) so that \( \Xi_1(x) > 0 \) for all \( x \in \mathbb{R} \). In particular \( \Xi_1(0)^{-1} \) exists.

**Corollary 3.2.3:** \( P(\alpha t) \) has spectrum

\[
0 < c_1(\alpha t)^{2/3} < c_2(\alpha t)^{2/3} \leq c_3(\alpha t)^{2/3} \leq \ldots \leq c_1(\alpha t)^{2/3} \leq \ldots
\]

**Proof of Lemma 3.2.2:** (i) \( P(1) \) has compact resolvent because \( V(x) = 9x^4 - 6x \to \infty \) as \( |x| \to \infty \) and is bounded from below. (cf. [RS] p249) It is positive because it is the restriction of the deformed Laplacian associated with the function \( x^3 \) on the invariant subspace \( L^2(\mathbb{R}) \).

(ii) Let \( \left( -\frac{d^2}{dx^2} + 9x^4 - 6x \right) f = 0 \)

Then

\[
f = c_1 e^{-x^3/2} + c_2 e^{-x^4/2} \int_0^x e^{2u^3} du
\]

(iii) The application of Feynman-Kac formula in (ii) shows that \( \Xi_1 \) can be chosen s.t. \( \Xi_1(x) > 0 \) almost everywhere.
Suppose \( \Xi_1(x_0) = 0 \) for some \( x_0 \). Since \( \Xi_1(x) > 0 \) almost everywhere, we have \( \Xi'_1(x_0) = 0 \).

Since \( \Xi_1 \) is an eigenfunction of \( P(1) \) corresponding to the eigenvalue \( \epsilon_1 \), we have

\[
\Xi''_1(x) = (9x^4 - 6x - \epsilon_1)\Xi_1(x)
\]

By Leibnitz Rule,

\[
\Xi^{(n+2)}_1(x) = \sum_{i=0}^{n} p_{n+2,i}(x)\Xi^{(i)}_1(x)
\]

where \( p_{n+2,i} \) are polynomials in \( x \) and \( \Xi^{(i)}_1 \) denotes the \( i \)-th derivative of \( \Xi_1 \).

Using the above equation together with \( \Xi_1(x_0) = \Xi'_1(x_0) = 0 \), one shows that

\[
\Xi^{(n)}_1(x_0) = 0 \quad \text{for all } n = 0, 1, 2, \cdots
\]

by induction on \( n \). Since \( \Xi_1 \) is analytic, we have \( \Xi_1 = 0 \), contrary to the assumption that \( \Xi_1 \) is a non-zero eigenfunction. □

Now, we have shown that the smallest eigenvalue of \( P(at) \) is \( \epsilon_1(|at|)^{2/3} > 0 \) and is simple. Since \( -\frac{d^2}{dx^2} + 9ax^2x + 6atx \) is conjugated to \( P(at) \) by an isometry \( R \), the same is true for its smallest eigenvalue. Hence \( -\frac{d^2}{dx^2} + 9ax^2x + 6atx[dx, \frac{dx}{dt}] \) on \( L^2(\Lambda^*(\mathbb{R})) \) has smallest eigenvalue \( \epsilon_1(|at|)^{2/3} \) of multiplicity 2, the corresponding eigenvectors are \( \Xi_1(x) (\in \Omega^0(\mathbb{R})) \) and \( \Xi_1(-x)dx (\in \Omega^1(\mathbb{R})) \).

Returning to the 'localized' operator,

\[
\overline{\Delta}(t) = \left\{ \Delta_{\mathbb{R}^{n-1}} + 4t^2(x_1^2 + \cdots + x_{n-1}^2) + 2t \sum_{i=1}^{n-1} \epsilon_i[d_{x_i}, i_{a_i}] \right\} \otimes \text{id} + \text{id} \otimes \left\{ \Delta_{\mathbb{R}} + 9at^2x_n + 6atx_n[d_{x_n}, i_{a_n}] \right\}
\]
The smallest eigenvalue is also $\epsilon_1(\cdot) 2/3$ and of multiplicity 2, whose eigenvectors are spanned by a k-form and a (k+1)-form.

$$\omega_k(t) = t^{(n-1)/4+1/6} \xi_1(t^{1/2}x_1) \ldots \xi_1(t^{1/2}x_{n-1}) \Xi_1(t^{1/3}x_n) dx_1 \wedge \ldots \wedge dx_k$$  \hspace{1cm} (3.59)

$$\omega_{k+1}(t) = t^{(n-1)/4+1/6} \xi_1(t^{1/2}x_1) \ldots \xi_1(t^{1/2}x_{n-1}) \Xi_1(-t^{1/3}x_n) dx_1 \wedge \ldots \wedge dx_k \wedge dx_n$$  \hspace{1cm} (3.60)

where $\xi_1(x)$ is the groundstate of $-\frac{d^2}{dx^2} + 4x^2$ i.e. $\xi_1(x) = e^{-tx^2}$.

**Sketch of Proof (of Theorem 3):** With the above observations, the proof of Theorem 3 follows essentially the arguments in \[S\](pp 219-222).

Let $C_{bd}$ be the set of birth-death critical points of $f$, $C_{nd}$ be the set of non-degenerate critical points of $f$.

$0 = \epsilon^k_1 = \ldots = \epsilon^k_{m_k}$ be the smallest eigenvalues of $\sum_{j \in C_{nd}} \Xi_j(1)$

$0 \leq \epsilon^k_{m_k+1} \leq \epsilon^k_{m_k+2} \leq \ldots$ be the eigenvalues of $\sum_{j \in C_{bd}} \Xi_j(1)$

Let $\{\Psi^k_l(1)\}_{l=1}^\infty$ be the eigenvectors corresponding to $\epsilon^k_l$ of $\sum_{j \in C_{crd}(f)} \Xi_j(1)$

More generally, let $\{\Psi^k_l(t)\}_{l=1}^\infty$ be the eigenvectors corresponding to $\epsilon^k_l t^{2/3}$ of the operator $\sum_{j \in C_{crd}(f)} \Xi_j(t)$.

Note that

$$\Psi^k_l(t) = \begin{cases} U(t^{1/2}) \Psi^k_l(1) & \text{if } 1 \leq l \leq m_k \\ U(t^{1/3}) \Psi^k_l(1) & \text{if } m_k + 1 \leq l \end{cases}$$

Then Theorem 3 is essentially equivalent to

$$\lim_{t \to \infty} \frac{E_l(t)}{t^{2/3}} = \epsilon^k_l$$  \hspace{1cm} (3.61)

The proof is divided into 2 steps,

(i) $\lim_{t \to \infty} \frac{E_l(t)}{t^{2/3}} \leq \epsilon^k_l$
This follows by similar arguments as in [S]. Let \( \{J_j\}_{j \in \mathcal{C}^{(n)}} \) be a partition of unity on \( M \) (cf.[S] p 27). Let \( \Psi^k_j(t) \) be an eigenvector of \( \Delta^k_j(t) \), define \( \varphi^k_j(t) = J_j(t) \Psi^k_j(t) \). Then \( \{\varphi^k_j(t)\} \) form a set of approximate eigenvectors in \( L^2(\Lambda^k(M)) \) with

\[
< \varphi^m(t), \Delta^k(t)\varphi^a(t) > = c^k \epsilon^{2/3} \delta_{nm} + o(\epsilon^{2/3})
\]  

By using the min-max principle, (i) follows.

(ii) \( \lim_{t \to \infty} \frac{E_i(t)}{\epsilon^{2/3}} \geq \epsilon^k_i \)

To prove (ii), one has to modify slightly the arguments in [S]. It suffices to construct, for \( e \in (e^k_1, e^k_{i+1}) \), a symmetric operator \( R(t) \) of rank \( l \) s.t.

\[
\Delta^k(t) \geq \epsilon^{2/3} e + R(t) + o(\epsilon^{2/3})
\]  

To construct \( R(t) \) define \( \Delta^k_j(t) : C^\infty(\Lambda^k(M)) \to C^\infty(\Lambda^k(M)) \)

\[
\Delta^k_j(t) = \Delta^k + \epsilon f_j + tA
\]  

\[
f_j(x) = \begin{cases} 
|df(x)|^2 & \text{if } x \in U_j \\
0 & \text{if } x \notin U_j
\end{cases}
\]

Let \( 0 \leq E^{(j)}_1(t) \leq E^{(j)}_2(t) \leq \ldots \leq E^{(j)}_{i}(t) \leq \ldots \) be the eigenvalues of \( \Delta^k_j(t) \).

\( \Psi^{(j)}_1(t), \Psi^{(j)}_2(t), \ldots, \Psi^{(j)}_i(t), \ldots \) be the corresponding eigenvectors of \( \Delta^k_j(t) \).

For \( j \in C_{\text{bd}} \), let \( 0 \leq e^{(j)}_1 \leq e^{(j)}_2 \leq \ldots \leq e^{(j)}_j \leq \ldots \) be the eigenvalues of \( \Delta^k_j(1) \), then one can show that

\[
\lim_{t \to \infty} \frac{E^{(j)}_i}{\epsilon^{2/3}} = e^{(j)}_i
\]  

Define \( n_j \) s.t. \( e^{(j)}_n_j < e < e^{(j)}_{n_j+1} \)

\[
P_j(t) = \begin{cases} 
\text{orthogonal projection onto } \text{span} \{ \Psi^{(j)}_i \}_{1 \leq i \leq n_j} & \text{if } j \in C_{\text{bd}} \\
\text{orthogonal projection onto smallest eigenvector of } \Delta^k_j(t) & \text{if } j \in C_{\text{int}}
\end{cases}
\]
\[ R_j(t) = \left( \Delta_j^k(t) - t^{2/3} c \right) P_j(t) \]

Define

\[ R(t) = \sum_{j \in \text{Crit}(f)} J_j R_j(t) J_j \] (3.66)

which is a symmetric operator of rank \( l \).

To verify (3.63), observe that by IMS localization formula (cf. [S] p28)

\[ \Delta_j^k(t) \geq t^{2/3} c J_j^2 + \sum_{j \in \text{Crit}(f)} J_j \Delta_j^k(t) J_j + O(1) \] (3.67)

Then (3.63) follows from the definition of \( R(t) \).

One finishes the proof by showing that

\[ \lim_{t \to \infty} E_1(t) = \ldots = \lim_{t \to \infty} E_{m_k}(t) = 0 \quad \square \]

### 3.3 Helffer-Sjöstrand Theory for a Generalized Morse Function

**Definition:** A pair \((f, g)\) is said to satisfy the Morse-Smale condition where \( f \) is a generalized Morse function if for any two critical points \( x \) and \( y \), the ascending manifold \( W_x^+ \) and the descending manifold \( W_y^- \), w.r.t. \(-\text{Grad}_g f\), intersect transversally.

In the case of a birth-death critical point \( y_j^k\) of index \( k \) such that (3.10) holds, define

\[ W_{y_j^k}^{+,0} = \{ x \in M | y_j(t) \in U_{y_j^k} \cap \mathbb{R}^{n-k-1} \text{ for some } t \in \mathbb{R} \} \]

where \( \mathbb{R}^{n-k-1} = \{(0, \ldots, 0, x_{k+1}, \ldots, x_{n-1}, 0) \in \mathbb{R}^n \} \)
\[ W_{y_j}^{+,1} = \{ x \in M | \gamma(t) \in U \cap (\mathbb{R}^{n-k-1} \times \mathbb{R}_{+}) \text{ for some } t \in \mathbb{R} \} \]

where \( \mathbb{R}^{n-k-1} \times \mathbb{R}_{+} = \{(0, \ldots, 0, x_{k+1}, \ldots, x_n) \in \mathbb{R}^n | x_n > 0 \} \)

while \( W_{y_j}^{-,0} \), \( W_{y_j}^{-,1} \) are defined similarly as in §3.1. Then the ascending and descending manifolds are defined as follows:

\[
W_{y_j}^+ = W_{y_j}^{+,0} \cup W_{y_j}^{+,1} \quad (3.68)
\]

\[
W_{y_j}^- = W_{y_j}^{-,0} \cup W_{y_j}^{-,1} \quad (3.69)
\]

**Proposition 3.3.1** For any pair \((f, y)\), there is a \( C^1 \) approximation \( y' \) such that \( y = y' \) in a neighbourhood of the critical points of \( f \) and \((f, y')\) satisfies the Morse-Smale condition.

**Proof:** The proof is the same as in [Sm].

**Definition:** Let \( f \) be a generalized Morse function. \( f \) is said to be self-indexing if

\[
\begin{cases}
  f(x_j^k) = k & \text{if } x_j^k \text{ is a non-degenerate critical point of index } k \\
  f(y_j^k) \in (k, k+1) & \text{if } y_j^k \text{ is birth-death critical point of index } k \\
  f(x_j^k - \kappa) = n & \text{if } x_j^k \text{ is a non-degenerate critical point of index } k \\
  f(y_j^k) \in (k, k+1) & \text{if } y_j^k \text{ is birth-death critical point of index } k
\end{cases}
\]

**Proposition 3.3.2** For any generalized Morse function \( f \), there exists a self-indexing generalized Morse function \( f' \) such that \( f \) and \( f' \) have the same critical points and corresponding indexes.

**Proof:** The proof is similar as in [M] §4.

**Definition:** A pair \((f, y)\) is called a generalized triangulation if

(i) \( f \) is a self-indexing generalized Morse function on \( M \) and in a neighbourhood \( U_r \) of any critical point \( c \), one can introduce local coordinates s.t. \( y = \delta_{ij} \) and
(a) if \( c \) is non-degenerate

\[
f(x) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2
\]  \hspace{1cm} (3.70)

(b) if \( c \) is of birth-death type

\[
f(x) = f(0) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + ax_n^3
\]  \hspace{1cm} (3.71)

(ii) \((f,g)\) satisfy the Morse-Smale condition.

Let \( f \) be a generalized Morse function, \( W_{x_j}^- \) be the descending manifold of a non-degenerate critical point. For a birth-death critical point \( y_j \), recall

\[
W_{y_j}^{0} = \{ x \in M | \gamma_x(t) \in U_{y_j} \cap \mathbb{R}^k \text{ for some } t \in \mathbb{R} \}
\]

where \( \gamma_x \) is the trajectory of Grad f s.t. \( \gamma_x(0) = x \}

\[
W_{y_j}^{-1} = \{ x \in M | \gamma_x(t) \in U_{y_j} \cap (\mathbb{R}^k \times \mathbb{R}_-) \text{ for some } t \in \mathbb{R} \}
\]

where \( \mathbb{R}^k \times \mathbb{R}_- = \{ (x_1, \ldots, x_k, 0, \ldots, 0, x_n) \in \mathbb{R}^n | x_n < 0 \} \}

Then we have

**Theorem 3.3.3:** Suppose \((f,g)\) is a generalized triangulation, then

(i) \( \{ W_{x_j}^-, W_{y_j}^{0}, W_{y_j}^{-1} \} \) is a CW-complex.

(ii) Let \((C_*(M, f), \partial)\) be the cellular chain complex of the above CW-complex (as described in §3.1), \((C^*(M, f), \delta)\) be its dual co-chain complex.

Then \( Int : (\Omega^*(M), d) \rightarrow (C^*(M, f), \delta) \)

\[
Int(\omega_k)(W^-) = \int_{W^-} \omega_k \quad \text{for } \omega_k \in \Omega^k(M), W^- \text{ of dimension } k
\]
is a morphism of cochain complexes.

**Proof:** A proof of this theorem in the case of a Morse function can be found in [L].

The same argument also works in the case of a generalized Morse function. However, a better argument for this is the following:

(i) One first verifies that the partition \( \{ W_{x_j}^+, W_{y_j}^-, W_{y_j}^{-,} \} \) is a stratification in the sense of Whitney (see [V2] for definition). Using the tubular neighbourhood theorem (Proposition 2.6 [V1]) and the fact that each stratum is diffeomorphic with an Euclidean space, one concludes that this partition is a CW-complex.

(ii) The fact that integration is well defined and represents a morphism of cochain complexes follows from Stokes theorem in the framework of integration theory on stratified sets (cf.[F],[V2]). □

As a consequence, the composition

\[
(\Omega^*(M, f, t), d(t)) \xrightarrow{\text{int}} (\Omega^*(M), d) \xrightarrow{\text{int}} (C^*(M, f), \delta)
\]

is also a morphism of cochain complexes.

Let

\[
M_{x_j} = M \setminus \left( \bigcup_{l \neq j} B(x_l^k, \eta) \right) \cup \left( \bigcup_l B(y_l^k, \eta) \right)
\]

(3.72)

where \( B(x_l^k, \eta) \) is the ball of radius \( \eta \) w.r.t the Agmon distance, centered at \( x_l^k \). Let \( \Delta_{M_{x_j}}(t) \) be the corresponding Laplace operator on \( M_{x_j} \) with Dirichlet boundary condition, \( \Psi_{x_j}(t) \) be an eigenvector corresponding to the smallest eigenvalue of \( \Delta_{M_{x_j}}^k(t) \) of norm one.

Similarly, let

\[
M_{y_j} = M \setminus \left( \bigcup_{l \neq j} B(x_l^k, \eta) \right) \cup \left( \bigcup_{l \neq j} B(y_l^k, \eta) \right)
\]

(3.73)
With $\Delta_{M,\epsilon}^k(t)$ similarly defined, let $\Psi_{y_j}^0(t)$, respectively $\Psi_{y_j}^1(t)$ be the smallest eigenvector of $\Delta_{M,\epsilon}^k(t), \Delta_{M,\epsilon}^{k+1}(t)$ of norm one.

Define $J_k(t) : C^k(M, f) \rightarrow \Omega^k(M)$ by

$$J_k(t) \left( e_{y_j}^i \right) = \Psi_{x_j}^i(t) \quad (3.74)$$

$$J_{k+1}(t) \left( e_{y_j}^i \right) = \Psi_{y_j}^i(t), \quad i = 0, 1 \quad (3.75)$$

where $\{ e_{y_j}^i, e_{y_j}^j \}$ is the dual basis of $\{ W_{x_j}^i, W_{y_j}^i \}$.

Let $Q_{k, \text{small}}(t), Q_{k+i, \text{large}}(t)$ be the orthogonal projection onto $\Omega_{\text{small}}^k(M, t)$ and $\Omega_{\text{large}, k+j}^{k+i}(M, t)$ respectively.

Define $Q_k(t) : J_k(t) \left( C^k(M, f) \right) \rightarrow \Omega^k_0(M, t)$ by

$$Q_k(t) \left( \Psi_{x_j}^i(t) \right) = Q_{k, \text{small}}(t) \left( \Psi_{x_j}^i(t) \right) \quad (3.76)$$

$$Q_{k+1}(t) \left( \Psi_{y_j}^i(t) \right) = Q_{k+i, \text{large}}(t) \left( \Psi_{y_j}^i(t) \right) \quad (3.77)$$

Let

$$H_k(t) = (Q_k(t) J_k(t))^* (Q_k(t) J_k(t)) \quad (3.78)$$

$$\tilde{J}_k(t) = Q_k(t) J_k(t) (H_k(t))^{-1/2} \quad (3.79)$$

Then $\tilde{J}_k(t) : C^k(M, f) \rightarrow \Omega^k_0(M, t)$ is an isometry.

Define $E_{x_j}^i(t) = \tilde{J}_k(t) \left( e_{x_j}^i \right), E_{y_j}^i(t) = \tilde{J}_k(t) \left( e_{y_j}^i \right)$.

Note that $E_{x_j}^i(t) \in \Omega_{\text{small}}^k(M, t), E_{y_j}^i(t) \in \Omega_{\text{large}, k+j}^{k+i}(M, t)$.

**Proposition 3.3.4** There exist neighbourhoods $U_{x_j}^k, U_{y_j}^k$ of $x_j^k, y_j^k$ contained in the chart of compatibility s.t.

$$E_{x_j}^i(t) = \left( \frac{2t}{\pi} \right)^{n/4} t^{-n/2} \left( dx_1 \wedge \ldots \wedge dx_k + O(t^{-1}) \right) \quad (3.80)$$
\[ E_{y_j}^0(t) = \left( \frac{2t}{\pi} \right)^{n-1/4} e^{-t(x_j^2 + \ldots + x_n^2)} |a|^{1/6} \Xi_1((at)^{1/3},x_n) \left( dx_1 \wedge \ldots \wedge dx_n + O(t^{-1}) \right) \]  

(3.81)

on \( U_{x_j} \) and \( U_{y_j} \) respectively.

**Remark:** Note that \( \left( \frac{2t}{\pi} \right)^{n-1/4} \) and \( |a|^{1/6} \) are the normalization constants for \( e^{-t(x_j^2 + \ldots + x_n^2)} \) and \( \Xi_1((at)^{1/3},x_n) \) respectively, i.e.

\[ || \left( \frac{2t}{\pi} \right)^{n-1/4} e^{-t(x_j^2 + \ldots + x_n^2)} || = || |a|^{1/6} \Xi_1((at)^{1/3},x_n) || = 1 \]  

(3.82)

**Proof:** One can follow the argument in [HS] or [BZ]. Note that the term \(|a|^{1/6} \Xi_1((at)^{1/3},x_n) \) is the ground state of \(-\frac{d^2}{dx^2} + 9a^2t^2x^4 - 6atx\). \( \square \)

Recall that we have defined \( \epsilon^{uw}_y \) for a generalized trajectory between two critical points and the incidence number \( I(x,y) \) between two critical points. See §3.1 for definitions. Also define \( Int_k = Int_{\mathbb{R}^k(M)} \). With these definitions, we have

**Proposition 3.3.5** (i)

\[ Int_k e^{IJ}(E_{y_j}(t)) = \left( \frac{2t}{\pi} \right)^{n-2k} e^{dk} \left( c_{x_j} + \sum I(y_j^k,x_j^k) c_{y_j^k} + O(t^{-1}) \right) \]

(ii)

\[ Int_k e^{IJ}(E_{y_j}^0(t)) = \left( \frac{2t}{\pi} \right)^{n-2k} e^{f(y_j^k)} \Xi_1(0) |a_j^{(k)}| t^{1/6} \left( c_{y_j^k} + \sum I(y_j^k,x_j^k) c_{y_j^k} + O(t^{-1}) \right) \]

(iii)

\[ Int_{k+1} e^{IJ}(E_{y_j}^1(t)) = \left( \frac{2t}{\pi} \right)^{n-2k} e^{f(y_j^k)} \Xi_1(0) |a_j^{(k)}| t^{1/6} \left( \delta \left( c_{y_j^k} + \sum I(y_j^k,x_j^k) c_{y_j^k} \right) + O(t^{-1}) \right) \]

**Proof:** We introduce the following notations. Let \( y_1^k, \ldots, y_{m_k}^k \) be all the birth-death critical points of index \( k \). Let

\[ f(y_1^k) = \ldots = f(y_r^k) < f(y_{r+1}^k) = \ldots = f(y_{r+r_2}^k) < f(y_{r+r_2+1}^k) < \ldots < \]

\[ < f(y_{r+r_2+\ldots+r_k-1}^k) = \ldots = f(y_{r+r_2+\ldots+r_k}) \]
where \( m_{k'} = r_1 + \ldots + r_k \).

Also, for \( 1 \leq q \leq l_k \) let

\[
E^{(k)}_{q} = \{ y^k \mid r_1 + \ldots + r_{q-1} + 1 \leq l \leq r_1 + \ldots + r_{q-1} + r_q \}
\]  
(3.83)

(i) It is clear from [HS],[BZ] that

\[
\int_{W_{x_j}} e^{iJ} E_{x_j}^k(t) = \left( \frac{2t}{\pi} \right)^{n-k} e^{ik} \left( 1 + O(t^{-1}) \right)
\]  
(3.84)

Suppose \( y^k_j \in L^{(k)} \), let

\[
\partial W_{y^k_j}^{-1} = W_{y^k_j}^{-0} + \sum \left( y^k_i, x^k_j \right) W_{x^k_i}^{-1} + \sum \left( y^k_i, y^k_i, y^k_i, \ldots \right) W_{y^k_i}^{-1}
\]  
(3.85)

where \( \left( y^k_i, x^k_j \right) \) is the (ordinary) incidence number between \( y^k_i \) and \( x^k_j \) defined in §3.1.

Denote the expression inside the parenthesis by \( R \), the remainder term.

Therefore,

\[
\int_{W_{y^k_j}^{-1}} e^{iJ} E_{x_j}^k(t) = \int_{W_{y^k_j}^{-0}} e^{iJ} E_{x_j}^k(t) + i \left( y^k_i, x^k_j \right) \int_{W_{x^k_i}^{-1}} e^{iJ} E_{x_j}^k(t) + \int_{R} e^{iJ} E_{x_j}^k(t)
\]  
(3.86)

But by Stoke’s Theorem,

\[
\int_{W_{y^k_j}^{-1}} e^{iJ} E_{x_j}^k(t) = \int_{W_{y^k_j}^{-1}} e^{iJ} \left( d(t) E_{x_j}^k(t) \right) = \int_{W_{y^k_j}^{-1}} e^{iJ} \left( \sum \lambda_i(t) E_{x_i}^{k+1}(t) \right)
\]  
(3.87)

for some exponentially decaying functions \( \lambda_i(t) \). Since \( |E_{x_i}^{k+1}(t)(x)| \) decreases as \( e^{-t|f(x)-f(x_{x_i}^{k+1})|} \), \( \int_{W_{y^k_j}^{-1}} e^{iJ} E_{x_j}^k(t) \) is of smaller order compared with \( \int_{W_{x_j}^{-1}} e^{iJ} E_{x_j}^k(t) \). The same is true for \( \int_{R} e^{iJ} E_{x_j}^k(t) \).

Hence,

\[
\int_{W_{y^k_j}^{-0}} e^{iJ} E_{x_j}^k(t) = -i \left( y^k_i, x^k_j \right) \int_{W_{x^k_i}^{-1}} e^{iJ} E_{x_j}^k(t) + O(t^{-1})
\]

\[
= I \left( y^k_i, x^k_j \right) \left( \frac{2t}{\pi} \right)^{n-k} e^{ik} + O(t^{-1})
\]  
(3.88)
One can show by using finite induction on \( q \) that for any \( y^k_j \in L^{(k)}_q \)

\[
\int_{\mathcal{W}_{-t}^0} e^{tJ} E^{0}_{y_j^k}(t) = I(y^k_j, x_j^k)(\frac{2t}{\pi})^{\frac{n-2k}{4}} e^{t^{1/6}} + O(t^{-1}) \tag{3.89}
\]

This proves (i).

(ii) A direct computation shows that

\[
\int_{\mathcal{W}_{-t}^0} e^{tJ} E^{0}_{y_j^k}(t) = \left(\frac{2t}{\pi}\right)^{\frac{n-1}{4}} e^{t^{1/6}} \Xi(0) \left| a_j^{(k)} t \right|^{1/6} \left(1 + O(t^{-1})\right) \tag{3.90}
\]

Next note that by choosing a coordinate system \((x_1^{(j)}, \ldots, x_k^{(j)})\) on \( W_{-t}^+, y^k_j \) and extending it to a neighbourhood of \( W_{-t}^+, y^k_j \), one can show as in \( [HS]\) p 276-8 that if \( x \neq y^k_j \),

\[
E^{0}_{y_j^k}(t) = \left(\frac{2t}{\pi}\right)^{\frac{n-1}{4}} |a t|^{1/6} e^{-t d(x, y^k_j)} \left( d_{x_1^{(j)}} \wedge \ldots \wedge d_{x_k^{(j)}} + O(t^{-1})\right) \tag{3.91}
\]

where \( d(x, y^k_j) \) is the Agmon distance between \( x \) and \( y^k_j \), i.e. w.r.t. the metric \( |df|^2 \, dg \), but it is not necessarily true for \( x = y^k_j \).

So we let

\[
E^{0}_{y_j^k}(t)(y^k_j) = \left(\frac{2t}{\pi}\right)^{\frac{n-1}{4}} |a t|^{1/6} e^{-t d(y^k_j, y^k_j)} c_{ij}(t) \tag{3.92}
\]

where \( c_{ij}(t) \in \Lambda^k(T^{\epsilon}_{y^k_j}(M)). \) By \([HS]\),

\[
|c_{ij}(t)| = O(e^{\epsilon t}) \text{ for any } \epsilon > 0 \tag{3.93}
\]

For \( x \in U_{y^k_j} \cap W_{-t}^0 \),

\[
f(x) = f(y^k_j) - (x_1^{(j)})^2 - \ldots - (x_k^{(j)})^2 \tag{3.94}
\]

Therefore,

\[
\int_{\mathcal{W}_{-t}^0} e^{tJ} E^{0}_{y_j^k}(t) = e^{t f(y^k_j)} \int_{U_{y^k_j} \cap \mathcal{W}_{-t}^0} e^{-t \left((x_1^{(j)})^2 + \ldots + (x_k^{(j)})^2\right)} E^{0}_{y_j^k}(t) + \text{lower order terms}
\]
By the stationary phase approximation formula ([D] p23–4), we have
\[ I_{W^{a}} (x) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} |\alpha|^{\frac{1}{2}} c^{-t/2k} e^{-i\alpha(x') \cdot \beta(x')} + \text{lower order terms} \]

\[ = \frac{1}{(2\pi)^{\frac{n-1}{2}}} |\alpha|^{\frac{1}{2}} c^{-t/2k} e^{if(y')c_{ij}(t)} e^{(i/\xi_{ij})} + \text{lower order terms} \]

since \( d(y^{k}, y'_{j}) = f(y^{k}) - f(y'_{j}) \) and for some \( \beta_{ij}(t) \). Hence (ii) is proved.

Note that (cf. [HS] p265, [HS1] p138)
\[ | \beta_{ij}(t) | = O(e^{-t}) \text{ for any } \epsilon > 0 \] (3.95)

(iii) Since \( d(t)E_{y}^{0}(t) = \lambda(t)E_{y}^{0}(t) \) for some \( \lambda(t) \neq 0 \) when \( t \) is sufficiently large (this follows from (3.105) below) and
\[ Int_{k+1} \left( e^{iF}d(t)E_{y}^{0}(t) \right) = \delta \left( Int_{k}e^{iF}E_{y}^{0}(t) \right) \] (3.96)

by (ii), we have
\[ Int_{k+1} e^{iF} \left( E_{y}^{0}(t) \right) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} e^{if(y')c_{ij}(t)} \lambda(t) \left( \delta(t, \beta_{ij}(t)c_{ij}^{0} + O(t^{-1})) \right) \]

(iii) is proved by noting that
\[ \lambda(t) = \sqrt{c_{ij} |\alpha|^{1/2}} + \text{lower order terms} \]

Using Proposition 3.3.5(i), we can prove

**Theorem 4':** There exist orthonormal bases \( \left\{ E_{y}^{0}(t) \right\} \) of \( \Omega_{\text{small}}^{k}(M, t) \), \( \left\{ E_{y}^{0}(t) \right\} \) of \( \Omega_{\text{large}, k, j}(M, t) \) s.t.
\[ < E_{y}^{k+1}(t), (t)E_{y}^{0}(t) > = e^{-t} \left( \sqrt{\frac{t}{\pi}} \sum_{e_{\gamma}^{\text{even}}} + O(t^{-1/2}) \right) \]
\[ < E_{y}^{0}(t), (t)E_{y}^{0}(t) > = \sqrt{c_{1}^{(k)} t^{1/3}} + O(t^{1/6}) \] (3.97) (3.98)
Also for \( t \) sufficiently large, we have

\[ < E_{y_{j_1}}^{i_1}(t), d(t) E_{y_{j_2}}^{i_2}(t) > = 0 \text{ if } j_1 \neq j_2 \]  

(3.99)

\( \frac{\partial}{\partial t} \) and

\[ < E_{x_{j}}(t), d(t) E_{y_{j}}(t) > = < E_{y_{j}}(t), d(t) E_{x_{j}}(t) > = 0 \]  

(3.100)

where \( \sum_\gamma e_{n \gamma} = I(x_{i+1}^k, x_j) \) is the incidence number between \( x_{i+1}^k \) and \( x_j^k \) defined in §3.1.

**Proof:** We prove the first and second equalities, the others are obvious.

Let \( d(t) E_{x_{j}}(t) = \sum_i \lambda_{ij}(t) E_{x_{j+1}}(t) \) for some \( \lambda_{ij}(t) \).

Since

\[ \text{Int}_{k+1} \text{tr} \left( d(t) E_{x_{j}}(t) \right) = \delta \text{Int}_{k+1} \text{tr} \left( E_{x_{j}}(t) \right) \]  

(3.101)

By Proposition 3.3.5(i), we have

\[ \sum_i \lambda_{ij}(t) \left( \frac{2t}{\pi} \right)^{\frac{n-2k}{4}} e^{(l+1)} \left( c_{x_{j+1}} + \sum I(y_t^k, x_{j+1}) c_{y_t^k} + O(t^{-1}) \right) \]

(3.102)

By comparing the coefficients of \( c_{x_{j+1}} \),

\[ \lambda_{ij}(t) \left( \frac{2t}{\pi} \right)^{-1/2} e^l = \delta_{ij} + \sum I(y_t^k, x_{j+1}) c_{y_t^k} + O(t^{-1}) \]  

(3.103)

But by definition of \( I(x, y) \),

\[ I(x_{i+1}^k, x_j^k) = \delta_{ij} + \sum I(y_t^k, x_{j+1}) c_{y_t^k} = \sum \varepsilon_{n \gamma} \]  

(3.104)

Hence the first equality is proved. For the second equality, note that \( E_{y_{j}}^i(t) \) are normalized eigenforms in \( \Omega_{\text{large}, k, j}^i(M, t) \) and

\[ ||d(t) E_{y_{j}}^i(t)||^2 = \varepsilon_{y_{j}}^0(t), \Delta^k(t) E_{y_{j}}^i(t) > = c_1 | a_j^{(k)} | t^{2/3} + \text{lower order terms} \]  

(3.105)
So,
\[ < c_{y_j^k}^1(t), d(t) E_{y_j^k}^0(t) > = \pm \sqrt{2} |a_j^{(k)}| t^{1/3} + \text{lower order terms} \]

In view of Proposition 3.3.5, define \( f^k(t) : \Omega^k_0(M, t) \to C^k(M, f) \) s.t.
\[
\begin{align*}
  f^k(t) \left( E_{x_j^k}^0(t) \right) &= \left( \frac{\pi}{2t} \right)^{n-k} e^{-k t} \text{Int}_k c^{(f)} \left( E_{x_j^k}^0(t) \right) \\
  f^k(t) \left( E_{y_j^k}^0(t) \right) &= \text{Int}_k c^{(f)} \left( E_{y_j^k}^0(t) \right) \\
  f^{k+1}(t) \left( E_{y_j^k}^1(t) \right) &= \text{Int}_{k+1} c^{(f)} \left( E_{y_j^k}^1(t) \right)
\end{align*}
\]

Let
\[
\left( \Omega^*_0(M, t), \tilde{d}(t) \right) = \left( \Omega^*_\text{small}(M, t), c^l(\pi/2t)^{1/2}d(t) \right) \\
\left( \\text{subject to} \right) \left( \Omega^*_\text{ar平坦,k-j}(M, t), d(t) \right)
\]

Also define
\[
\begin{align*}
  \tilde{c}_{x_j^k} &= c_{x_j^k} + \sum_i I(x_i^k, y_j^k) c_{y_i^k}^0 \\
  \tilde{c}_{y_j^k} &= c_{y_j^k}^0 \\
  \tilde{c}_{y_j^k}^1 &= \delta(c_{y_j^k}^0)
\end{align*}
\]

Then we have

**Proposition 3.3.6** \( f^*(t) : (\Omega^*_0(M, t), \tilde{d}(t)) \to (C^*(M, f), \delta) \) is a morphism of co-chain complexes s.t.
\[
f^k(t) \left( E_{x_j^k}^0(t) \right) = \tilde{c}_{x_j^k} + O(t^{-1}) \quad (3.106)
\]

with \( f^k(t) \left( E_{y_j^k}^0(t) \right), f^{k+1}(t) \left( E_{y_j^k}^1(t) \right) \) given by (ii) and (iii) in Proposition 3.3.5.

Let the matrix associated to the linear map \( f^k(t) \) w.r.t. the bases
\[
\{ E_{x_j^k}(t), E_{y_j^k}^0(t), E_{y_j^k}^1(t) \} \quad \text{and} \quad \{ \tilde{c}_{x_j^k}, c_{y_j^k}^0, c_{y_j^k}^1 \}
\]

be
\[
F^k(t) = \begin{pmatrix}
  I & O(\epsilon^k t) & N_{k_1}^k(t) \\
  O(t^{-1}) & M^k(t) & N_{k_2}^k(t) \\
  O(t^{-1}) & O(\epsilon^k t) & N_{k_3}^k(t)
\end{pmatrix} \quad (3.107)
\]
where $O(t^{-1})$ in a certain entry of the matrix means that the corresponding entry is of the order $O(t^{-1})$. Here $M^k(t)$ is \((2\pi)^{-\frac{1}{4}} \Xi_1(0)\) times the following matrix

\[
\begin{pmatrix}
| a_1^{(k)} t | \frac{1}{6} e^{f(y_1^k)} & | a_2^{(k)} t | \frac{1}{6} e^{f(y_2^k)} \beta_{32}(t) & \cdots & | a_m^{(k)} t | \frac{1}{6} e^{f(y_m^k)} \beta_{m_{m_k}^r}(t) \\
| a_1^{(k)} t | \frac{1}{6} e^{f(y_1^k)} \beta_{21}(t) & | a_2^{(k)} t | \frac{1}{6} e^{f(y_2^k)} & \cdots & | a_m^{(k)} t | \frac{1}{6} e^{f(y_m^k)} \beta_{m_{m_k}^r}(t) \\
\vdots & \vdots & \ddots & \vdots \\
| a_1^{(k)} t | \frac{1}{6} e^{f(y_1^k)} \beta_{m_{m_k}^r-1}(t) & | a_2^{(k)} t | \frac{1}{6} e^{f(y_2^k)} \beta_{m_{m_k}^r-2}(t) & \cdots & | a_m^{(k)} t | \frac{1}{6} e^{f(y_m^k)} \beta_{m_{m_k}^r}(t)
\end{pmatrix}
\]

Note that we have used the fact that the birth-death points are indexed such that

\[f(y_1^k) \leq f(y_2^k) \leq \cdots \leq f(y_{m_k}^k)\]

so that the above matrix is approximately lower triangular. Hence, $M^k(t)$ is invertible for sufficiently large $t$.

Also, let

\[A^k(t) = \text{diag} \left( \left( \sqrt{c_1} \right) | a_1^{(k)} t |^{1/3} \right)^{-1}, \ldots, \left( \sqrt{c_1} \right) | a_{m_k}^{(k)} t |^{1/3} \right)^{-1} \right) \quad (3.108)\]

Define for $1 < j < m_k$,

\[
\hat{E}^0_{y_j^k}(t) = \left( M^k(t) \right)^{-1} \left( E^0_{y_j^k}(t) \right) \quad (3.109)
\]

\[
\hat{E}^1_{y_j^k}(t) = \left( M^k(t)A^k(t) \right)^{-1} E^1_{y_j^k}(t) \quad (3.110)
\]

Observe that $\{ \hat{E}^j_{y_j^k}(t) \}$ is approximately orthogonal whose elements are still localized at the corresponding birth-death points.

Let

\[
B^k(t) = \begin{pmatrix}
I & 0 & 0 \\
0 & \left( M^k(t) \right)^{-1} & 0 \\
0 & 0 & \left( M^{k-1}(t)A^{k-1}(t) \right)^{-1}
\end{pmatrix} \quad (3.111)
\]
Then the matrix associated to $f^k(t)$ w.r.t. the new bases

$$\left\{ E_{x_j^k}(t), \tilde{E}_{y_{j_k}}^0(t), \tilde{E}_{y_{j_k}}^1(t) \right\} \text{ and } \left\{ \tilde{v}_{x_j^k}, e_{y_{j_k}}^0, e_{y_{j_k}}^1 \right\}$$

is

$$F^k(t)B^k(t) = \begin{pmatrix} I & O(t^k) & N_k^k(t) \\ O(t^{-1}) & M_k^k(t) & N_k^0(t) \\ O(t^{-1}) & O(t^k) & N_k^k(t) \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & (M_k(t))^{-1} & 0 \\ 0 & 0 & (M_k^{-1}(t)A^{-1}(t))^{-1} \end{pmatrix}$$

Suppose that

$$\delta^{(k-1)} = \begin{pmatrix} \delta_{11}^{(k-1)} & \delta_{12}^{(k-1)} & \delta_{13}^{(k-1)} \\ \delta_{21}^{(k-1)} & \delta_{22}^{(k-1)} & \delta_{23}^{(k-1)} \\ \delta_{31}^{(k-1)} & \delta_{32}^{(k-1)} & \delta_{33}^{(k-1)} \end{pmatrix}$$

w.r.t. the bases $\left\{ \tilde{v}_{x_j^k}, e_{y_{j_k}}^0, e_{y_{j_k}}^1 \right\}$.

Then,

$$F^k(t)B^k(t) = \begin{pmatrix} I & O(t^k) & \delta^{(k-1)} + O(t^{-1}) \\ O(t^{-1}) & I & \delta^{(k-1)} + O(t^{-1}) \\ O(t^{-1}) & O(t^{-1}) & \delta^{(k-1)} + O(t^{-1}) \end{pmatrix}$$

To see this, it suffices to show

$$\text{Int}_{k^I}(E_{y_{j_{k-1}}}^1(t)) = \delta(e_{y_{j_{k-1}}}^0) + O(t^{-1})$$

But by Proposition 3.3.5(iii),

$$\text{Int}_{k^I}(E_{y_{j_{k-1}}}^1) = \delta \left( \sum_t (M_k^{-1}(t)A_k^{-1}(t))_{ij} e_{y_{j_{k-1}}}^0 + O(t^{-1}) \right)$$

Using the definition of $\tilde{E}_{y_{j_{k-1}}}^1(t)$, (3.114) follows.

Hence, with the definition of $\tilde{v}_{y_{j_k}}$ on p111, we finally have

**Theorem 4:** $f^*(t) : (\Omega_0(M,t), d(t)) \rightarrow (C^*(M,f), \delta)$ is a morphism of cochain complexes such that

$$f^*(t) = I + O(t^{-1})$$
w.r.t. the bases \( \{ E_{x_i}'(t), \dot{E}_{y_j}'(t) \} \) and \( \{ \dot{c}_x, \dot{c}_y \} \).

Inside \((C^*(M, f), \delta)\), there is a subcomplex \((C^*_{nd}(M, f), \delta)\) such that

\[
\dim C^k_{nd}(M, f) = m_k
\]

where \( m_k \) is the number of non-degenerate critical points of index \( k \). This subcomplex can be obtained by application of the following Lemma.

**Lemma 3.3.7** Suppose \((C^*, \delta)\) is a cochain complex such that

\[
C^* = \begin{cases} 
\hat{C}^* & \text{if } * \neq k, k + 1 \\
\hat{C}^* \oplus \mathbb{R} & \text{if } * = k \text{ or } k + 1
\end{cases}
\]

Let \( \dim(\hat{C}^q) = n_q \), for \( 0 \leq q \leq n \),

\[
\{ c_{x_1}, \ldots, c_{x_{n_q}} \} \text{ be a basis of } \hat{C}^q
\]

so that

\[
\{ c_{x_1}, \ldots, c_{x_{k}}, c^0_y \} \text{ is a basis of } \hat{C}^k \oplus \mathbb{R}
\]

and

\[
\{ c_{x_1}, \ldots, c_{x_{k+1}}, c^1_y \} \text{ is a basis of } \hat{C}^{k+1} \oplus \mathbb{R}
\]

and w.r.t. the above bases,

\[
\delta^{(k)} = \begin{pmatrix}
\delta^{(k)}(x_1) & \cdots & \delta^{(k)}(x_k) \\
\vdots & \ddots & \vdots \\
\delta^{(k)}(x_{n_k}) & \cdots & \delta^{(k)}(x_{n_k}) \\
\end{pmatrix}
\]

Then with the following change of bases in \( C^k \) and \( C^{k+1} \),

\[
\{ c_x^0, c_y^0 \}_{1 \leq i \leq n_k} \quad \longrightarrow \quad \{ c_x^i - \delta x, c_y^0, c_y^0 \}_{1 \leq i \leq n_k}
\]
\[
\left\{ e_{r^k_{i+1}, e^1_y} \right\}_{1 \leq i \leq n_{k+1}} \longrightarrow \left\{ e_{r^k_{i+1}, \delta e^0_y} = e^1_y + \sum_{j=1}^{n_{k+1}} i(y, x^k_j, y) c_{r^k_{j+1}} \right\}_{1 \leq i \leq n_{k+1}}
\]

we have

(i)
\[
\delta^{(k)} = \begin{pmatrix}
\delta^{(k+1)} & \delta^{(k)} & \cdots & \delta^{(k)} \\
0 & \delta^{(k+1)} & \cdots & \delta^{(k)} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \delta^{(k+1)}
\end{pmatrix}
\]

where \( i'(x^k_{i+1}, x_j^k) = i(x^k_{i+1}, x_j^k) - i(x^k_{i+1}, y)i(y, x_j^k) \).

(ii)
\[
\delta^{(k-1)} = \begin{pmatrix}
\delta^{(k+1)} & \delta^{(k)} & \cdots & \delta^{(k)} \\
0 & \delta^{(k+1)} & \cdots & \delta^{(k)} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \delta^{(k+1)}
\end{pmatrix}
\]

Corollary 3.3.8: Let
\[
(C^*)^* = \begin{cases}
C^* & \text{if } * \neq k \\
\text{Span}\{ e_{r^k_{i+1}, e^0_y} \}_{1 \leq i \leq n_k} & \text{if } * = k
\end{cases}
\]

\[
(C''^*)^* = \begin{cases}
0 & \text{if } * \neq k, k + 1 \\
e_{r^0_y} & \text{if } * = k \\
\delta e^0_y & \text{if } * = k + 1
\end{cases}
\]

Then
\[
(C^*, \delta) = \left((C^*)^*, \delta \right) \oplus \left((C''^*)^*, \delta \right)
\]

In particular, \( (C^*, \delta) \) is a subcomplex of \( (C^*, \delta) \). Both of them calculate the same cohomology.
Remark: For the application of Lemma 3.3.7, it is clear that $\hat{C}^*$ need not be generated only by cells corresponding to non-degenerate critical points, but it can also be generated by cells corresponding to birth-death points. Let

$$k < f(y_1^k) \leq \ldots \leq f(y_m^k) < k + 1$$

We eliminate $y_1^k$ first by applying Lemma 3 and obtain a subcomplex $\left((C^{(1)})^*, \delta \right)$. Note that

$$\begin{cases} \epsilon_{y_2^k}^0 = \epsilon_{y_2^k}^0 - i(y_1^k, y_2^k) \epsilon_{y_2^k}^0 \in (C^{(1)})^k \\ \epsilon_{y_2^k}^1 \in (C^{(1)})^{k+1} \\ \delta(\epsilon_{y_2^k}^0) = \epsilon_{y_2^k}^1 + \ldots \end{cases}$$

Therefore, the assumptions in Lemma 3.3.7 are satisfied and we can apply Lemma 3.3.7 to eliminate $y_2^k$ and obtain $\left((C^{(2)})^*, \delta \right)$. Hence by applying Lemma 3.3.7 repeatedly, $(C^*_{nd}(M, f), \delta)$ is obtained.

Proof of Lemma 3.3.7: The lemma can be proved by direct calculation.

Theorem 4': $f^*(t) \mid_{\nu_{\text{small}}(M, t)}: \left(\Omega^*_{\text{small}}(M, t), \tilde{d}(t)\right) \longrightarrow (C^*(M, f), \delta)$ is an injective homomorphism of cochain complexes whose image complex converges to $(C^*_{nd}(M, f), \delta)$ in $(C^*(M, f), \delta)$ as $t \rightarrow \infty$, more precisely,

$$f^k(f) \left(E_{s_j^k}(t)\right) = \hat{c}_{s_j^k} + O(t^{-1}) \text{ in } C^*(M, f) \quad (3.121)$$

Remark: One can show by induction that

$$\hat{c}_{s_j^k} = c_{s_j^k} + \sum_l I(y_l^k, x_j^k) \epsilon_{y_l^k}^0 \in C^k_{nd}(M, f) \quad (3.122)$$
Appendix A

Appendix to §3.3

Recall that we introduced $\beta_{ij}(t)$ in Proposition 3.3.5 and we claimed that

**Proposition A1:**

$$| \beta_{ij}(t) | = O(e^{\epsilon t}) \quad \text{for any } \epsilon > 0 \quad (A.1)$$

It is the purpose of this Appendix to prove the above assertion. For simplicity, we shall prove the corresponding assertion for the operator

$$P(h) = h^2 \Delta + V(x) : C^\infty(M) \to C^\infty(M) \quad (A.2)$$

where $V(x) = | df(x) |^2$ and $f$ is a generalized Morse function on $M$.

Let $\{ x \in M \mid V(x) = 0 \} = \{ u_j \}_{1 \leq j \leq r}$ where $r$ is the number of critical points (degenerate and non-degenerate) of $f$. Also, let $m$, respectively $m'$, be the number of non-degenerate, respectively degenerate, critical points of $f$.

Let $M_j = M \setminus \bigcup_{k \neq j} B(u_k, \eta)$.

Let $P_{M_j}(h)$ be the Dirichlet realization of $P(h)$ on the manifold with boundary $M_j$, $\Phi_j$ be the normalized eigenvector corresponding to the smallest eigenvalue $\mu_j$ of $P_{M_j}(h)$. 
Let $\theta_j \in C^\infty(M)$ be s.t. $\text{supp} \ \theta_j \subset B(u_j, 2\eta)$ and it is equal to 1 in $B(u_j, \eta)$, and let $\chi_j = 1 - \sum_{k \neq j} \theta_k$.

Let $\Psi_j = \chi_j \Phi_j$ and let

$$P\Psi_j = \mu_j \Psi_j + r_j$$  \hspace{1cm} (A.3)

for some $r_j$.

Let $0 \leq \lambda_1(h) \leq \lambda_2(h) \leq \cdots$ be all the eigenvalues of $P(h)$. Then

$$0 \leq \max_{1 \leq j \leq m} \{\lambda_j(h)\} \leq C e^{-t}$$  \hspace{1cm} (A.4)

for some $C > 0$ and for $t$ sufficiently large. By the argument of [S] p219-222,

$$0 < C_1 h^{1/3} < \min_{m+1 \leq j \leq m+m'} \{\lambda_j(h)\} \leq \lambda_j(h) \leq \max_{m+1 \leq j \leq m+m'} \{\lambda_j(h)\}$$

$$< C_2 h^{1/3} < C_3 h^{1/3} < \lambda_{m+m'+1}(h) \leq \cdots$$  \hspace{1cm} (A.5)

Let $F$ be the direct sum of the eigenspaces of $P(h)$ corresponding to the eigenvalues $\{\lambda_j(h)\}_{m+1 \leq j \leq m+m'}$, $\Pi_F$ be the associated orthogonal projection onto $F$, and

$$v_j = \Pi_F \Psi_j$$

Let $\Gamma(h)$ be a contour enclosing $\{\lambda_j\}_{m+1 \leq j \leq m+m'}$ s.t.

$$C_1 h^{1/3} < \min_{z \in \Gamma(h)} \{Re \ z\} < \max_{z \in \Gamma(h)} \{Re \ z\} < C_2 h^{1/3}$$  \hspace{1cm} (A.6)

and for any $z \in \Gamma(h)$, $\epsilon > 0$, there exists $C' \epsilon > 0$ s.t.

$$\text{dis}(z, \text{Spec}(P)) \geq C' \epsilon e^{-\frac{h}{\epsilon}}$$  \hspace{1cm} (A.7)

which implies for any $\epsilon > 0$,

$$\|(P - z)^{-1}\| = O(e^{-\frac{h}{\epsilon}})$$  \hspace{1cm} (A.8)
Then one can easily show that

\[ v_j = \Psi_j + \int_{\Gamma(h)} \frac{1}{\mu_j - z} (P - z)^{-1} r_j dz \]  

(A.9)

**Lemma A2** For any \( z \in \Gamma(h) \), let \( v_j^z = (P - z)^{-1} r_j \). Then

\[ v_j^z = \mathcal{O}(e^{-\frac{1}{k} \delta_j(x)}) \]  

(A.10)

where \( \delta_j(x) = \inf_{k \neq j} \{ d(x, u_k) + d(u_k, u_j) \} \).

Before we prove the above lemma, we need the following

**Definition:** Let \( I \subset (0, 1] \) be s.t. \( 0 \in \bar{I} \) and let \( \{ A_h : L^2(M) \to H^1(M) \}_{h \in I} \) be a family of operators. Let \( f \in C^0(M \times M) \). Then we say that

the kernel \( A_h(x, y) \) of \( A_h \) is \( \mathcal{O}(e^{-\frac{1}{k} f(x, y)}) \)

if for any \( x_0, y_0 \in M, \epsilon > 0 \), there exists neighbourhoods \( U \) of \( x_0 \), \( V \) of \( y_0 \) and constant \( C, \epsilon > 0 \) s.t.

\[ \| A_h u \|_{H^1(V)} \leq C e^{-\frac{1}{k} f(x_0, y_0)} \| u \|_{L^2(U)} \]

(A.11)

for all \( h \in I, u \in L^2(U) \) with \( \text{supp} u \subset U \).

**Lemma A2'** For any \( z \in \Gamma(h) \), the kernel of \( (P - z)^{-1} \) is \( \mathcal{O}(e^{-\frac{1}{k} d(x, y)}) \).

**Proof:** Recall the following “energy inequality” (cf. [HS] p):

\[ Re < (P_M - z)u, e^{\frac{2\pi i}{h}} u > = h^2 \| \nabla (e^{\frac{2\pi i}{h}} u) \|^2 + \int_M (V - Rez - | \nabla \phi |^2) e^{\frac{2\pi i}{h}} u \bar{u} dx \]  

(A.12)

Here \( P_M \) represents the Dirichlet realization of \( P \) if \( M \) is a manifold with boundary and \( u \) is a function s.t. \( u |_{\partial M} = 0 \). In our case \( \partial M = \emptyset, P_M = P \) and (1) holds for any \( u \in C^2(M) \).
Now, let \( x_0, y_0 \in M, \phi(x) = d(x_0, x) \), then \( V(x) = |\nabla \phi(x)|^2 \).

It follows from (1) that

\[
\|(P - z)u\|\|c^{\frac{2\phi}{h}}u\| \geq Rc < (P - z)u, c^{\frac{2\phi}{h}}u > = h^2\|\nabla (e^{\frac{\phi}{h}}u)\|^2 - Rcz \int_M e^{\frac{2\phi}{h}}u\phi dx \quad (A.13)
\]

Let \( U = B(x_0, \epsilon) \) and \( u \) be s.t. \( \text{supp } u \subset U \).

Then

\[
\|(P - z)\|\|c^{\frac{2\phi}{h}}u\| \geq h^2\|\nabla (e^{\frac{\phi}{h}}u)\|^2 - Rcz \int_M e^{\frac{2\phi}{h}}u\phi dx \quad (A.14)
\]

But from (1), we have

\[
\|(P - z)^{-1}u\| \leq B \|c^{\frac{2\phi}{h}}u\| \text{ for some } B, \epsilon > 0 \quad (A.15)
\]

which implies

\[
\|u\| \leq B \|c^{\frac{2\phi}{h}}\|(P - z)u\| \quad (A.16)
\]

Therefore

\[
B \|c^{\frac{2\phi}{h}}\|(P - z)u\|^2 \geq h^2\|\nabla (e^{\frac{\phi}{h}}u)\|^2 - Rcz \int_M e^{\frac{2\phi}{h}}u\phi dx \quad (A.17)
\]

But

\[
\|\nabla (e^{\frac{\phi}{h}}u)\|^2 = \|e^{\frac{\phi}{h}}u\|^2 + \frac{1}{h^2}\|e^{\frac{\phi}{h}}u\nabla \phi\|^2 + \frac{2}{h} < e^{\frac{\phi}{h}}u, e^{\frac{\phi}{h}}u\nabla \phi > \quad (A.18)
\]

and

\[
\frac{2}{h} < e^{\frac{\phi}{h}}u, e^{\frac{\phi}{h}}u\nabla \phi > = \frac{1}{h} \int_M e^{\frac{2\phi}{h}}\nabla \phi \nabla u^2 dx = \frac{1}{2} \int_M \nabla (e^{\frac{2\phi}{h}}) \nabla u^2 dx = \frac{1}{2} \int_M u^2 \Delta (e^{\frac{2\phi}{h}}) dx \quad (A.19)
\]

where we have applied the Green's formula in the last equality.

Since \( \Delta (e^{\frac{2\phi}{h}}) = -(\frac{2}{h})^2 e^{\frac{2\phi}{h}} |\nabla \phi|^2 + \frac{2}{h} e^{\frac{2\phi}{h}} \Delta \phi \), therefore

\[
\frac{2}{h} < \phi h \nabla u, e^{\frac{2\phi}{h}}u \nabla \phi > = \frac{2}{h^2}\|e^{\frac{2\phi}{h}}u\nabla \phi\|^2 + \frac{1}{h} \int_M e^{\frac{2\phi}{h}}(\Delta \phi) u^2 dx \quad (A.20)
\]
Hence

\[\|\nabla (e^\frac{x}{h} u)\|^2 = \|e^\frac{x}{h} \nabla u\|^2 - \frac{1}{h^2} \|e^\frac{x}{h} u \nabla \phi\|^2 + \frac{1}{h} \int_I e^{\frac{2x}{h}} (\Delta \phi) u^2 \, dx\]  
(A.21)

Let \( C = \max_{x \in M} |\nabla \phi(x)| + h \max_{x \in M} |\Delta \phi(x)| + |Rc\, z| \). Therefore we have

\[Bc^\frac{x}{h} \|(P - z)u\|^2 \geq h^2 \|e^\frac{x}{h} \nabla u\|^2 - C\|e^\frac{x}{h} u\|^2\]  
(A.22)

That is we have

\[h^2 \|e^\frac{x}{h} \nabla u\|^2 \leq Bc^\frac{x}{h} \|(P - z)u\|^2 + C\|e^\frac{x}{h} u\|^2\]  
(A.23)

However

\[\|e^\frac{x}{h} u\|^2 \leq e^\frac{x}{h} \|u\|^2 \leq B^2 c^\frac{x}{h} \|(P - z)u\|^2\]  
(A.24)

Hence

\[h^2 \|e^\frac{x}{h} \nabla u\|^2 \leq Bc^\frac{x}{h} \|(P - z)u\|^2 + CB^2 e^{\frac{x}{h}} \|(P - z)u\|^2\]  
(A.25)

Therefore

\[h^2 \|e^\frac{x}{h} \nabla u\|^2_{L^2(V)} \leq Bc^\frac{x}{h} \|(P - z)u\|^2\]  
\[\Rightarrow h^2 e^{\frac{2(d(\ell, u_0) - 1)}{h}} \|\nabla u\|^2_{L^2(V)} \leq Bc^\frac{x}{h} \|(P - z)u\|^2\]  
(A.26)

\[\Rightarrow \|\nabla u\|^2_{L^2(V)} \leq Bc^\frac{2(d(\ell, u_0) + \ell_0)}{h} \|(P - z)u\|^2\]  
(A.27)

Similarly, we have

\[\|u\|^2_{L^2(V)} \leq Bc^\frac{2(d(\ell, u_0) + \ell_0)}{h} \|(P - z)u\|^2\]  
(A.28)

Hence

\[\|u\|^2_{H^1(V)} \leq Cc^{\frac{2(d(\ell, u_0) + \ell_0)}{h}} \|(P - z)u\|^2_{L^2(V)}\]  
(A.29)

which implies

\[\|(P - z)^{-1} u\|_{H^1(V)} \leq Cc^{-\frac{d(\ell, u_0) - 1}{h}} \|u\|_{L^2(V)}\]  
(A.29)
This proves Lemma A2'. □

**Proof of Lemma A2:** Since \( r_j(x) = \tilde{O}\left(e^{-\frac{1}{h} \delta_j(x)}\right)\), by Lemma A2'.

\[
v_j^*(x) = (P - z)^{-1} r_j = \tilde{O}\left(\left.e^{-\frac{1}{h} \delta_j(x)}\right|_{\inf(d(x,y) + \delta_j(y))}\right)
\]  
(A.30)

Since \( \delta_j(x) \leq d(x, y) + \delta_j(y) \),

\[
v_j^*(x) = \tilde{O}\left(\left.e^{-\frac{1}{h} \delta_j(x)}\right|_{\inf(d(x,y) + \delta_j(y))}\right)
\]  
(A.31)

This proves Lemma A2. □

**Proof of Proposition A1:** By Lemma A2,

\[
v_j(x) = \tilde{O}\left(\left.e^{-\frac{1}{h} \delta_j(x)}\right|_{\inf(d(x,y) + \delta_j(y))}\right)
\]  
(A.32)

In particular, we have

\[
v_j(u_j) = \tilde{O}\left(\left.e^{-\frac{1}{h} \delta_j(u_j)}\right|_{\inf(d(u_j, u_j))}\right) = \tilde{O}\left(\left.e^{-\frac{1}{h} d(u_j, u_j)}\right|_{\inf(d(x,y) + \delta_j(y))}\right)
\]  
(A.33)

This implies Proposition A1.
BIBLIOGRAPHY


[L] Laudenbach,F.: Appendix in [BZ].


